MATH FOR EDUCATORS (MATH 010A AND 010B CID120)



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1.1: Importance of the Teacher's Skills

> "Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country."
> — David Hilbert

Teachers' Math Skills and Knowledge are Crucial in the Classroom

Foundation for Student Success

Teachers with strong math skills and knowledge provide a solid foundation for students, fostering a deep understanding of mathematical concepts and principles. A teacher's expertise in math is critical because it directly impacts the quality of instruction and the learning experiences of students. When teachers have a robust understanding of math, they can present information more clearly, connect new ideas to previously learned concepts, and anticipate areas where students might struggle. This solid foundation not only helps students grasp complex ideas but also builds their confidence and competence in mathematics, paving the way for future academic success.

Building Confidence

Knowledgeable teachers can confidently address students' questions and misconceptions, creating an environment where students feel secure to explore and make mistakes. When students see that their teacher is well-versed in math, they feel more at ease to ask questions and express their misunderstandings. This open and supportive environment encourages students to engage deeply with the material, try out different problem-solving strategies, and learn from their errors without fear of judgment. As students' confidence grows, so does their willingness to tackle challenging problems and persist in their learning efforts.

Role Models

Teachers serve as role models for students. Demonstrating strong math skills inspires students to value and pursue excellence in mathematics. When teachers exhibit enthusiasm and proficiency in math, they set a positive example for their students. This modeling of expertise and passion for the subject can ignite a similar enthusiasm in students, motivating them to take their math studies seriously and strive for their own excellence. Seeing their teachers as knowledgeable and competent in math also helps students believe that they, too, can master the subject with effort and dedication.

Importance of Effective Math Teaching Techniques

Engaging Students

Effective teaching techniques make math interesting and engaging, helping students to see the relevance and application of math in everyday life. Using a variety of instructional strategies, such as hands-on activities, real-world problem-solving scenarios, and interactive technology, teachers can capture students' interest and demonstrate how math is used outside the classroom. Engaging lessons make math more relatable and enjoyable, which can significantly enhance students' motivation to learn and their overall attitudes toward the subject.

Addressing Diverse Learning Needs

Utilizing a variety of teaching methods ensures that all students, regardless of their learning styles or abilities, can grasp mathematical concepts. Effective teachers recognize that students learn in different ways and at different paces. By incorporating diverse instructional approaches, such as visual aids, manipulatives, collaborative group work, and differentiated tasks, teachers can provide multiple pathways for understanding. This inclusive approach ensures that every student has the opportunity to succeed in math, regardless of their individual strengths and challenges.

Promoting Long-term Retention

Techniques that involve active learning and critical thinking help students retain and apply mathematical knowledge beyond the classroom. Active learning strategies, such as problem-based learning, math discussions, and exploratory activities, engage students in the learning process and require them to think critically about the material. These methods not only help students understand and remember mathematical concepts more effectively but also develop their ability to apply what they have learned to new situations. This focus on deep understanding and application prepares students for future academic pursuits and real-world problem-solving.

Setting the Stage for Students' Mathematical Journeys

Initial Impressions

Teachers often provide students with their first formal experiences in mathematics. Positive experiences can spark a lifelong interest in the subject. The way math is introduced to students can shape their attitudes and perceptions for years to come. When teachers create a welcoming and stimulating environment, where math is presented as an exciting and valuable subject, students are more likely to develop a positive relationship with math. These initial experiences can ignite curiosity and a desire to learn more, setting the foundation for a lifelong appreciation of mathematics.

Teaching Correct Information the First Time Students' are Learning

It is equally important that the information taught is accurate and reliable. Teaching students incorrect information can have significant negative effects on their mathematical development. When students learn incorrect methods or concepts, it can create confusion and misunderstanding, making it difficult for them to build upon these foundations later on. Misconceptions can become deeply ingrained, requiring substantial effort to correct and reteach. Additionally, students may lose confidence in their abilities if they find out that what they have learned is wrong, leading to frustration and disengagement.

Impact of Teaching Incorrect Information

1. **Confusion and Misunderstanding**: Learning incorrect methods or concepts can cause confusion, making it challenging for students to understand more advanced topics that build on these foundational ideas. For example, if a student learns an incorrect method for solving equations, it can hinder their ability to solve more complex algebraic problems in the future.
2. **Erosion of Confidence**: Discovering that previously learned information is incorrect can be discouraging and undermine students' confidence in their abilities. They may begin to doubt their capacity to understand and succeed in math, leading to decreased motivation and engagement.
3. **Difficulty in Unlearning Misconceptions**: Once misconceptions are formed, they can be difficult to correct. Students may need to unlearn incorrect information and relearn the correct concepts, which can be a time-consuming and challenging process. This can also create gaps in their knowledge that may affect their overall mathematical proficiency.
4. **Impact on Future Learning**: Incorrect foundational knowledge can create a ripple effect, impacting students' ability to learn and understand future mathematical concepts. This can result in ongoing struggles and hinder their academic progress in math.
5. **Negative Experience when Student is Corrected:** When students are taught incorrect information and corrected later in their lives, they may feel embarrassed, especially if the correction is made publicly. This embarrassment can lead to a negative emotional response to learning math. Students may become reluctant to participate in class, ask questions, or engage with the material for fear of making mistakes and being corrected again. This hesitation can further impede their learning and contribute to a negative attitude towards mathematics and other subjects.

Ensuring Accurate Teaching To avoid these negative consequences, it is crucial for teachers to:

- Continuously improve their own mathematical knowledge and stay current with best practices in math education.
- Use reliable and accurate resources and teaching materials.
- Encourage a classroom culture where students feel comfortable asking questions and expressing doubts, ensuring that misconceptions can be identified and addressed promptly.
- Engage in professional development and collaborate with colleagues to enhance their understanding and teaching of mathematical concepts.

By providing accurate and reliable mathematical instruction, teachers can ensure that students build a solid foundation, develop confidence in their abilities, and maintain a positive attitude towards learning math.

Guidance and Support

Teachers guide students through complex mathematical concepts, offering support and encouragement along the way. Learning math can be challenging, and students need guidance to navigate difficult topics. Teachers play a crucial role in breaking down complex ideas into manageable parts, providing clear explanations, and offering strategies for solving problems. Through encouragement and support, teachers help students build resilience and perseverance, essential qualities for overcoming obstacles and achieving success in math.

Impact on Student Attitudes Towards Math

Shaping Perceptions

Teachers' attitudes towards math can significantly influence students' perceptions and attitudes. Enthusiastic and confident teachers can instill a love for math in their students. The enthusiasm and confidence that teachers bring to the classroom are contagious. When teachers demonstrate a positive attitude towards math, express excitement about the subject, and show confidence in their own abilities, they inspire similar feelings in their students. This positive influence can help students develop a love for math, a willingness to engage with challenging problems, and a belief in their own mathematical abilities.

Overcoming Challenges

Teachers play a crucial role in helping students overcome math anxiety and develop a positive mindset towards tackling mathematical problems. Math anxiety is a common issue that can hinder students' performance and enjoyment of the subject. Teachers can help by creating a supportive and non-threatening learning environment, using teaching methods that build confidence, and providing reassurance and encouragement. By addressing students' fears and helping them develop a growth mindset, teachers empower students to face challenges with confidence and resilience, transforming their approach to learning math.

The importance of teachers' math skills and knowledge cannot be overstated. A teacher’s expertise in mathematics is fundamental to the quality of instruction and the learning experiences of students. When teachers possess a deep understanding of mathematical concepts, they can present information clearly, connect new ideas to previously learned concepts, and anticipate areas where students might struggle. This solid foundation not only helps students grasp complex ideas but also builds their confidence and competence in mathematics, setting the stage for future academic success.

Additionally, knowledgeable teachers can confidently address students' questions and misconceptions, creating an environment where students feel secure to explore and make mistakes. This supportive atmosphere encourages students to engage deeply with the material, try out different problem-solving strategies, and learn from their errors without fear of judgment. As students’ confidence grows, so does their willingness to tackle challenging problems and persist in their learning efforts.

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1.2: Understanding Mathematical Content Knowledge

"The beauty of mathematics only shows itself to more patient followers."
- Maryam Mirzakhani

What is Mathematical Content Knowledge?

Mathematical content knowledge encompasses the understanding of fundamental principles and concepts in mathematics, including number sense, algebraic thinking, geometry, and data analysis. It involves recognizing the connections between different mathematical ideas, how they interrelate, and how they build upon each other. This depth of knowledge enables teachers to explain concepts clearly, identify and address student misconceptions, and provide relevant examples that enhance understanding.

Core Concepts: This involves a deep understanding of fundamental mathematical principles such as number sense, algebraic thinking, geometry, and data analysis. Number sense includes concepts like place value and arithmetic operations, while algebraic thinking covers patterns, functions, and equations. Geometry involves shapes, spatial reasoning, and properties of space, whereas data analysis includes statistical reasoning and probability.

Connections Between Concepts: Recognizing how different mathematical ideas interrelate and build upon each other is crucial. For instance, understanding how arithmetic leads to algebra, or how geometric concepts can be applied in algebraic contexts, helps to create a cohesive understanding of mathematics. This interconnected knowledge allows teachers to present math as an integrated subject rather than a series of isolated topics.

Difference Between Content Knowledge and Pedagogical Content Knowledge

Content Knowledge

Content knowledge refers to the understanding of mathematical facts, concepts, and procedures. It is about knowing the "what" and "how" of mathematics—the foundational principles, theorems, and algorithms that make up the subject. This includes mastering basic arithmetic operations, understanding the properties of geometric shapes, and grasping the concepts of algebraic expressions and equations. A solid grasp of content knowledge means being familiar with the various branches of mathematics, such as calculus, statistics, and discrete math, and understanding how to apply these principles to solve real-world problems.

Furthermore, content knowledge involves the ability to reason mathematically and make logical connections between different mathematical ideas. It requires a deep comprehension of mathematical language and symbols, enabling one to interpret and construct mathematical arguments accurately. For instance, understanding the Pythagorean theorem is not just about knowing the formula $a^2 + b^2 = c^2$, but also about knowing how to apply it in various contexts, such as in solving problems involving right triangles or in real-life scenarios like architecture and engineering.

Additionally, content knowledge includes staying current with mathematical advancements and continually expanding one's understanding of the subject. This ongoing learning process ensures that teachers can provide students with the most up-to-date information and methodologies, helping to foster a deeper appreciation and understanding of mathematics in their students.

Example 1.2.1

Example of Content Knowledge:

Grade Level: 2nd Grade
Topic: Addition and Subtraction within 20

Concept: Understanding that addition is combining two quantities to find the total, and subtraction is finding the difference between two quantities.

Solution

Example Explanation:

- **Addition:** If you have 7 apples and you get 5 more apples, you add them together to find out how many apples you have in total.
  - **Example Problem:** 7 + 5 = 12
  - **Explanation:** Start with 7 apples, add 5 more, and you will have 12 apples.
- **Subtraction:** If you have 12 apples and you give away 4 apples, you subtract to find out how many apples are left.
  - **Example Problem:** 12 - 4 = 8
  - **Explanation:** Start with 12 apples, take away 4 apples, and you will have 8 apples left.

Visual Aid: A number line where students can count up or down to solve addition and subtraction problems.

Pedagogical Content Knowledge

Pedagogical content knowledge goes beyond knowing mathematics to include the ability to teach these concepts effectively. It involves understanding how students learn math, the common misconceptions they might have, and strategies for making mathematical ideas understandable and accessible. This knowledge encompasses the awareness that students often struggle with abstract concepts and need concrete examples to bridge their understanding. For instance, when teaching fractions, a teacher with strong pedagogical content knowledge might use visual aids like fraction bars or pie charts to illustrate how fractions represent parts of a whole.

Additionally, pedagogical content knowledge includes knowing how to present concepts in various ways to accommodate different learning styles. Some students might grasp a concept better through hands-on activities, while others might benefit from visual representations or verbal explanations. A teacher skilled in this area can switch between these methods seamlessly, ensuring that all students have the opportunity to understand the material. For example, when introducing the concept of slope in algebra, the teacher might first explain it using a graph, then demonstrate it through a physical activity involving inclined planes, and finally provide algebraic equations to show its calculation.

Understanding common misconceptions is another crucial aspect of pedagogical content knowledge. Teachers must anticipate where students are likely to make errors and address these misunderstandings proactively. For example, students often confuse the properties of multiplication and addition, thinking that distributive property applies inappropriately in both operations. A teacher aware of this can design lessons that specifically highlight and correct these misconceptions, using targeted questions and problem sets to reinforce the correct concepts.

Moreover, effective teaching involves creating a classroom environment that encourages exploration and critical thinking. This means fostering a growth mindset, where mistakes are seen as learning opportunities rather than failures. Teachers with strong pedagogical content knowledge are adept at creating such an environment by encouraging questions, facilitating discussions, and providing constructive feedback. They might use open-ended questions that promote deep thinking and allow students to explore multiple solutions to a problem, thereby deepening their understanding and engagement with the material.

Pedagogical content knowledge is about understanding how to deliver mathematical content in a way that is engaging, comprehensible, and meaningful to students. It requires a deep awareness of the learning process, the ability to anticipate and address misconceptions, and the flexibility to use a variety of instructional methods to meet the diverse needs of learners.

Example 1.2.2

Example of Pedagogical Content Knowledge:

Grade Level: 2nd Grade
Topic: Addition and Subtraction within 20

Teaching Strategy: Using manipulatives and visual aids to help students grasp addition and subtraction concepts.

Solution

Example Teaching Activity:

- **Activity:** “Apple Basket Game”
  - **Materials:** Small apple counters or pictures of apples, a basket, and a number line.

  Activity Instructions:

  - Place a certain number of apple counters in the basket (e.g., 8 apples).
  - Ask students to add a certain number of apples to the basket (e.g., 3 apples) and have them count the total number of apples.
  - Then, ask students to take away a certain number of apples from the basket (e.g., 5 apples) and count how many are left.

  Teaching Tips:

  - Use the number line to show how you count up for addition and count down for subtraction.
  - Encourage students to verbalize their thought process: “I started with 8 apples, added 3 more, so now I have 11 apples. Then I took away 5 apples, so now I have 6 apples.”

Visual Aid: A number line and counters.

Teaching Reflection:

- “Did students understand the concepts of addition and subtraction through the manipulative activity?”
- “Were the visual aids and manipulatives effective in helping students visualize the math problems?”

Summary Table

Type	Definition	Example
Content Knowledge	Understanding mathematical concepts and facts	**Topic:** Addition and Subtraction within 20 **Concept:** Combining or separating quantities **Example Problem:** 7 + 5 = 12 and 12 - 4 = 8
Pedagogical Content Knowledge	Techniques and strategies for teaching math concepts effectively	**Topic:** Addition and Subtraction within 20 **Strategy:** Apple Basket Game **Activity:** Use counters and number line for addition and subtraction exercises

These examples illustrate the difference between knowing the math concepts themselves (content knowledge) and knowing how to teach those concepts in a way that helps students learn effectively (pedagogical content knowledge).

Benefits of a Strong Foundation in Math Concepts

Enhanced Teaching Effectiveness

Teachers with a deep understanding of math can explain concepts more clearly and accurately, making the learning process more effective for students. This deep understanding allows them to break down complex ideas into simpler, more digestible parts, ensuring that each step of the explanation is clear and logical. For instance, when teaching calculus, a teacher with a strong grasp of the subject can explain the concept of derivatives by connecting it to students' prior knowledge of slopes and rates of change, using real-world examples to illustrate abstract ideas.

Such teachers are better equipped to anticipate student difficulties and misconceptions. They understand common pitfalls and can proactively address them, thus preventing students from developing misunderstandings. For example, when teaching fractions, they know students often mistakenly believe that a larger denominator means a larger fraction. By anticipating this misconception, the teacher can design activities that clarify how fraction sizes actually work, such as using visual aids and comparison exercises.

Confidence in Addressing Student Questions

A thorough knowledge base allows teachers to answer questions confidently and accurately, creating a classroom atmosphere where students feel secure in their learning journey. When teachers can respond to inquiries with precision and depth, it reassures students that their teacher is a reliable source of information, which enhances their trust in the educational process. For instance, when a student asks a challenging question about the application of the Pythagorean theorem, a teacher with a solid understanding can provide a detailed explanation, including practical examples and different problem-solving approaches.

This confidence fosters a positive learning environment where students feel comfortable asking questions and engaging deeply with the material. When students see their teacher confidently handling complex topics, they are more likely to feel safe expressing their uncertainties and curiosities. This open communication is crucial for effective learning, as it encourages students to explore the subject matter more thoroughly and to seek clarification whenever they encounter difficulties. For example, in a calculus class, if a student is confused about the concept of limits, they are more likely to ask for help if they feel assured that their teacher will provide a clear and supportive response.

Confidence in the subject matter also allows teachers to present content in a more engaging and dynamic manner, which can inspire students. When teachers are well-versed in their subject, they can incorporate a variety of teaching methods to make lessons more interesting and relatable. They can draw connections to real-world applications, use interactive tools, and adapt their teaching style to meet the needs of their students. For example, a teacher who deeply understands statistics can bring the subject to life by using real data sets from current events, encouraging students to analyze and interpret this data, thus making the learning experience more relevant and exciting.

Moreover, a confident teacher can use storytelling, humor, and real-life scenarios to make abstract concepts more accessible and engaging. By relating mathematical principles to everyday experiences, teachers can help students see the value and applicability of what they are learning. For instance, explaining exponential growth through examples like population growth or interest rates can make the concept more tangible and interesting for students.

In addition to enhancing engagement, this dynamic presentation of content can inspire students to develop a genuine interest in mathematics. When teachers demonstrate passion and enthusiasm for the subject, it can be contagious, motivating students to approach math with curiosity and a positive attitude. This inspiration is vital for fostering a long-term interest in the subject, encouraging students to pursue further studies in mathematics and related fields.

Teachers who are confident in their subject matter can handle unexpected questions and classroom challenges more effectively. Their ability to think on their feet and provide thoughtful responses to spontaneous inquiries can enrich classroom discussions and deepen students' understanding. This adaptability and responsiveness contribute to a vibrant and stimulating learning environment where students feel intellectually challenged and supported.

A thorough knowledge base enables teachers to answer questions confidently and accurately, fostering a positive learning environment where students feel comfortable engaging with the material. This confidence allows teachers to present content dynamically and engagingly, inspiring students and enhancing their overall learning experience. By creating a classroom atmosphere of trust, curiosity, and enthusiasm, teachers can significantly impact students' attitudes toward mathematics and their academic success.

Example 1.2.3

Student's Question: "Why is 3 times 4 the same as 4 times 3?"

Teacher Response #1

"Well, um, multiplication is kind of tricky. I think you just get the same answer if you switch the numbers. It's like a rule, I guess. Let's move on to the next problem."

Teacher Response #2

"That's a fantastic question! The reason 3 times 4 is the same as 4 times 3 is because of a special property of multiplication called the commutative property. It means that no matter the order of the numbers you multiply, the result is the same. Let's use some blocks to see why this works. If we make 3 groups of 4 blocks and then switch to 4 groups of 3 blocks, we'll see that we end up with the same total number of blocks each time."

Think About

Which response would you prefer if you were the student?

Solution

Response #1 Impact of Lack of Confidence:

- **Student's Understanding:** The student might remain confused about why 3 times 4 equals 4 times 3 and may not understand the commutative property of multiplication.
- **Classroom Atmosphere:** The student might feel hesitant to ask questions in the future and might sense that the teacher isn't very sure about the material.
- **Learning Experience:** The missed opportunity to clarify a fundamental concept in multiplication could affect the student's grasp of more complex math topics later.

Response #2 Impact of Confidence:

- **Student's Understanding:** The student learns about the commutative property of multiplication, gaining a clearer understanding of why 3 times 4 equals 4 times 3.
- **Classroom Atmosphere:** The student feels encouraged to ask questions, seeing that the teacher is knowledgeable and eager to help.
- **Learning Experience:** The student's grasp of multiplication concepts is strengthened, providing a solid foundation for understanding more advanced math concepts.

Connection to Effective Teaching and Student Outcomes

Improved Student Achievement

Research shows that teachers' strong content knowledge is linked to higher student achievement in mathematics, demonstrating that a deep understanding of math by teachers translates into better outcomes for students. When teachers possess a robust grasp of mathematical concepts, they are able to provide clear, precise explanations that help students grasp challenging ideas. This clarity is crucial for student success, as it ensures that students receive accurate and comprehensive instruction on fundamental mathematical principles.

For example, a teacher who thoroughly understands algebraic functions can break down complex topics such as polynomial equations into manageable parts, offering step-by-step explanations that reveal the underlying processes. This thorough explanation helps students not only follow the steps to solve problems but also understand why those steps are taken, which strengthens their overall mathematical comprehension. Research supports this by showing that students taught by knowledgeable teachers perform better on assessments that measure both procedural skills and conceptual understanding.

Effective problem-solving strategies are another benefit of a teacher's deep mathematical content knowledge. Teachers who are well-versed in math can introduce students to a range of problem-solving techniques and guide them in choosing the most effective strategies for different types of problems. For instance, a teacher who is familiar with various methods for solving systems of equations can demonstrate techniques like substitution, elimination, and graphing, helping students to see multiple approaches to finding solutions. This exposure to diverse strategies empowers students to approach problems from different angles, fostering critical thinking and adaptability in their problem-solving efforts.

Moreover, meaningful feedback is a critical component of effective math instruction, and it is greatly enhanced by a teacher's deep content knowledge. A knowledgeable teacher can provide specific, actionable feedback that addresses students' mistakes and misconceptions. For example, if a student misinterprets the concept of quadratic functions, a teacher who understands the nuances of this topic can offer targeted feedback that clarifies the student's errors and guides them towards the correct understanding. This type of feedback helps students to learn from their mistakes, reinforces their understanding, and supports

their progress in mastering mathematical concepts.

The relationship between teachers' content knowledge and student achievement extends beyond immediate instructional benefits. Strong content knowledge allows teachers to design lessons that are logically sequenced and aligned with curricular goals, which contributes to long-term student success. For instance, a teacher who understands how early arithmetic concepts lead into more advanced topics like algebra and calculus can create a curriculum that builds upon students' previous knowledge and prepares them for future mathematical challenges. Research indicates that well-structured, coherent instruction that builds on students' prior knowledge results in better student performance over time.

Additionally, a teacher's deep understanding of math enables them to identify and address students' learning needs more effectively. Teachers who know their subject well are better equipped to recognize common misconceptions and adjust their teaching methods accordingly. For example, if students struggle with the concept of fractions, a teacher who deeply understands fractions can offer additional resources, adjust instructional methods, and provide support that addresses these specific challenges. This ability to adapt instruction based on a deep knowledge of math supports students in overcoming obstacles and achieving academic success.

Furthermore, teachers with strong content knowledge can foster a classroom environment where students are encouraged to explore and discuss mathematical ideas. By modeling enthusiasm for math and demonstrating its relevance, these teachers can motivate students to engage more deeply with the subject. For instance, a teacher who connects mathematical concepts to real-world problems, such as budgeting or engineering challenges, can inspire students to see the practical applications of math and encourage them to pursue further study in the field.

Research shows that strong content knowledge among teachers is crucial for higher student achievement in mathematics. Teachers' deep understanding of math enables them to provide clear explanations, introduce effective problem-solving strategies, and offer meaningful feedback, all of which contribute to students' success. This expertise also allows teachers to design well-structured lessons, address students' learning needs, and create a motivating classroom environment, leading to improved student performance and long-term success in mathematics.

Long-term Success

Students who are taught by knowledgeable teachers are more likely to develop a strong mathematical foundation, which serves as a critical asset for their academic and career success. This solid foundation is not merely about mastering specific mathematical techniques but also about cultivating a comprehensive understanding of mathematical principles that supports future learning and growth.

A knowledgeable teacher provides students with more than just procedural knowledge; they offer a deep and thorough understanding of fundamental mathematical concepts. This involves teaching students how to think mathematically, explore mathematical ideas, and connect different areas of math. For example, a teacher who effectively teaches the concept of functions not only covers how to solve equations but also explores how functions model real-world scenarios and relate to other mathematical concepts like graphs and rates of change. This foundational knowledge helps students build a robust framework for understanding more complex mathematical topics in the future.

A strong mathematical foundation is essential for success in subsequent math courses. When students grasp core concepts like algebraic manipulation, geometric reasoning, and statistical analysis early on, they are better prepared for advanced topics in higher-level math courses. For instance, a solid understanding of algebraic principles from pre-algebra and algebra 1 will make learning calculus concepts like limits and derivatives more accessible. Research shows that early success in mathematics leads to better performance in advanced math courses, creating a positive feedback loop of academic achievement. Students who start with a strong foundation are not only better equipped for future coursework but also more likely to enjoy continued success in challenging subjects.

A deep understanding of math extends beyond immediate academic success to influence future educational and career opportunities. Many fields, from engineering and science to economics and data analysis, require strong mathematical skills. For example, a student with a solid grasp of statistical methods might pursue a career in data science, where they can apply these skills to analyze and interpret data. Moreover, the analytical thinking and problem-solving abilities developed through rigorous math education are highly valued in many professions. Employers often seek individuals who can approach complex problems systematically, think critically, and devise innovative solutions—all skills that are nurtured through a strong foundation in mathematics.

Mathematics education fosters critical thinking and problem-solving skills that are applicable in various fields beyond academia. Teachers who emphasize conceptual understanding help students develop the ability to approach problems from different angles, evaluate solutions, and reflect on their methods. For example, learning to solve algebraic equations teaches students to break down problems into smaller parts, apply logical reasoning, and check their work for accuracy. These skills are transferable to real-world situations, such as analyzing business problems, designing experiments, or creating technology solutions.

Beyond immediate academic and career goals, a strong mathematical foundation supports lifelong learning. Students who develop a love for math and a curiosity about how mathematical principles apply to the world are more likely to engage in continuous learning throughout their lives. For instance, a student who enjoys solving complex math problems might pursue advanced studies or explore math-related hobbies like cryptography or mathematical art. This lifelong engagement with math can lead to personal fulfillment and ongoing intellectual growth.

A well-established mathematical foundation prepares students for complex challenges in higher education and the workforce. Understanding mathematical concepts deeply allows students to approach new and intricate problems with confidence. For example, a student who has mastered the basics of calculus will find it easier to tackle advanced topics such as differential equations or mathematical modeling. This preparedness is crucial for tackling the complex issues faced in scientific research, technological innovation, and analytical professions.

Teachers who impart a strong mathematical foundation also help instill a growth mindset in their students. When students experience success in math through effective instruction, they are more likely to believe in their ability to tackle difficult problems and persist through challenges. For example, a student who overcomes initial struggles with geometry through supportive teaching is more likely to approach future difficulties with resilience and a belief in their capacity for growth. This mindset is invaluable for achieving long-term academic and professional goals.

A deep mathematical foundation boosts students' confidence in their ability to handle future academic challenges. When students have a strong grasp of math concepts, they are more likely to approach new problems with confidence rather than fear. For example, a student who excels in algebra and geometry is more likely to embrace advanced topics like calculus or statistics with a positive attitude. This self-assurance supports their ability to tackle complex academic tasks and adapt to new learning experiences.

Finally, a thorough understanding of math encourages students to explore new ideas and pursue innovative solutions. Teachers who foster deep mathematical thinking inspire students to question existing methods and seek creative approaches to problems. For example, students who understand the principles of geometry might experiment with geometric design in art or engineering projects. This creative exploration leads to new discoveries and advancements in various fields, demonstrating the broad impact of a strong mathematical foundation.

Students who receive instruction from knowledgeable teachers are more likely to develop a strong mathematical foundation that benefits them throughout their academic and professional lives. This foundation supports success in future math courses, opens doors to various career opportunities, and develops critical thinking and problem-solving skills that are valuable in many fields. A deep understanding of math fosters a growth mindset, builds confidence, and encourages innovative thinking, all of which contribute to long-term academic and career success. By investing in effective math instruction, we equip students with the tools they need to thrive in their future endeavors.

Understanding mathematical content knowledge is essential for teaching math effectively. It enables teachers to present concepts clearly, address student questions with confidence, and create a learning environment that fosters curiosity and engagement. This strong foundation in math benefits both teachers and students, leading to improved learning outcomes and long-term success.

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1.3: Pedagogical Content Knowledge

> "Every child deserves a champion—an adult who will never give up on them, who understands the power of connection and insists that they become the best that they can possibly be."
> — Rita Pierson

Pedagogical content knowledge (PCK) is a crucial concept for educators, particularly those teaching mathematics. It represents the blending of content knowledge—the understanding of the subject matter—and pedagogical knowledge—the understanding of how to teach. This unique intersection allows teachers to present material in ways that are both accessible and engaging to students. Let's delve into the components and strategies for developing this essential skill set.

Definition: Pedagogy

Pedagogy refers to the art and science of teaching specifically children. It encompasses the methods and practices used by educators to facilitate learning and ensure that students understand and retain information. Pedagogy involves understanding how students learn, designing effective instructional strategies, creating engaging learning environments, and assessing student progress. It is a comprehensive approach that combines both theoretical knowledge and practical application to promote effective teaching and meaningful learning experiences.

Pedagogy is another word for education, the profession and science of teaching. Pedagogy and pedagogue come from the Greek paidos "boy, child" plus agogos "leader." Pedagogy refers to the teaching profession as well as the science of education, for example as an elementary, high school, or college subject.

Definition: Pedagogical Content Knowledge

Pedagogical content knowledge (PCK) is a framework in education that describes a teacher's specialized understanding of how to effectively teach a specific subject or content to students. It's a combination of content knowledge (CK) and pedagogical knowledge (PK) that's unique to each teacher:

- Content knowledge - What the teacher knows about the subject matter, such as facts, structures, and relationships between concepts
- Pedagogical knowledge - What the teacher knows about teaching, such as how to apply their content knowledge to their teaching practices

Pedagogical content knowledge is the foundation upon which effective teaching is built. It encompasses several key elements that enable educators to transform their understanding of mathematics into teaching practices that foster student learning.

Integrating Content Knowledge with Pedagogy

Teaching Strategies

Definition: Teaching Strategy

A **teaching strategy** is a deliberate and systematic plan or approach used by educators to facilitate learning and achieve educational goals. It encompasses the methods, techniques, and practices employed to deliver content, engage students, and promote understanding and skill development. Teaching strategies are designed to address the diverse needs of students, support effective instruction, and create an environment conducive to learning.

Effective teaching strategies are essential for presenting mathematical concepts in ways that are understandable and engaging for students. To achieve this, teachers must blend their deep understanding of mathematical content with insights into how

students learn. This section explores several methods that can make complex ideas more accessible and engaging.

Understanding How Students Learn

Before delving into specific teaching strategies, it's important to recognize the diversity in students' learning styles. Students learn in various ways—some are visual learners, others are kinesthetic learners, and some prefer auditory methods. Understanding these differences allows teachers to tailor their instruction to meet the needs of all students, ensuring that each one can grasp and retain mathematical concepts effectively.

Explaining Complex Ideas

Explaining complex ideas in a clear and relatable manner is a cornerstone of effective teaching. Teachers can break down difficult concepts into smaller, more manageable parts and use simple language to explain them. Analogies and metaphors are also powerful tools that can help students relate new information to something they already understand. For instance, comparing the flow of electricity in a circuit to water flowing through pipes can make the abstract concept of electrical circuits more tangible.

Visual Aids

Definition: Visual Aid

A **visual aid** is a tool or resource that uses visual elements to support and enhance the delivery of information or concepts in an educational or presentation setting.

Visual aids are invaluable in helping students understand and retain mathematical concepts. These can include diagrams, charts, graphs, and models. For example, when teaching geometry, using visual representations of shapes, angles, and transformations can make the concepts more concrete. Interactive whiteboards and digital tools also allow for dynamic visual presentations, which can further enhance understanding. Visual aids are designed to make complex ideas more accessible, clarify content, and engage learners by providing visual representations of the material.

- **Diagrams and Charts:** These can simplify complex information and show relationships between different concepts. For instance, a Venn diagram can help students understand set theory and the relationships between different sets.
- **Graphs and Models:** Visualizing data through graphs and creating physical models of geometric shapes can aid in comprehending abstract ideas. For example, a graph can illustrate the relationship between variables in algebra, while a 3D model can help students grasp the properties of different geometric solids.

Hands-On Activities

Hands-on activities engage students actively in the learning process, making abstract concepts more tangible. Manipulatives, such as blocks, fraction tiles, or algebra tiles, allow students to explore mathematical ideas through physical interaction. These activities can be particularly beneficial in teaching concepts like fractions, ratios, and algebraic expressions.

- **Manipulatives:** Physical objects that students can manipulate to understand abstract concepts. For instance, using fraction tiles to visualize parts of a whole can help students better understand fractions.
- **Experiments and Simulations:** Conducting experiments or using computer simulations can provide hands-on experience with mathematical principles. For example, using a simulation to model population growth can help students understand exponential functions.

Real-World Examples

Connecting mathematical concepts to real-world applications makes learning more relevant and interesting for students. By demonstrating how math is used in everyday life, teachers can motivate students and show the practical value of what they are learning.

- **Everyday Applications:** Relating math to everyday activities, such as cooking, shopping, or sports, can make abstract

concepts more concrete. For example, measuring ingredients for a recipe can teach students about ratios and proportions.
- **Interdisciplinary Approaches:** Integrating math with other subjects, such as science or economics, can provide a broader context for understanding mathematical concepts. For instance, using statistics to analyze scientific data can show students the real-world relevance of mathematical principles.

Interactive Technology

Definition: Interactive Technology

Interactive technology refers to digital tools and platforms that enable users to actively engage with and influence the content or environment they are interacting with. Unlike traditional technology which often provides a passive experience, interactive technology allows users to interact, manipulate, and respond to the information or systems in real time.

Incorporating technology into math instruction can enhance engagement and understanding. Interactive software, online tutorials, and educational games provide dynamic and interactive ways for students to explore mathematical concepts. Technology can also offer personalized learning experiences, catering to the individual needs of each student.

- **Interactive Software:** Programs that allow students to explore mathematical concepts through interactive simulations and exercises. For example, software that lets students manipulate geometric shapes to understand their properties.
- **Educational Games:** Games that make learning math fun and engaging. These can include puzzles, challenges, and competitions that motivate students to practice and apply their math skills.

Collaborative Learning

Definition: Collaborative Learning

Collaborative learning is an educational approach in which students work together in groups or teams to achieve common learning goals and complete tasks. This method emphasizes the interaction and cooperation among learners, leveraging diverse perspectives and skills to enhance understanding and problem-solving.

Collaborative learning encourages students to work together to solve problems and understand concepts. Group activities and peer discussions can foster a deeper understanding of mathematical ideas and help students develop critical thinking and communication skills.

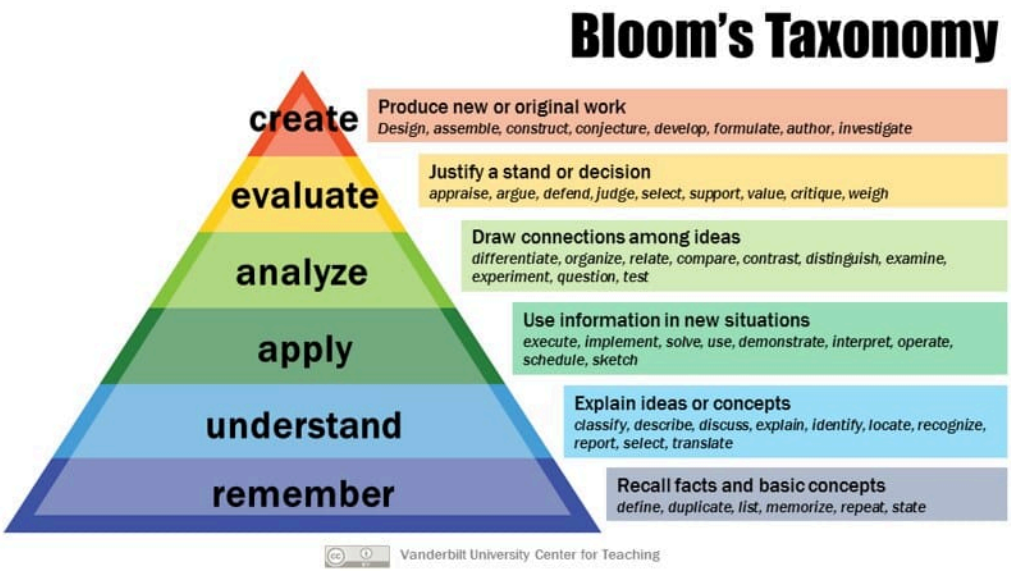
- **Group Projects:** Collaborative projects that require students to apply mathematical concepts to real-world problems. For example, a group project that involves creating a budget for a school event can teach students about financial math.
- **Peer Teaching:** Allowing students to explain concepts to each other can reinforce their understanding and build confidence. Peer teaching can also provide different perspectives and approaches to solving problems.

Utilizing a variety of teaching strategies is essential for presenting mathematical concepts in ways that are both understandable and engaging for students. By understanding how students learn, using visual aids, hands-on activities, real-world examples, interactive technology, and collaborative learning, teachers can make abstract ideas more concrete and foster a deeper understanding of mathematics.

Student Levels of Learning Using Bloom's Taxonomy

> “It is the supreme art of the teacher to awaken joy in creative expression and knowledge.”
> — Albert Einstein

Bloom's Taxonomy is a framework that categorizes educational goals into levels of complexity and specificity. Developed by Benjamin Bloom in 1956 and revised in 2001 by Lorin Anderson and David Krathwohl, Bloom's Taxonomy helps educators structure their teaching and assessment strategies to promote higher-order thinking skills. The taxonomy consists of six levels, each representing a different stage of cognitive development. This section explores these levels and how they can be applied to enhance student learning.





Remembering

This foundational level involves the ability to recall facts and basic concepts. It's about retrieving previously learned information from memory. At this stage, students are expected to remember terminology, specific details, and basic principles. Activities like memorizing formulas, listing key terms, and reciting definitions fall under this category. It serves as the building block for more complex cognitive tasks.

Examples of Activities:

- Memorizing multiplication tables.
- Reciting mathematical definitions.
- Listing the steps in a problem-solving process.

Teaching Strategies:

- Use flashcards for quick recall of facts.
- Implement quizzes and practice tests.
- Encourage note-taking and summarization.



Understanding

The next level focuses on comprehending the material. Students demonstrate understanding when they can explain ideas or concepts in their own words, interpret information, and summarize content. This stage moves beyond mere memorization, requiring students to grasp the meaning of the material. They might explain the reasoning behind a math solution, describe the plot of a story, or interpret the significance of a scientific experiment.

Examples of Activities:

- Interpreting a graph or chart.
- Explaining the reasoning behind a mathematical solution.
- Summarizing a lesson in their own words.

Teaching Strategies:

- Use concept maps to visualize relationships.
- Ask students to paraphrase information.
- Engage in discussions and Q&A sessions.



Applying

At this level, students use the knowledge they have acquired in new and concrete situations. Applying involves using information, concepts, and skills to solve problems or carry out tasks. This might include using a formula in a new context, applying rules in a practical situation, or performing a laboratory experiment. The focus is on the practical application of knowledge to real-world scenarios.

Examples of Activities:

- Solving word problems using mathematical concepts.
- Applying a formula to calculate an unknown variable.
- Using geometry to design a simple structure.

Teaching Strategies:

- Provide real-world problem-solving tasks.
- Use case studies and simulations.
- Encourage project-based learning.



Analyzing

Analysis requires students to break down information into its component parts and understand its structure. This involves identifying relationships, recognizing patterns, and examining the underlying principles. Students might compare and contrast different theories, dissect arguments to evaluate their validity, or identify errors in reasoning. Analysis encourages critical thinking and a deeper understanding of the material.

Examples of Activities:

- Identifying the components of a mathematical argument.
- Comparing and contrasting different methods to solve a problem.
- Analyzing errors in a solution to understand misconceptions.

Teaching Strategies:

- Use Venn diagrams to compare concepts.
- Teach students to identify patterns and relationships.
- Encourage critical thinking through open-ended questions.



Evaluating (or Synthesis)

Evaluation involves making judgments based on criteria and standards. Students at this level assess the validity of ideas or the quality of work, determine the effectiveness of solutions, and make informed decisions. They might critique a piece of literature, evaluate the reliability of sources, or judge the validity of scientific findings. Evaluation requires a higher level of thinking, combining knowledge, understanding, and analysis to make informed judgments.

Examples of Activities:

- Critiquing the logic of a mathematical argument.
- Assessing the efficiency of different problem-solving methods.
- Evaluating the relevance of data in a given context.

Teaching Strategies:

- Use rubrics to guide self-assessment and peer reviews.
- Engage students in debates and discussions.
- Teach students to use criteria to make judgments.



Creating

The highest level of Bloom's Taxonomy is creating, where students generate new or original work. This involves combining elements in novel ways, designing projects, or developing new theories. Creativity requires synthesis and innovation, drawing on all previous levels of learning. Students might design an experiment, create a piece of art, or develop a unique solution to a complex problem. This stage represents the pinnacle of cognitive learning, where students use their knowledge and skills to produce something original and valuable.

Examples of Activities:

- Designing an original experiment to test a mathematical hypothesis.
- Developing a new method for solving a complex problem.
- Creating a project that integrates multiple mathematical concepts.

Teaching Strategies:

- Encourage brainstorming and idea generation.
- Use project-based assignments that require innovation.
- Provide opportunities for independent research and exploration.

Understanding the different levels of learning, as outlined in Bloom's Taxonomy, is essential for teachers because it enables them to design and implement more effective instructional strategies. By recognizing the cognitive complexity of various tasks—ranging from simple recall to complex creation—educators can structure their lessons to progressively build students' knowledge and skills. This structured approach ensures that students move from basic understanding to higher-order thinking in a systematic way. Additionally, it helps teachers create well-aligned assessments that accurately measure student progress at each level of learning, providing valuable feedback for both students and educators.

Moreover, a grasp of these levels promotes critical thinking by challenging students to analyze, evaluate, and apply concepts beyond rote memorization. This fosters deeper engagement and develops essential problem-solving skills. Teachers can also

use this understanding to differentiate instruction, tailoring activities to accommodate diverse learning needs and abilities. By setting clear, measurable objectives that correspond to different cognitive levels, teachers ensure that their goals are focused and achievable. Furthermore, this understanding encourages self-reflection, allowing educators to evaluate and refine their teaching practices to better support student learning. Overall, incorporating Bloom's Taxonomy into teaching practices enhances the educational experience, making it more dynamic, engaging, and effective.

Applying Bloom's Taxonomy in the Classroom

Educators can use Bloom's Taxonomy to structure lessons, activities, and assessments that promote higher-order thinking. Here are some strategies for applying the taxonomy in the classroom:

1. **Align Objectives with Taxonomy Levels:** Clearly define learning objectives for each lesson and align them with the appropriate level of Bloom's Taxonomy. This ensures that activities and assessments are designed to meet these objectives.
2. **Progress from Lower to Higher Levels:** Begin with foundational knowledge and gradually progress to more complex cognitive tasks. This scaffolding approach helps students build their understanding and skills systematically.
3. **Use Diverse Assessment Methods:** Incorporate various assessment methods that target different levels of the taxonomy. For example, use quizzes and tests for lower-level recall and understanding, and projects and presentations for higher-level analysis and creation.
4. **Encourage Metacognition:** Teach students to reflect on their thinking processes and self-assess their understanding. Metacognitive activities help students become more aware of their learning and develop strategies for improvement.
5. **Foster a Growth Mindset:** Encourage students to view challenges as opportunities for growth. Emphasize the importance of effort, persistence, and learning from mistakes, which aligns with the higher levels of Bloom's Taxonomy.

Bloom's Taxonomy provides a valuable framework for understanding and promoting student learning at different cognitive levels. By incorporating the taxonomy into lesson planning, teaching strategies, and assessment methods, educators can create a more structured and effective learning environment. This approach not only helps students develop a deeper understanding of mathematical concepts but also equips them with the critical thinking and problem-solving skills necessary for success in their academic and professional lives.

So, how do we gauge students' progress? By evaluating their learning through assessments. Assessments serve the crucial purpose of measuring students' understanding, identifying areas of improvement, and guiding future instruction. They provide valuable feedback to both students and educators, ensuring that learning objectives are being met and informing adjustments to teaching strategies as needed.

Assessment Techniques

Assessment is a critical component of teaching, as it provides valuable information about student understanding and progress. Effective assessment techniques help teachers identify areas where students struggle, monitor their development, and adjust instruction to meet their needs. This section explores different types of assessments and how they can be used to support student learning.

Formative Assessments

Formative assessments play a pivotal role in the learning process by offering continuous feedback to both students and educators. Unlike summative assessments, which evaluate learning at the end of an instructional period, formative assessments are integrated throughout the teaching process. Their primary purpose is diagnostic; they help identify students' strengths and areas for improvement in real-time. This ongoing feedback allows teachers to adjust their instructional strategies to better meet students' needs and address any misunderstandings or gaps in knowledge.

Formative assessments come in many forms, each designed to capture different aspects of student learning. For example, quizzes and short tests provide immediate insights into students' grasp of recent material, helping to pinpoint areas that may require additional review. Classroom discussions offer a platform for students to articulate their understanding and pose questions, revealing both their comprehension and any lingering confusion. Exit tickets, where students write down what they've learned and any remaining questions, help teachers gauge the effectiveness of a lesson and plan future instruction accordingly.

Other forms of formative assessment include observations of student work and interactions, which provide insights into their

problem-solving approaches and engagement levels. Homework assignments also serve as a tool for teachers to evaluate how well students apply what they've learned independently. By employing a variety of formative assessment techniques, teachers can maintain a responsive and supportive learning environment, fostering a continuous cycle of improvement and ensuring that instructional adjustments are made promptly to enhance student success.

Formative assessments are typically low stakes. Some examples are:

- **Quizzes and Short Tests:** These quick assessments provide immediate feedback on students' understanding of recent material. They help teachers gauge whether students have grasped specific concepts and identify topics that may need further review.
- **Classroom Discussions:** Engaging students in discussions allows teachers to assess their understanding through their contributions and questions. Discussions can reveal misconceptions and provide opportunities for peer learning.
- **Exit Tickets:** At the end of a lesson, students write down one thing they learned and one question they still have. This helps teachers assess the effectiveness of the lesson and plan future instruction.
- **Observations:** Teachers can observe students as they work on tasks, noting their problem-solving strategies and interactions with peers. Observations provide insights into students' thought processes and engagement.
- **Homework Assignments:** Reviewing homework allows teachers to see how well students can apply what they've learned independently. Homework can also highlight areas where students need more practice.

Summative Assessments

Summative assessments are critical tools for evaluating student learning at the conclusion of an instructional period, such as a unit, semester, or academic year. Unlike formative assessments, which provide ongoing feedback and guide the learning process, summative assessments are designed to measure and summarize students' overall achievement and mastery of content. These assessments are typically high stakes, meaning they often have significant implications for students' grades, progression to the next level, or graduation.

Summative assessments come in various forms, including final exams, end-of-unit tests, standardized tests, and major projects or assignments. They are intended to provide a comprehensive evaluation of what students have learned and how well they have met the learning objectives of the course or curriculum. For instance, a final exam may cover all the material taught throughout a semester, assessing students' retention and understanding of the content as a whole. Similarly, end-of-unit tests evaluate students' grasp of specific concepts and skills learned during that period.

The results of summative assessments are used to make important decisions about students' academic progress and performance. They often contribute significantly to final grades and can impact students' future educational opportunities, such as placements in advanced courses or college admissions. As such, while summative assessments are essential for measuring overall achievement, it is important for educators to ensure that they are aligned with the instructional goals and content covered, and that they provide a fair and accurate reflection of students' learning.

Summative assessments typically have higher stakes. Some examples are:

- **Standardized Tests:** These tests measure students' performance against a common standard and are often used for accountability purposes. Standardized tests provide data on student achievement across different populations.
- **End-of-Unit Tests:** These tests assess students' understanding of the material covered in a specific unit. They help teachers evaluate whether students have met the learning objectives.
- **Final Exams:** Comprehensive exams at the end of a course assess students' cumulative knowledge and understanding of the subject matter.
- **Projects and Portfolios:** Long-term projects and portfolios allow students to demonstrate their learning through practical applications and creative work. These assessments can showcase students' abilities to integrate and apply knowledge.

Effective Assessment Techniques

To maximize the benefits of assessment, teachers should use a variety of techniques and ensure that assessments are aligned with learning objectives. Effective assessment techniques include:

- **Clear Criteria and Rubrics:** Providing students with clear criteria and detailed rubrics helps them understand expectations and how their work will be evaluated. Rubrics also ensure consistency and fairness in grading.
- **Timely Feedback:** Providing prompt and specific feedback helps students understand their mistakes and learn from them.

Feedback should be constructive and focus on how students can improve.
- **Differentiated Assessments:** Recognizing that students have different strengths and learning styles, teachers can use differentiated assessments to give all students the opportunity to demonstrate their understanding. This might include offering choices in how students complete an assignment or allowing for different formats (e.g., written, oral, visual).
- **Self-Assessment and Peer Assessment:** Encouraging students to assess their own work and that of their peers fosters reflection and critical thinking. Self-assessment helps students take ownership of their learning, while peer assessment can provide additional perspectives and feedback.
- **Data Analysis:** Analyzing assessment data allows teachers to identify patterns and trends in student performance. This information can inform instructional decisions and help target areas that need reinforcement.

Adjusting Instruction Based on Assessment

Effective assessment techniques are not just about measuring student performance but also about using the results to enhance teaching and learning. Based on assessment data, teachers can:

- **Differentiate Instruction:** Tailor instruction to meet the diverse needs of students. This might involve providing additional support for struggling students or offering enrichment activities for those who have mastered the content.
- **Review and Reteach:** Identify topics that need to be revisited and provide additional practice and explanations to ensure students achieve mastery.
- **Adjust Pacing:** Modify the pace of instruction based on students' progress. If students are advancing quickly, teachers can introduce more challenging material. If they are struggling, teachers can slow down and provide more practice.
- **Incorporate Student Feedback:** Use feedback from students about their learning experiences to improve teaching strategies and classroom practices.

Effective assessment techniques are essential for understanding student progress and guiding instruction. By using a combination of formative and summative assessments, providing clear criteria and timely feedback, and analyzing assessment data, teachers can create a responsive and supportive learning environment that helps all students succeed.

Strategies for Developing Pedagogical Content Knowledge

Developing strong PCK is a continuous process that requires dedication and a commitment to professional growth. Here are some strategies that educators can use to enhance their pedagogical content knowledge:

Continuous Professional Development

It might seem like math is one of those subjects that doesn't change much over time, but when it comes to teaching it certainly does. Continuous professional development (CPD) is vital for educators to stay current with best practices, deepen their content knowledge, and enhance their teaching skills. It involves engaging in various learning opportunities and activities that contribute to professional growth throughout an educator's career. This section explores the importance of CPD, the different forms it can take, and strategies for effective professional development.

Importance of Continuous Professional Development

1. **Staying Current with Best Practices:** Education is a dynamic field with ongoing research and new pedagogical strategies emerging regularly. CPD helps teachers stay informed about the latest developments, ensuring their teaching methods are effective and up-to-date.
2. **Improving Teaching Skills:** Professional development provides opportunities for teachers to refine their instructional techniques, learn new strategies, and integrate innovative tools and technologies into their classrooms.
3. **Enhancing Subject Knowledge:** Deepening their understanding of the subjects they teach allows educators to present content more effectively and confidently, enriching the learning experience for students.
4. **Meeting Diverse Student Needs:** CPD equips teachers with strategies to address the diverse learning needs of their students, including those with special needs or English language learners, ensuring an inclusive learning environment.
5. **Boosting Professional Confidence:** Engaging in professional development activities builds teachers' confidence in their abilities, leading to improved job satisfaction and a positive impact on student outcomes.

Forms of Continuous Professional Development

CPD can take many forms, each offering unique benefits and opportunities for growth. Here are some common types of professional development activities:

Workshops and Courses

- **In-Person Workshops:** These provide hands-on learning experiences where teachers can interact with experts and peers, practice new strategies, and receive immediate feedback.
- **Online Courses:** Flexible and accessible, online courses allow teachers to learn at their own pace and convenience. These courses often cover a wide range of topics, from instructional strategies to advanced content knowledge.

Conferences and Seminars

- **Education Conferences:** Large gatherings where educators can attend keynote presentations, participate in workshops, and network with colleagues. Conferences often feature the latest research, innovative teaching practices, and emerging trends in education.
- **Seminars:** Smaller, focused sessions that delve deeply into specific topics. Seminars provide opportunities for in-depth learning and discussion with experts in the field.

Collaboration with Colleagues

- **Professional Learning Communities (PLCs):** Regularly scheduled meetings where teachers collaborate, share resources, and discuss instructional practices. PLCs create a supportive environment for continuous learning and improvement.
- **Team Teaching:** Partnering with colleagues to plan and deliver lessons can provide valuable insights and foster collaborative problem-solving.

Mentorship Programs

- **Mentor-Mentee Relationships:** Experienced teachers mentor novice educators, offering guidance, support, and sharing their expertise. Mentorship helps new teachers acclimate to the profession and develop their skills more rapidly.
- **Peer Mentoring:** Colleagues at similar stages in their careers support each other, sharing ideas and providing mutual encouragement.

Research and Inquiry

- **Action Research:** Teachers conduct research in their own classrooms to investigate specific questions or challenges. This reflective practice allows educators to systematically improve their teaching based on evidence.
- **Reading Professional Literature:** Staying informed by reading journals, books, and articles related to education and their specific subject areas helps teachers stay current with new research and practices.

Technology Integration

- **EdTech Training:** Professional development focused on integrating technology into the classroom can help teachers use digital tools effectively to enhance student learning.
- **Webinars and Online Workshops:** Virtual sessions on various topics related to educational technology and digital pedagogy provide convenient and relevant professional development.

Continuous professional development is essential for educators to maintain high standards of teaching and adapt to the evolving educational landscape. By engaging in diverse learning opportunities, setting clear goals, and fostering a collaborative culture, teachers can continuously improve their skills and enhance student learning outcomes.

Pedagogical content knowledge is a dynamic and essential aspect of effective teaching. By integrating content knowledge with pedagogy, utilizing a variety of teaching strategies, and engaging in continuous professional development, educators can enhance their ability to teach mathematics in ways that are both understandable and engaging for students. Collaboration with colleagues and participation in math education communities further support this development, creating a robust framework for ongoing

professional growth.

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1.4: Common Core Standards for Mathematics

Common Core State Standards (CCSS)

The Common Core State Standards (CCSS) initiative was launched in 2009 as a collaborative effort to establish a clear and consistent framework for education across the United States. The initiative was spearheaded by the National Governors Association (NGA) and the Council of Chief State School Officers (CCSSO), with the aim of ensuring that all students, regardless of where they live, receive a high-quality education and are prepared for success in college and careers.

Prior to the Common Core, educational standards varied significantly from state to state, leading to disparities in the quality of education and student achievement. The CCSS initiative sought to address these inconsistencies by creating a set of shared goals and expectations for student learning in English Language Arts (ELA) and Mathematics.

To develop the standards, the NGA and CCSSO engaged a diverse group of stakeholders, including educators, administrators, researchers, and experts in various fields. The development process also included input from the public and feedback from a range of organizations and individuals. The goal was to create standards that were research-based, aligned with college and career expectations, and reflective of the best practices in education.

The final version of the Common Core State Standards was released in June 2010. Following their release, states were encouraged to adopt the standards voluntarily. By 2011, the majority of states, along with the District of Columbia, had adopted the CCSS for both ELA and Mathematics.

Adoption of the CCSS marked the beginning of a significant shift in educational practices across the country. States and school districts began the process of aligning their curricula, instructional materials, and assessments with the new standards. This involved extensive professional development for teachers and administrators to ensure effective implementation.

Core Principles

The Common Core State Standards are built on several key principles:

1. **Focus and Coherence:** The standards emphasize a clear focus on essential concepts and skills, ensuring coherence in the progression of learning across grades.
2. **Rigor and Application:** The standards aim to promote a deeper understanding of key concepts, encouraging students to apply their knowledge to real-world problems and scenarios.
3. **Clarity and Consistency:** The standards provide clear and consistent expectations for what students should know and be able to do at each grade level.
4. **Preparation for College and Careers:** The standards are designed to ensure that students graduate from high school with the skills and knowledge necessary for success in postsecondary education and the workforce.

Focus Areas

Mathematics standards have focus areas to ensure students develop a deep and comprehensive understanding of key concepts and skills essential for their academic and future career success. Concentrating on specific focus areas allows students to gain a deeper understanding of fundamental mathematical concepts, rather than a superficial grasp of many topics. This depth of understanding is critical for applying knowledge to solve complex problems and building on these concepts in future learning. Focus areas ensure a coherent curriculum with a logical progression of mathematical concepts from one grade level to the next. This helps students build on their prior knowledge and develop a strong foundation in mathematics. For example, understanding basic arithmetic is crucial before advancing to more complex topics like algebra and geometry. Certain mathematical skills and concepts are foundational and have wide-ranging applications in both academic settings and everyday life. Focus areas prioritize these key skills, ensuring all students develop strong competence in essential areas such as number sense, operations, fractions, and problem-solving. These focus areas align with the skills and knowledge students need for success in college and their future careers. By emphasizing these areas, the standards help prepare students for the demands of higher education and the workforce, where mathematical literacy is increasingly important. Given limited instructional time, it's important to concentrate on the most critical areas that significantly impact students' mathematical development. By identifying and emphasizing these focus areas,

teachers can use their time and resources more efficiently, providing targeted instruction that meets students' needs. When students understand the relevance and importance of what they are learning, they are more likely to be engaged and motivated. Focus areas help highlight the practical applications of mathematical concepts, demonstrating how math is used in real-world situations and making the subject more interesting and relevant to their lives. Focus areas provide a consistent framework for mathematics education across different grade levels and schools. This consistency ensures all students, regardless of where they are educated, have access to high-quality mathematics education and the opportunity to develop the same essential skills and knowledge. Clear focus areas offer guidance for educators in designing their instruction and assessment practices. Teachers can use the focus areas to plan lessons, identify key learning objectives, and create assessments that accurately measure student understanding of essential concepts. By having well-defined focus areas, mathematics standards aim to provide a structured and effective approach to mathematics education, ensuring students develop the skills and knowledge they need to succeed in their academic pursuits and beyond.

The defined focus areas are:

- **K-5:** The standards for kindergarten through fifth grades focus on building a solid foundation in basic mathematics. Students learn fundamental concepts such as addition, subtraction, multiplication, division, fractions, and basic geometry.
- **6-8:** In middle school, the standards build on this foundation by introducing more complex concepts such as ratios, proportions, algebraic thinking, and advanced geometry. The goal is to prepare students for high school mathematics.
- **9-12:** High school standards are designed to prepare students for college and careers. They cover advanced topics such as algebra, geometry, statistics, and calculus. The emphasis is on applying mathematical concepts to real-world problems and developing higher-order thinking skills.

Becoming Familiar with the Standards

Elementary school math teachers should become familiar with all grade Common Core Standards for several important reasons. By knowing the standards across all grade levels, teachers can better understand how mathematical concepts build on each other from one grade to the next. This helps them ensure that their instruction is aligned with the necessary prerequisites for future learning. It also allows them to identify gaps in student understanding that may need to be addressed to ensure success in subsequent grades. Familiarity with the entire spectrum of standards helps teachers provide a coherent and seamless educational experience. They can plan lessons that bridge the gap between what students have already learned and what they will need to know in the future. This coherence ensures that students see the connections between different mathematical concepts and understand their relevance. Understanding the full range of standards allows teachers to better differentiate instruction to meet the diverse needs of their students. They can provide appropriate challenges for advanced students who are ready for more complex concepts while offering targeted support for those who need to reinforce foundational skills. When all teachers are familiar with the full range of standards, they can work together to ensure a consistent and equitable educational experience for all students. This consistency helps to reduce learning disparities and ensures that all students have access to high-quality math instruction, regardless of their teacher or school. Knowledge of all grade-level standards helps teachers create more effective assessments that accurately measure student understanding and progress. Teachers can provide meaningful feedback that not only addresses current learning goals but also prepares students for future expectations. When elementary school teachers understand the standards across all grades, they can more effectively communicate and collaborate with their colleagues. This collaboration can lead to a more integrated and supportive learning environment, where teachers share strategies and resources to enhance student learning. Familiarity with the full range of standards can inform teachers' professional development goals and activities. Teachers can seek out training and resources that help them address specific standards and improve their instructional practices across the entire K-5 curriculum. Ultimately, understanding the full spectrum of standards helps teachers prepare their students for long-term success in mathematics. By ensuring that students master the foundational skills needed for higher-level math, teachers set their students on a path to academic achievement and future career readiness. Becoming familiar with all grade Common Core Standards empowers elementary school math teachers to provide more effective, coherent, and equitable instruction. It allows them to better support their students' learning journeys, from mastering basic concepts to developing the skills needed for future academic and career success.

These standards provide a comprehensive framework for math instruction, guiding educators in what content to teach and ensuring a consistent approach to mathematics education nationwide. In this chapter, specific Common Core Standards are cited using both their abbreviated notation and descriptive text. For comprehensive details on the standards, please consult the Common Core State Standards for Mathematics (CCSS).

Example: Standard Notation 1.4.1

Standard 3.NF.A.1: “Understand a fraction as a number on the number line; represent fractions on a number line diagram.”

Standard 3.NF.A.1 Notation Breakdown

- **3:** This refers to the grade level, which is Grade 3. The Common Core Standards are organized by grade level to specify what students should learn in each grade.
- **NF:** This stands for "Number and Operations—Fractions." It indicates the domain of the mathematics standards that deals with understanding and working with fractions. The NF domain includes all the standards related to fractions for that grade level.
- **A:** This letter represents the “cluster” within the domain. Clusters are groups of related standards within a domain. In this case, “A” denotes the cluster of standards related to understanding fractions.
- **1:** This is the specific standard within the cluster. It identifies the particular standard or expectation that students should meet. In this case, “1” refers to the standard that focuses on understanding a fraction as a number on the number line and representing fractions on a number line diagram.

Provide Good Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Mathematical Practices are a set of essential processes and proficiencies that guide students in developing a deep and resilient understanding of mathematics. These practices begin with making sense of problems and persevering in solving them, encouraging learners to engage with challenges thoughtfully and persistently. Reasoning abstractly and quantitatively enables students to move fluidly between real-world contexts and mathematical representations. Constructing viable arguments and critiquing the reasoning of others fosters a collaborative and critical thinking environment, essential for refining ideas. Modeling with mathematics allows students to use mathematical concepts to interpret and solve real-world problems. The strategic use of tools enhances problem-solving by selecting the most effective resources, while attending to precision ensures clarity and accuracy in communication and computation. Finally, recognizing and utilizing structure and regularity in repeated reasoning helps students identify patterns and apply them to solve complex problems efficiently. These practices form the foundation for becoming proficient in mathematics, equipping students with the skills needed for both academic success and practical application in everyday life.

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1.4.1: Kindergarten Standards

In Kindergarten, instructional time should focus on two critical areas: (1) representing and comparing whole numbers, initially with sets of objects; (2) describing shapes and space. More learning time in Kindergarten should be devoted to number than to other topics.

1. Students use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set; counting out a given number of objects; comparing sets or numerals; and modeling simple joining and separating situations with sets of objects, or eventually with equations such as $5 + 2 = 7$ and $7 - 2 = 5$. (Kindergarten students should see addition and subtraction equations, and student writing of equations in kindergarten is encouraged, but it is not required.) Students choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away.
2. Students describe their physical world using geometric ideas (e.g., shape, orientation, spatial relations) and vocabulary. They identify, name, and describe basic two-dimensional shapes, such as squares, triangles, circles, rectangles, and hexagons, presented in a variety of ways (e.g., with different sizes and orientations), as well as three-dimensional shapes such as cubes, cones, cylinders, and spheres. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.

Domain: Counting & Cardinality

Know number names and the count sequence.

CCSS.Math.Content.K.CC.A.1
Count to 100 by ones and by tens.

CCSS.Math.Content.K.CC.A.2
Count forward beginning from a given number within the known sequence (instead of having to begin at 1).

CCSS.Math.Content.K.CC.A.3
Write numbers from 0 to 20. Represent a number of objects with a written numeral 0-20 (with 0 representing a count of no objects).

Count to tell the number of objects.

CCSS.Math.Content.K.CC.B.4
Understand the relationship between numbers and quantities; connect counting to cardinality.

CCSS.Math.Content.K.CC.B.4.a
When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.

CCSS.Math.Content.K.CC.B.4.b
Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.

CCSS.Math.Content.K.CC.B.4.c
Understand that each successive number name refers to a quantity that is one larger.

CCSS.Math.Content.K.CC.B.5
Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1-20, count out that many objects.

Compare numbers.

CCSS.Math.Content.K.CC.C.6
Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.[1]

CCSS.Math.Content.K.CC.C.7
Compare two numbers between 1 and 10 presented as written numerals.

[1] Include groups with up to ten objects.

Domain: Operations & Algebraic Thinking

Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

CCSS.Math.Content.K.OA.A.1
Represent addition and subtraction with objects, fingers, mental images, drawings[1], sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.

CCSS.Math.Content.K.OA.A.2
Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.

CCSS.Math.Content.K.OA.A.3
Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).

CCSS.Math.Content.K.OA.A.4
For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.

CCSS.Math.Content.K.OA.A.5
Fluently add and subtract within 5.

Domain: Number & Operations in Base Ten

Work with numbers 11-19 to gain foundations for place value.

CCSS.Math.Content.K.NBT.A.1
Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (such as 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

Domain: Measurement & Data

Describe and compare measurable attributes.

CCSS.Math.Content.K.MD.A.1
Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.

CCSS.Math.Content.K.MD.A.2
Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*

Classify objects and count the number of objects in each category.

CCSS.Math.Content.K.MD.B.3
Classify objects into given categories; count the numbers of objects in each category and sort the categories by count.[1]

Domain: Geometry

Identify and describe shapes.

CCSS.Math.Content.K.G.A.1
Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.

CCSS.Math.Content.K.G.A.2
Correctly name shapes regardless of their orientations or overall size.

CCSS.Math.Content.K.G.A.3
Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").

Analyze, compare, create, and compose shapes.

CCSS.Math.Content.K.G.B.4
Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).

CCSS.Math.Content.K.G.B.5
Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.

CCSS.Math.Content.K.G.B.6
Compose simple shapes to form larger shapes. *For example, "Can you join these two triangles with full sides touching to make a rectangle?"*

Sources:

Common Core State Standards: Kindergarten

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1.4.2: Grade 1 Standards

In Grade 1, instructional time should focus on four critical areas: (1) developing understanding of addition, subtraction, and strategies for addition and subtraction within 20; (2) developing understanding of whole number relationships and place value, including grouping in tens and ones; (3) developing understanding of linear measurement and measuring lengths as iterating length units; and (4) reasoning about attributes of, and composing and decomposing geometric shapes.

1. Students develop strategies for adding and subtracting whole numbers based on their prior work with small numbers. They use a variety of models, including discrete objects and length-based models (e.g., cubes connected to form lengths), to model add-to, take-from, put-together, take-apart, and compare situations to develop meaning for the operations of addition and subtraction, and to develop strategies to solve arithmetic problems with these operations. Students understand connections between counting and addition and subtraction (e.g., adding two is the same as counting on two). They use properties of addition to add whole numbers and to create and use increasingly sophisticated strategies based on these properties (e.g., "making tens") to solve addition and subtraction problems within 20. By comparing a variety of solution strategies, children build their understanding of the relationship between addition and subtraction.
2. Students develop, discuss, and use efficient, accurate, and generalizable methods to add within 100 and subtract multiples of 10. They compare whole numbers (at least to 100) to develop understanding of and solve problems involving their relative sizes. They think of whole numbers between 10 and 100 in terms of tens and ones (especially recognizing the numbers 11 to 19 as composed of a ten and some ones). Through activities that build number sense, they understand the order of the counting numbers and their relative magnitudes.
3. Students develop an understanding of the meaning and processes of measurement, including underlying concepts such as iterating (the mental activity of building up the length of an object with equal-sized units) and the transitivity principle for indirect measurement.[1]
4. 4. Students compose and decompose plane or solid figures (e.g., put two triangles together to make a quadrilateral) and build understanding of part-whole relationships as well as the properties of the original and composite shapes. As they combine shapes, they recognize them from different perspectives and orientations, describe their geometric attributes, and determine how they are alike and different, to develop the background for measurement and for initial understandings of properties such as congruence and symmetry.

Domain: Operations & Algebraic Thinking

Represent and solve problems involving addition and subtraction.

CCSS.Math.Content.1.OA.A.1
Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.[1]

CCSS.Math.Content.1.OA.A.2
Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

Understand and apply properties of operations and the relationship between addition and subtraction.

CCSS.Math.Content.1.OA.B.3
Apply properties of operations as strategies to add and subtract.[2] *Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Associative property of addition.)*

CCSS.Math.Content.1.OA.B.4
Understand subtraction as an unknown-addend problem. *For example, subtract 10 - 8 by finding the number that makes 10 when added to 8.*

Add and subtract within 20.

CCSS.Math.Content.1.OA.C.5
Relate counting to addition and subtraction (e.g., by counting on 2 to add 2).

CCSS.Math.Content.1.OA.C.6
Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 - 4 = 13 - 3 - 1 = 10 - 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 - 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13).

Work with addition and subtraction equations.

CCSS.Math.Content.1.OA.D.7
Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. For example, which of the following equations are true and which are false? 6 = 6, 7 = 8 - 1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2.

CCSS.Math.Content.1.OA.D.8
Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 + ? = 11, 5 = _ - 3, 6 + 6 = _.*

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[1] See Glossary, Table 1

[2] Students need not use formal terms for these properties.

Domain: Number & Operations in Base Ten

Extend the counting sequence.

CCSS.Math.Content.1.NBT.A.1
Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral.

Understand place value.

CCSS.Math.Content.1.NBT.B.2
Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:

CCSS.Math.Content.1.NBT.B.2.a
10 can be thought of as a bundle of ten ones — called a "ten."

CCSS.Math.Content.1.NBT.B.2.b
The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.

CCSS.Math.Content.1.NBT.B.2.c
The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).

CCSS.Math.Content.1.NBT.B.3
Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols >, =, and <.

Use place value understanding and properties of operations to add and subtract.

CCSS.Math.Content.1.NBT.C.4
Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or

the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.

CCSS.Math.Content.1.NBT.C.5
Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.

CCSS.Math.Content.1.NBT.C.6
Subtract multiples of 10 in the range 10-90 from multiples of 10 in the range 10-90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Domain: Measurement & Data

Measure lengths indirectly and by iterating length units.

CCSS.Math.Content.1.MD.A.1
Order three objects by length; compare the lengths of two objects indirectly by using a third object.

CCSS.Math.Content.1.MD.A.2
Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. *Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps*.

Tell and write time.

CCSS.Math.Content.1.MD.B.3
Tell and write time in hours and half-hours using analog and digital clocks.

Represent and interpret data.

CCSS.Math.Content.1.MD.C.4
Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.

Domain: Operations & Algebraic Thinking

Reason with shapes and their attributes.

CCSS.Math.Content.1.G.A.1
Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes.

CCSS.Math.Content.1.G.A.2
Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.[1]

CCSS.Math.Content.1.G.A.3
Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

[1] Students should apply the principle of transitivity of measurement to make indirect comparisons, but they need not use this technical term.

Source: Common Core State Standards: Grade 1

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1.4.3: Grade 2 Standards

In Grade 2, instructional time should focus on four critical areas: (1) extending understanding of base-ten notation; (2) building fluency with addition and subtraction; (3) using standard units of measure; and (4) describing and analyzing shapes.

1. Students extend their understanding of the base-ten system. This includes ideas of counting in fives, tens, and multiples of hundreds, tens, and ones, as well as number relationships involving these units, including comparing. Students understand multi-digit numbers (up to 1000) written in base-ten notation, recognizing that the digits in each place represent amounts of thousands, hundreds, tens, or ones (e.g., 853 is 8 hundreds + 5 tens + 3 ones).
2. Students use their understanding of addition to develop fluency with addition and subtraction within 100. They solve problems within 1000 by applying their understanding of models for addition and subtraction, and they develop, discuss, and use efficient, accurate, and generalizable methods to compute sums and differences of whole numbers in base-ten notation, using their understanding of place value and the properties of operations. They select and accurately apply methods that are appropriate for the context and the numbers involved to mentally calculate sums and differences for numbers with only tens or only hundreds.
3. Students recognize the need for standard units of measure (centimeter and inch) and they use rulers and other measurement tools with the understanding that linear measure involves an iteration of units. They recognize that the smaller the unit, the more iterations they need to cover a given length.
4. Students describe and analyze shapes by examining their sides and angles. Students investigate, describe, and reason about decomposing and combining shapes to make other shapes. Through building, drawing, and analyzing two- and three-dimensional shapes, students develop a foundation for understanding area, volume, congruence, similarity, and symmetry in later grades.

Domain: Operations & Algebraic Thinking

Represent and solve problems involving addition and subtraction.

CCSS.Math.Content.2.OA.A.1
Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[1]

Add and subtract within 20.

CCSS.Math.Content.2.OA.B.2
Fluently add and subtract within 20 using mental strategies.[2] By end of Grade 2, know from memory all sums of two one-digit numbers.

Work with equal groups of objects to gain foundations for multiplication.

CCSS.Math.Content.2.OA.C.3
Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.

CCSS.Math.Content.2.OA.C.4
Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.

[1] See Glossary, Table 1.
[2] See standard 1.OA.6 for a list of mental strategies.

Domain: Number & Operations in Base Ten

Understand place value.

CCSS.Math.Content.2.NBT.A.1
Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:

CCSS.Math.Content.2.NBT.A.1.a
100 can be thought of as a bundle of ten tens — called a "hundred."

CCSS.Math.Content.2.NBT.A.1.b
The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).

CCSS.Math.Content.2.NBT.A.2
Count within 1000; skip-count by 5s, 10s, and 100s.

CCSS.Math.Content.2.NBT.A.3
Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.

CCSS.Math.Content.2.NBT.A.4
Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using >, =, and < symbols to record the results of comparisons.

Use place value understanding and properties of operations to add and subtract.

CCSS.Math.Content.2.NBT.B.5
Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

CCSS.Math.Content.2.NBT.B.6
Add up to four two-digit numbers using strategies based on place value and properties of operations.

CCSS.Math.Content.2.NBT.B.7
Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

CCSS.Math.Content.2.NBT.B.8
Mentally add 10 or 100 to a given number 100-900, and mentally subtract 10 or 100 from a given number 100-900.

CCSS.Math.Content.2.NBT.B.9
Explain why addition and subtraction strategies work, using place value and the properties of operations.[1]

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[1] Explanations may be supported by drawings or objects.

Domain: Measurement & Data

Measure and estimate lengths in standard units.

CCSS.Math.Content.2.MD.A.1
Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

CCSS.Math.Content.2.MD.A.2
Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.

CCSS.Math.Content.2.MD.A.3
Estimate lengths using units of inches, feet, centimeters, and meters.

CCSS.Math.Content.2.MD.A.4
Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

Relate addition and subtraction to length.

CCSS.Math.Content.2.MD.B.5
Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

CCSS.Math.Content.2.MD.B.6
Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, ..., and represent whole-number sums and differences within 100 on a number line diagram.

Work with time and money.

CCSS.Math.Content.2.MD.C.7
Tell and write time from analog and digital clocks to the nearest five minutes, using a.m. and p.m.

CCSS.Math.Content.2.MD.C.8
Solve word problems involving dollar bills, quarters, dimes, nickels, and pennies, using $ and ¢ symbols appropriately. Example: If you have 2 dimes and 3 pennies, how many cents do you have?

Represent and interpret data.

CCSS.Math.Content.2.MD.D.9
Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object. Show the measurements by making a line plot, where the horizontal scale is marked off in whole-number units.

CCSS.Math.Content.2.MD.D.10
Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems[1] using information presented in a bar graph.

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[1] See Glossary, Table 1

Domain: Geometry

Reason with shapes and their attributes.

CCSS.Math.Content.2.G.A.1
Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.[1] Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.

CCSS.Math.Content.2.G.A.2
Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

CCSS.Math.Content.2.G.A.3
Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words halves, thirds, half of, a third of, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

[1] Sizes are compared directly or visually, not compared by measuring.

Source: Common Core Standards: Grade 2

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1.4.4: Grade 3 Standards

In Grade 3, instructional time should focus on four critical areas: (1) developing understanding of multiplication and division and strategies for multiplication and division within 100; (2) developing understanding of fractions, especially unit fractions (fractions with numerator 1); (3) developing understanding of the structure of rectangular arrays and of area; and (4) describing and analyzing two-dimensional shapes.

1. Students develop an understanding of the meanings of multiplication and division of whole numbers through activities and problems involving equal-sized groups, arrays, and area models; multiplication is finding an unknown product, and division is finding an unknown factor in these situations. For equal-sized group situations, division can require finding the unknown number of groups or the unknown group size. Students use properties of operations to calculate products of whole numbers, using increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving single-digit factors. By comparing a variety of solution strategies, students learn the relationship between multiplication and division.
2. Students develop an understanding of fractions, beginning with unit fractions. Students view fractions in general as being built out of unit fractions, and they use fractions along with visual fraction models to represent parts of a whole. Students understand that the size of a fractional part is relative to the size of the whole. For example, 1/2 of the paint in a small bucket could be less paint than 1/3 of the paint in a larger bucket, but 1/3 of a ribbon is longer than 1/5 of the same ribbon because when the ribbon is divided into 3 equal parts, the parts are longer than when the ribbon is divided into 5 equal parts. Students are able to use fractions to represent numbers equal to, less than, and greater than one. They solve problems that involve comparing fractions by using visual fraction models and strategies based on noticing equal numerators or denominators.
3. Students recognize area as an attribute of two-dimensional regions. They measure the area of a shape by finding the total number of same-size units of area required to cover the shape without gaps or overlaps, a square with sides of unit length being the standard unit for measuring area. Students understand that rectangular arrays can be decomposed into identical rows or into identical columns. By decomposing rectangles into rectangular arrays of squares, students connect area to multiplication, and justify using multiplication to determine the area of a rectangle.
4. Students describe, analyze, and compare properties of two-dimensional shapes. They compare and classify shapes by their sides and angles, and connect these with definitions of shapes. Students also relate their fraction work to geometry by expressing the area of part of a shape as a unit fraction of the whole.

Domain: Operations & Algebraic Thinking

Represent and solve problems involving multiplication and division.

CCSS.Math.Content.3.OA.A.1
Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. *For example, describe a context in which a total number of objects can be expressed as 5 × 7.*

CCSS.Math.Content.3.OA.A.2
Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8.*

CCSS.Math.Content.3.OA.A.3
Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[1]

CCSS.Math.Content.3.OA.A.4
Determine the unknown whole number in a multiplication or division equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 × ? = 48, 5 = _ ÷ 3, 6 × 6 = ?*

Understand properties of multiplication and the relationship between multiplication and division.

CCSS.Math.Content.3.OA.B.5

Apply properties of operations as strategies to multiply and divide.[2] *Examples: If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known. (Commutative property of multiplication.) 3 × 5 × 2 can be found by 3 × 5 = 15, then 15 × 2 = 30, or by 5 × 2 = 10, then 3 × 10 = 30. (Associative property of multiplication.) Knowing that 8 × 5 = 40 and 8 × 2 = 16, one can find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56. (Distributive property.)*

CCSS.Math.Content.3.OA.B.6

Understand division as an unknown-factor problem. *For example, find 32 ÷ 8 by finding the number that makes 32 when multiplied by 8*.

Multiply and divide within 100.

CCSS.Math.Content.3.OA.C.7

Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

Solve problems involving the four operations, and identify and explain patterns in arithmetic.

CCSS.Math.Content.3.OA.D.8

Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.[3]

CCSS.Math.Content.3.OA.D.9

Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. *For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends*.

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[1] See Glossary, Table 2.

[2] Students need not use formal terms for these properties.

[3] This standard is limited to problems posed with whole numbers and having whole-number answers; students should know how to perform operations in conventional order when there are no parentheses to specify a particular order (Order of Operations).

Domain: Number & Operations in Base Ten

Use place value understanding and properties of operations to perform multi-digit arithmetic.[1]

CCSS.Math.Content.3.NBT.A.1

Use place value understanding to round whole numbers to the nearest 10 or 100.

CCSS.Math.Content.3.NBT.A.2

Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

CCSS.Math.Content.3.NBT.A.3

Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations.

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[1] A range of algorithms may be used.

Domain: Number & Operations—Fractions[1]

Develop understanding of fractions as numbers.

CCSS.Math.Content.3.NF.A.1
Understand a fraction 1/*b* as the quantity formed by 1 part when a whole is partitioned into *b* equal parts; understand a fraction *a*/*b* as the quantity formed by *a* parts of size 1/*b*.

CCSS.Math.Content.3.NF.A.2
Understand a fraction as a number on the number line; represent fractions on a number line diagram.

CCSS.Math.Content.3.NF.A.2.a
Represent a fraction 1/*b* on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into *b* equal parts. Recognize that each part has size 1/*b* and that the endpoint of the part based at 0 locates the number 1/*b* on the number line.

CCSS.Math.Content.3.NF.A.2.b
Represent a fraction *a*/*b* on a number line diagram by marking off a lengths 1/*b* from 0. Recognize that the resulting interval has size *a*/*b* and that its endpoint locates the number *a*/*b* on the number line.

CCSS.Math.Content.3.NF.A.3
Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

CCSS.Math.Content.3.NF.A.3.a
Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

CCSS.Math.Content.3.NF.A.3.b
Recognize and generate simple equivalent fractions, e.g., 1/2 = 2/4, 4/6 = 2/3. Explain why the fractions are equivalent, e.g., by using a visual fraction model.

CCSS.Math.Content.3.NF.A.3.c
Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. *Examples: Express 3 in the form 3 = 3/1; recognize that 6/1 = 6; locate 4/4 and 1 at the same point of a number line diagram*.

CCSS.Math.Content.3.NF.A.3.d
Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

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[1] Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.

Domain: Measurement & Data

Solve problems involving measurement and estimation.

CCSS.Math.Content.3.MD.A.1
Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

CCSS.Math.Content.3.MD.A.2
Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l).[1] Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.[2]

Represent and interpret data.

CCSS.Math.Content.3.MD.B.3
Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. *For example, draw a bar graph in which each square in the bar graph might represent 5 pets*.

CCSS.Math.Content.3.MD.B.4
Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units— whole numbers, halves, or quarters.

Geometric measurement: understand concepts of area and relate area to multiplication and to addition.

CCSS.Math.Content.3.MD.C.5
Recognize area as an attribute of plane figures and understand concepts of area measurement.

CCSS.Math.Content.3.MD.C.5.a
A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.

CCSS.Math.Content.3.MD.C.5.b
A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.

CCSS.Math.Content.3.MD.C.6
Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

CCSS.Math.Content.3.MD.C.7
Relate area to the operations of multiplication and addition.

CCSS.Math.Content.3.MD.C.7.a
Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.

CCSS.Math.Content.3.MD.C.7.b
Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.

CCSS.Math.Content.3.MD.C.7.c
Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.

CCSS.Math.Content.3.MD.C.7.d
Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.

Geometric measurement: recognize perimeter.

CCSS.Math.Content.3.MD.D.8
Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

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[1] Excludes compound units such as cm^3 and finding the geometric volume of a container

[2] Excludes multiplicative comparison problems (problems involving notions of “times as much”; see Glossary, Table 2).

Domain: Geometry

Reason with shapes and their attributes.

CCSS.Math.Content.3.G.A.1
Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.

CCSS.Math.Content.3.G.A.2
Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. *For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape*.

Source: Common Core State Standards: Grade 3, Introduction

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1.4.5: Grade 4 Standards

In Grade 4, instructional time should focus on three critical areas: (1) developing understanding and fluency with multi-digit multiplication, and developing understanding of dividing to find quotients involving multi-digit dividends; (2) developing an understanding of fraction equivalence, addition and subtraction of fractions with like denominators, and multiplication of fractions by whole numbers; (3) understanding that geometric figures can be analyzed and classified based on their properties, such as having parallel sides, perpendicular sides, particular angle measures, and symmetry.

1. Students generalize their understanding of place value to 1,000,000, understanding the relative sizes of numbers in each place. They apply their understanding of models for multiplication (equal-sized groups, arrays, area models), place value, and properties of operations, in particular the distributive property, as they develop, discuss, and use efficient, accurate, and generalizable methods to compute products of multi-digit whole numbers. Depending on the numbers and the context, they select and accurately apply appropriate methods to estimate or mentally calculate products. They develop fluency with efficient procedures for multiplying whole numbers; understand and explain why the procedures work based on place value and properties of operations; and use them to solve problems. Students apply their understanding of models for division, place value, properties of operations, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multi-digit dividends. They select and accurately apply appropriate methods to estimate and mentally calculate quotients, and interpret remainders based upon the context.
2. Students develop understanding of fraction equivalence and operations with fractions. They recognize that two different fractions can be equal (e.g., 15/9 = 5/3), and they develop methods for generating and recognizing equivalent fractions. Students extend previous understandings about how fractions are built from unit fractions, composing fractions from unit fractions, decomposing fractions into unit fractions, and using the meaning of fractions and the meaning of multiplication to multiply a fraction by a whole number.
3. Students describe, analyze, compare, and classify two-dimensional shapes. Through building, drawing, and analyzing two-dimensional shapes, students deepen their understanding of properties of two-dimensional objects and the use of them to solve problems involving symmetry.

Domain: Operations & Algebraic Thinking

Use the four operations with whole numbers to solve problems.

CCSS.Math.Content.4.OA.A.1
Interpret a multiplication equation as a comparison, e.g., interpret 35 = 5 × 7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

CCSS.Math.Content.4.OA.A.2
Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison.[1]

CCSS.Math.Content.4.OA.A.3
Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Gain familiarity with factors and multiples.

CCSS.Math.Content.4.OA.B.4
Find all factor pairs for a whole number in the range 1-100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1-100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1-100 is prime or composite.

Generate and analyze patterns.

CCSS.Math.Content.4.OA.C.5
Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. *For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.*

[1] See Glossary, Table 2.

Domain: Number & Operations in Base Ten

Generalize place value understanding for multi-digit whole numbers.

CCSS.Math.Content.4.NBT.A.1
Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. *For example, recognize that 700 ÷ 70 = 10 by applying concepts of place value and division.*

CCSS.Math.Content.4.NBT.A.2
Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.

CCSS.Math.Content.4.NBT.A.3
Use place value understanding to round multi-digit whole numbers to any place.

Use place value understanding and properties of operations to perform multi-digit arithmetic.

CCSS.Math.Content.4.NBT.B.4
Fluently add and subtract multi-digit whole numbers using the standard algorithm.

CCSS.Math.Content.4.NBT.B.5
Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

CCSS.Math.Content.4.NBT.B.6
Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

[1] Grade 4 expectations in this domain are limited to whole numbers less than or equal to 1,000,000.

Domain: Number & Operations—Fractions[1]

Extend understanding of fraction equivalence and ordering.

CCSS.Math.Content.4.NF.A.1
Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

CCSS.Math.Content.4.NF.A.2
Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Build fractions from unit fractions.

CCSS.Math.Content.4.NF.B.3
Understand a fraction *a*/*b* with *a* > 1 as a sum of fractions 1/*b*.

CCSS.Math.Content.4.NF.B.3.a
Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.

CCSS.Math.Content.4.NF.B.3.b
Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model. *Examples: 3/8 = 1/8 + 1/8 + 1/8 ; 3/8 = 1/8 + 2/8 ; 2 1/8 = 1 + 1 + 1/8 = 8/8 + 8/8 + 1/8*.

CCSS.Math.Content.4.NF.B.3.c
Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.

CCSS.Math.Content.4.NF.B.3.d
Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.

CCSS.Math.Content.4.NF.B.4
Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

CCSS.Math.Content.4.NF.B.4.a
Understand a fraction *a*/*b* as a multiple of 1/*b*. *For example, use a visual fraction model to represent 5/4 as the product 5 × (1/4), recording the conclusion by the equation 5/4 = 5 × (1/4).*

CCSS.Math.Content.4.NF.B.4.b
Understand a multiple of a/b as a multiple of 1/b, and use this understanding to multiply a fraction by a whole number. *For example, use a visual fraction model to express 3 × (2/5) as 6 × (1/5), recognizing this product as 6/5. (In general, n × (a/b) = (n × a)/b.)*

CCSS.Math.Content.4.NF.B.4.c
Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem. *For example, if each person at a party will eat 3/8 of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?*

Understand decimal notation for fractions, and compare decimal fractions.

CCSS.Math.Content.4.NF.C.5
Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[2] *For example, express 3/10 as 30/100, and add 3/10 + 4/100 = 34/100.*

CCSS.Math.Content.4.NF.C.6
Use decimal notation for fractions with denominators 10 or 100. *For example, rewrite 0.62 as 62/100; describe a length as 0.62 meters; locate 0.62 on a number line diagram.*

CCSS.Math.Content.4.NF.C.7
Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual model.

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[1] Grade 4 expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.

[2] Students who can generate equivalent fractions can develop strategies for adding fractions with unlike denominators in general. But addition and subtraction with unlike denominators in general is not a requirement at this grade.

Domain: Measurement & Data

Solve problems involving measurement and conversion of measurements.

CCSS.Math.Content.4.MD.A.1
Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. *For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), ...*

CCSS.Math.Content.4.MD.A.2
Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

CCSS.Math.Content.4.MD.A.3
Apply the area and perimeter formulas for rectangles in real world and mathematical problems. *For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.*

Represent and interpret data.

CCSS.Math.Content.4.MD.B.4
Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Solve problems involving addition and subtraction of fractions by using information presented in line plots. *For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*

Geometric measurement: understand concepts of angle and measure angles.

CCSS.Math.Content.4.MD.C.5
Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:

CCSS.Math.Content.4.MD.C.5.a
An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles.

CCSS.Math.Content.4.MD.C.5.b
An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees.

CCSS.Math.Content.4.MD.C.6
Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.

CCSS.Math.Content.4.MD.C.7
Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

Domain: Geometry

Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

CCSS.Math.Content.4.G.A.1
Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

CCSS.Math.Content.4.G.A.2
Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

CCSS.Math.Content.4.G.A.3
Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

Source: Common Core State Standards: Grade 4

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1.4.6: Grade 5 Standards

In Grade 5, instructional time should focus on three critical areas: (1) developing fluency with addition and subtraction of fractions, and developing understanding of the multiplication of fractions and of division of fractions in limited cases (unit fractions divided by whole numbers and whole numbers divided by unit fractions); (2) extending division to 2-digit divisors, integrating decimal fractions into the place value system and developing understanding of operations with decimals to hundredths, and developing fluency with whole number and decimal operations; and (3) developing understanding of volume.

1. Students apply their understanding of fractions and fraction models to represent the addition and subtraction of fractions with unlike denominators as equivalent calculations with like denominators. They develop fluency in calculating sums and differences of fractions, and make reasonable estimates of them. Students also use the meaning of fractions, of multiplication and division, and the relationship between multiplication and division to understand and explain why the procedures for multiplying and dividing fractions make sense. (Note: this is limited to the case of dividing unit fractions by whole numbers and whole numbers by unit fractions.)
2. Students develop understanding of why division procedures work based on the meaning of base-ten numerals and properties of operations. They finalize fluency with multi-digit addition, subtraction, multiplication, and division. They apply their understandings of models for decimals, decimal notation, and properties of operations to add and subtract decimals to hundredths. They develop fluency in these computations, and make reasonable estimates of their results. Students use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain why the procedures for multiplying and dividing finite decimals make sense. They compute products and quotients of decimals to hundredths efficiently and accurately.
3. Students recognize volume as an attribute of three-dimensional space. They understand that volume can be measured by finding the total number of same-size units of volume required to fill the space without gaps or overlaps. They understand that a 1-unit by 1-unit by 1-unit cube is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating and measuring volume. They decompose three-dimensional shapes and find volumes of right rectangular prisms by viewing them as decomposed into layers of arrays of cubes. They measure necessary attributes of shapes in order to determine volumes to solve real world and mathematical problems.

Domain: Operations & Algebraic Thinking

Write and interpret numerical expressions.

CCSS.Math.Content.5.OA.A.1
Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.

CCSS.Math.Content.5.OA.A.2
Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. *For example, express the calculation "add 8 and 7, then multiply by 2" as 2 × (8 + 7). Recognize that 3 × (18932 + 921) is three times as large as 18932 + 921, without having to calculate the indicated sum or product.*

Analyze patterns and relationships.

CCSS.Math.Content.5.OA.B.3
Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. *For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6" and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.*

Domain: Number & Operations in Base Ten

Understand the place value system.

CCSS.Math.Content.5.NBT.A.1
Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.

CCSS.Math.Content.5.NBT.A.2
Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.

CCSS.Math.Content.5.NBT.A.3
Read, write, and compare decimals to thousandths.

CCSS.Math.Content.5.NBT.A.3.a
Read and write decimals to thousandths using base-ten numerals, number names, and expanded form, e.g., 347.392 = 3 × 100 + 4 × 10 + 7 × 1 + 3 × (1/10) + 9 × (1/100) + 2 × (1/1000).

CCSS.Math.Content.5.NBT.A.3.b
Compare two decimals to thousandths based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.

CCSS.Math.Content.5.NBT.A.4
Use place value understanding to round decimals to any place.

Perform operations with multi-digit whole numbers and with decimals to hundredths.

CCSS.Math.Content.5.NBT.B.5
Fluently multiply multi-digit whole numbers using the standard algorithm.

CCSS.Math.Content.5.NBT.B.6
Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

CCSS.Math.Content.5.NBT.B.7
Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Domain: Number & Operations—Fractions

Use equivalent fractions as a strategy to add and subtract fractions.

CCSS.Math.Content.5.NF.A.1
Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. *For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12. (In general, a/b + c/d = (ad + bc)/bd.)*

CCSS.Math.Content.5.NF.A.2
Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. *For example, recognize an incorrect result 2/5 + 1/2 = 3/7, by observing that 3/7 < 1/2.*

Apply and extend previous understandings of multiplication and division.

CCSS.Math.Content.5.NF.B.3
Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. *For example, interpret 3/4 as the result of dividing 3 by 4, noting that 3/4 multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size 3/4. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?*

CCSS.Math.Content.5.NF.B.4
Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

CCSS.Math.Content.5.NF.B.4.a
Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$. *For example, use a visual fraction model to show (2/3) × 4 = 8/3, and create a story context for this equation. Do the same with (2/3) × (4/5) = 8/15. (In general, (a/b) × (c/d) = (ac)/(bd).*

CCSS.Math.Content.5.NF.B.4.b
Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

CCSS.Math.Content.5.NF.B.5
Interpret multiplication as scaling (resizing), by:

CCSS.Math.Content.5.NF.B.5.a
Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

CCSS.Math.Content.5.NF.B.5.b
Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

CCSS.Math.Content.5.NF.B.6
Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

CCSS.Math.Content.5.NF.B.7
Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.[1]

CCSS.Math.Content.5.NF.B.7.a
Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. *For example, create a story context for (1/3) ÷ 4, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that (1/3) ÷ 4 = 1/12 because (1/12) × 4 = 1/3.*

CCSS.Math.Content.5.NF.B.7.b
Interpret division of a whole number by a unit fraction, and compute such quotients. *For example, create a story context for 4 ÷ (1/5), and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that 4 ÷ (1/5) = 20 because 20 × (1/5) = 4.*

CCSS.Math.Content.5.NF.B.7.c
Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem. *For example, how much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 1/3-cup servings are in 2 cups of raisins?*

[1] Students able to multiply fractions in general can develop strategies to divide fractions in general, by reasoning about the relationship between multiplication and division. But division of a fraction by a fraction is not a requirement at this grade.

Domain: Measurement & Data

Convert like measurement units within a given measurement system.

CCSS.Math.Content.5.MD.A.1
Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

Represent and interpret data.

CCSS.Math.Content.5.MD.B.2
Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Use operations on fractions for this grade to solve problems involving information presented in line plots. *For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*

Geometric measurement: understand concepts of volume.

CCSS.Math.Content.5.MD.C.3
Recognize volume as an attribute of solid figures and understand concepts of volume measurement.

CCSS.Math.Content.5.MD.C.3.a
A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume.

CCSS.Math.Content.5.MD.C.3.b
A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.

CCSS.Math.Content.5.MD.C.4
Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.

CCSS.Math.Content.5.MD.C.5
Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume.

CCSS.Math.Content.5.MD.C.5.a
Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.

CCSS.Math.Content.5.MD.C.5.b
Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.

CCSS.Math.Content.5.MD.C.5.c
Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.

Domain: Geometry

Graph points on the coordinate plane to solve real-world and mathematical problems.

CCSS.Math.Content.5.G.A.1
Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., *x*-axis and *x*-coordinate, *y*-axis and *y*-coordinate).

CCSS.Math.Content.5.G.A.2
Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.

Classify two-dimensional figures into categories based on their properties.

CCSS.Math.Content.5.G.B.3
Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.

CCSS.Math.Content.5.G.B.4
Classify two-dimensional figures in a hierarchy based on properties.

Source: Common Core State Standards: Grade 5

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1.4.7: Grade 6 Standards

In Grade 6, instructional time should focus on four critical areas: (1) connecting ratio and rate to whole number multiplication and division and using concepts of ratio and rate to solve problems; (2) completing understanding of division of fractions and extending the notion of number to the system of rational numbers, which includes negative numbers; (3) writing, interpreting, and using expressions and equations; and (4) developing understanding of statistical thinking.

1. Students use reasoning about multiplication and division to solve ratio and rate problems about quantities. By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative size of quantities, students connect their understanding of multiplication and division with ratios and rates. Thus students expand the scope of problems for which they can use multiplication and division to solve problems, and they connect ratios and fractions. Students solve a wide variety of problems involving ratios and rates.
2. Students use the meaning of fractions, the meanings of multiplication and division, and the relationship between multiplication and division to understand and explain why the procedures for dividing fractions make sense. Students use these operations to solve problems. Students extend their previous understandings of number and the ordering of numbers to the full system of rational numbers, which includes negative rational numbers, and in particular negative integers. They reason about the order and absolute value of rational numbers and about the location of points in all four quadrants of the coordinate plane.
3. Students understand the use of variables in mathematical expressions. They write expressions and equations that correspond to given situations, evaluate expressions, and use expressions and formulas to solve problems. Students understand that expressions in different forms can be equivalent, and they use the properties of operations to rewrite expressions in equivalent forms. Students know that the solutions of an equation are the values of the variables that make the equation true. Students use properties of operations and the idea of maintaining the equality of both sides of an equation to solve simple one-step equations. Students construct and analyze tables, such as tables of quantities that are in equivalent ratios, and they use equations (such as $3x = y$) to describe relationships between quantities.
4. Building on and reinforcing their understanding of number, students begin to develop their ability to think statistically. Students recognize that a data distribution may not have a definite center and that different ways to measure center yield different values. The median measures center in the sense that it is roughly the middle value. The mean measures center in the sense that it is the value that each data point would take on if the total of the data values were redistributed equally, and also in the sense that it is a balance point. Students recognize that a measure of variability (interquartile range or mean absolute deviation) can also be useful for summarizing data because two very different sets of data can have the same mean and median yet be distinguished by their variability.

   Students learn to describe and summarize numerical data sets, identifying clusters, peaks, gaps, and symmetry, considering the context in which the data were collected. Students in Grade 6 also build on their work with area in elementary school by reasoning about relationships among shapes to determine area, surface area, and volume. They find areas of right triangles, other triangles, and special quadrilaterals by decomposing these shapes, rearranging or removing pieces, and relating the shapes to rectangles. Using these methods, students discuss, develop, and justify formulas for areas of triangles and parallelograms. Students find areas of polygons and surface areas of prisms and pyramids by decomposing them into pieces whose area they can determine. They reason about right rectangular prisms with fractional side lengths to extend formulas for the volume of a right rectangular prism to fractional side lengths. They prepare for work on scale drawings and constructions in Grade 7 by drawing polygons in the coordinate plane.

Domain: Ratios & Proportional Relationships

Understand ratio concepts and use ratio reasoning to solve problems.

CCSS.Math.Content.6.RP.A.1
Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. *For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."*

CCSS.Math.Content.6.RP.A.2
Understand the concept of a unit rate a/b associated with a ratio a:b with b ≠ 0, and use rate language in the context of a ratio relationship. *For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is 3/4 cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."*[1]

CCSS.Math.Content.6.RP.A.3
Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

CCSS.Math.Content.6.RP.A.3.a
Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

CCSS.Math.Content.6.RP.A.3.b
Solve unit rate problems including those involving unit pricing and constant speed. *For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?*

CCSS.Math.Content.6.RP.A.3.c
Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.

CCSS.Math.Content.6.RP.A.3.d
Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

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[1] Expectations for unit rates in this grade are limited to non-complex fractions.

Domain: The Number System

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

CCSS.Math.Content.6.NS.A.1
Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. *For example, create a story context for (2/3) ÷ (3/4) and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that (2/3) ÷ (3/4) = 8/9 because 3/4 of 8/9 is 2/3. (In general, (a/b) ÷ (c/d) = ad/bc.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?.*

Compute fluently with multi-digit numbers and find common factors and multiples.

CCSS.Math.Content.6.NS.B.2
Fluently divide multi-digit numbers using the standard algorithm.

CCSS.Math.Content.6.NS.B.3
Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

CCSS.Math.Content.6.NS.B.4
Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1-100 with a common factor as a multiple of a sum of two whole numbers with no common factor. *For example, express 36 + 8 as 4 (9 + 2).*.

Apply and extend previous understandings of numbers to the system of rational numbers.

CCSS.Math.Content.6.NS.C.5
Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.

CCSS.Math.Content.6.NS.C.6
Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

CCSS.Math.Content.6.NS.C.6.a
Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., -(-3) = 3, and that 0 is its own opposite.

CCSS.Math.Content.6.NS.C.6.b
Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.

CCSS.Math.Content.6.NS.C.6.c
Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.

CCSS.Math.Content.6.NS.C.7
Understand ordering and absolute value of rational numbers.

CCSS.Math.Content.6.NS.C.7.a
Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram. *For example, interpret -3 > -7 as a statement that -3 is located to the right of -7 on a number line oriented from left to right.*

CCSS.Math.Content.6.NS.C.7.b
Write, interpret, and explain statements of order for rational numbers in real-world contexts. *For example, write -3 ^{o}C > -7 ^{o}C to express the fact that -3 ^{o}C is warmer than -7 ^{o}C.*

CCSS.Math.Content.6.NS.C.7.c
Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. *For example, for an account balance of -30 dollars, write |-30| = 30 to describe the size of the debt in dollars.*

CCSS.Math.Content.6.NS.C.7.d
Distinguish comparisons of absolute value from statements about order. *For example, recognize that an account balance less than -30 dollars represents a debt greater than 30 dollars.*

CCSS.Math.Content.6.NS.C.8
Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

Domain: Expressions & Equations

Apply and extend previous understandings of arithmetic to algebraic expressions.

CCSS.Math.Content.6.EE.A.1
Write and evaluate numerical expressions involving whole-number exponents.

CCSS.Math.Content.6.EE.A.2
Write, read, and evaluate expressions in which letters stand for numbers.

CCSS.Math.Content.6.EE.A.2.a
Write expressions that record operations with numbers and with letters standing for numbers. *For example, express the calculation "Subtract y from 5" as 5 - y.*

CCSS.Math.Content.6.EE.A.2.b
Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. *For example, describe the expression 2 (8 + 7) as a product of two factors; view (8 + 7) as both a single entity and a sum of two terms.*

CCSS.Math.Content.6.EE.A.2.c
Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). *For example, use the formulas $V = s^3$ and $A = 6 s^2$ to find the volume and surface area of a cube with sides of length s = 1/2.*

CCSS.Math.Content.6.EE.A.3
Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3 (2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6 (4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.*

CCSS.Math.Content.6.EE.A.4
Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). *For example, the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for..*

Reason about and solve one-variable equations and inequalities.

CCSS.Math.Content.6.EE.B.5
Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

CCSS.Math.Content.6.EE.B.6
Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

CCSS.Math.Content.6.EE.B.7
Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and x are all nonnegative rational numbers.

CCSS.Math.Content.6.EE.B.8
Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x <$ c have infinitely many solutions; represent solutions of such inequalities on number line diagrams.

Represent and analyze quantitative relationships between dependent and independent variables.

CCSS.Math.Content.6.EE.C.9
Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation d = 65t to represent the relationship between distance and time.

Domain: Geometry

Solve real-world and mathematical problems involving area, surface area, and volume.

CCSS.Math.Content.6.G.A.1
Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

CCSS.Math.Content.6.G.A.2
Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = l\ w\ h$ and $V = b\ h$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

CCSS.Math.Content.6.G.A.3
Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.

CCSS.Math.Content.6.G.A.4
Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

Domain: Statistics & Probability

Develop understanding of statistical variability.

CCSS.Math.Content.6.SP.A.1
Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers. *For example, "How old am I?" is not a statistical question, but "How old are the students in my school?" is a statistical question because one anticipates variability in students' ages.*

CCSS.Math.Content.6.SP.A.2
Understand that a set of data collected to answer a statistical question has a distribution which can be described by its center, spread, and overall shape.

CCSS.Math.Content.6.SP.A.3
Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.

Summarize and describe distributions.

CCSS.Math.Content.6.SP.B.4
Display numerical data in plots on a number line, including dot plots, histograms, and box plots.

CCSS.Math.Content.6.SP.B.5
Summarize numerical data sets in relation to their context, such as by:

CCSS.Math.Content.6.SP.B.5.a
Reporting the number of observations.

CCSS.Math.Content.6.SP.B.5.b
Describing the nature of the attribute under investigation, including how it was measured and its units of measurement.

CCSS.Math.Content.6.SP.B.5.c
Giving quantitative measures of center (median and/or mean) and variability (interquartile range and/or mean absolute deviation), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered.

CCSS.Math.Content.6.SP.B.5.d
Relating the choice of measures of center and variability to the shape of the data distribution and the context in which the data were gathered.

Source: Common Core State Standards: Grade 6

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1.4.8: Grade 7 Standards

In Grade 7, instructional time should focus on four critical areas: (1) developing understanding of and applying proportional relationships; (2) developing understanding of operations with rational numbers and working with expressions and linear equations; (3) solving problems involving scale drawings and informal geometric constructions, and working with two- and three-dimensional shapes to solve problems involving area, surface area, and volume; and (4) drawing inferences about populations based on samples.

1. Students extend their understanding of ratios and develop understanding of proportionality to solve single- and multi-step problems. Students use their understanding of ratios and proportionality to solve a wide variety of percent problems, including those involving discounts, interest, taxes, tips, and percent increase or decrease. Students solve problems about scale drawings by relating corresponding lengths between the objects or by using the fact that relationships of lengths within an object are preserved in similar objects. Students graph proportional relationships and understand the unit rate informally as a measure of the steepness of the related line, called the slope. They distinguish proportional relationships from other relationships.
2. Students develop a unified understanding of number, recognizing fractions, decimals (that have a finite or a repeating decimal representation), and percents as different representations of rational numbers. Students extend addition, subtraction, multiplication, and division to all rational numbers, maintaining the properties of operations and the relationships between addition and subtraction, and multiplication and division. By applying these properties, and by viewing negative numbers in terms of everyday contexts (e.g., amounts owed or temperatures below zero), students explain and interpret the rules for adding, subtracting, multiplying, and dividing with negative numbers. They use the arithmetic of rational numbers as they formulate expressions and equations in one variable and use these equations to solve problems.
3. Students continue their work with area from Grade 6, solving problems involving the area and circumference of a circle and surface area of three-dimensional objects. In preparation for work on congruence and similarity in Grade 8 they reason about relationships among two-dimensional figures using scale drawings and informal geometric constructions, and they gain familiarity with the relationships between angles formed by intersecting lines. Students work with three-dimensional figures, relating them to two-dimensional figures by examining cross-sections. They solve real-world and mathematical problems involving area, surface area, and volume of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes and right prisms.
4. Students build on their previous work with single data distributions to compare two data distributions and address questions about differences between populations. They begin informal work with random sampling to generate data sets and learn about the importance of representative samples for drawing inferences.

Domain: Ratios & Proportional Relationships

Analyze proportional relationships and use them to solve real-world and mathematical problems.

CCSS.Math.Content.7.RP.A.1
Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. *For example, if a person walks 1/2 mile in each 1/4 hour, compute the unit rate as the complex fraction* $^{1/2}/_{1/4}$ *miles per hour, equivalently 2 miles per hour.*

CCSS.Math.Content.7.RP.A.2
Recognize and represent proportional relationships between quantities.

CCSS.Math.Content.7.RP.A.2.a
Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

CCSS.Math.Content.7.RP.A.2.b
Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.

CCSS.Math.Content.7.RP.A.2.c
Represent proportional relationships by equations. *For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as t = pn.*

CCSS.Math.Content.7.RP.A.2.d
Explain what a point (*x*, *y*) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and (1, *r*) where r is the unit rate.

CCSS.Math.Content.7.RP.A.3
Use proportional relationships to solve multistep ratio and percent problems. Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.

Domain: The Number System

Apply and extend previous understandings of operations with fractions.

CCSS.Math.Content.7.NS.A.1
Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.

CCSS.Math.Content.7.NS.A.1.a
Describe situations in which opposite quantities combine to make 0. *For example, a hydrogen atom has 0 charge because its two constituents are oppositely charged.*

CCSS.Math.Content.7.NS.A.1.b
Understand $p + q$ as the number located a distance $|q|$ from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.

CCSS.Math.Content.7.NS.A.1.c
Understand subtraction of rational numbers as adding the additive inverse, $p - q = p + (-q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.

CCSS.Math.Content.7.NS.A.1.d
Apply properties of operations as strategies to add and subtract rational numbers.

CCSS.Math.Content.7.NS.A.2
Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.

CCSS.Math.Content.7.NS.A.2.a
Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as (-1)(-1) = 1 and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.

CCSS.Math.Content.7.NS.A.2.b
Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-(p/q) = (-p)/q = p/(-q)$. Interpret quotients of rational numbers by describing real-world contexts.

CCSS.Math.Content.7.NS.A.2.c
Apply properties of operations as strategies to multiply and divide rational numbers.

CCSS.Math.Content.7.NS.A.2.d
Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

CCSS.Math.Content.7.NS.A.3
Solve real-world and mathematical problems involving the four operations with rational numbers.[1]

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[1] Computations with rational numbers extend the rules for manipulating fractions to complex fractions.

Domain: Expressions & Equations

Use properties of operations to generate equivalent expressions.

CCSS.Math.Content.7.EE.A.1
Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.

CCSS.Math.Content.7.EE.A.2
Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. *For example, a + 0.05a = 1.05a means that "increase by 5%" is the same as "multiply by 1.05."*

Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

CCSS.Math.Content.7.EE.B.3
Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. *For example: If a woman making $25 an hour gets a 10% raise, she will make an additional 1/10 of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar 9 3/4 inches long in the center of a door that is 27 1/2 inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.*

CCSS.Math.Content.7.EE.B.4
Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

CCSS.Math.Content.7.EE.B.4.a
Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. *For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?*

CCSS.Math.Content.7.EE.B.4.b
Solve word problems leading to inequalities of the form $px + q > r$ or $px + q < r$, where p, q, and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem. *For example: As a salesperson, you are paid $50 per week plus $3 per sale. This week you want your pay to be at least $100. Write an inequality for the number of sales you need to make, and describe the solutions.*

Domain: Geometry

Draw construct, and describe geometrical figures and describe the relationships between them.

CCSS.Math.Content.7.G.A.1
Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

CCSS.Math.Content.7.G.A.2
Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

CCSS.Math.Content.7.G.A.3
Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

CCSS.Math.Content.7.G.B.4
Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

CCSS.Math.Content.7.G.B.5
Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.

CCSS.Math.Content.7.G.B.6
Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Domain: Statistics & Probability

Use random sampling to draw inferences about a population.

CCSS.Math.Content.7.SP.A.1
Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.

CCSS.Math.Content.7.SP.A.2
Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. *For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.*

Draw informal comparative inferences about two populations.

CCSS.Math.Content.7.SP.B.3
Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. *For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean absolute deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable.*

CCSS.Math.Content.7.SP.B.4
Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. *For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book.*

Investigate chance processes and develop, use, and evaluate probability models.

CCSS.Math.Content.7.SP.C.5
Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around 1/2 indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.

CCSS.Math.Content.7.SP.C.6
Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. *For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times*.

CCSS.Math.Content.7.SP.C.7
Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.

CCSS.Math.Content.7.SP.C.7.a
Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events. *For example, if a student is selected at random from a class, find the probability that Jane will be selected and the probability that a girl will be selected*.

CCSS.Math.Content.7.SP.C.7.b
Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. *For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies?*

CCSS.Math.Content.7.SP.C.8
Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.

CCSS.Math.Content.7.SP.C.8.a
Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs.

CCSS.Math.Content.7.SP.C.8.b
Represent sample spaces for compound events using methods such as organized lists, tables and tree diagrams. For an event described in everyday language (e.g., "rolling double sixes"), identify the outcomes in the sample space which compose the event.

CCSS.Math.Content.7.SP.C.8.c
Design and use a simulation to generate frequencies for compound events. *For example, use random digits as a simulation tool to approximate the answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood?*

Source: Common Core State Standards: Grade 7

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1.4.9: Grade 8 Standards

In Grade 8, instructional time should focus on three critical areas: (1) formulating and reasoning about expressions and equations, including modeling an association in bivariate data with a linear equation, and solving linear equations and systems of linear equations; (2) grasping the concept of a function and using functions to describe quantitative relationships; (3) analyzing two- and three-dimensional space and figures using distance, angle, similarity, and congruence, and understanding and applying the Pythagorean Theorem.

1. Students use linear equations and systems of linear equations to represent, analyze, and solve a variety of problems. Students recognize equations for proportions ($y/x = m$ or $y = mx$) as special linear equations ($y = mx + b$), understanding that the constant of proportionality (m) is the slope, and the graphs are lines through the origin. They understand that the slope (m) of a line is a constant rate of change, so that if the input or x-coordinate changes by an amount A, the output or y-coordinate changes by the amount $m \cdot A$. Students also use a linear equation to describe the association between two quantities in bivariate data (such as arm span vs. height for students in a classroom). At this grade, fitting the model, and assessing its fit to the data are done informally. Interpreting the model in the context of the data requires students to express a relationship between the two quantities in question and to interpret components of the relationship (such as slope and *y*-intercept) in terms of the situation.

   Students strategically choose and efficiently implement procedures to solve linear equations in one variable, understanding that when they use the properties of equality and the concept of logical equivalence, they maintain the solutions of the original equation. Students solve systems of two linear equations in two variables and relate the systems to pairs of lines in the plane; these intersect, are parallel, or are the same line. Students use linear equations, systems of linear equations, linear functions, and their understanding of slope of a line to analyze situations and solve problems.

2. Students grasp the concept of a function as a rule that assigns to each input exactly one output. They understand that functions describe situations where one quantity determines another. They can translate among representations and partial representations of functions (noting that tabular and graphical representations may be partial representations), and they describe how aspects of the function are reflected in the different representations.
3. Students use ideas about distance and angles, how they behave under translations, rotations, reflections, and dilations, and ideas about congruence and similarity to describe and analyze two-dimensional figures and to solve problems. Students show that the sum of the angles in a triangle is the angle formed by a straight line, and that various configurations of lines give rise to similar triangles because of the angles created when a transversal cuts parallel lines. Students understand the statement of the Pythagorean Theorem and its converse, and can explain why the Pythagorean Theorem holds, for example, by decomposing a square in two different ways. They apply the Pythagorean Theorem to find distances between points on the coordinate plane, to find lengths, and to analyze polygons. Students complete their work on volume by solving problems involving cones, cylinders, and spheres.

Domain: The Number System

Know that there are numbers that are not rational, and approximate them by rational numbers.

CCSS.Math.Content.8.NS.A.1
Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

CCSS.Math.Content.8.NS.A.2
Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). *For example, by truncating the decimal expansion of √2, show that √2 is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.*

Domain: Expressions & Equations

Expressions and Equations Work with radicals and integer exponents.

CCSS.Math.Content.8.EE.A.1

Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$.

CCSS.Math.Content.8.EE.A.2

Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

CCSS.Math.Content.8.EE.A.3

Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other. *For example, estimate the population of the United States as 3 times 10^8 and the population of the world as 7 times 10^9, and determine that the world population is more than 20 times larger.*

CCSS.Math.Content.8.EE.A.4

Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology

Understand the connections between proportional relationships, lines, and linear equations.

CCSS.Math.Content.8.EE.B.5

Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.

CCSS.Math.Content.8.EE.B.6

Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation y = mx for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

Analyze and solve linear equations and pairs of simultaneous linear equations.

CCSS.Math.Content.8.EE.C.7

Solve linear equations in one variable.

CCSS.Math.Content.8.EE.C.7.a

Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers).

CCSS.Math.Content.8.EE.C.7.b

Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.

CCSS.Math.Content.8.EE.C.8

Analyze and solve pairs of simultaneous linear equations.

CCSS.Math.Content.8.EE.C.8.a

Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.

CCSS.Math.Content.8.EE.C.8.b
Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example, 3x + 2y = 5 and 3x + 2y = 6 have no solution because 3x + 2y cannot simultaneously be 5 and 6.*

CCSS.Math.Content.8.EE.C.8.c
Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.*

Domain: Functions

Define, evaluate, and compare functions.

CCSS.Math.Content.8.F.A.1
Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.[1]

CCSS.Math.Content.8.F.A.2
Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*

CCSS.Math.Content.8.F.A.3
Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. *For example, the function $A = s^2$ giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4) and (3,9), which are not on a straight line.*

Use functions to model relationships between quantities.

CCSS.Math.Content.8.F.B.4
Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

CCSS.Math.Content.8.F.B.5
Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

[1] Function notation is not required for Grade 8.

Domain: Geometry

Understand congruence and similarity using physical models, transparencies, or geometry software.

CCSS.Math.Content.8.G.A.1
Verify experimentally the properties of rotations, reflections, and translations:

CCSS.Math.Content.8.G.A.1.a
Lines are taken to lines, and line segments to line segments of the same length.

CCSS.Math.Content.8.G.A.1.b
Angles are taken to angles of the same measure.

CCSS.Math.Content.8.G.A.1.c
Parallel lines are taken to parallel lines.

CCSS.Math.Content.8.G.A.2
Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

CCSS.Math.Content.8.G.A.3
Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.

CCSS.Math.Content.8.G.A.4
Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.

CCSS.Math.Content.8.G.A.5
Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. *For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so*.

Understand and apply the Pythagorean Theorem.

CCSS.Math.Content.8.G.B.6
Explain a proof of the Pythagorean Theorem and its converse.

CCSS.Math.Content.8.G.B.7
Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

CCSS.Math.Content.8.G.B.8
Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

CCSS.Math.Content.8.G.C.9
Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

Domain: Statistics & Probability

Investigate patterns of association in bivariate data.

CCSS.Math.Content.8.SP.A.1
Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and nonlinear association.

CCSS.Math.Content.8.SP.A.2
Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line.

CCSS.Math.Content.8.SP.A.3
Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept. *For example, in a linear model for a biology experiment, interpret a slope of 1.5 cm/hr as meaning that an additional hour of sunlight each day is associated with an additional 1.5 cm in mature plant height.*

CCSS.Math.Content.8.SP.A.4
Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables. *For example, collect data from students in your class on whether or not they have a curfew on school nights and whether or not they have assigned chores at home. Is there evidence that those who have a curfew also tend to have chores?*

Source: Common Core State Standards: Grade 8, Introduction

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1.4.10: High School Number and Quantity Standards

Numbers and Number Systems.

During the years from kindergarten to eighth grade, students must repeatedly extend their conception of numbers. At first, "number" means "counting number": 1, 2, 3, ... Soon after that, 0 is used to represent "none" and the whole numbers are formed by counting numbers together with zero. The next extension is fractions. At first, fractions are barely numbers and tied strongly to pictorial representations. Yet by the time students understand the division of fractions, they have a strong concept of fractions as numbers and have connected them, via their decimal representations, with the base-ten system used to represent the whole numbers. During middle school, fractions are augmented by negative fractions to form the rational numbers. In Grade 8, students extend this system once more, augmenting the rational numbers with the irrational numbers to form the real numbers. In high school, students will be exposed to yet another extension of number, when the real numbers are augmented by the imaginary numbers to form the complex numbers.

This ascent through number systems makes it fair to ask: what does the word number mean that it can mean all of these things? One possible answer is that a number is something that can be used to do mathematics: calculate, solve equations, or represent measurements.

Although the notion of number changes, the four operations stay the same in important ways. The commutative, associative, and distributive properties extend the properties of operations to the integers, rational numbers, real numbers, and complex numbers. Extending the properties of exponents leads to new and productive notation; for example, since the properties of exponents suggest that $(5^{1/3})^3 = 5^{1/3 \cdot 3} = 5^1 = 5$, we define $5^{\frac{1}{3}}$ to be the cube root of 5.

Calculators are useful in this strand to generate data for numerical experiments, to help understand the workings of matrix, vector, and complex number algebra, and to experiment with non-integer exponents.

Quantities.

In their work in measurement up through Grade 8, students primarily measure commonly used attributes such as length, area, and volume. In high school, students encounter a wider variety of units in modeling, e.g. acceleration, currency conversions, derived quantities such as person-hours and heating degree days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages. They also encounter novel situations in which they themselves must conceive the attributes of interest. For example, to find a good measure of overall highway safety, they might propose measures such as fatalities per year, fatalities per year per driver, or fatalities per vehicle-mile traveled. Such a conceptual process might be called quantification. Quantification is important for science, as when surface area suddenly "stands out" as an important variable in evaporation. Quantification is also important for companies, which must conceptualize relevant attributes and create or choose suitable measures for them.

Domain: The Real Number System

Extend the properties of exponents to rational exponents.

CCSS.Math.Content.HSN.RN.A.1
Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.*

CCSS.Math.Content.HSN.RN.A.2
Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.

CCSS.Math.Content.HSN.RN.B.3
Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Domain: Quantities

Reason quantitatively and use units to solve problems.

CCSS.Math.Content.HSN.Q.A.1
Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

CCSS.Math.Content.HSN.Q.A.2
Define appropriate quantities for the purpose of descriptive modeling.

CCSS.Math.Content.HSN.Q.A.3
Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

Domain: The Complex Number System

Perform arithmetic operations with complex numbers.

CCSS.Math.Content.HSN.CN.A.1
Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

CCSS.Math.Content.HSN.CN.A.2
Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

CCSS.Math.Content.HSN.CN.A.3
(+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Represent complex numbers and their operations on the complex plane.

CCSS.Math.Content.HSN.CN.B.4
(+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

CCSS.Math.Content.HSN.CN.B.5
(+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example, $(-1 + \sqrt{3}\, i)^3 = 8$ because $(-1 + \sqrt{3}\, i)$ has modulus 2 and argument 120°.*

CCSS.Math.Content.HSN.CN.B.6
(+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Use complex numbers in polynomial identities and equations.

CCSS.Math.Content.HSN.CN.C.7
Solve quadratic equations with real coefficients that have complex solutions.

CCSS.Math.Content.HSN.CN.C.8
(+) Extend polynomial identities to the complex numbers. *For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.*

CCSS.Math.Content.HSN.CN.C.9
(+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Domain: Vector and Matrix Quantities

Represent and model with vector quantities.

CCSS.Math.Content.HSN.VM.A.1
(+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\boldsymbol{v}$, $|\boldsymbol{v}|$, $||\boldsymbol{v}||$, v).

CCSS.Math.Content.HSN.VM.A.2
(+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

CCSS.Math.Content.HSN.VM.A.3
(+) Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on vectors.

CCSS.Math.Content.HSN.VM.B.4
(+) Add and subtract vectors.

CCSS.Math.Content.HSN.VM.B.4.a
Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.

CCSS.Math.Content.HSN.VM.B.4.b
Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.

CCSS.Math.Content.HSN.VM.B.4.c
Understand vector subtraction $\boldsymbol{v} - \boldsymbol{w}$ as $\boldsymbol{v} + (-\boldsymbol{w})$, where $-\boldsymbol{w}$ is the additive inverse of $\boldsymbol{w}$, with the same magnitude as $\boldsymbol{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

CCSS.Math.Content.HSN.VM.B.5
(+) Multiply a vector by a scalar.

CCSS.Math.Content.HSN.VM.B.5.a
Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.

CCSS.Math.Content.HSN.VM.B.5.b
Compute the magnitude of a scalar multiple $c\boldsymbol{v}$ using $||c\boldsymbol{v}|| = |c|\boldsymbol{v}$. Compute the direction of $c\boldsymbol{v}$ knowing that when $|c|\boldsymbol{v} \neq 0$, the direction of $c\boldsymbol{v}$ is either along $\boldsymbol{v}$ (for $c > 0$) or against $\boldsymbol{v}$ (for $c < 0$).

Perform operations on matrices and use matrices in applications.

CCSS.Math.Content.HSN.VM.C.6
(+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.

CCSS.Math.Content.HSN.VM.C.7
(+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.

CCSS.Math.Content.HSN.VM.C.8
(+) Add, subtract, and multiply matrices of appropriate dimensions.

CCSS.Math.Content.HSN.VM.C.9
(+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.

CCSS.Math.Content.HSN.VM.C.10
(+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

CCSS.Math.Content.HSN.VM.C.11
(+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

CCSS.Math.Content.HSN.VM.C.12
(+) Work with 2 × 2 matrices as a transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Source: Common Core State Standards: High School: Number and Quantity, Introduction

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1.4.11: High School Algebra Standards

Expressions.

An expression is a record of a computation with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.

Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a different but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price p. Rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying the price by a constant factor.

Algebraic manipulations are governed by the properties of operations and exponents, and the conventions of algebraic notation. At times, an expression is the result of applying operations to simpler expressions. For example, $p + 0.05p$ is the sum of the simpler expressions p and $0.05p$. Viewing an expression as the result of operation on simpler expressions can sometimes clarify its underlying structure.

A spreadsheet or a computer algebra system (CAS) can be used to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave.

Equations and inequalities.

An equation is a statement of equality between two expressions, often viewed as a question asking for which values of the variables the expressions on either side are in fact equal. These values are the solutions to the equation. An identity, in contrast, is true for all values of the variables; identities are often developed by rewriting an expression in an equivalent form.

The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs of numbers, which can be plotted in the coordinate plane. Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system.

An equation can often be solved by successively deducing from it one or more simpler equations. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions.

Some equations have no solutions in a given number system, but have a solution in a larger system. For example, the solution of $x + 1 = 0$ is an integer, not a whole number; the solution of $2x + 1 = 0$ is a rational number, not an integer; the solutions of $x^2 - 2 = 0$ are real numbers, not rational numbers; and the solutions of $x^2 + 2 = 0$ are complex numbers, not real numbers.

The same solution techniques used to solve equations can be used to rearrange formulas. For example, the formula for the area of a trapezoid, $A = ((b_1+b_2)/2)h$, can be solved for h using the same deductive process. Inequalities can be solved by reasoning about the properties of inequality. Many, but not all, of the properties of equality continue to hold for inequalities and can be useful in solving them.

Connections to Functions and Modeling. Expressions can define functions, and equivalent expressions define the same function. Asking when two functions have the same value for the same input leads to an equation; graphing the two functions allows for finding approximate solutions of the equation. Converting a verbal description to an equation, inequality, or system of these is an essential skill in modeling.

Domain: Seeing Structure Expressions

Interpret the structure of expressions.

CCSS.Math.Content.HSA.SSE.A.1
Interpret expressions that represent a quantity in terms of its context.*

CCSS.Math.Content.HSA.SSE.A.1.a
Interpret parts of an expression, such as terms, factors, and coefficients.

CCSS.Math.Content.HSA.SSE.A.1.b
Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P.*

CCSS.Math.Content.HSA.SSE.A.2
Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$*, thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.

Write expressions in equivalent forms to solve problems.

CCSS.Math.Content.HSA.SSE.B.3
Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.*

CCSS.Math.Content.HSA.SSE.B.3.a
Factor a quadratic expression to reveal the zeros of the function it defines.

CCSS.Math.Content.HSA.SSE.B.3.b
Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

CCSS.Math.Content.HSA.SSE.B.3.c
Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

CCSS.Math.Content.HSA.SSE.B.4
Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.**

Domain: Arithmetic with Polynomials and Rational Functions

Perform arithmetic operations on polynomials.

CCSS.Math.Content.HSA.APR.A.1
Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Understand the relationship between zeros and factors of polynomials.

CCSS.Math.Content.HSA.APR.B.2
Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

CCSS.Math.Content.HSA.APR.B.3
Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

CCSS.Math.Content.HSA.APR.C.4

Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*

CCSS.Math.Content.HSA.APR.C.5

(+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[1]

Rewrite rational expressions.

CCSS.Math.Content.HSA.APR.D.6

Rewrite simple rational expressions in different forms; write $^{a(x)}/_{b(x)}$ in the form $q(x) + {}^{r(x)}/_{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

CCSS.Math.Content.HSA.APR.D.7

(+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

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[1] The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.

Domain: Creating Equations*

Create equations that describe numbers or relationships.

CCSS.Math.Content.HSA.CED.A.1

Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*

CCSS.Math.Content.HSA.CED.A.2

Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

CCSS.Math.Content.HSA.CED.A.3

Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*

CCSS.Math.Content.HSA.CED.A.4

Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law* $V = IR$ *to highlight resistance* R.

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* Modeling Standards: Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appears throughout the high school standards indicated by a star symbol (*).

Domain: Reasoning with Equations and Inequalities

Understand solving equations as a process of reasoning and explain the reasoning.

CCSS.Math.Content.HSA.REI.A.1
Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

CCSS.Math.Content.HSA.REI.A.2
Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Solve equations and inequalities in one variable.

CCSS.Math.Content.HSA.REI.B.3
Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

CCSS.Math.Content.HSA.REI.B.4
Solve quadratic equations in one variable.

CCSS.Math.Content.HSA.REI.B.4.a
Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

CCSS.Math.Content.HSA.REI.B.4.b
Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Solve systems of equations.

CCSS.Math.Content.HSA.REI.C.5
Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

CCSS.Math.Content.HSA.REI.C.6
Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

CCSS.Math.Content.HSA.REI.C.7
Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.

CCSS.Math.Content.HSA.REI.C.8
(+) Represent a system of linear equations as a single matrix equation in a vector variable.

CCSS.Math.Content.HSA.REI.C.9
(+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3 × 3 or greater).

Represent and solve equations and inequalities graphically.

CCSS.Math.Content.HSA.REI.D.10
Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

CCSS.Math.Content.HSA.REI.D.11
Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

CCSS.Math.Content.HSA.REI.D.12
Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Source: Common Core State Standards: High School: Algebra, Introduction

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1.4.12: High School Functions Standards

Functions describe situations where one quantity determines another. For example, the return on $\$10,000$ invested at an annualized percentage rate of 4.25 is a function of the length of time the money is invested. Because we continually make theories about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models.

In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car's speed in miles per hour, *v*; the rule $T(v) = 100/v$ expresses this relationship algebraically and defines a function whose name is T.

The set of inputs to a function is called its domain. We often infer the domain to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context.

A function can be described in various ways, such as by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, "I'll give you a state, you give me the capital city;" by an algebraic expression like $f(x) = a + bx$; or by a recursive rule. The graph of a function is often a useful way of visualizing the relationship of the function models, and manipulating a mathematical expression for a function can throw light on the function's properties.

Functions presented as expressions can model many important phenomena. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. Linear functions with a constant term of zero describe proportional relationships.

A graphing utility or a computer algebra system can be used to experiment with the properties of these functions and their graphs and to build computational models of functions, including recursively defined functions.

Connections to Expressions, Equations, Modeling, and Coordinates.

Determining an output value for a particular input involves evaluating an expression; finding inputs that yield a given output involves solving an equation. Questions about when two functions have the same value for the same input lead to equations, whose solutions can be visualized from the intersection of their graphs. Because functions describe relationships between quantities, they are frequently used in modeling. Sometimes functions are defined by a recursive process, which can be displayed effectively using a spreadsheet or other technology.

Domain: Interpreting Functions

Understand the concept of a function and use function notation.

CCSS.Math.Content.HSF.IF.A.1
Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If *f* is a function and *x* is an element of its domain, then *f*(*x*) denotes the output of *f* corresponding to the input *x*. The graph of *f* is the graph of the equation *y* = *f*(*x*).

CCSS.Math.Content.HSF.IF.A.2
Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

CCSS.Math.Content.HSF.IF.A.3
Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by f(0) = f(1) = 1, f(n+1) = f(n) + f(n-1) for n ≥ 1.*

Interpret functions that arise in applications in terms of the context.

CCSS.Math.Content.HSF.IF.B.4
For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*[*]

CCSS.Math.Content.HSF.IF.B.5
Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*[*]

CCSS.Math.Content.HSF.IF.B.6
Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.[*]

Analyze functions using different representations.

CCSS.Math.Content.HSF.IF.C.7
Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.[*]

CCSS.Math.Content.HSF.IF.C.7.a
Graph linear and quadratic functions and show intercepts, maxima, and minima.

CCSS.Math.Content.HSF.IF.C.7.b
Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

CCSS.Math.Content.HSF.IF.C.7.c
Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

CCSS.Math.Content.HSF.IF.C.7.d
(+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

CCSS.Math.Content.HSF.IF.C.7.e
Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

CCSS.Math.Content.HSF.IF.C.8
Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

CCSS.Math.Content.HSF.IF.C.8.a
Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

CCSS.Math.Content.HSF.IF.C.8.b
Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)12^t$, $y = (1.2)^t/10$, and classify them as representing exponential growth or decay.

CCSS.Math.Content.HSF.IF.C.9
Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Domain: Building Functions

Build a function that models a relationship between two quantities.

CCSS.Math.Content.HSF.BF.A.1
Write a function that describes a relationship between two quantities.*

CCSS.Math.Content.HSF.BF.A.1.a
Determine an explicit expression, a recursive process, or steps for calculation from a context.

CCSS.Math.Content.HSF.BF.A.1.b
Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*

CCSS.Math.Content.HSF.BF.A.1.c
(+) Compose functions. *For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.*

CCSS.Math.Content.HSF.BF.A.2
Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.*

Build new functions from existing functions.

CCSS.Math.Content.HSF.BF.B.3
Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\ f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

CCSS.Math.Content.HSF.BF.B.4
Find inverse functions.

CCSS.Math.Content.HSF.BF.B.4.a
Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. *For example, $f(x) = 2x^3$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$.*

CCSS.Math.Content.HSF.BF.B.4.b
(+) Verify by composition that one function is the inverse of another.

CCSS.Math.Content.HSF.BF.B.4.c
(+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

CCSS.Math.Content.HSF.BF.B.4.d
(+) Produce an invertible function from a non-invertible function by restricting the domain.

CCSS.Math.Content.HSF.BF.B.5
(+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Domain: Linear, Quadratic, and Exponential Models*

Construct and compare linear, quadratic, and exponential models and solve problems.

CCSS.Math.Content.HSF.LE.A.1
Distinguish between situations that can be modeled with linear functions and with exponential functions.

CCSS.Math.Content.HSF.LE.A.1.a
Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

CCSS.Math.Content.HSF.LE.A.1.b
Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

CCSS.Math.Content.HSF.LE.A.1.c
Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

CCSS.Math.Content.HSF.LE.A.2
Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

CCSS.Math.Content.HSF.LE.A.3
Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.

CCSS.Math.Content.HSF.LE.A.4
For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Interpret expressions for functions in terms of the situation they model.

CCSS.Math.Content.HSF.LE.B.5
Interpret the parameters in a linear or exponential function in terms of a context.

Domain: Trigonometric Functions

Extend the domain of trigonometric functions using the unit circle.

CCSS.Math.Content.HSF.TF.A.1
Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

CCSS.Math.Content.HSF.TF.A.2
Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

CCSS.Math.Content.HSF.TF.A.3
(+) Use special triangles to determine geometrically the values of sine, cosine, tangent for π/3, π/4 and π/6, and use the unit circle to express the values of sine, cosine, and tangent for x, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

CCSS.Math.Content.HSF.TF.A.4
(+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions.

CCSS.Math.Content.HSF.TF.B.5
Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.*

CCSS.Math.Content.HSF.TF.B.6
(+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

CCSS.Math.Content.HSF.TF.B.7
(+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.*

Prove and apply trigonometric identities.

CCSS.Math.Content.HSF.TF.C.8
Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

CCSS.Math.Content.HSF.TF.C.9
(+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Source: Common Core State Standards: High School: Functions

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1.4.13: High School Modeling Standards

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.Some examples of such situations might include:

- Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing stopping distance for a car.
- Modeling savings account balance, bacterial colony growth, or investment growth.
- Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyzing risk in situations such as extreme sports, pandemics, and terrorism.
- Relating population statistics to individual predictions.

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.

The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them.

Choices, assumptions, and approximations are present throughout this cycle.

In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model—for example, graphs of global temperature and atmospheric CO2 over time. Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.

Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena (e.g., the behavior of polynomials) as well as physical phenomena.

Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (*).

Source: Common Core State Standards: High School: Modeling, Introduction

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1.4.14: High School Geometry Standards

An understanding of the attributes and relationships of geometric objects can be applied in diverse contexts—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.

Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). Euclidean geometry is characterized most importantly by the Parallel Postulate, that through a point not on a given line there is exactly one parallel line. (Spherical geometry, in contrast, has no parallel lines.)

During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs. Later in college some students develop Euclidean and other geometries carefully from a small set of axioms.

The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.

In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. For triangles, congruence means the equality of all corresponding pairs of sides and all corresponding pairs of angles. During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures.

Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of "same shape" and "scale factor" developed in the middle grades. These transformations lead to the criterion for triangle similarity that two pairs of corresponding angles are congruent.

The definitions of sine, cosine, and tangent for acute angles are founded on right triangles and similarity, and, with the Pythagorean Theorem, are fundamental in many real-world and theoretical situations. The Pythagorean Theorem is generalized to non-right triangles by the Law of Cosines. Together, the Laws of Sines and Cosines embody the triangle congruence criteria for the cases where three pieces of information suffice to completely solve a triangle. Furthermore, these laws yield two possible solutions in the ambiguous case, illustrating that Side-Side-Angle is not a congruence criterion.Analytic geometry connects algebra and geometry, resulting in powerful methods of analysis and problem solving. Just as the number line associates numbers with locations in one dimension, a pair of perpendicular axes associates pairs of numbers with locations in two dimensions. This correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof. Geometric transformations of the graphs of equations correspond to algebraic changes in their equations.

Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same way as computer algebra systems allow them to experiment with algebraic phenomena.

Connections to Equations

The correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof.

Domain: Congruence

Experiment with transformations in the plane

CCSS.Math.Content.HSG.CO.A.1
Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

CCSS.Math.Content.HSG.CO.A.2
Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

CCSS.Math.Content.HSG.CO.A.3
Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

CCSS.Math.Content.HSG.CO.A.4
Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

CCSS.Math.Content.HSG.CO.A.5
Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions

CCSS.Math.Content.HSG.CO.B.6
Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

CCSS.Math.Content.HSG.CO.B.7
Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

CCSS.Math.Content.HSG.CO.B.8
Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Prove geometric theorems

CCSS.Math.Content.HSG.CO.C.9
Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*

CCSS.Math.Content.HSG.CO.C.10
Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

CCSS.Math.Content.HSG.CO.C.11
Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

Make geometric constructions

CCSS.Math.Content.HSG.CO.D.12
Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*

CCSS.Math.Content.HSG.CO.D.13
Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Domain: Similarity, Right Triangles, and Trigonometry

Understand similarity in terms of similarity transformations

CCSS.Math.Content.HSG.SRT.A.1
Verify experimentally the properties of dilations given by a center and a scale factor:

CCSS.Math.Content.HSG.SRT.A.1.a
A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.

CCSS.Math.Content.HSG.SRT.A.1.b
The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

CCSS.Math.Content.HSG.SRT.A.2
Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

CCSS.Math.Content.HSG.SRT.A.3
Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

Prove theorems involving similarity

CCSS.Math.Content.HSG.SRT.B.4
Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*

CCSS.Math.Content.HSG.SRT.B.5
Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Define trigonometric ratios and solve problems involving right triangles

CCSS.Math.Content.HSG.SRT.C.6
Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

CCSS.Math.Content.HSG.SRT.C.7
Explain and use the relationship between the sine and cosine of complementary angles.

CCSS.Math.Content.HSG.SRT.C.8
Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.*

Apply trigonometry to general triangles

CCSS.Math.Content.HSG.SRT.D.9
(+) Derive the formula $A = 1/2\ ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

CCSS.Math.Content.HSG.SRT.D.10
(+) Prove the Laws of Sines and Cosines and use them to solve problems.

CCSS.Math.Content.HSG.SRT.D.11
(+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Domain: Circles

Understand and apply theorems about circles

CCSS.Math.Content.HSG.C.A.1
Prove that all circles are similar.

CCSS.Math.Content.HSG.C.A.2
Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

CCSS.Math.Content.HSG.C.A.3
Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

CCSS.Math.Content.HSG.C.A.4
(+) Construct a tangent line from a point outside a given circle to the circle.

Find arc lengths and areas of sectors of circles

CCSS.Math.Content.HSG.C.B.5
Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Domain: Expressing Geometric Properties with Equations

Translate between the geometric description and the equation for a conic section

CCSS.Math.Content.HSG.GPE.A.1
Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

CCSS.Math.Content.HSG.GPE.A.2
Derive the equation of a parabola given a focus and directrix.

CCSS.Math.Content.HSG.GPE.A.3
(+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

Use coordinates to prove simple geometric theorems algebraically

CCSS.Math.Content.HSG.GPE.B.4
Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0, 2).*

CCSS.Math.Content.HSG.GPE.B.5
Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

CCSS.Math.Content.HSG.GPE.B.6
Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

CCSS.Math.Content.HSG.GPE.B.7
Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.*

Domain: Geometric Measurement and Dimension

Explain volume formulas and use them to solve problems

CCSS.Math.Content.HSG.GMD.A.1
Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments*.

CCSS.Math.Content.HSG.GMD.A.2
(+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

CCSS.Math.Content.HSG.GMD.A.3
Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.*

Visualize relationships between two-dimensional and three-dimensional objects

CCSS.Math.Content.HSG.GMD.B.4
Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Domain: Modeling with Geometry

Apply geometric concepts in modeling situations

CCSS.Math.Content.HSG.MG.A.1
Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).*

CCSS.Math.Content.HSG.MG.A.2
Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).*

CCSS.Math.Content.HSG.MG.A.3
Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).*

Source: Common Core State Standards: High School: Geometry, Introduction

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1.4.15: High School Statistics and Probability Standards

Decisions or predictions are often based on data—numbers in context. These decisions or predictions would be easy if the data always sent a clear message, but the message is often obscured by variability. Statistics provides tools for describing variability in data and for making informed decisions that take it into account.

Data are gathered, displayed, summarized, examined, and interpreted to discover patterns and deviations from patterns. Quantitative data can be described in terms of key characteristics: measures of shape, center, and spread. The shape of a data distribution might be described as symmetric, skewed, flat, or bell shaped, and it might be summarized by a statistic measuring center (such as mean or median) and a statistic measuring spread (such as standard deviation or interquartile range). Different distributions can be compared numerically using these statistics or compared visually using plots. Knowledge of center and spread are not enough to describe a distribution. Which statistics to compare, which plots to use, and what the results of a comparison might mean, depend on the question to be investigated and the real-life actions to be taken.

Randomization has two important uses in drawing statistical conclusions. First, collecting data from a random sample of a population makes it possible to draw valid conclusions about the whole population, taking variability into account. Second, randomly assigning individuals to different treatments allows a fair comparison of the effectiveness of those treatments. A statistically significant outcome is one that is unlikely to be due to chance alone, and this can be evaluated only under the condition of randomness. The conditions under which data are collected are important in drawing conclusions from the data; in critically reviewing uses of statistics in public media and other reports, it is important to consider the study design, how the data were gathered, and the analyses employed as well as the data summaries and the conclusions drawn.

Random processes can be described mathematically by using a probability model: a list or description of the possible outcomes (the sample space), each of which is assigned a probability. In situations such as flipping a coin, rolling a number cube, or drawing a card, it might be reasonable to assume various outcomes are equally likely. In a probability model, sample points represent outcomes and combine to make up events; probabilities of events can be computed by applying the Addition and Multiplication Rules. Interpreting these probabilities relies on an understanding of independence and conditional probability, which can be approached through the analysis of two-way tables.

Technology plays an important role in statistics and probability by making it possible to generate plots, regression functions, and correlation coefficients, and to simulate many possible outcomes in a short amount of time.

Connections to Functions and Modeling

Functions may be used to describe data; if the data suggest a linear relationship, the relationship can be modeled with a regression line, and its strength and direction can be expressed through a correlation coefficient.

Domain: Interpreting Categorical and Quantitative Data

Summarize, represent, and interpret data on a single count or measurement variable

CCSS.Math.Content.HSS.ID.A.1
Represent data with plots on the real number line (dot plots, histograms, and box plots).

CCSS.Math.Content.HSS.ID.A.2
Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.

CCSS.Math.Content.HSS.ID.A.3
Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).

CCSS.Math.Content.HSS.ID.A.4
Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.

Summarize, represent, and interpret data on two categorical and quantitative variables

CCSS.Math.Content.HSS.ID.B.5
Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.

CCSS.Math.Content.HSS.ID.B.6
Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.

CCSS.Math.Content.HSS.ID.B.6.a
Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.

CCSS.Math.Content.HSS.ID.B.6.b
Informally assess the fit of a function by plotting and analyzing residuals.

CCSS.Math.Content.HSS.ID.B.6.c
Fit a linear function for a scatter plot that suggests a linear association.

Interpret linear models

CCSS.Math.Content.HSS.ID.C.7
Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.

CCSS.Math.Content.HSS.ID.C.8
Compute (using technology) and interpret the correlation coefficient of a linear fit.

CCSS.Math.Content.HSS.ID.C.9
Distinguish between correlation and causation.

Domain: Making Inferences and Justifying Conclusions

Understand and evaluate random processes underlying statistical experiments

CCSS.Math.Content.HSS.IC.A.1
Understand statistics as a process for making inferences about population parameters based on a random sample from that population.

CCSS.Math.Content.HSS.IC.A.2
Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?*

Make inferences and justify conclusions from sample surveys, experiments, and observational studies

CCSS.Math.Content.HSS.IC.B.3
Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.

CCSS.Math.Content.HSS.IC.B.4
Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.

CCSS.Math.Content.HSS.IC.B.5
Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.

CCSS.Math.Content.HSS.IC.B.6
Evaluate reports based on data.

Domain: Conditional Probability and the Rules of Probability

Understand independence and conditional probability and use them to interpret data

CCSS.Math.Content.HSS.CP.A.1
Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").

CCSS.Math.Content.HSS.CP.A.2
Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent.

CCSS.Math.Content.HSS.CP.A.3
Understand the conditional probability of *A* given *B* as *P*(*A* and *B*)/*P*(*B*), and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*.

CCSS.Math.Content.HSS.CP.A.4
Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.*

CCSS.Math.Content.HSS.CP.A.5
Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.*

Use the rules of probability to compute probabilities of compound events.

CCSS.Math.Content.HSS.CP.B.6
Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model.

CCSS.Math.Content.HSS.CP.B.7
Apply the Addition Rule, P(A or B) = P(A) + P(B) - P(A and B), and interpret the answer in terms of the model.

CCSS.Math.Content.HSS.CP.B.8
(+) Apply the general Multiplication Rule in a uniform probability model, P(A and B) = P(A)P(B|A) = P(B)P(A|B), and interpret the answer in terms of the model.

CCSS.Math.Content.HSS.CP.B.9
(+) Use permutations and combinations to compute probabilities of compound events and solve problems.

Domain: Using Probability to Make Decisions

Calculate expected values and use them to solve problems

CCSS.Math.Content.HSS.MD.A.1
(+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.

CCSS.Math.Content.HSS.MD.A.2
(+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.

CCSS.Math.Content.HSS.MD.A.3
(+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. *For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.*

CCSS.Math.Content.HSS.MD.A.4
(+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. *For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?*

Use probability to evaluate outcomes of decisions

CCSS.Math.Content.HSS.MD.B.5
(+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

CCSS.Math.Content.HSS.MD.B.5.a
Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.*

CCSS.Math.Content.HSS.MD.B.5.b
Evaluate and compare strategies on the basis of expected values. *For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.*

CCSS.Math.Content.HSS.MD.B.6
(+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

CCSS.Math.Content.HSS.MD.B.7
(+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).

Source: Common Core State Standards: High School: Statistics & Probability

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1.5: Teaching Techniques in Mathematics

> “I do not teach math. I teach my students how to learn math.”
> — Jaime Escalante

Exploration of Effective Teaching Techniques

Inquiry-based Learning

Inquiry-based learning is a pedagogical approach that fosters student curiosity and promotes deep understanding through exploration. This technique involves posing open-ended questions that encourage students to investigate mathematical concepts independently or in groups. By engaging in inquiry-based learning, students are empowered to discover solutions on their own, developing critical thinking skills and a deeper grasp of mathematical principles. This method places emphasis on the process of problem-solving rather than merely focusing on arriving at the correct answer. As students explore different strategies and approaches, they gain a better appreciation for the underlying logic of mathematical ideas and develop a more robust problem-solving toolkit.

In an inquiry-based classroom, the role of the teacher shifts from being a source of knowledge to a facilitator of learning. The teacher's primary task is to create an environment where students feel comfortable asking questions, making mistakes, and exploring various pathways to understanding. This involves designing activities that are open-ended and rich in opportunities for investigation, providing just enough guidance to keep students on track while allowing them the freedom to make their own discoveries.

One of the key benefits of inquiry-based learning is that it helps students develop a growth mindset. As they encounter challenges and work through them, students learn to view difficulties as opportunities for growth rather than as obstacles. This fosters resilience and perseverance, qualities that are essential for success in mathematics and beyond. Additionally, by working collaboratively, students can share their ideas and strategies, learning from each other's perspectives and approaches.

Inquiry-based learning also aligns well with real-world applications of mathematics. By engaging in authentic problem-solving activities, students can see the relevance of mathematical concepts to everyday life. This not only enhances their motivation to learn but also helps them develop skills that are valuable in a variety of contexts. For instance, an inquiry-based lesson on fractions might involve students figuring out how to evenly distribute ingredients for a recipe, thereby connecting classroom learning to practical, real-life situations.

Ultimately, inquiry-based learning transforms the mathematics classroom into a dynamic space where curiosity and creativity are at the forefront. Students are not just passive recipients of information; they are active participants in their own learning journey. This approach not only deepens their understanding of mathematical concepts but also prepares them to tackle complex problems with confidence and ingenuity.

Common Core: 1st Grade

- **CCSS.MATH.CONTENT.1.G.A.1:** "Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size) and sort objects into categories based on their defining attributes."
- **CCSS.MATH.CONTENT.1.G.A.2:** "Compose two-dimensional shapes (e.g., triangles, squares, rectangles, circles) or three-dimensional shapes (e.g., cubes, cones, cylinders, and spheres) to form a composite shape, and compose new shapes from the composite shape."

These standards focus on helping students identify and categorize shapes based on their properties, as well as understand how shapes can be combined to form new shapes. For more advanced exploration of shapes, students would build on these foundational skills in higher grades.

Example 1.5.1

For first graders, an effective inquiry-based learning activity in math could involve exploring shapes and their properties through a hands-on scavenger hunt. Here's how it could be structured:

Activity: Shape Scavenger Hunt

1. **Introduction:** Begin by introducing the basic shapes—circle, square, triangle, and rectangle—using simple, colorful cutouts. Show the students examples and discuss their characteristics (e.g., number of sides, angles).
2. **Exploration:** Explain that they will go on a scavenger hunt to find these shapes around the classroom or outside. Provide each student or group with a shape checklist and a small bag or container to collect shape samples.
3. **Investigation:** Allow the students to explore their surroundings and search for objects that match the shapes on their checklist. For instance, they might find a round clock (circle), a square book (square), a triangular flag (triangle), or a rectangular door (rectangle).
4. **Discussion:** After the scavenger hunt, gather the students and have them share what they found. Ask questions like:
   - "What objects did you find that were circles?"
   - "Can you describe a rectangle you saw?"
   - "Which shape was the easiest to find? Why?"
5. **Reflection:** Have students draw or list the shapes they found and their corresponding objects. Discuss the different places where shapes can be found in everyday life and what makes each shape unique.
6. **Extension:** As an extension, introduce a sorting activity where students group objects based on their shapes and explain their reasoning.

This inquiry-based activity encourages curiosity and exploration, helps students understand the properties of shapes through real-world examples, and allows them to engage actively in their learning process.

Use of Manipulatives and Visual Aids

Manipulatives and visual aids are crucial tools in making abstract mathematical concepts more tangible and accessible. Concrete representations, such as physical objects and hands-on activities, help students bridge the gap between abstract ideas and their practical applications. For instance, using blocks to teach addition and subtraction or geometric shapes to explore properties of space allows students to interact with the concepts they are learning physically. This hands-on approach engages multiple senses and provides a more comprehensive understanding of the material, making it easier for students to grasp and retain complex concepts.

Visual tools, including charts, graphs, and diagrams, further aid in this process by visualizing complex information. These aids help students to better understand and retain mathematical concepts by breaking down information into more digestible formats and illustrating relationships between ideas. For example, a bar graph can visually convey the difference between quantities in a way that is more immediately comprehensible than numerical data alone. Similarly, a Venn diagram can clearly show the overlap and distinctions between different sets, aiding in the understanding of set theory and logic.

The use of manipulatives and visual aids also caters to diverse learning styles. Some students may struggle with traditional numerical and symbolic representations of mathematics but thrive when given the opportunity to work with physical objects or visual representations. By incorporating these tools into the classroom, teachers can provide multiple entry points to understanding, ensuring that all students have the opportunity to succeed.

Moreover, manipulatives and visual aids promote active learning and engagement. When students are manipulating objects or interpreting visual data, they are actively involved in the learning process rather than passively receiving information. This active participation can lead to deeper understanding and greater retention of mathematical concepts. Additionally, these tools can foster collaboration and discussion among students, as they work together to solve problems and explore concepts.

Incorporating manipulatives and visual aids into mathematics instruction also supports the development of critical thinking and problem-solving skills. For example, when students use fraction strips to compare different fractions, they must think critically about the relationships between the parts and the whole. Similarly, when interpreting a graph, students must analyze the data, draw conclusions, and justify their reasoning. These activities encourage students to think deeply about mathematical concepts and develop their analytical skills.

Ultimately, manipulatives and visual aids are essential components of effective mathematics instruction. They help demystify abstract concepts, making them more accessible and understandable for all students. By providing concrete and visual representations of mathematical ideas, these tools support a more inclusive and engaging learning environment, fostering a deeper and more lasting understanding of mathematics.

Common Core: 4th Grade

- **CCSS.MATH.CONTENT.4.NF.A.1:** "Explain why a fraction a/b is equivalent to a fraction (n × a)/(n × b) by using visual fraction models. Use this principle to recognize and generate equivalent fractions."
- **CCSS.MATH.CONTENT.4.NF.A.2:** "Compare two fractions with different numerators and different denominators by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole."
- **CCSS.MATH.CONTENT.4.NF.B.3:** "Understand a fraction a/b with a > 1 as a sum of fractions 1/b."
- **CCSS.MATH.CONTENT.4.NF.B.4:** "Apply and extend previous understandings of multiplication to multiply a fraction by a whole number."

These standards aim to deepen students' understanding of fractions by focusing on equivalence, comparison, and operations with fractions.

Example 1.5.2

For a 4th grade math topic, such as fractions, using manipulatives and visual aids can make abstract concepts more concrete and accessible. Here's an example of an activity that leverages these tools:

Activity: Fraction Pizza Party

Introduction: Start by discussing what fractions are and how they represent parts of a whole. Explain that fractions can be used to describe different portions of a whole object, such as a pizza.

Manipulatives: Provide students with paper or foam pizza slices that can be assembled into whole pizzas. Each pizza is divided into different fractions—quarters, thirds, and halves. You might use pizza templates where the fractions are pre-divided, or let students cut their own pizzas into the desired fraction parts.

Visual Aids: Use a large visual aid, such as a chart or interactive whiteboard, to display different fractions and their corresponding visual representations. Show examples of how a whole pizza can be divided into various fractions and illustrate these with colored diagrams

Activity: Give each student or group a set of pizza slices and a scenario, such as a pizza party where they need to share pizzas with different fractions. For example:

- "You have 2 pizzas, each divided into 8 slices. You need to give 3/8 of a pizza to each friend. How many slices will you give each friend?"
- "If you have a pizza divided into 4 equal parts and you eat 2 parts, what fraction of the pizza is left?"

Hands-On Practice: Have students physically manipulate their pizza slices to model the fractions they're working with. They can combine slices to make a whole pizza, divide pizzas into different fractions, and match their real-life experience with the fractions displayed on the visual aids.

Discussion: After the hands-on activity, discuss with the students what they learned about fractions. Ask questions like:

- "How did dividing the pizza into different parts help you understand fractions better?"
- "Can you show me 3/4 of a pizza using your slices?"

Extension: As an extension, you could incorporate additional fraction concepts, such as equivalent fractions, by having students compare different fraction sizes using their pizza slices.

This activity uses manipulatives (pizza slices) and visual aids (fraction charts) to help students grasp the concept of fractions in a tangible and interactive way. It allows them to see and physically handle the fractions, which reinforces their understanding and makes the learning experience more engaging.

Differentiated Instruction to Meet Diverse Learning Needs

Differentiated instruction is a teaching strategy designed to address the diverse needs of students by tailoring lessons to accommodate various learning styles and abilities. This approach involves adapting teaching methods, materials, and assessments to ensure that all students can access and engage with the content effectively. By recognizing and valuing the unique differences among students, differentiated instruction aims to create an equitable learning environment where every student can thrive.

Flexible grouping is one of the key techniques used in differentiated instruction. This involves organizing students into various groups based on their skill levels, interests, or specific learning needs. Groups can be fluid and change as needed, allowing teachers to provide targeted instruction and support to different students at different times. For example, a teacher might form small groups for guided reading sessions, where each group works on texts that are appropriate for their reading level. This ensures that all students are challenged appropriately and can progress at their own pace while receiving the necessary guidance and support.

Another crucial element of differentiated instruction is the use of varied instructional strategies. Teachers might employ a mix of direct instruction, collaborative learning, hands-on activities, and independent work to cater to different learning preferences. For instance, a lesson on fractions might include a direct explanation of the concept, followed by a group activity where students use fraction tiles to solve problems, and then individual practice with worksheets. This variety not only keeps students engaged but also allows them to interact with the material in multiple ways, deepening their understanding.

Differentiated instruction also involves adapting materials and resources to meet students' needs. This could mean providing reading materials at different levels of complexity, offering visual aids and manipulatives for students who benefit from concrete representations, or incorporating technology to provide interactive and personalized learning experiences. For example, in a math class, some students might use digital tools that offer immediate feedback and adaptive challenges, while others might work with traditional manipulatives to build their conceptual understanding.

Assessment is another area where differentiation is essential. Instead of using a one-size-fits-all approach, differentiated assessment involves using a range of methods to evaluate student learning. This might include traditional tests, quizzes, projects, presentations, and formative assessments like exit tickets or observation checklists. By using diverse assessment strategies, teachers can gain a more comprehensive understanding of each student's progress and provide feedback that is specific and meaningful.

By implementing differentiated instruction, educators can create a more inclusive learning environment that meets the individual needs of each student. This approach not only helps to close achievement gaps but also fosters a classroom culture where all students feel valued and supported. It encourages a growth mindset, as students are given opportunities to succeed based on their unique abilities and learning styles. Ultimately, differentiated instruction empowers students to take ownership of their learning and develop the skills and confidence they need to succeed academically and beyond.

Common Core: 2nd Grade

- **CCSS.MATH.CONTENT.2.MD.A.1:** "Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes."
- **CCSS.MATH.CONTENT.2.MD.A.2:** "Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen."

These standards align with the differentiated instruction activity by addressing the need to measure and compare lengths using various tools and units, providing appropriate support for different learning styles and levels.

Example 1.5.3

2nd Grade: Understanding and Comparing Lengths

Objective: Help students measure and compare lengths of objects using non-standard and standard units of measurement.

Visual Learners:

- **Activity:** Provide measurement charts with pictures of objects and their lengths represented in both non-standard units (e.g., paper clips) and standard units (e.g., inches).
- **Support:** Use a ruler and a visual guide that shows how to line up objects with measurement tools.

Kinesthetic Learners:

- **Activity:** Allow students to measure objects in the classroom using non-standard units such as blocks or their own hand spans. Have them physically move objects and measure them with a tape measure.
- **Support:** Provide hands-on measurement tools and allow students to work in pairs or small groups to compare their measurements.

Auditory Learners:

- **Activity:** Conduct a measurement discussion where students explain how they measured objects and describe their findings aloud. Use measurement-related songs or rhymes to reinforce the concept.
- **Support:** Offer verbal instructions and cues during activities to help students understand measurement concepts.

Struggling Learners:

- **Activity:** Use measurement tools with built-in guides or markers (e.g., ruler with labeled inches) to help students accurately measure objects. Provide one-on-one or small group instruction to practice measuring with concrete examples.
- **Support:** Offer additional practice with step-by-step guidance and visual aids to reinforce understanding.

Advanced Learners:

- **Activity:** Challenge students to measure and compare lengths of objects using different measurement units and convert between units (e.g., inches to feet). Have them create a simple measurement project where they measure and compare various classroom items.
- **Support:** Provide more complex measurement problems and opportunities to apply their skills to real-world scenarios.

A Quick Overview of Growth Mindset

> "In a growth mindset, challenges are exciting rather than threatening. So rather than thinking, oh, I'm going to reveal my weaknesses, you say, wow, here's a chance to grow."
> — Carol S. Dweck

A growth mindset is a concept developed by psychologist Carol Dweck that refers to the belief that abilities and intelligence can be developed through dedication, hard work, and perseverance. This mindset contrasts with a fixed mindset, where individuals believe that their abilities and intelligence are static and unchangeable.

Key Characteristics

1. **Belief in Development:** People with a growth mindset believe that their abilities can be improved with effort and practice. They view intelligence and talent as traits that can be developed over time through dedication and learning.
2. **Emphasis on Effort:** In a growth mindset, effort is seen as a crucial component of success. Individuals understand that hard work and persistence are essential for overcoming challenges and achieving their goals.
3. **Embracing Challenges:** Those with a growth mindset are more likely to embrace challenges rather than avoid them. They

see difficult tasks as opportunities to learn and grow, rather than as threats to their self-esteem.
4. **Learning from Feedback:** Constructive criticism is welcomed by individuals with a growth mindset. They view feedback as valuable information that helps them improve, rather than as a personal attack or a reflection of their abilities.
5. **Resilience in the Face of Setbacks:** A growth mindset fosters resilience. Individuals are more likely to persist through setbacks and failures, viewing these experiences as learning opportunities rather than as reflections of their abilities.
6. **Focus on the Process:** People with a growth mindset are more focused on the process of learning rather than just the end result. They value the journey of acquiring new skills and knowledge and understand that mastery comes with time and practice.

Implications for Education

1. **Encouraging a Growth Mindset in Students:** Educators can help students develop a growth mindset by praising effort rather than inherent talent. For example, instead of saying "You're so smart," teachers might say, "You worked really hard on this problem." This shift in language emphasizes the value of persistence and effort.
2. **Creating a Positive Learning Environment:** Classrooms that promote a growth mindset encourage students to take risks, make mistakes, and learn from them. Teachers can create an environment where mistakes are seen as a natural part of the learning process and where students feel safe to explore and experiment.
3. **Setting Goals and Reflection:** Teachers can encourage students to set personal learning goals and reflect on their progress. By helping students track their efforts and celebrate their growth, educators reinforce the idea that improvement is the result of dedication and hard work.
4. **Modeling a Growth Mindset:** Teachers and educators can model a growth mindset by sharing their own experiences with challenges and how they overcame them. Demonstrating resilience and a commitment to learning can inspire students to adopt a similar approach.

Benefits of a Growth Mindset

1. **Improved Academic Achievement:** Students with a growth mindset are more likely to achieve higher academic outcomes because they are more persistent, resilient, and open to learning from mistakes.
2. **Enhanced Motivation:** A growth mindset can lead to increased motivation, as students who believe in their ability to improve are more likely to engage with challenging tasks and persist through difficulties.
3. **Greater Resilience:** Individuals with a growth mindset are better equipped to handle setbacks and failures, as they view these experiences as opportunities for growth rather than as indicators of their limitations.
4. **Lifelong Learning:** Adopting a growth mindset encourages a lifelong love of learning and a willingness to continually seek out new knowledge and skills, contributing to personal and professional development throughout life.

In summary, a growth mindset fosters a positive and proactive approach to learning and personal development. By embracing challenges, valuing effort, and learning from feedback and setbacks, individuals with a growth mindset are better prepared to succeed in their academic, personal, and professional lives.

Encouraging Perseverance and Resilience in Problem-solving

Promoting a growth mindset in mathematics involves fostering an attitude of perseverance and resilience. One key aspect is emphasizing the importance of effort in achieving success. By reinforcing the idea that persistent effort and hard work are essential for mastering mathematical concepts, students are encouraged to tackle challenging problems with determination. This message can be communicated through positive reinforcement, such as praising students for their effort and perseverance rather than just their correct answers. When students see that their hard work is recognized and valued, they are more likely to develop a resilient approach to learning.

Creating a classroom environment where mistakes are viewed as valuable learning opportunities rather than failures is another critical component of fostering resilience. When students understand that errors are an integral part of the learning process, they are more likely to approach problems with confidence and a willingness to learn from their experiences. Teachers can encourage this mindset by openly discussing mistakes and demonstrating how to learn from them. For example, when a student makes an error in solving a math problem, the teacher can guide the class in analyzing the mistake and exploring different strategies to find the correct solution. This practice not only normalizes making mistakes but also teaches students how to think critically and reflect on their problem-solving processes.

Modeling resilience and perseverance is also essential. Teachers can share their own experiences with struggling through difficult

problems and how they overcame those challenges. By showing students that even experts encounter difficulties and must work hard to solve problems, teachers can help demystify the process of learning and build students' confidence in their abilities. Additionally, providing opportunities for students to work on increasingly complex problems can help them develop a sense of accomplishment and the belief that they can overcome obstacles through persistence.

Incorporating collaborative learning activities is another effective way to encourage perseverance and resilience. When students work together on challenging tasks, they can support and motivate each other. Collaboration allows students to see different perspectives and approaches to problem-solving, which can be particularly beneficial when they encounter difficulties. Group work also fosters a sense of community and shared responsibility, helping students to feel more supported and less isolated in their learning journey.

Teachers can also use goal-setting as a tool to promote perseverance. By helping students set specific, achievable goals, teachers can provide a clear path for students to follow and celebrate their progress along the way. This approach helps students to stay motivated and focused, even when they encounter challenges. Regularly reviewing and adjusting goals based on progress can further reinforce the importance of effort and persistence.

Ultimately, encouraging perseverance and resilience in problem-solving helps students to develop a growth mindset that will serve them well not only in mathematics but in all areas of their lives. By teaching students to embrace challenges, learn from mistakes, and persist through difficulties, educators can empower them to become confident, capable problem-solvers who are prepared to face future challenges with determination and resilience.

Techniques for Fostering a Positive Attitude Towards Math

To foster a positive attitude towards math, it is important to celebrate student achievements and provide positive reinforcement. Recognizing and celebrating progress, no matter how small, helps to build students' confidence and motivation. This could involve acknowledging improvements, efforts, and successes in various forms, such as praise, awards, or showcasing work. For example, a teacher might highlight a student's creative problem-solving approach during a class discussion or display a student's exemplary work on a bulletin board. These acknowledgments make students feel valued and proud of their accomplishments, encouraging them to continue striving for success.

Positive reinforcement, such as verbal encouragement and rewards for effort, further enhances students' self-esteem and enthusiasm for mathematics. Teachers can use phrases like "Great job on solving that difficult problem!" or "I can see how much effort you put into your homework!" to provide immediate and specific feedback. Reward systems, such as earning points for participation or receiving certificates for improvement, can also motivate students to engage with math more eagerly. By consistently reinforcing positive behaviors and achievements, educators help students associate math with positive experiences and emotions.

Creating a supportive and encouraging classroom environment is crucial in helping students develop a love for math. This involves fostering a culture where all students feel safe to express their ideas, ask questions, and take risks without fear of ridicule or failure. Teachers can establish this environment by setting clear expectations for respectful behavior, actively listening to students, and addressing any negative attitudes or behaviors promptly. Encouraging peer support and collaboration can also contribute to a positive classroom atmosphere. When students work together and support each other's learning, they build a sense of community and shared purpose.

Incorporating engaging and relevant activities into math lessons can also help foster a positive attitude towards the subject. Real-world applications, hands-on projects, and interactive games can make math more interesting and enjoyable. For instance, a teacher might use a math game to reinforce multiplication skills or design a project where students apply geometry concepts to create a model of a playground. By showing students how math is relevant to their lives and can be fun, teachers can increase their interest and enthusiasm for the subject.

Providing opportunities for student choice and autonomy in math can further enhance their positive attitude. When students have some control over their learning, such as choosing which problems to solve or selecting a project topic, they are more likely to feel invested and motivated. Allowing students to set personal goals and track their progress can also give them a sense of ownership and accomplishment.

Lastly, helping students develop a growth mindset is essential for fostering a positive attitude towards math. Teachers can explicitly teach students that intelligence and abilities can be developed through effort and perseverance. Using growth mindset language, such as praising effort rather than innate talent and encouraging students to view challenges as opportunities for growth, can help shift their mindset. Sharing stories of mathematicians who overcame difficulties and achieved great things can

also inspire students and help them see that success in math is attainable through hard work and determination.

By implementing these techniques, educators can create a classroom environment that nurtures a positive attitude towards math. This not only enhances students' current learning experiences but also lays the foundation for lifelong confidence and success in mathematics.

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1.5.1: NUMBER AND OPERATIONS

- Topic hierarchy

1.5.1.1: Introduction

1.5.1.2: Addition- Dots and Boxes

1.5.1.3: Subtraction- Dots and Boxes

1.5.1.4: Multiplication- Dots and Boxes

1.5.1.5: Division- Dots and Boxes

1.5.1.6: Number Line Model

1.5.1.7: Area Model for Multiplication

1.5.1.8: Properties of Operations (Part 1)

1.5.1.9: Properties of Operations (Part 2)

1.5.1.10: Division Explorations

1.5.1.11: Problem Bank

The “Dots and Boxes” approach to understanding operations used in this part (and throughout this book) comes from James Tanton, and is used with his permission. See his development of these and other ideas at http://gdaymath.com/.

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1.5.1.1: Introduction

When learning and teaching about arithmetic, it helps to have mental and physical *models* for what the operations mean. That way, when you are presented with an unfamiliar problem or a question about why something is true, you can often work it out using the model — this might mean drawing pictures, using physical materials (manipulatives), or just thinking about the model to help you reason out the answer.

Think / Pair / Share

Write down your mental models for each of the four basic operations. What do they actually *mean*? How would you explain them to a second grader? What pictures could you draw for each operation? Think about each one separately, as well as how they relate to each other:

- addition
- subtraction
- multiplication, and
- division.

After writing down you own ideas, share them with a partner. Do you and your partner have the same models for each of the operations or do you think about them differently?

Teachers should have lots of mental models — lots of ways to explain the same concept. In this chapter, we'll look at some different ways to understand the four basic arithmetic operations. First, let's define some terms:

Definition

Counting numbers are literally the numbers we use for counting: 1, 2, 3, 4, 5... These are sometimes called the *natural numbers* by mathematicians, and they are represented by the symbol $\mathbb{N}$.

Whole numbers are the counting numbers together with zero.

Integers include the positive and negative whole numbers, and mathematicians represent these with the symbol $\mathbb{Z}$. (This comes from German, where the word for "number" is "zählen.")

We already have a natural model for thinking about counting numbers: a number is a quantity of dots. Depending on which number system you use — Roman numerals, base ten, binary, etc. — you might write down the number in different ways. But the quantity of dots is a counting number, however you write it down.

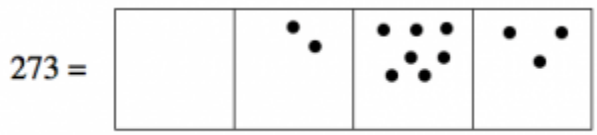
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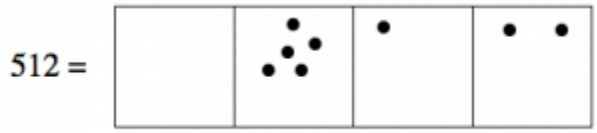
1.5.1.2: Addition- Dots and Boxes

Addition as combining

For now, we'll focus on the base-10 system. Here's how we think about the number 273 in that system:

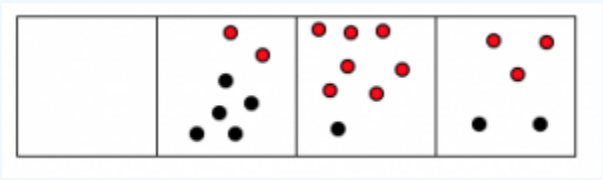


And here is the number 512:



273+512

We can add these in the natural way: just combine the piles of dots. Since they're already in place-value columns, we can combine dots from the two numbers that are in the same place-value box.



We can count up the answer: there are 7 dots in the hundreds box, 8 dots in the tens box, and 5 dots in the ones box.

$$\begin{array}{r} 273 \\ +\ 512 \\ \hline 785 \end{array}$$

And saying out the long way we have:

- Two hundreds plus five hundreds gives 7 hundreds.
- Seven tens plus one ten gives 8 tens.
- Three ones plus two ones gives 5 ones.

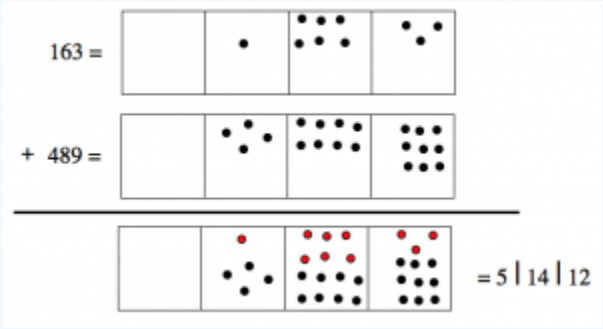
This gives the answer: 785.

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View this YouTube Video online

: 163+489

Let's do another one. Consider 163+489.



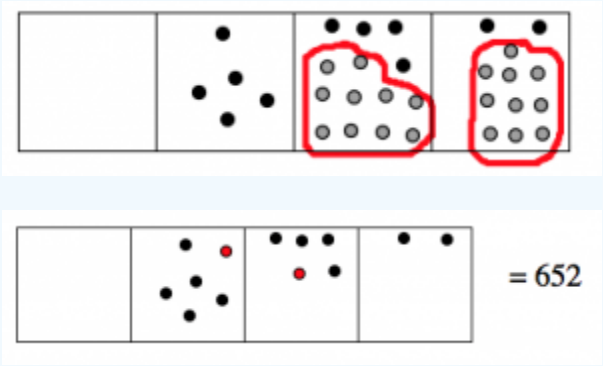
$$\begin{array}{r} 1\ \ 6\ \ 3 \\ +\,4\ \ 8\ \ 9 \\ \hline 5|14|12 \end{array}$$

And this is absolutely correct:

- One hundred plus four hundreds is 5 hundreds.
- Six tens plus eight tens is 14 tens.
- Three ones plus nine ones is 12 ones.

The answer is 5 | 14 | 12, which we might try to pronounce as "five hundred and fourteeny-tenty twelvety." The trouble with this answer is that most of the rest of the world wouldn't understand what we are talking about.

Since this is a base 10 system, we can do some explosions.



The answer is "six hundred fifty two." Okay, the world can understand this one!

$$\begin{array}{r} 1\ \ 6\ \ 3 \\ +\,4\ \ 8\ \ 9 \\ \hline 5|14|12 = 652 \end{array}$$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Think / Pair / Share

Solve the following exercises by thinking about the dots and boxes. (You can draw the pictures, or just imagine them.) Then translate the answer into something the rest of the world can understand.

$$\begin{array}{r} 148 \\ +\ 323 \\ \hline \end{array} \qquad \begin{array}{r} 567 \\ +\ 271 \\ \hline \end{array} \qquad \begin{array}{r} 310462872 \\ +\ 389107123 \\ \hline \end{array}$$

Problem 1

Use the dots and boxes technique to solve these problems. *Do not covert to base 10! Try to work directly in the base given.* It might help to actually draw the pictures.

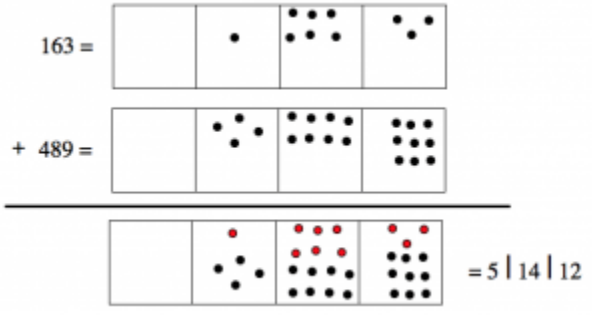
$$\begin{array}{r} 20413_{five} \\ +\ 13244_{five} \\ \hline \end{array} \qquad \begin{array}{r} 4052_{nine} \\ +\ 6288_{nine} \\ \hline \end{array} \qquad \begin{array}{r} 3323_{seven} \\ +\ 3555_{seven} \\ \hline \end{array}$$

The Standard Algorithm for Addition

Let's go back to the example 163+489. Some teachers don't like writing:

$$\begin{array}{r} 1\ \ 6\ \ 3 \\ +\ 4\ \ 8\ \ 9 \\ \hline 5|14|12 = 652 \end{array}$$

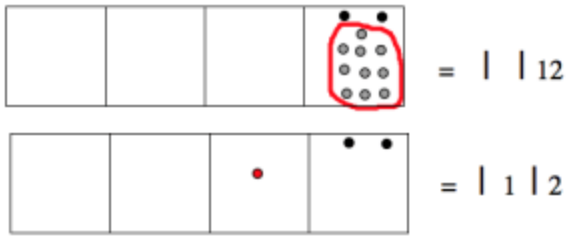
They prefer to teach their students to start with the 3 and 9 at the end and sum those to get 12. This is of course correct — we got 12 as well.



But they don't want students to write or think "twelvety," so they have their students write something like this:

$$\begin{array}{r} 1 \\ 163 \\ +\ 489 \\ \hline 2 \end{array}$$

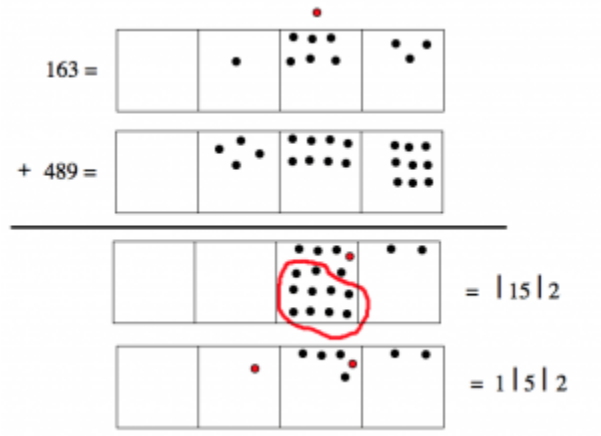
This can seem completely mysterious. What's really going on? They are exploding ten dots, of course!



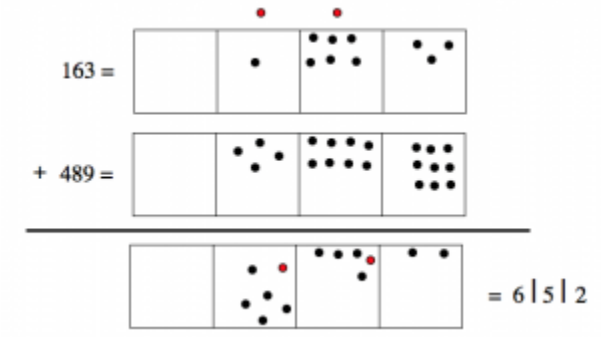
Now we carry on with the problem and add the tens. Students are taught to write:

$$\begin{array}{r} {\scriptstyle 1} \\ 163 \\ +\,489 \\ \hline 52 \end{array}$$

But what this means is better shown in this next picture. Notice the “exploded” (or regrouped) dot at the very top, which is added to the tens box in the answer.



And now we finish the problem by combining the dots in the hundreds boxes:



$$\begin{array}{r} {\scriptstyle 1} \\ 163 \\ +\,489 \\ \hline 652 \end{array}$$

In the standard algorithm, we work from right to left, doing the “explosions” as we go along. This means that we start adding at the ones place and work towards the left-most place value, “carrying” digits that come from the explosions. (This is really not carrying; a better term for it is *regrouping*. Ten ones become one ten. Ten tens become one hundred. And so on.)

In the dots and boxes method, we add in any direction or order we like and then we do the explosions at the end.

- **Why do we like the standard algorithm?** Because it is efficient.
- **Why do we like the dots and boxes method?** Because it is easy to understand.

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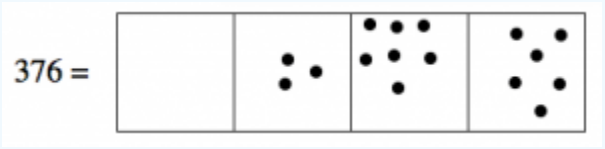
1.5.1.3: Subtraction- Dots and Boxes

Subtraction as Take-Away

To model addition, we started with two collections of dots (two numbers), and we *combined* them to form one bigger collection. That's pretty much the definition of addition: combining two collections of objects. In subtraction, we start with one collection of dots (one number), and we take some dots away.

: 376 – 125

Suppose we want to find 376-125 in the dots and boxes model. We start with the representation of 376:



Since we want to "take away" 125, that means:

- We take away one dot from the hundreds box, leaving two dots.
- We take away two dots from the tens box, leaving five dots.
- And we take away five dots from the ones box, leaving one dot.



So the answer is:

$$\begin{array}{r} 376 \\ -\ 125 \\ \hline 251 \end{array}$$

And saying it out the long way we have:

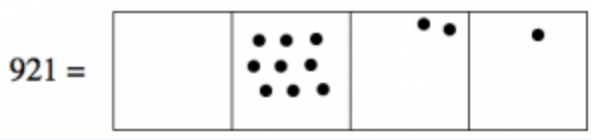
- Three hundreds take away one hundred leaves 2 hundreds.
- Seven tens take away two tens gives 5 tens.
- Six ones take away five ones gives 1 one.

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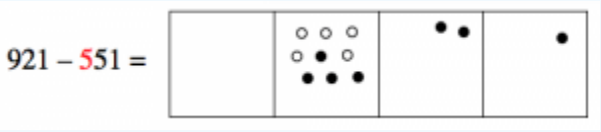
View this YouTube Video online

: 921 – 551

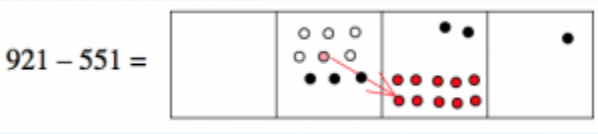
Let's try a somewhat harder example: 921-551. We start with the representation of 921:

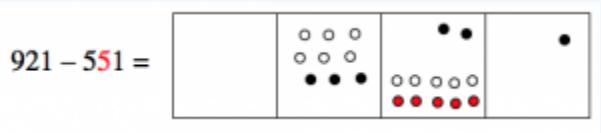


Since we want to “take away” 551, that means we take away five dots from the hundreds box, leaving four dots.



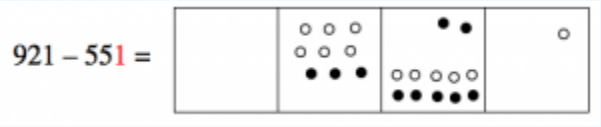
Now we want to take away five dots from the tens box, but we can’t do it! There are only two dots there. What can we do? Well, we still have some hundreds, so we can “unexplode” a hundreds dot, and put ten dots in the tens box instead. Then we’ll be able to take five of them away, leaving seven.





(Notice that we also have one less dot in the hundreds box; there’s only three dots there now.)

Now we want to take one dot from the ones box, and that leaves no dots there.



So the answer is:

$$\begin{array}{r} 921 \\ -\ 551 \\ \hline 370 \end{array}$$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Think / Pair / Share

Solve the following exercises by thinking about dots and boxes. (You can draw pictures, or just imagine them.)

$$\begin{array}{r} 323 \\ -\ 148 \\ \hline \end{array} \qquad \begin{array}{r} 567 \\ -\ 271 \\ \hline \end{array} \qquad \begin{array}{r} 389107123 \\ -\ 310462872 \\ \hline \end{array}$$

Problem 2

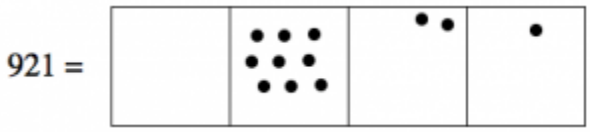
Use the dots and boxes technique to solve these problems. *Do not covert to base 10! Try to work directly in the base given.* It might help to actually draw the pictures.

$$\begin{array}{r} 30413_{five} \\ -\ 13244_{five} \\ \hline \end{array} \qquad \begin{array}{r} 6252_{nine} \\ -\ 4088_{nine} \\ \hline \end{array} \qquad \begin{array}{r} 4323_{seven} \\ -\ 3524_{seven} \\ \hline \end{array}$$

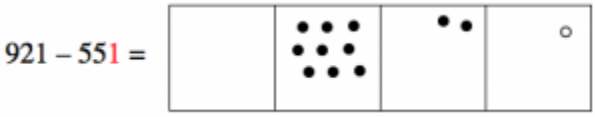
The Standard Algorithm for Subtraction

Just like in addition, the standard algorithm for subtraction requires you to work from right to left, and "borrow" (this is really *regrouping*!) whenever necessary. Notice that in the dots and boxes approach, you don't need to go in any particular order when you do the subtraction. You just "unexplode" the dots as necessary when computing.

Here's how the standard algorithm looks with the dots and boxes model for 921 – 551: Start with 921 dots.

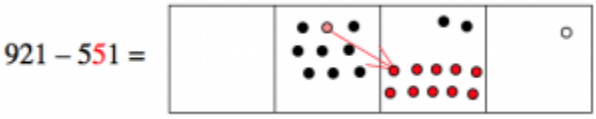


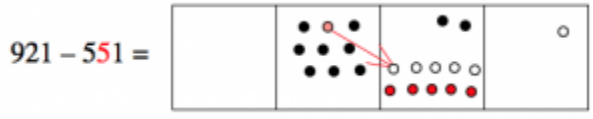
Then take away one dot from the ones box.



$$\begin{array}{r} 921 \\ -\ 551 \\ \hline 0 \end{array}$$

Now we want to take away five dots from the tens box. But there aren't five dots there. So we "unexplode" one of the hundreds dots to get more tens:



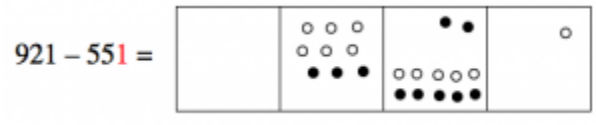


In the standard algorithm, we show the unxplosion as a regrouping, subtracting one from the hundreds place of 921 and adding ten to the tens place. So we are rewriting

$$9|2|1 = 8|12|1.$$

$$\begin{array}{r} 8\ 12 \\ 9\ 2\ \ 1 \\ -\ 5\ 5\ \ 1 \\ \hline 7\ \ 0 \end{array}$$

Finally, we want to take away five from the eight dots left in the hundreds column.



$$\begin{array}{r} {}^{8}\,{}^{12} \\ 9\,2\;1 \\ -\,5\,5\;1 \\ \hline 3\,7\;0 \end{array}$$

YouTube Video

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1.5.1.4: Multiplication- Dots and Boxes

Problem 3

Jenny was asked to compute 243192×4. She wrote:

$$243192 \times 4 = 8 \mid 16 \mid 12 \mid 4 \mid 36 \mid 8.$$

1. What was Jenny thinking about? Is her answer correct?
2. Translate Jenny's answer into a number that the rest of the world can understand.
3. Use Jenny's method to find the answers to these multiplication exercises. Be sure to translate your answers into familiar base 10 numbers.

$$156 \times 3 = \qquad 2873 \times 2 = \qquad 71181 \times 5 = \qquad 3726510392 \times 2$$

Problem 4

Can you adapt Jenny's method to solve these problems? Write your answers in base eight. Try to work directly in base eight rather than converting to base 10 and back again!

$$156_{eight} \times 3_{eight} =$$

$$2673_{eight} \times 4_{eight} =$$

$$36255772_{eight} \times 2_{eight} =$$

Jenny might have been thinking about multiplication as repeated addition. If we have some number N and we multiply that number by 4, what we mean is:

$$4 \cdot N = N + N + N + N.$$

If we take the number 243192 and add it to itself four times using the "combining method," we get

- 2 + 2 + 2 + 2 = 8 ones,
- 9 + 9 + 9 + 9 = 36 tens,
- 1 + 1 + 1 + 1 = 4 hundreds,
- and so on.

Notation

Notice that we have used both × and · to represent multiplication. It's a bit awkward to use × when you're also using variables. Is it the letter x? Or the multiplication symbol ×? It can be hard to tell! In this case, the symbol · is more clear.

We can even simplify the notation further, writing 4N instead of 4 · N. But of course we only do that when we are multiplying *variables* by some quantity. (We wouldn't want 34 to mean 3 · 4, would we?)

Problem 5

Here is a strange addition table. Use it to solve the following problems. Important: Don't try to assign numbers to A, B, and C. Solve the problems just using what you know about the operations!

+	A	B	C
A	C	A	B
B	A	B	C
C	B	C	A

$A+B$ $B+C$ $2A$ $5C$ $3A+4B$

Think / Pair / Share

How does an addition table help you solve multiplication problems like 5C?

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1.5.1.5: Division- Dots and Boxes

Quotative Model of Division

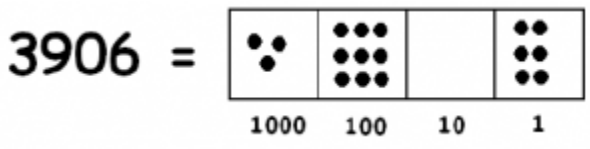
Suppose you are asked to compute 3906 : 3. One way to interpret this question (there are others) is:

"How many groups of 3 fit into 3906?"

Definition

In the **quotative model of division**, you are given a **dividend** (here it is 3906), and you are asked to split it into equal-sized groups, where the size of the group is given by the **divisor** (here it is 3).

In our dots and boxes model, the dividend 3906 looks like this:



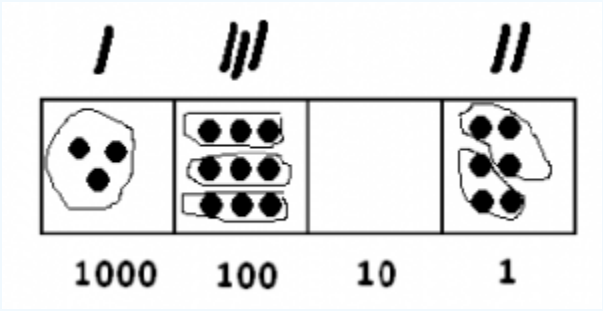
and three dots looks like this: 

So we are really asking:

"How many groups of  fit into the picture of 3906?"

: 3906 ÷ 3

There is one group of 3 at the thousands level, and three at the hundreds level, none at the tens level, and two at the ones level.



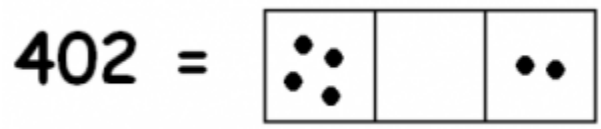
Notice what we have in the picture:

- One group of 3 in the thousands box.
- Three groups of 3 in the hundreds box.
- Zero groups of 3 in the tens box.
- Two groups of 3 in the ones box.

This shows that 3 goes into 3906 one thousand, three hundreds and two ones times. That is,

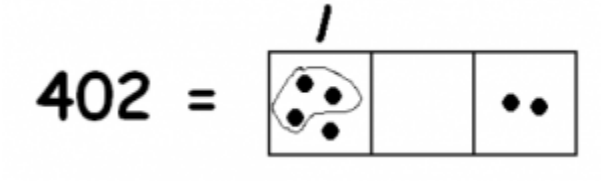
$$3906 \div 3 = 1302.$$

Let's try a harder one! Consider 402 : 3. Here's the picture:



We are still looking for groups of three dots: 

There is certainly one group at the 100's level.



and now it seems we are stuck there are no more groups of three!

Think / Pair / Share

What can we do now? Are we really stuck? Can you finish the division problem?

: 402 ÷ 3

Here are the details worked out for 402 : 3. But don't read this until you've thought about it yourself!

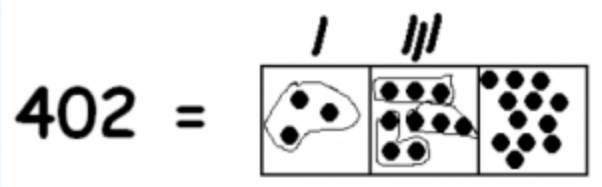
Since each dot is worth ten dots in the box to the right we can write:



Now we can find more groups of three:



There is still a troublesome extra dot. Let's unexplode it too



This gives us more groups of three:



In the picture we have:

- One group of 3 in the hundreds box.
- Three groups of 3 in the tens box.
- Four groups of 3 in the ones box.

Finally we have the answer!

$$402 \div 3 = 134.$$

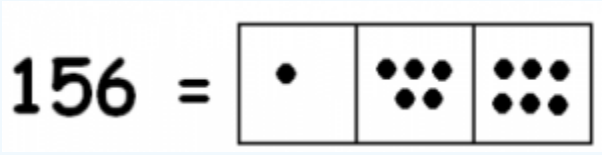
Think / Pair / Share

Solve each of these exercises using the dots and boxes method:

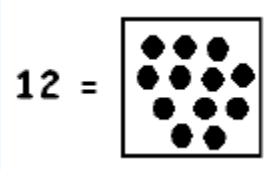
$$62124 : 3 \qquad 61230 : 5$$

: 156 ÷ 12

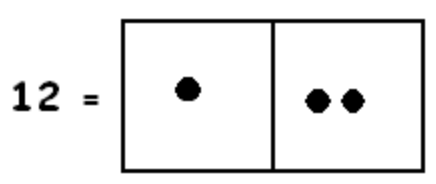
Let's turn up the difficulty a notch. Consider 156 : 12. Here we are looking for groups of 12 in this picture:



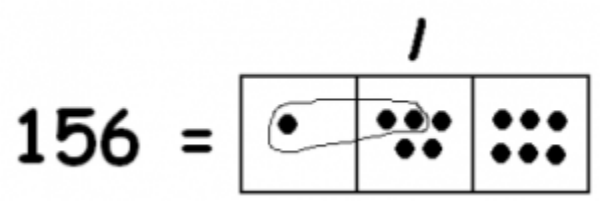
What does 12 look like? It can be twelve dots in a single box:



But most often we would write 12 this way, as a ten and 2 ones:

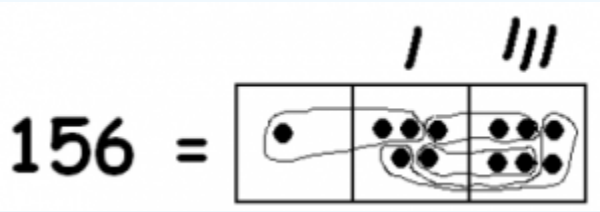


We certainly see some of these in the picture. There is certainly one at the tens level:



Note: With an unexplosion this would be twelve dots in the tens box, so we mark one group of 12 above the tens box.

We also see three groups of twelve ones:



So in the picture we have:

- One group of 12 dots in the tens box.
- Three groups of 12 dots in the ones box.

That means

$$156 : 12 = 13.$$

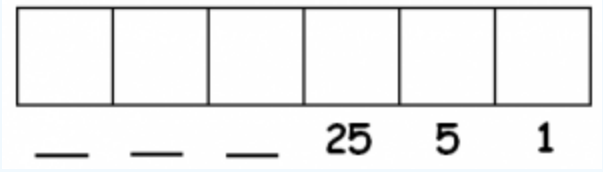
Problem 6

Use the dots and boxes model to compute each of the following:

$$13453 : 11$$
$$4853 : 23$$
$$214506 : 102$$

Problem 7

Remember that base five numbers are in a $1 \leftarrow 5$ dots-and-boxes system. What are the place values in the $1 \leftarrow 5$ system? Fill in the blanks:



1. Draw a dots-and-boxes picture of the number 424_{five}.
2. Draw a dots-and-boxes picture of the number 11_{five}.
3. Use the dots and boxes method to find $424_{five} \div 11_{five}$.
4. Rewrite the division sentence $424_{five} \div 11_{five} = 34_{five}$ in base ten, and check that it's correct.
5. Use dots-and-boxes to find $2021_{five} \div 12_{five}$. *Don't convert to base 10!*

Think / Pair / Share

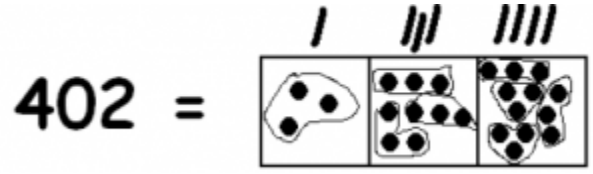
- Use dots and boxes to compute these.

$$\begin{array}{c} 2130 : 10 \\ 41300 : 100 \end{array} \qquad (1.5.1.5.1)$$

- What pictures did you use for 10 and for 100? Can you describe in words what happens when dividing by 10 and by 100 and why?

The Standard Algorithm for Division

We used dots and boxes to show that 402 : 3 = 134.

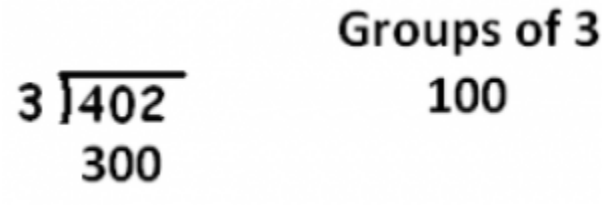


In elementary school, you might have learned to solve this division problem by using a diagram like the following:

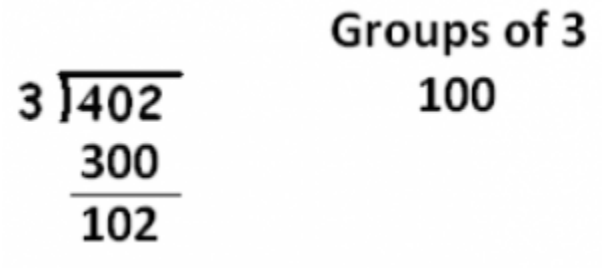
$$\begin{array}{r} 134 \\ 3\overline{)402} \\ \underline{3\downarrow} \\ 10 \\ \underline{9\downarrow} \\ 12 \\ \underline{12} \\ 0 \end{array}$$

At first glance this seems very mysterious, but it is really no different from the dots and boxes method. Here is what the table means.

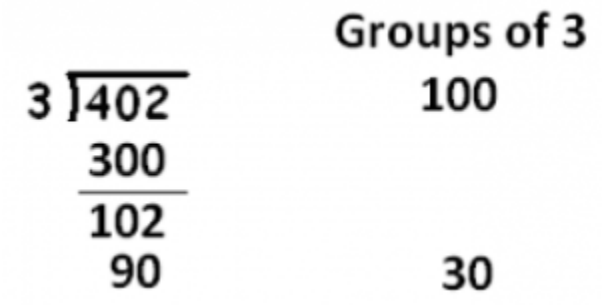
To compute 402 : 3, we first make a big estimation as to how many groups of 3 there are in 402. Let's guess that there are 100 groups of three.



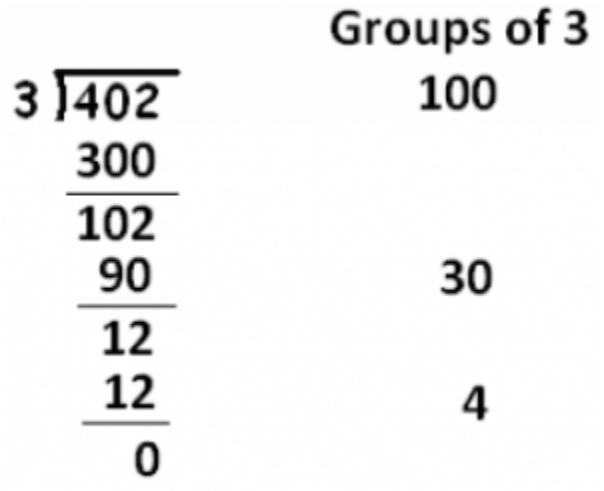
How much is left over after taking away 100 groups of 3? We subtract to find that there is 102 left.



How many groups of 3 are in 102? Let's try 30:



How many are left? There are 12 left and there are four groups of 3 in 12.

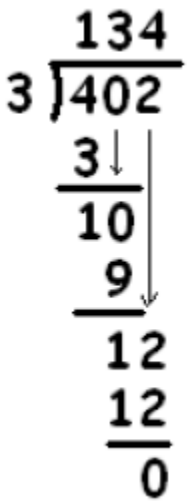


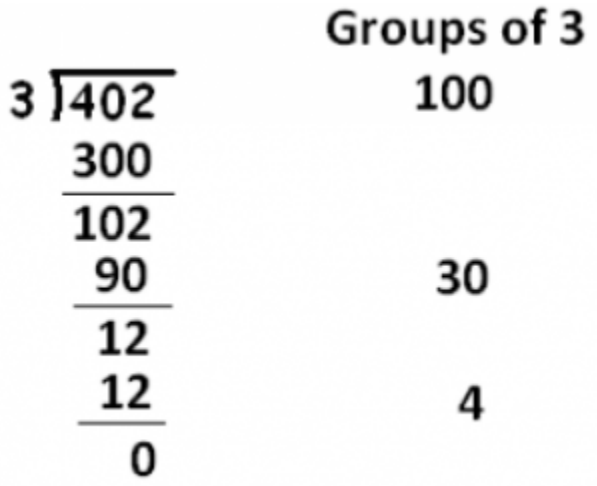
That accounts for entire number 402. And where do we find the final answer? Just add the total count of groups of three that we tallied:

$$402 : 3 = 100 + 30 + 4 = 134.$$

Think / Pair / Share

- Compare the two division diagrams below. In what way are they the same? In what way are they different?
- Also look at the dots and boxes method. In what way is it the same or different from the two diagrams?







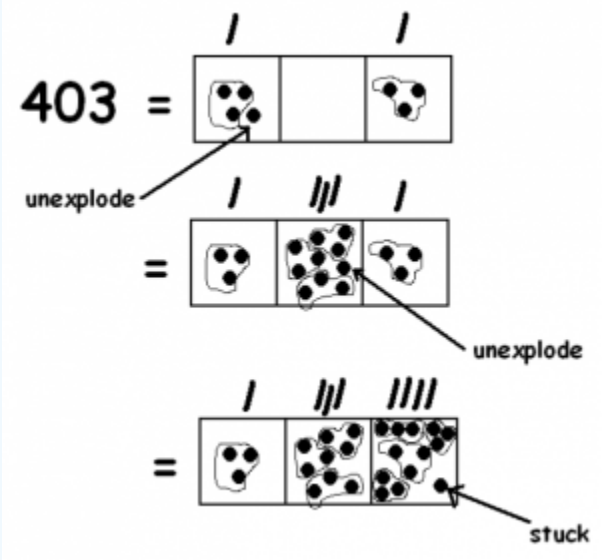
- **Why do we like the standard algorithm?** Because it is quick, not too much to write down, and it works every time.
- **Why do we like the dots and boxes method?** Because it easy to understand. (And drawing dots and boxes is kind of fun!)

Division with Remainders

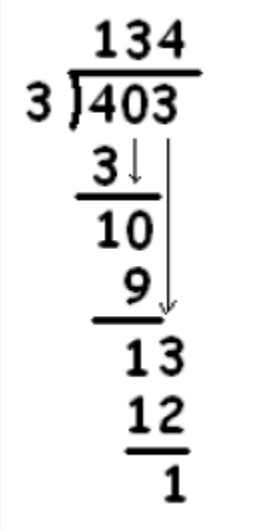
We saw that 402 is evenly divisible by 3: 402 : 3 = 134. This means that 403, one more, shouldn't be divisible by three. It should be one dot too big.

: 403 ÷ 3

Do we see the extra dot if we compute 402 : 3 with dots and boxes?



Yes we do! We have one dot left at the end that can't be divided. This is how it looks in the standard algorithm.



In school, we say that we have a *remainder* of one and sometimes write:

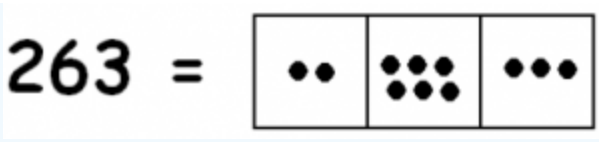
$$403 \div 3 = 134\ \mathrm{R}1.$$

But what does that really mean? It means that we have 134 groups of three with one dot left over. So

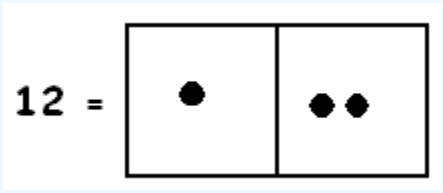
$$402 = 134 \cdot 3 + 1.$$

: 263 ÷ 12

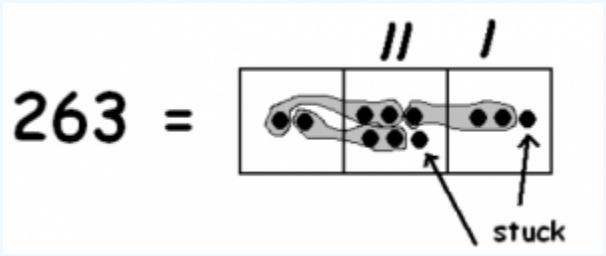
Let's try another one: 263 ÷ 12. Here's what we have:



And we are looking for groups like this:



Here goes!



Unexploding won't help any further and we are indeed left with one remaining dot in the tens position and a dot in the ones position. We have 21 groups of twelve, and a remainder of eleven.

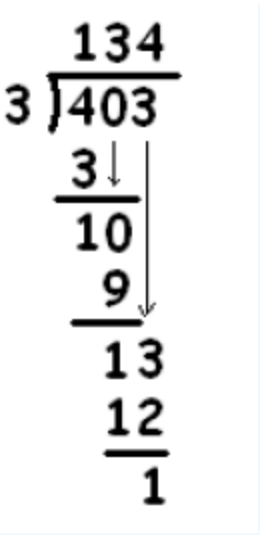
$$263 = 21 \cdot 12 + 11.$$

Think / Pair / Share

- Use the dots and boxes method to compute each quotient and remainder:

$$\begin{gathered} 5210 : 4 \\ 4857 : 23 \\ 31533 : 101 \end{gathered} \qquad (1.5.1.5.2)$$

- Now use the standard algorithm (an example is shown below) to compute each of the quotients and remainders above.



$$402 = 134 \cdot 3 + 1.$$

- Which method do you like better: dots and boxes or the standard algorithm method? Or does it depend on the problem you are doing?

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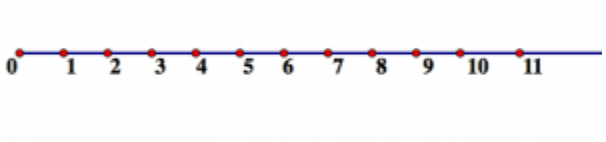
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1.5.1.6: Number Line Model

Another way we often think about numbers is as abstract quantities that can be measured: length, area, and volume are all examples.

In a measurement model, you have to pick a *basic unit*. The basic unit is a quantity — length, area, or volume — that you assign to the number one. You can then assign numbers to other quantities based on how many of your basic unit fit inside.

For now, we'll focus on the quantity length, and we'll work with a number line where the basic unit is already marked off.



Addition and Subtraction on the Number Line

Imagine a person — we'll call him Zed — who can stand on the number line. We'll say that the distance Zed walks when he takes a step is exactly one unit.



When Zed wants to add or subtract with whole numbers on the number line, he always starts at 0 and faces the positive direction (towards 1). Then what he does depends on the calculation.

If Zed wants to *add* two numbers, he walks forward (to the right of the number line) however many steps are indicated by the first number (the first *addend*). Then he walks forward (to your right on the number line) the number of steps indicated by the second number (the second *addend*). Where he lands is the *sum* of the two numbers.

: 3 + 4

If Zed wants to add 3 + 4, he starts at 0 and faces towards the positive numbers. He walks forward 3 steps, then he walks forward 4 more steps.

Zed ends at the number 7, so the sum of 3 and 4 is 7. 3 + 4 = 7. (But you knew that of course! The point right now is to make sense of the *number line model*.)

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When Zed wants to *subtract* two numbers, he he walks forward (to the right on the number line) however many steps are indicated by the first number (the *minuend*). Then he walks *backwards* (to the left on the number line) the number of steps indicated by the second number (the *subtrahend*). Where he lands is the *difference* of the two numbers.

: 11 – 3

If Zed wants to subtract 11 – 3, he starts at 0 and faces the positive numbers (the right side of the number line). He walks forward 11 steps on the number line, then he walks backwards 3 steps.

Zed ends at the number 8, so the difference of 11 and 3 is 8. 11 – 3 = 8. (But you knew that!)

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Think / Pair / Share

- Work out each of these exercises on a number line. You can actually pace it out on a life-sized number line or draw a picture:

$$4+5 \qquad 6+9 \qquad 10-7 \qquad 8-1 \tag{1.5.1.6.1}$$

- Why does it make sense to walk forward for addition and walk backwards for subtraction? In what way is this the same as "combining" for addition and "take away" for subtraction"?
- What happens if you do these subtraction problems on a number line? Explain your answers.

$$6-9 \qquad 1-7 \qquad 4-11 \qquad 0-1 \tag{1.5.1.6.2}$$

- Could you do the subtraction problems above with the dots and boxes model?

Multiplication and Division on the Number Line

Since multiplication is really repeated addition, we can adapt our addition model to become a multiplication model as well. Let's think about 3 × 4. This means to add four to itself three times (that's simply the definition of multiplication!):

$$3 \times 4 = 4 + 4 + 4.$$

So to multiply on the number line, we do the process for addition several times.

To multiply two numbers, Zed starts at 0 as always, and he faces the positive direction. He walks forward the number of steps given by the second number (the second *factor*). He repeats that process the number of times given by the first number (the first *factor*). Where he lands is the *product* of the two numbers.

: 3 × 4

If Zed wants to multiply 3 × 4, he can think of it this way:

$$\begin{array}{ccc} 3 & \times & 4 \\ \downarrow & & \downarrow \\ \text{how many times to repeat it} & & \text{how many steps to take forward} \end{array}$$

Zed starts at 0, facing the positive direction. The he repeats this three times: take four steps forward.

He ends at the number 12, so the product of 3 and 4 is 12. That is, 3 × 4 = 12.

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Remember our quotative model of division: One way to interpret 15 : 5 is:

> How many groups of 5 fit into 15?

Thinking on the number line, we can ask it this way:

> Zed takes 5 steps at a time. If Zed lands at the number 15, how many times did he take 5 steps?

To calculate a division problem on the number line, Zed starts at 0, facing the positive direction. He walks forward the number of steps given by the second number (the *divisor*). He repeats that process until he lands at the first number (the *dividend*). The number of times he repeated the process gives the *quotient* of the two numbers.

: 15 ÷ 5

If Zed wants to compute 15 : 5, he can think of it this way:

He starts at 0, facing the positive direction.

- Zed takes 5 steps forward. He is now at 5, not 15. So he needs to repeat the process.
- Zed takes 5 steps forward again. He is now at 10, not 15. So he needs to repeat the process.
- Zed takes 5 more steps forward. He is at 15, so he stops.

Since he repeated the process three times, we see there are 3 groups of 5 in 15. So the quotient of 15 and 5 is 3. That is, 15 : 5 = 3.

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Think / Pair / Share

- Work out each of these exercises on a number line. You can actually pace it out on a life-sized number line or draw a picture:

$$2 \times 5 \qquad 7 \times 1 \qquad 10 : 2 \qquad 6 : 1 \tag{1.5.1.6.3}$$

- Can you think of a way to interpret these multiplication problems on a number line? Explain your ideas.

$$4 \times 0 \qquad 0 \times 5 \qquad 3 \times (-2) \qquad 2 \times (-1) \tag{1.5.1.6.4}$$

- What happens if you try to solve these division problems on a number line? Can you do it? Explain your ideas.

$$0 : 2 \qquad 0 : 10 \qquad 3 : 0 \qquad 5 : 0 \tag{1.5.1.6.5}$$

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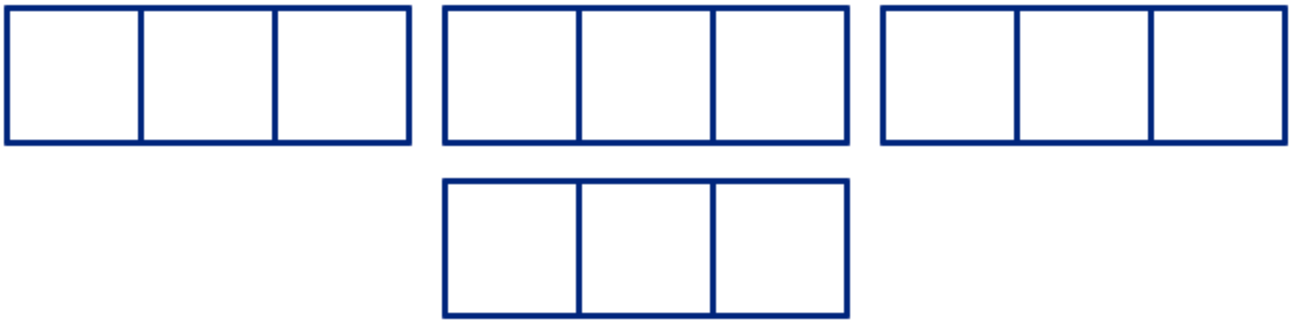
1.5.1.7: Area Model for Multiplication

So far we have focused on a linear measurement model, using the number line. But there's another common way to think about multiplication: using area.

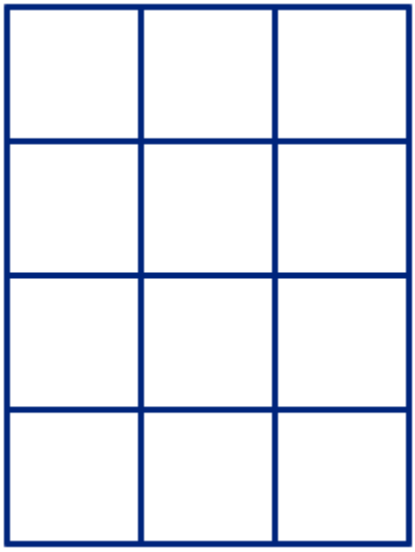
For example, suppose our basic unit is one square:



We can picture 4 × 3 as 4 groups, with 3 squares in each group, all lined up:



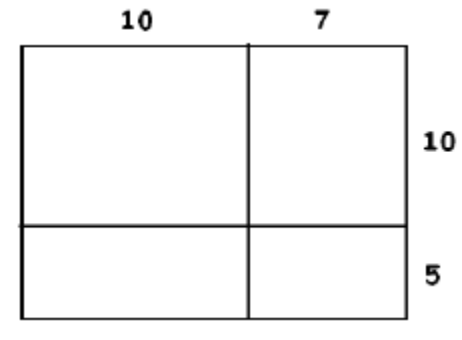
But we can also picture them stacked up instead of lined up. We would have 4 rows, with 3 squares in each row, like this:



So we can think about 4 × 3 as a rectangle that has length 3 and width 4. The product, 12, is the total number of squares in that rectangle. (That is also the area of the rectangle, since each square was one unit!)

Think / Pair / Share

Vera drew this picture as a model for 15 × 17. Use her picture to help you compute 15 × 17. Explain your work.



Problem 8

Draw pictures like Vera's for each of these multiplication exercises. Use your pictures to find the products without using a calculator or the standard algorithm.

23×37 8×43 371×42

The Standard Algorithm for Multiplication

How were you taught to compute 83 × 27 in school? Were you taught to write something like the following?

$$\begin{array}{r} 83 \\ \times\ 27 \\ \hline 21 \\ 56 \\ 6 \\ 16 \\ \hline 2241 \end{array}$$

Or maybe you were taught to put in the extra zeros rather than leaving them out?

$$\begin{array}{r} 83 \\ \times \quad 27 \\ \hline 21 \\ 560 \\ 60 \\ 1600 \\ \hline 2241 \end{array}$$

This is really no different than drawing the rectangle and using Vera's picture for calculating!

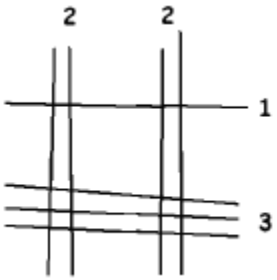
Think / Pair / Share

- Use the example above to explain why Vera's rectangle method and the standard algorithm are really the same.
- Calculate the products below using both methods. Explain where you're computing the same pieces in each algorithm.

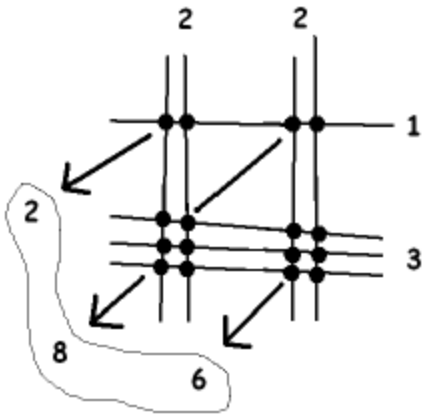
$$23 \times 14 \qquad 106 \times 21 \qquad 213 \times 31 \qquad (1.5.1.7.1)$$

Lines and Intersections

Here's an unusual way to perform multiplication. To compute 22 × 13, for example, draw two sets of vertical lines, the left set containing two lines and the right set two lines (for the digits in 22) and two sets of horizontal lines, the upper set containing one line and the lower set three (for the digits in 13).

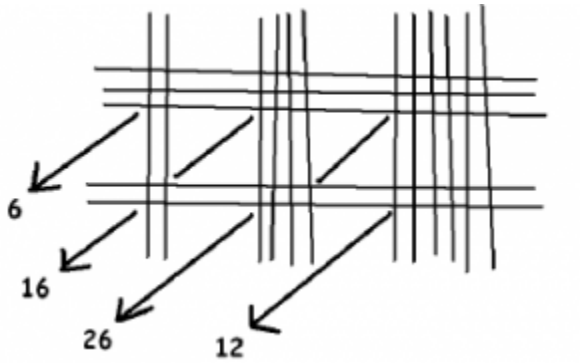


There are four sets of intersection points. Count the number of intersections in each and add the results diagonally as shown:



The answer 286 appears!

There is one possible glitch as illustrated by the computation 246 × 32:



Although the answer 6 thousands, 16 hundreds, 26 tens, and 12 ones is absolutely correct, one needs to carry digits and translate this as 7,872.

Problem 9

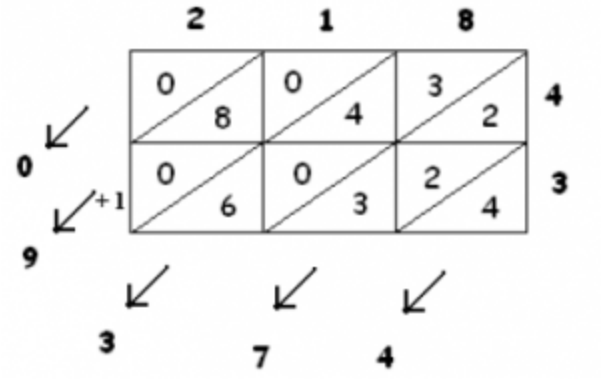
1. Compute 131 × 122 via this method. Check your answer using another method.
2. Compute 15 × 1332 via this method. Check your answer using another method.
3. Can you adapt the method to compute 102 × 3054? (Why is some adaptation necessary?)
4. Why does the method work in general?

Lattice Multiplication

In the 1500s in England, students were taught to compute multiplication using following galley method, now more commonly known as the *lattice method*.

To multiply 43 and 218, for example, draw a 2 × 3 grid of squares. Write the digits of the first number along the right side of the grid and the digits of the second number along the top.

Divide each cell of the grid diagonally and write in the product of the column digit and row digit of that cell, separating the tens from the units across the diagonal of that cell. (If the product is a one digit answer, place a 0 in the tens place.)



To get the answer, add the entries in each diagonal, carrying tens digits over to the next diagonal if necessary. In our example, we have

$$218 \times 43 = 9374.$$

Problem 10

1. Compute 5763 × 345 via the lattice method.
2. Explain why the lattice method is really the standard algorithm in disguise.
3. What is the specific function of the diagonal lines in the grid?

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1.5.1.8: Properties of Operations (Part 1)

So far, you have seen a couple of different *models* for the operations: addition, subtraction, multiplication, and division. But we haven't talked much about the operations themselves — how they relate to each other, what properties they have that make computing easier, and how some special numbers behave. There's lots to think about!

The goal in this section is to use the models to understand why the operations behave according to the rules you learned back in elementary school. We're going to keep asking ourselves "Why does it work this way?"

Think / Pair / Share

Each of these models lends itself to thinking about the operation in a slightly different way. Before we really dig in to thinking about the operations, discuss with a partner:

- Of the models we discussed so far, do you prefer one of them?
- How well do the models we discussed match up with how you usually think about whole numbers and their operations?
- Which models are useful for computing? Why?
- Which models do you think will be useful for explaining how the operations work? Why?

Connections Between the Operations

We defined addition as combining two quantities and subtraction as "taking away." But in fact, these two operations are intimately tied together. These two questions are exactly the same:

$$27 - 13 = \_\_\_ \qquad 27 = 13 + \_\_\_\_.$$

More generally, for any three whole numbers a, b, and c, these two equations express the same fact. (So either both equations are true or both are false. Which is the case depends on the values you choose for a, b, and c!)

$$c - b = a \qquad c = a + b.$$

In other words, we can think of every subtraction problem as a "missing addend" addition problem. Try it out!

Problem 11

Here is a strange addition table. Use it to solve the following problems. Justify your answers. Important: Don't try to assign numbers to A, B, and C. Solve the problems just using what you know about the operations!

+	A	B	C
A	C	A	B
B	A	B	C
C	B	C	A

$$A + C \qquad B + C \qquad A - C \qquad C - A \qquad A - A \qquad B - C$$

Think / Pair / Share

How does an addition table help you solve subtraction problems?

We defined multiplication as repeated addition and division as forming groups of equal size. But in fact, these two operations are also tied together. These two questions are exactly the same:

$$27 : 3 = \_\_\_ \qquad 27 = \_\_\_ \times 3.$$

More generally, for any three whole numbers a, b, and c, these two equations express the same fact. (So either both equations are true or both are false. Which is the case depends on the values you choose for a, b, and c!)

$$c : b = a \qquad c = a \cdot b.$$

In other words, we can think of every division problem as a "missing factor" multiplication problem. Try it out!

Problem 12

Rewrite each of these division questions as a "missing factor" multiplication question. Which ones can you solve and which can you not solve? Explain your answers.

$$9 : 3 \qquad 100 : 25 \qquad 0 : 3 \qquad 9 : 0 \qquad 0 : 0$$

Problem 13

Here's a multiplication table.

$\times$	A	B	C	D	E
A	A	A	A	A	A
B	A	B	C	D	E
C	A	C	E	B	D
D	A	D	B	E	C
E	A	E	D	C	B

- Use the table to solve the problems below. Justify your answers. Important: Don't try to assign numbers to the letters. Solve the problems just using what you know about the operations!

$$C \times D \qquad C \times A \qquad A \times A \qquad C : D \qquad D : C \qquad D : E \tag{1.5.1.8.1}$$

- Can you use the table to solve these problems? Explain your answers. Recall that x^n means n copies of x multiplied together, $x \cdot x \cdot x \cdot \ldots \cdot x$

$$D^2 \qquad C^3 \qquad A : C \qquad A : D \qquad D : A \qquad A : A \tag{1.5.1.8.2}$$

Think / Pair / Share

How does a multiplication table help you solve division (and exponentiation) problems?

Throughout this course, our focus is on explanation and justification. As teachers, you need to know what is true in mathematics, but you also need to know *why* it is true. And you will need lots of ways to explain *why*, since different explanations will make sense to different students.

Think / Pair / Share

Arithmetic Fact: *a* + *b* = *c* and *c* – *b* = *a* are the same mathematical fact.

Why is this *not* a good explanation?

- "I can check that this is true! For example, 2+3 = 5 and 5 – 3 = 2. And 3 + 7 = 10 and 10 – 7 = 3. It works for whatever numbers you try."

Addition and Subtraction: Explanation 1

Arithmetic Fact:

a + *b* = *c* and *c* – *b* = *a* are the same mathematical fact.

Why It's True, Explanation 1:

First we'll use the definition of the operations.

Suppose we know *c* – *b* = *a* is true. Subtraction means "take away." So

$$c - b = a$$

means we start with quantity *c* and take away quantity *b*, and we end up with quantity *a*. Start with this equation, and imagine adding quantity *b* to both sides.

On the left, that mans we started with quantity *c*, took away *b*things, and then put those *b* things right back! Since we took away some quantity and then added back the exact same quantity, there's no overall change. We're left with quantity *c*.

On the right, we would be combining (adding) quantity *a* with quantity *b*. So we end up with: c = a + b.

On the other hand, suppose we know the equation *a* + *b* = *c* is true. Imagine taking away (subtracting) quantity *b* from both sides of this equation: a + b = c.

On the left, we started with *a* things and combined that with *b* things, but then we immediately take away those *b* things. So we're left with just our original quantity of *a*.

On the right, we start with quantity *c* and take away *b* things. That's the very definition of *c* – *b*. So we have the equation:

$$a = c - b.$$

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Why It's True, Explanation 2:

Let's use the measurement model to come up with another explanation.

The equation *a* + *b* = *c* means Zed starts at 0, walks forward *a* steps, and then walks forward *b* steps, and he ends at *c*.

If Zed wants to compute *c* – *b*, he starts at 0, walks forward *c* steps, and then walks backwards *b* steps. But we know that to walk forward *c* steps, he can first walk forward *a* steps and then walk forward *b* steps. So Zed can compute *c* – *b* this way:

- Start at 0.
- Walk forward a steps.
- Walk forward b steps. (Now at c, since $a + b = c$.)
- Walk backwards b steps.

The last two sets of steps cancel each other out, so Zed lands back at a. That means $c - b = a$.

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On the other hand, the equation $c - b = a$ means that Zed starts at 0, walks forward c steps, then walks backwards b steps, and he ends up at a.

If Zed wants to compute $a + b$, he starts at 0, walks forward a steps, and then walks forwards b additional steps. But we know that to walk forward a steps, he can first walk forward c steps and then walk backwards b steps. So Zed can compute $a + b$ this way:

- Start at 0.
- Walk forward c steps.
- Walk backwards b steps. (Now at a, since $c - b = a$.)
- Walk forward b steps.

The last two sets of steps cancel each other out, so Zed lands back at c. That means $a + b = c$.

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Think / Pair / Share

- Read over the two explanations in the example above. Do you think either one is more clear than the other?
- Come up with your own **explanation** (not examples!) to explain:

$$c : b = a \quad \text{is the same fact as} \quad c = a \times b. \tag{1.5.1.8.3}$$

Properties of Addition and Subtraction

You probably know several properties of addition, but you may never have stopped to wonder: *Why is that true?!* Now's your chance! In this section, you'll use the definition of the operations of addition and subtraction and the models you've learned to explain why these properties are always true.

Here are the three properties you'll think about:

- Addition of whole numbers is *commutative*.

- Addition of whole numbers is *associative*.
- The number 0 is an *identity* for addition of whole numbers.

For each of the properties, we don't want to confuse these three ideas:

- what the property is called and what it means (the definition),
- some examples that *demonstrate* the property, and
- an explanation for *why* the property holds.

Notice that *examples* and *explanations* are not the same! It's also very important not to confuse the *definition* of a property with the *reason* it is true!

These properties are all universal statements — statements of the form "for all," "every time," "always," etc. That means that to show they are true, you either have to check every case or find a *reason why* it must be so.

Since there are infinitely many whole numbers, it's impossible to check every case. You'd never finish! Our only hope is to look for *general explanations*. We'll work out the explanation for the first of these facts, and you will work on the others.

ADDITION IS COMMUTATIVE

Example: Commutative Law

Property:

Addition of whole numbers is commutative.

What it Means (words):

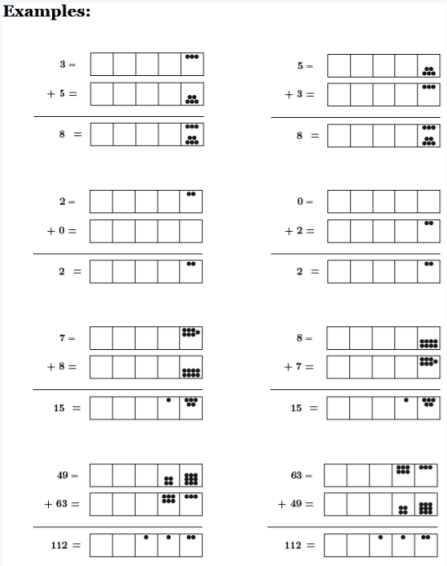
When I add two whole numbers, the order I add them doesn't affect the sum.

What it Means (symbols):

For any two whole numbers *a* and *b*,

$$a + b = b + a.$$

Examples:



Now we need a *justification*. *Why* is addition of whole numbers commutative?

Why It's True, Explanation 1:

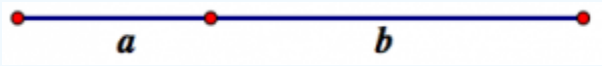
Let's think about addition as combining two quantities of dots.

- To add $a + b$, we take a dots and b dots, and we combine them in a box. To keep things straight, lets imagine the a dots are colored red and the b dots are colored blue. So in the box we have a red dots, b blue dots and $a + b$ total dots.
- To add $b + a$, let's take b blue dots and a red dots, and put them all together in a box. We have b blue dots, a red dots and $b + a$total dots.
- But the total number of dots are the same in the two boxes! How do we know that? Well, there are a red dots in each box, so we can match them up. There are b blue dots in each box, so we can match them up. That's it! If we can match up the dots one-for-one, there must be the same number of them!

• That means $a + b = b + a$.

Why It's True, Explanation 2:

We can also use the measurement model to explain why $a + b = b + a$ no matter what numbers we choose for a and b. Imagine taking a segment of length a and combining it linearly with a segment of length b. That's how we get a length of $a + b$.



But if we just rotate that segment so it's upside down, we see that we have a segment of length b combined with a segment of length a, which makes a length of $b + a$.



But of course it's the same segment! We just turned it upside down! So the lengths must be the same. That is, $a + b = b + a$.

ADDITION IS ASSOCIATIVE

Your turn! You'll answer the question, "Why is addition of whole numbers associative?"

Property: Addition of whole numbers is associative.

What it Means (words): When I add three whole numbers in a given order, the way I group them (to add two at a time) doesn't affect the sum.

What it Means (symbols): For any three whole numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

Problem 14

1. Come up with at least three *examples* to demonstrate associativity of addition.
2. Use our models of addition to come up with an *explanation*. Why does associativity hold in *every case*? **Note:** your explanation should not use specific numbers. It is not an example!

0 IS AN IDENTITY FOR ADDITION

Property: The number 0 is an *identity* for addition of whole numbers.

What it Means (words): When I add any whole number to 0 (in either order), the sum is the very same whole number I added to 0.

What it Means (symbols): For any whole numbers n,

$$n + 0 = n \quad \text{and} \quad 0 + n = n.$$

Problem 15

1. Come up with at least three *examples* to demonstrate that 0 is an identity for addition.
2. Use our models of addition to come up with an *explanation*. Why does this property of 0 hold in *every possible case*?

PROPERTIES OF SUBTRACTION

Since addition and subtraction are so closely linked, it's natural to wonder if subtraction has some of the same properties as addition, like commutativity and associativity.

Example: Is subtraction commutative?

Justin asked if the operation of subtraction is commutative. That would mean that the difference of two whole numbers doesn't depend on the order in which you subtract them.

In symbols: *for every choice* of whole numbers a and b we would have $a - b = b - a$.

Jared says that subtraction is *not* commutative since $4 - 3 = 1$, but $3 - 4 \neq 1$. (In fact, $3 - 4 = -1$.)

Since the statement "subtraction is commutative" is a *universal statement*, one counterexample is enough to show it's not true. So Jared's counterexample lets us say with confidence:

Subtraction is **not** commutative.

Think / Pair / Share

Can you find any examples of whole numbers a and b where $a - b = b - a$ is true? Explain your answer.

Problem 16

Lyle asked if the operation of subtraction is associative.

1. State what it would mean for subtraction to be associative. You should use words and symbols.
2. What would you say to Lyle? Decide if subtraction is associative or not. Carefully explain how you made your decision and *how you know you're right*.

Problem 17

Jess asked if the number 0 is an identity for subtraction.

1. State what it would mean for 0 to be an identity for subtraction. You should use words and symbols.
2. What would you say to Jess? Decide if 0 is an identity for subtraction or not. Carefully explain how you made your decision and *how you know you're right*

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1.5.1.9: Properties of Operations (Part 2)

Properties of Multiplication and Division

Now we're going to turn our attention to familiar properties of multiplication and division, with the focus still on explaining why these properties are always true.

Here are the four properties you'll think about:

- Multiplication of whole numbers is *commutative*.
- Multiplication of whole numbers is *associative*.
- Multiplication of whole numbers *distributes over addition*
- The number 1 is an *identity* for multiplication of whole numbers

For each of the properties, remember to keep straight:

- what the property is called and what it means (the definition),
- some examples that *demonstrate* the property, and
- an explanation for *why* the property holds.

Once again, it's important to distinguish between *examples* and *explanations*. They are not the same! Since there are infinitely many whole numbers, it's impossible to check every case, so examples will never be enough to explain why these properties hold. You have to figure out *reasons* for these properties to hold, based on what you know about the operations.

1 IS AN IDENTITY FOR MULTIPLICATION

We'll work out the explanation for the last of these facts, and you will work on the others.

Example: 1 is an Identity for multiplication

Property:

The number 1 is an identity for multiplication of whole numbers.

What it Means (words):

When I multiply a number by 1 (in either order), the product is that number.

What it Means (symbols):

For any whole number *m*,

$$m \times 1 = m \quad \text{and} \quad 1 \times m = m.$$

Examples:

$$1 \times 5 = 5, \qquad 19 \times 1 = 19, \qquad \text{and } 1 \times 1 = 1.$$

Why does the number 1 act this way with multiplication?

Why It's True, Explanation 1:

Let's think first about the definition of multiplication as repeated addition:

- *m* × 1 means to add the number one to itself m times:

$$\underbrace{1+1+\cdots+1}_{m \text{ times}} \tag{1.5.1.9.1}$$

So we see that $m \times 1 = m$ for any whole number m.

- On the other hand, $1 \times m$ means to add the number m to itself just one time. So $1 \times m = m$ also.

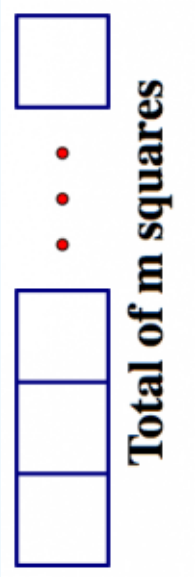
Why It's True, Explanation 2:

We can also use the number line model to create a justification. If Zed calculates $1 \times m$, he will start at 0 and face the positive direction. He will then take m steps forward, and he will do it just one time. So he lands at m, which means $1 \times m = m$.

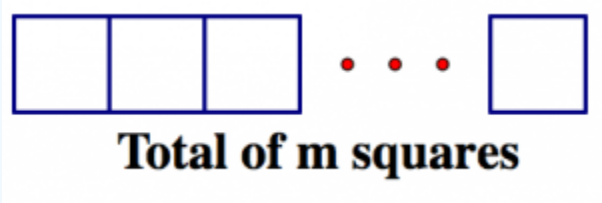
If Zed calculates $m \times 1$, he starts at 0 and faces the positive direction. Then he takes one step forward, and he repeats that $m$times. So he lands at m. We see that $m \times 1 = m$.

Why It's True, Explanation 3:

In the area model, $m \times 1$ represents m rows with one square in each row. That makes a total of m squares. So $m \times 1 = m$.



Similarly, $1 \times m$ represents one row of m squares. That's also a total of m squares. So $1 \times m = m$.



Think / Pair / Share

The example presented several different explanations. Do you think one is more convincing than the others? Or more clear and easier to understand?

MULTIPLICATION IS COMMUTATIVE

Property: Multiplication whole numbers is commutative.

What it Means (words): When I multiply two whole numbers, switching the order in which I multiply them does not affect the product.

What it Means (symbols): For any two whole numbers *a* and *b*,

$$a \cdot b = b \cdot a.$$

Problem 18

1. Come up with at least three *examples* to demonstrate the commutativity of multiplication.
2. Use our models of multiplication to come up with an *explanation*. Why does commutativity hold in *every case*? **Note:** Your explanation should not use particular numbers. It is not an example!

MULTIPLICATION IS ASSOCIATIVE

Property: Multiplication of whole numbers is associative.

What it Means (words): When I multiply three whole numbers in a given order, the way I group them (to multiply two at a time) doesn't affect the product.

What it Means (symbols): For any three whole numbers *a*, *b*, and *c*,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

Problem 19

1. Come up with at least three *examples* to demonstrate the associativity of multiplication.
2. Use our models of multiplication to come up with an *explanation*. Why does associativity hold in *every case*?

MULTIPLICATION DISTRIBUTES OVER ADDITION

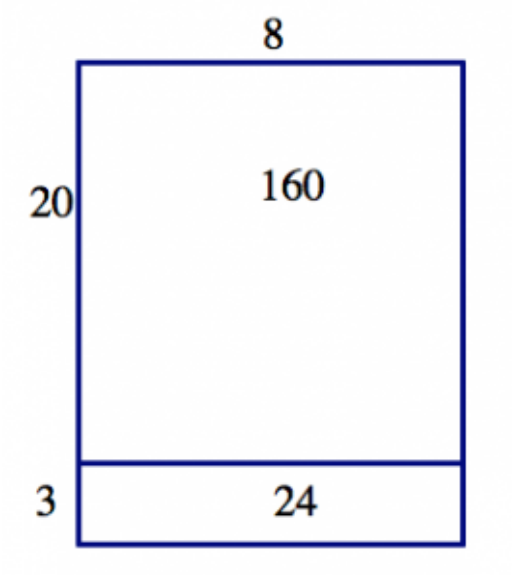
Property: Multiplication distributes over addition.

What it means: The distributive law for multiplication over addition is a little hard to state in words, so we'll jump straight to the symbols. For any three whole numbers *x*, *y*, and *z*:

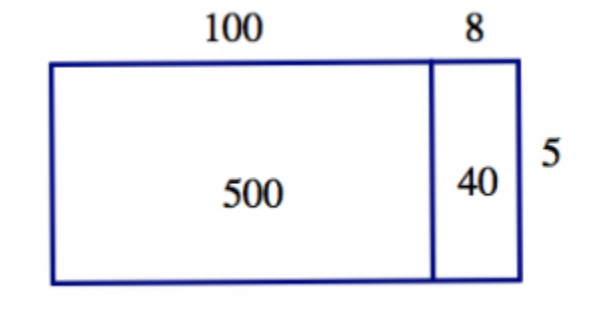
$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

Examples: We actually did calculations very much like the examples above, when we looked at the area model for multiplication.

$$8 \cdot (23) = 8 \cdot (20 + 3) = 8 \cdot 20 + 8 \cdot 3 = 160 + 24 = 184$$



$$5 \cdot (108) = 5 \cdot (100 + 8) = 5 \cdot 100 + 5 \cdot 8 = 500 + 40 = 540$$



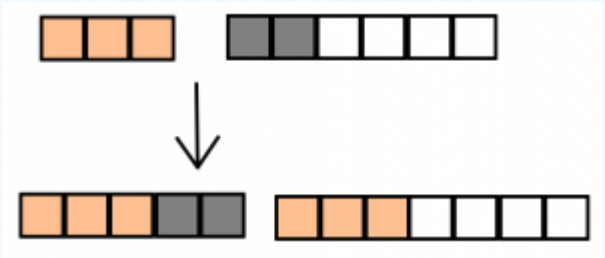
Problem 20

Which of the following pictures best represents the distributive law in the equation

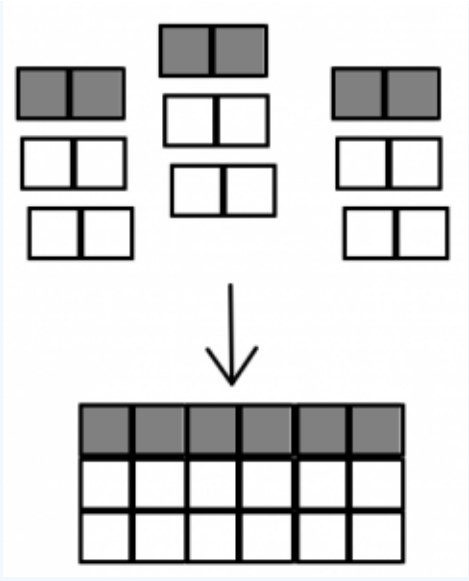
$$3 \cdot (2 + 4) = 3 \cdot 2 + 3 \cdot 4?$$

Explain your choice.

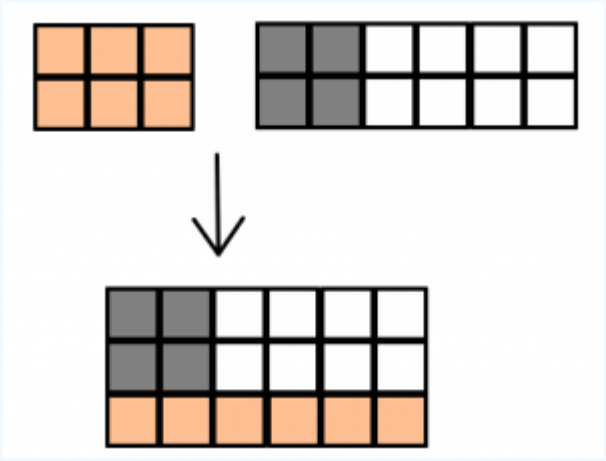
a.



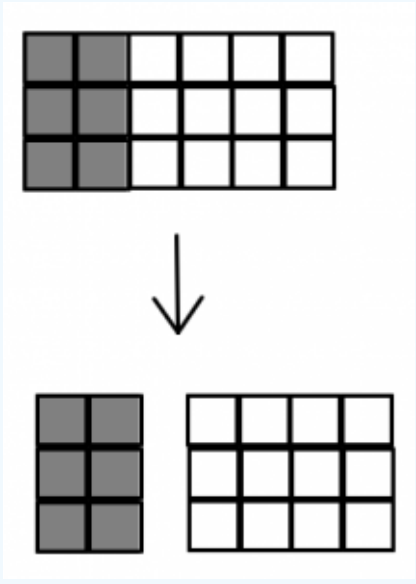
b.



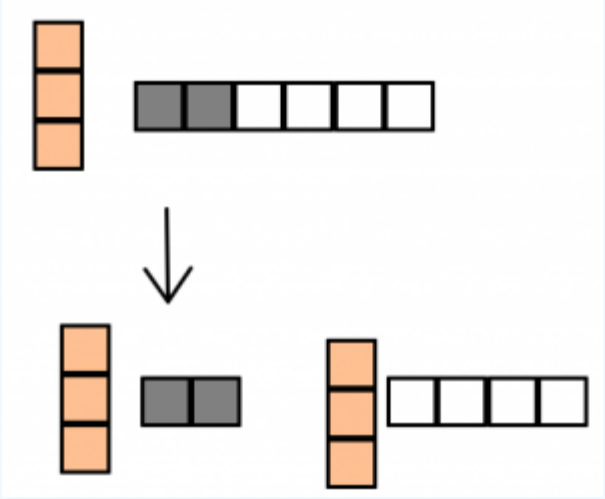
c.



d.



e.



Problem 21

Use the distributive law to easily compute each of these in your head (no calculators!). Explain your solutions.

$$45 \times 11 \qquad 63 \times 101 \qquad 172 \times 1001$$

Think / Pair / Share

Use one of our models for multiplication and addition to explain why the distributive rule works every time.

PROPERTIES OF DIVISION

It's natural to wonder which, if any, of these properties also hold for division (since you know that the operations of multiplication and division are connected).

Example: Is Division Associative?

If division were associative, then for any choice of three whole numbers *a*, *b*, and *c*, we would have

$$a : (b : c) = (a : b) : c.$$

Remember, the parentheses tell you which two numbers to divide first.

Let's try the example $a = 9$, $b = 3$, and $c = 1$. Then we have:

$$9 : (3 : 1) = 9 : 3 = 3$$

and

$$(9 : 3) : 1 = 3 : 1 = 3.$$

So is it true? Is division associative? Well, we can't be sure. This is just one example. But "division is associative" is a *universal statement*. If it's true, it has to work for *every possible example*. Maybe we just stumbled on a good choice of numbers, but it won't always work.

Let's keep looking. Try $a = 16$, $b = 4$, and $c = 2$.

$$16 : (4 : 2) = 16 : 2 = 8$$

and

$$(16 : 4) : 2 = 4 : 2 = 2.$$

That's all we need! A single counterexample lets us conclude:

Division is **not** associative.

What about the other properties? It's your turn to decide!

Problem 22

1. State what it would mean for division to be commutative. You should use words and symbols.
2. Decide if division is commutative or not. Carefully explain how you made your decision and *how you know you're right*.

Problem 23

1. State what it would mean for division to distribute over addition. You definitely want to use symbols!
2. Decide if division distributes over addition or not. Carefully explain how you made your decision and *how you know you're right*.

Problem 24

1. State what it would mean for the number 1 to be an identity for division. You should use words and symbols.
2. Decide if 1 is an identity for division or not. Carefully explain how you made your decision and *how you know you're right*.

ZERO PROPERTY FOR MULTIPLICATION AND DIVISION

Problem 25

You probably know another property of multiplication that hasn't been mentioned yet:

If I multiply any number times 0 (in either order), the product is 0. This is sometimes called the zero property of multiplication. Notice that the *zero property* is very different from the property of being an identity!

1. Write what the zero property means using both words and symbols:

   For every whole number *n* . . .

2. Give at least three examples of the zero property for multiplication.

3. Use one of our models of multiplication to explain why the zero property holds.

Think / Pair / Share

- For each division problem below, turn it into a multiplication problem. Solve those problems if you can. If you can't, explain what is wrong.

$$5 : 0 \qquad 0 : 5 \qquad 7 : 0 \qquad 0 : 7 \qquad 0 : 0 \tag{1.5.1.9.2}$$

- Use your work to explain why we say that *division by* 0 *is undefined*.
- Use one of our models of division to explain why *division by* 0 *is undefined*.

Four Fact Families

In elementary school, students are often encouraged to memorize "four fact families," for example:

$$2+3=5 \quad 5-3=2$$
$$3+2=5 \quad 5-2=3$$

Here's a different "four fact family":

$$2 \cdot 3 = 6 \quad 6 : 3 = 2$$
$$3 \cdot 2 = 6 \quad 6 : 2 = 3$$

Think / Pair / Share

- In what sense are these groups of equations "families"?
- Write down at least two more addition / subtraction four fact families.
- Use properties of addition and subtraction to explain *why* these four fact families are each really one fact.
- Write down at least two more multiplication / division four fact families.
- Use properties of multiplication and division to explain *why* these four fact families are each really one fact.

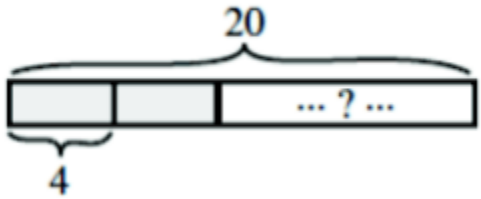
Problem 26

1. Here's a true fact in base six: $2_{six} + 3_{six} = 5_{six}$. Write the rest of this four fact family.
2. Here's a true fact in base six: $11_{six} - 5_{six} = 2_{six}$. Write the rest of this four fact family.

Going Deeper with Division

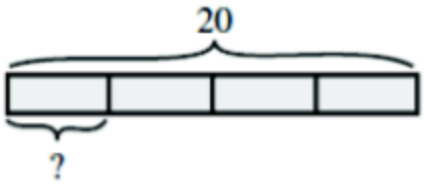
So far we've been thinking about division in what's called the *quotative model*. In the quotative model, we want to make groups of equal size. We know the *size of the group*, and we ask *how many groups*. For example, we think of 20 ÷ 4 as:

How many groups of 4 are there in a group of 20?



Thinking about four fact families, however, we realize we can turn the question around a bit. We could think about the *partitive model* of division. In the partitive model, we want to make an equal number of groups. We know *how many groups*, and we ask the *size of the group*. In the partitive model, we think of 20 ÷ 4 as:

20 is 4 groups of what size?



When we know the original amount and the number of parts, we use partitive division to find the size of each part.

When we know the original amount and the size of each part, we use quotative division to find the number of parts.

Here are some examples in word problems:

Partitive	Quotative
number of groups known find the number in each group	number in each group known find the number of groups
A movie theater made $6450 in one night of ticket sales. 430 people purchased a ticker. How much does one ticket cost?	A movie theater made $6450 in one night of ticket sales. Each ticket cost $12.50. How many people purchased a ticket?

Think / Pair / Share

For each word problem below:

- Draw a picture to show what the problem is asking.
- Use your picture to help you decide if it is a quotative or a partitive division problem.
- Solve the problem using any method you like.

1. David made 36 cookies for the bake sale. He packaged the cookies in boxes of 9. How many boxes did he use?
2. David made 36 cookies to share with his friends at lunch. There were 12 people at his lunch table (including David). How many cookies did each person get?
3. Liz spent one summer hiking the Appalachin trail. She completed 1,380 miles of the trail and averaged 15 miles per day. How many days was she out hiking that summer?
4. On April 1, 2012, **Chase Norton** became the first person to hike the entire Koʻolau summit in a single trip. (True story!) It took him eight days to hike all 48 miles from start to finish. If he kept a steady pace, how many miles did he hike each day?

Think / Pair / Share

Write your own word problems: Write one partitive division problem and one quotative division problem. Choose your numbers carefully so that the answer works out nicely. Be sure to solve your problems!

Why think about these two models for division? You won't be teaching the words *partitive* and *quotative* to your students. But recognizing the two kinds of division problems (and being able to come up with examples of each) will make you a better teacher.

It's important that your students are exposed to both ways of thinking about division, and to problems of both types. Otherwise,

they may think about division too narrowly and not really understand what's going on. If you understand the two kinds of problems, you can more easily diagnose and remedy students' difficulties.

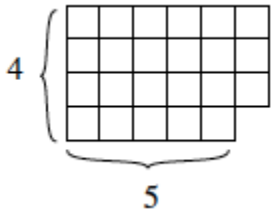
Most of the division problems we've looked at so far have come out evenly, with no remainder. But of course, that doesn't always happen! Sometimes, a whole number answer makes sense, and the context of the problem should tell you which whole number is the right one to choose.

Problem 27

What is 43 : 4?

1. Write a problem that uses the computation 43 : 4 and gives 10 as the correct answer.
2. Write a problem that uses the computation 43 : 4 and gives 11 as the correct answer.
3. Write a problem that uses the computation 43 : 4 and gives 10.75 as the correct answer.

We can think about division with remainder in terms of some of our models for operations. For example, we can calculate that 23 : 4 = 5 R3. We can picture it this way:



$$23 = 5 \cdot 4 + 3.$$

Think / Pair / Share

- Explain how the picture above illustrates 23 = 5 · 4 + 3. Where do you see the remainder of 3 in the picture?
- Explain the connection between these two equations.

$$23 : 4 = 5 \text{ R}3 \quad \text{and} \quad 23 = 5 \cdot 4 + 3. \tag{1.5.1.9.3}$$

- How could you use the number line model to show the calculation 23 = 5 · 4 + 3? What does a "remainder" look like in this model?
- Draw area models for each of these division problems. Find the quotient and remainder.

$$40 : 12 \qquad 59 : 10 \qquad 91 : 16 \tag{1.5.1.9.4}$$

Answer available at the end of this section.

Answers

Think / Pair / Share

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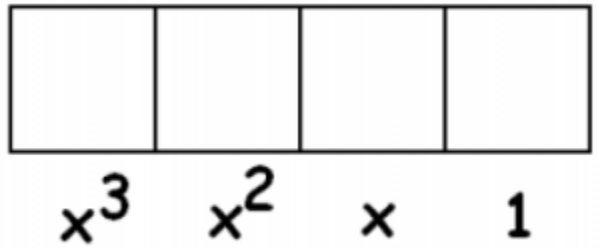
1.5.1.10: Division Explorations

Anu refuses to tell anyone if she is working in a 1←10 system, or a 1←5 system, or any other system. She makes everyone call it a 1 ← *x* system but won't tell anyone what *x* stands for.

We know that boxes in a 1←10 have values that are powers of ten: 1, 10, 100, 1000, 10000...

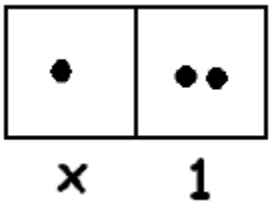
And boxes in a 1←5 system are powers of five: 1, 5, 25, 125, 625...

So Anu's system, whatever it is, must be powers of *x*: $1, x, x^2, x^3, x^4 \ldots$



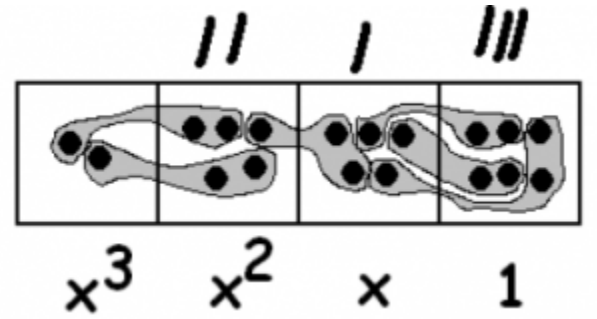
$$2x^3 + 5x^2 + 5x + 6.$$

And when she writes 12_x she means:



$$x + 2.$$

Anu decides to compute $2556_x \div 12_x$.



She obtains:

$$(2x^3 + 5x^2 + 5x + 6) \div (x + 2) = 2x^2 + x + 3.$$

Problem 28

1. Check Anu's division by computing the product

$$(x+2)(2x^2+x+3). \tag{1.5.1.10.1}$$

Did it work?

2. Use Anu's method to find

$$(3x^2+7x+2) \div (x+2). \tag{1.5.1.10.2}$$

3. Use Anu's method to find

$$(2x^4+3x^3+5x^2+4x+1) \div (2x+1). \tag{1.5.1.10.3}$$

4. Use Anu's method to find

$$(x^4+3x^3+6x^2+5x+3) \div (x^2+x+1). \tag{1.5.1.10.4}$$

Anu later tells use that she really was thinking of a $1 \leftarrow 10$ system so that *x*does equal ten. Then her number 2556 really was two thousand, five hundred and fifty six and 12 really was twelve. Her statement:

$$(2x^3+5x^2+5x+6) \div (x+2) = 2x^2+x+3.$$

is actually 2556 : 12 = 213.

Problem 29

1. Check that 2556 : 12 = 213 is correct in base 10.
2. Keeping with the $1 \leftarrow 10$ system, what division problems did you actually solve in parts (b), (c), and (d) of Problem 28? Check that your answers are correct.

Uh Oh! Anu has changed her mind. She now says she was thinking of a $1 \leftarrow 11$ system.

Now 2556_x means $2 \cdot 11^3 + 5 \cdot 11^2 + 5 \cdot 11 + 6 = 3328_{ten}$.

Similarly, 12_x means $1 \cdot 11 + 2 = 13_{ten}$, and 213_x means $2 \cdot 11^2 + 1 \cdot 11 + 3 = 256_{ten}$.

So Anu's computation $2556_x \div 12_x \ = 213_x$ is actually the (base 10) statement:

$$3328 : 13 = 256.$$

Problem 30

1. Check that 3328 : 13 = 256 is also correct in base ten.
2. Keeping with the $1 \leftarrow 11$ system, what division problems did you actually solve in parts (b), (c), and (d) of Problem 28? Check that they are correct.

Problem 31

1. Use Anu's method to show that

$$(x^4+4x^3+6x^2+4x+1) \div (x+1) = (x^3+3x^2+3x+1). \tag{1.5.1.10.5}$$

2. What is this saying for *x* = 10? Check that the division is correct.
3. What is this saying for *x* = 2? Check that the division is correct.
4. What is this saying for *x* equal to each of 3, 4, 5, 6, 7, 8, 9, and 11? Check that each division is correct.
5. What is this saying for *x* = 0?

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1.5.1.11: Problem Bank

Problem 32

Compute the following using dots and boxes:

64212 : 3

44793 : 21

6182 : 11

99916131 : 31

637824 : 302

2125122 : 1011

Problem 33

1. Fill in the squares using the digits 4, 5, 6, 7, 8, and 9 exactly one time each to make the largest possible sum:

$$\begin{array}{r} \square\square\square \\ +\square\square\square \\ \hline \end{array} \qquad (1.5.1.11.1)$$

2. Fill in the squares using the digits 4, 5, 6, 7, 8, and 9 exactly one time each to make the smallest possible (positive) difference:

$$\begin{array}{r} \square\square\square \\ -\square\square\square \\ \hline \end{array} \qquad (1.5.1.11.2)$$

Problem 34

1. Make a base six addition table.

+	0_{six}	1_{six}	2_{six}	3_{six}	4_{six}	5_{six}
0_{six}	0_{six}	1_{six}				
1_{six}						10_{six}
2_{six}						
3_{six}						
4_{six}						
5_{six}				12_{six}		

2. Use the table to solve these subtraction problems.

$$13_{six} - 5_{six} \qquad 12_{six} - 3_{six} \qquad 10_{six} - 4_{six}. \qquad (1.5.1.11.3)$$

Problem 35

Do these calculations in base four. Don't translate to base 10 and then calculate there — try to work in base four.

1. $33_{four} + 11_{four}$ (1.5.1.11.4)
2. $123_{four} + 22_{four}$ (1.5.1.11.5)
3. $223_{four} - 131_{four}$ (1.5.1.11.6)
4. $112_{four} - 33_{four}$ (1.5.1.11.7)

Problem 36

1. Make a base five multiplication table.

$\times$	0_{five}	1_{five}	2_{five}	3_{five}	4_{five}
0_{five}	0_{five}	0_{five}			
1_{five}					
2_{five}					
3_{five}			11_{five}		
4_{five}				22_{five}	

2. Use the table to solve these subtraction problems.

$$11_{five} \div 2_{five} \qquad 22_{five} \div 3_{five} \qquad 13_{five} \div 4_{five}. \tag{1.5.1.11.8}$$

Problem 37

1. Here is a true fact in base five:

$$2_{five} \cdot 3_{five} = 11_{five} \tag{1.5.1.11.9}$$

   Write the rest of this four fact family.
2. Here is a true fact in base five:

$$13_{five} \div 2_{five} = 4_{five} \tag{1.5.1.11.10}$$

   Write the rest of this four fact family.

Directions for AlphaMath Problems (Problems 38 – 41):

- Letters stand for digits 0–9.
- In a given problem, the same letter always represents the same digit, and different letters always represent different digits.
- There is no relation between problems (so "A" in part 1 and "A" in part 3 might be different).
- Two, three, and four digit numbers never start with a zero.
- Your job: Figure out what digit each letter stands for, so that the calculation shown is correct.

Problem 38

Notes: In part 2, "O" represents the letter "oh," not the digit zero.

1.
$$\begin{array}{r} A \\ A \\ +\,A \\ \hline H\,A \end{array} \tag{1.5.1.11.11}$$

2.
$$\begin{array}{r} O\,N\,E \\ +\,O\,N\,E \\ \hline T\,W\,O \end{array} \tag{1.5.1.11.12}$$

3.
$$\begin{array}{r} A\,B\,C \\ +\,A\,C\,B \\ \hline C\,B\,A \end{array} \tag{1.5.1.11.13}$$

Problem 39

Here's another AlphaMath problem.

$$\begin{array}{r} T\,E\,N \\ +\,N\,O\,T \\ \hline N\,I\,N\,E \end{array}$$

1. Solve this AlphaMath problem in base 10.
2. Now solve it in base 6.

Problem 40

Find all solutions to this AlphaMath problem **in base 9**.

Notes: Even though this is two calculations, it is a *single problem*. All T's in both calculations represent the same digit, all B's represent the same digit, and so on.

Remember that "O" represents the letter "oh" and not the digit zero, and that two and three digit numbers never start with the digit zero

$$\begin{array}{r} T\,O \\ -\,B\,E \\ \hline O\,R \end{array} \qquad \begin{array}{r} N\,O\,T \\ -\,T\,O \\ \hline B\,E \end{array}$$

Problem 41

This is a single AlphaMath problem. (So all G's represent the same digit. All A's represent the same digit. And so on.)

Solve the problem in **base 6**.

$$GALON = (GOO)^2 \qquad ALONG = (OOG)^2$$

Problem 42

A *perfect square* is a number that can be written as $a \cdot a$ or a^2 (some number times itself).

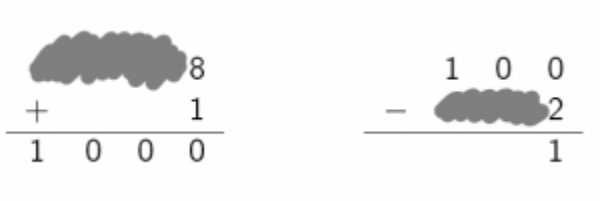
1. Which of the following *base seven numbers* are perfect squares? For each number, answer **yes** (it is a perfect square) or **no** (it is not a perfect square) and give a justification of your answer.

$$4_{seven} \qquad 25_{seven} \qquad 51_{seven} \tag{1.5.1.11.14}$$

2. For which choices of base b is the number b^2 a perfect square? Justify your answer.

Problem 43

Geoff spilled coffee on his homework. The answers were correct. Can you determine the missing digits and the bases?



Problem 44

1. Rewrite each subtraction problem as an addition problem:

$$x - 156 = 279 \qquad 279 - 156 = x \qquad a - x = b. \tag{1.5.1.11.15}$$

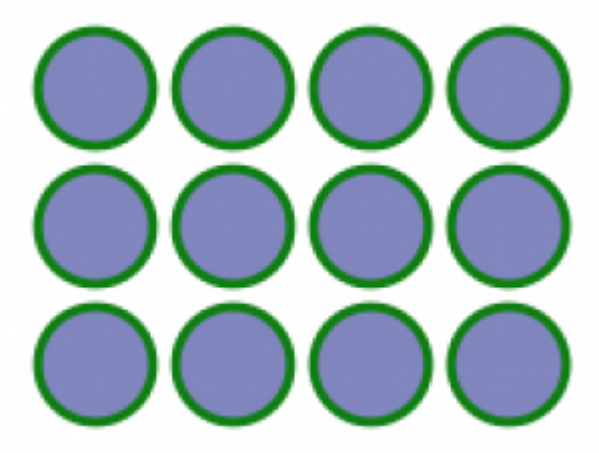
2. Rewrite each division problem as a multiplication problem:

$$24 \div x = 12 \qquad x \div 3 = 27 \qquad a \div b = x. \tag{1.5.1.11.16}$$

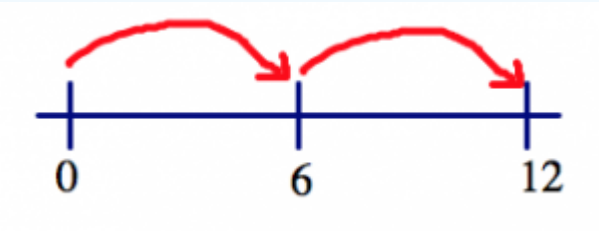
Problem 45

Which of the following models represent the same multiplication problem? Explain your answer.

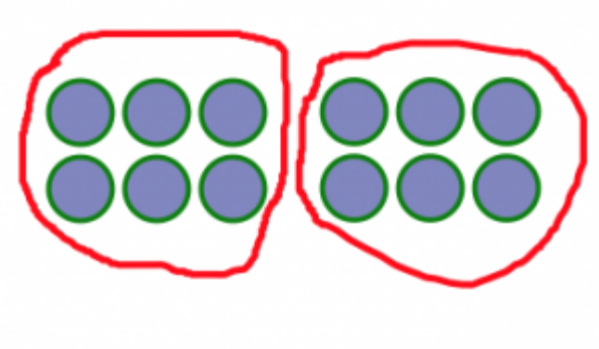
a.



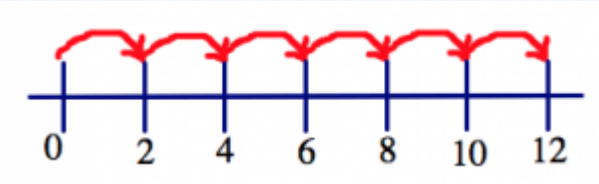
b.



c.



d.



Problem 46

Show an area model for each of these multiplication problems. Write down the standard computation next to the area model and see how it compares.

$$20 \times 33 \qquad 24 \times 13 \qquad 17 \times 11$$

Problem 47

Suppose the 2 key on your calculator is broken. How could you still use the calculator compute these products? Think about what properties of multiplication might be helpful. (Write out the calculation you would do on the calculator, not just the answer.)

$$1592 \times 3344 \qquad 2008 \times 999 \qquad 655 \times 525$$

Problem 48

Today is Jennifer's birthday, and she's twice as old as her brother. When will she be twice as old as him again? Choose the best answer and justify your choice.

1. Jennifer will always be twice as old as her brother.
2. It will happen every two years.
3. It depends on Jennifer's age.
4. It will happen when Jennifer is twice as old as she is now.
5. It will never happen again.

Problem 49

1. Find the quotient and remainder for each problem.

$$7 \div 3 \qquad 3 \div 7 \qquad 7 \div 1 \qquad 1 \div 7 \qquad (1.5.1.11.17)$$

2. How many possible remainders are there when dividing by these numbers? Justify what you say.

$$2 \qquad 12 \qquad 62 \qquad 23 \qquad (1.5.1.11.18)$$

Problem 50

Identify each problem as either partitive or quotative division and say why you made that choice. Then solve the problem.

1. Adriana bought 12 gallons of paint. If each room requires three gallons of paint, how many rooms can she paint?
2. Chris baked 15 muffins for his family of five. How many muffins does each person get?
3. Prof. Davidson gave three straws to each student for an activity. She used 51 straws. How many students are in her class?

Problem 51

Use the digits 1 through 9. Use each digit exactly once. Fill in the squares to make all of the equations true.

$$\begin{array}{c} \square - \square = \square \\ \times \\ \square \div \square = \square \\ = \\ \square + \square = \square \end{array}$$

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1.6: Building a Supportive Math Learning Environment

> "Without mathematics, there's nothing you can do. Everything around you is mathematics. Everything around you is numbers."
> Shakuntala Devi, Indian Mathematician

Creating a Classroom Culture that Values Math

Mathematics is more than just numbers and equations; it's a way of thinking and problem-solving that can enrich students' lives far beyond the classroom. To foster this understanding and appreciation, it's crucial to create a classroom culture that truly values math.

Establishing Norms and Expectations

Respect and Collaboration: Promoting a culture of respect and collaboration among students.

In a supportive math learning environment, respect and collaboration are paramount. Teachers should establish clear guidelines for how students interact with each other during math lessons. This includes active listening when others are speaking, offering constructive feedback, and valuing diverse problem-solving approaches.

To foster this culture of respect and collaboration, teachers can implement several strategies:

1. **Establish Clear Ground Rules**: At the beginning of the school year or semester, work with students to create a set of classroom norms. These might include rules like "We listen to understand, not to respond" or "We celebrate different approaches to problem-solving." By involving students in this process, they take ownership of these rules and are more likely to adhere to them.
2. **Model Respectful Behavior**: Teachers should consistently model the behavior they expect from their students. This includes demonstrating active listening, asking thoughtful questions, and showing appreciation for different ideas and approaches.
3. **Implement Structured Collaboration**: Use collaborative learning structures that require students to work together respectfully. For example, the "Think-Pair-Share" method encourages individual thinking, paired discussion, and then sharing with the larger group, ensuring all voices are heard.
4. **Encourage Peer Feedback**: Teach students how to give and receive constructive feedback. This could involve creating a feedback protocol that students follow, such as "Two Stars and a Wish" where peers identify two strengths and one area for improvement in a classmate's work.
5. **Celebrate Diverse Approaches**: Regularly highlight different methods for solving problems. This reinforces the idea that there's often more than one valid approach in mathematics and helps students appreciate diverse thinking.
6. **Address Disrespectful Behavior Promptly**: When instances of disrespect occur, address them immediately and use them as teaching moments to reinforce the importance of respect in the classroom.
7. **Create a Safe Space for Questions**: Encourage students to ask questions without fear of judgment. Implement a system where students can anonymously submit questions, ensuring that all students feel comfortable seeking clarification.

For example, when a student presents their solution to a problem, others should be encouraged to ask thoughtful questions and offer alternative perspectives respectfully. This not only reinforces mathematical concepts but also develops important social skills. A teacher might facilitate this by:

- Asking the presenting student to explain their thinking process.
- Inviting other students to ask clarifying questions.
- Encouraging students to offer alternative solutions prefaced with phrases like "Another way to approach this might be..." or "I saw it differently. Here's what I did..."
- Guiding the class in a discussion about the strengths of each approach and how they might be applied in different situations.

By consistently reinforcing these practices, teachers create an environment where students feel safe to take risks, share their ideas, and learn from one another. This collaborative atmosphere not only enhances mathematical understanding but also prepares students for future academic and professional environments where teamwork and respectful communication are crucial.

Moreover, a respectful and collaborative math classroom can help break down stereotypes and biases about who can excel in mathematics. When students see their peers from diverse backgrounds contributing valuable ideas and approaches, it challenges preconceptions and fosters a more inclusive view of mathematical ability.

In this supportive environment, students learn that mathematics is not just about finding the right answer, but about the process of exploration, reasoning, and communication. They develop resilience in the face of challenging problems, knowing that they have a supportive community to collaborate with and learn from.

High Expectations: Setting high expectations for all students and providing the necessary support to help them meet those expectations.

It's essential to communicate to all students that they are capable of succeeding in math. This means setting challenging yet attainable goals for each student, regardless of their current performance level. However, high expectations must be coupled with appropriate support.

Setting high expectations for all students is a powerful tool for creating an equitable and successful math classroom. Here's how teachers can effectively implement this approach:

1. **Growth Mindset Culture**: Foster a growth mindset in your classroom. Teach students that intelligence and math ability are not fixed traits, but can be developed through effort, good strategies, and support from others. Use phrases like "You haven't mastered this yet" instead of "You can't do this," emphasizing the potential for growth and improvement.
2. **Clear and Rigorous Learning Objectives**: Set clear, rigorous learning objectives for each lesson and unit. These objectives should be challenging but achievable with effort. Share these objectives with students so they understand what they're working towards.
3. **Scaffold Complex Tasks**: While maintaining high expectations, provide scaffolding for complex tasks. This might involve breaking down larger problems into smaller steps, providing worked examples, or using graphic organizers to help students visualize problem-solving processes.
4. **Personalized Goal Setting**: Work with each student to set personalized, ambitious goals. These goals should push students beyond their comfort zone while remaining achievable. Regularly review and adjust these goals as students progress.
5. **Emphasize Effort and Process**: When giving feedback, focus on effort and problem-solving processes rather than just correct answers. This encourages students to value the learning process and persevere through challenges.
6. **Challenging Questions**: Regularly ask higher-order thinking questions that require students to analyze, synthesize, and evaluate mathematical concepts. This pushes students to think deeply about mathematics and apply their knowledge in new ways.
7. **Real-World Applications**: Connect mathematical concepts to real-world applications to demonstrate the relevance and importance of what students are learning. This can motivate students to engage with challenging material.

Teachers can implement strategies such as differentiated instruction, where tasks are tailored to individual student needs, or provide additional resources like after-school tutoring or peer mentoring programs. Here's how these strategies can be effectively implemented:

1. **Differentiated Instruction**:
    - Use pre-assessments to identify students' current understanding and skill levels.
    - Create tiered assignments that address the same core concepts but at varying levels of complexity.
    - Implement flexible grouping, where students are grouped based on their needs for specific tasks or concepts.
    - Offer choice in how students demonstrate their learning (e.g., creating a video explanation, writing a report, or solving a set of problems).
2. **Additional Support Structures**:
    - Establish after-school math clubs or tutoring sessions where students can receive extra help.
    - Implement a peer mentoring program, pairing stronger math students with those who need additional support.
    - Create a math resource center in the classroom or school library with additional materials, manipulatives, and technology tools.
    - Utilize online learning platforms that allow students to practice at their own pace and receive immediate feedback.
3. **Collaborative Learning Structures**:
    - Use structures like "think-pair-share" or "jigsaw" to encourage peer learning and support.
    - Implement problem-based learning projects where students work in teams to solve complex, real-world math problems.
4. **Regular Formative Assessment**:
    - Use exit tickets, quick quizzes, or digital tools to regularly assess student understanding.

    - Use this data to adjust instruction, provide targeted support, and celebrate progress.
5. **Parent and Family Engagement**:
    - Provide resources and strategies for families to support math learning at home.
    - Host family math nights to engage parents in their child's math education and demonstrate high-level math thinking.
6. **Culturally Responsive Teaching**:
    - Incorporate diverse cultural perspectives and contexts into math problems and examples.
    - Recognize and value the mathematical knowledge students bring from their cultural backgrounds.

The key is to create an environment where students feel challenged but also supported in meeting those challenges. This balance is crucial: high expectations push students to reach their potential, while robust support structures ensure they have the resources and guidance needed to meet these expectations.

By implementing these strategies, teachers create a classroom where every student believes in their ability to succeed in math and has the support they need to achieve at high levels. This approach not only improves mathematical performance but also builds students' confidence, perseverance, and love for mathematical thinking.

Encouraging Collaborative Learning and Discussion

Group Work: Encouraging students to work together to solve problems and discuss mathematical concepts.

Collaborative learning is a powerful tool in mathematics education. When students work together, they're exposed to different problem-solving strategies and perspectives, which can deepen their understanding of mathematical concepts.

The benefits of group work in mathematics are numerous:

1. **Diverse Perspectives**: Students bring different approaches and insights to problem-solving, exposing the group to a variety of strategies.
2. **Verbalization of Thinking**: Explaining their reasoning to peers helps students clarify their own understanding and identify gaps in their knowledge.
3. **Peer Teaching**: Students often understand and relate to explanations from their peers in ways that differ from teacher explanations.
4. **Increased Engagement**: Group work can make math more enjoyable and less intimidating for some students.
5. **Real-World Skill Building**: Collaboration, communication, and teamwork are essential skills in many careers that use mathematics.

To effectively implement group work in the math classroom, teachers can consider the following strategies:

1. **Structured Collaborative Techniques**:
    - Think-Pair-Share: Students think about a problem individually, discuss with a partner, then share with the larger group.
    - Jigsaw: Each group member becomes an "expert" on one aspect of a problem, then teaches their part to the group.
    - Round Robin: Each group member contributes an idea or step in solving a problem, building on previous contributions.
2. **Roles and Responsibilities**: Assign specific roles within groups to ensure equal participation. Roles might include:
    - Facilitator: Keeps the group on task and ensures everyone participates.
    - Recorder: Writes down the group's ideas and solutions.
    - Checker: Verifies the accuracy of the group's work.
    - Reporter: Presents the group's findings to the class.
3. **Collaborative Problem-Solving**: Present complex, multi-step problems that require group effort to solve. This could involve:
    - Breaking down a large problem into parts, with each group member responsible for a section.
    - Using manipulatives or visual aids that require multiple hands to construct or manipulate.
    - Presenting problems with multiple correct approaches, encouraging discussion of different methods.
4. **Project-Based Learning**: Design projects that apply mathematical concepts to real-world situations. For example:
    - Planning a school event with a budget, requiring calculations of costs, proportions, and statistics.
    - Designing a garden, involving area, perimeter, and optimization concepts.
    - Analyzing local data to make recommendations for community improvements, incorporating statistical analysis and data visualization.

5. **Group Presentations**: Have groups present their solutions or projects to the class. This reinforces learning, develops communication skills, and exposes the whole class to different approaches.
6. **Technology Integration**: Utilize collaborative digital tools such as shared documents, digital whiteboards, or math-specific software that allows for group input and problem-solving.
7. **Reflection and Debriefing**: After group work, provide time for students to reflect on the process. Questions might include:
    - What strategy did your group use to solve the problem?
    - How did working in a group help you understand the concept better?
    - What challenges did you face, and how did you overcome them?
8. **Peer and Self-Assessment**: Incorporate peer and self-assessment into group work. This helps students develop critical thinking skills and take responsibility for their learning.

When implementing group work, it's crucial to consider potential challenges:

- **Unequal Participation**: Address this by assigning roles, using structured techniques, and monitoring group dynamics.
- **Grouping Strategies**: Vary group composition to balance skill levels and personalities. Consider both homogeneous and heterogeneous grouping depending on the task.
- **Time Management**: Provide clear timelines and checkpoints for group work to keep students on track.
- **Individual Accountability**: Include individual components or assessments alongside group work to ensure each student is actively learning.

Teachers can design group activities that require students to tackle complex problems together. For instance, a project-based learning approach where small groups work on real-world math applications can be highly effective. This not only reinforces math skills but also develops teamwork and communication abilities.

Example Project: "Designing an Eco-Friendly School"

In this project, students work in small groups to design an eco-friendly school building. The project incorporates various mathematical concepts:

1. Geometry: Designing the layout of the building and calculating areas and volumes.
2. Algebra: Creating and solving equations for energy consumption and savings.
3. Statistics: Analyzing data on energy usage and environmental impact.
4. Financial Mathematics: Budgeting and cost-benefit analysis of eco-friendly features.

Each group member could take on a specific aspect of the project, requiring them to become an expert in that area and then teach their teammates. The final product could be a presentation to the class, complete with scaled drawings, data visualizations, and a budget proposal.

By thoughtfully implementing group work strategies, teachers can create a collaborative learning environment that enhances mathematical understanding, develops crucial soft skills, and prepares students for future academic and professional challenges where teamwork and problem-solving are essential.

Math Talks: Facilitating discussions where students can share their thinking and reasoning.

Regular "math talks" or discussions can significantly enhance mathematical understanding. These sessions provide opportunities for students to articulate their thinking processes, justify their reasoning, and learn from their peers.

Math talks, also known as number talks or math discourse, are a powerful instructional strategy that promotes deeper mathematical thinking and communication. Here's an expanded look at how to implement effective math talks in the classroom:

1. **Structure of a Math Talk**:
    - Typically, math talks are short, focused discussions (usually 5-15 minutes) that can be incorporated into daily routines.
    - They often start with a carefully chosen problem or question that allows for multiple approaches or representations.
    - Students are given time to think and solve independently before sharing with the class.
    - The teacher facilitates a whole-class discussion, calling on students to share their thinking and encouraging peer-to-peer interaction.
2. **Types of Math Talk Problems**:
    - Computation problems with multiple solution strategies (e.g., 25 x 18)
    - Visual patterns or sequences to analyze and extend

- Open-ended problems with multiple correct answers
- Estimation challenges
- Real-world scenarios that require mathematical reasoning

3. **Teacher's Role in Facilitating Math Talks**:
   - Create a safe, respectful environment where all contributions are valued.
   - Ask probing questions to deepen student thinking, such as:
     - "Can you explain your reasoning?"
     - "How did you approach this problem?"
     - "Can anyone restate what [student] just said in their own words?"
   - Record student thinking on the board, using words, numbers, and visual representations.
   - Encourage students to make connections between different strategies or representations.
   - Highlight important mathematical ideas that emerge from the discussion.
4. **Developing Student Participation**:
   - Teach and reinforce norms for respectful discussion, such as active listening and constructive questioning.
   - Use think-pair-share to give students confidence before whole-class sharing.
   - Implement a "no hands up" policy, where the teacher calls on students randomly to ensure wide participation.
   - Encourage students to build on each other's ideas using sentence starters like "I agree/disagree with ___ because..."
5. **Questioning Strategies**:
   - Use a mix of questions that target different levels of thinking:
     - Recall: "What operation did you use first?"
     - Understanding: "Why did you choose that approach?"
     - Application: "How could we use this method for a similar problem?"
     - Analysis: "How does your method compare to [another student's] method?"
     - Evaluation: "Which method do you think is most efficient, and why?"
6. **Incorporating Visual Aids**:
   - Use manipulatives, diagrams, or technology to support discussions.
   - Encourage students to create visual representations of their thinking.
   - Display multiple representations side-by-side to facilitate comparisons.
7. **Addressing Misconceptions**:
   - Use incorrect answers as learning opportunities.
   - Ask students to analyze why a particular approach might not work.
   - Guide the class in identifying and correcting errors collaboratively.
8. **Connecting to Prior Knowledge**:
   - Relate new concepts to previously learned material.
   - Ask students to reflect on how new strategies connect to familiar ones.
9. **Differentiating Math Talks**:
   - Vary the complexity of problems to challenge all learners.
   - Use parallel tasks that address the same concept at different levels of difficulty.
   - Encourage multiple solution methods to cater to diverse learning styles.
10. **Assessing Through Math Talks**:
    - Use math talks as a form of formative assessment.
    - Take notes on student contributions to inform future instruction.
    - Look for growth in students' ability to articulate their thinking over time.

Example of a Math Talk in Action:

A teacher might present a problem and ask students to solve it individually. Then, in a whole-class discussion, different students can share their approaches. This not only exposes the class to various problem-solving methods but also helps students realize that there's often more than one valid way to solve a math problem.

For instance, consider this problem for a 5th-grade class:

"There are 24 students in our class. If we want to arrange them into equal groups, what are all the possible group sizes we could have?"

The teacher would give students a few minutes to think and solve independently. Then, the discussion might unfold like this:

Teacher: "Who would like to share their thinking?"

Student A: "I found that we could have groups of 1, 2, 3, 4, 6, 8, 12, or 24."

Teacher: "Thank you. Can you explain how you came up with these numbers?"

Student A: "I started with 1 and just tried dividing 24 by different numbers to see which ones worked without a remainder."

Teacher: "Interesting approach. Did anyone think about it differently?"

Student B: "I listed out the factors of 24. I knew 1 and 24 were factors, and then I multiplied numbers to see which ones equaled 24."

Teacher: "Great connection to factors! Can someone explain why all of these numbers are factors of 24?"

Student C: "Because when you divide 24 by any of these numbers, you get a whole number with no remainder."

Teacher: "Excellent observation. Now, let's think about this practically. If we were actually arranging our class, which of these group sizes might be most useful, and why?"

This example demonstrates how a simple problem can lead to rich discussion about factors, division, and practical applications of mathematics. It allows students to share different solution strategies, make connections between concepts (like the relationship between factors and division), and engage in higher-order thinking by evaluating the practicality of different solutions.

By regularly incorporating math talks into classroom routines, teachers can create a culture of mathematical discourse where students become confident in expressing their ideas, critiquing reasoning, and seeing mathematics as a creative and collaborative endeavor.

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1.7: Math Anxiety

> "Math anxiety is a learned condition, and it can be unlearned. Everyone can learn math; it's not about being born with a 'math brain' or not. The key is to change the messages we give to ourselves and others about our potential to succeed in math."
> ~ Dr. Jo Boaler, American mathematics educator at Stanford University

Addressing Math Anxiety and Building Confidence

Math anxiety is a common issue that can significantly hinder students' progress and enjoyment of mathematics. Addressing this anxiety and building confidence is crucial for creating a supportive math learning environment.

Identifying Signs of Math Anxiety

Recognizing Symptoms: Understanding the common signs of math anxiety, such as avoidance, nervousness, and frustration.

Math anxiety can manifest in various ways, and it's crucial for educators to be able to recognize these signs. By being attentive to these symptoms, teachers can intervene early and provide appropriate support. Let's delve deeper into the common symptoms and how they might present in the classroom:

1. **Avoidance of Math-Related Tasks or Discussions**
    - Students may:
        - Consistently arrive late to math class or try to skip it altogether
        - Avoid volunteering answers or participating in math discussions
        - Procrastinate on math homework or projects
        - Choose courses or career paths that minimize math involvement
    - Teachers should watch for:
        - Students who are engaged in other subjects but become withdrawn during math lessons
        - Frequent requests to leave the room during math class (e.g., bathroom breaks, nurse visits)
        - Lack of eye contact or physical turning away when math topics are introduced
2. **Physical Signs of Nervousness**
    - Symptoms may include:
        - Sweating, particularly of the palms
        - Increased heart rate or visible pulse in the neck
        - Shallow or rapid breathing
        - Fidgeting or restlessness
        - Facial tension or flushing
    - These signs might be particularly noticeable:
        - At the beginning of a math lesson
        - When a pop quiz is announced
        - As students prepare to present their work to the class
3. **Frustration or Anger**
    - This may manifest as:
        - Outbursts of anger when faced with challenging problems
        - Giving up quickly on tasks without real attempts
        - Tearing up papers or breaking pencils
        - Defensive or aggressive responses when asked about their work
    - Teachers should note:
        - The speed at which frustration sets in (anxiety may cause students to feel overwhelmed before they've truly engaged with the problem)
        - Whether the level of frustration seems disproportionate to the difficulty of the task
4. **Negative Self-Talk about Math Abilities**

- Students might say things like:
  - "I'm just not a math person"
  - "I'll never understand this"
  - "I'm stupid when it comes to math"
- This can also manifest non-verbally through:
  - Eye-rolling or sighing when math tasks are assigned
  - Slumped body language during math lessons
  - Nervous laughter or self-deprecating jokes about their math abilities
- Teachers should be aware that:
  - Some students may verbalize these thoughts, while others internalize them
  - Negative self-talk can become a self-fulfilling prophecy, reinforcing anxiety and poor performance

5. **Poor Performance on Math Tests Despite Adequate Preparation**
   - This might look like:
     - A significant discrepancy between homework/classwork performance and test scores
     - "Blanking out" during tests despite demonstrating knowledge in other settings
     - Making careless errors that are uncharacteristic of the student's typical work
   - Teachers should consider:
     - Whether test anxiety might be compounding math anxiety
     - If alternative assessment methods might better reflect the student's true abilities
6. **Physical Complaints**
   - Students with math anxiety might frequently report:
     - Headaches or stomachaches, particularly before or during math class
     - Feeling tired or dizzy when doing math
   - These complaints might:
     - Occur more frequently on days with math tests or important math lessons
     - Disappear when the math task is complete or avoided
7. **Perfectionism or Excessive Concern About Errors**
   - This can manifest as:
     - Excessive erasing and rewriting
     - Reluctance to show work unless it's "perfect"
     - Disproportionate emotional responses to small mistakes
   - Teachers might notice:
     - Incomplete work due to perfectionist tendencies
     - Students who take much longer than peers to complete tasks due to excessive checking
8. **Difficulty Explaining Mathematical Thinking**
   - Students might:
     - Struggle to verbalize their problem-solving process
     - Become flustered when asked to explain their work, even if the answer is correct
   - This could indicate:
     - A lack of confidence in their mathematical reasoning
     - Anxiety about being "put on the spot" in math contexts
9. **Overreliance on Procedural Knowledge**
   - Anxious students might:
     - Focus excessively on memorizing procedures without understanding concepts
     - Panic when faced with problems that deviate slightly from practiced examples
   - This suggests:
     - A coping mechanism to deal with anxiety by clinging to familiar procedures
     - A need for more emphasis on conceptual understanding to build confidence
10. **Comparison and Competition Anxiety**
    - Some students may:
      - Constantly compare their speed or performance to peers
      - Express anxiety about being "slower" or "worse at math" than classmates
    - Teachers should be aware that:
      - Classroom practices that encourage speed or competition may exacerbate anxiety for some students

By being attentive to these signs, teachers can intervene early and provide appropriate support. It's important to note that these symptoms may not always indicate math anxiety – they could be signs of other learning difficulties or personal issues. However, recognizing these patterns can help teachers initiate conversations with students, parents, and support staff to identify the root

causes and develop effective interventions.

Strategies for teachers to gather this information might include:

- Regular check-ins with students about their feelings towards math
- Observing body language and behavior during math lessons
- Analyzing patterns in test performance and homework completion
- Collaborating with school counselors or psychologists for professional insights
- Communicating with parents about any changes in the student's attitude towards math

Remember, the goal of recognizing these symptoms is not to label students, but to understand their experiences and provide targeted support to help them overcome math anxiety and build confidence in their mathematical abilities.

Open Communication: Encouraging students to express their feelings and concerns about math.

Creating an environment where students feel comfortable expressing their feelings about math is crucial. This open communication not only helps teachers understand their students' emotional states but also empowers students to recognize and manage their own feelings about mathematics. Here's an expanded look at how to foster open communication about math in the classroom:

1. **Regular Check-ins**
   - Implement routine emotional check-ins:
     - Use a "feelings thermometer" where students can indicate their comfort level with the current topic.
     - Start class with a quick "emoji check" where students select an emoji that represents their current feelings about math.
     - End each week with a brief reflection on what was challenging and what was rewarding in math class.
   - Benefits:
     - Helps students become more aware of their emotional responses to math.
     - Allows teachers to track emotional trends over time and adjust instruction accordingly.
     - Normalizes the idea that it's okay to have various feelings about math.
2. **Anonymous Surveys**
   - Conduct periodic anonymous surveys:
     - Use online tools or paper forms to gather honest feedback.
     - Ask questions about comfort levels with different math topics, sources of anxiety, and what helps students feel more confident.
     - Include open-ended questions to allow for detailed responses.
   - Implementation:
     - Administer surveys at the beginning of the year, before major units, and at the end of terms.
     - Share summarized results with the class to show that many students share similar feelings.
     - Use survey results to inform teaching strategies and classroom culture.
3. **One-on-One Conversations**
   - Schedule individual check-ins with students:
     - Set aside time for brief, regular conversations with each student.
     - Create a safe, non-judgmental space for students to share their feelings.
     - Use open-ended questions to encourage students to elaborate on their experiences.
   - Strategies:
     - Maintain a rotation to ensure you connect with every student over time.
     - Use active listening techniques to show students their feelings are valid and heard.
     - Collaboratively develop strategies to address specific concerns or anxieties.
4. **Math Journals**
   - Implement regular math journaling:
     - Provide prompts that encourage reflection on feelings and experiences in math class.
     - Allow for a mix of written and visual expressions (e.g., drawings, graphs of emotional states).
     - Make journaling a routine activity, perhaps at the start or end of each week.
   - Benefits:
     - Offers a private outlet for students to process their feelings.
     - Develops metacognitive skills as students reflect on their learning and emotions.

        - Provides teachers with insights into individual student experiences.
5. **Class Discussions about Math Feelings**
    - Facilitate open dialogues about emotions in math:
        - Share personal stories about overcoming math challenges.
        - Invite guest speakers (e.g., older students, professionals) to discuss their math journeys.
        - Use literature or videos that address math anxiety as discussion starters.
    - Implementation:
        - Establish ground rules for respectful, supportive discussion.
        - Emphasize that everyone's feelings are valid and that many people experience math anxiety.
        - Highlight strategies for coping with negative feelings about math.
6. **Visual Representation of Feelings**
    - Create visual tools for expressing emotions:
        - Use a "Math Mood Meter" poster where students can place markers to indicate their feelings.
        - Implement a "Confidence Corner" where students can post anonymous notes about their math confidence.
        - Design a class "Emotion Word Wall" specific to math feelings.
    - Benefits:
        - Provides a quick, accessible way for students to express themselves.
        - Makes emotional check-ins a visible, normalized part of the classroom culture.
7. **Parent-Teacher-Student Communication**
    - Involve parents in discussions about math feelings:
        - Send home surveys or reflection sheets to be completed with parents.
        - Include questions about math attitudes in parent-teacher conferences.
        - Provide resources for parents on how to discuss math positively at home.
    - Importance:
        - Helps align home and school support for students.
        - Increases awareness of how family attitudes can impact student feelings about math.
8. **Start-of-Unit Reflections**
    - Begin each new math unit with a reflection activity:
        - Ask students to write down their feelings about the upcoming topic.
        - Use prompts like "What excites you about this unit?" and "What concerns do you have?"
        - Encourage students to set personal goals for the unit.
    - Benefits:
        - Helps the teacher gauge the class's emotional state and potential areas of anxiety.
        - Encourages students to reflect on their own relationship with math.
        - Allows for proactive addressing of concerns before they become barriers to learning.
9. **Peer Support Systems**
    - Establish structures for peer emotional support:
        - Create "math buddies" who check in with each other regularly.
        - Implement a peer mentoring program where older or more confident students support those with math anxiety.
        - Use small group discussions to share feelings and coping strategies.
    - Benefits:
        - Builds a supportive classroom community.
        - Helps students realize they're not alone in their feelings.
        - Provides multiple channels for students to express themselves.
10. **Feedback on Teaching Methods**
    - Regularly seek student input on teaching approaches:
        - Use exit tickets to gather feedback on which teaching methods are most helpful or anxiety-inducing.
        - Conduct periodic "classroom climate" surveys that include questions about teaching style and pace.
        - Involve students in decision-making about classroom procedures and assessment methods.
    - Importance:
        - Empowers students by giving them a voice in their learning environment.
        - Helps teachers adjust their methods to better support students' emotional needs.
        - Demonstrates that the teacher values student input and experiences.

By implementing these strategies, teachers can create a classroom environment where open communication about math feelings is not only encouraged but becomes an integral part of the learning process. This openness can help destigmatize math anxiety, provide valuable insights for instructional decisions, and empower students to take an active role in managing their emotional responses to mathematics.

Remember, the goal is to create a safe, supportive space where all feelings about math are acknowledged and addressed constructively. By normalizing discussions about math emotions, teachers can help students develop healthier relationships with mathematics and build the confidence needed for long-term success in the subject.

Strategies for Helping Students Overcome Fears and Build Confidence

Positive Feedback: Providing constructive and positive feedback to build students' confidence.

Feedback is one of the most powerful tools a teacher has for building students' confidence, particularly in a subject like math, where anxiety and self-doubt can easily take root. Effective feedback goes beyond merely indicating right or wrong answers; it focuses on the learning process, emphasizes effort, and acknowledges the progress students are making. By doing so, feedback becomes a mechanism for growth rather than just an assessment of performance.

When giving feedback, it's crucial to highlight the effort students put into their work. This helps them understand that success in math is not solely about innate ability but about perseverance, practice, and a willingness to engage with challenging material. For example, a teacher might say, "I noticed how you worked hard to apply the new formula we discussed in class. That shows great initiative!" This type of comment reinforces the idea that effort leads to improvement, encouraging students to keep trying even when they encounter difficulties.

In addition to focusing on effort, effective feedback should also pinpoint specific strategies or thought processes that led to success. This not only reinforces the positive aspects of a student's approach but also helps them understand what they did right so they can replicate it in future problems. For instance, a teacher might say, "You did a great job identifying the key variables in this problem before starting your calculations. That's an important step in solving complex equations." By recognizing and naming these strategies, the teacher helps the student build a toolkit of skills they can rely on in future challenges.

It's also important to frame mistakes as valuable learning opportunities rather than failures. When a student makes an error, the feedback should guide them through the process of understanding what went wrong and how to correct it. Instead of simply marking an answer as incorrect, a teacher might say, "I see you've correctly set up the equation, which is a great start. Now let's look at the next step together to see how we can solve it." This approach acknowledges the student's correct thinking up to a certain point, providing a foundation to build upon, and gently leads them toward the correct solution. It shifts the focus from the mistake itself to the learning that can come from it.

Positive feedback also plays a crucial role in fostering a growth mindset, where students believe that their abilities can develop with time and effort. When students receive consistent, constructive feedback, they start to see challenges as opportunities to learn and grow, rather than as threats to their self-esteem. This shift in perspective can reduce math anxiety and increase resilience, as students become more willing to take risks and engage with difficult material.

Constructive and positive feedback is not just about correcting errors; it’s about building students’ confidence by recognizing their efforts, guiding their thinking, and framing mistakes as integral to the learning process. When done effectively, feedback can transform how students perceive their abilities and their relationship with math, leading to greater engagement, persistence, and ultimately, success.

Gradual Challenges: Introducing challenging problems gradually to help students build confidence and skills over time.

Building confidence in math is a gradual process that requires careful planning and thoughtful progression of challenges. One effective strategy for fostering this growth is the use of "scaffolding"—a teaching method where students are first introduced to simpler, more manageable problems that they can solve successfully before gradually moving on to more complex tasks. This approach allows students to build a strong foundation of understanding and skills, boosting their confidence as they progress.

When students are first introduced to a new concept, it's important to start with problems that are straightforward and directly related to the concept at hand. These initial problems should be designed to ensure that students can experience success early on, which is crucial for building their confidence. For instance, if the new concept is solving linear equations, the teacher might begin with problems where the equations have only one variable and require just a single step to solve. As students work through these simpler problems, they gain a sense of mastery and become more comfortable with the concept.

As students demonstrate proficiency with these basic problems, the difficulty level is gradually increased. The next step might involve introducing problems that require more than one step to solve or that incorporate additional variables. At this stage, the problems should still be within the students' reach but should require them to apply their knowledge in slightly more complex ways. This gradual increase in difficulty helps students to stretch their thinking and develop more advanced problem-solving skills without feeling overwhelmed.

Once students are comfortable with multi-step problems, the teacher can introduce word problems or scenarios that require deeper thinking and the application of multiple concepts. These problems often involve real-world situations, making the math more relevant and engaging. For example, students might be asked to solve a problem that involves both linear equations and inequalities, requiring them to integrate their knowledge of both topics. By this point, students have built up the confidence and skills necessary to tackle these more challenging tasks.

This scaffolding approach not only helps students build confidence but also encourages a growth mindset. As students encounter and overcome increasingly difficult problems, they begin to see challenges as opportunities to learn and grow. They understand that struggle is a natural part of the learning process and that their abilities can improve with effort and persistence. This mindset is crucial for success in math, where perseverance and problem-solving are key.

Moreover, gradually increasing the difficulty of problems allows students to develop a deeper understanding of mathematical concepts. As they move from simple to complex problems, they learn to recognize patterns, make connections between different concepts, and apply their knowledge in new and creative ways. This deeper understanding not only enhances their problem-solving skills but also prepares them for more advanced math topics in the future.

By implementing these strategies, educators can create a supportive math learning environment where students feel valued, respected, and confident in their abilities to tackle mathematical challenges. Gradual challenges help students build a solid foundation of skills, foster a love for learning, and develop the resilience needed to succeed in math and beyond.

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1.8: Assessing and Reflecting on Teaching Practices

The Importance of Reflection in Teaching

Self-Reflection: The Foundation of Growth

In the dynamic world of education, reflection is a critical tool for teacher growth and effectiveness. It's the engine that drives continuous improvement, allowing educators to refine their methods and better meet student needs.

Self-reflection in teaching is more than just a cursory glance at past performance; it's a deliberate and systematic examination of one's practices, beliefs, and impact. This introspective process enables teachers to gain deeper insights into their pedagogical approaches, identify areas for improvement, and celebrate successes.

The power of self-reflection lies in its ability to transform experience into learning. As educational theorist David Kolb posits in his experiential learning theory, reflection is a crucial step in the learning cycle. It bridges the gap between concrete experiences and the formulation of new concepts and strategies.

As John Dewey famously said, "We do not learn from experience... we learn from reflecting on experience."

For teachers, this reflection can take many forms:

1. Lesson Analysis: After each class, taking time to consider what worked well and what didn't. This might involve asking questions like: Did students engage with the material as expected? Were there moments of confusion or breakthrough? How could the lesson be improved for next time?
2. Student Outcomes Reflection: Regularly examining student performance data and work samples to gauge the effectiveness of teaching strategies. This could involve looking for patterns in student achievements or struggles and considering how instructional methods might be adapted to address these.
3. Pedagogical Philosophy Examination: Periodically revisiting and refining one's teaching philosophy. This deeper level of reflection involves questioning assumptions about learning, considering new educational research, and aligning practice with core values and beliefs about education.
4. Emotional Intelligence Reflection: Considering the emotional aspects of teaching, including teacher-student relationships, classroom atmosphere, and personal stress management. This type of reflection can lead to improved classroom dynamics and teacher well-being.

The benefits of consistent self-reflection are manifold. It can lead to:

- Increased self-awareness and professional growth
- Enhanced problem-solving skills in the classroom
- Greater job satisfaction and reduced burnout
- Improved student outcomes through more responsive and adaptive teaching
- A culture of continuous improvement within the school community

By making self-reflection a regular part of their practice, teachers can stay attuned to the ever-changing needs of their students and the evolving landscape of education. This ongoing process of self-examination and adjustment ensures that teachers remain effective, engaged, and inspired throughout their careers.

Moreover, self-reflection sets the stage for more formal professional development activities. When teachers approach workshops, training sessions, or collaborative meetings with a clear understanding of their strengths and areas for growth, they can engage more meaningfully with these opportunities and apply new insights more effectively in their classrooms.

In essence, self-reflection is not just a tool for improvement—it's a mindset that empowers teachers to take control of their professional growth, adapt to challenges, and continuously strive for excellence in their practice. As education continues to evolve, this capacity for thoughtful self-examination will be crucial in preparing teachers—and by extension, their students—for the demands of an ever-changing world.

Techniques for Self-Reflection

1. This technique involves listing what went well (Plus) and what could be improved (Delta) after each lesson. For example:
   - Plus: Students were engaged during the group problem-solving activity.
   - Delta: Need to allocate more time for questions at the end of class.
2. Spending just 10 minutes at the end of each day jotting down observations, challenges, and ideas creates a rich resource for self-assessment. Entries might include:
   - "Today's debate format really brought out critical thinking skills in my usually quiet students."
   - "The pacing of my lecture seemed off - students looked confused halfway through."

Setting Goals: Turning Reflection into Action

Setting goals based on these reflections is crucial. The SMART framework (Specific, Measurable, Achievable, Relevant, Time-bound) can help create effective goals.

Example of a SMART Goal

Instead of a vague goal like "improve student engagement," a teacher might set:

"Increase student participation in class discussions by implementing a think-pair-share technique twice per week for the next month."

Methods for Assessing Teaching Effectiveness

Student Feedback and Performance Data

Gathering Student Feedback

1. Short, anonymous responses collected at the end of a class provide immediate feedback. A simple prompt like "What was the muddiest point in today's lesson?" can reveal areas needing clarification.
2. Platforms like Google Forms or Mentimeter can make collecting and analyzing student feedback more efficient. For instance, a high school English teacher might use such a tool to gauge student preferences on essay topics or to collect peer feedback during writing workshops.

Analyzing Performance Data

Looking beyond test scores is crucial. For example, a science teacher might track lab report quality over time, noting improvements in specific areas like hypothesis formation or data analysis. This granular approach helps identify which teaching strategies are most effective for developing particular skills.

Peer Observations and Collaborative Reflections

Structured Peer Observations

1. Borrowed from medical training, this approach involves a group of teachers visiting multiple classrooms for short periods, focusing on a specific aspect of teaching, such as questioning techniques or student engagement strategies.

Collaborative Reflection Techniques

1. PLCs offer a structured approach where teachers meet regularly to discuss student work, share strategies, and plan interventions. For instance, a group of third-grade teachers might meet bi-weekly to analyze student writing samples, discussing effective methods for teaching paragraph structure and brainstorming ways to support struggling writers.
2. This technique involves a group of teachers collectively planning a lesson, observing its delivery, and then reflecting on its

effectiveness. For example, a team of high school physics teachers might collaborate on a unit about forces, with one teacher implementing the lesson while others observe student reactions and learning outcomes.

3. Platforms like Voxer allow teachers to share voice messages, creating an asynchronous dialogue about teaching practices. This can be particularly valuable for educators in different time zones or with conflicting schedules.

By incorporating these specific techniques and examples into their practice, educators can create a robust system for continuous improvement. Regular self-reflection, strategic goal-setting, diverse feedback mechanisms, and collaborative practices all contribute to a teacher's growth. This ongoing commitment to assessment and reflection not only enhances individual teaching quality but also fosters a culture of improvement within schools, ultimately leading to better educational experiences and outcomes for students.

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1.9: Common Mistakes Math Teachers Make

Chapter 5: Common Pitfalls in Elementary Math Instruction

Mathematics education in elementary school is the bedrock upon which a student's future mathematical understanding and success are built. It's a critical period where young minds are introduced to foundational concepts that will shape their relationship with numbers and problem-solving for years to come. However, even the most dedicated and well-intentioned teachers can inadvertently fall into common traps that may hinder their students' progress and enthusiasm for mathematics.

In this chapter, we'll explore five frequent mistakes made in elementary math instruction. By identifying these pitfalls and understanding their impact, educators can take proactive steps to avoid them. More importantly, we'll provide practical strategies to overcome these challenges, ensuring that students receive a robust and engaging mathematical education that prepares them for future academic and real-world success.

1. Prioritizing Procedure Over Concept

One of the most pervasive issues in math education is the tendency to focus on teaching procedures rather than fostering conceptual understanding. While procedures are undoubtedly important, students who only learn the "how" without grasping the "why" often struggle to apply their knowledge flexibly or to more advanced concepts.

This approach, often referred to as "algorithmic teaching," can lead to several problems:

- Students may be able to solve problems in familiar formats but struggle when presented with the same concept in a different context.
- Learners might memorize steps without understanding the underlying mathematical principles, leading to difficulties in more advanced math courses.
- It can foster a belief that mathematics is about following rules rather than understanding and reasoning, potentially diminishing students' interest and confidence in the subject.

Solution:

To address this issue, teachers should strive to unpack each standard to identify the core concepts. Use manipulatives and visual representations to build a concrete understanding before moving to abstract procedures. This approach, often referred to as the Concrete-Representational-Abstract (CRA) sequence, allows students to develop a solid conceptual foundation.

Example: Teaching Number Comparison

Instead of immediately introducing the greater than (>) and less than (<) symbols, start with physical objects:

1. Have students create two piles of objects (e.g., 8 blocks and 6 blocks).
2. Ask them to visually compare the piles and discuss which has more or less.
3. Introduce number lines as a representational tool, showing how the numbers' positions relate to their values.
4. Finally, introduce the symbols as a shorthand for expressing these comparisons.

This progression ensures students understand what these symbols truly represent, rather than just memorizing a rule about which way the "alligator mouth" points.

2. Insufficient Exploration Time

In the face of packed curricula and standardized testing pressures, teachers often feel compelled to rush through material, leaving limited time for hands-on exploration. This haste can result in superficial understanding and difficulty in applying concepts to new situations.

The consequences of insufficient exploration time include:

- Students may develop a fragile understanding that quickly fades after the unit is completed.
- Learners might struggle to see connections between different mathematical concepts.
- It can lead to math anxiety, as students feel pressured to quickly grasp ideas without adequate time to process and internalize them.

Solution:

Implement the Concrete-Representational-Abstract (CRA) framework rigorously. Ensure students have ample time to explore concepts with manipulatives before progressing to pictorial representations and abstract symbols. This might mean spending more time on fewer topics, but with deeper understanding.

Example: Teaching Fractions

A comprehensive exploration of fractions might look like this:

1. Concrete: Use fraction circles or bars to physically represent fractions. Have students manipulate these to compare, add, and subtract fractions.
2. Representational: Move to drawing fraction models, encouraging students to create visual representations of fraction problems.

3. Abstract: Introduce fraction notation and algorithms for operations, constantly referring back to the concrete and representational understandings.

This process might take longer, but it builds a robust understanding that serves students well in future mathematical endeavors.

3. Overloading with Strategies

In an admirable effort to cater to different learning styles and provide a toolkit for problem-solving, teachers sometimes introduce too many strategies simultaneously. This approach, while well-intentioned, can overwhelm students and prevent mastery of any single approach.

The pitfalls of strategy overload include:

- Students may become confused about which strategy to use when, leading to indecision and lack of confidence.
- Learners might rely on less efficient strategies because they haven't had time to fully understand and practice more sophisticated ones.
- It can create the misconception that there's always a "trick" to solving math problems, rather than encouraging deep thinking and problem-solving skills.

Solution:

Introduce strategies sequentially, allowing several days of practice for each. This gives students time to understand and internalize each method before moving on to the next. Focus on strategies that are most broadly applicable and build on each other.

Example: Teaching Addition Strategies

Instead of introducing all strategies at once, consider this sequence:

1. Start with counting on, using number lines and hundred charts for support.
2. Introduce doubles facts (2+2, 3+3, etc.) and practice until fluent.
3. Teach "doubles plus one" (7+8 as double 7 plus 1 more).
4. Move to making tens (9+6 as 9+1+5).
5. Finally, introduce the standard algorithm, showing how it relates to the previous strategies.

Spend several days on each strategy, ensuring students are comfortable before moving on. Regularly revisit previous strategies to maintain skills.

4. Neglecting Math Talks

Math talks—structured classroom discussions about mathematical thinking—are often overlooked in the rush to cover content. Yet, they are crucial for developing number sense, mathematical communication skills, and flexible thinking.

The consequences of neglecting math talks include:

- Students may develop the idea that math is about silent, individual work rather than collaborative problem-solving.
- Learners might struggle to articulate their mathematical thinking, hindering their ability to self-reflect and learn from others.
- Teachers miss valuable opportunities to assess student understanding and address misconceptions in real-time.

Solution:

Incorporate regular math talks into your routine. Use open-ended questions to encourage students to explain their reasoning and consider alternative approaches. Make math talks a daily practice, even if only for 5-10 minutes.

Example: Implementing Math Talks

Here's a simple structure for daily math talks:

1. Present a problem or number sentence (e.g., 25 + 18 = __).
2. Give students think time to solve mentally.
3. Ask for answers (but don't confirm correctness yet).
4. Call on students to share their strategies, encouraging multiple approaches.
5. Facilitate discussion about the efficiency and applicability of different strategies.
6. Summarize the key mathematical ideas that emerged from the discussion.

This routine not only improves mathematical thinking but also builds a classroom culture where mathematical discourse is valued and expected.

5. Losing Sight of Real-World Applications

When focused on skill acquisition and test preparation, it's easy to forget the practical applications of math in daily life. This disconnect can reduce student engagement and understanding, leading to the common question: "When will I ever use this in real life?"

The pitfalls of neglecting real-world applications include:

- Students may see math as an isolated subject, disconnected from their lives and interests.
- Learners might struggle to apply mathematical concepts to practical situations outside the classroom.
- It can diminish motivation and interest in mathematics, particularly for students who are more practically oriented.

Solution:

Regularly incorporate real-life scenarios and word problems that demonstrate the relevance of the math skills being taught. Look for opportunities to connect math to other subjects and to students' personal interests.

Example: Integrating Real-World Math

Here are some ways to bring real-world math into your classroom:

- Use actual flyers and catalogs for price comparison and percentage discount calculations.
- Create a class store or economy system to practice money skills and basic operations.
- Incorporate measurement into a class cooking project, scaling recipes up or down.
- Use sports statistics to explore data analysis and probability.
- Plan a hypothetical class trip, calculating distances, times, and budgets.

These activities not only make math more engaging but also help students see its relevance and importance in their daily lives.

By addressing these common pitfalls, elementary math teachers can create a more effective and engaging learning environment. Remember, the goal is not just to teach math, but to develop mathematical thinkers who can apply their skills confidently in various contexts.

Implementing these solutions may require adjustments to your teaching approach and possibly your curriculum pacing. However, the benefits—deeper understanding, increased engagement, and better long-term retention of mathematical concepts—far outweigh the challenges.

As you move forward, continually reflect on your teaching practices. Are you providing enough concrete experiences? Are your students getting opportunities to discuss their mathematical thinking? Are they seeing the connections between classroom math and the world around them? By keeping these questions in mind and actively working to avoid th

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2: HISTORICAL COUNTING SYSTEMS

2.1: Introduction

2.2: The Number and Counting System of the Inca Civilization

2.3: The Hindu-Arabic Number System

2.4: The Development and Use of Different Number Bases

2.5: The Mayan Numeral System

2.6: Addition and Subtraction with Other Bases

2.7: Exercises

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2.1: Introduction

As we begin our journey through the history of mathematics, one question to be asked is “Where do we start?” Depending on how you view mathematics or numbers, you could choose any of a number of launching points from which to begin. Howard Eves suggests the following list of possibilities [1].

Where to start the study of the history of mathematics...

- At the first logical geometric “proofs” traditionally credited to Thales of Miletus (600 BCE).
- With the formulation of methods of measurement made by the Egyptians and Mesopotamians/Babylonians.
- Where prehistoric peoples made efforts to organize the concepts of size, shape, and number.
- In pre-human times in the very simple number sense and pattern recognition that can be displayed by certain animals, birds, etc.
- Even before that in the amazing relationships of numbers and shapes found in plants.
- With the spiral nebulae, the natural course of planets, and other universe phenomena.

We can choose no starting point at all and instead agree that mathematics has always existed and has simply been waiting in the wings for humans to discover. Each of these positions can be defended to some degree and which one you adopt (if any) largely depends on your philosophical ideas about mathematics and numbers.

Nevertheless, we need a starting point. Without passing judgment on the validity of any of these particular possibilities, we will choose as our starting point the emergence of the idea of number and the process of counting as our launching pad. This is done primarily as a practical matter given the nature of this course. In the following chapter, we will try to focus on two main ideas. The first will be an examination of basic number and counting systems and the symbols that we use for numbers. We will look at our own modern (Western) number system as well those of a couple of selected civilizations to see the differences and diversity that is possible when humans start counting. The second idea we look at is base systems. By comparing our own base-ten (decimal) system with other bases, we quickly become aware that the system that we are so used to, when slightly changed, will challenge our notions about numbers and what symbols for those numbers actually mean.

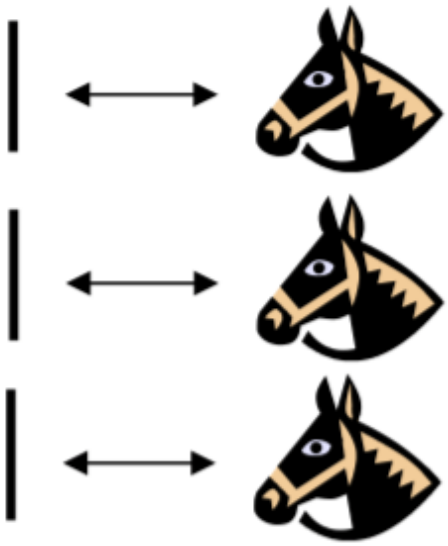
Recognition of More vs. Less

The idea of number and the process of counting goes back far beyond history began to be recorded. There is some archeological evidence that suggests that humans were counting as far back as 50,000 years ago [2]. However, we do not really know how this process started or developed over time. The best we can do is to make a good guess as to how things progressed. It is probably not hard to believe that even the earliest humans had some sense of *more* and *less*. Even some small animals have been shown to have such a sense. For example, one naturalist tells of how he would secretly remove one egg each day from a plover’s nest. The mother was diligent in laying an extra egg every day to make up for the missing egg. Some research has shown that hens can be trained to distinguish between even and odd numbers of pieces of food [3]. With these sorts of findings in mind, it is not hard to conceive that early humans had (at least) a similar sense of more and less. However, our conjectures about how and when these ideas emerged among humans are simply that; educated guesses based on our own assumptions of what might or could have been.

The Need for Simple Counting

As societies and humankind evolved, simply having a sense of more or less, even or odd, etc., would prove to be insufficient to meet the needs of everyday living. As tribes and groups formed, it became important to be able to know how many members were in the group, and perhaps how many were in the enemy’s camp. Certainly, it was important for them to know if the flock of sheep or other possessed animals were increasing or decreasing in size. “Just how many of them do we have, anyway?” is a question that we do not have a hard time imagining them asking themselves (or each other).

In order to count items such as animals, it is often conjectured that one of the earliest methods of doing so would be with “tally sticks.” These are objects used to track the numbers of items to be counted. With this method, each “stick” (or pebble, or whatever counting device being used) represents one animal or object. This method uses the idea of ***one-to-one correspondence***. In a one-to-one correspondence, items that are being counted are uniquely linked with some counting tool.



In the picture to the right, you see each stick corresponding to one horse. By examining the collection of sticks in hand one knows how many animals should be present. You can imagine the usefulness of such a system, at least for smaller numbers of items to keep track of. If a herder wanted to “count off” his animals to make sure they were all present, he could mentally (or methodically) assign each stick to one animal and continue to do so until he was satisfied that all were accounted for.

Of course, in our modern system, we have replaced the sticks with more abstract objects. In particular, the top stick is replaced with our symbol “1,” the second stick gets replaced by a “2” and the third stick is represented by the symbol “3,” but we are getting ahead of ourselves here. These modern symbols took many centuries to emerge.

Another possible way of employing the “tally stick” counting method is by making marks or cutting notches into pieces of wood, or even tying knots in string (as we shall see later). In 1937, Karl Absolom discovered a wolf bone that goes back possibly 30,000 years. It is believed to be a counting device [4]. Another example of this kind of tool is the Ishango Bone, discovered in 1960 at Ishango, and shown below [5]. It is reported to be between six and nine thousand years old and shows what appear to be markings used to do counting of some sort.

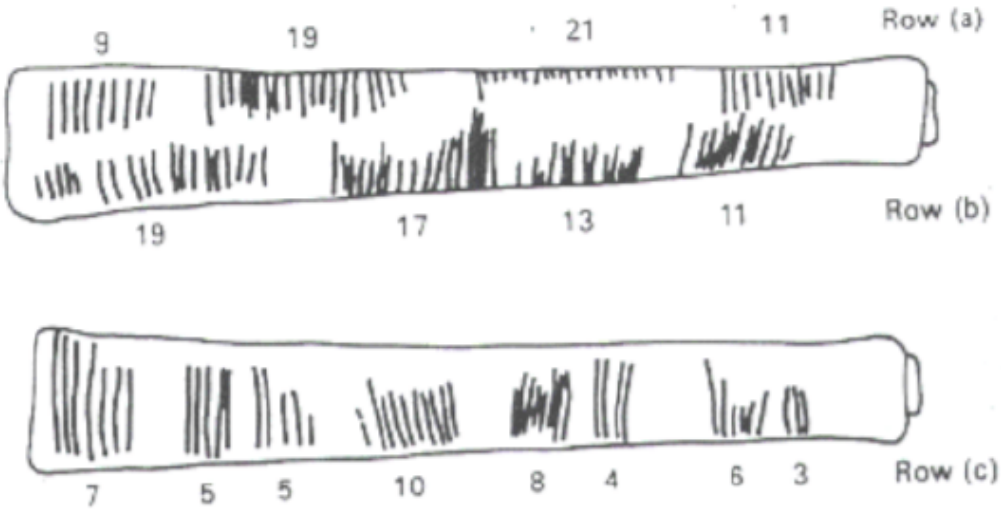


Figure 2.1.1: The markings on rows (a) and (b) each add up to 60. Row (b) contains the prime numbers between 10 and 20. Row (c) seems to illustrate for the method of doubling and multiplication used by the Egyptians. It is believed that this may also represent a lunar phase counter.

Spoken Words

As methods for counting developed, and as language progressed as well, it is natural to expect that spoken words for numbers would appear. Unfortunately, the developments of these words, especially those corresponding to the numbers from one through ten, are not easy to trace. Past ten, however, we do see some patterns:

Eleven comes from “ein lifon,” meaning “one left over.”

Twelve comes from “twe lif,” meaning “two left over.”

Thirteen comes from “Three and ten” as do fourteen through nineteen.

Twenty appears to come from “twe–tig” which means “two tens.”

Hundred probably comes from a term meaning “ten times.”

Written Numbers

When we speak of “written” numbers, we have to be careful because this could mean a variety of things. It is important to keep in mind that modern paper is only a little more than 100 years old, so “writing” in times past often took on forms that might look quite unfamiliar to us today.

As we saw earlier, some might consider wooden sticks with notches carved in them as writing as these are means of recording information on a medium that can be “read” by others. Of course, the symbols used (simple notches) certainly did not leave a lot of flexibility for communicating a wide variety of ideas or information.

Other mediums on which “writing” may have taken place include carvings in stone or clay tablets, rag paper made by hand (12th century in Europe, but earlier in China), papyrus (invented by the Egyptians and used up until the Greeks), and parchments from animal skins. And these are just a few of the many possibilities.

These are just a few examples of early methods of counting and simple symbols for representing numbers. Extensive books, articles, and research have been done on this topic and could provide enough information to fill this entire course if we allowed it to. The range and diversity of creative thought that has been used in the past to describe numbers and to count objects and people is staggering. Unfortunately, we don’t have time to examine them all, but it is fun and interesting to look at one system in more detail to see just how ingenious people have been.

Notes:

1) Eves, Howard; An Introduction to the History of Mathematics, p. 9.

2) Eves, p. 9.

3) McLeish, John; The Story of Numbers – How Mathematics Has Shaped Civilization, p. 7.

4) Bunt, Lucas; Jones, Phillip; Bedient, Jack; The Historical Roots of Elementary Mathematics, p. 2.

5) http://www.math.buffalo.edu/mad/Ancient-Africa/mad_zaire-uganda.html

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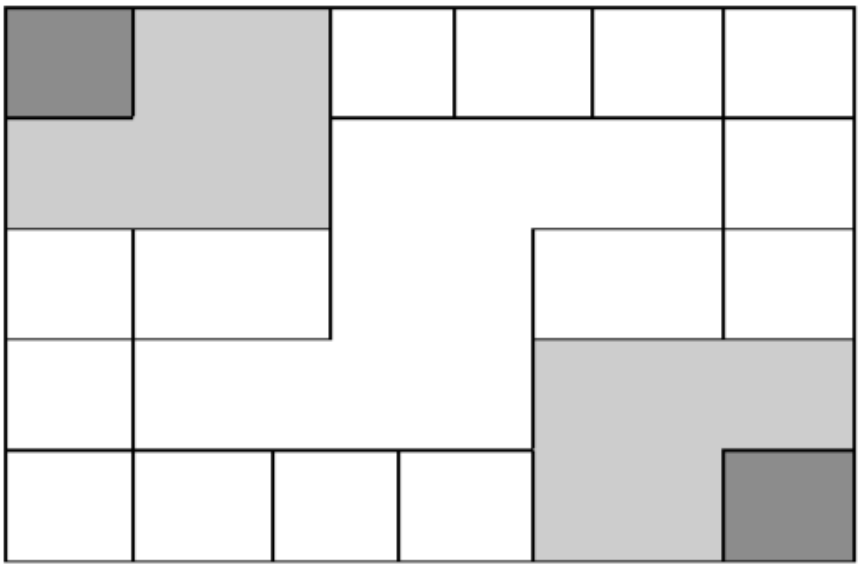
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2.2: The Number and Counting System of the Inca Civilization

Background

There is generally a lack of books and research material concerning the historical foundations of the Americas. Most of the "important" information available concentrates on the eastern hemisphere, with Europe as the central focus. The reasons for this may be twofold: first, it is thought that there was a lack of specialized mathematics in the American regions; second, many of the secrets of ancient mathematics in the Americas have been closely guarded [6]. The Peruvian system does not seem to be an exception here. Two researchers, Leland Locke and Erland Nordenskiold, have carried out research that has attempted to discover what mathematical knowledge was known by the Incas and how they used the Peruvian quipu, a counting system using cords and knots, in their mathematics. These researchers have come to certain beliefs about the quipu that we will summarize here.

Counting Boards

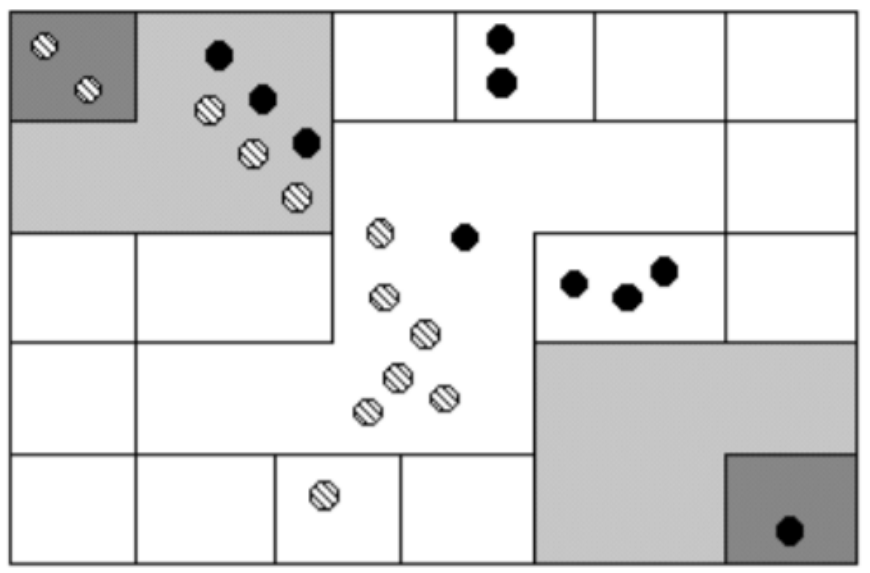


It should be noted that the Incas did not have a complicated system of computation. Where other peoples in the regions, such as the Mayans, were doing computations related to their rituals and calendars, the Incas seem to have been more concerned with the simpler task of record-keeping. To do this, they used what are called the "quipu" to record quantities of items. (We will describe them in more detail in a moment.) However, they first often needed to do computations whose results would be recorded on quipu. To do these computations, they would sometimes use a counting board constructed with a slab of stone. In the slab were cut rectangular and square compartments so that an octagonal (eight–sided) region was left in the middle. Two opposite corner rectangles were raised. Another two sections were mounted on the original surface of the slab so that there were actually three levels available. In the figure shown, the darkest shaded corner regions represent the highest, third level. The lighter shaded regions surrounding the corners are the second highest levels, while the clear white rectangles are the compartments cut into the stone slab.

Pebbles were used to keep accounts and their positions within the various levels and compartments gave totals. For example, a pebble in a smaller (white) compartment represented one unit. Note that there are 12 such squares around the outer edge of the figure. If a pebble was put into one of the two (white) larger, rectangular compartments, its value was doubled. When a pebble was put in the octagonal region in the middle of the slab, its value was tripled. If a pebble was placed on the second (shaded) level, its value was multiplied by six. And finally, if a pebble was found on one of the two highest corner levels, its value was multiplied by twelve. Different objects could be counted at the same time by representing different objects by different colored pebbles.

Example 2.2.1

Suppose you have the following counting board with two different kind of pebbles places as illustrated. Let the solid black pebble represent a dog and the striped pebble represent a cat. How many dogs are being represented?



Solution

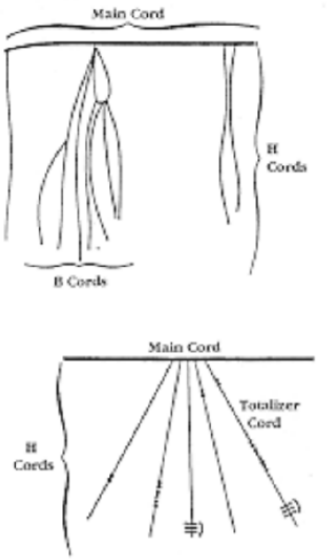
There are two black pebbles in the outer square regions...these represent 2 dogs.

There are three black pebbles in the larger (white) rectangular compartments. These represent 6 dogs. There is one black pebble in the middle region...this represents 3 dogs. There are three black pebbles on the second level...these represent 18 dogs. Finally, there is one black pebble on the highest corner level...this represents 12 dogs. We then have a total of 2+6+3+18+12 = 41 dogs.

Try It Now 1

How many cats are represented on this board?

The Quipu

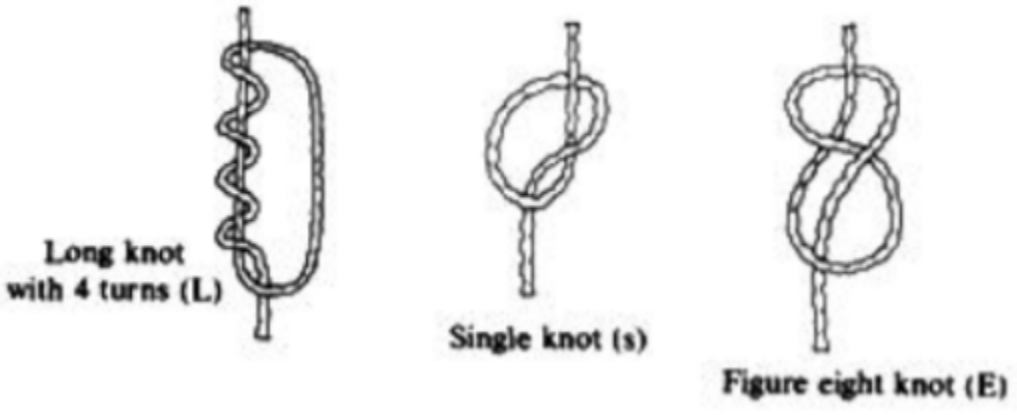


This kind of board was good for doing quick computations, but it did not provide a good way to keep a permanent recording of quantities or computations. For this purpose, they used the quipu. The quipu is a collection of cords with knots in them. These cords and knots are carefully arranged so that the position and type of cord or knot gives specific information on how to decipher the cord.

A quipu is made up of a main cord which has other cords (branches) tied to it. See pictures to the right [7].

Locke called the branches H cords. They are attached to the main cord. B cords, in turn, were attached to the H cords. Most of these cords would have knots on them. Rarely are knots found on the main cord, however, and tend to be mainly on the H and B cords. A quipu might also have a "totalizer" cord that summarizes all of the information on the cord group in one place.

Locke points out that there are three types of knots, each representing a different value, depending on the kind of knot used and its position on the cord. The Incas, like us, had a decimal (base-ten) system, so each kind of knot had a specific decimal value. The Single knot, pictured in the middle of the diagram [8] was used to denote tens, hundreds, thousands, and ten-thousands. They would be on the upper levels of the H cords. The figure-eight knot on the end was used to denote the integer "one." Every other integer from 2 to 9 was represented with a long knot, shown on the left of the figure. (Sometimes long knots were used to represents tens and hundreds.) Note that the long knot has several turns in it...the number of turns indicates which integer is being represented. The units (ones) were placed closest to the bottom of the cord, then tens right above them, then the hundreds, and so on.



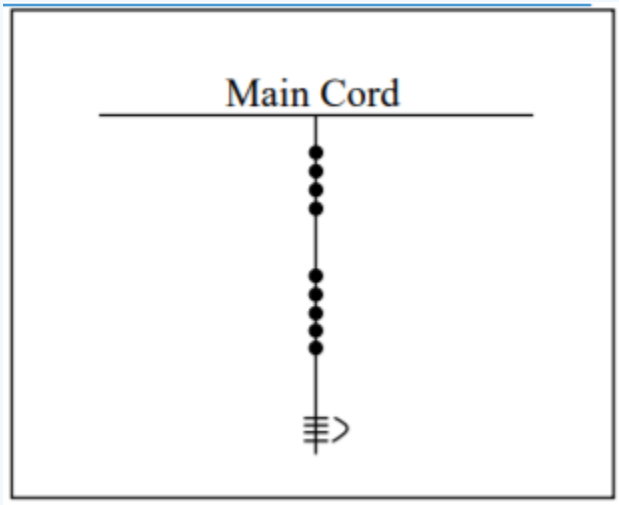
In order to make reading these pictures easier, we will adopt a convention that is consistent. For the long knot with turns in it (representing the numbers 2 through 9), we will use the following notation:

The four horizontal bars represent four turns and the curved arc on the right links the four turns together. This would represent the number 4.

We will represent the single knot with a large dot (•) and we will represent the figure-eight knot with a sideways eight (∞).

Example 2.2.1

What number is represented on the cord shown?

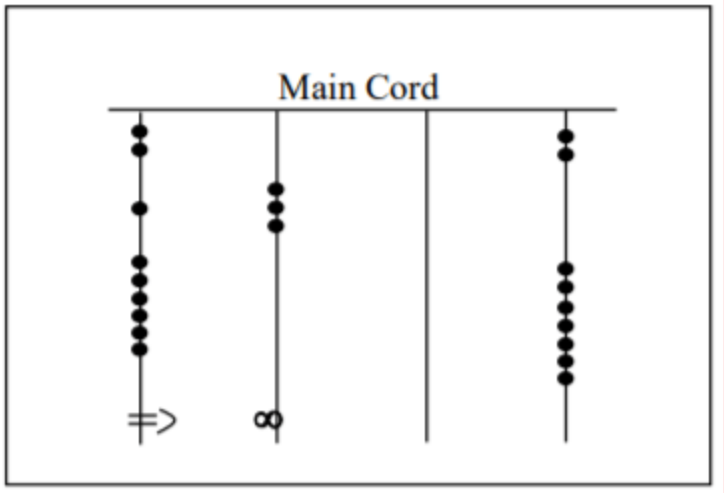


Solution

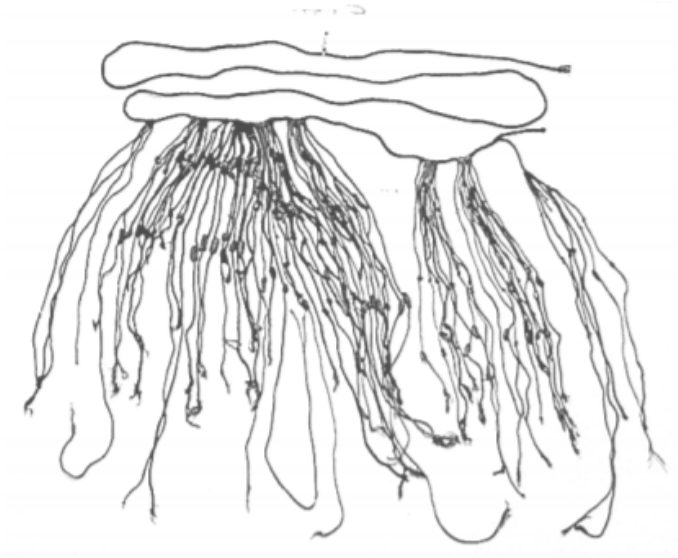
On the cord, we see a long knot with four turns in it...this represents four in the ones place. Then 5 single knots appear in the tens position immediately above that, which represents 5 tens, or 50. Finally, 4 single knots are tied in the hundreds, representing four 4 hundreds, or 400. Thus, the total shown on this cord is 454.

Try It Now 2

What numbers are represented on each of the four cords hanging from the main cord?



The colors of the cords had meaning and could distinguish one object from another. One color could represent llamas, while a different color might represent sheep, for example. When all the colors available were exhausted, they would have to be re–used. Because of this, the ability to read the quipu became a complicated task and specially trained individuals did this job. They were called Quipucamayoc, which means keeper of the quipus. They would build, guard, and decipher quipus.



As you can see from this photograph of an actual quipu, they could get quite complex.

There were various purposes for the quipu. Some believe that they were used to keep an account of their traditions and history, using knots to record history rather than some other formal system of writing. One writer has even suggested that the quipu replaced writing as it formed a role in the Incan postal system [9]. Another proposed use of the quipu is as a translation tool. After the conquest of the Incas by the Spaniards and subsequent “conversion” to Catholicism, an Inca supposedly could use the quipu to confess their sins to a priest. Yet another proposed use of the quipu was to record numbers related to magic and astronomy, although this is not a widely accepted interpretation.

The mysteries of the quipu have not been fully explored yet. Recently, Ascher and Ascher have published a book, The Code of the Quipu: A Study in Media, Mathematics, and Culture, which is “an extensive elaboration of the logical-numerical system of the quipu.[10]"

We are so used to seeing the symbols 1, 2, 3, 4, etc. that it may be somewhat surprising to see such a creative and innovative way to compute and record numbers. Unfortunately, as we proceed through our mathematical education in grade and high school, we receive very little information about the wide range of number systems that have existed and which still exist all over the world.

That's not to say our own system is not important or efficient. The fact that it has survived for hundreds of years and shows no sign of going away any time soon suggests that we may have finally found a system that works well and may not need further improvement, but only time will tell that whether or not that conjecture is valid or not. We now turn to a brief historical look at how our current system developed over history.

Notes:

6) Diana, Lind Mae; The Peruvian Quipu in Mathematics Teacher, Issue 60 (Oct., 1967), p. 623–28.

7) Diana, Lind Mae; The Peruvian Quipu in Mathematics Teacher, Issue 60 (Oct., 1967), p. 623–28.

8) http://wiscinfo.doit.wisc.edu/chaysimire/titulo2/khipus/what.htm

9) Diana, Lind Mae; The Peruvian Quipu in Mathematics Teacher, Issue 60 (Oct., 1967), p. 623–28.

10) http://www.cs.uidaho.edu/~casey931/seminar/quipu.html

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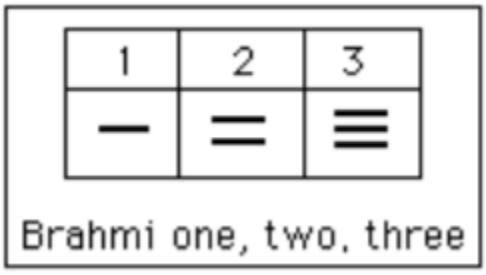
2.3: The Hindu-Arabic Number System

The Evolution of a System

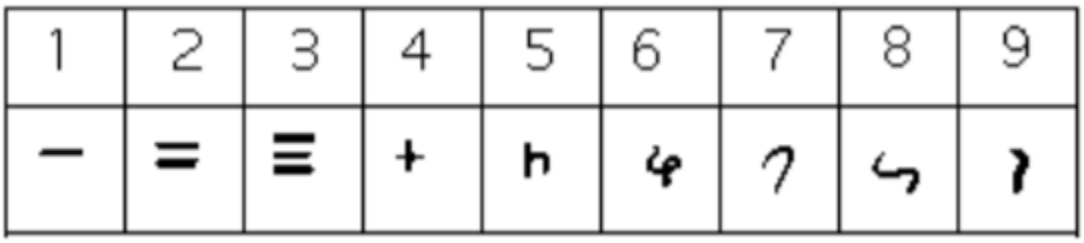
Our own number system, composed of the ten symbols {0,1,2,3,4,5,6,7,8,9} is called the **Hindu-Arabic system**. This is a base-ten (decimal) system since place values increase by powers of ten. Furthermore, this system is positional, which means that the position of a symbol has bearing on the value of that symbol within the number. For example, the position of the symbol 3 in the number 435,681 gives it a value much greater than the value of the symbol 8 in that same number. We'll explore base systems more thoroughly later. The development of these ten symbols and their use in a positional system comes to us primarily from India [11].

It was not until the 15th century that the symbols that we are familiar with today first took form in Europe. However, the history of these numbers and their development goes back hundreds of years. One important source of information on this topic is the writer al–Biruni, whose picture is shown here [12]. Al–Biruni, who was born in modern-day Uzbekistan, had visited India on several occasions and made comments on the Indian number system. When we look at the origins of the numbers that al–Biruni encountered, we have to go back to the third century B.C.E. to explore their origins. It is then that the Brahmi numerals were being used.

The Brahmi numerals were more complicated than those used in our own modern system. They had separate symbols for the numbers 1 through 9, as well as distinct symbols for 10, 100, 1000,..., also for 20, 30, 40,..., and others for 200, 300, 400, ..., 900. The Brahmi symbols for 1, 2, and 3 are shown below [13].



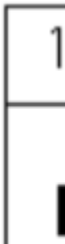
These numerals were used all the way up to the 4th century C.E., with variations through time and geographic location. For example, in the first century C.E., one particular set of Brahmi numerals took on the following form [14]:



From the 4th century on, you can actually trace several different paths that the Brahmi numerals took to get to different points and incarnations. One of those paths led to our current numeral system, and went through what are called the Gupta numerals. The Gupta numerals were prominent during a time ruled by the Gupta dynasty and were spread throughout that empire as they conquered lands during the 4th through 6th centuries. They have the following form [15]:

1	2	3	4	5	6	7	8	9
								

How the numbers got to their Gupta form is open to considerable debate. Many possible hypotheses have been offered, most of which boil down to two basic types [16]. The first type of hypothesis states that the numerals came from the initial letters of the names of the numbers. This is not uncommon...the Greek numerals developed in this manner. The second type of hypothesis states that they were derived from some earlier number system. However, there are other hypotheses that are offered, one of which is by the researcher Ifrah. His theory is that there were originally nine numerals, each represented by a corresponding number of vertical lines. One possibility is this [17]:

1	2	3	4	5	6	7	8	9
								

Because these symbols would have taken a lot of time to write, they eventually evolved into cursive symbols that could be written more quickly. If we compare these to the Gupta numerals above, we can try to see how that evolutionary process might have taken place, but our imagination would be just about all we would have to depend upon since we do not know exactly how the process unfolded.

The Gupta numerals eventually evolved into another form of numerals called the Nagari numerals, and these continued to evolve until the 11th century, at which time they looked like this [18]:

1	2	3	4	5	6	7	8	9	0
									

Note that by this time, the symbol for ০ has appeared! The Mayans in the Americas had a symbol for zero long before this, however, as we shall see later in the chapter.

These numerals were adopted by the Arabs, most likely in the eighth century during Islamic incursions into the northern part of India [19]. It is believed that the Arabs were instrumental in spreading them to other parts of the world, including Spain (see below).

Other examples of variations up to the eleventh century include:

Devangari, eighth century [20]:



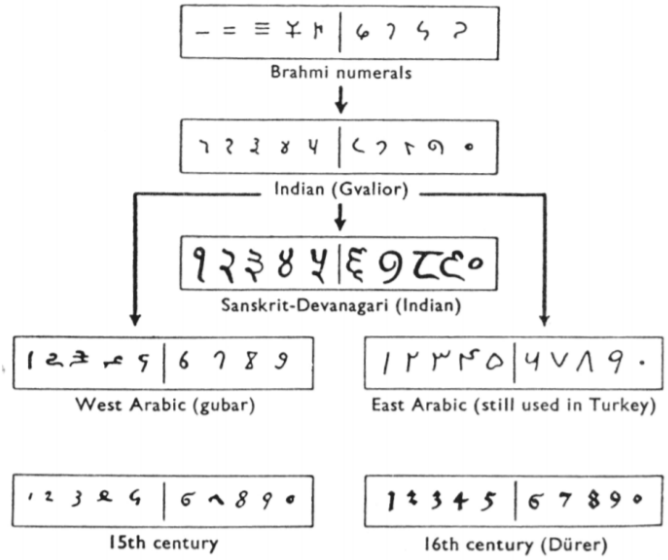
West Arab Gobar, tenth century [21]:



Spain, 976 B.C.E. [22]:



Finally, one more graphic [23] shows various forms of these numerals as they developed and eventually converged to the 15th century in Europe.



The Positional System

More important than the form of the number symbols is the development of the place value system. Although it is in slight dispute, the earliest known document in which the Indian system displays a positional system dates back to 346 C.E. However, some evidence suggests that they may have actually developed a positional system as far back as the first century C.E.

The Indians were not the first to use a positional system. The Babylonians (as we will see in Chapter 3) used a positional system with 60 as their base. However, there is not much evidence that the Babylonian system had much impact on later numeral systems, except with the Greeks. Also, the Chinese had a base-10 system, probably derived from the use of a counting board [24]. Some believe that the positional system used in India was derived from the Chinese system.

Wherever it may have originated, it appears that around 600 C.E., the Indians abandoned the use of symbols for numbers higher than nine and began to use our familiar system where the position of the symbol determines its overall value [25]. Numerous documents from the seventh century demonstrate the use of this positional system.

Interestingly, the earliest dated inscriptions using the system with a symbol for zero come from Cambodia. In 683, the 605th year

of the Saka era is written with three digits and a dot in the middle. The 608th year uses three digits with a modern 0 in the middle [26]. The dot as a symbol for zero also appears in a Chinese work (Chiu–chih li). The author of this document gives a strikingly clear description of how the Indian system works:

Using the [Indian] numerals, multiplication and division are carried out. Each numeral is written in one stroke. When a number is counted to ten, it is advanced into the higher place. In each vacant place a dot is always put. Thus, the numeral is always denoted in each place. Accordingly, there can be no error in determining the place. With the numerals, calculations are easy...[27]"

Transmission to Europe

It is not completely known how the system got transmitted to Europe. Traders and travelers of the Mediterranean coast may have carried it there. It is found in a tenth–century Spanish manuscript and may have been introduced to Spain by the Arabs, who invaded the region in 711 C.E. and were there until 1492.



In many societies, a division formed between those who used numbers and calculation for practical, every day business and those who used them for ritualistic purposes or for state business [28]. The former might often use older systems while the latter were inclined to use the newer, more elite written numbers. Competition between the two groups arose and continued for quite some time.

In a 14th century manuscript of Boethius' *The Consolations of Philosophy*, there appears a well-known drawing of two mathematicians. One is a merchant and is using an abacus (the "abacist"). The other is a Pythagorean philosopher (the "algorist") using his "sacred" numbers. They are in a competition that is being judged by the goddess of number. By 1500 C.E., however, the newer symbols and system had won out and has persevered until today. The Seattle Times recently reported that the Hindu–Arabic numeral system has been included in the book *The Greatest Inventions of the Past 2000 Years* [29].

One question to answer is why the Indians would develop such a positional notation. Unfortunately, an answer to that question is not currently known. Some suggest that the system has its origins with the Chinese counting boards. These boards were portable and it is thought that Chinese travelers who passed through India took their boards with them and ignited an idea in Indian mathematics [30]. Others, such as G. G. Joseph propose that it is the Indian fascination with very large numbers that drove them to develop a system whereby these kinds of big numbers could easily be written down. In this theory, the system developed entirely within the Indian mathematical framework without considerable influence from other civilizations.

Notes:

11) http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html

12) http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Al-Biruni.html

13) http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html

14) http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Indian_numerals.html

15) Ibid

16) Ibid

17) Ibid

18) Ibid

19) Katz, page 230

20) Burton, David M., History of Mathematics, An Introduction, p. 254–255

21) Ibid

22) Ibid

23) Katz, page 231.

24) Ibid, page 230

25) Ibid, page 231.

26) Ibid, page 232.

27) Ibid, page 232.

28) McLeish, p. 18

29) http://seattletimes.nwsource.com/news/health-science/html98/invs_20000201.html, Seattle Times, Feb. 1, 2000

30) Ibid, page 232.

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2.4: The Development and Use of Different Number Bases

Introduction and Basics

During the previous discussions, we have been referring to positional base systems. In this section of the chapter, we will explore exactly what a base system is and what it means if a system is "positional." We will do so by first looking at our own familiar, base-ten system and then deepen our exploration by looking at other possible base systems. In the next part of this section, we will journey back to Mayan civilization and look at their unique base system, which is based on the number 20 rather than the number 10.

A base system is a structure within which we count. The easiest way to describe a base system is to think about our own base-ten system. The base-ten system, which we call the "decimal" system, requires a total of ten different symbols/digits to write any number. They are, of course, 0, 1, 2, ..., 9.

The decimal system is also an example of a positional base system, which simply means that the position of a digit gives its place value. Not all civilizations had a positional system even though they did have a base with which they worked.

In our base-ten system, a number like 5,783,216 has meaning to us because we are familiar with the system and its places. As we know, there are six ones, since there is a 6 in the ones place. Likewise, there are seven "hundred-thousands," since the 7 resides in that place. Each digit has a value that is explicitly determined by its position within the number. We make a distinction between digit, which is just a symbol such as 5, and a number, which is made up of one or more digits. We can take this number and assign each of its digits a value. One way to do this is with a table, which follows:

5,000,000	= 5 × 1,000,000	= 5×10^6	Five million
+700,000	= 7 × 100,000	= 7×10^5	Seven hundred thousand
+80,000	= 8 × 10,000	= 8×10^4	Eighty thousand
+3,000	= 3 × 1000	= 3×10^3	Three thousand
+200	= 2 × 100	= 2×10^2	Two hundred
+10	= 1 × 10	= 1×10^1	Ten
+6	= 6 × 1	= 6×10^0	Six
5,783,216	Five million, seven hundred eighty-three thousand, two hundred sixteen		

From the third column in the table we can see that each place is simply a multiple of ten. Of course, this makes sense given that our base is ten. The digits that are multiplying each place simply tell us how many of that place we have. We are restricted to having at most 9 in any one place before we have to "carry" over to the next place. We cannot, for example, have 11 in the hundreds place. Instead, we would carry 1 to the thousands place and retain 1 in the hundreds place. This comes as no surprise to us since we readily see that 11 hundreds is the same as one-thousand, one-hundred. Carrying is a pretty typical occurrence in a base system.

However, base-ten is not the only option we have. Practically any positive integer greater than or equal to 2 can be used as a base for a number system. Such systems can work just like the decimal system except the number of symbols will be different and each position will depend on the base itself.

Other Bases

For example, let's suppose we adopt a base-five system. The only modern digits we would need for this system are 0,1,2,3 and 4. What are the place values in such a system? To answer that, we start with the ones place, as most base systems do. However, if we were to count in this system, we could only get to four (4) before we had to jump up to the next place. Our base is 5, after

all! What is that next place that we would jump to? It would not be tens, since we are no longer in base-ten. We're in a different numerical world. As the base-ten system progresses from 10^0 to 10^1, so does the base-five system moves from 5^0 to $5^1 = 5$. Thus, we move from the ones to the fives. After the fives, we would move to the 52 place, or the twenty-fives. Note that in base–ten, we would have gone from the tens to the hundreds, which is, of course, 10^2.

After the fives, we would move to the 5^2 place, or the twenty-fives. Note that in base–ten, we would have gone from the tens to the hundreds, which is, of course, 10^2.

Let's take an example and build a table. Consider the number 30412 in base five. We will write this as 30412_5, where the subscript 5 is not part of the number but indicates the base we're using. First off, note that this is NOT the number "thirty thousand, four hundred twelve." We must be careful not to impose the base-ten system on this number. Here's what our table might look like. We will use it to convert this number to our more familiar base-ten system.

	Base 5	This column coverts to base–ten	In Base–Ten
	3×5^4	$= 3 \times 625$	$= 1875$
+	0×5^3	$= 0 \times 125$	$= 0$
+	4×5^2	$= 4 \times 25$	$= 100$
+	1×5^1	$= 1 \times 5$	$= 5$
+	2×5^0	$= 2 \times 1$	$= 2$
		Total	1982

As you can see, the number 30412_5 is equivalent to 1,982 in base–ten. We will say $30412_5 = 1982_{10}$. All of this may seem strange to you, but that's only because you are so used to the only system that you've ever seen.

Example 2.4.1

Convert 6234_7 to a base-10 number

Solution

We first note that we are given a base-7 number that we are to convert. Thus, our places will start at the ones (7^0), and then move up to the 7's, 49's (7^2), etc. Here's the breakdown:

	Base 7	Convert	Base 10
	$= 6 \times 7^3$	$= 6 \times 343$	$= 2058$
+	$= 2 \times 7^2$	$= 2 \times 49$	$= 98$
+	$= 3 \times 7$	$= 3 \times 7$	$= 21$
+	$= 4 \times 1$	$= 4 \times 1$	$= 4$
		Total	2181

Thus, $6234_7 = 2181_{10}$

Try It Now 3

Convert 41065_7 to a base-10 number.

Converting from Base 10 to Other Bases

Converting from an unfamiliar base to the familiar decimal system is not that difficult once you get the hang of it. It's only a matter of identifying each place and then multiplying each digit by the appropriate power. However, going the other direction can be a little trickier. Suppose you have a base-ten number and you want to convert to base-five. Let's start with some simple examples before we get to a more complicated one.

Example 2.4.2

Convert twelve to a base-five number.

Solution

We can probably easily see that we can rewrite this number as follows: 12 = (2 × 5) + (2 × 1)

Hence, we have two fives and 2 ones. Hence, in base five, we would write twelve as 22_5. Thus, $12_{10} = 22_5$.

Example 2.4.3

Convert sixty-nine to a base-five number.

Solution

We can see now that we have more than 25, so we rewrite sixty-nine as follows: 69 = (2 × 25) + (3 × 5) + (4 × 1)

Here, we have two twenty-fives, 3 fives, and 4 ones. Hence, in base five we have 234. Thus, $69_{10} = 234_5$.

Example 2.4.4

Convert the base-seven number 32617 to base 10.

Solution

The powers of 7 are

$7^0 = 1$

$7^1 = 7$

$7^2 = 49$

$7^3 = 343$

Etc...

$3261_7 = (3{\times}343) + (2{\times}49) + (6{\times}7) + (1{\times}1) = 1170_{10}$.

Thus $3261_7 = 1170_{10}$.

Try It Now 4

Convert 143 to a base-5 number.

Try It Now 5

Convert the base-3 number 21021_3 to base 10.

In general, when converting from base-ten to some other base, it is often helpful to determine the highest power of the base that will divide into the given number at least once. In the last example, $5^2 = 25$ is the largest power of five that is present in 69, so that was our starting point. If we had moved to $5^3 = 125$, then 125 would not divide into 69 at least once.

Converting from Base 10 to Base *b*

1. Find the highest power of the base *b* that will divide into the given number at least once and then divide.
2. Write down the whole number part, then use the remainder from division in the next step.
3. Repeat step two, dividing by the next highest power of the base b, writing down the whole number part (including 0), and using the remainder in the next step.
4. Continue until the remainder is smaller than the base. This last remainder will be in the "ones" place.
5. Collect all your whole number parts to get your number in base *b* notation

Example 2.4.5

Convert the base-ten number 348 to base-five.

Solution

The powers of five are

$5^0 = 1$

$5^1 = 5$

$5^2 = 25$

$5^3 = 125$

$5^4 = 625$

Etc...

Since 348 is smaller than 625, but bigger than 125, we see that 5^3=125 is the highest power of five present in 348. So, we divide 125 into 348 to see how many 125's divide into 348:

348÷125 = 2 with remainder 98

We write down the whole part, 2, and continue with the remainder. There are 98 left over, so we see how many 25's (the next smallest power of five) there are in the remainder:

98÷25 = 3 with remainder 23

We write down the whole part, 2, and continue with the remainder. There are 23 left over, so we look at the next place, the 5's:

23÷5 = 4 with remainder 3

This leaves us with 3, which is less than our base, so this number will be in the "ones" place. We are ready to assemble our base-five number:

$348 = (2\times5^3) + (3\times5^2) + (4\times5^1) + (3\times1)$

Hence, our base-five number is 2343. We'll say that $348_{10} = 2343_5$.

Example 2.4.6

Convert the base-ten number 4,509 to base-seven.

Solution

The powers of 7 are

$7^0 = 1$

$7^1 = 7$

$7^2 = 49$

$7^3 = 343$

$7^4 = 2401$

$7^5 = 16807$

Etc...

The highest power of 7 that will divide into 4,509 is $7^4 = 2401$.

With division, we see that it will go in 1 time with a remainder of 2108. So we have 1 in the 7^4 place.

The next power down is $7^3 = 343$, which goes into 2108 six times with a new remainder of 50. So we have 6 in the 7^3 place.

The next power down is $7^2 = 49$, which goes into 50 once with a new remainder of 1. So there is a 1 in the 7^2 place.

The next power down is 7^1 but there was only a remainder of 1, so that means there is a 0 in the 7's place and we still have 1 as a remainder.

That, of course, means that we have 1 in the ones place.

$$4{,}509 \div 7^4 = 1 \text{ R } 2108$$

$$2108 \div 7^3 = 6 \text{ R } 50$$

$$50 \div 7^2 = 1 \text{ R } 1$$

$$1 \div 7^1 = 0 \text{ R } 1$$

$$1 \div 7^0 = 1$$

$$4{,}509_{10} = 16101_7$$

Putting all of this together means that $4{,}509_{10} = 16101_7$.

Try It Now 6

Convert 657_{10} to a base 4 number.

Try It Now 7

Convert 8377_{10} to a base 8 number.

Another Method for Converting from Base 10 to Other Bases

As you read the solution to this last example and attempted the "Try it Now" problems, you may have had to repeatedly stop and think about what was going on. The fact that you are probably struggling to follow the explanation and reproduce the process yourself is mostly due to the fact that the non-decimal systems are so unfamiliar to you. In fact, the only system that you are probably comfortable with is the decimal system.

As budding mathematicians, you should always be asking questions like "How could I simplify this process?" In general, that is one of the main things that mathematicians do...they look for ways to take complicated situations and make them easier or more familiar. In this section, we will attempt to do that.

To do so, we will start by looking at our own decimal system. What we do may seem obvious and maybe even intuitive but that's the point. We want to find a process that we readily recognize works and makes sense to us in a familiar system and then use it to extend our results to a different, unfamiliar system.

Let's start with the decimal number, 4863_{10}. We will convert this number to base 10. Yeah, I know it's already in base 10, but if you carefully follow what we're doing, you'll see it makes things work out very nicely with other bases later on. We first note that the highest power of 10 that will divide into 4863 at least once is $10^3 = 1000$. *In general, this is the first step in our new process; we find the highest power that a given base that will divide at least once into our given number.*

We now divide 1000 into 4863:

$$4863 \div 1000 = 4.863$$

This says that there are four thousands in 4863 (obviously). However, it also says that there are 0.863 thousands in 4863. This fractional part is our remainder and will be converted to lower powers of our base (10). If we take that decimal and multiply by 10 (since that's the base we're in) we get the following:

$$0.863 \times 10 = 8.63$$

Why multiply by 10 at this point? We need to recognize here that 0.863 thousands is the same as 8.63 hundreds. Think about that until it sinks in.

$$(0.863)(1000) = 863$$

$$(8.63)(100) = 863$$

These two statements are equivalent. So, what we are really doing here by multiplying by 10 is rephrasing or converting from one place (thousands) to the next place down (hundreds).

$$0.863 \times 10 \Rightarrow 8.63$$

$$\text{(Parts of Thousands)} \times 10 \Rightarrow \text{Hundreds}$$

We now have 8 hundreds and a remainder of 0.63 hundreds, which is the same as 6.3 tens. We can do this again with the 0.63 that remains after this first step.

$$0.63 \times 10 \Rightarrow 6.3$$

$$\text{Hundreds} \times 10 \Rightarrow \text{Tens}$$

So, we have six tens and 0.3 tens, which is the same as 3 ones, our last place value.

Now here's the punch line. Let's put all of the together in one place:

$$4863 \div 1000 = 4.863$$

$$0.863 \times 10 = 8.63$$

$$0.63 \times 10 = 6.3$$

$$0.3 \times 10 = 3.0$$

Note that in each step, the remainder is carried down to the next step and multiplied by 10, the base. Also, at each step, the whole number part, which is circled, gives the digit that belongs in that particular place. What is amazing is that this works for any base! So, to convert from a base 10 number to some other base, *b*, we have the following steps we can follow:

Converting from Base 10 to Base *b*: Another Method

1. Find the highest power of the base *b* that will divide into the given number at least once and then divide.
2. Keep the whole number part, and multiply the fractional part by the base *b*.
3. Repeat step two, keeping the whole number part (including 0), carrying the fractional part to the next step until only a whole number result is obtained.
4. Collect all your whole number parts to get your number in base *b* notation.

We will illustrate this procedure with some examples.

Example 2.4.7

Convert the base 10 number, 348_{10}, to base 5.

Solution

This is actually a conversion that we have done in a previous example. The powers of five are

$5^0 = 1$

$5^1 = 5$

$5^2 = 25$

$5^3 = 125$

$5^4 = 625$

Etc...

The highest power of five that will go into 348 at least once is 5^3. We divide by 125 and then proceed:

$$348 \div 5^3 = 2.784$$

$$0.784 \times 5 = 3.92$$

$$0.92 \times 5 = 4.6$$

$$0.6 \times 5 = 3.0$$

By keeping all the whole number parts, from top bottom, gives 2343 as our base 5 number. Thus, $2343_5 = 348_{10}$.

We can compare our result with what we saw earlier, or simply check with our calculator, and find that these two numbers really are equivalent to each other.

Example 2.4.8

Convert the base 10 number, 3007_{10}, to base 5.

Solution

The highest power of 5 that divides at least once into 3007 is $5^4 = 625$. Thus, we have:

$$3007 \div 625 = 4.8112$$

$$0.8112 \times 5 = 4.056$$

$$0.056 \times 5 = 0.28$$

$$0.28 \times 5 = 1.4$$

$$0.4 \times 5 = 2.0$$

This gives us that $3007_{10} = 44012_5$. Notice that in the third line that multiplying by 5 gave us 0 for our whole number part. We don't discard that! The zero tells us that a zero in that place. That is, there are no 5^2's in this number.

This last example shows the importance of using a calculator in certain situations and taking care to avoid clearing the calculator's memory or display until you get to the very end of the process.

Example 2.4.9

Convert the base 10 number, 63201_{10}, to base 7.

Solution

The powers of 7 are

$7^0 = 1$

$7^1 = 7$

$7^2 = 49$

$7^3 = 343$

$7^4 = 2401$

$7^5 = 16807$

Etc...

The highest power of 7 that will divide at least once into 63201 is 7^5. When we do the initial division on a calculator, we get the following:

$63201 \div 7^5 = 3.760397453$

The decimal part actually fills up the calculators display and we don't know if it terminates at some point or perhaps even repeats down the road. So, if we clear our calculator at this point, we will introduce error that is likely to keep this process from ever ending. To avoid this problem, we leave the result in the calculator and simply **subtract 3** from this to get the fractional part all by itself. DO NOT ROUND OFF! Subtraction and then multiplication by seven gives:

$$63201 \div 7^5 = 3.760397453$$

$$0.760397453 \times 7 = 5.322782174$$

$$0.322782174 \times 7 = 2.259475219$$

$$0.259475219 \times 7 = 1.816326531$$

$$0.816326531 \times 7 = 5.714285714$$

$$0.714285714 \times 7 = 5.000000000$$

Yes, believe it or not, that last product is exactly 5, *as long as you don't clear anything out on your calculator*. This gives us our final result: $63201_{10} = 352155_7$. If we round, even to two decimal places in each step, clearing our calculator out at each step along the way, we will get a series of numbers that do not terminate, but begin repeating themselves endlessly. (Try it!) We end up with something that doesn't make any sense, at least not in this context. So be careful to use your calculator cautiously on these conversion problems.

Also, remember that if your first division is by 7^5, then you expect to have 6 digits in the final answer, corresponding to the places for 7^5, 7^4, and so on down to 7^0. If you find yourself with more than 6 digits due to rounding errors, you know something went wrong.

Try It Now 8

Convert the base-10 number, 9352_{10}, to base 5.

Try It Now 9

Convert the base-10 number, 1500, to base 3.

Be careful not to clear your calculator on this one. Also, if you're not careful in each step, you may not get all of the digits you're looking for, so move slowly and with caution.

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2.5: The Mayan Numeral System

Background



1. Chichen Itza
2. Uxmal
3. Tulum
4. Palenque
5. Bonampak, Yaxchilan
6. Tikal
7. Altun Ha
8. Copán

As you might imagine, the development of a base system is an important step in making the counting process more efficient. Our own base-ten system probably arose from the fact that we have 10 fingers (including thumbs) on two hands. This is a natural development. However, other civilizations have had a variety of bases other than ten. For example, the Natives of Queensland used a base–two system, counting as follows: “one, two, two and one, two two’s, much.” Some Modern South American Tribes have a base-five system counting in this way: “one, two, three, four, hand, hand and one, hand and two,” and so on. The Babylonians used a base-60 (sexigesimal) system. In this chapter, we wrap up with a specific example of a civilization that actually used a base system other than 10.

The Mayan civilization is generally dated from 1500 B.C.E to 1700 C.E. The Yucatan Peninsula (see map [31]) in Mexico was the scene for the development of one of the most advanced civilizations of the ancient world. The Mayans had a sophisticated ritual system that was overseen by a priestly class. This class of priests developed a philosophy with time as divine and eternal [32]. The calendar, and calculations related to it, were thus very important to the ritual life of the priestly class, and hence the Mayan people. In fact, much of what we know about this culture comes from their calendar records and astronomy data. Another important source of information on the Mayans is the writings of Father Diego de Landa, who went to Mexico as a missionary in 1549.

There were two numeral systems developed by the Mayans – one for the common people and one for the priests. Not only did these two systems use different symbols, but they also used different base systems. For the priests, the number system was governed by ritual. The days of the year were thought to be gods, so the formal symbols for the days were decorated heads [33], like the sample to the left [34] Since the basic calendar was based on 360 days, the priestly numeral system used a mixed base system employing multiples of 20 and 360. This makes for a confusing system, the details of which we will skip.

The Mayan Number System

Instead, we will focus on the numeration system of the “common” people, which used a more consistent base system. As we stated earlier, the Mayans used a base–20 system, called the “vigesimal” system. Like our system, it is positional, meaning that the position of a numeric symbol indicates its place value. In the following table, you can see the place value in its vertical format [35].

Powers	Base–Ten Value	Place Name
20^7	12,800,000,000	Hablat

20^6	64,000,000	Alau
20^5	3,200,000	Kinchil
20^4	160,000	Cabal
20^3	8,000	Pic
20^2	400	*Bak*
20^1	20	Kal
20^0	1	Hun

In order to write numbers down, there were only three symbols needed in this system. A horizontal bar represented the quantity 5, a dot represented the quantity 1, and a special symbol (thought to be a shell) represented zero. The Mayan system may have been the first to make use of zero as a placeholder/number. The first 20 numbers are shown in the table to the right [36].

Number	Vertical Form	Number	Vertical Form
0		10	
1		11	
2		12	
3		13	
4		14	
5		15	
6		16	
7		17	
8		18	
9		19	

Unlike our system, where the ones place starts on the right and then moves to the left, the Mayan systems places the ones on the bottom of a vertical orientation and moves up as the place value increases.

When numbers are written in vertical form, there should never be more than four dots in a single place. When writing Mayan numbers, every group of five dots becomes one bar. Also, there should never be more than three bars in a single place...four bars would be converted to one dot in the next place up. It's the same as 10 getting converted to a 1 in the next place up when we carry during addition.

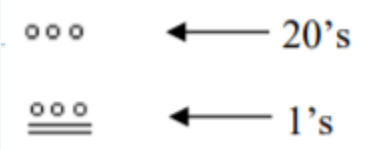
Example 2.5.1

What is the value of this number, which is shown in vertical form?



Solution

Starting from the bottom, we have the ones place. There are two bars and three dots in this place. Since each bar is worth 5, we have 13 ones when we count the three dots in the ones place. Looking to the place value above it (the twenties places), we see there are three dots so we have three twenties.



Hence, we can write this number in base ten as

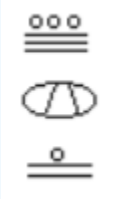
$(3 \times 20^1) + (13 \times 20^0) = (3 \times 20) + (13 \times 1)$

$= 60 + 13$

$= 73$

Example 2.5.2

What is the value of the following Mayan number?



Solution

This number has 11 in the ones place, zero in the 20's place, and 18 in the $20^2 = 400$'s place. Hence, the value of this number in base-ten is

$(18 \times 400) + (0 \times 20) + (11 \times 1) = 7211.$

Try It Now 10

Convert the Mayan number below to base 10.



Example 2.5.3

Convert the base-10 number 3575_{10} to Mayan numerals.

Solution

This problem is done in two stages. First, we need to convert to a base 20 number. We will do so using the method provided in the last section of the text. The second step is to convert that number to Mayan symbols.

The highest power of 20 that will divide into 3575 is $20^2 = 400$, so we start by dividing that and then proceed from there:

$3575 \div 400 = 8.9375$

$0.9375 \times 20 = 18.75$

$0.75 \times 20 = 15.0$

This means that $3575_{10} = 8,18,15_{20}$

The second step is to convert this to Mayan notation. This number indicates that we have 15 in the ones position. That's three bars at the bottom of the number. We also have 18 in the 20's place, so that's three bars and three dots in the second position. Finally, we have 8 in the 400's place, so that's one bar and three dots on the top. We get the following



Note that in the previous example a new notation was used when we wrote $8,18,15_{20}$. The commas between the three numbers 8, 18, and 15 are now separating place values for us so that we can keep them separate from each other. This use of the comma is slightly different than how they're used in the decimal system. When we write a number in base 10, such as 7,567,323, the commas are used primarily as an aide to read the number easily but they do not separate single place values from each other. We will need this notation whenever the base we use is larger than 10.

Writing Numbers with Bases Bigger Than 10

When the base of a number is larger than 10, separate each "digit" with a comma to make the separation of digits clear.

For example, in base 20, to write the number corresponding to $17\times20^2 + 6\times20^1 + 13\times20^0$, we'd write $17,6,13_{20}$.

Try It Now 11

Convert the base 10 number 10553_{10} to Mayan numerals.

Try It Now 12

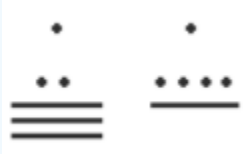
Convert the base 10 number 5617_{10} to Mayan numerals.

Adding Mayan Numbers

When adding Mayan numbers together, we'll adopt a scheme that the Mayans probably did not use but which will make life a little easier for us.

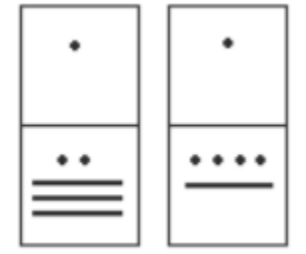
Example 2.5.4

Add, in Mayan, the numbers 37 and 29 [37]:

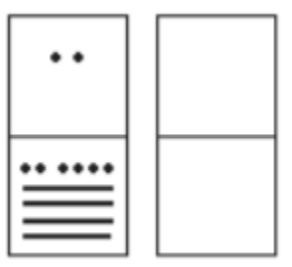


Solution

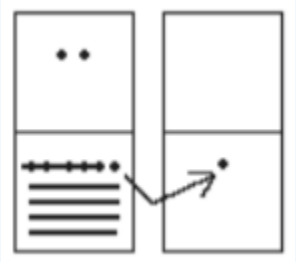
First draw a box around each of the vertical places. This will help keep the place values from being mixed up.



Next, put all of the symbols from both numbers into a single set of places (boxes), and to the right of this new number draw a set of empty boxes where you will place the final sum:

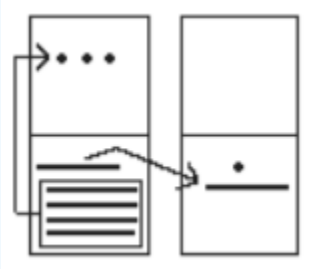


You are now ready to start carrying. Begin with the place that has the lowest value, just as you do with Arabic numbers. Start at the bottom place, where each dot is worth 1. There are six dots, but a maximum of four are allowed in any one place; once you get to five dots, you must convert to a bar. Since five dots make one bar, we draw a bar through five of the dots, leaving us with one dot which is under the four-dot limit. Put this dot into the bottom place of the empty set of boxes you just drew:

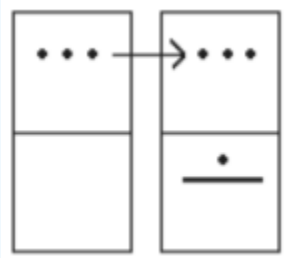


Now look at the bars in the bottom place. There are five, and the maximum number the place can hold is three. *Four bars are equal to one dot in the next highest place.*

Whenever we have four bars in a single place we will automatically convert that to a *dot* in the next place up. We draw a circle around four of the bars and an arrow up to the dots' section of the higher place. At the end of that arrow, draw a new dot. That dot represents 20 just the same as the other dots in that place. Not counting the circled bars in the bottom place, there is one bar left. One bar is under the three-bar limit; put it under the dot in the set of empty places to the right.



Now there are only three dots in the next highest place, so draw them in the corresponding empty box.



We can see here that we have 3 twenties (60), and 6 ones, for a total of 66. We check and note that 37 + 29 = 66, so we have done this addition correctly. Is it easier to just do it in base–ten? Probably, but that's only because it's more familiar to you. Your task here is to try to learn a new base system and how addition can be done in slightly different ways than what you have seen in the past. Note, however, that the concept of carrying is still used, just as it is in our own addition algorithm.

Try It Now 13

Try adding 174 and 78 in Mayan by first converting to Mayan numbers and then working entirely within that system. Do not add in base-ten (decimal) until the very end when you *check* your work.

Notes:

31) http://www.gorp.com/gorp/location/latamer/map_maya.htm

32) Bidwell, James; Mayan Arithmetic in Mathematics Teacher, Issue 74 (Nov., 1967), p. 762–68.

33) http://www.ukans.edu/~lctls/Mayan/numbers.html

34) http://www.ukans.edu/~lctls/Mayan/numbers.html

35) Bidwell

36) http://www.vpds.wsu.edu/fair_95/gym/UM001.html

37) http://forum.swarthmore.edu/k12/mayan.math/mayan2.html

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2.6: Addition and Subtraction with Other Bases

Addition in Other Bases

As we saw in the previous section with the Mayan numeration system, we can add or subtract in other bases. Below are a series of steps, but, overall, we add as usual while finding its equivalent number from base 10 to the new base.

Adding in Base *b*

1. Rewrite the addition vertically, if not already.
2. Start in the ones place (as usual), but find the number the sum represents in base *b*.
3. If the sum is larger than base b, then carry over to the b^1 place value.
4. Repeat steps 2 and 3 for the b^2, b^3, ... place values.

Example 2.6.1

Add in base two: $111_{two} + 11_{two}$

Solution

Writing this vertically, we get

$$111_{two}$$

$$+\underline{11_{two}}$$

Let's add the ones place as usual. If we were in base 10, 1+1=2; 2 in base 10 is equivalent to 0 in base 2 and we carry 1 over to the 2's place. Recall, base two is {0,1}. Hence, the ones place is 0:

$$11^{+1}1_{two}$$

$$+\underline{1\quad 1_{two}}$$

$$1\quad 0_{two}$$

Adding in the 2's place, 1+1+1=3 in base 10, but 3 is 1 in base 2 and we carry 1 to the 2^2's place:

$$1^{+1}11_{two}$$

$$+\underline{\ 11_{two}}$$

$$0\quad 10_{two}$$

Adding in the 2^2's place, we get 1+1=2 in base 10, but 2 in base 10 is 0 in base 2 and we carry a 1 over to the 2^3's place:

$$+1\ 1\ 11_{two}$$

$$+\underline{\ \ 11_{two}}$$

$$1\quad 010_{two}$$

Hence, $111_{two} + 11_{two} = 1010_{two}$. Note, another way to do this is convert each number to base 10 and add as usual, then convert the result back into base 2.

Example 2.6.2

Add in base five: $44_{five} + 42_{five}$

Solution

Let's try this by converting each number to base 10, adding them, then converting the sum back into base 5:

$$44_{five} = 4(5) + 4(1) = 24_{ten}$$

$$42_{five} = 4(5) + 2(1) = 22_{ten}$$

Next, we add the base 10 numbers:

$$24 + 22 = 46_{ten}$$

Converting 46 in base ten into a number in base 5, we use the previous sections' technique and obtain 141_{five}. Thus, 46 in base 10 is equivalent to 141 in base 5.

Subtraction in Other Bases

Subtracting in Base *b*

Rewrite the subtraction vertically, if not already.

Start in the ones place (as usual), but find the number the difference represents in base *b*.

If the ones place of the minuend is smaller than the ones place of the subtrahend, then borrow from the place value to the left in that base *b*. Then subtract as usual.

Repeat steps 2 and 3 for the b^2, b^3 place values.

Example 2.6.3

Subtract in base 5: $240_{five} - 40_{five}$

Solution

$$\begin{array}{r} 240_{five} \\ -\underline{40_{five}} \end{array}$$

If we take the ones place, 0-0 =0, which is 0 in base 5, then the ones place stays 0.

$$\begin{array}{r} 240_{five} \\ -\underline{40_{five}} \\ 0 \end{array}$$

Now let's take the 5's place: 4-4=0. Since 0 is in base 5, then the 5's place is 0.

$$240_{five}$$

$$\underline{-40_{five}}$$

$$00$$

Let's look at the 5^2's place. Notice, we can just drop the two down and get

$$240_{five}$$

$$\underline{-40_{five}}$$

$$200_5$$

Thus, $240_{five} - 40_{five} = 200_{five}$.

What if we have to borrow? We can subtract with borrowing easily in base 10, but what if we wanted to subtract two numbers that included borrowing? Let's see.

Example 2.6.4

Subtract in base 5: $404_{five} - 323_{five}$

Solution

Rewriting this vertically, we get

$$404_{five}$$

$$\underline{-323_{five}}$$

Subtracting in the ones place, we get 4-3=1. Since 1 is in base 5, then the ones place is 1.

$$404_{five}$$

$$\underline{-323_{five}}$$

$$1$$

Looking at the 5's place, notice that 0 is less than 2 and we have to borrow from the 52 's place. Recall, we are in base 5, so when we borrow, we still reduce 4 to 3, but we carry 5 since we are in base 5:

$$4^3 0^5 4_{five}$$

$$\underline{-3\ 2\ 3_{five}}$$

$$0\ \ 3\ \ 1_{five}$$

Now, we subtract as usual: 5-2 = 3. Since 3 is in base 5, then the 5's place is 3. Subtracting in the 5^2's place, we get 3-3=0; hence, the 5^2's place is 0.

Thus, $404_{five} - 323_{five} = 31_{five}$.

Conclusion

In this chapter, we have briefly sketched the development of numbers and our counting system, with the emphasis on the "brief" part. There are numerous sources of information and research that fill many volumes of books on this topic. Unfortunately, we cannot begin to come close to covering all of the information that is out there.

We have only scratched the surface of the wealth of research and information that exists on the development of numbers and counting throughout human history. It is important to note that the system that we use every day is a product of thousands of years of progress and development. It represents contributions by many civilizations and cultures. It does not come down to us from the sky, a gift from the gods. It is not the creation of a textbook publisher. It is indeed as human as we are, as is the rest of mathematics. Behind every symbol, formula, and rule there is a human face to be found, or at least sought.

Furthermore, we hope that you now have a basic appreciation for just how interesting and diverse number systems can get. Also, we're pretty sure that you have also begun to recognize that we take our own number system for granted so much that when we try to adapt to other systems or bases, we find ourselves truly having to concentrate and think about what is going on.

Try It Now Answers

1. 1+6×3+3×6+2×12 = 61 cats.

2. From left to right:

    Cord 1 = 2,162

    Cord 2 = 301

    Cord 3 = 0

    Cord 4 = 2,070

3. $41065_7 = 9994_{10}$

4. $143_{10} = 1033_5$

5. $21021_3 = 196_{10}$

6. $657_{10} = 22101_4$

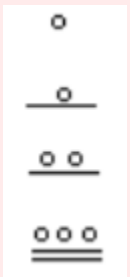
7. $8377_{10} = 20271_8$

8. $9352_{10} = 244402_5$

9. $1500_{10} = 2001120_3$

10. 1562

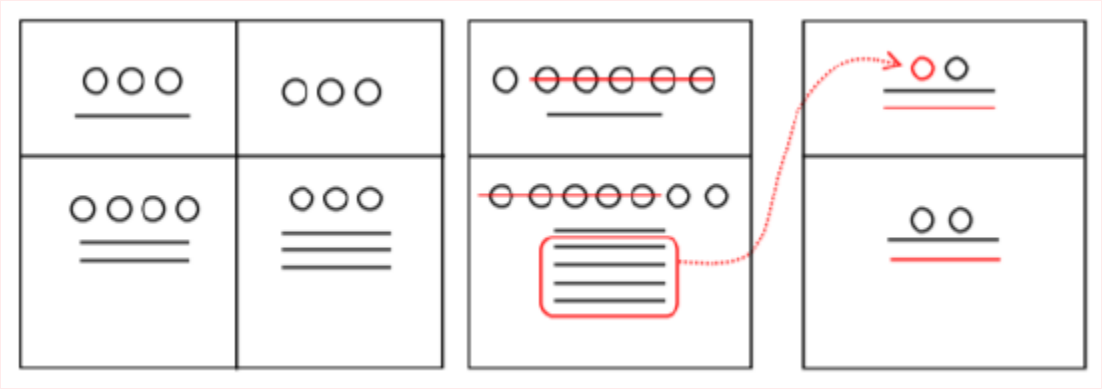
11. $10553_{10} = 1,6,7,13_{20}$



12. $5617_{10} = 14,0,17_{20}$. Note that there is a zero in the 20's place, so you'll need to use the appropriate zero symbol in between the ones and 400's places.



13. A sample solution is shown.



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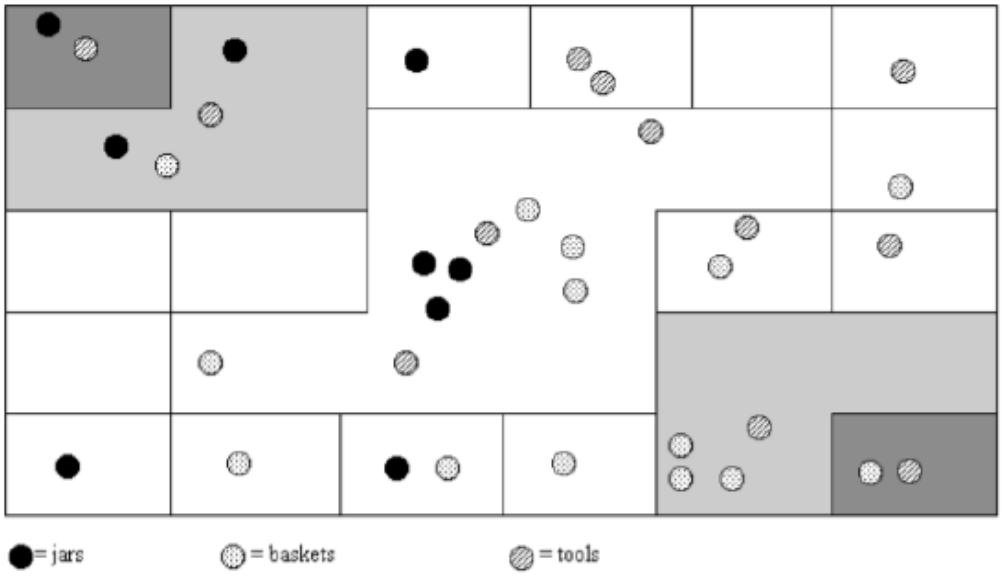
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2.7: Exercises

Skills

Counting Board and Quipu

1. In the following Peruvian counting board, determine how many of each item is represented. Please show all of your calculations along with some kind of explanation of how you got your answer. Note the key at the bottom of the drawing.



2. Draw a quipu with a main cord that has branches (H cords) that show each of the following numbers on them. (You should produce one drawing for this problem with the cord for part a on the left and moving to the right for parts b through d.)

a.232 b. 5065

c.23,451 d. 3002

Basic Base Conversions

3. 423 in base 5 to base 10 4. 3044 in base 5 to base 10

5. 387 in base 10 to base 5 6. 2546 in base 10 to base 5

7. 110101 in base 2 to base 10 8. 110100001 in base 2 to base 10

9. 100 in base 10 to base 2 10. 2933 in base 10 to base 2

11. Convert 653 in base 7 to base 10. 12. Convert 653 in base 10 to base 7

13. 3412 in base 5 to base 2 14. 10011011 in base 2 to base 5

(Hint: convert first to base 10 then to the final desired base)

The Caidoz System

Suppose you were to discover an ancient base–12 system made up twelve symbols. Let's call this base system the Caidoz system. Here are the symbols for each of the numbers 0 through 11:

0 = ♈	6 = ♎
1 = ♉	7 = ♏
2 = ♊	8 = ♐
3 = ♋	9 = ♑
4 = ♌	10 = ♒
5 = ♍	11 = ♓

Convert each of the following numbers in Caidoz to base 10.

15. ♏♉♒ 16. ♐♋♈♓

17. ♎♌♊ 18. ♍♉♒♈

Convert the following base 10 numbers to Caidoz, using the symbols shown above.

19. 175 20. 3030

21. 10,000 22. 5507

Mayan Conversions

Convert the following numbers to Mayan notation. Show your calculations used to get your answers.

23. 135 24. 234

25. 360 26. 1,215

27. 10,500 28. 1,100,000

Convert the following Mayan numbers to decimal (base-10) numbers. Show all calculations.

29. 30. 31. 32.



James Bidwell has suggested that Mayan addition was done by "simply combining bars and dots and carrying to the next higher place." He goes on to say, "After the combining of dots and bars, the second step is to exchange every five dots for one bar in the same position." After converting the following base 10 numbers into vertical Maya notation (in base 20, of course), perform the indicated addition:

33. 32 + 11 34. 82 + 15

35. 35 + 148 36. 2412 + 5000

37. 450 + 844 38. 10,000 + 20,000

39. 4,500 + 3,500 40. 130,000 + 30,000

41. Use the fact that the Mayans had a base-20 number system to complete the following multiplication table. The table entries should be in Mayan notation. Remember: Their zero looked like this...  . *Xerox and then cut out the table below, fill it in, and paste it onto your homework assignment if you do not want to duplicate the table with a ruler.* (To think about but not write up: Bidwell claims that only these entries are needed for "Mayan multiplication." What does he mean?)

×							
							
							
							
							
							
							

Binary and Hexadecimal Conversions

*Modern computers operate in a world of "on" and "off" electronic switches, so use a **binary** counting system – base 2, consisting of only two digits: 0 and 1.*

Convert the following binary numbers to decimal (base–10) numbers.

42. 1001 43. 1101

44. 110010 45. 101110

Convert the following base-10 numbers to binary.

46. 7 47. 12

48. 36 49. 27

Four binary digits together can represent any base-10 number from 0 to 15. To create a more human-readable representation of binary-coded numbers, hexadecimal numbers, base 16, are commonly used. Instead of using the $8,13,12_{16}$ notation used earlier, the letter A is used to represent the digit 10, B for 11, up to F for 15, so $8,13,12_{16}$ would be written as 8DC.

Convert the following hexadecimal numbers to decimal (base–10) numbers.

50. C3 51. 4D

52. 3A6 53. BC2

Convert the following base-10 numbers to hexadecimal.

54. 152 55. 176

56. 2034 57. 8263

Add or subtract in base 5 or in base 2.

58. $110_{five} - 34_{five}$ 59. $314_{five} - 32_{five}$

60. $30_{five} - 12_{five}$ 61. $110_{two} + 10_{two}$

62. $10011_{two} + 10000_{two}$ 63. $20_{five} + 42_{five}$

64. $14_{five} + 24_{five}$ 65. $44_{five} - 34_{five}$

Exploration

66. What are the advantages and disadvantages of bases other than ten.

67. Supposed you are charged with creating a base-15 number system. What symbols would you use for your system and why? Explain with at least two specific examples how you would convert between your base–15 system and the decimal system.

68. Describe an interesting aspect of Mayan civilization that we did not discuss in class. Your findings must come from some source such as an encyclopedia article, or internet site and you must provide reference(s) of the materials you used (either the publishing information or Internet address).

69. For a Papuan tribe in southeast New Guinea, it was necessary to translate the bible passage John 5:5 “And a certain man was there, which had an infirmity 30 and 8 years” into “A man lay ill one man, both hands, five and three years.” Based on your own understanding of bases systems (and some common sense), furnish an explanation of the translation. Please use complete sentences to do so. (Hint: To do this problem, I am asking you to think about how base systems work, where they come from, and how they are used. You won’t necessarily find an “answer” in readings or such…you’ll have to think it through and come up with a reasonable response. Just make sure that you clearly explain why the passage was translated the way that it was.)

70. The Mayan calendar was largely discussed leading up to December 2012. Research how the Mayan calendar works, and how the counts are related to the number based they use.

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3: THE REAL NUMBER SYSTEM

3.1: Introduction to the Real Number System
3.2: Whole and Natural Numbers
3.3: Integer Numbers
3.4: Rational Numbers
3.5: Irrational Numbers
3.6: Real numbers and the Number Line
3.7: Absolute Value
3.8: Adding and Subtracting Integers
3.9: Multiplying and Dividing Integers
3.10: Fractions
3.11: Review of Decimals and Percents
3.12: Exponents and Square Roots
3.13: Order of Operations
3.14: Scientific Notation
3.15: Approximation and Estimation
3.16: Real Numbers - Algebra Essentials
3.17: Challenges and Misconceptions
3.18: Summary and Review

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3.1: Introduction to the Real Number System

Understanding the Real Numbers

For elementary school teachers, it's important to introduce our students to the real number system, which forms the backbone of all mathematical learning. The real number system includes every number that can be placed on a number line, covering everything from simple counting numbers to more complex decimals and fractions. By helping our students understand this system, we lay a strong foundation for their future math studies.

Introducing the real number system at an early age helps students develop a comprehensive understanding of numbers and their relationships. This system forms the basis for all mathematical operations and concepts they will encounter in their academic journey. By familiarizing students with the different types of numbers, we can enhance their numerical literacy and prepare them for more advanced topics in mathematics.

Components of the Real Number System

1. **Natural Numbers:** These are the positive integers used for counting, such as 1, 2, 3, and so on.
2. **Whole Numbers:** This set includes all natural numbers plus zero (0, 1, 2, 3, ...).
3. **Integers:** These are whole numbers and their negatives (..., -3, -2, -1, 0, 1, 2, 3, ...).
4. **Rational Numbers:** These numbers can be expressed as a fraction of two integers, like 1/2, 3/4, or -5/6. Rational numbers include whole numbers, fractions, and both repeating and terminating decimals.
5. **Irrational Numbers:** These cannot be expressed as a simple fraction. Their decimal forms are non-terminating and non-repeating, such as $\sqrt{7}$ or π.

Why Teach the Real Number System?

Teaching the real number system is crucial because it helps students understand a variety of math concepts and operations. It enables them to perform arithmetic, grasp geometric principles, and eventually delve into more advanced areas like algebra and calculus. Familiarity with the real number system also enhances their problem-solving abilities and prepares them for real-life applications of math.

Applications in Everyday Life

Understanding the real number system also has practical applications in everyday life. Whether it's managing money, measuring objects, or analyzing data, real numbers are used in a variety of real-world situations. By teaching students about these numbers, we help them develop critical life skills that they will use beyond the classroom.

Real numbers are a part of many daily activities:

- **Finance:** Understanding money, making purchases, budgeting, and calculating interest.
- **Measurement:** Measuring length, area, volume, and other quantities.
- **Science:** Describing quantities such as speed, temperature, and energy.
- **Technology:** Using numbers in programming, algorithms, and data analysis.

Goals for Students

By learning about the real number system, our students will:

1. **Perform Arithmetic Operations:** Add, subtract, multiply, and divide various types of real numbers with confidence.
2. **Understand Number Properties:** Apply properties of real numbers, such as commutativity, associativity, and distributivity, to solve problems.
3. **Solve Real-World Problems:** Use real numbers to model and solve everyday problems.
4. **Prepare for Advanced Math:** Build a solid foundation for future studies in algebra, geometry, and beyond.

Introducing the real number system to our students is an essential part of their math education. By understanding the different types of real numbers and how to work with them, students will be well-prepared for more complex mathematical concepts and practical applications. This chapter will guide you through teaching these concepts effectively, ensuring that your students develop a comprehensive understanding of the real number system.

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3.2: Whole and Natural Numbers

Understanding Whole and Natural Numbers

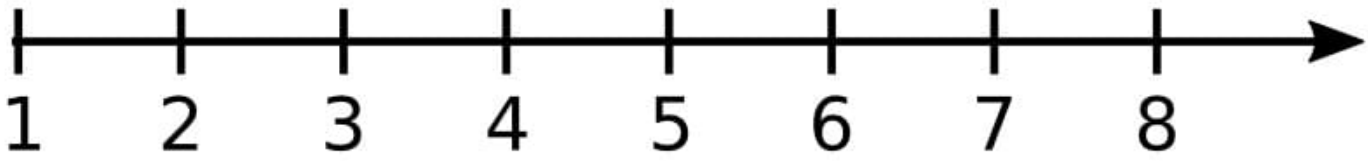
Mathematics is all around us, from the time we spend budgeting our expenses to the measurements we make in cooking. A strong foundation in basic number concepts, such as whole and natural numbers, is essential for more complex mathematical thinking. This section will refresh your understanding of these fundamental concepts, explore their properties, and demonstrate their real-world applications.

What Are Natural Numbers?

Definition: Natural numbers, (we use the symbol $\mathbb{N}$ to represent the natural numbers), are the set of positive numbers used for counting and ordering. They start at 1 and continue infinitely. This set of numbers is the most basic form of numbers that we use daily. **Natural Numbers:** 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ...

Visual Representation:

Imagine a number line that begins at 1 and continues to the right indefinitely. This number line represents natural numbers. They are sequential and used to count objects, rank items, and perform basic arithmetic.



Examples of using Natural Numbers:

- **Counting Items:** You have 5 apples.
- **Ordering:** Your book is number 7 on the list.

Important Note:

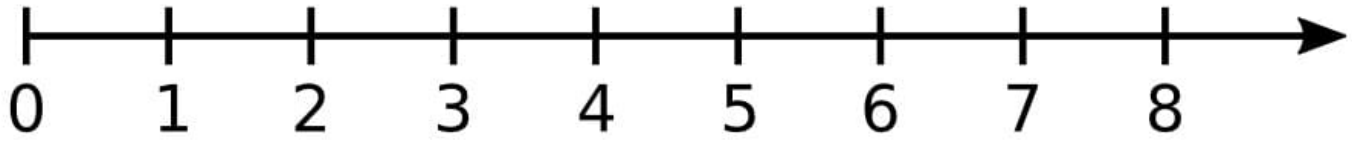
Natural numbers do not include zero or negative numbers. They are always positive integers starting from 1.

What Are Whole Numbers?

Definition: Whole numbers, (we use the symbol $\mathbb{N}_0$ or $\mathbb{W}$ to represent whole numbers), include all natural numbers plus zero. This set represents the numbers we use for counting as well as indicating the absence of quantity. **Whole Numbers:** 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ...

Visual Representation:

Consider a number line that starts at 0 and continues to the right indefinitely. This line represents whole numbers, showing that they include zero and extend infinitely.



Examples of Whole Numbers:

- **Count of Objects:** You have 3 books on the table.
- **Zero Scenario:** There are 0 apples in the basket.

Important Note:

Whole numbers include zero, which signifies a null quantity, while natural numbers do not include zero.

Comparing Natural and Whole Numbers

Key Differences:

Feature	Natural Numbers	Whole Numbers
Includes Zero	No	Yes
Starting Point	1	0
Examples	1, 2, 3, 4, 5	0, 1, 2, 3, 4, 5
Application	Counting, Ordering	Counting, Ordering, Indicating absence

Exercise 3.2.1

Determine which of the following numbers are natural numbers and which are whole numbers:

$7, 0, -3, 12, \frac{3}{4}$

Answer available at the end of this section.

Exercise 3.2.2

Determine which of the following numbers are natural numbers and which are whole numbers:

$\frac{1}{3}, -15, -\frac{4}{9}, 0.1, 20, 1, 783, -4.65$

Answer available at the end of this section.

Comparison of Properties

Whole numbers and natural numbers are foundational concepts in mathematics. Here's a comparison of their properties to help you understand their differences and similarities:

Property	Natural Numbers	Whole Numbers
Starting Point	1	0
Zero	Not included	Included
Infinite	Yes	Yes
Used For	Counting and ordering	Counting, ordering, and representing quantities (including none)
Includes Negative Numbers	No	No
Includes Fractions/Decimals	No	No

What does it mean to be 'Closed'

A set is **closed** under an operation if applying that operation to any elements of the set results in another element that is also in the set. This means if you apply certain operations on natural numbers the result will also be a natural number. Likewise, if you perform certain operations on whole numbers the result will also be a whole number.

Property	Natural Numbers	Whole Numbers
Addition	Closed under addition	Closed under addition
Subtraction	Not closed	Not closed
Multiplication	Closed under multiplication	Closed under multiplication
Division	Not closed	Not closed
Commutative Property	Yes (for addition and multiplication)	Yes (for addition and multiplication)
Associative Property	Yes (for addition and multiplication)	Yes (for addition and multiplication)
Distributive Property	Yes	Yes

The Closure Property

The **closure property** for natural numbers means that if you take any two natural numbers and perform a specific mathematical operation on them, the result will always be a natural number. This only happens with addition and multiplication for natural numbers. (This also applies to whole numbers.)

Closure Property of Addition

Addition: Adding two natural or whole numbers results in a larger natural or whole number.

Example for Natural Numbers: $3 + 4 = 7$

Example for Whole Numbers: $0 + 6 = 6$

Closure Property of Multiplication

Multiplication: Multiplying two natural or whole numbers always results in a natural or whole number.

Example for Natural Numbers: $3 \times 4 = 12$, the result will always be a larger natural number than the numbers you are multiplying

Example for Whole Numbers: $0 \times 12 = 0$, the result will always be zero or a larger natural number than the numbers you are multiplying

The Natural Numbers are NOT closed on subtraction or division

You might find some cases where the result is a natural or whole number but to be 'closed' it must result in a natural or

whole number in all cases.

Subtraction: Subtracting a natural number from a natural number may result in a number that is not always natural.

Example for Natural Numbers: $10 - 7 = 3$, is a natural number BUT $3 - 12 = -9$, is not a natural number, therefore, natural numbers are not closed on subtraction

Example for Whole Numbers: $13 - 0 = 13$, which is in the set of whole numbers BUT $0 - 8 = -8$, is not in the set of whole numbers, therefore, whole numbers are not closed on subtraction

Division: Dividing two natural or whole numbers can result in a fraction or decimal.

Example for Natural Numbers: $9 \div 12 = 0.75$, is not a natural number

Example for Whole Numbers: $6 \div 0 = undefined$, not even a number

The Commutative Property

The **commutative property** states that the order in which two numbers are added or multiplied does not change the result.

Commutative Property of Addition: $a + b = b + a$

Example: $7 + 13 = 13 + 7$, both equal 20

Commutative Property of Multiplication: $a \times b = b \times a$

Example: $5 \times 6 = 6 \times 5$. both equal 30

The Natural Numbers are NOT commutative on subtraction or division

Subtraction: $a - b \neq b - a$

Example: $5 - 3 \neq 3 - 5$

Division: $a \div b \neq b \div a$

Example: $10 \div 2 \neq 2 \div 10$

Exercise 3.2.3

Show how using the commutative property can be helpful with adding this list of numbers

$3 + 15 + 4 + 17 + 20 + 19 + 5 + 1 + 16$

Answer available at the end of this section.

Exercise 3.2.4

Show how using the commutative property can be helpful with adding this list of numbers.

$200 + 4 + 600 + 3 + 5 + 100 + 6$

Answer available at the end of this section.

The Associative Property

The **associative property** is a key mathematical concept that shows how the grouping of numbers in addition and multiplication does not affect the result of these operations.

Associative Property of Addition: $a + (b + c) = (a + b) + c$

Example:
$2 + (5 + 9) = (2 + 5) + 9$
$2 + 14 = 7 + 9$
$16 = 16$

Associative Property of Multiplication: $a \times (b \times c) = (a \times b) \times c$

Example:
$4 \times (6 \times 2) = (4 \times 6) \times 2$
$4 \times 12 = 24 \times 2$
$48 = 48$

Associative Property does not apply to subtraction and division

Example:
$2 - (5 - 9) = (2 - 5) - 9$
$2 - (-4) = (-3) - 9$
$2 + 4 = -3 - 9$
$6 \neq -12$

Example:
$2 \div (10 \div 3) = (2 \div 10) \div 3$
$2 \div \frac{10}{3} = \frac{2}{10} \div 3$
$2 \times \frac{3}{10} = \frac{2}{10} \times \frac{1}{3}$
$\frac{6}{10} \neq \frac{2}{30}$
$\frac{3}{5} \neq \frac{1}{15}$

Exercise 3.2.5

Using the associative property show an equivalent expression for $3 \times (4 \times 11)$.

Answer available at the end of this section.

Exercise 3.2.6

Using the associative property show an equivalent expression for $(20 + 94) + 356$.

Answer available at the end of this section.

The Distributive Property

The **distributive property** states that for any numbers a, b, and c:

$$a \times (b + c) = (a \times b) + (a \times c)$$

This means that when you multiply a number by a sum, you can distribute the multiplication to each term in the sum and then add the results.

Similarly, it applies to subtraction:

$a \times (b - c) = (a \times b) - (a \times c)$

This means that when you multiply a number by a difference, you can distribute the multiplication to each term in the difference and then subtract the results.

Exercise 3.2.7

Using the distribution property show an equivalent expression for $6 \times (7 + 3)$.

Answer available at the end of this section.

Exercise 3.2.8

Using the distribution property show an equivalent expression for $(10 \times 15) + (10 \times 9)$

Answer available at the end of this section.

Why is it Helpful to Know About Operations Being 'Closed'

Understanding that natural numbers are closed under addition and multiplication is fundamental for both practical arithmetic and advanced mathematical concepts. This closure property simplifies problem-solving because you can confidently perform these operations knowing the results will always be natural numbers. For example, adding 3 and 4 gives you 7, a natural number, and multiplying 2 and 5 gives you 10, another natural number. This reliability supports the development of arithmetic algorithms, such as those for addition and multiplication, ensuring that calculations are consistent and accurate. Furthermore, this concept forms the foundation for learning more complex topics in mathematics, such as algebraic structures and proofs, where closure properties are crucial for understanding and establishing theorems. In practical applications, knowing that natural numbers are closed under these operations helps in creating reliable mathematical models, like budgeting, where calculations must stay within the set of natural numbers. Additionally, this knowledge underpins mathematical proofs, helping you to construct and understand arguments about number properties. Thus, the closure property not only makes arithmetic straightforward but also prepares you for deeper mathematical exploration and practical problem-solving.

Understanding the concept of closure in mathematics is essential for elementary school teachers for several reasons.

1. Building a Strong Mathematical Foundation

Understanding closure helps elementary school teachers lay a solid foundation for their students' future math education. When teachers grasp how basic operations like addition and multiplication work within specific sets of numbers, they can more effectively teach these concepts to young learners. For example, knowing that addition is closed for whole numbers allows teachers to confidently explain to students that adding two whole numbers always results in another whole number. This foundational understanding is crucial for teaching arithmetic, which is the basis for more advanced math concepts.

Example: If a teacher understands that whole numbers are closed under addition, they can confidently explain to students that adding two whole numbers, such as 3 and 4, will always give another whole number, such as 7. This concept helps students grasp that the results of their arithmetic operations will remain within the set of whole numbers.

2. Developing Effective Teaching Strategies

Teachers who understand closure can develop more effective teaching strategies and lesson plans. By recognizing which operations are closed for different sets of numbers, teachers can create appropriate exercises and problems for their students. For instance, if a teacher knows that subtraction is not closed for whole numbers, they can design activities to help students explore what happens when subtraction results in a negative number and explain that the set of whole numbers does not include negative results.

Example: A teacher might use a visual aid to show that subtracting 3 from 2 results in -1, which is not a whole number. This helps students understand that not all operations will always stay within the same set of numbers.

3. Fostering Critical Thinking Skills

Teaching the concept of closure encourages students to think critically about numbers and operations. When students understand that operations can be performed within a set but may not always stay within that set, they learn to question and explore mathematical properties. This critical thinking is a key component of problem-solving skills and helps students develop a deeper understanding of mathematics.

Example: By explaining that multiplication is closed for natural numbers, teachers can encourage students to explore and predict outcomes for various multiplication problems, which fosters critical thinking and problem-solving abilities.

4. Supporting Effective Problem-Solving Techniques

An understanding of closure equips teachers with the knowledge to help students solve problems correctly. For example, when students learn that the set of integers is closed under addition and multiplication, teachers can guide students in solving problems involving these operations, knowing that their answers will always be integers.

Example: A teacher can show students that adding or multiplying any two integers will always result in another integer, which helps students understand and solve problems involving these operations.

5. Bridging to More Advanced Concepts

Early exposure to the idea of closure prepares students for more advanced mathematical concepts in the future. When teachers introduce these concepts early on, they help students develop a mindset for exploring and understanding more complex mathematical ideas as they advance in their education.

Example: By demonstrating that addition is closed for whole numbers and introducing similar concepts, teachers help students build a framework for learning about algebraic structures and operations in higher grades.

Imagine teaching a child to play a game with rules. If you know the rules work the same way every time (like always getting a red card in the game means you stay in the game), you can teach the child the rules confidently and help them understand how the game works. Similarly, understanding closure helps teachers explain math rules clearly and consistently, which is crucial for effective teaching and learning.

For elementary school teachers, understanding the concept of closure is crucial because it underpins the way they teach basic arithmetic operations and mathematical properties. It helps them create effective lesson plans, support students in developing critical thinking skills, and lay the groundwork for future math concepts. By grasping how and why operations work within certain sets, teachers can provide clearer explanations and better prepare students for more advanced mathematics.

Citations on Attributions

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Answers

Exercise $3.2.1$

Natural Numbers: $7, 12$; Whole Numbers: $0, 7, 12$

Exercise $3.2.2$

Natural Numbers: $20, 1, 783$; Whole Numbers: $20, 1, 783$

Exercise $3.2.3$

Since the commutative property allows us to reorder numbers, we could reorder these in such a way that each pair of numbers adds to 20 then it would be easier to add several 20's together

$3 + 15 + 4 + 17 + 20 + 19 + 5 + 1 + 16 = 3 + 17 + 15 + 5 + 4 + 16 + 20 + 19 + 1$

So, $3 + 17 + 15 + 5 + 4 + 16 + 20 + 19 + 1 = 20 + 20 + 20 + 20 + 20 = 100$

Exercise $3.2.4$

Since the commutative property allows us to reorder numbers, we could reorder these in such a way that the smaller numbers are first and the larger numbers are

last

$200 + 4 + 600 + 3 + 5 + 100 + 6 = 3 + 4 + 5 + 6 + 200 + 600 + 100$

So, $3 + 4 + 5 + 6 + 200 + 600 + 100 = 7 + 11 + 800 + 100 = 18 + 900 = 918$

Exercise 3.2.5

Since the associative property (for multiplication) is $a \times (b \times c) = (a \times b) \times c$, let $a = 3$, $b = 4$, and $c = 11$, then the equivalent expression for $3 \times (4 \times 11)$ is $(3 \times 4) \times 11$.

Proof:
$3 \times (4 \times 11) = 3 \times 44 = 132$
$(3 \times 4) \times 11 = 12 \times 11 = 132$

Exercise 3.2.6

Since the associative property (for addition) is $a + (b + c) = (a + b) + c$, let $a = 20$, $b = 94$, and $c = 356$, then the equivalent expression for $(20 + 94) + 356$ is \20 + (94 + 356)\).

Proof:
$(20 + 94) + 356 = 114 + 356 = 470$
$20 + (94 + 356) = 20 + 450 = 470$

Exercise 3.2.7

Since the distributive property is $a \times (b + c) = (a \times b) + (a \times c)$, let $a = 6$, $b = 7$, and $c = 3$, then the equivalent expression for $6 \times (7 + 3)$ is $(6 \times 7) + (6 \times 3)$

Proof:
$6 \times (7 + 3) = 6 \times 10 = 60$
$(6 \times 7) + (6 \times 3) = 42 + 18 = 60$

Exercise 3.2.8

Since the distributive property is $(a \times b) + (a \times c) = a \times (b + c)$, let $a = 10$, $b = 15$, and $c = 9$, then the equivalent expression for $(10 \times 15) + (10 \times 9)$ is $10 \times (15 + 9)$

Proof:
$(10 \times 15) + (10 \times 9) = 150 + 90 = 240$
$10 \times (15 + 9) = 10 \times 24 = 240$

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3.3: Integer Numbers

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract integers
- Multiply and divide integers
- Simplify expressions with absolute value

Natural Numbers and Integers

Let's begin by reviewing some numbers. We are all familiar with the counting numbers, or as mathematicians say, natural numbers $\mathbb{N}$. These are $0, 1, 2, \ldots, 51, \ldots, 1000000, \ldots$

We can see these numbers on what is called a number line. Here is the number line with the natural numbers going from 0 to 5. The numbers are ascending going from left to right and descending from right to left. Larger numbers are to the right and smaller numbers are to the left.



We can add them together to get another natural number.

Example 3.3.1

Compute the following:

a. 1+2=
b. 3+5=
c. 43 + 27=
d. 900+716=

Answer available at the end of this section.

We say the result of addition is the sum For example, the sum of 1 and 2 is 3. (The sum is 3, not 1 or 2).

We can also come up with numbers that add up to a certain number.

Example 3.3.2

Come up with two natural numbers whose sum is 7.

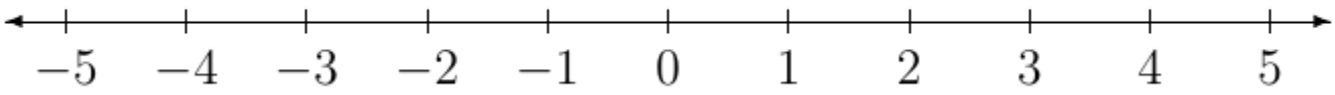
Possible answers: available at the end of this section.

What if we want to subtract? Sometimes we run into a new kind of number. For example 5-7=-2. This is a negative number and they were invented to keep track of debts. For example if you have 5 dollars and you need to pay me 7 dollars then you have -2

dollars, that is, you are 2 dollars in debt to me, the book.
You can actually think of subtraction as adding a negative, e.g, 5+(-7)=-2. So you really never lose money, you just get a negative money!
We can expand our view of numbers to include negative numbers to define the integers, $\mathbb{Z}$, by $\ldots, -1000000, \ldots, -51, \ldots, -2, -1, 0, 1, 2, \ldots, 51 \ldots, 1000000, \ldots$. That is, the integers are the negative AND positive versions of the natural numbers combined.

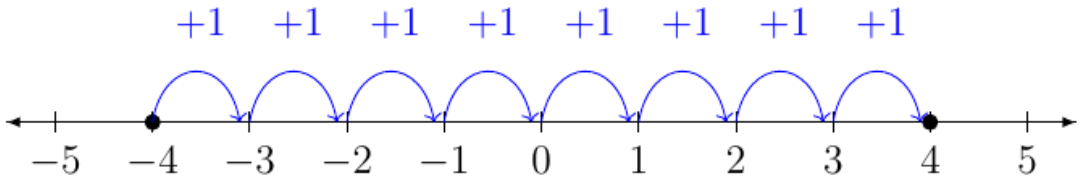
We also say that the result of subtraction is the difference. For example, the difference between 5 and 2 is 3. (The difference is 3, not 5 or 2).

Let's expand the number line to include the negatives. Again, larger numbers are to the right and smaller numbers are to the left.



Addition and subtraction may be easier to see with a number line. On a number line we go to the right for adding a positive number and to when adding a negative (subtraction).

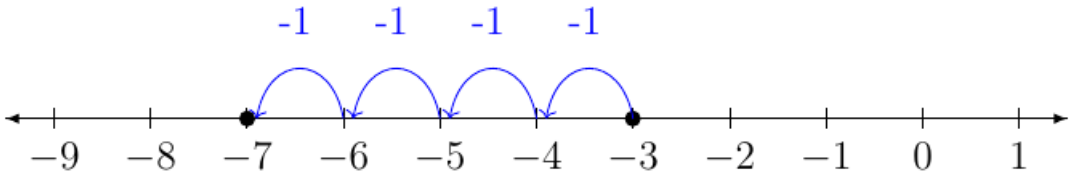
Consider -4+8=4. On a number line, this looks like



We start at -4 and then move 8 numbers to the right (because we are adding 8), landing at 4.

We can do the same for subtraction.

For example, let's do -3-4=-7



Now we start at -3 and move 4 numbers to the left (because we are subtracting 4), landing at -7.

With this in mind, let's try some examples.

Example 3.3.3

Compute the following:

1. 5-2=
2. 7-5=
3. 5-10=
4. -10+5=
5. -7+4=

Answer available at the end of this section.

We see that there is a relationship between addition and subtraction in the sense that one "undoes" the other. For example

$$\begin{aligned} 1+2&=3 \\ 1&=3-2 \end{aligned} \tag{3.3.1}$$

We see that the action of adding 2 to 1 to get 3 can be "undone" by subtracting 2 from 3.

Multiply and Divide Integers

Often in math and life, we would like to add quickly. Say you had 4 boxes with 3 phones in each box and I ask "how many phones do you have"? You could use addition with 3+3+3+3=12 or you could use multiplication with $3 \cdot 4 = 12$. So one could think of multiplication as adding the same number to itself in a succinct way. Multiplication is very useful when numbers get too inconvenient to add. For example, let's say you have 100 boxes now. It would be difficult to add up all those 3's but with multiplication we know that $3 \cdot 100 = 300$. We call the result of multiplication, the "product". For example, the product of 3 and 100 is 300. (300 is the product, not 3 or 100).

Example 3.3.4

1. $1 \cdot 2$
2. $5 \cdot 6$
3. $3 \cdot 9$
4. $12 \cdot 54$
5. Come up with two integers whose product is 6.

Answer available at the end of this section.

What if we want to do multiplication with negative numbers? An intuitive way of understanding multiplication with negative numbers is to consider debts again. Say you owe 3 dollars to 4 people, how much are you in debt? The answer is 12 dollars in debt. In math terms, this is $4 \cdot (-3) = -12$. So a positive number multiplied by a negative number is a negative number. (A negative number multiplied by a positive number is a negative number).

What about a negative number multiplied by a negative number? To understand this, let's consider an "I owe you" note. This note is a physical interpretation of debt or "negative money". If Tom has a note that reads "Tom owes (your name) 3 dollars" this means that Tom is in debt to you by the amount of 3 dollars. Ok, now let's say that you needs to give Tom 4 copies of the note above, we can say that you **owe** Tom those notes. How much money would you have if everyone paid off their debt? Well, each note reads "Tom owes (your name) 3 dollars" and you gives Tom 4 of those notes, then you will have 12 dollars once the debt is paid off. In math terms we have -4 (notes that you owes Tom) $\cdot -3$ (The amount of negative money each note carries)=12. That is to say "if you owe someone negative money then you are owed positive money". So a negative times a negative is a positive!

This also answers the question about why 4-(-3)=7. Since you can think of the subtraction of negative 3 as giving one of the notes above.

Division is a way of...dividing up a number into another number. To get an intuitive understanding let's consider some pizza. Let's say that your friends eat 2 slices of pizza each and you have a pizza with 8 slices. How many of your friends can you feed with that pizza? This question is asking to divide 8 (slices of pizza) by 2 (the amount of slices each friend will eat). So $8 \div 2 = 4$, you will be able to feed 4 friends with 1 pizza. We call the result of division, the "quotient". For example, the quotient of 8 by 2 is 4. (4 is the quotient, not 8 or 2). We see the same intuition with negative numbers works for division as well. That is, if you need to divide a debt of 6 dollars between 3 people evenly then each person would owe you 2 dollars: $\frac{-6}{-3} = 2$. Same intuition holds if you need to pay off a 6 dollar debt between 3 people evenly: $\frac{-6}{3} = -2$.

Example 3.3.5

1. $2 \div 2 =$
2. $30 \div 6 =$
3. $-27 \div 9 =$
4. $-16 \div -8 =$
5. $648 \div 54 =$

Answer available at the end of this section.

To recap

For multiplication and division of two signed numbers:

Rules for multiplication and division of two numbers

Same signs	Result
• Two positives	Positive
• Two negatives	Positive

If the signs are the same, the result is positive.

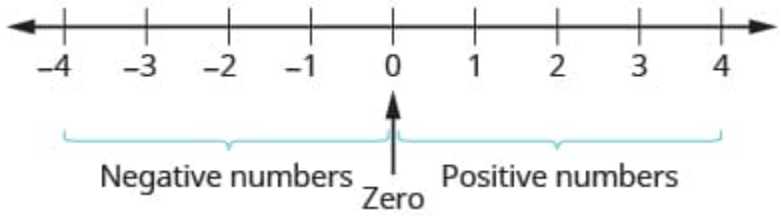
Rules for multiplication and division of two numbers

Different signs	Result
• Positive and negative	Negative
• Negative and positive	Negative

If the signs are different, the result is negative.

Sometimes we encounter a division that we can't get an integer out of. Say you have one slice of pizza and eat friend eats 2 slices. How many friends can you feed? You can feed half of a friend. In math this would look like this $\frac{1}{2}$. These are called fractions and we will be talking about them in the next section but first we should talk about when we do get an integer from division.

Simplify Expressions with Absolute Value



You may have noticed that, on the number line, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 2 and -2 are the same distance from zero, each one is called the **opposite** of the other. The opposite of 2 is -2, and the opposite of -2 is 2.

Definition: OPPOSITE

The **opposite** of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

This figure 3.3.2 illustrates the definition.

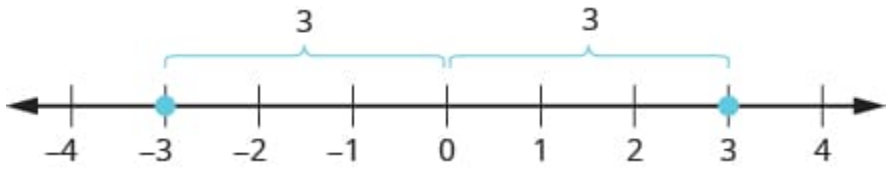


Figure 3.3.2***.*** *The opposite of 3 is* -3.

Definition: OPPOSITE NOTATION

$$-a \text{ means the opposite of the number } a \tag{3.3.2}$$
$$\text{The notation } -a \text{ is read as "the opposite of } a\text{."} \tag{3.3.3}$$

We saw that numbers such as 3 and –3 are opposites because they are the same distance from 0 on the number line. They are both three units from 0. The distance between 0 and any number on the number line is called the **absolute value** of that number.

Definition: ABSOLUTE VALUE

The **absolute value** of a number is its distance from 0 on the number line.

The absolute value of a number n is written as $|n|$ and $|n| \geq 0$ for all numbers.

Absolute values are always greater than or equal to zero.

For example,

$$-5 \text{ is 5 units away from 0, so } |-5| = 5. \tag{3.3.4}$$
$$5 \text{ is 5 units away from 0, so } |5| = 5. \tag{3.3.5}$$

Figure 3.3.3 illustrates this idea.



Figure 3.3.3: The numbers 5 and –5 are 5 units away from 0.

The absolute value of a number is never negative because distance cannot be negative. The only number with absolute value equal to zero is the number zero itself because the distance from 0 to 0 on the number line is zero units.

In the next example, we'll order expressions with absolute values.

EXAMPLE 3.3.6

Fill in $<$, $>$, or $=$ for each of the following pairs of numbers:

a. $|-5| \_ -|-5|$
b. $8 \_ -|-8|$
c. $-9 \_ -|-9|$
d. $-(-16) \_ |-16|$.

Answer available at the end of this section.

EXAMPLE 3.3.7

Fill in $<$, $>$, or $=$ for each of the following pairs of numbers:

a. $-9\_-|-9|$
b. $2\_-|-2|$
c. $-8\_|-8|$
d. $-(-9)\_|-9|$

Answer available at the end of this section.

Order of Addition and Multiplication

If we want to compute $2+3\cdot 2$ is this 10 or 8? It's ambiguous which operation we want to do first so mathematicians have some rules of which operations we do first. In order we do multiplication, division, addition, and finally subtraction. We will have exceptions like if the addition is inside parenthesis, like $(2+3)\cdot 2$, in this case, it would be 10. Expressions in parenthesis (or any grouping symbols) take priority.

We now add absolute value bars to our list of grouping symbols. When we use the order of operations, first we simplify inside the absolute value bars as much as possible, then we take the absolute value of the resulting number.

Definition: GROUPING SYMBOLS

Parentheses	()	Braces	{}
Brackets	[]	Absolute value	||

(3.3.6)

In the next example, we simplify the expressions inside absolute value bars first just like we do with parentheses.

EXAMPLE 3.3.8

Simplify: $24-|19-3(6-2)|$.

Answer available at the end of this section.

EXAMPLE 3.3.9

Simplify: $19-|11-4(3-1)|$.

Answer available at the end of this section.

EXAMPLE 3.3.10

Simplify: $9-|8-4(7-5)|$.

Answer available at the end of this section.

Key Concepts

- Numberlines help visualize addition and subtraction.
- The absolute value of a number is its distance from 0 on the number line.

The absolute value of a number *n* is written as $|n|$ and $|n| \geq 0$ for all numbers.

Absolute values are always greater than or equal to zero.

- $$\begin{array}{llll} \text{Parentheses} & () & \text{Braces} & \{\} \\ \text{Brackets} & [] & \text{Absolute value} & || \end{array} \tag{3.3.7}$$

- **Subtraction Property**
  $a - b = a + (-b)$
  Subtracting a number is the same as adding its negative.
- For multiplication and division of two signed numbers:

  Multiplication and division for two same signed numbers

Same signs	Result
• Two positives	Positive
• Two negatives	Positive

  If the signs are the same, the result is positive.

  Multiplication and division for two different signed numbers

Different signs	Result
• Positive and negative	Negative
• Negative and positive	Negative

  If the signs are different, the result is negative.
- **Multiplication by** -1

  $-1a = -a$

  Multiplying a number by -1 gives its opposite.

Glossary

absolute value
: The absolute value of a number is its distance from 0 on the number line.

integers
: The whole numbers and their opposites are called the integers.

negative numbers
: Numbers less than 0 are negative numbers.

Answers

Example 3.3.1

a. 1+2=3
b. 3+5=8
c. 43 + 27 = 70
d. 900+716=1616

Example 3.3.2

6 and 1
5 and 2
4 and 3
7 and 0
Any other combinations would be a repeat of those above.

Example 3.3.3

1. 5-2=3

2. 7-5=2
3. 5-10=-5
4. -10+5=-5
5. -7+4=-3

Example 3.3.4

1. 2
2. 30
3. 27
4. 648
5. An answer is $2 \cdot 3$

Example 3.3.5

1. 1
2. 5
3. -3
4. 2
5. 12

EXAMPLE 3.3.6

a. First, simplify each side:

$$|-5| = 5 \text{ and } -|-5| = -5$$

Then choose an inequality that makes the simplified statement true:

$$5 > -5$$

This is the inequality that makes the original statement true!

$$|-5| > -|-5|$$

b. First, simplify each side:

The left side is already simplified. The right side simplifies as $-|-8| = -8$

Then choose an inequality that makes the simplified statement true:

$$8 > -8$$

This is the inequality that makes the original statement true!

$$8 > -|-8|$$

c. First, simplify each side:

The left side is already simplified. The right side simplifies as $-|-9| = -9$

Then choose an inequality (or equality!) that makes the simplified statement true:

$$-9 = -9$$

This is the inequality or equality that makes the original statement true!

$$-9 = -|-9|$$

d. First, simplify each side:

$$-(-16) = 16 \text{ and } |-16| = 16$$

Then choose an inequality (or equality!) that makes the simplified statement true:

$$16 = 16$$

This is the inequality or equality that makes the original statement true!

$$-(-16) > |-16|$$

EXAMPLE 3.3.7

a. $>$
b. $>$
c. $<$
d. $=$

EXAMPLE 3.3.8

Work inside parentheses first -- subtract 2 from 6: $24 - |19 - 3(4)|$
Simplify the multiplication inside the absolute value next: $24 - |19 - 12|$
Simplify the inside of the absolute value bars -- subtract 12 from 19: $24 - |7|$
Take the absolute value: $24 - 7$
Subtract: 17

EXAMPLE $3.3.9$

16
Work inside parentheses first -- subtract 1 from 3: $19 - |11 - 4(2)|$
Simplify the multiplication inside the absolute value next: $19 - |11 - 8|$
Simplify the inside of the absolute value bars -- subtract 8 from 11: $19 - |3|$
Take the absolute value: $19 - 3$
Subtract: 16

EXAMPLE $3.3.10$

9

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3.4: Rational Numbers



Figure 3.4.1 Stock gains and losses are often represented as percentages.(credit: "stock market quotes in newspaper" by Andreas Poike/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

- Define and identify numbers that are rational.
- Simplify rational numbers and express in lowest terms.
- Add and subtract rational numbers.
- Convert between improper fractions and mixed numbers.
- Convert rational numbers between decimal and fraction form.
- Multiply and divide rational numbers.
- Apply the order of operations to rational numbers to simplify expressions.
- Apply density property of rational numbers.
- Solve problems involving rational numbers.
- Use fractions to convert between units.
- Define and apply percent.
- Solve problems using percent.

We are often presented with percentages or fractions to explain how much of a population has a certain feature. For example, the 6-year graduation rate of college students at public institutions is 57.6%, or 72/125. That fraction may be unsettling. But without the context, the percentage is hard to judge. So how does that compare to private institutions? There, the 6-year graduation rate is 65.4%, or 327/500. Comparing the percentages is straightforward, but the fractions are harder to interpret due to different denominators. For more context, historical data could be found. One study reported that the 6-year graduation rate in 1995 was 56.4%. Comparing that historical number to the recent 6-year graduation rate at public institutions of 57.6% shows that there hasn't been much change in that rate.

Definition: Rational Number

A **rational number** is any number that can be expressed as the ratio of two integers, where the denominator is not zero. In mathematical terms, a rational number is any number of the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. This includes integers (e.g., 5 can be written as $\frac{5}{1}$, fractions (e.g., $\frac{3}{4}$, and repeating or terminating decimals (e.g., 0.75 or 0.333...).

Forms of Rational Numbers

1. **Fraction Form:** A rational number can be written as a fraction, where the numerator and denominator are integers and the denominator is not zero.
   - Example: $\frac{6}{11}$, $-\frac{5}{7}$, $\frac{2}{1}$
2. **Decimal Form:** A rational number can be expressed as decimals, which can either terminate or repeat.
   - **Terminating Decimal:** A decimal that ends after a finite number of digits.
     - Example: 0.75 (which is $\frac{3}{4}$)
   - **Repeating Decimal:** A decimal where one or more digits repeat indefinitely.
     - Example: 0.333... (which is $\frac{1}{3}$)
3. **Integer Form:** An integer is a rational number because it can be expressed as a fraction with a denominator of 1.
   - Example: 5 (which can be written as $\frac{5}{1}$, -2 (which can be written as -$\frac{2}{1}$)
4. **Mixed Number Form:** A mixed number consists of a whole number and a proper fraction.
   - Example: $2\frac{1}{4}$ (which is $\frac{9}{4}$)
5. **Ratio Form:** Rational numbers can also be expressed as a ratio of two integers.
   - Example: 7:2 (which represents $\frac{7}{2}$)
6. **Percentage Form:** A rational number can be expressed as a percentage, which is essentially a fraction with a denominator of 100.
   - Example: 50% (which is $\frac{50}{100}$ or in reduced form $\frac{1}{2}$)
7. **Set Notation Form:** Rational numbers can be described using set notation, where the set contains all possible ratios of integers.
   - Example: { a/b | a, b ∈ ℤ, b ≠ 0 }

Summary of Forms

Form	Example	Representation
Fraction	`3/4`	Ratio of two integers
Terminating Decimal	`0.75`	Ends after a finite number of digits
Repeating Decimal	`0.333...`	Digits repeat indefinitely
Integer	`5`	Whole number (can be `5/1`)
Mixed Number	`2 1/4`	Whole number and a fraction
Ratio	`7:2`	Ratio of two integers
Percentage	`50%`	Fraction with a denominator of 100
Set Notation	`{ a/b | a, b ∈ ℤ, b ≠ 0 }`	Set of all ratios of integers

Defining and Identifying Numbers That Are Rational

A rational number (called rational since it is a ratio) is just a fraction where the numerator is an integer and the denominator is a

non-zero integer. As simple as that is, they can be represented in many ways. It should be noted here that any integer is a rational number. An integer, n, written as a fraction of two integers is $\frac{n}{1}$.

In its most basic representation, a rational number is an integer divided by a non-zero integer, such as $\frac{3}{12}$ [redacted](Figure 3.4.2). Similarly, if in a group of 20 people, 5 are wearing hats, then $\frac{5}{20}$ (Figure 3.4.3).

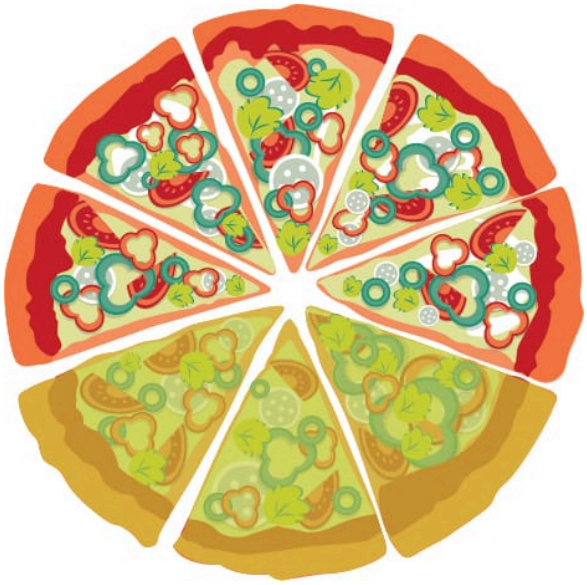


Figure 3.4.2 Pizza cut in 8 slices, with 3 slices highlighted



Figure 3.4.3 Group of 20 people, with 5 people wearing hats

Another representation of rational numbers is as a mixed number, such as $2\frac{5}{8}$ (Figure 3.4.4). This represents a whole number (2 in this case), plus a fraction (the $\frac{5}{8}$).



Figure 3.4.4 Two whole pizzas and one partial pizza

Rational numbers may also be expressed in decimal form; for instance, as 1.34. When 1.34 is written, the decimal part, 0.34, represents the fraction $\frac{34}{100}$, and the number 1.34 is equal to $1\frac{34}{100}$. However, not all decimal representations are rational numbers.

A number written in decimal form where there is a last decimal digit (after a given decimal digit, all following decimal digits are 0) is a **terminating decimal**, as in 1.34 above. Alternately, any decimal numeral that, after a finite number of decimal digits, has digits equal to 0 for all digits following the last non-zero digit.

All numbers that can be expressed as a terminating decimal are rational. This comes from what the decimal represents. The decimal part is the fraction of the decimal part divided by the appropriate power of 10. That power of 10 is the number of decimal digits present, as for 0.34, with two decimal digits, being equal to $\frac{34}{100}$.

Another form that is a rational number is a decimal that repeats a pattern, such as 67.1313... When a rational number is expressed in decimal form and the decimal is a repeated pattern, we use special notation to designate the part that repeats. For example, if we have the repeating decimal 4.3636..., we write this as $4.\overline{36}$. The bar over the 36 indicates that the 36 repeats forever.

Video 3.4.1

Introduction to Fractions

If the decimal representation of a number does not terminate or form a repeating decimal, that number is not a rational number.

Square Roots

One class of numbers that is not rational is the **square roots** of integers or rational numbers that are not **perfect squares**, such as $\sqrt{10}$ and $\sqrt{\frac{25}{6}}$ is the square root of the number a if $a = b^2$. The notation for this if $b = \sqrt{a}$, where the symbol, $sqrt$, is the square root sign. An integer perfect square is any integer that can be written as the square of another integer. A rational perfect square is any rational number that can be written as a fraction of two integers that are perfect squares.

Sometimes you may be able to identify a perfect square from memory. Another process that may be used is to factor the number into the product of an integer with itself. Or a calculator (such as Desmos) may be used to find the square root of the number. If the calculator yields an integer, the original number was a perfect square.

Tech Check 3.4.1

Using Desmos to Find the Square Root of a Number

When Desmos is used, there is a tab at the bottom of the screen that opens the keyboard for Desmos. The keyboard is shown below. On the keyboard (Figure 3.4.5) is the square root symbol $(\sqrt{})$. To find the square root of a number, click the square root key, and then type the number. Desmos will automatically display the value of the square root as you enter the number.

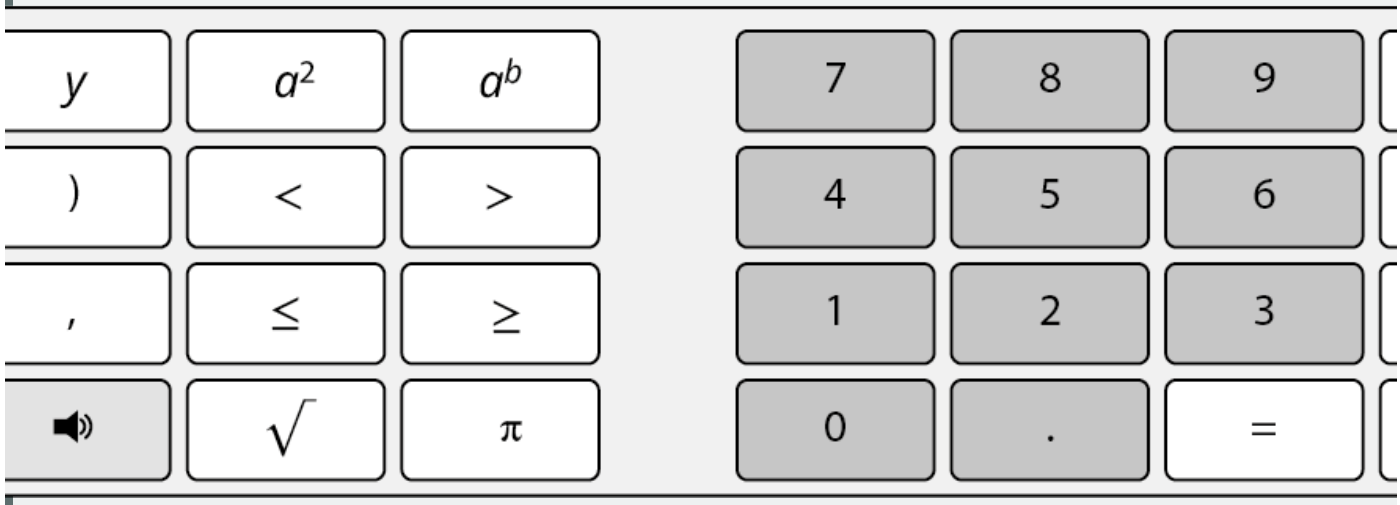


Figure 3.4.5 Desmos keyboard with square root key

Example 3.4.1

Identifying Perfect Squares

Which of the following are perfect squares?

1. 45
2. 144

Solution

1. We could attempt to find the perfect square by factoring. Writing all the factor pairs of 45 results in $1 \times 45, 3 \times 15$, and 5×9. None of the pairs is a square, so 45 is not a perfect square. Using a calculator to find the square root of 45, we obtain 6.708 (rounded to three decimal places). Since this was not an integer, the original number was not a perfect square.
2. We could attempt to find the perfect square by factoring. Writing all the factor pairs of 144 results in $1 \times 144, 2 \times 72, 3 \times 48, 6 \times 24, 8 \times 18$, and 12×12. Since the last pair is an integer multiplied by itself, 144 is a perfect square. Alternately, using Desmos to find the square root of 144, we obtain 12. Since the square root of 144 is an integer, 144 is a perfect square.

Your Turn 3.4.1

Determine if the following are perfect squares:

1. 94
2. 441

Example 3.4.2

Identifying Rational Numbers

Determine which of the following are rational numbers:

1. $\sqrt{73}$
2. 4.556
3. $3\frac{1}{5}$
4. $\frac{41}{17}$
5. $5.\overline{64}$

Solution

1. Since 73 is not a perfect square, its square root is not a rational number. This can also be seen when a calculator is used. Entering $\sqrt{73}$ into a calculator results in 8.544003745317 (and then more decimal values after that). There is no repeated pattern, so this is not a rational number.
2. Since 4.556 is a decimal that terminates, this is a rational number.
3. $3\frac{1}{5}$ is a mixed number, so it is a rational number.
4. $\frac{41}{17}$ is an integer divided by an integer, so it is a rational number.
5. $5.646464\ldots$ is a decimal that repeats a pattern, so it is a rational number.

Your Turn 3.4.2

Determine which of the following are rational numbers:

1. $\sqrt{13}$
2. $-13.\overline{21}$
3. $\frac{-48}{-16}$
4. $-4\frac{18}{19}$
5. 1.1131

Simplifying Rational Numbers and Expressing in Lowest Terms

A rational number is one way to express the division of two integers. As such, there may be multiple ways to express the same value with different rational numbers. For instance, $\frac{4}{5}$ and $\frac{12}{15}$ are the same value. If we enter them into a calculator, they both equal 0.8. Another way to understand this is to consider what it looks like in a figure when two fractions are equal.

In Figure 3.4.6, we see that $\frac{3}{5}$ of the rectangle and $\frac{9}{15}$ of the rectangle are equal areas.

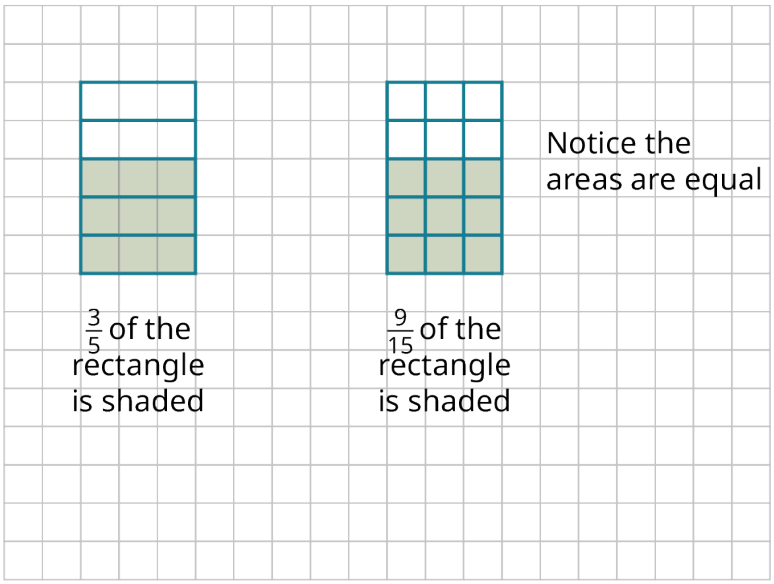


Figure 3.4.6 Two Rectangles with Equal Areas

They are the same proportion of the area of the rectangle. The left rectangle has 5 pieces, three of which are shaded. The right rectangle has 15 pieces, 9 of which are shaded. Each of the pieces of the left rectangle was divided equally into three pieces. This was a multiplication. The numerator describing the left rectangle was 3 but it becomes 3×3, or 9, as each piece was divided into three. Similarly, the denominator describing the left rectangle was 5, but became 5×3, or 15, as each piece was divided into 3. The fractions $\frac{3}{5}$ and $\frac{9}{15}$ are **equivalent** because they represent the same portion (often loosely referred to as equal).

This understanding of equivalent fractions is very useful for conceptualization, but it isn't practical, in general, for determining when two fractions are equivalent. Generally, to determine if the two fractions [redacted]$\frac{a}{b}$ and [redacted]$\frac{c}{d}$ are equivalent, we check to see that $a \times d = b \times c$. If those two products are equal, then the fractions are equal also.

Example 3.4.3

Determining If Two Fractions Are Equivalent

Determine if $\frac{12}{30}$ and $\frac{14}{35}$ are equivalent fractions.

Solution

Applying the definition, $a = 12, b = 30, c = 14$ and $d = 35$. So, $a \times d = 12 \times 35 = 420$. Also, $b \times c = 30 \times 14 = 420$. Since these values are equal, the fractions are equivalent.

Your Turn 3.4.3

Determine if $\frac{8}{14}$ and $\frac{12}{26}$ are equivalent fractions.

That $a \times d = b \times c$ indicates the fractions $\frac{a}{b}$ [redacted] and $\frac{c}{d}$ are equivalent is due to some algebra. One property of natural numbers, integers, and rational numbers (also irrational numbers) is that for any three numbers a, b, and c with $c \neq 0$, if $a = b$, then $a/c = b/c$. In other words, when two numbers are equal, then dividing both numbers by the same non-zero number, the two newly obtained numbers are also equal. We can apply that to $a \times d$ and $b \times c$, to show that $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if $a \times d = b \times c$.

If $a \times d = b \times c$, and $c \neq 0, d \neq 0$, we can divide both sides by c and obtain the following: $\frac{a \times d}{c} = \frac{b \times c}{c}$. We can divide out the c on the right-hand side of the equation, resulting in $\frac{a \times d}{c} = b$. Similarly, we can divide both sides of the equation by d and obtain the following: $\frac{a \times d}{d} = \frac{b \times c}{d}$. We can divide out d the on the left-hand side of the equation, resulting in $a = \frac{b \times c}{d}$. So, the rational numbers $\frac{a}{c}$ and $\frac{b}{d}$ are equivalent when $a \times d = b \times c$.

Video 3.4.2

Equivalent Fractions

Recall that a **common divisor** or **common factor** of a set of integers is one that divides all the numbers of the set of numbers being considered. In a fraction, when the numerator and denominator have a common divisor, that common divisor can be **divided out**. This is often called **canceling the common factors** or, more colloquially, as **canceling**.

To show this, consider the fraction $\frac{36}{63}$. The numerator and denominator have the common factor 3. We can rewrite the fraction as $\frac{36}{63} = \frac{12 \times 3}{21 \times 3}$. The common divisor 3 is then divided out, or canceled, and we can write the fraction as $\frac{12 \times \cancel{3}}{21 \times \cancel{3}} = \frac{12}{21}$. The 3s have been crossed out to indicate they have been divided out. The process of dividing out two factors is also referred to as **reducing the fraction**.

If the numerator and denominator have no common positive divisors other than 1, then the rational number is in **lowest terms**.

The process of dividing out common divisors of the numerator and denominator of a fraction is called **reducing the fraction**.

One way to reduce a fraction to lowest terms is to determine the GCD of the numerator and denominator and divide out the GCD. Another way is to divide out common divisors until the numerator and denominator have no more common factors.

Example 3.4.4

Reducing Fractions to Lowest Terms

Express the following rational numbers in lowest terms:

1. $\frac{36}{48}$
2. $\frac{100}{250}$
3. $\frac{51}{136}$

Solution

1. One process to reduce $\frac{36}{48}$ to lowest terms is to identify the GCD of 36 and 48 and divide out the GCD. The GCD of 36 and 48 is 12.

   Step 1: We can then rewrite the numerator and denominator by factoring 12 from both.

   $\frac{36}{48} = \frac{12\times 3}{12\times 4}$

   Step 2: We can now divide out the 12s from the numerator and denominator.

   $\frac{36}{48} = \frac{\cancel{12}\times 3}{\cancel{12}\times 4} = \frac{3}{4}$

   So, when $\frac{36}{48}$ is reduced to lowest terms, the result is $\frac{3}{4}$.

   Alternately, you could identify a common factor, divide out that common factor, and repeat the process until the remaining fraction is in lowest terms.

   Step 1: You may notice that 4 is a common factor of 36 and 48.

   Step 2: Divide out the 4, as in $\frac{36}{48} = \frac{4\times 9}{4\times 12} = \frac{\cancel{4}\times 9}{\cancel{4}\times 12} = \frac{9}{12}$.

   Step 3: Examining the 9 and 12, you identify 3 as a common factor and divide out the 3, as in $\frac{9}{12} = \frac{\cancel{3}\times 3}{\cancel{3}\times 4} = \frac{3}{4}$. The 3 and 4 have no common positive factors other than 1, so it is in lowest terms.

   So, when $\frac{36}{48}$ is reduced to lowest terms, the result is $\frac{3}{4}$.

2. **Step 1:** To reduce $\frac{100}{250}$ to lowest terms, identify the GCD of 100 and 250. This GCD is 50.

   Step 2: We can then rewrite the numerator and denominator by factoring 50 from both.

   $\frac{100}{250} = \frac{50\times 2}{50\times 5}$.

   Step 3: We can now divide out the 50s from the numerator and denominator.

   $\frac{100}{250} = \frac{\cancel{50}\times 2}{\cancel{50}\times 5} = \frac{2}{5}$

   So, when $\frac{100}{250}$ is reduced to lowest terms, the result is $\frac{2}{5}$.

3. **Step 1:** To reduce $\frac{51}{136}$ to lowest terms, identify the GCD of 51 and 136. This GCD is 17.

   Step 2: We can then rewrite the numerator and denominator by factoring 17 from both.

   $\frac{51}{136} = \frac{17\times 3}{17\times 8}$

   Step 3: We can now divide out the 17s from the numerator and denominator.

$\frac{51}{136} = \frac{\cancel{17}\times 3}{\cancel{17}\times 8} = \frac{3}{8}$

So, when $\frac{51}{136}$ is reduced to lowest terms, the result is $\frac{3}{8}$.

Your Turn 3.4.4

Express $\frac{252}{840}$ and $\frac{17}{51}$ in lowest terms.

Video 3.4.3

Reducing Fractions to Lowest Terms

Tech Check 3.4.2

Using Desmos to Find Lowest Terms

Desmos is a free online calculator. Desmos supports reducing fractions to lowest terms. When a fraction is entered, Desmos immediately calculates the decimal representation of the fraction. However, to the left of the fraction, there is a button that, when clicked, shows the fraction in reduced form.

Video 3.4.4

Using Desmos to Reduce a Fraction

Adding and Subtracting Rational Numbers

Adding or subtracting rational numbers can be done with a calculator, which often returns a decimal representation, or by finding a common denominator for the rational numbers being added or subtracted.

Tech Check 3.4.3

Using Desmos to Add Rational Numbers in Fractional Form

To create a fraction in Desmos, enter the numerator, then use the division key (/) on your keyboard, and then enter the denominator. The fraction is then entered. Then click the right arrow key to exit the denominator of the fraction. Next, enter the arithmetic operation (+ or –). Then enter the next fraction. The answer is displayed dynamically (calculates as you enter). To change the Desmos result from decimal form to fractional form, use the fraction button (Figure 3.26) on the left of the line that contains the calculation:



Figure 3.4.7 Fraction button on the Desmos keyboard

Example 3.4.5

Adding Rational Numbers Using Desmos

Calculate $\frac{23}{42} + \frac{9}{56}$ using Desmos.

Solution

Enter $\frac{23}{42} + \frac{9}{56}$ in Desmos. The result is displayed as 0.70833333333 (which is $0.708\overline{3}$). Clicking the fraction button to the left on the calculation line yields $\frac{17}{24}$.

Your Turn 3.4.5

Calculate $\frac{124}{297} + \frac{3}{125}$ in lowest terms.

Performing addition and subtraction without a calculator may be more involved. When the two rational numbers have a common denominator, then adding or subtracting the two numbers is straightforward. Add or subtract the numerators, and then place that value in the numerator and the common denominator in the denominator. Symbolically, we write this as $\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$ Figure 3.4.8 , which shows

$$\frac{3}{20} + \frac{4}{20} = \frac{7}{20} \tag{3.4.1}$$

.

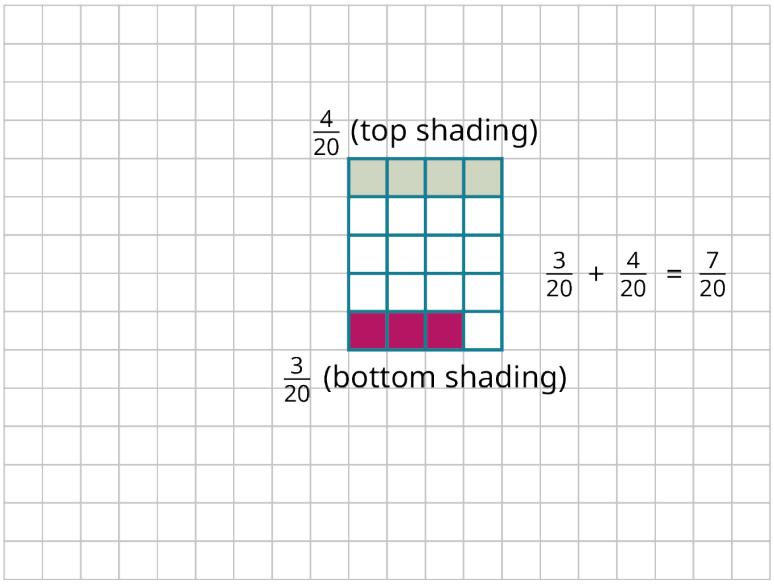


Figure 3.4.8 Partially Shaded Rectangle

It is customary to then write the result in lowest terms.

FORMULA

If c is a non-zero integer, then $\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$

Example 3.4.6

Adding Rational Numbers with the Same Denominator

Calculate $\frac{13}{28} + \frac{7}{28}$.

Solution

Since the rational numbers have the same denominator, we perform the addition of the numerators, $13 + 7$, and then place the result in the numerator and the common denominator, 28, in the denominator. $\frac{13}{28} + \frac{7}{28} = \frac{13+7}{28} = \frac{20}{28}$

Once we have that result, reduce to lowest terms, which gives $\frac{20}{28} = \frac{4\times5}{4\times7} = \frac{\cancel{4}\times5}{\cancel{4}\times7} = \frac{5}{7}$.

Your Turn 3.4.6

Calculate $\frac{38}{73} + \frac{7}{73}$.

Example 3.4.7

Subtracting Rational Numbers with the Same Denominator

Calculate $\frac{45}{136} - \frac{17}{136}$.

Solution

Since the rational numbers have the same denominator, we perform the subtraction of the numerators, $45 - 17$, and then place the result in the numerator and the common denominator, 136, in the denominator. $\frac{45}{136} - \frac{17}{136} - \frac{45-17}{136} = \frac{28}{136}$

Once we have that result, reduce to lowest terms, this gives $\frac{28}{136} = \frac{4\times7}{4\times34} = \frac{\cancel{4}\times7}{\cancel{4}\times34} = \frac{7}{34}$.

Your Turn 3.4.7

Calculate $\frac{21}{40} - \frac{8}{40}$.

When the rational numbers do not have common denominators, then we have to transform the rational numbers so that they do have common denominators. The common denominator that reduces work later in the problem is the LCM of the numerator and denominator. When adding or subtracting the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, we perform the following steps.

Step 1: Find $LCM(b, d)$.

Step 2: Calculate $n = \frac{LCM(b,d)}{b}$ and $m = \frac{LCM(b,d)}{d}$.

Step 3: Multiply the numerator and denominator of $\frac{a}{b}$ by n, yielding $\frac{a\times n}{b\times n}$.

Step 4: Multiply the numerator and denominator of $\frac{c}{d}$ by m, yielding $\frac{c\times m}{d\times m}$.

Step 5: Add or subtract the rational numbers from Steps 3 and 4, since they now have the common denominators.

You should be aware that the common denominator is $LCM(b, d)$. For the first denominator, we have $b \times n = b \times \frac{LCM(b,d)}{b} = LCM(b, d)$, since we multiply and divide $LCM(b, d)$ by the same number. For the same reason, $d \times m = d \times \frac{LCM(b,d)}{b} = LCM(b, d)$.

Example 3.4.8

Adding Rational Numbers with Unequal Denominators

Calculate $\frac{11}{18} + \frac{2}{15}$.

Solution

The denominators of the fractions are 18 and 15, so we label $b = 18$ and $d = 15$.

Step 1: Find LCM(18,15). This is 90.

Step 2: Calculate n and m. $n = \frac{90}{18} = 5$ and $m = \frac{90}{15} = 6$.

Step 3: Multiplying the numerator and denominator of $\frac{11}{18}$ by $n = 5$ yields $\frac{11\times5}{18\times5} = \frac{55}{90}$.

Step 4: Multiply the numerator and denominator of $\frac{2}{15}$ by $m = 6$ yields $\frac{2\times6}{15\times6} = \frac{12}{90}$.

Step 5: Now we add the values from Steps 3 and 4: $\frac{55}{90} + \frac{12}{90} = \frac{67}{90}$.

This is in lowest terms, so we have found that $\frac{11}{18} + \frac{2}{15} = \frac{67}{90}$.

Your Turn 3.4.8

Calculate $\frac{4}{9} + \frac{7}{12}$.

Example 3.4.9

Subtracting Rational Numbers with Unequal Denominators

Calculate $\frac{14}{25} - \frac{9}{70}$.

Solution

The denominators of the fractions are 25 and 70, so we label $b = 25$ and $d = 70$.

Step 1: Find LCM(25,70). This is 350.

Step 2: Calculate n and m: $n = \frac{350}{25}$ =14 and $m = \frac{350}{70} = 5$.

Step 3: Multiplying the numerator and denominator of $\frac{14}{25}$ by n =14 yields $\frac{14\times14}{25\times14} = \frac{196}{350}$.

Step 4: Multiplying the numerator and denominator of $\frac{9}{70}$ by $m = 5$ yields $\frac{9\times5}{70\times5} = \frac{45}{350}$.

Step 5: Now we subtract the value from Step 4 from the value in Step 3: $\frac{196}{350} - \frac{45}{350} = \frac{151}{350}$.

This is in lowest terms, so we have found that $\frac{14}{25} - \frac{9}{70} = \frac{151}{350}$.

Your Turn 3.4.9

Calculate $\frac{10}{99} - \frac{17}{300}$.

Video 3.4.5

Adding and Subtracting Fractions with Different Denominators

Converting Between Improper Fractions and Mixed Numbers

One way to visualize a fraction is as parts of a whole, as in $\frac{5}{12}$ of a pizza. But when the numerator is larger than the denominator, as in $\frac{23}{12}$, then the idea of parts of a whole seems not to make sense. Such a fraction is an **improper fraction.** That kind of fraction could be written as an integer plus a fraction, which is a **mixed number**. The fraction $\frac{23}{12}$ rewritten as a mixed number would be $1\frac{11}{12}$. Arithmetically, $1\frac{11}{12}$ is equivalent to $1 + \frac{11}{12}$, which is read as “one and 11 twelfths.”

Improper fractions can be rewritten as mixed numbers using division and remainders. To find the mixed number representation of an improper fraction, divide the numerator by the denominator. The quotient is the integer part, and the remainder becomes

the numerator of the remaining fraction.

Example 3.4.10

Rewriting an Improper Fraction as a Mixed Number

Rewrite $\frac{48}{13}$ as a mixed number.

Solution

When 48 is divided by 13, the result is 3 with a remainder of 9. So, we can rewrite $\frac{48}{13}$ as $3\frac{9}{13}$.

Your Turn 3.4.10

Rewrite $\frac{96}{26}$ as a mixed number.

Video 3.4.6

Converting an Improper Fraction to a Mixed Number Using Desmos

Similarly, we can convert a mixed number into an improper fraction. To do so, first convert the whole number part to a fraction by writing the whole number as itself divided by 1, and then add the two fractions.

Alternately, we can multiply the whole number part and the denominator of the fractional part. Next, add that product to the numerator. Finally, express the number as that product divided by the denominator.

Example 3.4.11

Rewriting a Mixed Number as an Improper Fraction

Rewrite $5\frac{4}{9}$ as an improper fraction.

Solution

Step 1: Multiply the integer part, 5, by the denominator, 9, which gives $5 \times 9 = 45$.

Step 2: Add that product to the numerator, which gives $45 + 4 = 49$.

Step 3: Write the number as the sum, 49, divided by the denominator, 9, which gives $\frac{49}{9}$.

Your Turn 3.4.11

Rewrite $9\frac{11}{12}$ as an improper fraction.

Tech Check 3.4.4

Using Desmos to Rewrite a Mixed Number as an Improper Fraction

Desmos can be used to convert from a mixed number to an improper fraction. To do so, we use the idea that a mixed number, such as $5\frac{6}{11}$, is another way to represent $5 + \frac{6}{11}$. If $5 + \frac{6}{11}$ is entered in Desmos, the result is the decimal form of the number. However, clicking the fraction button to the left will convert the decimal to an improper fraction, $\frac{61}{11}$. As an added bonus, Desmos will automatically reduce the fraction to lowest terms.

Converting Rational Numbers Between Decimal and Fraction Forms

Understanding what decimals represent is needed before addressing conversions between the fractional form of a number and its **decimal form**, or writing a number in **decimal notation**. The decimal number 4.557 is equal to $4\frac{557}{1,000}$. The decimal portion, .557, is 557 divided by 1,000. To write any decimal portion of a number expressed as a terminating decimal, divide the decimal number by 10 raised to the power equal to the number of decimal digits. Since there were three decimal digits in 4.557, we divided 557 by $10^3 = 1000$.

Decimal representations may be very long. It is convenient to **round off** the decimal form of the number to a certain number of decimal digits. To round off the decimal form of a number to n (decimal) digits, examine the $(n+1)$st decimal digit. If that digit is 0, 1, 2, 3, or 4, the number is rounded off by writing the number to the nth decimal digit and no further. If the $(n+1)$st decimal digit is 5, 6, 7, 8, or 9, the number is rounded off by writing the number to the nth digit, then replacing the nth digit by one more than the nth digit.

Example 3.4.12

Rounding Off a Number in Decimal Form to Three Digits

Round 5.67849 to three decimal digits.

Solution

The third decimal digit is 8. The digit following the 8 is 4. When the digit is 4, we write the number only to the third digit. So, 5.67849 rounded off to three decimal places is 5.678.

Your Turn 3.4.12

Round 5.1082 to three decimal places.

Example 3.4.13

Rounding Off a Number in Decimal Form to Four Digits

Round 45.11475 to four decimal digits.

Solution

The fourth decimal digit is 7. The digit following the 7 is 5. When the digit is 5, we write the number only to the fourth decimal digit, 45.1147. We then replace the fourth decimal digit by one more than the fourth digit, which yields 45.1148. So, 45.11475 rounded off to four decimal places is 45.1148.

Your Turn 3.4.13

Round 18.6298 to two decimal places.

To convert a rational number in fraction form to decimal form, use your calculator to perform the division.

Example 3.4.14

Converting a Rational Number in Fraction Form into Decimal Form

Convert $\frac{47}{25}$ into decimal form.

Solution

Using a calculator to divide 47 by 25, the result is 1.88.

Your Turn 3.4.14

Convert $\frac{48}{30}$ into decimal form.

Converting a terminating decimal to the fractional form may be done in the following way:

Step 1: Count the number of digits in the decimal part of the number, labeled n.

Step 2: Raise 10 to the nth power.

Step 3: Rewrite the number without the decimal.

Step 4: The fractional form is the number from Step 3 divided by the result from Step 2.

This process works due to what decimals represent and how we work with mixed numbers. For example, we could convert the number 7.4536 to fractional from. The decimal part of the number, the .4536 part of 7.4536, has four digits. By the definition of decimal notation, the decimal portion represents $\frac{4,536}{10^4} = \frac{4,536}{10,000}$. The decimal number 7.4536 is equal to the improper fraction $7\frac{4,536}{10,000}$. Adding those to fractions yields $\frac{74,536}{10,000}$.

Example 3.4.15

Converting from Decimal Form to Fraction Form with Terminating Decimals

Convert 3.2117 to fraction form.

Solution

Step 1: There are four digits after the decimal point, so $n = 4$.

Step 2: Raise 10 to the fourth power, $10^4 = 10,000$.

Step 3: When we remove the decimal point, we have 32,117.

Step 4: The fraction has as its numerator the result from Step 3 and as its denominator the result of Step 2, which is the fraction $\frac{32,117}{10,000}$.

Your Turn 3.4.15

Convert 17.03347 to fraction form.

The process is different when converting from the decimal form of a rational number into fraction form when the decimal form is a repeating decimal. This process is not covered in this text.

Multiplying and Dividing Rational Numbers

Multiplying rational numbers is less complicated than adding or subtracting rational numbers, as there is no need to find common denominators. To multiply rational numbers, multiply the numerators, then multiply the denominators, and write the numerator product divided by the denominator product. Symbolically, $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$. As always, rational numbers should be reduced to lowest terms.

FORMULA

If b and d are non-zero integers, then $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.

Example 3.4.16

Multiplying Rational Numbers

Calculate $\frac{12}{25} \times \frac{10}{21}$.

Solution

Multiply the numerators and place that in the numerator, and then multiply the denominators and place that in the denominator.

$\frac{12}{25} \times \frac{10}{21} = \frac{12 \times 10}{25 \times 21} = \frac{120}{525}$

This is not in lowest terms, so this needs to be reduced. The GCD of 120 and 525 is 15.

$\frac{120}{525} = \frac{15 \times 8}{15 \times 35} = \frac{8}{35}$

Your Turn 3.4.16

Calculate $\frac{45}{88} \times \frac{28}{75}$.

Video 3.4.7

Multiplying Fractions

As with multiplication, division of rational numbers can be done using a calculator.

Example 3.4.17

Dividing Decimals with a Calculator

Calculate 3.45 ÷ 2.341 using a calculator. Round to three decimal places if necessary.

Solution

Using a calculator, we obtain 1.473729175565997. Rounding to three decimal places we have 1.474.

Your Turn 3.4.17

Calculate $45.63 \div 17.13$ using a calculator. Round to three decimal places, if necessary.

Before discussing the division of fractions without a calculator, we should look at the reciprocal of a number. The **reciprocal** of a number is 1 divided by the number. For a fraction, the reciprocal is the fraction formed by switching the numerator and denominator. For the fraction $\frac{a}{b}$, the reciprocal is $\frac{b}{a}$. An important feature of a number and its reciprocal is that its product is 1.

When dividing two fractions by hand, find the reciprocal of the **divisor** (the number that is being divided into the other number). Next, replace the divisor by its reciprocal and change the division into multiplication. Then, perform the multiplication. Symbolicallly, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}$. As before, reduce to lowest terms.

FORMULA

If b, c and d are non-zero integers, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}$.

Example 3.4.18

Dividing Rational Numbers

1. Calculate $\frac{4}{21} \div \frac{6}{35}$.
2. Calculate $\frac{1}{8} \div \frac{5}{28}$.

Solution

1. **Step 1:** Find the reciprocal of the number being divided by $\frac{6}{35}$. The reciprocal of that is $\frac{35}{6}$.

   Step 2: Multiply the first fraction by that reciprocal.

   $\frac{4}{21} \div \frac{6}{35} = \frac{4}{21} \times \frac{35}{6} = \frac{140}{126}$

   The answer, $\frac{140}{126}$ is not in lowest terms. The GCD of 140 and 126 is 14. Factoring and canceling gives $\frac{140}{126} = \frac{14 \times 10}{14 \times 9} = \frac{10}{9}$.

2. **Step 1:** Find the reciprocal of the number being divided by, which is $\frac{5}{28}$. The reciprocal of that is $\frac{28}{5}$.

   Step 2: Multiply the first fraction by that reciprocal: $\frac{1}{8} \div \frac{5}{28} = \frac{1}{8} \times \frac{28}{5} = \frac{28}{40}$

   The answer, $\frac{28}{40}$, is not in lowest reduced form. The GCD of 28 and 40 is 4. Factoring and canceling gives $\frac{28}{40} = \frac{4 \times 7}{4 \times 10} = \frac{7}{10}$.

Your Turn 3.4.18

1. Calculate $\frac{46}{175} \div \frac{69}{285}$.
2. Calculate $\frac{3}{40} \div \frac{42}{55}$

Video 3.4.8

Dividing Fractions

Summary of the Operations for Rational Numbers

Operation	Rule	Example
Addition	Fractions must have the same denominator. Add the numerators and keep the common denominator.	$\frac{3}{5} + \frac{2}{5} = \frac{3+2}{5} = \frac{5}{5} = 1$
Subtraction	Fractions must have the same denominator. Subtract the numerators and keep the common denominator.	$\frac{7}{10} - \frac{2}{10} = \frac{7-2}{10} = \frac{5}{10} = \frac{1}{2}$
Addition (Different Denominators)	Find a common denominator, convert each fraction, then add.	$\frac{1}{4} + \frac{1}{6} = \frac{1 \cdot 6 + 1 \cdot 4}{4 \cdot 6} = \frac{6+4}{24} = \frac{10}{24} = \frac{5}{12}$
Subtraction (Different Denominators)	Find a common denominator, convert each fraction, then subtract.	$\frac{5}{8} - \frac{1}{4} = \frac{5 \cdot 2 - 1 \cdot 8}{8 \cdot 2} = \frac{10-8}{16} = \frac{2}{16} = \frac{1}{8}$
Multiplication	Multiply the numerators together and the denominators together.	$\frac{2}{3} \times \frac{4}{5} = \frac{2 \cdot 4}{3 \cdot 5} = \frac{8}{15}$

Operation	Rule	Example
Division	Multiply by the reciprocal of the second fraction.	$\frac{3}{4} \div \frac{2}{5} = \frac{3}{4} \times \frac{5}{2} = \frac{3 \cdot 5}{4 \cdot 2} = \frac{15}{8}$
Simplifying	Divide both the numerator and the denominator by their greatest common divisor.	$\frac{12}{16} = \frac{12 \div 4}{16 \div 4} = \frac{3}{4}$

Applying the Order of Operations to Simplify Expressions

The order of operations for rational numbers is the same as for integers, as discussed in Order of Operations. The order of operations makes it easier for anyone to correctly calculate and represent. The order follows the well-known acronym PEMDAS:

P	Parentheses
E	Exponents
M/D	Multiplication and division
A/S	Addition and subtraction

The first step in calculating using the order of operations is to perform operations inside the parentheses. Moving down the list, next perform all exponent operations moving from left to right. Next (left to right once more), perform all multiplications and divisions. Finally, perform the additions and subtractions.

Example 3.4.19

Applying the Order of Operations with Rational Numbers

Correctly apply the rules for the order of operations to accurately compute $\left(\frac{5}{7} - \frac{2}{7}\right) \times 2^3$.

Solution

Step 1: To calculate this, perform all calculations within the parentheses before other operations.

$\left(\frac{5}{7} - \frac{2}{7}\right) \times 2^3 = \left(\frac{3}{7}\right) \times 2^3$

Step 2: Since all parentheses have been cleared, we move left to right, and compute all the exponents next.

$\left(\frac{3}{7}\right) \times 2^3 = \left(\frac{3}{7}\right) \times 8$

Step 3: Now, perform all multiplications and divisions, moving left to right.

$\left(\frac{3}{7}\right) \times 8 = \frac{24}{7}$

Your Turn 3.4.19

Correctly apply the rules for the order of operations to accurately computer $\frac{3}{16} + \left(\frac{7}{16}\right)^2 + \frac{1}{5} \div \frac{3}{10}$.

Example 3.4.20

Applying the Order of Operations with Rational Numbers

Correctly apply the rules for the order of operations to accurately compute $4 + \frac{2}{3} \div \left(\left(\frac{5}{9}\right)^2 - \left(\frac{2}{3} + 5\right)\right)^2$.

Solution

To calculate this, perform all calculations within the parentheses before other operations. Evaluate the innermost parentheses first. We can work separate parentheses expressions at the same time.

Step 1: The innermost parentheses contain $\frac{2}{3}+5$. Calculate that first, dividing after finding the common denominator.

$$4+\tfrac{2}{3}\div\left(\left(\tfrac{5}{9}\right)^2-\left(\tfrac{2}{3}+5\right)\right)^2$$
$$=4+\tfrac{2}{3}\div\left(\left(\tfrac{5}{9}\right)^2-\left(\tfrac{2}{3}+\tfrac{5}{1}\right)\right)^2$$
$$=4+\tfrac{2}{3}\div\left(\left(\tfrac{5}{9}\right)^2-\left(\tfrac{2}{3}+\tfrac{15}{3}\right)\right)^2$$
$$=4+\tfrac{2}{3}\div\left(\left(\tfrac{5}{9}\right)^2-\left(\tfrac{17}{3}\right)\right)^2$$

Step 2: Calculate the exponent in the parentheses, $\left(\frac{5}{9}\right)^2$.

$$4+\tfrac{2}{3}\div\left(\left(\tfrac{5}{9}\right)^{\mathbf{2}}-\left(\tfrac{17}{3}\right)\right)^2$$
$$=4+\tfrac{2}{3}\div\left(\left(\tfrac{25}{81}\right)-\left(\tfrac{17}{3}\right)\right)^2$$

Step 3: Subtract inside the parentheses is done, using a common denominator.

$$4+\tfrac{2}{3}\div\left(\left(\tfrac{25}{81}\right)-\left(\tfrac{17}{3}\right)\right)^2$$
$$4+\tfrac{2}{3}\div\left(\left(\tfrac{25}{81}\right)-\left(\tfrac{17\times27}{3\times27}\right)\right)^2$$
$$4+\tfrac{2}{3}\div\left(\left(\tfrac{25}{81}\right)-\left(\tfrac{459}{81}\right)\right)^2$$
$$4+\tfrac{2}{3}\div\left(\left(\tfrac{-434}{81}\right)\right)^2$$

Step 4: At this point, evaluate the exponent and divide.

$$4+\tfrac{2}{3}\div\left(\left(\tfrac{-434}{81}\right)\right)^2$$
$$4+\tfrac{2}{3}\div\left(\tfrac{188{,}356}{6{,}561}\right)$$
$$=4+\tfrac{2}{3}\times\left(\tfrac{6{,}561}{188{,}356}\right)$$
$$=4+\tfrac{2{,}187}{94{,}178}$$

Step 5: Add.

$$4+\tfrac{2{,}187}{94{,}178}$$
$$=\tfrac{378{,}899}{94{,}178}$$

Had this been done on a calculator, the decimal form of the answer would be 4.0232 (rounded to four decimal places).

Your Turn 3.4.20

Correctly apply the rules for the order of operations to accurately compute $\left(\frac{3}{5}+2\right)\times\left(\frac{4}{5}-\frac{1}{2}\right)^2\div\frac{11}{15}$.

Video 3.4.9

Order of Operations Using Fractions

Applying the Density Property of Rational Numbers

For any two distinct rational numbers a and b where $a<b$, there exists a rational number c such that $a<c<b$. This is called the

density property of the rational numbers.

This means that no matter how close two rational numbers are, you can always find another rational number between them. In fact, there are infinitely many rational numbers that are possible between a and b.

To find one of those numbers:

Step 1: Add the two rational numbers.

Step 2: Divide that result by 2.

The result is always a rational number. This follows what we know about rational numbers. If two fractions are added, then the result is a fraction. Also, when a fraction is divided by a fraction (and 2 is a fraction), then we get another fraction. This two-step process will give a rational number, provided the first two numbers are rational.

Example 3.4.21

Applying the Density Property of Rational Numbers

Demonstrate the density property of rational numbers by finding a rational number between $\frac{4}{11}$ and $\frac{7}{12}$.

Solution

To find a rational number between $\frac{4}{11}$ and $\frac{7}{12}$:

Step 1: Add the fractions.

$$\frac{4}{11}+\frac{7}{12}=\frac{4\times12}{11\times12}+\frac{7\times11}{12\times11}=\frac{48}{132}+\frac{77}{132}=\frac{125}{132}$$

Step 2: Divide the result by 2. Recall that to divide by 2, you multiply by the reciprocal of 2. The reciprocal of 2 is $\frac{1}{2}$, as seen below.

$$\frac{125}{132}\div2=\frac{125}{132}\times\frac{1}{2}=\frac{125}{264}$$

So, one rational number between $\frac{4}{11}$ and $\frac{7}{12}$ is $\frac{125}{264}$.

We could check that the number we found is between the other two by finding the decimal representation of the numbers. Using a calculator, the decimal representations of the rational numbers are 0.363636..., 0.473484848..., and 0.5833333.... Here it is clear that $\frac{125}{264}$ is between $\frac{4}{11}$ and $\frac{7}{12}$.

Your Turn 3.4.20

Demonstrate the density property of rational numbers by finding a rational number between $\frac{27}{13}$ and $\frac{21}{10}$.

Solving Problems Involving Rational Numbers

Rational numbers are used in many situations, sometimes to express a portion of a whole, other times as an expression of a ratio between two quantities. For the sciences, converting between units is done using rational numbers, as when converting between gallons and cubic inches. In chemistry, mixing a solution with a given concentration of a chemical per unit volume can be solved with rational numbers. In demographics, rational numbers are used to describe the distribution of the population. In dietetics, rational numbers are used to express the appropriate amount of a given ingredient to include in a recipe. As discussed, the application of rational numbers crosses many disciplines.

Example 3.4.22

Mixing Soil for Vegetables

James is mixing soil for a raised garden, in which he plans to grow a variety of vegetables. For the soil to be suitable, he determines that $\frac{2}{5}$ of the soil can be topsoil, but $\frac{2}{5}$ needs to be peat moss and $\frac{1}{5}$ has to be compost. To fill the raised garden bed with 60 cubic feet of soil, how much of each component does James need to use?

Solution

In this example, we know the proportion of each component to mix, and we know the total amount of the mix we need. In this kind of situation, we need to determine the appropriate amount of each component to include in the mixture. For each component of the mixture, multiply 60 cubic feet, which is the total volume of the mixture we want, by the fraction required of the component.

Step 1: The required fraction of topsoil is $\frac{2}{5}$, so James needs $60 \times \frac{2}{5}$ cubic feet of topsoil. Performing the multiplication, James needs $60 \times \frac{2}{5} = \frac{120}{5} = 24$ (found by treating the fraction as division, and 120 divided by 5 is 24) cubic feet of topsoil.

Step 2: The required fraction of peat moss is also $\frac{2}{5}$, so he also needs $60 \times \frac{2}{5}$ cubic feet, or $60 \times \frac{2}{5} = \frac{120}{5} = 24$ cubic feet of peat moss.

Step 3: The required fraction of compost is $\frac{1}{5}$. For the compost, he needs $60 \times \frac{1}{5} = \frac{60}{5} = 12$ cubic feet.

Your Turn 3.4.22

Ashley wants to study for 10 hours over the weekend. She plans to spend half the time studying math, a quarter of the time studying history, an eighth of the time studying writing, and the remaining eighth of the time studying physics. How much time will Ashley spend on each of those subjects?

Example 3.4.23

Determining the Number of Specialty Pizzas

At Bella's Pizza, one-third of the pizzas that are ordered are one of their specialty varieties. If there are 273 pizzas ordered, how many were specialty pizzas?

Solution

One-third of the whole are specialty pizzas, so we need one-third of 273, which gives $\frac{1}{3} \times 273 = \frac{273}{3} = 91$, found by dividing 273 by 3. So, 91 of the pizzas that were ordered were specialty pizzas.

Your Turn 3.4.23

Danny, a nutritionist, is designing a diet for her client, Callum. Danny determines that Callum's diet should be 30% protein. If Callum consumes 2,400 calories per day, how many calories of protein should Danny tell Callum to consume?

Video 3.4.10

Finding a Fraction of a Total

Using Fractions to Convert Between Units

A common application of fractions is called **unit conversion**, or **converting units**, which is the process of changing from the units used in making a measurement to different units of measurement.

For instance, 1 inch is (approximately) equal to 2.54 cm. To convert between units, the two equivalent values are made into a fraction. To convert from the first type of unit to the second type, the fraction has the second unit as the numerator, and the first unit as the denominator.

From the inches and centimeters example, to change from inches to centimeters, we use the fraction $\frac{2.54\,\text{cm}}{1\,\text{in}}$. If, on the other hand,

we wanted to convert from centimeters to inches, we'd use the fraction $\frac{1\text{ in}}{2.54\text{ cm}}$. This fraction is multiplied by the number of units of the type you are converting *from*, which means the units of the denominator are the same as the units being multiplied.

Example 3.4.24

Converting Liters to Gallons

It is known that 1 liter (L) is 0.264172 gallons (gal). Use this to convert 14 liters into gallons.

Solution

We know that 1 liter = 0.264172 gal. Since we are converting from liters, when we create the fraction we use, make sure the liter part of the equivalence is in the denominator. So, to convert the 14 liters to gallons, we multiply 14 by $\frac{1\text{ gal}}{0.264172\text{ gal}/1\text{ liter}}$1liter. Notice the gallon part is in the numerator since we're converting *to* gallons, and the liter part is in the denominator since we are converting *from* liters. Performing this and rounding to three decimal places, we find that 14 liters is $14\text{ liter} \times \frac{0.264172\text{ gal}}{1\text{ liter}} = 3.69841\text{ gal}$.

Your Turn 3.4.24

One mile is equal to 1.60934 km. Convert 200 miles to kilometers. Round off the answer to three decimal places.

Example 3.4.25

Converting Centimeters to Inches

It is known that 1 inch is 2.54 centimeters. Use this to convert 100 centimeters into inches.

Solution

We know that 1 inch = 2.54 cm. Since we are converting from centimeters, when we create the fraction we use, make sure the centimeter part of the equivalence is in the denominator, $\frac{1\text{ in}}{2.54\text{ cm}}$. To convert the 100 cm to inches, multiply 100 by $\frac{1\text{ in}}{2.54\text{ cm}}$. Notice the inch part is in the numerator since we're converting *to* inches, and the centimeter part is in the denominator since we are converting *from* centimeters. Performing this and rounding to three decimal places, we obtain $100\text{ cm} \times \frac{1\text{ in}}{2.54\text{ cm}} = 39.370\text{ in}$. This means 100 cm equals 39.370 in.

Your Turn 3.4.25

It is known that 4 quarts equals 3.785 liters. If you have 25 quarts, how many liters do you have? Round off to three decimal places.

Video 3.4.11

Converting Units

Defining and Applying Percent

A **percent** is a specific rational number and is literally per 100. n percent, denoted $n\%$, is the fraction [redacted]n100n100.

Example 3.4.26

Rewriting a Percentage as a Fraction

Rewrite the following as fractions:

1. 31%
2. 93%

Solution

1. Using the definition and $n = 31$, 31% in fraction form is $\frac{31}{100}$.
2. Using the definition and $n = 93$, 93% in fraction form is $\frac{93}{100}$.

Your Turn 3.4.26

Rewrite the following as fractions:

1. 4
2. 50

Example 3.4.27

Rewriting a Percentage as a Decimal

Rewrite the following percentages in decimal form:

1. 54%
2. 83%

Solution

1. Using the definition and $n = 54$, 54% in fraction form is $\frac{54}{100}$. Dividing a number by 100 moves the decimal two places to the left; 54% in decimal form is then 0.54.
2. Using the definition and $n = 83$, 83% in fraction form is $\frac{83}{100}$. Dividing a number by 100 moves the decimal two places to the left; 83% in decimal form is then 0.83.

Your Turn 3.4.27

Rewrite the following percentages in decimal form:

1. 14
2. 7

You should notice that you can simply move the decimal two places to the left without using the fractional definition of percent.

Percent is used to indicate a fraction of a total. If we want to find 30% of 90, we would perform a multiplication, with 30% written in either decimal form or fractional form. The 90 is the **total**, 30 is the **percentage**, and 27 (which is 0.30×90) is the **percentage of the total**.

FORMULA

$n\%$ of x items is $\frac{n}{100} \times x$. The x is referred to as the **total**, the n is referred to as the **percent** or **percentage**, and the value obtained from $\frac{n}{100} \times x$ is the **part** of the total and is also referred to as the **percentage of the total**.

Example 3.4.28

Finding a Percentage of a Total

1. Determine 40% of 300.
2. Determine 64% of 190.

Solution

1. The total is 300, and the percentage is 40. Using the decimal form of 40% and multiplying we obtain $0.40 \times 300 = 120$.
2. The total is 190, and the percentage is 64. Using the decimal form of 64% and multiplying we obtain $0.64 \times 190 = 121.6$.

Your Turn 3.4.28

1. Determine 25% of 1,200.
2. Determine 53% of 1,588.

In the previous situation, we knew the total and we found the percentage of the total. It may be that we know the percentage of the total, and we know the percent, but we don't know the total. To find the total if we know the percentage the percentage of the total, use the following formula.

FORMULA

If we know that n% of the total is x, then the total is $\frac{100 \times x}{n}$.

Example 3.4.29

Finding the Total When the Percentage and Percentage of the Total Are Known

1. What is the total if 28% of the total is 140?
2. What is the total if 6% of the total is 91?

Solution

1. 28 is the percentage, so n =28. 28% of the total is 140, so x =140 . Using those we find that the total was $\frac{100 \times 140}{28} = 500$.
2. 6 is the percentage, so $n = 6$. 6% of the total is 91, so x =91x=91. Using those we find that the total was $\frac{100 \times 91}{6} = 1{,}516.6$.

Your Turn 3.4.29

1. What is the total if 25% of the total is 30?
2. What is the total if 45% of the total is 360?

The percentage can be found if the total and the percentage of the total is known. If you know the total, and the percentage of the total, first divide the part by the total. Move the decimal two places to the right and append the symbol %. The percentage may be found using the following formula.

FORMULA

The percentage, n, of b that is a is $\frac{a}{b} \times 100\%$.

Example 3.4.30

Finding the Percentage When the Total and Percentage of the Total Are Known

Find the percentage in the following:

1. Total is 300, percentage of the total is 60.
2. Total is 440, percentage of the total is 176.

Solution

1. The total is 300; the percentage of the total is 60. Calculating yields 0.2. Moving the decimal two places to the right gives 20. Appending the percentage to this number results in 20%. So, 60 is 20% of 300.
2. The total is 440; the percentage of the total is 176. Calculating yields 0.4. Moving the decimal two places to the right gives 40. Appending the percentage to this number results in 40%. So, 176 is 40% of 440.

Your Turn 3.4.30

Find the percentage in the following:

1. Total is 1,000, percentage of the total is 70.
2. Total is 500, percentage of the total is 425.

Solve Problems Using Percent

In the media, in research, and in casual conversation percentages are used frequently to express proportions. Understanding how to use percent is vital to consuming media and understanding numbers. Solving problems using percentages comes down to identifying which of the three components of a percentage you are given, the total, the percentage, or the percentage of the total. If you have two of those components, you can find the third using the methods outlined previously.

Example 3.4.31

Percentage of Students Who Are Sleep Deprived

A study revealed that 70% of students suffer from sleep deprivation, defined to be sleeping less than 8 hours per night. If the survey had 400 participants, how many of those participants had less than 8 hours of sleep per night?

Solution

The percentage of interest is 70%. The total number of students is 400. With that, we can find how many were in the percentage of the total, or, how many were sleep deprived. Applying the formula from above, the number who were sleep deprived was $0.70 \times 400 = 280$; 280 students on the study were sleep deprived.

Your Turn 3.4.31

Riley has a daily calorie intake of 2,200 calories and wants to take in 20% of their calories as protein. How many calories of protein should be in their daily diet?

Example 3.4.32

Amazon Prime Subscribers

There are 126 million users who are U.S. Amazon Prime subscribers. If there are 328.2 million residents in the United States, what percentage of U.S. residents are Amazon Prime subscribers?

Solution

We are asked to find the percentage. To do so, we divide the percentage of the total, which is 126 million, by the total, which is 328.2 million. Performing this division and rounding to three decimal places yields $\frac{126}{328.2} = 0.384$. The decimal is moved to the right by two places, and a % sign is appended to the end. Doing this shows us that 38.4% of U.S. residents are Amazon Prime subscribers.

Your Turn 3.4.32

A small town has 450 registered voters. In the primaries, 54 voted. What percentage of registered voters in that town voted in the primaries?

Example 3.4.33

Finding the Percentage When the Total and Percentage of the Total Are Known

Evander plays on the basketball team at their university and 73% of the athletes at their university receive some sort of scholarship for attending. If they know 219 of the student-athletes receive some sort of scholarship, how many student-athletes are at the university?

Solution

We need to find the total number of student-athletes at Evander's university.

Step 1: Identify what we know. We know the percentage of students who receive some sort of scholarship, 73%. We also know the number of athletes that form the part of the whole, or 219 student-athletes.

Step 2: To find the total number of student-athletes, use $\frac{100 \times x}{n}$, with $x = 219$ and $n = 73$. Calculating with those values yields $\frac{100 \times 219}{73} = 300$.

So, there are 300 total student-athletes at Evander's university

Your Turn 3.4.33

A store declares a deep discount of 40% for an item, which they say will save $30. What was the original price of the item?

Check Your Understanding

1. Identify which of the following are rational numbers.
   a. -41
   b. $\sqrt{13}$
   c. $\frac{4}{3}$
   d. 2.75
   e. $0.2\overline{13}$
2. Express $\frac{8}{30}$ in lowest terms.
3. Calculate $\frac{3}{8} + \frac{5}{12}$ and express in lowest terms.
4. Convert 0.34 into fraction form.
5. Convert $\frac{47}{12}$ into a mixed number.
6. Calculate $\frac{2}{9} \times \frac{21}{22}$ and express in lowest terms.
7. Calculate $\frac{2}{5} \div \frac{3}{10} + \frac{1}{6}$.
8. Identify a rational number between $\frac{7}{8}$ and $\frac{20}{21}$.
9. Convert $\frac{47}{12}$ into a mixed number.
10. Lina decides to save $\frac{1}{8}$ of her take-home pay every paycheck. Her most recent paycheck was for $882. How much will she save from that paycheck?
11. Determine 38% of 600.
12. A microchip factory has decided to increase its workforce by 10%. If it currently has 70 employees, how many new employees will the factory hire?

    For the following exercises, identify which of the following are rational numbers.<\li>

13. 4.598
14. $\sqrt{144}$
15. $\sqrt{131}$

For the following exercises, reduce the fraction to lowest terms

16. $\frac{8}{10}$
17. $\frac{30}{105}$
18. $\frac{36}{539}$
19. $\frac{231}{490}$
20. $\frac{750}{17875}$

For the following exercises, do the indicated conversion. If it is a repeating decimal, use the correct notation.

21. Convert $\frac{25}{6}$ to a mixed number.
22. Convert $\frac{240}{53}$ to a mixed number.
23. Convert $2\frac{3}{8}$ to an improper fraction.
24. Convert $15\frac{7}{30}$ to an improper fraction.
25. Convert $\frac{4}{9}$ to decimal form.
26. Convert $\frac{13}{20}$ to decimal form.
27. Convert $\frac{27}{625}$ to decimal form.
28. Convert $\frac{11}{14}$ to decimal form.
29. Convert 0.23 to fraction form and reduce to lowest terms.
30. Convert 3.8874 to fraction form and reduce to lowest terms.

For the following exercises, perform the indicated operations. Reduce to lowest terms.

31. $\frac{3}{5}+\frac{3}{10}$
32. $\frac{3}{14}+\frac{8}{21}$
33. $\frac{13}{36}-\frac{14}{99}$
34. $\frac{13}{24}-\frac{4}{117}$
35. $\frac{3}{7}\times\frac{21}{48}$
36. $\frac{48}{143}\times\frac{77}{120}$
37. $\frac{14}{27}\div\frac{7}{12}$
38. $\frac{44}{75}\div\frac{484}{285}$
39. $\left(\frac{3}{5}+\frac{2}{7}\right)\times\frac{10}{21}$
40. $\frac{3}{8}\times\left(\frac{13}{12}-\frac{35}{36}\right)$
41. $\left(\frac{3}{7}+\frac{5}{16}\right)^2-\frac{5}{12}$
42. $\frac{3}{8}\times\left(\frac{4}{9}-\frac{1}{8}\right)^2$
43. $\left(\frac{2}{5}\times\left(\frac{7}{8}-\frac{2}{3}\right)\right)^2\div\left(\frac{4}{9}+\frac{5}{6}\right)+\frac{7}{12}$
44. $\left(\frac{1}{5}\div\left(\frac{3}{10}+\frac{11}{15}\right)\right)\times\left(\frac{2}{21}+\frac{5}{9}\right)-\left(\frac{8}{15}\div\frac{4}{33}\right)^2$
45. Find a rational number between $\frac{8}{17}$ and $\frac{15}{28}$.
46. Find a rational number between $\frac{3}{50}$ and $\frac{13}{98}$.
47. Find two rational numbers between $\frac{3}{10}$ and $\frac{19}{45}$.
48. Find three rational numbers between $\frac{5}{12}$ and $\frac{175}{308}$.
49. Convert 24% to fraction form and reduce completely.
50. Convert 95% to fraction form and reduce completely.
51. Convert 0.23 to a percentage.
52. Convert 1.22 to a percentage.
53. Determine 30% of 250.
54. Determine 75% of 600.
55. If 25% of a group is 41 members, how many members total are in the group?
56. If 80% of the total is 60, how much is in the total?
57. 13 is what percent of 20?
58. 80 is what percent of 320?
59. Professor Donalson's history of film class has 60 students. Of those students, $\frac{2}{5}$, say their favorite movie genre is comedy. How many of the students in Professor Donalson's class name comedy as their favorite movie genre?
60. Naia's dormitory floor has 80 residents. Of those, $\frac{3}{8}$, play *Fortnight* for at least 15 hours per week. How many students on Naia's floor play *Fortnight* at least 15 hours per week?

61. In Tara's town there are 24,000 people. Of those, $\frac{13}{100}$,are food insecure. How many people in Tara's town are food insecure?
62. Roughly $\frac{39}{50}$ of air is nitrogen. If an enclosure holds 2,000 liters of air, how many liters of nitrogen should be expected in the enclosure?
63. To make the dressing for coleslaw, Maddie needs to mix it with $\frac{3}{5}$ mayonnaise and $\frac{2}{5}$ apple cider vinegar. If Maddie wants to have 8 cups of dressing, how many cups of mayonnaise and how many cups of apple cider vinegar does Maddie need?
64. Malik is figuring out his schedule. He wishes to spend $\frac{4}{15}$ of his time sleeping, $\frac{1}{3}$ of his time studying and going to class, $\frac{1}{5}$ of his time at work, and $\frac{2}{15}$ of his time doing other activities, such as entertainment or exercising. There are 168 hours in a week. How many hours in a week will Malik spend:
    a. Sleeping?
    b. Studying and going to class?
    c. Not sleeping?
65. Roughly 20.9% of air is oxygen. How much oxygen is there in 200 liters of air?
66. 65% of college students graduate within 6 years of beginning college. A first-year cohort at a college contains 400 students. How many are expected to graduate within 6 years?
67. A 20% discount is offered on a new laptop. How much is the discount if the new laptop originally cost $700?
68. Leya helped at a neighborhood sale and was paid 5% of the proceeds. If Leya is paid $171.25, what were the total proceeds from the neighborhood sale?
69. **Unit Conversion.** 1 kilogram (kg) is equal to 2.20462 pounds. Convert 13 kg to pounds. Round to three decimal places, if necessary.
70. **Unit Conversion.** 1 kilogram (kg) is equal to 2.20462 pounds. Convert 200 pounds to kilograms. Round to three decimal places, if necessary.
71. **Unit Conversion.** There are 12 inches in a foot, 3 feet in a yard, and 1,760 yards in a mile. Convert 10 miles to inches. To do so, first convert miles to yards. Next, convert the yards to feet. Last, convert the feet to inches.
72. **Unit Conversion.** There are 1,000 meters (m) in a kilometer (km), and 100 centimeters (cm) in a meter. Convert 4 km to centimeters.
73. **Markup.** In this exercise, we introduce the concept of **markup**. The **markup** on an item is the difference between how much a store sells an item for and how much the store paid for the item. Suppose Wegmans (a northeastern U.S. grocery chain) buys cereal at $1.50 per box and sells the cereal for $2.29.
    a. Determine the markup in dollars.
    b. The markup is what percent of the original cost? Round the percentage to one decimal place.
74. In this exercise, we explore what happens when an item is marked up by a percentage, and then marked down using the same percentage. Wegmans purchases an item for $5 per unit. The markup on the item is 25%.
    a. Calculate the markup on the item, in dollars.
    b. What is the price for which Wegmans sells the item? This is the price Wegmans paid, plus the markup.
    c. Suppose Wegmans then offers a 25% discount on the sale price of the item (found in part b). In dollars, how much is the discount?
    d. Determine the price of the item after the discount (this is the sales price of the item minus the discount). Round to two decimal places.
    e. Is the new price after the markup and discount equal to the price Wegmans paid for the item? Explain.
75. **Repeated Discounts.** In this exercise, we explore applying more than one discount to an item. Suppose a store cuts the price on an item by 50%, and then offers a coupon for 25% off any sale item. We will find the price of the item after applying the sale price and the coupon discount.
    a. The original price was $150. After the 50% discount, what is the price of the item?
    b. The coupon is applied to the discount price. The coupon is for 25%. Find 25% of the sale price (found in part a).
    c. Find the price after applying the coupon (this is the value from part a minus the value from part b).
    d. The total amount saved on the item is the original price after all the discounts. Determine the total amount saved by subtracting the final price paid (part c) from the original price of the item.
    e. Determine the effective discount percentage, which is the total amount saved divided by the original price of the item.
    f. Was the effective discount percentage equal to 75%, which would be the 50% plus the 25%? Explain.

76. **Converting Repeating Decimals to a Fraction**
    It was mentioned in the section that repeating decimals are rational numbers. To convert a repeating decimal to a rational number, perform the following steps:
    Step 1: Label the original number S.
    Step 2: Count the number of digits, n, in the repeating part of the number.
    Step 3: Multiply S by 10^n, and label this as $10^n \times S$.
    Step 4: Determine $10^n - 1$.
    Step 5: Calculate $10^n \times S - S$. If done correctly, the repeating part of the number will cancel out.

Step 6: If the result from Step 5 has decimal digits, count the number of decimal digits in the number from Step 5. Label this m.
Step 7: Remove the decimal from the result of Step 5.
Step 8: Add m zeros to the end of the number from Step 4.
Step 9: Divide the result from Step 7 by the result from Step 8. This is the fraction form of the repeating decimal.

a. Convert $0.\overline{7}$ to fraction form.
b. Convert $0.\overline{45}$ to fraction form.
c. Convert $3.1\overline{5}$ to fraction form.
d. Convert $2.71\overline{94}$ to fraction form.

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3.5: Irrational Numbers



Figure 3.5.1 The Pythagoreans were a philosophical sect of ancient Greece, often associated with mathematics. (credit: Fedor Andreevich Bronnikov (1827-1902) “Hymn of the Pythagoreans to the Rising Sun,” 1877, oil on canvas/Wikimedia, public domain)

Learning Objectives

After completing this section, you should be able to:

- Define and identify numbers that are irrational.
- Simplify irrational numbers and express in lowest terms.
- Add and subtract irrational numbers.
- Multiply and divide irrational numbers.
- Rationalize fractions with irrational denominators.

he Pythagoreans were a philosophical sect in ancient Greece. Their philosophy included reincarnation and purifying the mind through the study and contemplation of mathematics and science. One of their principles was the cosmos is ruled by order, specifically mathematics and music. They even held mystic beliefs about specific numbers and figures. For example, the number 1 was associated with the mind and essence. Four represented justice, as it is the first product of two even numbers. Most famously, though, is the association with the Pythagorean Theorem, which states that in a right triangle, the sum of the squares of the shorter sides of the triangle (the legs) equals the square of the longer side (the hypotenuse). Even the ancient Egyptians used this relationship, as triangles with side measures 3, 4, and 5 were often used in surveying following the flooding of the Nile.

There is a story of a Pythagorean, Hippasus, discovering that not all numbers could be expressed as fractions. In other words, not all numbers were rational numbers. The story ends with Hippasus, who shared this, or in some versions discovered it, put to death by drowning for sharing this fact, that not all quantities could be expressed as the ratio of two natural numbers.

As colorful as that story may be, it is most likely false, as there are no contemporary sources to corroborate it. But it does seem to mark the discovery that not all quantities or measures were fractions of numbers. And so, irrational numbers were discovered.

Video 3.5.1

The Philosophy of the Pythagoreans

Defining and Identifying Numbers That Are Irrational

We defined rational numbers in the last section as numbers that could be expressed as a fraction of two integers. **Irrational numbers** are numbers that cannot be expressed as a fraction of two integers. Recall that rational numbers could be identified as those whose decimal representations either terminated (ended) or had a repeating pattern at some point. So irrational numbers must be those whose decimal representations do not terminate or become a repeating pattern.

One collection of irrational numbers is **square roots** of numbers that aren't **perfect squares**. x is the square root of the number a, denoted $\sqrt{a}$, if x^2 =ax2=a. The number a is the perfect square of the integer n if a =n2a=n2. The rational number [redacted] abab is a perfect square if both a and b are perfect squares.

One method of determining if an integer is a perfect square is to examine its prime factorization. If, in that factorization, all the prime factors are raised to even powers, the integer is a perfect square. Another method is to attempt to factor the integer into an integer squared. It is possible that you recognize the number as a perfect square (such as 4 or 9). Or, if you have a calculator at hand, use the calculator to determine if the square root of the integer is an integer.

Example 3.5.1

Identifying Perfect Squares

Determine which of the following are perfect squares.

1. 45
2. 81
3. $\frac{9}{28}$
4. $\frac{144}{400}$

Solution

1. The prime factorization of 45 is $45 = 3^2 \times 5$. Since the 5 is not raised to an even power, 45 is not a perfect square.
2. The prime factorization of 81 is 3^4. All the prime factors are raised to even powers, so 81 is a perfect square.
3. We must determine if both the numerator and denominator of $\frac{9}{28}$ are perfect squares for the rational number to be a perfect square. The numerator is 9, and as mentioned above, 9 is a perfect square (it is 3 squared). Now we check the prime factorization of the denominator, 28, which is $28 = 2^2 \times 7$. Since 7 is not raised to an even power, 28 is not a perfect square. Since the denominator is not a perfect square, $\frac{9}{28}$ is not a perfect square.
4. We must determine if both the numerator and denominator of $\frac{144}{400}$ are perfect squares for the rational number to be a perfect square. The numerator is 144. The prime factorization of 144 is $144 = 2^4 \times 3^2$. Since all the prime factors of 144 are raised to even powers, 144 is a perfect square. Now we check the prime factorization of the denominator, 400, which is $400 = 2^4 \times 5^2$. Since all the prime factors of 400 are raised to even powers, 400 is a perfect square. Since the numerator and denominator of $\frac{144}{400}$ are perfect squares, $\frac{144}{400}$ is a perfect square.

Your Turn 3.5.1

Determine which of the following are perfect squares.

1. 36
2. 27
3. $\frac{9}{49}$
4. $\frac{12}{221}$

Tech Check 3.5.1

Using Desmos to Determine if a Number Is a Perfect Square

Desmos may be used to determine if a number is a perfect square by using its square root function. When Desmos is opened, there is a tab in the lower left-hand corner of the Desmos screen. This tab opens the Desmos keypad, shown in Figure 3.29.

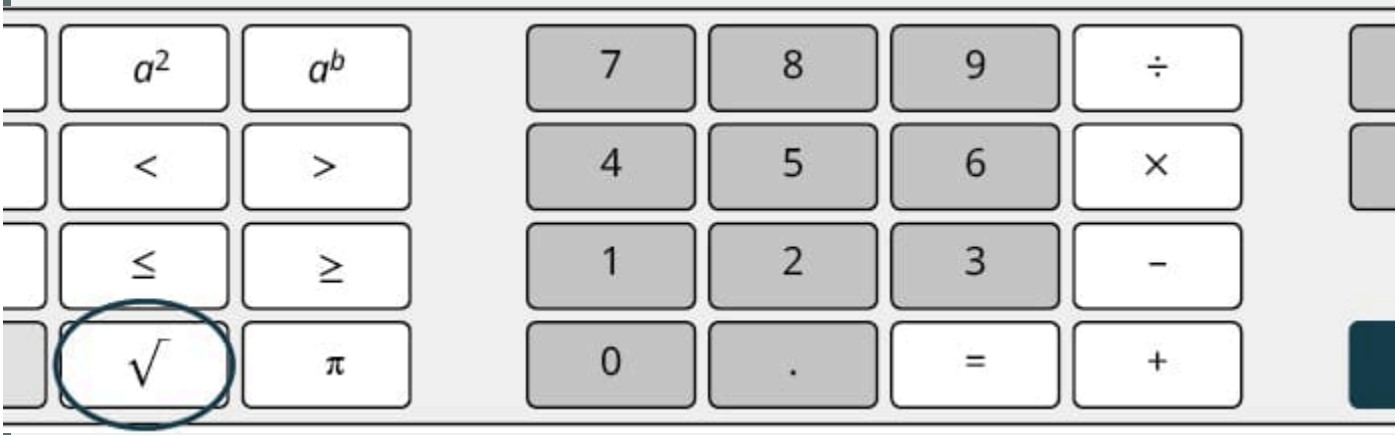


Figure 3.29 Desmos keyboard with square root key circles

There you find the key for the square root, which is circled in Figure 3.29. To find the square root of a number, click the square root key, which begins a calculation, and then enter the value for which you want a square root. If the result is an integer, then the number is a perfect square.

Video 3.5.2

Using Desmos to Find the Square Root of a Number

Another collection of irrational numbers is based on the special number, **pi**, denoted by the Greek letter π

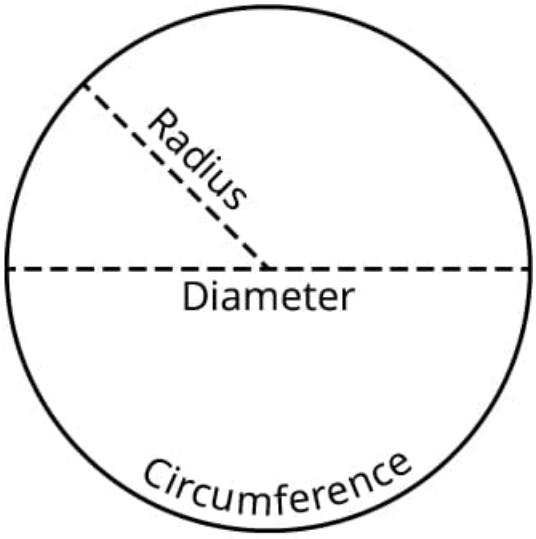


Figure 3.30 Circle with radius, diameter, and circumference labeled

Any multiple or power of [redacted]ππ is an irrational number.

Any number expressed as a rational number times an irrational number is an irrational number also. When an irrational number takes that form, we call the rational number the **rational part**, and the irrational number the **irrational part**. It should be noted that a rational number plus, minus, multiplied by, or divided by any irrational number is an irrational number.

Example 3.5.2

Identifying Irrational Numbers

Identify which of the following numbers are irrational.

1. $\sqrt{35}$
2. $0.\overline{15}$
3. $\sqrt{121}$
4. [redacted]4π4π

Solution

1. 35 can be factored as 5×7, showing that 35 is not the square of an integer or a rational number. This mean its square root is an irrational number.
2. Since $0.\overline{15}$ is a decimal with a repeating pattern, it is rational, so it is not an irrational number.
3. $121 = 11^2$. Since 121 is the square of an integer, its square root is a rational number.
4. Since [redacted]4π4π is a multiple of pi, it is irrational. In this case, the rational part of the number is 4, while the irrational part is [redacted]ππ.

Your Turn 3.5.2

Identify which of the following numbers are irrational.

1. $\sqrt{225}$
2. $3\sqrt{5}$
3. $\sqrt{80}$
4. $20 - 3\pi$

Who Knew?

Euler-Mascheroni Constant

Determining if a number is rational or irrational is not trivial. There are numbers that defied such classification for quite a long time. One such is the Euler-Mascheroni constant, denoted as γ. The Euler-Mascheroni constant is used in mathematics, and is primarily associated with the natural logarithm, which is a mathematical function. The constant has been around since around 1790. However, it was unknown if this constant was rational or irrational until 2013, at which point it was proven to be irrational.

Simplifying Square Roots and Expressing Them in Lowest Terms

To **simplify a square root** means that we rewrite the square root as a rational number times the square root of a number that has no perfect square factors. The act of changing a square root into such a form is simplifying the square root.

The number inside the square root symbol is referred to as the **radicand**. So in the expression $\sqrt{a}$ the number a is referred to as the radicand.

Before discussing how to simplify a square root, we need to introduce a rule about square roots. The square root of a product of numbers equals the product of the square roots of those number. Written symbolically, $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.

FORMULA

For any two numbers a and b, $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.

Using this formula, we can factor an integer inside a square root into a perfect square times another integer. Then the square root can be applied to the perfect square, leaving an integer times the square root of another integer. If the number remaining under the square root has no perfect square factors, then we've simplified the irrational number into lowest terms. To simplify the irrational number into lowest terms when n is an integer:

Step 1: Determine the largest perfect square factor of n, which we denote a^2.

Step 2: Factor n into $a^2 \times b$.

Step 3: Apply $sqrta^2 \times b = \sqrt{a^2} \times \sqrt{b}$.

Step 4: Write $\sqrt{n}$ in its simplified form, $a\sqrt{b}$.

When a square root has been simplified in this manner, a is referred to as the rational part of the number, and $\sqrt{b}$ is referred to as the irrational part.

Example 3.5.3

Simplifying a Square Root

Simplify the irrational number $\sqrt{180}$ and express in lowest terms. Identify the rational and irrational parts.

Solution

Begin by finding the largest perfect square that is a factor of 180. We can do this by writing out the factor pairs of 180:

$$1 \times 180 \quad 2 \times 90 \quad 3 \times 60 \quad 4 \times 45 \quad 5 \times 36 \quad 6 \times 30 \quad 9 \times 20 \quad 10 \times 18 \quad 12 \times 15$$

Looking at the list of factors, the perfect squares are 4, 9, and 36. The largest is 36, so we factor the into $36 \times 5 = 6^2 \times 5$. In the formula, $a = 6$ and $b = 5$. Apply $sqrta^2 \times b = \sqrt{a^2} \times \sqrt{b}$.

$\sqrt{6^2 \times 5} = \sqrt{6^2} \times \sqrt{5}$

The simplified form of $\sqrt{180}$ is $6\sqrt{5}$. In this example, the 6 is the rational part, and the $\sqrt{5}$ is the irrational part.

Your Turn 3.5.3

Simplify the irrational number $\sqrt{550}$ and express in lowest terms. Identify the rational and irrational parts.

Video 3.5.3

Simplifying Square Roots

Example 3.5.4

Simplifying a Square Root

Simplify the irrational number $\sqrt{330}$ and express in lowest terms. Identify the rational and irrational parts.

Solution

Begin by finding the largest perfect square that is a factor of 330. We can do this by writing out the factor pairs of 330:

$$1 \times 330 \quad 2 \times 165 \quad 3 \times 110 \quad 5 \times 66 \quad 6 \times 55 \quad 10 \times 33 \quad 11 \times 30 \quad 15 \times 22$$

Looking at the list of factors, there are no perfect squares other than 1, which means $\sqrt{330}$ is already expressed in lowest terms. In this case, 1 is the rational part, and $\sqrt{330}$ is the irrational part. **Though we could write this as $1\sqrt{330}$, but the product of 1 and any other number is just the number.**

Your Turn 3.5.4

Simplify the irrational number $\sqrt{773}$ and express in lowest terms. Identify the rational and irrational parts.

Example 3.5.5

Simplifying a Square Root

Simplify the irrational number $\sqrt{2{,}548}$ and express in lowest terms. Identify the rational and irrational parts.

Solution

Begin by finding the largest perfect square that is a factor of 2,548. We can do this by writing out the factor pairs of 2,548:

$$1 \times 2548 \quad 2 \times 1274 \quad 4 \times 637 \quad 7 \times 364 \quad 13 \times 196 \quad 14 \times 182 \quad 26 \times 98 \quad 28 \times 91 \quad 49 \times 52$$

Looking at the list of factors, the perfect squares are 4, 49, and 196. The largest is 196, so we factor the 2,548 into $196 \times 13 = 14^2 \times 13$. In the formula, $a = 14$ and $b = 5$. Apply $sqrta^2 \times b = \sqrt{a^2} \times \sqrt{b}$.

$$\sqrt{14^2 \times 13} = \sqrt{14^2} \times \sqrt{13}$$

The simplified form of $\sqrt{2{,}548}$ is $14\sqrt{13}$. In this example, 14 is the rational part, and $\sqrt{13}$ is the irrational part.

Your Turn 3.5.5

Simplify the irrational number $\sqrt{1{,}815}$.

Video 3.5.4

Simplifying Square Roots

Adding and Subtracting Irrational Numbers

Just like any other number we've worked with, irrational numbers can be added or subtracted. When working with a calculator, enter the operation and a decimal representation will be given. However, there are times when two irrational numbers may be added or subtracted without the calculator. This can happen only when the irrational parts of the irrational numbers are the same.

To add or subtract two irrational numbers that have the same irrational part, add or subtract the rational parts of the numbers, and then multiply that by the common irrational part.

FORMULA

Let our first irrational number be $a \times x$, where a is the rational and x the irrational parts.

Let the other irrational number be $b \times x$, where b is the rational and x the irrational parts.

Then $a \times x \pm b \times x = (a \pm b) \times x$.

Example 3.5.6

Subtracting Irrational Numbers with Similar Irrational Parts

If possible, subtract the following irrational numbers without using a calculator. If this is not possible, state why.

$$3\sqrt{7} - 8\sqrt{7}$$

Solution

Since these two irrational numbers have the same irrational part, $\sqrt{7}$, we can subtract without using a calculator. The rational part of the first number is 3. The rational part of the second number is 8. Using the formula yields $3\sqrt{7} - 8\sqrt{7} = (3 - 8) \times \sqrt{7} = -5\sqrt{7}$.

Your Turn 3.5.6

If possible, subtract the following irrational numbers without using a calculator. If this is not possible, state why.

$41\sqrt{15} - 23\sqrt{15}$

Example 3.5.7

Adding Irrational Numbers with Similar Irrational Parts

If possible, add the following irrational numbers without using a calculator. If this is not possible, state why.

$35\pi + 17\pi$

Solution

Since these two irrational numbers have the same irrational part,π, the addition can be performed without using a calculator. The rational part of the first number is 35. The rational part of the second number is 17. Using the formula yields $35\pi + 17\pi = (35 + 17) \times \pi = 52\pi$.

Your Turn 3.5.7

If possible, add the following irrational numbers without using a calculator. If this is not possible, state why.
$4.1\pi + 3.2\pi$

Example 3.5.8

Subtracting Irrational Numbers with Different Irrational Parts

If possible, subtract the following irrational numbers without using a calculator. If this is not possible, state why.

$19\sqrt{3} - 5.6\sqrt{7}$

Solution

The two numbers being subtracted do not have the same irrational part, so the operation cannot be performed.

Your Turn 3.5.8

If possible, subtract the following irrational numbers without using a calculator. If this is not possible, state why.
$2.1\sqrt{45} - 3.7\sqrt{5}$

Multiplying and Dividing Irrational Numbers

Just like any other number that we've worked with, irrational numbers can be multiplied or divided. When working with a calculator, enter the operation and a decimal representation will be given. Sometimes, though, you may want to retain the form of the irrational number as a rational part times an irrational part. The process is similar to adding and subtracting irrational numbers when they are in this form. We do not need the irrational parts to match. Even though they need not match, they do need to be similar, such as both irrational parts are square roots, or both irrational parts are multiples of pi. Also, if the irrational parts are square roots, we may need to reduce the resulting square root to lowest terms.

When multiplying two square roots, use the following formula. It is the same formula presented during the discussion of simplifying square roots.

FORMULA

For any two positive numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$.

When dividing two square roots, use the following formula.

FORMULA

For any two positive numbers a and b, with b not equal to 0, $\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.

To multiply or divide irrational numbers with similar irrational parts, do the following:

Step 1: Multiply or divide the rational parts.

Step 2: If necessary, reduce the result of Step 1 to lowest terms. This becomes the rational part of the answer.

Step 3: Multiply or divide the irrational parts.

Step 4: If necessary, reduce the result from Step 3 to lowest terms. This becomes the irrational part of the answer.

Step 5: The result is the product of the rational and irrational parts.

Example 3.5.9

Dividing Irrational Numbers with Similar Irrational Parts

Perform the following operations without a calculator. Simplify if possible.

1. $3\sqrt{15} \div \left(8\sqrt{3}\right)$
2. $14.7\sqrt{135} \div \left(3\sqrt{5}\right)$.

Solution

1. In this division problem, $3\sqrt{15} \div \left(8\sqrt{3}\right)$, notice that the irrational parts of these numbers are similar. They are both square roots, so follow the steps given above.

Step 1: Divide the rational parts. $3 \div 8 = \frac{3}{8}$

Step 2: If necessary, reduce the result of Step 1 to lowest terms. The 3 and 8 have no common factors, so $\frac{3}{8}$ is already in lowest terms.

Step 3: Divide the irrational parts. $\sqrt{15} \div \sqrt{3} = \frac{\sqrt{15}}{\sqrt{3}} = \sqrt{\frac{15}{3}}$

Step 4: If necessary, reduce the result from Step 3 to lowest terms. The radicand can be reduced, which yields $\sqrt{5}$.

Step 5: The result is the product of the rational and irrational parts, which is $\frac{3}{8}\sqrt{5}$.

2. In this division problem, $14.7\sqrt{135} \div \left(3\sqrt{5}\right)$, notice that the irrational parts of these numbers are similar. They are both square roots, so follow the steps given above.

Step 1: Divide the rational parts. $14.7 \div 3 = 4.9$

Step 2: If necessary, reduce the result of Step 1 to lowest terms. This rational number is expressed as a decimal so will not be reduced.

Step 3: Divide the irrational parts. $\sqrt{135} \div \sqrt{5} = \frac{\sqrt{135}}{\sqrt{5}} = \sqrt{\frac{135}{5}}$

Step 4: If necessary, reduce the result from Step 3 to lowest terms. The radicand can be reduced, which yields $\sqrt{\frac{135}{5}} = \sqrt{27} = \sqrt{9 \times 3} = 3\sqrt{3}$.

Step 5: The result is the product of the rational and irrational parts, which is $4.9 \times 3\sqrt{3} = 14.7\sqrt{3}$.

Your Turn 3.5.9

Perform the following operations without a calculator. Simplify if possible.

1. $84\sqrt{132} \div \left(14\sqrt{11}\right)$
2. $57\sqrt{792} \div \left(25\sqrt{2}\right)$

Example 3.5.10

Multiplying Irrational Numbers with Similar Irrational Parts

Perform the following operations without a calculator. Simplify if possible.

1. $\left(19\sqrt{3}\right) \times \left(5.6\sqrt{12}\right)$
2. $13\pi \times 8\pi$

Solution

1. In this multiplication problem, $\left(19\sqrt{3}\right) \times \left(5.6\sqrt{12}\right)$, notice that the irrational parts of these numbers are similar. They are both square roots. Follow the process above.

Step 1: Multiply the rational parts. $19 \times 5.6 = 106.4$

Step 2: If necessary, reduce the result of Step 1 to lowest terms. This rational number is expressed as a decimal and will not be reduced.

Step 3: Multiply the irrational parts. $\sqrt{3} \times \sqrt{12} = \sqrt{3 \times 12} = \sqrt{36}$

Step 4: If necessary, reduce the result from Step 3 to lowest terms. The radicand is 36, which is the square of 6. The irrational part reduces to $\sqrt{36} = 6$.

Step 5: The result is the product of the rational and irrational parts, which is $106.4 \times 6 = 638.4$.

Notice that sometimes multiplying or dividing irrational numbers can result in a rational number.

2. In this multiplication problem, $13\pi \times 8\pi$, notice that the irrational parts of these numbers are the same, π. Follow the process above.

Step 1: Multiply the rational parts. $13 \times 8 = 104$

Step 2: If necessary, reduce the result of Step 1 to lowest terms. That result is an integer.

Step 3: Multiply the irrational parts. $\pi \times \pi = \pi^2$.

Step 4: If necessary, reduce the result from Step 3 to lowest terms. This cannot be reduced.

Step 5: The result is the product of the rational and irrational parts, which is $104\pi^2$.

Your Turn 3.5.10

Perform the following operations without a calculator. Simplify if possible.

1. $\left(1.2\sqrt{21}\right) \times \left(4.5\sqrt{14}\right)$
2. $38\pi \div (2\pi)$

Rationalizing Fractions with Irrational Denominators

Fractions often represent that some amount is being equally divided into some number of parts. But to conceptualize a fraction in that manner, the denominator needs to be an integer. An irrational number in the denominator interferes with that interpretation of a fraction. Fractions that have denominators that are just the square root of an integer can be altered into fractions with integer denominators using a process called **rationalizing the denominator**. The process relies on the following property of square roots: $\sqrt{a} \times \sqrt{a} = a$ and the following property of fractions: $\frac{a}{b} = \frac{ac}{bc}$ for any non-zero number c.

Using these two properties, when a fraction has a square root in the denominator, we can eliminate that square root. Multiply the numerator and denominator by that square root from the denominator, $\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{\sqrt{b}\times\sqrt{b}}$. Then apply $\sqrt{a} \times \sqrt{a} = a$ to the denominator, yielding $\frac{a\sqrt{b}}{\sqrt{b}\times\sqrt{b}} = \frac{a\sqrt{b}}{b}$. Notice that there is no longer a square root in the denominator, which allows for interpreting the fraction as dividing a whole into equal parts.

Video 3.5.5

Rationalizing the Denominator

Example 3.5.11

Rationalizing the Denominator

Rationalize the denominator of the following:

1. $\frac{5}{\sqrt{7}}$
2. $\frac{3\sqrt{6}}{2\sqrt{10}}$

Solution

1. The square root in the denominator is $\sqrt{7}$. In order to rationalize the denominator of $\frac{5}{\sqrt{7}}$, we need to multiply the numerator and denominator by $\sqrt{7}$ and simplify.
   $\frac{5}{\sqrt{7}} = \frac{5\sqrt{7}}{\sqrt{7} \times \sqrt{7}} = \frac{5\sqrt{7}}{7}$
   The square root is in simplified form, so the final answer is $\frac{5\sqrt{7}}{7}$.
2. The square root in the denominator is $\sqrt{10}$.
   Step 1: In order to rationalize the denominator of $\frac{3\sqrt{6}}{2\sqrt{10}}$, we need to multiply the numerator and denominator by $\sqrt{10}$ and simplify.
   $\frac{3\sqrt{6}}{2\sqrt{10}} = \frac{3\sqrt{6} \times \sqrt{10}}{2\sqrt{10} \times \sqrt{10}} = \frac{3\sqrt{60}}{2 \times 10} = \frac{3\sqrt{60}}{20}$

   Step 2: The 60 under the square root can be factored into the following factor pairs:

   $1 \times 60 \quad 2 \times 30 \quad 3 \times 20 \quad 4 \times 15 \quad 5 \times 12 \quad 6 \times 10$

   Step 3: The largest square factor of 60 is 4, so we simplify the $\sqrt{60}$ in the numerator into $2\sqrt{15}$. We also cancel any common factors.

   $\frac{3\sqrt{60}}{20} = \frac{3 \times 2\sqrt{15}}{20} = \frac{6\sqrt{15}}{20} = \frac{3\sqrt{15}}{10}$

   This is completely simplified.

Your Turn 3.5.11

Rationalize the denominator of the following:

1. $\frac{24}{\sqrt{15}}$
2. $\frac{11\sqrt{14}}{6\sqrt{21}}$

There are occasions when the denominator is irrational but is the sum of two numbers where one or both involve square roots. For instance, $\frac{5}{4+\sqrt{3}}$. The process used earlier required that the denominator was the square root of a number and would not work here. However, this type of denominator can be rationalized. In order to rationalize such a denominator, we will multiply the numerator and denominator of the fraction by the **conjugate** of the denominator. The conjugate of $a+b$ is $a-b$. We say that $a+b$ and $a-b$ are **conjugate numbers**.

So, the conjugate of $-3+\sqrt{10}$ is just $-3-\sqrt{10}$. But why is this of interest? The reason is because it leads to the **difference of squares** formula, which is used to factor the difference of two squares. Or, for our purposes, in reverse it allows us to eliminate a square root.

FORMULA

For any two numbers, a and b, $a^2 - b^2 = (a-b)(a+b)$.

Looking at that formula, you should see that the two factors on the right-hand side of the equals sign are conjugates of one another. So, for our purposes, we're interested in $(a-b)(a+b) = a^2 - b^2$. This tells us that when we multiply $a+b$ by its conjugate, we get a squared minus b squared, or |(a^2 - b^2\). But how is this useful? Let's return to the fraction above, $\frac{5}{4+\sqrt{3}}$. The denominator is $4+\sqrt{3}$. Its conjugate is $4-\sqrt{3}$. According to the formula, and letting $a=4$ and $b=\sqrt{3}$, we see that $\left(4+\sqrt{3}\right)\left(4-\sqrt{3}\right) = 4^2 - \left(\sqrt{3}\right)^2$. But $\left(\sqrt{3}\right)^2$ is just 3. That means the product is $16-3$ or 13. This no longer has a square root. We use this to rationalize the denominator.

We will also need the **distributive property** of numbers.

FORMULA

For any three numbers a, b, and c, $a \times (b \pm c) = a \times b \pm a \times c$. This is called the distributive property.

Example 3.5.12

Rationalizing the Denominator Using Conjugates

Rationalize the denominator of $\frac{4}{6+\sqrt{10}}$.

Solution

Step 1: We recognize that the denominator is the sum of two numbers where one or both involve square roots. This means the conjugate can be used to remove the square root from the denominator.

Step 2: To do so, we multiply the numerator and the denominator each by the conjugate of the denominator. Since the denominator is $6+\sqrt{10}$, the conjugate we will use is $6-\sqrt{10}$.

Step 3: The conjugate is multiplied by the numerator and the denominator.

$$\frac{4}{6+\sqrt{10}} \times \frac{6-\sqrt{10}}{6-\sqrt{10}}$$

Step 4: Remembering how a number times its conjugate works, this becomes

$$\frac{4}{6+\sqrt{10}} \times \frac{6-\sqrt{10}}{6-\sqrt{10}} = \frac{4 \times (6-\sqrt{10})}{6^2 - \left(\sqrt{10}\right)^2}.$$

Step 5: In the numerator, we apply the distributive property. Using it yields
$\frac{4 \times \left(6-\sqrt{10}\right)}{6^2 - \left(\sqrt{10}\right)^2} = \frac{24-4\sqrt{10}}{36-10} = \frac{24-4\sqrt{10}}{26}$.

Step 6: Notice that the denominator no longer contains a square root. It has been rationalized. If desired, this can then be written as a rational number minus an irrational number, by recalling that $\frac{a-b}{c} = \frac{a}{c} - \frac{a}{c}$.

Applying that to the answer, we have $\frac{4}{6+\sqrt{10}} = \frac{24-4\sqrt{10}}{26} = \frac{24}{26} - \frac{4\sqrt{10}}{26}$.

Step 7: With a bit of cancellation, this reduces to $\frac{4}{6+\sqrt{10}} = \frac{24}{26} - \frac{4\sqrt{10}}{26} = \frac{12}{13} - \frac{2\sqrt{10}}{13}$.

Your Turn 3.5.12

Rationalize the denominator of $\frac{15}{5-\sqrt{13}}$.

Video 3.5.6

Rationalizing the Denominator

Check Your Understanding

1. Simplify the following square root: $\sqrt{500}$.
2. Perform the following operation: $3\sqrt{7} - 10\sqrt{7}$.
3. Perform the following operation:$8\sqrt{10} \times 3\sqrt{2}$.
4. Rationalize the denominator of the following:$\frac{4}{\sqrt{7}}$.

Section Exercises

1. Identify which of the following numbers are irrational: $\sqrt{144}, 4.33, \sqrt{70}, 5+9\pi$
2. Identify which of the following numbers are irrational: $\frac{13}{\sqrt{46}}, 4+13\pi, \sqrt{144}, \frac{5}{9}$

For the following exercises, simplify the square root by expressing it in lowest terms.

3. $\sqrt{12}$
4. $\sqrt{75}$
5. $\sqrt{605}$
6. $\sqrt{45}$
7. $\sqrt{112}$
8. $\sqrt{396}$
9. $\sqrt{2{,}940}$
10. $\sqrt{2{,}400}$
11. $\sqrt{3{,}240}$
12. $\sqrt{5{,}472}$

For the following exercises, perform the arithmetic operations without a calculator, if possible. If it is not possible, state why.

13. $4\sqrt{3}+2\sqrt{3}$
14. $8\sqrt{5}+3\sqrt{5}$
15. $9\sqrt{7}-15\sqrt{7}$
16. $\sqrt{13}-15\sqrt{13}$
17. $8\pi-13\sqrt{2}$
18. $7\sqrt{5}+6\sqrt{14}$
19. $7.2\pi+8.6\pi$
20. $14.5\pi-5.8\pi$
21. $19.8\sqrt{12}-6.1\sqrt{3}$
22. $7.3\sqrt{45}-6.8\sqrt{20}$
23. $\left(4\sqrt{15}\right)\times\left(3\sqrt{10}\right)$
24. $\left(7\sqrt{33}\right)\times\left(8\sqrt{66}\right)$
25. $\left(4.5\sqrt{154}\right)\div\left(3\sqrt{77}\right)$
26. $\left(70\sqrt{30}\right)\div\left(14\sqrt{6}\right)$

For the exercises (27 - 31), rationalize the denominators of the fractions, and then simplify.

27. $\frac{3}{\sqrt{2}}$
28. $\frac{8}{\sqrt{6}}$
29. $\frac{5}{\sqrt{13}}$
30. $\frac{9}{\sqrt{35}}$
31. $\frac{12}{\sqrt{40}}$
32. Determine the conjugate of $5+\sqrt{6}$.
33. Determine the conjugate of $10-\sqrt{13}$.
34. Determine the conjugate of $4+3\sqrt{5}$.
35. Determine the conjugate of $5\sqrt{15}+8\sqrt{13}$.
36. Find the product of $3+2\sqrt{7}$ and its conjugate.
37. Find the product of $2\sqrt{3}-5$ and its conjugate.

For the following exercises, rationalize the denominator of the fraction, and then simplify the fraction.

38. $\frac{4}{1+\sqrt{3}}$
39. $\frac{6}{5+\sqrt{7}}$
40. $\frac{-4}{5-\sqrt{10}}$
41. $\frac{10}{\sqrt{6}+7}$

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3.6: Real numbers and the Number Line

Learning Objectives

- Construct a number line and graph points on it.
- Use a number line to determine the order of real numbers.
- Determine the opposite of a real number.
- Determine the absolute value of a real number.

Definitions

A **set** is a collection of objects, typically grouped within braces { }, where each object is called an **element**. For example, $\{\text{red, green, blue}\}$ is a set of colors. A **subset** is a set consisting of elements that belong to a given set. For example, $\{\text{green, blue}\}$ is a subset of the color set above. A set with no elements is called the **empty set** and has its own special notation, { } or $\varnothing$.

When studying mathematics, we focus on special sets of numbers. The set of **natural (or counting) numbers**, denoted $\mathbb{N}$, is

$$\{1, 2, 3, 4, 5, ...\} \quad \textit{Natural Numbers}$$

The three periods $(...)$ is called an ellipsis and indicates that the numbers continue without bound. The set of **whole numbers**, denoted $\mathbb{W}$, is the set of natural numbers combined with zero.

$$\{0, 1, 2, 3, 4, 5, ...\} \quad \textit{Whole Numbers}$$

The set of **integers**, denoted $\mathbb{Z}$, consists of both positive and negative whole numbers, as well as zero.

$$\{..., -3, -2, -1, 0, 1, 2, 3, ...\} \quad \textit{Integers}$$

Notice that the sets of natural and whole numbers are both subsets of the set of integers.

Rational numbers, denoted $\mathbb{Q}$, are defined as any number of the form $\frac{a}{b}$, where a and b are integers and b is nonzero. Decimals that repeat or terminate are rational. For example,

$$0.7 = \tfrac{7}{10} \quad \text{and} \quad 0.\overline{3} = 0.3333\cdots = \tfrac{1}{3}$$

The set of integers is a subset of the set of rational numbers because every integer can be expressed as a ratio of the integer and 1. In other words, any integer can be written over 1 and can be considered a rational number. For example,

$$5 = \tfrac{5}{1}$$

Irrational numbers are defined as any number that cannot be written as a ratio of two integers. Nonterminating decimals that do not repeat are irrational. For example,

$$\pi = 3.14159\ldots \quad \text{and} \quad \sqrt{2} = 1.41421\ldots$$

The set of **real numbers**, denoted $\mathbb{R}$, is defined as the set of all rational numbers combined with the set of all irrational numbers. Therefore, all the numbers defined so far are subsets of the set of real numbers. In summary,

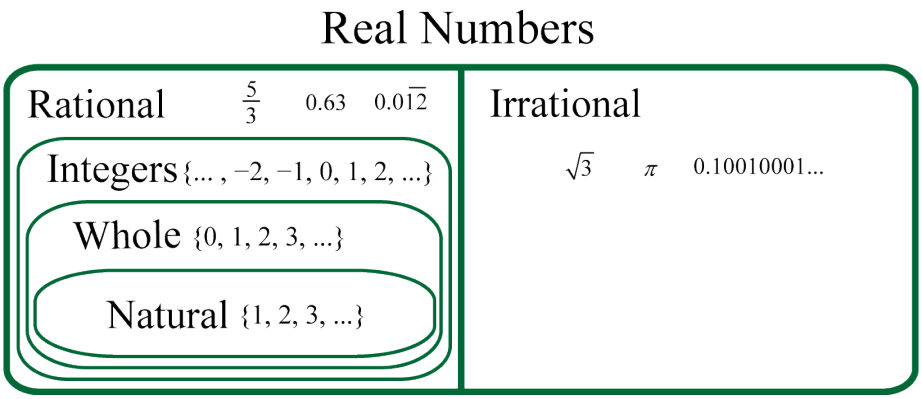


Figure 3.6.1: Real Numbers

Number Line

A **real number line**, or simply number line, allows us to visually display real numbers by associating them with unique points on a line. The real number associated with a point is called a **coordinate**. A point on the real number line that is associated with a coordinate is called its **graph**.

To construct a number line, draw a horizontal line with arrows on both ends to indicate that it continues without bound. Next, choose any point to represent the number zero; this point is called the **origin**.

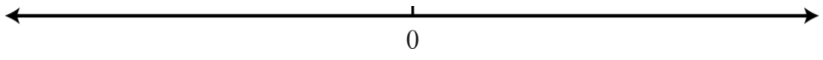


Figure 3.6.2

Mark off consistent lengths on both sides of the origin and label each tick mark to define the scale. Positive real numbers lie to the right of the origin and negative real numbers lie to the left. The number zero (0) is neither positive nor negative. Typically, each tick represents one unit.

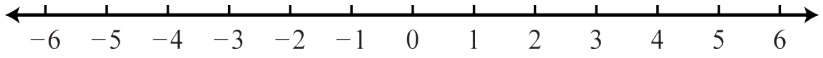


Figure 3.6.3

As illustrated below, the scale need not always be one unit. In the first number line, each tick mark represents two units. In the second, each tick mark represents $\frac{1}{7}$ of a unit.

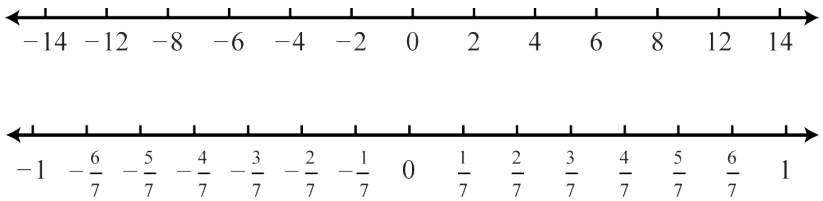


Figure 3.6.4

The graph of each real number is shown as a dot at the appropriate point on the number line. A partial graph of the set of integers $\mathbb{Z}$ follows:

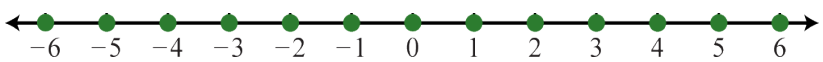


Figure 3.6.5

Example 3.6.1

Graph the following set of real numbers:

Solution

Graph the numbers on a number line with a scale where each tick mark represents $\frac{1}{2}$ of a unit.

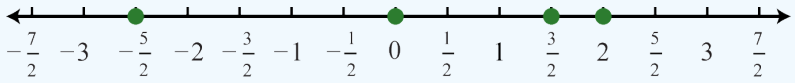


Figure 3.6.6

Ordering Real Numbers

When comparing real numbers on a number line, the larger number will always lie to the right of the smaller one. It is clear that 15 is greater than 5, but it may not be so clear to see that -1 is greater than -5 until we graph each number on a number line.

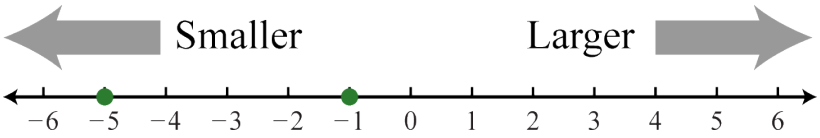


Figure 3.6.7

We use symbols to help us efficiently communicate relationships between numbers on the number line. The symbols used to describe an **equality relationship** between numbers follow:

$=$ *is equal to*
$\neq$ *is not equal to*
$\approx$ *is approximately equal to*

These symbols are used and interpreted in the following manner:

$5 = 5$ *5 is equal to 5*
$0 \neq 5$ *0 is not equal to 5*
$\pi \approx 3.14$ *pi is approximately equal to 3.14*

We next define symbols that denote an **order relationship** between real numbers.

$<$ *Less than*
$>$ *Greater than*
$\leq$ *Less than or equal to*
$\geq$ *Greater than or equal to*

These symbols allow us to compare two numbers. For example,

Since the graph of -120 is to the left of the graph of -10 on the number line, that number is less than -10. We could write an equivalent statement as follows:

Similarly, since the graph of zero is to the right of the graph of any negative number on the number line, zero is greater than any negative number.

The symbols $<$ and $>$ are used to denote **strict inequalities**, and the symbols and are used to denote **inclusive inequalities**. In some situations, more than one symbol can be correctly applied. For example, the following two statements are both true:

In addition, the “or equal to” component of an inclusive inequality allows us to correctly write the following:

The logical use of the word “or” requires that only one of the conditions need be true: the “less than” or the “equal to.”

Fill in the blank with $<$, $=$, or $>$: -2 ___ -12.

Solution

Use > because the graph of -2 is to the right of the graph of -12 on a number line. Therefore, $-2 > -12$, which reads "negative two is greater than negative twelve."

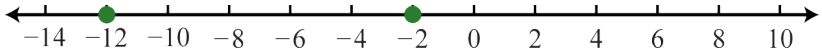


Figure 3.6.8

Answer:

$-2 > -12$

In this text, we will often point out the equivalent notation used to express mathematical quantities electronically using the standard symbols available on a keyboard. We begin with the equivalent textual notation for inequalities:

$$\begin{array}{ll} \geq & ">=" \\ \leq & "<=" \\ \neq & "!=" \end{array}$$

Many calculators, computer algebra systems, and programming languages use this notation.

Opposites

The **opposite** of any real number a is $-a$. Opposite real numbers are the same distance from the origin on a number line, but their graphs lie on opposite sides of the origin and the numbers have opposite signs.

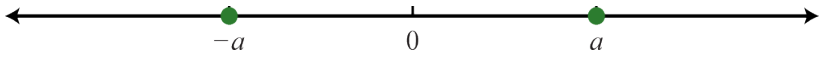


Figure 3.6.9

For example, we say that the opposite of 10 is -10.

Next, consider the opposite of a negative number. Given the integer -7, the integer the same distance from the origin and with the opposite sign is $+7$, or just 7.

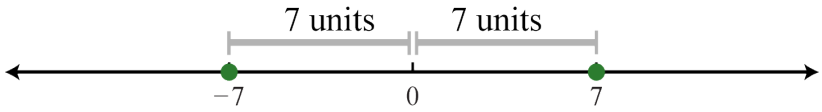


Figure 3.6.10

Therefore, we say that the opposite of -7 is $-(-7) = 7$. This idea leads to what is often referred to as the **double-negative property**. For any real number a,

$$-(-a) = a$$

Example 3.6.3

What is the opposite of $-\frac{3}{4}$?

Solution

Here we apply the double-negative property.

$$-(-\tfrac{3}{4}) = \tfrac{3}{4}$$

Example 3.6.4

Simplify $-(-(4))$

Solution

Start with the innermost parentheses by finding the opposite of $+4$.

$$\begin{aligned} -(-(4)) &= -(-(4)) \\ &= -(-4) \\ &= 4 \end{aligned}$$

Answer

4

Example 3.6.5

Simplify: $-(-(-2))$.

Solution

Apply the double-negative property starting with the innermost parentheses.

$$\begin{aligned} -(-(-2)) &= -(-(-2)) \\ &= -(2) \\ &= -2 \end{aligned}$$

Answer

-2

tip

If there is an even number of consecutive negative signs, then the result is positive. If there is an odd number of consecutive negative signs, then the result is negative.

Try this!

Exercise 3.6.1

Simplify: $-(-(-(5)))$.

Answer available at the end of this section.

Absolute Value

The **absolute value** of a real number a, denoted $|a|$, is defined as the distance between zero (the origin) and the graph of that real number on the number line. Since it is a distance, it is always positive. For example,

$$|-4| = 4 \quad \text{and} \quad |4| = 4$$

Both 4 and -4 are four units from the origin, as illustrated below:

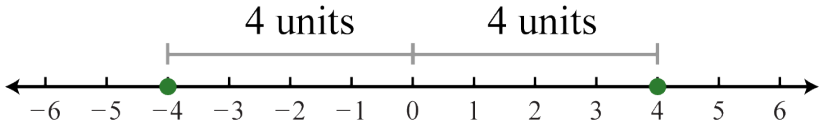


Figure 3.6.11

Example 3.6.6

Simplify:

a. $|-12|$

b.$|12|$

Solution

Both -12 and 12 are twelve units from the origin on a number line. Therefore,

$$|-12| = 12 \quad \text{and} \quad |12| = 12$$

Answer

a.12 b.12

Also, it is worth noting that

$$|0| = 0$$

The absolute value can be expressed textually using the notation abs(a). We often encounter negative absolute values, such as $-|3|$ or $-$ abs(3). Notice that the negative sign is in front of the absolute value symbol. In this case, work the absolute value first and then find the opposite of the result.

Try not to confuse this with the double-negative property, which states that $-(-7) = +7$.

Simplfy: $-|-(-7)|$.

Solution

First, find the opposite of -7 inside the absolute value. Then find the opposite of the result.

$$\begin{aligned} -|-(-7)| &= -|7| \\ &= -7 \end{aligned}$$

Answer

-7

At this point, we can determine what real numbers have a particular absolute value. For example,

$$|?| = 5$$

Think of a real number whose distance to the origin is 5 units. There are two solutions: the distance to the right of the origin and the distance to the left of the origin, namely, $\{\pm 5\}$. The symbol $(\pm)$ is read “plus or minus” and indicates that there are two answers, one positive and one negative.

$$|-5| = 5 \quad \text{and} \quad |5| = 5$$

Now consider the following:

$$|?| = -5$$

Here we wish to find a value for which the distance to the origin is negative. Since negative distance is not defined, this equation has no solution. If an equation has no solution, we say the solution is the empty set: $\varnothing$.

Key Takeaways

- Any real number can be associated with a point on a line.
- Create a number line by first identifying the origin and marking off a scale appropriate for the given problem.
- Negative numbers lie to the left of the origin and positive numbers lie to the right.
- Smaller numbers always lie to the left of larger numbers on the number line.
- The opposite of a positive number is negative and the opposite of a negative number is positive.
- The absolute value of any real number is always positive because it is defined to be the distance from zero (the origin) on a number line.
- The absolute value of zero is zero.

Exercise 3.6.2

Use set notation to list the described elements.

1. The hours on a clock.
2. The days of the week.
3. The first ten whole numbers.
4. The first ten natural numbers.
5. The first five positive even integers.
6. The first five positive odd integers.

Answer available at the end of this section.

Exercise 3.6.3

Determine whether the following real numbers are integers, rational, or irrational.

1. 12
2. -3
3. 4.5
4. -5
5. $0.3\overline{6}$
6. $0.\overline{3}$
7. $1.001000100001\ldots$
8. $1.00\overline{1}$
9. $e = 2.71828\ldots$
10. $\sqrt{7} = 2.645751\ldots$
11. -7
12. 3.14
13. 227
14. 1.33
15. 0
16. $8,675,309$

Answer available at the end of this section.

Exercise 3.6.4

True or false.

1. All integers are rational numbers.
2. All integers are whole numbers.
3. All rational numbers are whole numbers.
4. Some irrational numbers are rational.
5. All terminating decimal numbers are rational.
6. All irrational numbers are real.

Answer available at the end of this section.

Exercise 3.6.5

Choose an appropriate scale and graph the following sets of real numbers on a number line.

1. $\{-3, 0, 3\}$
2. $\{-2, 2, 4, 6, 8, 10\}$
3. $\{-2,\ -1/3,\ 2/3,\ 5/3\}$
4. $\{-5/2,\ -1/2,\ 0,\ 1/2,\ 2\}$
5. $\{-5/7,\ 0,\ 2/7,\ 1\}$
6. $\{-5,\ -2,\ -1,\ 0\}$
7. $\{-3,\ -2,\ 0,\ 2,\ 5\}$
8. $\{-2.5, -1.5, 0, 1, 2.5\}$
9. $\{0, 0.3, 0.6, 0.9, 1.2\}$
10. $\{-10, 30, 50\}$
11. $\{-6, 0, 3, 9, 12\}$
12. $\{-15, -9, 0, 9, 15\}$

Answer available at the end of this section.

Exercise 3.6.6

Fill in the blank with $<$, $=$, or $>$.

1. -7 ___ 0
2. 30 ___ 2
3. 10 ___ -10
4. -150 ___ -75
5. -0.5 ___ -1.5
6. 0 ___ 0
7. -500 ___ 200
8. -1 ___ -200
9. -10 ___ -10
10. -40 ___ -41

Answer available at the end of this section.

Exercise 3.6.7

True or false.

1. $5 \neq 7$
2. $4 = 5$
3. $1 \neq 1$
4. $-5 > -10$
5. $4 \leq 4$
6. $-12 \geq 0$
7. $-10 = -10$
8. $3 > 3$
9. $-1000 < -20$
10. $0 = 0$

Answer available at the end of this section.

Exercise 3.6.8

List the numbers.

1. List three integers less than -5.
2. List three integers greater than -10.
3. List three rational numbers less than zero.
4. List three rational numbers greater than zero.
5. List three integers between -20 and -5.
6. List three rational numbers between 0 and 1.

Answer available at the end of this section.

Exercise 3.6.9

Translate each statement into an English sentence.

1. $10 < 20$
2. $-50 \leq -10$
3. $-4 \neq 0$
4. $30 \geq -1$
5. $0 = 0$
6. $e \approx 2.718$

Answer available at the end of this section.

Exercise 3.6.10

Translate the following into a mathematical statement.

1. Negative seven is less than zero.
2. Twenty-four is not equal to ten.
3. Zero is greater than or equal to negative one.
4. Four is greater than or equal to negative twenty-one.
5. Negative two is equal to negative two.
6. Negative two thousand is less than negative one thousand.

Answer available at the end of this section.

Exercise 3.6.11

Simplify.

1. $-(-9)$
2. $-(-35)$
3. $-(10)$
4. $-(3)$
5. $-(5)$
6. $-(34)$
7. $-(-1)$
8. $-(-(-1))$
9. $-(-(1))$
10. $-(-(-3))$
11. $-(-(-(-11)))$

Answer available at the end of this section.

Exercise 3.6.12

Answer the following questions.

1. What is the opposite of -12
2. What is the opposite of π ?
3. What is the opposite -0.01?
4. Is the opposite of -12 smaller or larger than -11?
5. Is the opposite of 7 smaller or larger than -6?

Answer available at the end of this section.

Exercise 3.6.13

Fill in the blank with $<$, $=$, or $>$.

1. -7 ___ $-(-8)$
2. 6 ___ $-(6)$
3. 13 ___ $-(-12)$
4. $-(-5)$ ___ $-(-2)$
5. -100 ___ $-(-(-50))$
6. 44 ___ $-(-44)$

Answer available at the end of this section.

Exercise 3.6.14

Simplify.

1. $|20|$
2. $|-20|$
3. $|-33|$
4. $|-0.75|$
5. $|-\frac{3}{5}|$
6. $|38|$

7. $|0|$
8. $|1|$
9. $-|12|$
10. $-|-20|$
11. $-|20|$
12. $-|-8|$
13. $-|7|$
14. $-|-316|$
15. $-(-|\frac{8}{9}|)$
16. $|-(-2)|$
17. $-|-(-3)|$
18. $-(-|5|)$
19. $-(-|-45|)$
20. $-|-(-21)|$
21. $\text{abs}(6)$
22. $\text{abs}(-7)$
23. $-\text{abs}(5)$
24. $-\text{abs}(-19)$
25. $-(\ -\text{abs}(9))$
26. $-\text{abs}(-(-12))$

Answer available at the end of this section.

Exercise 3.6.15

Determine the unknown.

1. $|\ ?\ | = 9$
2. $|\ ?\ | = 15$
3. $|\ ?\ | = 0$
4. $|\ ?\ | = 1$
5. $|\ ?\ | = -8$
6. $|\ ?\ | = -20$
7. $|?| - 10 = -2$
8. $|?| + 5 = 14$

Answer available at the end of this section.

Exercise 3.6.16

Fill in the blank with $<$, $=$, or $>$.

1. $|-2|$ ____ 0
2. $|-7|$ ____ $|-10|$
3. -10 ____ $-|-2|$
4. $|-6|$ ____ $|-(-6)|$
5. $-|3|$ ____ $|-(-5)|$
6. 0 ____ $-|-(-4)|$

Answer available at the end of this section.

Exercise 3.6.17

Discussion Board Topics.

1. Research and discuss the history of the number zero.
2. Research and discuss the various numbering systems throughout history.
3. Research and discuss the definition and history of π.
4. Research the history of irrational numbers. Who is credited with proving that the square root of 2 is irrational and what happened to him?
5. Research and discuss the history of absolute value.
6. Discuss the “just make it positive” definition of absolute value

Answers

Exercise 3.6.1

-5

Procedure:

$$\begin{aligned} -(-(-(5))) &= -(-(-(5))) \\ &= -(-(-5)) \\ &= -(5) \\ &= -5 \end{aligned}$$

Video Solution:

(click to see video)

Exercise 3.6.2

1. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

3. $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

5. $\{2, 4, 6, 8, 10\}$

Exercise 3.6.3

1: Integer, Rational

3: Rational

5: Rational

7: Irrational

9: Irrational

11: Integer, Rational

13: Rational

15: Integer, Rational

Exercise 3.6.4

1: True

3: False

5: True

Exercise 3.6.5

1. $\{-3, 0, 3\}$

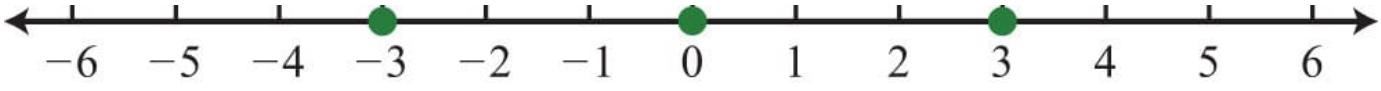


Figure 3.6.12

3. $\{-2, -1/3, 2/3, 5/3\}$

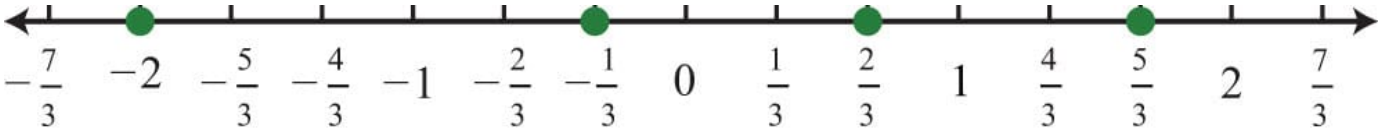


Figure 3.6.13

5. $\{-5/7, 0, 2/7, 1\}$

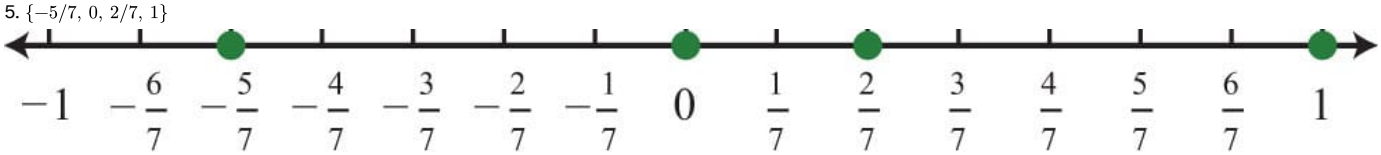


Figure 3.6.14

7. $\{-3, -2, 0, 2, 5\}$

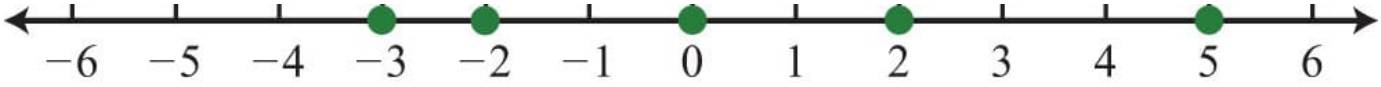


Figure 3.6.15

9. $\{0, 0.3, 0.6, 0.9, 1.2\}$

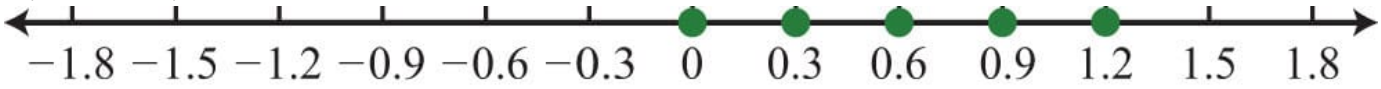


Figure 3.6.16

11. $\{-6, 0, 3, 9, 12\}$

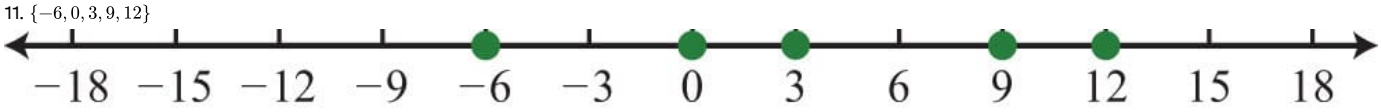


Figure 3.6.17

Exercise 3.6.6

1. $<$
3. $>$
5. $>$
7. $<$
9. $=$

Exercise 3.6.7

1. True
3. False
5. True
7. True
9. True

Exercise 3.6.8

1. $-10, -7, -6$ (answers may vary)
3. $-1, -2/3, -1/3$ (answers may vary)
5. $-15, -10, -7$ (answers may vary)

Exercise 3.6.9

1. Ten is less than twenty.
3. Negative four is not equal to zero.
5. Zero is equal to zero.

Exercise 3.6.10

1. $-7 < 0$
3. $0 \geq -1$
5. $-2 = -2$

Exercise 3.6.11

1. 9
3. -10
5. -5
7. 1
9. 1
11. 11

Exercise 3.6.12

2. $-\pi$
4. Larger

Exercise 3.6.13

1. $<$
3. $>$
5. $<$

Exercise 3.6.14

1. 20
3. 33
5. $\frac{3}{5}$
7. 0
9. -12
11. -20
13. -7

15. $\frac{8}{9}$
17. -3
19. 45
21. 6
23. -5
25. 9

Exercise $3.6.15$

1. ± 9
3. 0
5. $\varnothing$, No solution
7. ± 8

Exercise $3.6.16$

1. $>$
3. $<$
5. $<$

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3.7: Absolute Value

Learning Objectives

- understand the geometric and algebraic definitions of absolute value

Geometric Definition of Absolute Value

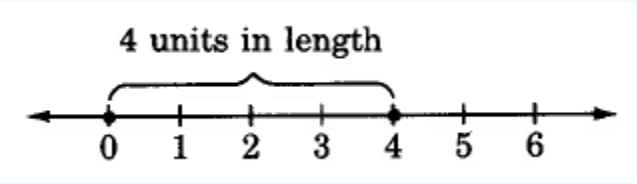
Absolute Value-Geometric Approach
Geometric definition of absolute value:
The **absolute value** of a number a, denoted $|a|$, is the distance from a to 0 on the number line.

Absolute value answers the question of "how far," and not "which way." The phrase "how far" implies "length" and *length is always a nonnegative quantity*. Thus, the absolute value of a number is a nonnegative number.

Sample Set A

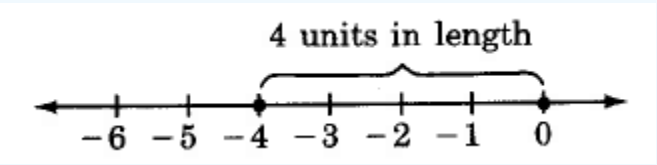
Determine each value.

$|4| = 4$



Sample Set A

$|-4| = 4$



Sample Set A

$|0| = 0$

Sample Set A

$-|5| = -5$. The quantity on the left side of the equal sign is read as "negative the absolute value of 5." The absolute value of 5 is 5. Hence, negative the absolute value of 5 is -5.

Sample Set A

$-|-3| = -3$. The quantity on the left side of the equal sign is read as "negative the absolute value of -3." The absolute value of -3 is 3. Hence, negative the absolute value of -3 is $-(3) = -3$.

Practice Set A

By reasoning geometrically, determine each absolute value.

$|7|$

Answer available at the end of this section.

Practice Set A

$|-3|$

Answer available at the end of this section.

Practice Set A

$|12|$

Answer available at the end of this section.

Practice Set A

$|0|$

Answer available at the end of this section.

Practice Set A

$-|9|$

Answer available at the end of this section.

Practice Set A

$-|-6|$

Answer available at the end of this section.

Algebraic Definition of Absolute Value

From the problems in Sample Set A, we can suggest the following algebraic definition of absolute value. Note that the definition has two parts.

Absolute Value—Algebraic Approach

Algebraic definition of absolute value
The absolute value of a number *a* is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

The algebraic definition takes into account the fact that the number aa could be either positive or zero $(a \geq 0)$ or negative $(a < 0)$.

1. If the number a is positive or zero $(a \geq 0)$, the upper part of the definition applies. The upper part of the definition tells us that if the number enclosed in the absolute value bars is a nonnegative number, the absolute value of the number is the number itself.
2. The lower part of the definition tells us that if the number enclosed within the absolute value bars is a negative number, the absolute value of the number is the opposite of the number. The opposite of a negative number is a positive number.

Note

The definition says that the vertical absolute value lines may be eliminated only if we know whether the number inside is positive or negative.

Sample Set B

Use the algebraic definition of absolute value to find the following values.

$|8|$. The number enclosed within the absolute value bars is a nonnegative number, so the upper part of the definition applies. This part says that the absolute value of 8 is 8 itself.

$|8| = 8$

Sample Set B

$|-3|$. The number enclosed within absolute value bars is a negative number, so the lower part of the definition applies. This part says that the absolute value of -3 is the opposite of -3, which is $-(-3)$. By the definition of absolute value and the double-negative property,

$|-3| = -(-3) = 3$

Practice Set B

Use the algebraic definition of absolute value to find the following values.

$|7|$

Answer available at the end of this section.

Practice Set B

$|9|$

Answer available at the end of this section.

Practice Set B

$|-12|$

Answer available at the end of this section.

Practice Set B

$|-5|$

Answer available at the end of this section.

Practice Set B

$-|8|$

Answer available at the end of this section.

Practice Set B

$-|1|$

Answer available at the end of this section.

Practice Set B

$-|-52|$

Answer available at the end of this section.

Practice Set B

$-|-31|$

Answer available at the end of this section.

Exercises

Determine each of the values.

Exercise 3.7.1

$|5|$

Answer available at the end of this section.

Exercise 3.7.2

$|3|$

Exercise 3.7.3

$|6|$

Answer available at the end of this section.

Exercise 3.7.4

$|-9|$

Exercise 3.7.5

$|-1|$

Answer available at the end of this section.

Exercise 3.7.6

$|-4|$

Exercise 3.7.7

$-|3|$

Answer available at the end of this section.

Exercise 3.7.8

$-|7|$

Exercise 3.7.9

$-|-14|$

Answer available at the end of this section.

Exercise 3.7.10

$|0|$

Exercise 3.7.11

$|-26|$

Answer available at the end of this section.

Exercise 3.7.12

$-|-26|$

Exercise 3.7.13

$-(-|4|)$

Answer available at the end of this section.

Exercise 3.7.14

$-(-|2|)$

Exercise 3.7.15

$-(-|-6|)$

Answer available at the end of this section.

Exercise 3.7.16

$-(-|-42|)$

Exercise 3.7.17

$|5|-|-2|$

Answer available at the end of this section.

Exercise 3.7.18

$|-2|^3$

Exercise 3.7.19

$-|(2 \cdot 3)|$

Answer available at the end of this section.

Exercise 3.7.20

$|-2|-|-9|$

Exercise 3.7.21

$(|-6|+|4|)^2$

Answer available at the end of this section.

Exercise 3.7.22

$(|-1|-|1|)^3$

Exercise 3.7.23

$(|4|+|-6|)^2-(|-2|)^3$

Answer available at the end of this section.

Exercise 3.7.24

$-[|-10|-6]^2$

Exercise 3.7.25

$-\{-[-|-4|+|-3|]^3\}^2$

Answer available at the end of this section.

Exercise 3.7.26

A Mission Control Officer at Cape Canaveral makes the statement “lift-off, *T* minus 50 seconds.” How long is it before lift-off?

Exercise 3.7.27

Due to a slowdown in the industry, a Silicon Valley computer company finds itself in debt $2,400,000. Use absolute value notation to describe this company’s debt.

Answer available at the end of this section.

Exercise 3.7.28

A particular machine is set correctly if upon action its meter reads 0. One particular machine has a meter reading of -1.6 upon action. How far is this machine off its correct setting?

Exercises for Review

Exercise 3.7.29

Find the sum: $\frac{9}{70}+\frac{5}{21}+\frac{8}{15}$.

Answer available at the end of this section.

Exercise 3.7.30

Find the value of $\frac{\frac{3}{10}+\frac{4}{12}}{\frac{19}{20}}$.

Exercise 3.7.31

Convert $3.2\frac{3}{5}$ to a fraction

Answer available at the end of this section.

Exercise 3.7.32

The ratio of acid to water in a solution is $\frac{3}{8}$. How many mL of acid are there in a solution that contain 112 mL of water?

Exercise 3.7.33

Find the value of $-6-(-8)$.

Answer available at the end of this section.

Answers

Practice Set A

7

Practice Set A

3

Practice Set A

12

Practice Set A

0

Practice Set A

-9

Practice Set A

-6

Practice Set B

7

Practice Set B

9

Practice Set B

12

Practice Set B

5

Practice Set B

-8

Practice Set B

-1

Practice Set B

-52

Practice Set B

-31

Exercise 3.7.1

5

Exercise 3.7.3

6

Exercise 3.7.5

1

Exercise 3.7.7

-3

Exercise 3.7.9

-14

Exercise 3.7.11

26

Exercise 3.7.13

4

Exercise 3.7.15

6

Exercise 3.7.17

3

Exercise 3.7.19

6

Exercise 3.7.21

100

Exercise 3.7.23

92

Exercise 3.7.25

-1

Exercise 3.7.27

-$|-2,400,000|

Exercise 3.7.29

$\frac{9}{10}$

Exercise 3.7.31

$3\frac{13}{50}$ or $\frac{163}{50}$

Exercise 3.7.33

2

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3.8: Adding and Subtracting Integers

Learning Objectives

- Add and subtract signed integers.
- Translate English sentences involving addition and subtraction into mathematical statements.
- Calculate the distance between two numbers on a number line.

Addition and Subtraction $(+, -)$

Visualize adding $3 + 2$ on the number line by moving from zero three units to the right then another two units to the right, as illustrated below:

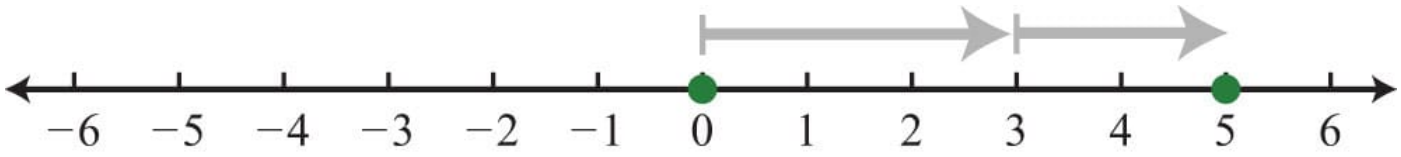


Figure 3.8.1

The illustration shows that $3 + 2 = 5$. Similarly, visualize adding two negative numbers $(-3) + (-2)$ by first moving from the origin three units to the left and then moving another two units to the left.

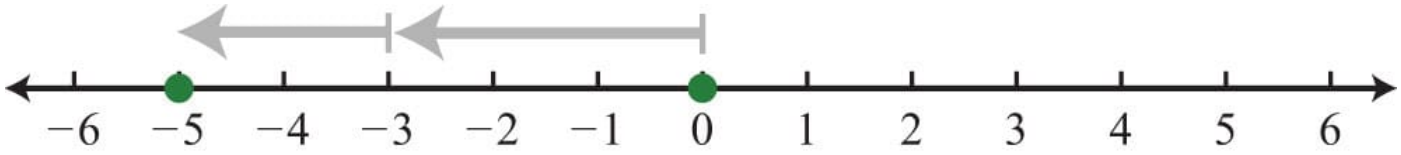


Figure 3.8.2

In this example, the illustration shows $(-3) + (-2) = -5$, which leads to the following two properties of real numbers.

$$\textit{positive number} + \textit{positive number} = \textit{positive number}$$
$$\textit{negative number} + \textit{negative number} = \textit{negative number}$$

Next, we will explore addition of numbers with unlike signs. To add $3 + (-7)$, first move from the origin three units to the right, then move seven units to the left as shown:

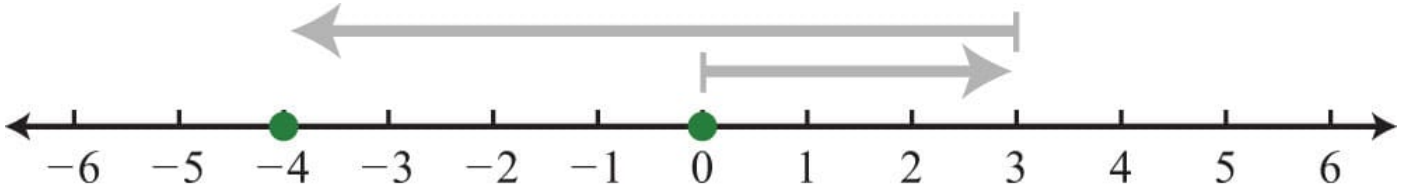


Figure 3.8.3

In this case, we can see that adding a negative number is equivalent to subtraction:

$$3 + (-7) = 3 - 7 = -4$$

It is tempting to say that a positive number plus a negative number is negative, but that is not always true: $7 + (-3) = 7 - 3 = 4$. The result of adding numbers with unlike signs may be positive or negative. The sign of the result is the same as the sign of the number with the greatest distance from the origin. For example, the following results depend on the sign of the number 12 because it is farther from zero than 5:

$$12 + (-5) = 7$$
$$-12 + 5 = -7$$

Example 3.8.1

Simplify: $14 + (-25)$.

Solution

Here -25 is the greater distance from the origin. Therefore, the result is negative.

$$\begin{aligned} 14 + (-25) &= 14 - 25 \\ &= -11 \end{aligned}$$

Answer

-11

Properties of Addition

Given any real numbers a, b, and c, we have the following properties of addition:

- **Additive Identity Property**:

$$a + 0 = 0 + a = a \tag{3.8.1}$$

- **Additive Inverse Property**:

$$a + (-a) = (-a) + a = 0 \tag{3.8.2}$$

- **Associative Property**:

$$(a + b) + c = a + (b + c) \tag{3.8.3}$$

- **Commutative Property**:

$$a + b = b + a \tag{3.8.4}$$

Below are some examples of these properties in action.

Example 3.8.2

Simplify:

a. $5 + 0$

b. $10 + (-10)$

Solution

a. Adding zero to any real number results in the same real number.

$$5 + 0 = 5$$

b. Adding opposites results in zero.

Answer

a. 5; b. 0

Simplify:

a. $(3+7)+4$

b. $3+(7+4)$

Solution

Parentheses group the operations that are to be performed first.

a.

$$\begin{aligned}(3+7)+4 &= 10+4\\ &= 14\end{aligned}$$

b.

$$\begin{aligned}3+(7+4) &= 3+10\\ &= 14\end{aligned}$$

These two examples both result in 14: changing the grouping of the numbers does not change the result.

$$(3+7)+4 = 3+(7+4) = 14$$

Answer

a. 14; b. 14

At this point, we highlight the fact that addition is commutative: the order in which we add does not matter and yields the same result.

$$\begin{aligned}2+9 &= 9+2\\ 11 &= 11\end{aligned}$$

On the other hand, subtraction is not commutative.

$$\begin{aligned}2-9 &\neq 9-2\\ -7 &\neq 7\end{aligned}$$

We will use these properties, along with the double-negative property for real numbers, to perform more involved sequential operations. To simplify things, we will make it a general rule to first replace all sequential operations with either addition or subtraction and then perform each operation in order from left to right.

Example 3.8.4

Simplify: $4-(-10)+(-5)$.

Solution

Replace the sequential operations and then perform them from left to right.

$$\begin{aligned}4-(-10)+(-5) &= 4+10-5 \qquad \begin{array}{l}\textit{Replace } -(-) \textit{ with addition } (+).\\ \textit{Replace } +(-) \textit{ with addition } (-).\end{array}\\ &= 14-5\\ &= 9\end{aligned}$$

Answer

9

Example 3.8.5

Simplify: $-3+(-8)-(-7)$.

Solution

$$\begin{aligned} -3+(-8)-(-7) &= -3-8+7 \qquad \begin{array}{l} \textit{Replace } +(-) \textit{ with } (-). \\ \textit{Replace } -(-) \textit{ with } (+). \end{array} \\ &= -11+7 \\ &= -4 \end{aligned}$$

Answer

-4

Example 3.8.6

Try this!

Simplify: $12-(-9)+(-6)$.

Solution

(click to see video)

Often we find the need to translate English sentences involving addition and subtraction to mathematical statements. Listed below are some key words that translate to the given operation.

Table 3.8.2

Key Words	Operation
Sum, increased by, more than, plus, added to, total	$+$
Difference, decreased by, subtracted from, less, minus	$-$

Example 3.8.7

What is the difference of 7 and -3?

Solution

The key word “difference” implies that we should subtract the numbers.

$$\begin{aligned} 7-(-3) &= 7+3 \\ &= 10 \end{aligned}$$

Answer

The difference of 7 and -3 is 10.

Example 3.8.8

What is the sum of the first five positive integers?

Solution

The initial key word to focus on is “sum”; this means that we will be adding the five numbers. The first five positive integers are $\{1, 2, 3, 4, 5\}$. Recall that 0 is neither positive nor negative.

$$1 + 2 + 3 + 4 + 5 = 15$$

Answer

The sum of the first five positive integers is 15.

Example 3.8.9

What is 10 subtracted from the sum of 8 and 6?

Solution

We know that subtraction is not commutative; therefore, we must take care to subtract in the correct order. First, add 8 and 6 and then subtract 10 as follows:

$$\overbrace{(8 \quad + \quad 6)}^{\textit{the sum of 8 and 6}} \qquad \overbrace{- \qquad 10}^{\textit{subtract 10 from the sum}}$$

It is important to notice that the phrase “10 subtracted from” does not translate to a mathematical statement in the order it appears. In other words, $10 - (8 + 6)$ would be an incorrect translation and leads to an incorrect answer. After translating the sentence, perform the operations.

$$\begin{aligned}(8 + 6) - 10 &= 14 - 10 \\ &= 4\end{aligned}$$

Answer

Ten subtracted from the sum of 8 and 6 is 4.

Distance on a Number Line

One application of the absolute value is to find the distance between any two points on a number line. For real numbers a and b, the **distance formula for a number line** is given as,

$$d = |b - a|$$

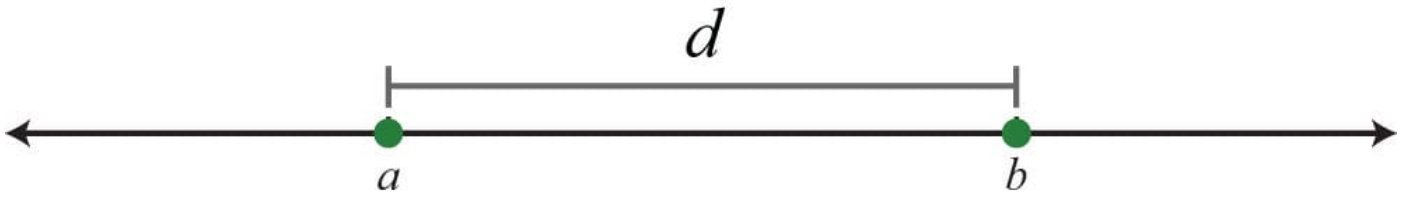


Figure 3.8.4

Example 3.8.10

Determine the distance between 2 and 7 on a number line.

Solution

On the graph we see that the distance between the two given integers is 5 units.

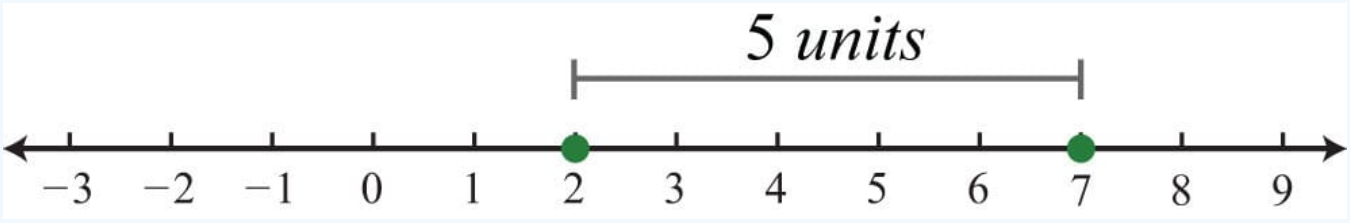


Figure 3.8.5

Using the distance formula we obtain the same result.

$$\begin{aligned} d &= |7-2| \\ &= |5| \\ &= 5 \end{aligned}$$

Answer

5 units

Example 3.8.11

Determine the distance between -4 and 7 on a number line.

Solution

Use the distance formula for a number line $d = |b-a|$, where $a = -4$ and $b = 7$.

$$\begin{aligned} d &= |7-(-4)| \\ &= |7+4| \\ &= |11| \\ &= 11 \end{aligned}$$

Answer

11 units

It turns out that it does not matter which points are used for a and$b$; the absolute value always ensures a positive result.

Table 3.8.3

Using $a = -4$ and $b = 7$	**Using $a = 7$ and $b = -4$**

$d = \|7 - (-4)\| = \|7 + 4\| = \|11\| = 11$	$d = \|-4 - 7\| = \|-11\| = 11$

Exercise 3.8.1

Determine the distance between -12 and -9 on the number line.

Answer available at the end of this section.

Key Takeaways

- A positive number added to a positive number is positive. A negative number added to a negative number is negative.
- The sign of a positive number added to a negative number is the same as the sign of the number with the greatest distance from the origin.
- Addition is commutative and subtraction is not.
- When simplifying, it is a best practice to first replace sequential operations and then work the operations of addition and subtraction from left to right.
- The distance between any two numbers on a number line is the absolute value of their difference. In other words, given any real numbers *a* and *b*, use the formula $d = |b - a|$ to calculate the distance *d* between them.

Exercise 3.8.2

Add and subtract.

1. $24 + (-18)$
2. $9 + (-11)$
3. $-31 + 5$
4. $-12 + 15$
5. $-30 + (-8)$
6. $-50 + (-25)$
7. $-7 + (-7)$
8. $-13 - (-13)$
9. $8 - 12 + 5$
10. $-3 - 7 + 4$
11. $-1 - 2 - 3 - 4$
12. $6 - (-5) + (-10) - 14$
13. $-5 + (-3) - (-7)$
14. $2 - 7 + (-9)$
15. $-30 + 20 - 8 - (-18)$
16. $10 - (-12) + (-8) - 20$
17. $5 - (-2) + (-6)$
18. $-3 + (-17) - (-13)$
19. $-10 + (-12) - (-20)$
20. $-13 + (-5) - (-25)$
21. $20 - (-4) - (-5)$
22. $17 + (-12) - (-2)$

Answer available at the end of this section.

Exercise 3.8.3

Translate each sentence to a mathematical statement and then simplify.

1. Find the sum of 3, 7, and -8.

2. Find the sum of -12, -5, and 7.
3. Determine the sum of the first ten positive integers.
4. Determine the sum of the integers in the set $\{-2, -1, 0, 1, 2\}$.
5. Find the difference of 10 and 6.
6. Find the difference of 10 and -6.
7. Find the difference of -16 and -5.
8. Find the difference of -19 and 7.
9. Subtract 12 from 10.
10. Subtract -10 from -20.
11. Subtract 5 from -31.
12. Subtract -3 from 27.
13. Two less than 8.
14. Five less than -10.
15. Subtract 8 from the sum of 4 and 7.
16. Subtract -5 from the sum of 10 and -3.
17. Subtract 2 from the difference of 8 and 5.
18. Subtract 6 from the difference of -1 and 7.
19. Mandy made a $\$200$ deposit into her checking account on Tuesday. She then wrote 4 checks for $\$50.00$, $\$125.00$, $\$60.00$, and $\$45.00$. How much more than her deposit did she spend?
20. The quarterback ran the ball three times in last Sunday's football game. He gained 7 yards on one run but lost 3 yards and 8 yards on the other two. What was his total yardage running for the game?
21. The revenue for a local photographer for the month is $\$1,200$. His costs include a studio rental of $\$600$, props costing $\$105$, materials fees of $\$135$, and a make-up artist who charges $\$120$. What is his total profit for the month?
22. An airplane flying at $30,000$ feet lost $2,500$ feet in altitude and then rose $1,200$ feet. What is the new altitude of the plane?
23. The temperature was $22\,^\circ$ at $6:00$ p.m. and dropped $26\,^\circ$ by midnight. What was the temperature at midnight?
24. A nurse has 30 milliliters of saline solution but needs 75 milliliters of the solution. How much more does she need?
25. The width of a rectangle is 2 inches less than its length. If the length measures 16 inches, determine the width.
26. The base of a triangle is 3 feet shorter than its height. If the height measures 5 feet, find the length of the base.

Answer available at the end of this section.

Exercise 3.8.4

Find the distance between the given numbers on a number line.

1. -3 and 12
2. 8 and -13
3. -25 and -10
4. -100 and -130
5. -7 and -20
6. 0 and -33
7. -10 and 10
8. -36 and 36
9. The coldest temperature on earth, $-129\,^\circ$F, was recorded in 1983 at Vostok Station, Antarctica. The hottest temperature on earth, $136\,^\circ$F, was recorded in 1922 at Al 'Aziziyah, Libya. Calculate earth's temperature range.
10. The daily high temperature was recorded as $91\,^\circ$F and the low was recorded as $63\,^\circ$F. What was the temperature range for the day?
11. A student earned 67 points on his lowest test and 87 points on his best. Calculate his test score range.
12. On a busy day, a certain website may have $12,500$ hits. On a slow day, it may have as few as 750 hits. Calculate the range of the number of hits.

Answer available at the end of this section.

Exercise 3.8.5

Discussion Board Topics

1. Share an example of adding signed numbers in a real-world application.
2. Demonstrate the associative property of addition with any three real numbers.
3. Show that subtraction is not commutative

Answers

Exercise 3.8.1

3

Video Solution

(click to see video)

Exercise 3.8.2

1: 6
3: −26
5: −38
7: −14
9: 1
11: −10
13: −1
15: 0
17: 1
19: −2
21: 29

Exercise 3.8.3

1: 2
3: 55
5: 4
7: −11
9: −2
11: −36
13: 6
15: 3
17: 1
19: $80
21: $240
23: −4 °
25: 14 inches

Exercise 3.8.4

1: 15 units
3: 15 units
5: 13 units
7: 20 units
9: 265 °F
11: 20 points

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3.9: Multiplying and Dividing Integers

Learning Objectives

- Multiply and divide signed integers.
- Translate English sentences involving multiplication and division into mathematical statements.
- Determine the prime factorization of composite numbers.
- Interpret the results of quotients involving zero.

Multiplication and Division

We begin with a review of what it means to multiply and divide signed numbers. The result of multiplying real numbers is called the **product** and the result of dividing is called the **quotient**. Recall that multiplication is equivalent to adding:

$$3 \cdot 4 = 4 + 4 + 4 = 12$$

Clearly, the product of two positive numbers is positive. Similarly, the product of a positive number and negative number can be written as shown:

We see that the product of a positive number and a negative number is negative. Next, explore the results of multiplying two negative numbers. Consider the products in the following illustration and try to identify the pattern:

$$\left.\begin{aligned} 3(-3) &= -9 \\ 2(-3) &= -6 \\ 1(-3) &= -3 \end{aligned}\right\} \quad \textit{The products increase by 3.}$$

$$0(-3) = 0 \quad \textit{Zero times any real number is zero.}$$

$$\left.\begin{aligned} (-1)(-3) &= 3 \\ (-2)(-3) &= 6 \\ (-3)(-3) &= 9 \end{aligned}\right\} \quad \textit{The pattern continues by increasing the products by 3.}$$

This shows that the product of two negative numbers is positive. To summarize,

$$\begin{aligned} \textit{positive} \times \textit{positive} &= \textit{positive} \\ \textit{positive} \times \textit{negative} &= \textit{negative} \\ \textit{negative} \times \textit{negative} &= \textit{negative} \end{aligned}$$

The rules for division are the same because division can always be rewritten as multiplication:

The rules for multiplication and division should not be confused with the fact that the sum of two negative numbers is negative.

Example 3.9.1

Simplify:

a. $(-3) + (-5)$

b. $(-3)(-5)$

Solution

Here we add and multiply the same two negative numbers.

a. The result of adding two negative numbers is negative.

$$\begin{aligned}(-3)+(-5) &= -3-5\\ &= -8\end{aligned}$$

b. The result of multiplying two negative numbers is positive.

$$(-3)(-5) = 15$$

Answer

a, -8 b.15

Given any real numbers a, b, and c, we have the following properties of multiplication:

Zero factor property:	$a \cdot 0 = 0 \cdot a = 0$
Multiplicative identity property:	$a \cdot 1 = 1 \cdot a = a$
Associative property:	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Commutative property:	$a \cdot b = b \cdot a$

Example 3.9.2

Simplify:

a. $5 \cdot 0$

b. $10 \cdot 1$

Solution

a. Multiplying by zero results in zero.

$$5 \cdot 0 = 0$$

b. Multiplying any real number by one results in the same real number.

$$10 \cdot 1 = 10$$

Answers:

a. 0 b. 10

Example 3.9.3

Simplify:

a. $(3 \cdot 7) \cdot 2$

b. $3 \cdot (7 \cdot 2)$

Solution

a.

$$(3 \cdot 7) \cdot 2 = 21 \cdot 2$$
$$= 42$$

b.

$$3 \cdot (7 \cdot 2) = 3 \cdot 14$$
$$= 42$$

The value of each expression is 42. Changing the grouping of the numbers does not change the result.

$$(3 \cdot 7) \cdot 2 = 3 \cdot (7 \cdot 2) = 42$$

Answer

a. 42 b. 42

At this point, we highlight that multiplication is commutative: the order in which we multiply does not matter and yields the same result.

$$2 \cdot 9 = 9 \cdot 2$$
$$18 = 18$$

On the other hand, division is not commutative.

$$10 \div 5 \neq 5 \div 10$$
$$2 \neq \frac{1}{2}$$

Use these properties to perform sequential operations involving multiplication and division. When doing so, it is important to perform these operations in order from left to right.

Example 3.9.4

Simplify: $3(-2)(-5)(-1)$

Solution

Multiply two numbers at a time as follows:

$$\begin{aligned} 5(-3)(-2)(-4) &= \underbrace{5(-3)}(-2)(-4) \\ &= \underbrace{-15(-2)}(-4) \\ &= \underbrace{30(-4)} \\ &= -120 \end{aligned}$$

Figure 3.9.1

Answer

-120

Because multiplication is commutative, the order in which we multiply does not affect the final answer. When sequential operations involve multiplication and division, order does matter; hence we must work the operations from left to right to obtain a correct result.

Example 3.9.5

Simplify: $10 \div (-2)(-5)$

Solution

Perform the division first; otherwise the result will be incorrect.

Correct!

$$10 \div (-2)(-5) = \underbrace{10 \div (-2)}_{\textit{division first}} (-5)$$
$$= -5(-5)$$
$$= 25 \quad \checkmark$$

Incorrect!

$$10 \div (-2)(-5) = 10 \div \underbrace{(-2)(-5)}_{\textit{multiplication first}}$$
$$= 10 \div 10$$
$$= 1 \quad \times$$

Figure 3.9.2

Notice that the order in which we multiply and divide does affect the result. Therefore, it is important to perform the operations of multiplication and division as they appear from left to right.

Answer

25

Notice that the order in which we multiply and divide does affect the final result. Therefore, it is important to perform the operations of multiplication and division as they appear from left to right.

Example 3.9.6

Simplify: $-6(3) \div (-2)(-3)$

Solution

Work the operations one at a time from left to right.

$$\begin{aligned} &-6(2) \div (-2)(-3) \\ =&-18 \div (-2)(-3) \\ =&9(-3) \\ =&-27 \end{aligned}$$

Example 3.9.7

Try this! Simplify: $-5 \div 5 \cdot 2(-3)$

Video Solution:

(click to see video)

Within text-based applications, the symbol used for multiplication is the **asterisk** (*) and the symbol used for division is the forward slash (/).

$$5 * 3 \text{ and } 14/2 = 7$$

The set of **even integers** is the set of all integers that are evenly divisible by 2. We can also obtain the set of even integers by multiplying each integer by 2.

$$\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\} \quad \textit{Even integers}$$

The set of **odd integers** is the set of all integers that are not evenly divisible by 2.

$$\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\} \quad \textit{Odd integers}$$

A **prime number** is an integer greater than 1 that is divisible only by 1 and itself. The smallest prime number is 2 and the rest are necessarily odd.

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\} \quad \textit{Prime numbers}$$

Any integer greater than 1 that is not prime is called a **composite number** and can be written as a product of primes. When a composite number, such as 30, is written as a product, $30 = 2 \cdot 15$, we say that $2 \cdot 15$ is a **factorization** of 30 and that 2 and 15 are **factors**. Note that factors divide the number evenly. We can continue to write composite factors as products until only a product of primes remains.

$$\begin{aligned} &30 \\ &= 2 \cdot 15 \\ &= 2 \cdot 3 \cdot 5 \end{aligned}$$

Figure $3.9.3$

The **prime factorization** of 30 is $2 \cdot 3 \cdot 5$.

Example $3.9.8$

Determine the prime factorization of 70.

Solution

Begin by writing 70 as a product with 2 as a factor. Then express any composite factor as a product of prime numbers.

$$\begin{aligned} 70 &= 2 \cdot 35 \\ &= 2 \cdot 5 \cdot 7 \end{aligned}$$

Since the prime factorization is unique, it does not matter how we choose to initially factor the number because the end result is the same.

Answer available at the end of this section.

$$\begin{aligned} 70 &= 7 \cdot 10 \\ &= 7 \cdot 2 \cdot 5 \\ &= 2 \cdot 5 \cdot 7 \end{aligned}$$

Answer

The prime factorization of 70 is $2 \cdot 5 \cdot 7$.

Some tests (called divisibility tests) useful for finding prime factors of composite numbers follow:

1. If the integer is even, then 2 is a factor.
2. If the sum of the digits is evenly divisible by 3, then 3 is a factor.

3. If the last digit is a 5 or 0, then 5 is a factor.

Often we find the need to translate English sentences that include multiplication and division terms to mathematical statements. Listed below are some key words that translate to the given operation.

Key Words	Operation
Product, multiplied by, of, times	* or $\cdot$
Quotient, divided by, ratio, per	/ or $\div$

Example 3.9.9

Calculate the quotient of 20 and -10.

Solution

The key word "quotient" implies that we should divide.

$$20 \div (-10) = -2$$

Answer:

The quotient of 20 and -10 is -2.

Example 3.9.10

What is the product of the first three positive even integers?

Solution

The first three positive even integers are {2, 4, 6} and the key word "product" implies that we should multiply.

$$\begin{aligned} 2 \cdot 4 \cdot 6 &= 8 \cdot 6 \\ &= 48 \end{aligned}$$

Answer

The product of the first three positive even integers is 48.

Example 3.9.11

Joe is able to drive 342 miles on 18 gallons of gasoline. How many miles per gallon of gas is this?

Solution

The key word "per" indicates that we must divide the number of miles driven by the number of gallons used:

$$\frac{342 \text{ miles}}{18 \text{ gallons}} = 19 \text{ miles per gallon (mpg)}$$

Answer

Joe gets 19 miles per gallon from his vehicle.

In everyday life, we often wish to use a single value that typifies a set of values. One way to do this is to use what is called the **arithmetic mean** or **average**. To calculate an average, divide the sum of the values in the set by the number of values in that set.

Example 3.9.12

A student earns 75, 86, and 94 on his first three exams. What is the student's test average?

Solution

Add the scores and divide the sum by 3.

$$\begin{aligned}\frac{75+86+94}{3} &= \frac{255}{3}\\ &= 85\end{aligned}$$

Answer

The student's test average is 85.

Zero and Division

Recall the relationship between multiplication and division:

$$\underbrace{12}_{\text{dividend}} \div \underbrace{6}_{\text{divisor}} = \underbrace{2}_{\text{quotient}} \quad \text{because} \quad 6 \cdot 2 = 12$$

Figure 3.9.4

In this case, the **dividend** 12 is evenly divided by the **divisor** 6 to obtain the quotient, 2. It is true in general that if we multiply the divisor by the quotient we obtain the dividend. Now consider the case where the dividend is zero and the divisor is nonzero:

$$0 \div 6 = 0 \text{ Since } 6 \cdot 0 = 0$$

This demonstrates that zero divided by any nonzero real number must be zero. Now consider a nonzero number divided by zero:

$$12 \div 0 = ? \text{ or } 0 \cdot ? = 0$$

Here any real number seems to work. For example, $0 \cdot 5 = 0$ and $0 \cdot 3 = 0$. Therefore, the quotient is uncertain or **indeterminate**.

$$0 \div 0 = \frac{0}{0} \; Is\ indeterminate.$$

In this course, we state that $0 \div 0$ is undefined.

Key Takeaways

- A positive number multiplied by a negative number is negative. A negative number multiplied by a negative number is positive.
- Multiplication is commutative and division is not.
- When simplifying, work the operations of multiplication and division in order from left to right.
- Even integers are numbers that are evenly divisible by 2 or multiples of 2, and all other integers are odd.
- A prime number is an integer greater than 1 that is divisible only by 1 and itself.
- Composite numbers are integers greater than 1 that are not prime. Composite numbers can be written uniquely as a product of primes.
- The prime factorization of a composite number is found by continuing to divide it into factors until only a product of primes remains.
- To calculate an average of a set of numbers, divide the sum of the values in the set by the number of values in the set.
- Zero divided by any nonzero number is zero. Any number divided by zero is undefined.

Exercise 3.9.1

Multiply and divide.

1. $5(-7)$
2. $-3(-8)$
3. $2(-4)(-9)$
4. $-3 \cdot 2 \cdot 5$
5. $-12(3)(0)$
6. $0(-12)(-5)$
7. $(-1)(-1)(-1)(-1)$
8. $(-1)(-1)(-1)$
9. $-100 \div 25$
10. $25 \div 5(-5)$
11. $-15(-2) \div 10(-3)$
12. $-5 \cdot 10 \div 2(-5)$
13. $(-3)(25) \div (-5)$
14. $6 * (-3)/(-9)$
15. $20/(-5) * 2$
16. $-50/2 * 5$
17. Determine the product of 11 and -3.
18. Determine the product of -7 and -22.
19. Find the product of 5 and -12.
20. Find the quotient of negative twenty-five and five.
21. Determine the quotient of -36 and 3.
22. Determine the quotient of 26 and -13.
23. Calculate the product of 3 and -8 divided by -2.
24. Calculate the product of -1 and -3 divided by 3.
25. Determine the product of the first three positive even integers.
26. Determine the product of the first three positive odd integers.

Answer available at the end of this section.

Exercise 3.9.2

Determine the prime factorization of the following integers.

1. 105
2. 78
3. 138
4. 154
5. 165
6. 330

Answer available at the end of this section.

Exercise 3.9.3

Calculate the average of the numbers in each of the following sets.

1. $\{50, 60, 70\}$
2. $\{9, 12, 30\}$
3. $\{3, 9, 12, 30, 36\}$
4. $\{72, 84, 69, 71\}$

5. The first four positive even integers.
6. The first four positive odd integers.

Answer available at the end of this section.

Exercise 3.9.4

The distance traveled D is equal to the average rate r times the time traveled t at that rate: $D = rt$. Determine the distance traveled given the rate and the time.

1. 60 miles per hour for 3 hours
2. 55 miles per hour for 3 hours
3. 15 miles per hour for 5 hours
4. 75 feet per second for 5 seconds
5. 60 kilometers per hour for 10 hours
6. 60 meters per second for 30 seconds
7. A student club ran a fund-raiser in the quad selling hot dogs. The students sold 122 hot dog meals for $\$3.00$ each. Their costs included $\$50.00$ for the hot dogs and buns, $\$25.00$ for individually wrapped packages of chips, and $\$35.00$ for the sodas. What was their profit?
8. A 230-pound man loses 4 pounds each week for 8 weeks. How much does he weigh at the end of 8 weeks?
9. Mary found that she was able to drive 264 miles on 12 gallons of gas. How many miles per gallon does her car get?
10. After filling his car with gasoline, Bill noted that his odometer reading was $45,346$ miles. After using his car for a week, he filled up his tank with 14 gallons of gas and noted that his odometer read $45,724$ miles. In that week, how many miles per gallon did Bill's car get?

Answer available at the end of this section.

Exercise 3.9.5

Perform the operations.

1. $0 \div 9$
2. $15 \div 0$
3. $4(-7) \div 0$
4. $7(0) \div (-15)$
5. $-5(0) \div 9(0)$
6. $5 \cdot 2(-3)(-5)$
7. $-8 - 5 + (-13)$
8. $-4(-8) \div 16(-2)$
9. $50 \div (-5) \div (-10)$
10. $49 \div 7 \div (-1)$
11. $3 \cdot 4 \div 12$
12. $0 - (-8) - 12$
13. $-8 \cdot 4(-3) \div 2$
14. $0/(-3 * 8 * 5)$
15. $(-4 * 3)/(2 * (-3))$
16. $-16/(-2 * 2) * 3$
17. $-44/11 * 2$
18. $-5 * 3/(-15)$
19. $4 * 3 * 2/6$
20. $-6 * 7/(-2)$
21. During 5 consecutive winter days, the daily lows were $-7\,°$, $-3\,°$, $0\,°$, $-5\,°$, and $-10\,°$. Calculate the average low temperature.
22. On a very cold day the temperature was recorded every 4 hours with the following results: $-16\,°$, $-10\,°$, $2\,°$, $6\,°$, $-5\,°$, and $-13\,°$. Determine the average temperature.

23. A student earns 9, 8, 10, 7, and 6 points on the first 5 chemistry quizzes. What is her quiz average?
24. A website tracked hits on its homepage over the Thanksgiving holiday. The number of hits for each day from Thursday to Sunday was $12,250$; $4,400$; $7,750$; and $10,200$, respectively. What was the average number of hits per day over the holiday period?

Answer available at the end of this section.

Exercise 3.9.1

Discussion Board Topics.

1. Demonstrate the associative property of multiplication with any three real numbers.
2. Show that division is not commutative.
3. Discuss the importance of working multiplication and division operations from left to right. Make up an example where order does matter and share the solution.
4. Discuss division involving 0. With examples, explain why the result is sometimes 0 and why it is sometimes undefined.
5. Research and discuss the fundamental theorem of arithmetic.
6. Research and discuss other divisibility tests. Provide an example for each test.
7. The arithmetic mean is one way to typify a set of values. Research other methods used to typify a set of values

Answers

Example 3.9.8

Exercise 3.9.1

1: -35
3: 72
5: 0
7: 1
9: -4
11: -9
13: 15
15: -8
17: -33
19: -60
21: -12
23: 12
25: 48

Exercise 3.9.2

1: $3 \cdot 5 \cdot 7$
3: $2 \cdot 3 \cdot 23$
5: $3 \cdot 5 \cdot 11$

Exercise 3.9.3

1: 60
2: 18
5: 5

Exercise 3.9.4

1: 180 miles
3: 75 miles
5: 600 kilometers
7: $\$256.00$
9: 22 miles per gallon
11: 0

Exercise 3.9.5

1: 0
3: Undefined
5: 0

7: -26
9: 1
11: 1
13: 48
15: 2
17: -8
19: 4
21: -5 °
23: 8 points

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3.10: Fractions

Learning Objectives

- Reduce a fraction to lowest terms.
- Multiply and divide fractions.
- Add and subtract fractions.

Reducing

A **fraction** is a real number written as a quotient, or ratio, of two integers a and b, where $b \neq 0$.



Figure 3.10.1

The integer above the fraction bar is called the **numerator** and the integer below is called the **denominator**. The numerator is often called the "part" and the denominator is often called the "whole." **Equivalent fractions** are two equal ratios expressed using different numerators and denominators. For example,

$$\frac{50}{100} = \frac{1}{2}$$

Fifty parts out of 100 is the same ratio as 1 part out of 2 and represents the same real number. Consider the following factorizations of 50 and 100:

$$\begin{aligned} 50 &= 2 \cdot 25 \\ 100 &= 4 \cdot 25 \end{aligned}$$

The numbers 50 and 100 share the factor 25. A shared factor is called a common factor. We can rewrite the ratio $\frac{50}{100}$ as follows:

$$\frac{50}{100} = \frac{2 \cdot 25}{4 \cdot 25}$$

Making use of the multiplicative identity property and the fact that $\frac{25}{25} = 1$, we have

$$\frac{50}{100} = \frac{2 \cdot \cancel{25}}{4 \cdot \cancel{25}} = \frac{2}{4} \cdot 1 = \frac{2}{4}$$

Figure 3.10.2

Dividing $\frac{25}{25}$ and replacing this factor with a 1 is called **canceling**. Together, these basic steps for finding equivalent fractions define the process of **reducing**. Since factors divide their product evenly, we achieve the same result by dividing both the numerator and denominator by 25 as follows:

$$\frac{50 \div 25}{100 \div 25} = \frac{2}{4}$$

Figure 3.10.3

Finding equivalent fractions where the numerator and denominator have no common factor other than 1 is called **reducing to lowest terms**. When learning how to reduce to lowest terms, it is helpful to first rewrite the numerator and denominator as a product of primes and then cancel. For example,

$$\begin{aligned} \frac{50}{100} &= \frac{\cancel{2}^{1} \cdot \cancel{5}^{1} \cdot \cancel{5}^{1}}{2 \cdot \cancel{2}_{1} \cdot \cancel{5}_{1} \cdot \cancel{5}_{1}} \\ &= \frac{1}{2} \end{aligned}$$

Figure 3.10.4

We achieve the same result by dividing the numerator and denominator by the **greatest common factor** (GCF). The GCF is the largest number that divides both the numerator and denominator evenly. One way to find the GCF of 50 and 100 is to list all the factors of each and identify the largest number that appears in both lists. Remember, each number is also a factor of itself.

$$\begin{aligned} &\{1, 2, 5, 10, 25, 50\} && \textit{Factors of } 50 \\ &\{1, 2, 4, 5, 10, 20, 25, 50, 100\} && \textit{Factors of } 100 \end{aligned}$$

Common factors are listed in bold, and we see that the greatest common factor is 50. We use the following notation to indicate the GCF of two numbers: $\text{GCF}(50, 100) = 50$. After determining the GCF, reduce by dividing both the numerator and the denominator as follows:

$$\frac{50 \div 50}{100 \div 50} = \frac{1}{2}$$

Figure 3.10.5

Example 3.10.1

Reduce to lowest terms: $\frac{105}{300}$.

Solution

Rewrite the numerator and denominator as a product of primes and then cancel.

$$\frac{105}{300} = \frac{\overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{5}} \cdot 7}{2 \cdot 2 \cdot \underset{1}{\cancel{3}} \cdot \underset{1}{\cancel{5}} \cdot 5}$$
$$= \frac{7}{2 \cdot 2 \cdot 5}$$
$$= \frac{7}{20}$$

Figure 3.10.6

Alternatively, we achieve the same result if we divide both the numerator and denominator by the GCF$(105, 300)$. A quick way to find the GCF of the two numbers requires us to first write each as a product of primes. The GCF is the product of all the common prime factors.

$$\left.\begin{aligned} 105 &= 3 \cdot 5 \cdot 7 \\ 300 &= 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \end{aligned}\right\} \quad \text{GCF}(105, 300) = 3 \cdot 5 = 15$$

Figure 3.10.7

In this case, the common prime factors are 3 and 5 and the greatest common factor of 105 and 300 is 15.

$$\frac{105}{300} = \frac{105 \div 15}{300 \div 15} = \frac{7}{20}$$

Figure 3.10.8

Answer:

$\frac{7}{20}$

Example 3.10.2

Try this! Reduce to lowest terms: $\frac{32}{96}$.

Video Solution:

(click to see video)

An **improper fraction** is one where the numerator is larger than the denominator. A mixed number is a number that represents the sum of a whole number and a fraction. For example, $5\frac{1}{2}$ is a mixed number that represents the sum $5 + \frac{1}{2}$. Use long division to convert an improper fraction to a mixed number; the remainder is the numerator of the fractional part.

Example 3.10.3

Write $\frac{23}{5}$ as a mixed number.

Solution

Notice that 5 divides into 23 four times with a remainder of 3.

$$\begin{array}{r} 4 \\ 5\overline{)\,23} \\ -20 \\ \hline 3 \end{array}$$

Figure 3.10.9

We then can write

$$\begin{aligned} \frac{23}{5} &= 4 + \frac{3}{5} \\ &= 4\frac{3}{5} \end{aligned}$$

Note that the denominator of the fractional part of the mixed number remains the same as the denominator of the original fraction.

Answer

$4\frac{3}{5}$

To convert mixed numbers to improper fractions, multiply the whole number by the denominator and then add the numerator; write this result over the original denominator.

Example 3.10.4

Write $3\frac{5}{7}$ as an improper fraction.

Solution

Obtain the numerator by multiplying 7 times 3 and then add 5.

$$\begin{aligned} 3\frac{5}{7} &= \frac{7 \cdot 3 + 5}{7} \\ &= \frac{21 + 5}{7} \\ &= \frac{26}{7} \end{aligned}$$

Answer

$\frac{26}{7}$

It is important to note that converting to a mixed number is not part of the reducing process. We consider improper fractions, such as 267, to be reduced to lowest terms. In algebra it is often preferable to work with improper fractions, although in some applications, mixed numbers are more appropriate.

Example 3.10.5

Try this! Convert $10\frac{1}{2}$ to an improper fraction.

Solution

(click to see video)

Multiplying and Dividing Fractions

In this section, assume that a, b, c, and d are all nonzero integers. The product of two fractions is the fraction formed by the product of the numerators and the product of the denominators. In other words, to multiply fractions, multiply the numerators and multiply the denominators:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Example 3.10.6

Multiply: $\frac{2}{3} \cdot \frac{5}{7}$

Solution

Multiply the numerators and multiply the denominators.

$$\begin{aligned} \frac{2}{3} \cdot \frac{5}{7} &= \frac{2 \cdot 5}{3 \cdot 7} \\ &= \frac{10}{21} \end{aligned}$$

Answer:

$\frac{10}{21}$

Example 3.10.7

Multiply: $\frac{5}{9}\left(-\frac{1}{4}\right)$

Solution

Recall that the product of a positive number and a negative number is negative.

$$\begin{aligned} \frac{5}{9}\left(-\frac{1}{4}\right) &= -\frac{5 \cdot 1}{9 \cdot 4} \\ &= -\frac{5}{36} \end{aligned}$$

Answer:

$-\frac{5}{36}$

Example 3.10.8

Multiply: $\frac{2}{3} \cdot 5\frac{3}{4}$

Solution

Begin by converting $5\frac{3}{4}$ to an improper fraction.

$$\begin{aligned} \frac{2}{3} \cdot 5\frac{3}{4} &= \frac{2}{3} \cdot \frac{23}{4} \\ &= \frac{\cancel{2}^{1}}{3} \cdot \frac{23}{\cancel{4}_{2}} \\ &= \frac{23}{6} \\ &= 3\frac{5}{6} \end{aligned}$$

Figure 3.10.10

In this example, we noticed that we could reduce before we multiplied the numerators and the denominators. Reducing in this way is called **cross canceling**, and can save time when multiplying fractions.

Answer

$3\frac{5}{6}$

Two real numbers whose product is 1 are called **reciprocals**. Therefore, $\frac{a}{b}$ and $\frac{b}{a}$ are reciprocals because $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab} = 1$. For example,

$$\frac{2}{3} \cdot \frac{3}{2} = \frac{6}{6} = 1$$

Because their product is 1, $\frac{2}{3}$ and $\frac{3}{2}$ are reciprocals. Some other reciprocals are listed below:

$$\frac{5}{8} \text{ and } \frac{8}{5} \qquad 7 \text{ and } \frac{1}{7} \qquad -\frac{4}{5} \text{ and } -\frac{5}{4}$$

This definition is important because dividing fractions requires that you multiply the dividend by the reciprocal of the divisor.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Example 3.10.9

Divide: $\frac{2}{3} \div \frac{5}{7}$

Solution

Multiply $\frac{2}{3}$ by the reciprocal of $\frac{5}{7}$.

$$\begin{aligned}\frac{2}{3} \div \frac{5}{7} &= \frac{2}{3} \cdot \frac{7}{5} \\ &= \frac{2 \cdot 7}{3 \cdot 5} \\ &= \frac{14}{15}\end{aligned}$$

Answer:

\(\frac{14}{15\)

You also need to be aware of other forms of notation that indicate division: / and —. For example,

$$5/(1/2) = 5 * (2/1) = (5/1) * (2/1) = 10/1 = 10$$

Or

$$\frac{\frac{7}{8}}{\frac{2}{3}} = \frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \cdot \frac{3}{2} = \frac{21}{16}$$

The latter is an example of a **complex fraction**, which is a fraction whose numerator, denominator, or both are fractions.

Note

Students often ask why dividing is equivalent to multiplying by the reciprocal of the divisor. A mathematical explanation comes from the fact that the product of reciprocals is 1. If we apply the multiplicative identity property and multiply numerator and denominator by the reciprocal of the denominator, then we obtain the following:

$$\frac{\frac{7}{8}}{\frac{2}{3}} = \frac{\frac{7}{8}}{\frac{2}{3}} \cdot 1 = \frac{\frac{7}{8}}{\frac{2}{3}} \cdot \frac{\frac{3}{2}}{\frac{3}{2}} = \frac{\frac{7}{8} \cdot \frac{3}{2}}{1} = \frac{7}{8} \cdot \frac{3}{2}$$

Figure 3.10.11

Before multiplying, look for common factors to cancel; this eliminates the need to reduce the end result.

Example 3.10.10

Divide: $\frac{\frac{5}{2}}{\frac{7}{4}}$.

Solution

$$\begin{aligned}\frac{\frac{5}{2}}{\frac{7}{4}} &= \frac{5}{2} \div \frac{7}{4} \\ &= \frac{5}{2} \cdot \frac{4}{7} \\ &= \frac{5}{\cancel{2}_{1}} \cdot \frac{\cancel{4}^{2}}{7} \\ &= \frac{5 \cdot 2}{1 \cdot 7} \\ &= \frac{10}{7}\end{aligned}$$

Figure 3.10.12

Answer

$\frac{10}{7}$

When dividing by an integer, it is helpful to rewrite it as a fraction over 1.

Example 3.10.11

Divide: $\frac{2}{3} \div 6$

Solution

Rewrite 6 as $\frac{6}{1}$ and multiply by its reciprocal.

$$\begin{aligned}\frac{2}{3} \div 6 &= \frac{2}{3} \div \frac{6}{1} \\ &= \frac{2}{3} \cdot \frac{1}{6} \\ &= \frac{\cancel{2}^{1}}{3} \cdot \frac{1}{\cancel{6}_{3}} \\ &= \frac{1 \cdot 1}{3 \cdot 3} \\ &= \frac{1}{9}\end{aligned}$$

Figure 3.10.13

Answer:

$\frac{1}{9}$

Also, note that we only cancel when working with multiplication. Rewrite any division problem as a product *before* canceling.

Example 3.10.12

Try this! Divide: $5 \div 2\frac{3}{5}$

Video Solution:

(click to see video)

Adding and Subtracting Fractions

Negative fractions are indicated with the negative sign in front of the fraction bar, in the numerator, or in the denominator. All such forms are equivalent and interchangeable.

$$\frac{-3}{4} = -\frac{3}{4} = \frac{3}{-4}$$

Adding or subtracting fractions requires a **common denominator**. In this section, assume the common denominator c is a nonzero integer.

It is good practice to use positive common denominators by expressing negative fractions with negative numerators. In short, avoid negative denominators.

Example 3.10.13

Subtract: $\frac{12}{15} - \frac{3}{15}$

Solution

The two fractions have a common denominator 15. Therefore, subtract the numerators and write the result over the common denominator:

$$\begin{aligned} \frac{12}{15} - \frac{3}{15} &= \frac{12-3}{15} && \textit{Subtract the numerators.} \\ &= \frac{9}{15} \\ &= \frac{9 \div 3}{15 \div 3} && \textit{Reduce.} \\ &= \frac{3}{5} \end{aligned}$$

Answer

$\frac{3}{5}$

Most problems that you are likely to encounter will have **unlike denominators**. In this case, first find equivalent fractions with a common denominator before adding or subtracting the numerators. One way to obtain equivalent fractions is to divide the numerator and the denominator by the same number. We now review a technique for finding equivalent fractions by multiplying the numerator and the denominator by the same number. It should be clear that $5/5$ is equal to 1 and that 1 multiplied times any number is that number:

$$\frac{1}{2} = \frac{1}{2} \cdot 1 = \frac{1}{2} \cdot \frac{5}{5} = \frac{5}{10}$$

We have equivalent fractions $\frac{1}{2} = \frac{5}{10}$. Use this idea to find equivalent fractions with a common denominator to add or subtract fractions. The steps are outlined in the following example.

Example 3.10.14

Subtract: $\frac{7}{15} - \frac{3}{10}$

Solution

Step 1: Determine a common denominator. To do this, use the **least common multiple** (LCM) of the given denominators. The LCM of 15 and 10 is indicated by LCM$(15, 10)$. Try to think of the smallest number that both denominators divide into evenly. List the multiples of each number:

$$\{10, 20, \mathbf{30}, 40, 50, \mathbf{60}, \ldots\} \quad \textit{Multiples of 10}$$
$$\{15, \mathbf{30}, 45, \mathbf{60}, 75, 90, \ldots\} \quad \textit{Multiples of 15}$$

Common multiples are listed in bold, and the least common multiple is 30.

$$\text{LCM}(10, 15) = 30$$

Step 2: Multiply the numerator and the denominator of each fraction by values that result in equivalent fractions with the determined common denominator.

$$\begin{aligned} \frac{7}{15} - \frac{3}{10} &= \frac{7 \cdot 2}{15 \cdot 3} - \frac{3 \cdot 3}{10 \cdot 3} \\ &= \frac{14}{30} - \frac{9}{30} \end{aligned}$$

Step 3: Add or subtract the numerators, write the result over the common denominator and then reduce if possible.

$$\begin{aligned} \frac{14}{30} - \frac{9}{30} &= \frac{14 - 9}{30} \\ &= \frac{5}{30} \\ &= \frac{5 \div 5}{30 \div 5} \\ &= \frac{1}{6} \end{aligned}$$

Answer:

$\frac{1}{6}$

The least common multiple of the denominators is called the **least common denominator** (LCD). Finding the LCD is often the difficult step. It is worth finding because if any common multiple other than the least is used, then there will be more steps involved when reducing.

Example 3.10.15

Add: $\frac{5}{10} + \frac{1}{18}$

Solution

First, determine that the LCM$(10, 18)$ is 90 and then find equivalent fractions with 90 as the denominator.

$$\begin{aligned}\frac{5}{10}+\frac{1}{18} &= \frac{5\cdot 9}{10\cdot 9}+\frac{1\cdot 5}{18\cdot 5}\\ &= \frac{45}{90}+\frac{5}{90}\\ &= \frac{45+5}{90}\\ &= \frac{50}{90}\\ &= \frac{50\div 10}{90\div 10}\\ &= \frac{5}{9}\end{aligned}$$

Answer

$\frac{5}{9}$

Example 3.10.16

Try this! Add: $\frac{2}{30}+\frac{5}{21}$

Video Solution:

(click to see video)

Example 3.10.17

Simplify: $2\frac{1}{3}+\frac{3}{5}-\frac{1}{2}$

Solution

Begin by converting $2\frac{1}{3}$ to an improper fraction.

$$\begin{aligned}2\frac{1}{3}+\frac{3}{5}-\frac{1}{2} &= \frac{7}{3}+\frac{3}{5}-\frac{1}{2} && \textit{Convert to improper fractions.}\\ &= \frac{7\cdot 10}{3\cdot 10}+\frac{3\cdot 6}{5\cdot 6}-\frac{1\cdot 15}{2\cdot 15} && \text{LCM}(3,\ 5,\ 2)\ =\ 30\\ &= \frac{70}{30}+\frac{18}{30}-\frac{15}{30} && \textit{Equivalent fractions with a common denominator}\\ &= \frac{70+18-15}{30}\\ &= \frac{73}{30}\\ &= 2\frac{13}{30}\end{aligned}$$

Answer:

$2\frac{13}{30}$

In general, it is preferable to work with improper fractions. However, when the original problem involves mixed numbers, if appropriate, present your answers as mixed numbers. Also, mixed numbers are often preferred when working with numbers on a number line and with real-world applications.

Example 3.10.18

Subtract: $\frac{5}{7} - 2\frac{1}{7}$

Example 3.10.19

How many $\frac{1}{2}$ inch thick paperback books can be stacked to fit on a shelf that is $1\frac{1}{2}$ feet in height?

Solution

First, determine the height of the shelf in inches. To do this, use the fact that there are 12 inches in 1 foot and multiply as follows:

$$\begin{aligned} 1\frac{1}{2}\text{ ft} &= 1\frac{1}{2}\text{ ft} \cdot \left(\frac{12\text{ in}}{1\text{ ft}}\right) \\ &= \frac{3}{\cancel{2}_{1}} \cdot \frac{\cancel{12}^{6}}{1}\text{ in} \\ &= 18\text{ in} \end{aligned}$$

Next, determine how many notebooks will fit by dividing the height of the shelf by the thickness of each book.

$$\begin{aligned} 18\text{ in} \div \frac{1}{2}\text{ in} &= 18 \cdot \frac{2}{1} \\ &= 18 \cdot 2 \\ &= 36 \end{aligned}$$

Answer

36 books can be stacked on the shelf.

Key Takeaways:

- Fractions are not unique; there are many ways to express the same ratio. Find equivalent fractions by multiplying or dividing the numerator and the denominator by the same real number.
- Equivalent fractions in lowest terms are generally preferred. It is a good practice to always reduce.

- In algebra, improper fractions are generally preferred. However, in real-life applications, mixed number equivalents are often preferred. We may present answers as improper fractions unless the original question contains mixed numbers, or it is an answer to a real-world or geometric application.
- Multiplying fractions does not require a common denominator; multiply the numerators and multiply the denominators to obtain the product. It is a best practice to cancel any common factors in the numerator and the denominator before multiplying.
- Reciprocals are rational numbers whose product is equal to 1. Given a fraction $\frac{a}{b}$, its reciprocal is $\frac{b}{a}$.
- Divide fractions by multiplying the dividend by the reciprocal of the divisor. In other words, multiply the numerator by the reciprocal of the denominator.
- Rewrite any division problem as a product *before* canceling.
- Adding or subtracting fractions requires a common denominator. When the denominators of any number of fractions are the same, simply add or subtract the numerators and write the result over the common denominator.
- Before adding or subtracting fractions, ensure that the denominators are the same by finding equivalent fractions with a common denominator. Multiply the numerator and the denominator of each fraction by the appropriate value to find the equivalent fractions.
- Typically, it is best to convert all mixed numbers to improper fractions before beginning the process of adding, subtracting, multiplying, or dividing.

Exercise 3.10.1

Reduce each fraction to lowest terms.

1. $\frac{5}{30}$
2. $\frac{6}{24}$
3. $\frac{30}{70}$
4. $\frac{18}{27}$
5. $\frac{44}{84}$
6. $\frac{54}{90}$
7. $\frac{135}{30}$
8. $\frac{105}{300}$
9. $\frac{18}{6}$
10. $\frac{256}{16}$
11. $\frac{126}{45}$
12. $\frac{52}{234}$
13. $\frac{54}{162}$
14. $\frac{2000}{3000}$
15. $\frac{270}{360}$

Answer available at the end of this section.

Exercise 3.10.2

Rewrite as an improper fraction.

1. $4\frac{3}{4}$
2. $2\frac{1}{2}$
3. $5\frac{7}{15}$
4. $1\frac{1}{2}$
5. $3\frac{5}{8}$
6. $1\frac{3}{4}$
7. $-2\frac{1}{2}$
8. $-1\frac{3}{4}$

Answer available at the end of this section.

Exercise 3.10.3

Rewrite as a mixed number.

1. $\frac{15}{2}$
2. $\frac{9}{2}$
3. $\frac{40}{13}$
4. $\frac{103}{25}$
5. $\frac{73}{10}$
6. $\frac{-52}{7}$
7. $\frac{-59}{6}$

Answer available at the end of this section.

Exercise 3.10.4

Multiply and reduce to lowest terms.

1. $\frac{2}{3} \cdot \frac{5}{7}$
2. $\frac{1}{5} \cdot \frac{4}{8}$
3. $\frac{1}{2} \cdot \frac{1}{3}$
4. $\frac{3}{4} \cdot \frac{20}{9}$
5. $\frac{5}{7} \cdot \frac{49}{10}$
6. $\frac{2}{3} \cdot \frac{9}{12}$
7. $\frac{6}{14} \cdot \frac{21}{12}$
8. $\frac{44}{15} \cdot \frac{15}{11}$
9. $3\frac{3}{4} \cdot 2\frac{1}{3}$
10. $2\frac{7}{10} \cdot 5\frac{5}{6}$
11. $\frac{3}{11}(-\frac{5}{2})$
12. $-\frac{4}{5}(\frac{9}{5})$
13. $(-\frac{9}{5}(-\frac{3}{10})$
14. $\frac{6}{7}(-\frac{14}{3})$
15. $(-\frac{9}{12})(-\frac{4}{8})$
16. $-\frac{3}{8}(-\frac{4}{15})$
17. $\frac{1}{7} \cdot \frac{1}{2} \cdot \frac{1}{3}$
18. $\frac{3}{5} \cdot \frac{15}{21} \cdot \frac{7}{27}$
19. $\frac{2}{5} \cdot 3\frac{1}{8} \cdot \frac{4}{5}$
20. $2\frac{4}{9} \cdot \frac{2}{5} \cdot 2\frac{5}{11}$

Answer available at the end of this section.

Exercise 3.10.5

Determine the reciprocal of the following numbers.

1. $\frac{1}{2}$
2. $\frac{8}{5}$
3. $-\frac{2}{3}$
4. $-\frac{4}{3}$
5. 10
6. -4
7. $2\frac{1}{3}$
8. $1\frac{5}{8}$

Answer available at the end of this section.

Exercise 3.10.6

Divide and reduce to lowest terms.

1. $\frac{1}{2} \div \frac{2}{3}$
2. $\frac{5}{9} \div \frac{1}{3}$
3. $\frac{5}{8} \div (-\frac{4}{5})$
4. $(-\frac{2}{5}) \div \frac{15}{3}$
5. $\frac{-\frac{6}{7}}{-\frac{6}{7}}$
6. $\frac{-\frac{1}{2}}{\frac{1}{4}}$
7. $\frac{-\frac{10}{3}}{-\frac{5}{20}}$
8. $\frac{\frac{2}{3}}{\frac{9}{2}}$
9. $\frac{\frac{30}{50}}{\frac{5}{3}}$
10. $\frac{\frac{1}{2}}{2}$
11. $\frac{5}{\frac{2}{5}}$
12. $\frac{-6}{\frac{5}{4}}$
13. $2\frac{1}{2} \div \frac{5}{3}$
14. $4\frac{2}{3} \div 3\frac{1}{2}$
15. $5 \div 2\frac{3}{5}$
16. $4\frac{3}{5} \div 23$

Answer available at the end of this section.

Exercise 3.10.7

Add or subtract and reduce to lowest terms.

1. $\frac{17}{20} - \frac{5}{20}$
2. $\frac{4}{9} - \frac{13}{9}$
3. $\frac{3}{5} + \frac{1}{5}$
4. $\frac{11}{15} + \frac{9}{15}$
5. $\frac{5}{7} - 2\frac{1}{7}$
6.
7. $\frac{1}{2} + \frac{1}{3}$
8. $\frac{1}{5} - \frac{1}{4}$
9. $\frac{3}{4} - \frac{5}{2}$
10. $\frac{3}{8} + \frac{7}{16}$
11. $\frac{7}{15} - \frac{3}{10}$
12. $\frac{3}{10} + \frac{2}{14}$
13. $\frac{2}{30} + \frac{5}{21}$
14. $\frac{3}{18} - \frac{1}{24}$
15. $5\frac{1}{2} + 2\frac{1}{3}$
16. $1\frac{3}{4} + 2\frac{1}{10}$
17. $\frac{1}{2} + \frac{1}{3} + \frac{1}{6}$
18. $\frac{2}{3} + \frac{3}{5} - \frac{2}{9}$
19. $\frac{7}{3} - \frac{3}{2} + \frac{2}{15}$
20. $\frac{9}{4} - \frac{3}{2} + \frac{3}{8}$
21.
22. $\frac{2}{3} - 4\frac{1}{2} + 3\frac{1}{6}$
23. $1 - \frac{6}{16} + \frac{3}{18}$
24. $3 - \frac{1}{21} - \frac{1}{15}$

Answer available at the end of this section.

Exercise 3.10.8

Perform the operations. Reduce answers to lowest terms.

1. $\frac{3}{14} \cdot \frac{7}{3} \div \frac{1}{8}$
2. $\frac{1}{2} \cdot (-\frac{4}{5}) \div \frac{14}{15}$
3. $\frac{1}{2} \div \frac{3}{4} \cdot \frac{1}{5}$
4. $-\frac{5}{9} \div \frac{5}{3} \cdot \frac{5}{2}$
5.
6.
7. $\frac{4}{5} \div 4 \cdot \frac{1}{2}$
8. $\frac{5}{3} \div 15 \cdot \frac{2}{3}$
9. What is the product of $\frac{3}{16}$ and $\frac{4}{9}$?
10. What is the product of $-\frac{24}{5}$ and $\frac{25}{8}$?
11. What is the quotient of $\frac{5}{9}$ and $\frac{25}{3}$?
12. What is the quotient of $-\frac{16}{5}$ and 32?
13. Subtract $\frac{1}{6}$ from the sum of $\frac{9}{2}$ and $\frac{2}{3}$.
14. Subtract $\frac{1}{4}$ from the sum of $\frac{3}{4}$ and $\frac{6}{5}$.
15. What is the total width when 3 boards, each with a width of $2\frac{5}{8}$ inches, are glued together?
16. The precipitation in inches for a particular 3-day weekend was published as $\frac{3}{10}$ inches on Friday, $1\frac{1}{2}$ inches on Saturday, and $\frac{3}{4}$ inches on Sunday. Calculate the total precipitation over this period.
17. A board that is $5\frac{1}{4}$ feet long is to be cut into 7 pieces of equal length. What is length of each piece?
18. How many $\frac{3}{4}$ inch thick notebooks can be stacked into a box that is 2 feet high?
19. In a mathematics class of 44 students, one-quarter of the students signed up for a special Saturday study session. How many students signed up?
20. Determine the length of fencing needed to enclose a rectangular pen with dimensions $35\frac{1}{2}$ feet by $20\frac{2}{3}$ feet.
21. Each lap around the track measures $\frac{1}{4}$ mile. How many laps are required to complete a $2\frac{1}{2}$ mile run?

22. A retiree earned a pension that consists of three-fourths of his regular monthly salary. If his regular monthly salary was $\$5,200$, then what monthly payment can the retiree expect from the pension plan?

Answer available at the end of this section.

Exercise 3.10.1

Discussion Board Topics

1. Does 0 have a reciprocal? Explain.
2. Explain the difference between the LCM and the GCF. Give an example.
3. Explain the difference between the LCM and LCD.
4. Why is it necessary to find an LCD in order to add or subtract fractions?
5. Explain how to determine which fraction is larger, $\frac{7}{16}$ or $\frac{1}{2}$.

Answers

Exercise 3.10.1

1: $1/6$
3: $3/7$
5: $11/21$
7: $9/2$
9: 3
11: $14/5$
13: $1/3$
15: $3/4$

Exercise 3.10.2

1: $5/2$
3: $3/2$
5: $7/4$
7: $-7/4$

Exercise 3.10.3

2: $4\frac{1}{2}$
4: $4\frac{3}{25}$
6: $-7\frac{3}{7}$

Exercise 3.10.4

1: $10/21$
3: $1/6$
5: $7/2$
7: $3/4$
9: 834
11: $-15/22$
13:$27/50$
15: $3/8$
17: $1/42$
19: 1

Exercise 3.10.5

1: 2
3: $-3/2$
5: $1/10$
7: $3/7$
9: $3/4$

Exercise 3.10.6

1: $3/4$
3: $-25/32$
5: 1
7: $40/3$

9: $9/25$
11: $25/2$
13: $1\frac{1}{2}$
15: $1\frac{12}{13}$

Exercise 3.10.7

1: $3/5$
3: $4/5$
5: $-1\frac{3}{7}$
7: $5/6$
9: $-7/4$
11: $1/6$
13: $32/105$
15: $7\frac{5}{6}$
17: 1
19: $29/30$
21: $2\frac{2}{3}$
23: $19/24$

Exercise 3.10.8

1: 4
3: $2/15$
5: $9/28$
7: $1/10$
9: $1/12$
11: $1/15$
13: 5
15: $7\frac{7}{8}$ inches
17: $\frac{3}{4}$ feet
19: 11 students
21: 10 laps

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3.11: Review of Decimals and Percents

Skills to Develop

- Convert fractions to decimals and back.
- Perform operations with decimals.
- Round off decimals to a given place.
- Define a percent.
- Convert percents to decimals and back.
- Convert fractions to percents and back.

Decimals

In this section, we provide a brief review of the decimal system. A real number in decimal form, a **decimal** consists of a decimal point, digits (0 through 9) to the left of the decimal point representing the whole number part, and digits to the right of the decimal point representing the fractional part. The digits represent powers of 10 as shown in the set $\{\ldots, 1,000, 100, 10, 1, 1/10, 1/100, 1/1,000, \ldots\}$according to the following diagram:

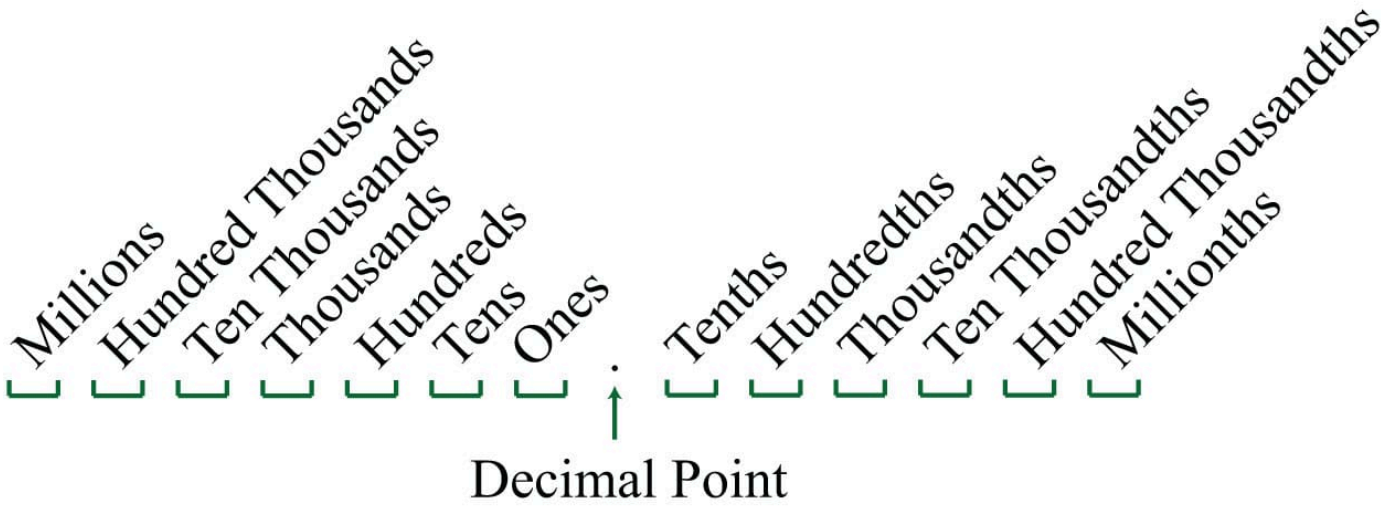


Figure 1.5.1

For example, the decimal 538.3 can be written in the following expanded form:

$$5 \cdot 100 + 3 \cdot 10 + 8 \cdot 1 + 3 \cdot \tfrac{1}{10}$$

After simplifying, we obtain the mixed number $528\frac{3}{10}$. Use this process to convert decimals to mixed numbers.

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3.12: Exponents and Square Roots

Learning Objectives

- Interpret exponential notation with positive integer exponents.
- Calculate the nth power of a real number.
- Calculate the exact and approximate value of the square root of a real number.

Exponential Notation and Positive Integer Exponents

If a number is repeated as a factor numerous times, then we can write the product in a more compact form using exponential notation. For example,

$$5 \cdot 5 \cdot 5 \cdot 5 = 5^4$$

The base is the factor, and the positive integer exponent indicates the number of times the base is repeated as a factor. In the above example, the base is 5 and the exponent is 4. In general, if a is the base that is repeated as a factor n times, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors of } a}$$

Figure $3.12.1$

When the exponent is 2, we call the result a square. For example,

$$3^2 = 3 \cdot 3 = 9$$

The number 3 is the base and the integer 2 is the exponent. The notation 3^2 can be read two ways: “three squared” or “3 raised to the second power.” The base can be any real number.

It is important to study the difference between the ways the last two examples are calculated. In the example $(-7)^2$, the base is -7 as indicated by the parentheses. In the example -5^2, the base is 5, not -5, so only the 5 is squared and the result remains negative. To illustrate this, write

$-5^2 = -1 \cdot 5^2 = -1 \cdot 5 \cdot 5 = -25$

This subtle distinction is very important because it determines the sign of the result.

The textual notation for exponents is usually denoted using the caret $()$ symbol as follows:

$$8^2 = 8 \wedge 2 = 8 * 8 = 64$$
$$-5.1^2 = -5.1 \wedge 2 = -5.1 * 5.1 = -26.01$$

The square of an integer is called a perfect square. The ability to recognize perfect squares is useful in our study of algebra. The squares of the integers from 1 to 15 should be memorized. A partial list of perfect squares follows:

$$\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, \ldots\}$$

Exercise 3.12.1

Simplify

$$(-12)^2.$$

Answer available at the end of this section.

When the exponent is 3 we call the result a cube. For example,

$$3^3 = 3 \cdot 3 \cdot 3 = 27$$

The notation 3^3 can be read two ways: "three cubed" or "3 raised to the third power." As before, the base can be any real number.

Note that the result of cubing a negative number is negative. The cube of an integer is called a perfect cube. The ability to recognize perfect cubes is useful in our study of algebra. The cubes of the integers from 1 to 10 should be memorized. A partial list of perfect cubes follows:

$\{0, 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...\}$

Exercise 3.12.2

Simplify $(-2)^3$.

Answer available at the end of this section.

If the exponent is greater than 3, then the notation an is read "a raised to the nth power."

$$\begin{aligned} 10^6 &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 1,000,000 \\ (-1)^4 &= (-1)(-1)(-1)(-1) = 1 \\ \left(\frac{1}{3}\right)^5 &= \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{243} \end{aligned}$$

Notice that the result of a negative base with an even exponent is positive. The result of a negative base with an odd exponent is negative. These facts are often confused when negative numbers are involved. Study the following four examples carefully:

Table 3.12.1

The base is (-2)	The base is 2
$(-2)^4 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) = +16$ $(-2)^3 = (-2) \cdot (-2) \cdot (-2) = -8$	$-2^4 = -2 \cdot 2 \cdot 2 \cdot 2 = -16$ $-2^3 = -2 \cdot 2 \cdot 2 = -8$

The parentheses indicate that the negative number is to be used as the base.

Example 3.12.1

Calculate:

a. $\left(-\frac{1}{3}\right)^3$
b. $\left(-\frac{1}{3}\right)^4$

Solution:

The base is $-\frac{1}{3}$ for both problems.

a. Use the base as a factor three times.

$$\begin{aligned}\left(-\frac{1}{3}\right)^3 &= \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\\ &= -\frac{1}{27}\end{aligned}$$

b. Use the base as a factor four times.

$$\begin{aligned}\left(-\frac{1}{3}\right)^4 &= \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\\ &= +\frac{1}{81}\end{aligned}$$

Answer:

a. $-\frac{1}{27}$; b. $\frac{1}{81}$

Exercise 3.12.3

Simplify:

-10^4 and $(-10)^4$.

Answer available at the end of this section.

Square Root of a Real Number

Think of finding the square root of a number as the inverse of squaring a number. In other words, to determine the square root of 25 the question is, "What number squared equals 25?" Actually, there are two answers to this question, 5 and -5.

$$5^2 = 25 \quad \text{and} (-5)^2 = 25$$

When asked for the square root of a number, we implicitly mean the principal (nonnegative) square root. Therefore we have,

$\sqrt{a^2} = a$, if $a \geq 0$ or more generally $\sqrt{a^2} = |a|$

As an example, $\sqrt{25} = 5$, which is read "square root of 25 equals 5." The symbol $\sqrt{}$ is called the radical sign and 25 is called the radicand. The alternative textual notation for square roots follows:

$$\sqrt{16} = textsqrt(16) = 4$$

It is also worthwhile to note that

$$\sqrt{1} = 1 \quad \text{and} \quad \sqrt{0} = 0$$

This is the case because $1^2 = 1$ and $0^2 = 0$.

Example 3.12.2

Simplify:

$\sqrt{10,000}$.

Solution:

$10,000$ is a perfect square because $100 \cdot 100 = 10,000$.

$$\begin{aligned}\sqrt{10,000} &= \sqrt{(100)^2} \\ &= 100\end{aligned}$$

Answer:

$$100$$

Example 3.12.3

Simplify:

$$\sqrt{\tfrac{1}{9}}.$$

Solution:

Here we notice that $\frac{1}{9}$ is a square because $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

$$\begin{aligned}\sqrt{\frac{1}{9}} &= \sqrt{\left(\frac{1}{3}\right)^2} \\ &= \frac{1}{3}\end{aligned}$$

Answer:

$$\tfrac{1}{3}$$

Given a and b as positive real numbers, use the following property to simplify square roots whose radicands are not squares:

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

The idea is to identify the largest square factor of the radicand and then apply the property shown above. As an example, to simplify $\sqrt{8}$ notice that 8 is not a perfect square. However, $8 = 4 \cdot 2$ and thus has a perfect square factor other than 1. Apply the property as follows:

$$\begin{aligned}\sqrt{8} &= \sqrt{4 \cdot 2} \\ &= \sqrt{4} \cdot \sqrt{2} \\ &= 2 \cdot \sqrt{2} \\ &= 2\sqrt{2}\end{aligned}$$

Here $2\sqrt{2}$ is a simplified irrational number. You are often asked to find an approximate answer rounded off to a certain decimal place. In that case, use a calculator to find the decimal approximation using either the original problem or the simplified equivalent.

$$\sqrt{8} = 2\sqrt{2} \approx 2.83$$

On a calculator, try $2.83 \wedge 2$. What do you expect? Why is the answer not what you would expect?

It is important to mention that ***the radicand must be positive***. For example, $\sqrt{-9}$ is undefined since there is no real number that when squared is negative. Try taking the square root of a negative number on your calculator. What does it say?

Note

Taking the square root of a negative number is defined later in the course.

Example 3.12.4

Simplify and give an approximate answer rounded to the nearest hundredth:

$$\sqrt{75}.$$

Solution:

The radicand 75 can be factored as $25 \cdot 3$ where the factor 25 is a perfect square.

$$\begin{aligned} \sqrt{75} &= \sqrt{25 \cdot 3} && \textit{The largest perfect square} \\ &= \sqrt{25} \cdot \sqrt{3} && \textit{factor of } 75 \textit{ is } 25. \\ &= 5 \cdot \sqrt{3} \\ &= 5\sqrt{3} && \textit{Exact answer} \\ &\approx 8.66 && \textit{Approximate answer} \end{aligned}$$

Answer:

$$\sqrt{75} \approx 8.66$$

As a check, calculate (\sqrt{75}\) and $5\sqrt{3}$ on a calculator and verify that the both results are approximately 8.66.

Example 3.12.5

Simplify:

$$\sqrt{180}.$$

Solution:

$$\begin{aligned} \sqrt{180} &= \sqrt{36 \cdot 5} \\ &= \sqrt{36} \cdot \sqrt{5} \\ &= 6 \cdot \sqrt{5} \\ &= 6\sqrt{5} \end{aligned}$$

Since the question did not ask for an approximate answer, we present the exact answer.

Answer:

$$6\sqrt{5}$$

Example 3.12.6

Simplify:

$$-5\sqrt{162}.$$

Solution:

$$\begin{aligned} -5\sqrt{162} &= -5 \cdot \sqrt{81 \cdot 2} \\ &= -5 \cdot \sqrt{81} \cdot \sqrt{2} \\ &= -5 \cdot 9 \cdot \sqrt{2} \\ &= -45 \cdot \sqrt{2} \\ &= -45\sqrt{2} \end{aligned}$$

Answer:

$$-45\sqrt{2}$$

Exercise 3.12.4

Simplify and give an approximate answer rounded to the nearest hundredth:

$$\sqrt{128}.$$

Answer available at the end of this section.

A right triangle is a triangle where one of the angles measures $90\,°$. The side opposite the right angle is the longest side, called the hypotenuse, and the other two sides are called legs. Numerous real-world applications involve this geometric figure. The Pythagorean theorem states that given any right triangle with legs measuring a and b units, the square of the measure of the hypotenuse c is equal to the sum of the squares of the measures of the legs: $a^2 + b^2 = c^2$. In other words, the hypotenuse of any right triangle is equal to the square root of the sum of the squares of its legs.

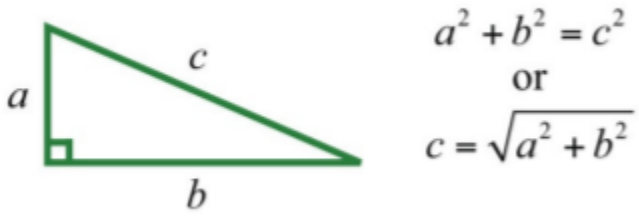


Figure 3.12.1

Example 3.12.7

If the two legs of a right triangle measure 3 units and 4 units, then find the length of the hypotenuse.

Solution:

Given the lengths of the legs of a right triangle, use the formula $c = \sqrt{a^2 + b^2}$ to find the length of the hypotenuse.

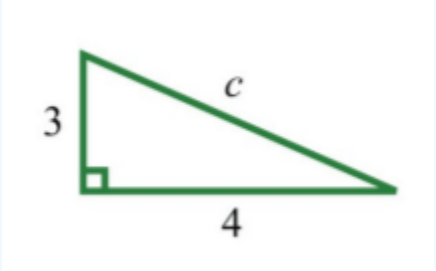


Figure $3.12.2$

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ c &= \sqrt{3^2 + 4^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Answer:

$c = 5$ units

When finding the hypotenuse of a right triangle using the Pythagorean theorem, the radicand is not always a perfect square.

Example $3.12.8$

If the two legs of a right triangle measure 2 units and 6 units, find the length of the hypotenuse.

Solution:

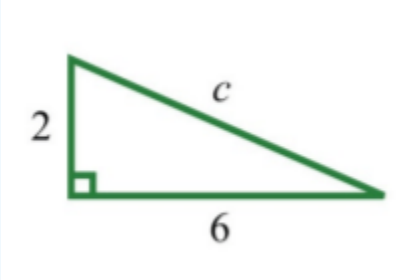


Figure $3.12.3$

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ &= \sqrt{2^2 + 6^2} \\ &= \sqrt{4 + 36} \\ &= \sqrt{40} \\ &= \sqrt{4 \cdot 10} \\ &= \sqrt{4} \cdot \sqrt{10} \\ &= 2 \cdot \sqrt{10} \end{aligned}$$

Answer:

$c = 2\sqrt{10}$ units

Key Takeaways

- When using exponential notation a^n, the base a is used as a factor n times.
- When the exponent is 2, the result is called a square. When the exponent is 3, the result is called a cube.
- Memorize the squares of the integers up to 15 and the cubes of the integers up to 10. They will be used often as you progress in your study of algebra.
- When negative numbers are involved, take care to associate the exponent with the correct base. Parentheses group a negative number raised to some power.
- A negative base raised to an even power is positive.

- A negative base raised to an odd power is negative.
- The square root of a number is a number that when squared results in the original number. The principal square root is the positive square root.
- Simplify a square root by looking for the largest perfect square factor of the radicand. Once a perfect square is found, apply the property $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$, where a and b are nonnegative, and simplify.
- Check simplified square roots by calculating approximations of the answer using both the original problem and the simplified answer on a calculator to verify that the results are the same.
- Find the length of the hypotenuse of any right triangle given the lengths of the legs using the Pythagorean theorem.

Exercise 3.12.5 Square of a Number

Simplify.

1. 10^2
2. 12^2
3. $(-9)^2$
4. -12^2
5. 11^2
6. $(-20)^2$
7. 0^2
8. 1^2
9. $-(-8)^2$
10. $-(13)^2$
11. $(\frac{1}{2})^2$
12. $(-\frac{2}{3})^2$
13. 0.5^2
14. 1.25^2
15. $(-2.6)^2$
16. $-(-5.1)^2$
17. $(2\frac{1}{3})^2$
18. $(5\frac{3}{4})^2$

Answer available at the end of this section.

Exercise 3.12.6 Square of a Number

If s is the length of the side of a square, then the area is given by $A = s^2$.

1. Determine the area of a square given that a side measures 5 inches.
2. Determine the area of a square given that a side measures 2.3 feet.
3. List all the squares of the integers 0 through 15.
4. List all the squares of the integers from -15 to 0.
5. List the squares of all the rational numbers in the set $\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\}$.
6. List the squares of all the rational numbers in the set $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}\}$.

Answer available at the end of this section.

Exercise 3.12.7 Integer Exponents

Simplify.

1. 5^3
2. 2^6
3. $(-1)^4$

4. $(-3)^3$
5. -1^4
6. $(-2)^4$
7. -7^3
8. $(-7)^3$
9. $-(-3)^3$
10. $-(-10)^4$
11. $(-1)^{20}$
12. $(-1)^{21}$
13. $(-6)^3$
14. -3^4
15. 1^{100}
16. 0^{100}
17. $-(\frac{1}{2})^3$
18. $(\frac{1}{2})^6$
19. $(\frac{5}{2})^3$
20. $(-\frac{3}{4})^4$
21. List all the cubes of the integers -5 through 5.
22. List all the cubes of the integers from -10 to 0.
23. List all the cubes of the rational numbers in the set $\{-\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}\}$.
24. List all the cubes of the rational numbers in the set $\{-\frac{3}{7}, -\frac{1}{7}, 0, \frac{1}{7}, \frac{3}{7}\}$.

Answer available at the end of this section.

Exercise 3.12.8 Square Root of a Number

Determine the exact answer in simplified form.

1. $\sqrt{121}$
2. $\sqrt{81}$
3. $\sqrt{100}$
4. $\sqrt{169}$
5. $-\sqrt{25}$
6. $-\sqrt{144}$
7. $\sqrt{12}$
8. $\sqrt{27}$
9. $\sqrt{45}$
10. $\sqrt{50}$
11. $\sqrt{98}$
12. $\sqrt{2000}$
13. $\sqrt{\frac{1}{4}}$
14. $\sqrt{\frac{9}{16}}$
15. $\sqrt{\frac{5}{9}}$
16. $\sqrt{\frac{8}{36}}$
17. $\sqrt{0.64}$
18. $\sqrt{0.81}$
19. $\sqrt{30^2}$
20. $\sqrt{15^2}$
21. $\sqrt{(-2)^2}$
22. $\sqrt{(-5)^2}$
23. $\sqrt{-9}$
24. $\sqrt{-16}$
25. $3\sqrt{16}$
26. $5\sqrt{18}$

27. $-2\sqrt{36}$
28. $-3\sqrt{32}$
29. $6\sqrt{200}$
30. $10\sqrt{27}$

Answer available at the end of this section.

Exercise 3.12.9 Square Root of a Number

Approximate the following to the nearest hundredth.

1. $\sqrt{2}$
2. $\sqrt{3}$
3. $\sqrt{10}$
4. $\sqrt{15}$
5. $2\sqrt{3}$
6. $5\sqrt{2}$
7. $-6\sqrt{5}$
8. $-4\sqrt{6}$
9. $\sqrt{79}$
10. $\sqrt{54}$
11. $-\sqrt{162}$
12. $-\sqrt{86}$
13. If the two legs of a right triangle measure 6 units and 8 units, then find the length of the hypotenuse.
14. If the two legs of a right triangle measure 5 units and 12 units, then find the length of the hypotenuse.
15. If the two legs of a right triangle measure 9 units and 12 units, then find the length of the hypotenuse.
16. If the two legs of a right triangle measure $\frac{3}{2}$ units and 2 units, then find the length of the hypotenuse.
17. If the two legs of a right triangle both measure 1 unit, then find the length of the hypotenuse.
18. If the two legs of a right triangle measure 1 unit and 5 units, then find the length of the hypotenuse.
19. If the two legs of a right triangle measure 2 units and 4 units, then find the length of the hypotenuse.
20. If the two legs of a right triangle measure 3 units and 9 units, then find the length of the hypotenuse.

Answer available at the end of this section.

Exercise 3.12.10 Discussion Board Topics

1. Why is the result of an exponent of 2 called a square? Why is the result of an exponent of 3 called a cube?
2. Research and discuss the history of the Pythagorean theorem.
3. Research and discuss the history of the square root.
4. Discuss the importance of the principal square root.

Answer available at the end of this section.

Answers

Exercise 3.12.1

144

Exercise 3.12.2

-8

Exercise 3.12.3

$-10,000$ and $10,000$

Exercise 3.12.4

$8\sqrt{2} \approx 11.31$

Exercise 3.12.5 **Square of a Number**

1. 100

3. 81
5. 121
7. 0
9. -64
11. $\frac{1}{4}$
13. $.25$
15. 6.76
17. $5\frac{4}{9}$

Exercise $3.12.6$ **Square of a Number**

1. 25 square inches
3. $\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225\}$
5. $\{0, \frac{1}{9}, \frac{4}{9}, 1, \frac{16}{9}, \frac{25}{9}, 4\}$

Exercise $3.12.7$ **Integer Exponents**

1. 125
3. 1
5. -1
7. -343
9. 27
11. 1
13. -216
15. 1
17. $-\frac{1}{8}$
19. $\frac{12}{58}$
21. $\{-125, -64, -27, -8, -1, 0, 1, 8, 27, 64, 125\}$
23. $\{-\frac{8}{27}, -\frac{1}{27}, 0, \frac{1}{27}, \frac{8}{27}\}$

Exercise $3.12.8$ **Square Root of a Number**

1. 11
3. 10
5. -5
7. $2\sqrt{3}$
9. $3\sqrt{5}$
11. $7\sqrt{2}$
13. $\frac{1}{2}$
15. $5\sqrt{3}$
17. 0.8
19. 30
21. 2
23. Not real
25. 12
27. -12
29. $60\sqrt{2}$

Exercise $3.12.9$ **Square Root of a Number**

1. 1.41
3. 3.16
5. 3.46
7. -13.42
9. 8.89
11. -12.73
13. 10 units
15. 15 units
17. $\sqrt{2}$ units
19. $2\sqrt{5}$ units

Exercise $3.12.10$ **Discussion Board Topics**

1. Answers may vary
3. Answers may vary

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3.13: Order of Operations

Learning Objectives

- Identify and work with grouping symbols.
- Understand the order of operations.
- Simplify using the order of operations.

Grouping Symbols

In a computation where more than one operation is involved, grouping symbols help tell us which operations to perform first. The grouping symbols commonly used in algebra are

$$\begin{array}{ll} (\ ) & \textit{Parentheses} \\ [\ ] & \textit{Brackets} \\ \{\ \} & \textit{Braces} \\ \text{—} & \textit{Fraction bar} \end{array}$$

All of the above grouping symbols, as well as absolute value, have the same order of precedence. Perform operations inside the innermost grouping symbol or absolute value first.

Example 3.13.1

Simplify:

$$5-(4-12).$$

Solution:

Perform the operations within the parentheses first. In this case, first subtract 12 from 4.

$$\begin{aligned} 5-(4-12) &= 5-(-8) \\ &= 5+8 \\ &= 13 \end{aligned}$$

Answer:

$$13$$

Example 3.13.2

Simplify:

$$3\{-2[-(-3-1)]\}.$$

Solution:

Answer:

-24

Example 3.13.3

Simplify:

$$\frac{5-|4-(-3)|}{|-3|-(5-7)}.$$

Solution:

$$\begin{aligned}\frac{5-|4-(-3)|}{|-3|-(5-7)} &= \frac{5-|4+3|}{|-3|-(-2)} \\ &= \frac{5-|7|}{|-3|+2} \\ &= \frac{5-7}{3+2} \\ &= \frac{-2}{5} \\ &= -\frac{2}{5}\end{aligned}$$

Answer:

$$-\frac{2}{5}$$

Exercise 3.13.1

Simplify:

$$-[-3(2+3)].$$

Answer available at the end of this section.

Order of Operations

When several operations are to be applied within a calculation, we must follow a specific order to ensure a single correct result.

1. Perform all calculations within the innermost **parentheses** or grouping symbols.
2. Evaluate all **exponents**.
3. Perform **multiplication and division** operations from left to right.
4. Finally, perform all remaining **addition and subtraction** operations from left to right.

Note

Caution: Note that multiplication and division operations ***must*** be worked from ***left to right***.

Example 3.13.4

Simplify:

$$5^2 - 4 \cdot 3 \div 12.$$

Solution:

First, evaluate 5^2 and then perform multiplication and division as they appear from left to right.

$$\begin{aligned} 5^2 - 4 \cdot 3 \div 12 &= 25 - \underbrace{4 \cdot 3} \div 12 \\ &= 25 - \underbrace{12 \div 12} \\ &= \underbrace{25} - 1 \\ &= 24 \end{aligned}$$

Because multiplication and division operations should be worked from left to right, it is sometimes correct to perform division before multiplication.

Example 3.13.5

Simplify:

$$2^4 - 12 \div 3 \cdot 2 + 11.$$

Solution:

Begin by evaluating the exponent, $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$.

$$\begin{aligned} 2^4 - 12 \div 3 \cdot 2 + 11 &= 16 - \underbrace{12 \div 3} \cdot 2 + 11 \\ &= 16 - \underbrace{4 \cdot 2} + 11 \\ &= \underbrace{16 - 8} + 11 \\ &= \underbrace{8 + 11} \\ &= 19 \end{aligned}$$

Multiplying first leads to an incorrect result.

$$\begin{aligned} 2^4 - 12 \div 3 \cdot 2 + 11 &= 16 - 12 \div \underbrace{3 \cdot 2}_{Incorrect} + 11 \\ &= 16 - \underbrace{12 \div 6} + 11 \\ &= \underbrace{16 - 2} + 11 \\ &= \underbrace{14 + 11} \\ &= 25 \quad x \end{aligned}$$

Answer:

$$19$$

Example 3.13.6

Simplify:

$$-3 - 5^2 + (-7)^2.$$

Solution:

Take care to correctly identify the base when squaring.

$$\begin{aligned} -3 - 5^2 + (-7)^2 &= -3 - 25 + 49 \\ &= -28 + 49 \\ &= 21 \end{aligned}$$

Answer:

$$21$$

Example 3.13.7

Simplify:

$$5-3[2^3-5+7(-3)].$$

Solution:

It is tempting to first subtract $5-3$, but this will lead to an incorrect result. The order of operations requires us to simplify within the brackets first.

$$\begin{aligned} 5-3[2^3-5+7(-3)] &= 5-3[8-5-21] \\ &= 5-3[-18] \\ &= 5+54 \\ &= 59 \end{aligned}$$

Subtracting $5-3$ first leads to an incorrect result.

$$\begin{aligned} 5-3[2^3-5+7(-3)] &= \underbrace{5-3}_{Incorrect}\,[8-5-21] \\ &= 2[-18] \\ &= -36 \quad x \end{aligned}$$

Answer:

$$59$$

Example 3.13.8

Simplify:

$$-3^2-[5-(4^2-10)].$$

Solution:

Perform the operations within the innermost parentheses first.

Answer:

-8

Example 3.13.9

Simplify:

$$\left(-\tfrac{2}{3}\right)^2 \div \left[\tfrac{5}{3}-\left(-\tfrac{1}{2}\right)^3\right].$$

Solution:

$$\begin{aligned}\left(-\frac{2}{3}\right)^2 \div \left[\frac{5}{3}-\left(-\frac{1}{2}\right)^3\right] &= \left(-\frac{2}{3}\right)^2 \div \left[\frac{5}{3}-\left(-\frac{1}{8}\right)\right] \\ &= \left(-\frac{2}{3}\right)^2 \div \left[\frac{5}{3}+\frac{1}{8}\right] \\ &= \left(-\frac{2}{3}\right)^2 \div \left[\frac{40}{24}+\frac{3}{24}\right] \\ &= \left(-\frac{2}{3}\right)^2 \div \frac{43}{24} \\ &= \frac{4}{\cancel{9}_{3}} \cdot \frac{\cancel{24}^{8}}{43} \\ &= \frac{32}{129}\end{aligned}$$

Answer:

$\frac{32}{129}$

We are less likely to make a mistake if we work one operation at a time. Some problems may involve an absolute value, in which case we assign it the same order of precedence as parentheses.

Example 3.13.10

Simplify:

$$2-4|-4-3|+(-2)^4.$$

Solution:

We begin by evaluating the absolute value and then the exponent $(-2)^4=(-2)(-2)(-2)(-2)=+16$.

Answer:

-10

Exercise 3.13.2

Simplify:

$$10 \div 5 \cdot 2|(-4)+|-3|| + (-3)^2.$$

Answer available at the end of this section.

Key Takeaways

- Grouping symbols indicate which operations to perform first. We usually group mathematical operations with parentheses, brackets, braces, and the fraction bar. We also group operations within absolute values. All groupings have the same order of precedence: the operations within the innermost grouping are performed first.
- When applying operations within a calculation, follow the order of operations to ensure a single correct result.
  - Address innermost parentheses or groupings first.
  - Simplify all exponents.
  - Perform multiplication and division operations from left to right.
  - Finally, perform addition and subtraction operations from left to right.
- It is important to highlight the fact that multiplication and division operations should be applied as they appear from left

to right. It is a common mistake to always perform multiplication before division, which, as we have seen, in some cases produces incorrect results.

Exercise 3.13.3 Order of Operations

Simplify.

1. $-7-3\cdot 5$
2. $3+2\cdot 3$
3. $-3(2)-6^2$
4. $2(-3)^2+5(-4)$
5. $\frac{6}{3}*2$
6. $6/(3*2)$
7. $-\frac{1}{2}-\frac{3}{5}\cdot\frac{2}{3}$
8. $\frac{5}{8}\div\frac{1}{2}-\frac{5}{6}$
9. $3.2^2-6.9\div 2.3$
10. $8.2-3\div 1.2\cdot 2.1$
11. $2+3(-2)-7$
12. $8\div 2-3\cdot 2$
13. $3+6^2\div 12$
14. $5-4^2\div(-8)$
15. $-9-3\cdot 2\div 3(-2)$
16. $-2-3^2+(-2)^2$
17. $12\div 6\cdot 2-2^2$
18. $4\cdot 3\div 12\cdot 2-(-2)^2$
19. $(-5)^2-2(5)^2\div 10$
20. $-3(4-7)+2$
21. $(-2+7)^2-10^2$
22. $10-7(3+2)+7^2$
23. $-7-3(4-2\cdot 8)$
24. $5-3[6-(2+7)]$
25. $1+2[(-2)^3-(-3)^2]$
26. $-3[2(7-5)\div 4\cdot(-2)+(-3)^3]$
27. $-7^2-[-20-(-3)^2]-(-10)$
28. $4.7-3.2(4-1.2^3)$
29. $-5.4(6.1-3.1\div 0.1)-8.2^2$
30. $-7.3^2+(-9.3)^2-37.8\div 1.8$
31. $2-7(3^2-3+4\cdot 3)$
32. $(\frac{1}{2})^2-(-\frac{2}{3})^2$
33. $(\frac{1}{2})^3+(-2)^3$
34. $(-\frac{1}{3})^2-(-\frac{2}{3})^3$
35. $\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{5}$
36. $\frac{5}{8}\div\frac{3}{2}\cdot\frac{14}{15}$
37. $5\cdot\frac{2}{15}-(\frac{1}{2})^3$
38. $\frac{5}{17}(\frac{3}{5}-\frac{4}{35})$
39. $\frac{3}{16}\div(\frac{5}{12}-\frac{1}{2}+\frac{2}{3})\cdot 4$
40. $(\frac{2}{3})^2-(\frac{1}{2})^2$
41. $\frac{1}{2}[\frac{3}{4}\cdot(-4)^2-2]^2$
42. $6\cdot[(\frac{2}{3})^2-(\frac{1}{2})^2]\div(-2)^2$
43. $(-5)^2+\frac{3}{2}-\frac{4}{2}+2\cdot 7$
44. $(-3.2-3.3)(8.7-4.7)(-4.7+3.9+2.1)$
45. $2-[3-(5-7)^2]3(6-32)$
46. $\frac{2+3\cdot 6-4\cdot 32}{2-32}$
47. $(2+7)\cdot 2-\frac{23}{10}+\frac{9}{2}+3^3$
48. $\frac{(-1-3)^2-1}{5-3\cdot(-7+22)-5}$
49. $(7+4*(-2))/(-3+(-2)^2)$

50. $4+3*((-3)^3+5^2)/6-2^2$

Answer available at the end of this section.

Exercise 3.13.4 Order of Operations

1. Mary purchased 14 bottles of water at $\$0.75$ per bottle, 4 pounds of assorted candy at $\$3.50$ per pound, and 16 packages of microwave popcorn costing $\$0.50$ each for her party. What was her total bill?
2. Joe bought four 8-foot 2-by-4 boards for $\$24.00$. How much did he spend per linear foot?
3. Margaret bought two cases of soda at the local discount store for $\$23.52$. If each case contained 24 bottles, how much did she spend per bottle?
4. Billy earns $\$12.00$ per hour and "time and a half" for every hour he works over 40 hours a week. What is his pay for 47 hours of work this week?
5. Audry bought 4 bags of marbles each containing 15 assorted marbles. If she wishes to divide them up evenly between her 3 children, how many will each child receive?
6. Mark and Janet carpooled home from college for the Thanksgiving holiday. They shared the driving, but Mark drove twice as far as Janet. If Janet drove 135 miles, then how many miles was the entire trip?

Answer available at the end of this section.

Exercise 3.13.5 Order of Operations with Absolute Values

Simplify.

1. $3+2|-5|$
2. $9-4|-3|$
3. $-(-|2|+|-10|)$
4. $-(|-6|-|-8|)$
5. $|-(40-|-22|)|$
6. $||-5|-|10||$
7. $-(|-8|-5)^2$
8. $(|-1|-|-2|)^2$
9. $-4+2|2^2-3^2|$
10. $-10-|4-5^2|$
11. $-|(-5)^2+4^2\div 8|$
12. $-(-3-[6-|-7|])$
13. $-2[7-(4+|-7|)]$
14. $3-7|-2-3|+4^3$
15. $7-5|6^2-5^2|+(-7)^2$
16. $(-4)^2-|-5+(-2)^3|-3^2$
17. $\frac{2}{3}-|\frac{1}{2}-(-\frac{4}{3})^2|$
18. $-30|\frac{10}{3}-\frac{1}{2}\div\frac{1}{5}|$
19. $(-4)^3-(2-|-4|)\div|-3^2+7|$
20. $[10-3(6-|-8|)]\div 4-5^2$

Answer available at the end of this section.

Exercise 3.13.6 Order of Operations with Absolute Values

Find the distance between the given numbers on a number line.

1. $\frac{1}{2}$ and $-\frac{1}{4}$
2. $-\frac{3}{4}$ and $-\frac{2}{3}$
3. $-\frac{5}{8}$ and $-\frac{3}{4}$

4. $-\frac{7}{5}$ and $\frac{3}{7}$
5. -0.5 and 8.3
6. 10.7 and -2.8
7. $3\frac{1}{5}$ and $-2\frac{1}{3}$
8. $5\frac{3}{4}$ and 0

Answer available at the end of this section.

Exercise 3.13.7 Discussion Board Topics

1. Convert various examples in this section to equivalent expressions using text-based symbols.
2. What is PEMDAS and what is it missing?
3. Discuss the importance of proper grouping and give some examples.
4. Experiment with the order of operations on a calculator and share your results.

Answer available at the end of this section.

Answers

Exercise 3.13.1

15

Exercise 3.13.2

13

Exercise 3.13.3 **Order of Operations**

1. -22
3. -42
5. 4
7. $-\frac{9}{10}$
9. 7.24
11. -11
13. 6
15. -5
17. 0
19. 20
21. -75
23. 29
25. -33
27. -10
29. 67.22
31. -124
33. $-\frac{63}{8}$
35. $\frac{7}{30}$
37. $\frac{13}{24}$
39. $\frac{9}{7}$
41. 50
43. -17
45. $-\frac{1}{3}$
47. $\frac{5}{59}$
49. -1

Exercise 3.13.4 **Order of Operations**

1. $\$32.50$
3. $\$0.49$
5. 20 marbles

Exercise 3.13.5 **Order of Operations with Absolute Values**

1. 13
3. -8
5. 18

7. -9
11. 6
13. -27
15. 8
17. 1
19. $-\frac{11}{18}$
21. -63

Exercise $3.13.6$ **Order of Operations with Absolute Values**

1. $\frac{3}{4}$ unit
3. $\frac{1}{8}$ unit
5. 8.8 units
7. $5\frac{8}{15}$ units

Exercise $3.13.7$ **Discussion Board Topics**

1. Answers may vary
3. Answers may vary

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3.14: Scientific Notation

Learning Objectives

1. Write numbers in standard or scientific notation.
2. Convert numbers between standard and scientific notation.
3. Add and subtract numbers in scientific notation.
4. Multiply and divide numbers in scientific notation.
5. Use scientific notation in computing real-world applications.

The amount of information available on the Internet is simply incomprehensible. One estimate for the amount of data that will be on the Internet by 2025 is 175 Zettabytes. A single zettabyte is one billion trillion. Written out, it is 1,000,000,000,000,000,000,000. One estimate is that we're producing 2.5 quintillion bytes of data per day. A quintillion is a trillion trillion, or, written out, 1,000,000,000,000,000,000. To determine how many days it takes to increase the amount of information that is on the Internet by 1 zettabyte, divide these two numbers, a zettabyte being 1,000,000,000,000,000,000,000, and 2.5 quintillion, being 2,500,000,000,000,000,000, shows it takes 400 days to generate 1 zettabyte of information. But doing that calculation is awkward with a calculator. Keeping track of the zeros can be tedious, and a mistake can easily be made.

$$\frac{5 \text{ mol C}}{1} \times \frac{6.02 \times 10^{23} \text{ atoms C}}{1 \text{ mol C}} = 3.01 \times 10^{24} \text{ atoms C}$$

Figure 3.14.1: Calculations in the sciences often involve numbers in scientific notation form.

On the other end of the scale, a human red blood cell has a diameter of 7.8 micrometers. One micrometer is one millionth of a meter. Written out, 7.8 micrometers is 0.0000078 meters. Smaller still is the diameter of a virus, which is about 100 nanometers in diameter, where a nanometer is a billionth of a meter. Written out, 100 nanometers is 0.0000001 meters. To compare that to engineered items, a single transistor in a computer chip can be 14 nanometers in size (0.000000014 meters). Smaller yet is the diameter of an atom, at between 0.1 and 0.5 nanometers.

Sometimes we have numbers that are incredibly big, and so have an incredibly large number of digits, or sometimes numbers are incredibly small, where they have a large number of digits after the decimal. But using those representations of the names of the sizes makes comparing and computing with these numbers problematic. That's where scientific notation comes in.

Writing Numbers in Standard or Scientific Notation Form

When we say that a number is in scientific notation, we are specifying the form in which that number is written. That form begins with an integer with an absolute value between 1 and 9, then perhaps followed the decimal point and then some more digits. This is then multiplied by 10 raised to some power. When the number only has one non-zero digit, the scientific notation form is the digit multiplied by 10 raised to an exponent. When the number has more than one non-zero digit, the scientific notation form is a single digit, followed by a decimal, which is then followed by the remaining digits, which is then multiplied by 10 to a power.

The following numbers are written in scientific notation:

- 1.45×10^3
- -8.345×10^{-4}
- 3×10^2
- 3.14159×10^0

The following numbers are not written in scientific notation:

- 1.45 because it isn't multiplied by 10 raised to a power
- -50.053×10^7 because the absolute value of −50.053 is not at least 1 and less than 10
- 41.7×10^9 because 41.7 is not at least 1 and less than 10
- 0.036×10^{-3} because 0.036 is not at least one and less than 10

Example 3.14.1: Identifying Numbers in Scientific Notation

Which of the following numbers are in scientific notation? If the number is not in scientific notation, explain why it is not.

1. -9.67×10^{20}
2. 145×10^{-8}
3. 1.45

Answer available at the end of this section.

Your Turn 3.14.1

Which of the following numbers are in scientific notation? If the number is not in scientific notation, explain why it is not.

1. 42.67×10^{13}
2. 7.113×10^{-2}
3. -80.91

Some numbers are so large or so small that it is impractical to write them out fully. Avogadro's number is important in chemistry. It represents the number of units in 1 mole of any substance. The substance many be electrons, atoms, molecules, or something else. Written out, the number is: 602,214,076,000,000,000,000,000. Another example of a number that is impractical to write out fully is the length of a light wave. The wavelength of the color blue is about 0.000000450 to 0.000000495 meters. Such numbers are awkward to work with, and so scientific notation is often used. We need to discuss how to convert numbers into scientific notation, and also out of scientific notation.

Recall that multiplying a number by 10 adds a 0 to the end of the number or moves the decimal one place to the right, as in $43 \times 10 = 430$ or $3.89 \times 10 = 38.9$. And if you multiply by 100, it adds two zeros to the end of the number or moves the decimal two places to the right, and so on. For example, $38 \times 100{,}000 = 3{,}800{,}000$ and $32.998 \times 10{,}000 = 329{,}980$. Multiplying a number by 1 followed by some number of zeros just adds that many zeros to the end of the number or moves the decimal place that many places to the right. Numbers written as 1 followed by some zeros are just powers of 10, as in $10^1 = 10$, $10^2 = 100$, $10^3 = 1{,}000$, etc. Generally, [redacted]10n=10…0‾‾n10n=10…0‾‾n.

We can use this to write very large numbers. For instance, Avogadro's number is 602,214,076,000,000,000,000,000, which can be written as $6.02214076 \times 10^{23}$. The multiplication moves the decimal 23 places to the right.

Similarly, when we divide by 10, we move the decimal one place to the left, as in $\frac{46.7}{10} = 4.67$. If we divide by 100, we move the decimal two places to the left, as in $\frac{3.456}{100} = 0.03456$. In general, when you divide a number by a 1 followed by n zeros, you move the decimal n places to the left, as in $\frac{8{,}244.902}{1{,}000{,}000} = 0.008244902$. This denominator could be written as 10^6. If we use that in the expression and allow for negative exponents, rewrite the number as $\frac{8{,}244.902}{1{,}000{,}000} = \frac{8{,}244.902}{10^6} = 8{,}244.902 \times 10^{-6}$. With this, we can write division by a 1 followed by n zeros as multiplication by 10 raised to $-n$.

Using that information, we can demonstrate how to convert from a number in standard form into scientific notation form.

Case 1: The number is a single-digit integer.

In this case, the scientific notation form of the number is [redacted]digit×10digit×10.

Case 2: The absolute value of the number is less than 1.

Follow the process below.

- **Step 1:** Count the number of zeros between the decimal and the first non-zero digit. Label this n.
- **Step 2:** Starting with the first non-zero digit of the number, write the digits. If the number was negative, include the negative sign.
- **Step 3:** If there is more than one digit, place the decimal after the first digit from Step 2.
- **Step 4:** Multiply the number from Step 3 by [redacted]10n+110n+1.

Case 3: The absolute value of the number is 10 or larger.

Follow the process below.

- **Step 1:** Count the number of digits that are to the left of the decimal point. Label this n.
- **Step 2:** Write the digits of the number without the decimal place, if one was present. If the number was negative, include the negative sign.
- **Step 3:** If there is more than one digit, place the decimal point after the first digit.
- **Step 4:** Multiply the number from Step 3 by [redacted]10n−110n−1.

Example3.14.2: Writing a Number in Scientific Notation

Write the following numbers in scientific notation form:

1. 428.9
2. −0.00000981
3. 8

Answer available at the end of this section.

Your Turn 3.14.2

Write the following numbers in scientific notation form:

1. -38300
2. 0.0045
3. 1

When we write numbers in scientific notation form, we can manipulate the representation of the number by moving the decimal around, and making an appropriate change to the exponent of the 10. For instance, let's look at 145.8141×10^8. If we wanted to move the decimal one place to the left, we'd have to increase the power of 10, as shown here: $145.8141 \times 10^8 = 14.58141 \times 10^9$. Since we moved the decimal one to the left, we balance that with moving the exponent up by one. Similarly, if we move the decimal one place to the right, we have to balance that by moving the exponent one to the left, or subtracting one from the exponent, as shown here: $145.8141 \times 10^8 = 1458.141 \times 10^7$. Generally, for a number in the form [redacted]number×10nnumber×10n:

- If you move the decimal to the left by k digits, you increase the exponent by k.
- If you move the decimal to the right by k digits, you decrease the exponent by k digits.

Example 3.14.3: Increasing the Exponent

Change 456.142×10^5 by moving the decimal two places to the left.

Answer available at the end of this section.

Your Turn 3.14.3

Change 46.113×10^{8} by moving the decimal four places to the left.

Example 3.14.4: Decreasing the Exponent

Change 12.3×10^{2} by moving the decimal five places to the right.

Answer available at the end of this section.

Your Turn 3.14.4

Change 149.11×10^{-4} by moving the decimal two places to the right

Converting Numbers from Scientific Notation to Standard Form

In the previous section, converting a number from standard form to scientific notation was explored. Now, we explore converting from scientific notation back into standard form. Doing so involves moving the decimal according to the power of the 10. The decimal is moved a number of steps equal to the exponent of the 10. As demonstrated previously, when the exponent of the 10 is negative, the decimal is moved to the left and when the exponent of the 10 is positive, the decimal is moved to the right.

Example 3.14.5: Converting from Scientific Notation to Standard Form

Convert the following into standard form:

1. 2.78×10^{9}
2. 9.04×10^{-8}

Answer available at the end of this section.

Your Turn 3.14.5

Convert the following into standard form:

1. 1.02×10^{6}
2. 4.09×10^{-5}

Video

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Converting from Standard Form to Scientific Notation Form

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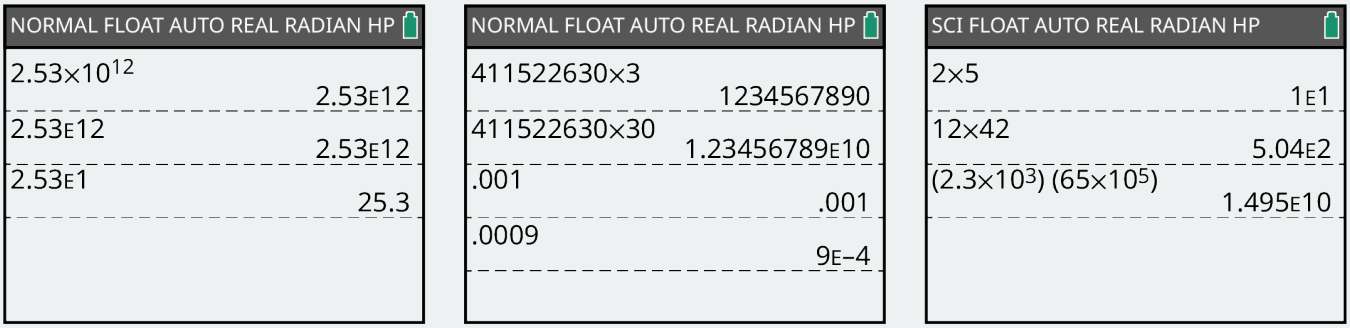
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Converting from Scientific Notation Form to Standard Form

Tech Check: Scientific Notation on a Calculator

Most scientific and graphing calculators come with the ability to directly convert from standard form to scientific notation. On the TI-83, it is accessed through the MODE menus. For a commonly used, free phone scientific calculator, the calculator can be forced to work in scientific notation mode through its settings.

Some calculators, such as the Desmos online calculator, display scientific notation as a number times 10 to a power as you've seen in this section. However, some calculators indicate scientific notation by replacing the [redacted]×10nFigure 3.47 shows what you may see on a TI-84.



Accessing E | Using normal mode | Using sci mode

Figure 3.14.2: Calculator screens

Adding and Subtracting Numbers in Scientific Notation

To add or subtract numbers in scientific notation, the numbers first need to have the same exponent for the 10s. It is possible to add the following since the powers of 10 match: $4.5 \times 10^4 + 3.15 \times 10^4 = 7.65 \times 10^4$

Notice that the number parts were added, but the exponent part remained the same. This is due to the distributive property of the real numbers. The 10^4 is factored from the two terms, as shown: $4.5 \times 10^4 + 3.15 \times 10^4 = (4.5 + 3.15) \times 10^4 = 7.65 \times 10^4$

Numbers in scientific notation can be added or subtracted directly using a calculator. Simply enter the values in scientific form and set your calculator to display scientific notation.

Example 3.14.6: Adding and Subtracting Numbers in Scientific Notation with the Same Powers of 10

Calculate the following:

1. $3.8 \times 10^{-3} + 1.006 \times 10^{-3}$
2. $9.61 \times 10^8 - 3.85 \times 10^8$

Answer available at the end of this section.

Your Turn 3.14.6

Calculate the following:

- $7.57 \times 10^{13} + 2.031 \times 10^{13}$
- $3.03 \times 10^{-6} - 1.5 \times 10^{-6} 7.57 \times 10^{13} + 2.031 \times 10^{13}$

Adding and subtracting in scientific notation is straightforward when the exponents are the same. There are two issues that can arise. The first issue is what to do if after adding or subtracting the result is not in scientific notation.

Example 3.14.7: Correcting an Answer to Scientific Notation After Adding or Subtracting

Calculate the following:

1. $7.03 \times 10^{13} + 8.5 \times 10^{13}$
2. $4.3 \times 10^{21} - 4.613 \times 10^{21}$

Answer available at the end of this section.

Your Turn 3.14.7

Calculate the following:

1. \(5.08 \times {10^3} + 6.9 \times {10^3}\(
2. \(8.968 \times {10^{ - 38}} - 8.761 \times {10^{ - 38}}\(

The second issue that might be encountered when adding or subtracting is that the powers of 10 do not match. In that case, one of the numbers must be changed so that the powers of 10 match. It is easiest to make the smaller power of 10 larger to match the other power of 10.

For example, to perform the following, $4.5 \times 10^5 + 3.9 \times 10^3$, we'd change the 3.9×10^3 so that the power of 10 is 5. To do so, we need to increase the power of 10 and move the decimal in the number part two places to the left. That would alter 3.9×10^3 into 0.039×10^5. We would use 0.039×10^5 in the addition problem, so that the exponents match, allowing the addition to occur. $4.5 \times 10^5 + 3.9 \times 10^3 = 4.5 \times 10^5 + 0.039 \times 10^5 = (4.5 + 0.039) \times 10^5 = 4.539 \times 10^5$

The steps to take when the exponents of the 10s are not equal are:

Step 1: Increase the smaller exponent to equal the larger exponent. Label the amount increased as n.

Step 2: For the number with the smaller power of 10, move the decimal point of the number part to the left n places.

Step 3: Perform the addition or subtraction.

Step 4: If the result is not in scientific notation, adjust the number to be in scientific notation.

Example 3.14.18: Adding Numbers in Scientific Notation with Different Powers of 10

Calculate the following:

$$6.1 \times 10^4 + 4.8 \times 10^5$$

Answer available at the end of this section.

Your Turn 3.14.8

Calculate the following:

$$1.14 \times 10^{-43} + 2.56 \times 10^{-46} \tag{3.14.1}$$

Example 3.14.9: Subtracting Numbers in Scientific Notation with Different Powers of 10

Calculate the following:

$$7.9 \times 10^{-15} - 6.8 \times 10^{-13} \tag{3.14.2}$$

Answer available at the end of this section.

Your Turn 3.14.9

Calculate the following:

$$9.15 \times 10^{28} - 7.23 \times 10^{26} \tag{3.14.3}$$

Multiplying and Dividing Numbers in Scientific Notation

Multiplying and dividing numbers in scientific notation is somewhat easier than adding or subtracting, because the exponents of the 10s do not have to match. However, it is much more likely that the result will not be in scientific notation, and so that will have to be adjusted at the end. Generally, we multiply or divide the number parts of the two values, and then apply exponent rules to the 10 raised to the powers.

To multiply two numbers in scientific notation:

Step 1: Multiply the number parts.

Step 2: Add the exponents of the 10s.

Step 3: The result is the answer from Step 1 times 10 raised to the answer from Step 2.

Step 4: If the number is not in scientific notation, adjust it appropriately.

Example 3.14.10: Multiplying Numbers in Scientific Notation

Calculate the following:

1. $(4.3 \times 10^{3}) \times (1.8 \times 10^{7})$
2. $(5 \times 10^{-13}) \times (7.3 \times 10^{6})$

Answer available at the end of this section.

Your Turn 3.14.10

Calculate the following:

1. $(2.29 \times 10^{3}) \times (3 \times 10^{4})$
2. $(6.91 \times 10^{-3}) \times (9.1 \times 10^{5})$

Video

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Multiplying Numbers in Scientific Notation

Dividing Numbers in Scientific Notation

To divide two numbers that are in scientific notation:

Step 1: Divide the number parts.

Step 2: Subtract the exponent of the denominator from the exponent of the numerator.

Step 3: The answer is the result from Step 1 times 10 raised to the result from Step 2.

Step 4: If the number is not in scientific notation, adjust it appropriately.

Example 3.14.11: Dividing Numbers in Scientific Notation

Calculate the following:

1. $(8.4 \times 10^{31}) / (2.1 \times 10^{7})$
2. $(4.14 \times 10^{-13}) / (8.28 \times 10^{9})$

Answer available at the end of this section.

Your Turn 3.14.11

Calculate the following:

1. $(3.6 \times 10^{-2}) / (1.5 \times 10^{3})$
2. $(1.8 \times 10^{4}) / (4.8 \times 10^{3})$

Video

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Dividing Numbers in Scientific Notation

Using Scientific Notation in Computing Real-World Applications

As noted at the start of this section, scientific notation is useful when the standard representation of a number is awkward or impractical, which occurs when the numbers being used are extremely large or extremely small. For example, Venus is 67,667,000 miles from the sun. In scientific notation, this is 6.7667×10^7. Planetary and galaxy distances is one set of numbers that is easier to express using scientific notation.

Example 3.14.12: Calculating Distances

How much farther from the sun is Earth compared to Venus if Venus is 6.7667×10^7 miles from the sun and Earth is 9.1692×10^7 miles from the sun?

Answer available at the end of this section.

Your Turn 3.14.21

Earlier we saw that a single transistor in a computer chip 0.000000014 meters, or 1.4×10^{-8} m, in size, and that the diameter of an atom could be 0.2 nanometers, or 2×10^{-10} m in size. How much larger is the transistor than the atom?

Example 3.14.13: Calculating Probability

The probability of winning the Mega Millions lottery is published as 3.304693×10^{-9}. The probability of being hit by lightning is approximated to be 2×10^{-6}. How many times more likely are you to be hit by lightning than win the Mega Millions?

Answer available at the end of this section.

Your Turn 3.14.31

Mercury is about 3.114×10^7 miles from the sun. Neptune is about 2.781×10^9 miles from the sun. How many times further is Neptune from the sun than Mercury?

Example 3.14.14: Calculating Time and Length

Sometimes it is entertaining to determine the time it takes for something to happen. Fingernails grow about 8.032×10^{-11} km per minute. How many kilometers long would fingernails be after 6×10^4 minutes?

Answer available at the end of this section.

Your Turn 3.14.14

There are approximately 1×10^{12} grains of sand in a cubic meter. If the number of grains of sand on the Australian coastline is roughly 7.5×10^{21} grains, roughly how many cubic meters of sand is there on the Australian coastline? of sand on the Australian coastline is roughly 7.5×10^{21} grains, roughly how many cubic meters of sand is there on the Australian coastline?

Example 3.14.15: Calculating Data Generated

As mentioned in the opening to this section, it is estimated that we're producing 2.5 quintillion bytes of data per day. A good estimate is that there are 7.674 billion people on the planet. Convert both of those numbers to scientific notation, and then determine how much data is being generated per person each day.

Answer available at the end of this section.

Your Turn $3.14.15$

Humans collectively exhale approximately 6.4235×10^{12} pounds of carbon dioxide per year. There are approximately 7.647×10^{9} humans currently living on Earth. How many pounds of carbon dioxide does a single human, on average, exhale per year?

Video

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Application of Scientific Notation

What Numbers Could Be Considered “Too Big” or “Too Small”?

One wonders when the numbers we represent become too large or small for consideration. Perhaps the following examples put limits on what is meaningful. The number of particles in the known universe has been estimated at 4×10^{80} particles. The smallest distance that has been measured is $1 \times 10^{-}$18m1×10−18m, though the theoretical smallest measurable value is $1 \times 10^{-}$35m1×10−35m. The distance across the universe is 4.4×10^{26} m. Considering what those numbers represent, the extreme largest and extreme smallest, they might be numbers that constrain what we should reasonably be expected to deal with.

Check Your Understanding

1. Write 0.00456 in scientific notation.
2. Write 5.67×10^{8} in standard form.
3. Calculate $4.5 \times 10^{3} + 9.8 \times 10^{2}$.
4. Calculate $2.5 \times 10^{5} - 9.8 \times 10^{6}$.
5. Calculate $(7.4 \times 10^{4}) \times (4.8 \times 10^{3})$.
6. Calculate \(\frac
   ParseError: EOF expected (click for details)

   ```
   Callstack:
       at (Bookshelves/Applied_Mathematics/Contemporary_Mathematics_(OpenStax)/03:_Real_Number
       at wiki.page()
       at (Courses/Coalinga_College/Math_for_Educators_(MATH_010A_and_010B_CID120)/03:_The_Rea
   ```

   ParseError: EOF expected (click for details)

   ```
   Callstack:
       at (Bookshelves/Applied_Mathematics/Contemporary_Mathematics_(OpenStax)/03:_Real_Number
       at wiki.page()
       at (Courses/Coalinga_College/Math_for_Educators_(MATH_010A_and_010B_CID120)/03:_The_Rea
   ```

   \).
7. The distance from Earth to the moon is 1.514×10^{10} inches. The thickness of a dollar bill is 4.3×10^{-3} inches. How many dollar bills must be stacked so the pile reaches the moon?

Answers

Example 3.14.1**: Identifying Numbers in Scientific Notation**

1. The number -9.67×10^{20} is in scientific notation because the absolute value of −9.67 is at least 1 and less than 10.
2. The number 145×10^{-8} is not in scientific notation because 145 is not at least 1 and less than 10.
3. The number 1.45 is not in scientific notation form. Even though it is at least 1 but less than 10, it is not multiplied by 10 raised to a power.

Example3.14.2**: Writing a Number in Scientific Notation**

1. Since the absolute value of 428.9 is 10 or larger, so we use the process from Case 3, above.
   Step 1: There are three digits to the left of the decimal point, so $n = 3$.
   Step 2: Write the digits of the number without the decimal place, which is 4289.
   Step 3: Since there is more than one digit, place the decimal point after the first digit. We now have 4.289.
   Step 4: Since $n = 3$, we multiply 4.289 by 10 raised to the second power, 4.289×10^2.
   The scientific notation form of 428.9 is 4.289×10^2.
2. Since the absolute value of −0.00000981 is less than 1, we use the process from Case 2.
   Step 1: The number of zeros between the decimal and the first non-zero digit is 5, so $n = 5$.
   Step 2: We write the non-zero digits, including the negative sign, yielding −981.
   Step 3: The decimal gets placed to the right of the first digit, resulting in −9.81.
   Step 4: Since $n = 5$, we multiply −9.81 by 10 raised to the fourth power, -9.81×10^{-6}.
   The scientific notation form of −0.00000981 is 9.81×10^{-6}.
3. Since 8 is a single-digit integer, apply Case 1. The scientific notation form of 8 is 8×10^1.

Example 3.14.3**: Increasing the Exponent**

Since we are moving the decimal to the left by two places, we increase the exponent of 10 by 2, so that the exponent is now 7. This gives us $456.142 \times 10^5 = 4.56142 \times 10^7$.

Example 3.14.4**: Decreasing the Exponent**

Since we are moving the decimal to the right by five places, we decrease the exponent of 10 by 5, so that exponent is now −3. This give us $12.3 \times 10^2 = 1230000.0 \times 10^{-3}$.

Example 3.14.5**: Converting from Scientific Notation to Standard Form**

1. Since the exponent is positive, the decimal moves nine places to the right, so 2.78×10^9 is $2{,}780{,}000{,}000$.
2. Since the exponent is negative, the decimal moves eight places to the left, so 9.04×10^{-8} is 0.0000000904.

Example 3.14.6**: Adding and Subtracting Numbers in Scientific Notation with the Same Powers of 10**

1. Since the powers of 10 match, we use the distributive property of real numbers to factor 10^{-3} from the numbers. We then add the number parts separately to get 4.806. $3.8 \times 10^{-3} + 1.006 \times 10^{-3} = (3.8 + 1.006) \times 10^{-3} = 4.806 \times 10^{-3}$
2. Since the powers of 10 match, we use the distributive property of real numbers to factor 10^8 from the numbers. We then subtract the number parts separately to get 5.76.

Example 3.14.7**: Correcting an Answer to Scientific Notation After Adding or Subtracting**

1. Since the powers of 10 match, we add the number parts and multiply that by 10^{13}: $7.03 \times 10^{13} + 8.5 \times 10^{13} = (7.03 + 8.5) \times 10^{13} = 15.53 \times 10^{13}$.
   However, 15.53×10^{13} is not in scientific notation because the absolute value of 15.53 is more than 10. To put this number in scientific notation, the decimal needs to move one to the left. To balance that move, the power of 10 must be increased by 1. So, the answer in scientific notation is 1.553×10^{14}.
2. Since the powers of 10 match, we add the number parts: $4.3 \times 10^{21} - 4.613 \times 10^{21} = (4.3 - 4.613) \times 10^{21} = -0.313 \times 10^{21}$

Example 3.14.18**: Adding Numbers in Scientific Notation with Different Powers of 10**

Step 1: The lower exponent is 4. To make this equal to the larger exponent, we increased it by 1.

Step 2: Since the smaller exponent was increased by 1, move the decimal one to the left, so the addition become $6.1 \times 10^4 + 4.8 \times 10^5 = 0.61 \times 10^5 + 4.8 \times 10^5$.

Step 3: Now add the numbers, $0.61 \times 10^5 + 4.8 \times 10^5 = 5.41 \times 10^5$

Step 4: The result is in scientific notation, so no additional adjustment is necessary.

$6.1 \times 10^4 + 4.8 \times 10^5 = 5.41 \times 10^5$

Example 3.14.9**: Subtracting Numbers in Scientific Notation with Different Powers of 10**

Step 1: The lower exponent is −15 and the larger is −13. To make −15 equal to the larger exponent, we increased it by 2.

Step 2: Since the smaller exponent increased by 2, move the decimal two to the left. The subtraction changes to $7.9 \times 10^{-15} - 6.8 \times 10^{-13} = 0.079 \times 10^{-13} - 6.8 \times 10^{-13}$.

Step 3: Subtract the numbers, $0.079 \times 10^{-13} - 6.8 \times 10^{-13} = -6.721 \times 10^{-13}$.

Step 4: The result is in scientific notation, so no additional adjustment is necessary.

$7.9 \times 10^{-15} - 6.8 \times 10^{-13} = -6.721 \times 10^{-13}$

Example 3.14.10**: Multiplying Numbers in Scientific Notation**

1. **Step 1:** Multiply the number parts to get $4.3 \times 1.8 = 7.74$.
   Step 2: Add the exponents of the 10s to get $3 + 7 = 10$.
   Step 3: The result is then 7.74×10^{10}.
   Step 4: This number is already in scientific notation, so no additional adjustment is necessary, $(4.3 \times 10^3) \times (1.8 \times 10^7) = 7.74 \times 10^{10}$.
2. **Step 1:** Multiply the number parts to get $5 \times 7.3 = 36.5$.
   Step 2: Add the exponents of the 10s to get $-13 + 6 = -7$.
   Step 3: The result then is 36.5×10^{-7}.
   Step 4: Since the number is not in scientific notation, it must be adjusted. To put 36.5×10^{-7} into scientific notation, the decimal moves one to the left, so the

exponent would be increased by 1, giving 3.65×10^{-6}.

$(5 \times 10^{-13}) \times (7.3 \times 10^{6}) = 3.65 \times 10^{-6}$

Example $3.14.11$**: Dividing Numbers in Scientific Notation**

1. **Step 2:** Subtract the exponent of the denominator from the exponent of the numerator to get $37 - 7 = 24$.

   Step 3: The result is then 4×10^{24}.

   Step 4: This number is already in scientific notation, so no adjustment is necessary. $(8.4 \times 10^{31}) \ / \ (2.1 \times 10^{7}) = 4 \times 10^{24}$
2. **Step 2:** Subtract the exponent of the denominator from the exponent of the numerator to get $-13 - 9 = -22$.

   Step 3: The result then is 0.5×10^{-22}.

   Step 4: Since this number is not in scientific notation, it must be adjusted. To put 0.5×10^{-22} into scientific notation, the decimal needs to move one to the right, so the exponent is decreased by 1, giving 5×10^{-23}. $(4.14 \times 10^{-13}) \ / \ (8.28 \times 10^{9}) = 5 \times 10^{-23}$

Example $3.14.12$**: Calculating Distances**

To determine how much farther Earth is compared to Venus, we'd subtract the distances.
$9.1692 \times 10^{7} - 6.7667 \times 10^{7} = 2.4025 \times 10^{7}$.

So, Earth is 2.4025×10^{7} miles farther from the sun than Venus.

Example $3.14.13$**: Calculating Probability**

To find out how many times more likely you are to be hit by lightning, divide the probability of being hit by lightning by the probability of winning the Mega Millions.

Step 1: Divide the number parts to get = 0.6052 (rounded to the fourth digit).

Step 2: Subtract the exponent of the denominator from the exponent of the numerator to get $-6 - (-9) = 3$.

Step 3: The result then is 0.6052×10^{3}.

Step 4: Since this number is not in scientific notation, it must be adjusted. To put 0.6052×10^{3} into scientific notation, the decimal needs to move one place to the right, so the exponent is decreased by 1, giving 6.052×10^{2}.

You are 6.052×10^{2}, or 605.2, times more likely to be hit by lightning than you are to win the Mega Millions.

Example $3.14.14$**: Calculating Time and Length**

To find the length of the fingernails after the specified time, we multiply their rate of growth and the time they've grown. $(8.032 \times 10^{-11}) \times (6 \times 10^{4}) = 48.192 \times 10^{-7} = 4.8192 \times 10^{-6}$

So, after 6×10^{4} minutes, the fingernails would be 4.8192×10^{-6} km long. To put this in perspective, 1×10^{-6} km is a millimeter, and 6×10^{4} minutes is about 4.16 days. So, after about 4.16 days, fingernails have grown about 4.8 millimeters.

Example $3.14.15$**: Calculating Data Generated**

Written in standard form, 2.5 quintillion is 2,500,000,000,000,000,000. Changing that to scientific notation, move the decimal 18 places, so 2.5 quintillion bytes = 2.5×10^{18} bytes. Writing 7.647 billion in scientific notation would be 7.647×10^{9} because a billion is 1,000,000,000 = 10^{9}. So, to find out how much data is being produced daily per person, we would divide these two numbers. $\frac{2.5 \times 10^{18}}{7.647 \times 10^{9}} = 0.327 \times 10^{9} = 3.27 \times 10^{8}$

In standard form, that's 327,000,000 bytes per person, so 327 million bytes of data daily are being produced per person.

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3.15: Approximation and Estimation

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3.16: Real Numbers - Algebra Essentials

Learning Objectives

- Classify a real number as a natural, whole, integer, rational, or irrational number.
- Perform calculations using order of operations.
- Use the following properties of real numbers: commutative, associative, distributive, inverse, and identity.
- Evaluate algebraic expressions.
- Simplify algebraic expressions.

It is often said that mathematics is the language of science. If this is true, then the language of mathematics is numbers. The earliest use of numbers occurred 100 centuries ago in the Middle East to count, or enumerate items. Farmers, cattlemen, and tradesmen used tokens, stones, or markers to signify a single quantity—a sheaf of grain, a head of livestock, or a fixed length of cloth, for example. Doing so made commerce possible, leading to improved communications and the spread of civilization.

Three to four thousand years ago, Egyptians introduced fractions. They first used them to show reciprocals. Later, they used them to represent the amount when a quantity was divided into equal parts.

But what if there were no cattle to trade or an entire crop of grain was lost in a flood? How could someone indicate the existence of nothing? From earliest times, people had thought of a “base state” while counting and used various symbols to represent this null condition. However, it was not until about the fifth century A.D. in India that zero was added to the number system and used as a numeral in calculations.

Clearly, there was also a need for numbers to represent loss or debt. In India, in the seventh century A.D., negative numbers were used as solutions to mathematical equations and commercial debts. The opposites of the counting numbers expanded the number system even further.

Because of the evolution of the number system, we can now perform complex calculations using these and other categories of real numbers. In this section, we will explore sets of numbers, calculations with different kinds of numbers, and the use of numbers in expressions.

Classifying a Real Number

The numbers we use for counting, or enumerating items, are the natural numbers: $1, 2, 3, 4, 5$ and so on. We describe them in set notation as $\{1, 2, 3, ...\}$ where the ellipsis $(\cdots)$ indicates that the numbers continue to infinity. The natural numbers are, of course, also called the counting numbers. Any time we enumerate the members of a team, count the coins in a collection, or tally the trees in a grove, we are using the set of natural numbers. The set of whole numbers is the set of natural numbers plus zero: $\{0, 1, 2, 3, ...\}$.

The set of integers adds the opposites of the natural numbers to the set of whole numbers: $\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$. It is useful to note that the set of integers is made up of three distinct subsets: negative integers, zero, and positive integers. In this sense, the positive integers are just the natural numbers. Another way to think about it is that the natural numbers are a subset of the integers.

$$\overbrace{\cdots, -3, -2, -1}^{\text{negative integers}}, \underbrace{0}_{\text{zero}}, \overbrace{1, 2, 3, \cdots}^{\text{positive integers}}$$

The set of rational numbers is written as $\{\frac{m}{n} | \text{m and n are integers and } n \neq 0\}$.Notice from the definition that rational numbers are fractions (or quotients) containing integers in both the numerator and the denominator, and the denominator is never 0. We can also see that every natural number, whole number, and integer is a rational number with a denominator of 1.

Because they are fractions, any rational number can also be expressed in decimal form. Any rational number can be represented as either:

1. a terminating decimal: $\frac{15}{8} = 1.875$, or
2. a repeating decimal: $\frac{4}{11} = 0.36363636\cdots = 0.\overline{36}$

We use a line drawn over the repeating block of numbers instead of writing the group multiple times.

Example 3.16.1: Writing Integers as Rational Numbers

Write each of the following as a rational number. Write a fraction with the integer in the numerator and 1 in the denominator.

a. 7
b. 0
c. -8

Solution

a. $7 = \frac{7}{1}$

b. $0 = \frac{0}{1}$

c. $-8 = \frac{-8}{1}$

Exercise 3.16.1

Write each of the following as a rational number.

a. 11
b. 3
c. -4

Answer available at the end of this section.

Example 3.16.2: Identifying Rational Numbers

Write each of the following rational numbers as either a terminating or repeating decimal.

a. $-\frac{5}{7}$
b. $\frac{15}{5}$
c. $\frac{13}{25}$

Solution

a. a repeating decimal

b. $\frac{15}{5} = 3$(or 3.0), a terminating decimal

c. $\frac{13}{25} = 0.52$, a terminating decimal

Exercise 3.16.2

Write each of the following rational numbers as either a terminating or repeating decimal.

a. $\frac{68}{17}$
b. $\frac{8}{13}$
c. $-\frac{13}{25}$

Answer available at the end of this section.

Irrational Numbers

At some point in the ancient past, someone discovered that not all numbers are rational numbers. A builder, for instance, may have found that the diagonal of a square with unit sides was not 2 or even 32, but was something else. Or a garment maker might have observed that the ratio of the circumference to the diameter of a roll of cloth was a little bit more than 3, but still not a rational number. Such numbers are said to be irrational because they cannot be written as fractions. These numbers make up the set of irrational numbers. Irrational numbers cannot be expressed as a fraction of two integers. It is impossible to describe this set of numbers by a single rule except to say that a number is irrational if it is not rational. So we write this as shown.

$$\{h \mid h \text{ is not a rational number}\} \tag{3.16.1}$$

Example 3.16.3: Differentiating Rational and Irrational Numbers

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

a. $\sqrt{25}$
b. $\frac{33}{9}$
c. $\sqrt{11}$
d. $\frac{17}{34}$
e. $0.3033033303333\ldots$

Solution

a. $\sqrt{25}$: This can be simplified as $\sqrt{25}=5$. Therefore,$\sqrt{25}$is rational.
b. $\frac{33}{9}$: Because it is a fraction,$\frac{33}{9}$ is a rational number. Next, simplify and divide.

$$\frac{33}{9}=\frac{\cancel{33}}{\cancel{9}}$$

So, $\frac{33}{9}$ is rational and a repeating decimal.
c. $\sqrt{11}$: This cannot be simplified any further. Therefore, $\sqrt{11}$ is an irrational number.
d. $\frac{17}{34}$: Because it is a fraction, $\frac{17}{34}$ is a rational number. Simplify and divide.

$$\frac{17}{34}=0.5$$

So, $\frac{17}{34}$ is rational and a terminating decimal.
e. $0.3033033303333\ldots$ is not a terminating decimal. Also note that there is no repeating pattern because the group of $3s$ increases each time. Therefore it is neither a terminating nor a repeating decimal and, hence, not a rational number. It is an irrational number.

Exercise 3.16.3

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

a. $\frac{7}{77}$
b. $\sqrt{81}$
c. $4.27027002700027\ldots$
d. $\frac{91}{13}$
e. $\sqrt{39}$

Answer available at the end of this section.

Real Numbers

Given any number n, we know that n is either rational or irrational. It cannot be both. The sets of rational and irrational numbers together make up the set of real numbers. As we saw with integers, the real numbers can be divided into three subsets: negative real numbers, zero, and positive real numbers. Each subset includes fractions, decimals, and irrational numbers according to their algebraic sign (+ or –). Zero is considered neither positive nor negative.

The real numbers can be visualized on a horizontal number line with an arbitrary point chosen as 0, with negative numbers to the left of 0 and positive numbers to the right of 0. A fixed unit distance is then used to mark off each integer (or other basic value) on either side of 0. Any real number corresponds to a unique position on the number line.The converse is also true: Each location on the number line corresponds to exactly one real number. This is known as a one-to-one correspondence. We refer to this as the real number line as shown in Figure $3.16.1$.

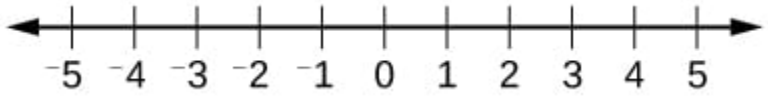


Figure $3.16.1$: The real number line.

Example $3.16.4$: Classifying Real Numbers

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

a. $-\frac{10}{3}$
b. $\sqrt{5}$
c. $-\sqrt{289}$
d. -6π
e. $0.615384615384\ldots$

Solution

a. $-\frac{10}{3}$ is negative and rational. It lies to the left of 0 on the number line.
b. $\sqrt{5}$ is positive and irrational. It lies to the right of 0.
c. $-\sqrt{289} = -\sqrt{17^2} = -17$ is negative and rational. It lies to the left of 0.
d. -6π is negative and irrational. It lies to the left of 0.
e. $0.615384615384\ldots$ is a repeating decimal so it is rational and positive. It lies to the right of 0.

Exercise $3.16.4$

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

a. $\sqrt{73}$
b. $-11.411411411\ldots$
c. $\frac{47}{19}$
d. $-\frac{\sqrt{5}}{2}$
e. 6.210735

Answer available at the end of this section.

Sets of Numbers as Subsets

Beginning with the natural numbers, we have expanded each set to form a larger set, meaning that there is a subset relationship between the sets of numbers we have encountered so far. These relationships become more obvious when seen as a diagram, such as Figure$(3.16.2)$.

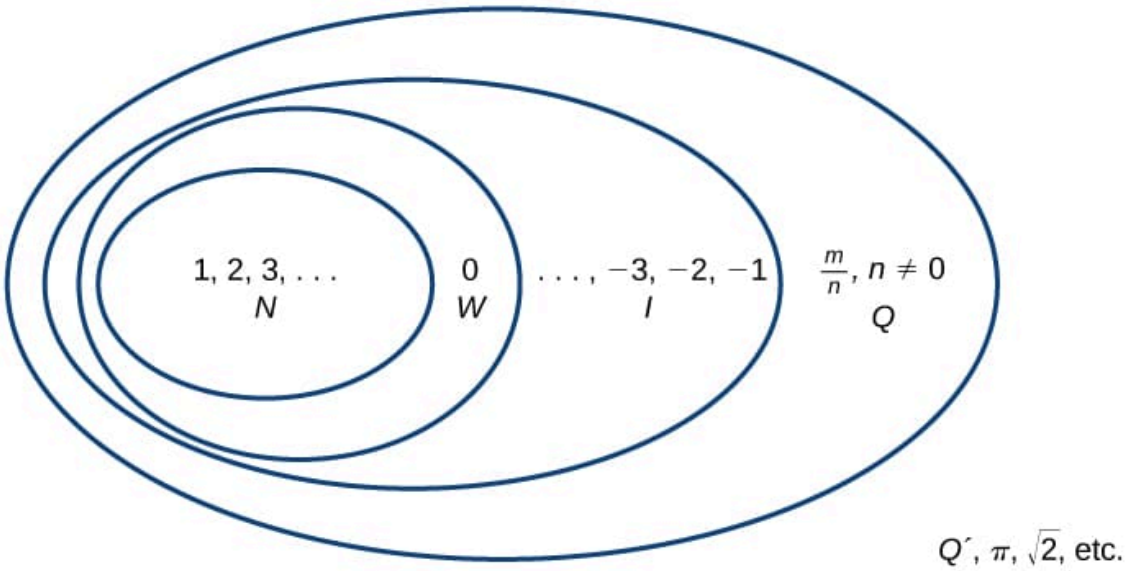


Figure 3.16.2: Sets of numbers. **N**: the set of natural numbers. **W**: the set of whole numbers. **I**: the set of integers. **Q**: the set of rational numbers. **Q´**: the set of irrational numbers

SETS OF NUMBERS

The set of natural numbers includes the numbers used for counting: $\{1, 2, 3, ...\}$.

The set of whole numbers is the set of natural numbers plus zero: $\{0, 1, 2, 3, ...\}$.

The set of integers adds the negative natural numbers to the set of whole numbers: $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.

The set of rational numbers includes fractions written as $\{\frac{m}{n} | \text{m and n are integers and } n \neq 0\}$.

The set of irrational numbers is the set of numbers that are not rational, are nonrepeating, and are nonterminating: $\{h \parallel \text{h is not a rational number}\}$.

Example 3.16.5: Differentiating the Sets of Numbers

Classify each number as being a natural number (N), whole number (W), integer (I), rational number (Q), and/or irrational number (Q′).

a. $\sqrt{36}$
b. $\frac{8}{3}$
c. $\sqrt{73}$
d. -6
e. $3.2121121112\ldots$

Solution

	N	W	I	Q	Q'
a. $\sqrt{36} = 6$	X	X	X	X	
b. $\frac{8}{3} = 2.\overline{6}$				X	
c. $\sqrt{73}$					X
d. -6			X	X	

	N	W	I	Q	Q'
e. $3.2121121112\ldots$					X

Exercise 3.16.5

Classify each number as being a natural number (N), whole number (W), integer (I), rational number (Q), and/or irrational number (Q′).

a. $-\frac{35}{7}$
b. 0
c. $\sqrt{169}$
d. $\sqrt{24}$
e. $4.763763763\ldots$

Answer available at the end of this section.

Performing Calculations Using the Order of Operations

When we multiply a number by itself, we square it or raise it to a power of 2. For example, $4^2 = 4 \times 4 = 16$. We can raise any number to any power. In general, the exponential notation an means that the number or variable a is used as a factor n times.

$$a^n = a \cdot a \cdot a \cdots a \qquad \text{n factors}$$

In this notation, a^n is read as the nth power of a, where a is called the base and n is called the exponent. A term in exponential notation may be part of a mathematical expression, which is a combination of numbers and operations. For example, $24 + 6 \times \frac{2}{3} - 4^2$ is a mathematical expression.

To evaluate a mathematical expression, we perform the various operations. However, we do not perform them in any random order. We use the order of operations. This is a sequence of rules for evaluating such expressions.

Recall that in mathematics we use parentheses ( ), brackets [ ], and braces { } to group numbers and expressions so that anything appearing within the symbols is treated as a unit. Additionally, fraction bars, radicals, and absolute value bars are treated as grouping symbols. When evaluating a mathematical expression, begin by simplifying expressions within grouping symbols.

The next step is to address any exponents or radicals. Afterward, perform multiplication and division from left to right and finally addition and subtraction from left to right.

Let's take a look at the expression provided.

$$24 + 6 \times \frac{2}{3} - 4^2$$

There are no grouping symbols, so we move on to exponents or radicals. The number 4 is raised to a power of 2, so simplify 4^2 as 16.

$$24 + 6 \times \frac{2}{3} - 4^2$$

$$24 + 6 \times \frac{2}{3} - 16$$

Next, perform multiplication or division, left to right.

$$24 + 6 \times \frac{2}{3} - 16$$

$$24 + 4 - 16$$

Lastly, perform addition or subtraction, left to right.

$$24 + 4 - 16$$

$$28 - 16$$

$$12$$

Therefore,

$$24 + 6 \times \frac{2}{3} - 4^2 = 12$$

For some complicated expressions, several passes through the order of operations will be needed. For instance, there may be a radical expression inside parentheses that must be simplified before the parentheses are evaluated. Following the order of operations ensures that anyone simplifying the same mathematical expression will get the same result.

ORDER OF OPERATIONS

Operations in mathematical expressions must be evaluated in a systematic order, which can be simplified using the acronym **PEMDAS**:

- **P**(arentheses)
- **E**(xponents)
- **M(**ultiplication) and **D**(ivision)
- **A**(ddition) and **S**(ubtraction)

HOW TO: Given a mathematical expression, simplify it using the order of operations.

1. Simplify any expressions within grouping symbols.
2. Simplify any expressions containing exponents or radicals.
3. Perform any multiplication and division in order, from left to right.
4. Perform any addition and subtraction in order, from left to right.

Example 3.16.6: Using the Order of Operations

Use the order of operations to evaluate each of the following expressions.

a. $(3 \times 2)^2 - 4 \times (6 + 2)$

b. $\dfrac{5^2 - 4}{7} - \sqrt{11 - 2}$

c. $6 - |\, 5 - 8 \,| + 3 \times (4 - 1)$

d. $\dfrac{14 - 3 \times 2}{2 \times 5 - 3^2}$

e. $7 \times (5 \times 3) - 2 \times [(6 - 3) - 4^2] + 1$

Solution

a.

$$\begin{aligned}(3\times 2)^2-4\times(6+2) &= (6)^2-4\times(8) && \text{Simplify parentheses}\\ &= 36-4\times 8 && \text{Simplify exponent}\\ &= 36-32 && \text{Simplify multiplication}\\ &= 4 && \text{Simplify subtraction}\end{aligned}$$

b.

$$\begin{aligned}\frac{5^2-4}{7}-\sqrt{11-2} &= \frac{5^2-4}{7}-\sqrt{9} && \text{Simplify grouping symbols (radical)}\\ &= \frac{5^2-4}{7}-3 && \text{Simplify radical}\\ &= \frac{25-4}{7}-3 && \text{Simplify exponent}\\ &= \frac{21}{7}-3 && \text{Simplify subtraction in numerator}\\ &= 3-3 && \text{Simplify division}\\ &= 0 && \text{Simplify subtraction}\end{aligned}$$

Note that in the first step, the radical is treated as a grouping symbol, like parentheses. Also, in the third step, the fraction bar is considered a grouping symbol so the numerator is considered to be grouped.

c.

$$\begin{aligned}6-|\,5-8\,|+3\times(4-1) &= 6-|-3|+3\times 3 && \text{Simplify inside grouping symbols}\\ &= 6-3+3\times 3 && \text{Simplify absolute value}\\ &= 6-3+9 && \text{Simplify multiplication}\\ &= 3+9 && \text{Simplify subtraction}\\ &= 12 && \text{Simplify addition}\end{aligned}$$

d.

$$\begin{aligned}\frac{14-3\times 2}{2\times 5-3^2} &= \frac{14-3\times 2}{2\times 5-9} && \text{Simplify exponent}\\ &= \frac{14-6}{10-9} && \text{Simplify products}\\ &= \frac{8}{1} && \text{Simplify differences}\\ &= 8 && \text{Simplify quotient}\end{aligned}$$

In this example, the fraction bar separates the numerator and denominator, which we simplify separately until the last step.

e.

$$\begin{aligned}7\times(5\times 3)-2\times[(6-3)-4^2]+1 &= 7\times(15)-2\times[(3)-4^2]+1 && \text{Simplify inside parentheses}\\ &= 7\times(15)-2\times(3-16)+1 && \text{Simplify exponent}\\ &= 7\times(15)-2\times(-13)+1 && \text{Subtract}\\ &= 105+26+1 && \text{Multiply}\\ &= 132 && \text{Add}\end{aligned}$$

Exercise 3.16.6

Use the order of operations to evaluate each of the following expressions.

a. $\sqrt{5^2-4^2}+7\times(5-4)^2$
b. $1+\dfrac{7\times 5-8\times 4}{9-6}$
c. $|1.8-4.3|+0.4\times\sqrt{15+10}$
d. $\dfrac{1}{2}\times[5\times 3^2-7^2]+\dfrac{1}{3}\times 9^2$
e. $[(3-8^2)-4]-(3-8)$

Answer available at the end of this section.

Using Properties of Real Numbers

For some activities we perform, the order of certain operations does not matter, but the order of other operations does. For example, it does not make a difference if we put on the right shoe before the left or vice-versa. However, it does matter whether we put on shoes or socks first. The same thing is true for operations in mathematics.

Commutative Properties

The commutative property of addition states that numbers may be added in any order without affecting the sum.

$$a + b = b + a \tag{3.16.2}$$

We can better see this relationship when using real numbers.

$$(-2) + 7 = 5 \text{ and } 7 + (-2) = 5$$

Similarly, the **commutative property of multiplication** states that numbers may be multiplied in any order without affecting the product.

$$a \times b = b \times a \tag{3.16.3}$$

Again, consider an example with real numbers.

$$(-11) \times (-4) = 44 \text{ and } (-4) \times (-11) = 44$$

It is important to note that neither subtraction nor division is commutative. For example, $17 - 5$ is not the same as $5 - 17$. Similarly, $20 \div 5 \neq 5 \div 20$.

Associative Properties

The associative property of multiplication tells us that it does not matter how we group numbers when multiplying. We can move the grouping symbols to make the calculation easier, and the product remains the same.

$$a(bc) = (ab)c \tag{3.16.4}$$

Consider this example.

$$(3 \times 4) \times 5 = 60 \text{ and } 3 \times (4 \times 5) = 60$$

The associative property of addition tells us that numbers may be grouped differently without affecting the sum.

$$a + (b + c) = (a + b) + c \tag{3.16.5}$$

This property can be especially helpful when dealing with negative integers. Consider this example.

$$[15 + (-9)] + 23 = 29 \text{ and } 15 + [(-9) + 23] = 29$$

Are subtraction and division associative? Review these examples.

$$\begin{aligned} 8 - (3 - 15) &\stackrel{?}{=} (8 - 3) - 15 \\ 8 - (-12) &\stackrel{?}{=} 5 - 15 \\ 20 &\neq 10 \end{aligned}$$

$$\begin{aligned} 64 \div (8 \div 4) &\stackrel{?}{=} (64 \div 8) \div 4 \\ 64 \div 2 &\stackrel{?}{=} 8 \div 4 \\ 32 &\neq 2 \end{aligned}$$

As we can see, neither subtraction nor division is associative.

Distributive Property

The distributive property states that the product of a factor times a sum is the sum of the factor times each term in the sum.

$$a \times (b + c) = a \times b + a \times c \tag{3.16.6}$$

This property combines both addition and multiplication (and is the only property to do so). Let us consider an example.

$$\begin{aligned} 4 \cdot [12 + (-7)] &= 4 \cdot 12 + 4 \cdot (-7) \\ &= 48 + (-28) \\ &= 20 \end{aligned}$$

Note that 4 is outside the grouping symbols, so we distribute the 4 by multiplying it by 12, multiplying it by -7, and adding the products.

Example 3.16.7

To be more precise when describing this property, we say that multiplication distributes over addition. The reverse is not true, as we can see in this example.

$$\begin{aligned} 6 + (3 \times 5) &\stackrel{?}{=} (6 + 3) \times (6 \times 5) \\ 6 + (15) &\stackrel{?}{=} (9) \times (11) \\ 21 &\neq 99 \end{aligned}$$

Multiplication does not distribute over subtraction, and division distributes over neither addition nor subtraction.

A special case of the distributive property occurs when a sum of terms is subtracted.

$$a - b = a + (-b) \tag{3.16.7}$$

For example, consider the difference $12 - (5 + 3)$. We can rewrite the difference of the two terms 12 and $(5 + 3)$ by turning the subtraction expression into addition of the opposite. So instead of subtracting $(5 + 3)$, we add the opposite.

$$12 + (-1) \times (5 + 3)$$

Now, distribute -1 and simplify the result.

$$\begin{aligned} 12 - (5 + 3) &= 12 + (-1) \times (5 + 3) \\ &= 12 + [(-1) \times 5 + (-1) \times 3] \\ &= 12 + (-8) \\ &= 4 \end{aligned}$$

This seems like a lot of trouble for a simple sum, but it illustrates a powerful result that will be useful once we introduce algebraic terms. To subtract a sum of terms, change the sign of each term and add the results. With this in mind, we can rewrite the last example.

$$\begin{aligned} 12 - (5 + 3) &= 12 + (-5 - 3) \\ &= 12 - 8 \\ &= 4 \end{aligned}$$

Identity Properties

The **identity property of addition** states that there is a unique number, called the additive identity (0) that, when added to a number, results in the original number.

$$a + 0 = a \tag{3.16.8}$$

The **identity property of multiplication** states that there is a unique number, called the multiplicative identity (1) that, when multiplied by a number, results in the original number.

$$a \times 1 = a \tag{3.16.9}$$

For example, we have $(-6) + 0 = -6$ and $23 \times 1 = 23$. There are no exceptions for these properties; they work for every real number, including 0 and 1.

Inverse Properties

The inverse property of addition states that, for every real number a, there is a unique number, called the additive inverse (or opposite), denoted $-a$, that, when added to the original number, results in the additive identity, 0.

$$a + (-a) = 0 \tag{3.16.10}$$

For example, if $a = -8$, the additive inverse is 8, since $(-8) + 8 = 0$.

The inverse property of multiplication holds for all real numbers except 0 because the reciprocal of 0 is not defined. The property states that, for every real number a, there is a unique number, called the multiplicative inverse (or reciprocal), denoted $1a$, that, when multiplied by the original number, results in the multiplicative identity, 1.

$$a \times \frac{1}{a} = 1 \tag{3.16.11}$$

For example, if $a = -\frac{2}{3}$, the reciprocal, denoted $\frac{1}{a}$, is $-\frac{3}{2}$ because

$$a \cdot \frac{1}{a} = \left(-\frac{2}{3}\right) \times \left(-\frac{3}{2}\right) = 1$$

PROPERTIES OF REAL NUMBERS

The following properties hold for real numbers a, b, and c.

Table $3.16.1$

	Addition	Multiplication
Commutative Property	$a + b = b + a$	$a \times b = b \times a$
Associative Property	$a + (b + c) = (a + b) + c$	$a(bc) = (ab)c$
Distributive Property	$a \times (b + c) = a \times b + a \times c$	
Identity Property	There exists a unique real number called the additive identity, 0, such that, for any real number a $a + 0 = a$	

There exists a unique real number called the multiplicative identity, 1, such that, for any real number a

$$a \times 1 = a$$

Inverse Property

Every real number a has an additive inverse, or opposite, denoted –a, such that

$$a + (-a) = 0$$

Every nonzero real number a has a multiplicative inverse, or reciprocal, denoted 1a , such that

$$a \times \left(\frac{1}{a}\right) = 1$$

Example 3.16.8: Using Properties of Real Numbers

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

a. $3 \times 6 + 3 \times 4$
b. $(5 + 8) + (-8)$
c. $6 - (15 + 9)$
d. $\frac{4}{7} \times \left(\frac{2}{3} \times \frac{7}{4}\right)$
e. $100 \times [0.75 + (-2.38)]$

Solution

a.

$$\begin{aligned} 3 \times 6 + 3 \times 4 &= 3 \times (6 + 4) && \text{Distributive property} \\ &= 3 \times 10 && \text{Simplify} \\ &= 30 && \text{Simplify} \end{aligned}$$

b.

$$\begin{aligned} (5 + 8) + (-8) &= 5 + [8 + (-8)] && \text{Associative property of addition} \\ &= 5 + 0 && \text{Inverse property of addition} \\ &= 5 && \text{Identity property of addition} \end{aligned}$$

c.

$$\begin{aligned} 6 - (15 + 9) &= 6 + [(-15) + (-9)] && \text{Distributive property} \\ &= 6 + (-24) && \text{Simplify} \\ &= -18 && \text{Simplify} \end{aligned}$$

d.

$$\begin{aligned} \frac{4}{7} \times \left(\frac{2}{3} \times \frac{7}{4}\right) &= \frac{4}{7} \times \left(\frac{7}{4} \times \frac{2}{3}\right) && \text{Commutative property of multiplication} \\ &= \left(\frac{4}{7} \times \frac{7}{4}\right) \times \frac{2}{3} && \text{Associative property of multiplication} \\ &= 1 \times \frac{2}{3} && \text{Inverse property of multiplication} \\ &= \frac{2}{3} && \text{Identity property of multiplication} \end{aligned}$$

e.

$$\begin{aligned} 100 \times [0.75 + (-2.38)] &= 100 \times 0.75 + 100 \times (-2.38) && \text{Distributive property} \\ &= 75 + (-238) && \text{Simplify} \\ &= -163 && \text{Simplify} \end{aligned}$$

Exercise 3.16.7

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

a. $\left(-\frac{23}{5}\right) \times \left[11 \times \left(-\frac{5}{23}\right)\right]$
b. $5 \times (6.2 + 0.4)$
c. $18 - (7 - 15)$
d. $\frac{17}{18} + \left[\frac{4}{9} + \left(-\frac{17}{18}\right)\right]$
e. $6 \times (-3) + 6 \times 3$

Answer available at the end of this section.

Evaluating Algebraic Expressions

So far, the mathematical expressions we have seen have involved real numbers only. In mathematics, we may see expressions such as $x+5$, $\frac{4}{3}\pi r^3$, or $\sqrt{2m^3n^2}$. In the expression $x+5$, 5 is called a **constant** because it does not vary and x is called a **variable** because it does. (In naming the variable, ignore any exponents or radicals containing the variable.) An **algebraic expression** is a collection of constants and variables joined together by the algebraic operations of addition, subtraction, multiplication, and division.

We have already seen some real number examples of exponential notation, a shorthand method of writing products of the same factor. When variables are used, the constants and variables are treated the same way.

$$(-3)^5 = (-3)\times(-3)\times(-3)\times(-3)\times(-3) \Rightarrow x^5 = x\times x\times x\times x\times x$$
$$(2\times 7)^3 = (2\times 7)\times(2\times 7)\times(2\times 7) \quad \Rightarrow (yz)^3 = (yz)\times(yz)\times(yz)$$

In each case, the exponent tells us how many factors of the base to use, whether the base consists of constants or variables.

Any variable in an algebraic expression may take on or be assigned different values. When that happens, the value of the algebraic expression changes. To evaluate an algebraic expression means to determine the value of the expression for a given value of each variable in the expression. Replace each variable in the expression with the given value, then simplify the resulting expression using the order of operations. If the algebraic expression contains more than one variable, replace each variable with its assigned value and simplify the expression as before.

Example 3.16.9: Describing Algebraic Expressions

List the constants and variables for each algebraic expression.

a. $x+5$
b. $\frac{4}{3}\pi r^3$
c. $\sqrt{2m^3n^2}$

Solution

	Constants	Variables
a. $x+5$	5	x
b. $\frac{4}{3}\pi r^3$	$\frac{4}{3}$, π	r
c. $\sqrt{2m^3n^2}$	2	m,n

Exercise 3.16.8

List the constants and variables for each algebraic expression.

a. $2(L+W)$
b. $4y^3+y$

Answer available at the end of this section.

Example 3.16.10: Evaluating an Algebraic Expression at Different Values

Evaluate the expression $2x-7$ for each value for x.

a. $x = 0$
b. $x = 1$
c. $x = 12$
d. $x = -4$

Solution

a. Substitute 0 for x.

$$\begin{aligned} 2x - 7 &= 2(0) - 7 \\ &= 0 - 7 \\ &= -7 \end{aligned}$$

b. Substitute 1 for x.

$$\begin{aligned} 2x - 7 &= 2(1) - 7 \\ &= 2 - 7 \\ &= -5 \end{aligned}$$

c. Substitute $\frac{1}{2}$ for x.

$$\begin{aligned} 2x - 7 &= 2\left(\frac{1}{2}\right) - 7 \\ &= 1 - 7 \\ &= -6 \end{aligned}$$

d. Substitute -4 for x.

$$\begin{aligned} 2x - 7 &= 2(-4) - 7 \\ &= -8 - 7 \\ &= -15 \end{aligned}$$

Exercise 3.16.9

Evaluate the expression $11 - 3y$ for each value for y.

a. $y = 2$
b. $y = 0$
c. $y = \frac{2}{3}$
d. $y = -5$

Answer available at the end of this section.

Example 3.16.11: Evaluating Algebraic Expressions

Evaluate each expression for the given values.

a. $x + 5$ for $x = -5$
b. $\frac{t}{2t - 1}$ for $t = 10$
c. $\frac{4}{3}\pi r^3$ for $r = 5$
d. $a + ab + b$ for $a = 11$, $b = -8$
e. $\sqrt{2m^3n^2}$ for $m = 2$, $n = 3$

Solution

a. Substitute -5 for x.

$$\begin{aligned} x+5 &= (-5)+5 \\ &= 0 \end{aligned}$$

b. Substitute 10 for t.

$$\begin{aligned} \frac{t}{2t-1} &= \frac{(10)}{2(10)-1} \\ &= \frac{10}{20-1} \\ &= \frac{10}{19} \end{aligned}$$

c. Substitute 5 for r.

$$\begin{aligned} \frac{4}{3}\pi r^3 &= \frac{4}{3}\pi(5)^3 \\ &= \frac{4}{3}\pi(125) \\ &= \frac{500}{3}\pi \end{aligned}$$

d. Substitute 11 for a and -8 for b.

$$\begin{aligned} a+ab+b &= (11)+(11)(-8)+(-8) \\ &= 11-88-8 \\ &= -85 \end{aligned}$$

e. Substitute 2 for m and 3 for n.

$$\begin{aligned} \sqrt{2m^3n^2} &= \sqrt{2(2)^3(3)^2} \\ &= \sqrt{2(8)(9)} \\ &= \sqrt{144} \\ &= 12 \end{aligned}$$

Exercise 3.16.10

Evaluate each expression for the given values.

a. $\dfrac{y+3}{y-3}$ for $y=5$
b. $7-2t$ for $t=-2$
c. $\dfrac{1}{3}\pi r^2$ for $r=11$
d. $(p^2q)^3$ for $p=-2$, $q=3$
e. $4(m-n)-5(n-m)$ for $m=\dfrac{2}{3}$ $n=\dfrac{1}{3}$

Answer available at the end of this section.

Formulas

An equation is a mathematical statement indicating that two expressions are equal. The expressions can be numerical or algebraic. The equation is not inherently true or false, but only a proposition. The values that make the equation true, the solutions, are found using the properties of real numbers and other results. For example, the equation $2x + 1 = 7$ has the unique solution of 3 because when we substitute 3 for x in the equation, we obtain the true statement $2(3) + 1 = 7$.

A formula is an equation expressing a relationship between constant and variable quantities. Very often, the equation is a means of finding the value of one quantity (often a single variable) in terms of another or other quantities. One of the most common examples is the formula for finding the area A of a circle in terms of the radius r of the circle: $A = \pi r^2$. For any value of r, the area A can be found by evaluating the expression πr^2.

Example 3.16.12: Using a Formula

A right circular cylinder with radius r and height h has the surface area S (in square units) given by the formula $S = 2\pi r(r + h)$. See Figure 3.16.3. Find the surface area of a cylinder with radius 6 in. and height 9 in. Leave the answer in terms of π.

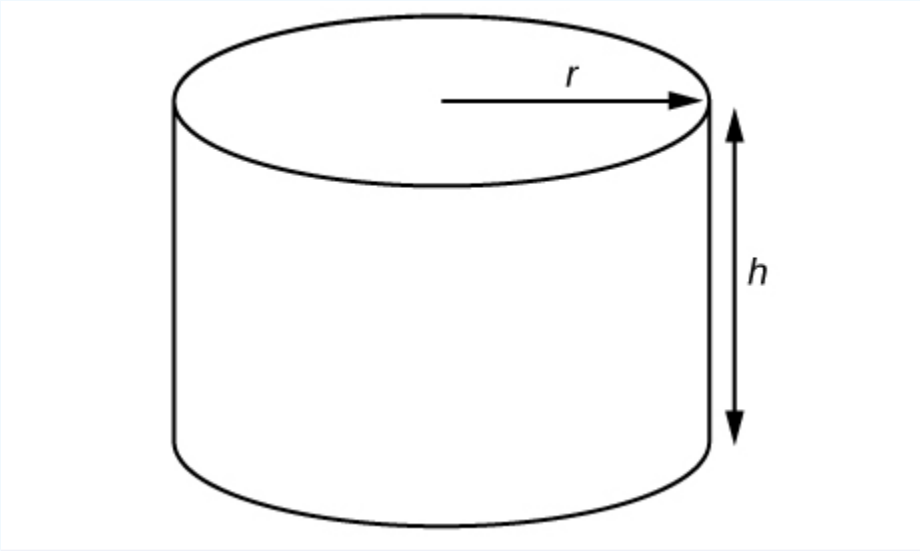


Figure 3.16.3

Evaluate the expression $2\pi r(r + h)$ for $r = 6$ and $h = 9$.

Solution

$$\begin{aligned} S &= 2\pi r(r + h) \\ &= 2\pi(6)[(6) + (9)] \\ &= 2\pi(6)(15) \\ &= 180\pi \end{aligned}$$

The surface area is 180π square inches.

Exercise 3.16.11

A photograph with length L and width W is placed in a matte of width 8 centimeters (cm). The area of the matte (in square centimeters, or cm^2 is found to be $A = (L + 16)(W + 16) - L \cdot W$. See Figure 3.16.4. Find the area of a matte for a photograph with length 32cm and width 24cm.

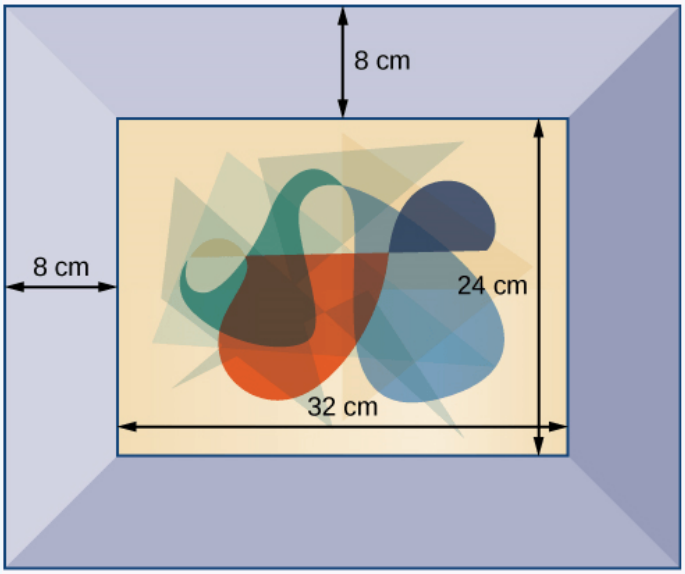


Figure 3.16.4

Answer available at the end of this section.

Simplifying Algebraic Expressions

Sometimes we can simplify an algebraic expression to make it easier to evaluate or to use in some other way. To do so, we use the properties of real numbers. We can use the same properties in formulas because they contain algebraic expressions.

Example 3.16.13: Simplifying Algebraic Expressions

Simplify each algebraic expression.

a. $3x - 2y + x - 3y - 7$
b. $2r - 5(3 - r) + 4$
c. $\left(4t - \frac{5}{4}s\right) - \left(\frac{2}{3}t + 2s\right)$
d. $2mn - 5m + 3mn + n$

Solution

a.
$$\begin{aligned} 3x - 2y + x - 3y - 7 &= 3x + x - 2y - 3y - 7 && \text{Commutative property of addition} \\ &= 4x - 5y - 7 && \text{Simplify} \end{aligned}$$

b.
$$\begin{aligned} 2r-5(3-r)+4 &= 2r-15+5r+4 && \text{Distributive property} \\ &= 2r+5y-15+4 && \text{Commutative property of addition} \\ &= 7r-11 && \text{Simplify} \end{aligned}$$

c.
$$\begin{aligned} \left(4t-\frac{5}{4}s\right)-\left(\frac{2}{3}t+2s\right) &= 4t-\frac{5}{4}s-\frac{2}{3}t-2s && \text{Distributive property} \\ &= 4t-\frac{2}{3}t-\frac{5}{4}s-2s && \text{Commutative property of addition} \\ &= \frac{10}{3}t-\frac{13}{4}s && \text{Simplify} \end{aligned}$$

d.
$$\begin{aligned} 2mn-5m+3mn+n &= 2mn+3mn-5m+n && \text{Commutative property of addition} \\ &= 5mn-5m+n && \text{Simplify} \end{aligned}$$

Exercise 3.16.12

Simplify each algebraic expression.

a. $\frac{2}{3}y-2\left(\frac{4}{3}y+z\right)$
b. $\frac{5}{t}-2-\frac{3}{t}+1$
c. $4p(q-1)+q(1-p)$
d. $9r-(s+2r)+(6-s)$

Answer available at the end of this section.

Example 3.16.14: Simplifying a Formula

A rectangle with length L and width W has a perimeter P given by $P=L+W+L+W$. Simplify this expression.

Solution

$$\begin{aligned} P &= L+W+L+W \\ P &= L+L+W+W && \text{Commutative property of addition} \\ P &= 2L+2W && \text{Simplify} \\ P &= 2(L+W) && \text{Distributive property} \end{aligned}$$

Exercise 3.16.13

If the amount P is deposited into an account paying simple interest r for time t, the total value of the deposit A is given by $A=P+Prt$. Simplify the expression. (This formula will be explored in more detail later in the course.)

Answer available at the end of this section.

Access these online resources for additional instruction and practice with real numbers.

- Simplify an Expression
- Evaluate an Expression1
- Evaluate an Expression2

Key Concepts

- Rational numbers may be written as fractions or terminating or repeating decimals. See Example and Example.
- Determine whether a number is rational or irrational by writing it as a decimal. See Example.
- The rational numbers and irrational numbers make up the set of real numbers. See Example. A number can be classified as natural, whole, integer, rational, or irrational. See Example.
- The order of operations is used to evaluate expressions. See Example.
- The real numbers under the operations of addition and multiplication obey basic rules, known as the properties of real numbers. These are the commutative properties, the associative properties, the distributive property, the identity properties, and the inverse properties. See Example.
- Algebraic expressions are composed of constants and variables that are combined using addition, subtraction, multiplication, and division. See Example. They take on a numerical value when evaluated by replacing variables with constants. See Example,Example, and Example
- Formulas are equations in which one quantity is represented in terms of other quantities. They may be simplified or evaluated as any mathematical expression. See Example and Example.

Glossary

algebraic expression
: constants and variables combined using addition, subtraction, multiplication, and division

associative property of addition
: the sum of three numbers may be grouped differently without affecting the result; in symbols,$a+(b+c)=(a+b)+c$

associative property of multiplication
: the product of three numbers may be grouped differently without affecting the result; in symbols,$a \cdot (b \cdot c)=(a \cdot b) \cdot c$

base
: in exponential notation, the expression that is being multiplied

commutative property of addition
: two numbers may be added in either order without affecting the result; in symbols,$a+b=b+a$

commutative property of multiplication
: two numbers may be multiplied in any order without affecting the result; in symbols,$a \cdot b=b \cdot a$

constant
: a quantity that does not change value

distributive property
: the product of a factor times a sum is the sum of the factor times each term in the sum; in symbols,$a \cdot (b+c)=a \cdot b+a \cdot c$

equation
: a mathematical statement indicating that two expressions are equal

exponent
: in exponential notation, the raised number or variable that indicates how many times the base is being multiplied

exponential notation
: a shorthand method of writing products of the same factor

formula
: an equation expressing a relationship between constant and variable quantities

identity property of addition
: there is a unique number, called the additive identity, 0, which, when added to a number, results in the original number; in symbols,$a+0=a$

identity property of multiplication
: there is a unique number, called the multiplicative identity, 1, which, when multiplied by a number, results in the original number; in symbols,a · 1=a

integers
: the set consisting of the natural numbers, their opposites, and 0:{...,–3,–2,–1,0,1,2,3,...}

inverse property of addition
: for every real numbera,there is a unique number, called the additive inverse (or opposite), denoted–a,which, when added to the original number, results in the additive identity, 0; in symbols,a+(–a)=0

inverse property of multiplication
: for every non-zero real numbera,there is a unique number, called the multiplicative inverse (or reciprocal), denoted1a,which, when multiplied by the original number, results in the multiplicative identity, 1; in symbols,a · 1a=1

irrational numbers
: the set of all numbers that are not rational; they cannot be written as either a terminating or repeating decimal; they cannot be expressed as a fraction of two integers

natural numbers
: the set of counting numbers:{1,2,3,...}

order of operations
: a set of rules governing how mathematical expressions are to be evaluated, assigning priorities to operations

rational numbers
: the set of all numbers of the formmn,wheremandnare integers andn≠0.Any rational number may be written as a fraction or a terminating or repeating decimal.

real number line
: a horizontal line used to represent the real numbers. An arbitrary fixed point is chosen to represent 0; positive numbers lie to the right of 0 and negative numbers to the left.

real numbers
: the sets of rational numbers and irrational numbers taken together

variable
: a quantity that may change value

whole numbers
: the set consisting of 0 plus the natural numbers:{0,1,2,3,...}

Answers

Exercise 3.16.1

a. $\frac{11}{1}$
b. $\frac{3}{1}$
c. $-\frac{4}{1}$

Exercise 3.16.2

a. 4 (or 4.0), terminating
b. $0.\overline{615384}$, repeating
c. -0.85, terminating

Exercise 3.16.3

a. rational and repeating;
b. rational and terminating;
c. irrational;
d. rational and terminating;
e. irrational

Exercise 3.16.4

a. positive, irrational; right
b. negative, rational; left
c. positive, rational; right
d. negative, irrational; left
e. positive, rational; right

Exercise 3.16.5

	N	W	I	Q	Q'
a. $-\frac{35}{7}$			X	X	
b. 0		X	X	X	
c. $\sqrt{169}$	X	X	X	X	
d. $\sqrt{24}$					X
e. $4.763763763\ldots$				X	

Exercise 3.16.6

a. 10
b. 2
c. 4.5
d. 25
e. -60

Exercise 3.16.7

1. 11, commutative property of multiplication
2. 33, distributive property
3. 26, distributive property
4. $\frac{4}{9}$, commutative property of addition, associative property of addition, inverse property of addition, identity property of addition
5. 0, distributive property, inverse property of addition, identity property of addition

Exercise 3.16.8

	Constants	Variables
a. $2\pi r(r+h)$	$2,\pi$	r,h
b. $2(L+W)$	2	L, W
c. $4y^3+y$	4	y

Exercise 3.16.9

1. 11
2. 26

Exercise 3.16.10

a. 4
b. 11
c. $\frac{121}{3}\pi$
d. 1728
e. 3

Exercise 3.16.11

$1152cm^2$

Exercise 3.16.12

a. $-2y-2z$ or $-2(y+z)$
b. $\frac{2}{t}-1$
c. $3pq-4p+q$
d. $7r-2s+6$

Exercise 3.16.13

$$A = P(1 + rt)$$

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3.17: Challenges and Misconceptions

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3.18: Summary and Review

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4: NUMBER THEORY

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4.1: Introduction to Number Theory

Importance of Number Theory in Elementary Education

Number theory, often referred to as "higher arithmetic," is a branch of mathematics that deals with the properties and relationships of numbers, especially the integers. For elementary teachers, understanding the basics of number theory is crucial because it lays the foundation for many mathematical concepts that students will encounter throughout their education. By integrating number theory into the elementary curriculum, teachers can help students develop strong numerical and problem-solving skills, which are essential for their overall mathematical literacy.

Key Concepts

In this chapter, we will explore several fundamental concepts of number theory that are particularly relevant for elementary education. These include:

1. Prime and Composite Numbers
2. Even and Odd Numbers
3. Factors and Greatest Common Factor
4. The Sieve of Eratosthenes
5. Least Common Multiple and other topics
6. The Euclidean Algorithm
7. Lame's Theorem
8. The Meaning of Division
9. Digital Roots and Divisibility
10. Division Algorithms
11. The Meaning of Multiplication
12. Multiplication Algorithms
13. Modular Arithmetic
14. Arithmetic Sequences
15. Geometric Sequences and Series

Goals for Elementary Teachers

As an elementary teacher, your goal is to:

1. **Foster a Strong Number Sense:** Help students develop an intuitive understanding of numbers and their properties.
2. **Encourage Mathematical Thinking:** Promote logical reasoning and problem-solving skills through number theory concepts.
3. **Integrate Number Theory into Everyday Math:** Use real-world examples and practical applications to make number theory relevant and engaging for students.
4. **Address Common Misconceptions:** Identify and correct misunderstandings about numbers and their properties to build a solid mathematical foundation.

By mastering the basics of number theory, you will be better equipped to teach these important concepts to your students, helping them to become confident and proficient in mathematics. This chapter will provide the knowledge and tools needed to effectively incorporate number theory into your teaching practice.

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4.2: Primes and Composite Numbers

Introduction

In the world of numbers, some hold a special status due to their unique properties. Two such categories are prime numbers and composite numbers. Understanding these concepts is fundamental to number theory and provides a foundation for many advanced mathematical ideas.

Definition: Factor

A factor is a whole number that divides another number evenly (with no remainder).

Let $a, b \in \mathbb{Z}$ (where $\mathbb{Z}$ represents the set of integers).

We say that a is a factor of b if and only if:

$\exists k \in \mathbb{Z} \text{ such that } b = a * k$

Note: $\exists$ means there exists

Prime Numbers

Definition: Prime Number

A **prime number** (or a **prime**) is a natural number greater than 1 with two unique factors specifically 1 and itself.

Let $p \in \mathbb{N}$ (where $\mathbb{N}$ represents the set of natural numbers).

p is prime if and only if:

1. $p > 1$
2. $\forall\ a, b \in \mathbb{N},\ \text{ if } p = ab, \text{ then } a = 1 \text{ or } b = 1$ (but not both)

Note: $\forall$ means for every

Prime numbers are the building blocks of arithmetic. A prime number is a natural number greater than 1 with exactly two (unique) factors: 1 and itself. In other words, it can only be divided evenly by 1 and itself.

Examples of prime numbers include:

- 2 (the first prime number and the only even prime number)
- 3
- 5
- 7
- 11
- 13

Prime numbers have fascinated mathematicians for centuries. They appear to be distributed randomly among the natural numbers, yet they follow certain patterns. The study of prime numbers has led to important applications in various fields, including cryptography and computer science.

Fun Fact: While prime numbers are fundamental to mathematics, their significance extends far beyond the realm of pure arithmetic. One of the most fascinating real-world applications of prime numbers can be found in the life cycles of certain species of cicadas.

Cicada Life Cycles

Cicadas are insects known for their periodic emergence from the ground in large numbers. While many cicada species have annual life cycles, some species, known as "periodical cicadas," have much longer life cycles.

The most famous of these are the Magicicada species, which include:

- 13-year cicadas
- 17-year cicadas

Notice something special about these numbers? Both 13 and 17 are prime numbers.

Composite Numbers

Definition: Composite Numbers

A natural number greater than 1 is considered composite if it can be formed by multiplying two natural numbers, both of which are smaller than the number itself and greater than 1.

Let $n \in \mathbb{N}$ (where $\mathbb{N}$ represents the set of natural numbers).

n is composite if and only if:

1. $n > 1$
2. $\exists\ a, b \in \mathbb{N} \text{ such that } n = ab, \text{ where } 1 < a < n \text{ and } 1 < b < n$

Composite numbers, on the other hand, are natural numbers greater than 1 that are not prime. They have more than two factors.

Examples of composite numbers include:

- 4 (factors: 1, 2, 4)
- 6 (factors: 1, 2, 3, 6)
- 8 (factors: 1, 2, 4, 8)
- 9 (factors: 1, 3, 9)
- 12 (factors: 1, 2, 3, 4, 6, 12)

Properties of Composite Numbers

Composite numbers possess several interesting properties that distinguish them from prime numbers and play a crucial role in number theory. Understanding these properties helps students develop a deeper grasp of number relationships and lays the foundation for more advanced mathematical concepts.

Key Properties

1. **Multiple Factors**: Every composite number has at least three factors: 1, itself, and at least one other number between 1 and itself.
2. **Product of Primes**: Every composite number can be expressed as a product of prime numbers. This is known as its prime factorization.
3. **Divisibility**: A composite number is always divisible by at least one number other than 1 and itself.

4. **Even Numbers**: All even numbers greater than 2 are composite.
5. **Square Numbers**: All square numbers greater than 1 are composite, except for the squares of prime numbers.
6. **Abundance**: There are infinitely many composite numbers.
7. **Consecutive Composites**: Between any two prime numbers (except between 2 and 3), there is at least one composite number.
8. **Product Property**: The product of any two natural numbers greater than 1 is always a composite number.
9. **Sum of Primes**: Any even number greater than 2 can be expressed as the sum of two prime numbers (Goldbach's Conjecture - not yet proven for all numbers but verified for very large numbers).
10. **Factors in Pairs**: The factors of a composite number always come in pairs. If a is a factor of composite number n, then n/a is also a factor of n.

Teaching Implications

Understanding these properties can help students:

- Develop strategies for identifying composite numbers
- Understand the relationship between prime and composite numbers
- Explore patterns in number sequences
- Build a foundation for more advanced topics like number theory and algebra

Educators can use these properties to create engaging activities, such as:

- Finding all factors of a given composite number
- Exploring prime factorization
- Investigating patterns in even and odd composite numbers
- Challenging students to find consecutive composite numbers

By mastering these properties, students will enhance their number sense and problem-solving skills, preparing them for more complex mathematical concepts in the future.

Common Misconceptions

A common misconception is that 1 is a prime number.

This misconception likely arises because:

1. 1 is only divisible by 1 and itself, which seems to fit the basic definition of a prime number.
2. 1 is often grouped with other small numbers like 2 and 3, which are prime.
3. The special status of 1 isn't always clearly explained in elementary mathematics.

However, 1 is not considered a prime number for several important mathematical reasons:

1. Fails the definition of a prime number
    1. The formal definition of a prime number includes the clause "greater than 1". 1 is not "greater than 1"; 1 is equal to 1.
    2. The formal definition of a prime number states a number can be written as a product of two natural numbers which 1 can be written as such $1 = 1 \times 1$ but the definition is clear that the natural numbers must be 1 and itself and unique.
2. Historical Development: Mathematicians debated this issue for centuries before reaching a consensus in the early 20th century to exclude 1 from the set of prime numbers.

To address this misconception, it's helpful to:

- Clearly state and explain why 1 is neither prime nor composite when introducing these concepts.
- Emphasize that prime numbers must have exactly two factors, not just be divisible by 1 and themselves.
- Provide examples of how including 1 as prime would complicate other areas of mathematics.

A common misconception is believing all odd numbers are prime.

A common misconception among students learning about prime numbers is the belief that all odd numbers are prime. This misunderstanding likely stems from two true observations:

1. All even numbers greater than 2 are composite.
2. The first few prime numbers (3, 5, 7) are odd.

While it's true that all prime numbers greater than 2 are odd, the reverse is not true: not all odd numbers are prime.

1. **Definition of Prime Numbers**: A prime number is a natural number greater than 1 that has exactly two factors: 1 and itself.
2. **Odd Composite Numbers**: Many odd numbers have factors other than 1 and themselves, making them composite. For example:
   - 9 = 3 × 3
   - 15 = 3 × 5
   - 21 = 3 × 7
   - 25 = 5 × 5
   - 27 = 3 × 3 × 3
3. **Pattern of Odd Composites**: In fact, there are infinitely many odd composite numbers. Every odd number that's the product of two different odd primes (like 15 = 3 × 5) or the cube of a prime (like $27 = 3^3$) is composite.
4. **Density**: As numbers get larger, odd composite numbers become more common than odd prime numbers.

To address this misconception:

1. **Explicit Discussion**: Directly address the misconception in class.
2. **Examples**: Provide plenty of examples of odd composite numbers.
3. **Factor Trees**: Use factor trees to demonstrate the composition of odd composite numbers.
4. **Pattern Recognition**: Help students recognize patterns in odd composite numbers (e.g., odd multiples of 3, 5, 7, etc.)
5. **Hands-on Activities**: Use manipulatives or visual aids to factor odd numbers.

By understanding that not all odd numbers are prime, students can develop a more nuanced understanding of number properties and improve their overall number sense.

A common misconception if there is a formula that exists to generate all prime numbers

Some students (and even some adults) believe that there exists a simple formula that can generate all prime numbers. This misconception often arises from exposure to certain mathematical formulas that can generate some prime numbers, or from a desire for a straightforward solution to identifying primes.

While there are formulas that can generate some prime numbers, there is no known formula that can generate all prime numbers and only prime numbers.

1. **Complexity of Prime Numbers**: The distribution of prime numbers among the natural numbers is highly complex and not fully understood, despite centuries of mathematical research.
2. **Existing Formulas**: Some formulas can generate prime numbers, but they all have limitations:
   - Euclid's formula: 2^n - 1 (where n is prime) can generate some prime numbers (Mersenne primes), but not all.
   - Euler's quadratic formula: n^2 + n + 41 generates primes for n = 0 to 39, but not for all n.
   - Mills' constant formula: $\lfloor \theta^{(3^n)} \rfloor$ (where $\theta \approx 1.3063778838$) can theoretically generate primes, but θ is irrational and can't be precisely calculated.
3. **Fundamental Theorem of Arithmetic**: This theorem states that every integer greater than 1 is either prime or can be uniquely factored into primes. This suggests that primes are fundamentally "random" in their occurrence.
4. **Computational Complexity**: If a simple formula to generate all primes existed, many complex problems in number theory and cryptography would be easily solvable, which is not the case.
5. **Ongoing Research**: The search for patterns in prime numbers is an active area of mathematical research, with many open questions remaining.

To address this misconception:

1. **Historical Context**: Discuss the historical search for prime number patterns and formulas.
2. **Explore Known Formulas**: Examine formulas that generate some primes, discussing their limitations.
3. **Sieve of Eratosthenes**: Introduce this ancient method to find primes, emphasizing its algorithmic nature rather than a formula.
4. **Prime Number Theorem**: For advanced students, introduce this theorem which describes the approximate distribution of primes.
5. **Computational Exploration**: Use technology to explore large prime numbers and the challenges in finding them.

Understanding that there's no simple formula for all primes helps students appreciate the complexity and mystery still present in mathematics, potentially inspiring further interest in number theory.

Activity: Exploring Primes and Composites

You will need: ***Centimeter Strips*** *(Material Cards 16A-16L)*

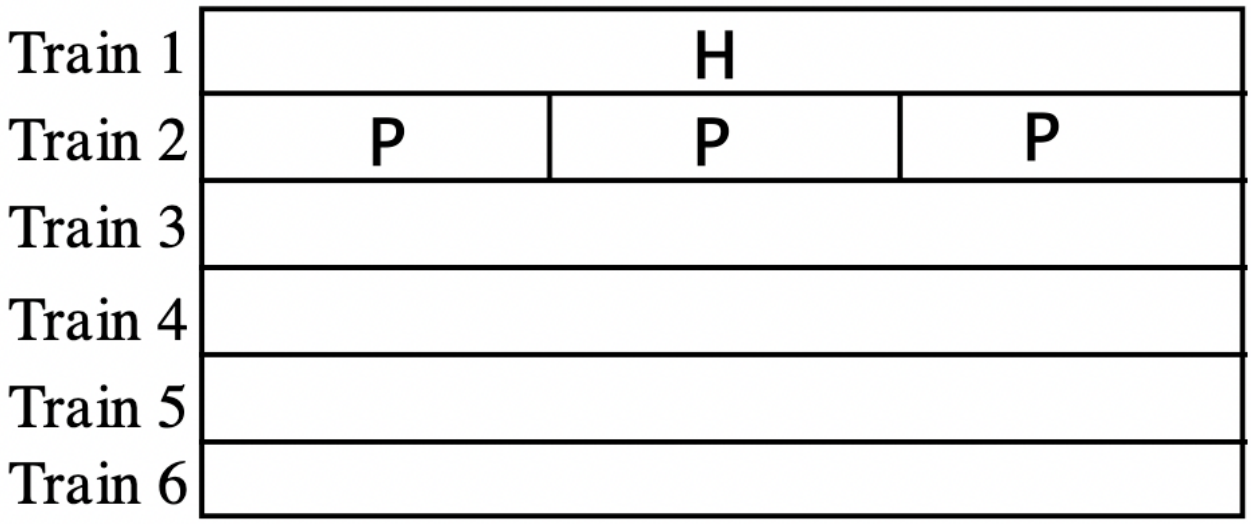
Prime Number Squares *(Material Cards 19A-19B)*

Composite Number Squares *(Material Cards 20A-20B)*

In these first few exercises, you'll be using C-strips to explore divisors, factors, prime numbers, and composite numbers.

Exercise 1

a. Take out the Hot Pink (H) C-strip. Use your C-strips (one color at a time) to see if you can form a train made up of C-strips of the same color equal in length to the hot pink C-strip in other words, see if you can do it with all whites (always possible for any train length), or all reds, or all light greens, etc. You should be able to make six different trains each train is made up of a single color. Draw a picture of each of these trains under the Hot Pink one shown. I've drawn two trains for you already one is simply the hot pink strip (a train consisting of just one C-strip), and a second is made up of three purple C-strips.



b. Take each train and form it into a rectangle. Then, find a C-strip that fits across the width of the rectangle, which is the top if the C-strips of the train made into a rectangle are placed vertically. From this, you should be able to tell from which multiplication problem each train was formed. From this information, write an equation using C-strips and then translate to an equation with numbers. For instance, for train 2, first I would make a rectangle out of the three purple C-strips. Next, I would try to find a C-strip to fit across the top, which would be light green. Therefore, train 2 was formed from the multiplication L

$\times P$: remember that since the train is formed with purple C-strips, P is the *second* letter in the multiplication. So, the equation in C-strips is $L \times P = H$, and the numerical equivalent is $3 \times 4 = 12$. Follow this same procedure for the other four trains you made in part a.

Train 1 illustrates the multiplication $W \times H = H$, or $1 \times 12 = 12$. (Note that if the hot pink strip is placed vertically, the white C-strip fits across the top.)

Train 2 illustrates the multiplication $L \times P = H$, or $3 \times 4 = 12$.

Train 3 illustrates the multiplication ____ × ____ = H, or ____ × ____ = 12.

Train 4 illustrates the multiplication ____ × ____ = H, or ____ × ____ = _____

Train 5 illustrates the multiplication ____ × ____ = H, or ____ × ____ = _____

Train 6 illustrates the multiplication ____ × ____ = H, or ____ × ____ = _____

In exercise part 1b, the set of numbers placed in the blanks before the equal sign in the equations with numbers are called **factors** or **divisors** of 12. Note, that because of the commutative property of multiplication, each factor or divisor was listed twice.

c. List the numbers that are factors of 12 (length of the hot pink C-strip). Only list each number once.

Exercise 2

Use the same procedure used in exercise 1 to make all possible rectangles from a train having each of the following lengths. Use the C-strip(s) shown in the parenthesis to make a train having the given length. Then, after discovering the possible rectangles that can be made, list the factors (the actual numbers) of each number. Every number greater than 1 has at least two factors.

a. Factors of 2 (red C-strip): ____
b. Factors of 3 (light green C-strip): ____
c. Factors of 4 (purple C-strip): ____
d. Factors of 5 (yellow C-strip): ____
e. Factors of 6 (dark green C-strip): ____
f. Factors of 7 (black C-strip): ____
g. Factors of 8 (brown C-strip): ____
h. Factors of 9 (blue C-strip): ____
i. Factors of 10 (orange C-strip): ____
j. Factors of 11 (silver C-strip): ____
k. Factors of 13 (brown + yellow): ____
l. Factors of 14 (orange + purple): ____
m. Factors of 15 (black + brown): ____
n. Factors of 16 (blue + black): ____

Exercise 3

a. List the numbers from exercise 2 that have **exactly** 2 factors: _____

b. What do you notice about the 2 factors each of these number have? Is there a pattern or anything they have in common with each other?

c. List numbers from exercise 2 that have an odd number (3 or 5) of factors: _____

d. What do all of these numbers have in common?

Definition: A whole number that has exactly two different factors is called a **prime** number.

If a number is prime, its only factors are 1 and itself. In other words, if a number, p, is prime, its only factors are 1 and p. If you are using the C-strips and try to make a rectangle out of a train that has a length that is a prime number, the only possibility is when the width is 1 and the length is the length of the original train. That is because those are the only factors, and no other number divides into it.

NOTE: **1 is NOT prime** since it doesn't have two different factors *it only has one 1.*

Definition: Any whole number larger than 1 that is not prime is called a **composite** number.

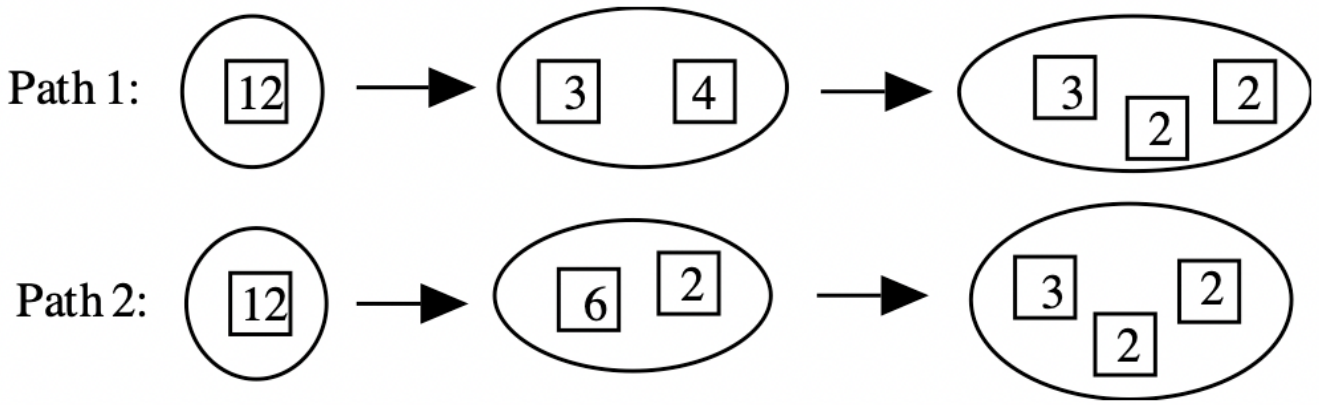
The whole numbers, 0 and 1, are neither prime nor composite. Any whole number larger than 1 is either prime or composite.

If a number is composite, it means that it has more than 2 factors, and can be written as a product of factors less than itself. For instance, 12 is not a prime number. It can be written as $2 \cdot 6$ or $3 \cdot 4$. This is called factoring, and $2 \cdots 6$ and $3 \cdots 4$ are only two ways of factoring 12.

Get out your prime number squares and composite number squares. If you haven't already done so, color the prime number squares so that each prime number has a different color the 2's could be yellow, the 3's could be blue, the 5's could be green, the 7's could be purple, the 11's could be red, the 13's could be orange, the 17's could be pink, and so on. All of the composite number squares should remain white.

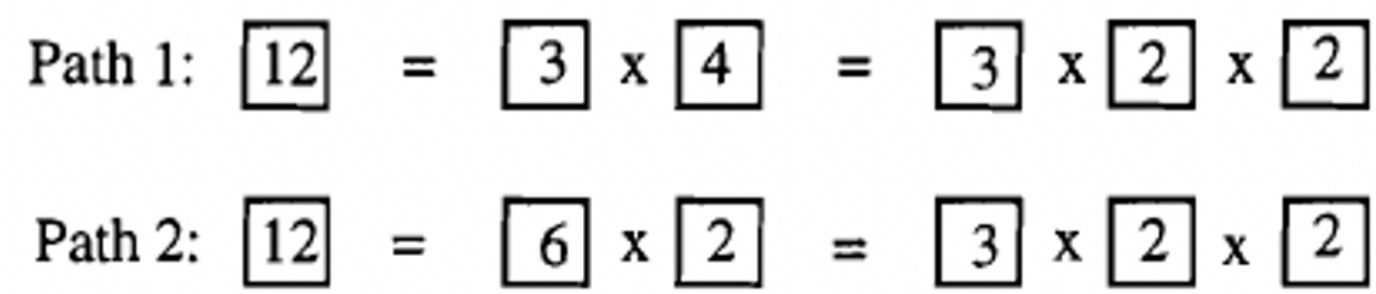
What we are going to do is factor numbers. Anytime we have a white number, we know it is composite and can be factored further. The ultimate goal will be to factor numbers into a product of primes, which means there will only be colored squares to represent each number.

Let's begin by factoring 12. Take out a white number square that says 12, and put it in front of you. Since it is white, it can be factored into a product of two smaller numbers. Someone might choose 3 and 4; another might choose 2 and 6. Replace 12 with the two squares you chose. For 12, it so happens that no matter which combination of two numbers you choose, one of the squares you choose will be colored, which means it is prime and can't be factored (or broken down) any further. But the other square will be white. Replace the white one with two other number squares by using smaller factors. In your pile, instead of a 12, you should have three squares, a 2, a 2 and a 3. There is no order it's just a group of three squares that if multiplied together represent 12. Below are two paths one might have taken to come up with the final solution. The arrows show one step after the other.



In the first path, the 12 was replaced with 3 and 4, and then the 4 was replaced with two 2's. In the second path, the 12 was

replaced with a 6 and a 2, and then the 6 was replaced with a 2 and a 3. Using this model to prime factor, the numbers you end up should all be prime, and the understanding is that when the final (prime) factors are MULTIPLIED together, we obtain the original number we tried to factor. In other words, the prime factorization for 12 is $3 \cdot 2 \cdot 2$. Because of the commutative and associative properties of multiplication, the order of the factors is insignificant. Sometimes, for consistency, the factors are written in ascending or descending order, but this is not necessary unless you are instructed to write it in a particular order. If you are asked to write the prime factorization of 12, you might write $12 = 2 \cdot 2 \cdot 3$ (or $12 = 2^2 \cdot 3$). To check, multiply the prime factors to make sure that the product really is the number you set out to prime factor. For large numbers, you should use a calculator. Here is another way to show on paper the replacement process going on for factoring 12 using the number squares.



Example: Use the number squares to prime factor 210, and show the individual steps.

Note: Since there is not a number square for 210, write 210 on a blank square to begin.

Solution 1: $210 = 21 \cdots 10 = 3 \cdots 7 \cdots 2 \cdots 5$

Solution 2: $210 = 3 \cdots 70 = 2 \cdots 105 = 2 \cdots 3 \cdots 35 = 2 \cdots 3 \cdots 5 \cdots 7$

Solution 3: $210 = 30 \cdots 7 = 5 \cdots 6 \cdots 7 = 5 \cdots 2 \cdots 3 \cdots 7$

There are many other ways one might go about factoring 210, but in the end, there are 4 prime factors that when multiplied together equal 210. Because of the commutative and associative properties of multiplication, $3 \cdots 7 \cdots 2 \cdots 5 = 2 \cdots 3 \cdots 5 \cdots 7 = 5 \cdots 2 \cdots 3 \cdots 7$. Usually, the primes are written in ascending order (from the smallest factor to the largest factor. We write that the prime factorization of 210 is $2 \cdots 3 \cdots 5 \cdots 7$.

This leads to a very important theorem. You can think of the prime numbers as building blocks for all whole numbers greater than 1. Every whole number greater than 1 is prime, or it can be expressed as the product of prime factors (called the prime factorization). The fact that any composite number can be written as a **unique** product of primes is so important that it is called the **Fundamental Theorem of Arithmetic**.

The Fundamental Theorem of Arithmetic:

Every composite number has exactly one unique prime factorization (except for the order in which you write the factors.)

Note again that the order in which the factors are written doesn't matter. However, for consistency, they are usually written in ascending order (from smallest to largest). Also, exponents may be used if a factor is repeated in the prime factorization. For instance, the prime factorization of 12 is usually written as $2 \cdots 2 \cdots 3$ or $2^2 \cdots 3$

Exercise 4

Use the number squares to prime factor each of the following composite numbers into a product of primes. Write the prime factorization so the factors are written in ascending order (from smallest to largest). None of these numbers is prime. **Show each of the individual steps one at a time, not just the final product.**

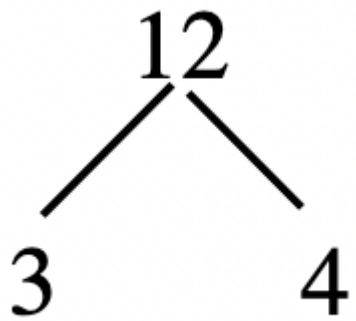
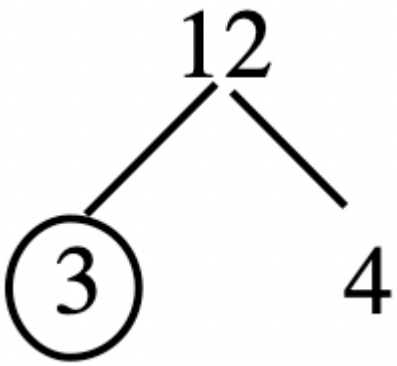
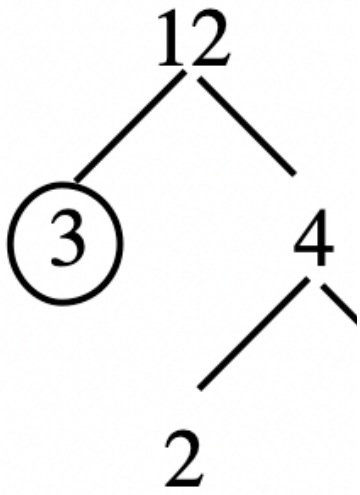
a. 45 = _____
b. 65 = _____

c. 200 = _____
d. 91 = _____
e. 76 = _____
f. 350 = _____
g. 189 = _____
h. 74 = _____
i. 512 = _____
j. 147 = _____

There are other methods commonly used to find the prime factorization. One uses a FACTOR TREE, which is similar to what you did with the prime and composite number squares. The difference is that it is done on paper, as opposed to using manipulatives. The number you are trying to factor is called the root, and is at the top. So, it's actually an upside down tree. If a number is not prime, you draw two branches down from that number and factor it as the product of any two factors. At the end of each branch is a smaller factor, which is called a leaf. If a leaf is prime, circle it, it is one of the factors in the prime factorization of the number. Otherwise, branch off again. When all of the leaves are circled prime numbers, you are done. The prime factorization of the root is the product of the leaves. Below is one way we factored 12, on the previous page, using squares.

Path 1: $\boxed{12} = \boxed{3} \times \boxed{4} = \boxed{3} \times \boxed{2} \times \boxed{2}$

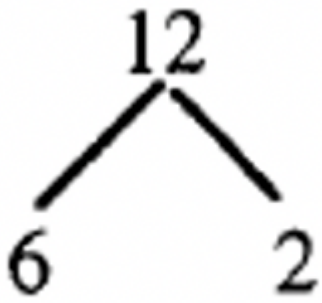
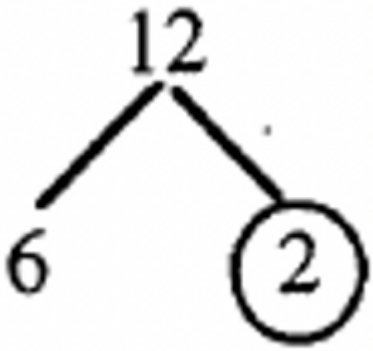
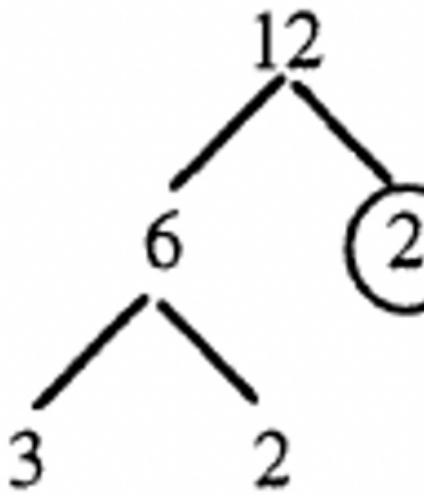
Here are the individual steps showing how one might use a factor tree to factor 12, similar to how it was factored using the squares, via Path 1. Note the similarity.

Step 1:	Step 2:	Step 3
Factor 12 as 3d4	Circle 3 since it is prime	Circle the 2s since they are
		

Below is the other way we factored 12 using squares.

Path 2: $\boxed{12} = \boxed{6} \times \boxed{2} = \boxed{3} \times \boxed{2} \times \boxed{2}$

Here are the individual steps showing how one might use a factor tree to factor 12, similar to how it was factored using the squares, via Path 2. Again, note the similarity.

Step 1:	Step 2:	Step 3:
Facrtor 12 as 6×2	Circle 2 since it is prime	Factor 6 as 2×3
		

If you aren't sure whether a number is prime or composite and don't know how to start factoring, use the divisibility tests. See if it is possible to divide by 2, 3, 5, 7, 11, etc. Make **sure** there are no factors before you circle it and decide it's prime. You are going to repeat exercise 4 again, but this time, use a factor tree.

Exercise 4

Use a factor tree to factor each of the following composite numbers into a product of primes. Write the prime factorization so the factors are written in ascending order (from smallest to largest). None of these numbers is prime. Show the individual steps. Show the factor tree under each problem.

a. 45 = ____	c. 200 = ____
b. 65 = ____	d. 91 = ____
e. 76 = ____	h. 74 = ____
f. 350 = ____	i. 512 = ____
g. 189 = ____	j. 147 = ____

The problem with trying to find the prime factorization is that sometimes it isn't obvious if a number you are trying to factor is prime or not. For instance, it is not immediately obvious whether or not 517 is prime or composite. This is where divisibility tests come in handy. Actually, if you want to find the prime factorization of 517, you only need to check if 517 has any prime numbers as factors. You'll need to try one prime at a time. Before going on, let's list the first several prime numbers starting with 2. Note: 2

is the only even prime number. To make a list, start with 2, 3, 5, and check to see if the next odd number is prime or not. It is not prime if one of the prime numbers listed earlier is a factor.

Exercise 5

List all of the prime numbers less than 100. You only need to use the divisibility tests for 2, 3, 5 and 7 at the most to check if any odd number less than 100 is prime. In other words, any composite number less than 100 has 2, 3, 5 or 7 as a factor. We'll discuss why after this exercise.

Consider the possible ways to factor 54 as a product of 2 factors:

$1 \cdots 54$, $2 \cdots 27$, $3 \cdots 18$, $6 \cdots 9$, $9 \cdots 6$, $18 \cdots 3$, $27 \cdots 2$, $54 \cdots 1$

Note that if you start with the smallest factor (1) as the left factor, you start repeating half-way through the list. This "half-way" mark happens after you get to the square root of the number you are factoring. The square root of 54 is between 7 and 8. So if you list a factor bigger than 7, it would have shown up earlier in the list as a factor less than 7. So that means if I am trying to find the prime factorization of 54, I only need to check prime numbers up to and including at most 7.

So why does any composite number less than 100 have 2, 3, 5 or 7 as a factor? The next prime number after 7 is 11. Since 11^2, or 121, is greater than 100, if there was a prime number greater than or equal to 11 that was a factor of a number less than 100, then the other factor would have to be less than 11. Think about it. If both factors were bigger than 11, the product would be more than 121! This realization makes finding the prime factorization of a number much easier. This is how it works.

Since $3^2 = 9$, any composite number < 9 has 2 as a factor.

Since $5^2 = 25$, any composite number < 25 has 2 or 3 as a factor.

Since $7^2 = 49$, any composite number < 49 has 2, 3 or 5 as a factor.

Since $11^2 = 121$, any composite number < 121 has 2, 3, 5 or 7 as a factor.

Since $13^2 = 169$, any composite number < 169 has 2, 3, 5, 7 or 11 as a factor.

Since $17^2 = 289$, any composite number < 289 has 2, 3, 5, 7, 11 or 13 as a factor.

Since $19^2 = 361$, any composite number < 361 has 2, 3, 5, 7, 11, 13 or 17 as a factor.

Since $23^2 = 529$, any composite number < 529 has 2, 3, 5, 7, 11, 13, 17 or 19 as a factor.

Now, we'll verify that some of the above statements are true.

Example

Since $3^2 = 9$, any composite number < 9 has 2 as a factor.

This can be verified by listing the composite numbers less than 9 and showing that 2 is a factor of each of those numbers: the only composite numbers less than 9 are 4, 6, and 8. Verify that 2 is a factor of each of these numbers: $4 = 2 \cdots 2, 6 = 2 \cdots 3, 8 = 2 \cdots 4$

Example

Since $5^2 = 25$, any composite number < 25 has 2 or 3 as a factor.

The previous example verified that the composite numbers less than 9 have 2 or 3 as a factor (since we've shown they all have 2 as a factor). So we only need to list and verify that the composite numbers less than 25 have 2 or 3 as a factor. The only composite numbers between 9 and 25 are 10, 12, 14, 15, 16, 18, 20, 21, 22, and 24. Mentally verify that 2 or 3 is a factor of each of these numbers. Note: each number only needs to have 2 OR 3 as a factor, although it may have both. For instance, 2 is a factor of 10, 3 is a factor of 15, and both 2 and 3 are factors of 18. You only have to make sure at least one of those factors is a factor of each composite number.

Exercise 6

Since $7^2 = 49$, any composite number < 49 has 2, 3 or 5 as a factor. List the composite numbers between 25 and 49, and mentally verify that 2, 3, or 5 is a factor of each of these numbers.

Exercise 7

Since $11^2 = 121$, any composite number < 121 has 2, 3, 5 or 7 as a factor. List the composite numbers between 49 and 121, and mentally verify that 2, 3, 5 or 7 is a factor of each of these numbers.

It gets tedious trying to verify for larger numbers, but the pattern continues. In other words, if you listed all of the composite numbers less than 361, which is 19^2, then you could actually verify that every one of them has 2, 3, 5, 7, 11 or 13 as a factor.

So, to find if 517 is prime or composite, since 517 is less than 23^2, I only need to check if any of the following are factors: 2, 3, 5, 7, 11, 13, 17 or 19. Instead of checking every number up to 517, only these eight need to be checked. You can use divisibility tests for the first five primes, then use a calculator for the last three. If you find a factor early on, there is no need to keep going.

Exercise 8

Find the prime factorization of 517. Do this by using the divisibility tests up to 11, if necessary. If you still don't find a factor of 517, use a calculator to see if 13, 17 or 19 is a factor. Write the prime factorization here:

Because I wrote the squares of the first several prime numbers on the previous page, I knew I didn't have to check any primes higher than 19. One way to figure out the highest prime number you **might** have to check is to take the square root of the number you are trying to prime factor.

Exercise 9

On your calculator, find the square root of 517 rounded to the nearest tenth:

You should have gotten 22.7. This tells us 517 is not a perfect square. Next, we know that if 517 is composite, it must have a prime factor less than 22.7. So, we need to determine the highest whole number that is prime that is less than 22.7. Start with 22 not prime; then 21 not prime; then 20 not prime; then 19 prime! Therefore, we at most need to check the primes up to 17: 2, 3, 5, 7, 11, 13, 17 and 19. When you actually prime factored 517, did you notice 11 was a factor? Did you use the divisibility test for 11?

So, $517 = 11 \cdots 47$. Then, you look at the other factor, 47, and note it is prime, so you are done! By the way, 5 is the highest prime you'd have to check to see if 47 is prime!

What we have just described is a theorem called the Prime Factor Test. This is the formal way of stating that theorem.

Prime Factor Test: To test for prime factors of a number n, one need only search for prime factors p of n, where $p^2 \leq n$ (or $p \leq \sqrt{n}$)

Exercise 10

For each number, determine the highest prime that might need to be checked to find the prime factorization of the number. Then, find the prime factorization. If it is prime, simply write the number itself, since that ***is*** the prime factorization.

a. 149 highest prime to check: ____ Prime factorization for 149:
b. 273 highest prime to check: ____ Prime factorization for 149:
c. 381 highest prime to check: ____ Prime factorization for 149:
d. 437 highest prime to check: ____ Prime factorization for 149:
e. 509 highest prime to check: ____ Prime factorization for 149:
f. 613 highest prime to check: ____ Prime factorization for 149:
g. 787 highest prime to check: ____ Prime factorization for 149:

Now, let's look at a range of numbers and figure out how to determine which ones are prime. For instance, let's determine which numbers between 350 and 370 are prime. First of all, only odd numbers in this range are prime. So, begin by listing the odd numbers as possibilities: 351, 353, 355, 357, 359, 361, 363, 365, 367 and 369. Next, note 355 and 365 can't be prime since it is divisible by 5. Now, you might use the divisibility test for 3 to cross off 351, 357, 363 and 369 note you can cross of multiples of 3 by crossing off every third odd number if you start at a multiple of 3. Now, our list is down to these possibilities: 353, 359, 361, 367 and 369. The highest prime you'd have to check is the prime number that is less than the square root of 369, which is 19. So, simply check the rest of the primes (7, 11, 13, 17 and 19 at most) on each of these numbers to determine which, if any, are prime. 353 is prime; 359 is prime; 361 is 19, 367 is prime. Therefore, 353, 359 and 367 are the numbers between 350 and 370 that are prime. Furthermore, you should be able to write the prime factorization for all the numbers between 350 and 370 that are composite.

Exercise 11

Find the prime factorization for all the numbers between 280 and 295. If a number is prime, simply write the number itself.

a. 280 = _____	i. 288 = _____
b. 281 = _____	j. 289 = _____
c. 282 = _____	k. 290 = _____
d. 283 = _____	l. 291 = _____
e. 284 = _____	m. 292 = _____
f. 285 = _____	n. 293 = _____
g. 286 = _____	o. 294 = _____
h. 287 = _____	p. 295 = _____

Twin primes are two consecutive odd numbers that are prime. For instance, 5 and 7 are twin primes, 11 and 13 are twin primes, 17 and 19 are twin primes. There is no pattern to determine how often twin primes come up. One unsolved question in mathematics is if there are a finite number of or infinitely many sets of twin primes. Nobody knows.

Exercise 12

From your work in exercise 11, list any sets of twin primes: ____

Exercise 13

Is it possible to have three odd numbers in a row that are prime? Why or why not?

One use of prime factoring a set of numbers is so you can find the **greatest common factor** (GCF) and **least common multiple** (LCM) of a set of numbers. Many people have trouble distinguishing between the greatest common factor and the least common multiple (LCM) because they don't think about what the words actually mean. The greatest common factor of a number is obviously a **factor**, but the adjective **common** describes that you want a factor that is common to all the numbers, and the adjective **greatest** describes that you want the very largest of the common factors of the numbers.

We are going to explore ways of finding the greatest common factor of two numbers, a and b. The notation to express this is GCF(a, b). It doesn't matter which number you list first in the parentheses – it's not an ordered pair. **The greatest common factor of a set of numbers is the largest number that is a factor of each number.**

We are going to explore different ways of finding the greatest common factor of two numbers.

First, we are going to explore one way to find the greatest common factor of 42 and 72.The notation to express this is GCF(42, 70). Remember: it doesn't matter which number you list first in the parentheses, *it's not an ordered pair*. GCF(42, 70) means the same thing as GCF(70, 42). **The greatest common factor of 42 and 70 is the largest number that is a factor of both 42 and 70.**

One way of doing this is to list every single factor of each number and then pick the biggest one that is a factor of each.

Exercise 14

a. List all of the factors of 42 in ascending order: ____

b. List all of the factors of 70 in ascending order: ____
c. List all of the factors that are common to both 42 and 70: ____
d. List the greatest common factor of 42 and 70: ____
e. Fill in the blank: GCF(42, 70) = ____

Listing all of the factors of a given number is sometimes a difficult task. For instance, it's easy to miss a factor. One remedy is to prime factor the number first. To list all of the factors of 42, one might first prime factor 42 like this: $2 \cdots 3 \cdots 7$. For a number to be a factor of 42, it must be composed of the prime factors listed. Of course, 1 is always a factor. Next, you'd check 2, and then 3, which are both factors. 4 is not a factor because if it were, $2 \cdots 2$ would be in the prime factorization! It's clear to see 5 is not a factor. 6 is a factor, since $2 \cdots 3$ is in the prime factorization. Continuing on, 7 is a factor, but 8 is not because $2 \cdots 2 \cdots 2$ is not in the prime factorization of 42. Neither is 9 $(3 \cdots 3), 10(2 \cdots 5), 11, 12(2 \cdots 2 \cdots 3)$, or 13. But 14 is a factor of 42 since $2 \cdots 7$ is in the prime factorization. You can use this strategy as you check every number up to 42, but that is still a lot of numbers to check. Eventually, you'd get this list: 1, 2, 3, 6, 7, 14, 21, 42.

Here's a way to shorten the process a little more. Starting with the smallest factor 1, immediately list the other factor you'd have to multiply by that factor to get 42. So, we start with 1, 42. We check the next number, 2, and note it is a factor. To get the other factor that pairs up with 2, either divide 2 into 42, or simply look at the prime factorization of 42, with the 2 missing. There is $3 \cdots 7$ left, which is the other factor. So the list is now 1, 42, 2, 21. Continuing, we note 3 is a factor. To get the other factor, either divide 42 by 3, or do it the easy way, which is to see what factors are left in the prime factorization of 42 with the 3 missing. Since there is a 2 and a 7, then the number that pairs up with 3 is 14.

The list is now 1, 42, 2, 21, 3, 14. Next, we note 4 and 5 are not factors. 6 is a factor since 2 and 3 $(2 \cdots 3)$ is in the prime factorization. 7 is the number that pairs up with 6. So, the list is now: 1, 42, 2, 21, 3, 14, 6, 7. If you were to continue, the next number to check would be 7. Since 7^2 > 42, you can stop. All the factors that are larger than 7 will already be in the list because there would have been a smaller factor that it paired up with already. Now, put the list of factors of 42 in ascending order: 1, 2, 3, 6, 7, 14, 21, 42.

The same procedure can be used to list all the factors of 70. First, write the prime factorization of 70: $2 \cdots 5 \cdots 7$. You would start off with 1 and 70: 1, 70. Next, it's clear 2 is a factor that pairs up with 35. The list is now: 1, 70, 2, 35. Next discard 3 and 4 as factors, and note 5 is a factor. The factors left are 2 and 7, which multiplied together is 14. So the list is 1, 70, 2, 35, 5, 14. Continuing on, note 6 is not a factor, and 7 is. 7 pairs up with 10. The list is now: 1, 70, 2, 35, 5, 14, 7, 10. Continuing on, note that 8 is not a factor. The next number to check would be 9. But 9^2 > 70, so all the factors higher are already in the list. Writing the list in ascending order, we get: 1, 2, 5, 7, 10, 14, 35, 70.

Make a note that in both of these examples, 42 and 70 each had exactly 3 prime numbers in the prime factorization. Consider the prime factorization of 220: $2 \cdots 2 \cdots 5 \cdots 11$. Note that as you list factors, there may be one, two, or three other factors to multiply together to get the pair. 2 is a factor; its pair is $2 \cdots 5 \cdots 11$, or 110. 4 $(2 \cdots 2)$ is a factor; its pair is 5 $\cdots$ 11, or 55. 5 is a factor; its pair is $2 \cdots 2 \cdots 11$, or 44. 10 $(2 \cdots 5)$ is a factor; its pair is $2 \cdot 11$, or 22. 11 is a factor; its pair is $2 \cdots 2 \cdots 5$, or 20. As I check numbers larger than 11, I stop at 15 since 152 > 220. So, the list is: 1, 220, 2, 110, 4, 55, 5, 44, 10, 22, 11, 20. Writing these in ascending order, we get: 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110, 220.

By the way, it's a lot more time-consuming for me to explain (and for you to read) how to list all the factors by prime factoring. But doing it is a lot easier! Now, do the next exercises. If you wish, use prime factorization to do parts a and b.

Exercise 15

The goal of this problem is to find the greatest common factor of 92 and 115.

a. List all of the factors of 92 in ascending order:
b. List all of the factors of 115 in ascending order:
c. List all of the factors that are common to both 92 and 115:

d. List the **greatest** common factor of 92 and 115:
e. Fill in the blank: GCF(92, 115) =

Exercise 16

The goal of this problem is to find the greatest common factor of 48, 54 and 63.

a. List all of the factors of 48 in ascending order:
b. List all of the factors of 54 in ascending order:
c. List all of the factors of 63 in ascending order:
d. List all of the factors that are common to 48, 54 and also 63:
e. List the **greatest** common factor of 48, 54 and 63:
f. Fill in the blank: GCF(48, 54 and 63) =

As you might have realized, listing factors can still be a very time-consuming way to find the greatest common factor, especially if the numbers are very large or have a lot of factors.

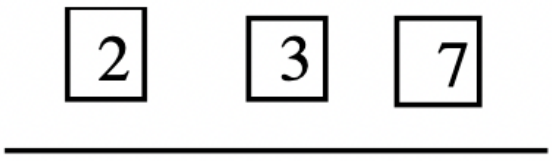
It's time to get out your colored prime number squares and white composite number squares again. We'll be using these to prime factor sets of numbers, which can then be used to find the greatest common factor of a set of numbers. This method is usually faster than the one we just used.

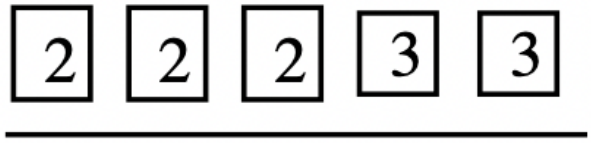
The goal of this next example is to find the greatest common factor of 42 and 72 using prime factorization with the prime and composite number squares.

Step 1: Use the squares to prime factor 42 and 72.

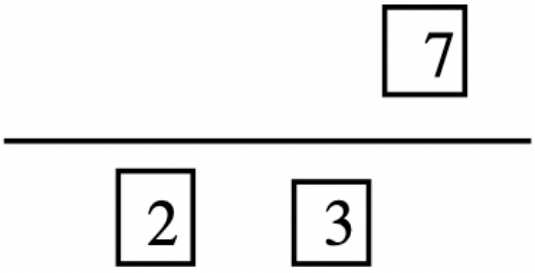
$42 = 2 \cdots 3 \cdots 7$ and $72 = 2 \cdots 2 \cdots 2 \cdots 3 \cdots 3$

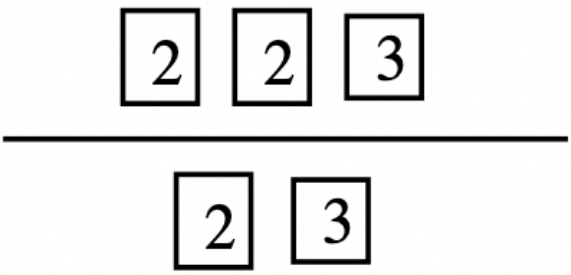
Step 2: Draw a line and put the prime factors of 42 above it. Next to that, draw a line and put the prime factors of 72 above it.





Step 3: Look at the prime factorization of the numbers. Move every factor they have in common under the line. They should have the same exact factors under the line, and they should have no common factors left above the line.





Step 4: The product of the factors under a line is the greatest common factor of the numbers. In this case, 2 ⋯ 3, or 6, is the

greatest common factor of 42 and 72.

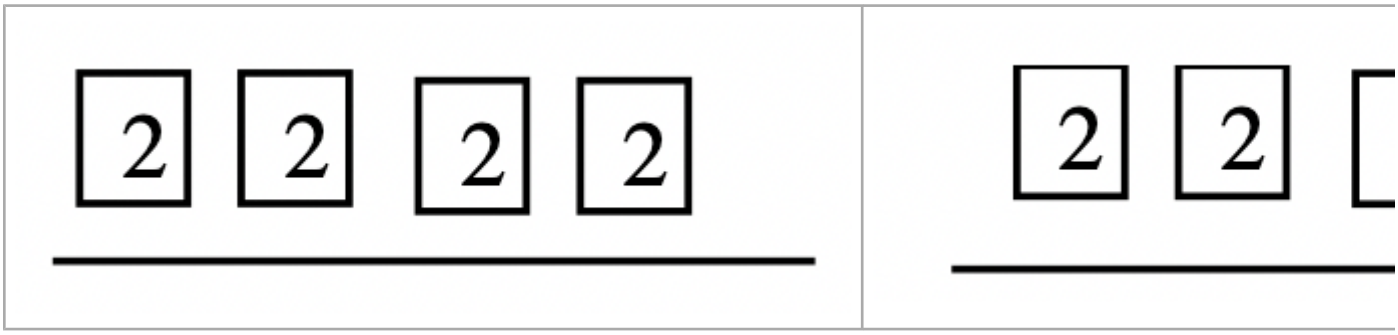
Step 5: Use the correct notation to write the answer: GCF(42, 70) = 6.

This next example used the same steps to find the greatest common factor of 16, 24 and 36.

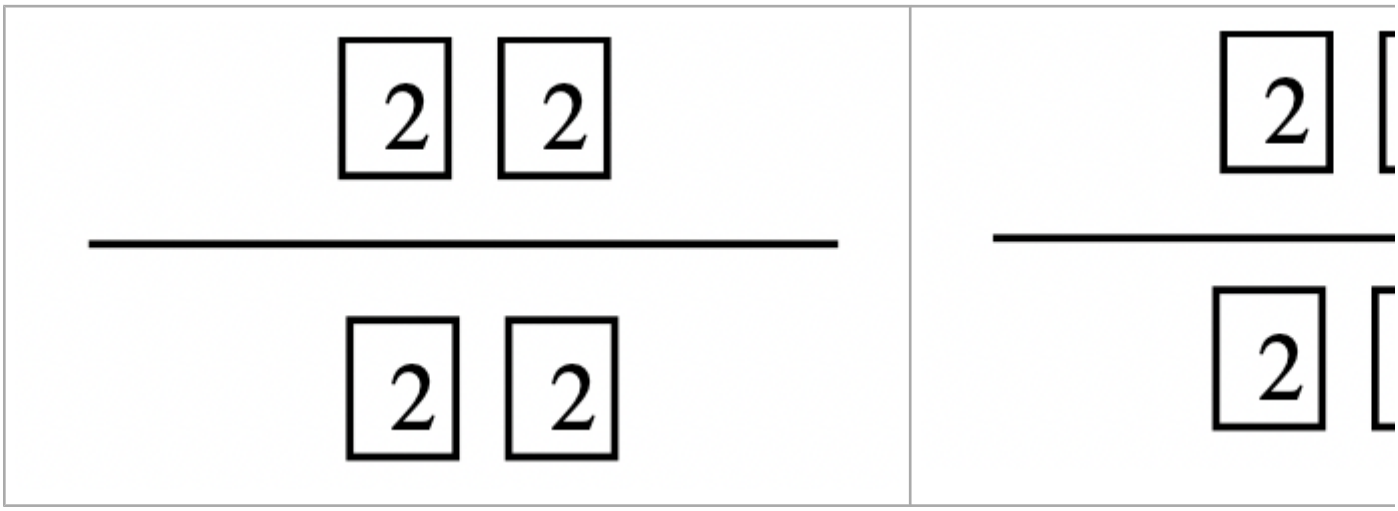
Step 1: Use the squares to prime factor 16, 24 and 36.

$16 = 2 \cdots 2 \cdots 2 \cdots 2, 24 = 2 \cdots 2 \cdots 2 \cdots 3$ and $36 = 2 \cdots 2 \cdots 3 \cdots 3$

Step 2: Draw a line for each number and put the prime factors of each number above it.



Step 3: Look at the prime factorization of the numbers. Move every factor that all three numbers have in common. There should be no factors left above the line that all three numbers have in common.



Step 4: The product of the factors under a line is the greatest common factor of the numbers. In this case, $2 \cdots 2$, or 4, is the greatest common factor of 16, 24 and 36.

Step 5: Use the correct notation to write the answer: GCF(16, 24, 36) = 4.

Exercise 17

Use the steps shown in the previous two examples to find the greatest common factor of each problem. Show a picture of how the problem looks at step 3 where the prime factorization of the greatest common factor is shown under the line of each number you first prime factored.

a. GCF(42, 70) = ____ (This answer should agree with Exercise 14e.) Show work below:
b. GCF(92, 115) = ____ (This answer should agree with Exercise 15e.) Show work below:
c. GCF(48, 54, 63) = ____ (This answer should agree with Exercise 16f.) Show work below:
d. GCF(306, 340) = ____ Show work below:
e. GCF(125, 275, 400) = ____ Show work below:
f. GCF(126, 168, 210) = ____ Show work below:

Exercise 18

Let's say someone prime factored three large numbers, X, Y and Z like this:

$X = 2^5 \cdots 3^4 \cdots 7^2 \cdots 11^8 \cdots 13^3$	$Y = 2^4 \cdots 3^5 \cdots 5^3 \cdots 7^7 \cdots 13^2$	$Z = 2^6 \cdots 3 \cdots 5^5 \cdots 11^3 \cdots 13^4$

a. State the prime factorization (exponential notation is okay) of the greatest common factor of X, Y and Z. GCF(X, Y, Z) =

b. Explain how you did part a.

One way to do the previous exercise is to begin by listing the common prime factors (without exponents) of X, Y and Z. They are 2, 3 and 13. So, as a start, write 2 ⋯ 3 ⋯ 13. Next, determine how many of each prime factor is common to X, Y and Z. Since X has 5 factors of 2, Y has 4 factors of 2, and Z has 6 factors of 2, they only have 4 factors of 2 in common. Similarly, they only have one factor of 3 in common, and 2 factors of 13 in common. Put these exponents on the factors and you've got the GCF of the three numbers: GCF(X, Y, Z) = $2^4 \cdots 3 \cdots 13^2$

Did you notice that if you list the factors they have in common without exponents, you put on the smallest exponent they have in common for each prime?

Exercise 19

Write the prime factorization of the greatest common factor of the set of numbers. For those that are factored with letters, assume each letter represents a different prime number.

a. If $X = 2^4 \cdots 3^2 \cdots 7^6 \cdots 11^3 \cdots 13^2$ and $Y = 2^5 \cdots 3^6 \cdots 5^4 \cdots 7^6 \cdots 13^3$

GCF(X, Y) = _____

b. If $X = 3^4 \cdots 5^2 \cdots 7^6$ and $Y = 2^5 \cdots 3^6 \cdots 5^6 \cdots 7^3$ and $Z = 2^4 \cdots 3^5 \cdots 5^4$

GCF(X, Y, Z) = _____

c. If $X = a^4 \cdots b^2 \cdots c^6 \cdots d^3 \cdots e^2$ and $Y = a^5 \cdots c^3 \cdots d^4 \cdots e^6 \cdots f^3$

GCF(X, Y) = _____

d. If $X = a^4 \cdots b \cdots c^4 \cdots d^3$ and $Y = a^5 \cdots b^3 \cdots d^4 \cdots e^6$ and $Z = a^2 \cdots c^3 \cdots d^7 \cdots f^3$

GCF(X, Y, Z) = _____

If two numbers have no factors in common, they are called **relatively prime**. In other words, **if GCF(a, b) = 1, then a and b are relatively prime**. The numbers, a and b, might both be prime, both be composite, or one might be prime and the other composite.

Exercise 20

Give an example of two composite numbers that are relatively prime:

Exercise 21

Write a prime number and composite number that are relatively prime:

Exercise 22

Assume GCF(28, x) = 1

a. List any prime numbers that are not factors of x: _____

b. Give three examples (numbers) of what x could equal: _____

Exercise 23

If x and y are different prime numbers, GCF(x, y) = _____

Exercise 24

If m is a whole number, find the following:

a. GCF(2m, 3m) = _____	b. GCF (4m, 10m) = _____
c. GCF(m, m) = _____	d. GCF(m, 1) = _____
e. GCF(m, 0) = _____	

Trying to find the greatest common factor of two large numbers by prime factorization is sometimes quite time-consuming. There are two other algorithms you can use that we'll use. One is called **The Old Chinese Method**, and the other is **The Euclidean Algorithm**. Both of these methods use a fact we can prove using what we know about divisibility. First, let's look at some examples.

Exercise 25

Compute each of the following:

a. List the common factors of 42 and 72: _____
b. List the common factors of 42 and 30 (which is 72 – 42): _____
c. List the common factors of 30 and 12 (which is 42 – 30): _____
d. List the common factors of 12 and 18 (which is 30 – 12): _____
e. List the common factors of 12 and 6 (which is 18 – 12): _____
f. List the common factors of 6 and 6 (which is 12 – 6): _____
g. List the common factors of 6 and 0 (which is 6 – 6): _____

From your work in exercise 25, compute the following:

a. GCF(42, 72) = ___	b. GCF(42, 30) = ___
c. GCF(30, 12) = ___	d. GCF(12, 18) = ___
e. GCF(12, 6) = ___	f. GCF(6, 6) = ___
g. GCF(6, 0) = ___	

The previous two exercises illustrate that the greatest common factor of two numbers is equal to the greatest common factor of the smaller number, and the difference of the original two numbers; i.e., **if $x \geq y$, then GCF(x, y) = GCF(y, x – y)**.

Prove the following theorem: Let $a \geq b$. If c|a and c|b, then c|(a – b).

Solution: If c|a, then cn=a for some whole number, n. If c|b, then cm=b for some whole number, m. Using these substitutions for a and b, we get that c|(a – b) is true if c|(cn – cm) which is true if c is a factor of cn – cm. Factor: cn – cm = c(n – m). This clearly shows that c is indeed a factor of cn – cm. Therefore, if c|a and c|b, then c|(a – b).

This theorem can be used to show that **if $a \geq b$, then GCF(a, b) = GCF(b, a – b)**. The above theorem states that if c is a factor of two numbers, then it is also a factor of their difference. Hence, if c is a common factor of a and b, where $a \geq b$, then c is also a common factor of b and a – b. Since every common factor of a and b is also a common factor of ba and a – b, the pairs (a, b) and (b, a – b) have the same common factors. So, GCF(a, b) and GCF(b, a – b) must also be the same number.

The **Old Chinese Method** employs the fact that **GCF(a, b) = GCF(b, a – b)**.

Note three more properties:

GCF(x, x) = x: GCF(x, x) states that the greatest common factor of the same two numbers is itself. That should be clear since x is the greatest factor of each number, so each has x as the greatest common factor.

GCF(x, 0) = x: GCF(x, 0) = x, is true since every number is a factor of zero. So, since x is the greatest factor of x, then x is the greatest common factor of x and 0.

GCF(x, 1) = 1: GCF(x, 1) = 1 is true since 1 is a factor of every number, including x, and 1 is the only factor of 1. Therefore, 1 must be the greatest common factor of 1 and x.

Old Chinese Method of finding the greatest common factor of two numbers:

Write the GCF of the two numbers in parentheses (remember the order of the numbers is irrelevant). Let that equal the GCF of the smaller of the two numbers, and the difference of the original two numbers. If the numbers in the parentheses are the same, that number is the GCF; if one the new numbers is 1, 1 is the GCF. Otherwise, repeat the process until the two numbers are the same, or 1 is one of the numbers. An example is shown below. On the right is an explanation of how I obtained the two new numbers in parentheses. The new numbers are underlined in the explanation. You do not need to write that out.

Example

GCF(546, 390)	= GCF(390, 156)	(*390* is smaller than 546, *156* = 546 – 390)
	= GCF(156, 234)	(*156* is smaller than 390, *234* = 390 – 156)
	= GCF(156, 78)	(*156* is smaller than 234, *78* = 234 – 156)
	= GCF(78, 78)	(*78* is smaller than 156, *78* = 156 – 78)
	= ***78***	78 is the GCF(78, 78); The answer is a number!

Therefore, GCF(546, 390) = 78.

Check: First, make sure 78 is a factor of 546 and 390: $546 = 78 \cdots 7$ and $390 = 78 \cdots 5$. Second, check to make sure the other factors of each (7 and 5) are relatively prime. If so, then 78 is not only a factor of 546 and 390, but is in fact the greatest common factor of each since they have no other common factors (because 7 and 5 have no common factors).

Another example is shown below. On the right is an explanation of how I obtained the two new numbers (underlined) in parentheses. You do not need to write that out.

Example

GCF(1200, 504)	= GCF(504, 696)	(*504* is smaller than 1200, *696* = 1200 – 504)
	= GCF(504, 192)	(*504* is smaller than 696, *192* = 696 – 504)
	= GCF(192, 312)	(*192* is smaller than 504, *312* = 504 – 192)
	= GCF(192, 120)	(*192* is smaller than 312, *120* = 312 – 192)
	= GCF(120, 72)	(*120* is smaller than 192, *72* = 192 – 120)
	= GCF(72, 48)	(*72* is smaller than 120, *48* = 120 – 72)
	= GCF(48, 24)	(*48* is smaller than 72, *24* = 72 – 48)
	= GCF(24, 24)	(*24* is smaller than 48, *24* = 48 – 24)
	= ***24***	24 is the GCF(24, 24); The answer is a number!

Therefore, GCF(1200, 504) = 24.

Check: First, make sure 24 is a factor of 1200 and 504: 1200 = $24 \cdots 50$ and $502 = 24 \cdots 21$. Second, check to make sure the other factors of each (25 and 21) are relatively prime. If so, then 24 is not only a factor of 1200 and 504, but is in fact the greatest common factor of each since they have no other common factors (because 25 and 21 have no common factors.)

Here is one more example:

Example

GCF(667, 437)	= GCF(437, 230)	
	= GCF(230, 207)	
	= GCF(207, 23)	
	= GCF(23, 184)	
	= GCF(23, 161)	
	= GCF(23, 138)	
	= GCF(23, 115)	
	= GCF(23, 92)	
	= GCF(23, 69)	
	= GCF(23, 46)	
	= GCF(23, 23)	
	= ***23***	Remember: The answer is a number!

Therefore, GCF(667, 437) = 23.

Check: First, make sure 23 is a factor of 667 and 437: $667 = 23 \cdots 29$ and $437 = 23 \cdots 19$. Second, check to make sure the other factors of each (29 and 19) are relatively prime. If so, then 23 is not only a factor of 667 and 437, but is in fact the greatest common factor of each since they have no other common factors (because 29 and 19 have no common factors.)

A comment: You could have obtained the greatest common factor of the previous three examples by prime factoring. Often, this is a lengthy process for large numbers that look like they might be prime, as in the last example. Therefore, the Old Chinese provides an alternative way to obtain the greatest common factor. After doing a few using this method, we'll explore another alternate method, called the Euclidean Algorithm, which is related to, but usually takes less steps than, the Old Chinese Method.

Exercise 27

Use the Old Chinese Method to compute the greatest common factor of the numbers given. Use correct notation, and show each step. State the answer. Then, show how you check your answer. Use the previous example as a model to do these problems.

a.

GCF(143, 91)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= ____	

b.

GCF(468, 378)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= ____	

c.

GCF(504, 180)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= ____	

Using the Chinese Method could be quite tedious. Take a look at the following example:

Example

GCF(1200, 504)	= GCF(504, 696)	(*504* is smaller than 1200, *696* = 1200 – 504)

	= GCF(504, 192)	(*504* is smaller than 696, *192* = 696 – 504)
	= GCF(192, 312)	(*192* is smaller than 504, *312* = 504 – 192)
	= GCF(192, 120)	(*192* is smaller than 312, *120* = 312 – 192)
	= GCF(120, 72)	(*120* is smaller than 192, *72* = 192 – 120)
	= GCF(72, 48)	(*72* is smaller than 120, *48* = 120 – 72)
	= GCF(48, 24)	(*48* is smaller than 72, *24* = 72 – 48)
	= GCF(24, 24)	(*24* is smaller than 48, *24* = 48 – 24)
	= ***24***	24 is the GCF(24, 24); The answer is a number!

In the beginning of this example, GCF(1200, 504), we had to subtract 504 twice until we got GCF(504, 192). Then, notice once we wrote GCF(504, 192), we had to subtract 192 twice until we got GCF(192, 120). Basically, we do repeated subtraction until we get a number smaller than the one we are subtracting. Repeated subtraction is actually division. Note that $1200 \div 504 = 2$ remainder 192. On the second step, we see GCF(504, 192), which has the **smaller** of the numbers in the original parentheses (504), and the **remainder** after dividing the larger number (1200) by the smaller number (504). Note that $504 \div 192 = 2$ remainder 120. Look at the fourth step: we see GCF(192, 120), which has the **smaller** of the numbers in GCF(504, 192), and the **remainder** after dividing the larger number (504) by the smaller number (192). The Euclidean Algorithm uses division instead of repeated subtraction to shorten the steps.

How to use the Euclidean Algorithm to find the greatest common factor of two numbers:

Write the GCF of the two numbers in parentheses (remember the order of the numbers is irrelevant). The smaller of the two numbers will be one of the numbers in the next parentheses. To get the other number, divide the larger number by the smaller number, and put the **remainder** in parentheses. If the the smaller number is a factor of the larger number, that means it will divide evenly, so there will be no remainder. That means the remainder is 0. Remember to put the remainder in the parentheses, not the quotient! If one of the new numbers in the parentheses is zero, the other number is the GCF; if one the new numbers is 1, 1 is the GCF. Otherwise, repeat the process until one of the two numbers is 0 or 1. An example is shown below. This is the first example we did using the Old Chinese Method. You might want to look back and compare. On the right is an explanation of how I obtained the two new numbers in parentheses.

Example

GCF(546, 390)	= GCF(390, 156)	(*390* is smaller than 546, $546 \div 390$ = 1 r *156*)
	= GCF(156, 78)	(*156* is smaller than 390, $390 \div 156$ = 2 r *78*)
	= GCF(78, 0)	(78 is smaller than 156, $156 \div 78$ = 2 r 0)
	= ***78***	***Put the remainder (0), NOT 2, in the parentheses!***
		78 is the GCF(78, 78); The answer is a number!

Therefore, GCF(546, 390) = 78

Check: First, make sure 78 is a factor of 546 and 390: $546 = 78 \backslash = \cdots 7$ and $390 = 78 \cdots 5$. Second, check to make sure the other factors of each (7 and 5) are relatively prime. If so, then 78 is not only a factor of 546 and 390, but is in fact the greatest common factor of each since they have no other common factors (because 7 and 5 have no common factors

Below is another example – we did this earlier using the Old Chinese Method. On the right is an explanation of how I obtained the two new numbers in parentheses. You do not need to write that out.

Example

GCF(667, 437)	= GCF(437, 230)	(*437* is smaller than 667, $667 \div 437 = 1$ r *230*)
	= GCF(230, 207)	(*230* is smaller than 437, $437 \div 230 = 1$ r *207*)
	= GCF(207, 23)	(*207* is smaller than 230, $230 \div 207$ = 1 r *23*)
	= GCF(23, 0)	(*23* is smaller than 207, $207 \div 23$ = 9 r 0)
	= ***23***	***Put the remainder (0), NOT 9, in the parentheses!***
		Remember: The answer is a number!

Therefore, GCF(667, 437) = 23

Check: First, make sure 23 is a factor of 667 and 437: 667 = $23 \cdots 29$ and $437 = 23 \cdots 19$. Second, check to make sure the other factors of each (29 and 19) are relatively prime. If so, then 23 is not only a factor of 667 and 437, but is in fact the greatest common factor of each since they have no other common factors (because 29 and 19 have no common factors.)

A comment: You could have obtained the greatest common factor of the previous three examples by prime factoring, the Old Chinese Method or the Euclidean Algorithm.

CAUTION: THE MOST COMMON MISTAKE PEOPLE MAKE WHEN USING THE EUCLIDEAN ALGORITHM IS ON THE LAST STEP, WHEN THE SMALLER NUMBER DIVIDES EVENLY INTO THE LARGER NUMBER, WHICH *MEANS THE REMAINDER IS ZERO*. IT DOESN'T MATTER WHAT THE QUOTIENT IS – IT'S THE *REMAINDER* THAT MATTERS – 0 GOES IN THE PARENTHESES!!! Also, zero is not a factor of any number except zero, so the GCF can't be zero. On the other hand, every number is a factor of zero. So, when zero is one of the numbers in parentheses, the other number is the GCF. **REMEMBER TO ALWAYS CHECK YOUR Answer!!!**

Before going on, I'm going to remind you of a quick and easy way to find out the quotient and remainder by using a simple calculator when doing these division problems, where you need to find the remainder. If you can already do it easily or your calculator figures it out for you, skip on down to Exercise 29. Let's say you wanted to divide $5263 \div 360$. When you do this on your calculator, it shows up something like 14.619444. This indicates that there are 14 360's in 5263, but the remainder isn't evident. At least, you know 14 is the quotient. To find the remainder on your calculator, key in 14×360 - 5263 and the number showing is the remainder if you ignore the negative sign! In this case, the remainder is 223. Remember that the remainder must be less than what you originally divided by – less than 360 in this case. Think about why this process works and try it on the next few problems.

Exercise 28

Use a calculator to find the quotient and remainder for these division problems.

a. $9876 \div 255$ =	b. $1509 \div 164$ =
c. $333 \div 46$ =	d. $4657 \div 579$ =

Exercise 29

Use the Euclidean Algorithm to compute the greatest common factor of the numbers given. Use correct notation, and show each step. State the answer. Then, show how you check your answer. Use the previous example as a model to do these problems. These are the same exercises you did using the Old Chinese Method in exercise 27. You might want to compare the two methods when you are done. Of course, the answer should be the same.

a.

GCF(143, 91)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= ____	

b.

GCF(468, 378)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= ____	

c.

GCF(504, 180)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= ____	

Exercise 30

Find GCF(418, 88) using the three methods discussed:

a) by prime factorization;
b) by the Old Chinese Method;
c) by the Euclidean Algorithm
d) Write the answer, and show how to check the answer.

a. Use prime factorization	c. Use the Euclidean Algorithm
b. Use the Old Chinese Method	d. Answer: GCF(418, 88) = _____ Check your answer

Exercise 31

Find GCF(527, 465) using the three methods discussed:

a) by prime factorization;
b) by the Old Chinese Method;
c) by the Euclidean Algorithm
d) Write the answer, and show how to check the answer.

a. Use prime factorization	c. Use the Euclidean Algorithm
b. Use the Old Chinese Method	d. Answer: GCF(527, 465) = _____ Check your answer

Exercise 32

Find GCF(353, 213) using the three methods discussed:

a) by prime factorization;
b) by the Old Chinese Method;
c) by the Euclidean Algorithm
d) Write the answer, and show how to check the answer.

a. Use prime factorization	c. Use the Euclidean Algorithm
b. Use the Old Chinese Method	d. Answer: GCF(353, 213) = _____ Check your answer

Now that we've covered a lot about prime numbers and factoring, we are going to revisit the divisibility tests one more time.

Exercise 33

Note that the divisibility test for 6 utilized two other divisibility tests, the one for 2 and the one for 3. Also, note that the divisibility test for 15 utilized two other divisibility tests, the one for 5 and 3. Why do you think that works? What is the procedure for figuring out what tests to use?

Fact: If each prime in the prime factorization of a composite number, c, is listed only once, the divisibility test for that composite number is this: c|n if each of the prime factors of c divide n.

The prime factorization for 6 is $2 \cdot 3$. Since both primes are only listed once, the divisibility test for 6 is the union of the divisibility tests for 2 and 3.

The prime factorization for 15 is $5 \cdot 3$. Since both primes are only listed once, the divisibility test for 15 is the union of the divisibility tests for 5 and 3.

Exercise 34

Write the divisibility test for 14:

Exercise 35

Use the divisibility test for 14 to see which of the following is true. Show work.

a. 14|742

b. 14|968
c. 14|483

Exercise 36

The prime factorization for 20 is $2 \cdots 2 \cdots 5$. The divisibility test for 2, 2 and 5 does not work since 20|70 is false even though 2|70 and 2|70 and 5|70. Why doesn't it work?

Exercise 37

This is the divisibility test for 20: 20|n if 4|n and 5|n. Why do you think this works?

Exercise 38

Try to write the divisibility tests for each of the following numbers:

a. 12|n if
b. 18|n if

Divisibility Test for a composite number: **Assume P_1, P_2, P_3, P_4, P_5,... are all different prime numbers. (You can think of P_1 as the first prime (2), P_2 as the second prime (3), P_3 as the third prime (5), P_4 as the fourth prime (7), P_5 as the fifth prime (11), etc. Assume the prime factorization of a number, c, is $(P_a)^x \cdots (P_b)^y \cdots \ldots \cdots (P_c)^z$ where $P_a \neq P_b \neq P_c$, etc. Then, c|n if $(P_a)^x$|n AND $(P_b)^y$|n AND $(P_c)^z$|n, etc.**

Example

The divisibility test for the following numbers is done by first factoring them.

1) Divisibility test for 26: Since 26 = $2 \cdots 13$: **26|n if 2|n and 13|n.**

2) Divisibility test for 12: Since 12 = $2^2 \cdots 3$: **12|n if 4|n and 3|n.**

3) Divisibility test for 24: Since 24 = $2^3 \cdots 3$: **24|n if 8|n and 3|n.**

4) Divisibility test for 45: Since 45 = $3^2 \cdots 5$: **45|n if 9|n and 5|n.**

5) Divisibility test for a number, c, whose prime factorization is $2^3 \cdots 3^2 \cdots 5^2 \cdots 11$: **c|n if 8|n AND 9|n and 25|n and 11|n**

7) Divisibility test for a number, b, whose prime factorization is $3^2 \cdots 5 \cdots 7^2 \cdots 11^2$: **b|n if 9|n AND 5|n and 49|n and 121|n**

Exercise 39

Write the divisibility test for the following numbers:

a. 35: ___
b. 28: ___
c. 75: ___

d. 56: ____
e. A number, c, whose prime factorization is $2^2 \cdots 3^3 \cdots 5^2 \cdots 11$: ____
f. A number, d, whose prime factorization is $2^4 \cdots 3 \cdots 5 \cdots 11$: ____

Exercise 40

This exercise is for you to think about what you'd do if someone asked you to find the LEAST common factor of a set of numbers, instead of the greatest common factor.

a. Find the least common factor of 55 and 66: ____
b. Find the least common factor of 10 and 12: ____
c. Find the least common factor of 1 and 8: ____
d. Find the least common factor of 3, 6, and 9: ____
e. Find the least common factor of m and n: ____
f. Do you think it is a useful question to find the least common factor of a set of numbers? Why or why not? Explain your answer.
g. What is the least common factor of any set of numbers? ____

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4.3: The Sieve of Eratosthenes

Imagine a clever way to find all the hidden treasure (prime numbers) on a number island. That's exactly what the Sieve of Eratosthenes does! This ingenious method, named after the ancient Greek mathematician Eratosthenes who lived over 2,200 years ago, is like a number detective's toolkit. It helps us discover all the prime numbers up to any limit we choose.

What makes the Sieve of Eratosthenes special is its simplicity and visual nature. Picture crossing off numbers on a giant calendar - that's essentially how it works! While mathematicians have developed other ways to identify prime numbers, the Sieve remains a favorite for its intuitive approach and its ability to find all primes within a range, not just determine if a single number is prime.

This method isn't just an old relic - it's a powerful tool that can spark curiosity and deepen understanding of numbers in your future classroom. It turns abstract math into a fun, hands-on adventure that your students will remember!

The Sieve of Eratosthenes Algorithm

The Sieve of Eratosthenes is an efficient algorithm for finding all prime numbers up to a given limit. Here's a step-by-step breakdown of the algorithm:

1. **Create a list**:
   - Start with a list of all integers from 2 to your chosen limit, n.
   - Initially, consider all numbers in this list as potential primes.
2. **Start with the first prime number, 2**:
   - 2 is the smallest and first prime number.
3. **Mark multiples as composite**:
   - Starting from 2^2 = 4 (as 2 * 2 is the first multiple of 2 that hasn't been crossed off yet), mark every multiple of 2 up to n as composite (not prime).
   - You can do this by counting in steps of 2 from 4 to n.
4. **Find the next unmarked number**:
   - The next unmarked number in the list is prime.
   - In this case, it would be 3.
5. **Repeat the marking process**:
   - Again, mark all multiples of this number (3 in this case) as composite.
   - Start from 3^2 = 9 (as all smaller multiples of 3 have already been marked by smaller primes), and count in steps of 3 up to n.
6. **Continue the process**:
   - Keep repeating steps 4 and 5 until you've processed all numbers up to √n.
   - You only need to go up to √n because any composite number larger than √n must have at least one prime factor less than or equal to √n.
7. **Collect the results**:
   - All unmarked numbers in your list are prime.

Example 4.3.1

Let's find all primes up to 30:

List: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30

1. Start with 2, mark multiples: 2, 3,
2.
3. 4, 5, ~~6~~, 7, ~~8~~, 9, ~~10~~, 11, ~~12~~, 13, ~~14~~, 15, ~~16~~, 17, ~~18~~, 19, ~~20~~, 21, ~~22~~, 23, ~~24~~, 25, ~~26~~, 27, ~~28~~, 29, ~~30~~
4. Move to 3, mark multiples: 2, 3,
5.
6.

~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~, 11, ~~12~~, 13, ~~14~~, ~~15~~, ~~16~~, 17, ~~18~~, 19, ~~20~~, ~~21~~, ~~22~~, 23, ~~24~~, 25, ~~26~~, ~~27~~, ~~28~~, 29, ~~30~~

7. Move to 5 (next unmarked), mark multiples: 2, 3,
8.
9.

~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~, 11, ~~12~~, 13, ~~14~~, ~~15~~, ~~16~~, 17, ~~18~~, 19, ~~20~~, ~~21~~, ~~22~~, 23, ~~24~~, ~~25~~, ~~26~~, ~~27~~, ~~28~~, 29, ~~30~~

10. We're done because $\sqrt{30} \approx 5.5$, and we've processed all numbers up to 5.

The unmarked numbers (2, 3, 5, 7, 11, 13, 17, 19, 23, 29) are all the primes up to 30.

This algorithm efficiently finds all primes in a given range, making it a powerful tool for number theory and various applications in computer science and mathematics.

Try It! Sieve of Eratosthenes Widget

Instructions:

1. Choose a number up to which you want to find all prime numbers.
2. On a piece of paper, try to apply the Sieve of Eratosthenes method to find these primes.
3. Write down your process and the prime numbers you identified.
4. Once you're done, enter your chosen number in the box below and click "Run Sieve".
5. Compare your work with the steps and result shown by the widget.
6. If your answer matches, great job! If not, review the steps to see where you might have made a mistake.

Run Sieve

How Can Understanding the Sieve Help in Teaching Math?

1. **Visualization of Abstract Concepts**: The sieve turns the abstract idea of prime numbers into a visual, tangible process.
2. **Pattern Recognition**: Students can observe patterns in the multiples being crossed out, reinforcing multiplication facts.
3. **Problem-Solving Skills**: It introduces a systematic approach to solving a complex problem (finding all primes).
4. **Differentiated Learning**: The sieve can be adapted for different skill levels by changing the number range.
5. **Cross-Curricular Connections**: You can integrate history (ancient Greece) and technology (coding simple sieve programs) into math lessons.
6. **Foundation for Advanced Concepts**: It lays the groundwork for understanding factors, multiples, and even basic cryptography.

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4.4: Even and Odd Numbers

Understanding even and odd numbers is a fundamental concept in mathematics that lays the groundwork for more advanced topics. This seemingly simple classification of integers provides students with their first glimpse into number theory and helps develop critical thinking skills. By mastering the concepts of even and odd numbers, students enhance their ability to recognize patterns, make predictions, and solve problems in various mathematical contexts. These notes aim to provide educators with a comprehensive overview of even and odd numbers, including definitions, key concepts, properties, and effective teaching strategies to engage students in this essential mathematical concept.

Definition: Even Numbers

Even numbers are integers that can be divided by 2 without a remainder.

In mathematical terms, an integer n is even if and only if there exists an integer k such that:

$n = 2k$

This means that any even number can be expressed as 2 multiplied by some integer k.

Definition: Odd Numbers

Odd numbers are integers that cannot be divided by 2 without a remainder.

An integer n is odd if and only if there exists an integer k such that:

$n = 2k + 1$ or $n = 2k - 1$

This means that any odd number can be expressed as 2 multiplied by some integer k, plus or minus 1.

Alternatively, we can say that an integer n is odd if and only if there exists an integer m such that:

$n = 2m \pm 1$

Where m is any integer and \(\pm\) means "plus or minus".

General Concepts for Even and Odd Numbers

- Even numbers end in 0, 2, 4, 6, or 8 in their ones place.
  This pattern emerges from the nature of even numbers being divisible by 2. When we divide by 2, the only possible remainders are 0 and 1. Numbers ending in 0, 2, 4, 6, or 8 are the only ones that yield a remainder of 0 when divided by 2. This concept provides a quick visual cue for identifying even numbers, especially useful for younger learners or rapid mental calculations.
- Odd numbers end in 1, 3, 5, 7, or 9 in their ones place.
  Complementary to even numbers, odd numbers are those that aren't divisible by 2. They always leave a remainder of 1 when divided by 2. This pattern in the ones place offers an easy way to spot odd numbers. It's important to note that this rule applies in base-10 number system, which is our standard decimal system.
- Zero is considered an even number.
  The evenness of zero often confuses students because zero divided by two equals zero, with no remainder. However, zero fits the formal definition of even numbers (0 = 2 × 0). Understanding zero as an even number is crucial in many areas of mathematics, including parity operations in computer science and properties of functions in higher mathematics.
- The set of integers alternates between even and odd numbers.
  This concept illustrates the beautiful symmetry in our number system. Starting from any integer and moving in either direction (adding or subtracting 1), we alternate between even and odd numbers. This pattern is infinite in both directions

on the number line. This alternating property is fundamental to many mathematical proofs and helps in understanding sequences and series.

Properties of Even and Odd Numbers

- Even + Even = Even
- Odd + Odd = Even
- Even + Odd = Odd
- Even × Even = Even
- Odd × Odd = Odd
- Even × Odd = Even

Why is this Important?

Understanding and recognizing the key concepts related to even and odd numbers is crucial for math teachers for several reasons:

1. Foundational Knowledge: Even and odd numbers form a fundamental concept in mathematics. A solid grasp of these concepts allows teachers to build a strong foundation for students' mathematical understanding. This knowledge serves as a stepping stone to more advanced topics like number theory, algebra, and computational thinking.
2. Pattern Recognition: The ability to recognize patterns in numbers is a critical mathematical skill. Understanding even and odd numbers helps teachers guide students in identifying and creating number patterns, which is essential for developing mathematical thinking and problem-solving skills.
3. Algebraic Thinking: The properties of even and odd numbers (such as even + even = even, odd + odd = even) introduce students to basic algebraic thinking. Teachers who understand these concepts can help students transition more smoothly into formal algebra.
4. Error Identification: Recognizing these key concepts helps teachers quickly identify and correct common student misconceptions. For example, students might initially think that zero is neither even nor odd, or struggle with identifying large even or odd numbers.
5. Real-world Applications: Understanding even and odd numbers allows teachers to provide relevant real-world examples, making math more engaging and applicable to students' daily lives. This could include concepts like pairing objects, alternating patterns, or even basic principles of computer science.
6. Cross-curricular Connections: Even and odd numbers have applications beyond math, including in science (e.g., electron pairing in chemistry) and computer science (e.g., parity checks in data transmission). Teachers who understand these concepts can make meaningful cross-curricular connections.
7. Differentiated Instruction: A deep understanding of these concepts enables teachers to present the material in multiple ways, catering to different learning styles and levels. They can move flexibly between concrete examples, visual representations, and abstract concepts.
8. Assessment Design: Teachers who thoroughly understand even and odd numbers can create more effective assessments, ranging from basic identification tasks to complex problem-solving questions that apply these concepts.
9. Mathematical Communication: Proficiency with these concepts allows teachers to explain ideas clearly and precisely, using correct mathematical language. This models good mathematical communication for students.
10. Curriculum Development: Understanding the importance and applications of even and odd numbers helps teachers in curriculum planning, ensuring that adequate time and resources are allocated to this fundamental topic.

Try It! Even or Odd Number

Instructions:

1. Think of any whole number you want to check.

2. Before using the widget, try to determine if it's even or odd on your own.
3. Remember: Even numbers are divisible by 2 with no remainder, odd numbers aren't.
4. Once you've made your guess, enter the number in the box below and click "Check".
5. Compare your answer with the widget's result.
6. Try several numbers to reinforce your understanding of even and odd numbers.

Check

By mastering these key concepts, math teachers can provide a richer, more comprehensive mathematical education that goes beyond rote learning to foster true mathematical understanding and appreciation.

Additional Considerations for Elementary School Teachers

As an elementary school teacher, there are several additional aspects of even and odd numbers that are important to understand and incorporate into your teaching:

1. Concrete Representations: Young learners often grasp concepts better with tangible examples. Use manipulatives like counters, blocks, or even students themselves to demonstrate evenness and oddness. For instance, have students pair up – if everyone has a partner, the number is even; if one is left out, it's odd.
2. Visual Patterns: Introduce visual representations of even and odd numbers. For example, even numbers can always be arranged in equal rows or pairs, while odd numbers will always have one left over.
3. Skip Counting: Connect even and odd numbers to skip counting. Counting by 2s starting from 0 gives even numbers, while starting from 1 gives odd numbers. This helps reinforce number sense and mental math skills.
4. Number Lines: Use number lines to show the alternating pattern of even and odd numbers. This visual aid can help students predict whether a number is even or odd based on its position.
5. Place Value Understanding: Help students understand that the ones digit determines whether a number is even or odd, regardless of its size. This is particularly important when dealing with larger numbers.
6. Everyday Examples: Incorporate examples from daily life, such as pairs of shoes, bicycle wheels (even), or tripods (odd). This helps make the concept more relatable and memorable.
7. Game-Based Learning: Utilize games that reinforce even and odd concepts. For example, students can play "Even or Odd" with dice rolls or card draws.
8. Odd and Even Sums/Differences: Introduce simple rules about adding or subtracting even and odd numbers. For instance, even + even = even, odd + odd = even, etc. This lays groundwork for algebraic thinking.
9. Connection to Multiplication: As students progress, show how multiplying by an even number always results in an even product, while multiplying by an odd number maintains the even/odd quality of the other factor.
10. Digital Root Connection: Introduce the concept of digital roots (sum of digits) and how it relates to evenness and oddness. This can be a fun way to check large numbers quickly.
11. Cultural Mathematics: Explore how different cultures have used even and odd numbers in their traditions or number systems. This can foster cultural awareness and show the universality of mathematical concepts.
12. Estimating and Rounding: Discuss how rounding to the nearest ten always results in an even number. This connects even/odd concepts with estimation skills.
13. Problem-Solving Strategies: Use even/odd properties in simple problem-solving scenarios. For example, "If you have an odd number of candies, can you share them equally among two friends without cutting any?"
14. Connections to Fractions: As students advance, show how even numbers are divisible by 2, connecting to the concept of halves in fractions.
15. Technology Integration: Utilize educational technology and apps that focus on even and odd numbers to provide interactive learning experiences.

By incorporating these additional aspects into your teaching, you can provide a rich, multi-faceted understanding of even and odd numbers that will serve as a strong foundation for future mathematical learning.

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4.5: Factors and GCF

Factors are the fundamental building blocks of numbers. They can be likened to the ingredients in a recipe, each contributing to the whole. A factor is defined as a number that divides evenly into another number, leaving no remainder.

Definition: Factor

Given two integers a and b, a is said to be a factor (or divisor) of b if there exists an integer k such that $b = a \times k$.

In other words, a factor is an integer that divides another integer without leaving a remainder.

Factors can be positive or negative, but for most elementary purposes, we typically consider only positive factors. For a positive integer n, its factors are the positive integers that divide n without a remainder.

For example, the factors of 12 are 1, 2, 3, 4, 6, and 12. Each of these numbers divides into 12 without leaving a remainder.

To illustrate this concept, educators may use relatable examples such as dividing a pizza into equal slices. This tangible demonstration helps students visualize how factors work in practice.

Understanding factors is crucial as they form the basis for many mathematical concepts, including:

1. Prime factorization
2. Greatest Common Factor (GCF)
3. Least Common Multiple (LCM)
4. Divisibility rules
5. Simplification of fractions

Finding Factors

The process of finding factors can be approached systematically:

1. Begin with 1 and the number itself, as these are always factors.
2. Check all numbers between 1 and the number.
3. If division results in a whole number (no remainder), it is a factor.

Visual aids such as number lines or multiplication tables can be valuable tools in helping students recognize patterns in factors. Encouraging students to work in pairs to find factors of various numbers can promote discussion and deeper understanding of the concept.

Example 4.5.1

Find the all of the factors of 24.

1. Start with 1 and 24:
   - 1 is always a factor (24 ÷ 1 = 24, no remainder)
   - 24 is always a factor of itself (24 ÷ 24 = 1, no remainder)
2. Check numbers between 1 and 24:
   - 2: 24 ÷ 2 = 12 (no remainder, so 2 is a factor)
   - 3: 24 ÷ 3 = 8 (no remainder, so 3 is a factor)
   - 4: 24 ÷ 4 = 6 (no remainder, so 4 is a factor)
   - 5: 24 ÷ 5 = 4 remainder 4 (not a factor)
   - 6: 24 ÷ 6 = 4 (no remainder, so 6 is a factor)

- 7: 24 ÷ 7 = 3 remainder 3 (not a factor)
- 8: 24 ÷ 8 = 3 (no remainder, so 8 is a factor)
- 9: 24 ÷ 9 = 2 remainder 6 (not a factor)
- 10: 24 ÷ 10 = 2 remainder 4 (not a factor)
- 11: 24 ÷ 11 = 2 remainder 2 (not a factor)
- 12: 24 ÷ 12 = 2 (no remainder, so 12 is a factor)
- Numbers 13 through 23 will all leave remainders, so we can stop here.

3. Compile the list of factors: The factors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24.

Observations:

1. Pairing: Notice that the factors come in pairs. If a is a factor of 24, then 24 ÷ a is also a factor:
   - 1 and 24
   - 2 and 12
   - 3 and 8
   - 4 and 6
2. Efficiency: We only needed to check up to the square root of 24 (which is approximately 4.9) to find all unique factors. After finding 4 as a factor, we automatically knew 6 was also a factor (24 ÷ 4 = 6).
3. Even number: Since 24 is even, 2 is always a factor. This can be a helpful starting point for even numbers.
4. Divisibility rules: Knowledge of divisibility rules can speed up the process. For example, 24 is divisible by 3 (sum of digits 2+4 = 6, which is divisible by 3) and by 4 (last two digits, 24, form a number divisible by 4).

Visual aids such as factor trees or rectangular arrays can help represent the factors of a number, especially for visual learners. Encouraging students to work in pairs to find factors of various numbers can promote discussion and a deeper understanding of the concept.

Greatest Common Factor (GCF)

Definition: Greatest Common Factor

The **Greatest Common Factor** (**GCF**), also known as **the Greatest Common Divisor** (**GCD**), is the largest positive integer that divides each of the numbers without a remainder.

Formally, for two or more integers $a, b, c, \ldots$, not all zero, the GCF is the largest positive integer d such that d is a factor of each of the integers $a, b, c, \ldots$

Properties of GCF

1. The GCF of any number and zero is the absolute value of the number: $GCF(a, 0) = |a|$
2. The GCF of any two numbers is always positive, even if the numbers are negative.
3. The GCF of two prime numbers is always 1, unless the numbers are the same prime.
4. For any integers a and b: $GCF(a, b) = GCF(|a|, |b|)$

Methods to Find GCF

There are several methods to find the GCF. We'll explore three common approaches:

1. Listing Factors

This method involves listing all factors of each number and identifying the largest common factor.

Example 4.5.2

Find the GCF of 48 and 72

Factors of 48: 1, 2, 3, 4, 6, 8, 12, 16, 24, 48
Factors of 72: 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72

The common factors are: 1, 2, 3, 4, 6, 8, 12, 24 The greatest among these is 24.

Therefore, $GCF(48, 72) = 24$

2. Prime Factorization

This method involves breaking down each number into its prime factors and multiplying the common factors.

Example 4.5.3

Example: Find the GCF of 48 and 72

\(48 = 2^4 * 3 = **2** * **2** * **2** * 2 * **3** \ 72 = 2^3 * 3^2 = **2** * **2** * **2** * **3** * 3\)

Common factors: $2 * 2 * 2 * 3 = 2^3 * 3 = 8 * 3 = 24$

Therefore, $GCF(48, 72) = 24$

Try It! Prime Factorization Widget

Instructions:

1. Choose a number you want to factorize.
2. On a piece of paper, try to find its prime factorization on your own.
3. Write down your steps and final answer.
4. Once you're done, enter your number in the box below and click "Factorize".
5. Compare your work with the steps and results shown by the widget.
6. If your answer matches, great job! If not, try to identify where you went wrong.

Enter a positive integer to see its prime factorization and check your work:

Factorize

Applications of GCF

1. Simplifying Fractions: To simplify a fraction, divide both the numerator and denominator by their GCF. Example: Simplify 48/72 GCF(48, 72) = 24 48/72 = (48 ÷ 24) / (72 ÷ 24) = 2/3
2. Algebraic Simplification: GCF is used to factor algebraic expressions. Example: Factor 24x^2 + 36x GCF = 12x 24x^2 + 36x = 12x(2x + 3)
3. Problem Solving: GCF is useful in various real-world scenarios. Example: A carpenter has planks of wood 48 inches and 72 inches long. What is the longest piece that can be cut from each plank with no waste? Solution: GCF(48, 72) = 24 inches

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4.6: Euclidean Algorithm

The Euclidean Algorithm: An Ancient Method for Finding the Greatest Common Factor

Another way to calculate the Greatest Common Factor (GCF) is the Euclidean Algorithm. It is an efficient method, especially for larger numbers. The algorithm is based on a simple yet powerful principle: the GCF of two numbers is the same as the GCF of the smaller number and the remainder of the larger number divided by the smaller number.

This elegant method has a rich history that stretches back over two millennia. The algorithm is named after the ancient Greek mathematician Euclid, who first described it in his seminal work "Elements" around 300 BCE. This makes the Euclidean Algorithm one of the oldest numerical algorithms still in use today, a testament to the enduring power of mathematical thinking across the ages.

Euclid of Alexandria, often referred to as the "Father of Geometry," was a mathematician who lived in Egypt during the reign of Ptolemy I. While little is known about his life, his works have had a profound impact on mathematics. "Elements," his most famous treatise, is a comprehensive compilation of mathematical knowledge of his time, covering topics from plane geometry and number theory to irrational numbers.

The inclusion of the GCF algorithm in this ancient text highlights its fundamental importance in mathematics. The fact that we still use this algorithm today, over 2300 years later, in fields ranging from basic arithmetic to computer science and cryptography, showcases the timeless nature of mathematical discoveries.

As students apply this algorithm, they are not just solving a mathematical problem; they are participating in a tradition of logical thinking that spans civilizations and centuries. The Euclidean Algorithm serves as a beautiful example of how a simple idea, when properly formulated, can have incredible longevity and wide-ranging applications.

Steps of the Euclidean Algorithm:

1. Divide the larger number by the smaller number.
2. Take the remainder from step 1 and divide the smaller number by this remainder.
3. Repeat step 2, each time dividing the previous divisor by the previous remainder.
4. Continue this process until the remainder is zero.
5. The last non-zero remainder is the GCF.

Example 4.6.1

Find the GCF of 48 and 72 using the Euclidean Algorithm

Step 1: $72 \div 48$
$72 = 1 * \mathbf{48} + \mathbf{24}$ <-- use 42 and 24 for Step 2
Quotient: 1, Remainder: 24

Step 2: $48 \div 24$
$48 = 2 * \mathbf{24} + \mathbf{0}$ <-- stop since the remainder is 0
Quotient: 2, Remainder: 0

Since the remainder is now 0, we stop. The last non-zero remainder was 24. Therefore, $GCF(48, 72) = 24$

Let's try a more complex example to better illustrate the process.

Example 4.6.2

Example: Find the GCF of 270 and 192

Step 1: $270 \div 192$
$270 = 1 * \mathbf{192} + \mathbf{78}$ <-- use these numbers for Step 2
Quotient: 1, Remainder: 78

Step 2: $192 \div 78$
$192 = 2 * \mathbf{78} + \mathbf{36}$ <-- use these numbers for Step 3
Quotient: 2, Remainder: 36

Step 3: $78 \div 36$
$78 = 2 * \mathbf{36} + \mathbf{6}$ <-- use these numbers for Step 3
Quotient: 2, Remainder: 6

Step 4: $36 \div 6$
$36 = 6 * \mathbf{6} + \mathbf{0}$ <-- stop since the remainder is 0
Quotient: 6, Remainder: 0

The process stops here as the remainder is 0. The last non-zero remainder was 6. Therefore, $GCF(270, 192) = 6$.

Try It! Euclidean Algorithm Widget

Instructions:

1. Choose two positive integers for which you want to find the Greatest Common Factor (GCF).
2. On a piece of paper, try to apply the Euclidean Algorithm to find the GCF on your own.
3. Write down each step of your calculation.
4. Once you're done, enter your two numbers in the boxes below and click "Calculate GCF".
5. Compare your work with the steps and results shown by the widget.
6. If your answer matches, excellent work! If not, try to identify where you might have made a mistake.

Enter two positive integers to find their Greatest Common Factor (GCF) using the Euclidean Algorithm and check your work:

Calculate GCF

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4.7: Digital Roots and Divisibility

It's helpful to understand what is meant by the **DIGITAL ROOT** of a number because they are used in divisibility tests, and are also used for checking arithmetic problem. A **DIGITAL ROOT** of a number is *one* of these digits: **0, 1, 2, 3, 4, 5, 6, 7 or 8**.

Definition: The DIGITAL ROOT of a number is the remainder obtained when a number is divided by 9.

Exercise 1

Divide each of the following numbers by 9. Then, write the remainder.

a. 25	h. 8
b. 48	i. 54
c. 53	j. 74
d. 829	k. 481
e. 5402	l. 936
f. 3455	m. 8314
g. 47522	n. 647

Below is another easier way to find the **digital root** of a number.

Step 1: Add the individual digits of the number.

Step 2: If there is more than one digit after adding all the digits, repeat this process until you get a single digit. If the final sum is 9, write 0, because 9 and 0 are equivalent in digital roots (since the remainder is a number smaller than 9). That is why the digital root of a number is only one of these digits: 0, 1, 2, 3, 4, 5, 6, 7 or 8. Those are the only possible remainders that can be obtained when a number is divided by 9! It can't be 9. *The single digit you finally end up with is the **DIGITAL ROOT** of the number.*

Examples: Find the digital roots of the following numbers.

Example 1: 34

34: Add the digits: 3 + 4 = 7

*The digital root of 34 is **7***

Example 2: 321

321: Add the digits: 3 + 2 + 1 = 6

*The digital root of 321 is **6***

Example 3: 58

58: Add the digits: 5 + 8 = 13 Add the digits again: 1 + 3 = 4

*The digital root of 58 is **4***

Example 4: 97

97: Add the digits: 9 + 7 = 16 Add the digits again: 1 + 6 = 7

The digital root of 97 is **7**

Example 5: 72

72: Add the digits: 7 + 2 = 9 In digital roots, 9 is the same as 0

The digital root of 72 is **0**

Example 6: 346,721

346,721: Add the digits, we get 3 + 4 + 6 + 7 + 2 + 1= 23
Add the digits again: 2 + 3 = 5

The digital root of 346,725 is **5**

Note that you **should not write 97 = 16 = 7.** 97 **IS NOT EQUAL** to 7!! The *digital root* of 97 is equal to 7. I use dashes, colons or arrows to record the digital root, so it looks something like this: 346,721 $\rightarrow$ 23 $\rightarrow$ 5. **Don't use equal signs!!!**

Another method for finding digital roots uses the fact that 9 and 0 are equivalent in digital roots. The process of finding digital roots is also called **Casting out Nines**. When you add the digits, you don't have to add the digit 9 or any combination of numbers that add up to 9 (like 2 and 7, or 5 and 4, or 2 and 3 and 4, etc.) – you can "cast out" all 9's. Cross them off; then add the remaining digits together. The following example illustrates how "casting out nines" simplifies the process of finding the digital root of a large number.

Example

Find the digital root of 5,624,398.

Without casting out nines: 5 + 6 + 2 + 4 + 3 + 9 + 8 = 37. Add again: 3 + 7 = 10. Add again: 1 + 0 = **1**.

Casting out Nines: Since 5 and 4 add up to 9, cross them off: $\not{5}, 6, \not{2}\not{4}, \not{3}\not{9}\not{8}$. Since 6 and 3 add up to 9, also cross them off: $\not{5}, \not{6}, \not{2}\not{4}, \not{3}\not{9}\not{8}$. Also cross off the 9: $\not{5}, \not{6}, \not{2}\not{4}, \not{3}\not{9}\not{8}$. The only digits to add are the 2 and 8, which is 10. The digital root of 10 is 1. So **1** is the digital root of 5,624,398 which is the same answer obtained without first casting out nines.

Below, the digital roots for examples 4, 5 and 6 on the previous page is computed again using casting out nines. Note that the digital root remains the same.

Example 4: 97

97: Cross off the 9. Only the 7 remains. The digital root is **7**.

Example 5: 72

72: 7 + 2 = 9, so cross them off. Therefore, the digital root is **0**.

Example 6: 346,721

346,721: Cross off the 3 and 6, and also the 7 and 2. The only digits to add are the 4 and 1. Therefore, the digital root is **5**.

Exercise 2

Find the digital roots of the following numbers, using either method. Remember that 9 is not a digital root. Write zero (0) if the sum is 9 (cast out the 9).

a. 25	h. 8
b. 48	i. 54
c. 53	j. 74
d. 829	k. 481
e. 5402	l. 936
f. 3455	m. 8314
g. 47522	n. 647

Exercise 3

Did you obtain the same answers for exercises 1 and 2?

Note: This method of finding the digital root only has to do with the number 9. You can't find the remainder of a number when dividing by 7 by adding the digits.

Using Digital Roots is to Check Addition and Multiplication Problems.

To check an addition problem, add the digital roots of the addends. Then, check to see if the digital root of that sum is the same as the digital root of the actual sum of the addends. This works whether there are only two addends, or several addends.

Below are some examples of how to check addition. The actual addition is shown to the left. Arrows are used to show the digital roots of the addends and sum. To check, the digital roots of the addends are added together, and then the digital root of that sum is computed. Compare it to the digital root of the actual sum. If they are equal, put a check to indicate the answer is ***probably*** correct. *NOTE: There is a **slight** chance that the digital roots match, but the answer is still not correct due to some other mistake, like transposing digits. For instance, in example 1 below, it's possible someone might write down 1153 for the answer. The digital roots in the check would match. The possibility of this happening is slight, so we generally assume the problem was done correctly if the digital roots check.* On the other hand, if the digital roots do not check, you know it is wrong.

Example 1

$$\begin{array}{r} 723 \rightarrow 3 \\ +412 \rightarrow 7 \\ \hline 1135 \rightarrow 1 \end{array}$$

Ck: 3 + 7 = 10 $\rightarrow$ 1 Correct!

Explanation: To check, add the digital root of the addends (3 + 7 = 10); then find the digital root of 10 (1). Verify that this equals the digital root of the sum, 1135 (1). Since it does, the addition problem was probably done correctly.

Example 2

$$\begin{array}{r} 463 \rightarrow 4 \\ +529 \rightarrow 7 \\ \hline 1630 \rightarrow 1 \end{array}$$

Ck: 4 + 7 = 11 $\rightarrow$ 2 Wrong!

Explanation: To check, add the digital root of the addends (4 + 7 = 11); then find the digital root of 11 (2). This should equal the digital root of the sum, 1630 (1). Since it doesn't , there is a mistake and the addition problem was done incorrectly.

Exercise 4

Someone did the following addition problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Note that the procedure is the same if there are more than two addends. Add the digital roots of all the addends. Show Work!

a. $\begin{array}{r} 4983 \\ +6829 \\ \hline 10802 \end{array}$	b. $\begin{array}{r} 5567 \\ +4987 \\ \hline 10554 \end{array}$	c. $\begin{array}{r} 3467 \\ +2541 \\ \hline 5908 \end{array}$
d. $\begin{array}{r} 8972 \\ 8876 \\ +5873 \\ \hline 23721 \end{array}$	e. $\begin{array}{r} 5873 \\ 3674 \\ +4763 \\ \hline 13260 \end{array}$	f. $\begin{array}{r} 4789 \\ 9835 \\ +5301 \\ \hline 19925 \end{array}$

For a multiplication problem, the check is similar, except to check, you **multiply** the digital roots of the numbers you are multiplying. To check addition, you ADD the digital roots of the addends and check against the digital root of the sum. To check multiplication, you MULTIPLY the digital root of the numbers being multiplied and check against the digital root of the product. Here are some examples that provide an answer someone might have written down after doing the multiplication problem on another piece of paper.

Example 1

$$\begin{array}{r} 85 \rightarrow 13 \rightarrow 4 \\ \times 63 \rightarrow 0 \\ \hline 5355 \rightarrow 18 \rightarrow 0 \end{array}$$

Ck: $4 \times 0 = 0$ Correct!

Explanation: To check, multiply the digital roots of the numbers you are multiplying $(4 \times 0 = 0)$; then find the digital root of 0 (0). Verify that it equals the digital root of the product, 5355 (0). Since it does, the multiplication problem was probably done correctly.

Example 2

$$\begin{array}{r} 43 \rightarrow 7 \\ \times 52 \rightarrow 7 \\ \hline 2136 \rightarrow 3 \end{array}$$

Ck: $7 \times 7 = 49 \rightarrow 4$ Incorrect!

Explanation: To check, multiply the digital roots of the numbers you are multiplying ($7 \times 7 = 49$); then find the digital root of 49 (4). This should equal the digital root of the product, 2136 (3). Since it doesn't , there is a mistake. Therefore, the multiplication problem was done incorrectly.

Exercise 5

Someone did the following multiplication problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Show Work!

a. $\begin{array}{r} 74 \\ \times 53 \\ \hline 3922 \end{array}$	b. $\begin{array}{r} 29 \\ \times 36 \\ \hline 944 \end{array}$	c. $\begin{array}{r} 68 \\ \times 28 \\ \hline 1904 \end{array}$
d. $\begin{array}{r} 65 \\ \times 79 \\ \hline 5125 \end{array}$	e. $\begin{array}{r} 81 \\ \times 55 \\ \hline 4455 \end{array}$	f. $\begin{array}{r} 48 \\ \times 52 \\ \hline 2596 \end{array}$
g. $\begin{array}{r} 569 \\ \times 61 \\ \hline 34709 \end{array}$	h. $\begin{array}{r} 653 \\ \times 524 \\ \hline 242172 \end{array}$	i. $\begin{array}{r} 2333 \\ \times 1203 \\ \hline 2806599 \end{array}$

Exercise 6

Add or multiply the following as indicated, then use digital roots to check the answer to each problem. Show Work!

a. $\begin{array}{r} 7362 \\ +5732 \end{array}$	b. $\begin{array}{r} 8308 \\ +956 \end{array}$
d. $\begin{array}{r} 6784 \\ +6835 \end{array}$	e. $\begin{array}{r} 9994 \\ +8721 \end{array}$
f. $\begin{array}{r} 57 \\ \times 8 \end{array}$	g. $\begin{array}{r} 34 \\ \times 7 \end{array}$
g. $\begin{array}{r} 87 \\ \times 52 \end{array}$	h. $\begin{array}{r} 825 \\ \times 13 \end{array}$

Using Digital Roots is to Check Subtraction Problems.

To check subtraction, you usually add the difference (the answer) to the subtrahend (the number that was subtracted) and see if you get the minuend (the number you subtracted from). In other words, to check that 215 – 134 = 81 was done correctly, simply add 81 and 134, which is 215. To check subtraction using digital roots, simply add the *digital root* of the difference to the *digital root* of the subtrahend and see if you get the *digital root* of the minuend. To check that 215 – 134 = 81 was done correctly, simply add the digital roots 81 and 134: 0 + 8 = 8. Verify that 8 is the digital root of the minuend, 215. It is! Here is another example. Let's say someone did the following subtraction problem: 5462 – 2873 = 2589. To check, add the digital roots of 2589 and 2873, or 6 + 2 to get 8. Since 8 is also the digital root of 5462, the answer is probably correct.

Here are some more examples of how to check subtraction using digital roots.

Example 1

$$\begin{array}{r} 7362 \to 0 \\ -5732 \to 8 \\ \hline 1630 \to 1 \end{array}$$

Ck: 1 + 8 = 9 → 0 $\surd$ Correct!

Explanation: To check, add the digital root of the difference and subtrahend (1 + 8 = 9). The digital root of 9 is 0. Verify that this equals the digital root of the minuend, 7632 (0). Since it does, the subtraction was probably done correctly.

Example 2

$5073 \rightarrow 6$
$\underline{-878} \rightarrow 5$
$4205 \rightarrow 2$

Ck: 2 + 5 = 7 WRONG!

Explanation: To check, add the digital root of the difference and subtrahend (2 + 5 = 7). This should equal the digital root of the minuend, 5073 (6). Since it doesn't, the subtraction was not done correctly.

Exercise 7

Someone did the following subtractions problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Show Work!

a. $\begin{array}{r} 572 \\ -356 \\ \hline 216 \end{array}$	b. $\begin{array}{r} 296 \\ -189 \\ \hline 107 \end{array}$	c. $\begin{array}{r} 501 \\ -327 \\ \hline 284 \end{array}$
d. $\begin{array}{r} 3217 \\ -2391 \\ \hline 926 \end{array}$	e. $\begin{array}{r} 3546 \\ -1138 \\ \hline 2408 \end{array}$	f. $\begin{array}{r} 7502 \\ -4729 \\ \hline 2773 \end{array}$

Exercise 8

Subtract, add or multiply the following as indicated, then use digital roots to check the answer to each problem. Show Work!

a. $\begin{array}{r} 7362 \\ -5732 \end{array}$	b. $\begin{array}{r} 8308 \\ -956 \end{array}$
d. $\begin{array}{r} 6784 \\ -6335 \end{array}$	e. $\begin{array}{r} 9994 \\ -8721 \end{array}$
f. $\begin{array}{r} 557 \\ +348 \end{array}$	g. $\begin{array}{r} 834 \\ +767 \end{array}$
h. $\begin{array}{r} 48 \\ \times 6 \end{array}$	i. $\begin{array}{r} 13 \\ \times 29 \end{array}$

Using Digital Roots is to Check Division Problems.

To check division, you multiply the divisor (the number you are dividing by) by the quotient (the answer), and then add the remainder. It is correct if the answer obtained is equal to the dividend (the number you divided into). In other words, here is how to check this division, $431 \div 34 = 12$r. 23: Multiply 34 by 12 and add 23. Here are the steps: $34 \times 12 + 23 = 408 + 23 = 431$. Since 431 is the dividend, this problem was done correctly. To check division using digital roots, simply multiply the *digital root* of the divisor by the *digital root* of the quotient, and add the *digital root* of the remainder. If the digital root of that number equals the *digital root* of the dividend, it is probably correct.

Note: After the digital roots of the divisor and quotient are multiplied, you can first find the digital root of that product before adding the digital root of the remainder.

Here is how to check the division problem we did above by using digital roots. Multiply the digital root of the divisor, 34, by the

digital root of the quotient, 12: $7 \times 3 = 21$. Then add the digital root of the remainder, 23: 21 + 5 = 26. Determine the digital root of that number, 26, which is 8. Check to see if 8 matches the digital root of the dividend, 431. It does. As stated in the note, you could determine the digital root of 21 before adding the digital root of the remainder. In this case, it would have looked like this: $7 \times 3 = 21 \rightarrow 3$. Then add the digital root of the remainder, 23: 3 + 5 = 8. Verify that this equals the digital root of the dividend, 431. It does. Therefore, the division was probably done correctly. More examples follow. The check without digital roots is shown to the right. The digital root check is shown under the division problem.

Example 1

Example 1:

0 ← 45 r. 15 → 6

0 ← 27 | 1230 → 6

Digital Root Check:

0 x 0 = 0; 0 + 6 = 6 Correct!

Explanation: To check, multiply the digital root of the divisor, 27, by the digital root of the quotient, 45: 0 x 0 = 0. Add the digital root of the remainder, 15: 0 + 6 = 6. Since this equals the digital root of the dividend, 1230, the division was probably done correctly.

Check:

27
x 45
135
108
1215
+15
1230

Example 2

Example 2:

1 ← 19 r. 32 → 5

7 ← 34 | 678 → 21 → 3

Digital Root Check:

7 x 1 = 7; 7 + 5 = 12 --> 3 Correct!

Explanation: To check, multiply the digital root of the divisor, 34, by the digital root of the quotient, 19: 7 x 1 = 7. Add the digital root of the remainder, 32: 7 + 5 = 12 --> 3. Since this equals the digital root of the dividend, 678, the division was probably done correctly.

Check:

34
x 19
646
+32
678

Below is one more example for you to study.

Example 3

Example 3:

6 ← 213 r. 271 → 1

8 ← 458 | 97831 → 10 → 1

Digital Root Check:

8 x 6 = 48 --> 12 --> 3; 3 + 1 = 4 WRONG!

Explanation: To check, multiply the digital root of the divisor, 458, by the digital root of the quotient, 213: 8 x 6 = 48 --> 12 --> 3. Add the digital root of the remainder, 271: 3 + 1 = 4. Since 4 does not equal the digital root of the dividend, 97831, the division was done incorrectly.

Check:

458
x 213
97554
+271
97825

Incorrect!

Exercise 9

Someone did the following division problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Show work. You don't need to do the actual check as shown to the right of the last three examples. Show work!

a.

$$73\overline{)5609} \quad 76 \text{ r. } 61$$

$$123\overline{)6}$$

c.

$$26\overline{)5134} \quad 197 \text{ r. } 10$$

$$78\overline{)79}$$

One more note of caution about using digital roots:

In problem 9d, a common mistake some people make is to forget to put the zero in the quotient, 102. Instead, the answer might have been written as 12 r. 32. Using digital roots, the answer would check. So, you should also check the reasonableness of the answer by approximating. For instance, 78×12 is about 80 times 10 or 800, which is not even close to equaling the dividend, 7988. Another time digital roots might fail is when someone transposes the digits of a number. If the answer to a problem was 465, and it was written as 456, using digital roots wouldn't detect the mistake.

Exercise 10

Do the following division problem, and check the answer using digital roots. Show work.

$23972 \div 156$ = _____

Later in this module, we will explore prime numbers, composite numbers, the greatest common factor of two or more numbers, and the least common multiple of two or more numbers. Factoring is the method used to find the prime factorization of a composite number, and it can also be used to find the greatest common factor and least common multiple of a set of numbers. One of the problems in factoring large numbers is that sometimes it isn't clear if it is prime or composite. In other words, it isn't clear whether it has any factors other than 1 or itself. Most of us know that if the last digit of a numeral is even, then 2 will divide into it; or if it ends in 0, 10 will divide into it; or if it ends in 0 or 5, that 5 will divide into it. Sometimes, people even have trouble determining if relatively small numbers are prime. For instance, many people think 91 is prime, but in fact it is not. Knowing some divisibility tests makes the task easier, so we'll soon take some time to discuss divisibility tests for several numbers. First, we need to go over some notation concerning divisibility.

When you see 12/3, this means 12 "**divided by**" 3. The slash that slants to the right is another way to write the division sign, $\div$. 12/3 (or $12 \div 3$) is a division problem, and the answer is 4.

Here is something altogether different. If I say "3 **divides** 12", I am making a *statement*. "3 divides 12" is not a division problem that needs to be done. It is a statement that happens to be true. The symbol used to represent the word "divides" is a vertical line. So, "3 **divides** 12" can be written "3|12". Again, this is a statement, a fact, not a division problem. The way to express "does not divide" is to put a slash through the symbol: $\not|$

So, how do I know "3 ***divides*** 12" is a true statement? The definition of "divides" follows:

Definition: **"a divides b" if there exists a whole number, n, such that an = b.** This is simply saying that **"a divides b" means a is a factor of b!** In shorthand notation, this is written **"a|b if there exists a whole number, n, such that an = b, or a|b means that a is a factor of b."**

Okay, so what exactly does "there exists a whole number, n, such that an = b" mean? Well, it means that the first number times some whole number equals the second number. Or you can think "a divides b" is true if $b \div a$ is a whole number.

Let's get back to why "3 **divides** 12" is a true statement. Hmmm...you can be formal and ask yourself: Is there a whole number n such that 3n = 12? Or, you can simply ask yourself: Is 3 a factor (or divisor) of 12? In either case, the answer is yes, so the statement is true.

Let's work a little more on the difference between a statement using "divides", and an actual division problem.

Examples: Determine if each of the following is a statement or if it is a division problem. If it is a statement, state if it is true or false and back up your answer. If it is a division problem, state the answer to the division problem.

Example 1: 4|20

4|20

Solution

This is a true statement, because 4 is a factor of 20 (since $4 \cdot 5 = 20$)

Example 2: 20/6

20/6

Solution

This is a division problem. The answer is 3 r. 2

Example 3: 15|3

15|3

Solution

This is a false statement because 15 is not a factor of 3

Example 4: 6 divides 20

6 divides 20

Solution

This is a false statement because 6 is not a factor of 20

Example 5: 3 divides 21

3 divides 21

Solution

This is a true statement because 3 is a factor of 21 (since $3 \cdot 7 = 21$)

Exercise 11

Determine if each of the following is a statement or if it is a division problem. If it is a statement, then decide if it is true or false and back up your answer. If it is a division problem, state the answer to the division problem. Study the examples on the previous page if you need help getting started.

a. 35/7
b. 35|7
c. 7|35:
d. 40/7
e. 56|8
f. 7|40:
g. 12 divides 60
h. 80 divided by 30
i. 70 divided by 5

j. 42 divides 3
k. 6 divides 42
l. 80 divided by 10
m. 100/2
n. 4|100
o. 4|90:
p. 25|5

Let's go back to the formal definition of "divides":

Definition: **a|b if there exists a whole number, n, such that an = b.**

We need to use this formal notation in order to do some proofs. Consider the following:

Example 1

Is the following statement true? If a|b and a|c, then a|(b+c).

a|b and a|c means that a is a factor of b and a is also a factor of c. We must determine if that necessarily implies that a is also a factor of the sum, b + c. The first line of strategy is to test it out on a few numbers, and see if you can find a counterexample. If you find a counterexample, the answer is no and you are done. If you can't find a counterexample, maybe it is true. If you think it is true, you must PROVE it by being general and formal.

I would start by picking any number for a, like a = 3, and then choose numbers for b and c for which 3 is a factor, like 15 and 18. Plug them in: "If 3|15 and 3|18, does 3|(15+18)?" Since 3|33, the answer is yes. It looks like this statement might be true.

Exercise 12

Choose different numbers for a, b and c in "If a|b and a|c, then a|(b+c)" to see if the statement seems to be true.

The statement, "If a|b and a|c, then a|(b+c)", happens to be true. You must write a proof to prove that it is always true. Here is one way to write a formal proof:

Example 1

Is the following statement true?: If a|b and a|c, then a|(b+c).

Solution

If a|b, then an = b for some whole number, n. If a|c, then am = c for some whole number, m. Using these substitutions for b and c, we get that a|(b + c) is true if a|(an + am) which is true if a is a factor of an + am. Factor: an + am = a(n + m). This clearly shows that a is indeed a factor of an + am. Therefore, if a|b and a|c, then a|(b+c).

Exercise 13

Write a formal proof to show that the following is true: If x|y and x|z, then x|(y + z).

Example 2

Is the following statement true? If a|(b+c), then a|b and a|c.

a|(b + c) means that a is a factor of the sum, b + c. The question asks if that necessarily implies that a is a factor of b and also a factor of c. The first line of strategy is to test it out on a few numbers, and see if you can find a counterexample, or if it looks like it is true. If you find a counterexample, the answer is no and you are done. If you think it is true, you must PROVE it by being general and formal.

Start by picking any number for a, like a = 3 and picking a number for the sum b + c, for which 3 is a factor, like 15 or 18, etc. Let b + c = 15. Note that there are many combinations of numbers that add up to 15: 1 + 14, 2 + 13, 3 + 12, 7 + 8, etc. You are being asked if 3 divides into the sum of any two whole numbers, will it necessarily divide into each individual addend as well? For instance, if you broke 15 up as the sum of 9 and 6, this would be the statement: "If 3|(9 + 6), then 3|9 and 3|6." Using these numbers, it is true and you haven't found a counterexample. Try splitting up 15 another way, perhaps as the sum of 10 and 5. Then, the statement becomes: "If 3|(10 + 5), then 3|10 and 3|5." The answer is no, and therefore these numbers may be used as a counterexample, which proves the statement is false. Remember: In order for a statement to be true, it must be for all values of a and b. On the other hand, only one counterexample showing it is not true is sufficient to prove a statement is false.

Exercise 14

Consider the statement: "If a|(b+c), then a|b and a|c." Provide an example using numbers other than a = 3, b = 9 and c = 6 that makes it look like this statement might be true.

Exercise 15

Consider the statement: "If a|(b+c), then a|b and a|c." Provide a counterexample using numbers other than a = 3, b = 10 and c = 5 to show this statement is false.

Exercise 16

Consider the statement: "If a|c and b|c, then (a + b)|c." Provide a counterexample to show this statement is false.

Exercise 17

Consider the statement: "If (a + b)|c, then a|c and b|c." Provide a counterexample to show this statement is false.

Exercise 18

Prove that the following statement is true: "If a|b and a|c, then a|(bc)"

Example

Prove that the following statement is true: "If a|b and a|(b + c), then a|c"

Solution

If a|b, then am = b for some whole number, m. If a|(b + c), then an = b + c for some whole number, n. Keep in mind that since b and c are positive, then (b + c) > b, which means n > m. We are trying to prove that a is a factor of c. Since am = b, we can substitute am for b into the equation an = b + c, which means an = am + c. Solving for c, this is equivalent to an – am = c. So

if a is a factor of an – am, then a is a factor of c. Factor: an – am = a(n – m). This clearly shows a is a factor of an – am, which means a is a factor of c. Therefore, the statement "If a|b and a|(b + c), then a|c" is true. (Note: n – m must be a whole number since n and m are whole numbers and n > m.)

Exercise 19

Prove that the following statement is true: "If c|a and c|(a + b), then c|b"

Okay, finally on to the divisibility tests. We will use our new notation for divides (|).

All counting numbers can be considered factors of zero. In other words, for all counting numbers, m, m|0 is always true since there is always some number times m that equals zero, namely zero itself.

Assume n is a positive whole number. The following are divisibility tests to determine what numbers divide into n, or what numbers are factors of n.

Divisibility Test for 2: 2|n if the last digit of n is even (0, 2, 4, 6, or 8)

Examples

Use the divisibility test for 2 to determine if the following is true or false

1. 2|97

This is false because the last digit (7) is not even

2. 2|356

This is true because the last digit (6) is even

Divisibility Test for 4: (think of 4 as 2^2)

4|n if 4|(the number represented by the last two digits of n)

Examples

Use the divisibility test for 4 to determine if the following is true or false.

1. 4|527

This is false because 4 does not divide 27, (since 4 is not a factor of 27).

2. 4|25,356

This is true because 4|56.

3. 4|624

This is true because 4|24.

Divisibility Test for 8: (think of 8 as 2^3)

8|n if 8|(the number represented by the last three digits of n)

Examples

Use the divisibility test for 8 to determine if the following is true or false.

1. 8|42,527

This is false because 8 does not divide 527

2. 8|25,336

This is true because 8|336

3. 8|7,624

This is true because 8 does divide 624, (since 8 is a factor of 624)

It should be clear to you that if a number is not divisible by 2, it is not divisible by 4 or 8; and if it is not divisible by 4, it is not divisible by 8. Conversely, if a number is divisible by 8, then it is divisible by both 2 and 4; and if it is divisible by 4, it is divisible by 2.

Exercise 20

Use the divisibility tests for 2, 4 and 8 to determine if the following is true or false. Support your answer with a reason using the appropriate divisibility test.

a. 2|9,712 _____
b. 2|5,643 _____
c. 4|5,690 _____
d. 4|63,868 _____
e. 4|854,100 _____
f. 8|12,345,248 _____
g. 8| 54,094,422 _____

Divisibility Test for 5: 5|n if the last digit of n is 0 or 5

Examples

Use the divisibility test for 5 to determine if the following is true or false.

1. 5|527

This is false because 7 is the last digit of 527

2. 5|25,335

This is true because 5 is the last digit of 25,335

3. 5|7,620

This is true because 0 is the last digit of 7,620

Divisibility Test for 10: 10|n if the last digit of n is 0

Examples

Use the divisibility test for 10 to determine if the following is true or false.

1. 10|527

This is false because 7 is the last digit of 527

2. 10|25,335

This is false because 5 is the last digit of 25,335

3. 10|7,620

This is true because 0 is the last digit of 7,620

Exercise 21

Use the divisibility tests for 5 and 10 to determine if the following is true or false. Support your answer with a reason using the appropriate divisibility test.

a. 5|9,750 _____
b. 5|5,645 _____
c. 5|5,696 _____
d. 10|63,860 _____
e. 10|854,105 _____

Divisibility Test for 3: 3|n if 3|(the digital root of n)

Note: This is equivalent to saying 3|n if the digital root of n is 0, 3 or 6

Examples

Use the divisibility test for 3 to determine if the following is true or false.

1. 3|97

This is false because 3 does not divide 7, which is the digital root of 97

2. 3|356

This is false because 3 does not divide 5, which is the digital root of 356

3. 3|738

This is true because 3|0, where 0 is the digital root of 738

Exercise 22

Use the divisibility test for 3 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 3. Show work.

a. 3|9,750 _____
b. 3|5,645 _____
c. 3|5,696 _____
d. 3|63,860 _____
e. 3|854,115 _____

Divisibility Test for 9: 9|n if 9|(the digital root of n)

Note: **This is equivalent to saying that the digital root of n must equal zero.** Take a moment to think about this divisibility test for 9. In exercise 16, we discovered that the digital root of a number is the same as the remainder you obtain when you divide a number by 9. In order for a number to be divisible by 9, the remainder would be zero, which is exactly what the digital root would be.

Examples

Use the divisibility test for 9 to determine if the following is true or false.

1. 9|627

This is false because 9 does not divide 6, which is the digital root of 627.

2. 9|25,334

This is false since 9 does not divide 8, which is the digital root of 25,334.

3. 9|7,533

This is true because the digital root of 9 is 0.

Exercise 23

Use the divisibility test for 9 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 9. Show work.

a. 9|9,753 _____
b. 9|5,646 _____
c. 9|5,697 _____
d. 9|63,576 _____
e. 9|854,103 _____

Divisibility Test for 6: 6|n if 2|n AND 3|n. (Think of 6 as $2 \cdot 3$)

Important Note: A number is divisible by 6 only if it passes the divisibility test for **both** 2 and 3. If it doesn't pass one of the tests, then it is not divisible by 6. To show it is divisible by 6, you must show it passes both of the tests.

Examples

Use the divisibility test for 6 to determine if the following is true or false.

1. 6|627

This is false because 2 does not divide 7 (the last digit, which is not even).

2. 6|25,334

This is false because 3 does not divide 25,334, since 3 does not divide the digital root, which is 8.

3. 6|7,620

This is true because 2|7,620 (since the last digit, 0, is even) AND 3|7,620 (since 3|6, where 6 is the digital root of 7,620).

Exercise 24

Use the divisibility test for 6 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 6.

a. 6|9,753
b. 6|5,645
c. 6|5,696
d. 6|63,876
e. 6|854,103

Divisibility Test for 15: 15|n if 5|n AND 3|n. (Think of 15 as $5 \cdot 3$)

Important Note: A number is divisible by 15 only if it passes the divisibility test for both 5 and 3. If it doesn't pass one of the tests, then it is not divisible by 15. To show it is divisible by 15, you must show it passes both of the tests.

Examples

Use the divisibility test for 15 to determine if the following is true or false.

1. 15|623: This is false because 5 does not divide 623 (since the last digit is not 0 or 5).

2. 15|24,335: This is false because 3 does not divide 24,335, since 3 does not divide the digital root, which is 8.

3. 15|7,620: This is true because 5|7,620 (since the last digit is 0) AND 3|7,620 (since 3|6, where 6 is the digital root of 7,620).

Exercise 25

Use the divisibility test for 15 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 15.

a. 15|9,753
b. 15|6,645

c. 15|5,690
d. 15|63,872
e. 15|654,105

Divisibility Test for 7: This test isn't easy to describe. Below are the steps.

Step 1: Cross off the one's digit of the number to get a number with one less place value.

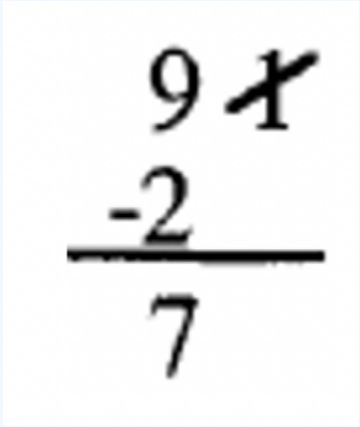
Step 2: Double the one's digit you crossed off and subtract from the new number obtained with the one's digit missing.

Step 3: If 7 divides the number you get after subtracting, it divides the original number. Otherwise, it doesn't. If you aren't sure, repeat the procedure on the new number by going back to step 1.

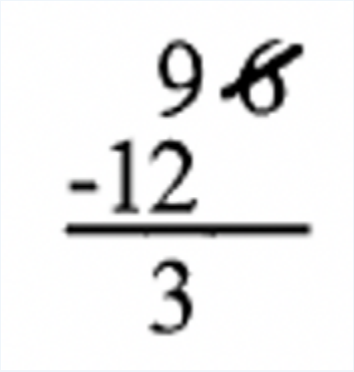
Examples

Use the divisibility test for 7 to determine if the following is true or false. The way you would show the steps using the numbers is shown to the right of the explanation.

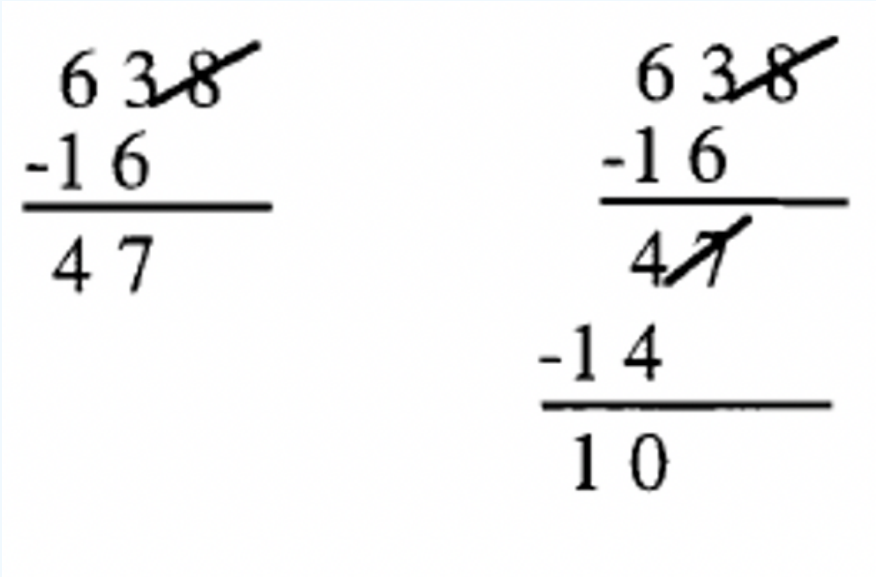
1. 7|91: Cross off the 1, double it (2), and subtract from what is left (9). The answer is 7 (9 – 2 = 7). Since 7|7, 7|91 is true.



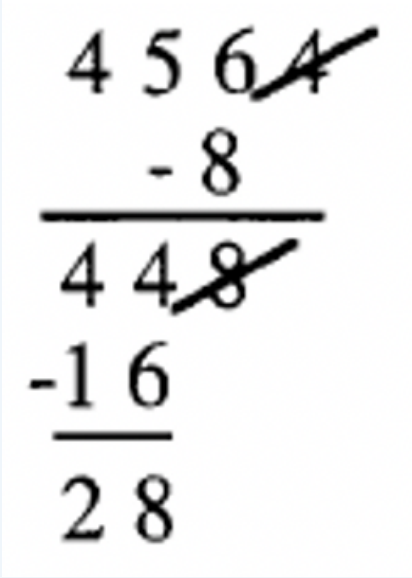
2. 7|96: Cross off the 6, double it (12), and subtract from what is left (9). When you subtract, ignore the sign (just do 12 – 9 = 3). Since 7 does not divide 3, 7|96 is false.



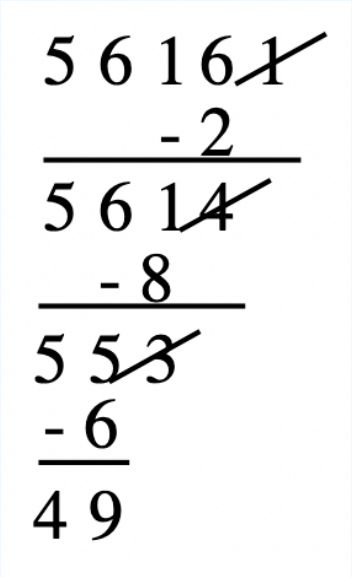
3. 7|638: Cross off the 8, double it (16) and subtract from what is left (63). The answer is 47 (63 – 16 = 47). Since 7 does not divide 47, then 7|638 is false. (If you weren't sure whether or not 7 divided 47, you can take it a step further; this is shown to the right.)



4. 7|4564: Cross off the 4, double it (8) and subtract from what is left (456). The answer is 448 (456 – 8 = 456). Cross off the 8, double it (16) and subtract from what is left (44). The answer is 28 (44 – 16 = 28). Since 7|28, then 7|4564 is true.



5. 7|56161: Cross off the 1, double it (2) and subtract from what is left (5616 – 2 = 5614). Cross off the 4, double it (8) and subtract from what is left (561 – 8 = 553). Cross off the 3, double it (6) and subtract from what is left (55 – 6 = 49). Since 7|49, then 7|56161 is true.



Exercise 26

Use the divisibility test for 7 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 7. Show work.

a. 7|833
b. 7|5,645
c. 7|4,795
d. 7|14,763

Divisibility Test for 11: 11|n if the difference between the sum of the digits in the places that are even powers of 10 and the sum of the digits in the places that are odd powers of 10 is divisible by 11.

This test is more confusing to describe than to do. What you do is add up every other digit. Then, add up the ones you skipped. Then, subtract these two numbers and see if 11 divides this number.

Examples

Use the divisibility test for 11 to determine if the following is true or false.

a. 11|4,365: Add 4 + 6 = 10 Add 3 + 5 = 8 Subtract 10 – 8 = 2 Since 11 doesn't divide 2, then 11 doesn't divide 4365. Therefore, 11|4365 is false.

b. 11|540,879,216: Add 5 + 0 + 7 + 2 + 6 = 20. Add 4 + 8 + 9 + 1 = 22. Subtract 22 – 20 = 2 Since 11 doesn't divide 2, 11|540,879,216 is false.

c. 11|542,879,216: Add 5 + 2 + 7 + 2 + 6 = 22 Add 4 + 8 + 9 + 1 = 22 Subtract 22 – 22 = 0 Since 11|0, then 11|542,879,216 is true.

d. 11|4,052,631: Add 4 + 5 + 6 + 1 = 16 Add 0 + 2 + 3 = 5 Subtract 16 – 5 = 11 Since 11|11, then 11|4,052,631 is true.

Exercise 27

Use the divisibility test for 11 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 11. Show work.

a. 11|9,053
b. 11|63,920,876
c. 11|568,696
d. 11|513,645
e. 11|803,003,808

Listed below are the divisibility tests that you should know. It's important to realize that each of the tests only work for the number specified. In other words, you can't use the divisibility test for 3 to determine is 7 divides a number. The divisibility test for 7 has nothing to do with digital roots!

Divisibility Test for 2: 2|n if the last digit of n is even (0, 2, 4, 6, or 8)

Divisibility Test for 3: 3|n if 3|(the digital root of n); OR 3|n if the digital root of n is 0, 3 or 6

Divisibility Test for 4: 4|n if 4|(the number represented by the last two digits of n)

Divisibility Test for 5: 5|n if the last digit of n is 0 or 5

Divisibility Test for 6: 6|n if 2|n AND 3|n. (Think of 6 as $2 \cdot 3$)

Divisibility Test for 7: The steps of this test are described below.

- Step 1: Cross off the one's digit of the number to get a number with one less place value.
- Step 2: Double the one's digit you crossed off and subtract from the new number without the one's digit.
- Step 3: If 7 divides the number you get after subtracting, it divides the original number. Otherwise, it doesn't. If you aren't sure, repeat the procedure on the new number by going back to step 1.

Divisibility Test for 8: 8|n if 8|(the number represented by the last three digits of n)

Divisibility Test for 9: 9|n if 9|(the digital root of n); OR 9|n if the digital root of n is zero

Divisibility Test for 10: 10|n if the last digit of n is 0

Divisibility Test for 11: 11|n if the difference between the sum of the digits in the places that are even powers of 10 and the sum of the digits in the places that are odd powers of 10 is divisible by 11

Divisibility Test for 15: 15|n if 5|n AND 3|n

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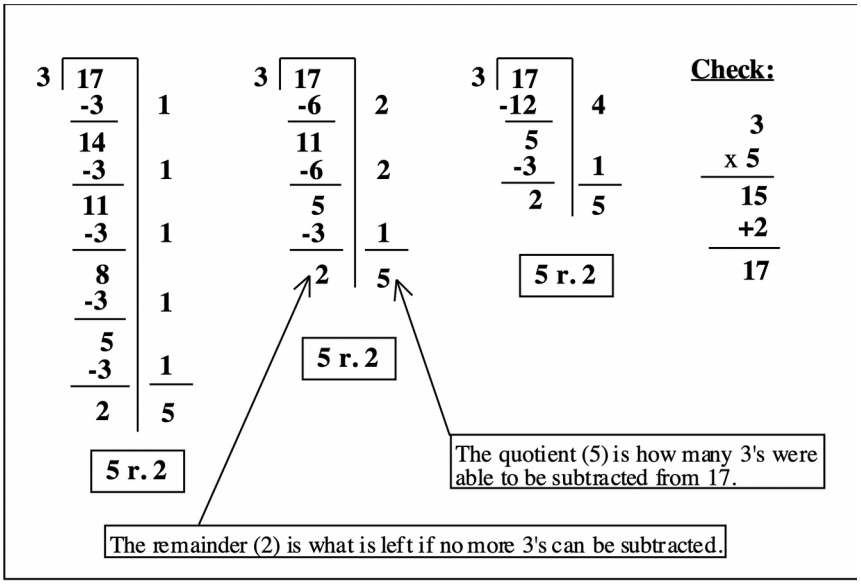
4.8: Division Algorithms

As we've seen, division can be viewed as the opposite of multiplication.

Another way to think about it is as **REPEATED SUBTRACTION**.

Let's do DIVISION the Easy Way!

That's right. Division is really just "repeated subtraction." If you are asked to divide 17 by 3, which is written $17 \div 3$, you have to figure out how many 3's can be subtracted from 17. You probably already know the answer is 5 r. 2, but let's see how this works by counting how many times 3 can be subtracted from 17. Look at the work below. Three different ways are shown. You make a scaffold (looks like a hangman) and keep track of how many 3's you subtract at each step to the right of the scaffold. In the first one, 3 is subtracted one at a time, and a total of five 3's were subtracted. In the second one, first two 3's are subtracted, then 1 more, then 2 more, for a total of five 3's being subtracted. In the third one, four 3's are subtracted, and then 1 more, again for a total of five 3's being subtracted. Each time you subtract, look to see what number is left. If it is 3 or more, you can subtract some more 3's. But when you get a number smaller than 3, you know 3 can't be subtracted anymore, so that is your remainder. For this problem, 2 was the remainder. You need to count how many 3's you subtracted. To the right of the scaffold, you write *how many* 3's you are subtracting as you go along. Simply add those numbers to find out how many 3's you subtracted. The answer to the problem is how many 3's you subtracted, and then you need to also write the remainder. In this case, $17 \div 3 = 5$ r. 2. Always check your answer my multiplying the divisor (3) times the quotient (5) and adding the remainder (2): $3 \times 5 + 2 = 15 + 2 = 17$. It checks!



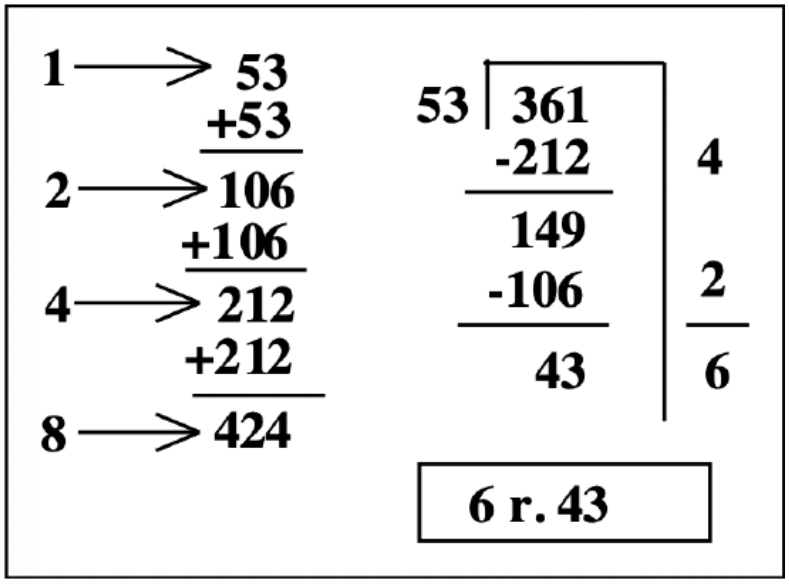
Doing long division with large numbers is often difficult because you have to multiply in your head, and it's easy to make mistakes. But, by using this new method of repeated subtraction, along with a partial multiplication you make up ahead of time using repeated addition, you never have to multiply in your head (or multiply at all), and you never have to guess how many times one number "goes into" another number. This method takes a little more paper, but it makes the problem easier and it's practically foolproof. Lots of people don't like to do long division, because it's hard to always guess the exact right number and sometimes they get a headache from thinking too hard. We're going to eliminate that problem. The important thing is that you know what division means (repeated subtraction), you know how to get the right answer using a method that makes sense to you, and you know how to check your answer to make sure it is right.

Okay, so let's say you had to do this division problem: $361 \div 53$.

YUK! This doesn't look like any fun. Actually, all this is asking is how many 53's can be subtracted from 361. We can subtract 53 one at a time, but it might take a long time. We might decide to subtract more than one 53 at a time.

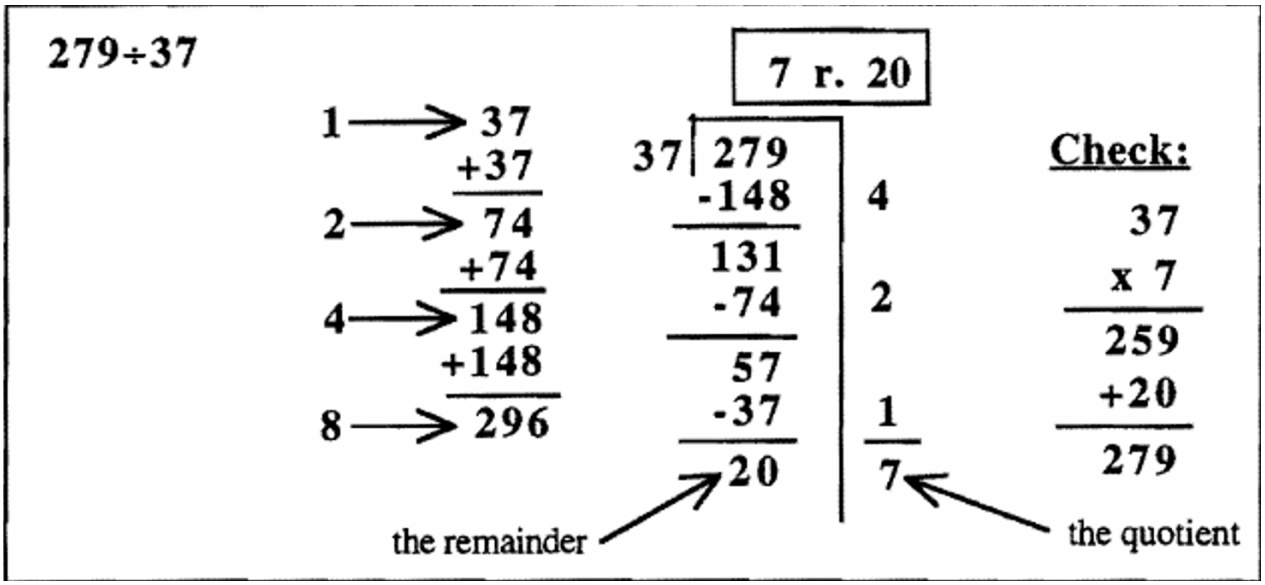
Before we doing the division, let's make a quick partial multiplication table up for 53, by using REPEATED ADDITION! This table is

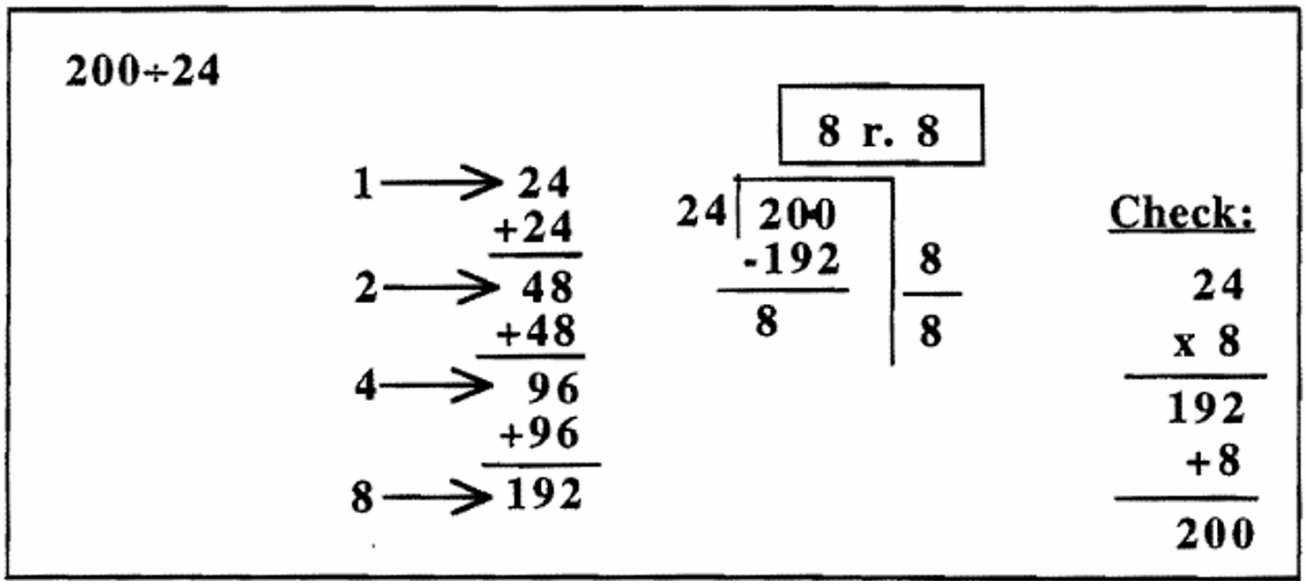
your prep work for division. A simple way to make up a partial multiplication table for 53 without ever multiplying entails doubling numbers. We know that $1 \times 53 = 53$. To figure out 2×53, just add two 53's together. Two 53's added to two more 53's would equal four 53's, or 4×53. Four 53's added to four more 53's would equal eight 53's, or 8×53. Okay, so how do you show this? Start with one 53 and double it to find two 53's. Double that number to find four 53's. Double that number to find eight 53's. Look in the left part of the box shown below. See how there is a 1 pointing to 53 that means 1×53. If you double the 53, (53 + 53), then you now know $2 \times 53 = 106$, so a 2 points to 106. If you double that (106 + 106), you now know $4 \times 53 = 212$, so a 4 points to 212. And if you double this number (212 + 212), then you now know $8 \times 53 = 424$, so an 8 points to 424. Isn't that an easy way to find 8×53 ? But the cool thing is that you also know $1 \times 53, 2 \times 53, 4 \times 53,$ AND 8×53! This is useful to know to do the long division. Instead of guessing how many times 53 "goes into" 361, we note how many 53's can be subtracted from 361 (underestimating is okay). No multiplication needs to be done in your head, either! The box to the right shows how to use repeated subtraction to do the division problem: $361 \div 53$. First, make up a partial multiplication table. Then, write the division problem using a scaffold. Look at the multiplication table and pick the biggest number you can subtract from 361, which is 212. Subtract 212 from 361 and write 4 on the right because you just subtracted 4 53's. The biggest number you can subtract from 149 is 106. If you do that, you've subtracted 2 more 53's, so write a 2 on the right. After subtracting, 43 is left and no more 53's can be subtracted. Therefore, 43 is the remainder. Add the numbers on the right to see how many 53's you subtracted. The answer is 6 r. 43. To check, multiply 53×6 and add 43. The answer is 361.

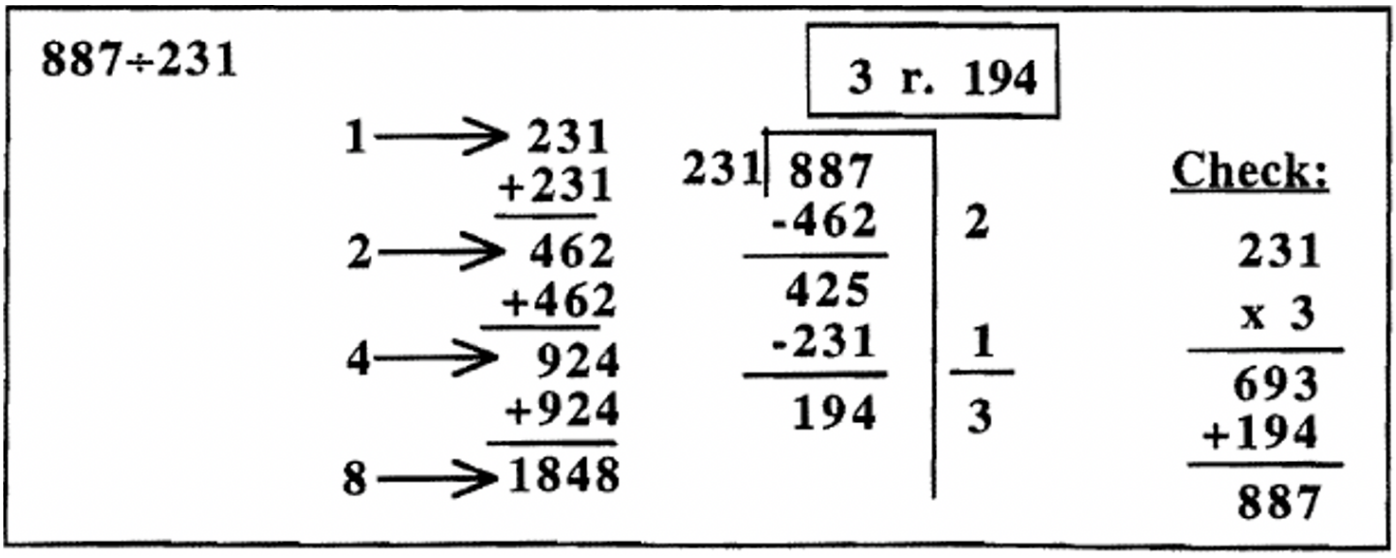


P.S. You can multiply 53 times 6 by adding 106 + 106 + 106, since that is two 53's added together 3 times, which is six 53's!!!

Here are some more examples for you to study. A partial multiplication table is made up first. This prep work which involves a few additions makes the division much easier. Then, the problem is done using repeated subtraction (with help from the partial multiplication table) using a scaffold. After adding up how many times the divisor was subtracted from the dividend (the numbers to the right of the scaffold), the answer (quotient and remainder) is written in a box up above the problem. Finally, the check is shown. Also, check to make sure the remainder is less than the divisor.

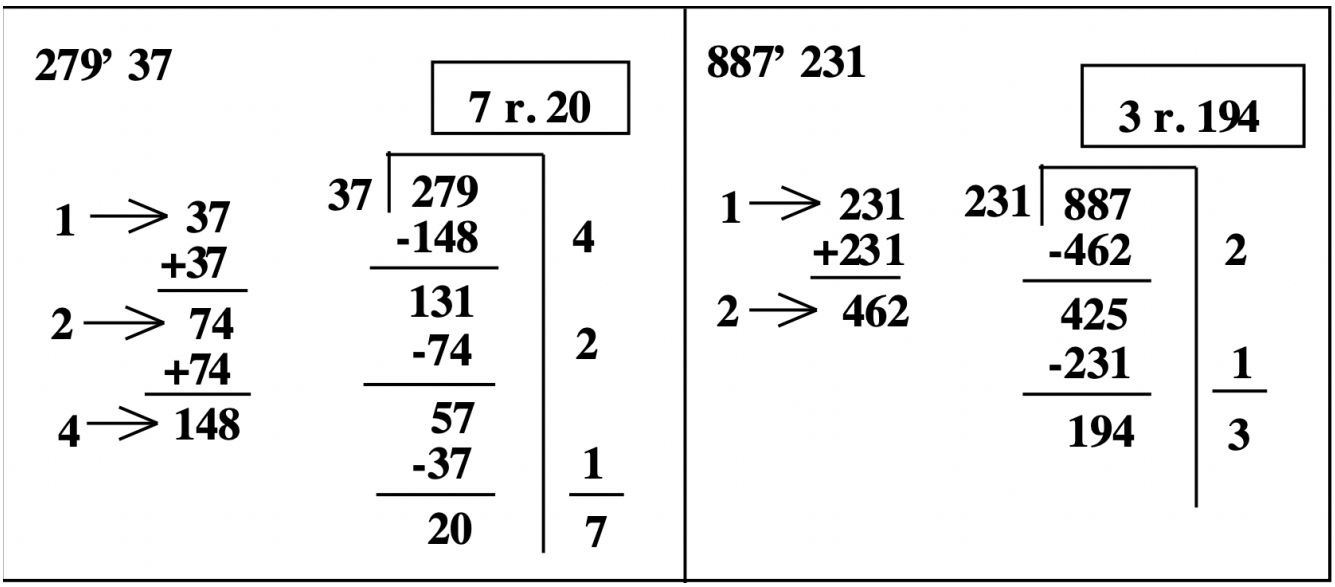






In the examples so far, all of the quotients were less than 10. It's possible to continue doubling the divisor in the partial

multiplication table, so that you can find the divisor times any power of 2 — 1, 2, 4, 8, 16, 32, 64, etc. You would stop doubling if you notice that doubling once more gives a number bigger than the quotient. In fact, in the first example on the previous page, $279 \div 37$, I really didn't have to double 148 to figure out 8×37, since doubling 148 gives me a number bigger than 279. In the third example, $887 \div 231$, I didn't have to double 462 to figure out 4×231, since doubling 462 gives a number bigger than 887. The problems could have been shortened slightly had I noticed that ahead of time. They would look like this (the check is omitted).



Okay, now that you've studied some examples, it's time for you to try using this method. The first four problems are the same exact ones done in the previous examples. If you get stuck, look back at them and copy how I did them on another piece of paper. Then, try again.

Exercise 1

For each division problem, do your prep work by using the doubling method (double as far as necessary) to make a partial multiplication table first. Then, draw a scaffold, and use repeated subtraction to do each division. When you are done, make sure the remainder is less than the divisor. Then, check the answer. All work should be shown in the space provided. After checking, write the answer on the space provided at the beginning of the problem.

a. $361 \div 53$ = _____

Partial multiplication table:	Scaffold:	Check:

b. $279 \div 37$ = _____

Partial multiplication table:	Scaffold:	Check:

c. $200 \div 24$ = _____

Partial multiplication table:	Scaffold:	Check:

d. $887 \div 231$ = _____

Partial multiplication table:	Scaffold:	Check:

e. $415 \div 72$ = _____

Partial multiplication table:	Scaffold:	Check:

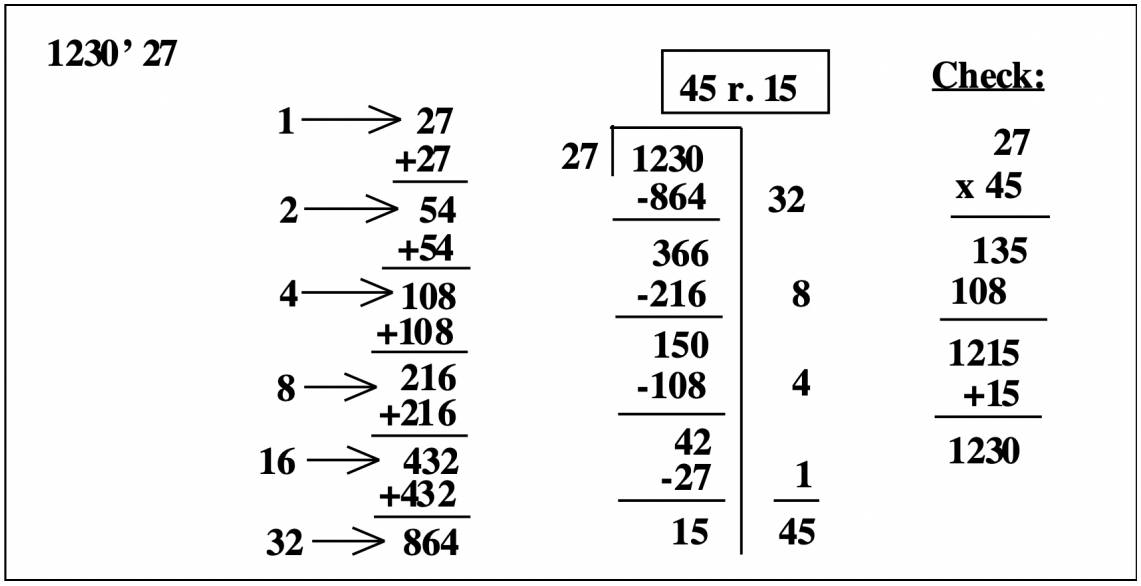
f. $1235 \div 214$ = _____

Partial multiplication table:	Scaffold:	Check:

g. $3128 \div 321$ = _____

Partial multiplication table:	Scaffold:	Check:

The following is an example to show that the doubling can continue in this way. You stop doubling if one more double gives a number bigger than the quotient. Below is $1230 \div 27$.



I thank and credit my son, Jakob, with this method of doubling to figure out more than eight times the divisor. I had always used a method that employed the use of place values and multiples of 10, which we'll get to after doing a few this way. Jakob, nine years old and in fourth grade, hated long division because it didn't make sense to him, and had trouble with the "guessing" involved. Before I was able to show him my method where you use the doubling method to make a partial multiplication table only up to 8 times the divisor, he did all of the problems like the way shown above. It finally made sense to him, and he did them all correctly!!! It opened up a whole new way of thinking for me. Kids and students can be great innovators! It's wonderful when you get a chance to explore and discover on your own, and find your own solution that makes sense. We have to remember that there is nothing sacred about any one algorithm for division, or any other operation.

Okay, it's time for you to try some more problems, using the doubling method, where the quotient is bigger than 10. The first problem is the same exact one done above. If you get stuck, look back at it and copy how I did it on another piece of paper. Then,

try again.

Exercise 2

For each division problem (starting on the next page), use the doubling method (double as far as necessary) to make a partial multiplication table. Then, draw a scaffold, and use repeated subtraction to do each division. Then, check the answer, and make sure the remainder is less than the dividend. All work should be shown. After checking, write the answer on the space provided at the beginning of the problem.

$$\begin{array}{r} 27 \\ \times 45 \\ \hline 1215 \\ +15 \\ \hline 1230 \end{array}$$

Check:

A calculator may be used to check the answer. For instance, to check that 45 r. 15 is the correct answer to $1230 \div 27$, you need to multiply 27 by 45 and add 15. The answer should be 1230. You can use the calculator to do the multiplication and addition, but write down the steps.. It would look the way it is shown to the right.

a. $1230 \div 27$ = _____

Partial multiplication table:	Scaffold:	Check:

b. $603 \div 48$ = _____

Partial multiplication table:	Scaffold:	Check:

c. $1346 \div 41$ = _____

Partial multiplication table:	Scaffold:	Check:

d. $3512 \div 43$ = _____

Partial multiplication table:	Scaffold:	Check:

You may have noticed the partial multiplication table could get long if you have to keep doubling for quite awhile. Also, it's not as easy to add the numbers up to the right of the scaffold. Another approach is to take advantage of the fact that multiplying a whole number by 10 or 100, etc., simply adds zeroes to the end of the number. For instance, if you know 53 times 2 is 106, then 53 times 20 is 1060, and 53 times 200 is 10600. We'll use this fact to divide using repeated subtraction when the quotient is more than 10.

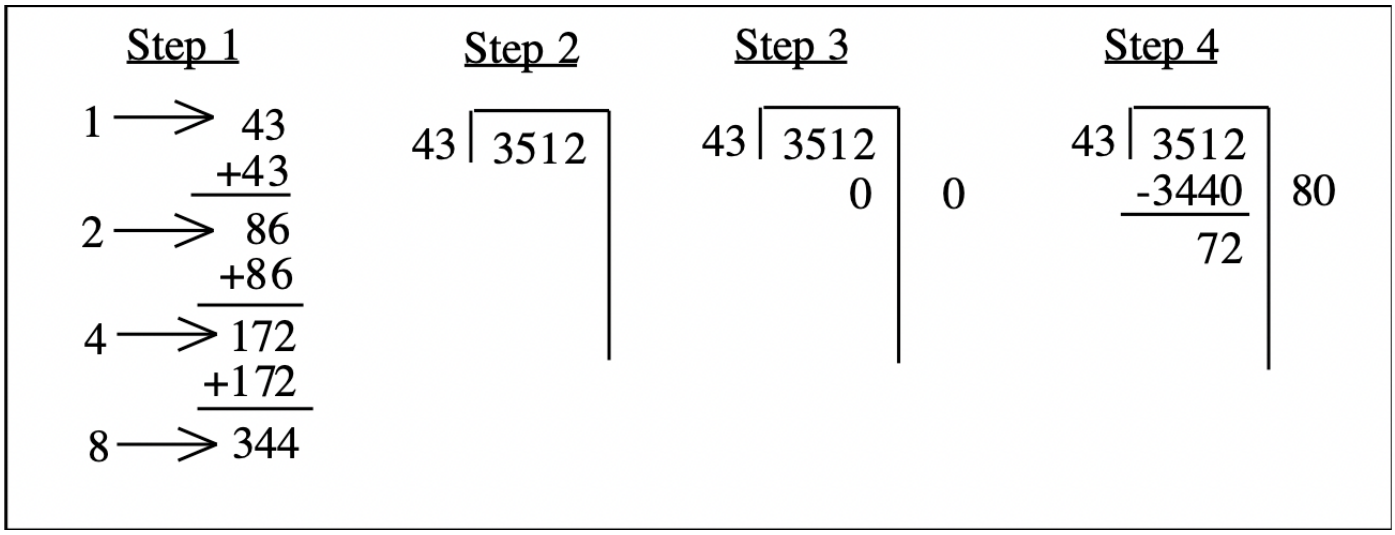
Look back at the last problem you did in Exercise 2: $3512 \div 43$. You had to double until you figured out what 64 times 43 was. In the box at the bottom of this page is a different way of doing the same problem. The explanation and steps are on this and the next page.

Step 1: Use doubling to make up a partial multiplication table up to 8 times 43.

Step 2: Draw the scaffold.

Step 3: Scan 3512 starting with the left-most digit until you get a number greater or equal to 43. Try 3, then 35, then 351. Note how many more digits are after 351. In this case there is only one (the 2). This means you can take a multiple of ten times 43 away at a time. (In the old standard algorithm, when you guess and put the number above the 1 in 3512, you are putting it in the tens place value.) So we put a zero under the extra numbers in the dividend (the 2) and the same number of zeroes to the right of the scaffold. Leave a space for a digit before the 0 on the right of the scaffold.

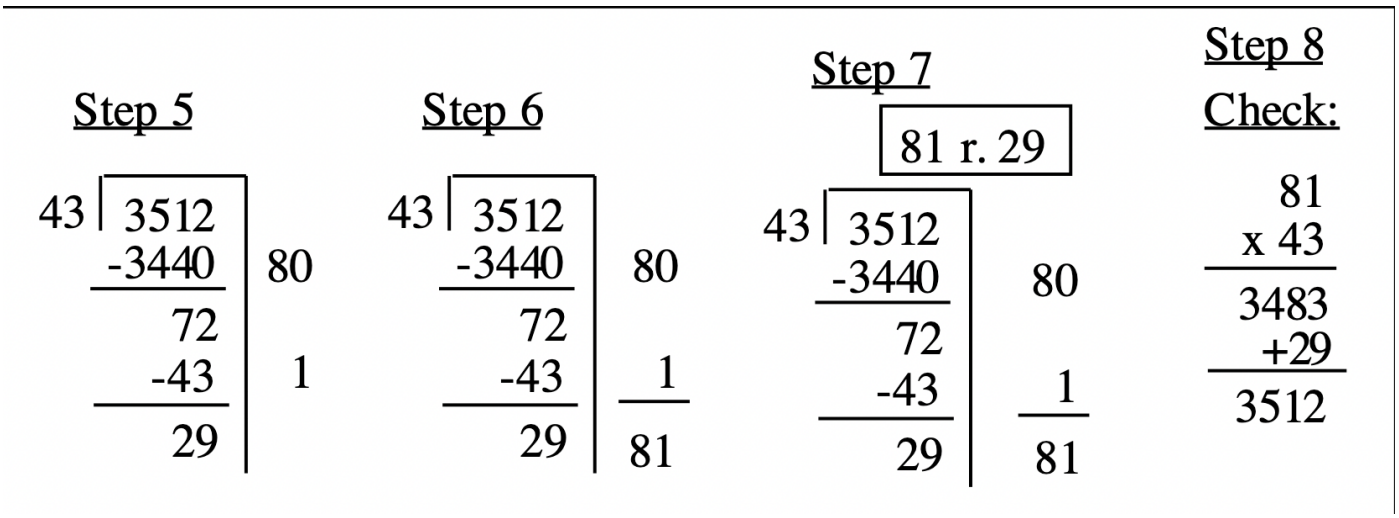
Step 4: Now, look at the partial multiplication table to determine the largest number you can subtract from 351. It is 344, which is 8 times 43. Note that if 8 times 43 is 344, then 80 times 43 is 3440. You've already put the extra zero under the 2, so just put the 344 in front of the 0 that you already put down in step 3. Note that 3440 is 80 times 43, and you need to keep track of that to the right of the scaffold. You've already written the zero down in step 3, so simply put an 8 in front of that 0. You've just subtracted eighty 43's from the dividend, since 80 times 43 is 3440. Subtract 3440 from 3512 to get 72.



Step 5: Now we have to see how many more 43's can be subtracted from 72. There are no extra zeroes to tack on this time, as in step 4. We can subtract 1 more 43 from 72, so write a 1 to the right of the scaffold, and subtract 43 from 72 to get 29.

Steps 6, 7 and 8: Since 29 is less than 43, that is the remainder, so add up the numbers on the right of the scaffold to get the quotient.Write the answer as a quotient and remainder in a box up at the top of the problem. Check the answer. You can use a calculator to multiply 81 times 43, which is 3483, and then add 29.

The last four steps are shown below.



If you were to show your work, this is what the whole problem would look like:

3512’ 43

Partial multiplication table

1 → 43
+43
2 → 86
+86
4 → 172
+172
8 → 344

Scaffold

81 r. 29

43 | 3512 | 80
-3440
72
-43 | 1
29 | 81

Check

81
x 43
3483
+29
3512

Here's what it would look like doing it the way you did it in exercise 2:

Partial multiplication table

1 → 43
+43
2 → 86
+86
4 → 172
+172
8 → 344
+344
16 → 688
+688
32 → 1376
+1376
64 → 2752

Scaffold

81 r. 29

43 | 3512 | 64
-2752
760 | 16
-688
72
-43 | 1
29 | 81

Check

81
x 43
3483
+29
3512

For both methods, when you add the numbers on the right of the scaffold to get the quotient, be careful to line up the place values! Both methods produce the same result and work equally well. What's important is that the procedure makes sense to you!

Two more examples using multiples of tens (putting in the extra zeroes) is shown below.

9082 ’ 58

Partial multiplication table

1 → 58
+58
2 → 116
+116
4 → 232
+232
8 → 464

Scaffold

156 r. 34

58	9082	
	-5800	100
	3282	
	-2320	40
	962	
	-580	10
	382	
	-232	4
	150	
	-116	2
	34	156

Check

156
x 58
9048
+34
9082

7977 ÷ 26

Partial multiplication table

1 → 26
+26
2 → 52
+52
4 → 104
+104
8 → 208

Scaffold

306 r. 21

26	7977	
	-5200	200
	2777	
	-2600	100
	177	
	-104	4
	73	
	-52	2
	21	306

Check

306
x 26
7956
+21
7977

In the above example, many people (children and adults) miss writing the zero in the tens place for the answer of 306 when they do it the short division way it is usually taught. A very common answer to this problem is 36 r. 21. In the first step above, writing the full number on the side really emphasizes the fact that you aren't just multiplying 26 by a 2 to get 52 you are multiplying 26 by **200** to get 5200. You know the answer to the problem will be more than 200 since you already subtracted 200 26's from 7977. When you have 177 left, you simply note you can only subtract a few more 26's. If one neatly writes the numbers on the right, there are no numbers in the tens place to add up. The confusion about the zero doesn't even come up!

Exercise 3

For each division problem, make a partial multiplication table up to 8 times the divisor. Then, draw a scaffold, and use repeated subtraction to do each division. Use multiples of tens (putting in extra zeroes where necessary) to do these problems. Then, check the answer. All work should be shown in the space provided. After checking, write the answer on the space provided at the beginning of the problem. A calculator may be used to check the answer, but write down the steps.

a. $1230 \div 27$ = _____

Partial multiplication table:	Scaffold:	Check:

b. $603 \div 48$ = _____

Partial multiplication table:	Scaffold:	Check:

c. $1346 \div 41$ = _____

Partial multiplication table:	Scaffold:	Check:

d. $3512 \div 43$ = _____

Partial multiplication table:	Scaffold:	Check:

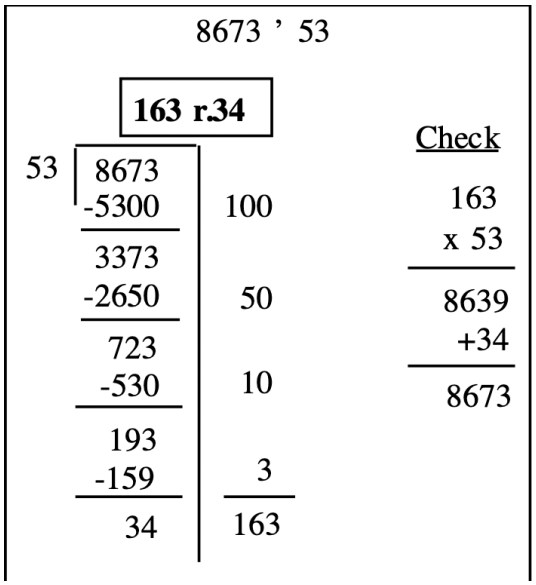
e. $66289 \div 325$ = _____

Partial multiplication table:	Scaffold:	Check:

f. $5222 \div 21$ = _____

Partial multiplication table:	Scaffold:	Check:

Doing the partial multiplication table on the side, in conjunction with the repeated subtraction method, is completely optional. But, if you make up the table first, you do not need to do any multiplying or estimating in your head. Another option is to skip doing the partial multiplication table altogether and still use the repeated subtraction method. In this case, if you aren't sure how many times the divisor can be subtracted, underestimate at each step that you aren't absolutely sure how many of the divisor to subtract. In other words, make sure you don't multiply something by the divisor and come out with a number too big to subtract from the dividend. Underestimating is okay with the repeated subtraction method because you just subtract some more at the next step. For instance, in the previous example, you could have subtracted 200 26's (5200) at the first step and then 100 more 26's at the second step, or you could have subtracted 300 26's at the first step by multiplying 26 by 3 in your head if you were *sure* that 26 times 3 wasn't too big. I like the partial multiplication table because it takes the guesswork out of the problem. In the example in the box below, the division is done using repeated subtraction without making up a table Instead, you estimate how many of the divisor can be subtracted at each step. In this case, the multiplication must be done in your head, or on the side, as you go along.

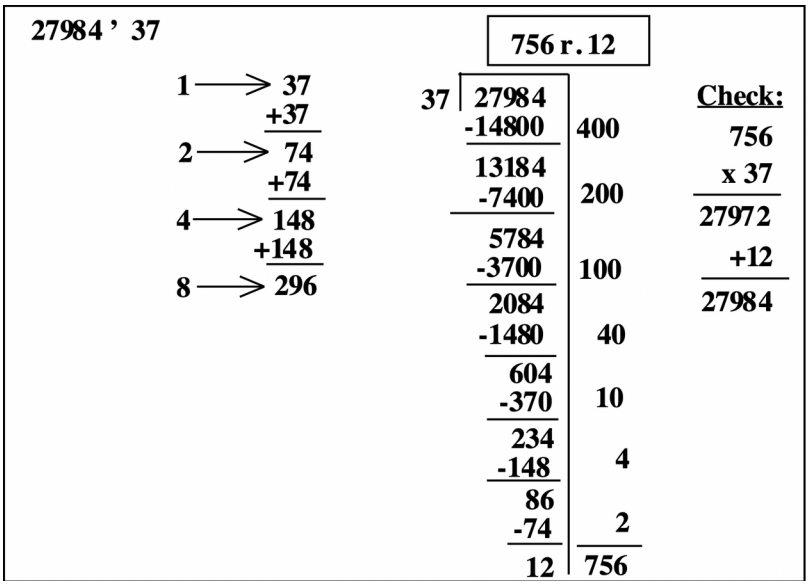


Here is a way to divide $8673 \div 53$ without the usual prep work of making up a partial multiplication table first. This means the multiplication must be done along the way. It's important to note that it is okay to underestimate when using the repeated subtraction method (unlike the traditional division method), because you can always subtract more at the next step. On the other hand, overestimating will cause you to go back and redo the problem. First I note that 53 goes into 86 one time, so I subtract 100 53's, or 5300, from 8673. Next, estimate $337 \div 53$. Since I'm not sure if it is 5 or 6 times, I play it safe and therefore, estimate 5. So, I subtract 50 times 53, or 2650, from 3373. (Note I have to do the multiplication in my head.) Well, I see another 53 can be subtracted from 72, so I subtract 10 times 53, or 530, from 723. Then, I estimate 53 goes into 193 at least 3 times, so I subtract 3 times 53 from 193. Since 34 is less than 53, 34 is the remainder. Add the numbers on the side to get the quotient of 163. Check by multiplying 53×163 and adding the remainder of 34 to get 8673.

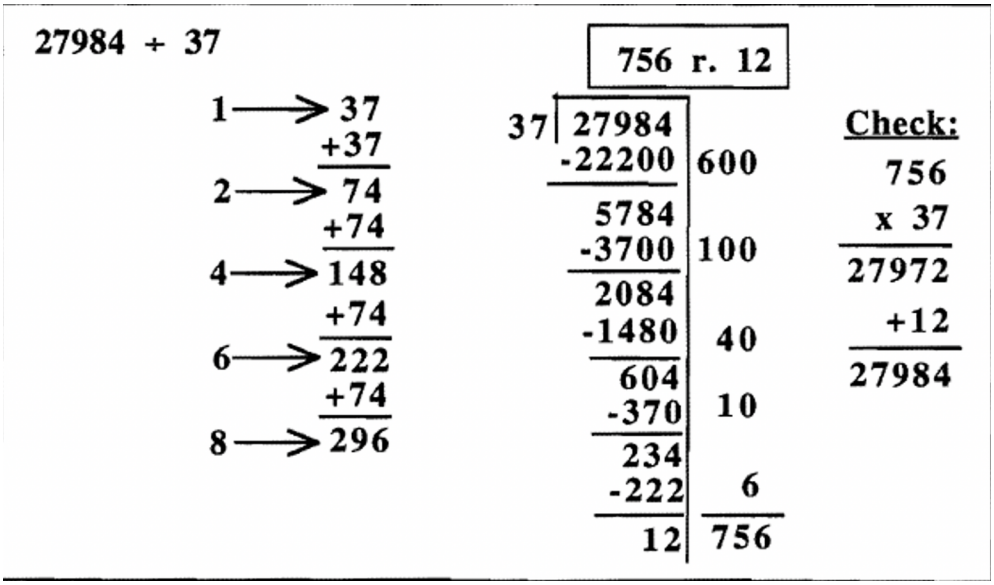
In the traditional division algorithm, there is no room for underestimating or overestimating. You have to go back and erase if you put the wrong digit in a particular place value. If you estimate incorrectly, the multiplication you do ends up being a waste of time and effort, and you have to start over. Also, most people don't think about what or why they are putting digits in a particular place. They don't think about the place values, or that division is really about repeated subtraction. I think the scaffold method, with or without making up a partial multiplication table, promotes a better understanding of division, and is less stressful than the traditional "guess" method where there is absolutely no room for error. The one advantage to the traditional algorithm is that it is a shortcut to the scaffold method, and it uses less space on paper.

In the end, it's really up to the individual doing the division to decide which way to do it.

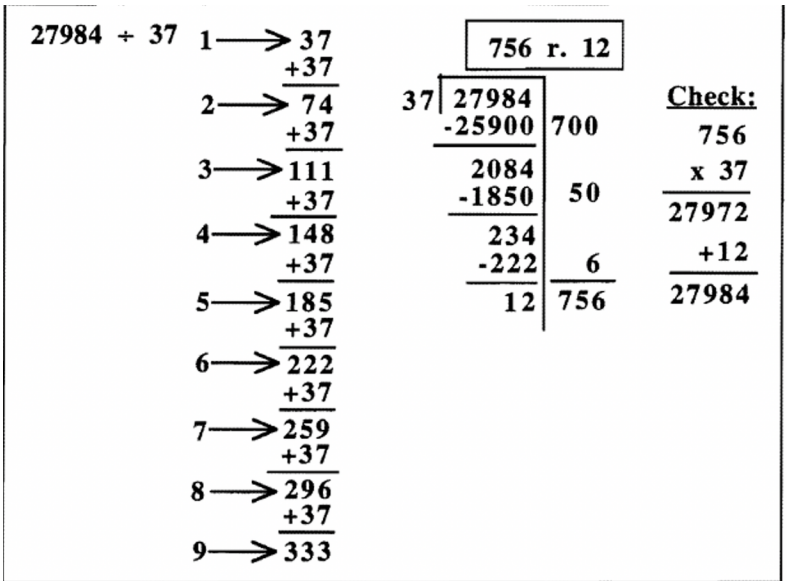
Here is an example of dividing $27984 \div 37$ using the doubling method.



When making up a multiplication table, it doesn't have to be partial. In base ten, doubling to get the divisor times 1, 2, 4 or 8 gives us close enough estimates most of the time. The worst possible case is that 7 is the correct number of times something can be subtracted, as in the above example. In that case, we would first subtract 4, then 2, then 1. That doesn't happen too often, but once in awhile it does. Another strategy is to add one more addition to the partial table so you also know what 6 times the divisor is. To do this, after figuring what 4 times the divisor is, add 2 more, as shown below (add 74 instead of doubling 148), then add 2 more again (add 74 again) to get 8 times the number. Look at the difference.



In fact, we could have used repeated addition to find out the answers to the divisor being multiplied by each digit from 1 to 9, in which case neither underestimating nor overestimating would ever occur. Although this leaves absolutely no guesswork at all, it's more work than necessary in most cases. Here is an example of how someone could do the same problem done on the previous page by making a complete multiplication table first. It's easy to do the division if you do this first, but there is a lot of prep work at the beginning to make the table.



Look back at the three different ways the division $27984 \div 37$ was done. The first way, shown on the previous page, had a short partial multiplication table (only three additions for prep work), but the repeated subtraction method took a little longer. It required seven subtractions to get the answer (look to the right of the scaffold), because the 700 took three steps 400, 200, 100; the 50 took two steps 40 and 10; and the 6 took two steps 4 and 6. The second way had one more addition for the prep of the multiplication table, and then only required five subtractions to get the answer (look to the right of the scaffold), because the 700 took two steps 600, and 100; the 50 took two steps 40 and 10; and the 6 took one step. The third way, shown at the top, required eight additions of prep work to make the multiplication table, but the division was a breeze, taking only three steps. If you do it this way, you may as well write the 756 up at the top, as you do when you do it using the traditional algorithm because you can never underestimate or overestimate.

Depending on the particular problem, one method may be faster than another, but many times you wouldn't know until you've completed the problem. I usually just do the doubling to 8 to get good enough estimates, but if you'd like to make a more complete table, it works just as well. You have to decide if you prefer to do more prep work up front to possibly make the division faster.

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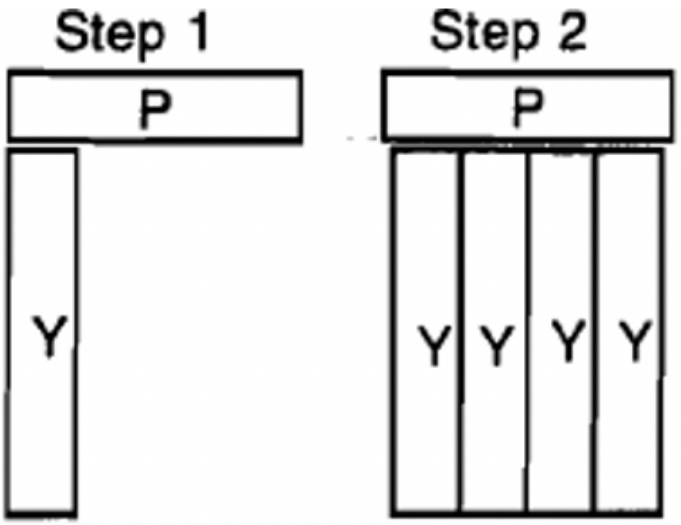
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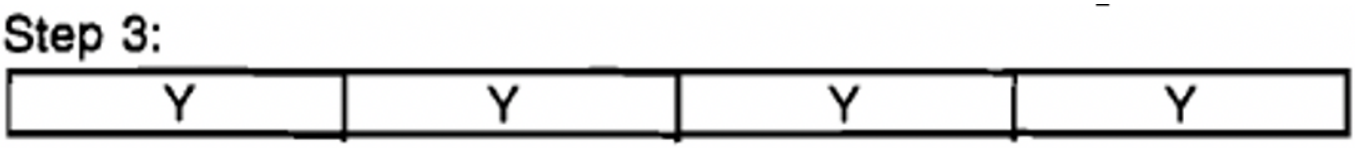
4.9: The Meaning of Multiplication

This exercise set presents a different, challenging way for you to look at multiplication. You will be using the Centimeter Strips (C-strips) to explore and discover the commutative, associative and distributive properties of multiplication. Below is a reminder of the different C-Strips we'll be working with along with the abbreviation used for each highlighted in **bold**:

1 cm by 1 cm	**W**HITE	1 cm by 2 cm	**R**ED
1 cm by 3 cm	**L**IGHT GREEN	1 cm by 4 cm	**P**URPLE
1 cm by 5 cm	YELLOW	1 cm by 6 cm	**D**ARK GREEN
1 cm by 7 cm	BLAC**K**	1 cm by 8 cm	BROW**N**
1 cm by 9 cm	**B**LUE	1 cm by 10 cm	**O**RANGE
1 cm by 11 cm	**S**ILVER	1 cm by 12 cm	HOT PINK

We will begin by defining a way to "multiply" two C-strips together. Consider multiplying the Purple C-strip by the Yellow C-Strip. Using abbreviations, we will write $P \times Y$, which we will read aloud as "P cross Y". The first step in this "multiplication" is to place the first strip (P) down in a horizontal position and then to place the second strip (Y) underneath the first strip so that it is perpendicular to the first strip. The second step is to fill in more yellow C-strips alongside the one already there so that you form a rectangle of yellow strips where the length is the yellow strip and the width is the purple strip. Last, remove the horizontal strip (purple in this case) and form a train of the vertical strip(s) (yellow in this case). This train is $P \times Y$ and is shown below. See the illustrations of the three steps.





We say $P \times Y$ is the train consisting of 4 yellow C-strips. *It is important to notice that the answer is a train that is formed with only yellow strips, which is the* ***second*** *letter in the multiplication.*

Exercise 1

a. Make the train $P \times Y$ as explained above. **Save it until you finish part d.**

b. Find $Y \times P$ by following the same procedure used to find $P \times Y$, by doing the first ***two*** steps and showing them below.

Step 1:	*Step 2:*

c. This train you eventually end up with should consist of only purple C-strips. How many purple strips are there? Draw a picture of the train $Y \times P$.

d. Place the two trains, $P \times Y$ and $Y \times P$, side by side and compare the lengths of the two trains. What do you notice?

To make notation simple, let's agree to say that **two trains are equal if they have the same length**. In other words, from what you discovered in Exercise 1, $P \times Y = Y \times P$.

For the following exercises, describe the trains formed by doing the following multiplications with C-strips. You should physically form each train by going through the same steps used to multiply $P \times Y$ on the previous page. For part a and b, fill in the first blank with a numeral (how many) and the second blank with a color. Save each train in order to answer part c. Use your observation to write the equations asked for in Exercise 1d. For instance, since you discovered $P \times Y = Y \times P$, this translates to $4 \times 5 = 5 \times 4$, since P is 4 cm long and Y is 5 cm long. Use this example to fill in blanks for exercises 2, 3 and 4.

Example a

Find $P \times Y$. This train consists of

Example b

Find $Y \times P$. This train consists of

Exercise 2

a. Find $R \times D$. This train consists of _____ C-strips

b. Find $D \times R$. This train consists of _____ C-strips

c. $R \times D$ is made up of ____ white strips and $D \times R$ is made up of _____ white strips.

d. What do you notice when you compare the lengths of the trains $R \times D$ and $D \times R$?

e. From your work in parts a - d, write an equation including C-strips R and D; then translate the equation to one with numbers.

Exercise 3

a. Find $K \times L$. This train consists of _____ C-strips

b. Find $L \times K$. This train consists of _____ C-strips

c. $K \times L$ is made up of ____ white strips and $L \times K$ is made up of _____ white strips.

d. What do you notice when you compare the lengths of the trains $K \times L$ and $L \times K$?

e. From your work in parts a - d, write an equation including C-strips K and L; then translate the equation to one with numbers.

Exercise 4

a. Find $H \times W$. This train consists of _____ C-strips

b. Find $W \times H$. This train consists of _____ C-strips

c. $H \times W$ is made up of ____ white strips and $W \times H$ is made up of ____ white strips.

d. What do you notice when you compare the lengths of the trains $H \times W$ and $W \times H$?

e. From your work in parts a - d, write an equation including C-strips H and W; then translate the equation to one with numbers.

The above problems illustrate the *commutative property of multiplication* for trains.

The commutative property of multiplication for trains states that if s and t are <u>any</u> two trains, then $s \times t = t \times s$

Note: A train can consist of one or more C-strips. In other words, a single white (or any other color) C-strip is actually a train. There doesn't have to be a caboose and engine!

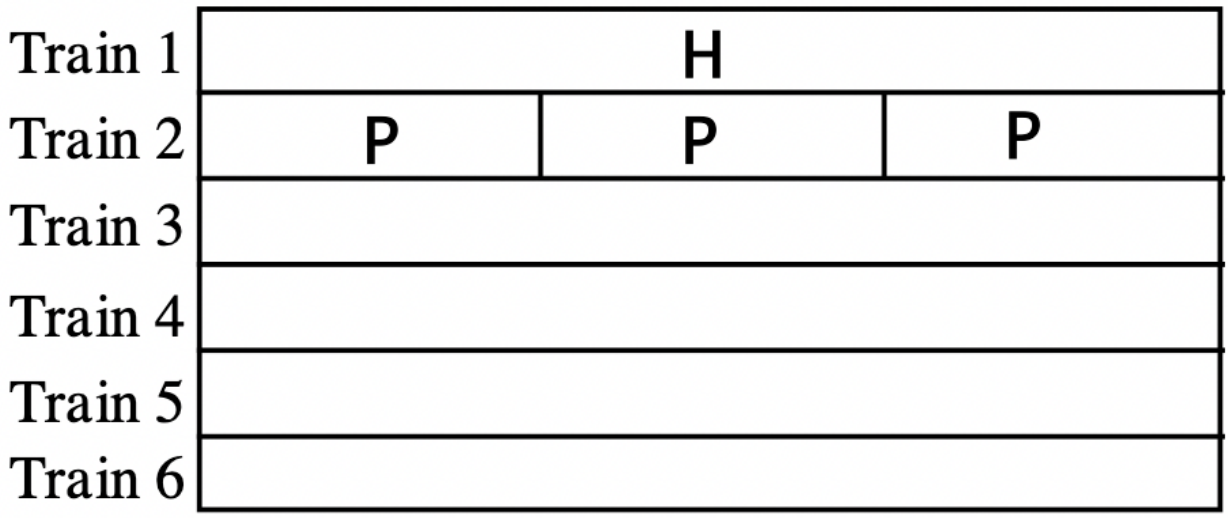
Exercise 5

Fill in the blanks to each of the following problems. To do this, take the color c-strip shown after the multiplication sign and make a train out of that color that is equal in length to the c-strip on the right side of the equal sign. Second, take the train and form it into a rectangle. Then find a c-strip that fits across the width of the rectangle. Write the abbreviation for that c-strip in the blank.

a. ____ $\times R = H$	b. ____ $\times P = P$	c. ____ $\times L = D$
d. ____ $\times R = N$	e. ____ $\times L = B$	f. ____ $\times W = K$
g. ____ $\times R = D$	h. ____ $\times D = H$	i. ____ $\times Y = O$

Exercise 6

a. Take out the Hot Pink (H) C-strip. Use your C-strips (one color at a time) to see if you can form a train made up of c-strips of the same color equal in length to the hot pink c-strip in other words, see if you can do it with all whites (always possible for any train), or all reds, or all light greens, etc. You should be able to make six different trains each train is made up of a single color. Draw a picture of each of these trains under the Hot Pink one shown. I've drawn two trains for you already one is simply the hot pink strip (a train consisting of just one c-strip), and a second is made up of three purple c-strips.



b. Your work from exercise 5 should help complete this exercise. Take each train, one at a time, and put each strip in a vertical position, but side by side to form a rectangle. Then, find a c-strip that fits across the width (or top) of the rectangle. This should tell you from which multiplication problem each train was formed. Write an equation using c-strips and then translate to an equation with numbers. For instance, for train 2, first I would make a rectangle out of the three purple c-strips. Next, I would try to find a c-strip to fit across the top, which would be light green. Therefore, train 2 was formed from the multiplication $L \times P$ remember that since the train is formed with purple c-strips, P is the *second* letter in the multiplication. So, the equation in c-strips is $L \times P = H$, and the numerical equivalent is $3 \times 4 = 12$. Follow this same procedure for the other four trains you made in part a.

Train 1 illustrates the multiplication $W \times H = H$, or $1 \times 12 = 12$. (Note that if the hot pink strip is vertical, the white c-strip fits across the top.)

Train 2 illustrates the multiplication $L \times P$ = H, or $3 \times 4 = 12$

Train 3 illustrates the multiplication ____ × ____ = H, or ____ × ____ = 12

Train 4 illustrates the multiplication ____ × ____ = H, or ____ × ____ = _____

Train 5 illustrates the multiplication ____ × ____ = H, or ____ × ____ = _____

Train 6 illustrates the multiplication ____ × ____ = H, or ____ × ____ = _____

Exercise 7

For train 2 in exercise 6, I made a train of purple c-strips equal in length to the hot pink c-strip. Now, make a train of purple c-strips equal in length to the brown (N) c-strip. Take the train of the purple c-strips and make it into a rectangle.

Which color c-strip measures across the width of this rectangle? _____

Put the answer (abbreviation for the c-strip) in the blank to solve the multiplication: ____ × P = N.

Now, we'll explore more about the equations formed in exercise 5. Consider part a. You should have written $D \times R = H$. This is analogous to the statement $6 \times 2 = 12$. But, if you actually look at the original train that was formed to find the answer, you would see that there were 6 red c-strips that made a train equal in length to the hot pink c-strip. This gives rise to one of the whole number definitions of multiplication. *Multiplication* is really just the same thing as ***Repeated Addition.***

Definition: Multiplication of Whole Numbers using the repeated addition approach

If a and b are whole numbers, then a × b = b + b+ b+ b + ... + b, where there are a addends of b in this sum.

I like to think of using the definition like this:

3 × 4 means 3 fours added together or 3 × 4 = 4 + 4 + 4 = 12

4 × 3 means 4 threes added together, or 4 × 3 = 3 + 3 + 3 + 3 = 12

5 × 2 means 5 twos added together, or 5 × 2 = 2 + 2 + 2 + 2 + 2 = 10

2 × 5 means 2 fives added together, or 2 × 5 = 5 + 5 = 10

Note that although 3 × 4 gives the same answer as 4 × 3, it does not *mean* the same thing! One means 4 + 4 + 4 and the other means 3 + 3 + 3 + 3. Before we delve any further into this discussion, I want you to use the repeated addition definition to compute the following:

Exercise 8

Use the repeated addition definition of multiplication to compute the following. First, write out the meaning of the multiplication, and then compute the answer.

a. 6 × 3 =

b. 2 × 9 =

c. 9 × 2 =

Now, it may seem obvious to you that 3 × 4 is the same answer as 4 × 3. That's because you've no doubt been using the commutative property of addition for years. Whether or not you really thought about why it was true is another thing. But, although 3 × 4 gives the same answer as 4 × 3, it does not mean the same thing. One means 4 + 4 + 4 and the other means 3 + 3 + 3 + 3. To a child just learning about addition and multiplication, there is no reason for them to conclude that one sum (4 + 4 + 4) happens to give the same answer as the other sum (3 + 3 + 3 + 3). For instance, if I was asked to compute the following two sums, it wouldn't be evident to me at all that the answer to each would be end up being the same:

17 + 17 + 17 + 17 and 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4

Actually, I find it quite amazing that they are equal. But, indeed they are, because the first sum comes from the multiplication problem 4 × 17 (since there are 4 seventeens added together), and the second sum comes from the multiplication problem 17 × 4 (since there are 17 fours added together). Later, when we discuss a more geometrical definition for multiplication, it becomes obvious why the commutative property of multiplication is true.

The commutative property of multiplication for whole numbers states that if a and b are *any* two numbers, then

$$a \times b = b \times a.$$

So, the very exciting thing about the commutative property of multiplication is that, given a problem like 100 × 7, where it would take a long time to write out and compute the answer using the strict definition of multiplication by adding 7 over and over again, we can use our knowledge of the commutative property, that is 100 × 7 = 7 × 100, and compute the second multiplication instead. Wouldn't you agree that 100 + 100 + 100 + 100 + 100 + 100 + 100 is quicker and easier to compute than adding together 100 sevens? Thank goodness for the commutative property of multiplication!!!

Before going on to another property, let's practice verifying that the commutative property of multiplication is true for some numbers by using the repeated addition definition of multiplication. The commutative property requires two numbers (they don't have to be two different numbers) and there are two sides of an equation to verify. The difference between the expressions a × b

and b × a is that the order of the numbers is different. Remember that the commutative property means there has been an order change. To verify the commutative property of multiplication, you need to use the repeated addition definition of multiplication and add from left to right, one number at a time. The next example shows how to verify the commutative property of multiplication, given a set of two numbers.

Example

Verify the commutative property of multiplication is true for the given set of numbers. You'll have to remember how to add in base six to do part b.

a. {5, 3}

Solution

I need to show $5 \times 3 = 3 \times 5$. Since 5×3 = 3 + 3 + 3 + 3 + 3 = 6 + 3 + 3 + 3 = 9 + 3 + 3 = 12 + 3 = 15 and since 3×5= 5 + 5 + 5 = 10 + 5 = 15, then $5 \times 3 = 3 \times 5$, because both equal 15.

b. $\{3_{\text{six}}, 4_{\text{six}}\}$

Solution

I need to show $3_{\text{six}} \times 4_{\text{six}} = 4_{\text{six}} \times 3_{\text{six}}$ Since $3_{\text{six}} \times 4_{\text{six}} = 4_{\text{six}} + 4_{\text{six}} + 4_{\text{six}} = 12_{\text{six}} + 4_{\text{six}} = 20_{\text{six}}$ and since $4_{\text{six}} \times 3_{\text{six}} = 3_{\text{six}} + 3_{\text{six}} + 3_{\text{six}} + 3_{\text{six}} = 10_{\text{six}} + 3_{\text{six}} + 3_{\text{six}} = 13_{\text{six}} + 3_{\text{six}} = 20_{\text{six}}$, then $3_{\text{six}} \times 4_{\text{six}} = 4_{\text{six}} \times 3_{\text{six}}$, because both equal 20_{six}.

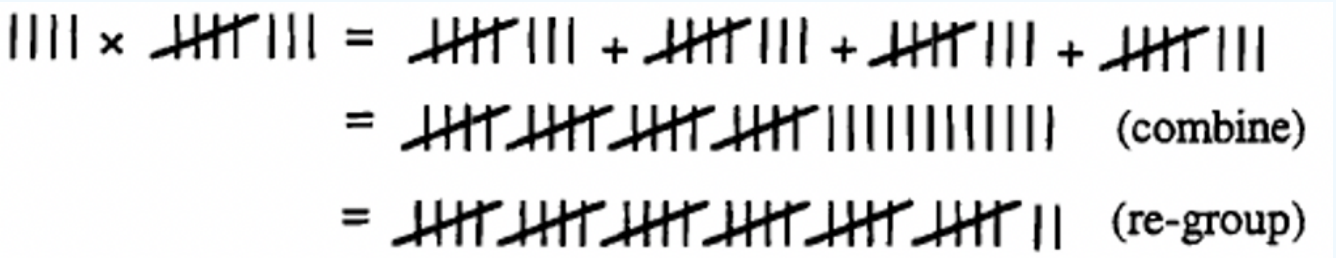
Exercise 9

Verify the commutative property of multiplication by writing an equation that must be true and using the repeated addition definition of multiplication is true using the given set of numbers. Simplify each expression separately; then show they are equal.

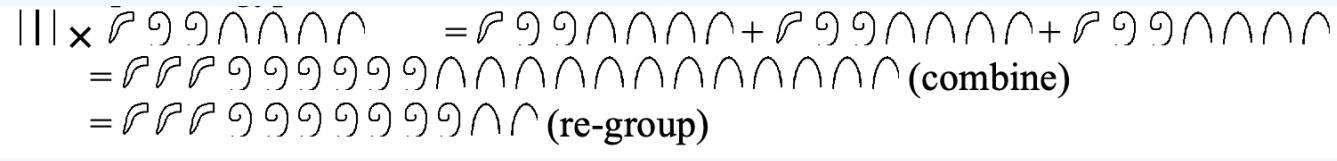
a. {4, 3}
b. $\{2_{\text{five}}, 3_{\text{five}}\}$
c. Make up a set in a base other than 10 and verify the commutative property of multiplication.

We can also use the **repeated addition definition** of multiplication to multiply in other numeration systems. Study the following examples and then work through the problems in exercise 10.

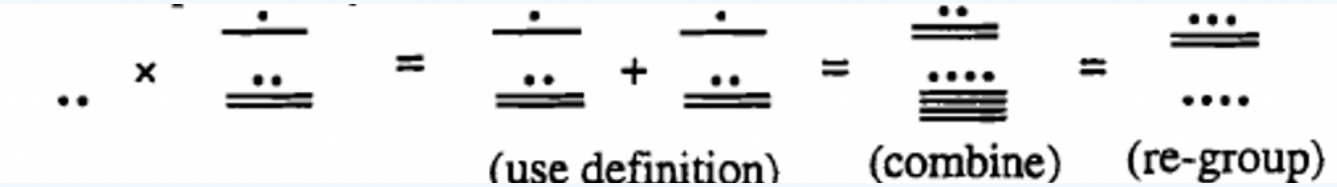
Example in Tally



Example in Egyptian



Example in Mayan



Notice that all of these examples are done completely in the numeration system given. All of the steps are shown in that numeration system. This is how you should do the problems in exercise 10. If you want, you can check the answer after you complete the problem by converting to base ten. For instance, the Mayan example in base ten is 2×132. So the answer should be 264. If you convert the final answer to base ten, it is also 264.

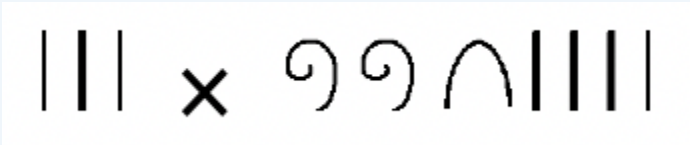
Exercise 10

Use the repeated addition definition of multiplication to compute the following. Make sure you write out the meaning of the multiplication **in the system given** (see examples on previous page), showing all of the work and exchanges (if necessary), to compute the answer. Do not do the problem in base ten, although you can use base ten to check the answer when you are done.

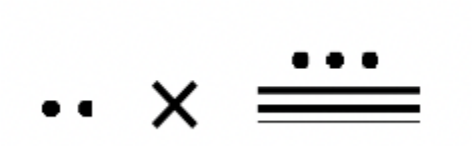
a.



b.



c.



Exercise 11

Next, we will form the trains for $(R \times L) \times P$ and for $R \times (L \times P)$. We can only work on one at a time. In both cases, *you must do what's inside the parentheses first.*

a. First, we'll do $(R \times L) \times P$. Begin by forming the train $R \times L$ to use in part b.

b. Next, we need to form the train $(R \times L) \times P$. The first part, $R \times L$, which you formed in part a, should be in a horizontal position, and the purple c-strip goes underneath it, perpendicular to the train $R \times L$. Note that although $R \times L$ is made up of more than one c-strip, it still forms one train across the top. In the space to the right, draw a picture of what this looks like so far.

c. Do the multiplication by filling in the rest of the purple c-strips in the vertical position under the train $R \times L$; then form a train out of the rectangle of purple c-strips. **Keep this train until you have finished all of exercise 11.** How many *purple c-strips* are in the train formed by doing the multiplication $(R \times L) \times P$?

d. Now, we'll do $R \times (L \times P)$. Begin by forming the train $L \times P$ to use in part e.

e. Next, form the train $R \times (L \times P)$. Put the first part, R, in the horizontal position. The second part, $L \times P$, formed in part d, should go in the vertical position as one long train, underneath and perpendicular to the train R. Don't separate the purple strips in $L \times P$ line up as one long vertical train (made up of more than one purple C-strip) under the red c-strip. To the right, draw a picture of what this looks like so far.

f. Do the multiplication by filling in the rest of the purple c-strips to form the rectangle; then make a train out of the rectangle of purple c-strips. How many purple c-strips are in the train formed by doing the multiplication $R \times (L \times P)$?

g. Compare the length of train $(R \times L) \times P$, from part c with length of train $R \times (L \times P)$, from part f. What do you notice?

h. From your observation, write an equation using the c-strips:

i. Translate the equation from part h to one with numbers:

j. Write the left side of the equation you wrote down in part i (which has numbers, not c-strips), and simplify by following the order of operations do what is in parentheses first. Then, do the same for the right side of the equation.

Left side:

Right side:

k. Are the two expressions you simplified in part (j) equal? If so, you have verified the equation from part (i) is true

Exercise 12

Next, we will form the trains for $(Y \times R) \times L$ and for $Y \times (R \times L)$.

a. First, we'll do $(Y \times R) \times L$. Begin by forming the train $Y \times R$.

b. Next, we need to form the train $(Y \times R) \times L$. The first part, $Y \times R$, which you formed in part a, should be in a horizontal position, and the light green c-strip should go underneath it, perpendicular to the train $Y \times R$. Note that although $Y \times R$ is made up of more than one c-strip, it still forms one train that is across the top. That's okay. Below, draw a picture of what this looks like so far.

c. Do the multiplication by filling in the rest of the light green c-strips and then making a train out of the rectangle of light green c-strips. **Keep this train until you have finished all of exercise 12.** How many light green c-strips are in the train formed by doing the multiplication $(Y \times R) \times L$?

d. Now, we'll do $Y \times (R \times L)$. Begin by forming the train $R \times L$.

e. Next, form the train $Y \times (R \times L)$. Put the first part, Y, in the horizontal position. The second part, $R \times L$, formed in part d, should go in the vertical position as one long train, underneath and perpendicular to the train Y. Don't separate the light green strips in $R \times L$ line up as one long vertical train (made up of more than one light green C-strip) under the yellow c-strip. To the right, draw a picture of what this looks like so far.

f. Do the multiplication by filling in the rest of the light green c-strips to form the rectangle, and then making a train out of the rectangle of light green c-strips. How many light green c-strips are in the train formed by the multiplication $Y \times (R \times L)$?

g. Compare the length of train $(Y \times R) \times L$, from part c with the length of train $Y \times (R \times L)$, from part f. What do you notice?

h. From your observation, write an equation using the c-strips:

i. Translate the equation from part h to one with numbers:

j. Write the left side of the equation you wrote down in part i (which has numbers, not c-strips), and simplify by following the order of operations do what is in parentheses first. Then, do the same for the right side of the equation.

Left side:

Right side:

k. Are the two expressions you simplified in part (j) equal? If so, you have verified the equation from part (i) is true.

Exercises 11 and 12 illustrate the *associative property of multiplication* for trains.

The *associative property of multiplication* for trains states that if r, s and t are *any* three trains, then $(r \times s) \times t = r \times (s \times t)$.

Before going on to any more properties, we'll practice verifying that the associative property of multiplication is true for some numbers in base ten later on, we'll do it in other bases, too. The associative property requires three numbers (they don't have to be three different numbers) and there are different expressions on each side of the equation. The difference between the left and right sides is that the parentheses are around a different pair of numbers. Remember that the associative property means there is a parentheses change, whereas the commutative property means there has been an order change. To verify, you need to use the order of operations when simplifying each side. If each expression simplifies to same thing, the equation is true. In the order of operations, you always simplify what is in parentheses first. The following are some examples of how to verify the associative property of multiplication, given a set of three numbers.

The *associative property of multiplication* for whole numbers states that if a, b and c are *any* three numbers, then $(a \times b) \times c = a \times (b \times c)$.

Example 1

Verify the associative property of multiplication is true using the three numbers {4, 5, 7} in your example.

Note: Since the associative property states $(a \times b) \times c = a \times (b \times c)$, one of the numbers given is put in for a, the other for b and the other for c. The choice is yours. For example, we can show any of the following six equations are true:

a. $(4 \times 5) \times 7 = 4 \times (5 \times 7)$ or d. $(4 \times 7) \times 5 = 4 \times (7 \times 5)$
b. $(5 \times 4) \times 7 = 5 \times (4 \times 7)$ or e. $(7 \times 4) \times 5 = 7 \times (4 \times 5)$
c. $(7 \times 5) \times 4 = 7 \times (5 \times 4)$ or f. $(5 \times 7) \times 4 = 5 \times (7 \times 4)$.

If this was a question on a test, you would only have to verify one of the six equations shown there are always six possibilities one might choose, depending on which numbers you put in for a, b or c. I will show you how to do it if you decided to verify the equation (e). I'll also verify it for another equation (f). Then, for practice, you get to verify the other four equations.

One Solution to Example 1: I'll show: $(7 \times 4) \times 5 = 7 \times (4 \times 5)$

Left side: $(7 \times 4) \times 5 = 28 \times 5 = 140$ and Right side: $7 \times (4 \times 5) = 7 \times 20 = 140$

Since I used the order of operations to simplify each expression, and both expressions are equal, the associative property is true using the numbers 4, 5 and 7.

Here is a different way to verify it using a different equation.

Another Solution to Example 1: I'll show: $(5 \times 7) \times 4 = 5 \times (7 \times 4)$

Left side: $(5 \times 7) \times 4 = 35 \times 4 = 140$ and Right side: $5 \times (7 \times 4) = 5 \times 28 = 140$

Since I used the order of operations to simplify each expression, and both expressions are equal, the associative property is true using the numbers 4, 5 and 7.

Exercise 13

Verify the associative property of multiplication is true using the numbers 4, 5 and 7, by doing the order of operations on each side. Do this four times, using the equations a, b, c and then d, as shown on the previous page.

a. Verify: $(4 \times 5) \times 7 = 4 \times (5 \times 7)$
b. Verify: $(5 \times 4) \times 7 = 5 \times (4 \times 7)$
c. Verify: $(7 \times 5) \times 4 = 7 \times (5 \times 4)$
d. Verify: $(4 \times 7) \times 5 = 4 \times (7 \times 5)$

To verify the associative property of multiplication is true by example, first you must write an equation that must be true using any three numbers (or by using three particular numbers given to you.) Then, by doing the order of operations on each side (each individual expression), you must show each side of the equation gives the same result.

Example 2

Verify the associative property is true using the numbers 10, 8 and 3.

a. I will show this equation is true: $(10 \times 8) \times 3 = 10 \times (8 \times 3)$

Since $(10 \times 8) \times 3 = 80 \times 3 = 240$ and $10 \times (8 \times 3) = 10 \times 24 = 240$, then the equation $(10 \times 8) \times 3 = 10 \times (8 \times 3)$ is true.

Exercise 14

Verify the associative property is true using the numbers given.

a. {9, 7, 6}

b. any three numbers of your choice.

Time to work on another property...

STEP 1: Form the train $P \times (R + L)$. To do this, the purple strip should be place in a horizontal position, and the train R + L should be placed in a vertical position underneath the purple c-strip. Then, the rectangle needs to filled in. Lastly, make a long train out of the rectangle. Note that this train is made up of 8 c-strips (4 reds and 4 light greens). **Save this train**.

STEP 2: Form two trains: $P \times R$ and $P \times L$. Then add the two trains together (in other words, $P \times R + P \times L$) to form one long train. Note that this train is made up of 8 c-strips (4 reds and 4 light greens). Compare the length of the train formed in Step 1 with this train. Are they are the same length? Are they made up of the same exact c-strips?

We have just shown and verified that $P \times (R + L) = P \times R + P \times L$.

Exercise 15

a. Form the train $L \times (P + Y)$ by first forming the train P + Y. Then do the multiplication $L \times (P + Y)$. Exactly what c-strips make up this train?

b. Form the train $L \times P$ and $L \times Y$. Now add them together to form one long train. Exactly what c-strips make up this train?

c. Compare the trains from part a and b, and write an equation (with c-strips).

d. Translate the equation from part c by writing an equation using numbers.

e. Verify the equation you wrote in part d by first simplifying the left hand side of the equation using the order of operations and then by simplifying the right hand side as well. If the answer to each expression is the same, you have verified the equation.

left side:

right side:

Exercise 16

a. Form the train $R \times (W + D)$ by first forming the train W + D. Then do the multiplication $R \times (W + D)$. Exactly what c-strips make up this train?

b. Form the train $R \times W$ and $R \times D$. Now add them together to form one long train. Exactly what c-strips make up this train?

c. Compare the trains from part a and b, and write an equation (with c-strips).

d. Translate the equation from part c by writing an equation using numbers.

e. Verify the equation you wrote in part d by first simplifying the left hand side of the equation using the order of operations and then by simplifying the right hand side as well. If the answer to each expression is the same, you have verified the equation.

left side:

right side:

The above problems illustrate the distributive property of multiplication over addition for trains.

The distributive property of multiplication over addition for trains states that if r, s and t are *any* three trains (each train may be made up of one or more c-strips)**, then** $r \times (s + t) = (r \times s) + (r \times t)$.

Now, it's time to practice verifying the distributive property of multiplication over addition is true for some numbers in base ten — later on, we'll do it in other bases, too. The distributive property requires three numbers (they don't have to be three different numbers) and there are two sides of an equation to verify. Note the difference between the left and right sides. To verify, you need to use the order of operations when simplifying each side. In the order of operations, you always simplify what is in parentheses first. If each expression simplifies to same thing, the equation is true. The following are some examples of how to verify the distributive property of multiplication, given a set of three numbers.

The distributive property of multiplication over addition for whole numbers states that if a, b and c are *any* three numbers, then $a \times (b + c) = (a \times b) + (a \times c)$.

Example 1

Verify the distributive property of multiplication is true using the three numbers {4, 5, 7} in your example.

Note: Since the distributive property states $a \times (b + c) = (a \times b) + (a \times c)$, one of the numbers given is put in for a, the other for b and the other for c. The choice is yours. For example, we can show any of the following six equations are true:

a. $4 \times (5 + 7) = (4 \times 5) + (4 \times 7)$ or d. $5 \times (7 + 4) = (5 \times 7) + (5 \times 4)$
b. $4 \times (7 + 5) = (4 \times 7) + (4 \times 5)$ or e. $7 \times (4 + 5) = (7 \times 4) + (7 \times 5)$
c. $5 \times (4 + 7) = (5 \times 4) + (5 \times 7)$ or f. $7 \times (5 + 4) = (7 \times 5) + (7 \times 4)$

If this was a question on a test, you would only have to verify one of the six equations shown. There are always six possibilities one might choose, depending on which numbers you put in for a, b or c. I will show you how to do it if you decided to verify the equation (c). I'll also verify it for another equation (f). Then, for practice, you get to verify the other four equations.

One Solution to Example 1: I'll show: $5 \times (4 + 7) = (5 \times 4) + (5 \times 7)$

Left side: $5 \times (4 + 7) = 5 \times (11) = 55$ and Right side: $(5 \times 4) + (5 \times 7) = 20 + 35 = 55$

Since I used the order of operations to simplify each expression, and both expressions are equal, the distributive property is true using the numbers 4, 5 and 7.

Here is a different way to verify it using a different equation.

Another Solution to Example 1: I'll show: $7 \times (5 + 4) = (7 \times 5) + (7 \times 4)$

Left side: $7 \times (5 + 4) = 7 \times 9 = 63$ and Right side: $(7 \times 5) + (7 \times 4) = 35 + 28 = 63$

Since I used the order of operations to simplify each expression, and both expressions are equal, the distributive property is true using the numbers 4, 5 and 7.

Exercise 17

Verify the distributive property of multiplication is true using the numbers 4, 5 and 7, by doing the order of operations on each side. Do this four times, using the equations a, b, d and then e, as shown on the previous page.

a. Verify: $4 \times (5 + 7) = (4 \times 5) + (4 \times 7)$
b. Verify: $4 \times (7 + 5) = (4 \times 7) + (4 \times 5)$
c. Verify: $5 \times (7 + 4) = (5 \times 7) + (5 \times 4)$
d. Verify: $7 \times (4 + 5) = (7 \times 4) + (7 \times 5)$

To verify the distributive property of multiplication is true by example, first you must write an equation that must be true using any three numbers (or by using three particular numbers given to you.) Then, by doing the order of operations on each side (each individual expression), you must show each side of the equation gives the same result.

Example 2

Verify the distributive property is true using the numbers 10, 8 and 5.

a. I will show this equation is true: $10 \times (8 + 5) = (10 \times 8) + (10 \times 5)$

Solution

Since $10 \times (8+5) = 10 \times 13$ and $(10 \times 8) + (10 \times 5) = 80 + 50$, then the equation $10 \times (8+5) = (10 \times 8) + (10 \times 5)$ is true.

Exercise 18

Verify the distributive property of multiplication over addition is true for the three numbers specified:

a. {7, 8, 12}
b. {28, 65, 35}
c. any three numbers of your choice.

Exercise 19

Use the definition of multiplication for trains to compute the following. Then translate to make an equation using numbers.

a. $W \times R$ = ____ translates to _____

b. $W \times B$ = ____ translates to _____

c. $W \times D$ = ____ translates to _____

d. $W \times S$ = ____ translates to _____

e. $N \times W$ = ____ translates to _____

f. $H \times W$ = ____ translates to _____

Exercise 19 illustrates that **1 is the identity element for multiplication**. In other words, for any number m, $\mathrm{m} \times 1 = \mathrm{m}$ **and** $1 \times \mathrm{m} = \mathrm{m}$.

Exercise 20

The identity element for multiplication if we are referring to trains is what you fill in the blank to make the following statement true. Fill in the blanks to find the identity.

For any train t, $\mathrm{t} \times$ **____** $= \mathrm{t}$ **and ____** $\times \mathrm{t} = \mathrm{t}$**.**

The distributive property of multiplication over addition is used quite often to simplify arithmetic problems. Consider the following examples:

Someone who needed to multiply 57×102 might find it easier to think of the problem as $57 \times (100+2) = (57 \times 100) + (57 \times 2) = 5700 + 114 = 5814$.

Someone who needed to compute $47 \times 38 + 47 \times 62$ might realize this is really just the same as $47 \times (38+62) = 47 \times 100 = 4700$.

This next example illustrates that multiplication also distributes over subtraction:

Someone who needed to multiply 38×99 might find it easier to think of the problem as $38 \times (100-1) = (38 \times 100) - (38 \times 1) = 3800 - 38 = 3762$.

Exercise 21

Use the distributive property of multiplication over addition (or subtraction) to rewrite the following problems. Then, simplify. For d, e and f, make up three of your own problems where it would be easier to use the distributive property first before simplifying. Show how to use the property and simplify.

a. 764×999
b. 324×102
c. $83 \times 74 + 83 \times 26$
d.
e.
f.

Exercise 22

So far, you have worked with three operations; addition, subtraction and multiplication. You have learned many properties. Take a little time to reflect on and review these by writing the properties or answering the questions as stated below:

a. State the associative property of multiplication and provide an example.
b. State the commutative property of addition and provide an example.
c. State the identity element for addition and what it means and provide an example.
d. State the commutative property of multiplication and provide an example.
e. State the distributive property of multiplication over addition and provide an example.
f. State the identity element for multiplication and provide an example.
g. Provide a counterexample to show that subtraction is not commutative
h. State the associative property of addition.
i. Provide a counterexample to show that subtraction is not associative.

There is another way to define multiplication using Set Theory. In order to do this, we have to remember how to take the **Cartesian product** of two sets, A and B.

The **Cartesian product** of set A with set B, which is written $A \times B$ and is read as "A cross B" is the set of *all possible* ordered pairs (a,b), where $a \times A$ and $b \times B$.

If you are finding the Cartesian product, the answer is a set that contains ordered pairs.

Example 1

If A = {x, y, z} and B = {a, b}, find $A \times B$.

Solution

$A \times B$ = {(x, a), (x, b), (y, a), (y, b), (z, a), (z, b)}

Example 2

If E = {1, 2} and F = {2, 3}, find $E \times F$.

Solution

$E \times F$ = {(1, 2), (1, 3), (2, 2), (2, 3)}

Exercise 23

Write the Cartesian product. Each answer is a set containing ordered pairs!

a. {3, 4} × {2, 6} = _____

b. {6, 7, 8, 9} × {5} = _____

c. {r, s, t} × { } = _____

d. {a} × {a} = _____

e. {x, y} × {x, y} = _____

f. {1, 3, 5} × {1, 3, 5} = _____

Set Theory Definition for Multiplying two whole numbers:

For any two sets A and B, $\mathrm{n(A) \cdot n(B) = n(A \times B)}$.

In other words, to multiply two numbers, a and b, write one set A which has a elements in it and another set B that has b elements in it. To find the product, write the Cartesian
product of A and B and count the elements in the set.
Note: There are no restrictions on what elements you choose for sets A and B. They can be disjoint or they can have elements in common. It's up to you.

Example 1

Use the set theory definition of multiplication to show that $3 \cdot 2 = 6$

Solution

Let A = {v, w, x} and B = {1, 2}. Since n(A) = 3 and n(B) = 2, then

$$\begin{aligned} 3 \cdot 2 &= n(A) \cdot n(B) && \text{by substituting n(A)for 3 and n(B)for 2} \\ &= n(A \times B) && \text{by the set theory definition of multiplication} \\ &= n(\{(v,1),(v,2),(w,1),(w,2),(x,1),(x,2)\}) && \text{by computing } A \times B \\ &= 6 && \text{by counting the elements in } A \times B \end{aligned}$$

Therefore, $3 \cdot 2 = 6$.

Example 2

Use the set theory definition of multiplication to show that $1 \cdot 3 = 3$

Solution

Let A = {v} and B = {1, 2, 3}. Since n(A) = 1 and n(B) = 3, then

$$\begin{aligned} 1 \cdot 3 &= n(A) \cdot n(B) && \text{by substituting n(A)for 1 and n(B)for 3} \\ &= n(A \times B) && \text{by the set theory definition of multiplication} \\ &= n(\{(v,1),(v,2),(v,3)\}) && \text{by computing } A \times B \\ &= 3 && \text{by counting the elements in } A \times B \end{aligned}$$

Therefore, $1 \cdot 3 = 3$

Example 3

Use the set theory definition of multiplication to show that $2 \cdot 0 = 0$

Solution

Let A = {v, w} and B = { }. Since n(A) = 2 and n(B) = 0, then

$$\begin{aligned} 2 \cdot 0 &= n(A) \cdot n(B) && \text{by substituting n(A)for 2 and n(B)for 0} \\ &= n(A \times B) && \text{by the set theory definition of multiplication} \\ &= n(\{\ \}) && \text{by computing } A \times B \\ &= 0 && \text{by counting the elements in } A \times B \end{aligned}$$

Therefore, $2 \cdot 0 = 0$

Example 4

Use the set theory definition of multiplication to show that $4 \cdot 3 = 12$

Solution

Let A = {e, f, g, h} and B = {g, h, i }. Since n(A) = 4 and n(B) = 3, then

$$\begin{aligned} 4 \cdot 3 &= n(A) \cdot n(B) && \text{by substituting n(A)for 2 and n(B)for 0} \\ &= n(A \times B) && \text{by the set theory definition of multiplication} \\ &= n(\{(e,g),(e,h),(e,i),(f,g),(f,h),(f,i),(g,g),(g,h),(g,i),(h,g),(h,h),(h,i)\}) && \text{by computing } A \times B \\ &= 12 && \text{by counting the elements in } A \times B \end{aligned}$$

Therefore, $4 \cdot 3 = 12$

Exercise 24

Use the set theory definition of multiplication to verify each multiplication. Show and justify each step. Only the solution to 24a is shown in the solutions.

a. $5 \cdot 2 = 10$

b. $3 \cdot 1 = 10$

c. $0 \cdot 2$

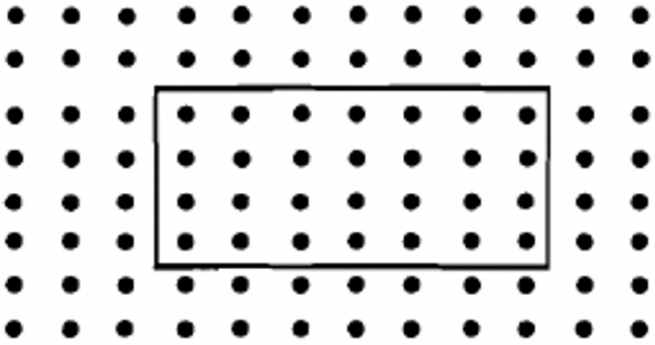
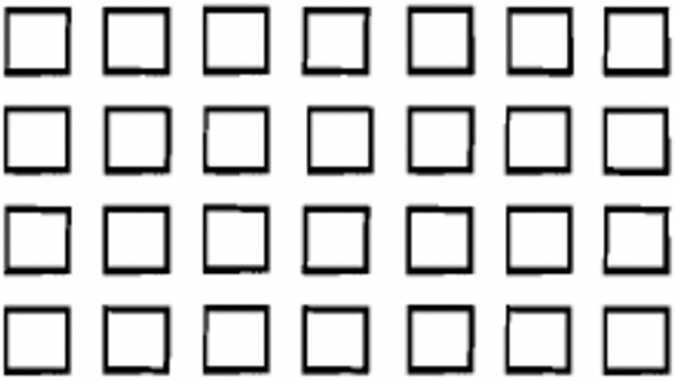
Here's a more geometrical approach to defining multiplication of two whole numbers.

Definition

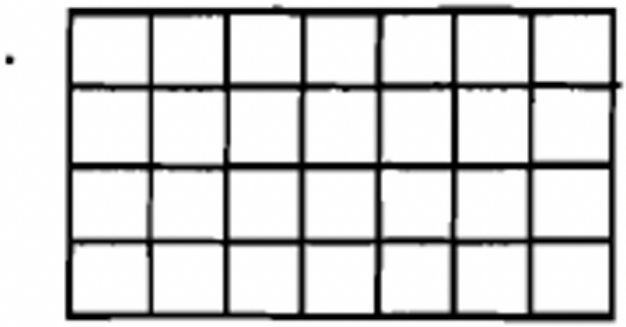
To find the product of any two whole numbers, m and n, make a rectangular array of objects having m rows and n columns. The product of m and n, written $m \times n$, equals the number of objects in the array.

The cool thing about using this geometrical approach is that *any* objects can be used, and one only needs to be able to count to get the answer. You don't need to use addition in order to multiply! I personally advocate having lots of graph paper around for anyone just learning to multiply. To compute 4×7, you or the learner can mark off a rectangle on the graph paper that has 4 rows and 7 columns of boxes. The number of boxes in the rectangle is the answer to the problem. This is a useful tool for first learning a particular multiplication fact, or for anyone (adults, too) who forgot a fact and needs to figure it out again. The "objects" used on a graph paper are the individual boxes on the graph paper.

Here is how six different students might geometrically show how to multiply 4×7.

<table>
<tr>
<td>This student took some dotted paper and marked off a rectangular array containing 4 rows and 7 columns of dots.
  </td>
<td>Thi
the
and
  </td>
</tr>
<tr>
<td>This student arranged four rows and seven columns of squares into a rectangular array.
  </td>
<td>This
seve
sign</td>
</tr>
</table>

This student drew a rectangular array having four rows and seven columns.



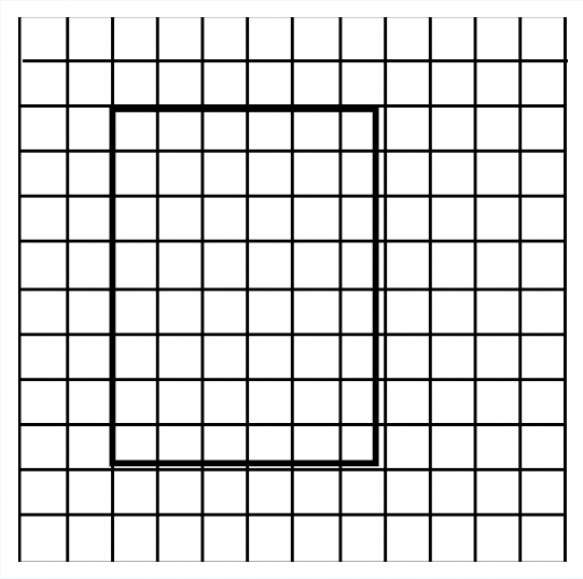
This student t
some dotted p
and then drew
rectangular ar
having four r
and seven col

There is no point to making anyone stress out if they forgot, didn't memorize or couldn't recall the answer to 4×7. The point is to *KNOW WHAT MULTIPLICATION MEANS* so you **can** figure a problem out again if you have to. Of course, there is merit to using flashcards and eventually memorizing basic multiplication facts like your multiplication tables. If not, it takes longer to do problems, you aren't able to focus on more advanced issues, and you might not trust your own mind. You shouldn't have to think about multiplication every time you have to multiply two numbers together, but you should *be able* to think about it. Knowing ***how*** to get the answer is better than having memorized a bunch of facts if you don't know where they came from, what multiplication really means, or how to figure it out again in case you forget the answer to some fact. And, as far as I'm concerned, nothing's wrong with using your fingers, either! Let people do what is comfortable for them, if they understand and can get the right answer, that's GREAT!

Exercise 25

Write the multiplication problem represented by each geometrical representation:

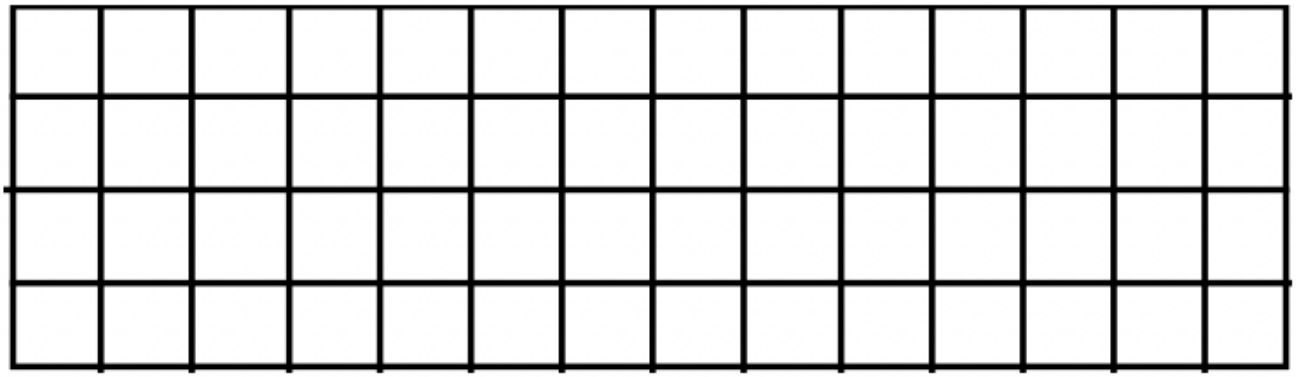
a. ___



c. ___



e. ____



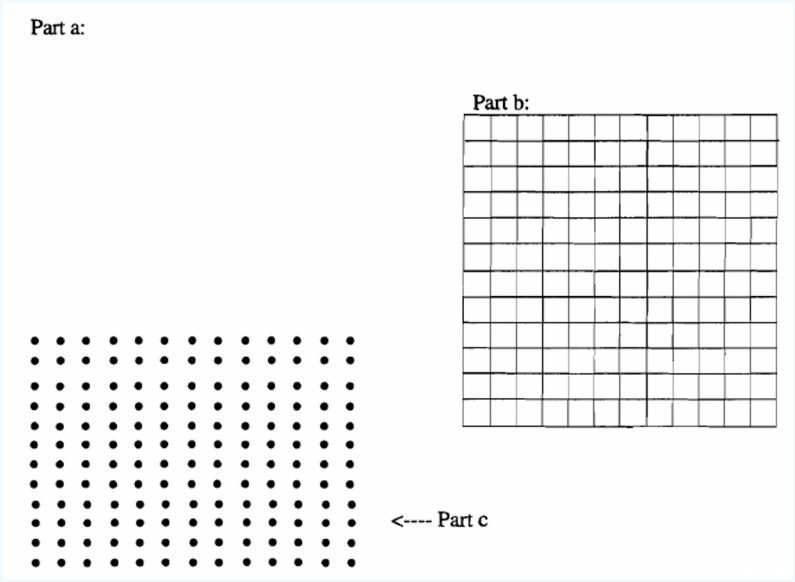
Exercise 26

Show two different geometrical representations for 4×3

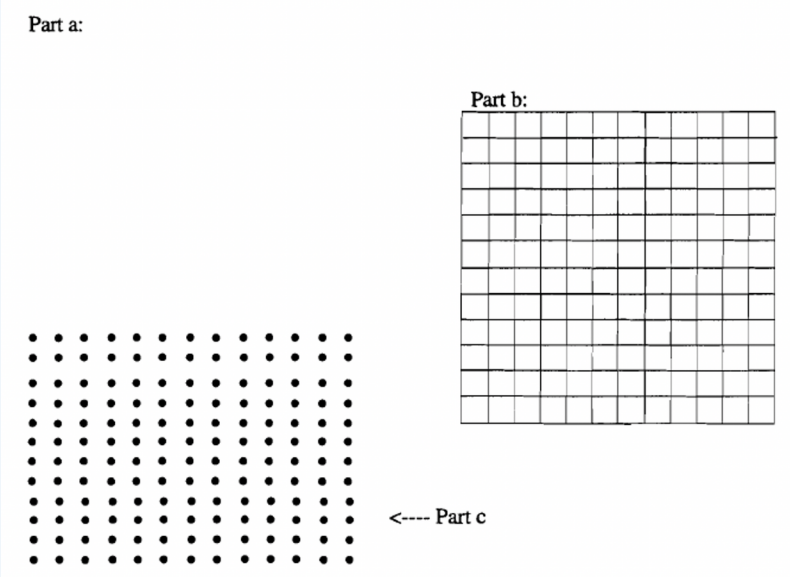
Exercise 27

For each multiplication, do the following to represent the multiplication: For part a, draw a rectangular array of objects, for part b use the graph paper, and for part c use the dotted paper. There are two ways you can choose to use the dotted paper.

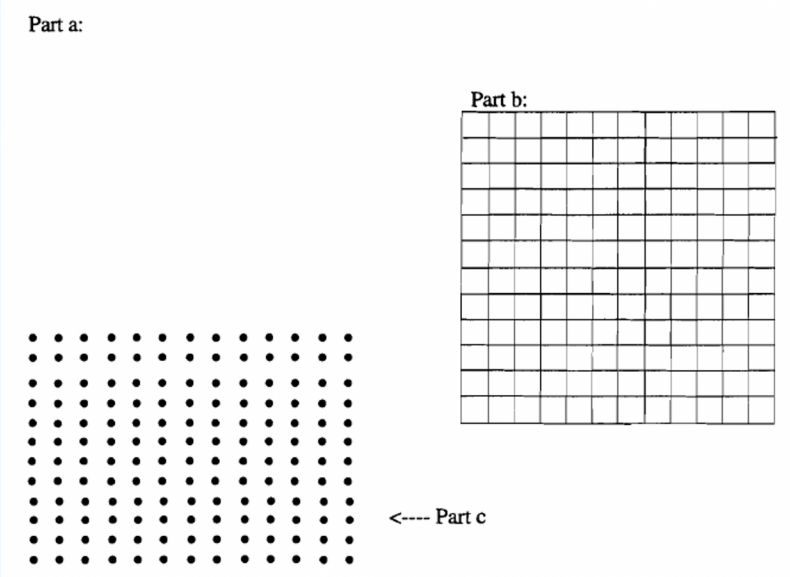
a. 2×6



b. 5×3

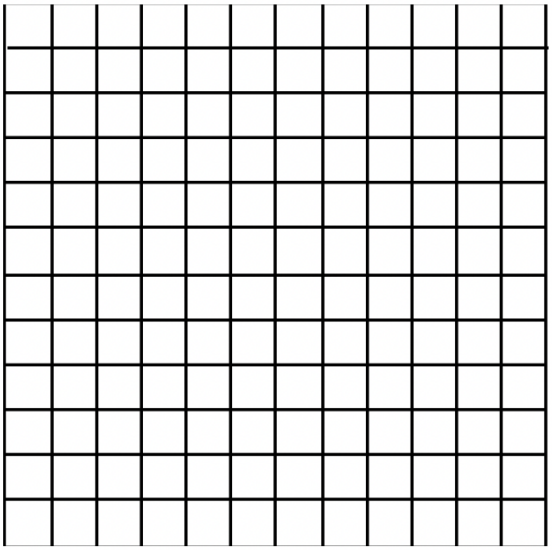


c. 3×7

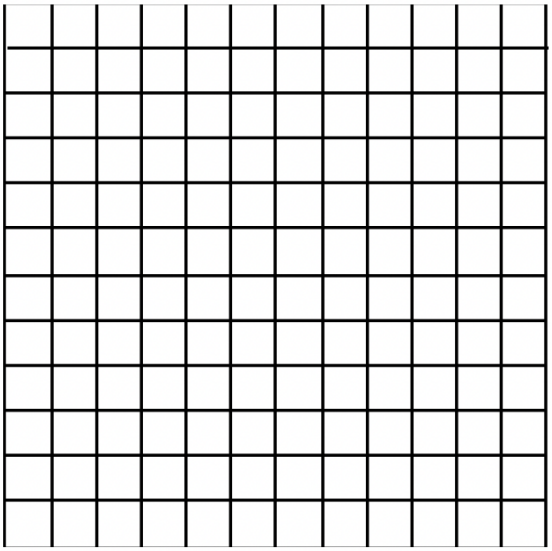


Exercise 28

a. Draw a rectangular array of boxes (a rectangle) on the graph paper provided to do the multiplication 4×6:



b. Turn your page sideways and look at the rectangle. Draw what the picture looks like:



c. What multiplication problem is now represented? _____

Exercise 28 should convince you that multiplication is commutative. In fact, the commutative property of multiplication is easiest to see using the geometrical approach. A rectangular array of objects having m rows and n columns represents the multiplication $m \times n$. Once the rectangle is rotated 90° (or looked at sideways), there are now n rows and m columns, which represents the multiplication $n \times m$. But of course, it's really the same rectangle of objects, so the number of objects in $m \times n$ equals the number of objects in $n \times m$. Therefore, $m \times n = n \times m$. Okay, just for fun, one more time:

The commutative property of multiplication for whole numbers states that if a and b are *any* two numbers, then $a \times b = b \times a$.

Wow! Isn't this exciting? There are several ways to think about and define multiplication, and we can show the commutative property still holds each time. Does it get any better than this? Well,...YES...but that's another story.

It's a little bit trickier to show on a two-dimensional page that the associative property of multiplication holds. With your work with the C-strips, you know multiplication is associative. Here's one way to geometrically model the multiplication of three numbers.

Definition

To find the product of any three whole numbers, $(a \times b) \times c$, first make a rectangular array of cubes (dice, sugar cubes, etc.) having a rows and b columns. Then, fill in c layers of the bottom rectangle to obtain a rectangular box, where the base of the box has width a and length b, and the height of the box is c.

Basically, you need to make (or visualize) a 3-dimensional box to do multiplication on three numbers. As it turns out, no matter how you construct it, if at the end, one dimension is a, one is b and the other is c, you've got the same box, and it holds the same number of cubes in it. Therefore, multiplication is associative. We won't be going through the motions of showing it in 3-dimensions, but elementary school children should have the benefit of working through it with cubes. I hope you'll model it this way if you become an elementary school teacher.

Since multiplication is both commutative and associative, any time more than two numbers are being multiplied together, the parentheses do not have to be shown, and the numbers can be multiplied together in any order.

For instance, notice how differently one might compute $4 \times 10 \times 3 \times 5$

Student 1: This person just goes left to right. First, 4×10 is 40, then 40×3 is 120, then $120 \times 5 = 600$.

Student 2: This person first multiplies 4×5 to get 20, then multiplies 10×3 to get 30, then multiplies the two answers together, 20×30 to get 600.

Student 3: This person does 4×5 first to get 20, then multiplies by 3 to get 60, then multiplies by 10 to get 600.

Student 4: This person multiplies 3×5 first to get 15, then multiplies that by 4 to get 60, and then multiplies by 10 to get 600.

And there are many more possibilities. The point is that you have the option of multiplying in any order and any combination you want. The following show how this is very convenient for certain computations:

Example

$5 \times 87 \times 2$: Well, it's easier to multiply the 5 and 2 first to get 10 and then multiply by 87 to get 870. Otherwise, you could do 5×87, which isn't so easy to do in your head.

Example

$8 \times 22 \times 5$: Well, I'd do 8 times 5 first to get 40 and then multiply by 22 to get 880.

Exercise 29

Make up a multiplication problem, where it would be easier to multiply by doing some rearranging first. Explain your steps and how to get the answer.

Exercise 30

We're going to show that $3 \times (4 + 2) = (3 \times 4) + (3 \times 2)$. Let's first work on the right side of the equation, which has two parts.

a. Draw a rectangular array of dots or stars to represent 3×4.

b. Now, draw a rectangular array of dots or stars to represent 3×2.

c. Since the right side of the equation says to join the two parts together (addition), then draw the array from part a below, and right next to it, draw the array from part b.

d. Now, let's do the left hand side of the equation, $3 \times (4 + 2)$. We first have to compute in the parentheses to get 3×6. So, make a rectangular array of dots or stars to represent 3×6.

e. Is the number of dots or starts in part c equal to the number of dots or starts in part d?

Exercise 31

What property did you just illustrate in exercise 30?

Well, I hope you realize you just showed that multiplication distributes over addition. In fact, it is also true that multiplication distributes over subtraction. In other words,

For any three whole numbers, a, b and c, where b > c, $a \times (b + c) = (a \times b) + (a \times c)$ and $a \times (b–c) = (a \times b)–(a \times c)$.

Exercise 32

Show that $2 \times (3 + 4) = (2 \times 3) + (2 \times 4)$ using a geometrical approach, like you did in exercise 30. Explain the steps.

Exercise 33

a. Form the train $P \times (Y–L)$ by first forming the train Y – L. Find out what C-strip Y – L is. Then do the multiplication $P \times (Y–L)$. Exactly how many white C-strips make up this train?

b. Form the train $P \times Y$ and $P \times L$. Now subtract the second train from the first train. Exactly how many white C-strips make up this train?

c. Compare the lengths of the trains from part a and b.

Exercise 33 showed that $P \times (Y–L) = P \times Y–P \times L$. This is one illustration that multiplication distributes over subtraction.

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4.10: Multiplication Algorithms

You will need: ***Base Blocks*** *(Material Cards 4-15)*

There are many algorithms for multiplication. Personally, I think the way most of us learned it in school is one of the hardest algorithms. It's an uphill battle getting new methods into the school system. But, it's worth pursuing. It's a good thing to know what multiplication *means*, and how to multiply any two numbers together. However, there is no reason that everyone should have to use the same method to obtain the answer. In this exercise set, you'll learn some new algorithms for multiplication. You'll also learn how to multiply in different bases.

Many people use repeated addition all the time to multiply. Consider the multiplication problem 26 times 17. Basically, this means add 26 seventeens added together. This could be done by adding them together one at a time. That would take a long time.

Another approach would be to add 10 seventeens together first to get 170, then add 10 more seventeens together to get 170 again, then add 5 more seventeens together which is 85, and then add one more seventeen. Adding 10 seventeens (170), 10 seventeens (170), 5 seventeens (85) and 1 seventeen (17) gives 170 + 170 + 85 + 17 = 442, which is exactly the answer to 26 times 17. That should be no surprise since we added 26 (10+10+5+1) seventeens together.

Someone else might add 10 seventeens (170), 10 seventeens (170), 3 seventeens (51) and 3 more seventeens (51) to get 170 + 170 + 51 + 51 = 340 + 102 =442.

There are several other ways people might break up the number 26 and still get the correct answer. This approach may be one you frequently or maybe this is the first time you've thought of it this way. In any case, it's different from just doing it the same old way.

Because of the commutative property, you could have done 17 times 26 instead. Maybe you'd add 10 twenty-sixes (260), then 5 more twenty-sixes (130) and 2 more twenty-sixes (52) to get 260 + 130 + 52 = 442.

One reason it's advantageous to do multiplication using repeated addition is that many problems can be done in your head this way. But, of course, every new method takes practice and getting used to, and it's common for people to revert back to using their old, familiar methods even if it takes longer and requires paper. That's okay, too. It is, however, very important that a teacher knows several ways so that he or she can provide more opportunities for students. What seems obvious to one student is foreign to another, so you give alternatives and present different ways of looking at things.

Exercise 1

Use some form of repeated addition to do the following multiplications. Explain how you do each one.

a. 23×13

b. 14×14

One new algorithm (for most of you reading this) is called **Duplation**. This is actually a very old algorithm used in ancient times by the Egyptians. It also uses the principle that multiplication is simply repeated addition. It is called duplation because it uses the idea of doubling numbers to get the answer. Let's look again at the first multiplication we did in this section. It was 26×17. Since 26 seventeens must be added together, the number 26 is broken down into powers of two. It's easy to get powers of two: simply start with 1, then double it to get 2, then double it to get 4, then double it to get 8, then double it to get 16, etc. To figure out how far you have to go in the doubling process, stop if doubling gives a number bigger than the number you are trying to break down (26, in this case). Now, all this sounds more complicated than it really is. Before explaining too much more, notice how you'll write it. Write 26 at the top and underline it. Underneath, start with 1, then, double continuously, stopping at 16 since if you doubled another time, you'd get a number bigger than 26. To the right of 26, write the number 17 and underline it.

Underneath it, first write 17, then double 17 to get 34, then double 34 to get 68, double 68 to get 136, etc., until you have the same amount of numbers under the 17 as you do under the 26.

$\underline{26}$	$\times$	$\underline{17}$
1	17	
2	34	
4	68	
8	136	
16	272	
……	……	

Do you see the corresponding numbers? 1 corresponds with 17, because 17 is 1×17, 2 corresponds with 34, because 34 is 2×17, 4 corresponds with 68, because 68 is 4×17, 8 corresponds with 136, because 136 is 8×17, and 16 corresponds with 272, because 272 is 8×17. Now, you may need to think about this for a few minutes. The left side keeps track of how many of some number you are adding together. So, the 1×17 should be clear. So, 2×17 is simply 17 doubled. Now if 2×17 is 34, then 4×17 is twice as many as 2×17, so double 34 to get 68. This continues on and on. So, if 8×17 is 136, then if we double 136, we should have 16×17. Isn't it neat how we know that $16 \times 17 = 272$ and we just double a few numbers to get there?

Okay, now we only need to add 26 seventeens together. Simply start at the bottom of the first column, and check off numbers that add up to 26 (this is like doing it in base two). After you check off the numbers in the left column, circle or point to their corresponding numbers in the right column. It's those numbers in the right column that you add together to get the answer. Watch how the rest of the problem is done:

26	×	17
1		17
✓2	----->	34
4		68
✓8	----->	136
✓16	----->	272

272 + 136 + 34 = **442** (yep, it's the same answer again!)

Let's do the same problem over again, but use the commutative property of multiplication. In other words, use the duplation method to compute: 17×26. This time, in the left column, you check off numbers that add up to 17 it's coincidental that I had to double to 16 again and stop. In the right hand column, start with 26 and double away.

17	×	26
✓1	----->	26
2		52
4		104
8		208
✓16	----->	416

416 + 26 = **442** (yep, it's the same answer again!)

Here are two more examples for you to study. I'll do 15×35 and 35×15.

15	×	35
✓1	----->	35
✓2	----->	70
✓4	----->	140
✓8	----->	280

280+140+70+35=**525**

35	×	15
✓1	----->	15
✓2	----->	30
4		60
8		120
16		240
✓32	----->	480

480+30+15=**525**

Of course, because of the commutative property of multiplication, the answer is 525, no matter which way you do it. Are you ready to try a couple of duplation problems of your own? Well, ready or not, here they come!

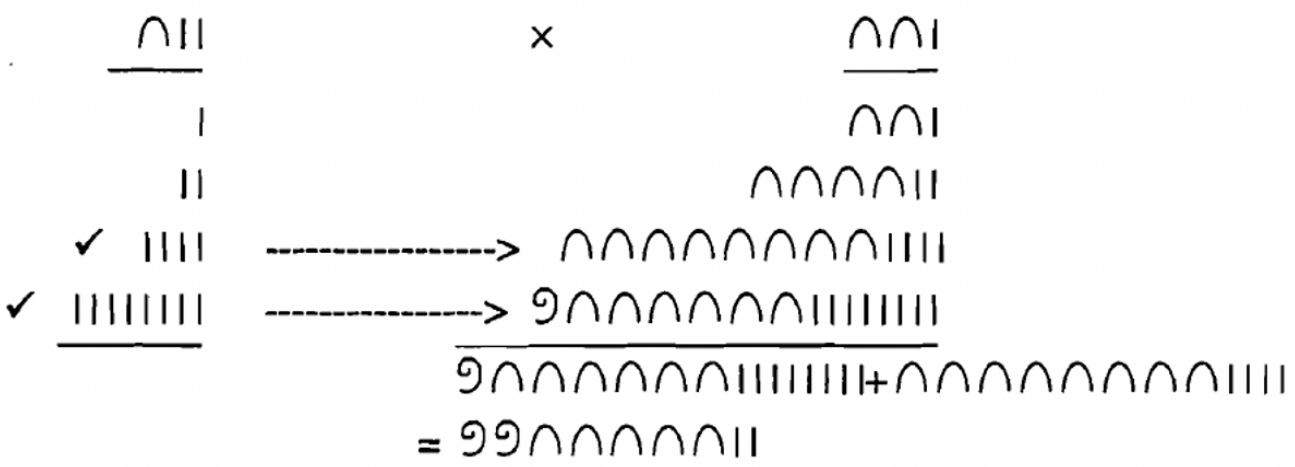
Exercise 2

Multiply 13×29 using duplation. Then, multiply 29×13 using duplation. Show all of your work. Make it clear what is added together for each problem.

Exercise 3

Use duplation to multiply 27 by 14, and then 14 by 27. Show your work.

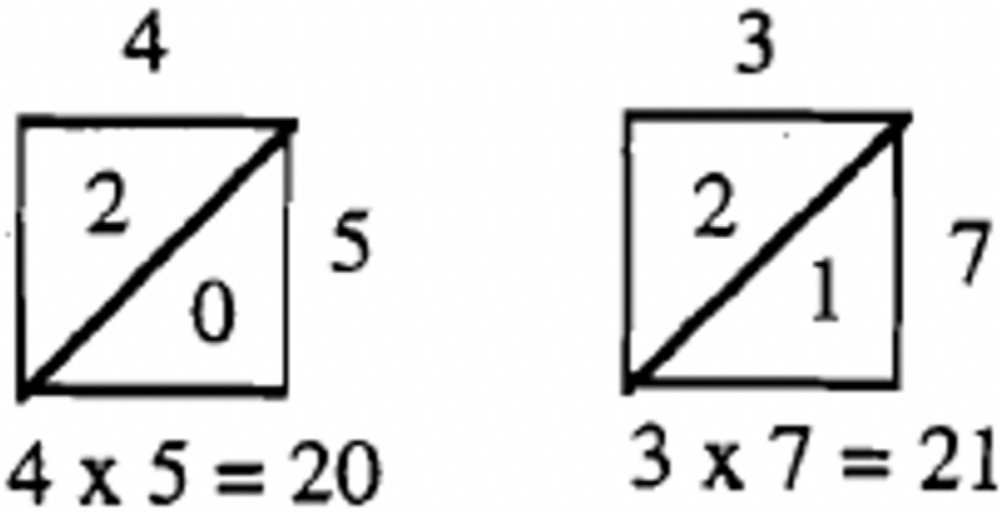
Here is an example of how the Egyptians multiplied using duplation.



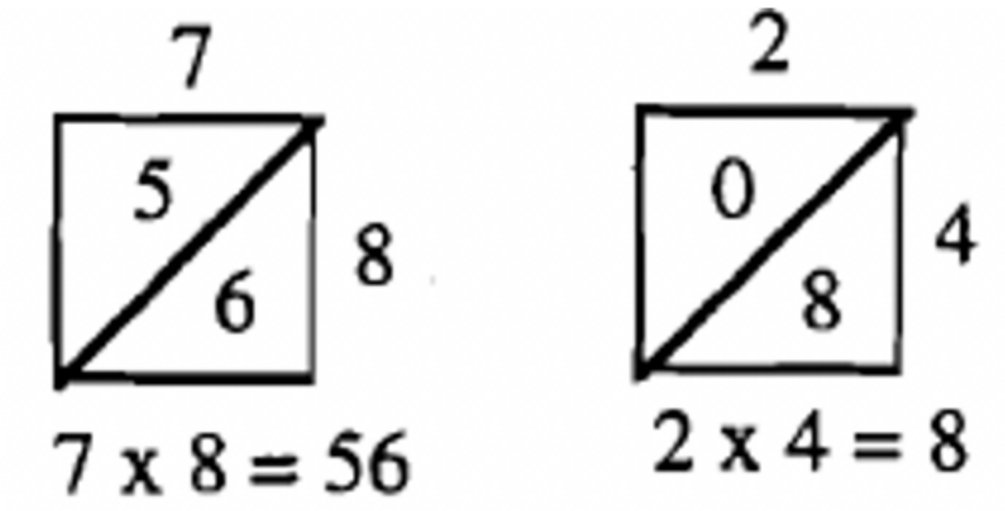
Exercise 4

Use duplation to multiply  .

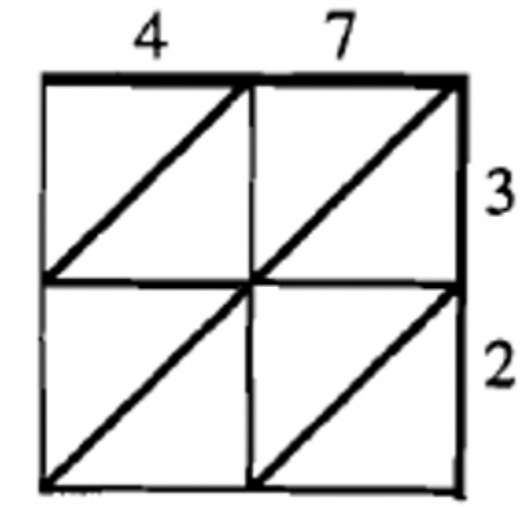
Do the problem entirely in Egyptian.



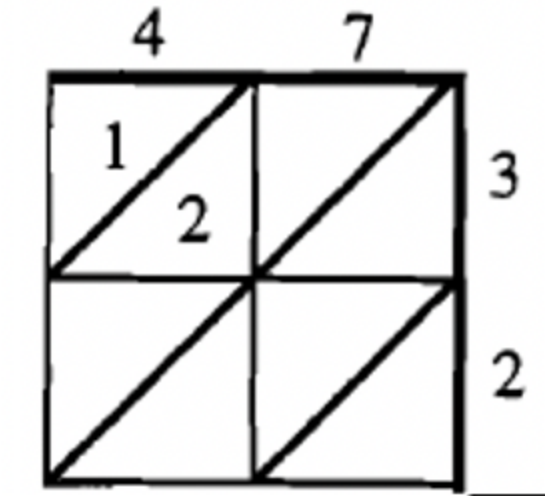
You are going to learn a cool and easy way to multiply numbers together. But first, I'll show you how to multiply 2 single digits together. Make a square with a diagonal in it. Put one number on top and the other on the right of the square. Multiply the two digits together and put the answer in the box with the diagonal. Look at the examples to the right.



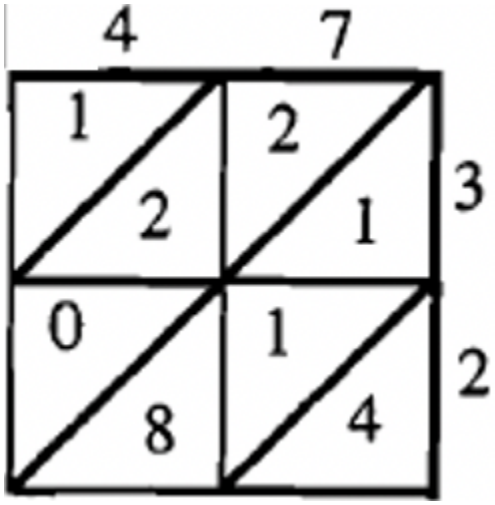
The method you are now going to learn is called the **LATTICE METHOD** and it could be used for multiplying 2 digit or 3 digit or even bigger numbers together. The example well do is 47×32.



STEP 1: To begin, put one of the numbers at the top (47) of a rectangle (that has a space for each digit in the number) and the other number along the side (32) of the same rectangle (that has a space for each digit). In this case, we need a rectangle that has two spaces across the top and 2 spaces along the side. Then, you make diagonals. Look to the right.

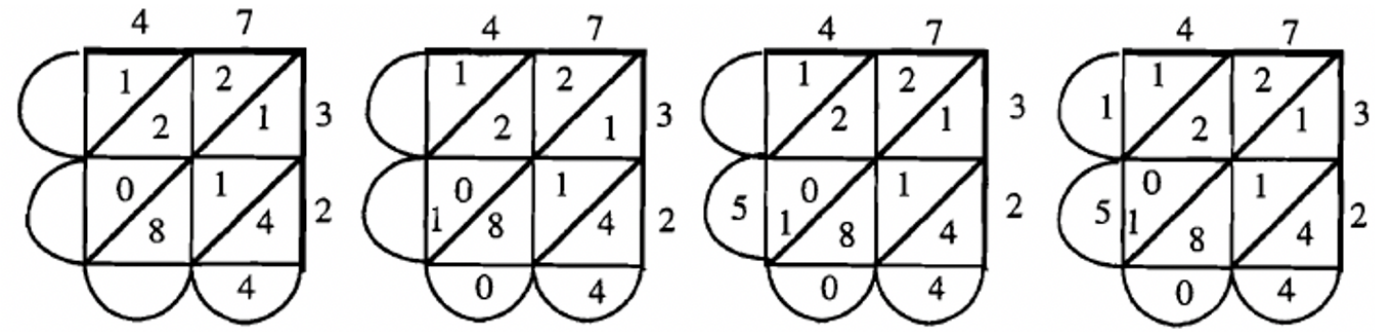


STEP 2: Multiply each of the digits on the top by each of the digits along the side, and put the answer where they would meet. For instance, look where the 12 goes when you multiply 4 times 3.



STEP 3: There are three other multiplications to do: 7 times 3, 4 times 2 and 7 times 2. Do you see how all the answers are filled in? It doesn't matter which ones you multiply and put in first.

STEP 4: There are four diagonals (made up of little triangles) slanting to the right inside the rectangle. Starting with the one in the right bottom corner, add the digits inside each diagonal and fill put the sum in the "cup" under its diagonal. In the first diagonal is 4, so 4 is put in the "cup" under its diagonal (bottom right of rectangle). In the next diagonal are three digits a 1, 1 and 8 which adds up to 10, so put a 0 in the cup under that diagonal (at the bottom to the left) and a 1 (for the carry) in one of the triangles to the left. The next diagonal has a 2, 2, 0 and 1, which adds up to 5, and goes in the "cup" under its diagonal (left, bottom). In the last diagonal, there is just a 1, so a 1 is put in the "cup"under its diagonal (left, top). Look at how I filled in the numbers one at a time.



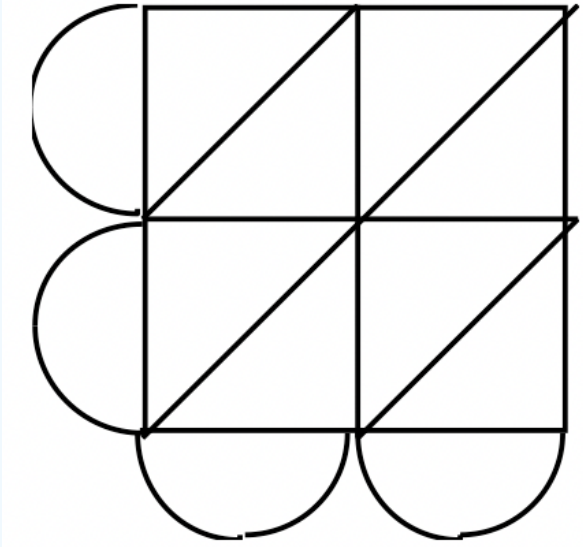
STEP 5: The numbers you filled in under each diagonal in the “cups” (outside the rectangle) form the answer. It is read by starting at the left top and reading down and then across the bottom. In this case, the answer is 1504. Isn’t that cool? That’s all there is to it!

Most of my students love this very old Lattice method of multiplying. It works as easily for small numbers as it does for large numbers. First, you simply do a series of simple single digit multiplications, and then you add, with carrying. Actually you could use the left to right (with underlining for carrying) method of addition for the last step. The main disadvantage is that you have to draw a lattice, which is a small price to pay to make multiplication easier and less complicated. Most likely, it takes less time than using the standard algorithm with carrying, because there's less chance of error and confusion with all that multiplying and carrying, etc. I propose all students learn multiplication this way first, and let them stay with it if they like it. Who needs to shorten it up and make it more complicated than it really is? Ah, well, I take it you get my point.

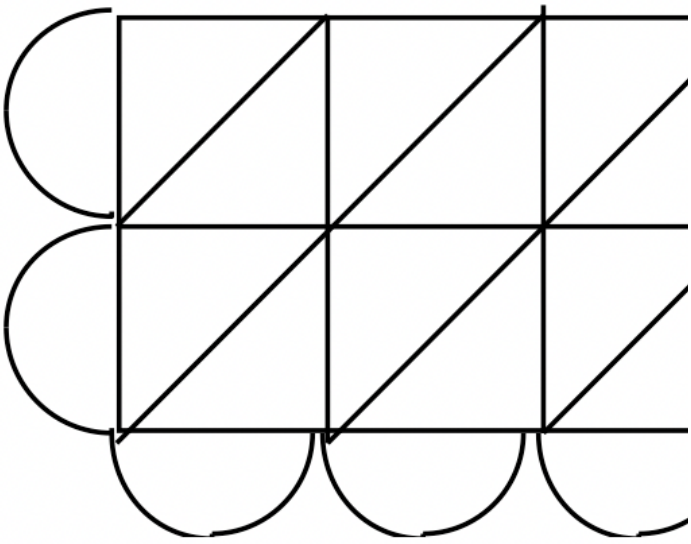
Exercise 5

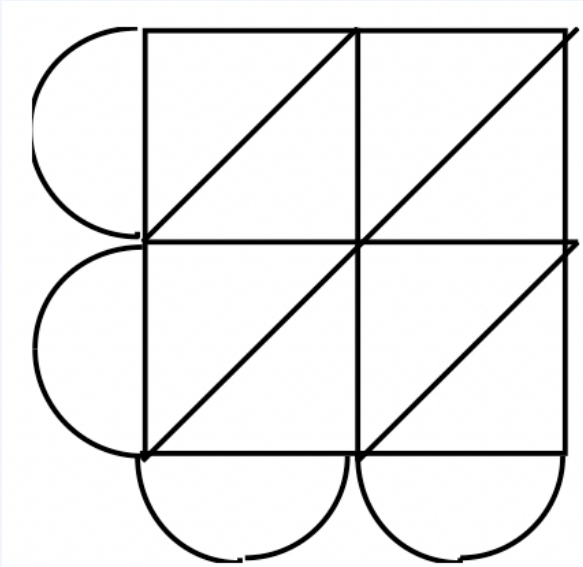
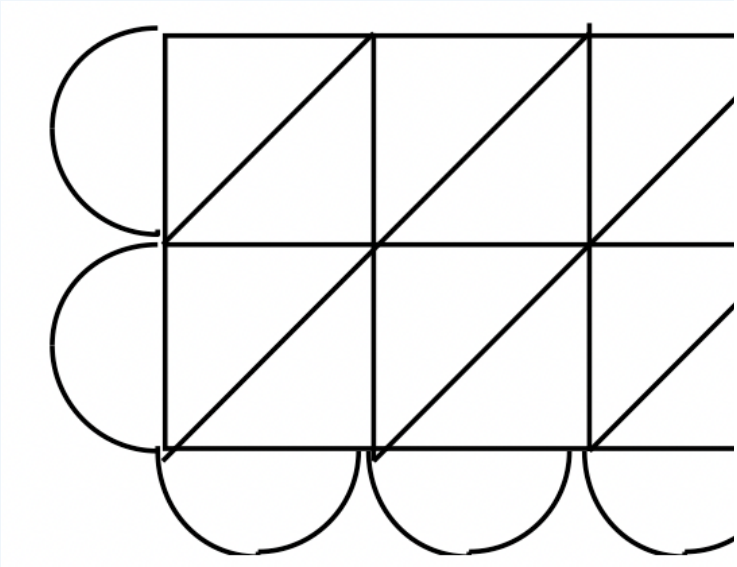
Use the lattices provided in a - d to multiply the given numbers. Then write the answer in the space provided. For e, draw your own lattice to compute the multiplication. By the way, you don't have to draw the little cups. That was my own little invention so it would be easy to identify the answer. There are two more problems (f and g) on the next page. Check your answers with a calculator

a. 63×57 = ____



b. 342×85 = ____



c. Make up a problem:	d. Make up a problem:
	
e. 58×67 = ___ Draw your own lattice.	

Exercise 6

a. Get out your Base Four Blocks. Take 41 units, and trade them in using your Base Four blocks. You should have some flats, longs and units which you can write as a numeral in Base Four.

Write the Base Four numeral here:

b. If you wanted to multiply 2 times that number, you would need to make two piles representing that number, join the two equal piles together (since multiplication is repeated addition) and then take the new pile and make trade-ins. Make two equal piles with the blocks, where each pile is the same as the one you formed from trade-ins in part a.Draw a picture of what the two piles look like.

c. Join the two piles together and make trade-ins to get a new base four numeral. Write this Base Four numeral here:

d. Fill in the first blank with what you got for part a and the second blank with what you got for part c. You just did a multiplication problem in Base Four using repeated addition!

$2_{\text{four}} \times$ _____ $_{\text{four}} =$ _____ $_{\text{four}}$

Exercise 7

a. Write the Base Four numeral from 6a here:

b. If you wanted to multiply 3 times that number, you would need to make three piles representing that number, join the three equal piles together (since multiplication is repeated addition) and then take the new pile and make trade ins. Make three equal piles with the blocks, where each pile is the same as the one you formed from trade-ins in part a. Draw a picture of what the three piles look like.

c. Join the three piles together and make trade-ins to get a new base four numeral. What Base Four numeral did you get?

d. Fill in the first blank with what you got for part a and the second blank with what you got for part c. You just did another multiplication problem in Base Four using repeated addition!

$3_{\text{four}} \times$ _____ four $=$ _____ four

Exercise 8

Using the same procedure as in 6 and 7, take out your base three blocks and do the following multiplication: $2_{\text{three}} \times 221_{\text{three}}$. Explain your work, and show pictures.

Exercise 9

If you wanted to do $10_{\text{three}} \times 221_{\text{three}}$, you would need to trade the long (10_{three}) in for three units, make three piles using Base three blocks representing 221_{three}, and do the same thing you did in exercise 7. Explain how you do this and show pictures.

This method of repeated addition with the blocks would be too cumbersome if we were working with higher numbers. For instance, if you were trying to use this method to multiply $321_{\text{four}} \times 312_{\text{four}}$, the first number when broken down into single units would consist of more than 50 units, and that would be way too many piles to make! Instead, we'll think about and figure out what it means to multiply using larger blocks than units. For instance, we'll figure out what a flat times a long is, or what a long times a flat is in any base! We'll be using these abbreviations: U for unit, L for long, F for flat, B for block, LB for long block, FB for flat block, and BB for block block.

Exercise 10

a. Get out your base three blocks. You're going to find the product of a long and a long similar to how you did products with the C-strips in Exercise Set 1. $L \times L$ is obtained by putting one long in the horizontal position and the second long in the vertical position, underneath and perpendicular to the first long. Then form a rectangle of longs underneath. Take the rectangle (don't include the top horizontal long) and trade up in base three, if possible. What is the product of long and a long in base three?

b. Get out your base four blocks. Find the product of long and a long in base four using the method explained in part a. What is the product of long and a long in base four?

c. Get out your base five blocks. Find the product of long and a long in base five using the method explained in part a. What is the product of long and a long in base five?

d. In any base, what is the product of a long and a long? $L \times L =$

Exercise 11

a. Get out your base three blocks. You're going to find the product of a flat and a long now. First, we need to convert the flat to longs so that we can make a horizontal train across the top of the T. Do this. Then, put a long in the vertical position, underneath and perpendicular to the long train of longs making the top of the T. Then form a rectangle of longs underneath. Take the rectangle (don't include any longs from the top horizontal part of the T) and trade up in base three, if possible. What is the product of flat and a long in base three?

b. Get out your base four blocks. Find the product of flat and a long in base four using the method explained in part a. What is the product of flat and a long in base four?

c. Get out your base six blocks. Find the product of flat and a long in base six using the method explained in part a. What is the product of flat and a long in base six?

d. In any base, what is the product of a flat and a long? $F \times L =$

Exercise 12

a. Get out your base three blocks. You're going to find the product of a block and a long now. First, we need to convert the block to flats, and then the flat to longs so that we can make a horizontal train across the top of the T. Do this. Then, put a long in the vertical position, underneath and perpendicular to the long train of longs making the top of the T. Then form a rectangle of longs underneath. Take the rectangle (don't include any longs from the top horizontal part of the T) and trade up in base three, if possible. What is the product of block and a long in base three?

b. Get out your base four blocks. Find the product of block and a long in base four using the method explained in part a. What is the product of block and a long in base four?

c. Get out your base two blocks. Find the product of block and a long in base two using the method explained in part a. What is the product of block and a long in base two?

d. In any base, what is the product of a block and a long? $B \times L =$

Exercise 13

Summarize the results from part d of exercises 10, 11 and 12.

a. $L \times L =$ _____

b. $L \times F =$ _____

c. $L \times B =$ _____

Use the commutative property of multiplication to compute the following:

d. $F \times L =$ _____

e. $B \times L =$ _____

Exercise 14

In your own words, what happens when something is multiplied by a long?

Did you notice that when something is multiplied by a long, it bumps up to the next place value. This is analogous to multiplying by 10 in base 10, since 10 is a long in base ten. For instance, in base 10, 347 represents 3 hundreds, 4 tens and 7 ones; when you multiply 347×10, you get 3470, 3 thousands, 4 hundreds and 7 tens, the place value for each digit moved up one place value.

Exercise 15

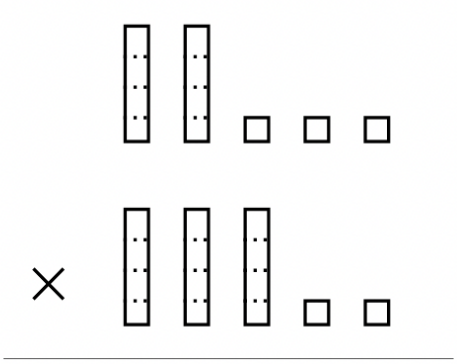
Consider multiplying $312_{\text{four}} \times 10_{\text{four}}$. This means you are multiplying a number in base four by a long in base four. Get out your base four blocks and represent the number 312_{four}.

a. You have ___ flats, ____ longs and ____ units.

b. If you multiply each of these by the long, now how many of each type of place value block do you have in base four?

To multiply a two-digit numeral, X, in a base by some number, Y, in that base, the idea of repeated addition can be used. For instance, if we represent the number with base blocks, we'll want to create an "answer pile" where we put the blocks representing the product as we multiply by the two-digit numeral. Every unit in X multiplied by Y gives us one set of base blocks contained in Y in the answer pile. Every long in X multiplied by Y gives us the next place value up for each of the base blocks contained in Y. This is much more complicated to explain than to do with the blocks. So, let's do an example.

Get out your Base Four Blocks. We'll multiply $23_{\text{four}} \times 32_{\text{four}}$. With your blocks, represent this as a multiplication problem. It should look something like this:



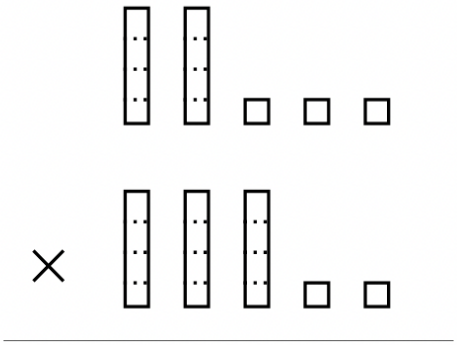
or you might abbreviate it on paper by writing it like this:

L L U U U

x L L L U U

Recall that each long in the bottom multiplied by the top makes each top piece move up to the next higher place value block. So, when the first bottom long is multiplied by the top number, 2 flats and 3 longs go into the answer pile. When the second bottom long is multiplied by the top number, another 2 flats and 3 longs go into the answer pile. When the third bottom long is multiplied by the top number, yet another 2 flats and 3 longs go into the answer pile. When I do this with the blocks, I usually move each part of the multiplier (bottom number) I've done so far to the right or out of the way. In any case, in your answer pile, you should now have 6 flats and 9 longs in the answer pile. We still have to multiply by each of the two units. Each unit multiplied by the top number gives a set of the top number; it's like multiplying by 1. So, when the first bottom unit is multiplied by the top number, 2 longs and 3 units go into the answer pile. When the second bottom unit is multiplied by the top number, another 2 longs and 3 units go into the answer pile. Create a place for the answer pile and do this with your blocks. Try it yourself, then check on the next page.

I'll show the answer pile below: each row shows what I got as I multiplied each bottom piece (3 longs and 2 units) by the top. By the way, you could have multiplied by the units first and then by the longs I'll do it that way for the next example. It all just eventually ends up in the answer pile and all exchanges have to made in the end, so you can write it in Base Four. This is $23_{\text{four}} \times 32_{\text{four}}$:

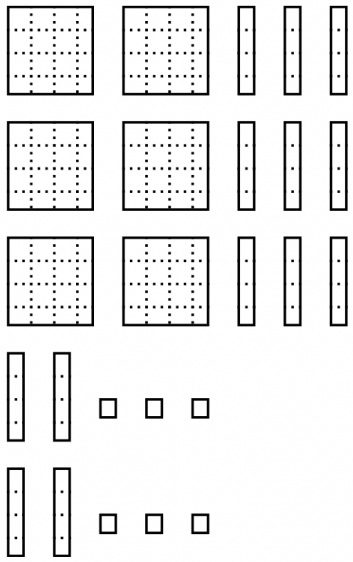


or you might abbreviate it on paper by writing it like this:

L L U U U

x L L L U U

THE Answer PILE IS SHOWN BELOW:

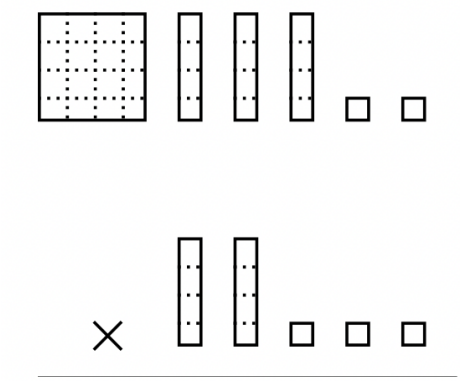


or you might abbreviate it on paper by writing it like this:

F F L L L
F F L L L
F F L L L
L L U U U
L L U U U

Now, make all your trade-ins with these pieces. You should end up with 2 blocks, 1 flat, 2 longs and 2 units, which is the number 2122_{four}, written in base four.

Next, we'll multiply $132_{\text{four}} \times 23_{\text{four}}$. With your blocks, represent this as a multiplication problem. It should look something like this:

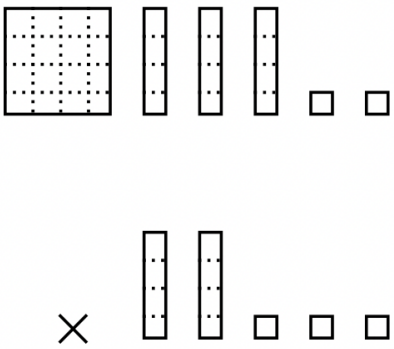


or you might abbreviate it on paper by writing it like this:

F L L L U U

x L L U U U

Each unit multiplied by the top number gives a set of the top number it's like multiplying by 1. So, when the first bottom unit is multiplied by the top number, a flat, 3 longs and 2 units go into the answer pile. When the second bottom unit is multiplied by the top number, another flat, 3 longs and 2 units go into the answer pile. When the third bottom unit is multiplied by the top number, yet another flat, 3 longs and 2 units go into the answer pile. Create a place for the answer pile and do this with your blocks. When I do this with the blocks, I usually move each part of the multiplier I've done so far to the right or out of the way. In any case, in your answer pile, you should now have 3 flats, 9 longs and 6 units in the answer pile. We still have to multiply by each of the two longs. Recall that each long multiplied by the top makes each top piece move up to the next higher place value block. So, when the first bottom long is multiplied by the top number, a block, 3 flats and 2 longs go into the answer pile. When the second bottom long is multiplied by the top number, another block, 3 flats and 2 longs go into the answer pile. I'll show the answer pile below: each row shows what I got as I multiplied each bottom piece (3 units and 2 longs) by the top. By the way, you could have multiplied by the longs first and then by the units. It all just eventually ends up in the answer pile and all exchanges have to made in the end, so you can write it in Base Four. This is $132_{\text{four}} \times 23_{\text{four}}$:

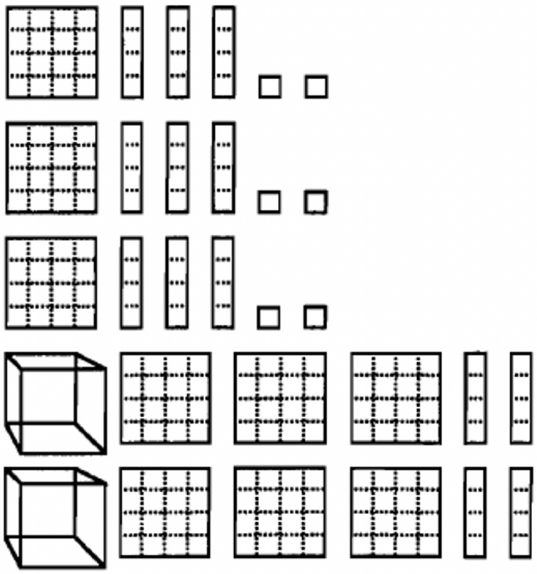


or you might abbreviate it on paper by writing it like this:

F L L L U U

x L L U U U

THE Answer PILE IS SHOWN BELOW:



or you might abbreviate it on paper by writing it like this:

F L L L U U

F L L L U U

F L L L U U

B F F F L L

B F F F L L

Now, make all your trade-ins with these pieces. You should end up with 1 long block, 1 block, 2 longs and 2 units, which is the number 11022_{four}, written in base four. Make sure you do and understand these last two problems before going on.

Exercise 16

Multiply each of the following using base blocks. Then explain and draw the steps in the space below as done in the previous two examples.

a. Use Base Four blocks to multiply $12_{\text{four}} \times 31_{\text{four}}$

b. Use Base Three blocks to multiply $112_{\text{three}} \times 21_{\text{three}}$

Exercise 17

a. Get out your base three blocks. You're going to find the product of a flat and a flat now. We need to convert the first flat to longs, make a train, and put this horizontal train across the top of the T. Do this. Then, convert the second flat to longs, make a train, put this train in the vertical position, underneath and perpendicular to the long train of longs making the top of the T. Then form a rectangle of a bunch of longs underneath. Take the rectangle (don't include any longs from the top horizontal part of the T) and trade up as much as possible in base three, if possible. What is the product of flat and a flat in base three?

b. Here's another way to find the product of a flat and a flat in base three. For each unit in the first flat, put a flat in an answer pile. Take all those flats in the answer pile and trade them in to see what you get. What is the product of flat and a flat in base three?

c. Get out your base two blocks. Find the product of flat and a flat in base two using the method explained in part a or b. What is the product of flat and a flat in base two?

d. Complete the following: In any base, $F \times F =$ _____

Another way to think about what happens when you multiply by a flat is to think of a flat as a long times a long. Let's say you are multiplying something by a flat. First, multiply it by a long, which means it moves up one place value, and then multiply that answer by a long once more, which means it moves up one more place value. So, anything multiplied by a long moves up one place value, and anything multiplied by a flat moves up two place values. Multiplying by a flat is analogous to multiplying by 100 in base ten. 100b in any base b represents a flat. Think about what happens when a block is multiplied by anything.

Exercise 18

Complete the following:

a. $U \times U =$ ____	f. $F \times U =$ _____
b. $U \times L =$ _____	g. $B \times U =$ _____
c. $U \times F =$ _____	h. $L \times L =$ _____
d. $U \times B =$ _____	i. $L \times F =$ _____
e. $L \times U =$ _____	j. $L \times B =$ _____

Exercise 19

Use base six blocks to multiply the following. $105_{\text{six}} \times 123_{\text{six}}$. If you have trouble, go on to the next page, where it is explained and then try it again. Show the steps and diagram.

If you had trouble with #19, below is an abbreviated picture of of the steps. The multiplication shown on the left is $105_{\text{six}} \times 123_{\text{six}}$ and the multiplication shown on the right is $123_{\text{six}} \times 105_{\text{six}}$ In both cases, I started multiplying with the biggest block at the bottom and moved left to right. In both cases, the final result is the same, and after all exchanges have been made, there is 1 long block, 3 blocks, 4 flats and 3 units, which is 13403_{six}, written in base six. It takes a lot of practice to really do this with the blocks. But it's fun once you get it!

	F	U	U	U	U
x	F	L	L	U	U
LB	F	F	F	F	F
B	L	L	L	L	L
B	L	L	L	L	L
F	U	U	U	U	U
F	U	U	U	U	U
F	U	U	U	U	U

	F	L	L	U	U
x	F	U	U	U	U
LB	B	B	F	F	F
F	L	L	U	U	U
F	L	L	U	U	U
F	L	L	U	U	U
F	L	L	U	U	U
F	L	L	U	U	U

Exercise 20

Use base three blocks to multiply $102_{\text{three}} \times 120_{\text{three}}$. Show steps like done in the example shown above.

We can shorten up this process of multiplying with the blocks even further. In exercise 18, you showed the products obtained when multiplying two base blocks together. The interesting thing is that a flat times a long is a block, no matter what base you are in. You can actually make up a multiplication chart for using blocks.

Exercise 21

Complete the rest of the multiplication chart below.

$\times$	**U**	**L**	**F**	**B**
U				
L	L		B	
F				FB
B		LB		

In our first example with the blocks, we computed $23_{\text{four}} \times 32_{\text{four}}$. Since we are multiplying 2 longs and 3 units by 3 longs and 2 units, the multiplication can be written as $(2L + 3U) \times (3L + 2U)$. Applying the distributive property (or FOIL), we get

$(2L + 3U) \times (3L + 2U)$

$= 2L \times 3L + 2L \times 2U + 3U \times 3L + 3U \times 2U$

$= (\mathbf{2 \times 3})(\mathbf{L \times L}) + (\mathbf{2 \times 2})(\mathbf{L \times U}) + (\mathbf{3 \times 3})(\mathbf{L \times U}) + (\mathbf{3 \times 2})(\mathbf{U \times U})$

= 6F + 4L + 9L + 6U

= 6F + 13L + 6U **

At this point exchanges have to be made. In Base Four,

6F = 1B + 2F, 13L = 3F + 1L and 6U = 1L + 2U

= 1B + 2F + 3F + 1L + 1L + 2U

= 1B + 5F + 2L + 2U

The 5 flats are now exchanged for a block and a flat

= 1B + 1B + 1F + 2L + 2U

= 2B + 1F + 2L + 2U

which is written as 2122_{four} in base four. This is the same answer we got before.

**If I were doing 23×32 in a different base, all of steps up to this point are exactly the same. For instance, on the next page, the same basic problem is done in base six. All of the steps are the same, but notice that once exchanges are made, they are done with base six blocks in mind.

Also, it isn't necessary to write out the step that is in bold type. That's an application of the commutative and associative properties, and I wanted to show that step for clarification.

Here is the same basic problem done in Base Six.

$$
\begin{aligned}
&23_{\text{six}} \times 32_{\text{six}} = (2L + 3U) \times (3L + 2U) \\
&= 2L \times 3L + 2L \times 2U + 3U \times 3L + 3U \times 2U \\
&= (2 \times 3)(L \times L) + (2 \times 2)(L \times U) + (3 \times 3)(L \times U) + (3 \times 2)(U \times U) \\
&= 6F + 4L + 9L + 6U \\
&= 6F + 13L + 6U \\
&= 1B + 2F + 1L + 1L \\
&= 1B + 2F + 2L \\
&= 1220_{\text{six}}
\end{aligned}
$$

Here is the same basic problem done in Base Nine.

$$
\begin{aligned}
&23_{\text{nine}} \times 32_{\text{nine}} = (2L + 3U) \times (3L + 2U) \\
&= 2L \times 3L + 2L \times U3U \times 3L + 3U \times 2U \\
&= 6F + 4L + 9L + 6U \\
&= 6F + 13L + 6U \\
&= 6F + 1F + 4L + 6U \\
&= 7F + 4L + 6U \\
&= 746_{\text{nine}}
\end{aligned}
$$

Exercise 22

Show the steps using the blocks as shown in the previous examples to multiply the following.

a. $23_{\text{five}} \times 32_{\text{five}}$

b. $42_{\text{eight}} \times 53_{\text{eight}}$

Here is an example of using this method for numerals with more than two digits each. You still use the distributive property. Each term in the first parentheses is multiplied by each term in the second parentheses.

$$
\begin{aligned}
&212_{\text{five}} \times 102_{\text{five}} = (2F + 1L + 2U) \times (1F + 2U) \\
&= 2F \times 1F + 2F \times 2U + 1L \times 1F + 1L \times 2U + 2U \times 1F + 2U \times 2U \\
&= 2LB + 4F + 1B + 2L + 2F + 4U \\
&= 2LB + 1B + 6F + 2L + 4U \\
&= 2LB + 1B + 1B + 1F + 2L + 4U \\
&= 2LB + 2B + 1F + 2L + 4U \\
&= 22124_{\text{five}}
\end{aligned}
$$

$212_{\text{eight}} \times 102_{\text{eight}} = (2F + 1L + 2U) \times (1F + 2U)$
$= 2F \times 1F + 2F \times 2U + 1L \times 1F + 1L \times 2U + 2U \times 1F + 2U \times 2U$
$= 2LB + 4F + 1B + 2L + 2F + 4U$
$= 2LB + 1B + 6F + 2L + 4U$
$= 21624_{\text{eight}}$

In this last example, we got away without having to make any exchanges. We aren't so lucky in this next example, which is in base three. Base Three is harder to work with since it only takes three of something before you have to make an exchange. Sometimes it

seems that all you are doing is exchanging!

$212_{\text{three}} \times 102_{\text{three}} = (2F + 1L + 2U) \times (1F + 2U)$
$= 2F \times 1F + 2F \times 2U + 1L \times 1F + 1L \times 2U + 2U \times 1F + 2U \times 2U$
$= 2LB + 4F + 1B + 2L + 2F + 4U$
$= 2LB + 1B + 6F + 2L + 4U$
$= 2LB + 1B + 2B + 2L + 1L + 1U$
$= 2LB + 3B + 3L + 1U$
$= 2LB + 1LB + 1F + 1U$
$= 3LB + 1F + 1U$
$= 1FB + 1F + 1U$
$= 100101_{\text{three}}$

Exercise 23

Show the steps using the blocks as shown in the previous examples to multiply:

$212_{\text{four}} \times 102_{\text{four}}$

Exercise 24

Show the steps using the blocks as shown in the previous examples to multiply:

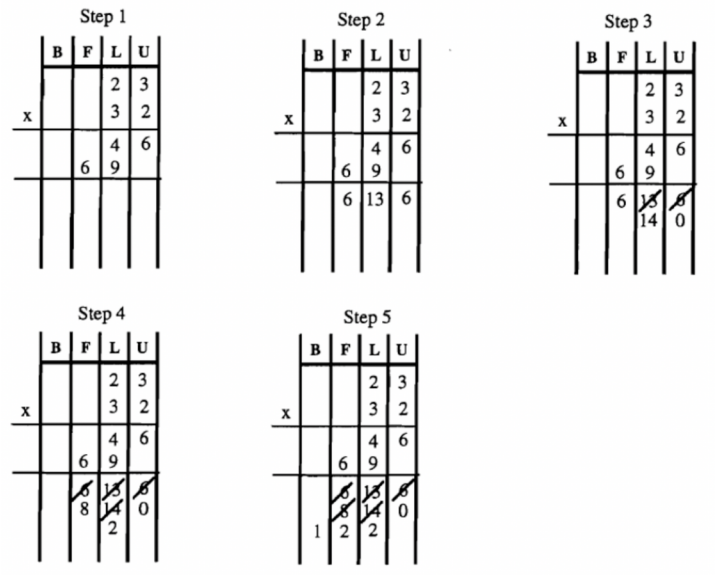
a. $361_{\text{nine}} \times 15_{\text{nine}}$

b. $111_{\text{two}} \times 11_{\text{two}}$

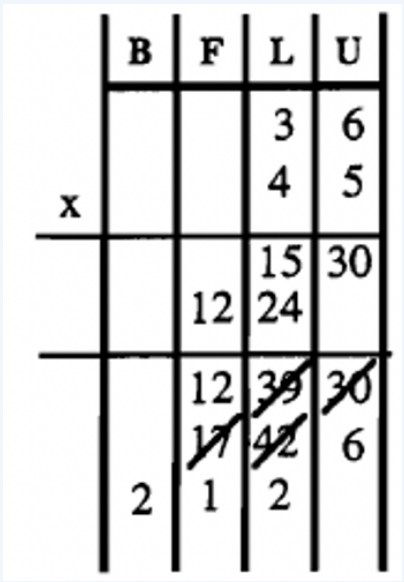
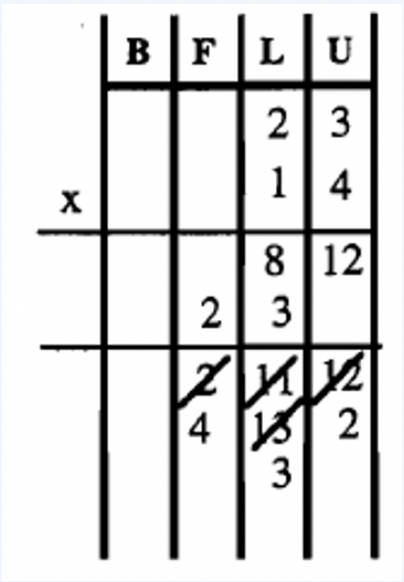
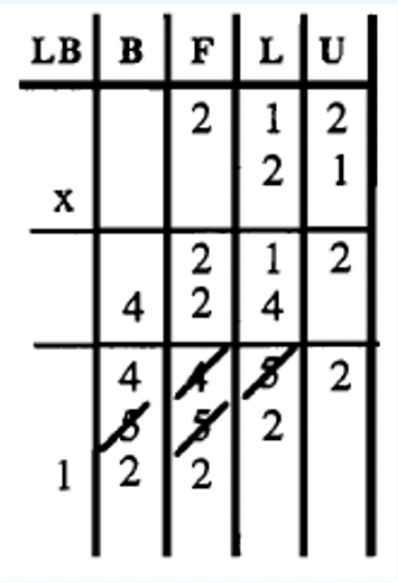
Another way to multiply with the blocks is by using a chart. This is similar to how we did it for addition. It's actually very similar to doing it using the regular multiplication algorithm, except we don't have to write the carry as we go. You simply multiply each digit of one numeral by each digit in the other numeral, and put it in the proper column. Then, you add the like terms, and still do exchanges. Here is an example writing it out with the distributive property. Below, contrast with the method using a chart.

$23_{\text{six}} \times 32_{\text{six}} = (2L + 3U) \times (3L + 2U)$
$= 2L \times 3L + 2L \times 2U + 3U \times 3L + 3U \times 2U$
$= 6F + 4L + 9L + 6U$
$= 6F + 13L + 6U$
$= 1B + 2F + 1L + 1L$
$= 1B + 2F + 2L$
$= 1220_{\text{six}}$

In step one, the individual multiplications are done. If you get a 2-digit numeral, just write it in. No need to carry. In step 2, the like terms are added. Then, exchanges are made. I did the exchanges by exchanging the units first. In step 3, When 6 units were exchanged, I had 0 units and 1 more long, so I crossed off the 13 and put 14 in the next column to the left. In step 4, I exchanged 14 longs for 2 flats and 2 longs, so I crossed off 6 in the flat column, and added 2 more to get 8, and crossed off 14 in the long column since there were only 2 left. In the last step, 8 flats are exchanged for 1 block and 2 flats. I've shown the individual steps, all the work is shown in Step 5. That's all the space it takes.

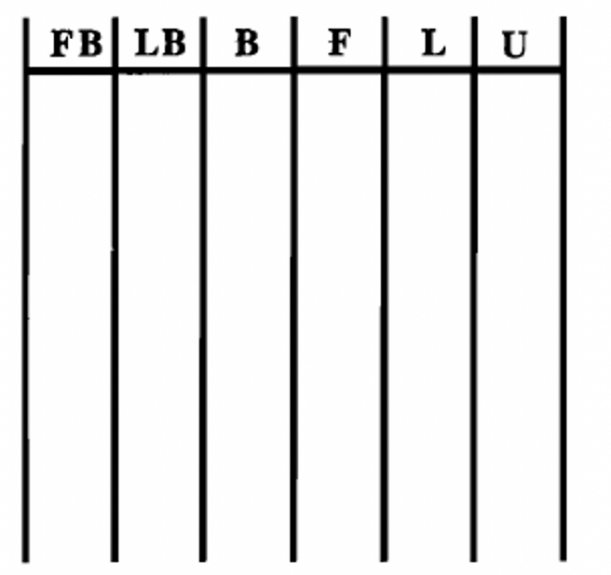


Here are three more examples of using the charts to multiply with the blocks.

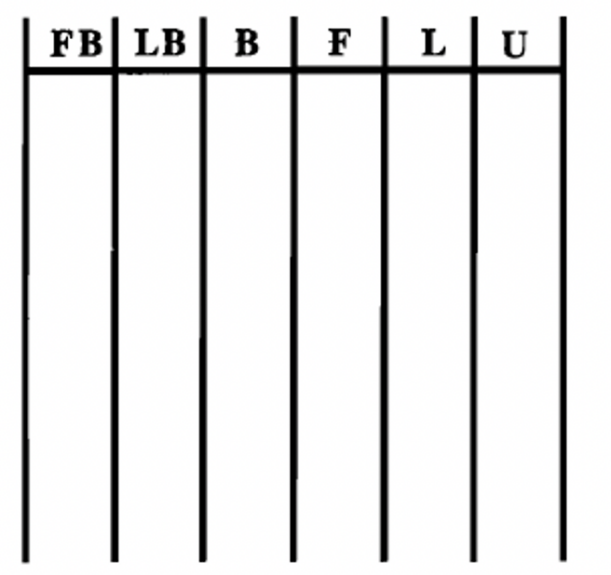
$36_{\text{eight}} \times 45_{\text{eight}}$	$23_{\text{five}} \times 14_{\text{five}}$	$212_{\text{three}} \times 22_{\text{three}}$
		
$= 2126_{\text{eight}}$	$= 432_{\text{five}}$	$= 12222_{\text{three}}$

Exercise 25

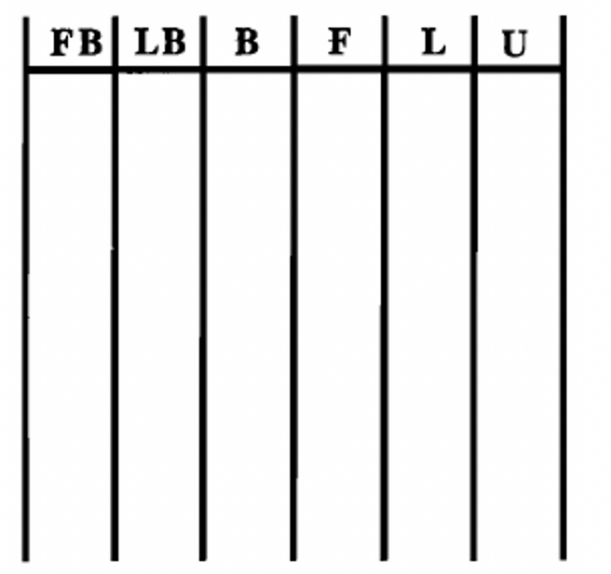
Use the base block charts, writing the numbers in terms of actual base blocks at first, as shown in the above examples, to compute the following. Show all your work and steps below.



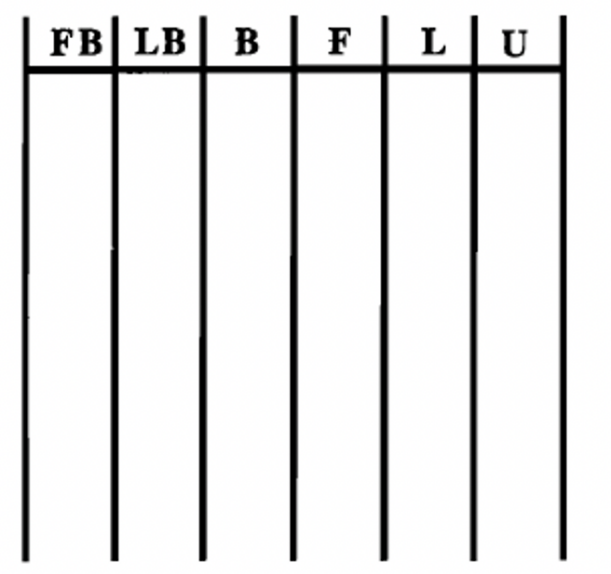
a. $34_{\text{six}} \times 25_{\text{six}}$ =



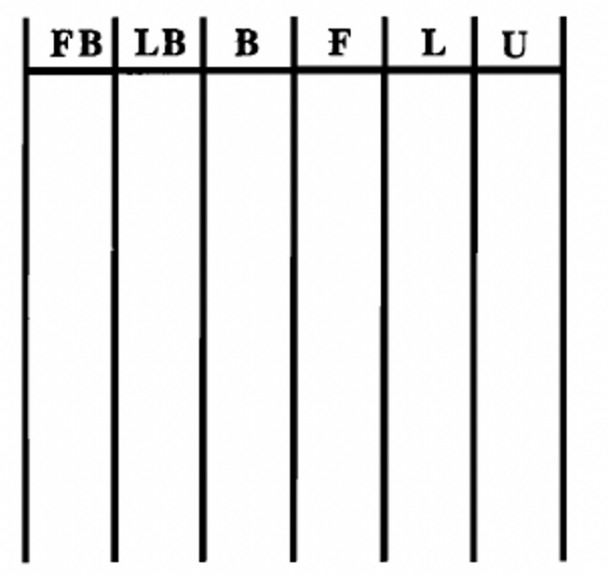
b. $32_{\text{eleven}} \times 4T_{\text{eleven}}$ =



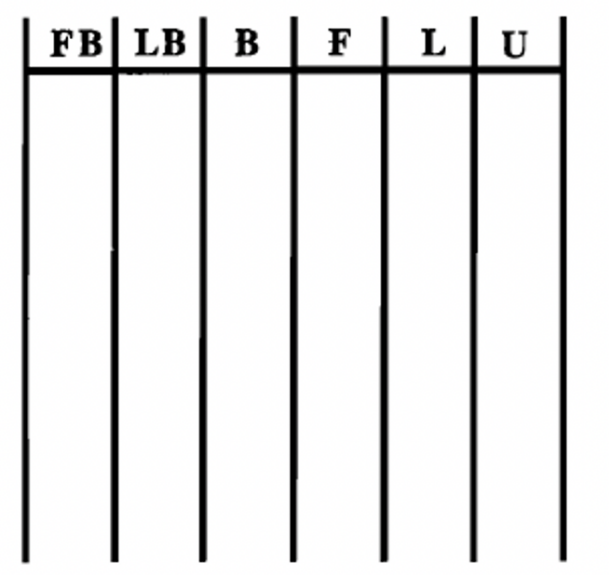
c. $111_{\text{two}} \times 101_{\text{two}}$ =



d. $1E_{\text{twelve}} \times 53_{\text{twelve}}$ =



e. $212_{\text{three}} \times 22_{\text{three}}$ =



f. $23_{\text{four}} \times 32_{\text{four}}$ =

Eventually, it would be nice to be able to do the multiplications without having to think in terms of the actual blocks. The traditional algorithm could be used. To do this, you have to make sure you immediately convert to the given base before writing the answer. For instance, $4_{\text{five}} \times 4_{\text{five}} = 31_{\text{five}}$ because 4 units times 4 units is 16 units, which in base five converts to 3 longs and 1 unit, which is written as 31_{five} in base five. A good start is to make up some multiplication tables in other bases. A base seven multiplication table is shown on the next page. Keep in mind that there are seven digits in base seven, so you'll have a table to fill in that is 7 rows by 7 columns. Because of the commutative property, there is symmetry in the table, which means many answers are duplicated. Furthermore, multiplying by 0 or 1 is trivial. At the top of the table, we'll indicate it is in base seven, and then leave off writing the base to the right and below each numeral.

Table $4.10.1$: Base Seven Multiplication Table

$\times$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	31	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

Make sure you don't read these as base ten numerals. 26 means "two six, base seven", which is 2 longs and 6 units, or 20 base ten units! An easy way to get started is to do the two rows and two columns where you multiply by 0 and 1, then, go down the diagonal, from upper left to bottom right. Then, fill in the numbers above the diagonal. Using the commutative property, fill in the bottom of the diagonal.

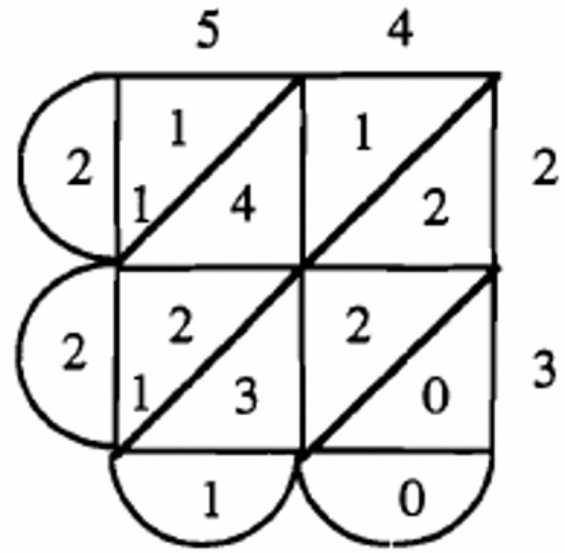
Exercise 26

Make up a multiplication table for each base indicated.

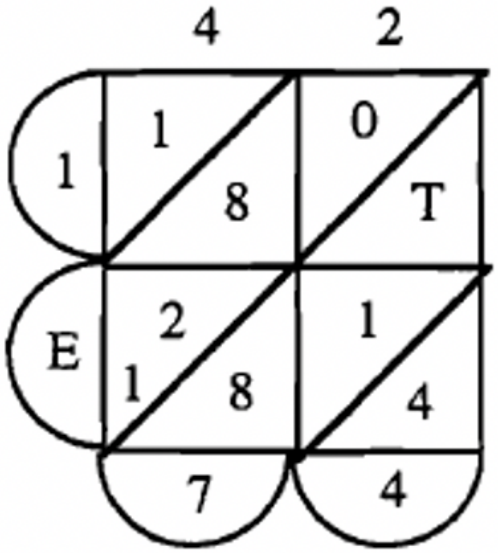
a. Base Two Multiplication Table	b. Base Three Multiplication Table
c. Base Six Multiplication Table	d. Base Eight Multiplication Table
e. Base Twelve Multiplication Table	

If you can quickly figure out the answer to multiplying two single digits in any given base, you can now use a lattice or the standard algorithm to do multiplication in different bases. Here are some examples of multiplying numbers in different bases using the lattice method.

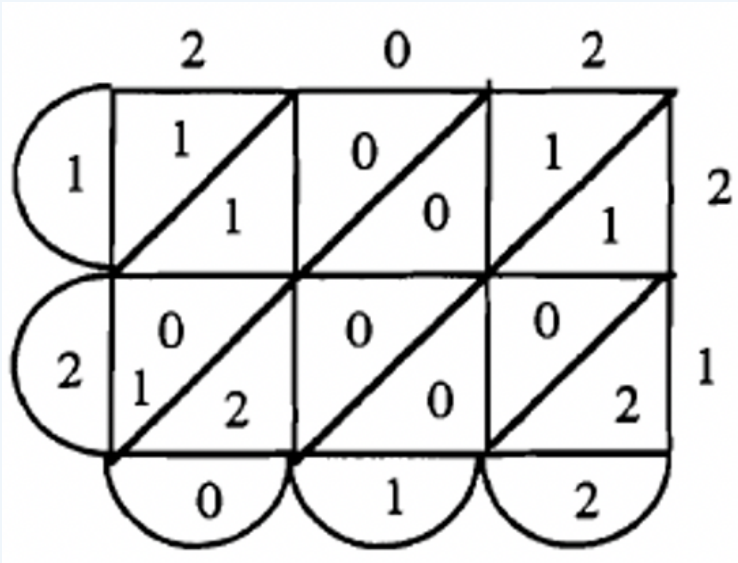
$54_{\text{six}} \times 23_{\text{six}} = 2210_{\text{six}}$



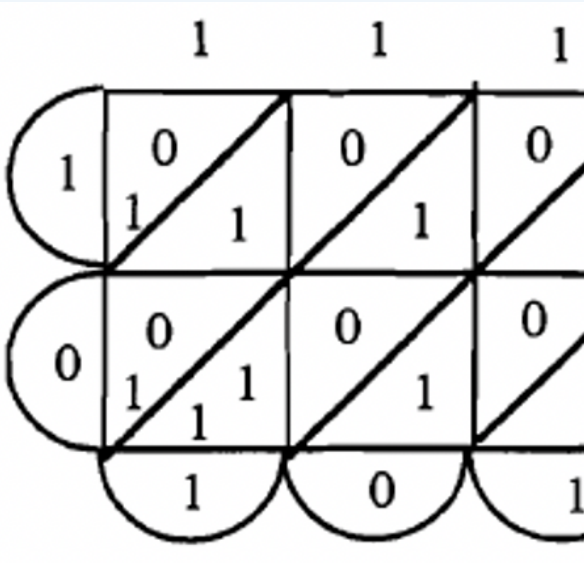
$42_{\text{twelve}} \times 58_{\text{twelve}} = 1E74_{\text{twelve}}$



$202_{\text{three}} \times 21_{\text{three}} = 12012_{\text{three}}$

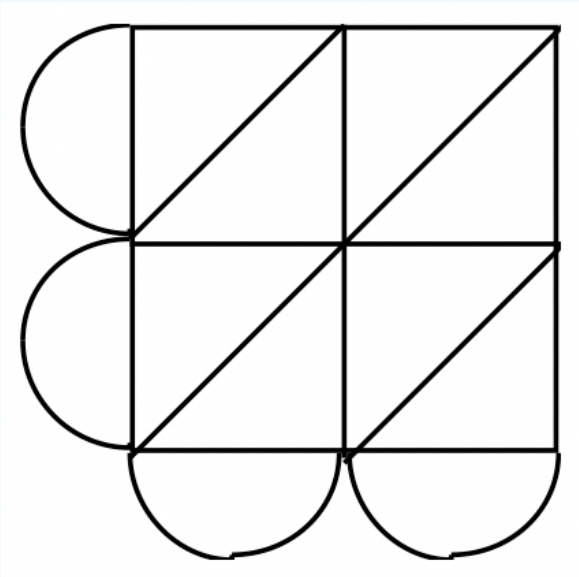


$111_{\text{two}} \times 11_{\text{two}} = 10101_{\text{two}}$

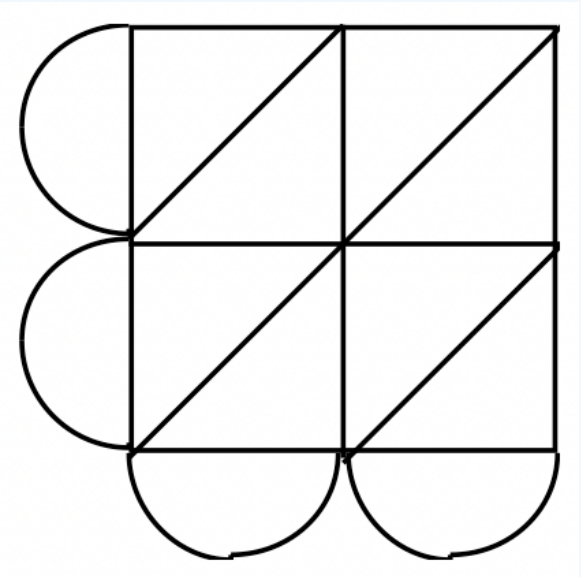


Exercise 27

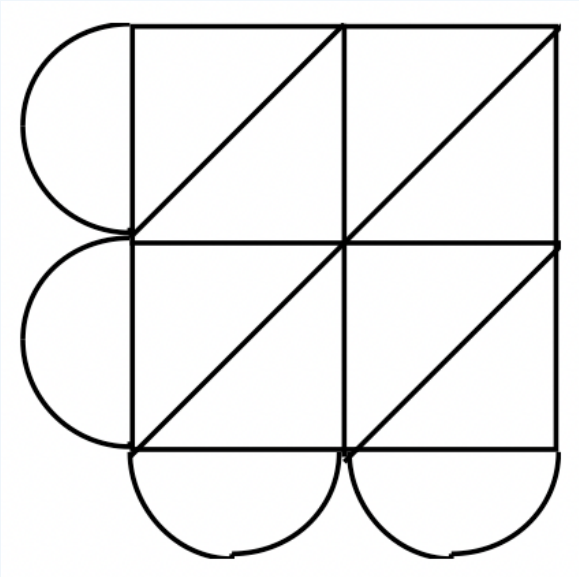
Use a lattice to multiply each of the numbers in the base given.



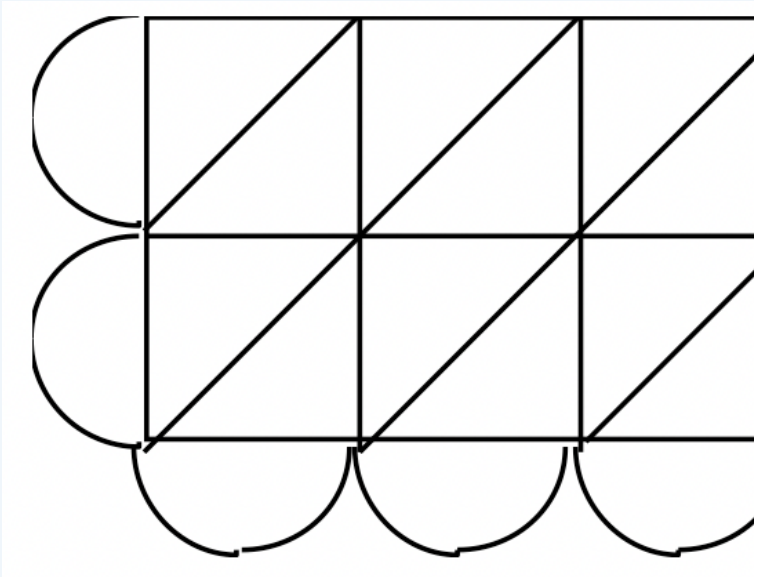
a. $32_{\text{five}} \times 14_{\text{five}} =$



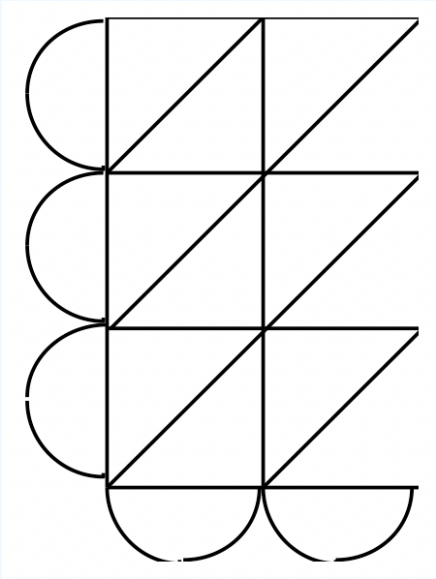
b. $93_{\text{thirteen}} \times 27_{\text{thirteen}} =$



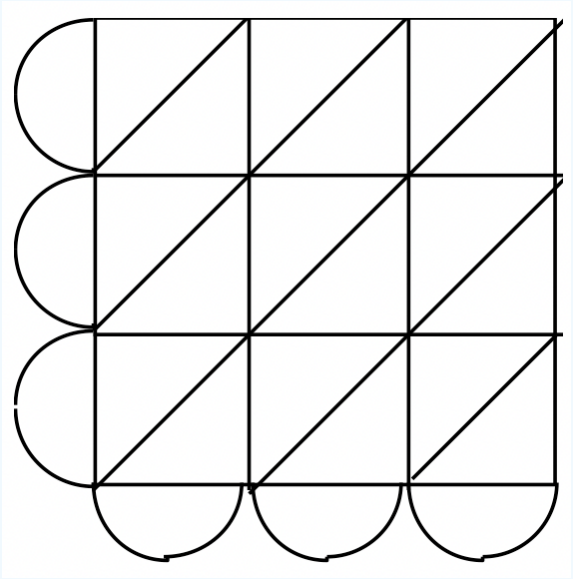
c. $12_{\text{three}} \times 21_{\text{three}} =$



d. $524_{\text{seven}} \times 23_{\text{seven}} =$



e. $23_{\text{seven}} \times 524_{\text{seven}} =$



f. $212_{\text{four}} \times 132_{\text{four}} =$

Finally, you could use the standard multiplication algorithm with carrying. This is the shortest algorithm in terms of space on the paper, but you have to be fairly adept at mental calculations, which doesn't come naturally to most of us when working in other bases.

Before going on to the standard multiplication algorithm, we need to notice a few more things. First of all, remember when we were working with the blocks to figure out a $L \times L$, etc.? When something was multiplied by a long, everything moved up one place value. In base ten, when you multiply a whole number by 10, the effect is that the answer is the original number with a zero tacked on to the end. That is, every digit moved up a place value. This is true in all bases. For example, $524_{\text{seven}} \times 10_{\text{seven}} = 5240_{\text{seven}}$ and $1T6_{\text{twelve}} \times 10_{\text{twelve}} = 1T60_{\text{twelve}}$. If you are multiplying by 100 in any base, two zeroes are tacked on to the end, and so on. This principle can be used to multiply 2000×400 by taking advantage of the commutative and associative properties of multiplication as shown below:

$$2000 \times 400 = 2 \times 1000 \times 4 \times 100 = 2 \times 4 \times 1000 \times 100 = 8 \times 100000 = 800000.$$

Of course, there is no need to write all the steps out. Simply multiply 2 times 4 and tack on five zeroes. Here's one in base eight: $3000_{\text{eight}} \times 500_{\text{eight}} = 1700000_{\text{eight}}$.

The **Partial Products Algorithm** uses the above fact. Doing the partial products algorithm is similar to how we used the distributive property to multiply with the blocks (back in exercise 22). For instance, to multiply, rewrite the problem like this:

$$\begin{aligned} 346 \times 72 &= (300 + 40 + 6) \times (70 + 2) \\ &= (300 + 40 + 6) \times 70 + (300 + 40 + 6) \times 2 \\ &= 300 \times 70 + 40 \times 70 + 6 \times 70 + 300 \times 2 + 40 \times 2 + 6 \times 2 \\ &= 21000 + 2800 + 420 + 600 + 80 + 12 \\ &= 24912 \end{aligned}$$

The partial products are shown in the third line in the example above. They are:

$$300 \times 70, 40 \times 70, 6 \times 70, 300 \times 2, 40 \times 2 \text{ and } 6 \times 2$$

It's easier to write this in a vertical format. I'll show this problem two ways. It doesn't matter which partial products you multiply first. The second way it is shown is more similar to our standard algorithm, where we start with the unit's value of the bottom number and eventually work up to the larger place values.

346	
$\times\underline{72}$	
21000	(300×70)
2800	(40×70)
420	(6×70)
600	(300×2)
80	(40×2)
$\underline{12}$	(6×2)
24912	

346	
$\times\underline{72}$	
12	(2×6)
80	(2×40)
600	(2×300)
420	(70×6)
2800	(70×40)
$\underline{21000}$	(70×300)
24912	

On the left the first three partial products are 346×70 and the second three partial products are 346×2. On the right, the first three partial products are 2×346 and the second three partial products are 70×346.

Our standard algorithm is simply a shortening up of the partial products algorithm. We don't write all the zeroes and we doing the carrying involved with adding more than one partial product at a time in our head. Sure, it's shorter, but it is easier to make mistakes.

346	
$\times\underline{72}$	
692	(2×346)
$\underline{24220}$	(70×346)
24912	

Usually we just move over one place to the left to account for the zero in the one's place and don't write the zero in 24220. We "think" 7 times 346.

Exercise 28

For each problem, write out the partial products in a vertical format to multiply the numbers together. Show your work, identifying the multiplication for each partial product as shown in the example on the bottom of the previous page.

a. 538 $\times\underline{34}$	b. 257 $\times\underline{941}$

Finally, I'll provide some examples of what the multiplication looks like when the standard algorithm is used to multiply numbers in different bases. You have to remember to write down the number and all carries in the base you are working in. We tend to always put the numeral with the most digits on the top, but this is not necessary. Here are three ways to multiply $563_{\text{eight}} \times 24_{\text{eight}}$.

IMPORTANT NOTE:

Each numeral below is in base eight, even though the "eight" isn't written in each intermediate step. E.g., 4×3 in the first multiplication below, is done in base 8; therefore $4 \times 3 = 14$.

563_{eight}	24_{eight}	563_{eight}	
$\times\underline{24_{\text{eight}}}$	$\times\underline{563_{\text{eight}}}$	$\times\underline{24_{\text{eight}}}$	
2714	74	14	(4×2)
$\underline{13460}$	1700	300	(4×60)
16374_{eight}	$\underline{14400}$	2400	(4×500)
	16374_{eight}	60	(20×3)
		1400	(20×60)
		$\underline{12000}$	(20×500)
		16374_{eight}	

Multiply the following numbers together using **two different algorithms** (or switching the order of the numbers around as shown above). Show your work. Do not do it by converting each number to base ten, multiplying them together, and then convert back to the given base. Use each algorithm at least one time; lattice, partial products, standard algorithm, using the blocks with charts,

using the distributive property with or without the blocks. You have lots of choices, but always keep in mind what base you are working in. You need to know all the algorithms for the exam so be sure to try them all. Okay, have fun!

Exercise 29

$23_{\text{five}} \times 43_{\text{five}}$

First way:	Second way:

Exercise 30

$612_{\text{seven}} \times 35_{\text{seven}}$

First way:	Second way:

Exercise 31

$1011_{\text{two}} \times 1101_{\text{two}}$

First way:	Second way:

Exercise 32

$2121_{\text{three}} \times 202_{\text{three}}$

First way:	Second way:

Exercise 33

$4T1_{\text{tweleve}} \times 27_{\text{tweleve}}$

First way:	Second way:

Exercise 34

$423_{textsix} \times 145_{\text{six}}$

First way:	Second way:

Exercise 35

$2212_{\text{three}} \times 210_{\text{three}}$

First way:	Second way:

Exercise 36

$5421_{\text{six}} \times 44_{\text{six}}$

First way:	Second way:

Exercise 37

$10110_{\text{two}} \times 1111_{\text{two}}$

First way:	Second way:

Exercise 38

$E4T_{\text{twelve}} \times E2_{\text{twelve}}$

First way:	Second way:

Exercise 39

$6661_{\text{eight}} \times 72_{\text{eight}}$

First way:	Second way:

Exercise 40

$1212_{\text{four}} \times 302_{\text{four}}$

First way:	Second way:

Exercise 41

$4234_{\text{five}} \times 244_{\text{five}}$

First way:	Second way:

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4.11: The Meaning of Division

You will need: ***Centimeter Strips*** *(Material Cards 16A-16L) and* ***Base Blocks*** *(Material Cards 3 - 15)*

In these first few exercises, you'll be using c-strips to explore division.

Exercise 1

In this first exercise, we will divide $H \div Y$. H, the part before the division sign, is called the **DIVIDEND**, and Y, the part after the division sign, is called the **DIVISOR**. One way to do this division is to find the maximum number of times the divisor (Y) can be subtracted from the dividend (H). Take out a hot pink c-strip (H). To figure out how many yellow c-strips could be subtracted, make a train consisting of as many yellow c-strips (Y) as possible such that the train you make is shorter or equal to the length of H, *and* if you were to add one more yellow c-strip, the train would be longer than the hot pink c-strip.

a. How many yellow c-strips did you use? _____

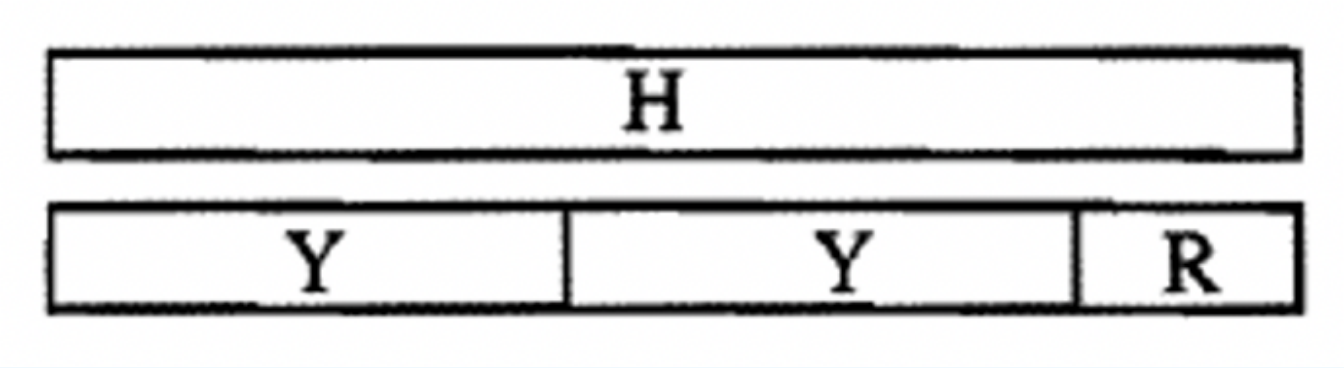
This number is called the **QUOTIENT**.

The **quotient** is the maximum number of times the **divisor** can be subtracted from the **dividend**.

b. If the train you made is shorter than H, what color c-strip could
you add to that train to make it equal in length to the hot pink c-strip? _____

This piece, which must be shorter than the divisor, is called the **REMAINDER**.

Put the hot pink strip next to the train you formed. It should look like this:



To compute $H \div Y$, we need to figure out what goes in the blanks to make the equation in part c, below, true. The first blank must contain the largest c-strip possible that will make the equation true, *and* the second blank must be shorter than the yellow c-strip. We made a train of yellows, with at most one extra piece to form the length of H. Notice this is a train of yellows (which can be formed into a rectangle) plus an extra piece. Form the two yellow strips into a rectangle and find the c-strip that goes across the width. That c-strip goes in the first blank, and the extra piece (R) goes in the second blank. Fill in the blanks of the equation below. This equation should make sense by looking at the picture above showing H and the train are the same length.

c. H = ___ $\cdots Y$ + ___

The **QUOTIENT** is in bold, and the **REMAINDER** is underlined.

We can now conclude that: $H \div Y = \mathbf{R}$ remainder $\underline{R}$, since $H = \mathbf{R} \cdots Y + \underline{R}$

This can be written as a statement about numbers by translating each c-strip to a number.

Using numbers in the above equation, $12 \div 5 = \mathbf{2}$ remainder $\underline{2}$, since 12 = $\mathbf{2} \cdots 5 + \underline{2}$

Exercise 2

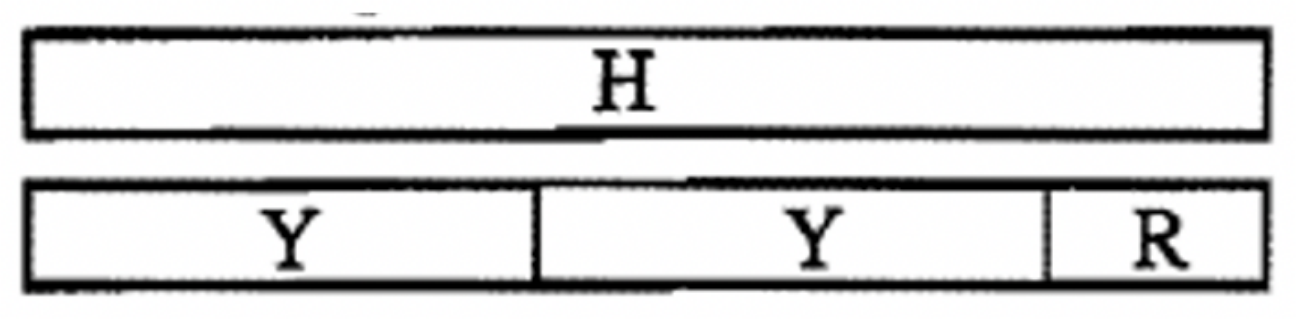
For a - e, do the division using the c-strips to make a train of as many of the divisor as possible such that the length of the train you make is less than or equal to the length of the dividend, *and* if you were to add one more of the divisor to the train, it would be longer than the dividend. If the train ends up being the same length as the dividend, there will be no remainder and the second blank will be empty. Draw a diagram showing the c-strip that is the dividend and under it, draw the train you formed. Do this the same way we did it for exercise 1. Follow the example below.

Example: $H \div Y$

$H \div Y = \mathbf{R}$ remainder $\underline{R}$, since H = $\mathbf{R} \cdots$ Y + $\underline{R}$

$12 \div 5 = \mathbf{2}$ remainder $\underline{2}$, since 12 = $\mathbf{2} \cdots$ 5 + $\underline{2}$

Show diagram below:



a. $B \div R$:

$B \div R$ = __ remainder __ since B = __ ··· R + __

__ ÷ __ = __ remainder __ since __ = __ ··· __ + __

Show diagram below:

b. S ÷ L:

S ÷ L = __ remainder __ since S = __ ··· L + __

__ ÷ __ = __ remainder __ since __ = __ ··· __ + __

Show diagram below:

c. $N \div P$:

$N \div P$ = __ remainder __ since N = __ ··· P + __

__ ÷ __ = __ remainder __ since __ = __ ··· __ + __

Show diagram below:

d. $K \div P$:

$K \div P$ = __ remainder __ since K = __ ··· P + __

__ ÷ __ = __ remainder __ since __ = __ ··· __ + __

Show diagram below:

e. $D \div L$:

$D \div L$ = __ remainder __ since D = __ ··· L + __

__ ÷ __ = __ remainder __ since __ = __ ··· __ + __

Show diagram below:

Exercise 3

Count out 36 unit blocks. How would you divide the 36 blocks by 2? Explain how you would do it, and show a picture of what you did. What is the answer?

Exercise 4

Get out your base four blocks now, and trade in 36 units for a number in base four.

a. Write 36 as a base four numeral: _____

Decide how to divide this by 2 using base four blocks.

b. Explain how you would do it, and show a picture of what you did.

c. What is the answer in base four? _____

d. Write the division problem and the quotient (answer) in base four: _____$\div 2_{\text{four}}$ = _____

Exercise 5

Get out your base three blocks now, and trade in 36 units for a number in base three.

a. Write 36 as a base three numeral: _____

Decide how to divide this by 2 using base three blocks.

b. Explain how you would do it, and show a picture of what you did.

c. What is the answer in base three? _____

d. Write the division problem and the quotient (answer) in base three: _____$\div 2_{\text{three}}$ = _____

Exercise 6

Count out 36 unit blocks. How would you divide the 36 blocks by 3? Explain how you would do it, and show a picture of what you did. What is the answer?

Exercise 7

Get out your base four blocks now, and trade in 36 units for a number in base four.

a. Write 36 as a base four numeral: _____

Decide how to divide this by 3 using base four blocks.

b. Explain how you would do it, and show a picture of what you did.

c. What is the answer in base four? _____

d. Write the division problem and the quotient (answer) in base four: _____$\div 3_{\text{four}}$ = _____

Exercise 8

Count out 48 unit blocks. How would you divide the 48 blocks by 16? Explain how you would do it, and show a picture of what you did. What is the answer?

There are two basic ways of thinking about division. The two word problems below both require the division $18 \div 3$, but the problems are really quite different.

Word Problem A:

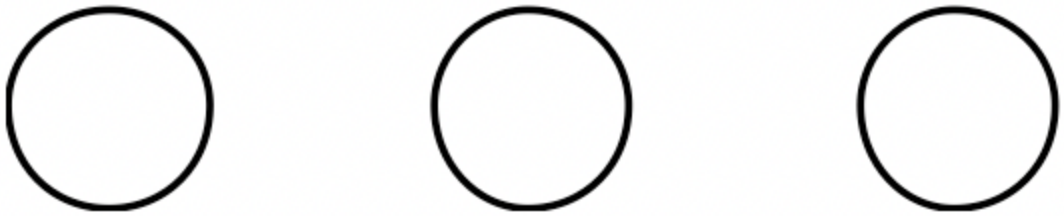
If you have 18 cookies and want to divide the cookies up equally amongst your three children, how many cookies should you give to each child?

Word Problem B:

If you have 18 cookies and want to make baggies having three cookies in each to put in lunch boxes, how many baggies will you be able to make?

Both require the division problem $18 \div 3$, and the answer to each problem is 6. But the picture used to solve each is very different.

In Word Problem A, *you are told how many kids there are, and you are asked to figure out how many cookies each child should be given.* Imagine 3 children in front of you. You need to disburse the cookies amongst the 3 children so that each receives the same number of cookies. To begin, you might draw a model to represent three kids. It might look something like this:



You know there are 3 kids and each will be given the same number of cookies so that the total number of cookies they all have is 18. After disbursing, you might show the picture like this:



This is a way to illustrate that each of the 3 children has 6 cookies. If you prefer, you can actually draw 6 dots or 6 cookies in each circle. It's up to you.

Note that if you look at the model drawn, it is simply the multiplication problem 3 ··· 6, since there are 3 sets of 6 added together. The *interpretation* of division Word Problem A represents is called:

Partitioning into (Equal) Subsets

In this interpretation, *the divisor is the number of subsets in which to disburse something*. The dividend, 18, must be divided amongst 3 equal subsets (3 is the divisor). The quotient (answer to the division problem) will be the number of elements (6) placed in each subset (the kids, in this case.) The task is to **partition 18 cookies into 3 equal subsets**. In this interpretation, you are given how many subsets there are (from reading the problem in this case, each kid represents one subset), but you do not how many are in each subset (that is the question being asked in this case, how many cookies each child should receive). You do the division to figure out how many elements will end up in each subset. After doing the division, **the number of elements *IN* each subset is the answer to the problem.**

One way to distinguish that a problem is suggesting this interpretation is to consider whether the information in the problem provides you with the *number of subsets*. If it does, as in Word Problem A, then it is the *partitioning into equal subsets* interpretation.

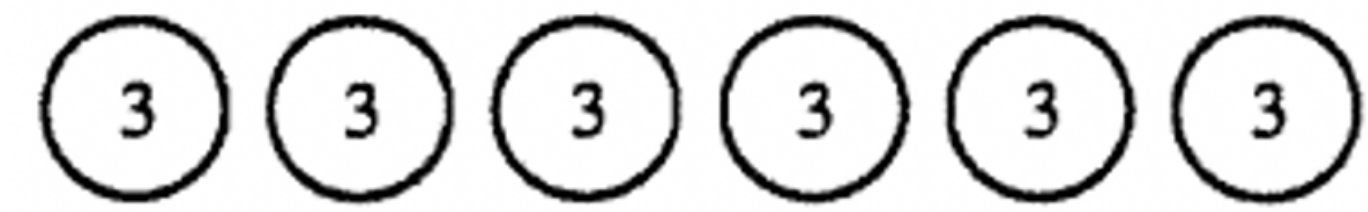
Okay, let's look at Word Problem B now.

Word Problem B:

If you have 18 cookies and want to make baggies having three cookies in each to later put in lunch boxes, how many baggies will you be able to make?

In Word Problem B, *you don't know how many subsets there will be*. In fact, you are being ***asked*** how many subsets you will be able to make, so the partitioning into subsets interpretation wouldn't apply.

What you **do** know is how many of something will be in each subset. You would place three cookies in a baggie, then 3 more in another baggie, and so on. You won't know how many subsets you will have until you finish disbursing the cookies. You would take three cookies away at a time and put them in a baggie, until you ran out of cookies. After doing this, a picture you might draw to illustrate this procedure might look like this:



Again, if you prefer to show three dots in each circle, as opposed to just writing the numeral 3, in each circle, that is perfectly fine. Either way, it should be clear from the drawing what procedure was used.

After disbursing all the cookies, you would count up how many baggies you were able to make, and that would give you the answer. Note the answer is 6, in this case. The model shown represents the multiplication $6 \cdots 3$, since there are 6 baggies of 3 cookies each.

The *interpretation* of division Word Problem B represents is called:

Repeated Subtraction

In this interpretation, *the divisor is the number of elements that will go into each subset*. Each time the divisor (3) is subtracted from the dividend (18), a subset is formed. After all elements have been disbursed, the quotient (answer) is found by counting how many subsets were formed. This interpretation is used when you are given how many elements are in each subset, and you are trying to find out how many subsets can be made (that is the question being asked). You do the division (by using repeated subtraction) to figure out how many subsets can be made. After doing the division, **the number of subsets you made is the answer to the problem.**

One way to distinguish that a problem is suggesting this interpretation is to consider whether the information in the problem provides you with *how many elements* go into a subset. If so, as in Word Problem B, then it is the *repeated subtraction* interpretation.

Word Problems requiring division generally fall into the category of Repeated Subtraction or Partitioning into Subsets. After reading a word problem, you should be able to analyze and interpret it as a problem that requires **Partitioning into Subsets** or **Repeated Subtraction** to solve the problem, and then be able to draw a picture using that interpretation. Also, for any division problem given, you should be able to write two distinct word problems, one for each interpretation, and draw a picture for each interpretation, and be able to explain and show all the steps necessary to solve the problem.

Exercise 9

Terry wanted to divide 100 marbles amongst his three best friends and himself. How many marbles would each of them get?

a. What division problem do you need to do to find the answer?
b. What interpretation of division does this word problem represent?
c. Explain how to use the correct interpretation to draw a picture to find the answer.
d. Draw a picture using the correct interpretation to help find the answer.
e. Write the multiplication problem shown by the picture:
f. Explain how you use the picture to find the answer to the problem. Be specific.
g. What is the answer to this problem? _____

Exercise 10

Jordan made 80 ounces of jello, and wanted to put them in 8 ounce containers. How many containers can she fill?

a. What division problem do you need to do to find the answer?
b. What interpretation of division does this word problem represent?
c. Explain how to use the correct interpretation to draw a picture to find the answer.

d. Draw a picture using the correct interpretation to help find the answer.
e. Write the multiplication problem shown by the picture:
f. Explain how you use the picture to find the answer to the problem. Be specific.
g. What is the answer to this problem? _____

Exercise 11

Angie had 144 pages of blank paper. She was making copies of a 16 page short story she wrote to send to magazines. How many complete copies could she make?

a. What division problem do you need to do to find the answer?
b. What interpretation of division does this word problem represent?
c. Explain how to use the correct interpretation to draw a picture to find the answer.
d. Draw a picture using the correct interpretation to help find the answer.
e. Write the multiplication problem shown by the picture:
f. Explain how you use the picture to find the answer to the problem. Be specific.
g. What is the answer to this problem? _____

Exercise 12

A group of four friends pooled their money together for some lottery tickets and won $500. How much does each person receive from the winnings?

a. What division problem do you need to do to find the answer?
b. What interpretation of division does this word problem represent?
c. Explain how to use the correct interpretation to draw a picture to find the answer.
d. Draw a picture using the correct interpretation to help find the answer.

Exercise 13

Sheila wanted to make as many cakes from a recipe requiring $1\frac{1}{3}$ cups of flour for each cake. She had 8 cups of flour. How many cakes could she make?

a. What division problem do you need to do to find the answer?
b. What interpretation of division does this word problem represent?
c. Explain how to use the correct interpretation to draw a picture to find the answer.
d. Draw a picture using the correct interpretation to help find the answer.
e. Write the multiplication problem shown by the picture:
f. Explain how you use the picture to find the answer to the problem. Be specific.
g. What is the answer to this problem? _____

Exercise 14

Tom won 65 baseballs and decided to divide them up evenly amongst the 13 boys on his baseball team. How many baseballs will each kid get?

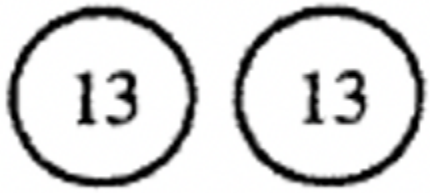
a. What division problem do you need to do to find the answer?
b. What interpretation of division does this word problem represent?
c. Explain how to use the correct interpretation to draw a picture to find the answer.
d. Draw a picture using the correct interpretation to help find the answer.
e. Write the multiplication problem shown by the picture:
f. Explain how you use the picture to find the answer to the problem. Be specific.

g. What is the answer to this problem? _____

If you are *not* presented with a word problem, but are simply given a division problem, you can use either interpretation to think about and solve the problem. Many times, it is easier to use one interpretation over another, especially if you need to draw a picture to show how you solved the problem.

Consider the following division problem: $26 \div 2$. Let's say the 26 represented 2 pennies. Would it be easier to partition 26 pennies into 2 piles and count how many are in each pile; or would it be easier to make several equal subsets containing 2 pennies each from the 26 pennies, and then to count how many subsets there are? The picture you draw to represent each interpretation is quite different. Look at the difference:

A picture using partitioning into equal subsets interpretation is shown below:



A picture using the Repeated Subtraction interpretation is shown below:



Exercise 15

For parts a - h of this exercise, consider the division problem: $48 \div 12$.

a. Which interpretation do you think would be easier to use if you needed to draw a picture?
b. **Explain why** the interpretation you chose in part a would be easier to use.
c. Make up a word problem requiring the division $48 \div 12$, such that the partitioning into subsets interpretation is used. Make sure your problem asks a question!
d. Draw a picture to solve your problem using the Partitioning into Subsets interpretation:
e. The multiplication for the picture shown in part d is: _____
f. Make up a word problem requiring the division $48 \div 12$, such that the repeated subtraction interpretation is used. Make sure your problem asks a question!
g. Draw a picture to solve your problem using the Repeated Subtraction interpretation:
h. The multiplication for the picture shown in part g is: _____

Exercise 16

For parts a - h of this exercise, consider the division problem: $200 \div 4$.

a. Which interpretation do you think would be easier to use if you needed to draw a picture?
b. **Explain why** the interpretation you chose in part a would be easier to solve this division problem. Contrast it with the other interpretation.
c. Make up a word problem requiring the division $200 \div 4$, such that the partitioning into subsets interpretation is used. Make sure your problem asks a question!
d. Draw a picture to solve your problem using the Partitioning into Subsets interpretation:

e. The multiplication for the picture using Partitioning into Subsets is: _____
f. Make up a word problem requiring the division $200 \div 4$, such that the repeated subtraction interpretation is used. Make sure your problem asks a question!
g. **Explain** what you would have to do if you had to draw a picture to solve your problem using the Repeated Subtraction interpretation. ***Do not actually draw the picture* — just explain.**
h. The multiplication for a picture using Repeated Subtraction would be:

Exercise 17

For parts a - h of this exercise, consider the division problem: $150 \div 50$.

a. Which interpretation do you think would be easier to use if you needed to draw a picture?
b. **Explain why** the interpretation you chose in part a would be easier to solve this division problem. Contrast it with the other interpretation.
c. Make up a word problem requiring the division $150 \div 50$, such that the partitioning into subsets interpretation is used. Make sure your problem asks a question!
d. **Explain** what you would have to do if you had to draw a picture to solve your problem using the Partitioning into Subsets interpretation. ***Do not actually draw the picture* — just explain!**
e. The multiplication for a picture using Partitioning into Subsets would be: ____
f. Make up a word problem requiring the division $150 \div 50$, such that the repeated subtraction interpretation is used. Make sure your problem asks a question!
g. Draw a picture to solve your problem using the Repeated Subtraction interpretation:
h. The multiplication for the picture using Repeated Subtraction is: _____

Exercise 18

For parts a - h of this exercise, consider the division problem: $\mathbf{140 \div 35}$.

a. Which interpretation do you think would be easier to use if you needed to draw a picture?
b. **Explain why** the interpretation you chose in part a would be easier to solve this division problem. Contrast it with the other interpretation.
c. Make up a word problem requiring the division $140 \div 35$, such that the partitioning into subsets interpretation is used. Make sure your problem asks a question!
d. **Explain** what you would have to do if you had to draw a picture to solve your problem using the Partitioning into Subsets interpretation. ***Do not actually draw the picture* — just explain!**
e. The multiplication for a picture using Partitioning into Subsets would be: ____
f. Make up a word problem requiring the division $140 \div 35$, such that the repeated subtraction interpretation is used. Make sure your problem asks a question!
g. Draw a picture to solve your problem using the Repeated Subtraction interpretation:
h. The multiplication for the picture using Repeated Subtraction is:

Exercise 19

For parts a - h of this exercise, consider the division problem: $\mathbf{95 \div 5}$.

a. Which interpretation do you think would be easier to use if you needed to draw a picture?
b. **Explain why** the interpretation you chose in part a would be easier to solve this division problem. Contrast it with the other interpretation.
c. Make up a word problem requiring the division $95 \div 5$, such that the partitioning into subsets interpretation is used. Make sure your problem asks a question!
d. Draw a picture to solve your problem using the Partitioning into Subsets interpretation:
e. The multiplication for the picture using Partitioning into Subsets is: _____

f. Make up a word problem requiring the division $95 \div 5$, such that the repeated subtraction interpretation is used. Make sure your problem asks a question!
g. **Explain** what you would have to do if you had to draw a picture to solve your problem using the Repeated Subtraction interpretation. ***Do not actually draw the picture* — just explain!**
h. The multiplication for a picture using Repeated Subtraction would be: _____

Exercise 20

For parts a - h of this exercise, consider the division problem: $800 \div 160$.

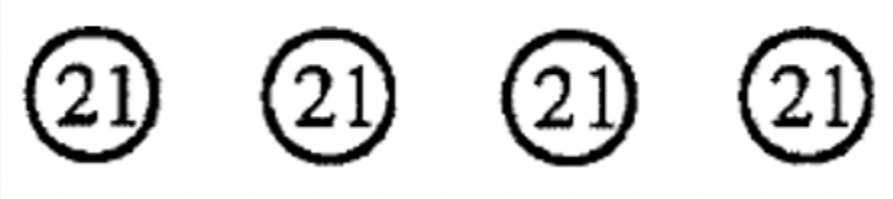
a. Which interpretation do you think would be easier to use if you needed to draw a picture?
b. **Explain why** the interpretation you chose in part a would be easier to solve this division problem. Contrast it with the other interpretation.
c. Make up a word problem requiring the division $800 \div 160$, such that the partitioning into subsets interpretation is used. Make sure your problem asks a question!
d. **Explain** what you would have to do if you had to draw a picture to solve your problem using the Partitioning into Subsets interpretation. ***Do not actually draw the picture* — just explain!**
e. The multiplication for a picture using Partitioning into Subsets would be: ___
f. Make up a word problem requiring the division $800 \div 160$, such that the repeated subtraction interpretation is used. Make sure your problem asks a question!
g. Draw a picture to solve your problem using the Repeated Subtraction interpretation:
h. The multiplication for the picture using Repeated Subtraction is:

Exercise 21

For each division problem, a picture is shown. Based on the picture, state which interpretation of division was used.

a. $84 \div 21$

Interpretation: _____



b. $85 \div 5$

Interpretation: _____



c. $30 \div 5$

Interpretation: _____



d. $30 \div 6$

Interpretation: _____



Exercise 22

For each picture, a particular interpretation of division was used to solve a division problem. Based on the interpretation and picture, decide what the original division problem was, and circle the correct one of the two choices given.

a. Repeated Subtraction



Circle the correct division problem: $56 \div 8$ or $56 \div 7$

a. Repeated Subtraction

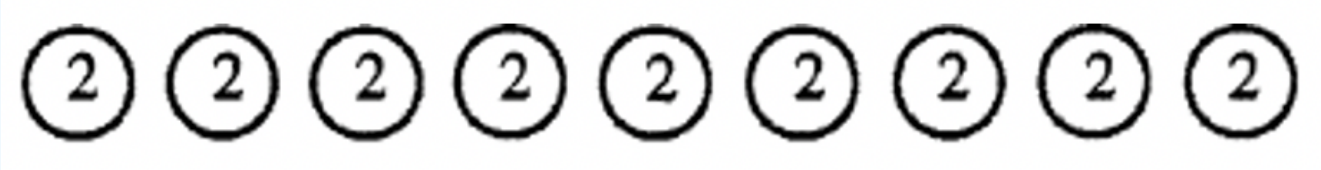


Circle the correct division problem: $56 \div 8$ or $56 \div 7$

Exercise 23

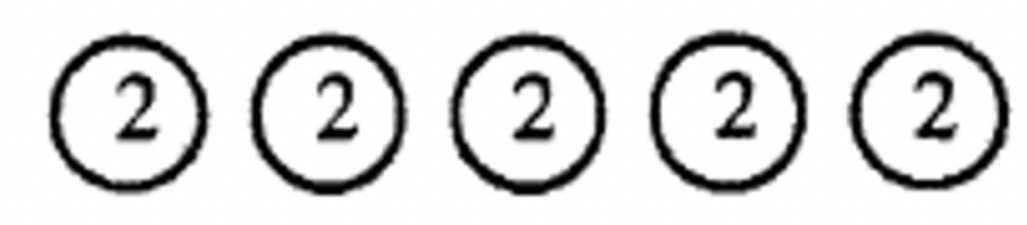
For each picture, a particular interpretation of division was used to solve a division problem. Based on that interpretation and picture, state the original division problem.

a. Partitioning into Subsets



If partitioning into subsets was used, state the division problem: _____

b. Repeated Subtraction



If repeated subtraction was used, state the division problem: _____

Exercise 24

If you drew a picture for the multiplication problem, a · b, which letter represents how subsets are made, and which letter represents how many are in each subset?

Division is the operation that undoes mutiplication. Students can rely on this fact to learn basic division facts. For instance, since 6 · 7 = 42, then $42 \div 6 = 7$ and $42 \div 7 = 6$. ***Below is a formal definition of division***

Definition: a ÷ b is defined to be the unique number, c, under the following conditions: $a \div b = c$ **if and only if a = b · c.** If there is no number, c, that exists, or if there is more than one unique number that can be put in for c, then $a \div b$ is said to be ***undefined***.

Instead of writing c, it is sometimes easier to write a blank or box in the equations to see if a unique solution exists. If you can put a unique solution in the blank to the multiplication problem, then you can go back and fill in the solution to the division problem.

Example

Use the definition of division to find the answer to $21 \div 3$.

First, note that $21 \div 3$ = ___ means 21 = 3 · ____ Second, try to fill in the blank in the multiplication problem with a unique number to make the equation, 21 = 3 · ____, true: 21 = 3 ·$\underline{7}$. If the number filled in is the only possible solution (unique), then that is the answer to the division problem. Therefore, $21 \div 3 = \underline{7}$. *You do not need to write this paragraph for your solution. Below is how to write the solution.*

Solution

Since $21 = 3 \cdot \underline{7}$, then $21 \div 3 = \underline{7}$.

Exercise 25

Use the definition of division to find the following quotients:

a. $32 \div 8$
b. $56 \div 8$
c. $32 \div 2$
d. $0 \div 13$
e. $12 \div 1$
f. $X \div 1$ (Assume X is any number.)
g. $0 \div Y$ (Assume Y is any nonzero number.

Exercise 25f illustrates that any number divided by 1 is itself.

In other words, $\mathrm{a} \div 1 = \mathrm{a}$ **for all values of a**.

Exercise 25g illustrates that zero divided by any *nonzero* whole number is 0.

In other words, $0 \div \mathrm{a} = 0$**, for all nonzero values of a**.

In the above problem, all of the answers were whole numbers. If you were asked to use the definition to find only whole number solutions, then some divisions would be undefined under whole numbers. There may, however, be rational or irrational solutions depending on the problem.

Example

Use the definition of division to find if $21 \div 5$ is defined under whole numbers. Show the solution if it is defined, or explain why not if it is not defined.

First, note that $21 \div 5$ = ___ means 21 = 5 · ____ Second, try to fill in the blank in the multiplication problem with a unique whole number to make the equation, 21 = 5 · ____, true. There is no whole number that can be put in the blank to make that equation true. Therefore, $21 \div 5$ is not defined under whole numbers. Here is how you would show the answer.

Solution

Since there is no whole number solution to make the equation, 21 = 5 · ___ true, $21 \div 5$ is not defined under whole numbers.

Exercise 26

Use the definition of division to find which of the following quotients are defined under whole numbers. Show the solution if it is defined, or explain why not if it is not defined. Part a and b are done for you. Use these as examples.

a. $45 \div 5$

Solution: Since $45 = 5 \cdot \underline{9}$, then $45 \div 5 = \underline{9}$.

b. $42 \div 9$

Solution: Since there is no whole number solution to make the equation,

42 = 9 · ___ true, $42 \div 9$ is not defined under whole numbers. *(Yes, you should write all of this out!)*

c. $48 \div 6$

Solution:

d. $35 \div 4$

Solution:

e. $48 \div 1$

Solution:

f. $55 \div 7$

Solution:

g. $0 \div 8$

Solution:

Let's explore what happens when we divided by zero.

Example

Use the definition of division to find the quotient: $8 \div 0$.

First, note that $8 \div 0$ = ___ means 8 = 0 · ____ Second, try to fill in the blank in the multiplication problem with a unique number to make the equation, 8 = 0 · ____, true. There is no number that can be put in the blank to make that equation true since any number times 0 is 0, and will never equal 8. This is how you could write the solution:

Solution

There is no number that will make the equation, 8 = 0 · ___ true, since any number put in the blank will make the right hand side of the equation zero, which can never equal the left-hand side of the equation, which is 8. Therefore, $8 \div 0$ is not defined.

Exercise 27

Use the definition of division to show that the following quotients are undefined:

a. $6 \div 0$

b. $18 \div 0$

c. $M \div 0$ (Assume $M \neq 0$)

In exercise 27, you showed that dividing a nonzero number by 0 was undefined. Now, let's explore what happens if we divide zero by zero.

Example

Use the definition of division to find the quotient: $0 \div 0$.

First, note that $0 \div 0$ = ___ means 0 = 0 · ____ Second, try to fill in the blank in the multiplication problem with a unique number to make the equation, 0 = 0 · ____, true. There is no unique number that can be put in the blank to make that equation true since any number put in the blank will make the equation true, and therefore there is no unique number that will work. This is how you could write the solution:

Solution

There is no unique number that can be put in the blank to make the equation 0 = 0 · ___ true, since any number will make it true. Therefore, $0 \div 0$ is called indeterminate.

In exercise 27, you showed that a nonzero number divided by zero is undefined. The next example showed that zero divided by zero is indeterminate.

To prove that division by zero is undefined, two cases must be shown. Below is a complete proof to show that division by zero is undefined.

Problem: Prove that division by zero is undefined

Solution: The goal is to show that $a \div 0$ is undefined for all values of a. There are two cases to consider Case 1: $a \neq 0$; Case 2: a = 0.

Case 1: Let $a \neq 0$. There is no unique number that could be put in the blank to make the equation, a = 0 · ___, true because the left-hand side of the equation is not zero (since $a \neq 0$), and any number put in the blank will make the right hand side of the equation zero. Therefore, when $a \neq 0$, $a \div 0$ is not defined.

Case 2: Let a = 0. There is no unique number that can be put in the blank to make the equation 0 = 0 · ___ true, since any number will make it true. Therefore, $0 \div 0$ is indeterminate.

It takes practice writing this proof. You'll need to practice writing it several times to master the problem. One option is to memorize the proof shown above by writing it out a few times until you really get the idea. This would make a good exam question.

Exercise 28

Prove that division by zero is undefined.

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4.12: LCM and other Topics

You will need: ***Prime Number Squares*** *(Material Cards 19A-19B)*

Composite Number Squares *(Material Cards 20A-20B)*

There are a ***finite*** *number of factors* for any given number. On the other hand, there are an ***infinite*** *number of nonzero multiples* of a number. For instance the list of multiples of 2 are all of the even numbers. It's impossible to list all of them, because this is an infinite set. A multiple of a number c is nc where n is a whole number. In other words, to find a multiple of c, multiply c by a whole number. Although 0 is a multiple of every number, we usually omit listing it as a multiple. The (nonzero) multiples of 6 are listed here: 6, 12, 18, 24, 30, 36, 42, ...

To list the multiples of a number, you begin with the number itself, then the number times 2, then the number times 3, etc. Since multiplication is repeated addition, you could repeatedly add the number to get the next multiple.

Exercise 1

a. List the first 10 multiples of 8:
b. List the first 10 multiples of 12:
c. Of the list you produced in parts a and b, list the multiples that 8 and 12 have in common:
d. From part c, what is the *smallest multiple* that 8 and 12 have in common?
e. Is there a greatest common multiple that 8 and 12 have in common? Is so, what is it? Before you answer, remember that in a and b, you only listed the *first 10* multiples of 8 and 12.

Just like with the greatest common factor (GCF), many people don't really think about what **LEAST COMMON MULTIPLE *means***. Every number has an infinite number of multiples. Every set of numbers has an infinite number of multiples in common. The **LEAST COMMON MULTIPLE** is the ***smallest*_multiple** they have in **common**.

In exercise 1, we write LCM(8, 12) = 24. The order of the numbers in the parentheses is irrelevant. So, LCM(8, 12) = LCM(12, 8). It is NOT an ordered pair. Also, you can find the least common multiple of a large set of numbers. Soon, you'll find how easy it is to find the least common multiple of several numbers; e.g., LCM(2, 3, 4, 5, 6, 7, 8, 9, 10). In this case, listing several multiples of each of these numbers until you find the smallest one all of them have in common is not the efficient way to find it, so you'll be exploring another way to find it using prime factorization.

Exercise 2

a. List the first 15 multiples of 4:
b. List the first 15 multiples of 6:
c. List the first 15 multiples of 10:
d. Of the list you produced in parts a, b and c, list the multiples all three numbers (4, 6 and 10) have in common:
e. From part d, complete the following: LCM(4, 6, 10) =

The problem with trying to find the least common multiple this way is that you may have to write a lengthy list of multiples of each number until you finally find a multiple they all have in common. For instance, if you were asked to find the LCM(59,61), you wouldn't find a multiple they each have in common until you wrote 61 multiples of 59 and 59 multiples of 61. That is because these are prime numbers, so the least common multiple is their product: $59 \cdots 61$. This is the case with any numbers that are **relatively prime** as well. (Remember 4 and 15 are relatively prime, although neither is prime.)

If x and y have no factors in common, then GCF(x,y) = 1 and LCM(x,y) = xy.

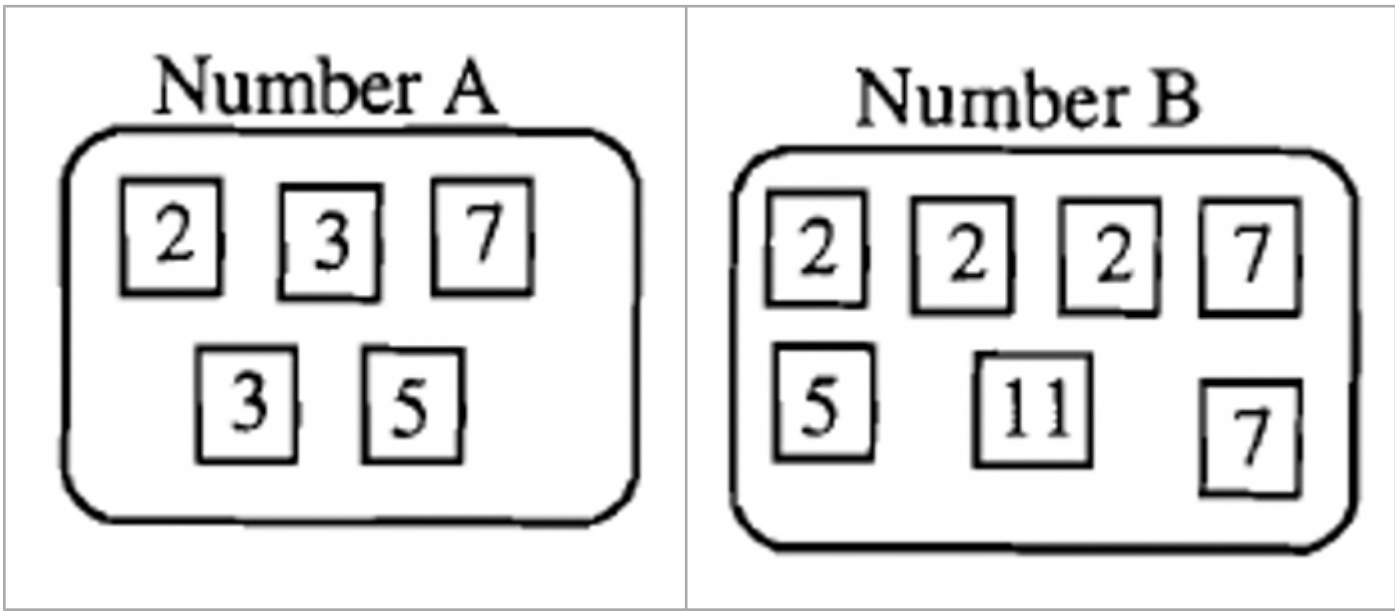
The following is also true:

If GCF(x,y) = 1, then x and y have no factors in common, and LCM(x,y) = xy.

If LCM(x,y) = xy, then x and y have no factors in common, and GCF(x,y) = 1.

A more efficient way to find the least common multiple of a set of numbers is to find the prime factorization of each number, and then BUILD the least common multiple. Get out your prime number squares to do the following exercises.

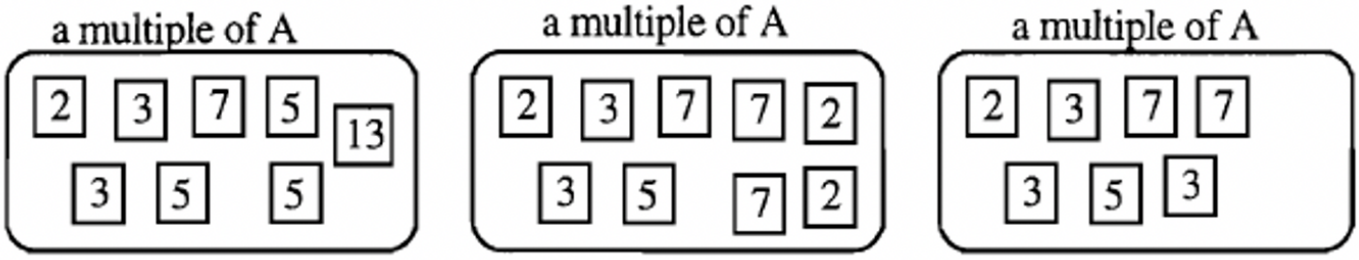
Below is the prime factorization of three numbers, A, B and C.



Below is how to write each of these numbers in prime factorization form:

$A = 2 \cdots 3^2 \cdots 5 \cdots 7$	$B = 2^3 \cdots 5 \cdots 7^2 \cdots 11$	$C = 2^2 \cdots 3^2 \cdots 5 \cdots 7 \cdots 11 \cdots 13$

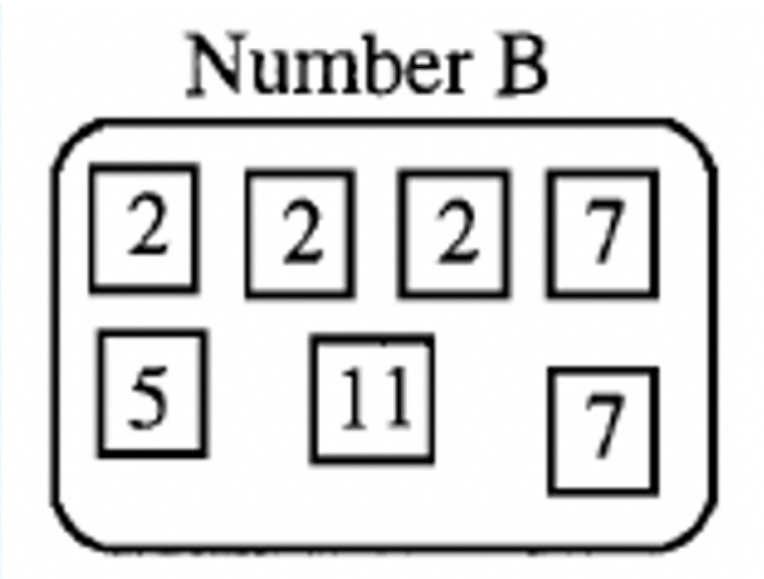
One way to make a multiple of A is to simply add extra factors to factors of A. *Note: A itself is a multiple of A, so you don't* ***have*** *to add any extra factors to get a multiple*. There are an infinite number of possibilities for multiples of A. Below are three multiples of A:



In prime factorization form, the first multiple of A above is written: $2 \cdots 3^2 \cdots 5^3 \cdots 7 \cdots 13$

Exercise 3

Below is the prime factorization of a number, B



a. Form the number B with your prime number squares. Form a multiple of B by "throwing in" one or more prime number squares as factors to the original prime factorization of B shown above. Show a picture of the multiple of B that you formed.

Write the prime factorization of this multiple: ____

b. Form the number B with your prime number squares. Form a multiple of B, *different* from the one you formed in A, by "throwing in" one or more prime number squares as factors to the original prime factorization of B shown above. Show a picture of the multiple of B that you formed:

Write the prime factorization of this multiple of B: ____

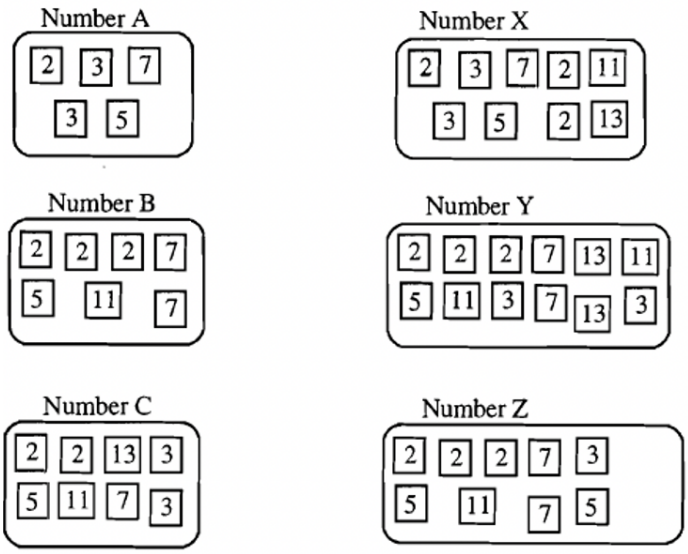
c. Form the number B with your prime number squares. Form another multiple of B by "throwing in" one or more prime number squares as factors to the original prime factorization of B shown above. Write the prime factorization of this multiple of B: _____

Number C is shown below toward the middle of the page. Use it to do exercise 4.

Exercise 4

a. Form the number C with your prime number squares. Form a multiple of C by adding one or more prime number squares as factors. Write the prime factorization of the multiple of C you formed:
b. Form the number C with your prime number squares. Form a different multiple of C by adding one or more prime number squares as factors. Write the prime factorization of the multiple of C you formed:
c. Form the number C with your prime number squares. Form another multiple of C by adding one or more prime number squares as factors. Write the prime factorization of the multiple of C you formed:

NOTE: X is a multiple of M if M is a factor of X. So, you have a multiple of a number if the prime factors of the number itself are factors in the multiple. It's as if you can "see" the number in a multiple. We are going to determine if any of the numbers shown on the right are multiples of A, B or C.

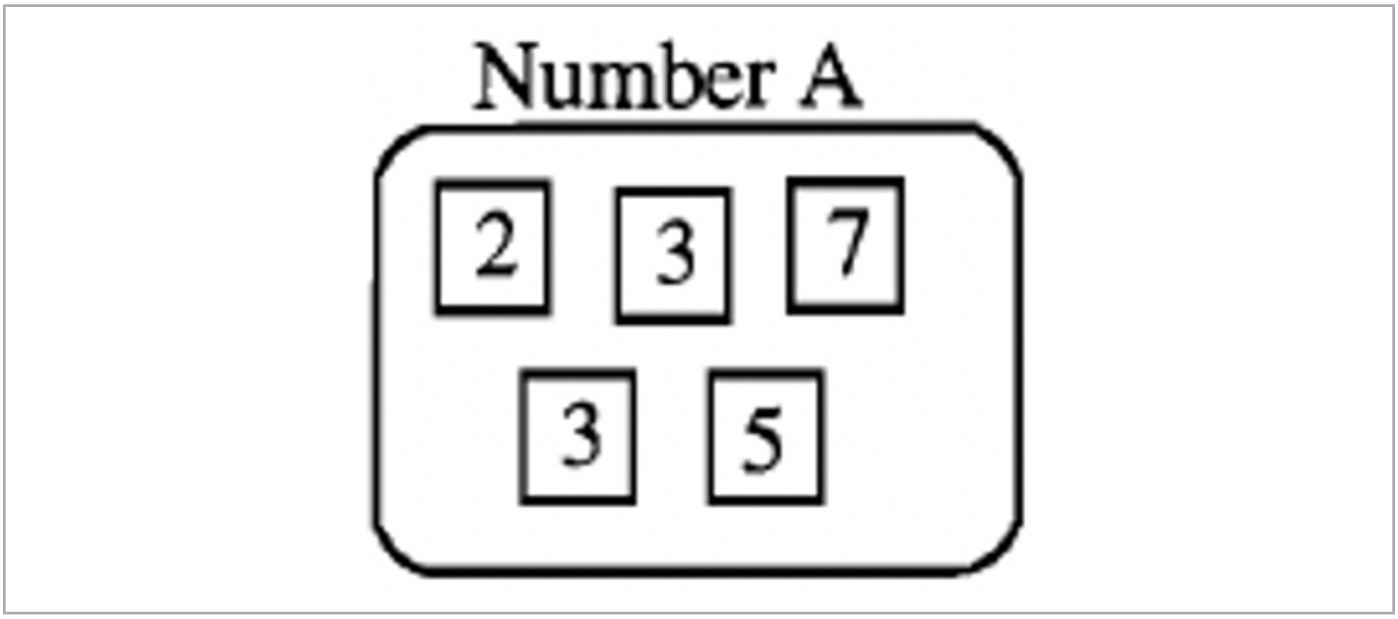


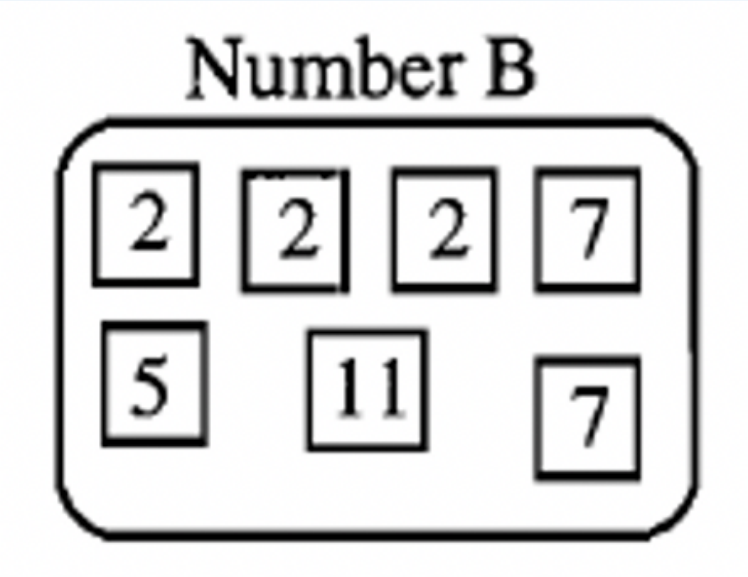
To decide if X, Y or Z is a multiple of A, see if each of X, Y or Z has the prime factors of A as a subset. In other words, A has one factor of 2, two factors of 3, one factor of 5 and one factor of 7. Any number containing those factors is a multiple of A. A multiple of A may have more factors of A, but can't be missing any factors of A.

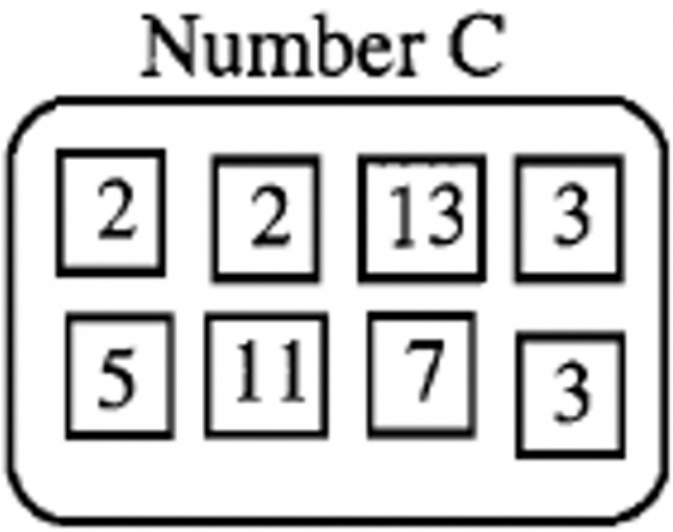
Exercise 5

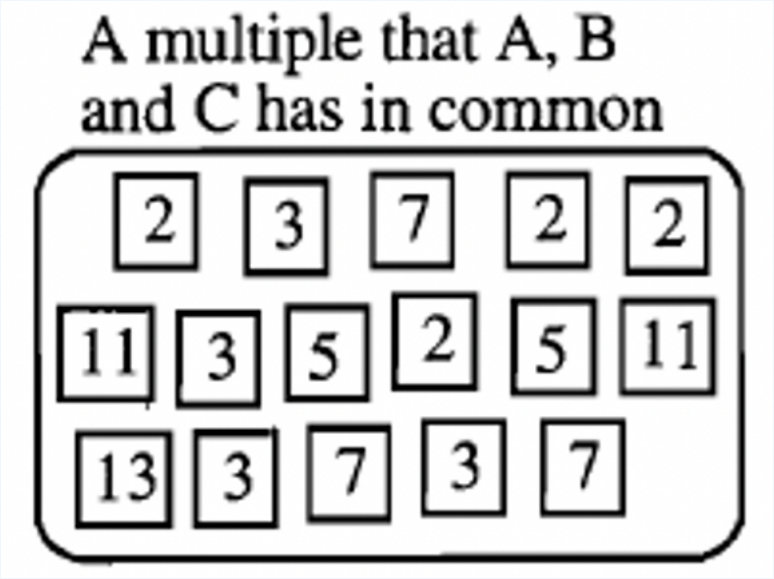
a. List any number on the right (X, Y and/or Z) that is a multiple of A:
b. List any number on the right (X, Y and/or Z) that is a multiple of B:
c. List any number on the right (X, Y and/or Z) that is a multiple of C:

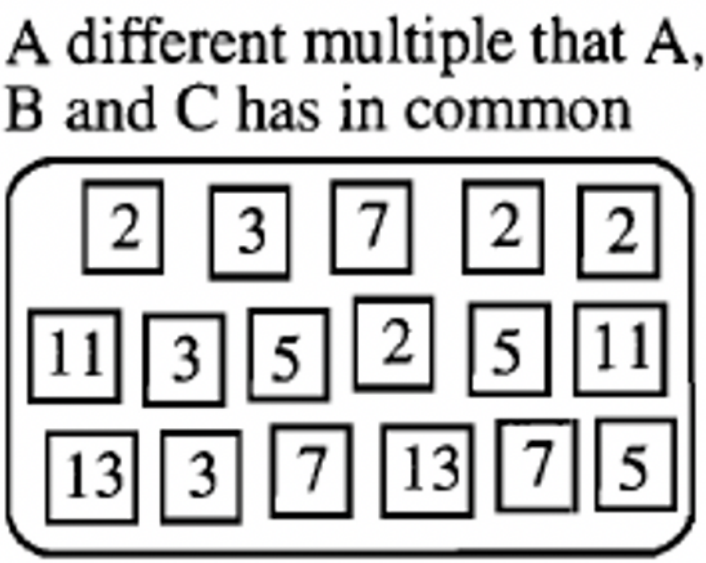
None of the numbers X, Y or Z was a multiple of all three of the numbers, A, B and C. Numbers A, B and C are shown below. Below that are only two examples of multiples that A, B and C have in common.




Number B
2
2
2
7
5
11
7


Number C
2
2
13
3
5
11
7
3





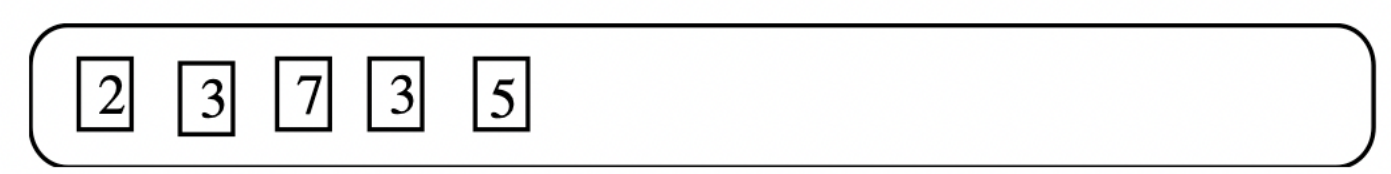
Exercise 6

Use prime number squares to form three other multiples that A, B and C have in common. Write the prime factorization of each of the multiples you formed.

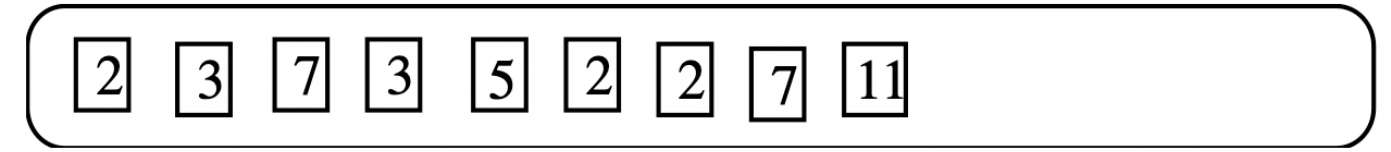
a. _____
b. _____
c. _____

Now, we are going to use the prime factorization to build the ***LEAST* COMMON MULTIPLE** of a set of numbers. The least common multiple is a multiple that the numbers have in common, but that has the least number of factors possible. The multiples formed above have 16 and 17 factors, respectively. To build a least common multiple, we only add in a factor if it is necessary.

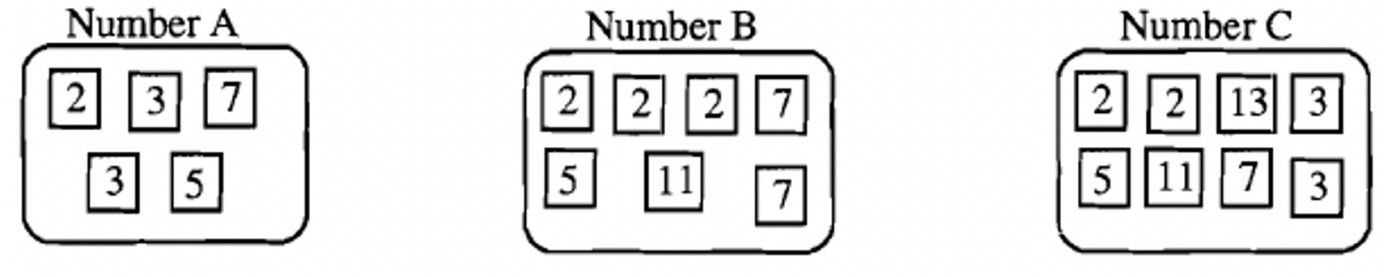
Let's say we want to build the least common multiple of A, B and C. In order to be a multiple of A, we need to have the factors of A. So, we start ***building*** the multiple by putting in the factors of A:



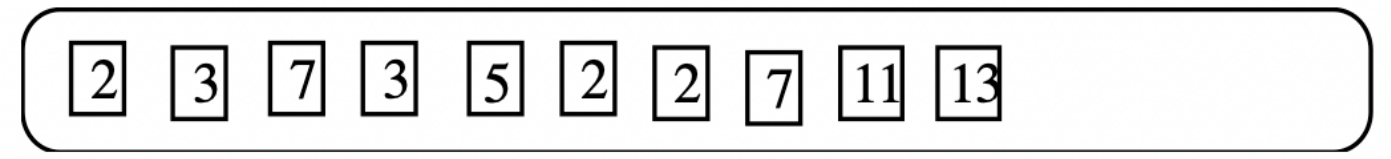
The factors of B must also be in the multiple. B has three factors of 2, but there is only one factor of 2 in the multiple so far. So, the common multiple will need two more factors of 2. B has two factors of 7, but there is only one factor of 7 in the multiple so far. So, the common multiple will need one more factor of 7. B has one factor of 5, which is already there. B also has one factor of 11. That is not there, so must be put in. Therefore, two factors of 2, one factor of 7 and a factor of 11 must be joined with the other factors in the common multiple. Once this is done, we have built the least common multiple of A and B:



The prime factorizations of A, B and C are shown below for your convenience.



We have built the least common multiple of A and B. To find the least common multiple of A, B and C, we need to make sure the factors of C are also in the multiple. C has two factors of 2, which are already in the multiple, two factors of 3, which are already in the multiple, one factor each of 5, 7 and 11, each of which is already in the multiple, and a factor of 13, which is not in the multiple. So 13 is the only factor that needs to be joined with the factors already in the multiple we are building.



The above is a multiple of A, B and C. We only have we built a multiple, we actually built the **LEAST** COMMON MULTIPLE of A, B and C. Note it is a multiple of A, B *and* C; therefore it is a *common* multiple. It is the *least* common multiple because if any of the factors were removed, it wouldn't be a multiple of one of the numbers. For instance, if a factor of 2 was removed, it would not be a multiple of A. If a factor of 3 was removed, it would not be a multiple of A or C. If a factor of 7 was removed, it would not be a multiple of B, etc. If a factor of 11 was removed, it would not be a multiple of B or C. If a factor of 13 was removed, it would not be a multiple of C.

We write: LCM(A, B, C) = $2^3 \cdots 3^2 \cdots 5 \cdots 7^2 \cdots 11 \cdots 13$

Exercise 7

Build the least common multiple of A and B using prime number squares. Then, write the prime factorization of the least common multiple of A and B. Let $A = 2^2 \cdots 3^5 \cdots 5 \cdots 13$ and $B = 2^2 \cdots 3^2 \cdots 7^3 \cdots 11^2 \cdots 13$

LCM(A, B) = ___

Exercise 8

Build the least common multiple of A, B and C using prime number squares. Then, write the prime factorization of the least common multiple of A and B.

Let A = $2^2 \cdots 11 \cdots 19$ and $B = 2 \cdots 3^2 \cdots 7 \cdots 11^2$ and $C = 2^2 \cdots 3^4 \cdots 7^3 \cdots 13^2 \cdots 19$

LCM(A, B, C) = ____

Exercise 9

Build the least common multiple of A, B and C using prime number squares. Then, write the prime factorization of the least common multiple of A and B.

Let $A = 2 \cdots 5^3 \cdots 11 \cdots 19$ and $B = 2^4 \cdots 3^6 \cdots 5^2 \cdots 23^2$ and $C = 5^4 \cdots 7^6 \cdots 11^2 \cdots 23$

LCM(A, B, C) = ____

Look back at the answer from exercise 9.

Let $A = 2 \cdots 5^3 \cdots 11 \cdots 19$ and $B = 2 \cdots 3^6 \cdots 5^2 \cdots 23^2$ and $C = 5^4 \cdots 7^6 \cdots 11^2 \cdots 23$

Answer: LCM(A, B, C) = $2 \cdots 3^6 \cdots 5^4 \cdots 7^6 \cdots 11^2 \cdots 19 \cdots 23^2$

If the numbers are prime factored using exponents, then the least common multiple contains each of the prime factors shown in any of the numbers. The exponent on each of the prime numbers is the highest exponent found on that prime number factor in the prime factorization of the numbers.

For instance, the prime numbers in the prime factorization of A, B and C are 2, 3, 5, 7, 11, 19 and 23. So begin by writing the product of these prime numbers: $2 \cdots 3 \cdots 5 \cdots 7 \cdots 11 \cdots 19 \cdots 23$

2 is a factor in A and B. The highest power of 2 found in either prime factorization of A and B is 1. 3 is a factor found in B, as 3^6. So, the highest power of 3 is 6. 5 is a factor of all three numbers. In A, the exponent on 5 is 3; in B, the exponent on 5 is 2; in C, the exponent on 5 is 4. So, the highest power of 5 is 4. Similarly, do the same for all of the other prime factors. Write the highest exponent on the factor.

Therefore, LCM(A, B, C) = $2 \cdots 3^6 \cdots 5^4 \cdots 7^6 \cdots 11^2 \cdots 19 \cdots 23^2$

Exercise 10

Find the greatest common factor and least common multiple for each set of numbers, written in prime factorization. Assume a, b, c, d, and e are different prime numbers.

a. Let $A = 2^2 \cdots 3^5 \cdots 5 \cdots 13$ and $B = 2^2 \cdots 3^2 \cdots 7^3 \cdots 11^2 \cdots 13$

GCF(A, B) = _____

LCM(A, B) = _____

b. Let $A = 2^2 \cdots 11 \cdots 19$ and $B = 2 \cdots 3^2 \cdots 7 \cdots 11^2$ and $C = 2^2 \cdots 3^4 \cdots 7^3 \cdots 13^2 \cdots 19$

GCF(A, B, C) = _____
LCM(A, B, C) = _____

c. Let $X = a^5 \cdots b^4 \cdots c^5 \cdots d$ and $Y = b^2 \cdots c^3 \cdots d^2 \cdots e^2$ and $Z = a^2 \cdots c^4 \cdots d^3 \cdots e^2$

GCF(X, Y, Z) = _____
LCM(X, Y, Z) = _____

d. Let $X = a^6 \cdots b^3 \cdots c \cdots d$ and $Y = b^4 \cdots c^4 \cdots d^3 \cdots e^7$ and $Z = a^2 \cdots c^4 \cdots d^2 \cdots e^3$

GCF(X, Y, Z) = _____
LCM(X, Y, Z) = _____

So far, the prime factorization of numbers has been given, and all you had to do was to build the prime factorization of the least common multiple. Now, your job will be to first prime factor numbers; then you can build the least common multiple from the prime factorizations. In the end, the least common multiple is a number. Multiply the factors in the least common multiple to find the one number that is the least common multiple.

Example

Find the least common multiple of 15, 18 and 20

Solution

Prime factor each of these numbers. $15 = 3 \cdots 5, 18 = 2 \cdots 3 \cdots 3$, and $20 = 2 \cdots 2 \cdots 5$

Build the least common multiple by first "throwing in" the prime factors that make up the prime factorization of 15; then "throw in" any prime factors needed for 18; next, "throw in" any prime factors needed for 20. We get

LCM(15, 18, 20) = $3 \cdots 5 \cdots 2 \cdots 3 \cdots 2 = 2^2 \cdots 3^2 \cdots 5$ = **180**

Exercise 11

Find the least common multiples. Show the prime factorization of each number, and how you use it to build the least common multiple.

a. LCM (6, 8, 10) = _____

b. LCM (25, 35, 40) = _____

c. LCM (49, 91, 26) = _____

d. LCM(56, 24, 30) = _____

e. LCM(22, 34, 55) = _____

Note how easy it is to find the least common multiple of a larger set of numbers using prime factorization. We'll find: LCM(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14)

We don't need to expressly write down the prime factorization of the 12 numbers because it's easy enough to do that in our head. For instance, the prime factorization of 4 is $2 \cdots 2$, the prime factorization of 10 is $2 \cdots 5$, and the prime factorization of 13 is 13.

We build up the LCM of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14 by first making sure the prime factorization of 2 is there.

Step 1: Make sure the prime factorization 2 is there. **So far, LCM: 2**

Step 2: Make sure the prime factorization 3 is there. We need to "throw in" a 3 to the LCM. **So far, LCM:** $2 \cdots 3$

Step 3: Make sure the prime factorization 4 is there. We need to "throw in" a 2 to the LCM. **So far, LCM:** $2 \cdots 3 \cdots 2$

Step 4: Make sure the prime factorization 5 is there. We need to "throw in" a 5 to the LCM. **So far, LCM:** $2 \cdots 3 \cdots 2 \cdots 5$

Step 5: Make sure the prime factorization 6 is there. We don't need to "throw in" any factor(s) to the LCM, so there is no change to the LCM. **So far, LCM:** $2 \cdots 3 \cdots 2 \cdots 5$.

Step 6: Make sure the prime factorization 7 is there. We need to "throw in" a 7 to the LCM. **So far, LCM:** $2 \cdots 3 \cdots 2 \cdots 5 \cdots 7$

Step 7: Make sure the prime factorization 8 is there. We need to "throw in" a 2 to the LCM. **So far, LCM:** $2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2$

Step 8: Make sure the prime factorization 9 is there. We need to "throw in" a 3 to the LCM. **So far, LCM:** $2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2 \cdots 3$

Step 9: Make sure the prime factorization 10 is there. We don't need to "throw in" any factor(s) to the LCM, so there is no change to the LCM. **So far, LCM:** $2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2 \cdots 3$

Step 10: Make sure the prime factorization 11 is there. We need to "throw in" an 11 to the LCM. **So far, LCM:** $2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2 \cdots 3 \cdots 11$

Step 11: Make sure the prime factorization 12 is there. We don't need to "throw in" any factor(s) to the LCM, so there is no change to the LCM. **So far, LCM:** $2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2 \cdots 3 \cdots 11$

Step 12: Make sure the prime factorization 13 is there. We need to "throw in" a 13 to the LCM. **So far, LCM:** $2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2 \cdots 3 \cdots 11 \cdots 13$

Step 13: Make sure the prime factorization 14 is there. We don't need to "throw in" any factor(s) to the LCM, so there is no change to the LCM. **So far, LCM:** $2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2 \cdots 3 \cdots 11 \cdots 13$

That's it! You have now built up the LEAST common multiple of all 12 numbers! You can check to make sure that the prime factorization of each of the 12 numbers is indeed in the prime factorization of the least common multiple you built. Furthermore, if any of the prime factors were removed, it wouldn't be a multiple of all 12 numbers. Therefore, the prime factorization of the LCM is as shown below:

LCM(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14) = $2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2 \cdots 3 \cdots 11 \cdots 13$.

If these factors are multiplied together, LCM(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14) = 360, 360.

The prime factorization of the LCM may have been written with the factors in ascending order using exponents: $2^3 \cdots 3^2 \cdots 5 \cdots 7 \cdots 11 \cdots 13$

Exercise 12

Use prime factorization to build the least common multiple. Show the prime factorization as the number is built. Then, multiply the factors to find the answer.

a. LCM(3, 4, 6, 7, 9, 10, 12, 14, 15, 18, 20) = _____ = _____

b. LCM(2, 3, 9, 11, 14, 15, 16, 17, 18, 22) = _____ = _____

c. LCM(5, 6, 7, 8, 12, 14, 15, 17, 18, 25) = _____ = _____

d. LCM(15, 18, 20, 25, 30, 35, 42, 45) = _____ = _____

Let's say you know the greatest common factor of 165 and some other number was 3, and the least common multiple of the same two numbers was 15,015. How would you figure out what the other number was?

First, it's a good idea to write down what you know. Let N be the other number.

Then, GCF(165, N) = 3 and LCM (165, N) = 15,015.

Since 3 is a factor, and in fact the greatest common factor, of both 165 and N, then each number can be written as 3 times something, and the other factor you obtain for 165 should be relatively prime to the other factor you obtain for N, *since 3 is the* ***greatest*** *common factor of 165 and N.*

So, let's rewrite, LCM (165,N) = 15,015 like this: LCM (3×55, $3 \times$ _____) = 15,015

In other words, I know N = 3 $\times$ _____, but I have to figure out what goes in the blank to figure out what N is. If you want to introduce another variable, like M, instead of writing a blank, that works just as well. It's up to you.

To find the LCM of 3×55 and 3 $\times$ _____ , where the 55 and the number on the blank have no common factors, you would multiply $3 \times 55 \times$ _____. But we know the product should be 15,015. So, the number that must go on the blank must be 91, since $3 \times 55 \times 91 = 15,015$. So, now we can figure out what N is: $N = 3 \times 91 = 273$.

Let's see if this makes sense by first rewriting 165 and 273 either in prime factored form or as the GCF(165,273) times something; then we'll figure out the GCF and LCM from the factored form, and see if it agrees with our original problem.

$165 = 3 \times 55$ and 273 = 3×91. First, make sure 3 is really a factor of each number!

Now, convince yourself that 3 really is the **greatest** common factor of 165 and 273 by checking to see that the other factor of one number is relatively prime (no factors in common) with the other factor of the other number. So, ask yourself if 55 and 91 have any factors in common (besides 1) if you aren't sure, then prime factor each of these numbers first ($55 = 5 \times 11$ and $91 = 7 \times 13$); note that they have no factors in common.

Since 3 is the greatest common factor, then the least common multiple is obtained by multiplying $3 \times 55 \times 91$, which is 15,015 which is what it should be according to the original information given. Therefore, 273 really was the number we were looking for.

N = ***273***

There is another way to do the above problem using the following property:

For any *two* numbers, m and n, the following is always true:

$$m \times n = GCF(m, n) \times LCM(m, n)$$

This equation is *not true* for more than two numbers $r \times s \times t \neq GCF(r, s, t) \times LCM(r, s, t)$

Example

Verify the above property for numbers 15 and 18.

Since GCF(15, 18) = 3 and LCM(15, 18) = 90, we want to verify that the product of the two numbers, 15 and 18, equals the product of the GCF and LCM of the two numbers.

$15 \cdots 18 = 270$; GCF(15, 18) $\cdots$ LCM(15, 18) = $3 \cdots 90 = 270$

Therefore, $15 \cdots 18 =$ GCF(15, 18) $\cdots$ LCM(15, 18)

Example

Show that the above property does not work for three numbers.

Counterexample: Use the numbers 4, 6 and 8. GCF(4, 6, 8) = 2 and LCM(4, 6, 8) = 24

$4 \cdots 6 \cdots 8 = 192$; GCF(4, 6, 8) ··· LCM(4, 6, 8) = $2 \cdots 24 = 48$

Clearly, $4 \cdots 6 \cdots 8 \neq$ GCF(4, 6, 8) ··· LCM(4, 6, 8)

Example

Find N if GCF(165, N) = 3 and LCM (165, N) = 15,015

In this case, the two numbers are 165 and N, the GCF is 3 and the LCM is 15,015. Plug these values into the equation shown above in bold.

$165N = 3 \times 15,015$. Divide both sides by 165 to find N: N = $\frac{3 \times 15015}{165} = 273$

The property can also be used to find the LCM of two numbers if you know the GCF. For instance, if you were asked to find the GCF and LCM of 24 and 30. You can easily find the GCF, which is 6. To find the LCM, multiply the two numbers together, and divide by the GCF. (This should make sense to you intuitively if you think about it: You wouldn't list the GCF twice as you build the LCM. Also, the GCF will cancel into either of the two numbers since it is a factor of each.) So, the LCM(24, 30) = $(24 \cdot 30) \div 6 = 120$.

Exercise 13

If GCF(1176, 288) = 24, find the LCM(1176, 288)

Exercise 14

Find X if GCF(2940, X) = 105 and LCM(2940, X) = 79,380

Exercise 15

If the greatest common factor of 3,211 and another number is 247 and the least common multiple of the same two numbers is 48,165, then what is the other number?

Exercise 16

If you aren't given at least one of the numbers, there might be more that one possible solution. Figure out the possibilities for a and b if all you know is GCF(a, b) = 2 and LCM(a, b) = 20

The next few problems are a review of GCF and LCM. You now have several methods you can use to find the GCF and LCM.

17-20 Find the greatest common factor of each of the following pairs of numbers using prime factorization, the Old Chinese Method or the Euclidean Algorithm. Then, find the LCM of each pair. Show all work.

Exercise 17

a. GCF (693, 546) = _____
b. LCM (693, 546) = _____

Exercise 18

a. GCF (2117, 2555) = _____
b. LCM (2117, 2555) =

Exercise 19

a. GCF (1369, 10693) = _____
b. LCM (1369, 10693) =

Exercise 20

a. GCF (24300, 14406) = _____
b. LCM (24300, 14406) =

Exercise 21

Make up a problem where the GCF of 2 different 3-digit numbers is 32. For instance, GCF (x, y) = 32. Find an x and y that will work.

Exercise 22

Make up a problem where the GCF of two different 4-digit numbers is 32. In other words, find 2 numbers, a and b, such that GCF (a, b) = 32.

Exercise 23

Make up a problem where the GCF of 2 different 3-digit numbers is 35. For instance, GCF (x, y) = 35. Find an x and y that will work.

Exercise 24

Make up a problem where the GCF of two different 4-digit numbers is 28. In other words, find 2 numbers, a and b, such that GCF (a, b) = 28.

All even numbers are multiples of 2. Therefore, every even number can be written as the product of 2 and an integer. Symbolically, we can write every even number can be written in the form: 2k, where k is an integer. For example, 12 = 2(6), 20 = 2(10), 58 = 2(29), etc.

All odd numbers can be written as one more than an even number. Since an even number can be written as 2k, then symbolically, we write that every odd number can be written in the form: 2k + 1, where k is an integer. For example, 15 = 2(7) + 1, 41 = 2(20) + 1, etc.

Any integer that can be written in the form 2k is even and any integer that cannot be written in the form 2k is not even. Any integer that can be written in the form 2k + 1 is odd and any integer that cannot be written in the form 2k + 1 is not odd.

We'll be doing some proofs about even and odd numbers. The only way to express an even number, in general, is by 2k. The variable can be any letter. If you want to express more than one even number, you must use a new variable like 2m or 2n. The same is true for odd numbers.

For the examples and exercises that follow, assume all variables are integers.

EXAMPLES: State which of the following always represents an even number, which of the following always represents an odd number, and which are sometimes even and sometimes odd.

Example 1: 6n+14

6n + 14

Solution

6n + 14 = 2(3n + 7), which is in the form of an even number. Therefore, 6n + 14 will always represent an even number (since n is an integer).

Example 2: 4n+23

4n + 23

Solution

4n + 22 + 1 = 2(2n + 11) + 1, which is in the form of an odd number. Therefore, 6n + 14 will always represent an odd number.

Example 3: 5n+2

5n + 2

Solution

It is impossible to write 5n + 2 in the form 2k or 2k + 1. Therefore, it can't be determined. It is sometimes even and sometimes odd.

Note: You can verify these answers by plugging in an even number for n and then an odd number for n, and see if the conclusion makes sense. In example 1, if n = 2, then 6n + 14 = 6(2) + 14 = 28 (even). If n = 3, then 6(3) + 14 = 32 (even). So, the result was even when n was replaced by either an even or an odd number. Do the same verification for example 2 and 3 in the space below:

Exercise 25

State which of the following always represents an even number, which of the following always represents an odd number, and which are sometimes even and sometimes odd. Justify your answer.

a. 8n + 20

b. 10k + 9

c. 5x + 2

Example 1

Formally prove that the sum of two odd numbers is even.

IMPORTANT Note: Don't define the two odd numbers to be the same odd number. You must be general, and assume they may be two different odd numbers! Use different variables to be the most general.

Solution

Let 2n+1 = one odd number, and let 2m+1 = another odd number.

The sum is: 2n+1 + 2m+1= 2n+2m+2 = 2(n+m+1), which is in the form of an even number. Therefore, the sum of 2 odd numbers is even.

Example 2

Formally prove that the product of two odd numbers is odd.

Solution

Let 2n+1 = one odd number, and let 2m+1 = another odd number.

The sum is: (2n+1) (2m+1)= 4nm + 2m + 2n + 1 = 2(2nm + n + m) +1, which is in the form of an odd number. Therefore, the product of 2 odd numbers is odd.

Exercise 26

Formally prove that the sum of two even numbers is even.

Exercise 27

Formally prove that the sum of two odd numbers is even.

Exercise 28

Formally prove that the sum of an even number and an odd number is odd.

Exercise 29

Formally prove that the product of two even numbers is even.

Exercise 30

Formally prove that the product of two odd numbers is odd.

Exercise 31

Formally prove that the product of an even number and an odd number is even.

We are going to explore a way to find the sum of several consecutive whole numbers.

Example 1

1 + 2 + 3 + 4 + 5 + 6

Example 2

The sum of the first 100 counting numbers would take a long time to write out and compute. We write it in the following way:
1 + 2 + 3 + ... + 98 + 99 + 100

Example 3

55 + 56 + 57 + ... + 128 + 129 + 130

The first example is fairly easy to compute. But as more and more numbers are added, the computation becomes cumbersome. The first two examples start with the number, 1. Our first goal will be to find a pattern, and then a formula for adding any set of consecutive counting numbers starting with 1.

Even though we know the answer to example 1, we'll use that to find a pattern for the sum for other sums. Let X = the sum we are looking for (1 + 2 + 3 + 4 + 5 + 6). Notice what happens if X is written down twice – first in ascending order, then in descending order – and then the two rows are added by adding lined-up columns:

$$\begin{aligned} X &= 1+2+3+4+5+6 \\ +X &= \underline{6}+\underline{5}+\underline{4}+\underline{3}+\underline{2}+\underline{1} \\ 2X &= 7+7+7+7+7+7 \end{aligned}$$

The left side equals 2X, which is twice the actual sum we want. How many columns of numbers are on the right hand side of the equals sign? ____ Note: the number of columns is the same as the amount of numbers in the actual sum we are trying to find. What does each column on the right side of the equals sign add up to? _____ Since the right hand side of the equal sign is repeated addition, you can obtain the answer by using multiplication. What multiplication problem is this? _____ Since the left side equals the right side (2X = 42), then X = 21. Even if you didn't represent the sum using a variable, you would divide the right side by 2 because the sum is twice as large since the numbers in the sum were added twice.

Let's use this same technique to add the first 100 consecutive counting numbers. Let N = the actual sum. You fill in the third row by adding the left side, and then adding all of the columns on the right side.

$$\begin{array}{rccccccccccc} N = & 1 & + & 2 & + & 3 & + & \ldots & + & 98 & + & 99+ & 100 \\ +N = & \underline{100} & + & \underline{99} & + & \underline{98} & + & \ldots & + & \underline{3} & + & \underline{2}+ & \underline{1} \end{array}$$

How many columns (it's the same answer as the amount of numbers in the sum) are there on the right? _____ Since each column adds up to the same number, this is repeated addition. State the number that the right side adds up to: _____ That is twice as big as the actual sum, so what does the sum (1 + 2 + 3 + ... + 98 + 99 + 100) equal? _____

Hopefully, you got the answer of 5,050!

Exercise 32

Use the previous technique to find the sum of the first 80 counting numbers. Show work

Let's try to find a formula for finding the first n counting numbers using the same technique. Before writing the sum, what is the counting number preceding n? _____ What is the counting number preceding that number? _____

Let X = the sum of the first n counting numbers. Below, we'll use the same techique to find the sum. Fill in the sum on the left side of the equals side and the sum of each column on the right.

What does each column on the right add up to? _____ Again, the right side is a repeated addition again. Multiply the number of columns (which is equal to the amount of numbers in the sum of the first n counting numbers) by the number each column adds up to.

What does the right side equal? _____ This is twice as large as the actual sum we want.

What does the actual sum of the first n counting numbers equal? _____

Hopefully, you got this answer: $\frac{n(n+1)}{2}$

Of course, the answer depends on n. Let's using this formula to compute the sum of the first six counting numbers, which was the first example we did. In this example, n = 6. So, plugging n = 6 into the formula, we get $(6 \cdot 7)$ / 2 = 21. Same answer!

Use the formula to find the sum of the first 80 counting numbers.

Exercise 34

Use the formula to find this sum: 1 + 2 + 3 + ... + 248 + 249 + 250.

What if you wanted to find this sum? 55 + 56 + 57 + ... + 128 + 129 + 130

If you used the formula, then you would have the sum of all the numbers from 1 to 130 instead of only the ones from 55 on. One strategy is to use the formula and then subtract off the extra numbers you added. For instance, if you add all the numbers from 1 - 130, the extra numbers added that are not part of the sum are: 1 + 2 + 3 + ... + 52 + 53 + 54. But this sum is easy to compute since we can use the formula to get this sum! So here is the strategy:

55 + 56 + 57 + ... + 128 + 129 + 130

= (the sum of the first 130 counting numbers) – (the sum of the first 54 counting numbers)

= (1 + 2 + 3 + ... + 128 + 129 + 130) – (1 + 2 + 3 + ... + 52 + 53 + 54)

= $\frac{130(131)}{2} - \frac{54(55)}{2}$ = 8,515 - 1,485 = 7,030

Example

Find the sum: 81 + 82 + 83 + ... + 198 + 199 + 200

Solution

Subtract the sum of the first 80 counting numbers from the sum of the first 200 counting numbers: 200(201)/2 – 80(81)/2 = 20,100 – 3,240 = 16,860.

Exercise 35

Find the following sums using the strategy just shown. Show work.

a. 51 + 52 + 53 + ... + 98 + 99 + 100

b. 146 + 147 + 148 + ... + 561 + 562 + 563

c. 500 + 501 + 502 + ... + 798 + 799 + 800

There is a different strategy we can use to find the sum of consecutive whole numbers that do not begin with the number, 1. Let's look at another way to compute the sum of these whole numbers: 55 + 56 + 57 + ... + 128 + 129 + 130. We'll begin by using the same type of strategy we used at the beginning of this topic. First, let X = the sum. Write the sum down twice – first in ascending order, then in descending order – and then add the two rows by adding the individual lined-up columns. This is shown on the next page.

$$\begin{array}{rrrrrrr} X = & 55+ & 56+ & 57+ & \ldots+ & 128+ & 129+ & 130 \\ \underline{X} = & \underline{130}+ & \underline{129}+ & \underline{128}+ & \ldots+ & \underline{57}+ & \underline{56}+\underline{55} & \\ 2X = & 185+ & 185+ & 185+ & \ldots+ & 185+ & 185+ & 185 \end{array}$$

At this point, we can see the similarity to how we derived the sum of the first n counting numbers, but there is one big difference here. It's not clear exactly how many addends of 185 are on the right-hand side of the equals sign. Most people will say that there are 130 – 55, or 75 of them, but actually that reasoning (subtracting the first number from the last number) is not correct. For instance, if you were adding the numbers from 1 to 130, most people will agree that there are 130 numbers being added, which is NOT the same answer you would get if you subtracted the first number from the last number since 130 – 1 only equals 129. Look at simpler example: 17 + 18 + 19 + 20. It's clear there are 4 numbers in the sum, but if you did the subtraction 20 – 17, you would get 3, which is the incorrect answer. One way to figure out how many numbers are in the sum is to decide how many numbers are missing from the sum if it did begin with 1. Look at the sum again: 17 + 18 + 19 + 20 has the first 16 numbers missing, so instead of 20 numbers in the sum (which is how many there would be if we started at 1), there are 20 – 16 numbers in the sum. Okay, back to figuring out how many numbers are actually in the sum above.

How many numbers are in the sum: 55 + 56 + 57 + ... + 128 + 129 + 130? _____

So, the right side of the equation has 76 185's added together, which is 76 ··· 185, which equals 14,060. But, this is twice as big as the actual sum, so after dividing by 2, we get the actual sum of 7,030. This is the same answer we got when we did this problem on the previous page using a different strategy.

Exercise 36

Find the following sums using the strategy just shown. Show work.

a. 51 + 52 + 53 + ... + 98 + 99 + 100

b. 146 + 147 + 148 + ... + 561 + 562 + 563

c. 500 + 501 + 502 + ... + 798 + 799 + 800

d. Challenge: k + (k + 1) + (k + 2) + ... + (n - 2) + (n - 1) + n

So far, you have learned how to easily find the sum of several consecutive whole numbers. Let's take it one step further. What if the sum you want to find are numbers that are not consecutive. Depending on the the sequence of numbers, the sum may or may not be easy to find. We are going to look at sums that have consecutive numbers added together in disguise. For instance, look

at the following sum:

$$7 + 14 + 21 + \ldots + 693 + 700$$

Exercise 37

What do you notice about the numbers added together?

Exercise 38

Hopefully, you noticed that all of the numbers were multiples of seven, or perhaps you noticed you add seven to each number to get the next number. Take the sum and rewrite the problem by factoring out a 7; just fill in the blanks below:

7 + 14 + 21 + ...+ 693 + 700 = 7 (___ + ___ + ___ + ... + ___ + ___)

Exercise 39

Now, you should be able to figure out the sum of the numbers in parentheses. ***Show your work to figure out the sum***. Then answer a, b and c.

a. What is the sum of the numbers in parentheses? _____

b. So, the sum 7 + 14 + 21 + 28 + ... + 700 becomes 7 _____

c. Therefore, 7 + 14 + 21 + 28 + ... + 700 = _____

Exercise 40

Find the following sums. ***Show your work.***

a. 8 + 16 + 24 + ... + 992 + 1000 = _____

b. 11 + 22 + 33 + ... + 935+ 946 = _____

c. 20 + 40 + 60 + 80 + ... + 2980 + 3000 = _____

Okay, just one more twist...this puts it all together. Consider this sum:

$$112 + 116 + 120 + \ldots + 524 + 528$$

Exercise 41

This time, what do you notice about the numbers added together?

Exercise 42

Hopefully, you noticed all the numbers were multiples of four, or that you added four to each number to get the next number. Take the sum and rewrite the problem by factoring out a 4; just fill in the blanks below:

112 + 116 + 120 + ... + 524 + 528 = 4 (___ + ___ + ___ + ... + ____ + ____)

Exercise 43

Now, you should be able to figure out the sum of the numbers in parentheses. Note that the sum does not start with a 1. ***Show your work*** to figure out the sum. Then answer a, b and c.

a. What is the sum of the numbers in parentheses? _____

b. So, the sum 112 + 116 + 120 + ... + 524 + 528 becomes 4 _____

c. Therefore, 112 + 116 + 120 + ... + 524 + 528 = _____

Exercise 44

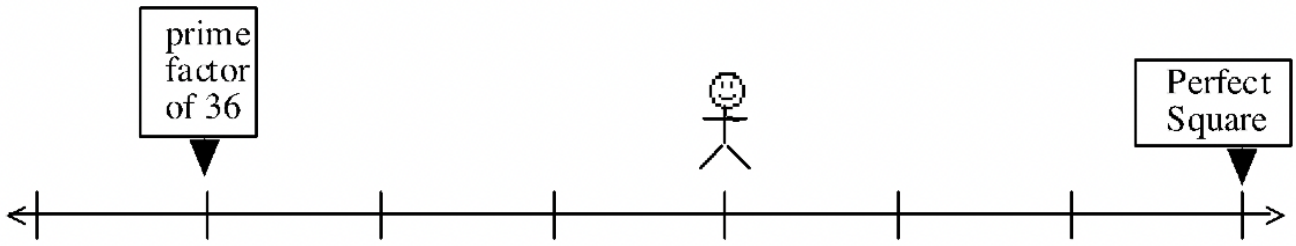
Find the following sums. ***Show your work.***

a. 85 + 90 + 95 + ... + 735 + 740 = _____

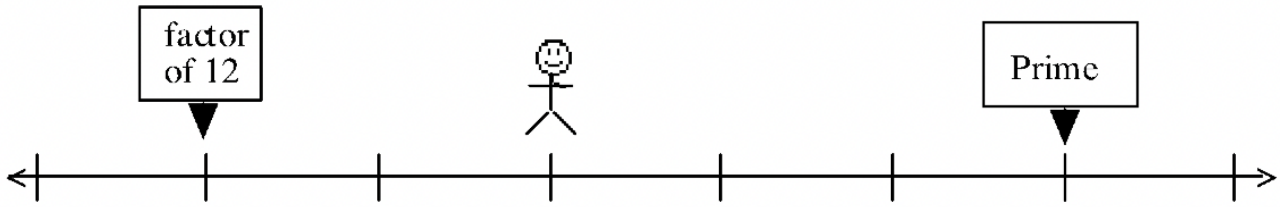
b. 430 + 473 + 516 + ... + 2838+ 2881 = _____

Here are some problems for you to figure out what numbers are on the number line. For each problem, figure out where on the number line (what number) the man might be standing. Where there is more than one possibility, only list numbers between 1 and 1000.

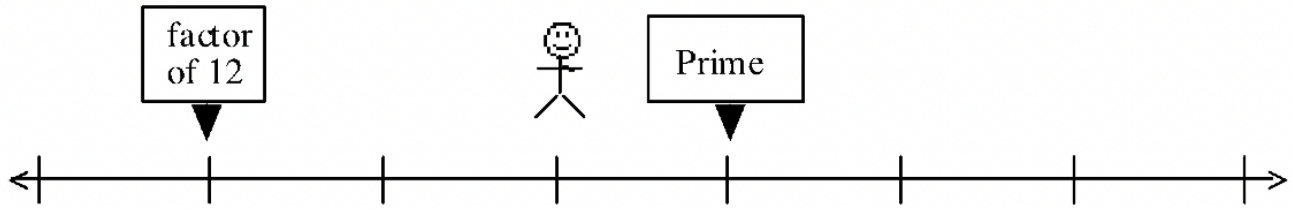
Exercise 45



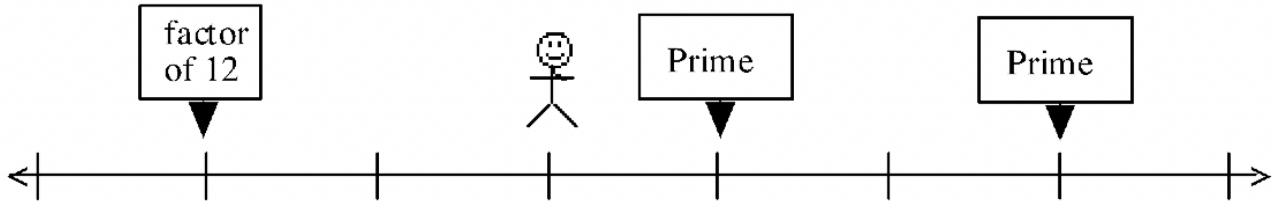
Exercise 46



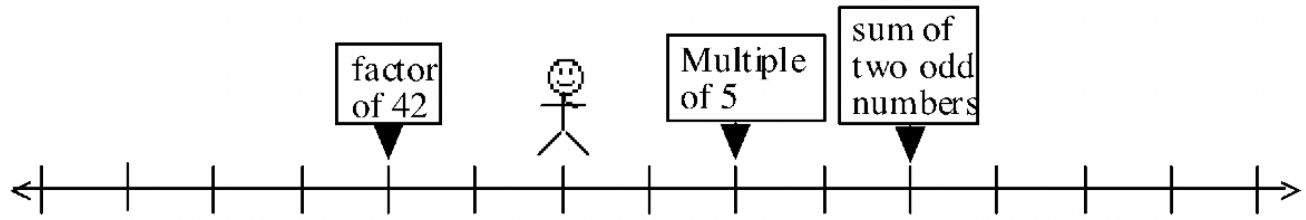
Exercise 47



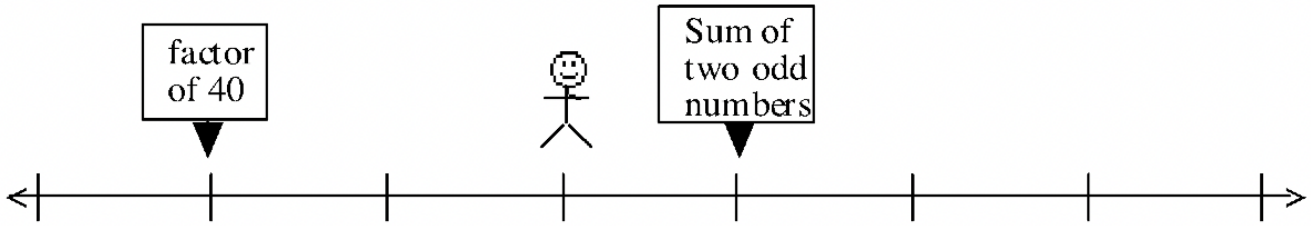
Exercise 48



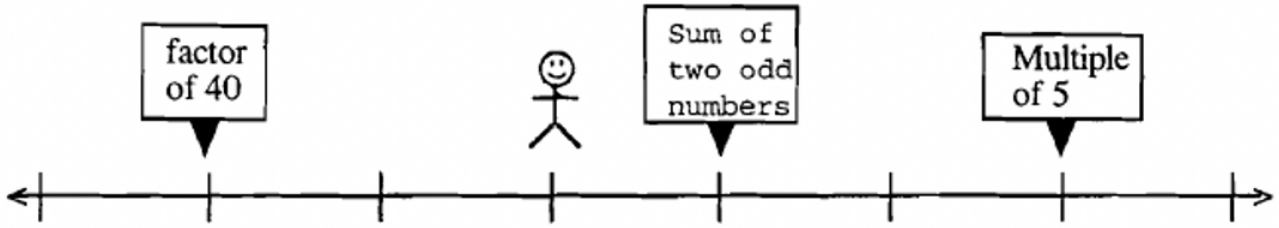
Exercise 49



Exercise 50

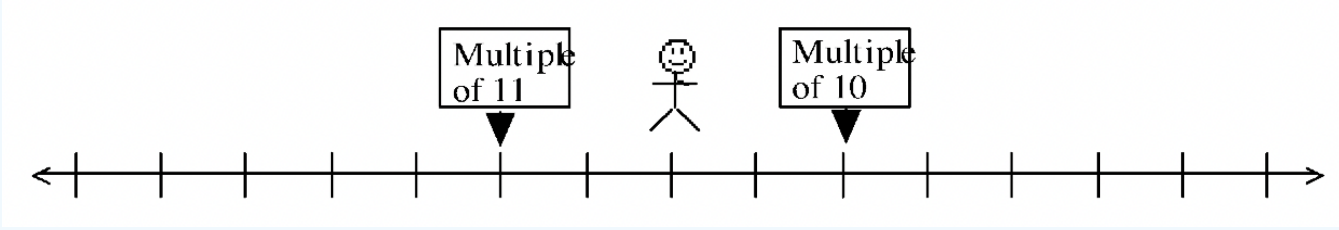


Exercise 51

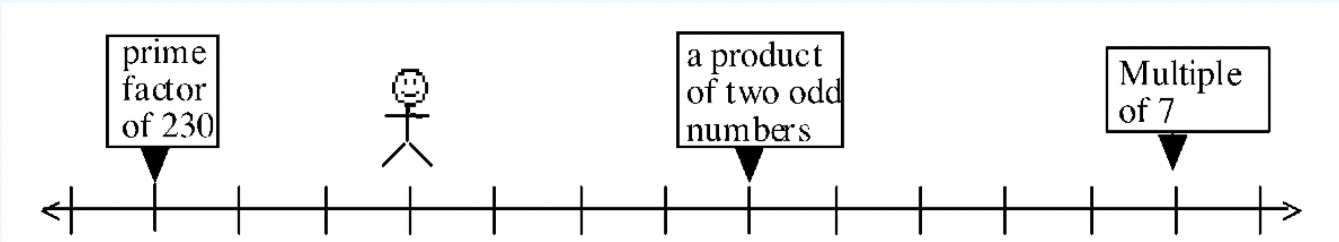


Exercise 52

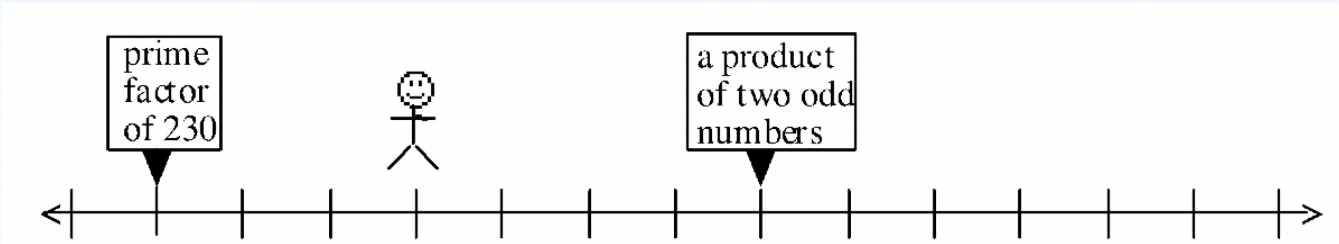
(There are actually nine possibilities between 1 and 1000 for this problem.)



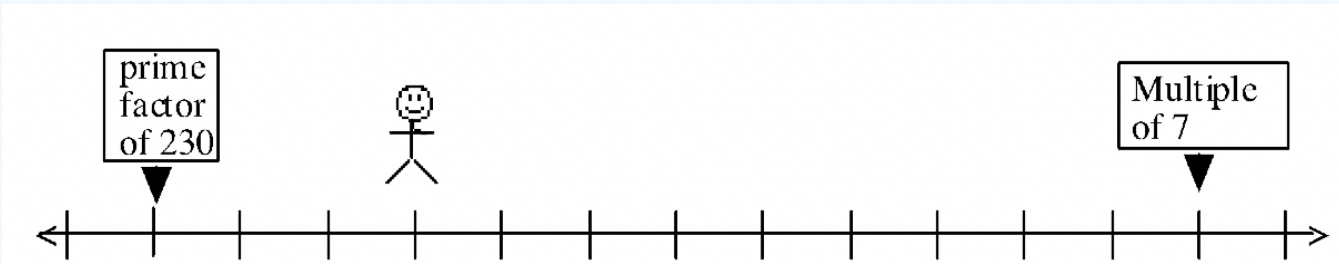
Exercise 53



Exercise 54



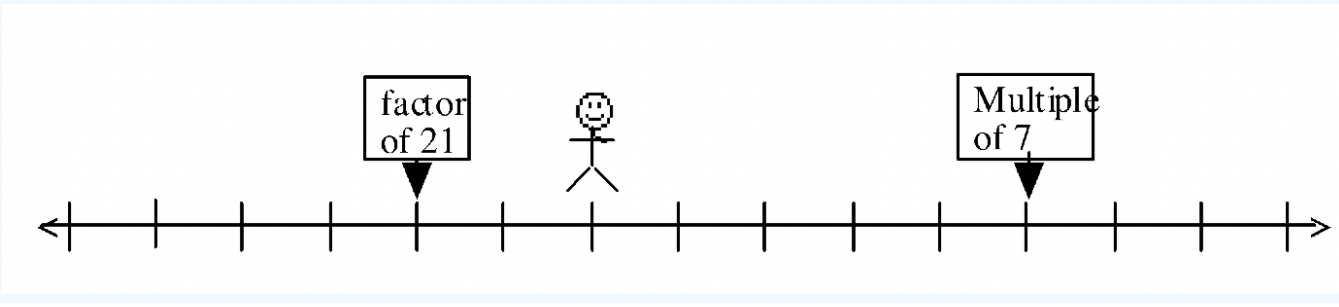
Exercise 55



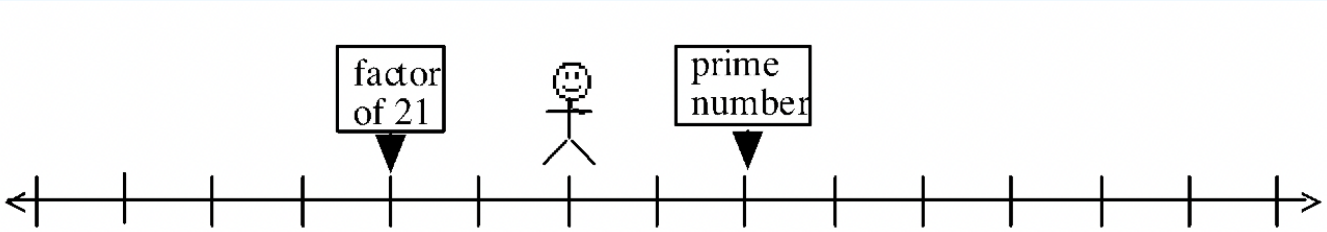
Exercise 56

Which of the clues in #53 was was not needed if the man is only standing on one single number? #54 and #55 should help you answer this. For instance, if you got more than one possibility for #54 or #55, then that problem didn't provide enough clues. For the one that gave exactly one possibility, that was enough clues, so the clue from #53 that was missing wasn't really needed.

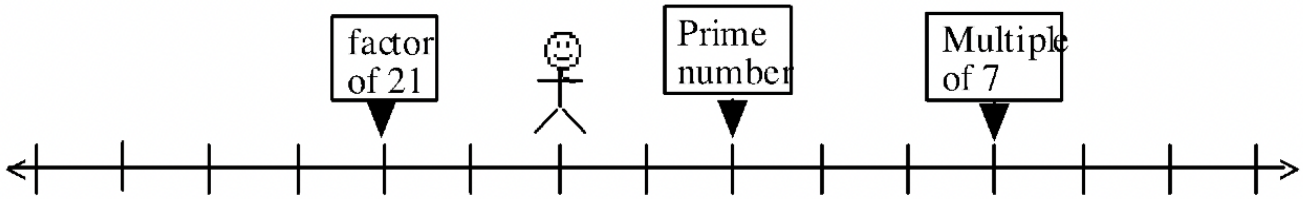
Exercise 57



Exercise 58



Exercise 59



Exercise 60

If the man in exercise #59 is standing on only one number in this problem, are all three clues given needed?

Yes or No:_____ Explain:

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4.13: Modular Arithmetic



Figure 3.41 If a credit card number is entered incorrectly, error checking algorithms will often catch the mistake. (credit: modification of work “Senior couple at home checking finance on credit card from above” by Nenad Stojkovic/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

- Add, subtract, and multiply using modulo (clock) arithmetic.
- Apply modular (clock) arithmetic to calculate real-world applications.

Online shopping requires you to enter your credit card number, which is then sent electronically to the vendor. Using an ATM involves sliding your bank card into a reader, which then reads, sends, and verifies your card. Swiping or tapping for a purchase in a brick–and-mortar store is how your card sends its information to the machine, which is then communicated to the store’s computer and your credit card company. This information is read, recorded, and transferred many times. Each instance provides one more opportunity for error to creep into the process, a misrecorded digit, transposed digits, or missing digits. Fortunately, these card numbers have a built-in error-checking system that relies on modular arithmetic, which is often referred to as clock arithmetic. In this section, we explore clock, or modular, arithmetic.

Video 4.13.1

Determining the Day of the Week for Any Date in History

Adding, Subtracting, and Multiplying Using Modulo Arithmetic

When we do arithmetic, numbers can become larger and larger. But when we work with time, specifically with clocks, the numbers cycle back on themselves. It will never be 49 o’clock. Once 12 o’clock is reached, we go back to 1 and repeat the numbers. If it's 11 AM and someone says, “See you in four hours,” you know that 11 AM plus 4 hours is 3 PM, not 15 AM (ignoring military time for now). Math worked on the clock, where numbers restart after passing 12, is called clock arithmetic.

Clock arithmetic hinges on the number 12. Each cycle of 12 hours returns to the original time (Figure 3.42). Imagine going around the clock one full time. Twelve hours pass, but the time is the same. So, if it is 3:00, 14 hours later and two hours later both read the same on the clock, 5:00. Adding 14 hours and adding 2 hours are identical. As is adding 26 hours. And adding 38 hours.

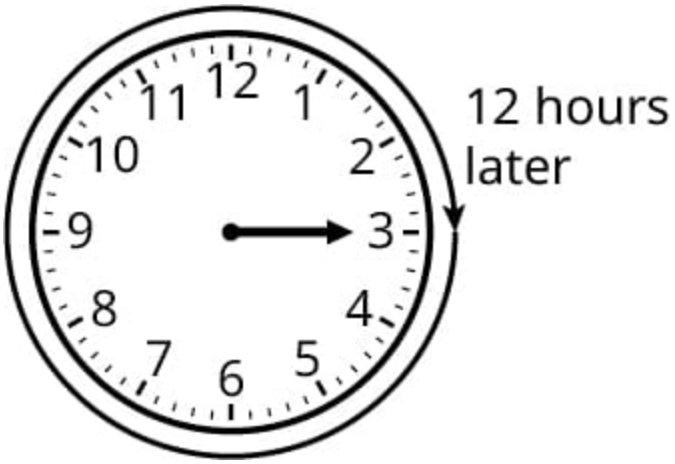


Figure 3.42 Clock showing 3:00 with arrow going around the clock one full time, or 12 hours

What do 2, 14, 26, and 38 have in common in relation to 12? When they are divided by 12, they each have a remainder of 2. That's the key. When you add a number of hours to a specific time on the clock, first divide the number of hours being added by 12 and determine the remainder. Add that remainder to the time on the clock to know what time it will be.

A good visualization is to wrap a number line around the clock, with the 0 at the starting time. Then each time 12 on the number line passes, the number line passes the starting spot on the clock. This is referred to as modulo 12 arithmetic. Even though the process says to divide the number being added by 12, first perform the addition; the result will be the same if you add the numbers first, and then divide by 12 and determine the remainder.

In general terms, let n be a positive integer. Then n modulo 12, written (n mod 12), is the remainder when n is divided by 12. If that remainder is x, we would write $n = x$ (mod 12).

Checkpoint 4.13.1

Caution: 12 mod 12 is 0. So, if a mod 12 problem ends at 0, that would be 12 on the clock.

Example 4.13.1

Determining the Value of a Number modulo 12

Find the value of the following numbers modulo 12:

1. 34
2. 539
3. 156

Answer available at the end of this section.

Your Turn 4.13.1

Find the value of the following numbers modulo 12.

1. 93
2. 387

Tech Check 4.13.1

Using Desmos to Determine the Value of a Number module 12

Desmos may be used to determine the value of a number modulo 12. It is flexible enough to find the value of a number modulo of any other integer you want. To determine the value of [redacted]nFigure 3.43.

Figure 3.43 Display of 539 modulo 12

Clock arithmetic is modulo 12 arithmetic but applied to time. As time is divided into 12 hours that repeat a cycle, we use modulo 12 for clock arithmetic.

Video 4.13.1

Clock Arithmetic

Example 4.13.2

Adding with Clock Arithmetic

If it's 3:00, what time will it be in 89 hours?

Answer available at the end of this section.

Your Turn 4.13.2

If it is 9:00 now, what time will it be in 43 hours?

Subtracting time on the clock works in much the same way as addition. Find the value of the number of hours being subtracted modulo 12, then subtract that from the original time.

Example 4.13.3

Subtracting with Clock Arithmetic

If it is 4:00 now, what time was it 67 hours ago?

Answer available at the end of this section.

Your Turn 4.13.3

If it is 7:00 now, what time was it 34 hours ago?

Recall that clock arithmetic was referred to as modulo 12 arithmetic. Multiplying in modulo 12 also relies on the remainder when dividing by 12. To multiply modulo 12 is just to multiply the two numbers, and then determine the remainder when divided by 12.

Example 4.13.4

Multiplying modulo 12

What is the product of 11 and 45 modulo 12?

Answer available at the end of this section.

Your Turn 4.13.4

What is the product of 4 and 19 modulo 12?

Try It! Clock Arithmetic

Instructions:

1. Choose a modulus (clock size) between 2 and 24.
2. Enter two numbers you want to add, subtract, or multiply in clock arithmetic.
3. Before using the widget, try to calculate the result on your own.
4. Select the operation (addition, subtraction, or multiplication) you want to perform.
5. Click "Calculate" to see the result and the step-by-step explanation.
6. Compare your answer with the widget's result and review the explanation.
7. Try different numbers and operations to reinforce your understanding of clock arithmetic.

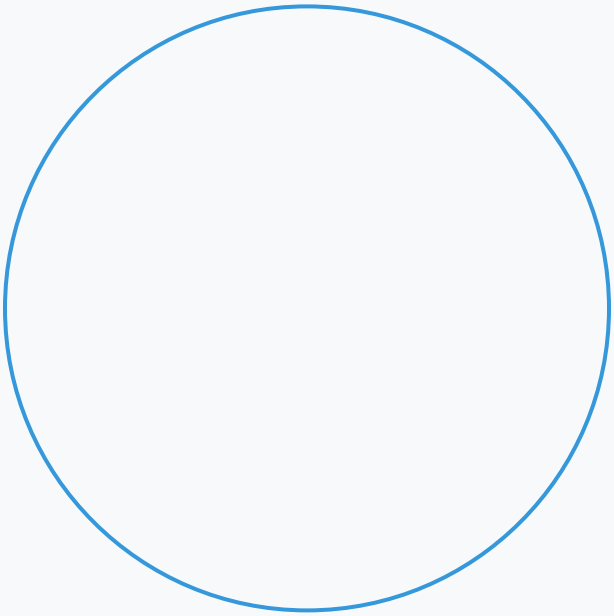
Modulus (Clock Size): 12

First Number:

Second Number:

Operation: Addition

Calculate



Calculating Real-World Applications with Clock Arithmetic

Example 4.13.5

Applying Clock Arithmetic

Suppose it is 3:00, and you decide to check your email every 5 hours. What time will it be when you check your email the ninth time?

Answer available at the end of this section.

Your Turn 4.13.5

You have agreed to text your friend every 3 hours while driving across the country. You began your trip at 8 AM. What time will it be when you text your friend the 15th time?

Clock arithmetic processes can be applied to days of the week. Every 7 days the day of the week repeats, much like every 12 hours the time on the clock repeats. The only difference will be that we work with remainders after dividing by 7. In technical terms, this is referred to as modulo 7. More generally, let n be a positive integer. Then n modulo 7, written n mod 7, is the remainder when n is divided by 7. If that value is x, we may write $n = x$ (mod 7).

Example 4.13.6

Applying Clock Arithmetic to Days of the Week

Your family has a cat, and no one wants to empty the litter box. However, it has to be done daily. The six of you agree to take turns, so everyone has to empty the litter box every 6 days. You empty the box on a Thursday. What day will you empty the box for the 10th time?

Answer available at the end of this section.

Your Turn 4.13.6

Your family shares the cooking duties in the home. You've agreed to prepare the meal every 5 days. The last time you prepared dinner was a Tuesday. What day of the week will it be after you've prepared the meals 20 more times?

Check Your Understanding

For the following exercises, use clock arithmetic to perform the following:

1. $7 + 19$
2. \(8-31\)
3. \(5 \times 37\)
4. Calene calls her mother every fourth day. She calls on a Monday. What day of the week will it be on Calene's eighth time calling after that?

Section $PageIndex$ Exercises

1. Explain what modulo 12 means.
2. Explain what modulo 7 means.
3. What is 75 modulo 12?
4. What is 139 modulo 12?
5. What is 38 modulo 7?
6. What is 83 modulo 7?

For the following exercises, use clock arithmetic (mod 12), to perform the indicated calculation.

7. \(7 + 13\)
8. \(8 + 19\)
9. \(4 + 27\)
10. \(3 + 100\)
11. \(9 - 15\)
12. \(6 - 27\)
13. \(4 \times 18\)
14. \(7 \times 29\)
15. \(11 \times 38\)
16. \(6 \times 23\)
17. It is 8:00. What time will it be in 70 hours?
18. It is a Thursday. What day of the week will it be in 100 days?
19. It is Monday. What day of the week will it be in 58 days?
20. It is 3:00. What time of the day will it be in 150 hours?
21. It is 6:00. What time was it 34 hours ago?
22. It is 2:00. What time was it 100 hours ago?

23. A trucker passes through Kokomo, Indiana, once every 9 days. They come through Kokomo on a Wednesday. What day of the week will the driver pass through Kokomo after 8 more visits?

24. Jason checks his email every 5 hours. He checks it at 6 PM one day. What time of the day will it be when he checks his mail the 50th time after that 6 PM check?
25. Mickey gets a new prescription of a drug that she needs to take every day. The prescription is for 250 days. She takes the first pill of the new bottle on a Friday. What day of the week will her prescription run out?
26. Zainab visits the nursing home every 5 days. She visits on a Sunday. What day of the week will it be when she visits it for the 7th time after that?
27. Micaela has to check in with her boss every 14 hours. If she checks in at 3:00, what time will it be when she checks in the 10th time after that?
28. Tracy has an alarm set for every 4 hours. It goes off at 3:00. What time will it be when the alarm goes off the 20th time after that?
29. Dejan must check his blood sugar every 5 hours. He checks his blood sugar at 4:00. What time will it be when Dejan checks their his sugar the 40th time after that?
30. Latanjana is in the hospital, where her blood pressure is checked every 3 hours. If her blood pressure is checked at 5:00, what time will it be when her blood pressure is checked the 13th time after that?Months come in twelves, just as hours do. This means that months can be calculated using modulo 12, just like hours. For the following exercises, calculate what month it will be for each exercise.
31. Micaela works for a sprinkler maintenance company and runs a routine check on the Harris's sprinkler system every third month. If Micaela checks the system in an April, what month will it be when Micaela returns to the Harris's for the 11th time after that?
32. Dana runs a half marathon every 5 months. She runs one in a May. What month will it be when she runs her 8th marathon after that?

Answers

Example 4.13.1

To determine the value of a number modulo 12, divide the number by 12 and record the remainder.

1. To find the value 34 modulo 12:

   Step 1: Determine the remainder when 34 is divided by 12 using long division. The largest multiple of 12 that is less than or equal to 34 is 24, which is the product of 12 and 2.

   $$\begin{array}{r} 2 \\ 12\overline{)34} \\ \underline{24} \end{array}$$

   Step 2: Performing the subtraction yields 10.

   $$\begin{array}{r} 2 \\ 12\overline{)34} \\ \underline{24} \\ 10 \end{array}$$

   Since that subtraction resulted in a number less than 12, that is the remainder, 10. The value of 34 modulo 12 is 10, or $34 = 10 \pmod{12}$.

2. To find the value 539 modulo 12:

   Step 1: Determine the remainder when 539 is divided by 12 using long division. We first look to the first two digits of 539, 53. The largest multiple of 12 that is less than or equal to 53 is 48, which is the product of 12 and 4.

   $$\begin{array}{r} 4 \\ 12\overline{)539} \\ \underline{48} \end{array}$$

   Step 2: Performing the subtraction results in 5.

   $$\begin{array}{r} 4 \\ 12\overline{)539} \\ \underline{48} \\ 5 \end{array}$$

   Step 3: Now, the 9 is brought down.

   $$\begin{array}{r} 4 \\ 12\overline{)539} \\ \underline{48} \\ 59 \end{array}$$

   Step 4: The largest multiple of 12 that is less than or equal to 59 is once more 48 itself, which is 12×4.

   $$\begin{array}{r} 44 \\ 12\overline{)539} \\ \underline{48} \\ 59 \\ \underline{48} \end{array}$$

   Step 5: Finishing the process, the 48 is subtracted from the 59, yielding 11.

$$\begin{array}{r} 44 \\ 12\overline{)539} \\ \underline{48} \\ 59 \\ \underline{48} \\ 11 \end{array}$$

We've used all the digits of 539, and the last subtraction resulted in a number less than 12, so that number, 11, is the remainder. The value of 539 modulo 12 is 11, or, 539 = 11 (mod 12).

3. To find the value 156 modulo 12:

Step 1: Determine the remainder when 156 is divided by 12 using long division. We first look to the first two digits of 156, 15. The largest multiple of 12 that is less than or equal to 15 is 12 itself, which is the product of 12 and 1.

$$\begin{array}{r} 1 \\ 12\overline{)156} \\ \underline{12} \end{array}$$

Step 2: Performing the subtraction results in 3.

$$\begin{array}{r} 1 \\ 12\overline{)156} \\ \underline{12} \\ 3 \end{array}$$

Step 3: Now, the 6 is brought down.

$$\begin{array}{r} 1 \\ 12\overline{)156} \\ \underline{12} \\ 36 \end{array}$$

Step 4: The largest multiple of 12 that is less than or equal to 36 is 36 itself, which is 12×3.

$$\begin{array}{r} 13 \\ 12\overline{)156} \\ \underline{12} \\ 36 \\ \underline{36} \end{array}$$

Step 5: Finishing the process, the 36 is subtracted from the 36, yielding 0.

$$\begin{array}{r} 13 \\ 12\overline{)156} \\ \underline{12} \\ 36 \\ \underline{36} \\ 0 \end{array}$$

We've used all the digits of 156, and the last subtraction resulted in a number less than 12, so that number, 0, is the remainder. The value of 156 modulo 12 is 0, or, 156 = 0 (mod 12).

We should note here that, had we been speaking of time, the 0 would be interpreted as 12:00.

Example 4.13.2

To find that future time, we may determine the value of 89 (mod 12), either by long division or by using a calculator, such as Desmos. Then add the result to 3:00. Entering **mod(89,12)** in Desmos results in 5. Adding 5 hours, which was 89 (mod12), to 3:00 results in 8:00.

Example 4.13.3

To find that past time, we may determine the value of 67 (mod 12), either by long division or by using a calculator, such as Desmos. Then subtract the result to 4:00. Entering **mod(67,12)** in Desmos results in 7. Subtracting 7 hours from 4:00 results in –3:00. We know, though, that time is not represented with negative times. This value, –3:00, indicates three hours before 12:00, which is 9:00. So, 67 hours before 4:00 was 9:00. We see this in the Figure 3.44.

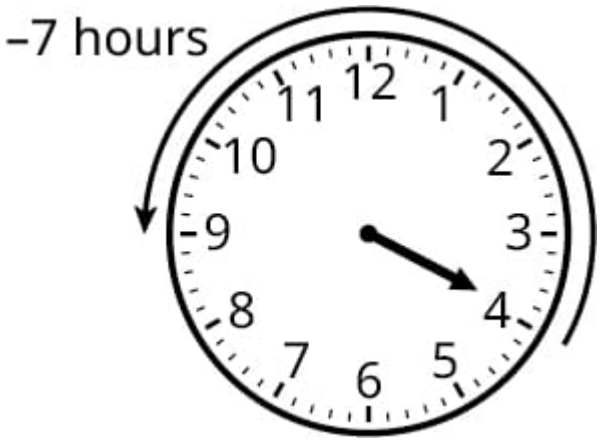


Figure 3.44 Clock showing 7 hours subtracted from 4:00

Example 4.13.4

We begin by multiplying 11 and 45, which is 495. Next, we find 495 modulo 12, either by dividing the result by 12 to determine the remainder, or by using a calculator. Entering **mod(495,12)** in Desmos yields 3. Had long division been used, the remainder would be 3. So $11 \times 45 = 3$ modulo 12.

Example 4.13.5

If you check your email every 5 hours nine times, that ninth check will occur 45 hours after 3:00, which is an addition of 45 hours to 3:00. So, we find 45 modulo

12, which is 9. Nine hours after 3:00 is 12:00. It will be 12:00 when you check your email the ninth time.

Example 4.13.6

The first time you emptied the litter box was on a Thursday. So,the 10th time you empty the litter box will be 9 times later (you've already had your first turn, so 9 turns left!). This will happen 54 (9 times 6) days later. Finding the value of 54 modulo 7, using division to determine the remainder or using a calculator to find the value of 54 modulo 7 gives the answer 5. Five days after a Thursday is Tuesday.

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4.14: Arithmetic Sequences

Learning Objectives

After completing this section, you should be able to:

- Identify arithmetic sequences.
- Find a given term in an arithmetic sequence.
- Find the nth term of an arithmetic sequence.
- Find the sum of a finite arithmetic sequence.
- Use arithmetic sequences to solve real-world applications

As we saw in the previous section, we are adding about 2.5 quintillion bytes of data per day to the Internet. If there are 550 quintillion bytes of data today, then there will be 552.5 quintillion bytes tomorrow, and 555 quintillion bytes in 2 days. This is an example of an arithmetic sequence. There are many situations where this concept of fixed increases comes into play, such as raises or table arrangements.

Identifying Arithmetic Sequences

A sequence of numbers is just that, a list of numbers in order. It can be a short list, such as the number of points earned on each assignment in a class, such as {10, 10, 8, 9, 10, 6, 10}. Or it can be a longer list, even infinitely long, such as the list of prime numbers. For example, here's a sequence of numbers, specifically, the squares of the first 12 natural numbers.

{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144}

Each value in the sequence is called a term. Terms in the list are often referred to by their location in the sequence, as in the [redacted]Figure 3.48).

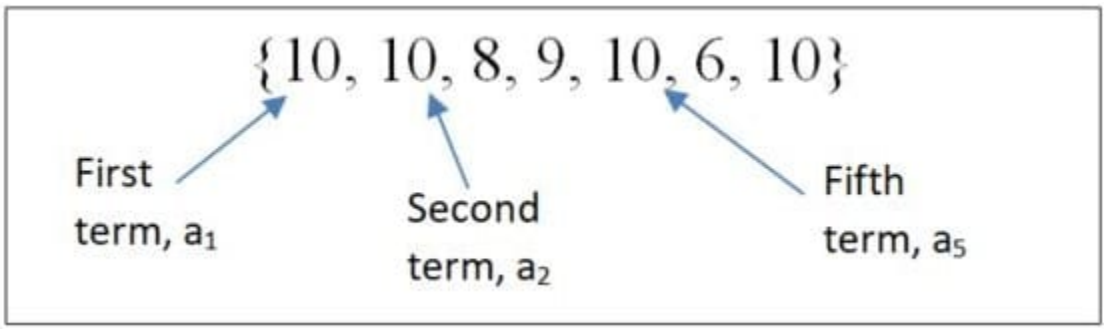


Figure 3.48 Sequence showing first, second, and fifth terms

The notation we use with sequences is a letter, which represents a term in the sequence, and a subscript, which indicates what place the term is in the sequence. For the sequence {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144}, we will use the letter a as a value in the sequence, and so (\a_5\) would be the term in the sequence at the fifth position. That number is 25, so we can write $a_5 = 25$.

In this section, we focus on a special kind of sequence, one referred to as an arithmetic sequence. Arithmetic sequences have terms that increase by a fixed number or decrease by a fixed number, called the constant difference (denoted by d), provided that value is not 0. This means the next term is always the previous term plus or minus a specified, constant value. Another way to say this is that the difference between any consecutive terms of the sequence is always the same value.

To see a constant difference, look at the following sequence: {7, 15, 23, 31, 39, 47, 55, 63, 71, 79, 87}. Figure 3.49 illustrates that each term of the sequence is the previous term plus 8. Eight is the constant difference here.

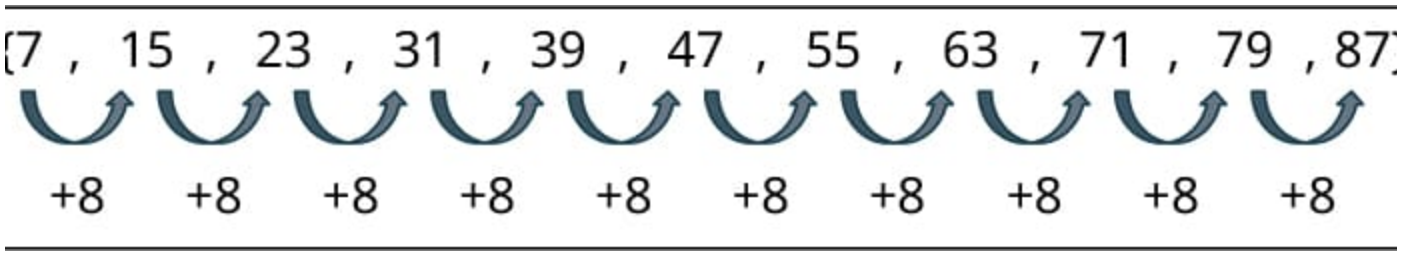


Figure 3.49 Sequence of numbers with 8 added to each term

Example 4.14.1

Identifying Arithmetic Sequences

Determine if the following sequences are arithmetic sequences. Explain your reasoning.

1. $\{4, 7, 10, 13, 16, 19, 22, 25, \ldots\}$
2. $\{20, 40, 80, 160, 320, 640\}$
3. $\{7, 1, -5, -11, -17, -23, -29, -34, -40\}$

Answer available at the end of this section.

Your Turn 4.14.1

Determine if the following sequences are arithmetic sequences. Explain your reasoning.

1. {7.6, 5.4, 3.2, 1.0, -1.2, -3.4, -5.6, -7.8, -10.0}
2. {14, 16, 18, 22, 28, 40, 32, 0}
3. {14, 20, 26, 32, 38, 44, 50, 56, 62, 68, 74, 80, ...}

Arithmetic sequences can be expressed with a formula. When we know the first term of an arithmetic sequence, which we label a_1 [redacted]a1a1, and we know the constant difference, which is denoted d, we can find any other term of the arithmetic sequence. The formula for the [redacted]ith term of an arithmetic sequence is $a_i = a_1 + d \times (i - 1)$.

FORMULA 4.14.1

If we have an arithmetic sequence with the first term a_1 and constant difference d, then the ith term of the arithmetic sequence is $a_i = a_1 + d \times (i - 1)$.

Let's examine the formula with this arithmetic sequence: $\{4, 7, 10, 13, 16, 19, 22, 25, \ldots\}$. In this sequence a_1 =4 and $d = 3$. The table below shows the values calculated.

i, Place in Sequence	[redacted]ai, [redacted]ith Term	Value of Term	Term Written as $a_1 + 3 \times (i - 1)$
1	[redacted]a1a1	4	$4 + 3 \times 0$
2	[redacted]a2a2	7	$4 + 3 \times 1$
3	[redacted]a3a3	10	$4 + 3 \times 2$
4	[redacted]a4a4	13	$4 + 3 \times 3$
5	[redacted]a5a5	16	$4 + 3 \times 4$
i	[redacted]aiai		$4 + 3 \times (i - 1)$

We can see how the [redacted]ithith term can be directly calculated. In this sequence, the formula is $a_1 + 3 \times (i - 1)$ where the first term, [redacted]a1a1, is 4 and the constant difference d is 3. We can then determine the [redacted]47th47th term of this

sequence: [redacted]a47=4+3×(47−1)=4+3×46=4+138=142a47=4+3×(47−1)=4+3×46=4+138=142.

Example 4.14.2

Calculating a Term in an Arithmetic Sequence

Identify [redacted]a1a1 and d for the following arithmetic sequence. Use this information to determine the [redacted]60th60th term.

$$\{18, 31, 44, 57, 70, 83, \ldots\}$$

Answer

Inspecting the sequence shows that a_1 =18a1=18 and d =13d=13. We use those values in the formula, with i = 60i=60.

[redacted]

a60=a1+d×(i−1)=18+13×(60−1)=18+13×59=18+767=785a60=a1+d×(i−1)=18+13×(60−1)=18+13×59=18+767=785

Your Turn 4.14.2

Identify a_1 and d for the following arithmetic sequence. Use this information to determine the 86th term.

{4.5,8.1,11.7,15.3,18.9,22.5,26.1,...}

Video 4.14.1

Arithmetic Sequences

If we know two terms of the sequence, it is possible to determine the general form of an arithmetic sequence, $a_i = a_1 + d \times (i - 1)$.

FORMULA 4.14.2

If we have the ith term of an arithmetic sequence, [redacted]aiai, and the jth term of the sequence, [redacted]ajaj, then the constant difference is $d =$ [redacted]aij−id=aj−aij−i and the first term of the sequence is $a_1 = a_i - d(i - 1)$.

Example 4.14.3

Determining First Term and Constant Difference Using Two Terms

A sequence is known to be arithmetic. Two of its terms are a_7 =56a7=56 and [redacted]a19=104a19=104. Use that information to find the constant difference, the first term, and then the [redacted]50th50th term of the sequence.

Answer available at the end of this section.

Your Turn 4.14.3

A sequence is known to be arithmetic. Two of the terms are \(a_{14} = 41\) and \(a_{38} = 161\). Use that information to find the constant difference and the first term. Then determine the \(151st\) term of the sequence.

Video 4.14.2

Finding the First Term and Constant Difference for an Arithmetic Sequence

Finding the Sum of a Finite Arithmetic Sequence

Sometimes we want to determine the sum of the numbers of a finite arithmetic sequence. The formula for this is fairly straightforward.

FORMULA 4.14.3

The sum of the first n terms of a finite arithmetic sequence, written [redacted]snsn, with first and last term [redacted]a1a1 and [redacted]anan, respectively, is $s_n = n($[redacted]an2)sn=n(a1+an2).

Example 4.14.4

Finding the Sum of a Finite Arithmetic Sequence

What is the sum of the first 60 terms of an arithmetic sequence with $a_1 = 4.5$ and $d = 2.5$?

Answer available at the end of this section.

Your Turn 4.14.4

What is the sum of the first 101 terms of an arithmetic sequence with $a_1 = 13$ and \(d = 2.25\)?

Video 4.14.3

Finding the Sum of a Finite Arithmetic Sequence

Using Arithmetic Sequences to Solve Real-World Applications

Applications of arithmetic sequences occur any time some quantity increases by a fixed amount at each step. For instance, suppose someone practices chess each week and increases the amount of time they study each week. The first week the person practices for 3 hours, and vows to practice 30 more minutes each week. Since the amount of time practicing increases by a fixed number each week, this would qualify as an arithmetic sequence.

Example 4.14.5

Applying an Arithmetic Sequence

Jordan has just watched *The Queen's Gambit* and decided to hone their skills in chess. To really improve at the game, Jordan decides to practice for 3 hours the first week, and increase their time spent practicing by 30 minutes each week. How many hours will Jordan practice chess in week 20?

Answer available at the end of this section.

Your Turn 4.14.5

Christina decides to save money for after graduation. Christina starts by setting aside $10. Each week, Christina increases the amount she saves by $5. How much money will Christina save in week 52?

Example 4.14.6

Finding the Sum of a Finite Arithmetic Sequence

Let's check back in on Jordan. Recall, Jordan had just watched *The Queen's Gambit* and decided to hone their skills, practicing for 3 hours the first week, and increasing the time spent practicing by 30 minutes each week. How many hours total will Jordan have practiced chess after 30 weeks of practice?

Answer available at the end of this section.

Your Turn 4.14.6

In a theater, the first row has 24 seats. Each row after that has 2 more seats. How many total seats are there if there are 40 rows of seat in the theater?

Who Knew?

The Fibonacci Sequence

Not all sequences are arithmetic. One special sequence is the Fibonacci sequence, which is the sequence that has as its first two terms 1 and 1. Every term thereafter is the sum of the previous two terms. The first nine terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, and 34.

This sequence is found in nature, architecture, and even music! In nature, the Fibonacci sequence describes the spirals of sunflower seeds, certain galaxy spirals, and flower petals. In music, the band Tool used the Fibonacci sequence in the song "Lateralus." The Fibonacci sequence even relates to architecture, as it is closely related to the golden ratio.

Video 4.14.4

Fibonacci Sequence and "Lateralus"

Check Your Understanding

1. Is the following an arithmetic sequence? Explain.
   {3, 6, 9, 15, 25, 39, 90}
2. What is the 7th term of the following sequence?
   {1, 5, 7, 100, 4, -17, 8, 100, 19, 7.6, 345}
3. In an arithmetic sequence, the first term is 10 and the constant difference is 4.5. What is the 135th term?
4. If the eighth term of an arithmetic sequence is 35 and the 40th term is 131, what is the constant difference and the first term of the sequence?
5. What is the sum of the first 100 terms of the arithmetic sequence with first term 4 and constant difference 7?
6. A new marketing firm began with 30 people in its survey group. The firm adds 4 people per day. How many people will be in their survey group after 100 days?

Section 3.10 Exercises

For the following exercises, determine if the sequence is an arithmetic sequence.

1. {3,7,11,15,25,100,...}
2. {27,24,21,18,15,12,9,...}
3. {6, - 1, - 8, - 15, - 23, - 31, - 39,...}
4. { - 5,4,13,22,31,40,49,58,67,...}
5. {14,19,24,29,34,50,60}
6. {3.9,2.3,0.7, - 0.9, - 2.5, - 4.1, - 5.7,...}
7. {4, - 8,12, - 16,20, - 24,28, - 32,...}
8. {1,2,3,5,8,13,21,34,55,...}

For the following exercises, the sequences given are arithmetic sequences. Determine the constant difference for each sequence. Verify that each term is the previous term plus the constant difference.

1. {18,68,118,168,218,268,...}
2. {13,35,57,79,101,123,145,167,...}
3. {14,11,8,5,2, - 1, - 4,...}
4. {4.5,1.9, - 0.7, - 3.3, - 5.9,...}
5. { - 27, - 13,1,15,29,43,57,71,...}
6. {3.8,10.6,17.4,24.2,31,37.8,44.6,...}

For the following exercises, the first term and the constant difference of an arithmetic sequence is given. Using that information, determine the indicated term of the sequence.

1. $a_1 = 12$, $d = 11$, find a_{20}.
2. $b_1 = 5$, $d = 8$, find b_{38}.
3. $c_1 = 48$, $d = -7$, find c_{50}.
4. $a_1 = 110$, $d = -16$, find a_{27}.
5. $t_1 = 15.3$, $d = 4.2$, find t_{17}.
6. $b_1 = 23.8$, $d = 11.7$, find b_{120}.
7. $b_1 = 27.45$, $d = -3.67$, find b_{40}.
8. $a_1 = 67.4$, $d = -12.3$, find a_{200}.

For the following exercises, two terms of an arithmetic sequence are given. Using that information, identify the first term and the constant difference.

1. $a_5 = 27$, $a_{15} = 77$
2. $b_{10} = 47$, $b_{25} = 137$
3. $a_9 = 38$, $a_{45} = 189.2$
4. $a_6 = 43$, $a_{41} = -377$
5. $a_4 = -12.3$, $a_{54} = -106.5$
6. $a_{12} = 45.9$, $a_{60} = -563.7$

For the following exercises, the first term and the constant difference is given for an arithmetic sequence. Use that information to find the sum of the first n terms of the sequence, s_n.

1. $a_1 = 15$, $d = 7$, calculate s_{10}.
2. $a_1 = 2$, $d = 13$, calculate s_{20}.
3. $a_1 = 105$, $d = 0.3$, calculate s_{15}.
4. $a_1 = 56.2$, $d = 1.1$, calculate s_{35}.
5. $a_1 = 450$, $d = -20$, calculate s_{20}.
6. $a_1 = 1400$, $d = -35$, calculate s_{40} .

For the following exercises, apply your knowledge of arithmetic sequences to these real-world scenarios.

1. A collection is taken up to support a family in need. The initial amount in the collection is $135. Everyone places $20 in the collection. When the 35th person puts their $20 in the collection, how much is present in the collection?

2. There are 50 songs on a playlist. Every minute, 3 more songs are added to the playlist. How many songs are on the playlist after 40 minutes have passed?
3. One genre on Netflix has 1,000 shows. Every week, 20 shows are added to that genre. After 15 weeks, how many shows are in that genre?
4. A new local band has 10 people come to their first show. News of the band spreads afterwards. Each week, 4 more people attend their show than the previous week. After 50 weeks, how many people are at their show?
5. The Jester Comic book store is going out of business and is taking in no new inventory. Its inventory is currently 13,563 titles. Each day after, they sell or give away 250 titles. After 15 days, how many titles are left?
6. Jasmyn has decided to train for a marathon. In week one, Jasmyn runs 5 miles. Each week, Jasmyn increased the running distance by 2 miles. How many miles will Jasmyn run in week 13 of the training schedule?
7. A 42-gallon bathtub sits with 14 gallons in it. The faucet is turned on and is now being filled at the rate of 2.2 gallons per minute, but is draining slowly, at 1.8 gallons per minute. After 20 minutes, how many gallons are in the tub?
8. A trained diver is 250 feet deep. The diver is nearly out of air and needs to surface. However, the diver can only comfortably ascend 30 feet per minute. How deep is the diver after ascending for 5 minutes?
9. Jaclyn, an investor, begins a start-up to revitalize homes in South Bend, Indiana. She begins with $10,000, making her investor 1. Each investor that joins will invest $500 more than the previous investor. How much does the 50th investor invest in the project? With that 50th investor, what is the total amount invested in the project?
10. Jasmyn has decided to train for a marathon. In week one, Jasmyn runs 5 miles. Each week, Jasmyn increased the running distance by 2 miles. After training for 14 weeks, how many total miles will Jasmyn have run?
11. The base of a pyramidal structure has 144 blocks. Each level above has 5 fewer blocks than the previous level. How many total blocks are there if the pyramidal structure has 25 levels?
12. As part of a deal, a friend tells you they will give you $10 on day 1, $20 on day 2, $30 on day 3, for all 30 days of a month. At the end of that month, what is the total amount your friend has given you?

Answers

Example 4.14.1

1. In the sequence $\{4, 7, 10, 13, 16, 19, 22, 25, ...\}$, every term is the previous term plus 3. The ellipsis indicates that the pattern continues, which means keep adding 3 to the previous term to get the new term. Therefore, this is an infinite arithmetic sequence.
2. In the sequence $\{20, 40, 80, 160, 320, 640\}$, terms increase by various amounts, for instance from term 1 to term 2, the sequence increases by 20, but from term 2 to term 3 the sequence increases by 40. So, this is not an arithmetic sequence.
3. In the sequence $\{7, 1, -5, -11, -17, -23, -29, -34, -40\}$, every term is the previous term minus 6, so this is an arithmetic sequence.

Example 4.14.3

To find the constant difference, use $d =$ [redacted]aij–id=aj–aij–i. The location of the terms is given by the subscript of the two a terms, $i = 7$ and $j =$19j=19. So, the constant difference can be calculated as such:

$$d = \frac{104 - 56}{19 - 7} = \frac{48}{12} = 4$$

.

The constant difference of 4 is then used to find [redacted]a1a1.

$$a_1 = a_i - d(i - 1) = a_7 - 4(7 - 1) = 56 - 4 \times 6 =$$

32a1=ai–d(i–1)=a7–4(7–1)=56–4×6=32.

So $d = 4$ and $a_1 =$32a1=32.

With this information, the [redacted]50th50th term can be found.

[redacted]

a50=a1+d×(i–1)=32+4×(50–1)=32+4×49=32+196=228a50=a1+d×(i–1)=32+4×(50–1)=32+4×49=32+196=228.

The [redacted]50th term is [redacted]\(a_{50}=228\).

Example 4.14.4

The formula requires the first and last terms of the sequence. The first term is given, $a_1 = 4.5$. The [redacted]60th60th term is needed. Using the formula $a_1 = a_i + d(i - 1)$ provides the value for the [redacted]60th60th term.

[redacted]

a60=4.5+2.5(60–1)=4.5+2.5×59=4.5+147.5=152a60=4.5+2.5(60–1)=4.5+2.5×59=4.5+147.5=152.

Applying the formula $s_n = n($[redacted]an2)sn=n(a1+an2) provides the sum of the first 60 terms.

[redacted]

s60=60(4.5+1522)=60×156.52=4,695s60=60(4.5+1522)=60×156.52=4,695.

The sum of the first 60 terms is 4,695.

Example 4.14.5

Jordan's practice scheme is an arithmetic sequence, as it increases by a fixed amount each week. The first week there are 3 hours of practice. This means $a_1 =$ 3a1=3. Jordan increases the time spent practicing by 30 minutes, or half an hour, each week. This means $d = 0.5$. Using those values, and that we want to know the amount of time Jordan will study in week 20, we determine the time in week 20 using $a_i = a_1 + d \times (i - 1)$.

$$a_{20} = 3 + 0.5 \times (20 - 1) = 3 + 0.5 \times 19 = 3 + 9.5 = 12.5$$

So, Jordan will practice 12.5 hours in week 20.

Example 4.14.6

To calculate the total amount of time that Jordan practiced, we need to use $s_n = n($[redacted]an2)sn=n(a1+an2). The formula requires the first and last terms of the sequence. Since Jordan practiced 3 hours in the first week, the first term is a_1 =3a1=3. Because we want the total practice time after 30 weeks, we need the [redacted]30th30th term. Because the constant difference is $d = 0.5$, the [redacted]30th30th term is [redacted] a30=3+0.5(30−1)=3+0.5×29=3+14.5=17.5a30=3+0.5(30−1)=3+0.5×29=3+14.5=17.5.

Applying the formula $s_n = n($[redacted]an2)sn=n(a1+an2) provides the sum of the first 30 terms.

[redacted]

s30=30(3+17.52)=60×20.52=615s30=30(3+17.52)=60×20.52=615.

This means that Jordan practiced a total of 615 hours after 30 weeks.

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4.15: Geometric Sequences and Series

Learning Objectives

1. Identify geometric sequences.
2. Find a given term in a geometric sequence.
3. Find the nth term of a geometric sequence.
4. Find the sum of a finite geometric sequence.
5. Use geometric sequences to solve real-world applications.

One of the concerns when investing is the doubling time, which is length of time it takes for the value of the investment to be twice, or double, that of its starting value. A shorter doubling times means the investment gets bigger, sooner. For example, if you invest $\$200$ in an account with an 8 -year doubling time, then in 8 years the value of the account will be double the starting amount, or $2 \times \$200 = \$400$. After another 8 years (for a total of 16 years) the investment would be twice its value after the first 8 years, or $2 \times (2 \times \$400) = 2 \times (\$400) = \$800$. Every 8 years, the investment would double again, so after the third 8 -year period, the investment would be worth $2 \times 2 \times (2 \times \$400) = \$1,600$. This process exhibits exponential growth, an application of geometric sequences, which is explored in this section.



Figure 4.15.1: Savings grows in a geometric sequence. (credit: modification of "A big part of financial freedom is having your heart and mind free from worry about the what-ifs of life. – Suze Orman" by Morgan/Flickr, CC BY 2.0)

Identifying Geometric Sequences

We know what a sequence is, but what makes a sequence a geometric sequence? In an arithmetic sequence, each term is the previous term plus the constant difference. So, you add a (possibly negative) number at each step. In a geometric sequence, though, each term is the previous term multiplied by the same specified value, called the common ratio. In the sequence $\{3, 6, 12, 24, 48, 96, 192, 384, 728, 1456\}$

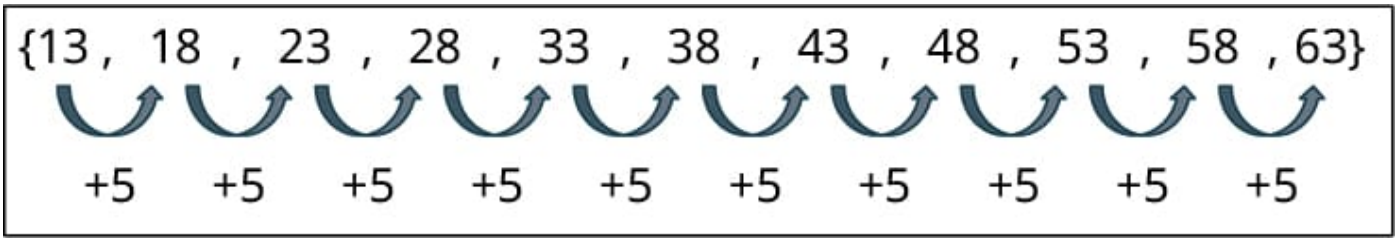


Figure 4.15.2: Arithmetic sequence

Each term in this arithmetic sequence is the previous term plus 5.

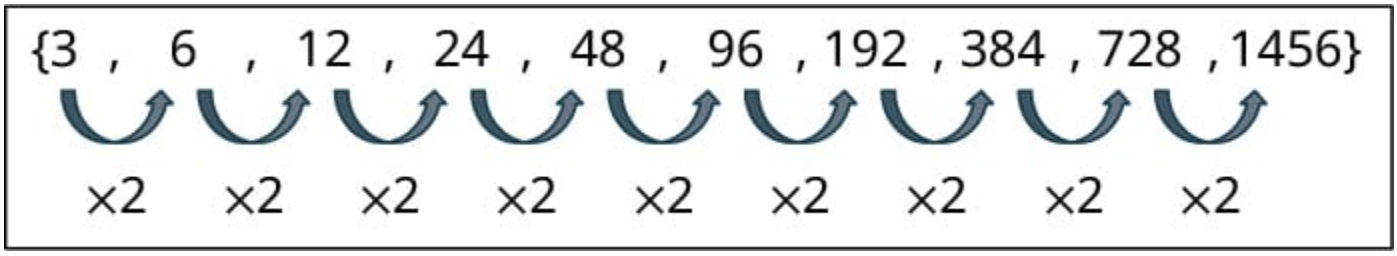


Figure 4.15.3: Geometric sequence

Each term in this geometric sequence is the previous term times 2.

In the sequence $\{3, 6, 12, 24, 48, 96, 192, 384, 728, 1456\}$, the numbers get big fairly quickly, and stay positive. However, that's not always the case with geometric sequences. Depending on the value of the common ratio, the terms could increase each time (like in the one shown in Figure 3.51), or the terms can get smaller each time, or the terms can alternate between positive and negative values. It all depends on the value of the common ratio, r.Math input error

Consider this geometric sequence:

$$\{5, 15, 45, 135, 405, 2025...\}$$

Each term is the previous term times 5, which means the common ratio is 5. This common ratio is larger than 1, and so the terms increase each time. Now, look at this geometric sequence:

$$\{2, -6, 18, -54, 162, -486, 1458...\}$$

Each term is the previous term times –3, and the sign of the terms alternate from positive to negative. Then, there's this geometric sequence:

$$\left\{9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}...\right\}$$

Each term is the previous term times $\frac{1}{3}$, and the terms decrease each time. What we should take away from these three examples is if the common ratio is a positive number larger than 1, then the sequence increases. If the common ratio is a negative number, then the sign of the terms alternates between positive and negative. If the common ratio is between 0 and 1, then the terms decrease.

Two special cases of geometric sequences are when the constant ratio is 1 and when the common ratio is 0. When the constant ratio is 1, every term of the sequence is the same, as in $\{3, 3, 3, 3, 3, 3, 3, 3, 3\}$. This is referred to as a constant sequence. When the constant ratio is 0, the first term can be any number, but every term after the first term is 0, as in $\{-43.2, 0, 0, 0, 0, 0, 0, 0\}$.

Example 4.15.1: Identifying Geometric Sequences

For each sequence, determine if the sequence is a geometric sequence. If so, identify the common ratio.

1. $\{5, 20, 80, 320, 1{,}280, 5{,}120, 20{,}480, ...\}$
2. $\{-3, 6, -12, 24, 11, 33\}$

3. $\{4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$

Answer available at the end of this section.

Your Turn 4.15.1

For each sequence, determine if the sequence is a geometric sequence. If so, identify the common ratio.

1. $\{-1, -5, -25, -125, -625, -3125, , \ldots\}$
2. $\{-3, 6, -12, 24, 11, 33\}$
3. $\{-500, 50, -5, \frac{1}{2}, -\frac{1}{2}0, \ldots\}$

As with arithmetic sequences, the first term of a geometric sequence is labeled [redacted]a1a1. The number that is multiplied by each term is called the common ratio and is denoted r. So, if the first term is known, a_1, and the common ratio is known, r, then the [redacted]nthnth term, [redacted]anan, can be calculated with the formula a_n =a1rn−1an=a1rn−1.

FORMULA

The nth term of the geometric sequence, [redacted]anan, with first term [redacted]a1a1 and common ratio r, is a_n = a1rn−1an=a1rn−1.

Return to the sequence $\{3, 6, 12, 24, 48, 96, 192, 384, 728, \ldots\}$. We observe that the first term is 3, so a_1 =3a1=3. We also found that the common ratio is 2, so $r = 2$. The table below shows how any term can be calculated using just [redacted]a1a1 and r.

i, Place in Sequence	a_i,ithai,ith,Term	Value of Term	Term Written as a1×ri−1a1×ri−1
1	[redacted]a1a1	3	3×2^0
2	[redacted]a2a2	6	3×2^1
3	[redacted]a3a3	12	3×2^2
4	[redacted]a4a4	24	3×2^3
5	[redacted]a5a5	48	3×2^4
i	[redacted]aiai		3×2i−13×2i−1

Example 4.15.2: Determining the Value of a Specific Term in a Geometric Sequence

In the following geometric sequences, determine the indicated term of the geometric sequence with a given first term and common ratio.

1. Determine the [redacted]9th term of the geometric sequence with [redacted]a1×6a1×6 and $r = 3$.
2. Determine the [redacted]11th term of the geometric sequence with a_1 =2a1=2 and $r = -5$.

Answer available at the end of this section.

Your Turn 4.15.2

In the following geometric sequences, determine the indicated term of the geometric sequence with a given first term and common ratio.

1. Determine the 12th

2. Determine the 5th

Video

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Geometric Sequences

Finding the Sum of a Finite Geometric Sequence

As with arithmetic sequences, it is possible to add the terms of the geometric sequence. Like arithmetic sequences, the formula for the finite sum of the terms of a geometric sequence has a straightforward formula.

FORMULA

The sum of the first n terms of a finite geometric sequence, written [redacted]snsn, with first term [redacted]a1a1 and common ratio r, is $s_n = a_1\left(\frac{1-r^{n-1}}{1-r}\right)$ provided that $r \neq 1$.

Example $4.15.3$: Calculating the Sum of a Finite Geometric Sequence

1. What is the sum of the first 13 terms of the geometric sequence with first term a_1 =5 and common ratio $r = 3$?
2. What is the sum of the first 7 terms of the geometric sequence with first term a_1 =16 and common ratio r =18?

Answer available at the end of this section.

Your Turn $4.15.3$

1. What is the sum of the first 10 terms of the geometric sequence with first term $a_1 = 7\ ''style = ''font - size : 1.1rem;$
2. What is the sum of the first 6 terms of the geometric sequence with first term $a_1 = 27$

Using Geometric Sequences to Solve Real-World Applications

Geometric sequences have a multitude of applications, one of which is compound interest. Compound interest is something that happens to money deposited into an account, be it savings or an individual retirement account, or IRA. The interest on the account is calculated and added to the account at regular intervals. This means the interest that was earned later gains its own interest. This allows the money to grow faster. If that interest is added every month, we say it is compounded monthly. If the interest is added daily, then we say it is compounded daily. The amount of money that is deposited into the account is called the principal and is denoted P. The account earns money on that principal. The amount it earns is a percentage of the money in the account. The interest rate, expressed as a decimal, is denoted r.

FORMULA

If you deposit P dollars in an account that earns interest compounded yearly, then the amount in the account, A, after t years is calculated with the formula: $A = P(1 + r)^t$. This is a geometric sequence, with constant ratio $(1 + r)$ and first term $a_1 =$ Pa1=P.

Example 4.15.4: Calculating Interest Compounded Yearly

Daryl deposits $1,000 in an account earning 4% interest compounded yearly. How much money is in the account after 25 years?

Answer available at the end of this section.

Your Turn 4.15.4

Sophia deposited $4,000 in an account that earns 5.5% interest compounded yearly. After 20 years, Sophia withdrew all the money in the account to pay for her child's college. How much money was in the account when Sophia withdrew the money?

Another application of geometric sequences is exponential growth. This arises in biology quite frequently, especially in relation to bacterial cultures, but also with other organism population models. In bacterial cultures, the time it takes the population to double is often recorded. This time to double is the same, regardless of how big the population gets. So, if the population doubles after 3 hours, it doubles again after another 3 hours, and again after another 3 hours, and so on. Put into geometric sequence language, it has a common ratio of 2.

Example 4.15.5: Doubling a Bacterial Culture

When *Escherichia coli* (*E. coli*) is in a broth culture at 37°C, the population of *E. coli* doubles in number with 30 organisms, how many *E. coli* bacteria are present in the culture after 16 hours?

Answer available at the end of this section.

Your Turn 4.15.5

When *Streptococcus lactis* (S. *lactis*) is in a milk culture at 37°C, the population of S. *lactis* doubles in number every 30 minutes. If the culture began with 15 organisms, how many S. *lactis* bacteria are present in the culture after 20 hours?

Example 4.15.6: Applying the Sum of a Finite Geometric Sequence

A player places one grain of rice on the first square of a chess board. On the second square, the player places 2 grains of rice. On the third square, the player places 4 grains of rice. On each successive square of the board, the player doubles the number of grains of rice placed on the chess board. When the player places the last rice on the 64th square, how many total grains of rice have been placed on the board?

Answer available at the end of this section.

Your Turn 4.15.6

You have a square 1 meter on each side. You begin by coloring one half of the square blue. Then you color half the remaining area blue. Then you color half the remaining area blue once more. At each step, you color half the remaining area. What is the total area you have colored blue after performing this process 15 times?

Video

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Sum of a Finite Geometric Sequence

Check Your Understanding

1. Is the following a geometric sequence? Explain. {3, 6, 12, 24, 48, 96, 192}
2. Find the common ratio of the geometric sequence {3, −30, 300, −3,000, ...}.
3. In a geometric sequence, the first term is 10 and the common ratio is 1.5. What is the 15th term?
4. What is the sum of the first 100 terms of the geometric sequence with first term 4 and common ratio 0.3?
5. $15,000 is deposited in an account the yields 4.2% interest compounded annually. How much is in the account after 17 years?

Answers

Example 4.15.1: Identifying Geometric Sequences

1. In the sequence $\{5, 20, 80, 320, 1{,}280, 5{,}120, 20{,}480, ...\}$, the jump from 5 to 20 is a multiplication by 4, as is the next jump to 80, and the next to 320. Each term is 4 times the previous term. Since each term is 4 times the previous, this is a geometric sequence. The common ratio is 4.
2. In the sequence $\{-3, 6, -12, 24, 11, 33\}$, notice that 6 is −3 times −2. The jump from 6 to −12 is another multiplication by negative. So, if this is a geometric sequence, each term should be the previous term times −2. But the change from 24 to 11 is not a multiplication by −2, This means the sequence is not a geometric sequence.
3. In the sequence $\left\{4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\right\}$, the change from 4 to 2 is a multiplication by $\frac{1}{2}$, as is the next jump, from 2 to 1, as is the next from 1 to $\frac{1}{2}$. Each term is $\frac{1}{2}$ times the previous term. Since each term is $\frac{1}{2}$ times the previous, this is a geometric sequence. The common ratio is $\frac{1}{2}$.

Example 4.15.2: Determining the Value of a Specific Term in a Geometric Sequence

1. Using a_n =a1rn−1an=a1rn−1 with a_1 =6a1=6, $r = 3$, and $n = 9$, we calculate

   $$a_9 =$$

   a1r9−1=6×(3)9−1=6×(3)8=6×6561=39366a9=a1r9−1=6×(3)9−1=6×(3)8=6×6561=39366

   .

   The [redacted]9th9th term of the geometric sequence with a_1 =6a1=6 and $r = 3$ is a_9 =39366a9=39366.
2. Using a_n =a1rn−1an=a1rn−1 with a_1 =2a1=2, $r = -5$, and $n =$, we calculate

   [redacted]

   a11=a1r11−1=2×(−5)11−1=2×(−5)10=2×9,765,625=19,531,250a11=a1r11−1=2×(−5)11−1=2×(−5)10=2×9,765,625=19,531,250

Example 4.15.3: Calculating the Sum of a Finite Geometric Sequence

1. The sum of the first 13 terms of this geometric sequence is 1,328,600.
2. Using a_1 =16, r =18, and $n = 7$, we find that the sum is:

   $$s_7 = a_1\left(\frac{1-r^{n-1}}{1-r}\right) = 16 \times \left(\frac{1-\left(\frac{1}{8}\right)^{7-1}}{1-\left(\frac{1}{8}\right)}\right) = 16 \times \left(\frac{1-\left(\frac{1}{8}\right)^{6}}{\frac{7}{8}}\right) = 16 \times \left(\frac{1-\frac{1}{262{,}144}}{\frac{7}{8}}\right) = 16 \times \left(\frac{\frac{262{,}143}{262{,}144}}{\frac{7}{8}}\right) = 16 \times \left(\frac{262{,}143}{229{,}376}\right) = \frac{262{,}143}{14{,}336} = 18.2856 \quad (4.15.1)$$

   The sum of the first 7 terms of this geometric sequence is 18.2856.

Example 4.15.4: Calculating Interest Compounded Yearly

Using $A = P(1 + r)^t$ with P =1000, $r = 0.04$, and t =25, we find that $A = P(1 + r)^t = 1{,}000 \times (1 + 0.04)^{25} = 1{,}000 \times (1.04)^{25} = 1{,}000 \times 2.66583633 = 2{,}665.85$. After 25 years, there is $\$2{,}665.84$ in the account.

Example $4.15.5$**: Doubling a Bacterial Culture**

Since the population is doubling every 20 minutes, this is a geometric sequence situation with common ratio $r = 2$. The culture begins with 30 organisms, so a_1 =30a1=30. The time,16 hours, is 48 twenty-minute periods, so we're looking for the 48th term in the sequence. Using these values in the geometric sequence formula gives

a48=a1rn−1=30×248−1=30×247=30×(1.40737×1014)=4.22212×1015a48=a1rn−1=30×248−1=30×247=30×(1.40737×1014)=4.22212×1015.

So, after 16 hours, the culture contains 4.22212×10^{15} *E. coli* organisms. That's more than 4,000 trillion bacteria.

Example $4.15.6$**: Applying the Sum of a Finite Geometric Sequence**

Since the number of grains of rice is doubled at each step, this is a geometric sequence with first term a_1 =1a1=1 and common ratio $r = 2$. Rice is placed on 64 total squares, so we want the sum of the first 64 terms. Using this information and the formula, the total number of grains of rice on the board will be: s64=a1(1−rn−11−r)=1×(1−264−11−2)=(1−263−1)=−(1−263)=−(−9.2233720369×1018)=9.2233720369×1018s64=a1(1−rn−11−r)=1×(1−264−11−2)=(1−263−1)=−(1−263

That's a 20-digit number!

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5: PROBLEM SOLVING

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5.1: Problem Solving

Math problem-solving is a crucial skill that helps people understand and deal with the complexities of the world. It's about more than just doing calculations; it involves interpreting problems, creating strategies, and using logical thinking to find solutions. Many influential educators and mathematicians have established the foundations of effective problem-solving in math, focusing on understanding the concepts behind the problems instead of just memorizing procedures. This approach not only improves math skills but also encourages critical thinking and creativity.

One key figure in developing problem-solving strategies is George Pólya, a Hungarian mathematician whose work has greatly impacted math education. In his famous book "How to Solve It," Polya introduced a four-step process for solving math problems: understanding the problem, making a plan, carrying out the plan, and looking back to review the solution. This structured method gives both students and teachers a clear framework, making problem-solving more approachable and organized. Polya's methods have been widely used in classrooms and remain fundamental in math education.

For younger students, problem-solving strategies need to match their developmental stages and thinking abilities. Educators like Marilyn Burns and John Van de Walle have made significant contributions by creating techniques and materials specifically for grade school children. Marilyn Burns promotes hands-on, inquiry-based learning, where students explore math concepts through real-world problems and manipulatives. John Van de Walle's resources focus on building a strong conceptual foundation, helping students understand the "why" behind math procedures.

Incorporating these strategies in the classroom involves creating a supportive environment where students feel safe trying different approaches. Teachers play a vital role in guiding students through the problem-solving process, asking thought-provoking questions, and providing support as needed. Using visual aids, interactive tools, and group activities can also increase students' engagement and understanding. By encouraging a growth mindset, educators can help students develop resilience and perseverance, which are essential for successful problem-solving.

Ultimately, the goal of teaching problem-solving is to equip students with the skills they need to face challenges confidently and effectively. Whether solving simple arithmetic problems or complex real-world issues, the strategies and habits they develop in the classroom will benefit them throughout their lives. By building on the insights and methods of pioneering educators, we can continue to improve math teaching, ensuring all students have the chance to become proficient problem solvers.

Unlike exercises, solving problems doesn't follow a simple recipe. You can improve your problem-solving skills by building up your background knowledge and practicing regularly. As you solve more problems and learn how others have solved them, you pick up strategies and techniques that can be useful. However, no single strategy works every time, so being adaptable and persistent is key.

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5.2: George Pólya's Strategy

George Pólya was a great champion in the field of *teaching* effective problem-solving skills. He was born in Hungary in 1887, received his Ph.D. at the University of Budapest, and was a professor at Stanford University (among other universities). He wrote many mathematical papers along with three books, most famously, “How to Solve it.” Pólya died at the age 98 in 1985.

In 1945, Pólya published the short book *How to Solve It*, which gave a four-step method for solving mathematical problems. It consists of four main steps:

George Pólya, a Hungarian mathematician, is renowned for his substantial contributions to problem-solving in mathematics. Born in 1887, Pólya had a prolific career that spanned several decades and included significant work in probability, combinatorics, and mathematical physics. However, he is perhaps best known for his book "How to Solve It," published in 1945, which has become a classic in mathematical education. Pólya's work laid the foundation for modern mathematical problem-solving techniques and has influenced generations of educators and students.

Pólya's problem-solving strategy is a systematic approach designed to tackle mathematical problems effectively. His method consists of four principal steps: understanding the problem, devising a plan, carrying out the plan, and looking back. This structured approach encourages critical thinking and fosters a deeper comprehension of the problem-solving process. Polya believed that by following these steps, students could develop their analytical skills and become more proficient in tackling a wide range of mathematical challenges.

Polya's Problem-Solving Strategy

1. **Understand the Problem**:
   - **Read the problem carefully**: Ensure you understand all the terms and the problem’s requirements.
   - **Identify what is given and what needs to be found**: Distinguish between the known and unknown variables.
   - **Restate the problem in your own words**: This helps clarify the problem and ensures you have grasped the main idea.
   - **Draw a diagram if necessary**: Visual aids can make complex problems more manageable and understandable.
2. **Devise a Plan**:
   - **Think of possible strategies**: Consider various approaches, such as working backwards, looking for a pattern, or breaking the problem into smaller parts.
   - **Choose the most promising strategy**: Select the method that seems most likely to lead to a solution.
   - **Relate the problem to similar problems**: Use insights from previously solved problems that are similar in nature.
3. **Carry Out the Plan**:
   - **Implement the chosen strategy**: Apply the steps of your plan systematically.
   - **Be thorough and patient**: Carefully execute each step without rushing.
   - **Keep track of your work**: Document each part of the process to make it easier to review and correct if necessary.
4. **Look Back**:
   - **Review the solution**: Check the results to ensure they make sense and the problem is fully solved.
   - **Reflect on the process**: Consider what worked well and what didn’t, which helps improve future problem-solving skills.
   - **Generalize the solution**: Think about how the strategy could be applied to other problems.

George Pólya's problem-solving strategy remains a cornerstone of mathematical education. By providing a clear, step-by-step method for approaching problems, Polya empowers students to develop their analytical skills and confidence in their mathematical abilities. His emphasis on understanding, planning, executing, and reflecting offers a comprehensive framework that is as applicable today as it was when he first introduced it. Polya’s legacy endures in classrooms around the world, where his strategies continue to inspire and guide students in their mathematical journeys.

Problem Solving Strategy 1 (Guess and Test)

Make a guess and test to see if it satisfies the demands of the problem. If it doesn't, alter the guess appropriately and check again. Keep doing this until you find a solution.

Example 5.2.1

Mr. Jones has a total of 25 chickens and cows on his farm. How many of each does he have if all together there are 76 feet?

Solution

Step 1: Understanding the problem

We are given in the problem that there are 25 chickens and cows.

All together there are 76 feet.

Chickens have 2 feet and cows have 4 feet.

We are trying to determine how many cows and how many chickens Mr. Jones has on his farm.

Step 2: Devise a plan

Going to use Guess and test along with making a tab

Many times the strategy below is used with guess and test.

Make a table and look for a pattern:

Procedure: Make a table reflecting the data in the problem. If done in an orderly way, such a table will often reveal patterns and relationships that suggest how the problem can be solved.

Step 3: Carry out the plan:

Table 5.2.1

Chickens	Cows	Number of chicken feet	Number of cow feet	Total number of feet
20	5	40	20	60
21	4	42	16	58
Notice we are going in the wrong direction! The total number of feet is decreasing!				
19	6	38	24	62
Better! The total number of feet are increasing!				
15	10	30	40	70
12	13	24	52	76

Step 4: Looking back:

Check: 12 + 13 = 25 heads

24 + 52 = 76 feet.

We have found a solution to this problem. I could use this strategy when there are a limited number of possible answers and when two items are the same but they have one characteristic that is different.

Check in question 1:

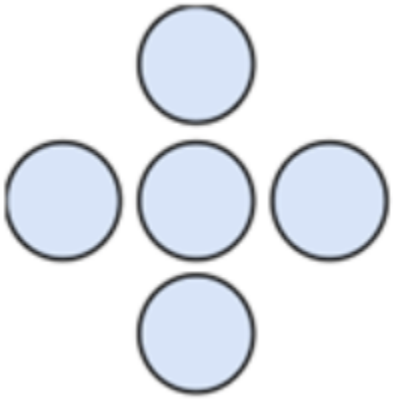


Figure 5.2.1

Place the digits 8, 10, 11, 12, and 13 in the circles to make the sums across and vertically equal 31.

Check in question 2:

Old McDonald has 250 chickens and goats in the barnyard. Altogether there are 760 feet. How many of each animal does he have? Make sure you use Polya's 4 problem-solving steps.

Problem Solving Strategy 2

Example 5.2.2

Problem:

A bakery sells 24 cupcakes in a day. If each cupcake costs $2, how much money does the bakery make from selling cupcakes in a day?

Solution

Step 1: Understand the Problem:

- We need to find out how much money the bakery makes from selling cupcakes in a day.
- Given information: The bakery sells 24 cupcakes in a day, and each cupcake costs $2.

Here's how a teacher might approach this step:

1. **Read the Problem Aloud:** The teacher reads the problem aloud to the class, ensuring that all students are listening and paying attention.
2. **Clarify the Goal:** The teacher asks questions to clarify the goal of the problem. For example, "What are we trying to find out in this problem?"
3. **Identify the Given Information:** The teacher helps students identify the information given in the problem. For example, "What do we know about the number of cupcakes sold and the price per cupcake?"
4. **Discuss the Context:** The teacher discusses the context of the problem to make it more relatable for students. For example, "Imagine you own a bakery and you sell cupcakes. How much money would you make if each cupcake costs $2 and you sell 24 cupcakes?"
5. **Encourage Student Questions:** The teacher encourages students to ask questions if they are unsure about any part of the problem.
6. **Summarize the Problem:** The teacher summarizes the problem in simple terms, reiterating the goal and the given information. For example, "So, we want to find out how much money the bakery makes if they sell 24 cupcakes at $2 each."

By approaching the problem in this way, the teacher helps students understand the context and purpose of the problem, making it easier for them to devise a plan for solving it.

Step 2: Devise a Plan:

- We can multiply the number of cupcakes sold by the price per cupcake to find the total amount of money made.

Here's how a teacher might approach this step:

1. **Brainstorm Strategies:** The teacher leads a class discussion to brainstorm different strategies that could be used to solve the problem. For example, "How can we figure out the total amount of money the bakery makes from selling cupcakes?"
2. **Discuss Strategies:** The teacher discusses the pros and cons of each strategy, helping students understand when each strategy might be most effective. For example, "We could multiply the number of cupcakes by the price per cupcake, or we could add $2 for each cupcake sold. Which strategy do you think would be easier?"
3. **Choose a Strategy:** The teacher helps students choose a strategy that they think will be most effective for solving the problem. For example, "Let's use multiplication to find the total amount of money made by the bakery."
4. **Plan the Solution:** The teacher guides students in planning how they will use their chosen strategy to solve the problem. For example, "We will multiply 24 cupcakes by $2 to find the total amount of money made."
5. **Encourage Student Input:** The teacher encourages students to share their ideas and thoughts about the chosen strategy, fostering a collaborative problem-solving environment.

By approaching the problem in this way, the teacher helps students think critically about the problem and choose a strategy that will lead to a successful solution.

Step 3: Carry Out the Plan:

- Number of cupcakes sold = 24
- Price per cupcake = $2
- Total money made = Number of cupcakes sold × Price per cupcake = 24 × $2 = $48

Here's how a teacher might approach this step:

1. **Demonstrate the Solution:** The teacher demonstrates how to use the chosen strategy to solve the problem. For example, the teacher might write out the multiplication problem 24 × $2 on the board and solve it step by step.
2. **Model the Process:** The teacher models the process of solving the problem, explaining each step as they go. For example, "First, we multiply 24 by 2 to find out how much money the bakery makes from selling one cupcake. Then, we multiply that amount by 24 to find the total amount."
3. **Provide Guided Practice:** The teacher provides guided practice opportunities for students to solve similar problems with support. For example, the teacher might give students a similar problem to solve in pairs or small groups, offering guidance and feedback as needed.
4. **Encourage Independent Practice:** Once students have a good understanding of how to use the chosen strategy, the teacher encourages them to solve similar problems independently.
5. **Review the Solution:** After students have solved the problem, the teacher reviews the solution with the class, ensuring that everyone understands how the problem was solved and why.

By approaching the problem in this way, the teacher helps students develop confidence in their problem-solving abilities and reinforces the importance of using a systematic approach to problem solving.

Step 4: Look Back and Review the Solution:

- Does $48 make sense as the answer? Yes, because if each cupcake costs $2 and the bakery sells 24 cupcakes, the total amount should be 24 × $2 = $48.

Here's how a teacher might approach this step:

1. **Check the Answer** The teacher asks students to check their answer to ensure that it makes sense in the context of the problem. For example, "Does $48 seem like a reasonable answer for the total amount of money made by selling cupcakes?"

2. **Review the Steps:** The teacher reviews the steps that were taken to solve the problem, asking students to explain each step in their own words. For example, "Can you explain why we multiplied 24 by $2 to find the total amount of money made?"
3. **Discuss Alternative Strategies:** The teacher discusses alternative strategies that could have been used to solve the problem, encouraging students to think about different approaches. For example, "Could we have solved this problem by adding $2 twenty-four times instead of multiplying?"
4. **Verify the Solution:** The teacher helps students verify that their solution is correct by checking their calculations and reasoning. For example, the teacher might ask students to recalculate the total amount of money made to double-check their answer.
5. **Reflect on the Process:** Finally, the teacher encourages students to reflect on their problem-solving process, asking questions such as, "What did you learn from solving this problem? Is there anything you would do differently next time?"

By approaching the problem in this way, the teacher helps students develop a deeper understanding of the problem-solving process and reinforces the importance of checking their work and reflecting on their strategies.

In this problem, we applied Van de Walle's problem-solving strategy by understanding the problem, devising a plan, carrying out the plan, and reviewing the solution to find out how much money the bakery makes from selling cupcakes in a day.

Check in question 3:

Find the 320th term of 7, 10, 13, 16 ...

Problem Solving Strategy 3 (Working Backwards)

This is considered a strategy in many schools. If you are given an answer, and the steps that were taken to arrive at that answer, you should be able to determine the starting point.



Example 5.2.3

Karen is thinking of a number. If you double it and subtract 7, you obtain 11. What is Karen's number?

Solution

1. We start with 11 and work backward.

2. The opposite of subtraction is addition. We will add 7 to 11. We are now at 18.

3. The opposite of doubling something is dividing by 2. 18/2 = 9

4. This should be our answer. Looking back:

9 x 2 = 18 -7 = 11

5. We have the right answer.

Check in question 4:

Christina is thinking of a number.

If you multiply her number by 93, add 6, and divide by 3, you obtain 436. What is her number? Solve this problem by working backward. (5 points)

Problem Solving Strategy 4 (Looking for a Pattern)

Definition: Sequence

A sequence is a pattern involving an ordered arrangement of numbers.

We first need to find a pattern.

Ask yourself as you search for a pattern – are the numbers growing steadily larger? Steadily smaller? How is each number related?

Example 5.2.4

1, 4, 7, 10, 13... Find the next 2 numbers.

Solution

The pattern is each number is increasing by 3. The next two numbers would be 16 and 19.

Example 5.2.5

1, 4, 9, 16 ... Find the next 2 numbers.

Solution

It looks like each number is a perfect square. $1^2 = 1$, $2^2 = 4$

$3^2 = 9$

$4^2 = 16$

So the next numbers would be

$5^2 = 25$

$6^2 = 36$

Example 5.2.6

10, 7, 4, 1, -2... Find the next 2 numbers.

Solution

In this sequence, the numbers are decreasing by 3. So the next 2 numbers would be

-2 -3 = -5

-5 – 3 = -8

Example 5.2.7

1, 2, 4, 8 ...Ffind the next two numbers.

Solution

This example is a little bit harder. The numbers are increasing but not by a constant. Maybe a factor?

1 x 2 = 2

2 x 2 = 4

4 x 2 =8

So each number is being multiplied by 2.

8 x 2 = 16

16 x 2 = 32

Problem Solving Strategy 5 (Make a List or a Table)

Example 5.2.8

Can perfect squares end in a 2 or a 3?

Solution

List all the squares of the numbers 1 to 20.

1 4 9 16 25 36 49 64 81 100 121 144 169 196 225 256 289 324 361 400.

Now look at the number in the ones digits. Notice they are 0, 1, 4, 5, 6, or 9. Notice none of the perfect squares end in 2, 3, 7, or 8. This list suggests that perfect squares cannot end in a 2, 3, 7 or 8. But note that this is just an observation. To answer this question, one would need a mathematically rigorous proof.

Example 5.2.9

How many different amounts of money can you have in your pocket if you have only three coins including only dimes and quarters?

q = quarter = 25 cents
d = dime = 10 cents

Solution

Table 5.2.2

Quarters	Dimes	Calculation	Total Cents
0	3	$0q + 3d = 0 \times 25 + 3 \times 10$	30 cents
1	2	$1q + 2d = 1 \times 25 + 2 \times 10$	45 cents
2	1	$2q + 1d = 2 \times 25 + 1 \times 10$	60 cents
3	0	$3q + 0d = 3 \times 25 + 0 \times 10$	75 cents

Check in question 5:

How many ways can you make 23 cents using only pennies, nickels, and dimes? (10 points)

Problem Solving Strategy 6 (Process of Elimination)

This strategy can be used when there is only one possible solution.

Example 5.2.10

I'm thinking of a number.

- The number is odd.
- It is more than 1 but less than 100.
- It is greater than 20.
- It is less than 5 times 7.
- The sum of the digits is 7.
- It is evenly divisible by 5.

Solution

- We know it is an odd number between 1 and 100.
- It is greater than 20 but less than 35
  - 21, 23, 25, 27, 29, 31, 33, 35. These are the possibilities.
- The sum of the digits is 7
  - 21 (2+1=3) No
  - 23 (2+3 = 5) No
  - 25 (2 + 5= 7) Yes

Using the same process we see there are no other numbers that meet this criteria. Also, we notice 25 is divisible by 5. By using the strategy of elimination, we have found our answer.

Reference

1. References (2)

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5.3: John Van de Walle's Strategy

John Van de Walle was an influential mathematics educator whose work significantly impacted how mathematics is taught in schools. Born in 1944, Van de Walle was a professor emeritus at Virginia Commonwealth University, where he dedicated his career to improving mathematics education. He is best known for his book "Elementary and Middle School Mathematics: Teaching Developmentally," which offers a comprehensive guide for teachers on how to develop students' mathematical thinking. Van de Walle emphasized the importance of understanding mathematical concepts rather than simply memorizing procedures. His approach to problem-solving focuses on developing students' ability to think critically and solve problems in a way that is meaningful and connected to their experiences.

Van de Walle's problem-solving strategy is centered around fostering a deep understanding of mathematical concepts through active engagement and exploration. His approach involves encouraging students to explore multiple strategies, make connections, and reflect on their thinking. By promoting a classroom environment where students are encouraged to discuss and share their ideas, Van de Walle's strategy helps students develop a robust understanding of mathematics. His methods are designed to be student-centered, focusing on the development of problem-solving skills through real-world applications and collaborative learning.

Van de Walle's Problem-Solving Strategy

1. **Understand the Problem**:
   - **Engage students in the problem**: Use real-world contexts that are meaningful to students to capture their interest.
   - **Clarify the problem**: Ensure that students understand the problem by discussing it and rephrasing it in their own words.
   - **Identify what is known and unknown**: Help students distinguish between the given information and what needs to be determined.
2. **Explore and Plan**:
   - **Encourage multiple strategies**: Promote the use of various strategies to solve the problem, such as drawing diagrams, using manipulatives, or looking for patterns.
   - **Facilitate discussion and collaboration**: Create an environment where students feel comfortable sharing their ideas and strategies with peers.
   - **Guide students to make connections**: Help students relate the current problem to previously learned concepts and experiences.
3. **Implement the Plan**:
   - **Support students as they work**: Provide guidance and encouragement as students carry out their chosen strategies.
   - **Monitor progress and provide feedback**: Observe students' work and offer constructive feedback to help them stay on track.
   - **Encourage perseverance**: Foster a growth mindset by encouraging students to keep trying even if they encounter difficulties.
4. **Reflect and Discuss**:
   - **Review the solution**: Discuss the solution as a class, highlighting different strategies used and their effectiveness.
   - **Reflect on the process**: Encourage students to think about what they learned and how they approached the problem.
   - **Generalize learning**: Help students apply the problem-solving strategies they used to new and different problems.

John Van de Walle's problem-solving strategy emphasizes understanding, exploration, and reflection, making it a powerful approach for developing students' mathematical thinking. By focusing on real-world contexts, multiple strategies, and collaborative learning, Van de Walle's methods encourage students to engage deeply with mathematical concepts. His approach not only helps students solve individual problems but also fosters a broader understanding of mathematics that can be applied to various situations. Van de Walle's contributions to mathematics education continue to influence teaching practices and inspire educators to cultivate a love for mathematics in their students.

Example 5.3.1

Problem:

Sarah has a rectangular prism-shaped box that she wants to fill with small 1-unit by 1-unit by 1-unit cubes. The dimensions of the box are 5 units long, 3 units wide, and 4 units high. How many 1-unit cubes will Sarah need to completely fill the box?

Solution

Step 1: Understand the Problem

- We need to find the total number of 1-unit by 1-unit by 1-unit cubes required to fill a rectangular prism.
- The dimensions of the rectangular prism are given as 5 units (length), 3 units (width), and 4 units (height).

Here's how a teacher might approach this step:

1. **Introduce the Concept:** Begin with a discussion about volume and three-dimensional shapes. Use real-life examples, such as filling a box with cubes, to make the concept relatable.
2. **Clarify the Task:** Explain the problem clearly, ensuring that students understand the dimensions given (length, width, height) and the goal (filling the box with unit cubes).
3. **Ask Guiding Questions:** Prompt students with questions like, "What are we trying to find?" and "What information do we have?" to ensure they comprehend the problem.
4. **Use Visuals:** Provide diagrams or physical models of rectangular prisms to help students visualize the problem.

Step 2: Developing and Using Strategies

- To find the volume of the rectangular prism, we can use the formula for the volume of a rectangular prism:
  $Volume = Length \times Width \times Height$

- Here, the length is 5 units, the width is 3 units, and the height is 4 units.

Here's how a teacher might approach this step:

- **Discuss Possible Strategies:** Encourage students to brainstorm different strategies to find the volume. Discuss the formula for the volume of a rectangular prism and how it relates to multiplying the dimensions.
- **Model the Process:** Demonstrate how to apply the volume formula using a step-by-step approach. Show how each dimension is multiplied.
- **Encourage Estimation:** Ask students to estimate the volume before calculating it to develop their estimation skills and understanding of the problem's scale.
- **Collaborative Learning:** Allow students to work in pairs or groups to discuss and develop their strategies, fostering collaboration and peer learning.

Step 3: Implementing the Solution

- Using the volume formula:
  $Volume = 5\,units \times 3\,units \times 4\,units$
  $Volume = 60\ cubic\ units$
- Sarah needs 60 small 1-unit by 1-unit by 1-unit cubes to completely fill the box.

Here's how a teacher might approach this step:

- **Guide the Calculation:** Walk students through the calculation process. Ensure they understand each step and how it contributes to finding the final volume.
- **Use Manipulatives:** Provide unit cubes for students to physically fill a model of the box, reinforcing the concept of volume and the multiplication of dimensions.
- **Check for Understanding:** Frequently check in with students to ensure they are following along and comprehending each step. Ask questions to probe their understanding.

Step 4: Reflecting on the Process and Solution

- We checked that we used the correct formula for volume and applied the given dimensions correctly.
- We ensured that the solution (60 cubic units) makes sense for the given problem.

Here's how a teacher might approach this step:

- **Encourage Reflection:** After solving the problem, ask students to reflect on the process. Questions like, "What steps did we take to solve the problem?" and "Why did we use this method?" help deepen their understanding.
- **Discuss Errors:** If mistakes were made, discuss them openly. Analyze why the errors occurred and how to correct them, turning mistakes into learning opportunities.
- **Connect to Real Life:** Discuss how this problem-solving approach can be applied to other real-world scenarios, reinforcing the practical value of the concept.

Step 5: Building Conceptual Understanding

- This problem helps students understand that the volume of a three-dimensional shape can be determined by multiplying its dimensions.
- They also see that the volume represents the number of unit cubes that can fill the shape without gaps or overlaps.
- Understanding this concept is crucial for recognizing how to measure and calculate volume in various real-world and mathematical problems.

Here's how a teacher might approach this step:

- **Revisit the Concept:** Regularly revisit the concept of volume and its measurement in different contexts to reinforce understanding.
- **Use Varied Examples:** Provide different problems involving volume, including irregular shapes and real-world applications, to deepen students' conceptual understanding.
- **Encourage Exploration:** Allow students to explore and experiment with different three-dimensional shapes and their volumes, fostering curiosity and deeper learning.
- **Assess Understanding:** Use formative assessments, such as quizzes or quick checks, to gauge students' understanding and provide feedback to support their learning journey.

Example of Implementing Each Step:

1. **Understanding the Problem:**
   - **Teacher:** "Today, we're going to solve a problem involving the volume of a box. Can anyone tell me what volume means?"
   - **Students:** "It's how much space is inside an object."
   - **Teacher:** "Right! We have a box that is 5 units long, 3 units wide, and 4 units high. We need to find out how many small cubes will fit inside this box."
2. **Developing and Using Strategies:**
   - **Teacher:** "What strategies can we use to find the volume of this box?"
   - **Students:** "We can multiply the length, width, and height."
   - **Teacher:** "Exactly! The formula for volume is length times width times height. Let's apply this formula."
3. **Implementing the Solution:**
   - **Teacher:** "Let's calculate the volume together. What is 5 times 3?"
   - **Students:** "15."
   - **Teacher:** "And 15 times 4?"
   - **Students:** "60."
   - **Teacher:** "So, the volume is 60 cubic units. This means we need 60 small cubes to fill the box."
4. **Reflecting on the Process and Solution:**
   - **Teacher:** "Why did we multiply these numbers? What do each of these numbers represent in our problem?"
   - **Students:** "They represent the dimensions of the box."
   - **Teacher:** "Yes, and multiplying them gives us the total volume. How can we check our work to make sure it's correct?"
5. **Building Conceptual Understanding:**
   - **Teacher:** "Can you think of other objects that we could measure the volume of? How would you approach finding their volume?"
   - **Students:** "We could measure the volume of a fish tank or a carton of milk by using the same formula."
   - **Teacher:** "Great! Remember, understanding how to find volume helps us in many real-world situations."

By following these steps and approaches, teachers can effectively guide students through solving volume problems and help them build a solid understanding of three-dimensional measurement.

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5.4: Marilyn Burns' Strategy

Marilyn Burns is a highly respected mathematics educator, author, and founder of Math Solutions, an organization dedicated to improving math instruction. With over four decades of experience, Burns has significantly influenced the field of math education through her innovative teaching methods and commitment to making mathematics accessible and engaging for all students. Her work emphasizes the importance of developing a deep understanding of mathematical concepts rather than just memorizing procedures.

Burns' problem-solving strategy centers around the belief that students learn best when they actively engage with mathematical concepts. She advocates for a hands-on, inquiry-based approach where students are encouraged to explore, ask questions, and discover solutions on their own. This approach helps students develop critical thinking skills and a deeper understanding of the subject matter. Burns also emphasizes the importance of communication in mathematics, encouraging students to explain their thinking and reasoning, both verbally and in writing. This process not only reinforces their understanding but also helps teachers identify and address any misconceptions.

One of the key components of Burns' problem-solving strategy is the use of rich, open-ended problems that have multiple entry points and can be solved in various ways. These types of problems allow students of different abilities to participate and contribute, fostering a collaborative learning environment. Burns believes that presenting students with challenging problems encourages them to persevere and develop resilience, which are crucial skills not only in mathematics but in all areas of learning.

Additionally, Burns emphasizes the role of the teacher as a facilitator rather than a lecturer. She encourages teachers to create a classroom environment where students feel safe to take risks and make mistakes, as these are essential parts of the learning process. Teachers are guided to ask probing questions that stimulate thinking and to provide appropriate scaffolding to support students in their problem-solving efforts. By promoting a positive and supportive classroom culture, Burns' strategy helps students build confidence in their mathematical abilities and fosters a lifelong love for learning.

Overall, Marilyn Burns' problem-solving strategy is rooted in the belief that all students can be successful in mathematics if given the right opportunities and support. Her approach has had a lasting impact on math education, inspiring teachers to adopt more student-centered, inquiry-based methods that promote a deeper understanding and appreciation of mathematics.

Marilyn Burns' Problem-Solving Strategy

1. **Engage Students with Rich, Open-Ended Problems**
   - **Select Problems with Multiple Entry Points:** Choose problems that can be approached in different ways to accommodate diverse student abilities.
   - **Encourage Exploration:** Allow students to explore various methods and strategies for solving the problem.
   - **Promote Collaboration:** Facilitate group work where students can share ideas and learn from each other.
2. **Encourage Hands-On, Inquiry-Based Learning**
   - **Use Manipulatives and Visuals:** Incorporate physical tools and visual aids to help students understand abstract concepts.
   - **Foster Inquiry:** Encourage students to ask questions, make predictions, and test their ideas.
   - **Guide Discovery:** Allow students to discover solutions on their own rather than providing direct answers.
3. **Emphasize Communication and Reasoning**
   - **Verbal Explanation:** Have students explain their thinking and reasoning verbally.
   - **Written Justification:** Encourage students to write down their thought processes and solutions.
   - **Class Discussions:** Facilitate discussions where students can present and critique different approaches to the problem.
4. **Support Perseverance and Resilience**
   - **Present Challenging Problems:** Introduce problems that require sustained effort and critical thinking.
   - **Encourage Persistence:** Promote a classroom culture where persistence is valued and students are encouraged to keep trying despite difficulties.
   - **Recognize Effort:** Acknowledge and celebrate effort and perseverance as much as correct answers.
5. **Create a Positive and Supportive Classroom Environment**
   - **Safe Space for Mistakes:** Establish an environment where mistakes are viewed as learning opportunities.
   - **Positive Reinforcement:** Use positive reinforcement to build students' confidence and self-esteem.
   - **Individual Support:** Provide individualized support and scaffolding to help each student succeed.

6. **Teacher as Facilitator**
   - **Ask Probing Questions:** Use questions to guide students' thinking and deepen their understanding.
   - **Provide Scaffolding:** Offer hints and support without giving away the solution.
   - **Observe and Assess:** Continuously observe and assess students' progress to identify and address misconceptions.
7. **Develop Deep Conceptual Understanding**
   - **Focus on Concepts:** Ensure that students understand the underlying concepts rather than just memorizing procedures.
   - **Connect to Real-World Applications:** Relate mathematical concepts to real-world situations to make learning relevant and meaningful.
   - **Encourage Reflection:** Have students reflect on what they have learned and how they approached the problem.

Implementation Steps for Teachers

1. **Preparation**
   - Select appropriate open-ended problems.
   - Gather necessary materials and manipulatives.
   - Plan questions and scaffolding techniques.
2. **Introduction**
   - Introduce the problem to the class.
   - Explain the expectations for exploration and collaboration.
3. **Exploration**
   - Allow students to work on the problem individually or in groups.
   - Circulate the room to observe and ask guiding questions.
4. **Discussion**
   - Facilitate a class discussion where students share their approaches and solutions.
   - Highlight different strategies and reasoning used by students.
5. **Reflection**
   - Encourage students to reflect on their problem-solving process and what they learned.
   - Discuss how the strategies used can be applied to other problems.
6. **Assessment**
   - Assess students' understanding through their explanations, written work, and participation in discussions.
   - Provide feedback that reinforces conceptual understanding and problem-solving skills.

Example 5.4.1

Maria has 3/4 of a chocolate bar, and she wants to share it equally between herself and her friend. How much of the chocolate bar will each person get?

Solution

Solving the Problem Using Marilyn Burns' Strategy

1. **Engage Students with Rich, Open-Ended Problems**
   - **Select Problems with Multiple Entry Points:** Present the problem of sharing 3/4 of a chocolate bar, which can be approached in various ways (e.g., visual models, fraction division).
   - **Encourage Exploration:** Ask students how they might share 3/4 of a chocolate bar equally.
   - **Promote Collaboration:** Have students discuss their ideas in pairs or small groups.
2. **Encourage Hands-On, Inquiry-Based Learning**
   - **Use Manipulatives and Visuals:** Provide fraction strips or chocolate bar models for students to physically divide 3/4 into two equal parts.
   - **Foster Inquiry:** Encourage students to ask questions like, "What does it mean to share equally?" and "How can we divide 3/4?"
   - **Guide Discovery:** Allow students to discover that dividing 3/4 by 2 gives each person 3/8.
3. **Emphasize Communication and Reasoning**
   - **Verbal Explanation:** Have students explain their reasoning for how they divided 3/4 equally.
   - **Written Justification:** Ask students to write down their steps and reasoning.
   - **Class Discussions:** Facilitate a class discussion where students share different methods they used to solve the problem.

4. **Support Perseverance and Resilience**
    - **Present Challenging Problems:** Encourage students to solve similar problems with different fractions (e.g., sharing 5/6 or 7/8).
    - **Encourage Persistence:** Praise efforts and strategies even if they aren't immediately correct, emphasizing the importance of trying different methods.
    - **Recognize Effort:** Celebrate students' perseverance and the process of problem-solving.
5. **Create a Positive and Supportive Classroom Environment**
    - **Safe Space for Mistakes:** Establish a classroom culture where mistakes are seen as learning opportunities.
    - **Positive Reinforcement:** Use positive reinforcement to build confidence.
    - **Individual Support:** Provide additional support to students who may struggle with the concept.
6. **Teacher as Facilitator**
    - **Ask Probing Questions:** Use questions like, "How did you decide to divide 3/4?" and "What does each part represent?" to deepen understanding.
    - **Provide Scaffolding:** Offer hints and support as needed, such as demonstrating how to use fraction strips.
    - **Observe and Assess:** Continuously observe students' work and provide feedback to address misconceptions.
7. **Develop Deep Conceptual Understanding**
    - **Focus on Concepts:** Ensure students understand that dividing 3/4 by 2 means finding how much each person gets when 3/4 is shared equally.
    - **Connect to Real-World Applications:** Discuss real-life situations where dividing fractions is useful (e.g., cooking, sharing food).
    - **Encourage Reflection:** Have students reflect on what they learned and how they solved the problem.

Example of Implementing Each Step

1. Preparation

- **Select Problem:** "Today, we have a problem to solve. Maria has 3/4 of a chocolate bar, and she wants to share it equally between herself and her friend."
- **Gather Materials:** Prepare fraction circles and a visual representation of 3/4 of a chocolate bar.
- **Plan Questions:** "How do you think Maria can divide 3/4 of the chocolate bar so that she and her friend get the same amount?"

2. Introduction

- **Teacher:** "Maria has 3/4 of a chocolate bar. How can she divide it equally between herself and her friend?"
- **Discuss Objective:** "Our goal is to find out how much each person gets when 3/4 is divided equally."

3. Exploration

- **Teacher:** "Let's use these fraction circles to represent 3/4 of the chocolate bar. How would you divide it so both Maria and her friend get the same amount?"
- **Student A:** "We can divide it into two parts."
- **Teacher:** "Great! How do we divide 3/4 into two equal parts using the fraction circles?"

4. Guided Discussion

- **Teacher:** "Let's see how Student A divided it. Can you show us?"
- **Student A:** *Divides the fraction circles.* "I split it like this. Each part is 3/8."
- **Teacher:** "Why did you split it like that?"
- **Student A:** "Because when you divide 3/4 by 2, you get 3/8 for each person."
- **Teacher:** "Exactly! Does anyone have a different way to solve this?"

5. Whole-Class Sharing

- **Teacher:** "Let's hear from another group. How did you divide 3/4 of the chocolate bar?"
- **Student B:** *Shows a different method using fraction bars.* "We divided it into fourths first, and then each person gets 3/8."
- **Teacher:** "Interesting approach! Did anyone else use a similar method or have a different way?"

6. Reflection and Justification

- **Teacher:** "Now that we've seen different methods, why does dividing 3/4 by 2 give each person 3/8?"
- **Student C:** "Because when you divide, you're finding out how much each person gets when it's shared equally."
- **Teacher:** "That's right! How can we explain our answer to someone who doesn't understand fractions well?"

7. Application and Extension

- **Teacher:** "Let's try another problem. What if Maria had 5/6 of a chocolate bar? How would you divide it equally between three friends?"
- **Students:** *Discuss and use fraction circles to solve the problem.*

8. Assessment and Feedback

- **Teacher:** "I'm impressed with how well you divided the chocolate bar today. You all did a great job explaining your reasoning. Let's continue practicing dividing fractions to become even better!"

By engaging students in this interactive and guided manner, teachers can effectively use Marilyn Burns' problem-solving strategy to deepen their understanding of fraction division while promoting collaboration and critical thinking in the classroom.

Marilyn Burns' problem-solving strategy offers a transformative approach to math education, emphasizing exploration, communication, and resilience. By engaging students with rich, open-ended problems and fostering a classroom environment that values inquiry and collaboration, teachers can help students develop a deep conceptual understanding of mathematics. This approach not only enhances students' problem-solving skills but also builds their confidence and love for learning. Through Burns' methods, math becomes a dynamic and interactive subject where all students have the opportunity to succeed and thrive.

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5.5: Use a Problem-Solving Strategy

Learning Objectives

By the end of this section, you will be able to:

- Approach word problems with a positive attitude
- Use a problem-solving strategy for word problems
- Solve number problems

Note

Before you get started, take this readiness quiz.

1. Translate "6 less than twice x" into an algebraic expression.
   If you missed this problem, review Exercise 1.3.43.
2. Solve: $\frac{2}{3}x = 24$.
   If you missed this problem, review Exercise 2.2.10.
3. Solve: $3x + 8 = 14$.
   If you missed this problem, review Exercise 2.3.1.

Approach Word Problems with a Positive Attitude

"If you think you can... or think you can't... you're right."—Henry Ford

The world is full of word problems! Will my income qualify me to rent that apartment? How much punch do I need to make for the party? What size diamond can I afford to buy my girlfriend? Should I fly or drive to my family reunion? How much money do I need to fill the car with gas? How much tip should I leave at a restaurant? How many socks should I pack for vacation? What size turkey do I need to buy for Thanksgiving dinner, and then what time do I need to put it in the oven? If my sister and I buy our mother a present, how much does each of us pay?

Now that we can solve equations, we are ready to apply our new skills to word problems. Do you know anyone who has had negative experiences in the past with word problems? Have you ever had thoughts like the student below (Figure 5.5.1)?

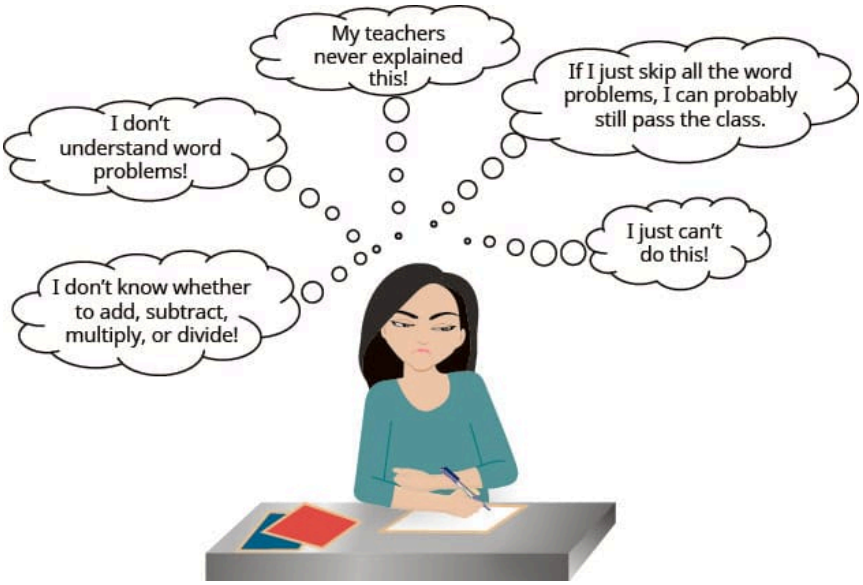


Figure 5.5.1: Negative thoughts can be barriers to success.

When we feel we have no control, and continue repeating negative thoughts, we set up barriers to success. We need to calm our fears and change our negative feelings.

Start with a fresh slate and begin to think positive thoughts. If we take control and believe we can be successful, we will be able to master word problems! Read the positive thoughts in Figure 5.5.2 and say them out loud.



Figure 5.5.2: Thinking positive thoughts is a first step towards success.

Think of something, outside of school, that you can do now but couldn't do 3 years ago. Is it driving a car? Snowboarding? Cooking a gourmet meal? Speaking a new language? Your past experiences with word problems happened when you were younger—now you're older and ready to succeed!

Use a Problem-Solving Strategy for Word Problems

We have reviewed translating English phrases into algebraic expressions, using some basic mathematical vocabulary and symbols. We have also translated English sentences into algebraic equations and solved some word problems. The word problems applied math to everyday situations. We restated the situation in one sentence, assigned a variable, and then wrote an equation to solve the problem. This method works as long as the situation is familiar and the math is not too complicated.

Now, we'll expand our strategy so we can use it to successfully solve any word problem. We'll list the strategy here, and then we'll use it to solve some problems. We summarize below an effective strategy for problem solving.

USE A PROBLEM-SOLVING STRATEGY TO SOLVE WORD PROBLEMS.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Example 5.5.1

Pilar bought a purse on sale for $\$18$, which is one-half of the original price. What was the original price of the purse?

Solution

Step 1. Read the problem. Read the problem two or more times if necessary. Look up any unfamiliar words in a dictionary or on the internet.

In this problem, is it clear what is being discussed? Is every word familiar?

Let p = the original price of the purse.

Step 2. Identify what you are looking for. Did you ever go into your bedroom to get something and then forget what you were looking for? It's hard to find something if you are not sure what it is! Read the problem again and look for words that tell you what you are looking for!

In this problem, the words "what was the original price of the purse" tell us what we need to find.

Step 3. Name what we are looking for. Choose a variable to represent that quantity. We can use any letter for the variable, but choose one that makes it easy to remember what it represents.

Step 4. Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Translate the English sentence into an algebraic equation.

Reread the problem carefully to see how the given information is related. Often, there is one sentence that gives this information, or it may help to write one sentence with all the important information. Look for clue words to help translate the sentence into algebra. Translate the sentence into an equation.

Restate the problem in one sentence with all the important information.	18 $\quad$ is $\quad$ one-half the original price.
Translate into an equation.	$18 \quad = \quad \frac{1}{2} \cdot p$

Step 5. Solve the equation using good algebraic techniques. Even if you know the solution right away, using good algebraic techniques here will better prepare you to solve problems that do not have obvious answers.

Solve the equation.	$18 = \frac{1}{2}p$
Multiply both sides by 2.	$2 \cdot 18 = 2 \cdot \frac{1}{2}p$
Simplify.	$36 = p$

Step 6. Check the answer in the problem to make sure it makes sense. We solved the equation and found that $p = 36$,which means "the original price" was $\$36$.

Does $36 make sense in the problem? Yes, because 18 is one-half of 36, and the purse was on sale at half the original price.

If this were a homework exercise, our work might look like this:

Pilar bought a purse on sale for $\$18$, which is one-half the original price. What was the original price of the purse?

Step 7. Answer the question with a complete sentence. The problem asked "What was the original price of the purse?"

The answer to the question is: "The original price of the purse was $36."

	Let $p =$ the original price.
	18 is one-half the original price.
	$18 = \frac{1}{2}p$
Multiply both sides by 2.	$2 \cdot 18 = 2 \cdot \frac{1}{2}p$
Simplify.	$36 = p$
Check. Is $\$36$ a reasonable price for a purse?	
Yes.	
Is 18 one half of 36?	
$18 \stackrel{?}{=} \frac{1}{2} \cdot 36$	
$18 = 18$✓	
	The original price of the purse was $\$36$.

Try It 5.5.2

Joaquin bought a bookcase on sale for $\$120$, which was two-thirds of the original price. What was the original price of the bookcase?

Answer available at the end of this section.

Try It 5.5.3

Two-fifths of the songs in Mariel's playlist are country. If there are 16 country songs, what is the total number of songs in the playlist?

Answer available at the end of this section.

Let's try this approach with another example.

Example 5.5.4

Ginny and her classmates formed a study group. The number of girls in the study group was three more than twice the number of boys. There were 11 girls in the study group. How many boys were in the study group?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	How many boys were in the study group?
Step 3. Name. Choose a variable to represent the number of boys.	Let $n =$ the number of boys.
Step 4. Translate. Restate the problem in one sentence with all the important information.	$\underbrace{\text{The number of girls}(11)}\ \underbrace{\text{was}}\ \underbrace{\text{three more than twice the number of boys}}$
Translate into an equation.	$11 \quad = \quad 2b+3$
Step 5. Solve the equation.	$11 = 2b + 3$
Subtract 3 from each side.	$11 \ -3 = 2b + 3 \ -3$
Simplify.	$8 = 2b$
Divide each side by 2.	$\frac{8}{2} = \frac{2b}{2}$
Simplify.	$4 = b$
Step 6. Check. First, is our answer reasonable? Yes, having 4 boys in a study group seems OK. The problem says the number of girls was 3 more than twice the number of boys. If there are four boys, does that make eleven girls? Twice 4 boys is 8. Three more than 8 is 11.	
Step 7. Answer the question.	There were 4 boys in the study group.

Try It 5.5.5

Guillermo bought textbooks and notebooks at the bookstore. The number of textbooks was 3 more than twice the number of notebooks. He bought 7 textbooks. How many notebooks did he buy?

Answer available at the end of this section.

Try It 5.5.6

Gerry worked Sudoku puzzles and crossword puzzles this week. The number of Sudoku puzzles he completed is eight more than twice the number of crossword puzzles. He completed 22 Sudoku puzzles. How many crossword puzzles did he do?

Answer available at the end of this section.

Solve Number Problems

Now that we have a problem solving strategy, we will use it on several different types of word problems. The first type we will work on is “number problems.” Number problems give some clues about one or more numbers. We use these clues to write an equation. Number problems don’t usually arise on an everyday basis, but they provide a good introduction to practicing the problem solving strategy outlined above.

Example 5.5.7

The difference of a number and six is 13. Find the number.

Solution

Step 1. Read the problem. Are all the words familiar?	
Step 2. Identify what we are looking for.	the number
Step 3. Name. Choose a variable to represent the number.	Let $n =$ the number.
Step 4. Translate. Remember to look for clue words like "difference... of... and..."	
Restate the problem as one sentence.	$\underbrace{\text{The difference of the number and 6}}\ \underbrace{\text{is}}\ \underbrace{13}$
Translate into an equation.	$n - 6 = 13$
Step 5. Solve the equation.	$n - 6 = 13$
Simplify.	$n = 19$
Step 6. Check.	
The difference of 19 and 6 is 13. It checks!	
Step 7. Answer the question.	The number is 19.

Try It 5.5.8

The difference of a number and eight is 17. Find the number.

Answer available at the end of this section.

Try It 5.5.9

The difference of a number and eleven is -7. Find the number.

Answer available at the end of this section.

Example 5.5.10

The sum of twice a number and seven is 15. Find the number.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the number
Step 3. Name. Choose a variable to represent the number.	Let $n =$ the number.
Step 4. Translate.	
Restate the problem as one sentence.	$\underbrace{\text{The sum of twice a number and 7}}\ \underbrace{\text{is}}\ \underbrace{15}$
Translate into an equation.	$2n + 7 = 15$
Step 5. Solve the equation.	$2n + 7 = 15$

Subtract 7 from each side and simplify.	$2n = 8$
Divide each side by 2 and simplify.	$n = 4$
Step 6. Check.	
Is the sum of twice 4 and 7 equal to 15?	
$2 \cdot 4 + 7 \stackrel{?}{=} 15$ $15 = 15\checkmark$	
Step 7. Answer the question.	The number is 4.

Did you notice that we left out some of the steps as we solved this equation? If you're not yet ready to leave out these steps, write down as many as you need.

Try It 5.5.11

The sum of four times a number and two is 14. Find the number.

Answer available at the end of this section.

Try It 5.5.12

The sum of three times a number and seven is 25. Find the number.

Answer available at the end of this section.

Some number word problems ask us to find two or more numbers. It may be tempting to name them all with different variables, but so far we have only solved equations with one variable. In order to avoid using more than one variable, we will define the numbers in terms of the same variable. Be sure to read the problem carefully to discover how all the numbers relate to each other.

Example 5.5.13

One number is five more than another. The sum of the numbers is 21. Find the numbers.

Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		We are looking for two numbers.
Step 3. Name. We have two numbers to name and need a name for each.		
Choose a variable to represent the first number.		Let $n = 1^{st}$ number.
What do we know about the second number?		One number is five more than another.
		$n + 5 = 2^{nd}$ number
Step 4. Translate. Restate the problem as one sentence with all the important information.		The sum of the 1st number and the 2nd number is 21.
Translate into an equation.		$\underbrace{1^{st}\text{ number}} + \underbrace{2^{nd}\text{ number}} \quad \underbrace{=} \quad \underbrace{21}$
Substitute the variable expressions.		$n + n + 5 = 21$

Step 5. Solve the equation.		$n+n+5=21$
Combine like terms.		$2n+5=21$
Subtract 5 from both sides and simplify.		$2n=16$
Divide by 2 and simplify.		$n=8$ 1st number
Find the second number, too.		$n+5$ 2nd number
		$8+5$
		13
Step 6. Check.		
Do these numbers check in the problem?		
Is one number 5 more than the other?	$13 \stackrel{?}{=} 8+5$	
Is thirteen 5 more than 8? Yes.	$13=13\checkmark$	
Is the sum of the two numbers 21?	$8+13 \stackrel{?}{=} 21$	
	$21=21\checkmark$	
Step 7. Answer the question.		The numbers are 8 and 13.

Try It 5.5.14

One number is six more than another. The sum of the numbers is twenty-four. Find the numbers.

Answer available at the end of this section.

Try It 5.5.15

The sum of two numbers is fifty-eight. One number is four more than the other. Find the numbers.

Answer available at the end of this section.

Example 5.5.16

The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.

Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		We are looking for two numbers.
Step 3. Name.		
Choose a variable.		Let $n=1^{st}$ number.

One number is 4 less than the other.		$n-4=2^{nd}$ number
Step 4. Translate.		
Write as one sentence.		The sum of the 2 numbers is negative 14.
Translate into an equation.		$\underbrace{1^{st}\text{ number}}+\underbrace{2^{nd}\text{ number}}\ \underbrace{\text{is}}\ \underbrace{\text{negative fourteen}}$
Step 5. Solve the equation.		$n + n-4 = -14$
Combine like terms.		$n+n-4=-14$
Add 4 to each side and simplify.		$2n-4=-14$
Simplify.		$2n=-10$
		$n=-5$ 1st number
		$n-4$ 2nd number
		$-5-4$
		-9
Step 6. Check.		
Is –9 four less than –5?	$-5-4\stackrel{?}{=}-9$	
	$-9=-9\checkmark$	
Is their sum –14?	$-5+(-9)\stackrel{?}{=}-14$	
	$-14=-14\checkmark$	
Step 7. Answer the question.		The numbers are –5 and –9.

Try It 5.5.17

The sum of two numbers is negative twenty-three. One number is seven less than the other. Find the numbers.

Answer available at the end of this section.

Try It 5.5.18

The sum of two numbers is -18. One number is 40 more than the other. Find the numbers.

Answer available at the end of this section.

Example 5.5.19

One number is ten more than twice another. Their sum is one. Find the numbers.

Solution

Step 1. Read the problem.		

Step 2. Identify what you are looking for.		We are looking for two numbers.
Step 3. Name.		
Choose a variable.		Let $x = 1^{st}$ number.
One number is 10 more than twice another.		$2x + 10 = 2^{nd}$ number
Step 4. Translate.		
Restate as one sentence.		Their sum is one.
		The sum of the two numbers is 1.
Translate into an equation.		$x + 2x + 10 = 1$
Step 5. Solve the equation.		
Combine like terms.		$x + 2x + 10 = 1$
Subtract 10 from each side.		$3x + 10 = 1$
Divide each side by 3.		$3x = -9$
		$x = -3$ 1st number
		$2x + 10$ 2nd number
		$2(-3) + 10$
		4
Step 6. Check.		
Is ten more than twice –3 equal to 4?	$2(-3) + 10 \stackrel{?}{=} 4$	
	$-6 + 10 \stackrel{?}{=} 4$	
	$4 = 4$✓	
Is their sum 1?	$-3 + 4 \stackrel{?}{=} 1$	
	$1 = 1$✓	
Step 7. Answer the question.		The numbers are –3 and 4.

Try It 5.5.20

One number is eight more than twice another. Their sum is negative four. Find the numbers.

Answer available at the end of this section.

Try It 5.5.21

One number is three more than three times another. Their sum is -5. Find the numbers.

Answer available at the end of this section.

Some number problems involve consecutive integers. *Consecutive integers* are integers that immediately follow each other. Examples of consecutive integers are:

$$\begin{aligned} &1, 2, 3, 4 \\ &-10, -9, -8, -7 \\ &150, 151, 152, 153 \end{aligned} \tag{5.5.1}$$

Notice that each number is one more than the number preceding it. So if we define the first integer as n, the next consecutive integer is $n+1$. The one after that is one more than $n+1$, so it is $n+1+1$, which is $n+2$.

$$\begin{aligned} &n && 1^{\text{st}} \text{ integer} \\ &n+1 && 2^{\text{nd}} \text{ consecutive integer} \\ &n+2 && 3^{\text{rd}} \text{ consecutive integer } \ldots \text{ etc.} \end{aligned} \tag{5.5.2}$$

Example 5.5.22

The sum of two consecutive integers is 47. Find the numbers.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	two consecutive integers
Step 3. Name each number.	Let $n = 1^{st}$ integer.
	$n+1 =$ next consecutive integer
Step 4. Translate.	
Restate as one sentence.	The sum of the integers is 47.
Translate into an equation.	$n + n + 1 = 47$
Step 5. Solve the equation.	$n + n + 1 = 47$
Combine like terms.	$2n + 1 = 47$
Subtract 1 from each side.	$2n = 46$
Divide each side by 2.	$n = 23$ 1st integer
	$n + 1$ next consecutive integer
	$23 + 1$
	24
Step 6. Check.	
$23 + 24 \stackrel{?}{=} 47$ $47 = 47\checkmark$	
Step 7. Answer the question.	The two consecutive integers are 23 and 24.

Try It 5.5.23

The sum of two consecutive integers is 95. Find the numbers.

Answer available at the end of this section.

Try It 5.5.24

The sum of two consecutive integers is –31. Find the numbers.

Answer available at the end of this section.

Example 5.5.25

Find three consecutive integers whose sum is –42.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	three consecutive integers
Step 3. Name each of the three numbers.	Let $n = 1^{st}$ integer.
	$n + 1 = 2^{nd}$ consecutive integer
	$n + 2 = 3^{rd}$ consecutive integer
Step 4. Translate.	
Restate as one sentence.	The sum of the three integers is -42.
Translate into an equation.	$n + n + 1 + n + 2 = -42$
Step 5. Solve the equation.	$n + n + 1 + n + 2 = -42$
Combine like terms.	$3n + 3 = -42$
Subtract 3 from each side.	$3n = -45$
Divide each side by 3.	$n = -15$ 1st integer
	$n + 1$ 2nd integer
	$-15 + 1$
	-14
	$n + 2$ 3rd integer
	$-15 + 2$
	-13
Step 6. Check.	
$-13 + (-14) + (-15) \stackrel{?}{=} -42$ $-42 = -42$✓	
Step 7. Answer the question.	The three consecutive integers are –13, –14, and –15.

Try It 5.5.26

Find three consecutive integers whose sum is –96.

Answer available at the end of this section.

Try It 5.5.27

Find three consecutive integers whose sum is –36.

Answer available at the end of this section.

Now that we have worked with consecutive integers, we will expand our work to include consecutive even integers and consecutive odd integers. *Consecutive even integers* are even integers that immediately follow one another. Examples of consecutive even integers are:

$$\begin{array}{l} 18, 20, 22 \\ 64, 66, 68 \\ -12, -10, -8 \end{array} \tag{5.5.3}$$

Notice each integer is 2 more than the number preceding it. If we call the first one n, then the next one is $n+2$. The next one would be $n+2+2$ or $n+4$.

$$\begin{array}{ll} n & 1^{\text{st}} \text{ even integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive even integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive even integer } \ldots \text{ etc.} \end{array} \tag{5.5.4}$$

Consecutive odd integers are odd integers that immediately follow one another. Consider the consecutive odd integers 77, 79, and 81.

$$\begin{array}{l} 77, 79, 81 \\ n, n+2, n+4 \end{array} \tag{5.5.5}$$

$$\begin{array}{ll} n & 1^{\text{st}} \text{ odd integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive odd integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive odd integer } \ldots \text{ etc.} \end{array} \tag{5.5.6}$$

Does it seem strange to add 2 (an even number) to get from one odd integer to the next? Do you get an odd number or an even number when we add 2 to 3? to 11? to 47?

Whether the problem asks for consecutive even numbers or odd numbers, you don't have to do anything different. The pattern is still the same—to get from one odd or one even integer to the next, add 2.

Example 5.5.28

Find three consecutive even integers whose sum is 84.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	three consecutive even integers
Step 3. Name the integers.	Let $n = 1^{st}$ even integers.
	$n + 2 = 2^{nd}$ consecutive even integer
	$n + 4 = 3^{rd}$ consecutive even integer
Step 4. Translate.	
Restate as one sentence.	The sum of the three even integers is 84.
Translate into an equation.	$n + n + 2 + n + 4 = 84$
Step 5. Solve the equation.	
Combine like terms.	$n + n + 2 + n + 4 = 84$
Subtract 6 from each side.	$3n + 6 = 84$
Divide each side by 3.	$3n = 78$
	$n = 26$ 1st integer
	$n + 2$ 2nd integer
	$26 + 2$
	28
	$n + 4$ 3rd integer
	$26 + 4$
	30
Step 6. Check.	
$26 + 28 + 30 \stackrel{?}{=} 84$	
$84 = 84$✓	
Step 7. Answer the question.	The three consecutive integers are 26, 28, and 30.

(5.5.7)

Try It 5.5.29

Find three consecutive even integers whose sum is 102.

Answer available at the end of this section.

Try It 5.5.30

Find three consecutive even integers whose sum is –24.

Answer available at the end of this section.

Example 5.5.31

A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	How much does the husband earn?
Step 3. Name.	

Choose a variable to represent the amount the husband earns.	Let $h =$ the amount the husband earns.
The wife earns $\$16,000$ less than twice that.	$2h - 16,000$ the amount the wife earns.
Step 4. Translate.	Together the husband and wife earn $\$110,000$.
Restate the problem in one sentence with all the important information.	The amount the husband earns plus the amount the wife earns is $110,000
Translate into an equation.	$h + 2h - 16,000 = 110,000$
Step 5. Solve the equation.	$h + 2h - 16,000 = 110,000$
Combine like terms.	$3h - 16,000 = 110,000$
Add $16,000$ to both sides and simplify.	$3h = 126,000$
Divide each side by 3.	$h = 42,000$
	$\$42,000$ amount husband earns
	$2h - 16,000$ amount wife earns
	$2(42,000) - 16,000$
	$84,000 - 16,000$
	$68,000$
Step 6. Check.	
If the wife earns $\$68,000$ and the husband earns $\$42,000$ is the total $\$110,000$? Yes!	
Step 7. Answer the question.	The husband earns $\$42,000$ a year.

Try It 5.5.32

According to the National Automobile Dealers Association, the average cost of a car in 2014 was $28,500. This was $1,500 less than 6 times the cost in 1975. What was the average cost of a car in 1975?

Answer available at the end of this section.

Try It 5.5.33

U.S. Census data shows that the median price of new home in the United States in November 2014 was $280,900. This was $10,700 more than 14 times the price in November 1964. What was the median price of a new home in November 1964?

Answer available at the end of this section.

Key Concepts

- **Problem-Solving Strategy**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.

3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

- **Consecutive Integers**
  Consecutive integers are integers that immediately follow each other.

$$\begin{array}{cc} n & 1^{\text{st}} \text{ integer} \\ n+1 & 2^{\text{nd}} \text{ consecutive integer} \\ n+2 & 3^{\text{rd}} \text{ consecutive integer } \ldots \text{ etc.} \end{array} \tag{5.5.8}$$

  Consecutive even integers are even integers that immediately follow one another.

$$\begin{array}{cc} n & 1^{\text{st}} \text{ integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive even integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive even integer } \ldots \text{ etc.} \end{array} \tag{5.5.9}$$

  Consecutive odd integers are odd integers that immediately follow one another.

$$\begin{array}{cc} n & 1^{\text{st}} \text{ integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive odd integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive odd integer } \ldots \text{ etc.} \end{array} \tag{5.5.10}$$

Answers

Try It 5.5.2
$\$180$

Try It 5.5.3
40

Try It 5.5.5
2

Try It 5.5.6
7

Try It 5.5.8
25

Try It 5.5.9
4

Try It 5.5.11
3

Try It 5.5.12
6

Try It 5.5.14
9, 15

Try It 5.5.15
27, 31

Try It 5.5.17
-15, -8

Try It 5.5.18
-29, 11

Try It 5.5.20
$-4,\ 0$

Try It 5.5.21
$-3,\ -2$

Try It 5.5.23
47, 48

Try It 5.5.24
-16, -15

Try It 5.5.26
-33, -32, -31

Try It 5.5.27
-13, -12, -11

Try It 5.5.29
32, 34, 36

Try It 5.5.30
−10,−8,−6

Try It 5.5.32
$5000

Try It 5.5.33
$19300

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5.6: Recognizing Patterns

Mathematics is often seen as a subject rooted in strict rules and procedures. However, one of the most powerful tools in a mathematician's toolkit is the ability to recognize patterns. Identifying patterns can simplify complex problems, reveal underlying structures, and lead to elegant solutions. This section will explore how recognizing patterns can enhance problem-solving skills and provide strategies for identifying and utilizing patterns in various mathematical contexts.

The Importance of Patterns in Mathematics

Patterns are fundamental to mathematics. They appear in numbers, shapes, algebraic expressions, and even in the way we solve problems. Recognizing patterns helps in:

1. **Predicting Future Events:** Understanding a pattern allows us to predict what comes next. This is crucial in sequences and series, as well as in real-world applications like predicting trends.
2. **Simplifying Problems:** Patterns can reduce the complexity of a problem by revealing relationships that are not immediately obvious.
3. **Enhancing Understanding:** Seeing the connections between different problems and solutions can deepen our understanding of mathematical concepts.
4. **Developing Strategies:** Recognizing patterns can lead to the development of general strategies for solving classes of problems.

Types of Patterns

Numerical Patterns

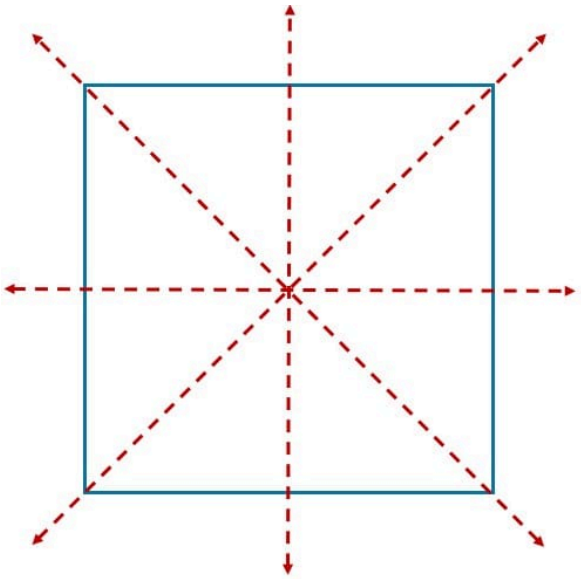
Numerical patterns involve sequences of numbers that follow a specific rule. For example:

- **Arithmetic Sequences:** Each term is obtained by adding a constant difference to the previous term.
  - Example: 2, 5, 8, 11, 14 (difference of 3)
- **Geometric Sequences:** Each term is obtained by multiplying the previous term by a constant factor.
  - Example: 3, 6, 12, 24, 48 (factor of 2)
- **Fibonacci Sequence:** Each term is the sum of the two preceding terms.
  - Example: 1, 1, 2, 3, 5, 8, 13

Visual Patterns

Visual patterns appear in shapes and arrangements. These are often used in geometry and can include:

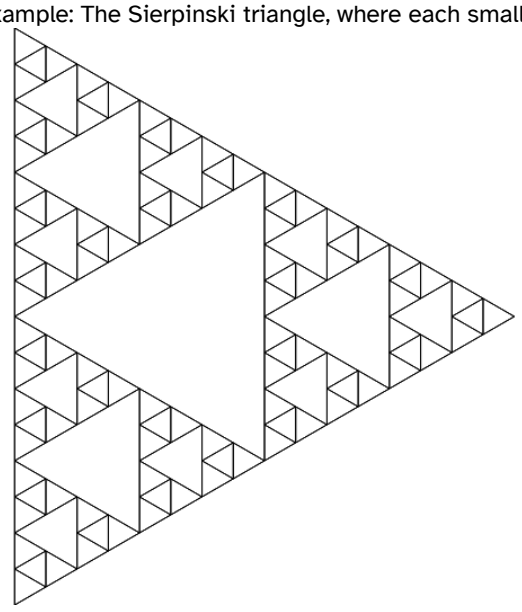
- **Symmetry:** Objects or shapes that are mirror images.
  - Example: A square has four lines of symmetry.



- **Tessellations:** Patterns of shapes that fit perfectly together without gaps.
  - Example: Hexagons tessellate to form a honeycomb pattern.



- **Fractals:** Complex patterns that are self-similar at different scales.
  - Example: The Sierpinski triangle, where each smaller triangle is a reduced-scale copy of the whole.



Algebraic Patterns

Algebraic patterns involve expressions and equations. Recognizing these can simplify solving equations and understanding functions.

- **Binomial Patterns:** Expansion of powers of binomials follows the binomial theorem.
  - Example: $(a+b)^2 = a^2 + 2ab + b^2$
- **Factoring Patterns:** Certain forms of expressions can be factored in standard ways.
  - Example:$a^2 - b^2 = (A+b)(a-b)$
- **Function Patterns:** Functions can exhibit specific types of behavior.
  - Example: Linear functions $f(x) = mx + b$ have a constant rate of change.

Strategies for Recognizing Patterns

Observation and Practice

The more you work with mathematical problems, the more you will start to notice patterns. Practice is essential. Consistently engaging with a variety of mathematical problems helps train your mind to recognize patterns and relationships more readily. When you work on diverse problems, make it a habit to pay attention to recurring themes and commonalities. Over time, this practice will sharpen your observational skills, allowing you to detect patterns more quickly and with greater accuracy.

Additionally, documenting the patterns you observe can be incredibly beneficial. Keep a journal or a log of the different types of problems you solve and note any patterns or strategies that emerge. This process not only reinforces your learning but also provides a valuable reference for future problem-solving tasks. Engaging with peers in discussions about problem-solving approaches and patterns can also enhance your ability to see different perspectives and uncover patterns you might have missed on your own.

Look for Regularities

Looking for regularities is a crucial aspect of recognizing patterns in mathematical problem-solving. Regularities are consistent and predictable aspects that appear repeatedly within problems, which can help simplify complex issues and provide insight into their solutions. To develop this skill, it's important to actively search for these regularities as you work through various mathematical tasks.

One effective strategy is to start by examining simpler versions of the problem at hand. Often, complex problems can be broken down into more manageable parts, and these simpler components may reveal underlying regularities. For example, when studying sequences, identifying the first few terms and their relationships can help you discern a pattern that applies to the entire sequence.

Another approach is to use visualization techniques. Drawing diagrams, graphs, or tables can make regularities more apparent. Visual representations often highlight relationships and patterns that might be less obvious in a purely numerical or algebraic form. For instance, plotting data points on a graph can reveal trends and regularities that facilitate the formulation of a general rule or function.

It's also beneficial to compare and contrast similar problems. By analyzing how different problems are solved and identifying common strategies, you can uncover regularities that apply across multiple contexts. This comparative approach not only reinforces your understanding of specific patterns but also broadens your ability to apply these patterns to new and unfamiliar problems.

Furthermore, consider using mathematical notation and symbols consistently. Standardized notation can help you see regularities more clearly and communicate your findings effectively. When you notice that a particular notation or symbolic representation is recurring, it can signal an underlying regularity worth exploring further.

Engaging in collaborative problem-solving can enhance your ability to identify regularities. Discussing problems with peers and sharing different perspectives can lead to new insights and uncover regularities that you might not have noticed on your own. Group work encourages a deeper exploration of patterns and often leads to a more comprehensive understanding of the problem-solving process.

Finally, regular reflection on your problem-solving methods and the regularities you encounter is crucial. After solving a problem, take the time to review your approach and identify any patterns or regularities that emerged. Reflecting on these aspects can consolidate your learning and improve your ability to recognize and apply regularities in future problems.

Looking for regularities involves actively seeking out consistent and predictable aspects of problems, using simplification, visualization, comparison, notation, collaboration, and reflection. By honing this skill, you can enhance your mathematical problem-solving abilities and approach complex problems with greater confidence and insight.

Use Tables and Diagrams

Tables and diagrams are powerful tools in mathematics for visualizing and identifying regularities in problem-solving. They provide structured ways to organize information, making patterns more apparent and aiding in the discovery of underlying relationships.

Tables:

Tables are useful for organizing data systematically, especially when dealing with numerical sequences, functions, or categorical data. By tabulating relevant variables or values, you can observe how different elements interact over time or across different conditions. For example, when analyzing the growth of a population over several years, a table can show how the numbers change annually and highlight any consistent rate of increase or decrease.

To effectively use tables to identify regularities, consider the following strategies:

1. **Pattern Recognition:** Arrange data in a table format and look for recurring numbers or sequences. For instance, in a multiplication table, noticing how each row and column follows a predictable pattern (e.g., increasing by the same multiple) can reveal underlying multiplication rules.
2. **Comparative Analysis:** Create multiple tables to compare different scenarios or conditions. By placing side-by-side tables of experimental results, you can identify trends or patterns that occur consistently across different sets of data. This comparative approach helps in drawing conclusions and formulating general rules.
3. **Function Representation:** Use tables to represent functions by mapping input values to corresponding output values. By examining how changes in input affect output, you can derive equations or formulas that describe the relationship between variables. This method is particularly useful in calculus and algebra, where tables can illustrate the behavior of functions graphically.

Diagrams:

Diagrams provide visual representations that can reveal patterns and relationships in a problem-solving context. They are especially effective for illustrating geometric properties, structural relationships, and process flows. Different types of diagrams include:

1. **Graphs and Charts:** Graphical representations like line graphs, bar charts, and pie charts visually display data relationships. For example, plotting points on a graph can show trends over time or correlations between variables, making patterns in data more apparent.
2. **Geometric Figures:** Diagrams such as geometric shapes (e.g., triangles, circles) or spatial arrangements (e.g., Venn diagrams) help in understanding spatial relationships and logical connections. These diagrams often highlight symmetries, proportions, or overlapping areas that signify regularities in geometric properties.
3. **Flowcharts:** Flowcharts depict sequences of actions or decision-making processes. They illustrate step-by-step procedures and help identify recurring steps or decision points. Analyzing flowcharts can reveal algorithmic patterns or procedural regularities in problem-solving strategies.

Strategies for Effective Use:

- **Label and Title:** Clearly label elements in tables and diagrams to enhance clarity and understanding.
- **Analyze Trends:** Look for trends or changes in data patterns as you study tables or interpret diagrams. Identifying consistent trends can lead to the discovery of underlying regularities.
- **Iterative Refinement:** Iterate on your tables and diagrams by refining them based on new observations or insights.

Adjusting variables or adding annotations can help in uncovering deeper patterns.

Tables and diagrams are versatile tools that complement mathematical reasoning by providing visual insights into problem structures and relationships. By leveraging these tools effectively, you can enhance your ability to recognize and utilize regularities in mathematical problem-solving, ultimately improving your proficiency and confidence in tackling complex problems.

Generalize from Examples

Generalizing from examples is a fundamental skill in mathematics that empowers elementary students to uncover patterns and rules from specific instances, fostering a deeper understanding of mathematical concepts.

Steps to Guide Students in Generalizing from Examples:

1. **Explore Specific Instances:** Start with concrete examples that students can easily grasp. For example, when introducing addition, use simple problems like $2 + 3$ and $4 + 1$. Encourage students to solve these problems using manipulatives or drawings.
2. **Identify Patterns:** Guide students to identify patterns or similarities among the examples. Ask questions like, "What do you notice about these addition problems?" Prompt them to recognize that in each case, they are combining two numbers to find a total.
3. **Formulate Simple Rules:** Help students formulate simple rules or strategies based on their observations. For addition problems, they might notice that they always combine two numbers to get a larger number.
4. **Apply to New Examples:** Provide additional examples or variations of the same concept. Ask students to apply the rule they formulated to solve new problems. For instance, after understanding $2 + 3 = 5$ and $4 + 1 = 5$, they can predict that $1 + 4 = 5$.
5. **Encourage Reflective Thinking:** Foster reflective thinking by asking students to explain their reasoning. Encourage them to articulate why their rule works and how they know it applies to different examples.
6. **Express Generalization:** Once students have tested their rule across multiple examples and variations, guide them to express their generalized understanding in simple terms. For example, they might say, "When you add two numbers together, the total is always bigger."

Benefits of Generalizing from Examples for Students:

- **Conceptual Understanding:** By exploring patterns and formulating rules, students develop a deeper conceptual understanding of mathematical operations.
- **Problem-Solving Skills:** Generalized rules serve as tools for problem-solving, enabling students to approach new problems with confidence.
- **Critical Thinking:** The process encourages critical thinking as students analyze patterns, make connections, and apply their understanding to different situations.
- **Building Mathematical Confidence:** Successfully generalizing from examples builds students' confidence in their mathematical abilities and promotes a positive attitude towards learning.

Strategies for Effective Teaching:

- **Use Concrete Examples:** Start with manipulatives, drawings, or real-world examples to make abstract concepts more tangible.
- **Provide Guided Practice:** Scaffold learning by guiding students through progressively more complex examples.
- **Encourage Discussion:** Foster collaborative learning by encouraging students to discuss their observations and reasoning with peers.
- **Connect to Real Life:** Relate generalized rules to everyday situations to show relevance and application.

By guiding elementary students through the process of generalizing from examples, teachers empower them to not only solve mathematical problems but also to develop a deeper appreciation and mastery of mathematical concepts from an early age.

Verify and Refine

Verifying and refining mathematical understanding is a crucial step in ensuring that elementary students develop solid foundations in math concepts. It involves helping students confirm the accuracy of their solutions, identify errors, and improve

their problem-solving strategies.

Steps to Guide Elementary Students in Verifying and Refining Mathematical Understanding:

1. **Check Solutions for Accuracy:** Teach students to check their solutions by verifying if their answers make sense in the context of the problem. For example, if solving $23 + 16$ encourage them to estimate the answer or use a different method (like subtraction or grouping) to confirm their result.
2. **Identify and Correct Errors:** Guide students to identify common types of errors, such as calculation mistakes, misinterpretation of the problem, or overlooking important details. Show them strategies to backtrack and find where mistakes occurred.
3. **Use Multiple Approaches:** Encourage students to solve problems using different methods or strategies. This not only reinforces understanding but also allows them to cross-verify their answers. For instance, after adding numbers, they could use subtraction to check their work.
4. **Refine Problem-Solving Strategies:** Help students reflect on their problem-solving processes. Ask questions like, "Was there a better or faster way to solve this problem?" Guide them to consider efficiency and accuracy when refining their strategies.
5. **Seek Alternative Solutions:** Promote creativity by encouraging students to find alternative solutions to the same problem. This fosters flexibility in thinking and deepens understanding of mathematical concepts.
6. **Provide Constructive Feedback:** Offer specific feedback that highlights both correct approaches and areas for improvement. Encourage peer feedback and discussions to promote a supportive learning environment.

Benefits of Verifying and Refining Mathematical Understanding for Students:

- **Enhanced Accuracy:** Students develop skills to ensure their solutions are correct and logical.
- **Improved Problem-Solving Skills:** The process encourages students to think critically and strategically about different ways to approach problems.
- **Building Resilience:** Learning to identify and correct mistakes fosters resilience and perseverance in mathematical tasks.
- **Deepened Conceptual Understanding:** By refining strategies and exploring alternatives, students gain a deeper understanding of mathematical concepts and relationships.

Strategies for Effective Teaching:

- **Model the Process:** Demonstrate how to verify solutions and refine strategies through examples and think-alouds.
- **Encourage Peer Collaboration:** Promote peer discussions where students can explain their reasoning and critique each other's solutions.
- **Use Real-World Examples:** Relate mathematical problems to real-life situations to make the verification process more meaningful and applicable.
- **Celebrate Mistakes:** Emphasize that mistakes are opportunities for learning and growth, encouraging students to embrace challenges.

By guiding elementary students through the process of verifying and refining their mathematical understanding, teachers empower them to become confident, reflective problem solvers who approach mathematics with curiosity and resilience.

Applying Pattern Recognition in Problem Solving

Example 1: Arithmetic Sequence

Problem: Find the 10th term of the sequence: 4, 9, 14, 19, ...

Solution

1. **Identify the Pattern:** Notice the difference between terms is 5.
2. **Write the Rule:** The nth term of an arithmetic sequence is given by $a_n = a_1 + (n-1)d$.
3. **Apply the Rule:** For the 10th term, $a_{10} = 4 + (10-1) \times 5 = 4 + 45 = 49$.

Example 2: Investigating Patterns in Bivariate Data

Problem: ou want to understand if there is a connection between students' favorite seasons and their preferred sports activities.

Solution

1. **Data Collection:** Ask each student in your class to share their favorite season (e.g., winter, spring, summer, fall) and their favorite sport (e.g., soccer, basketball, swimming, biking).
2. **Organize Data:** Create a table on the board or a worksheet with two columns: one for favorite seasons and another for favorite sports.

Favorite Season	Favorite Sport
Winter	Soccer
Spring	Basketball
Summer	Swimming
Fall	Biking
...	...

3. **Plotting Data:** Use a simple scatter plot on a large poster or interactive whiteboard. Place each student's favorite season along the x-axis and their favorite sport along the y-axis.
4. **Observing Patterns:** Look at the plotted points together as a class. Discuss what patterns or trends students notice. For example, are there groups of students who prefer winter and skiing?
5. **Identifying Association:** Guide students to identify any associations between favorite seasons and sports. Are there more students who prefer summer and swimming?
6. **Interpreting Results:** Help students interpret their findings. For instance, if many students who like fall also enjoy biking, discuss the association between these variables.
7. **Discussion:** Lead a class discussion on why these associations might exist. For example, why might students who like winter prefer skiing or snowboarding?

Example 3: Factoring Algebraic Expressions

Problem: Factor the expression $x^2 - 10x + 25$.

Solution

1. **Identify the Pattern:** Recognize the expression as a perfect square trinomial.
2. **Apply the Pattern:** $x^2 - 10x + 25 = (x - 5)^2$.

Conclusion

Recognizing patterns is a crucial skill in mathematics that enhances problem-solving abilities. By identifying and utilizing patterns, you can simplify complex problems, make accurate predictions, and develop deeper mathematical insights. Practice these strategies regularly to become proficient in recognizing patterns and applying them to various mathematical contexts.

Citations and Attributions

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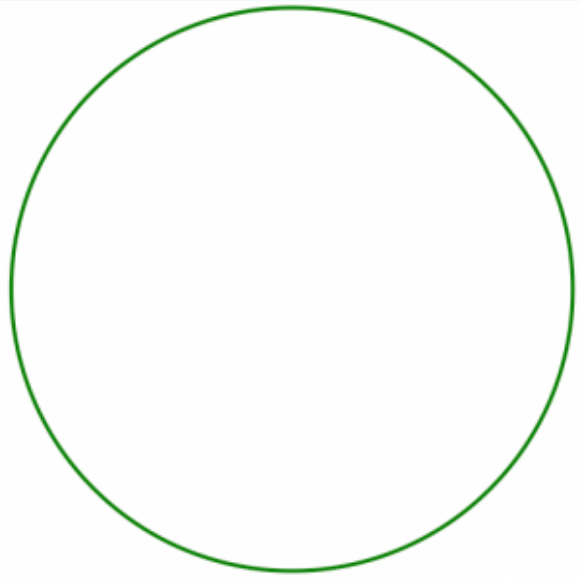
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5.7: Beware of Patterns!

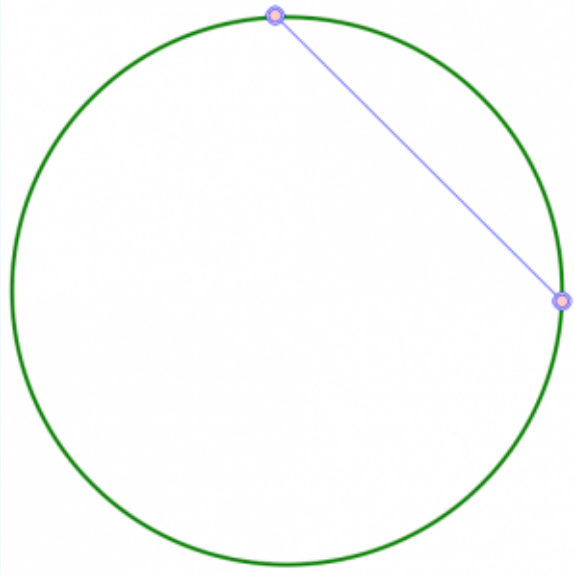
When exploring the “Look for Patterns” strategy but you have to be careful! Do not forget the “and Explain” part of the strategy., it's crucial for elementary students to understand that not all data will exhibit clear patterns. Sometimes, data might not show an obvious trend, which doesn't mean the data is incorrect but rather highlights the variability inherent in observations. Additionally, different interpretations of patterns can arise among individuals, emphasizing the importance of discussing findings collaboratively. It's essential to avoid premature conclusions based on initial observations alone; patterns may be coincidental or influenced by factors not immediately apparent. Encouraging ongoing exploration and inquiry helps students develop a deeper understanding of how to critically analyze data and draw meaningful conclusions grounded in evidence.

(Dots on a Circle)

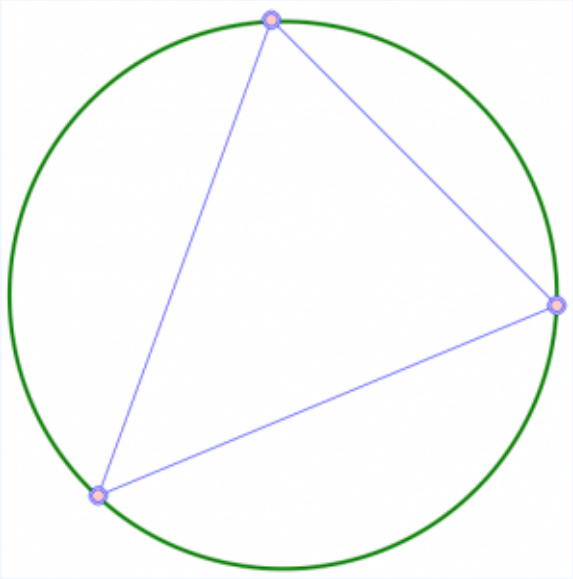
Start with a circle.



If I put two dots on the circle and connect them, the line divides the circle into two pieces.



If I put three dots on the circle and connect each pair of dots, the lines divides the circle into four pieces.



Suppose you put one hundred dots on a circle and connect each pair of dots, meaning every dot is connected to 99 other dots. How many pieces will you get? Lines may cross each other, but assume the points are chosen so that three or more lines never meet at a single point.

Think / Pair / Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What strategies did you try? What did you figure out? What questions do you still have?

The natural way to work on this problem is to use smaller numbers of dots and look for a pattern, right? If you have not already, try it. How many pieces when you have four dots? Five dots? How would you describe the pattern?

Now try six dots. You will want to draw a big circle and space out the six dots to make your counting easier. Then carefully count up how many pieces you get. It is probably a good idea to work with a partner so you can check each other's work. Make sure you

count every piece once and do not count any piece twice. How can you be sure that you do that?

Were you surprised? For the first several steps, it seems to be the case that when you add a dot you double the number of pieces. But that would mean that for six dots, you should get 32 pieces, and you only get 30 or 31, depending on how the dots are arranged. No matter what you do, you cannot get 32 pieces. The pattern simply does not hold up.

Mathematicians love looking for patterns and finding them. We get excited by patterns. But we are also very skeptical of patterns! If we cannot explain why a pattern would occur, then we are not willing to just believe it.

For example, if my number pattern starts out: 2, 4, 8, ... I can find lots of ways to continue the pattern, each of which makes sense in some contexts. Here are some possibilities:

- 2, 4, 8, 2, 4, 8, 2, 4, 8, 2, 4, 8, ...

This is a a repeating pattern, cycling through the numbers 2, 4, 8 and then starting over with 2.

- 2, 4, 8, 32, 256, 8192, ...

To get the next number, multiply the previous two numbers together.

- 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
- 2, 4, 8, 14, 22, 32, 44, 58, 74 ...

Think / Pair / Share

- For the last two patterns above, describe in words how the number sequence is being created.
- Find at least two other ways to continue the sequence 2, 4, 8, . . . that looks different from all the ones you have seen so far. Write your rule in words, and write the next five terms of the number sequence.

So how can you be sure your pattern fits the problem? You have to tie them together! Remember the “Squares on a Chess Board” problem? You might have noticed a pattern like this one:

If the chess board has 5 squares on a side, then there are

- 5 × 5 = 25 squares of size 1 × 1.
- 4 × 4 = 16 squares of size 2 × 2.
- 3 × 3 = 9 squares of side 3 × 3.
- 2 × 2 = 4 squares of size 4 × 4.
- 1 × 1 = 1 squares of size 5 × 5.

So there are a total of

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

squares on a 5 × 5 chess board. You can probably guess how to continue the pattern to any size board, but how can you be absolutely sure the pattern continues in this way? What if this is like “Dots on a Circle,” and the obvious pattern breaks down after a few steps? You have to tie the pattern to the problem, so that it is clear why the pattern must continue in that way.

The first step in explaining a pattern is writing it down clearly. This brings us to another problem solving strategy.

Problem Solving Strategy 11 (Use a Variable!).

One of the most powerful tools we have is the use of a variable. If you find yourself doing calculations on things like “the number of squares,” or “the number of dots,” give those quantities a name! They become much easier to work with.

Think/Pair/Share

For now, just work on describing the pattern with variables.

- Stick with a 5 × 5 chess board for now, and consider a small square of size $k \times k$. Describe the pattern: How many squares of size $k \times k$ fit on a chess board of size 5 × 5?
- What if the chess board is bigger? Based on the pattern above, how many squares of size $k \times k$ should fit on a chess board of size 10 × 10?
- What if you do not know how big the chess board is? Based on the pattern above, how many squares of size $k \times k$ should fit on a chess board of size $n \times n$?

Now comes the tough part: explaining the pattern. Let us focus on an 8 × 8 board. Since it measures 8 squares on each side, we can see that we get 8× 8 = 64 squares of size 1× 1. And since there is just a single board, we get just one square of size 8 × 8. But what about all the sizes in-between?

Think/Pair/Share

Using the Chess Board video in the previous chapter as a model, work with a partner to carefully explain why the number of 3 × 3 squares will be $6 \cdot 6 = 36$, and why the number of 4 × 4 squares will be $5 \cdot 5 = 25$.

There are many different explanations other than what is found in the video. Try to find your own explanation.

Here is what a final justification might look like (*watch the Chess Board video as a concrete example of this solution*):

Solution (Chess Board Pattern).

Let n be the side of the chess board and let k be the side of the square. If the square is going to fit on the chess board at all, it must be true that $k \leq n$. Otherwise, the square is too big.

If I put the $k \times k$ square in the upper left corner of the chess board, it takes up k spaces across and there are $(n - k)$ spaces to the right of it. So I can slide the $k \times k$ square to the right $(n - k)$ times, until it hits the top right corner of the chess board. The square is in $(n - k + 1)$ different positions, counting the starting position.

If I move the $k \times k$ square back to the upper left corner, I can shift it down one row and repeat the whole process again. Since there are $(n - k)$ rows below the square, I can shift it down $(n - k)$ times until it hits the bottom row. This makes $(n - k + 1)$ total rows that the square moves across, counting the top row.

So, there are $(n - k + 1)$ rows with $(n - k + 1)$ squares in each row. That makes $(n - k + 1)^2$ total squares.

Thus, the solution is the sum of $(n - k + 1)^2$ for all $k \leq n$. In symbols:

$$\text{number of squares on an n} \times \text{n board} = \sum_{k=1}^{n} (n - k + 1)^2.$$

Once we are sure the pattern continues, we can use it to solve the problem. So go ahead!

- How many squares on a 10 × 10 chess board?
- What calculation would you do to solve that problem for a 100 × 100 chess board?

There is a number pattern that describes the number of pieces you get from the “Dots on a Circle” problem. If you want to solve the problem, go for it! Think about all of your problem solving strategies. But be sure that when you find a pattern, you can explain *why it is the right pattern for this problem*, and not just another pattern that seems to work but might not continue.

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5.8: Careful Use of Language in Mathematics

This section might seem like a bit of a sidetrack from the idea of problem solving, but in fact it is not. Mathematics is a social endeavor. We do not just solve problems and then put them aside. Problem solving has (at least) three components:

1. Solving the problem. This involves a lot of scratch paper and careful thinking.
2. Convincing yourself that your solution is complete and correct. This involves a lot of self-check and asking yourself questions.
3. Convincing someone else that your solution is complete and correct. This usually involves writing the problem up carefully or explaining your work in a presentation.

If you are not able to do that last step, then you have not really solved the problem. We will talk more about how to write up a solution soon. Before we do that, we have to think about how mathematicians use language (which is, it turns out, a bit different from how language is used in the rest of life).

Mathematical Statements

Definition

A mathematical statement is a complete sentence that is either true or false, but not both at once.

So a "statement" in mathematics cannot be a question, a command, or a matter of opinion. It is a complete, grammatically correct sentence (with a subject, verb, and usually an object). It is important that the statement is either true or false, though you may not know which! (Part of the work of a mathematician is figuring out which sentences are true and which are false.)

Think / Pair / Share

For each English sentence below, decide if it is a mathematical statement or not. If it is, is the statement true or false (or are you unsure)? If it is not a mathematical statement, in what way does it fail?

1. Blue is the prettiest color.
2. 60 is an even number.
3. Is your dog friendly?
4. Honolulu is the capital of Hawaii.
5. This sentence is false.
6. All roses are red.
7. UH Manoa is the best college in the world.
8. 1/2 = 2/4.
9. Go to bed.
10. There are a total of 204 squares on an 8 × 8 chess board.

Now write three mathematical statements and three English sentences that fail to be mathematical statements.

Notice that "1/2 = 2/4" is a perfectly good mathematical statement. It does not look like an English sentence, but read it out loud. The subject is "1/2." The verb is "equals." And the object is "2/4." This is a very good test when you write mathematics: try to read it out loud. Even the equations should read naturally, like English sentences.

Statement (5) is different from the others. It is called a **paradox**: a statement that is self-contradictory. If it is true, then we conclude that it is false. (Why?) If it is false, then we conclude that it is true. (Why?) Paradoxes are no good as mathematical statements, because it cannot be true and it cannot be false.

And / or

Consider this sentence:

After work, I will go to the beach, or I will do my grocery shopping.

In everyday English, that probably means that if I go to the beach, I will not go shopping. I will do one or the other, but not both activities. This is called an "exclusive or."

We can usually tell from context whether a speaker means "either one or the other or both," or whether he means "either one or the other but not both." (Some people use the awkward phrase "and/or" to describe the first option.)

Remember that in mathematical communication, though, we have to be very precise. We cannot rely on context or assumptions about what is implied or understood.

Definition

In mathematics, the word "or" always means "one or the other or both."

The word "and" always means "both are true."

Think / Pair / Share

For each sentence below:

- Decide if the choice $x = 3$ makes the statement true or false.
- Choose a different value of that makes the statement true (or say why that is not possible).
- Choose a different value of that makes the statement false (or say why that is not possible).

1. x is odd or x is even.
2. x is odd and x is even.
3. x is prime or x is odd.
4. $x > 5$ or $x < 5$.
5. $x > 5$ and $x < 5$.
6. $x + 1 = 7$ or $x - 1 = 7$.
7. $x \cdot 1 = x$ or $x \cdot 0 = x$.
8. $x \cdot 1 = x$ and $x \cdot 0 = x$.
9. $x \cdot 1 = x$ or $x \cdot 0 = 0$.

Quantifiers

Problem 23 (All About the Benjamins)

You are handed an envelope filled with money, and you are told "Every bill in this envelope is a $100 bill."

- What would convince you beyond any doubt that the sentence is true? How could you convince someone else that the sentence is true?
- What would convince you beyond any doubt that the sentence is false? How could you convince someone else that the sentence is false?

Suppose you were given a different sentence: "There is a $100 bill in this envelope."

- What would convince you beyond any doubt that the sentence is true? How could you convince someone else that the sentence is true?

- What would convince you beyond any doubt that the sentence is false? How could you convince someone else that the sentence is false?

Think / Pair / Share

What is the difference between the two sentences? How does that difference affect your method to decide if the statement is true or false?

Some mathematical statements have this form:

- "Every time..."
- "For all numbers. . . "
- "For every choice. . . "
- "It's always true that. . . "

These are *universal* statements. Such statements claim that something is always true, no matter what.

- To prove a universal statement is false, you must find an example where it fails. This is called a **counterexample** to the statement.
- To prove a universal statement is true, you must either check every single case, or you must find a logical reason why it would be true. (Sometimes the first option is impossible, because there might be infinitely many cases to check. You would never finish!)

Some mathematical statements have this form:

- "Sometimes..."
- "There is some number. . . "
- "For some choice. . . "
- "At least once..."

These are *existential* statements. Such statements claim there is some example where the statement is true, but it may not always be true.

- To prove an existential statement is true, you may just find the example where it works.
- To prove an existential statement is false, you must either show it fails in every single case, or you must find a logical reason why it cannot be true. (Sometimes the first option is impossible!)

Think / Pair / Share

For each statement below, do the following:

- Decide if it is a universal statement or an existential statement. (This can be tricky because in some statements the quantifier is "hidden" in the meaning of the words.)
- Decide if the statement is true or false, and do your best to justify your decision.

1. Every odd number is prime.
2. Every prime number is odd.
3. For all positive numbers $x, x^3 > x$.
4. There is some number x such that $x^3 = x$.
5. The points (1, 1), (2, 1), and (3, 0) all lie on the same line.
6. Addition (of real numbers) is commutative.
7. Division (of real numbers) is commutative.

Look back over your work. you will probably find that some of your arguments are sound and convincing while others are less so. In some cases you may "know" the answer but be unable to justify it. That is okay for now! Divide your answers into four categories:

1. I am confident that the justification I gave is good.
2. I am not confident in the justification I gave.
3. I am confident that the justification I gave is not good, or I could not give a justification.
4. I could not decide if the statement was true or false.

Conditional Statements

Problem 24 (Card Logic)

You have a deck of cards where each card has a letter on one side and a number on the other side. Your friend claims: “If a card has a vowel on one side, then it has an even number on the other side.”

These cards are on a table.



Which cards must you flip over to be certain that your friend is telling the truth?

Think / Pair / Share

After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which cards you must check? Try to come to agreement on an answer you both believe.

Here is another very similar problem, yet people seem to have an easier time solving this one:

Problem 25 (IDs at a Party)

You are in charge of a party where there are young people. Some are drinking alcohol, others soft drinks. Some are old enough to drink alcohol legally, others are under age. You are responsible for ensuring that the drinking laws are not broken, so you have asked each person to put his or her photo ID on the table. At one table, there are four young people:

- One person has a can of beer, another has a bottle of Coke, but their IDs happen to be face down so you cannot see their ages.
- You can, however, see the IDs of the other two people. One is under the drinking age, the other is above it. They both have fizzy clear drinks in glasses, and you are not sure if they are drinking soda water or gin and tonic.

Which IDs and/or drinks do you need to check to make sure that no one is breaking the law?

Think / Pair / Share

After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which cards you must check? Compare these two problems. Which question is easier and why?

Definition

A **conditional statement** can be written in the form

If some statement then some statement.

Where the first statement is the **hypothesis** and the second statement is the **conclusion**.

Think / Pair / Share

These are each conditional statements, though they are not all stated in "if/then" form. Identify the hypothesis of each statement. (You may want to rewrite the sentence as an equivalent "if/then" statement.)

1. If the tomatoes are red, then they are ready to eat.
   The tomatoes are red. / The tomatoes are ready to eat.
2. An integer n is even if it is a multiple of 2.
   n is even. / n is a multiple of 2.
3. If n is odd, then n is prime.
   n is odd. / n is prime.
4. The team wins when JJ plays.
   The team wins. / JJ plays.

Remember that a mathematical statement must have a definite truth value. It is either true or false, with no gray area (even though we may not be sure which is the case). How can you tell if a conditional statement is true or false? Surely, it depends on whether the hypothesis and the conclusion are true or false. But how, exactly, can you decide?

The key is to think of a conditional statement like a promise, and ask yourself: under what condition(s) will I have broken my promise?

Example 5.8.1:

Here is a conditional statement:

If I win the lottery, then I'll give each of my students $1,000.

There are four things that can happen:

- **True hypothesis, true conclusion:** I do win the lottery, and I do give everyone in class $1,000. I kept my promise, so the conditional statement is TRUE.
- **True hypothesis, false conclusion:** I do win the lottery, but I decide not to give everyone in class $1,000. I broke my promise, so the conditional statement is FALSE.
- **False hypothesis, true conclusion:** I do not win the lottery, but I am exceedingly generous, so I go ahead and give everyone in class $1,000. I did not break my promise! (Do you see why?) So the conditional statement is TRUE.
- **False hypothesis, false conclusion:** I do not win the lottery, so I do not give everyone in class $1,000. I did not break my promise! (Do you see why?) So the conditional statement is TRUE.

What can we conclude from this? **A conditional statement is false only when the hypothesis is true and the conclusion is false.** In every other instance, the promise (as it were) has not been broken. If a mathematical statement is not false, it must be true.

Example 5.8.2:

Here is another conditional statement:

If you live in Honolulu, then you live in Hawaii.

Is this statement true or false? It seems like it should depend on who the pronoun "you" refers to, and whether that person lives in Honolulu or not. Let us think it through:

- Sookim lives in Honolulu, so the hypothesis is true. Since Honolulu is in Hawaii, she does live in Hawaii. The statement is true about Sookim, since both the hypothesis and conclusion are true.
- DeeDee lives in Los Angeles. The statement is true about DeeDee since the hypothesis is false.

So in fact it does not matter! The statement is true either way. The right way to understand such a statement is as a universal statement: "Everyone who lives in Honolulu lives in Hawaii."

This statement is true, and here is how you might justify it: "Pick a random person who lives in Honolulu. That person lives in Hawaii (since Honolulu is in Hawaii), so the statement is true for that person. I do not need to consider people who do not live in Honolulu. The statement is automatically true for those people, because the hypothesis is false!"

Example 5.8.3:

How do we show a (universal) conditional statement is false?

You need to give a specific instance where the hypothesis is true and the conclusion is false. For example:

If you are a good swimmer, then you are a good surfer.

Do you know someone for whom the hypothesis is true (that person is a good swimmer) but the conclusion is false (the person is not a good surfer)? Then the statement is false!

Think / Pair / Share

For each conditional statement, decide if it is true or false. Justify your answer.

1. If $2 \times 2 = 4$ then $1 + 1 = 3$.
2. If $2 \times 2 = 5$ then $1 + 1 = 3$.
3. If $\pi > 3$ then all odd numbers are prime.
4. If $\pi < 3$ then all odd numbers are prime.
5. If a number has a 4 in the one's place, then the number is even.
6. If a number is even, then the number has a 4 in the one's place.
7. If the product of two numbers is 0, then one of the numbers is 0.
8. If the sum of two numbers is 0, then one of the numbers is 0.

Think / Pair / Share (Two truths and a lie)

On your own, come up with two conditional statements that are true and one that is false. Share your three statements with a partner, but do not say which are true and which is false. See if your partner can figure it out!

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5.9: Exercises and Applications

Practice Makes Perfect

Use the Approach Word Problems with a Positive Attitude

In the following exercises, prepare the lists described.

Exercise 5.9.1

List five positive thoughts you can say to yourself that will help you approach word problems with a positive attitude. You may want to copy them on a sheet of paper and put it in the front of your notebook, where you can read them often.

Answer available at the end of this section.

Exercise 5.9.2

List five negative thoughts that you have said to yourself in the past that will hinder your progress on word problems. You may want to write each one on a small piece of paper and rip it up to symbolically destroy the negative thoughts.

Use a Problem-Solving Strategy for Word Problems

In the following exercises, solve using the problem solving strategy for word problems. Remember to write a complete sentence to answer each question.

Exercise 5.9.3

Two-thirds of the children in the fourth-grade class are girls. If there are 20 girls, what is the total number of children in the class?

Answer available at the end of this section.

Exercise 5.9.4

Three-fifths of the members of the school choir are women. If there are 24 women, what is the total number of choir members?

Exercise 5.9.5

Zachary has 25 country music CDs, which is one-fifth of his CD collection. How many CDs does Zachary have?

Answer available at the end of this section.

Exercise 5.9.6

One-fourth of the candies in a bag of M&M's are red. If there are 23 red candies, how many candies are in the bag?

Exercise 5.9.7

There are 16 girls in a school club. The number of girls is four more than twice the number of boys. Find the number of boys.

Answer available at the end of this section.

Exercise 5.9.8

There are 18 Cub Scouts in Pack 645. The number of scouts is three more than five times the number of adult leaders. Find the number of adult leaders.

Exercise 5.9.9

Huong is organizing paperback and hardback books for her club's used book sale. The number of paperbacks is 12 less than three times the number of hardbacks. Huong had 162 paperbacks. How many hardback books were there?

Answer available at the end of this section.

Exercise 5.9.10

Jeff is lining up children's and adult bicycles at the bike shop where he works. The number of children's bicycles is nine less than three times the number of adult bicycles. There are 42 adult bicycles. How many children's bicycles are there?

Exercise 5.9.11

Philip pays $1,620 in rent every month. This amount is $120 more than twice what his brother Paul pays for rent. How much does Paul pay for rent?

Answer available at the end of this section.

Exercise 5.9.12

Marc just bought an SUV for $54,000. This is $7,400 less than twice what his wife paid for her car last year. How much did his wife pay for her car?

Exercise 5.9.13

Laurie has $46,000 invested in stocks and bonds. The amount invested in stocks is $8,000 less than four times the amount invested in bonds. How much does Laurie have invested in bonds?

Answer available at the end of this section.

Exercise 5.9.14

Erica earned a total of $50,450 last year from her two jobs. The amount she earned from her job at the store was $1,250 more than three times the amount she earned from her job at the college. How much did she earn from her job at the college?

Solve Number Problems

In the following exercises, solve each number word problem.

Exercise 5.9.15

The sum of a number and eight is 12. Find the number.

Answer available at the end of this section.

Exercise 5.9.16

The sum of a number and nine is 17. Find the number.

Exercise 5.9.17

The difference of a number and 12 is three. Find the number.

Answer available at the end of this section.

Exercise 5.9.18

The difference of a number and eight is four. Find the number.

Exercise 5.9.19

The sum of three times a number and eight is 23. Find the number.

Answer available at the end of this section.

Exercise 5.9.20

The sum of twice a number and six is 14. Find the number.

Exercise 5.9.21

The difference of twice a number and seven is 17. Find the number.

Answer available at the end of this section.

Exercise 5.9.22

The difference of four times a number and seven is 21. Find the number.

Exercise 5.9.23

Three times the sum of a number and nine is 12. Find the number.

Answer available at the end of this section.

Exercise 5.9.24

Six times the sum of a number and eight is 30. Find the number.

Exercise 5.9.25

One number is six more than the other. Their sum is 42. Find the numbers.

Answer available at the end of this section.

Exercise 5.9.26

One number is five more than the other. Their sum is 33. Find the numbers.

Exercise 5.9.27

The sum of two numbers is 20. One number is four less than the other. Find the numbers.

Answer available at the end of this section.

Exercise 5.9.28

The sum of two numbers is 27. One number is seven less than the other. Find the numbers.

Exercise 5.9.29

The sum of two numbers is –45. One number is nine more than the other. Find the numbers.

Answer available at the end of this section.

Exercise 5.9.30

The sum of two numbers is –61. One number is 35 more than the other. Find the numbers.

Exercise 5.9.31

The sum of two numbers is –316. One number is 94 less than the other. Find the numbers.

Answer available at the end of this section.

Exercise 5.9.32

The sum of two numbers is –284. One number is 62 less than the other. Find the numbers.

Exercise 5.9.33

One number is 14 less than another. If their sum is increased by seven, the result is 85. Find the numbers.

Answer available at the end of this section.

Exercise 5.9.34

One number is 11 less than another. If their sum is increased by eight, the result is 71. Find the numbers.

Exercise 5.9.35

One number is five more than another. If their sum is increased by nine, the result is 60. Find the numbers.

Answer available at the end of this section.

Exercise 5.9.36

One number is eight more than another. If their sum is increased by 17, the result is 95. Find the numbers.

Exercise 5.9.37

One number is one more than twice another. Their sum is –5. Find the numbers.

Answer available at the end of this section.

Exercise 5.9.38

One number is six more than five times another. Their sum is six. Find the numbers.

Exercise 5.9.39

The sum of two numbers is 14. One number is two less than three times the other. Find the numbers.

Answer available at the end of this section.

Exercise 5.9.40

The sum of two numbers is zero. One number is nine less than twice the other. Find the numbers.

Exercise 5.9.41

The sum of two consecutive integers is 77. Find the integers.

Answer available at the end of this section.

Exercise 5.9.42

The sum of two consecutive integers is 89. Find the integers.

Exercise 5.9.43

The sum of two consecutive integers is –23. Find the integers.

Answer available at the end of this section.

Exercise 5.9.44

The sum of two consecutive integers is –37. Find the integers.

Exercise 5.9.45

The sum of three consecutive integers is 78. Find the integers.

Answer available at the end of this section.

Exercise 5.9.46

The sum of three consecutive integers is 60. Find the integers.

Exercise 5.9.47

Find three consecutive integers whose sum is –36.

Answer available at the end of this section.

Exercise 5.9.48

Find three consecutive integers whose sum is –3.

Exercise 5.9.49

Find three consecutive even integers whose sum is 258.

Answer available at the end of this section.

Exercise 5.9.50

Find three consecutive even integers whose sum is 222.

Exercise 5.9.51

Find three consecutive odd integers whose sum is 171.

Answer available at the end of this section.

Exercise 5.9.52

Find three consecutive odd integers whose sum is 291.

Exercise 5.9.53

Find three consecutive even integers whose sum is −36.

Answer available at the end of this section.

Exercise 5.9.54

Find three consecutive even integers whose sum is −84.

Exercise 5.9.55

Find three consecutive odd integers whose sum is −213.

Answer available at the end of this section.

Exercise 5.9.56

Find three consecutive odd integers whose sum is −267.

Everyday Math

Exercise 5.9.57

Sale Price Patty paid $35 for a purse on sale for $10 off the original price. What was the original price of the purse?

Answer available at the end of this section.

Exercise 5.9.58

Sale Price Travis bought a pair of boots on sale for $25 off the original price. He paid $60 for the boots. What was the original price of the boots?

Exercise 5.9.59

Buying in Bulk Minh spent $6.25 on five sticker books to give his nephews. Find the cost of each sticker book.

Answer available at the end of this section.

Exercise 5.9.60

Buying in Bulk Alicia bought a package of eight peaches for $3.20. Find the cost of each peach.

Exercise 5.9.61

Price before Sales Tax Tom paid $1,166.40 for a new refrigerator, including $86.40 tax. What was the price of the refrigerator?

Answer available at the end of this section.

Exercise 5.9.62

Price before Sales Tax Kenji paid $2,279 for a new living room set, including $129 tax. What was the price of the living room set?

Writing Exercises

Exercise 5.9.63

What has been your past experience solving word problems?

Answer available at the end of this section.

Exercise 5.9.64

When you start to solve a word problem, how do you decide what to let the variable represent?

Exercise 5.9.65

What are consecutive odd integers? Name three consecutive odd integers between 50 and 60.

Answer available at the end of this section.

Exercise 5.9.66

What are consecutive even integers? Name three consecutive even integers between −50 and −40.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
approach word problems with a positive attitude.			
use a problem solving strategy for word problems.			
solve number problems.			

ⓑ If most of your checks were:

...confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

...with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential—every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

...no—I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

Answers

Exercise 5.9.1
Answers will vary

Exercise 5.9.3
30

Exercise 5.9.5
125

Exercise 5.9.7
6

Exercise 5.9.9
58

Exercise 5.9.11
$750

Exercise 5.9.13
$13,500

Exercise 5.9.15
4

Exercise 5.9.17
15

Exercise 5.9.19
5

Exercise 5.9.21
12

Exercise 5.9.23
-5

Exercise 5.9.25
18, 24

Exercise 5.9.27
8, 12

Exercise 5.9.29

−18,−27

Exercise 5.9.31

−111,−205

Exercise 5.9.33

32, 46

Exercise 5.9.35

23, 28

Exercise 5.9.37

−2,−3

Exercise 5.9.39

4, 10

Exercise 5.9.41

38, 39

Exercise 5.9.43

−11,−12

Exercise 5.9.45

25, 26, 27

Exercise 5.9.47

−11,−12,−13

Exercise 5.9.49

84, 86, 88

Exercise 5.9.51

55, 57, 59

Exercise 5.9.53

−10,−12,−14

Exercise 5.9.55

−69,−71,−73

Exercise 5.9.57

$45

Exercise 5.9.59

$1.25

Exercise 5.9.61

$1080

Exercise 5.9.63

answers will vary

Exercise 5.9.65

Consecutive odd integers are odd numbers that immediately follow each other. An example of three consecutive odd integers between 50 and 60 would be 51, 53, and 55.

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6: MATHEMATICAL REASONING

6.1: Introduction on Mathematical Reasoning
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6.1: Introduction on Mathematical Reasoning

Mathematical reasoning plays a crucial role in problem-solving by providing a structured approach to analyzing and solving complex problems. It allows individuals to break down a problem into smaller, more manageable parts, identify patterns and relationships, and apply logical reasoning to reach a solution. This process helps in developing a systematic problem-solving strategy that can be applied to a wide range of problems, from simple arithmetic calculations to advanced mathematical concepts. Moreover, mathematical reasoning enhances decision-making skills by enabling individuals to evaluate options, consider different scenarios, and choose the most effective course of action based on logical reasoning and evidence.

Furthermore, mathematical reasoning is essential for understanding complex concepts by providing a framework for organizing and synthesizing information. It helps individuals make sense of abstract ideas and relationships that may initially seem confusing or overwhelming. By applying mathematical reasoning, individuals can identify underlying principles and patterns that govern complex systems, leading to a deeper understanding of the world around them. Additionally, mathematical reasoning fosters creativity by encouraging individuals to think outside the box and explore unconventional solutions to problems, which can lead to innovative breakthroughs and advancements in various fields.

What is Mathematical Reasoning?

Definition: Mathematical Reasoning

Mathematical reasoning is the process of thinking logically about mathematics. It involves analyzing mathematical concepts, making connections between them, and applying logical principles to solve problems and draw conclusions. Mathematical reasoning is not just about performing calculations; it is about understanding the underlying principles and relationships that govern mathematical systems. It requires careful thought and attention to detail, as well as the ability to construct logical arguments and justify conclusions based on evidence. Mathematical reasoning is essential for understanding advanced mathematical concepts and theories, as well as for solving complex problems in a wide range of fields, from science and engineering to economics and finance.

Mathematical reasoning plays a significant role in various aspects of everyday life, including cooking, sports, and shopping. In cooking, mathematical reasoning is used when measuring ingredients and adjusting recipes. When following a recipe, cooks must carefully measure ingredients to ensure the right balance of flavors and textures. This involves understanding fractions and proportions to scale ingredients up or down based on the desired serving size. Additionally, cooks often need to adjust recipes based on personal preferences or ingredient availability, requiring them to use mathematical reasoning to make accurate adjustments.

In sports, mathematical reasoning is used in calculating scores and analyzing statistics. For example, in basketball, players and coaches use statistics such as field goal percentage, free throw percentage, and points per game to evaluate performance and make strategic decisions. These statistics are calculated using mathematical formulas, requiring players and coaches to apply mathematical reasoning to interpret the data effectively. Mathematical reasoning is also used in sports betting, where individuals use probabilities and statistics to make informed decisions about which teams or players to bet on.

In shopping, mathematical reasoning is used when comparing prices and calculating discounts. When comparing prices, shoppers need to calculate the unit price of items to determine which offers the best value for money. This involves dividing the price of an item by its quantity to find the cost per unit. Similarly, when calculating discounts, shoppers need to understand percentages and apply mathematical reasoning to determine the final price of an item after the discount is applied. Overall, mathematical reasoning is an essential skill in everyday life, helping individuals make informed decisions and solve practical problems in a variety of contexts.

Why is Mathematical Reasoning Important?

Mathematical reasoning is a powerful tool for developing critical thinking skills because it requires individuals to analyze, evaluate, and solve complex problems using logical reasoning. When faced with a mathematical problem, individuals must first understand the problem, identify relevant information, and formulate a plan to solve it. This process involves critical thinking skills

such as analysis, interpretation, and inference. Additionally, mathematical reasoning requires individuals to evaluate different strategies for solving a problem and choose the most effective one based on logical reasoning and evidence. By practicing mathematical reasoning, individuals can enhance their critical thinking skills, which are essential for success in academics, careers, and everyday life.

Furthermore, mathematical reasoning encourages individuals to think creatively and explore multiple approaches to problem-solving. It teaches individuals to look beyond the obvious solution and consider alternative methods and perspectives. This ability to think outside the box is a key aspect of critical thinking, as it allows individuals to approach problems from different angles and develop innovative solutions. Additionally, mathematical reasoning teaches individuals to evaluate the validity of their conclusions and consider the implications of their decisions. This aspect of critical thinking is essential for making informed choices and solving complex problems in a wide range of contexts. Overall, mathematical reasoning is a valuable tool for developing critical thinking skills, as it challenges individuals to think logically, creatively, and analytically about mathematical concepts and problems.

Something to think about...

1st Grade: Mathematical reasoning is important when students learning to count and understand number relationships. For example, they use reasoning to understand that if they have three apples and give one away, they will have two left.

2nd Grade: Mathematical reasoning helps students understand place value and how numbers can be decomposed into tens and ones. For example, they use reasoning to understand that 47 is composed of 4 tens and 7 ones.

3rd Grade: Mathematical reasoning is crucial for understanding multiplication and division. Students use reasoning to figure out that if they know 5 x 3 = 15, then they also know that 15 ÷ 5 = 3.

4th Grade: Mathematical reasoning is important for understanding fractions and decimals. They use reasoning to compare fractions and understand that 0.25 is the same as 1/4.

5th Grade: Mathematical reasoning is essential for understanding more complex concepts like ratios and proportions. Students use reasoning to solve problems like if it takes 3 hours to mow 2 lawns, then it will take 6 hours to mow 4 lawns.

Helping Students Develop Mathematical Reasoning Skills

Elementary school teachers play a crucial role in helping students develop mathematical reasoning skills. Here are some strategies they can use:

- **Problem-solving**: Engage students in solving a variety of math problems that require reasoning and critical thinking. Encourage them to explain their thought process and justify their solutions.
- **Real-world applications**: Use real-world examples to illustrate mathematical concepts and their relevance. This helps students see the practical applications of math and how it is used in everyday life.
- **Mathematical discussions**: Encourage students to discuss their strategies and reasoning with their peers. This helps them gain new perspectives and refine their own reasoning skills.
- **Multiple representations**: Present mathematical concepts using different representations such as visual models, manipulatives, and symbolic notation. This helps students develop a deeper understanding of the concepts and strengthens their reasoning skills.
- **Open-ended tasks**: Assign open-ended tasks that allow for multiple approaches and solutions. This encourages students to think creatively and develop their own problem-solving strategies.
- **Scaffolding**: Provide support and guidance as students work on more complex problems. Gradually remove scaffolding as they become more confident in their reasoning abilities.
- **Reflective journaling**: Ask students to keep a journal where they reflect on their problem-solving processes, challenges they faced, and strategies they used. This helps them become more aware of their reasoning skills and how to improve them.
- **Promote a growth mindset:** Encourage students to see mistakes as opportunities for learning and to persevere when faced with challenging problems. Emphasize that mathematical reasoning is a skill that can be developed with practice and effort.
- **Use manipulatives and visual aids:** Use concrete objects and visual representations to help students understand abstract mathematical concepts. This can help them see the patterns and relationships that underlie mathematical ideas.
- **Encourage mathematical discussions:** Foster a classroom environment where students feel comfortable sharing their

ideas and reasoning processes. Encourage them to explain their thinking and listen to and learn from their peers.

- **Assign open-ended tasks:** Give students tasks that have multiple solutions or require them to explain their reasoning. This encourages them to think creatively and develop their own problem-solving strategies.
- **Provide feedback:** Give students feedback on their reasoning processes, highlighting both correct reasoning and areas for improvement. Encourage them to reflect on their thinking and consider different approaches.
- **Relate math to real life:** Help students see the relevance of mathematical reasoning by connecting it to real-world situations. This can help them understand why mathematical reasoning is important and how it can be applied outside the classroom.
- **Differentiate instruction:** Recognize that students may be at different levels of mathematical reasoning development and tailor instruction to meet their individual needs. Provide additional support or challenges as needed.
- **Model mathematical reasoning:** Demonstrate your own reasoning processes when solving problems or explaining concepts. This can help students see how mathematical reasoning is applied in practice and inspire them to develop their own reasoning skills.

Mathematical reasoning is a foundational skill that is essential for success in mathematics and many other areas of life. It enables individuals to think critically, solve complex problems, and make informed decisions based on evidence and logic. By developing their mathematical reasoning skills, students not only become better mathematicians but also better equipped to navigate the challenges of the modern world. It is therefore crucial for educators to foster the development of mathematical reasoning skills in their students through engaging and challenging activities that encourage them to think deeply and creatively about mathematical concepts. With strong mathematical reasoning skills, students can approach any problem with confidence and find innovative solutions that benefit themselves and society as a whole.

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6.2: Types of Reasoning

The study of reasoning is essential for developing a sound foundation in mathematics and critical thinking. It enables individuals to analyze complex problems, formulate logical arguments, and make informed decisions based on evidence. This textbook is designed to offer a structured and comprehensive approach to understanding the different types of reasoning, providing readers with the tools and knowledge necessary to apply these principles effectively in academic, professional, and everyday contexts. Through a systematic study of deductive, inductive, and abductive reasoning, readers will gain a deeper appreciation for the role of reasoning in problem-solving and decision-making processes.

Deductive Reasoning

Definition: Deductive Reasoning

Deductive reasoning is a fundamental aspect of mathematics. It involves starting with a set of premises or known facts and using logical principles to reach a conclusion. For example, if we know that all squares have four sides and that a rectangle is a type of square, we can deduce that all rectangles also have four sides. Deductive reasoning is used in mathematical proofs to establish the validity of mathematical statements and theorems.

Deductive reasoning is an important concept for elementary school students to understand as it lays the foundation for logical thinking and problem-solving skills. In elementary mathematics, deductive reasoning is used to draw conclusions based on known facts and rules. By teaching deductive reasoning, teachers can help students develop critical thinking skills and apply them to mathematical concepts.

Key Concepts:

- **Premises:** In elementary mathematics, premises are the facts, rules, or properties that students know to be true. For example, students may know that all squares have four sides.
- **Conclusion:** The conclusion is the logical result that can be drawn from the premises. For example, if students know that a shape has four sides and all shapes with four sides are squares, they can conclude that the shape is a square.
- **Logic Chains:** Deductive reasoning often involves creating a chain of logic, where each step in the reasoning process follows logically from the previous step. For example, in a geometry problem, students may use deductive reasoning to show that if one angle in a triangle is a right angle, then the other two angles must add up to 90 degrees.

Grade-Level Examples of Deductive Reasoning

- **1st Grade**: All birds have wings. A robin is a bird. Therefore, a robin has wings.
- **2nd Grade**: All squares have four sides. This shape is a square. Therefore, this shape has four sides.
- **3rd Grade**: All even numbers end in 0, 2, 4, 6, or 8. 14 is an even number. Therefore, 14 ends in 4.
- **4th Grade**: All multiples of 5 end in 0 or 5. 35 is a multiple of 5. Therefore, 35 ends in 0 or 5.
- **5th Grade**: All right angles measure 90 degrees. This angle measures 90 degrees. Therefore, this is a right angle.

Teaching Strategies:

- Use concrete examples and manipulatives to illustrate deductive reasoning. For example, use geometric shapes to show how deductive reasoning can be used to classify shapes.
- Encourage students to explain their reasoning process and justify their conclusions. This helps them develop a deeper understanding of the concept.
- Provide opportunities for students to practice deductive reasoning skills through problem-solving activities and puzzles.

Real-World Applications:

- Deductive reasoning is used in everyday life when making decisions based on known facts. For example, if a student knows

that all cats have fur and sees an animal with fur, they can deduce that the animal is a cat.
- Deductive reasoning is also used in fields such as science and engineering to make predictions and draw conclusions based on existing knowledge.

Teaching deductive reasoning in elementary mathematics helps students develop important critical thinking skills that are valuable in both mathematics and everyday life. By providing students with opportunities to practice deductive reasoning, teachers can help them become more effective problem solvers and logical thinkers.

Inductive Reasoning

Definition: Inductive Reasoning

Inductive reasoning is the process of making generalizations based on specific observations or patterns. It is used to predict outcomes or infer general rules from limited data. For example, if you observe that the sun has risen every morning for as long as you can remember, you might use inductive reasoning to conclude that the sun will rise tomorrow morning as well. Inductive reasoning is used in scientific research, data analysis, and everyday problem solving.

Inductive reasoning is a method of reasoning that involves making generalizations based on specific observations or patterns. In elementary mathematics, inductive reasoning is used to identify patterns, make predictions, and formulate hypotheses. By teaching inductive reasoning, teachers can help students develop their problem-solving skills and deepen their understanding of mathematical concepts.

Key Concepts:

- **Observations:** Inductive reasoning starts with specific observations or examples. For example, students may observe that the first five terms of a sequence are 2, 4, 6, 8, 10.
- **Pattern Recognition:** Students use inductive reasoning to recognize patterns in the observations. For example, students may notice that each term in the sequence is 2 more than the previous term.
- **Generalization:** Based on the observed pattern, students make a generalization or hypothesis about the entire sequence. For example, students may hypothesize that the nth term of the sequence is given by the formula 2n.

Grade-Level Examples of Inductive Reasoning

- **1st Grade**: The sun rose in the east yesterday. The sun rose in the east today. The sun will rise in the east tomorrow.
- **2nd Grade**: 2, 4, 6, and 8 are all even numbers and are all divisible by 2. Therefore, all even numbers are divisible by 2.
- **3rd Grade**: The number 3 is prime. The number 5 is prime. The number 7 is prime. Therefore, all odd numbers between 1 and 10 are prime.
- **4th Grade**: A square has four equal sides and four right angles. All squares we've seen fit this description. Therefore, all squares have four equal sides and four right angles.
- **5th Grade**: The fractions 1/2, 2/4, and 3/6 are all equivalent. Therefore, fractions with the same value for numerator/denominator ratios are equivalent.

Teaching Strategies:

- Encourage students to look for patterns in numbers, shapes, and other mathematical objects.
- Provide opportunities for students to make predictions based on observed patterns.
- Guide students in formulating hypotheses and testing them through further observations or calculations.

Real-World Applications:

- Inductive reasoning is used in scientific research to formulate hypotheses based on observed patterns or phenomena.
- It is also used in fields such as data analysis and market research to make predictions based on observed trends or patterns.

Teaching inductive reasoning in elementary mathematics helps students develop important problem-solving skills and enhances their ability to make informed decisions based on evidence. By providing students with opportunities to practice inductive

reasoning, teachers can help them become more effective learners and critical thinkers.

Abductive Reasoning

Definition: Abductive Reasoning

Abductive reasoning is a form of logical inference that seeks to find the simplest and most likely explanation for a set of observations. It involves forming hypotheses to explain observed phenomena and then testing those hypotheses to see if they hold true. For example, if you come home and find your front door ajar, you might use abductive reasoning to hypothesize that someone has broken into your house. Abductive reasoning is used in detective work, scientific inquiry, and problem-solving.

Abductive reasoning is a type of logical inference that involves forming hypotheses to explain observed phenomena. It is used to make educated guesses or predictions based on incomplete information. In elementary mathematics, abductive reasoning is used to make conjectures and formulate explanations for patterns or observations. By teaching abductive reasoning, teachers can help students develop their problem-solving skills and enhance their ability to think critically.

Key Concepts:

- **Observations:** Abductive reasoning starts with specific observations or data. For example, students may observe that a certain pattern exists in a sequence of numbers.
- **Hypothesis Formation:** Based on the observations, students form hypotheses or explanations for the observed pattern. For example, students may hypothesize that the pattern in the sequence is caused by a specific rule or formula.
- **Testing Hypotheses:** Students test their hypotheses by applying them to new examples or data. If the hypothesis holds true for the new examples, it may be accepted as a valid explanation for the observed pattern.

Grade-Level Examples of Abductive Reasoning

- **1st Grade**: If a classroom is like a beehive, the teacher is like the queen bee.
- **2nd Grade**: A pencil is to writing as a paintbrush is to painting.
- **3rd Grade**: If a day is like a story, then the morning is the beginning, the afternoon is the middle, and the night is the end.
- **4th Grade**: If fractions are like pieces of a pizza, then 1/2 is like cutting the pizza into two equal parts.
- **5th Grade**: If a plant's roots are like a foundation, then a plant's leaves are like the building's roof.

Teaching Strategies:

- Encourage students to think creatively and explore different possible explanations for patterns or observations.
- Provide opportunities for students to test their hypotheses through experimentation or by applying them to new examples.
- Guide students in evaluating the validity of their hypotheses based on evidence and logical reasoning.

Real-World Applications:

- Abductive reasoning is used in scientific research to generate hypotheses and theories to explain observed phenomena.
- It is also used in fields such as medicine and law enforcement to make educated guesses or predictions based on limited information.

Teaching abductive reasoning in elementary mathematics helps students develop their critical thinking skills and enhances their ability to formulate explanations and make predictions based on evidence. By providing students with opportunities to practice abductive reasoning, teachers can help them become more effective problem solvers and develop a deeper understanding of mathematical concepts.

Analogical Reasoning

Definition: Analogical Reasoning

Analogical reasoning involves using analogies, or comparisons between two things, to draw conclusions or solve problems. It is based on the idea that if two things are similar in some respects, they are likely to be similar in other respects as well. Analogical reasoning is often used to make predictions, solve unfamiliar problems, or gain new insights by comparing a known situation or concept to an unknown one.

Analogical reasoning is a type of reasoning that involves using analogies, or comparisons between two things, to draw conclusions or solve problems. It is based on the idea that if two things are similar in some respects, they are likely to be similar in other respects as well. In elementary mathematics, analogical reasoning is used to apply known concepts or strategies to new situations. By teaching analogical reasoning, teachers can help students develop their problem-solving skills and enhance their understanding of mathematical concepts.

Key Concepts:

- **Analogies:** Analogical reasoning involves identifying similarities between two things and using those similarities to draw conclusions or make predictions. For example, students may use the analogy of addition to understand multiplication as repeated addition.
- **Transfer of Knowledge:** Analogical reasoning involves transferring knowledge or strategies from a familiar context to an unfamiliar context. For example, students may use their knowledge of fractions to understand decimals.
- **Creative Thinking:** Analogical reasoning encourages students to think creatively and explore different ways of applying known concepts or strategies to new situations.

Grade-Level Examples of Analogical Reasoning

- **1st Grade**: A bee is like a bird because both can fly
- **2nd Grade**: A clock is like a calendar because both help us keep track of time.
- **3rd Grade**: A plant is like a sponge because both absorb water.
- **4th Grade**: An atom is like a solar system because both have a central nucleus with objects orbiting around it.
- **5th Grade**: A cell is like a factory because both have different parts that work together to perform functions.

Teaching Strategies:

- Encourage students to look for similarities between different mathematical concepts or problems.
- Provide opportunities for students to apply known concepts or strategies to new situations.
- Guide students in making connections between different mathematical ideas and using analogies to solve problems.

Real-World Applications:

- Analogical reasoning is used in everyday life to solve problems and make decisions based on similarities between different situations.
- It is also used in fields such as engineering and design to apply known principles or strategies to new problems.

Teaching analogical reasoning in elementary mathematics helps students develop their problem-solving skills and enhance their ability to transfer knowledge from one context to another. By providing students with opportunities to practice analogical reasoning, teachers can help them become more effective learners and develop a deeper understanding of mathematical concepts.

Comparing the Types of Reasoning

While deductive reasoning is often seen as the gold standard of reasoning because it leads to certain conclusions, inductive and abductive reasoning are also valuable tools. Inductive reasoning allows us to make educated guesses and predictions based on past experiences, while abductive reasoning helps us make sense of complex situations by forming plausible explanations. Each

type of reasoning has it's strengths and limitations, and knowing when to use each type is an important skill for mathematicians and critical thinkers.

Aspect	Deductive Reasoning	Inductive Reasoning	Abductive Reasoning	Analogical Reasoning
Starting Point	General Premise	Specific Observations	Observations or Data	Observations or Data
Process	General to Specific	Specific to General	Specific to General	Specific to General
Conclusion	Certain	Probable	Plausible	Plausible
Use	Proving	Predicting	Explaining	Applying
Example	All men are mortal. Socrates is a man. Therefore, Socrates is mortal.	Every cat I have seen has fur. Therefore, all cats have fur.	The grass is wet. It must have rained.	Understanding multiplication as repeated addition.
Application	Mathematics, Logic	Scientific Research	Problem-Solving	Problem-Solving

Deductive reasoning is about certainty, starting with general premises to reach specific conclusions; inductive reasoning deals with probability, starting with specific observations to form general conclusions; abductive reasoning is about plausibility, starting with observations to find the most likely explanation; and analogical reasoning involves using analogies to draw parallels between different concepts or situations. Each type of reasoning has its strengths and applications, and they can be used together to enhance problem-solving and decision-making skills.

Conclusion

Understanding the different types of reasoning in mathematics is essential for becoming a proficient problem solver and critical thinker. By mastering deductive, inductive, and abductive reasoning, you will be better equipped to tackle the challenges of mathematics and apply your reasoning skills to a wide range of real-world problems. I encourage you to think about how you use these types of reasoning in your own life and how you can continue to develop and refine your reasoning skills. Thank you for joining me today, and I look forward to our future discussions on this fascinating topic.

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6.3: Logical Connectives and Statements

In mathematics, logical connectives are used to combine or modify statements to form compound statements. Understanding logical connectives is essential for constructing logical arguments and solving problems in mathematics. This section will introduce you to common logical connectives, such as "and," "or," "not," and "if...then," and show you how to use them to form compound statements.

Definition: Statement

A **statement** is a declarative sentence that is either true or false, but not both. It is a sentence that makes a claim or expresses a fact. In the example "I will eat an apple and a banana but not an orange," the entire sentence is a statement because it expresses a specific intention or plan.

Logical Connectives

"And" (Conjunction)

Logical conjunction is a fundamental concept in logic and mathematics that plays a crucial role in combining statements to form more complex expressions. At its core, conjunction is a logical operation that connects two statements using the word "and." Understanding conjunction is essential for constructing precise statements and reasoning about relationships between different ideas.

Definition: Conjunction

A **conjunction** is a logical operation that combines two statements using the word "and." In logic, a conjunction is true only if both statements it connects are true. For example, the statement "It is sunny and I am going to the park" is a conjunction. It is true only if both parts, "It is sunny" and "I am going to the park," are true. If either part is false, the entire conjunction is false.

In logic, a conjunction is considered true only when both statements it connects are true. This concept is similar to the idea of a joint condition or requirement. For example, if we say "It is sunny and warm," we are expressing a conjunction where both the condition of being sunny and the condition of being warm must be true for the entire statement to be true. Understanding how conjunction works is key to building logical arguments and analyzing complex statements.

Grade-Level Examples for Conjunction

1st Grade: "I have 3 apples and 2 oranges. How many fruits do I have in total?"

- **Explanation:** This statement uses the word "and" to tell us that the person has both apples and oranges. We need to count all the fruits together to find out how many there are in total.

2nd Grade: "Sarah has 5 pencils and 4 erasers. How many school supplies does she have in all?"

- **Explanation:** Here, the word "and" tells us that Sarah has both pencils and erasers. To find out how many school supplies she has in total, we need to add the number of pencils to the number of erasers.

3rd Grade: "There are 7 red balls and 3 blue balls in a box. How many balls are there in total?"

- **Explanation:** The word "and" in this sentence tells us that there are both red balls and blue balls in the box. To find out how many balls there are in total, we need to add the number of red balls to the number of blue balls.

4th Grade: "A rectangle has a length of 5 units and a width of 3 units. What is the perimeter of the rectangle?"

- **Explanation:** In this sentence, "and" tells us that the rectangle has both a length and a width. To find the perimeter, we need to add up all the sides, which are the length and the width added together twice.

5th Grade: "A recipe requires 2 cups of flour and 1 cup of sugar. How many cups of ingredients are needed in total?"

- **Explanation:** The word "and" here tells us that the recipe needs both flour and sugar. To find out how many cups of ingredients are needed, we need to add the amount of flour to the amount of sugar.

Teaching Tips for Conjunction "AND":

1. Use visual aids such as pictures or diagrams to illustrate the concept of combining two things with "and."
2. Provide concrete examples that are relatable to students' experiences, such as "I have a cat and a dog" or "I like both pizza and ice cream."
3. Encourage students to create their own sentences using "and" to combine ideas or items.
4. Use hands-on activities where students physically combine objects to understand the concept of "and."

Real-World Application of Conjunction "AND":

- In cooking, you might use "and" to describe adding multiple ingredients to a recipe, like "Add flour and sugar to the bowl."
- In sports, you could use "and" to describe a player's skills, such as "She is fast and agile."
- In planning a trip, you might use "and" to list items to pack, like "Pack sunscreen and a hat for the beach."
- In science, you could use "and" to describe two conditions necessary for an experiment, such as "Heat and pressure are needed to create a diamond."

The conjunction "and" is a fundamental concept in language and mathematics that allows us to combine ideas, actions, or conditions. By understanding how "and" works, students can form more complex sentences, solve problems, and make logical connections between different concepts. The ability to use "and" effectively is not only important for communication but also for developing critical thinking skills. Through practice and application, students can master the use of "and" and enhance their overall understanding of language and mathematics.

"Or" (Disjunction)

Logical disjunction, represented by the word "or," is a fundamental concept in logic and mathematics. It allows us to express alternatives, choices, and possibilities in statements. In everyday language, we often use "or" to present options, and in logic, disjunction serves a similar purpose, indicating that at least one of the options is true.

Definition: Disjunction

Disjunction is a logical operation that combines two statements using the word "or." In logic, a disjunction is true if at least one of the statements it connects is true. If both statements are true, the disjunction is also true. The only case where a disjunction is false is when both statements are false.

Understanding logical disjunction is crucial for constructing and interpreting statements that involve multiple conditions or choices. It helps us analyze situations where outcomes depend on different possibilities. By mastering the concept of disjunction, we can make clearer arguments, solve complex problems, and reason effectively about various scenarios.

Grade-Level Examples for Disjunction

1st Grade: "You can have 2 cookies or 3 cookies for snack time."

- **Explanation:** This sentence uses the word "or" to give you a choice between two options for the number of cookies you can have.

2nd Grade: "You can solve the math problem by adding or subtracting."

- **Explanation:** Here, the word "or" tells you that you can choose between two different ways to solve the math problem.

3rd Grade: "The number is even or odd."

- **Explanation:** The word "or" in this sentence tells you that the number can be either even or odd, but not both.

4th Grade: "The shape is a triangle or a quadrilateral."

- **Explanation:** This sentence uses the word "or" to describe two possible shapes that the shape could be.

5th Grade: "You can choose to convert the fraction to a decimal or a percent."

- **Explanation:** The word "or" here tells you that you can choose between two different ways to convert the fraction.

Teaching Tips for Disjunction "OR":

1. Use everyday examples to explain the concept of "or," such as choosing between two snacks or deciding on activities for a weekend.
2. Provide opportunities for students to create their own "or" statements, encouraging creativity and critical thinking.
3. Use visual aids, such as Venn diagrams, to illustrate the concept of "or" and show how it represents overlapping or separate possibilities.
4. Reinforce the idea that "or" allows for more than one correct option or possibility.

Real-World Applications of Disjunction "OR":

- In shopping, you might choose between buying a shirt or a dress.
- In weather forecasting, it could be said that it will be sunny or rainy tomorrow.
- In sports, a team could win a game or lose it.
- In transportation, you might take a bus or a train to reach your destination.

The disjunction "or" is a versatile tool that allows us to express choices, alternatives, or possibilities. Understanding how "or" works helps us make decisions, analyze situations, and communicate effectively. By mastering the concept of "or," students can become more adept at problem-solving and critical thinking, both in mathematics and in everyday life.

"Not" (Negation)

Logical negation, often represented by the word "not," is a fundamental concept in logic and mathematics. It is used to express the opposite or denial of a statement. In everyday language, we frequently use negation to indicate a contrary or contradictory idea. For instance, if someone says, "I am going to the store," the negation of that statement would be "I am not going to the store."

Definition: Negation

Negation is a logical operation that reverses the truth value of a statement. In simple terms, it is the act of stating the opposite of a given statement. For example, the negation of the statement "It is sunny" is "It is not sunny."

Understanding logical negation is essential for constructing precise statements and reasoning about the truth of statements. It allows us to express disagreement, contradiction, or the absence of a condition. Mastery of negation enables us to analyze complex arguments, evaluate the validity of statements, and make logical deductions.

Grrade-Level Examples for Negation

1st Grade: "I do not have any apples."

- **Explanation:** This sentence uses "not" to show that the person does not have any apples.

2nd Grade: "The number is not greater than 10."

- **Explanation:** Here, "not" tells us that the number is less than or equal to 10.

3rd Grade: "The shape is not a circle."

- **Explanation:** The word "not" in this sentence tells us that the shape is not a circle.

4th Grade: "The answer is not divisible by 2."

- **Explanation:** This sentence uses "not" to indicate that the answer cannot be divided evenly by 2.

5th Grade: "The angle is not a right angle."

- **Explanation:** Here, "not" tells us that the angle is not 90 degrees.

Teaching Tips for Negation:

- Use visual aids, such as drawings or diagrams, to illustrate the concept of negation.
- Encourage students to use the word "not" in their own sentences to describe things that are not true.
- Provide examples where students must identify whether a statement is true or false based on the use of "not."

Real-World Applications of Negation:

- In cooking, a recipe might say "do not overmix the batter."
- In sports, a coach might say "do not step out of bounds."
- In science, a scientist might say "the substance is not soluble in water."

Negation, represented by the word "not," is used to express the opposite or absence of something. Understanding negation helps us make accurate statements and descriptions, both in math and in everyday life.

"If...Then" (Implication)

Logical implication is a foundational concept in logic and mathematics that expresses a conditional relationship between statements. It is used to infer one statement from another, indicating that if the antecedent is true, then the consequent must also be true. In everyday language, we often use implication to make predictions or draw conclusions based on certain conditions. For example, if someone says, "If it is raining, then I will bring an umbrella," they are implying that if it rains, they will bring an umbrella.

Definition: Implication

Implication is a logical relationship between two statements, where the truth of one statement (the consequent) follows from the truth of another statement (the antecedent). In logic, implication is typically represented by the symbol "$\rightarrow$" (arrow), which can be read as "if...then."

Understanding logical implication is essential for constructing logical arguments, making predictions, and drawing conclusions based on given conditions. It allows us to reason about cause and effect, establish relationships between statements, and evaluate the validity of logical arguments. Mastery of logical implication is key to developing strong analytical and problem-solving

Grade-Level Examples for Implication

1st Grade: "If I have 3 cookies, then I can share with a friend."

- **Explanation:** This sentence uses "if...then" to show that having 3 cookies allows for sharing with a friend.

2nd Grade: "If it is raining, then I will bring an umbrella."

- **Explanation:** Here, "if...then" tells us that if it is raining, the person will bring an umbrella.

3rd Grade: "If the number is even, then it is divisible by 2."

- **Explanation:** This sentence uses "if...then" to show that being even means the number can be divided by 2.

4th Grade: "If the shape has four sides, then it is a quadrilateral."

- **Explanation:** Here, "if...then" tells us that having four sides makes a shape a quadrilateral.

5th Grade: "If the angle is greater than 90 degrees, then it is obtuse."

- **Explanation:** This sentence uses "if...then" to show that angles larger than 90 degrees are obtuse.

Teaching Tips for Implication:

- Use visual aids, such as diagrams or drawings, to illustrate the concept of implication.
- Encourage students to create their own "if...then" statements based on their experiences.
- Provide examples where students must determine the outcome based on the given condition.

Real-World Applications of Implication:

- In weather forecasting, "if it is cloudy, then it might rain."
- In traffic signs, "if the light is red, then stop."
- In sports, "if the team scores a goal, then they win the game."

Implication, represented by "if...then," is used to show a relationship between two conditions or events. Understanding implication helps us make predictions and draw conclusions based on given conditions, both in math and in everyday life.

Combining Logical Connectives

Combining multiple logical connectives allows for the creation of more complex statements with nuanced meanings. For example, consider the statement: "If it is raining, then I will bring an umbrella and wear a raincoat." This statement combines the implication "if...then" with the conjunction "and." It suggests that two actions will be taken based on the condition of rain: bringing an umbrella and wearing a raincoat. Both actions are necessary if it is raining.

Another example is: "I will either go to the park or stay home, but only if it is sunny." This statement combines the disjunction "or" with the conjunction "but only if." It suggests that the speaker has two options—going to the park or staying home—but only if the condition of sunny weather is met. This statement illustrates how multiple connectives can be used to express conditional choices and requirements.

Grade-Level Examples of Combining Logical Statements

1st Grade: "I will eat an apple and a banana but not an orange."

- **Explanation:** This statement tells us that the person plans to eat both an apple and a banana, but they do not plan to eat an orange. They are making a choice about which fruits to eat, and they have decided to eat an apple and a banana, excluding the orange.

2nd Grade: "I will read a book or play outside, but only if it is sunny."

- **Explanation:** This statement combines the logical connectives "or" and "but only if" to indicate a choice between reading a book or playing outside, but only if the condition of sunny weather is met.

3rd Grade: "If I have 10 marbles and I give 5 to my friend, then I will have 5 left."

- **Explanation:** This statement combines the logical connectives "if...then" and "and" to show a sequence of actions: having 10 marbles, giving 5 to a friend, and being left with 5 marbles.

4th Grade: "I will study for the math test or the science test, but not both."

- **Explanation:** This statement combines the logical connectives "or" and "not" to indicate a choice between studying for the math test or the science test, but not both.

5th Grade: "If the number is divisible by 2 and 3, then it is divisible by 6."

- **Explanation:** This statement combines the logical connectives "if...then" and "and" to show a mathematical relationship: if a number is divisible by both 2 and 3, then it is also divisible by 6.

Teaching Tips for Complex Logical Statements

Teaching complex statements involving logical connectives to elementary students can be challenging but rewarding.

1. **Use Concrete Examples:** Use familiar, real-world examples that students can relate to. For example, "If it is sunny, then I will go to the park and play on the swings."
2. **Visual Aids:** Use visual aids such as drawings, diagrams, or manipulatives to illustrate complex statements. This can help students visualize the relationships between different parts of the statement.
3. **Step-by-Step Approach:** Break down complex statements into smaller, more manageable parts. Teach each connective separately before combining them in more complex statements.
4. **Interactive Activities:** Engage students in interactive activities that involve creating and analyzing complex statements. For example, have students create their own "if...then" statements using picture cards.
5. **Encourage Discussion:** Encourage students to discuss their understanding of complex statements with their peers. This can help them clarify their thinking and learn from each other.
6. **Provide Feedback:** Offer specific and constructive feedback on students' work. Highlight correct examples and gently correct any misunderstandings.
7. **Make it Fun:** Use games, puzzles, or storytelling to make learning about logical connectives enjoyable and engaging for students.
8. **Reinforce Learning:** Provide plenty of opportunities for practice and reinforcement. Use worksheets, games, or online activities to reinforce understanding.

By using these teaching tips, you can help elementary students develop a solid understanding of complex statements involving logical connectives.

Real-World Application of Complex Logical Statements

One real-world application of combining logical connectives is in computer programming and circuit design. In these fields, logical connectives are used to create complex logical operations that control the behavior of software and hardware.

For example, consider a simple security system that uses sensors to detect motion and a keypad to enter a code. The system may be programmed with the following logical conditions:

1. **If the motion sensor detects movement and the code entered on the keypad is correct, then the alarm should be**

deactivated.
   - Here, the logical connective "and" is used to combine the conditions that both the motion sensor detects movement and the correct code is entered on the keypad.
2. **If the motion sensor detects movement or the door is opened, then the alarm should be activated.**
   - Here, the logical connective "or" is used to combine the conditions that either the motion sensor detects movement or the door is opened.
3. **If the alarm is activated and the correct disarm code is not entered within 30 seconds, then an alert should be sent to the security company.**
   - Here, the logical connective "and" is again used to combine the conditions that the alarm is activated and the correct disarm code is not entered within 30 seconds.

Combining logical connectives in this way, complex behaviors can be programmed into the security system, allowing it to respond appropriately to different conditions and events.

Conclusion

Logical connectives are powerful tools that help us combine, modify, and reason about statements in mathematics and everyday life. Through the use of "and," "or," "not," and "if...then," we can express complex ideas, make decisions, and draw conclusions based on given conditions. Understanding logical connectives is essential for developing strong problem-solving skills and critical thinking abilities. By mastering these connectives, students can communicate more effectively, make better-informed decisions, and approach mathematical and logical problems with confidence.

Moreover, logical connectives play a crucial role in various real-world applications, from computer programming and engineering to decision-making and everyday communication. They help us express relationships, conditions, and possibilities in a clear and concise manner. By recognizing and applying logical connectives in different contexts, individuals can enhance their logical reasoning skills and navigate complex situations with greater ease and precision.

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6.4: Proof Techniques

Proof techniques are fundamental tools in mathematics that help us demonstrate the truth or validity of mathematical statements. While formal proofs may seem daunting, especially for elementary students, introducing basic proof techniques can help develop their logical reasoning skills and deepen their understanding of mathematical concepts.

Why Introduce Proof Techniques to Grade-School Students?

Introducing proof techniques to grade-school-age students is important for several reasons:

1. **Developing Critical Thinking:** Proof techniques encourage students to think critically and logically. They learn to analyze information, identify patterns, and construct reasoned arguments, skills that are valuable in all areas of life.
2. **Understanding the Foundations of Mathematics:** Proof techniques help students understand why mathematical concepts work the way they do. Instead of just memorizing formulas or rules, students learn the underlying principles behind them.
3. **Preparation for Higher Education:** Proof techniques are fundamental in advanced mathematics and many other fields. By introducing these concepts early, students are better prepared for higher education and future careers.
4. **Building Confidence:** Successfully constructing a proof can be very satisfying and can boost students' confidence in their mathematical abilities.
5. **Encouraging Exploration and Curiosity:** Proof techniques can be used to explore new ideas and concepts. They encourage students to ask questions, seek solutions, and think creatively.

Overall, introducing proof techniques to school-age students not only helps them become better mathematicians but also equips them with valuable skills and mindsets that are applicable across disciplines.

Definition: Proof

A **proof** is a rigorous and logical argument that establishes the truth of a mathematical statement. A proof demonstrates that a statement, proposition, or theorem is true based on a set of axioms, definitions, and previously proven statements or theorems.

Definition: Axiom

An **axiom** is a statement or proposition that is assumed to be true without requiring proof. Axioms are used as the basis for deducing and proving other statements or theorems within a mathematical system. They are considered to be self-evident or so fundamental that they do not need to be proven.

A few of the foundational proof techniques are Direct, Indirect, Contradiction, Induction, and Visual. Direct proof is perhaps the most intuitive, where the truth of a statement is established by straightforward logical reasoning. Indirect proof, also known as proof by contradiction, assumes the negation of the statement and then derives a contradiction, showing that the original statement must be true. Contradiction, similar to indirect proof, involves assuming the truth of a statement and then deriving a contradiction. Induction is a powerful technique used to prove statements about all natural numbers by proving a base case and then showing that if the statement is true for some value, it must also be true for the next value. Visual proofs, while less formal, use diagrams or visual representations to illustrate the truth of a statement, providing a more intuitive understanding of mathematical concepts.

Direct Proof

Direct proof is a method used in mathematics to establish the truth of a statement by providing a logical argument based on known facts or assumptions. It is a straightforward and intuitive approach that involves starting with the statement to be proved and then using logical deductions to arrive at a conclusion. In a direct proof, each step in the argument follows logically from the previous ones, leading to the final conclusion that the statement is true. Direct proof is a fundamental technique that is used in

various branches of mathematics, including algebra, geometry, and number theory. It is an essential skill for students to develop as it helps them build their logical reasoning skills and understand the underlying principles of mathematics. By mastering direct proof, students can become more confident in their ability to solve mathematical problems and develop a deeper appreciation for the beauty and elegance of mathematics.

Definition: Direct Proof

Direct Proof is a straightforward method where we start with given premises and use logical reasoning to reach a conclusion. We use known facts, definitions, and previously proven theorems to justify each step of the proof until we reach the desired conclusion.

Grade-Level Introduction and Example for Direct Proof

4th Grade:

In fourth grade, students can be introduced to direct proof through simple arithmetic statements. For example, to prove that the sum of two even numbers is even, you can use concrete examples such as:

- "Let's consider two even numbers, 4 and 6. We know that an even number is divisible by 2 without any remainder. Since 4 divided by 2 equals 2 and 6 divided by 2 equals 3, when we add them together (4 + 6), we get 10, which is also divisible by 2 without any remainder. Therefore, the sum of two even numbers is even." This example helps students understand the logic behind direct proof and demonstrates how it can be applied to solve mathematical problems.

5th Grade:

In fifth grade, students can further explore direct proof with more complex arithmetic and geometric concepts. For example, to prove that the sum of the interior angles of a triangle is 180 degrees, you can use the following approach:

- "Consider a triangle with three interior angles. We know that a straight line measures 180 degrees. If we extend one side of the triangle, we can form a straight line. Therefore, the sum of the three angles of the triangle must also be 180 degrees. Let's take a triangle with angles measuring 60, 70, and 50 degrees. When we add these angles together (60 + 70 + 50), we get 180 degrees, confirming that the sum of the interior angles of a triangle is always 180 degrees." This example introduces students to more abstract geometric concepts and demonstrates how direct proof can be used to establish mathematical truths about shapes and figures.

6th Grade:

In sixth grade, students can expand their understanding of direct proof to include concepts from algebra and number theory. For example, to prove that the square of an odd number is also odd, you can use the following approach:

- "Let's take an odd number, such as 5. We know that an odd number can be represented as 2n + 1, where n is an integer. If we square 5, we get 25, which can be represented as $(2n+1)^2$. When we expand this expression, we get $4n^2+4n+1$. Since $4n^2+4n$ is always even (as it is divisible by 2), adding 1 to this expression gives us an odd number. Therefore, the square of an odd number is also odd." This example demonstrates how direct proof can be used to prove properties of numbers and algebraic expressions, building on students' previous knowledge and expanding their mathematical reasoning skills.

Indirect Proof (also known as Proof by Contradiction)

Indirect proof, also known as proof by contradiction, is a powerful method used in mathematics to prove the truth of a statement by assuming that the statement is false and then deriving a contradiction. This method relies on the principle that if assuming the opposite of a statement leads to a logical contradiction, then the original statement must be true. Indirect proof is a valuable tool in mathematics because it allows us to prove statements that may be difficult or impossible to prove directly.

Definition: Indirect Proof or (Proof by Contradiction)

Indirect proof, also known as **proof by contradiction,** is a method of proof in mathematics where one assumes the negation of what is to be proved and then shows that this assumption leads to a contradiction. By demonstrating that the assumption of the negation of the statement results in an impossible or self-contradictory situation, one can conclude that the original statement must be true. Indirect proof is based on the principle of reductio ad absurdum, which means "reduction to absurdity," and is a powerful technique used to prove theorems and statements where a direct proof is difficult to construct.

Grade-Level Introduction and Example for Indirect Proof

Introduction to Indirect Proof for 4th Grade:

In mathematics, we often use something called "proof" to show that a statement is true. Sometimes, to prove that something is true, we use a method called "indirect proof." Indirect proof is like solving a puzzle by showing that the opposite of what we think is true leads to a problem, or a contradiction. It's a bit like saying, "If my answer is wrong, then something really strange would have to happen!" Let's explore how indirect proof works with some fun examples.

Example for 4th Grade (Sum of Consecutive Odd Numbers):

Statement: The sum of two consecutive odd numbers cannot be even.

Explanation:

- Let's say we have two consecutive odd numbers, like 3 and 5.
- If we add these numbers together, we get 8, which is even.
- But, we believe that the sum of two consecutive odd numbers should be odd, not even.
- So, we can say, "If the sum of two consecutive odd numbers is even, then something strange is happening!"
- By showing that something strange would have to happen for the sum to be even, we prove that the original statement is true.

Example for 6th Grade (Existence of Infinitely Many Patterns in Pascal's Triangle):

Statement: There are infinitely many patterns in Pascal's Triangle.

Explanation:

- Pascal's Triangle is a triangular arrangement of numbers where each number is the sum of the two numbers directly above it.
- Let's say we've found a few patterns in Pascal's Triangle, like the Fibonacci sequence appearing in the diagonal sums.
- Now, let's suppose there are only a finite number of patterns in Pascal's Triangle.
- We can keep exploring Pascal's Triangle and find new patterns, like the powers of 2 appearing in the rows.
- Since we can keep finding new patterns without end, there must be infinitely many patterns in Pascal's Triangle, contradicting our assumption.
- Therefore, there are infinitely many patterns in Pascal's Triangle.

Indirect proof is a fascinating method in mathematics that allows us to prove the truth of a statement by assuming its opposite and showing that this assumption leads to a contradiction. Through simple and engaging examples, we've seen how indirect proof can be a powerful tool for exploring mathematical concepts and discovering new truths.

We explored the idea that the sum of two consecutive odd numbers cannot be even. By assuming the sum is even and then reaching a contradiction, we demonstrated the truth of the original statement. This illustrates how indirect proof can be used to prove simple mathematical facts in a logical and intuitive way.

We delved into the existence of infinitely many patterns in Pascal's Triangle. By assuming there are only finitely many patterns and then finding new patterns, we showed that the original assumption was incorrect. This showcases how indirect proof can be applied to more complex mathematical ideas, leading to deeper insights and discoveries.

Overall, indirect proof is a valuable tool in the mathematician's toolkit, allowing us to unravel mysteries, prove theorems, and expand our understanding of the mathematical world.

Proof by Induction

Proof by induction is a powerful technique used in mathematics to prove that a statement holds for all natural numbers. It is like setting up a row of dominoes, where we show that if the first domino falls (the statement is true for the smallest number), then all the other dominoes will fall too (the statement is true for all numbers). This method is particularly useful for proving statements that follow a pattern or have a recursive nature. Let's explore how proof by induction works and see some examples to understand it better.

Definition: Proof by Induction (for older grades)

Proof by induction is a mathematical technique used to prove that a statement is true for all natural numbers greater than or equal to some starting point (usually 0 or 1). The proof consists of two steps:

1. **Base Case:** Prove that the statement is true for the starting point. This is typically the simplest case and serves as the base for the induction.
2. **Inductive Step:** Assume that the statement is true for some arbitrary natural number k (this is called the induction hypothesis) and then prove that it must also be true for the next natural number $k+1$. This step is crucial as it establishes the "domino effect" – if the statement is true for one number, it must be true for the next.

Since the base case establishes that the statement is true for the starting point, and the inductive step shows that if it is true for one number then it is true for the next, the statement is proven to be true for all natural numbers greater than or equal to the starting point.

Proof by induction is a powerful and widely used technique in mathematics, especially in areas such as number theory, combinatorics, and algebra.

Grade-Level Introduction and Example for Proof by Induction

Introduction to Proof by Induction for 6th Grade:

In mathematics, we often want to prove that a statement is true for all natural numbers. Proof by induction is a method that allows us to do this efficiently. It's like climbing a ladder: if we can show that we can reach the first rung (the statement is true for the smallest number), and if we can also show that if we can reach any rung, then we can reach the next one (if the statement is true for one number, it's true for the next), then we can conclude that we can reach any rung (the statement is true for all numbers). Let's explore how this method works with a fun example.

Example of Proof by Induction for 6th Grade:

Statement: The sum of the first $n$positive integers is $\frac{n(n+1)}{2}$ for all natural numbers n.

Explanation:

1. **Base Case (n = 1):** We start by checking if the statement is true for the smallest natural number, which is 1. The sum of the first positive integer (1) is indeed 1, and $\frac{1(1+1)}{2} = \frac{2}{2} = 1$ So, the statement holds for $n = 1$.
2. **Inductive Step:** Next, we assume that the statement is true for some arbitrary natural number k, i.e., the sum of the first k positive integers is $\frac{k(k+1)}{2}$.
3. **Proving for $k+1$:** We want to show that the statement is also true for $k+1$, i.e., the sum of the first $k+1$ positive integers is $\frac{(k+1)(k+1)+1)}{2}$. We can write the sum of the first $k+1$ positive integers as the sum of the first k positive integers plus $(k+1)$. By our assumption, the sum of the first k positive integers is $\frac{k(k+1)}{2}$, and adding $(k+1)$ to this gives us:

$$\frac{k(k+1)}{2} + (k+1)$$

$$\frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$\frac{k(k+1)+2(k+1)}{2}$$

$$\frac{k^2+k+2k+2}{2}$$

$$\frac{k^2+3k+2}{2}$$

$$\frac{(k+1)(k+2)}{2}$$

This matches our formula for the sum of the first $k+1$ positive integers. Therefore, if the statement is true for k, it is also true for $k+1$

By the principle of mathematical induction, the statement is true for all natural numbers.

Visual Proof

In mathematics, visual proof is a fascinating approach that uses diagrams, pictures, or geometric shapes to demonstrate the truth of a mathematical statement. Unlike traditional algebraic or symbolic proofs, visual proofs rely on the intuitive understanding of spatial relationships and geometric properties to convey mathematical ideas. This method is particularly effective for illustrating concepts that may be challenging to understand purely through equations or formulas. Let's explore some examples of visual proofs and see how they can help us gain deeper insights into mathematical truths.

Definition: Visual Proof

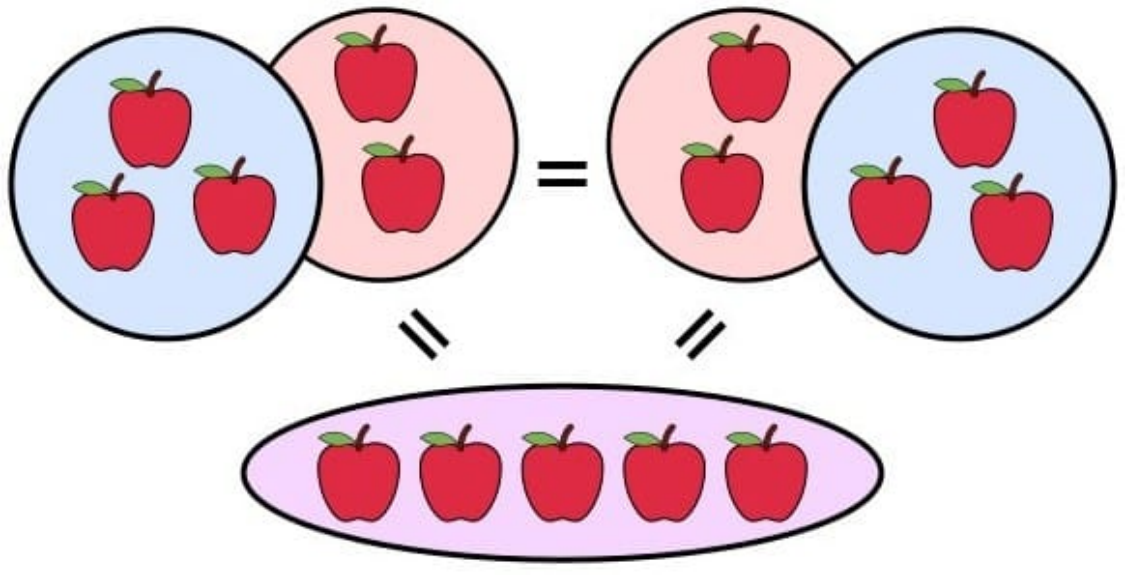
A visual proof, also known as a geometric proof or a pictorial proof, is a method of proving mathematical statements using visual diagrams or illustrations instead of or in addition to traditional algebraic or symbolic reasoning. Visual proofs often rely on the properties of geometric shapes, such as angles, lengths, and areas, to demonstrate the truth of a mathematical statement.

Grade-Level Examples of Visual Proofs

3rd Grade: Commutative Property of Addition

Statement: The order of numbers in an addition expression does not affect the sum. For example, $2+3=3+2$

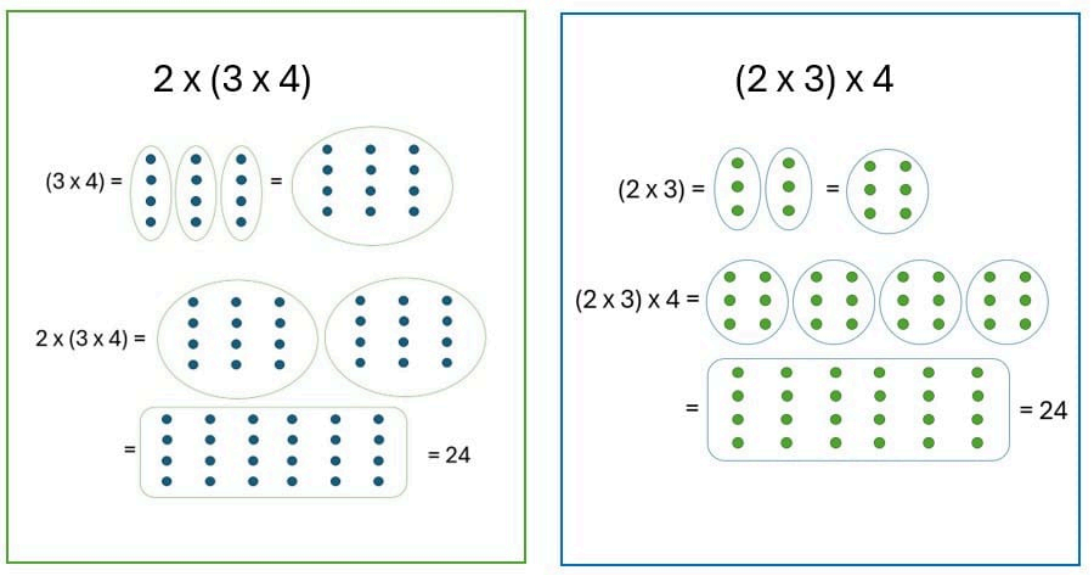
Visual Proof: Use a number line or a set of objects (like blocks or counters) to represent the numbers being added. Show that whether you add 2 and then 3 or 3 and then 2, you end up with the same total number of objects, illustrating the commutative property of addition visually.



4th Grade: Associative Property of Multiplication

Statement: The way in which numbers are grouped in a multiplication expression does not affect the product. For example, $(2 \times 3) \times 4 = 2 \times (3 \times 4)$.

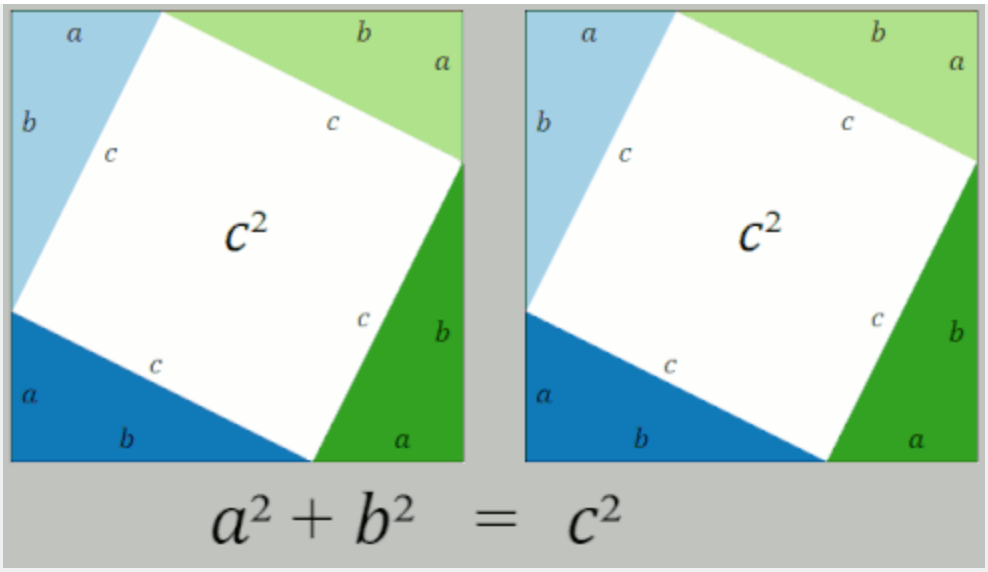
Visual Proof: Use an array of objects (like dots arranged in rows and columns) to represent the multiplication expression. Show that whether you group the objects as $2 \times (3 + 4)$ or $(2 \times 3) \times 4$, you end up with the same total number of objects arranged in the same way, illustrating the associative property of multiplication visually.



5th Grade: Pythagorean Theorem:

Statement: In a right triangle, the square of the length of the hypotenuse, c, is equal to the sum of the squares of the lengths of the other two sides a and b, or $a^2 + b^2 = c^2$.

Visual Proof: Draw a right triangle on a piece of paper. Cut out squares from paper of side lengths equal to the lengths of the three sides of the triangle. Arrange these squares to form a larger square. The area of the larger square (formed by c^2) is equal to the sum of the areas of the two smaller squares (formed by a^2 and b^2), visually demonstrating the Pythagorean Theorem.



Proof techniques are fundamental to the field of mathematics. From direct and indirect proofs to proof by induction and visual proofs, each technique offers a unique approach to establishing the truth of mathematical statements.

By mastering these proof techniques, students not only deepen their understanding of mathematics but also develop valuable problem-solving skills that are applicable across various disciplines. These techniques lay the foundation for further exploration in mathematics and empower students to think critically and analytically about the world around them.

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6.5: Common Logical Fallicies

Understanding logical fallacies is an essential part of developing strong mathematical reasoning skills in elementary school students. Logical fallacies are errors in reasoning that can lead to incorrect conclusions and misunderstandings. These fallacies often sneak into our thinking and arguments without us even realizing it. For young learners, recognizing and avoiding these fallacies can greatly enhance their ability to think critically and solve problems effectively.

In the context of elementary school math, logical fallacies can take various forms. For example, a student might commit an "ad hominem" fallacy by dismissing a peer's solution to a problem simply because they don't like them, rather than evaluating the solution on its own merits. Another common fallacy is the "appeal to authority," where students might accept a mathematical claim as true solely because it was stated by a teacher, without understanding the reasoning behind it. By learning to identify these and other fallacies, such as "false dilemmas" (presenting only two options when more exist) or "hasty generalizations" (making broad conclusions based on limited evidence), students can develop a more disciplined approach to mathematical thinking. This foundation not only supports their current learning but also prepares them for more complex reasoning in higher grades and everyday decision-making.

Fallacies of Relevance

Fallacies of relevance are logical errors that occur when the premises of an argument are not logically related to the conclusion. These fallacies distract from the core issue by introducing irrelevant or emotionally charged information that can mislead the audience. In the context of elementary school mathematics, recognizing fallacies of relevance is essential because they can lead to faulty reasoning and incorrect conclusions.

For young learners, understanding fallacies of relevance helps them stay focused on the logical structure of arguments rather than being swayed by extraneous details. For example, when solving math problems or participating in discussions, students might encounter arguments that divert attention from the actual problem by bringing up unrelated personal traits or emotional appeals. By learning to identify and avoid these fallacies, students can improve their ability to think critically and reason logically, which are fundamental skills not only in mathematics but also in their overall academic and everyday decision-making processes. This awareness fosters a more disciplined approach to evaluating information and constructing sound arguments, ensuring that their conclusions are based on relevant and valid reasoning.

Ad Hominem

Definition: Ad Hominem

The **ad hominem** fallacy occurs when an argument is directed against a person rather than addressing the point they are making. Instead of focusing on the logic or evidence behind the argument, this fallacy attacks the character, motive, or other attributes of the individual presenting the argument. This type of reasoning is flawed because it shifts attention away from the actual issue and does not engage with the argument's merits.

Examples of Grade School Age Situations of Ad Hominem Fallacy

1. **Classroom Discussion:**
   - **Situation:** During a math class, one student suggests a different method to solve a problem. Another student responds, "Why should we listen to you? You didn't even pass the last test."
   - **Explanation:** Instead of considering the validity of the suggested method, the response attacks the student's past performance, which is irrelevant to the current argument.
2. **Group Project:**
   - **Situation:** In a group project on geometry, one student proposes using a specific tool to measure angles. Another student replies, "Your idea can't be good because you always get confused with angles."
   - **Explanation:** The comment dismisses the idea based on the student's past difficulties rather than evaluating the usefulness of the tool.

3. **Playground Debate:**
   - **Situation:** On the playground, children are debating the rules of a new game. One child says, "Your rule doesn't make sense because you're always cheating in games."
   - **Explanation:** Instead of discussing the logic of the proposed rule, the argument attacks the child's character, which doesn't address the actual point about the rule.
4. **Science Fair:**
   - **Situation:** At a science fair, a student presents a project about the solar system. Another student comments, "No one should believe your project because you got a low grade in science last year."
   - **Explanation:** This response targets the student's past academic performance rather than engaging with the content and findings of the project.

These examples illustrate how ad hominem fallacies can appear in everyday situations for grade school students. Teaching children to recognize and avoid such fallacies encourages them to focus on the substance of arguments and engage in more constructive and logical reasoning.

Appeal to Authority

Definition: Appeal to Authority

The **appeal to authority** fallacy occurs when someone argues that a claim is true because an authority figure or expert says it is true, without providing further evidence or reasoning. While it can be reasonable to trust experts in their field, it is important to remember that authority figures can also be wrong or biased. Relying solely on an authority's opinion without considering additional evidence can lead to faulty conclusions.

Examples of Grade School Age Situations of Appeal to Authority Fallacy

1. **Science Class Discussion:**
   - **Situation:** During a discussion about the solar system, a student says, "Pluto is not a planet because my science book says so."
   - **Explanation:** The student is appealing to the authority of the science book without providing any additional evidence or reasoning.
2. **Math Homework Dispute:**
   - **Situation:** Two students are arguing about the correct way to solve a math problem. One student says, "My older brother, who is in high school, told me this is the right answer."
   - **Explanation:** The student is appealing to the authority of their older brother without providing a logical explanation for why the answer is correct.
3. **Sports Debate:**
   - **Situation:** During recess, students are discussing who the best basketball player is. One student says, "LeBron James is the best because my dad, who knows a lot about basketball, said so."
   - **Explanation:** The student is appealing to the authority of their dad without considering other factors or opinions.
4. **History Class Debate:**
   - **Situation:** In a discussion about historical events, a student says, "The Civil War started in 1861 because my history teacher told me."
   - **Explanation:** The student is appealing to the authority of their history teacher without providing any historical evidence to support the claim.

These examples demonstrate how the appeal to authority fallacy can manifest in grade school age situations. Teaching children to critically evaluate information and arguments, even if they come from authority figures, helps them develop stronger critical thinking skills and avoid relying on unsupported claims.

Appeal to Emotion

Definition: Appeal to Emotion

The appeal to emotion fallacy occurs when someone manipulates others' emotions—such as fear, pity, or joy—to win an argument, rather than using valid reasoning. This fallacy can be persuasive, as it plays on people's feelings rather than their ability to think critically. It is important to recognize this fallacy and base arguments on logic and evidence rather than emotional manipulation.

Examples of Grade School Age Situations of Appeal to Emotion Fallacy

1. **Persuasive Writing Assignment:**
   - **Situation:** A student writes an essay arguing for longer recess time, stating, "We deserve more playtime because it's not fair that we have to sit inside all day."
   - **Explanation:** The student is appealing to the emotion of fairness to support their argument, rather than presenting logical reasons for longer recess.
2. **Classroom Debate:**
   - **Situation:** During a debate on school uniforms, a student argues against uniforms by saying, "Uniforms are terrible because they make us all look the same, and no one wants that!"
   - **Explanation:** The student is appealing to the desire for individuality and uniqueness to persuade others, rather than addressing the practical reasons for or against uniforms.
3. **Sharing Time:**
   - **Situation:** A student shares a personal story during class about a difficult experience they had, unrelated to the current topic, in an attempt to gain sympathy or attention.
   - **Explanation:** The student is appealing to the emotions of sympathy or empathy to divert attention from the main discussion or lesson.
4. **Campaign for Class President:**
   - **Situation:** A student running for class president promises to bring a puppy to school every day if elected, saying, "Wouldn't it be amazing to have a cute puppy to play with during recess?"
   - **Explanation:** The student is appealing to the positive emotions associated with having a puppy to distract from more substantive campaign issues.

These examples show how the appeal to emotion fallacy can be used in grade school-age situations to sway opinions without relying on sound reasoning. Teaching children to recognize when emotions are being used to manipulate them can help them make more informed and rational decisions.

Straw Man

Definition: Straw Man

The straw man fallacy occurs when someone distorts or misrepresents their opponent's argument to make it easier to attack. Instead of addressing the actual argument, the person attacks a weaker or distorted version of it. This fallacy can be misleading and unfair, as it fails to engage with the real issues or points of contention.

Examples of Grade School Age Situations of Straw Man Fallacy

1. **Math Class Discussion:**
   - **Situation:** During a discussion about different problem-solving strategies, a student suggests using a particular method. Another student responds, "That method is too complicated and boring. We should just use the easier method everyone knows."
   - **Explanation:** The second student is misrepresenting the first student's argument by exaggerating its complexity and implying that it is unnecessary, making it easier to dismiss.
2. **Debate Club:**

- **Situation:** In a debate about the benefits of homework, one team argues that homework helps students practice and reinforce what they learn in class. The opposing team responds by saying, "The other team wants to give students hours of extra work to do at home every night, which is unfair and stressful."
- **Explanation:** The opposing team is misrepresenting the first team's argument by portraying it as a desire to overload students with excessive homework, making their own position seem more reasonable in comparison.

3. **Social Studies Project Presentation:**
   - **Situation:** A student presents a project arguing for the preservation of a historical site. Another student, who disagrees with the preservation, responds by saying, "So you're saying we should never build anything new and just keep everything old?"
   - **Explanation:** The second student is misrepresenting the first student's argument by presenting it as an extreme position, making it easier to argue against.
4. **Student Council Election:**
   - **Situation:** During a campaign speech, one candidate promises to improve school lunches by adding more healthy options. A rival candidate responds by saying, "The other candidate wants to take away all the foods you love and force you to eat nothing but vegetables."
   - **Explanation:** The rival candidate is misrepresenting their opponent's position by presenting it as an extreme and unappealing choice, making their own position seem more favorable.

Understanding fallacies of relevance is essential for elementary school students to develop strong critical thinking skills. By recognizing these fallacies, students can avoid being swayed by irrelevant or emotionally charged arguments. Teaching students to identify these fallacies empowers them to think critically and make reasoned judgments, both in mathematics and in their everyday lives.

Fallacies of Ambiguity

Fallacies of ambiguity arise when unclear or misleading language is used in an argument, causing confusion and misunderstanding. These fallacies occur because key terms or phrases can be interpreted in multiple ways, leading to incorrect conclusions. In elementary school mathematics, it is crucial for students to understand how ambiguity can affect their reasoning and problem-solving processes.

For young learners, recognizing and avoiding ambiguous language helps ensure clarity and precision in their mathematical thinking. When students encounter problems or explanations that use vague or dual-meaning terms, they might draw incorrect inferences or struggle to find the right solution. By focusing on clear definitions and consistent use of terminology, students can avoid these pitfalls and develop stronger, more accurate mathematical arguments. Understanding fallacies of ambiguity equips students with the skills to critically evaluate the language used in problems and discussions, fostering better communication and deeper comprehension in mathematics and beyond. This awareness not only enhances their current learning but also builds a solid foundation for tackling more complex concepts in higher grades.

Equivocation

Amphiboly

Fallacies of Presumption

Fallacies of presumption occur when an argument is based on assumptions that have not been proven or are taken for granted without sufficient evidence. These fallacies often lead to faulty conclusions because the underlying premises are not adequately supported. In elementary school mathematics, understanding fallacies of presumption is vital as they can lead students to incorrect solutions and hinder their ability to reason logically.

For young learners, recognizing fallacies of presumption involves being critical of the assumptions they and others make when solving problems or constructing arguments. It is essential for students to learn to question the validity of these assumptions and seek proper justification for them. By doing so, they can avoid drawing conclusions that are not grounded in solid reasoning. This critical approach not only improves their problem-solving skills in mathematics but also encourages a habit of thorough and reflective thinking. Developing an awareness of fallacies of presumption helps students build more robust and logically sound arguments, laying the groundwork for more advanced mathematical thinking and effective decision-making in everyday situations.

Begging the Question

False Cause

False Dilemma

Hasty Generalization

Slippery Slope

Definition: Hasty Generalization

Hasty Generalization is a fallacy where a conclusion is drawn from insufficient or biased evidence. It occurs when we make a general statement based on a limited number of examples.

Definition: False Cause

False Cause is a fallacy where a causal connection is assumed between two events that are not actually related. This fallacy often arises from a misunderstanding of correlation and causation.

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6.6: Strategies for Teaching Mathematical Reasoning

Mathematical reasoning is a fundamental skill that allows students to understand and make sense of mathematical concepts. It involves critical thinking, problem-solving, and the ability to justify and explain one's thinking. There are various strategies that elementary school teachers can use to foster mathematical reasoning in their students.

1. Encourage Exploration and Curiosity

Foster an environment where students feel comfortable exploring mathematical concepts and asking questions. Provide opportunities for open-ended exploration and encourage students to be curious and inquisitive.

Activity Example: Pattern Exploration

- Present students with various patterns (e.g., number patterns, geometric patterns) and ask them to describe and extend the patterns.
- Encourage students to create their own patterns and explain their reasoning.

2. Use Real-World Problems

Incorporate real-world problems that are relevant to students' lives. This helps them see the practical application of mathematics and engages them in meaningful problem-solving.

Activity Example: Grocery Store Math

- Give students a list of items with prices and a budget.
- Ask them to select items to purchase without exceeding the budget.
- Have them explain their choices and calculations.

3. Promote Mathematical Discussions

Create opportunities for students to discuss their mathematical thinking with peers. Encourage them to explain their reasoning, listen to others, and ask questions.

Activity Example: Math Talks

- Present a problem and have students solve it individually.
- Organize small group discussions where students share their solutions and reasoning.
- Facilitate a whole-class discussion to highlight different strategies and solutions.

4. Use Visual Representations

Incorporate visual aids such as diagrams, charts, and models to help students understand abstract concepts. Visual representations can make complex ideas more accessible and easier to grasp.

Activity Example: Fraction Models

- Use fraction bars or pie charts to represent different fractions.
- Ask students to compare fractions using the visual models.
- Have them explain their comparisons and reasoning.

5. Encourage Multiple Strategies

Encourage students to solve problems using different strategies. This helps them see that there are multiple ways to approach a

problem and deepens their understanding of mathematical concepts.

Activity Example: Addition Strategies

- Present an addition problem (e.g., 27 + 15).
- Ask students to solve it using at least two different strategies (e.g., breaking apart numbers, using a number line).
- Have them explain each strategy and discuss which one they prefer and why.

6. Provide Opportunities for Reflection

Give students time to reflect on their mathematical thinking and learning. Reflection helps them consolidate their understanding and recognize areas where they need further practice.

Activity Example: Math Journals

- Have students keep a math journal where they write about what they learned, strategies they used, and any challenges they faced.
- Encourage them to reflect on their progress and set goals for improvement.

Assessing Mathematical Reasoning

Formative Assessment

Use formative assessments to gauge students' understanding and reasoning during the learning process. These assessments provide immediate feedback and guide instruction.

Example: Exit Tickets

- At the end of a lesson, ask students to complete a quick problem or write a brief explanation of a concept.
- Review their responses to identify any misconceptions and adjust future lessons accordingly.

Summative Assessment

Summative assessments evaluate students' overall understanding and reasoning at the end of a unit or course. These assessments should include problems that require students to explain their thinking and justify their solutions.

Example: Performance Tasks

- Design tasks that involve real-world scenarios and require students to apply multiple mathematical concepts.
- Include questions that ask students to explain their reasoning and the steps they took to arrive at their solution.

Creating a Supportive Learning Environment

Build a Growth Mindset

Encourage a growth mindset by praising effort and perseverance rather than just correct answers. Help students see mistakes as opportunities for learning and growth.

Example: Positive Reinforcement

- Use phrases like "I can see you worked really hard on this problem" and "Great job trying a different strategy!"

Foster a Collaborative Culture

Promote collaboration and teamwork in the classroom. Create opportunities for students to work together, share ideas, and learn

from one another.

Example: Group Projects

- Assign group projects where students must work together to solve a complex problem.
- Encourage them to divide tasks, discuss strategies, and present their findings as a team.

Provide Differentiated Instruction

Recognize that students have different learning needs and provide differentiated instruction to support all learners. Offer various levels of support and challenge to ensure that every student can engage in mathematical reasoning.

Example: Tiered Assignments

- Create assignments with different levels of difficulty.
- Allow students to choose the level that challenges them appropriately and provides opportunities for growth.

By tailoring strategies to each grade level, you can effectively support the development of mathematical reasoning skills in your students. These activities and approaches will help them build a strong foundation in critical thinking and problem-solving, preparing them for future success in mathematics.

Teaching mathematical reasoning is essential for developing students' critical thinking and problem-solving skills. By using strategies that promote exploration, discussion, and reflection, teachers can create a learning environment that supports and enhances mathematical reasoning. Remember to assess students' reasoning regularly and provide differentiated instruction to meet their diverse needs. With these strategies, you can help your students build a strong foundation in mathematical reasoning that will serve them well throughout their education and beyond.

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6.7: Practical Applications

By understanding these practical applications, students can see the value of mathematical reasoning beyond the classroom and develop the skills they need to succeed in a variety of fields.

These examples illustrate how mathematical reasoning can be applied to real-world situations, helping students understand its practical significance in everyday life.

What type of reasoning skill does each situation use?

Exercise 6.7.1

First Grade Situations:

1. A student is sorting toys into two boxes. The teacher says, "Put the cars in the blue box and the dolls in the red box."
2. At a grocery store, there are two checkout lines. One line has three people waiting, and the other line has five people waiting. A student observes, "If I join the line with three people, I will wait less than if I join the line with five people."
3. A child notices that every time they add one more block to their tower, it gets taller. They think, "If I keep adding blocks, my tower will keep getting taller."
4. A student is counting apples. They start with 3 apples and their friend gives them 2 more. The student thinks, "If I start with 3 apples and get 2 more, I will have 5 apples."

Answers available at the end of this section.

Exercise 6.7.2

Second Grade Situations:

1. A student observes that every time they skip count by 2s, they get even numbers. They conclude, "If I skip count by 2s, I will always get even numbers."
2. A child is packing lunch and wants to make sandwiches using bread slices. They know they have two slices of bread for each sandwich and six slices of bread in total. The child thinks, "If I make three sandwiches, I will use all the bread."
3. A child is making a sandwich. Their parent says, "You can use peanut butter or jelly, but not both."
4. A teacher shows students a collection of shapes. There are 4 triangles and 4 squares. The teacher asks, "Do we have more than 5 shapes in total?"

Answers available at the end of this section.

Exercise 6.7.3

Third Grade Situations:

1. A group of friends notices that every time they share a pizza equally, each person gets a fair share. They think, "If we divide any pizza equally, everyone will get the same amount."
2. Students are solving a puzzle where they need to find numbers that are both even and greater than 10.
3. A group of friends wants to share 12 cookies equally. One friend says, "If we each take two cookies at a time, we will finish all the cookies."
4. A student observes that every time they add an even number to another even number, the result is always even. They check with 2 + 2 = 4, 4 + 6 = 10, and 8 + 12 = 20. They conclude, "Any time I add two even numbers, the result will always be even."

Answers available at the end of this section.

Exercise 6.7.4

Fourth Grade Situations:

1. A family is planning a road trip. They know that their car gets 30 miles per gallon of gas. The family says, "If we drive 300 miles, we will need 10 gallons of gas.".
2. A student notices that every time they multiply a number by 10, the digit in the ones place becomes 0. They conclude, "If I multiply any number by 10, the ones place digit will always be 0."
3. A student is baking cookies and the recipe says, "Add 2 cups of sugar if you want them sweet, or 1 cup of sugar if you want them less sweet."
4. A student is solving a problem where they need to determine if the sum of two odd numbers is always even. They check with 1 + 3 = 4 and 5 + 7 = 12. They conclude, "If I add two odd numbers, the result is always even."

Answers available at the end of this section.

Exercise 6.7.5

Fifth Grade Situations:

1. In a math class, the teacher says, "If a number is divisible by 4, then it is also divisible by 2."
2. A farmer has a field that is 60 meters long and 40 meters wide. The farmer thinks, "If I plant crops in rows that are 1 meter apart, I can fit 40 rows in the field."
3. A family goes grocery shopping and notices that every time they buy 1 gallon of milk, it costs $3. They think, "If we buy any number of gallons of milk, it will cost $3 per gallon."
4. A student is given a problem to prove that if a number is divisible by 6, then it is divisible by 3. They assume the opposite: "If a number is not divisible by 3, it is not divisible by 6." They check with 6 (divisible by both), 12 (divisible by both), and 14 (not divisible by 6, not divisible by 3). They conclude, "Since 14 is not divisible by 6 and not divisible by 3, our assumption holds."

Answers available at the end of this section.

Exercise 6.7.6

Sixth Grade Situations:

1. A baker observes that every time they double the recipe, they need to use twice the amount of each ingredient. They conclude, "If I double any recipe, I will need to use twice the amount of each ingredient."
2. A student is planning a trip and calculates the budget. They think, "I will spend $50 if I go to the amusement park, or $30 if I go to the zoo."
3. The family uses inductive reasoning to generalize that the cost of milk is $3 per gallon, regardless of the quantity purchased.
4. A student is asked to prove that the area of a rectangle is the same when calculated in different ways. They are given a rectangle with length 5 units and width 3 units. The teacher says, "Draw a rectangle with these dimensions and count the squares inside."

Answers available at the end of this section.

Exercises

Fallacies in Common Language

For each of the following statements, name the type of logical fallacy being used.

1. If you don't want to drive from Boston to New York, then you will have to take the train.
2. Every time I go to Dodger Stadium, the Dodgers win. I should go there for every game.
3. New England Patriots quarterback Tom Brady likes his footballs slightly underinflated. The "Cheatriots" have a history of bending or breaking the rules, so Brady must have told the equipment manager to make sure that the footballs were underinflated.
4. What you are doing is clearly illegal because it's against the law.
5. The county supervisor voted against the new education tax. He must not believe in education.
6. "Apples a day keeps doctors away." No one has said apples are bad, so this old saying must be true.
7. Wine has to be good for your health because... I mean, can you imagine a life without wine?
8. Studies show that slightly overweight senior citizens live longer than underweight ones. The extra weight must make people live longer.
9. Whenever our smoke detector beeps, my kids eat cereal for dinner. The loud beeping sound must make them want to eat cereal for some reason.
10. There is a scientist who works at a really good university, and he says there is no strong evidence for climate change, especially global warming. Some politicians also question climate change. So I don't really believe it.
11. My neighbor cheats on his tax returns. I don't believe anything he says.
12.

    A: "Don't fight over small things. Just let them go."

    B: "What exactly are 'small things'? How do I know what is small and what isn't?"

    A: "Well, small things are things you really don't want to fight over."

Propositions and Logic

13. List the set of integers that satisfy the following statement: A positive multiple of 5 and not a multiple of 2
14. List the set of integers that satisfy the following statement: Greater than 12 and less than or equal to 18
15. List the set of integers that satisfy the following statement: Even number less than 10 or odd number between 12 and 10
16. You qualify for a special discount if you are either

- a full-time student in the state of California or
- at least 18 and your income is less than $20,000 a year.

For each person below, determine if the person qualifies for this discount. If more information is needed, indicate that.

a. A 17-year-old full-time student at a California community college with no job
b. A 28-year-old man earning $50,000 a year
c. A 60-year-old grandmother who does not work and does not go to school
d. A boy in first grade
e. A mother making $18,000 a year and not enrolled in any college
f. A 22-year-old full-time student at Arizona State University with no job
g. A 18-year-old earning exactly $20,000 a year while attending UCLA part-time

17. Write the negation: Everyone failed the quiz today.
18. Write the negation: Someone in the car needs to use the restroom.
19. How can you prove this statement wrong?

    "Everyone who ate at that restaurant got sick."

20. How can you prove this statement wrong?

    "There was someone who ate at that restaurant and got sick."

21. How can you prove this statement wrong?

    "There is no baseball player who can excel at both pitching and hitting. Everyone must choose one or the other."

22. How can you prove this statement wrong?

"Every student must have an ID number before registering for classes."

Analyzing Arguments

39. Determine whether each of the following is an inductive or deductive argument:
    a. The new medicine works. We tried it on 100 patients, and all of them were cured.
    b. Every student has an ID number. Sandra is a student, so she has an ID number.
    c. Every angle of a rectangle is 90 degrees. A soccer field (pitch) is rectangular, so every corner is 90 degrees.
    d. Everything that goes up comes down. I throw a ball up. It must come down.
    e. Every time it rains, my grass grows fast. Rain speeds up the growth of grass.
    f. Sports makes a person strong. My daughter plays basketball. She will be strong.
40. Analyze the validity of the argument.

    Everyone who gets a degree in science will get a good job. I got a science degree. Therefore, I will get a good job.

41. Analyze the validity of the argument.

    If someone turns off the switch, the lights will not be on. The lights are off. Therefore, someone must have turned off the switch.

42. Suppose the statement: "If today is Dec. 25, then the store is closed."
    a. "Today is Dec. 25. Thus, the store is closed." Which property was used? Is this valid?
    b. "The store is not closed. So today is not Dec. 25." Which property was used? Is this valid?
    c. "The store is closed. Therefore, today must be Dec. 25." Which property was used? Is this valid?
    d. "Today is not Dec. 25. Thus, the store is open." Which property was used? Is this valid?

For the following questions, use a Venn diagram or a truth table to determine the validity.

43. If a person is on this reality show, they must be self-absorbed. Laura is not self-absorbed. Therefore, Laura cannot be on this reality show.
44. If you are a triathlete, then you have outstanding endurance. LeBron James is not a triathlete. Therefore, LeBron does not have outstanding endurance.
45. Jamie must scrub the toilets or hose down the garbage cans. Jamie refuses to scrub the toilets. Therefore, Jamie will hose down the garbage cans.
46. Some of these kids are rude. Jimmy is one of these kids. Therefore, Jimmy is rude!
47. Every student brought a pencil or a pen. Marcie brought a pencil. Therefore, Marcie did not bring a pen.
48. If a creature is a chimpanzee, then it is a primate. If a creature is a primate, then it is a mammal. Bobo is a mammal. Therefore, Bobo is a chimpanzee.
49. Every cripsee is a domwow. Mekep is not a domwow. Therefore, Mekep is not a cripsee. (This sentence has a lot of made-up words, but it is still possible to check for the validity of the argument. This is a good practice for abstract thinking.)
50. Whoever dephels a kipoc will be bopied. I did not dephel any kipoc. Therefore, I will not be bopied. (Again, you do not need to know the meaning of each word to do this exercise.)

Answers

Exercise 6.7.1

Solutions:

1. The student uses the connective "and" to follow the instructions correctly, putting cars in the blue box and dolls in the red box.
2. The student uses deductive reasoning to understand that joining the shorter line will likely result in less waiting time.
3. The child uses inductive reasoning to understand that adding more blocks increases the height of the tower.
4. The student uses a direct proof by adding the numbers directly: 3 + 2 = 5, showing they will indeed have 5 apples.

Exercise 6.7.2

Solutions:

1. The student uses inductive reasoning to generalize that skip counting by 2s always results in even numbers.
2. The child uses deductive reasoning to determine that making three sandwiches will use all six slices of bread.
3. The child uses the connective "or" to understand that they can choose either peanut butter or jelly for their sandwich, but not both.
4. The students use a visual proof by counting each shape. They see there are 8 shapes (4 triangles + 4 squares) which is more than 5, proving the statement visually.

Exercise 6.7.3

Solutions:

1. The friends use inductive reasoning to generalize that dividing any pizza equally among them will result in everyone getting the same amount.
2. The students use the connective "and" to identify numbers like 12, 14, and 16 that meet both criteria.
3. The friends use deductive reasoning to understand that taking two cookies each time will result in them finishing all 12 cookies.
4. The student uses inductive reasoning to generalize that adding any two even numbers results in an even number based on specific examples.

Exercise 6.7.4

Solutions:

1. The family uses deductive reasoning based on their car's mileage to calculate the amount of gas needed for a specific distance.
2. The student uses inductive reasoning to generalize that multiplying any number by 10 always results in the ones place digit being 0.
3. The student uses the connective "or" to decide how much sugar to add based on their preference for sweetness.
4. A student is solving a problem where they need to determine if the sum of two odd numbers is always even. They check with 1 + 3 = 4 and 5 + 7 = 12. They conclude, "If I add two odd numbers, the result is always even."

Exercise 6.7.5

Solutions:

1. The student uses the connective "if...then" to understand that any number divisible by 4 (like 8, 12, 16) will also be divisible by 2.
2. The farmer uses deductive reasoning to calculate the number of rows that can fit in the field based on its dimensions and the spacing between rows.
3. The family uses inductive reasoning to generalize that the cost of milk is $3 per gallon, regardless of the quantity purchased.
4. The student uses an indirect proof (contrapositive) to show that if a number is divisible by 6, it must be divisible by 3.

Exercise 6.7.6

Solutions:

1. The baker uses inductive reasoning to generalize that doubling any recipe requires using twice the amount of each ingredient.
2. A student is planning a trip and calculates the budget. They think, "I will spend $50 if I go to the amusement park, or $30 if I go to the zoo."
3. The baker uses deductive reasoning to determine that making 5 cakes will use all 15 cups of flour.
4. The student draws the rectangle and counts 15 squares inside (5 rows of 3 squares each). This visual proof shows that the area calculation (length × width) matches the count of squares inside the rectangle.

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7: LOGIC

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7.1: Introduction

What is logic? Logic is the study of reasoning, and it has applications in many fields, including philosophy, law, psychology, digital electronics, and computer science.

In law, constructing a well-reasoned, logical argument is extremely important. The main goal of arguments made by lawyers is to convince a judge and jury that their arguments are valid and well-supported by the facts of the case, so the case should be ruled in their favor. Think about Thurgood Marshall arguing for desegregation in front of the U.S. Supreme Court during the *Brown v. Board of Education of Topeka* lawsuit in 1954, or Ruth Bader Ginsburg arguing for equality in social security benefits for both men and women under the law during the mid-1970s. Both these great minds were known for the preparation and thoroughness of their logical legal arguments, which resulted in victories that advanced the causes they fought for. Thurgood Marshall and Ruth Bader Ginsburg would later become well respected justices on the U.S. Supreme Court themselves.



Figure 7.1.1: Logic is key to a well-reasoned argument, in both math and law. (credit: modification of work "Lady Justicia holding sword and scale bronze figurine with judge hammer on wooden table" by Jernej Furman/Flickr, CC BY 2.0)

In this chapter, we will explore how to construct well-reasoned logical arguments using varying structures. Your ability to form and comprehend logical arguments is a valuable tool in many areas of life, whether you're planning a dinner date, negotiating the purchase of a new car, or persuading your boss that you deserve a raise.

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7.2: Statements and Quantifiers



Figure 7.2.1: Construction of a logical argument, like that of a house, requires you to begin with the right parts. (credit: modification of work "Barn Raising" by Robert Stinnett/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Identify logical statements.
2. Represent statements in symbolic form.
3. Negate statements in words.
4. Negate statements symbolically.
5. Translate negations between words and symbols.
6. Express statements with quantifiers of all, some, and none.
7. Negate statements containing quantifiers of all, some, and none.

Have you ever built a club house, tree house, or fort with your friends? If so, you and your friends likely started by gathering some tools and supplies to work with, such as hammers, saws, screwdrivers, wood, nails, and screws. Hopefully, at least one member of your group had some knowledge of how to use the tools correctly and helped to direct the construction project. After all, if your house isn't built on a strong foundation, it will be weak and could possibly fall apart during the next big storm. This same foundation is important in logic.

In this section, we will begin with the parts that make up all logical arguments. The building block of any logical argument is a logical statement, or simply a statement. A logical statement has the form of a complete sentence, and it must make a claim that can be identified as being true or false.

When making arguments, sometimes people make false claims. When evaluating the strength or validity of a logical argument, you must also consider the truth values, or the identification of true or false, of all the statements used to support the argument. While a false statement is still considered a logical statement, a strong logical argument starts with true statements.

Identifying Logical Statements



Figure 7.2.2: Not all roses are red. (credit: “assorted pink yellow white red roses macro” by ProFlowers/Flickr, CC BY 2.0)

An example of logical statement with a false truth value is, “All roses are red.” It is a logical statement because it has the form of a complete sentence and makes a claim that can be determined to be either true or false. It is a false statement because not all roses are red: some roses are red, but there are also roses that are pink, yellow, and white. Requests, questions, or directives may be complete sentences, but they are not logical statements because they cannot be determined to be true or false. For example, suppose someone said to you, “Please, sit down over there.” This request does not make a claim and it cannot be identified as true or false; therefore, it is not a logical statement.

Exercise 7.2.1: Identifying Logical Statements

Determine whether each of the following sentences represents a logical statement. If it is a logical statement, determine whether it is true or false.

1. Tiger Woods won the Master’s golf championship at least five times.
2. Please, sit down over there.
3. All cats dislike dogs.

Answer available at the end of this section.

Your Turn 7.2.1

Determine whether each of the following sentences represents a logical statement. If it is a logical statement, determine whether it is true or false.

1. The Buffalo Bills defeated the New York Giants in Super Bowl XXV
2. Michael Jackson’s album *Thriller* was released in 1982.
3. Would you like some coffee or tea?

Representing Statements in Symbolic Form

When analyzing logical arguments that are made of multiple logical statements, symbolic form is used to reduce the amount of writing involved. Symbolic form also helps visualize the relationship between the statements in a more concise way in order to determine the strength or validity of an argument. Each logical statement is represented symbolically as a single lowercase letter,

usually starting with the letter p.

To begin, you will practice how to write a single logical statement in symbolic form. This skill will become more useful as you work with compound statements in later sections.

Exercise 7.2.2: Representing Statements Using Symbolic Form

Write each of the following logical statements in symbolic form.

1. Barry Bonds holds the Major League Baseball record for total career home runs.
2. Some mammals live in the ocean.
3. Ruth Bader Ginsburg served on the U.S. Supreme Court from 1993 to 2020.

Answer available at the end of this section.

Your Turn 7.2.2

Write each of the following logical statements in symbolic form.

1. The movie *Gandhi* won the Academy Award for Best Picture in 1982.
2. Soccer is the most popular sport in the world.
3. All oranges are citrus fruits.

Who Knew?

Mathematics is not the only language to use symbols to represent thoughts or ideas. The Chinese and Japanese languages use symbols known as Hanzi and Kanji, respectively, to represent words and phrases. At one point, the American musician Prince famously changed his name to a symbol representing love.

BBC Prince Symbol Article

Negating Statements

Consider the false statement introduced earlier, “All roses are red.” If someone said to you, “All roses are red,” you might respond with, “Some roses are not red.” You could then strengthen your argument by providing additional statements, such as, “There are also white roses, yellow roses, and pink roses, to name a few.”

The negation of a logical statement has the opposite truth value of the original statement. If the original statement is false, its negation is true, and if the original statement is true, its negation is false. Most logical statements can be negated by simply adding or removing the word *not*. For example, consider the statement, “Emma Stone has green eyes.” The negation of this statement would be, “Emma Stone does not have green eyes.” The table below gives some other examples.

Table 7.2.1

Logical Statement	Negation
Gordon Ramsey is a chef.	Gordon Ramsey is not a chef.
Tony the Tiger does not have spots.	Tony the Tiger has spots.

The way you phrase your argument can impact its success. If someone presents you with a false statement, the ability to rebut that statement with its negation will provide you with the tools necessary to emphasize the correctness of your position.

Exercise 7.2.3: Negating Logical Statements

Write the negation of each logical statement in words.

1. Michael Phelps was an Olympic swimmer.
2. Tom is a cat.
3. Jerry is not a mouse.

Answer available at the end of this section.

Your Turn 7.2.3

Write the negation of each logical statement in words.

1. Ted Cruz was not born in Texas
2. Adele has a beautiful singing voice.
3. Leaves convert carbon dioxide to oxygen during the process of photosynthesis.

The symbol for negation, or not, in logic is the tilde, ~. So, not p is represented as $\sim p$. To negate a statement symbolically, remove or add a tilde. The negation of not (not p) is p. Symbolically, this equation is $\sim(\sim p) = p$.

Exercise 7.2.4: Negating Logical Statements Symbolically

Write the negation of each logical statement symbolically.

1. p: Michael Phelps was an Olympic swimmer.
2. r: Tom is not a cat.
3. $\sim q$: Jerry is not a mouse.

Answer available at the end of this section.

Your Turn 7.2.4

Write the negation of each logical statement symbolically.

1. p: Ted Cruz was not born in Texas
2. q: Adele has a beautiful voice.
3. r: Leaves convert carbon dioxide to oxygen during the process of photosynthesis.

Translating Negations Between Words and Symbols

In order to analyze logical arguments, it is important to be able to translate between the symbolic and written forms of logical statements.

Exercise 7.2.5

Given the statements:

r: Elmo is a red Muppet.

p: Ketchup is not a vegetable.

1. Write the symbolic form of the statement, “Elmo is not a red Muppet.”
2. Translate the statement $\sim p$ into words.

Answer available at the end of this section.

Your Turn 7.2.5

Given the statements:
r: Woody and Buzz Lightyear are best friends.
p: Wonder Woman is not stronger than Captain Marvel.

1. Write the symbolic form of the statement, "Wonder Woman is stronger than Captain Marvel."
2. Translate the statement r into words

Expressing Statements with Quantifiers of All, Some, or None

A quantifier is a term that expresses a numerical relationship between two sets or categories. For example, all squares are also rectangles, but only some rectangles are squares, and no squares are circles. In this example, *all, some,* and *none* are quantifiers. In a logical argument, the logical statements made to support the argument are called premises, and the judgment made based on the premises is called the conclusion. Logical arguments that begin with specific premises and attempt to draw more general conclusions are called inductive arguments.

Consider, for example, a parent walking with their three-year-old child. The child sees a cardinal fly by and points it out. As they continue on their walk, the child notices a robin land on top of a tree and a duck flying across to land on a pond. The child recognizes that cardinals, robins, and ducks are all birds, then excitedly declares, "All birds fly!" The child has just made an inductive argument. They noticed that three different specific types of birds all fly, then synthesized this information to draw the more general conclusion that all birds can fly. In this case, the child's conclusion is false.

The specific premises of the child's argument can be paraphrased by the following statements:

- Premise: Cardinals are birds that fly.
- Premise: Robins are birds that fly.
- Premise: Ducks are birds that fly.

The general conclusion is: “All birds fly!”

All inductive arguments should include at least three specific premises to establish a pattern that supports the general conclusion. To counter the conclusion of an inductive argument, it is necessary to provide a counter example. The parent can tell the child about penguins or emus to explain why that conclusion is false.

On the other hand, it is usually impossible to prove that an inductive argument is true. So, inductive arguments are considered either strong or weak. Deciding whether an inductive argument is strong or weak is highly subjective and often determined by the background knowledge of the person making the judgment. Most hypotheses put forth by scientists using what is called the “scientific method” to conduct experiments are based on inductive reasoning.

In the following example, we will use quantifiers to express the conclusion of a few inductive arguments.

Exercise 7.2.6: Drawing General Conclusions to Inductive Arguments Using Quantifiers

For each series of premises, draw a logical conclusion to the argument that includes one of the following quantifiers: all, some, or none.

1. Squares and rectangles have four sides. A square is a parallelogram, and a rectangle is a parallelogram. What conclusion can be drawn from these premises?

2. $1+2=3, 6+7=13$, and $23+24=47$. Of these, 1 and 2,6 and 7 , and 23 and 24 are consecutive integers; 3,13 , and 47 are odd numbers. What conclusion can be drawn from these premises?

3. Sea urchins live in the ocean, and they do not breathe air. Sharks live in the ocean, and they do not breathe air. Eels live in the ocean, and they do not breath air. What conclusion can be drawn from these premises?

Answer available at the end of this section.

Your Turn 7.2.6

For each series of premises, draw a logical conclusion to the argument that includes one of the following quantifiers: all, some, or none.

1. $1+2=3$, $5+6=11$, and $14+15=29$. Of these, 1 and 2 are consecutive integers, 5 and 6 are consecutive integers, and 14 and 15 are consecutive integers. Also, their sums, 3, 11, and 29 are all prime numbers. Prime numbers are positive integers greater than one that are only divisible by one and the number itself. What conclusion can you draw from these premises?
2. A robin is a bird that lays blue eggs. A chicken is a bird that typically lays white and brown eggs. An ostrich is a bird that lays exceptionally large eggs. If a bird lays eggs, then they do not give live birth to their young. What conclusion can you draw from these premises?
3. All parallelograms have four sides. All rectangles are parallelograms. All squares are rectangles. What additional conclusion can you make about squares from these premises?

Checkpoint

It is not possible to prove definitively that an inductive argument is true or false in most cases.

Negating Statements Containing Quantifiers

Recall that the negation of a statement will have the opposite truth value of the original statement. There are four basic forms that logical statements with quantifiers take on.

- All $\mathbf{A}$ are B.
- Some A are B.
- No A are B.
- Some A are not B.

The negation of logical statements that use the quantifiers *all*, *some*, or *none* is a little more complicated than just adding or removing the word *not*.

For example, consider the logical statement, “All oranges are citrus fruits.” This statement expresses as a subset relationship. The set of oranges is a subset of the set of citrus fruit. This means that there are no oranges that are outside the set of citrus fruit. The negation of this statement would have to break the subset relationship. To do this, you could say, “At least one orange is not a citrus fruit.” Or, more concisely, “Some oranges are not citrus fruit.” It is tempting to say "No oranges are citrus fruit," but that would be incorrect. Such a statement would go beyond breaking the subset relationship, to stating that the two sets have nothing in common. The negation of "[redacted]A is a subset of B" would be to state that "A is not a subset of B," as depicted by the Venn diagram in Figure 7.2.3.

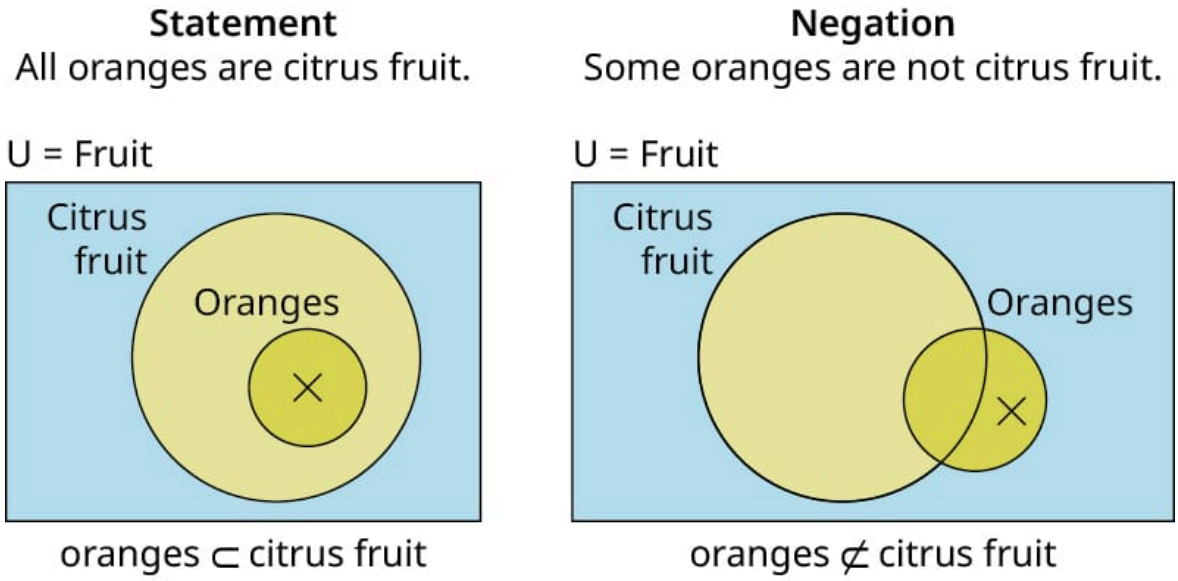


Figure 7.2.3: The statement, “All oranges are citrus fruit,” is true, so its negation, “Some oranges are not citrus fruit,” is false.

Now, consider the statement, “No apples are oranges.” This statement indicates that the set of apples is disjointed from the set of oranges. The negation must state that the two are not disjoint sets, or that the two sets have a least one member in common. Their intersection is not empty. TThe negation of the statement, “A intersection B is the empty set,” is the statement that " A intersection B is not empty," as depicted in the Venn diagram in Figure 7.2.4.

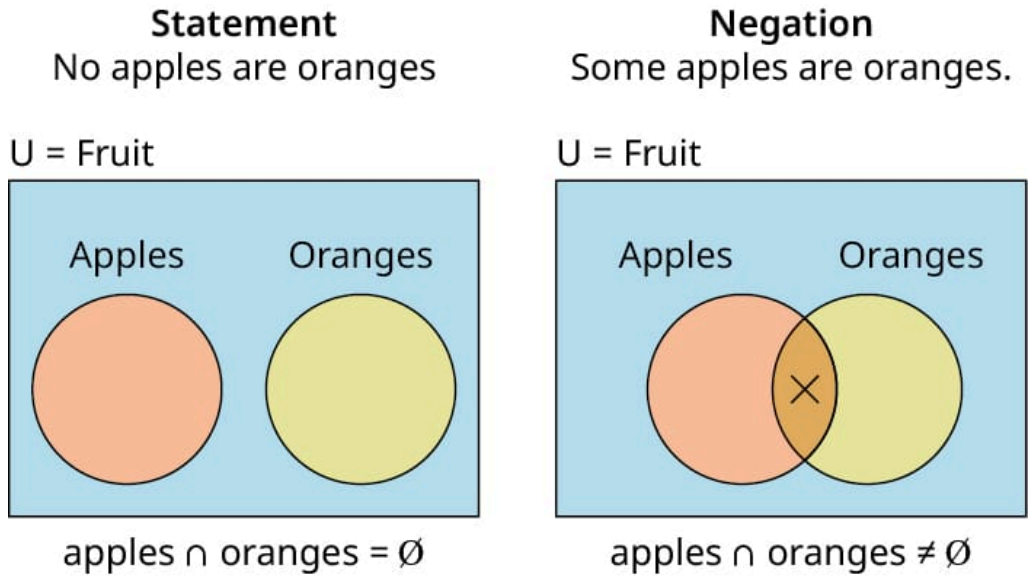


Figure 7.2.4:The negation of the true statement “No apples are oranges,” is the false statement, “Some apples are oranges.”

Table 7.2.2 summarizes the four different forms of logical statements involving quantifiers and the forms of their associated negations, as well as the meanings of the relationships between the two categories or sets A and B.

Table 7.2.2

Logical Statements with Quantifiers	Negation of Logical Statements w/Quantifiers
Form: All A are B. Means: A is a subset of B, $A \subset B$. All zebras have stripes. (True)	Form: Some A are not B. Means: A is not a subset of B, $A \not\subset B$. Some zebras do not have stripes. (False)

Logical Statements with Quantifiers	Negation of Logical Statements w/Quantifiers
Form: Some A are B. Means: A intersection B is not empty, $A \cap B \neq \emptyset$. Some fish are sharks. (True)	Form: No A are B. Means: A intersection B is empty, $A \cap B = \emptyset$. No fish are sharks. (False)
Form: No A are B. Means: A intersection B is empty, $A \cap B = \emptyset$. No trees are evergreens. (False)	Form: Some A are B. Means: A intersection B is not empty, $A \cap B \neq \emptyset$. Some trees are evergreens. (True)
Form: Some A are not B. Means: A is not a subset of B, $A \not\subset B$. Some horses are not mustangs. (True)	Form: All A are B. Means: A is a subset of B, $A \subset B$. All horses are mustangs. (False)

We covered sets in great detail in Chapter 1. To review, " A is a subset of B" means that every member of set A is also a member of set B. The intersection of two sets A and B is the set of all elements that they share in common. If A intersection B is the empty set, then sets A and B do not have any elements in common. The two sets do not overlap. They are disjoint.

YouTube Video

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Logic Part 1A: Logic Statements and Quantifiers

Exercise 7.2.7: Negating Statements Containing Quantifiers *All*, *Some*, or *None*

Given the statements:

p: All leopards have spots.
r: Some apples are red.
s: No lemons are sweet.

Write each of the following symbolic statements in words.

1. $\sim p$
2. $\sim r$

3. $\sim s$

Answer available at the end of this section.

Your Turn 7.2.7

Given the statements:
$\sim p$: Some apples are not sweet.
r : No triangles are squares.
s : Some vegetables are green. Write each of the following symbolic statements in words.

1. p
2. $\sim r$
3. $\sim s$

Check Your Understanding

1. A _____ _____ is a complete sentence that makes a claim that may be either true or false.
2. The _____ of a logical statement has the opposite truth value of the original statement.
3. If p represents the logical statement, “Marigolds are yellow flowers,” then _____ represents the statement, “Marigolds are not yellow flowers.”
4. The statement $(\ p)$ has the same truth value as the statement _____.
5. The logical statements used to support the conclusion of an argument are called _____.
6. _____ arguments attempt to draw a general conclusion from specific premises.
7. All, some, and none are examples of _____, words that assign a numerical relationship between two or more groups.
8. The negation of the statement, “All giraffes are tall,” is _____.

Answers

Exercise 7.2.1: Identifying Logical Statements

1. This is a logical statement because it is a complete sentence that makes a claim that can be identified as being true or false. As of 2021, this statement is true: Tiger Woods won the Master’s in 1997, 2001, 2002, 2005 and 2019.
2. This is not a logical statement because, although it is a complete sentence, this request does not make a claim that can be identified as being either true or false.
3. This is a logical statement because it is a complete sentence that makes a claim that can be identified as being true or false. This statement is false because some cats do like some dogs.

Exercise 7.2.2: Representing Statements Using Symbolic Form

1. *p* : Barry Bonds holds the Major League Baseball record for total career home runs. The statement is labeled with a p. Once the statement is labeled, use p as a replacement for the full written statement to write and analyze the argument symbolically.
2. *q* : Some mammals live in the ocean. The letter q is used to distinguish this statement from the statement in question 1, but any lower-case letter may be used.
3. *r* : Ruth Bader Ginsburg served on the U.S. Supreme Court from 1993 to 2020. When multiple statements are present in later sections, you will want to be sure to use a different letter for each separate logical statement.

Exercise 7.2.3: Negating Logical Statements

1. Add the word *not* to negate the affirmative statement: “Michael Phelps was not an Olympic swimmer.”
2. Add the word *not* to negate the affirmative statement: “Tom is not a cat.”
3. Remove the word *not* to negate the negative statement: “Jerry is a mouse.”

Exercise 7.2.4: Negating Logical Statements Symbolically

1. To negate an affirmative logical statement symbolically, add a tilde: $\sim p$.
2. Because the symbol for this statement is r, its negation is $\sim r$.
3. The symbol for this statement is $\sim q$, so to negate it we simply remove the ~, because $\sim(\sim q) = q$. The answer is q.

Exercise 7.2.5

1. “Elmo is not a red muppet” is the negation of “Elmo is a red muppet,” which is represented as *r*. Thus, we would write “Elmo is not a red muppet” symbolically as *~r*.
2. Because $p$is the symbol representing the statement, “Ketchup is not a vegetable,” *~p* is equivalent to the statement, “Ketchup is a vegetable.”

Exercise 7.2.6: Drawing General Conclusions to Inductive Arguments Using Quantifiers

1. The conclusion you would likely come to here is “Some four-sided figures are parallelograms.” However, it would be incorrect to say that all four-sided figures are parallelograms because there are some four-sided figures, such as trapezoids, that are not parallelograms. This is a false conclusion.

2. From these premises, you may draw the conclusion “All sums of two consecutive counting numbers result in an odd number.” Most inductive arguments cannot be proven true, but several mathematical properties can be. If we let n n represent our first counting number, then $n + 1$ would be the next counting number and $n + (n + 1) = 2 n + 1$. Because $2n$ is divisible by two, it is an even number, and if you add one to any even number the result is always an odd number. Thus, the conclusion is true!
3. Based on the premises provided, with no additional knowledge about whales or dolphins, you might conclude “No creatures that live in the ocean breathe air.” Even though this conclusion is false, it still follows from the known premises and thus is a logical conclusion based on the evidence presented. Alternatively, you could conclude “Some creatures that live in the ocean do not breathe air.” The quantifier you choose to write your conclusion with may vary from another person’s based on how persuasive the argument is. There may be multiple acceptable answers.

Exercise $7.2.7$**: Negating Statements Containing Quantifiers *All*, *Some*, or *None***

1. The statement "All leopards have spots" is p and has the form "All A are B." According to Table $7.2.2$, the negation will have the form "Some A are not B." The negation of p is the statement, "Some leopards do not have spots."
2. The statement "Some apples are red" has the form "Some A are B." This indicates that the categories A and B overlap or intersect. According to Table $7.2.2$, the negation will have the form, "No A are B," indicating that A and B do not intersect. This results in the opposite truth value of the original statement, so the negation of "Some apples are red" is the statement: "No apples are red."
3. Because s is the statement: "No lemons are sweet," s is asserting that the set of lemons does not intersect with the set of sweet things. The negation of $s, \sim s$, must make the opposite claim. It must indicate that the set of lemons intersects with the set of sweet things. This means at least one lemon must be sweet. The statement, "Some lemons are sweet" is $\sim s$. The negation of the statement, "No A are B," is the statement, "Some A are B," as indicated in Table $7.2.2$.

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7.3: Compound Statements



7.3.1:A person seeking their driver's license must pass two tests. A compound statement can be used to explain performance on both tests at once. (credit: modification of work “Drivers License -Teen driver” by State Farm/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Translate compound statements into symbolic form.
2. Translate compound statements in symbolic form with parentheses into words.
3. Apply the dominance of connectives.

Suppose your friend is trying to get a license to drive. In most places, they will need to pass some form of written test proving their knowledge of the laws and rules for driving safely. After passing the written test, your friend must also pass a road test to demonstrate that they can perform the physical task of driving safely within the rules of the law.

Consider the statement: "My friend passed the written test, but they did not pass the road test." This is an example of a compound statement, a statement formed by using a connective to join two independent clauses or logical statements. The statement, “My friend passed the written test,” is an independent clause because it is a complete thought or idea that can stand on its own. The second independent clause in this compound statement is, “My friend did not pass the road test.” The word "but" is the connective used to join these two statements together, forming a compound statement. So, did your friend acquire their driving license?.

This section introduces common logical connectives and their symbols, and allows you to practice translating compound statements between words and symbols. It also explores the order of operations, or dominance of connectives, when a single compound statement uses multiple connectives.

Common Logical Connectives

Understanding the following logical connectives, along with their properties, symbols, and names, will be key to applying the topics presented in this chapter. The chapter will discuss each connective introduced here in more detail.

The joining of two logical statements with the word "and" or "but" forms a compound statement called a conjunction. In logic, for a conjunction to be true, all the independent logical statements that make it up must be true. The symbol for a conjunction is $\wedge$. Consider the compound statement, “Derek Jeter played professional baseball for the New York Yankees, and he was a shortstop.” If p represents the statement, “Derrick Jeter played professional baseball for the New York Yankees,” and if q represents the

statement, “Derrick Jeter was a short stop,” then the conjunction will be written symbolically as $p \wedge q$.

The joining of two logical statements with the word “or” forms a compound statement called a disjunction. Unless otherwise specified, a disjunction is an inclusive *or* statement, which means the compound statement formed by joining two independent clauses with the word *or* will be true if a least one of the clauses is true. Consider the compound statement, "The office manager ordered cake for for an employee’s birthday or they ordered ice cream.” This is a disjunction because it combines the independent clause, “The office manager ordered cake for an employee’s birthday,” with the independent clause, “The office manager ordered ice cream,” using the connective, *or*. This disjunction is true if the office manager ordered only cake, only ice cream, or they ordered both cake and ice cream. Inclusive *or* means you can have one, or the other, or both!

Joining two logical statements with the word *implies*, or using the phrase “if *first statement*, then *second statement,”* is called a conditional or *implication*. The clause associated with the "if" statement is also called the hypothesis or *antecedent*, while the clause following the "then" statement or the word *implies* is called the **conclusion** or *consequent*. The conditional statement is like a one-way contract or promise. The only time the conditional statement is false, is if the hypothesis is true and the conclusion is false. Consider the following conditional statement, “If Pedro does his homework, then he can play video games.” The hypothesis/antecedent is the statement following the word *if*, which is “Pedro does/did his homework.” The conclusion/consequent is the statement following the word *then*, which is “Pedro can play his video games.”

Joining two logical statements with the connective phrase “if and only if” is called a biconditional. The connective phrase "if and only if" is represented by the symbol, $\leftrightarrow$. In the biconditional statement, $p \leftrightarrow q$, p is called the hypothesis or antecedent and q is called the conclusion or consequent. For a biconditional statement to be true, the truth values of p and q must match. They must both be true, or both be false.

The table below lists the most common connectives used in logic, along with their symbolic forms, and the common names used to describe each connective.

Table 7.3.1

Connective	Symbol	Name
and but	$\wedge$	conjunction
or	$\vee$	disjunction, inclusive or
not	~	negation
if ..., then implies	$\rightarrow$	conditional, implication
if and only if	$\leftrightarrow$	biconditional

Checkpoint

These connectives, along with their names, symbols, and properties, will be used throughout this chapter and should be memorized.

Exercise 7.3.1

Associate Connectives with Symbols and Names

For each of the following connectives, write its name and associated symbol.

1. or
2. implies
3. but

Answer available at the end of this section.

Your Turn 7.3.1

For each connective write its name and associated symbol.

1. not
2. and
3. if and only if

Translating Compound Statements to Symbolic Form

To translate a compound statement into symbolic form, we take the following steps.

1. Identify and label all independent affirmative logical statements with a lower case letter, such as p, q, or r.
2. Identify and label any negative logical statements with a lowercase letter preceded by the negation symbol, such as $\sim p$, $\sim q$, or $\sim r$.
3. Replace the connective words with the symbols that represent them, such as $\wedge, \vee, \rightarrow, \text{or } \leftrightarrow$.

Consider the previous example of your friend trying to get their driver's license. Your friend passed the written test, but they did not pass the road test. Let p represent the statement, "My friend passed the written test." And, let $\sim q$ represent the statement, "My friend did not pass the road test." Because the connective *but* is logically equivalent to the word *and*, the symbol for *but* is the same as the symbol for *and*; replace *but* with the symbol $\wedge$. The compound statement is symbolically written as: $p \wedge \sim q$. My friend passed the written test, but they did not pass the road test.

Checkpoint

When translating English statements into symbolic form, do not assume that every connective "and" means you are dealing with a compound statement. The statement, "The stripes on the U.S. flag are red and white," is a simple statement. The word "white" doesn't express a complete thought, so it is not an independent clause and does not get its own symbol. This statement should be represented by a single letter, such as s: The stripes on the U.S. flag are red and white.

Exercise 7.3.2: Translating Compound Statements into Symbolic Form

Let p represent the statement, "It is a warm sunny day," and let q represent the statement, "the family will go to the beach." Write the symbolic form of each of the following compound statements.

1. If it is a warm sunny day, then the family will go to the beach.
2. The family will go to the beach, and it is a warm sunny day.
3. The family will not go to the beach if and only if it is not a warm sunny day.
4. The family not go to the beach, or it is a warm sunny day.

Answer available at the end of this section.

Your Turn 7.3.2

Let p represent the statement, "Last night it snowed," and let q represent the statement, "Today we will go skiing." Write the symbolic form of each of the following compound statements:

1. Today we will go skiing, but last night it did not snow.
2. Today we will go skiing if and only if it snowed last night.
3. Last night is snowed or today we will not go skiing.
4. If it snowed last night, then today we will go skiing.

Translating Compound Statements in Symbolic Form with Parentheses into Words

When parentheses are written in a logical argument, they group a compound statement together just like when calculating numerical or algebraic expressions. Any statement in parentheses should be treated as a single component of the expression. If multiple parentheses are present, work with the inner most parentheses first.

Consider your friend's struggles to get their license to drive. Let p represent the statement, "My friend passed the written test," let q represent the statement, "My friend passed the road test," and let r represent the statement, "My friend received a driver's license." The statement $(p \wedge q) \rightarrow r$ can be translated into words as follows: the statement $p \wedge q$ is grouped together to form the hypothesis of the conditional statement and r is the conclusion. The conditional statement has the form "if $p \wedge q$, then r." Therefore, the written form of this statement is: "If my friend passed the written test and they passed the road test, then my friend received a driver's license."

Sometimes a compound statement within parentheses may need to be negated as a group. To accomplish this, add the phrase, "it is not the case that" before the translation of the phrase in parentheses. For example, using p, q, and r of your friend obtaining a license, let's translate the statement $\sim (p \wedge q) \rightarrow \sim r$ into words.
In this case, the hypothesis of the conditional statement is $\sim (p \wedge q)$ and the conclusion is $\sim r$. To negate the hypothesis, add the phrase "it is not the case" before translating what is in parentheses. The translation of the hypothesis is the sentence, "It is not the case that my friend passed the written test and they passed the road test," and the translation of the conclusion is, "My friend did not receive a driver's license." So, a translation of the complete conditional statement, $\sim (p \wedge q) \rightarrow \sim r$ is: "If it is not the case that my friend passed the written test and the road test, then my friend did not receive a driver's license."

Checkpoint

It is acceptable to interchange proper names with pronouns and remove repeated phrases to make the written statement more readable, as long the meaning of the logical statement is not changed.

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Logic Part 1B: Compound Statements, Connectives and Symbols

Exercise 7.3.3

Let p represent the statement, "My child finished their homework," let q represent the statement, "My child cleaned her room," let r represent the statement, "My child played video games," and let s represent the statement, "My child streamed a movie." Translate each of the following symbolic statements into words.

1. $\sim (p \wedge q)$
2. $(p \wedge q) \rightarrow (r \vee s)$
3. $\sim (r \vee s) \leftrightarrow \sim (p \wedge q)$

Answer available at the end of this section.

Your Turn 7.3.3

Let p represent the statement, "My roommates ordered pizza," let q represent the statement, "I ordered wings," and let r be the statement, "Our friends came over to watch the game." Translate the following statements into words.

1. $\sim r \rightarrow (p \vee q)$
2. $(p \wedge q) \rightarrow r$
3. $\sim (p \vee r)$

The Dominance of Connectives

The order of operations for working with algebraic and arithmetic expressions provides a set of rules that allow consistent results. For example, if you were presented with the problem $1 + 3 \times 2$, and you were not familiar with the order of operation, you might assume that you calculate the problem from left to right. If you did so, you would add 1 and 3 to get 4, and then multiply this

answer by 2 to get 8, resulting in an incorrect answer. Try inputting this expression into a scientific calculator. If you do, the calculator should return a value of 7, not 8.

Use the Scientific Calculator found under the Resources tab to the far left

The order of operations for algebraic and arithmetic operations states that all multiplication must be applied prior to any addition. Parentheses are used to indicate which operation—addition or multiplication—should be done first. Adding parentheses can change and/or clarify the order. The parentheses in the expression $1 + (3 \times 2)$ indicate that 3 should be multiplied by 2 to get 6, and then 1 should be added to 6 to get 7: $1 + (3 \times 2) = 7$.

As with algebraic expressions, there is a set of rules that must be applied to compound logical statements in order to evaluate them with consistent results. This set of rules is called the dominance of connectives. When evaluating compound logical statements, connectives are evaluated from least dominant to most dominant as follows:

1. Parentheses are the least dominant connective. So, any expression inside parentheses must be evaluated first. Add as many parentheses as needed to any statement to specify the order to evaluate each connective.
2. Next, we evaluate negations.
3. Then, we evaluate conjunctions and disjunctions from left to right, because they have equal dominance.
4. After evaluating all conjunctions and disjunctions, we evaluate conditionals.
5. Lastly, we evaluate the most dominant connective, the biconditional. If a statement includes multiple connectives of equal dominance, then we will evaluate them from left to right.

See Figure $7.3.2$ for a visual breakdown of the dominance of connectives.

Dominance	Connective	Symbol	Evaluate
Least Dominant ↓	Parentheses	( )	First ↓
	Negation	~	
	Disjunction/Conjunction	∨, ∧	Left to right or add parentheses to specify order because or/and have equal dominance. ↓
	Conditional	→	
Most Dominant	Biconditional	↔	Last

Figure $7.3.2$

Let's revisit your friend's struggles to get their driver's license. Let p represent the statement, "My friend passed the written test," let q represent the statement, "My friend passed the road test," and let r represent the statement, "My friend received a driver's license." Let's use the dominance of connectives to determine how the compound statement $p \wedge \sim q \rightarrow r$ should be evaluated.

Step 1: There are no parentheses, which is least dominant of all connectives, so we can skip over that.

Step 2: Because negation is the next least dominant, we should evaluate $\sim q$ first. We could add parentheses to the statement to make it clearer which connecting needs to be evaluated first: $p \wedge \sim q \rightarrow r$ is equivalent to $p \wedge (\sim q) \rightarrow r$.

Step 3: Next, we evaluate the conjunction, $\wedge$. $p \wedge (\sim q) \rightarrow r$ is equivalent to $(p \wedge (\sim q)) \rightarrow r$.

Step 4: Finally, we evaluate the conditional, $\rightarrow$, as this is the most dominant connective in this compound statement.

Who Knew?

When using spreadsheet applications, like Microsoft Excel or Google Sheets, have you noticed that core functions, such as sum, average, or rate, have the same name and syntax for use? This is not a coincidence. Various standards organizations have developed requirements that software developers must implement to meet the conditions set by governments and large customers worldwide. The OpenDocument Format OASIS Standard enables transferring data between different office productivity applications and was approved by the International Standards Organization (ISO) and IEC on May 6, 2006.

Prior to adopting these standards, a government entity, business, or individual could lose access to their own data simply because it was saved in a format no longer supported by a proprietary software product—even data they were required by law to preserve, or data they simply wished to maintain access to.

Just as rules for applying the dominance of connectives help maintain understanding and consistency when writing and interpreting compound logical statements and arguments, the standards adopted for database software maintain global integrity and accessibility across platforms and user bases.

Exercise 7.3.4

For each of the following compound logical statements, add parentheses to indicate the order to evaluate the statement. Recall that parentheses are evaluated innermost first.

1. $p \wedge \sim q \vee r$
2. $q \rightarrow \sim p \wedge r$
3. $\sim (p \vee q) \leftrightarrow \sim p \wedge \sim q$

Answer available at the end of this section.

Example 7.3.4

For each of the following compound logical statements, add parentheses to indicate the order in which to evaluate the statement. Recall that parentheses are evaluated innermost first.

1. $p \vee q \wedge \sim r$
2. $\sim p \rightarrow q \vee r$
3. $\sim p \vee \sim q \leftrightarrow \sim (p \wedge q)$

WORK IT OUT: Logic Terms and Properties – Matching Game

Materials: For every group of four students, include 30 cards (game, trading, or playing cards), 30 individual clear plastic gaming card sleeves, and 30 card-size pieces of paper. Prepare a list of 60 questions and answers ahead of time related to definitions and problems in Statements and Quantifiers and Compound Statements. Provide each group of four students with 20 questions and their associated answers. Instruct each group to select 15 of the 20 questions, and then, for each problem selected, create one card with the question and one card with the answer. Instruct the groups to then place each problem and answer in a separate card sleeve. Once they create 15 problem cards and 15 answer cards, have students shuffle the set of cards.

To play the game, groups should exchange their card decks to ensure no team begins playing with the deck that they created. Split each four-person group into teams of two students. After shuffling the cards, one team lays cards face down on their desk in a five-by-six grid. The other team will go first.

Each player will have a turn to try matching the question with the correct answer by flipping two cards to the face up position. If a team makes a match, they get to flip another two cards; if they do not get a match, they flip the cards face down and it is the other team's turn to flip over two cards. The game continues in this manner until teams match all question cards with their corresponding answer cards. The team with the most set of matching cards wins.

In the first module of this chapter, we evaluated the truth value of negations. In the following modules, we will discuss how to evaluate conjunctions, disjunctions, conditionals, and biconditionals, and then evaluate compound logical statements using truth tables and our knowledge of the dominance of connectives.

Check Your Understanding

1. A _____ _____ is a logical statement formed by combining two or more statements with connecting words, such as and, or, but, not, and if ..., then.
2. A ____ is a word or symbol used to join two or more logical statements together to form a compound statement.
3. The most dominant connective is the ____.
4. ____ are used to specify which logical connective should be evaluated first when evaluating a compound statement.
5. Both ____ and ____ have equal dominance and are evaluated from left to right when no parentheses are present in a compound logical statement.

Answers

Exercise 7.3.1

A compound statement formed with the connective word *or* is called a disjunction, and it is represented by the $\vee$ symbol.

A compound statement formed with the connective word *implies* or phrase *"if ..., then"* is called a conditional statement or implication and is represented by the $\rightarrow$ symbol.

A compound statement formed with the connective words *but* or *and* is called a conjunction, and it is represented by the $\wedge$ symbol.

Exercise 7.3.2**: Translating Compound Statements into Symbolic Form**

1. Replace "it is a warm sunny day" with p. Replace "the family will go to the beach." with q. Next. Next, because the connective is if ..., then place the conditional symbol, $\rightarrow$, between p and q. The compound statement is written symbolically as: $p \rightarrow q$.
2. Replace "The family will go to the beach" with q. Replace "it is a warm sunny day." with p. Next, because the connective is and, place the $\wedge$ symbol between q and p. The compound statement is written symbolically as: $q \wedge p$.
3. Replace "The family will not go to the beach. with $\sim q$. Replace "it is not a warm sunny day" with $\sim p$. Next, because the connective is or, if and only if, place the biconditional symbol, $\leftrightarrow$ between $\sim q$ and $\sim p$. The compound statement is written symbolically as: $\sim q \leftrightarrow \sim p$.
4. Replace "The family will not go to the beach" with $\sim q$. Replace "it is a warm sunny day" with p. Next, because the connective is or, place the $\vee$ symbol between $\sim q$ and p. The compound statement is written symbolically as: $\sim q \vee p$.

Exercise 7.3.3

1. Replace with "It is not the case," and $\wedge$ with "and." One possible translation is: "It is not the case that my child finished their homework and cleaned their room."
2. The hypothesis of the conditional statement is, "My child finished their homework and cleaned their room." The conclusion of the conditional statement is, "My child played video games or streamed a movie." One possible translation of the entire statement is: "If my child finished their homework and cleaned their room, then they played video games or streamed a movie."
3. The hypothesis of the biconditional statement is $\sim (r \vee s)$ and is written in words as: "It is not the case that my child played video games or streamed a movie." The conclusion of the biconditional statement is $\sim (p \wedge q)$, which translates to: "It is not the case that my child finished their homework and cleaned their room." Because the biconditional, $\leftrightarrow$ translates to if and only if, one possible translation of the statement is: "It is not the case that my child played video games or streamed a movie if and only if it is not the case that my child finished their homework and cleaned their room."

Exercise 7.3.4

1. Because negation is the least dominant connective, we evaluate it first: $p \wedge (\sim q) \vee r$. Because conjunction and disjunction have the same dominance, we evaluate them left to right. So, we evaluate the conjunction next, as indicated by the additional set of parentheses: $(p \wedge (\sim q)) \vee r$. The only remaining connective is the disjunction, so it is evaluated last, as indicated by the third set of parentheses. The complete solution is: $((p \wedge (\sim q)) \vee r)$.
2. Negation has the lowest dominance, so it is evaluated first: $q \rightarrow (\sim p) \wedge r$. The remaining connectives are the conditional and the conjunction. Because conjunction has a lower precedence than the conditional, it is evaluated next, as indicated by the second set of parentheses: $q \rightarrow ((\sim p) \wedge r)$. The last step is to evaluate the conditional, as indicated by the third set of parentheses: $(q \rightarrow ((\sim p) \wedge r))$.
3. This statement is known as De Morgan's Law for the negation of a disjunction. It is always true. Section 2.6 of this chapter will explore De Morgan's Laws in more detail.
   - First, we evaluate the negations on the right side of the biconditional prior to the conjunction.
   - Then, we evaluate the disjunction on the left side of the biconditional, followed by the negation of the disjunction on the left side.
   - Lastly, after completely evaluating each side of the biconditional, we evaluate the biconditional. It does not matter which side you begin with.

   The final solution is: $(\sim (p \vee q)) \leftrightarrow ((\sim p) \wedge (\sim q))$.

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7.4: Constructing Truth Tables



Figure 7.4.1: Just like solving a puzzle, a computer programmer must consider all potential solutions in order to account for each possible outcome. (credit: modification of work “Deadline Xmas 2010” by Allan Henderson/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Interpret and apply negations, conjunctions, and disjunctions.
2. Construct a truth table using negations, conjunctions, and disjunctions.
3. Construct a truth table for a compound statement and interpret its validity.

Are you familiar with the Choose Your Own Adventure book series written by Edward Packard? These gamebooks allow the reader to become one of the characters and make decisions that affect what happens next, resulting in different sequences of events in the story and endings based on the choices made. Writing a computer program is a little like what it must be like to write one of these books. The programmer must consider all the possible inputs that a user can put into the program and decide what will happen in each case, then write their program to account for each of these possible outcomes.

A truth table is a graphical tool used to analyze all the possible truth values of the component logical statements to determine the validity of a statement or argument along with all its possible outcomes. The rows of the table correspond to each possible outcome for the given logical statement identified at the top of each column. A single logical statement p has two possible truth values, true or false. In truth tables, a capital T will represent true values, and a capital F will represent false values.

In this section, you will use the knowledge built in Statements and Quantifiers and Compound Statements to analyze arguments and determine their truth value and validity. A logical argument is valid if its conclusion follows from its premises, regardless of whether those premises are true or false. You will then explore the truth tables for negation, conjunction, and disjunction, and use these truth tables to analyze compound logical statements containing these connectives.

Interpret and Apply Negations, Conjunctions, and Disjunctions

The negation of a statement will have the opposite truth value of the original statement. When p is true, $\sim p$ is false, and when p is false, $\sim p$ is true.

Exercise 7.4.1

For each logical statement, determine the truth value of its negation.

1. p: $3 + 5 = 8$.
2. q: All horses are mustangs.
3. $\sim r$: New Delhi is not the capital of India.

Answer available at the end of this section.

Your Turn 7.4.1

For each logical statement, determine the truth value of its negation.

1. p: $3 \times 5 = 14$.
2. q: Some houses are built with bricks
3. r: Abuja is the capital of Nigeria.

A conjunction is a logical *and* statement. For a conjunction to be true, both statements that make up the conjunction must be true. If at least one of the statements is false, the *and* statement is false.

1. $p \wedge q$
2. $\sim q \wedge r$
3. $\sim p \wedge q$

Answer available at the end of this section.

Your Turn 7.4.2

Given p : Yellow is a primary color, q : Blue is a primary color, and r : Green is a primary color, determine the truth value of each conjunction.

1. $p \wedge q$
2. $q \wedge r$
3. $\sim r \wedge p$.

Checkpoint

The only time a conjunction is true is if both statements that make up the conjunction are true.

A disjunction is a logical inclusive *or* statement, which means that a disjunction is true if one or both statements that form it are true. The only way a logical inclusive *or* statement is false is if both statements that form the disjunction are false.

1. $p \vee q$
2. $\sim q \vee r$
3. $\sim p \vee q$

Answer available at the end of this section.

Your Turn 7.4.3

Given p : Yellow is a primary color, q : Blue is a primary color, and r : Green is a primary color, determine the truth value of each disjunction.

1. $p \vee q$
2. $\sim p \vee r$
3. $q \vee r$

In the next example, you will apply the dominance of connectives to find the truth values of compound statements containing negations, conjunctions, and disjunctions and use a table to record the results. When constructing a truth table to analyze an argument where you can determine the truth value of each component statement, the strategy is to create a table with two rows. The first row contains the symbols representing the components that make up the compound statement. The second row contains the truth values of each component below its symbol. Include as many columns as necessary to represent the value of each compound statement. The last column includes the complete compound statement with its truth value below it.

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Logic Part 2: Truth Values of Conjunctions: Is an "AND" statement true or false?

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Logic Part 3: Truth Values of Disjunctions: Is an "OR" statement true or false?

1. $\sim p \wedge q \vee r$
2. $\sim p \vee q \wedge r$
3. $\sim (p \wedge r) \vee q$

Answer available at the end of this section.

Your Turn 7.4.4

Given p : Yellow is a primary color, q : Blue is a primary color, and r : Green is a primary color, determine the truth value of each compound statement, by constructing a truth table.

1. $\sim q \wedge p \vee r$
2. $p \vee q \wedge \sim r$
3. $\sim (p \wedge r) \wedge q$

Construct Truth Tables to Analyze All Possible Outcomes

Recall from Statements and Questions that the negation of a statement will always have the opposite truth value of the original statement; if a statement p is false, then its negation $\sim p$ is true, and if a statement p is true, then its negation $\sim p$ is false. To create a truth table for the negation of statement p, add a column with a heading of $\sim p$, and for each row, input the truth value with the opposite value of the value listed in the column for p, as depicted in the table below.

Negation	
p	**~ p**
T	F
F	T

Conjunctions and disjunctions are compound statements formed when two logical statements combine with the connectives “and” and “or” respectively. How does that change the number of possible outcomes and thus determine the number of rows in our truth table? The multiplication principle, also known as the *fundamental counting principle*, states that the number of ways you can select an item from a group of n items and another item from a group with m items is equal to the product of m and n . Because each proposition p and q has two possible outcomes, true or false, the multiplication principle states that there will be $2 \times 2 = 4$ possible outcomes: {TT, TF, FT, FF}.

The tree diagram and table in Figure $7.4.2$ demonstrate the four possible outcomes for two propositions p and q.

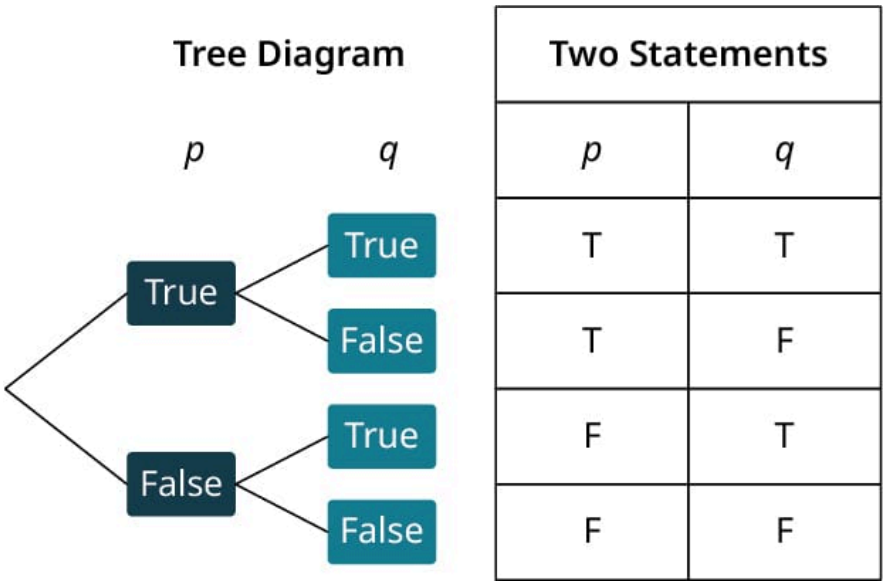


Figure 7.4.2:

A conjunction is a logical *and* statement. For a conjunction to be true, both the statements that make up the conjunction must be true. If at least one of the statements is false, the *and* statement is false.

A disjunction is a logical inclusive *or* statement. Which means that a disjunction is true if one or both statements that make it are true. The only way a logical inclusive *or* statement is false is if both statements that make up the disjunction are false.

Conjunction (AND)				**Disjunction (OR)**		
p	q	$p \wedge q$		p	q	$p \vee q$
T	T	T		T	T	T
T	F	F		T	F	T
F	T	F		F	T	T
F	F	F		F	F	F

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Logic Part 4: Truth Values of Compound Statements with "and", "or", and "not"

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Logic Part 5: What are truth tables? How do you set them up?

Exercise 7.4.5: Constructing Truth Tables to Analyze Compound Statements

Construct a truth table to analyze all possible outcomes for each of the following arguments.

1. $p \vee \sim q$
2. $\sim (p \wedge q)$
3. $(p \vee \sim q) \wedge r$

Answer available at the end of this section.

Your Turn 7.4.5

Construct a truth table to analyze all possible outcomes for each of the following arguments.

1. $p \wedge \sim q$
2. $\sim (p \vee q)$
3. $(p \wedge \sim q) \vee r$

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Logic Part 6: More on Truth Tables and Setting Up Rows and Column Headings

Determine the Validity of a Truth Table for a Compound Statement

A logical statement is valid if it is always true regardless of the truth values of its component parts. To test the validity of a compound statement, construct a truth table to analyze all possible outcomes. If the last column, representing the complete statement, contains only true values, the statement is valid.

Exercise 7.4.6: Determining the Validity of Compound Statements

Construct a truth table to determine the validity of each of the following statements.
1. $\sim p \wedge q$
2. $\sim (p \wedge \sim p)$

Answer available at the end of this section.

Your Turn7.4.6

Construct a truth table to determine the validity of each of the following statements.
1. $p \vee \sim p$
2. $\sim p \vee \sim q$

Check Your Understanding

1. A logical argument is ____ if its conclusion follows from its premises.
2. A logical statement is valid if it is always ____.
3. A ____ ____ is a tool used to analyze all the possible outcomes for a logical statement.

4. The truth table for the conjunction, $p \wedge q$, has ____ rows of truth values.
5. The truth table for the negation of a logical statement, p, has ____ rows of truth values.

Answers

Exercise 7.4.1

p is true because 3 + 5 does equal 8; therefore, the negation of p, ~ p : 3 + 5 ≠ 8 ~ p : 3 + 5 ≠ 8 , is false.

q is false because there are other types of horses besides mustangs, such as Clydesdales or Arabians; therefore, the negation of *q*, ~ *q* , is true.

~ *r* is false because New Delhi is the capital of India; therefore, the negation of *~r, r* , is true.

1. $p \wedge q$
2. $\sim q \wedge r$
3. $\sim p \wedge q$

1. p is true, and q is false. Because one statement is true, and the other statement is false, this makes the complete conjunction false.
2. q is false, so $\sim q$ is true, and r is true. Therefore, both statements are true, making the complete conjunction true.
3. $\sim p$ is false, and q is false. Because both statements are false, the complete conjunction is false.

1. **p is true, and q is false. One statement is true, and one statement is false, which makes the complete disjunction true.**
2. **q is false, so $\sim q$ is true, and r is true. Therefore, one statement is true, and the other statement is true, which makes the complete disjunction true.**
3. **$\sim p$ is false, and q is false. When all of the component statements are false, the disjunction is false.**

1. Step 1: The statement " $\sim p \wedge q \vee r$ " contains three basic logical statements, p, q, and r, and three connectives, $\sim, \wedge, \vee$. When we place parentheses in the statement to indicate the dominance of connectives, the statement becomes $(((\sim p) \wedge q) \vee r)$.
Step 2: After we have applied the dominance of connectives, we create a two row table that includes a column for each basic statement that makes up the compound statement, and an additional column for the contents of each parentheses. Because we have three sets of parentheses, we include a column for $\sim p$, the innermost parentheses, a column for $(\sim p) \wedge q$, the next set of parentheses, and $((\sim p) \wedge q) \vee r$ in the last column for the third parentheses.
Step 3: Once the table is created, we determine the truth value of each statement starting from left to right. The truth values of p, q, and r are true, false, and true, respectively, so we place T, F, and T in the second row of the table. Because p is true, $\sim p$ is false.
Step 4: Next, evaluate $\sim p \wedge q$ from the table: $\sim p$ is false, and q is also false, so $\sim p \wedge q$ is false, because a conjunction is only true if both of the statements that make it are true. Place an F in the table below its heading.
Step 5: Finally, using the table, we understand that $(\sim p \wedge q)$ is false and r is true, so the complete statement $(\sim p \wedge q) \vee r$ is false or true, which is true (because a disjunction is true whenever at least one of the statements that make it is true). Place a T in the last column of the table. The complete statement $\sim p \wedge q \vee r$ is true.

p	*q*	*r*	*-p*	(~p)∧q	((~p)∧q)∨r
T	F	T	F	F	T

2. Step 1: Applying the dominance of connectives to the original compound statement $\sim p \vee q \wedge r$, we get $(((\sim p) \vee q) \wedge r)$.
Step 2: The table needs columns for $p, q, r, \sim p, (\sim p) \vee q$, and $((\sim p) \vee q) \wedge r$.
Step 3:The truth values of p, q, r, and $\sim p$ are the same as in Question 1.
Step 4: Next, $\sim p \vee q$ is false or false, which is false, so we place an F below this statement in the table. This is the only time that a disjunction is false.
Step 5: Finally, $\sim p \vee q$ and r are the conjunction of the statements, $\sim p \vee q$ and r, and so the expressions evaluate to false and true, which is false. Recall that the only time an "and" statement is true is when both statements that form it are also true. The complete statement $\sim p \vee q \wedge r$ is false.

p	*q*	*r*	*-p*	(~p)∧q	((~p)∧q)∨r
T	F	T	F	F	F

3. Step 1: Applying the dominance of connectives to the original statement, we have: $((\sim (p \wedge r)) \vee q)$.
Step 2: So, the table needs the following columns: $p, q, r, p \wedge r, \sim (p \wedge r)$, and $\sim (p \wedge r) \vee q$.
Step 3: The truth values of p, q, and r are the same as in Questions 1 and 2.
Step 4: From the table it can be seen that $p \wedge r$ is true and true, which is true. So the negation of p and r is false, because the negation of a statement always has the opposite truth value of the original statement.
Step 5: Finally, $\sim (p \wedge r) \vee q$ is the disjunction of $\sim (p \wedge r)$ with q, and so we have false or false, which makes the complete statement false.

p	*q*	*r*	*-p*	(~p)∧q	((~p)∧q)∨r
T	F	T	T	F	F

Exercise 7.4.5**: Constructing Truth Tables to Analyze Compound Statements**

1. Step 1: Because there are two basic statements, p and q, and each of these has two possible outcomes, we will have $2(2) = 4$ rows in our table to represent all possible outcomes: TT, TF, FT, and FF. The columns will include $p, q, \sim q$, and $p \vee \sim q$.
Step 2: Every value in column $\sim q$ will have the opposite truth value of the corresponding value in column q : F, T, F, and T.
Step 3: To complete the last column, evaluate each element in column p with the corresponding element in column $\sim q$ using the connective or.

p	q	$\sim q$	$p \vee \sim q$.
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

2. Step 1: The columns will include $p, q, p \wedge q$, and $\sim (p \wedge q)$. Because there are two basic statements, p and q, the table will have four rows to account for all possible outcomes.
Step 2: The $p \wedge q$ column will be completed by evaluating the corresponding elements in columns p and q respectively with the and connective.
Step 3: The final column, $\sim (p \wedge q)$, will be the negation of the $p \wedge q$ column.

p	q	$p \wedge q$	$\sim (p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

3. Step 1: This statement has three basic statements, p, q, and r. Because each basic statement has two possible truth values, true or false, the multiplication principal indicates there are $2(2)(2) = 8$ possible outcomes. So eight rows of outcomes are needed in the truth table to account for each possibility. Half of the eight possibilities must be true for the first statement, and half must be false.
Step 2: So, the first column for statement p, will have four T's followed by four F's. In the second column for statement q, when p is true, half the outcomes for q must be true and the other half must be false, and the same pattern will repeat for when p is false. So, column q will have TT, FF, FF, FF.
Step 3: The column for the third statement, r, must alternate between T and F. Once, the three basic propositions are listed, you will need a column for $\sim, p \vee \sim q$, and $(p \vee \sim q) \wedge r$.
Step 4: The column for the negation of q, $\sim q$, will have the opposite truth value of each value in column q.
Step 5: Next, fill in the truth values for the column containing the statement $p \vee \sim q$. The or statement is true if at least one of p or $\sim q$ is true, otherwise it is false.
Step 6: Finally, fill in the column containing the conjunction $(p \vee \sim q) \wedge r$. To evaluate this statement, combine column $p \vee \sim q$ and column r with the and connective. Recall, that only time "and" is true is when both values are true, otherwise the statement is false. The complete truth table is:

p	q	r	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \wedge r$
T	T	T	F	T	T
T	T	F	F	T	F
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	T	T	T
F	F	F	T	T	F

Exercise $7.4.6$**: Determining the Validity of Compound Statements**

1. Step 1: Because there are two statements, p and q, and each of these has two possible outcomes, there will be $2(2) = 4$ rows in our table to represent all possible outcomes: TT, TF, FT, and FF.
Step 2: The columns, will include $p, q, \sim p$ and $\sim p \wedge q$. Every value in column $\sim p$ will have the opposite truth value of the corresponding value in column p.
Step 3: To complete the last column, evaluate each element in column $\sim p$ with the corresponding element in column q using the connective and. The last column contains at least one false, therefore the statement $\sim p \wedge q$ is not valid.

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

2. Step 1: Because the statement $\sim (p \wedge \sim p)$ only contains one basic proposition, the truth table will only contain two rows. Statement p may be either true or false.
Step 2: The columns will include $p, \sim p, p \wedge \sim p$, and $\sim (p \wedge \sim p)$. Evaluate column $p \wedge \sim p$ with the and connective, because the symbol $\wedge$ represents a conjunction or logical and statement. True and false is false, and false and true is also false.
Step 3: The final column is the negation of each entry in the third column, both of which are false, so the negation of false is true. Because all the truth values in the final column are true, the statement $\sim (p \wedge \sim p)$ is valid.

p	$\sim p$	$p \wedge \sim p$	$\sim (p \wedge \sim p)$
T	F	F	T
F	T	F	T

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7.5: Truth Tables for the Conditional and Biconditional

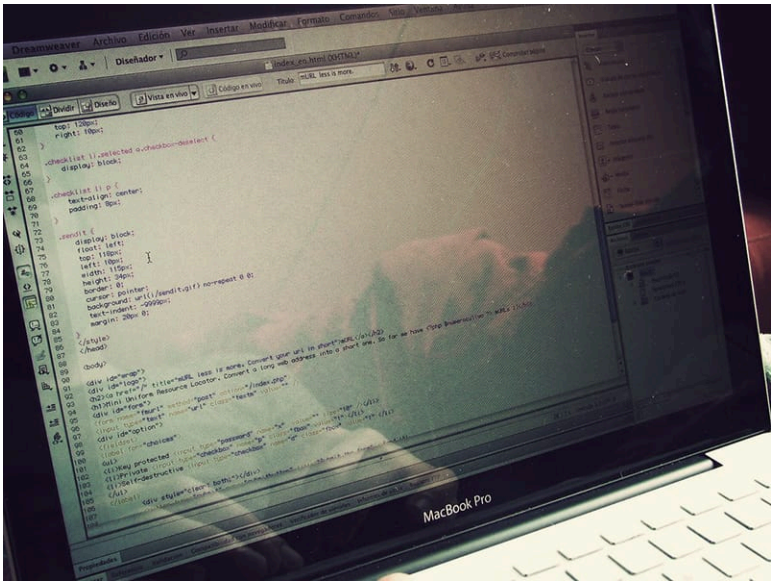


Figure 7.5.1: If-then statements use logic to execute directions. (credit: “Coding” by Carlos Varela/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Use and apply the conditional to construct a truth table.
2. Use and apply the biconditional to construct a truth table.
3. Use truth tables to determine the validity of conditional and biconditional statements.

Computer languages use if-then or if-then-else statements as decision statements:

- *If* the hypothesis is true, *then* do something.
- Or, *if* the hypothesis is true, *then* do something; else do something else.

For example, the following representation of computer code creates an if-then-else decision statement:

Check value of variable i.

If $i < 1$, then print "Hello, World!" else print "Goodbye".

In this imaginary program, the if-then statement evaluates and acts on the value of the variable i. For instance, if $i = 0$, the program would consider the statement $i < 1$ as true and “Hello, World!” would appear on the computer screen. If instead, $i = 3$, the program would consider the statement $i < 1$ as false (because 3 is greater than 1), and print “Goodbye” on the screen.

In this section, we will apply similar reasoning without the use of computer programs.

People in Mathematics

The Countess of Lovelace, Ada Lovelace, is credited with writing the first computer program. She wrote an algorithm to work with Charles Babbage’s Analytical Engine that could compute the Bernoulli numbers in 1843. In doing so, she became the first person to write a program for a machine that would produce more than just a simple calculation. The computer programming language ADA is named after her.

Reference: Posamentier, Alfred and Spreitzer Christian, “Chapter 34 Ada Lovelace: English (1815-1852)” pp. 272-278, *Math Makers: The Lives and Works of 50 Famous Mathematicians*, Prometheus Books, 2019.

Use and Apply the Conditional to Construct a Truth Table

A conditional is a logical statement of the form if p, then q. The conditional statement in logic is a promise or contract. The only time the conditional, $p \rightarrow q$, is false is when the contract or promise is broken.

For example, consider the following scenario. A child’s parent says, “If you do your homework, then you can play your video games.” The child really wants to play their video games, so they get started right away, finish within an hour, and then show their parent the completed homework. The parent thanks the child for doing a great job on their homework and allows them to play video games. Both the parent and child are happy. The contract was satisfied; true implies true is true.

Now, suppose the child does not start their homework right away, and then struggles to complete it. They eventually finish and show it to their parent. The parent again thanks the child for completing their homework, but then informs the child that it is too late in the evening to play video games, and that they must begin to get ready for bed. Now, the child is really upset. They held up their part of the contract, but they did not receive the promised reward. The contract was broken; true implies false is false.

So, what happens if the child does not do their homework? In this case, the hypothesis is false. No contract has been entered, therefore, no contract can be broken. If the conclusion is false, the child does not get to play video games and might not be happy, but this outcome is expected because the child did not complete their end of the bargain. They did not complete their homework. False implies false is true. The last option is not as intuitive. If the parent lets the child play video games, even if they did not do their homework, neither parent nor child are going to be upset. False implies true is true.

The truth table for the conditional statement below summarizes these results.

p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Checkpoint

Notice that the only time the conditional statement, $p \rightarrow q$, is false is when the hypothesis, p, is true and the conclusion, q, is false.

Video

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Logic Part 8: The Conditional and Tautologies

Exercise 7.5.1: Constructing Truth Tables for Conditional Statements

Assume both of the following statements are true: p: My sibling washed the dishes, and q: My parents paid them $5.00. Create a truth table to determine the truth value of each of the following conditional statements.

1. $p \rightarrow q$
2. $p \rightarrow \sim q$
3. $\sim p \rightarrow q$

Answer available at the end of this section.

Your Turn 7.5.1

Assume p is true and q is false. p: Kevin vacuumed the living room, and q: Kevin's parents did not let him borrow the car. Create a truth table to determine the truth value of each of the following conditional statements.

1. $p \rightarrow q$

2. $p \rightarrow \sim q$

3. $\sim p \rightarrow q$

Exercise 7.5.2: **Determining Validity of Conditional Statements**

Construct a truth table to analyze all possible outcomes for each of the following statements then determine whether they are valid.

1. $p \wedge q \rightarrow \sim q$
2. $p \rightarrow \sim p \vee q$

Answer available at the end of this section.

Your Turn 7.5.2

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $q \rightarrow \sim p \vee q$

2. $\sim p \rightarrow q \wedge p$

Use and Apply the Biconditional to Construct a Truth Table

The biconditional, $p \leftrightarrow q$, is a two way contract; it is equivalent to the statement $(p \rightarrow q) \wedge (q \rightarrow p)$. A biconditional statement, $p \leftrightarrow q$, is true whenever the truth value of the hypothesis matches the truth value of the conclusion, otherwise it is false.

The truth table for the biconditional is summarized below.

p	q	$p \leftrightarrow q$
T	T	**T**
T	F	**F**

p	q	$p \leftrightarrow q$
F	T	**F**
F	F	**T**

Video

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Logic Part 11B: Biconditional and Summary of Truth Value Rules in Logic

Exercise 7.5.3: Constructing Truth Tables for Biconditional Statements

Assume both of the following statements are true: p : The plumber fixed the leak, and q : The homeowner paid the plumber $\$150.00$. Create a truth table to determine the truth value of each of the following biconditional statements.

1. $p \leftrightarrow q$
2. $p \leftrightarrow \sim q$
3. $\sim p \leftrightarrow \sim q$

Answer available at the end of this section.

Your Turn 7.5.3

Assume p is true and q is false: p : The contractor fixed the broken window, and q : The homeowner paid the contractor $\$200$. Create a truth table to determine the truth value of each of the following biconditional statements.

1. $p \leftrightarrow q$
2. $p \leftrightarrow \sim q$
3. $\sim p \leftrightarrow q$

Checkpoint

The biconditional, $p \leftrightarrow q$, is true whenever the truth values of p and q match, otherwise it is false.

Video

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Logic Part 13: Truth Tables to Determine if Argument is Valid or Invalid

Exercise 7.5.1: Determining Validity of Biconditional Statements

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $p \wedge q \leftrightarrow p \wedge \sim q$
2. $p \vee q \leftrightarrow \sim p \vee q$
3. $p \rightarrow q \leftrightarrow \sim q \rightarrow \sim p$
4. $p \wedge q \rightarrow \sim r \leftrightarrow p \wedge q \wedge r$

Answer available at the end of this section.

Your Turn 7.5.1

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
2. $\sim p \leftrightarrow q \wedge p$
3. $p \rightarrow q \leftrightarrow \sim p \vee q$
4. $p \wedge q \rightarrow r \leftrightarrow \sim p \vee \sim q \vee r$

Check Your Understanding

1. 1. In logic, a conditional statement can be thought of as a _____.
2. If the hypothesis, p, of a conditional statement is true, the ____, q, must also be true for the conditional statement \($p \rightarrow q$ to be true.
3. If the _____ of a conditional statement is false, the conditional statement is true.
4. The symbolic form of the _____ statement is $p \leftrightarrow q$
5. The _____ statement is equivalent to the statement $(p \rightarrow q) \wedge (q \rightarrow p)$.
6. p if and only if q is _____ whenever the truth value of p matches the truth value of q, otherwise it is false.

Answers

Exercise 7.5.1: Constructing Truth Tables for Conditional Statements

Because p is true and q is true, the statement p → q is, “If my sibling washed the dishes, then my parents paid them $5.00.” My sibling did wash the dishes, since

p p is true, and the parents did pay the sibling $5.00, so the contract was entered and completed. The conditional statement is true, as indicated by the truth table representing this case:
T → T = T.

p	q	$p \rightarrow q$
T	T	**T**

p → ~ q p

2. p → ~ q p → ~ q translates to the statement, “If my sibling washed the dishes, then my parents did not pay them $5.00.” p p is true, but ~ q ~ q is false. The sibling completed their end of the contract, but they did not get paid. The contract was broken by the parents. The conditional statement is false, as indicated by the truth table representing this case:
T → F = F.

p	q	$\sim q$	$p \rightarrow \sim q$
T	T	F	**F**

3. ~ p → q translates to the statement, “If my sibling did not wash the dishes, then my parents paid them $5.00.” ~ p is false, but q is true. The sibling did not do the dishes. No contract was entered, so it could not be broken. The parents decided to pay them $5.00 anyway. The conditional statement is true, as indicated by the truth table representing this case: F → T = T.

p	q	$\sim p$	$\sim p \rightarrow q$
T	T	F	**T**

Exercise 7.5.2**: Determining Validity of Conditional Statements**

1. Applying the dominance of connectives, the statement $p \wedge q \rightarrow \sim q$ is equivalent to $(p \wedge q) \rightarrow (\sim q)$. So, the columns of the truth table will include p, q, $p \wedge q$, $\sim q$, and $p \wedge q \rightarrow \sim q$. Because there are only two basic propositions, p and q, the table will have 2 (2) = 4 2 (2) = 4 rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$p \wedge q$	$\sim q$	$p \wedge q \rightarrow \sim q$
T	T	T	F	**F**
T	F	F	T	**T**
F	T	F	F	**T**
F	F	F	T	**T**

Applying the dominance of connectives, the statement $p \rightarrow \sim p \vee q$ is equivalent to $(p) \rightarrow ((\sim p) \vee q)$. So, the columns of the truth table will include p , q , ~ p, ~ p ∨ q, and $p \rightarrow (\sim p \vee q)$. Because there are only two basic propositions, p and q , the table will have 2 (2) = 4 2 (2) = 4 rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$\sim p$	$\sim p \vee q$	$p \rightarrow (\sim p \vee q)$
T	T	F	T	**T**
T	F	F	F	**F**
F	T	T	T	**T**
F	F	T	T	**T**

Exercise 7.5.3**: Constructing Truth Tables for Biconditional Statements**

1. Because p p is true and q q is true, the statement p ↔ q p ↔ q is “The plumber fixed the leak if and only if the homeowner paid them $150.00.” Because both p p and q q are true, the leak was fixed and the plumber was paid, meaning both parties satisfied their end of the bargain. The biconditional statement is true, as indicated by the truth table representing this case: T ↔ T = T.

p	q	$p \leftrightarrow q$
T	T	**T**

2. p → ~ q p → ~ q translates to the statement, “The plumber fixed the leak if and only if the homeowner did not pay them $150.” If the plumber fixed the leak and

the homeowner did not pay them, the homeowner will have broken their end of the contract. The biconditional statement is false, as indicated by the truth table representing this case:
T ↔ F = F.

p	q	$\sim q$	$p \leftrightarrow \sim q$
T	T	F	**F**

3. ~ p ↔ ~ q ~ p ↔ ~ q translates to the statement, “The plumber did not fix the leak if and only if the homeowner did not pay them $150.” In this case, neither party—the plumber nor the homeowner—entered into the contract. The leak was not repaired, and the plumber was not paid. No agreement was broken. The biconditional statement is true, as indicated by the truth table representing this case: F ↔ F = T.

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow \sim q$
T	T	F	F	**T**

Exercise $7.5.1$**: Determining Validity of Biconditional Statements**

1. Applying the dominance of connectives, the statement $p \wedge q \leftrightarrow p \wedge \sim q$ is equivalent to $(p \wedge q) \leftrightarrow (p \wedge (\sim q))$. So, the columns of the truth table will include $p, q, p \wedge q, \sim q, p \wedge \sim q$ and $(p \wedge q) \leftrightarrow (p \wedge \sim q)$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$p \wedge q$	$\sim q$	$p \wedge \sim q$	$(p \wedge q) \leftrightarrow (p \wedge \sim q)$
T	T	T	F	F	**F**
T	F	F	T	T	**F**
F	T	F	F	F	**T**
F	F	F	T	F	**T**

2. Applying the dominance of connectives, the statement $p \vee q \leftrightarrow \sim p \vee q$ is equivalent to $(p \vee q) \leftrightarrow ((\sim p) \vee q)$. So, the columns of the truth table will include $p, q, p \vee q, \sim p, \sim p \vee q$, and $(p \vee q) \leftrightarrow (\sim p \vee q)$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$p \vee q$	$\sim p$	$\sim p \vee q$	$(p \vee q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	**T**
T	F	T	F	F	**F**
F	T	T	T	T	**T**
F	F	F	T	T	**F**

3. Applying the dominance of connectives, the statement $p \rightarrow q \leftrightarrow \sim q \rightarrow \sim p$ is equivalent to $(p \rightarrow q) \leftrightarrow ((\sim q) \rightarrow (\sim p))$. So, the columns of the truth table will include $p, q, p \rightarrow q, \sim q, \sim p, \sim q \rightarrow \sim p$, and $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$. Because there are only two basic propositions, p and q the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is valid because the last column is all true.

p	q	$p \rightarrow q$	$\sim q$	$\sim p$	$\sim q \rightarrow \sim p$	$(p \rightarrow \mathbf{q}) \leftrightarrow (\sim q \rightarrow \sim p)$
T	T	T	F	F	T	**T**
T	F	F	T	F	F	**T**
F	T	T	F	T	T	**T**
F	F	T	T	T	T	**T**

4. Applying the dominance of connectives, the statement $p \wedge q \rightarrow \sim r \leftrightarrow p \wedge q \wedge r$ is equivalent to $((p \wedge q) \rightarrow (\sim r)) \leftrightarrow ((p \wedge q) \wedge r)$. So, the columns of the truth table will include $p, q, r, \sim r, p \wedge q, (p \wedge q) \wedge \sim r, (p \wedge q) \wedge r$, and $((p \wedge q) \rightarrow (\sim r)) \leftrightarrow ((p \wedge q) \wedge r)$. Because there are three basic propositions, p, q, and r, the table will have $2(2)(2) = 8$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	r	$\sim r$	$p \wedge q$	$(p \wedge q) \rightarrow \sim r$	$(p \wedge q) \wedge r$	$(p \wedge q \rightarrow \sim r) \leftrightarrow (p \wedge q \wedge r)$
T	T	T	F	T	F	T	**T**
T	T	F	T	T	T	F	**T**

p	q	r	$\sim r$	$p \wedge q$	$(p \wedge q) \rightarrow \sim r$	$(p \wedge q) \wedge r$	$(p \wedge q \rightarrow \sim r) \leftrightarrow (p \wedge q \wedge r)$
T	F	T	F	F	T	F	**F**
T	F	F	T	F	T	F	**F**
F	T	T	F	F	T	F	**F**
F	T	F	T	F	T	F	**F**
F	F	T	F	F	T	F	**F**
F	F	F	T	F	T	F	**F**

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7.6: Equivalent Statements



Figure 7.6.1: How your logical argument is stated affects the response, just like how you speak when holding a conversation can affect how your words are received. (credit: modification of work by Goelshivi/Flickr, Public Domain Mark 1.0)

Learning Objectives

After completing this section, you should be able to:

1. Determine whether two statements are logically equivalent using a truth table.
2. Compose the converse, inverse, and contrapositive of a conditional statement

Have you ever had a conversation with or sent a note to someone, only to have them misunderstand what you intended to convey? The way you choose to express your ideas can be as, or even more, important than what you are saying. If your goal is to convince someone that what you are saying is correct, you will not want to alienate them by choosing your words poorly.

Logical arguments can be stated in many different ways that still ultimately result in the same valid conclusion. Part of the art of constructing a persuasive argument is knowing how to arrange the facts and conclusion to elicit the desired response from the intended audience.

In this section, you will learn how to determine whether two statements are **logically equivalent** using truth tables, and then you will apply this knowledge to compose logically equivalent forms of the conditional statement. Developing this skill will provide the additional skills and knowledge needed to construct well-reasoned, persuasive arguments that can be customized to address specific audiences.

Checkpoint

An alternate way to think about logical equivalence is that the truth values have to match. That is, whenever p is true, q is also true, and whenever p is false, q is also false.

Determine Logical Equivalence

Two statements, p and q, are logically equivalent when $p \leftrightarrow q$ is a valid argument, or when the last column of the truth table consists of only true values. When a logical statement is always true, it is known as a tautology. To determine whether two statements p

and q are logically equivalent, construct a truth table for $p \leftrightarrow q$ and determine whether it valid. If the last column is all true, the argument is a tautology, it is valid, and p is logically equivalent to q; otherwise, p is not logically equivalent to q.

Exercise 7.6.1: Determining Logical Equivalence with a Truth Table

Create a truth table to determine whether the following compound statements are logically equivalent.

1. $p \rightarrow q; \sim p \rightarrow \sim q$
2. $p \rightarrow q; \sim p \vee q$

Answer available at the end of this section.

Your Turn 7.6.1

Create a truth table to determine whether the following compound statements are logically equivalent.

1. $p \rightarrow q; q \rightarrow \sim p$
2. $p \rightarrow q; p \vee \sim q$

Compose the Converse, Inverse, and Contrapositive of a Conditional Statement

The converse, inverse, and contrapositive are variations of the conditional statement, $p \rightarrow q$.

- The converse is if q then p, and it is formed by interchanging the hypothesis and the conclusion. The converse is logically equivalent to the inverse.
- The inverse is if $\sim p$ then $\sim q$, and it is formed by negating both the hypothesis and the conclusion. The inverse is logically equivalent to the converse.
- The contrapositive is if $\sim q$ then $\sim p$, and it is formed by interchanging and negating both the hypothesis and the conclusion. The contrapositive is logically equivalent to the conditional.

The table below shows how these variations are presented symbolically.

				Conditional	**Contrapositive**	**Converse**	**Inverse**
p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$q \rightarrow p$	$\sim p \rightarrow \sim q$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

Exercise 7.6.2: Writing the Converse, Inverse, and Contrapositive of a Conditional Statement

Use the statements, p: Harry is a wizard and q: Hermione is a witch, to write the following statements:

1. Write the conditional statement, $p \rightarrow q$, in words.
2. Write the converse statement, $q \rightarrow p$, in words.
3. Write the inverse statement, $\sim p \rightarrow \sim q$, in words.
4. Write the contrapositive statement, $\sim q \rightarrow \sim p$, in words.

Answer available at the end of this section.

Your Turn 7.6.2

Use the statements, p : Elvis Presley wore capes and q : Some superheroes wear capes, to write the following statements:

1. Write the conditional statement, $p \rightarrow q$, in words.
2. Write the converse statement, $q \rightarrow p$, in words.
3. Write the inverse statement, $\sim p \rightarrow \sim q$, in words.
4. Write the contrapositive statement, $\sim q \rightarrow \sim p$, in words.

Exercise 7.6.3

Use the conditional statement, "If all dogs bark, then Lassie likes to bark," to identify the following.

1. Write the hypothesis of the conditional statement and label it with a p.
2. Write the conclusion of the conditional statement and label it with a q.
3. Identify the following statement as the converse, inverse, or contrapositive: "If Lassie likes to bark, then all dogs bark."
4. Identify the following statement as the converse, inverse, or contrapositive: "If Lassie does not like to bark, then some dogs do not bark."
5. Which statement is logically equivalent to the conditional statement?

Answer available at the end of this section.

Your Turn 7.6.3

Use the conditional statement, "If Dora is an explorer, then Boots is a monkey," to identify the following:

1. Write the hypothesis of the conditional statement and label it with a p.

2. Write the conclusion of the conditional statement and label it with a q.

3. Identify the following statement as the converse, inverse, or contrapositive: "If Dora is not an explorer, then Boots is not a monkey."

4. Identify the following statement as the converse, inverse, or contrapositive: "If Boots is a monkey, then Dora is an explorer."

5. Which statement is logically equivalent to the inverse?

Exercise 7.6.4

Assume the conditional statement, $p \rightarrow q$: "If Chadwick Boseman was an actor, then Chadwick Boseman did not star in the movie *Black Panther*" is false, and use it to answer the following questions.

1. Write the converse of the statement in words and determine its truth value.
2. Write the inverse of the statement in words and determine its truth value.
3. Write the contrapositive of the statement in words and determine its truth value.

Answer available at the end of this section.

Your Turn 7.6.4

Assume the conditional statement $p \rightarrow q$: "If my friend lives in San Francisco, then my friend does not live in California" is false, and use it to answer the following questions.

1. Write the converse of the statement in words and determine its truth value.
2. Write the inverse of the statement in words and determine its truth value.
3. Write the contrapositive of the statement in words and determine its truth value.

Check Your Understanding

1. Two statements p and q are logically equivalent to each other if the biconditional statement, $p \leftrightarrow q$ is _____.
2. The _____ statement has the form, "p then q."
3. The contrapositive is _____ _____ to the conditional statement, and has the form, "if q, then p."
4. The _____ of the conditional statement has the form, "if p, then q."
5. The _____ of the conditional statement is logically equivalent to the _____ and has the form, "if q then p."

Answers

Exercise 7.6.1: Determining Logical Equivalence with a Truth Table

1. Construct a truth table for the biconditional formed by using the first statement as the hypothesis and the second statement as the conclusion, (p → q) ↔ (~ p → ~ q).

p	q	$p \rightarrow q$	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$	$(p \rightarrow q) \leftrightarrow (\sim p \rightarrow \sim q)$
T	T	T	F	F	T	**T**
T	F	F	F	T	T	**F**
F	T	T	T	F	F	**F**
F	F	T	T	T	T	**T**

Because the last column it not all true, the biconditional is not valid and the statement $p \rightarrow q$ is not logically equivalent to the statement $\sim p \rightarrow \sim q$.

2. Construct a truth table for the biconditional formed by using the first statement as the hypothesis and the second statement as the conclusion, (p → q) ↔ (~ p ∨ q) .

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$	$(p \rightarrow q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	**T**
T	F	F	F	F	**T**
F	T	T	T	T	**T**
F	F	T	T	T	**T**

Because the last column is true for every entry, the biconditional is valid and the statement $p \rightarrow q$ is logically equivalent to the statement $\sim p \vee q$. Symbolically, $p \rightarrow q \equiv \sim p \vee q$.

Exercise 7.6.2: Writing the Converse, Inverse, and Contrapositive of a Conditional Statement

1. The conditional statement takes the form, "if p p , then q q ," so the conditional statement is: "If Harry is a wizard, then Hermione is a witch." Remember the *if ... then ...* words are the connectives that form the conditional statement.
2. The converse swaps or interchanges the hypothesis, p p , with the conclusion, q q . It has the form, "if q q , then p p ." So, the converse is: "If Hermione is a witch, then Harry is a wizard."
3. To construct the inverse of a statement, negate both the hypothesis and the conclusion. The inverse has the form, "if ~ p ~ p , then ~ q ~ q ," so the inverse is: "If Harry is not a wizard, then Hermione is not a witch."
4. The contrapositive is formed by negating and interchanging both the hypothesis and conclusion. It has the form, "if ~ q ~ q , then ~ p ~ p ," so the contrapositive statement is: "If Hermione is not a witch, then Harry is not a wizard."

Exercise 7.6.3

1. The hypothesis is the phrase following the *if*. The answer is *p*: All dogs bark. Notice, the word *if* is not included as part of the hypothesis.
2. The conclusion of a conditional statement is the phrase following the *then*. The word *then* is not included when stating the conclusion. The answer is: q: Lassie likes to bark.
3. "Lassie likes to bark" is *q* and "All dogs bark" is *p*. So, "If Lassie likes to bark, then all dogs bark," has the form "if *q*, then *p*," which is the form of the converse. "Lassie does not like to bark" is ~ *q* and "Some dogs do not bark" is ~ *p*. The statement, "If Lassie does not like to bark, then some dogs do not bark," has the form "if ~ *q*, then ~ *p*," which is the form of the contrapositive.
4. The contrapositive ~ q → ~ *p* is logically equivalent to the conditional statement *p* → q .

Exercise 7.6.4

The only way the conditional statement can be false is if the hypothesis, *p*: Chadwick Boseman was an actor, is true and the conclusion, q: Chadwick Boseman did not star in the movie *Black Panther*, is false. The converse is q → p, and it is written in words as: "If Chadwick Boseman did not star in the movie *Black Panther*, then Chadwick Boseman was an actor." This statement is true, because false → true is true.

The inverse has the form " ~ *p* → ~ *q*." The written form is: "If Chadwick Boseman was not an actor, then Chadwick Boseman starred in the movie *Black Panther*." Because *p* is true, and *q* is false, ~ *p* is false, and ~ *q* is true. This means the inverse is false → true, which is true. Alternatively, from Question 1, the converse is true, and because the inverse is logically equivalent to the converse it must also be true.

The contrapositive is logically equivalent to the conditional. Because the conditional is false, the contrapositive is also false. This can be confirmed by looking at the truth values of the contrapositive statement. The contrapositive has the form " ~ *q* → ~ *p*." Because q is false and p is true, ~ q is true and ~ *p* is false. Therefore, ~ *q* → ~ *p* is true → false, which is false. The written form of the contrapositive is "If Chadwick Boseman starred in the movie *Black Panther*, then Chadwick Boseman was not an actor."

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7.7: De Morgan's Laws

Figure 7.7.1: De Morgan's Laws were key to the rise of logical mathematical expression and helped serve as a bridge for the invention of the computer. (credit: modification of work "Golden Gate Bridge (San Francisco Bay, California, USA)" by James St. John/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Use De Morgan's Laws to negate conjunctions and disjunctions.
2. Construct the negation of a conditional statement.
3. Use truth tables to evaluate De Morgan's Laws.

The contributions to logic made by Augustus De Morgan and George Boole during the 19th century acted as a bridge to the development of computers, which may be the greatest invention of the 20th century. **Boolean** logic is the basis for computer science and digital electronics, and without it the technological revolution of the late 20th and early 21st centuries—including the creation of computer chips, microprocessors, and the Internet—would not have been possible. Every modern computer language uses Boolean logic statements, which are translated into commands understood by the underlying electronic circuits enabling computers to operate. But how did this logic get its name?

People in Mathematics: George Boole

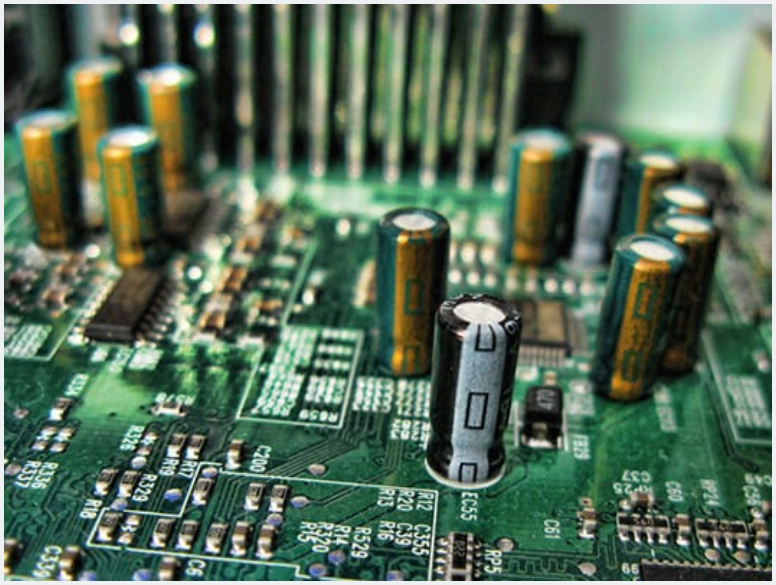


Figure 7.7.2: Boole's algebra of logic was foundational in the design of digital computer circuits. (credit: "Circuit Board" by Squeezyboy/ Flickr, CC BY 2.0)

George Boole was born in Lincolnshire, England in 1815. He was the son of a cobbler who provided him some initial education, but Boole was mostly self-taught. He began teaching at 16 years of age, and opened his own school at the age of 20. In 1849, at the age of 34, he was appointed Professor of Mathematics at Queens College in Cork, Ireland. In 1853, he published the paper, *An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities*, which is the treatise that the field of Boolean algebra and digital circuitry was built on.

Reference: Posamentier, Alfred and Spreitzer Christian, "Chapter 35 George Boole: English (1815-1864)" pp. 279-283, *Math Makers: The Lives and Works of 50 Famous Mathematicians*, Prometheus Books, 2019.

Negation of Conjunctions and Disjunctions

In Chapter 1, Example 1.37 used a Venn diagram to prove De Morgan's Law for set complement over union. Because the complement of a set is analogous to negation and union is analogous to an *or* statement, there are equivalent versions of De Morgan's Laws for logic.

FORMULA

De Morgan's Law for negation of a conjunction: $\sim(p \wedge q) \equiv \sim p \vee \sim q$

De Morgan's Law for the negation of a disjunction: $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Negation of a conditional: $\sim(p \rightarrow q) \equiv p \wedge \sim q$

Writing conditional as a disjunction: $p \rightarrow q \equiv \sim p \vee q$

Checkpoint

Recall that the symbol for logical equivalence is: $\equiv$.

De Morgan's Laws allow us to write the negation of conjunctions and disjunctions without using the phrase, "It is not the case that ..." to indicate the parentheses. Avoiding this phrase often results in a written or verbal statement that is clearer or easier to understand.

Exercise 7.7.1: Applying De Morgan's Law for Negation of Conjunctions and Disjunctions

Write the negation of each statement in words without using the phrase, "It is not the case that."

1. Kristin is a biomedical engineer and Thomas is a chemical engineer.
2. A person had cake or they had ice cream.

Answer available at the end of this section.

Your Turn 7.7.1

Write the negation of each statement in words without using the phrase, *it is not the case that*.

1. Jackie played softball or she ran track.

2. Brandon studied for his certification exam, and he passed his exam.

is not a chemical engineer."

Negation of a Conditional Statement

The negation of any statement has the opposite truth values of the original statement. The **negation of a conditional**, $\sim(p \rightarrow q)$, is the conjunction of p and not q, $p \wedge \sim q$. Consider the truth table below for the negation of the conditional.

p	q	$p \rightarrow q$	$\sim(p \rightarrow q)$
T	T	T	F

p	q	$p \to q$	$\sim(p \to q)$
T	F	F	T
F	T	T	F
F	F	T	F

The only time the negation of the conditional statement is true is when p is true, and q is false. This means that $\sim(p \to q)$ is logically equivalent to $p \wedge \sim q$, as the following truth table demonstrates.

p	q	$p \to q$	$\sim(p \to q)$	$\sim q$	$p \wedge \sim q$	$\sim(p \to q) \leftrightarrow (p \wedge \sim q)$
T	T	T	F	F	F	**T**
T	F	F	T	T	T	**T**
F	T	T	F	F	F	**T**
F	F	T	F	T	F	**T**

Exercise 7.7.2: Constructing the Negation of a Conditional Statement

Write the negation of each conditional statement.

1. If Adele won a Grammy, then she is a singer.
2. If Henrik Lundqvist played professional hockey, then he did not win the Stanley Cup.

Answer available at the end of this section.

Your Turn 7.7.2

Write the negation of each conditional statement.

1. If Edna Mode makes a new superhero costume, then it will not include a cape.

2. If I had pancakes for breakfast, then I used maple syrup.

Exercise 7.7.3: Constructing the Negation of a Conditional Statement with Quantifiers

Write the negation of each conditional statement.

1. If all cats purr, then my partner's cat purrs.
2. If a penguin is a bird, then some birds do not fly.

Answer available at the end of this section.

Your Turn 7.7.3

Write the negation of each conditional statement.

1. If some people like ice cream, then ice cream makers will make a profit.

2. If Raquel cannot play video games, then nobody will play video games.

Many of the properties that hold true for number systems and sets also hold true for logical statements. The following table

summarizes some of the most useful properties for analyzing and constructing logical arguments. These properties can be verified using a truth table.

Property	Conjunction (AND)	Disjunction (OR)
Commutative	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive	$p \wedge (q \wedge r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
De Morgan's	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
Conditional	$\sim(p \rightarrow q) \equiv p \wedge \sim q$	$p \rightarrow q \equiv \sim p \vee q$

Exercise 7.7.4

Negating a Conditional Statement with a Conjunction or Disjunction

Write the negation of each conditional statement applying De Morgan's Law.

1. If mom needs to buy chips, then Mike had friends over and Bob was hungry.
2. If Juan had pizza or Chris had wings, then dad watched the game.

Answer available at the end of this section.

Your Turn 7.7.4

Write the negation of each conditional statement applying De Morgan's Law.

1. If Eric needs to replace the light bulb, then Marcus left the light on all night or Dan broke the bulb.

2. If Trenton went to school and Regina went to work, then Merika cleaned the house.

Evaluating De Morgan's Laws with Truth Tables

In Chapter 1, you learned that you could prove the validity of De Morgan's Laws using Venn diagrams. Truth tables can also be used to prove that two statements are logically equivalent. If two statements are logically equivalent, you can use the form of the statement that is clearer or more persuasive when constructing a logical argument.

The next example will prove the validity of one of De Morgan's Laws using a truth table. The same procedure can be applied to any two logical statement that you believe are equivalent. If the last column of the truth table is a tautology, then the two statements are logically equivalent.

Exercise 7.7.5: Verifying De Morgan's Law for Negation of a Conjunction

Construct a truth table to verify De Morgan's Law for the negation of a conjunction, $\sim(p \wedge q) \equiv \sim p \vee \sim q$, is valid.

Answer available at the end of this section.

Your Turn 7.7.5

1. Construct a truth table to verify De Morgan's Law for the negation of a disjunction, $\sim(p \vee q) \equiv \sim p \wedge \sim q$, is valid.

Check Your Understanding

1. De Morgan's Law for the negation of a conjunction states that \($(p \wedge q)$is logically equivalent to _____.

2. De Morgan's Law for the negation of a disjunction states that $(p \vee q)$ is logically equivalent to _____.

3. The negation of the conditional statement, $(p \rightarrow q)$, is logically equivalent to _____.

$(\ (p \rightarrow q)) \equiv p \rightarrow q$, which means the conditional statement is logically equivalent to $(p \wedge\ q)$. Apply _____ to the statement $(p \wedge\ q)$to show that the conditional statement $p \rightarrow q \equiv\ p \vee q$.

Answers

Exercise 7.7.1**: Applying De Morgan's Law for Negation of Conjunctions and Disjunctions**

1. Kristin is a biomedical engineer and Thomas is a chemical engineer has the form " $p \wedge q$," where p is the statement, "Kristin is a biomedical engineer," and q is the statement, "Thomas is a chemical engineer." By De Morgan's Law, the negation of a conjunction, $\sim (p \wedge q)$, is logically equivalent to $\sim p \vee \sim q$. $\sim p$ is "Kristen is not a biomedical engineer," and $\sim q$ is "Thomas is not a chemical engineer." By De Morgan's Law, the solution has the form " $\sim p \vee \sim q$," so the answer is: "Kristin is not a biomedical engineer or Tom is not a chemical engineer."
2. A person had cake or they had ice cream has the form " $p \vee q$," where p is the statement, "A person had cake," and q is the statement, "A person had ice cream." By De Morgan's Law for the negations of a disjunction, $\sim (p \vee q) \equiv \sim p \wedge \sim q$. The solution is the statement: "A person did not have cake and they did not have ice cream."

Exercise 7.7.2**: Constructing the Negation of a Conditional Statement**

1. The negation of the conditional statement, $p \rightarrow q$, is the statement, $p \wedge \sim q$. The hypothesis of the conditional statement is p : "Adele won a Grammy," and conclusion of the conditional statement is q : "Adele is a singer." The negation of the conclusion, $\sim q$, is the statement: "She is not a singer." Therefore, the answer is $p \wedge \sim q$: "Adele won a Grammy, and she is not a singer."
2. The hypothesis is p : "Henrik Lundqvist played professional hockey," and the conclusion of the conditional statement is q : "He did not win the Stanley Cup." The negation of q is the statement: "He won the Stanley Cup." The negation of the conditional statement is equal to $p \wedge \sim q$: "Henrick Lundqvist played professional hockey, and he won the Stanley Cup."

Exercise 7.7.3**: Constructing the Negation of a Conditional Statement with Quantifiers**

1. The negation of the conditional statement $p \rightarrow q$ is the statement $p \wedge \sim q$. The hypothesis of the conditional statement is p : "All cats purr," and the conclusion of the conditional statement is q : "My partner's cat purrs." The negation of the conclusion, $\sim q$, is the statement: "My partner's cat does not purr." Therefore, the answer is $p \wedge \sim q$: "All cats purr, but my partner's cat does not purr."
2. The hypothesis is p : "A penguin is a bird," and the conclusion of the conditional statement is q : "Some birds do not fly." The negation of q is the statement: "All birds fly." Therefore, the negation of the conditional statement is equal to $p \wedge \sim q$: "A penguin is a bird, and all birds fly."

Exercise 7.7.4

1. The conditional has the form "If p then q or r," where p is "Mom needs to buy chips," q is "Mike had friends over," and r is "Bob was hungry." The negation of $p \rightarrow (q \wedge r)$ is $p \wedge \sim (q \wedge r)$. Applying De Morgan's Law to the statement $\sim (q \wedge r)$ the result is $\sim q \vee \sim r$, so our conditional statement becomes $p \wedge (\sim q \vee \sim r)$. By the distributive property for conjunction over disjunction, this statement is equivalent to $(p \wedge \sim q) \vee (p \wedge \sim r)$. Translating the statement $(p \wedge \sim q) \vee (p \wedge \sim r)$ into words, the solution is: "Mom needs to buy chips and Mike did not have friends over, or Mom needs to buy chips and Bob was not hungry."
2. The conditional has the form "If p or q, then r," where p is "Juan had pizza," q is "Chris had wings," and r is "Dad watched the game." The negation of $(p \vee q) \rightarrow r$ is $(p \vee q) \wedge \sim r$. By the distributive property for disjunction over conjuction, the statement is equivalent to $(p \vee \sim r) \wedge (q \vee \sim r)$. Translating the statement $(p \vee \sim r) \wedge (q \vee \sim r)$ into words, the solution is: "Juan had pizza or dad did not watch the game, and Chris had wings or dad did not watch the game."

Exercise 7.7.5**: Verifying De Morgan's Law for Negation of a Conjunction**

Step 1: To verify any logical equivalence, you must first replace the logical equivalence symbol, $\equiv$, with the biconditional symbol, $\leftrightarrow$. The statement $\sim (p \wedge q) \equiv \sim p \vee \sim q$ becomes $\sim (p \wedge q) \leftrightarrow \sim p \vee \sim q$
Step 2: Next, you create a truth table for the statement. Because we have two basic statements, p, and q, the truth table will have four rows to account for all the possible outcomes. The columns will be $p, q, \sim p, \sim q, p \wedge q, \sim (p \wedge q), \sim p \vee \sim q$, and the biconditional statement is $\sim (p \wedge q) \leftrightarrow \sim p \vee \sim q$.

p	q	$p \wedge q$	$\sim (p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
T	T	T	F	F	F	F	**T**
T	F	F	T	F	T	T	**T**
F	T	F	T	T	F	T	**T**
F	F	F	T	T	T	T	**T**

Step 3: Finally, verify that the statement is valid by confirming it is a tautology. In this instance, the last column is all true. Therefore, the statement is valid and De Morgan's Law for the negation of a conjunction is verified.

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7.8: Logical Arguments

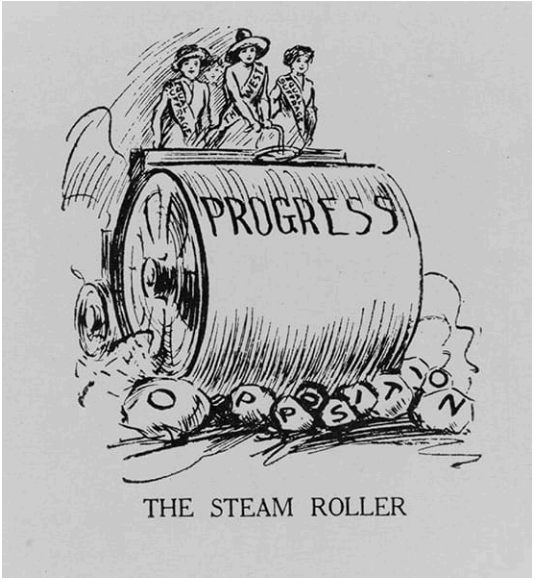


Figure 7.8.1: Not all logical arguments are valid, and the ongoing fight for equal rights proves that much progress has yet to be made. (credit: "The Steam Roller" by Library of Congress Prints and Photographs Division, public domain) ;

Learning Objectives

After completing this section, you should be able to:

1. Apply the law of detachment to determine the conclusion of a pair of statements.
2. Apply the law of denying the consequent to determine the conclusion for pairs of statements.
3. Apply the chain rule to determine valid conclusions for pairs of true statements.

The previous sections of this chapter provide the foundational skills for constructing and analyzing logical arguments. All logical arguments include a set of premises that support a claim or conclusion; but not all logical arguments are valid and sound. A logical argument is valid if its conclusion follows from the premises, and it is sound if it is valid and all of its premises are true. A false or deceptive argument is called a fallacy. Many types of fallacies are so common that they have been named.

Who Knew?

In 1936, Dale Carnegie published his first book, titled *How to Win Friends and Influence People*. It was marketed as training materials for the improvement of public speaking and negotiation skills, and the methods it presented are still used today. Carnegie famously said, “When dealing with people, remember you are not dealing with creatures of logic, but creatures of emotion.”

People who put forth fallacious logical arguments often take advantage of our susceptibility to emotional appeals, to try to convince us that what they are saying is true. The study of logic helps us combat this weakness through recognition and learning to focus on the facts and structure of the argument.

This section focuses on the two main forms that logical arguments can take. While inductive arguments attempt to draw a more general conclusion from a pattern of specific premises, deductive arguments attempt to draw specific conclusions from at least one or more general premises. Deductive arguments can be proven to be valid using Venn diagrams or truth tables.

Inductive arguments generally cannot be proven to be true. They are judged as being strong or weak, but, like any opinion, whether you believe an argument is strong or weak often depends on your knowledge of the topic being discussed along with the evidence

being provided in the premises. *Hasty generalization* is the name given to any fallacy that presents a weak inductive argument.

Checkpoint

Be careful! Premises may be true or false. If a premise is false, the claims made by the argument should be questioned.

Law of Detachment

The law of detachment is a valid form of a conditional argument that asserts that if both the conditional, $p \to q$, and the hypothesis, p, are true, then the conclusion q must also be true. The law of detachment is also called affirming the hypothesis (or antecedent) and modus ponens. Symbolically, it has the form $((p \to q) \wedge p) \to q$.

Law of Detachment	
Premise:	$p \to q$
Premise:	p
Conclusion:	$\therefore q$

Checkpoint

The $\therefore$ is read as the word, "therefore."

Looking at the truth table for the conditional statement, the only time the conditional is true is when the hypothesis p is also true. The only place this happens is in the first row, where q is also true, confirming that the law of detachment is a valid argument.

p	q	$p \to q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Another way to verify that the law of detachment is a valid argument is to construct a truth table for the argument $((p \to q) \wedge p) \to q$ and verify that it is a tautology.

p	q	$p \to q$	$(p \to q) \wedge p$	$((p \to q) \wedge p) \to q$
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	F	**T**
F	F	T	F	**T**

Venn diagrams may also be used to verify deductive arguments, which include conditional premises. Consider the statement $p \to q$:Then verify if the conclusion is also represented by the Venn diagram of the premises. If it is, the argument is valid. If it is not, the argument is not valid. The set of guitarists is drawn as a subset of the set of musicians to represent the premise.

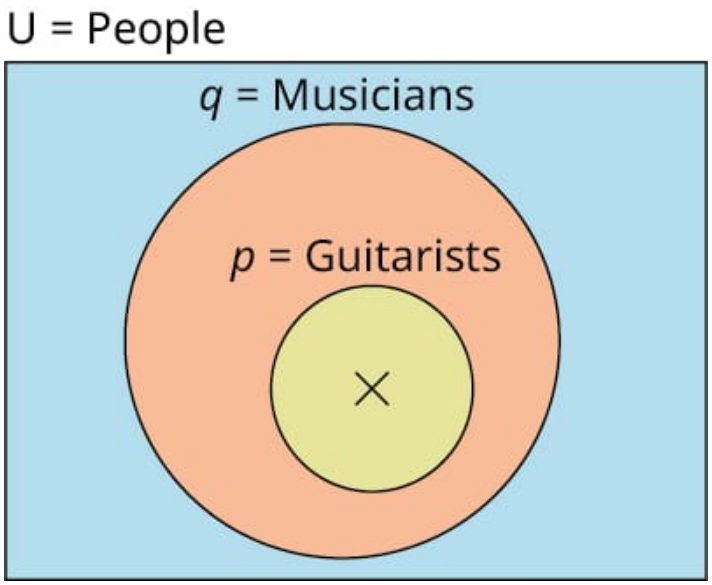


Figure 7.8.2

Now, examine the Venn diagram to verify if the conclusion is included in the picture. The conclusion is q. Because the $\times$ is in the set p, and p is a subset of q, $\times$ is also in q; therefore, the law of detachment is a valid argument.

Checkpoint

Remember that an argument can be valid without being true. For the argument to be proven true, it must be both valid and sound. An argument is sound if all its premises are true.

Video

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Logic Part 14: Common Argument Forms like Modus Ponens and Tollens

Exercise 7.8.1: Applying the Law of Detachment to Determine a Valid Conclusion

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of detachment to determine a valid conclusion.

1. If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*. Leonardo da Vinci was an artist.
2. If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player. Michael Jordan played for the Chicago Bulls.
3. If all fish have gills, then clown fish have gills. All fish have gills.

Answer available at the end of this section.

Your Turn 7.8.1

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of detachment to determine a valid conclusion.

1. If my classmate likes history, then some people like history. My classmate likes history.

2. If you do not like to read, then some people do not like reading. You do not like to read.

3. If the polygon has five sides, then it is not an octagon. The polygon has five sides.

Another form of a valid conditional argument is called the law of denying the consequent, or modus tollens. Recall, that the conditional statement, $p \to q$, is logically equivalent to the contrapositive, $\sim q \to \sim p$. So, if the conditional statement is true, then the contrapositive statement is also true. By the law of detachment, if $\sim q$ is also true, then it follows that $\sim p$ must also be true. Symbolically, it has the form $((p \to q) \wedge \sim q) \to \sim p$.

Law of Denying the Consequent	
Premise:	$p \to q$
Premise:	$\sim q$
Conclusion:	$\therefore \sim p$

Checkpoint

The conditional statement can also be described as, "If antecedent, then consequent."This is where the law of denying the consequent gets its name.

To verify if the law of denying the consequent is a valid argument, construct a truth table for the argument, $((p \to q) \wedge \sim q) \to \sim p$, and verify that it is a tautology.

p	q	$\sim p$	$\sim q$	$p \to q$	$(p \to q) \wedge \sim q$	$((p \to q) \wedge \sim q) \to \sim p$
T	T	F	F	T	F	**T**
T	F	F	T	F	F	**T**
F	T	T	F	T	F	**T**
F	F	T	T	T	T	**T**

To verify an argument of this form using a Venn diagram, again consider the premise: $p \to q$

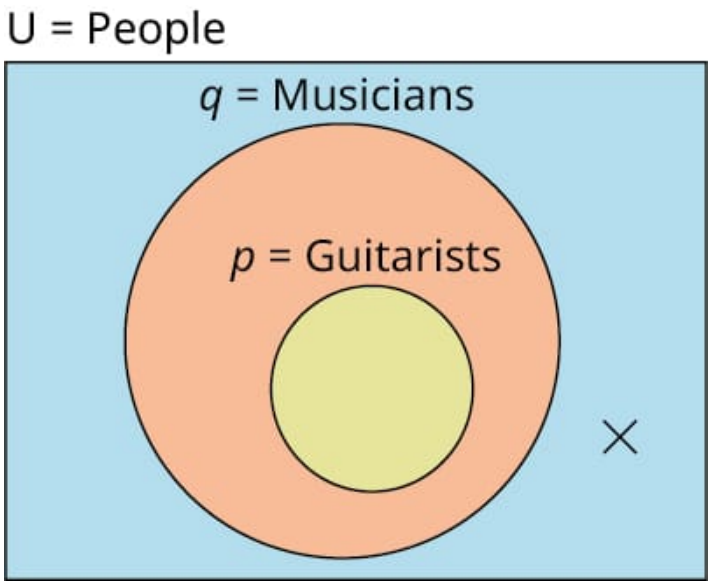


Figure 7.8.3

Because the $\times$ is also outside the set of guitarists, the statement $\sim p$ follows from the premises and the argument is valid.

Exercise 7.8.2: Applying the Law of Denying the Consequent to Determine a Valid Conclusion

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of denying the consequent to determine a valid conclusion.

1. If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*. Leonardo da Vinci did not paint the *Mona Lisa*.
2. If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player. Michael Jordan was a soccer player.
3. If all fish have gills, then clown fish have gills. Clown fish do not have gills.

Answer available at the end of this section.

Your Turn 7.8.2

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of denying the consequent to determine a valid conclusion.

1. If my classmate likes history, then some people like history. Nobody likes history.

2. If Homer does not like to read, then some people do not like reading. All people like reading.

3. If the polygon has five sides, then it is not an octagon. The polygon is an octagon.

Chain Rule for Conditional Arguments

The chain rule for conditional arguments is another form of a valid conditional argument. It is also called hypothetical syllogism or the transitivity of implication. Recall that the conditional statement $p \to q$ can also be read as p implies q. This is where the name *transitivity of implication* comes from. The transitive property for numbers states that, if $3 < 4$ and $4 < 5$, then it follows that $3 < 5$. The chain rule extends this property to conditional statements. If the premises of the argument consist of two conditional statements, with the form "$p \to q$" and "$q \to r$," then it follows that $p \to r$. Symbolically, it has the form $((p \to q) \wedge (q \to r)) \to (p \to r)$.

Chain Rule for Conditional Arguments	
Premise:	$p \to q$
Premise:	$q \to r$
Conclusion:	$\therefore p \to r$

To verify the chain rule for conditional arguments, construct a truth table for the argument, $((p \to q) \wedge (q \to r)) \to (p \to r)$, and verify that it is a tautology.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (p \to r)$	$p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$
T	T	T	T	T	T	T	**T**
T	T	F	T	F	F	F	**T**
T	F	T	F	T	F	T	**T**
T	F	F	F	T	F	F	**T**
F	T	T	T	T	T	T	**T**
F	T	F	T	F	F	T	**T**
F	F	T	T	T	T	T	**T**
F	F	F	T	T	T	T	**T**

To verify an argument of this form using a Venn diagram, again consider the premise $p \to q$:

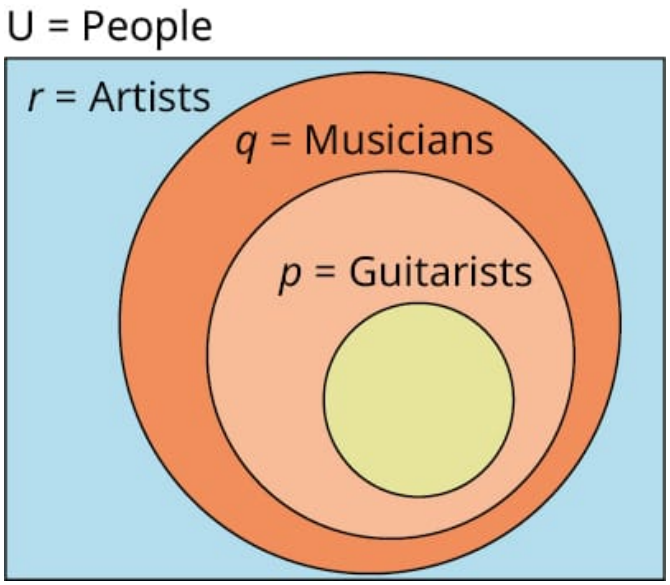


Figure 7.8.4

Exercise 7.8.3: Applying the Chain Rule for Conditional Arguments to Determine a Valid and Sound Conclusion

Each pair of statements represents true premises in a logical argument. Based on these premises, apply the chain rule for conditional arguments to determine a valid and sound conclusion.

1. If my roommate goes to work, then my roommate will get paid. If my roommate gets paid, then my roommate will pay their bills.

2. If robins can fly, then some birds can fly. If some birds can fly, then we will watch birds fly.
3. If Irma is a teacher, then Irma has a college degree. If Irma has a college degree, then Irma graduated from college.

Answer available at the end of this section.

Your Turn 7.8.3

Each pair of statements represent true premises in a logical argument. Based on these premises, apply the chain rule for conditional arguments to determine a valid and sound conclusion.

1. If my roommate does not go to work, then my roommate will not get paid. If my roommate does not get paid, then they will not be able to pay their bills.

2. If penguins cannot fly, then some birds cannot fly. If some birds cannot fly, then we will watch the news.

3. If Marcy goes to the movies, then Marcy will buy popcorn. If Marcy buys popcorn, then she will buy water.

Check Your Understanding

1. A _____ is a logical statement used as a fact to support the conclusion of an argument.
2. A logical argument is _____ if its conclusion follows from the premises.
3. A logical argument that attempts to draw a more general conclusion from a pattern of specific premises is called an _____ argument.
4. A _____ argument draws specific conclusions from more general premises.
5. Not all arguments are true. A false or deceptive argument is called a _____.
6. If an argument is valid and all of its premises are true, then it is considered _____.

Answers

Exercise 7.8.1: Applying the Law of Detachment to Determine a Valid Conclusion

1. The premises are $p \to q$: If Leonardo da Vinci was an artist, then he painted the Mona Lisa, and p : Leonardo da Vinci was an artist. This argument has the form of the law of detachment, so, the conclusion is q : Leonardo da Vinci painted the Mona Lisa.

2. The premises follow the form of the law of detachment, so a valid conclusion would be q. The premises are $p \to q$: If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player, and p : Michael Jordan played for the Chicago Bulls. The conclusion that follows from the premises is q : Michael Jordan was not a soccer player.

3. The premises are $p \to q$: If all fish have gills, then clown fish have gills, and p : All fish have gills. This argument has the form of the law of detachment, so the conclusion is q : clown fish have gills.

Exercise 7.8.2: Applying the Law of Denying the Consequent to Determine a Valid Conclusion

1. The premises are $p \to q$: If Leonardo da Vinci was an artist, then he painted the Mona Lisa, and $\sim q$: Leonardo da Vinci did not paint the Mona Lisa. This argument has the form of the law of denying the consequent, so the conclusion is $\sim p$: Leonardo da Vinci was not an artist.

2. The premises follow the form of the law of denying the consequent, so a valid conclusion would be $\sim p$. The premises are: $p \to q$: If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player, and $\sim q$: Michael Jordan was a soccer player. The conclusion that follows from the premises is $\sim p$: Michael Jordan did not play for the Chicago Bulls.

3. The premises are $p \to q$: If all fish have gills, then clown fish have gills, and $\sim q$: Clown fish do not have gills. This argument has the form of the law denying the consequent, so the conclusion is $\sim p$: Some fish do not have gills.

Exercise 7.8.3: Applying the Chain Rule for Conditional Arguments to Determine a Valid and Sound Conclusion

1. The premises are $p \to q$: "If my roommate goes to work, then they will get paid," and $q \to r$: "If my roommate gets paid, then my roommate will pay their bills." This argument has the form of the chain rule for conditional arguments, so the valid conclusion will have the form " $p \to r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If my roommate goes to work, then my roommate will pay their bills."

2. The premises are $p \to q$: "If robins can fly, then some birds can fly," and $q \to r$: "If some birds can fly, then we will watch them fly." This argument has the form of the chain rule for conditional arguments, so, the valid conclusion will have the form " $p \to r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If robins can fly, then we will watch birds fly."

3. The premises are $p \to q$: (see line 1 of solution 1 and 2 above) "If Irma is a teacher, then Irma has a college degree," and $q \to r$: "If Irma has a college degree, then Irma graduated from college." This argument has the form of the chain rule for conditional arguments, so the valid conclusion will have the form " $p \to r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If Irma is a teacher, then Irma graduated from college."

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7.9: CHAPTER SUMMARY

7.9.1: Key Terms

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7.9.1: Key Terms

Key Terms

2.1 Statements and Quantifiers

- logic
- logical statement
- truth values
- symbolic form
- negation of a logical statement
- quantifier
- premises
- conclusion
- inductive logical arguments

2.2 Compound Statements

- compound statement
- connective
- conjunction
- disjunction
- conditional
- hypothesis
- conclusion
- biconditional
- dominance of connectives

2.3 Constructing Truth Tables

- Truth table
- Multiplication principle
- Valid

2.5 Equivalent Statements

- logically equivalent
- tautology
- inverse
- converse
- contrapositive

2.6 De Morgan's Laws

- Boolean logic
- negation of a conditional

2.7 Logical Arguments

- sound

- fallacy
- deductive arguments
- law of detachment
- law of denying the consequent
- chain rule for conditional arguments

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7.9.2: Key Concepts

Key Concepts

2.1 Statements and Quantifiers

- Logical statements have the form of a complete sentence and make claims that can be identified as true or false.
- Logical statements are represented symbolically using a lowercase letter.
- The negation of a logical statement has the opposite truth value of the original statement.
- Be able to
  - Determine whether a sentence represents a logical statement.
  - Write and translate logical statements between words and symbols.
  - Negate logical statements, including logical statements containing quantifiers of *all, some, and none*.

2.2 Compound Statements

- Logical connectives are used to form compound logical statements by using words such as *and, or, and if ..., then*.
- A conjunction is a compound logical statement formed by combining two statements with the words “and” or “but.” If the two independent clauses are represented by p and q, respectively, then the conjunction is written symbolically as $p \wedge q$. For the conjunction to be true, both p and q must be true.
- A disjunction joins two logical statements with the *or* connective. In, logic *or* is inclusive. For an *or* statement to be true at least one statement must be true, but both may also be true.
- A conditional statement has the form if p, then q, where p and q are logical statements. The only time the conditional statement is false is when p is true, and q is false.
- The biconditional statement is formed using the connective if and only for the biconditional statement to be true, the true values of p and q, must match. If p is true then q must be true, if p is false, then q must be false.
- Translate compound statements between words and symbolic form.

Connective	Symbol	Name
and but	$\wedge$	conjunction
or	$\vee$	disjunction, inclusive or
not	~	negation
if ..., then implies	$\rightarrow$	conditional, implication
if and only if	$\leftrightarrow$	biconditional

The dominance of connectives explains the order in which compound logical statements containing multiple connectives should be interpreted.
The dominance of connectives should be applied in the following order

- Parentheses
- Negations
- Disjunctions/Conjunctions, left to right
- Conditionals
- Biconditionals

Dominance	Connective	Symbol	Evaluate
Least Dominant ↓ Most Dominant	Parentheses	( )	First ↓ Left to right or add parentheses to specify order because or/and have equal dominance. ↓ Last
	Negation	~	
	Disjunction/Conjunction	∨, ∧	
	Conditional	→	
	Biconditional	↔	

Figure 2.18

2.3 Constructing Truth Tables

- Determine the true values of logical statements involving negations, conjunctions, and disjunctions.
  - The negation of a logical statement has the opposite true value of the original statement.
  - A conjunction is true when both p and q are true, otherwise it is false.
  - A disjunction is false when both p and q are false, otherwise it is true.
- Know how to construct a truth table involving negations, conjunctions, and disjunctions and apply the dominance of connectives to determine the truth value of a compound logical statement containing, negations, conjunctions, and disjunctions.

Negation			Conjunction (AND)				Disjunction (OR)			
p	$\sim p$		p	q	$p \wedge q$		p	q	$p \vee q$	
T	F		T	T	T		T	T	T	
F	T		T	F	F		T	F	T	
			F	T	F		F	T	T	
			F	F	F		F	F	F	

A logical statement is valid if it is always true. Know how to construct a truth table for a compound statement and use it to determine the validity of compound statements involving negations, conjunctions, and disjunctions.

2.4 Truth Tables for the Conditional and Biconditional

- The conditional statement, if p then q, is like a contract. The only time it is false is when the contract has been broken. That is, when p is true, and q is false.

Conditional		
p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

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7.9.3: Videos

Videos

2.1 Statements and Quantifiers

YouTube Video

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- Logic Part 1A: Logic Statements and Quantifiers

2.2 Compound Statements

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- Logic Part 1B: Compound Statements, Connectives and Symbols

2.3 Constructing Truth Tables

YouTube Video

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- Logic Part 2: Truth Values of Conjunctions: Is an "AND" statement true or false?

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- Logic Part 3: Truth Values of Disjunctions: Is an "OR" statement true or false?

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- Logic Part 4: Truth Values of Compound Statements with "and", "or", and "not"

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- Logic Part 5: What are truth tables? How do you set them up?

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- Logic Part 6: More on Truth Tables and Setting Up Rows and Column Headings

2.4 Truth Tables for the Conditional and Biconditional

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- Logic Part 8: The Conditional and Tautologies

YouTube Video

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- Logic Part 11B Biconditional and Summary of Truth Value Rules in Logic

YouTube Video

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- Logic Part 13: Truth Tables to Determine if Argument is Valid or Invalid

2.7 Logical Arguments

YouTube Video

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- Logic Part 14: Common Argument Forms like Modus Ponens and Tollens

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7.9.4: Projects

Projects

Logic Gates

Logic gates are the basis for all digital circuits.

1. Research and document the following terms: logic gate, OR gate, AND gate, and NOT gate.
2. Construct a diagram of a NAND gate, NOR gate, and a XOR gate by using at least two of the following gates: AND, OR, and NOT.
3. Digital electronics use a 1 for true or on, and a 0 for false or off. Create a truth table documenting all possible cases using 0s and 1s for the NAND gate, NOR gate and XOR gate.
4. Use a truth table to explain how XOR is related to the biconditional statement.

Logical Fallacies

Fallacies are false or deceptive logical arguments.

1. Research and document the structure of five of the following named fallacies: hasty generalization, limited choice, false cause, appeal to popularity, appeal to emotion, appeal to authority, personal attack, gamblers' ruin, slippery slope, and circular reasoning.
2. Create a presentation highlighting one of the five fallacies researched in the previous question. The presentation must include an introductory slide with the title of the fallacy and the form or structure of the argument. The second slide must include an example of this fallacy as used in a commercial, a political cartoon or a current event or new article. The third slide must include an explanation of why the example on slide to is a representative example of the fallacy. The last slide must include citations for any materials used. No textbooks should be used as reference.

Careers in Logic

Lawyers, mathematicians, and computer programmers are a few of the careers that require knowledge of logic.

1. What career are you interested in? Research how knowledge of logic applies to your chosen field of study. Then, write a cover letter for a position in your field you'd like to apply to. In the cover letter, include how your knowledge of logic qualifies you for the position you are applying for. If you do not think logic is important for your given career choice, find a position where logic is an essential element of the position and complete the project by pretending you are writing a cover letter for that job.

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7.9.5: Chapter Review

Statements and Quantifiers

Fill in the blanks to complete the following sentences.

Exercise 7.9.5.1

The _____ of a logical statement has the opposite truth value of the original statement.

Exercise 7.9.5.2

_____ are logical statements presented as the facts used to support the conclusion of a logical argument.

Determine whether each of the following sentences represents a logical statement, also called a proposition. If it is a logical statement, determine whether it is true or false.

Exercise 7.9.5.3

Where is the restroom?

Exercise 7.9.5.4

No even numbers are odd numbers.

Exercise 7.9.5.5

4 + 3 = 8.

Write the negation of each following statement symbolically and in words.

Exercise 7.9.5.6

p : Pink Floyd's album *The Wall* is not a rock opera.

Exercise 7.9.5.7

q : Some dogs are Labrador retrievers.

Exercise 7.9.5.8

r : Some universities are not expensive.

Draw a logical conclusion to the following arguments, and include in both one of the following quantifiers: all, some, or none.

Exercise 7.9.5.9

Spaghetti noodles are made with wheat, ramen noodles are made with wheat, and lo mein noodles are made with wheat.

Exercise 7.9.5.10

A Porsche Boxster does not have four doors, a Volkswagen Beetle does not have four doors, and a Mazda Miata does not have four doors.

Compound Statements

Fill in the blanks to complete the following sentences.

Exercise 7.9.5.11

_____ are words or symbols used to join two or more logical statement together to from a compound statement.

Exercise 7.9.5.12

_____ and _____ have equal dominance and are evaluated from left to right when no parentheses are present in a compound logical statement.

Translate each compound statement below into symbolic form. Given: p : "Tweety Bird is a bird," q : "Bugs is a bunny," r : "Bugs says, 'What's up, Doc?'," s : "Sylvester is a cat," and t : "Sylvester chases Tweety Bird."

Exercise 7.9.5.13

If Tweety Bird is a bird, then Sylvester will not chase him.

Exercise 7.9.5.14

Tweety Bird is a bird and Sylvester chases him if and only if Bugs says, "What's up Doc?"

Translate the symbolic form of each compound logical statement below into words.

Given: p : "Tweety Bird is a bird," q : "Bugs is a bunny," r : "Bugs says, 'What's up, Doc?'," s : "Sylvester is a cat," and t : "Sylvester chases Tweety Bird."

Exercise 7.9.5.15

$\sim q \vee p \rightarrow \sim s$

Exercise 7.9.5.16

$\sim (p \wedge \sim s) \leftrightarrow q \rightarrow r$

For each of the following compound logical statements, apply the proper dominance of connectives by adding parentheses to indicate the order to evaluate the statement.

Exercise 7.9.5.17

$p \vee q \wedge r \rightarrow \sim s \wedge t$

Exercise 7.9.5.18

$\sim p \rightarrow q \vee r \leftrightarrow p \wedge s \rightarrow \sim t$

Constructing Truth Tables

Fill in the blanks to complete the sentences.

Exercise 7.9.5.19

A _____ is true if at least one of its component statements is true.

Exercise 7.9.5.20

For a _____ to be true, all of its component statements must be true.

Given the statements, p : "No fish are mammals," q : "All lions are cats," and r : "Some birds do not lay eggs," construct a truth table to determine the truth value of each compound statement below.

Exercise 7.9.5.21

$p \wedge \sim r$

Exercise 7.9.5.22

$\sim (p \vee \sim r)$

Exercise 7.9.5.23

$\sim p \vee q \wedge \sim (\sim r)$

Construct a truth table to analyze all the possible outcomes of the following statements, and determine whether the statements are valid.

Exercise 7.9.5.24

$\sim p \vee q \wedge p$

Exercise 7.9.5.25

$\sim p \vee q \vee \sim q$

Truth Tables for the Conditional and Biconditional

Fill in the blanks to complete the following sentences.

Exercise 7.9.5.26

If the , p, of a conditional statement is true, then the conclusion, q, must also be true for the conditional statement $p \rightarrow q$ to be true.

Exercise 7.9.5.27

The biconditional statement $p \leftrightarrow q$ is whenever the truth value of p matches the truth value of q, otherwise it is

Complete the truth tables below to determine the truth value of the proposition in the last column.

Exercise 7.9.5.28

p	q	r	$p \vee q$	$\sim (p \vee q)$	r	$\sim (p \vee q) \rightarrow \sim r$
F	F	T				

Exercise 7.9.5.29

p	q	q	$p \rightarrow q$	$\sim (p \rightarrow q)$	$p \wedge \sim q$	$\sim (p \rightarrow q) \leftrightarrow (p \wedge \sim q)$
T	F					

Assume the following statements are true. p : "Poof is a baby fairy," q : "Timmy Turner has fairly odd parents," r : "Cosmo and Wanda will grant Timmy's wishes," and t : "Timmy Turner is 10 years old." Translate each of the following statements into symbolic form, then determine its truth value.

Exercise 7.9.5.30

If Timmy Turner is 10 years old and Poof is not a baby fairy, then Timmy Turner has fairly odd parents.

Exercise 7.9.5.31

Cosmos and Wanda will not grant Timmy's wishes if and only if Timmy Turner is 10 years old or he does not have fairly odd parents.

Exercise 7.9.5.32

Construct a truth table to analyze all the possible outcomes and determine the validity of the following argument.

$p \vee q \leftrightarrow \quad q \rightarrow \quad p$

Equivalent Statements

Fill in the blanks to complete the sentences below. 33. The _____ is logically equivalent to the inverse $\sim p \rightarrow \sim q\ p \rightarrow \ q$.

Exercise 7.9.5.34

The _____ is logically equivalent to the conditional $p \rightarrow qp \rightarrow q$.

Use the conditional statement, $p \rightarrow qp \rightarrow q$: “If Novak makes the basket, then Novak’s team will win the game," to answer the following questions. 35. Write the conclusion of the conditional statement in words and label it appropriately.

Exercise 7.9.5.36

Write the hypothesis of the conditional statement in words and label it appropriately.

Exercise 7.9.5.37

Identify the following statement as the converse, inverse, or contrapositive: “If Novak does not make the basket, then his team will not win the game.”

Exercise 7.9.5.38

Identify the following statement as the converse, inverse, or contrapositive: “If Novak’s team wins the game, then he made the basket.”

De Morgan’s Laws

Fill in the blanks to complete the sentences.

Exercise 7.9.5.39

De Morgan’s Law for the negation of a disjunction states that \($(p \vee q) \equiv$_____.

Exercise 7.9.5.40

De Morgan’s Law for the negation of a conjunctions states that _____\($\equiv \quad p \ \vee \ q$.

Exercise 7.9.5.41

Apply De Morgan's Law to write the statement without parentheses: \(\sim (p \wedge q)\).

Exercise 7.9.5.42

Apply the property for the negation of a conditional to write the statement as a conjunction or disjunction: \(\sim (p \wedge q \rightarrow \sim r)\).

Exercise 7.9.5.43

Write the negation of the conditional statement in words: If Thomas Edison invented the phonograph, then albums are made of vinyl, or the transistor radio was the first portable music device.

Exercise 7.9.5.44

Construct a truth table to verify that the logical property is valid: $\sim (p \rightarrow \sim q) \equiv \sim p \wedge q$.

Logical Arguments

Fill in the blanks to complete the sentences below.

Exercise 7.9.5.45

The _____ ___ _____ is a valid logical argument with premises, $p \rightarrow q$ and p, used to support the conclusion, q.

Exercise 7.9.5.46

The chain rule for conditional arguments states that the _____ property applies to conditional arguments, so that: \((p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)\).

Assume each pair of statements represents true premises in a logical argument. Based on these premises, state a valid conclusion that is consistent with the form of the argument.

Exercise 7.9.5.47

If the Tampa Bay Buccaneers did not win Super Bowl LV, then Tom Brady was not their quarterback. Tom Brady was the Tampa Bay Buccaneers quarterback.

Exercise 7.9.5.48

If $\sim q$, then p and if r, then $\sim q$.

Exercise 7.9.5.49

If Kamala Harris is the vice president of the United States, then Kamala Harris is the president of the U.S. Senate. Kamala Harris is the vice president of the United States.

Exercise 7.9.5.50

Construct a truth table or Venn diagram to prove whether the following argument is valid. If the argument is valid, determine whether it is sound.

If all frogs are brown, then Kermit is not a frog. Kermit is a frog. Therefore, some frogs are not brown.

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7.9.6: Reasoning and Logic

Logic is a tool to solve problems. In mathematical logic, a PROPOSITION (or STATEMENT) is a sentence that is either true or false. It cannot be both. One way to think about it is that in order for it to be true, it must be always be true. There can be no qualifiers.

The following are NOT statements in mathematical logic:

- He has black hair. (This depends on who “he” is, so we can’t tell if it is true or false.)
- x + 3y = 6 (This depends on the values of x and y.)
- 7 + 16 (This is not even a sentence.)
- Go to the store. (This has nothing to do with being true or false.)
- Are you a woman? (This is a question.)
- Napoleon was short. (This depends on how someone defines short.)

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7.9.7: Basic Arguments- Using Logic

An argument requires a number of **premises** (facts or assumptions) which are followed by a **conclusion** (point of the argument). The premises are used as justification for a conclusion. A conclusion which is correctly supported by the premises is known as a **valid argument**, while a **fallacy** is a deceptive argument that can sound good but is not well supported by the premises.

We will look at examples where the first two statements are the premises, and the third statement is the conclusion.

Example 1

Determine if the following argument is valid.

All men are mortal.

John Smith is a man.

John Smith is mortal.

There are two premises (the first 2 sentences) and one conclusion (the last sentence). If we think of the premises as a and b, and the conclusion as c, then the argument in symbolic form is: $a \wedge b) \rightarrow c$. In order for the argument to be valid, we need this conditional statement to always be true. If there is ever a time, *even just one time*, when this conditional statement is false, then it is an **invalid argument**. Another way to think of this is to say that the conclusion must follow from the premises. If the premises are true, then the conclusion *must* be true in order for the argument to be valid.

Look at the argument – if we *assume* that a and b are both true, then does the conclusion have to follow? YES! If all men are mortal, and if John Smith is a man, then John Smith must be mortal. This is valid.

Example 2

Determine if the following argument is valid.

All dogs are yellow.

Flippy is a dog.

Flippy is yellow.

All dogs are yellow is equivalent to “If it is a dog then it is yellow.” That is equivalent to “If it is not yellow, then it is not a dog” by the contrapositive. Assume the premises are true. Does the conclusion have to follow? YES! So this is valid!

In both of the examples above, the first statement of the premises could be written as an “if-then” statement. “All dogs are yellow” means the same thing as “If it is a dog, it is yellow."

The above examples are examples of **Modus Ponens**, which is always a valid argument.

Format of **Modus Ponens** (which is a valid logical argument)

$p \rightarrow q$

p

q

Basically Modus Ponens states that if p implies q, and p is true, then q must also be true!

One could create a truth table to show Modus Tollens is true in all cases : $[(p \to q) \wedge p] \to q$

Example

Determine if the following argument is valid. (Hint: rewrite the "all" as "if-then", then also write the contrapositive)

All dogs are yellow.

Chipper is yellow.

Chipper is a dog.

"All dogs are yellow" is equivalent to "If it is a dog then it is yellow." or "If it is not yellow, then it is not a dog" by the contrapositive. Assume the premises are true. Does the conclusion have to follow? It states all dogs are yellow, but doesn't say anything about yellow things, or that everything yellow is a dog. It is possible to have something yellow (like a lemon) that is not a dog; that means the conclusion isn't necessarily true. This argument is invalid. Let p stand for "It is a dog." Let q stand for "It is yellow." The format of the above argument, shown below, is not Modus Ponens.

It is an example of **Fallacy by Converse Error**.

$p \to q$
p
q

Remember the example where p is "You live in Vista" and q is "You live in California"? Consider

If you live in Vista, then you live in California.	p → q
Johns lives in Vista.	p
So, John lives in California.	q

This is a valid logical statement because it is of the form **Modus Ponens**.

If you live in Vista, then you live in California.	p → q
John lives in California.	q
So, Johns lives in Vista.	p

This is an invalid argument, and is an example of **Fallacy by Converse Error**.

If you live in Vista, then you live in California.	p → q
Johns does not live in Vista.	~ p
So, John does not live in California.	~q

This is also an invalid argument, and is an example of **Fallacy by Inverse Error**.

If you live in Vista, then you live in California.	p → q
John does not live in California.	~ q
So, John does not live in Vista.	~ p

This is a valid argument, and is an example of **Modus Tollens**.

Modus Tollens is based on the contrapositive. Remember that p → q *is logically equivalent to* (~ q) → (~ p)

So the above argument could be written in four steps:

If you live in Vista, then you live in California.	p → q
Rewrite as the contrapositive:	
If you do not live in California, you do not live in Vista.	(~ q) → (~ p)
John does not live in California.	~ q
So, John does not live in Vista.	~ p

The last three statements LOOKS like Modus Ponens. But the original argument only had three lines. It wasn't written as the contrapositive. So it's not called Modus Ponens. One could create a truth table to show Modus Tollens is true in all cases: [(p → q) ∧ $q] \rightarrow \ p$

Another reasoning argument is called the **Chain Rule (transitivity)**. Below is an example. The first two sentences are the premises, and the last is the conclusion. If the first two are true, the conclusion is true.

If I have a bus pass, I will go to school.

If I go to school, I will attend class.

If I have a bus pass, I will attend class.

So the idea is that if "if p, then q" and "if q, then r" are both true, then "if p, then r" is also true.

Symbolically, the chain rule is: [(p → q) $\wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

One could create a truth table to show the truth table is true in all cases, but it's more complicated because there are 3 statements, hence 8 rows in the truth table.

The format for the Chain Rule where the first two lines are the premises and the third is the conclusion is:
p → q

q → r

p → r

Exercise 17

17. If the two statements below are premises, use the Chain Rule to state the conclusion.

If Mia doesn't study, then Mia does not pass the final.

If Mia does not pass the final, then Mia does not pass the class.

What about a logic statement where all of the outcomes of a formula are true in every situation? When this happens, it is called a **tautology**. Modus Ponens, Modus Tollens, and the Chain Rule (transitivity) are tautologies. A truth table will show the statement true in each row of the column for that statement. **A fallacy is when all the outcomes of a logic statement are false.** An example of a fallacy in words is "I called Jim and I did not call Jim." If p is "I called Jim," the logic statement in symbols for this fallacy is $p \wedge \ p$). A tautology would be "I called Jim or I did not call Jim," which is written as $p \vee \ p$)

You will create your own truth tables for Modus Ponens and Modus Tollens in the next exercises. Create intermediate columns so it is clear how you get the final column, which will show each is a tautology.

Exercise 18

18. Make a Truth Table showing Modus Ponens is a valid argument. In other words, create and fill out a truth table where the last column is $[(p \rightarrow q) \wedge p] \rightarrow q$, and show that in all four situations, it is true, which means it is a tautology

Exercise 19

19. Make a Truth Table showing Modus Tollens is a valid argument. In other words, create and fill out a truth table where the last column is $[(p \rightarrow q) \wedge q] \rightarrow \ p$, and show that in all four situations, it is true.

SUMMARY of arguments, where the first two statements are premises, and the third is the conclusion.

Format of **Modus Ponens** (which is a valid logical argument) $p \rightarrow q$ p q
Format of **Modus Tollens** (which is a valid logical argument) $p \rightarrow q$ $\sim q$ $\sim p$

Format of **Fallacy by the Converse Error** (an invalid argument)

$p \rightarrow q$

q

p

Format of **Fallacy by the Inverse Error** (an invalid argument)

$p \rightarrow q$

$\sim p$

$\sim q$

Format of **Chain Rule** (which is a valid logical argument)

$p \rightarrow q$

$q \rightarrow r$

$p \rightarrow r$

Exercise 20

20. Determine if the following arguments are valid or not. If they are valid, write if it is by Modus Ponens, Modus Tollens, or the Chain Rule. If it is not valid, write if it is by Fallacy by Converse Error, or Fallacy by Inverse Error, or neither. If it looks like the chain rule, but has a false conclusion, write the correct conclusion.

If you are a gambler, then you are not financially stable.

Sean isn't financially stable.

Sean is a gambler.

If you have a college degree, then you are not lazy.

Marsha has a college degree.

Marsha is not lazy.

If you have a college degree, then you are not lazy.

Shannon is lazy.

Shannon does not have a college degree.

If you have a college degree, then you are not lazy. Beth is not lazy. Beth has a college degree.
If you are a kitten, then you are a cat. If you are a cat, then you can purr. If you are a kitten, then you can purr.
If you are a comedian, then you are funny. If you are funny, then you are smart. If you are smart, then you are a comedian.

Exercise 21

21. Write a conclusion that would make each argument valid, and state if you used Modus Ponens or Modus Tollens.

If you are a gambler, then you are not financially stable. Hollis is financially stable.
If a defendant is innocent, then he does not go to jail. Podric went to jail

Exercise 22

22. Determine whether there is a problem with the person's thinking. Explain your reasoning.

A) John's mom told him "If you get home after 10pm, then you are grounded." John got home at 9:30pm and was grounded. He was really ticked off because he said that she lied to him. Did she?

B) Marcia told her daughter: "If you get home before 10pm, then I will give back your cell phone." Her daughter got home at 9:45pm, but her mom didn't give back the cell phone. Did her mother lie?

Exercise 23

23. Create a truth table for $p \vee (p \rightarrow q)$

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7.9.8: Review Exercises

Exercise 1

1. State George Polya's Four Step Problem Solving Process. Only list the 4 headings.

Exercise 2

2. State 4 strategies for Devising a Plan.

Exercise 3

3. Juan will give you $1 on the first day of April, $2 on the second day of April, $4 on the third day of April, and double the amount through April 15th, or he will give you $30,000 up front on the first day of April. Which is the better deal and why?

Exercise 4

4. Find the next three numbers in the sequence:

a. 0, 10, 10, 20, 20, 20, 30, 30, 30, 30, 40,

b. 5, 15, 45, 135,

Exercise 5

5. List the ways you can make change for $25 using $5, $10, and $20 bills.

Exercise 6

6. Explain the difference between inductive and deductive reasoning.

Exercise 7

7. State whether each of the following is a proposition are not. If not, state why not. If it is a proposition, state its truth value.

a. 3 + 4 = 7

b. California is not in the U.S.

c. Mary is pretty.

Exercise 8

8. Determine whether each quantified statement is true or false.

a. All men have black hair.

b. Some women have blond hair.

c. No boys like pigs.

Exercise 9

9. State the negation of each statement.

a. Grass is green.

b. Tim married Tom.

c. It is not cold outside.

Exercise 10

10. State the negation of each quantified statement.

a. Everyone plays Pokemon Go.

b. Some kids wear hats.

c. No dogs jump on the couch.

Exercise 11

11. If p is true and q is false, state the truth value of each of the following.

a. $p \wedge q$
b. $p \vee (q)$
c. p → q
d. q → p
e. ~$(p \vee q)$

Exercise 12

12. Negate each statement.

a. I am married if I wear a wedding ring.

b. It is hot outside and I am at the beach.

c. My name is Julie or my name is Beth.

Exercise 13

13. State the converse, inverse, and contrapositive of this conditional.

"If Omar doesn't buy cat food, then Omar owns a fish.

Exercise 14

14. Determine if the following arguments are valid or not. If they are valid, write if it is by Modus Ponens, Modus Tollens, or the Chain Rule. If it is not valid, write if it is by Fallacy by Converse Error, or Fallacy by Inverse Error, or neither. If it looks like the chain rule, but has a false conclusion, write the correct conclusion.

If you are a child then you are not a legal adult. Sean isn't a legal adult. Sean is a child.
If you drink wine, then you are at a bar. Marsha is at a bar. Marsha drinks wine.
If you eat hot dogs, then it is 4th of July. It is 4th of July. You are eating hot dogs.
If you like vegetables, then you are vegan. Beth is not vegan. Beth does not like vegetables.
If you are in college, then you have a car. If you have a car, then you know how to drive. If you are in college, then you know how to drive.

Exercise 15

15. Write a conclusion to make each argument valid. State if you used Modus Ponens or Modus Tollens.

If you eat worms, then you are a fish. Matt is not a fish.

If you have a certificate, then you have a job.

Isabel has a certificate.

Exercise 16

16. What is a tautology? What is a fallacy?

Exercise 17

17. Make a truth table for the following. State if it is a tautology, fallacy, or neither.

a. p $\rightarrow$ $(p \vee q)$
b. ~$(plandq) \rightarrow$ q

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7.9.9: Chapter Test

Chapter Test

Determine whether each of the following sentences represent a logical statement. If it is a logical statement, determine whether it is true or false.

1. $1+2-3=0$

2. Please, sit down over there.

3. All mammals lay eggs.

Write the negation of each statement below.

4. Some monkeys do not have tails.

5. $p \wedge \sim q$

6. If the plumber does not remove the clog, then the homeowner will not pay the plumber.

Given: p : Frodo is a hobbit, q : Gandalf is a wizard, r : Frodo and Samwise will take the ring to Mordor, and s : Gollum will help Frodo get into Mordor.
Translate the symbolic form of each compound logical statement into words.

7. $p \vee q \leftrightarrow r$

8. $\sim(\sim r \vee \sim s)$

Translate the written form of each compound logical statement into symbolic form.

9. Frodo and Samwise will take the ring to Mordor or Gandalf is not a wizard and Frodo is a hobbit.

10. If Gollum will not help Frodo get into Mordor, then Gandalf is not a wizard and Frodo is not a hobbit.

For each of the following compound logical statements, apply the proper dominance of connectives by adding parentheses to indicate the order in which the statement must be evaluated

11. $\sim p \rightarrow q \leftrightarrow r$

12. $\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q$

13. Complete the truth table to determine the truth value of the proposition in the last column.

p	q	$p \wedge q$	$\sim q$	$p \vee \sim q$	$(p \wedge q) \leftrightarrow (p \vee \sim q)$
F	T				

Given the true statements p : "A right triangle has one 90-degree angle," q : "The triangle is a right triangle," r : " $a^2+b^2=c^2,$ " and s : "The longest side of a triangle is c implies $a+b$ must be $>c$." Write each of the following compound statements in symbolic form, then construct a truth table to determine the truth value of the compound statement.

14. If a triangle is a right triangle, then it does not have one 90-degree angle or $a^2+b^2=c^2$.

15. The triangle is a right triangle, or a right triangle does not have a 90-degree angle, if and only if it is not the case that the longest side of a triangle is c implies $a+b$ must be $>c$.

Use the conditional statement, $p \rightarrow q$: "If Phil Mickelson is 50 years old, then Phil Mickelson won the Player's Championship," to answer the following questions.

16. Write the converse statement in words.

17. If the conditional statement is true, and the hypothesis is true, what is a valid conclusion to the argument?

18. If the conditional statement is true, and the conclusion is false, what is a valid conclusion to the argument?

19. Construct a truth table to analyze all the possible outcomes and determine the validity of the following argument.

$$\sim p \vee q \leftrightarrow q \rightarrow p$$

20. Construct a truth table or Venn diagram to prove whether the following argument is valid. If the argument is valid, determine whether it is sound.
If John Mayer played MTV unplugged, then some guitars are acoustic. John Mayer played MTV unplugged. Therefore, some guitars are acoustic.

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8: ALGEBRAIC THINKING

Algebraic Thinking

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8.1: The Language of Algebra

Learning Objectives

By the end of this section, you will be able to:

- Use variables and algebraic symbols
- Evaluate an expression
- Identify and combine like terms
- Translate an English phrase to an algebraic expression

Use Variables and Algebraic Symbols

Suppose this year Greg is 20 years old and Alex is 23. You know that Alex is 3 years older than Greg. When Greg was 12, Alex was 15. When Greg is 35, Alex will be 38. No matter what Greg's age is, Alex's age will always be 3 years more, right? In the language of algebra, we say that Greg's age and Alex's age are variables and the 3 is a constant. The ages change ("vary") but the 3 years between them always stays the same ("constant"). Since Greg's age and Alex's age will always differ by 3 years, 3 is the constant. In algebra, we use letters of the alphabet to represent variables. So if we call Greg's age g, then we could use $g+3$ to represent Alex's age. See Table $8.1.1$.

Table $8.1.1$

Greg's age	Alex's age
12	15
20	23
35	38
g	$g+3$

The letters used to represent these changing ages are called variables. The letters most commonly used for variables are $x, y, a, b,$ and c.

Definition: VARIABLE

A **variable** is a letter that represents a number whose value may change.

Definition: CONSTANT

A **constant** is a number whose value always stays the same.

To write algebraically, we need some operation symbols as well as numbers and variables. There are several types of symbols we will be using.

There are four basic arithmetic **operations: addition, subtraction, multiplication,** and **division**. We'll list the symbols used to indicate these operations below (Table $8.1.2$). You'll probably recognize some of them.

Table $8.1.2$

Operation	Notation	Say:	The result is...
Addition	$a+b$	a plus b	the sum of a and b

Operation	Notation	Say:	The result is...
Subtraction	$a-b$	a minus b	the difference of a and b
Multiplication	$a \cdot b, ab, (a)(b), (a)b, a(b)$	a times b	the product of a and b
Division	$a \div b, a/b, \frac{a}{b}, b\overline{)a}$	a divided by b	the quotient of a and b, a is called the dividend, and b is called the divisor

We perform these operations on two numbers. When translating from symbolic form to English, or from English to symbolic form, pay attention to the words “of” and “and.”

- The *difference* of 9 and 2 means subtract 9 and 2, in other words, 9 minus 2, which we write symbolically as $9-2$.
- The *product* of 4 and 8 means multiply 4 and 8, in other words 4 times 8, which we write symbolically as $4 \cdot 8$.

In algebra, the cross symbol, $\times$, is not used to show multiplication because that symbol may cause confusion. Does $3xy$ mean $3 \times y$ (‘three times y’) or $3 \cdot x \cdot y$ (three times x times y)? To make it clear, use $\cdot$ or parentheses for multiplication.

When two quantities have the same value, we say they are equal and connect them with an **equal sign**.

EQUALITY SYMBOL

$a=b$ is read “a is equal to b”

The symbol ″ $=$″ is called the **equal sign**.

On the number line, the numbers get larger as they go from left to right. The number line can be used to explain the symbols ″ $<$″ and ″ $>$ ″.

INEQUALITY

\(a

a is to the left of b on the number line

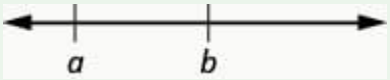


Figure $8.1.1$

$a>b$ is read "a is greater than b”

a is to the right of b on the number line

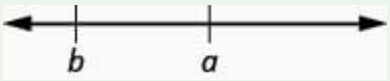


Figure $8.1.2$

The expressions $a<b$ or $a>b$ can be read from left to right or right to left, though in English we usually read from left to right Table $8.1.3$. In general, $a<b$ is equivalent to $b>a$. For example $7<11$ is equivalent to $11>7$. And $a>b$ is equivalent to $b<a$. For example $17>4$ is equivalent to $4<17$.

Table $8.1.3$

Inequality Symbols	Words
$a \neq b$	a is not equal to b
$a<b$	a is less than b

Inequality Symbols	Words
$a \leq b$	a is less than or equal to b
$a > b$	a is greater than b
$a \geq b$	a is greater than or not equal to b

Exercise 8.1.1

Translate from algebra into English:

1. $17 \leq 26$
2. $8 \neq 17 - 3$
3. $12 > 27 \div 3$
4. $y + 7 < 19$

Answer available at the end of this section.

Exercise 8.1.2

Translate from algebra into English:

1. $14 \leq 27$
2. $19 - 2 \neq 8$
3. $12 > 4 \div 2$
4. $x - 7 < 1$

Answer available at the end of this section.

Exercise 8.1.3

Translate from algebra into English:

1. $19 \leq 15$
2. $7 = 12 - 5$
3. $15 \div 3 < 8$
4. $y + 3 < 6$

Answer available at the end of this section.

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in English. They help to make clear which expressions are to be kept together and separate from other expressions. We will introduce three types now.

GROUPING SYMBOLS

Parentheses	$()$
Brackets	$[]$
Braces	$\{\}$

Here are some examples of expressions that include **grouping symbols**. We will simplify expressions like these later in this section.

$$8(14 - 8) \qquad 21 - 3[2 + 4(9 - 8)] \qquad 24 \div \{13 - 2[1(6 - 5) + 4]\}$$

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. “Running very fast” is a phrase, but “The football player was running very fast” is a sentence. A sentence has a subject and a verb. In algebra, we have expressions and *equations*.

EXPRESSION

An **expression** is a number, a variable, or a combination of numbers and variables using operation symbols.

An **expression** is like an English phrase. Here are some examples of expressions:

Table 8.1.4

Expression	Words	English Phrase
$3+5$	3 plus 5	the sum of three and five
$n-1$	n minus one	the difference of n and one
$6 \cdot 7$	6 times 7	the product of six and seven
$\frac{x}{y}$	x divided by y	the quotient of x and y

Notice that the English phrases do not form a complete sentence because the phrase does not have a verb. An **equation** is two expressions linked with an **equal sign**. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb.

Definition: EQUATION

An **equation** is two expressions connected by an equal sign.

Here are some examples of equations.

Table 8.1.5

Equation	English Sentence
$3+5=8$	the sum of three and five is equal to eight
$n-1=14$	n minus one equals fourteen
$6 \cdot 7=42$	The product of six ans seven is equal to forty-two
$x=53$	x is equal to fifty-three
$y+9=2y-3$	y plus nine is equal to two y minus three

Exercise 8.1.4

Determine if each is an expression or an equation:

1. $2(x+3)=10$
2. $4(y-1)+1$
3. $x \div 25$
4. $y+8=40$

Answer available at the end of this section.

Exercise 8.1.5

Determine if each is an expression or an equation:

1. $3(x-7)=27$
2. $5(4y-2)-7$

Answer available at the end of this section.

Exercise 8.1.6

Determine if each is an expression or an equation:

1. $y^3 \div 14$
2. $4x-6=22$

Answer available at the end of this section.

Evaluate an Expression

In the last few examples, we simplified expressions using the order of operations. Now we'll evaluate some expressions—again following the order of operations. To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

EVALUATE AN EXPRESSION

To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

To evaluate an expression, substitute that number for the variable in the expression and then simplify the expression.

Exercise 8.1.19

Evaluate $7x-4$, when

1. $x=5$
2. $x=1$

Answer available at the end of this section.

Exercise 8.1.20

Evaluate $8x-3$, when

1. $x=2$
2. $x=1$

Answer available at the end of this section.

Exercise 8.1.21

Evaluate $4y - 4$, when

1. $y = 3$
2. $y = 5$

Answer available at the end of this section.

Exercise 8.1.22

Evaluate $x = 4$, when

1. x^2
2. 3^x

Answer available at the end of this section.

Exercise 8.1.23

Evaluate $x = 3$, when

1. x^2
2. 4^x

Answer available at the end of this section.

Exercise 8.1.24

Evaluate $x = 6$, when

1. x^3
2. 2^x

Answer available at the end of this section.

Exercise 8.1.25

Evaluate $2x^2 + 3x + 8$ when $x = 4$.

Answer available at the end of this section.

Exercise 8.1.26

Evaluate $3x^2 + 4x + 1$ when $x = 3$.

Answer available at the end of this section.

Exercise 8.1.27

Evaluate $6x^2 - 4x - 7$ when $x = 2$.

Answer available at the end of this section.

Identify and Combine Like Terms

Algebraic expressions are made up of terms. A **term** is a constant, or the product of a constant and one or more variables.

TERM

A **term** is a constant, or the product of a constant and one or more variables.

Examples of terms are $7, y, 5x^2, 9a$, and b^5.

The constant that multiplies the variable is called the **coefficient**.

COEFFICIENT

The **coefficient** of a term is the constant that multiplies the variable in a term.

Think of the coefficient as the number in front of the variable. The coefficient of the term $3x$ is 3. When we write x, the coefficient is 1, since $x = 1 \cdot x$.

Exercise 8.1.28

Identify the coefficient of each term:

1. $14y$
2. $15x^2$
3. a

Answer available at the end of this section.

Exercise 8.1.29

Identify the coefficient of each term:

1. $17x$
2. $41b^2$
3. z

Answer available at the end of this section.

Exercise 8.1.30

Identify the coefficient of each term:

1. $9p$
2. $13a^2$
3. y^3

Answer available at the end of this section.

Some terms share common traits. Look at the following 6 terms. Which ones seem to have traits in common?

$$5x \quad 7 \quad n^2 \quad 4 \quad 3x \quad 9n^2$$

The 7 and the 4 are both constant terms.

The $5x$ and the $3x$ are both terms with x.

The n^2 and the $9n^2$ are both terms with n^2.

When two terms are constants or have the same variable and exponent, we say they are **like terms**.

- 7 and 4 are like terms.
- $5x$ and $3x$ are like terms.
- x^2 and $9x^2$ are like terms.

LIKE TERMS

Terms that are either constants or have the same variables raised to the same powers are called **like terms**.

Exercise 8.1.31

Identify the like terms: $y^3, 7x^2, 14, 23, 4y^3, 9x, 5x^2$.

Answer available at the end of this section.

Exercise 8.1.32

Identify the like terms: $9, 2x^3, y^2, 8x^3, 15, 9y, 11y^2$.

Answer available at the end of this section.

Exercise 8.1.33

Identify the like terms: $4x^3, 8x^2, 19, 3x^3, 24, 6x^3$.

Answer available at the end of this section.

Adding or subtracting terms forms an expression. In the expression $2x^2 + 3x + 8$, from Example, the three terms are $2x^2$, $3x$, and 8.

Exercise 8.1.34

Identify the terms in each expression.

1. $9x^2 + 7x + 12$

2. $8x + 3y$

Answer available at the end of this section.

Exercise 8.1.35

Identify the terms in the expression $4x^2 + 5x + 17$.

Answer available at the end of this section.

Exercise 8.1.36

Identify the terms in the expression $5x + 2y$.

Answer available at the end of this section.

If there are like terms in an expression, you can simplify the expression by combining the like terms. What do you think $4x + 7x + x$ would simplify to? If you thought $12x$, you would be right!

$$\begin{array}{c} 4x + 7x + x \\ x + x + x + x \quad + x + x + x + x + x + x + x \quad + x \\ 12x \end{array} \tag{8.1.1}$$

Add the coefficients and keep the same variable. It doesn't matter what x is—if you have 4 of something and add 7 more of the same thing and then add 1 more, the result is 12 of them. For example, 4 oranges plus 7 oranges plus 1 orange is 12 oranges. We will discuss the mathematical properties behind this later.

Simplify: $4x + 7x + x$

Add the coefficients. $12x$

Exercise 8.1.37: How To Combine Like Terms

Simplify: $2x^2 + 3x + 7 + x^2 + 4x + 5$

Answer available at the end of this section.

Exercise 8.1.38

Simplify: $3x^2 + 7x + 9 + 7x^2 + 9x + 8$.

Answer available at the end of this section.

Exercise 8.1.39

Simplify: $4y^2 + 5y + 2 + 8y^2 + 4y + 5$.

Answer available at the end of this section.

COMBINE LIKE TERMS.

1. Identify like terms.
2. Rearrange the expression so like terms are together.
3. Add or subtract the coefficients and keep the same variable for each group of like terms.

Translate an English Phrase to an Algebraic Expression

In the last section, we listed many operation symbols that are used in algebra, then we translated expressions and equations into English phrases and sentences. Now we'll reverse the process. We'll translate English phrases into algebraic expressions. The symbols and variables we've talked about will help us do that. Table 8.1.7 summarizes them.

Table 8.1.7

Operation	Phrase	Expression
Addition	a plus b the sum of a and b a increased by b b more than a the total of a and b b added to a	$a+b$
Subtraction	a minus b the difference of a and b a decreased by b b less than a b subtracted from a	$a-b$
Multiplication	a times b the product of a and b twice a	$a \cdot b, ab, a(b), (a)(b)$ $2a$
Division	a divided by b the quotient of a and b the ratio of a and b b divided into a	$a \div b, a/b, \frac{a}{b}, b\overline{)a}$

Look closely at these phrases using the four operations:

the **sum** *of a and b*

the **difference** *of a and b*

the **product** *of a and b*

the **quotient** *of a and b*

Figure 8.1.5

Each phrase tells us to operate on two numbers. Look for the words *of* and *and* to find the numbers.

Exercise 8.1.40

Translate each English phrase into an algebraic expression:

1. the difference of $17x$ and 5
2. the quotient of $10x^2$ and 7.

Answer available at the end of this section.

Exercise 8.1.41

Translate each English phrase into an algebraic expression:

1. the difference of $14x^2$ and 13
2. the quotient of $12x$ and 2.

Answer available at the end of this section.

Exercise 8.1.42

Translate each English phrase into an algebraic expression:

1. the sum of $17y^2$ and 19
2. the product of 7 and y.

Answer available at the end of this section.

How old will you be in eight years? What age is eight more years than your age now? Did you add 8 to your present age? Eight "more than" means 8 added to your present age. How old were you seven years ago? This is 7 years less than your age now. You subtract 7 from your present age. Seven "less than" means 7 subtracted from your present age.

Exercise 8.1.43

Translate the English phrase into an algebraic expression:

1. Seventeen more than y
2. Nine less than $9x^2$.

Answer available at the end of this section.

Exercise 8.1.44

Translate the English phrase into an algebraic expression:

1. Eleven more than x
2. Fourteen less than $11a$.

Answer available at the end of this section.

Exercise 8.1.45

Translate the English phrase into an algebraic expression:

1. 13 more than z
2. 18 less than $8x$.

Answer available at the end of this section.

Exercise 8.1.46

Translate the English phrase into an algebraic expression:

1. five times the sum of m and n
2. the sum of five times m and n.

Answer available at the end of this section.

Exercise 8.1.47

Translate the English phrase into an algebraic expression:

1. four times the sum of p and q
2. the sum of four times p and q.

Answer available at the end of this section.

Exercise 8.1.48

Translate the English phrase into an algebraic expression:

1. the difference of two times x and 8,
2. two times the difference of x and 8.

Answer available at the end of this section.

Later in this course, we'll apply our skills in algebra to solving applications. The first step will be to translate an English phrase to an algebraic expression. We'll see how to do this in the next two examples.

Exercise 8.1.49

The length of a rectangle is 6 less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Answer available at the end of this section.

Exercise 8.1.50

The length of a rectangle is 7 less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Answer available at the end of this section.

Exercise 8.1.51

The width of a rectangle is 6 less than the length. Let l represent the length of the rectangle. Write an expression for the width of the rectangle.

Answer available at the end of this section.

Exercise 8.1.52

June has dimes and quarters in her purse. The number of dimes is three less than four times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Answer available at the end of this section.

Exercise 8.1.53

Geoffrey has dimes and quarters in his pocket. The number of dimes is eight less than four times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Answer available at the end of this section.

Exercise 8.1.54

Lauren has dimes and nickels in her purse. The number of dimes is three more than seven times the number of nickels. Let n represent the number of nickels. Write an expression for the number of dimes.

Answer available at the end of this section.

Key Concepts

- **Notation**

	The result is...
• $a + b$	the sum of a and b
• $a - b$	the difference of a and b
• $a \cdot b, ab, (a)(b)(a)b, a(b)$	the product of a and b
• $a \div b, a/b, \frac{a}{b}, b)\overline{a}$	the quotient of a and b

- **Inequality**
  - $a < b$ is read "a is less than b'' a is to the left of b on the number line
  - $a > b$ is read "a is greater than b'' a is to the right of b on the number line
- **Inequality Symbols** **Words**
  - $a \neq b$ a is not equal to b
  - $a < b$ a is less than b
  - $a \leq b$ a is less than or equal to b
  - $a > b$ a is greater than b
  - $a \geq b$ a is greater than or equal to b
- **Combine Like Terms**
  1. Identify like terms.
  2. Rearrange the expression so like terms are together.
  3. Add or subtract the coefficients and keep the same variable for each group of like terms.

Glossary

coefficient
The coefficient of a term is the constant that multiplies the variable in a term.

constant
A constant is a number whose value always stays the same.

equality symbol
The symbol "$=$" is called the equal sign. We read $a = b$ as "a is equal to b."

equation
An equation is two expressions connected by an equal sign.

evaluate an expression
To evaluate an expression means to find the value of the expression when the variable is replaced by a given number.

expression
An expression is a number, a variable, or a combination of numbers and variables using operation symbols.

like terms
Terms that are either constants or have the same variables raised to the same powers are called like terms.

simplify an expression
To simplify an expression, do all operations in the expression.

term
A term is a constant or the product of a constant and one or more variables.

variable
A variable is a letter that represents a number whose value may change.

Answers

Exercise $8.1.1$

1. $17 \leq 26$, 17 is less than or equal to 26
2. $8 \neq 17 - 3$, 8 is not equal to 17 minus 3
3. $12 > 27 \div 3$, 12 is greater than 27 divided by 3
4. $y + 7 < 19$, y plus 7 is less than 19

Exercise $8.1.2$

1. 14 is less than or equal to 27
2. 19 minus 2 is not equal to 8
3. 12 is greater than 4 divided by 2
4. x minus 7 is less than 1

Exercise $8.1.3$

1. 19 is greater then or equal to 15

2. 7 is equal to 12 minus 5
3. 15 divided by 3 is less than 8
4. y plus 3 is greater than 6

Exercise 8.1.4

1. $2(x+3)=10$. This is an equation — two expressions are connected with an equal sign.
2. $4(y-1)+1$. This is an expression — no equal sign.
3. $x \div 25$. This is an expression — no equal sign.
4. $y+8=40$. This is an equation — two expressions are connected with an equal sign.

Exercise 8.1.5

1. equation
2. expression

Exercise 8.1.6

1. expression
2. equation

Exercise 8.1.19

1.

when $x=5$	$7x-4$
	$7(5)-4$
Multiply.	$35-4$
Subtract.	31

2.

when $x=1$	$7x-4$
	$7(1)-4$
Multiply.	$7-4$
Subtract.	3

Exercise 8.1.20

1. 13
2. 5

Exercise 8.1.21

1. 8
2. 16

Exercise 8.1.22

1.

	x^2
Replace x with 4.	$(4)^2$
Use definition of exponent.	$4 \cdot 4$
Simplify.	16

2.

	3^x
Replace x with 4.	\(3^ ParseError: invalid DekiScript (click for details) Callstack: at (Courses/Northeast_Wisconsin_Technical_College/College_Technical_Math_1A_(NWTC) at wiki.page() at (Courses/Coalinga_College/Math_for_Educators_(MATH_010A_and_010B_CID120)/08:_Al \)

Use definition of exponent.	$3 \cdot 3 \cdot 3 \cdot 3$
Simplify.	81

Exercise 8.1.23

1. 9
2. 64

Exercise 8.1.24

1. 216
2. 64

Exercise 8.1.25

	$2x^2 + 3x + 8$
Substitute $x = 4$.	$2x^2 + 3x + 8$ $2(4)^2 + 3(4) + 8$
Follow the order of operations.	$2(16) + 3(4) + 8$
	$32 + 12 + 8$
	52

Exercise 8.1.26

40

Exercise 8.1.27

9

Exercise 8.1.28

1. The coefficient of $14y$ is 14
2. The coefficient of $15x^2$ is 15
3. The coefficient of a is 1 since $a = 1a$.

Exercise 8.1.29

1. 14
2. 41
3. 1

Exercise 8.1.30

1. 9
2. 13
3. 1

Exercise 8.1.31

y^3 and $4y^3$ are like terms because both have y^3; the variable and the exponent match.

$7x^2$ and $5x^2$ are like terms because both have x^2; the variable and the exponent match.

14 and 23 are like terms because both are constants.

There is no other term like $9x$.

Exercise 8.1.32

9 and 15, y^2 and $11y^2$, $2x^3$ and $8x^3$

Exercise 8.1.33

19 and 24, $8x^2$ and $3x^2$, $4x^3$ and $6x^3$

Exercise 8.1.34

1. The terms of $9x^2 + 7x + 12$ are $9x^2, 7x$, and 12.
2. The terms of $8x + 3y$ are $8x$ and $3y$.

Exercise 8.1.35

$4x^2, 5x, 17$

Exercise 8.1.36

$5x, 2y$

Exercise 8.1.37**: How To Combine Like Terms**

Step 1. Identify the like terms.	$2x^2 + 3x + 7 + x^2 + 4x + 5$ $2x^2 + 3x + 7 + x^2 + 4x + 5$
Step 2. Rearrange the expression so the like terms are together.	$2x^2 + x^2 + 3x + 4x + 7 + 5$
Step 3. Combine like terms.	$3x^2 + 7x + 12$

Exercise 8.1.38

$10x^2 + 16x + 17$

Exercise 8.1.39

$12y^2 + 9y + 7$

Exercise 8.1.40

1. The key word is *difference*, which tells us the operation is subtraction. Look for the words *of* and *and* to find the numbers to subtract.

   the *difference of* $17x$ *and* 5

   $17x$ minus 5

   $17x - 5$

2. The key word is "quotient," which tells us the operation is division.

   the *quotient of* $10x^2$ *and* 7

   divide $10x^2$ by 7

   $10x^2 \div 7$

This can also be written $10x^2/7$ or $\frac{10x^2}{7}$.

Exercise 8.1.41

1. $14x^2 - 13$
2. $12x \div 2$

Exercise 8.1.42

1. $17y^2 + 19$
2. $7y$

Exercise 8.1.43

1. The key words are *more than*. They tell us the operation is addition. *More than* means "added to."

   Seventeen more than y

   Seventeen added to y

   $y + 17$

2. The key words are *less than*. They tell us to subtract. *Less than* means "subtracted from."

   Nine less than $9x^2$

   Nine subtracted from $9x^2$

   $9x^2 - 9$

Exercise 8.1.44

1. $x + 11$
2. $11a - 14$

Exercise 8.1.45

1. $z + 13$
2. $8x - 18$

Exercise 8.1.46

There are two operation words—*times* tells us to multiply and *sum* tells us to add.
1. Because we are multiplying 5 times the sum we need parentheses around the sum of m and n, $(m + n)$. This forces us to determine the sum first. (Remember the order of operations.)

$$\begin{array}{c}\text{five times the sum of } m \text{ and } n \\ 5(m+n)\end{array} \tag{8.1.7}$$

2. To take a sum, we look for the words “of” and “and” to see what is being added. Here we are taking the sum *of* five times m and \(n\.)

$$\begin{array}{c}\text{the sum of five times } m \text{ and } n \\ 5m+n\end{array} \tag{8.1.8}$$

Exercise 8.1.47

1. $4(p+q)$
2. $4p+q$

Exercise 8.1.48

1. $2x-8$
2. $2(x-8)$

Exercise 8.1.49

Write a phrase about the length of the rectangle.	6 less than the width
Substitute w for "the width."	6 less then w
Rewrite "less than" as "subtracted from."	6 subtracted from w
Translate the phrase into algebra.	$w-6$

(8.1.9)

Exercise 8.1.50

$w-7$

Exercise 8.1.51

$l-6$

Exercise 8.1.52

Write the phrase about the number of dimes.	three less than four times the number of quarters
Substitute q for the number of quarters.	3 less than 4 times q
Translate "4 times q ."	3 less than 4q
Translate the phrase into algebra.	4q - 3

(8.1.10)

Exercise 8.1.53

$4q-8$

Exercise 8.1.54

$7n+3$

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8.2: Solving Equations with Addition and Subtraction Properties of Equality

Learning Objectives

By the end of this section, you will be able to:

- Verify a solution of an equation
- Solve equations using the Subtraction and Addition Properties of Equality
- Solve equations that require simplification
- Translate to an equation and solve
- Translate and solve applications

Verify a Solution of an Equation

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that make each side of the equation the same – so that we end up with a true statement. Any value of the variable that makes the equation true is called a solution to the equation. It is the answer to the puzzle!

Definition: Solution of an Equation

A **solution of an equation** is a value of a variable that makes a true statement when substituted into the equation.

TO DETERMINE WHETHER A NUMBER IS A Solution TO AN EQUATION

1. Substitute the number in for the variable in the equation.
2. Simplify the expressions on both sides of the equation.
3. Determine whether the resulting equation is true (the left side is equal to the right side)
   - If it is true, the number is a solution.
   - If it is not true, the number is not a solution.

Exercise 8.2.1

Determine whether $x = \frac{3}{2}$ is a solution of $4x - 2 = 2x + 1$.

Answer available at the end of this section.

Exercise 8.2.2

Is $y = \frac{4}{3}$ a solution of $9y + 2 = 6y + 3$?

Answer available at the end of this section.

Exercise 8.2.3

Is $y = \frac{7}{5}$ a solution of $5y + 3 = 10y - 4$?

Answer available at the end of this section.

Solve Equations Using the Subtraction and Addition Properties of Equality

We are going to use a model to clarify the process of solving an equation. An envelope represents the variable – since its contents are unknown – and each counter represents one. We will set out one envelope and some counters on our workspace, as shown in Figure 8.2.1. Both sides of the workspace have the same number of counters, but some counters are "hidden" in the envelope. Can you tell how many counters are in the envelope?

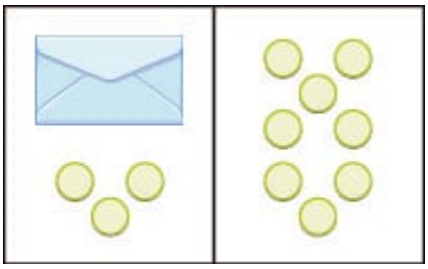


Figure 8.2.1: The illustration shows a model of an equation with one variable. On the left side of the workspace is an unknown (envelope) and three counters, while on the right side of the workspace are eight counters.

What are you thinking? What steps are you taking in your mind to figure out how many counters are in the envelope?

Perhaps you are thinking: "I need to remove the 3 counters at the bottom left to get the envelope by itself. The 3 counters on the left can be matched with 3 on the right and so I can take them away from both sides. That leaves five on the right—so there must be 5 counters in the envelope." See Figure 8.2.2 for an illustration of this process.

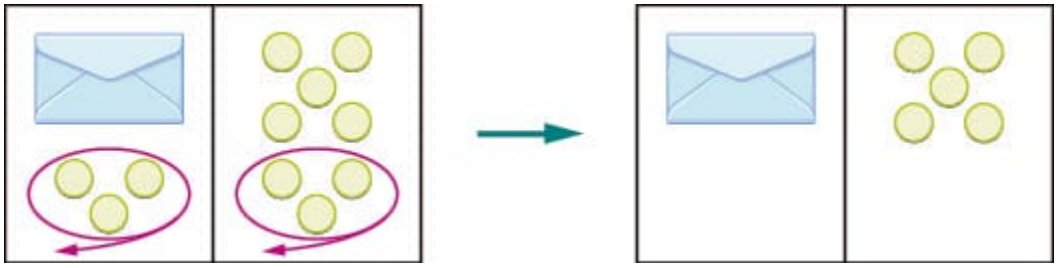


Figure 8.2.2: The illustration shows a model for solving an equation with one variable. On both sides of the workspace remove three counters, leaving only the unknown (envelope) and five counters on the right side. The unknown is equal to five counters.

What algebraic equation would match this situation? In Figure 8.2.3 each side of the workspace represents an expression and the center line takes the place of the equal sign. We will call the contents of the envelope x.

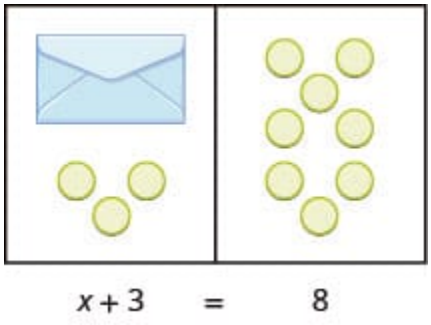


Figure 8.2.3: The illustration shows a model for the equation $x + 3 = 8$.

Let's write algebraically the steps we took to discover how many counters were in the envelope:

Table 8.2.1

	$x+3=8$
First, we took away three from each side.	$x+3-3=8-3$
Then we were left with five.	$x=5$

Check:

Five in the envelope plus three more does equal eight!

$$5+3=8 \tag{8.2.1}$$

Our model has given us an idea of what we need to do to solve one kind of equation. The goal is to isolate the variable by itself on one side of the equation. To solve equations such as these mathematically, we use the **Subtraction Property of Equality**.

SUBTRACTION PROPERTY OF EQUALITY

For any numbers *a*, *b*, and *c*,

$$\begin{array}{ll} \text{If} & a=b \\ \text{then} & a-c=b-c \end{array} \tag{8.2.2}$$

When you subtract the same quantity from both sides of an equation, you still have equality.

Note

Doing the Manipulative Mathematics activity "Subtraction Property of Equality" will help you develop a better understanding of how to solve equations by using the **Subtraction Property of Equality**.

Let's see how to use this property to solve an equation. Remember, the goal is to isolate the variable on one side of the equation. And we check our solutions by substituting the value into the equation to make sure we have a true statement.

Exercise 8.2.4

Solve: $y+37=-13$.

Answer available at the end of this section.

Exercise 8.2.5

Solve: $x+19=-27$.

Answer available at the end of this section.

Exercise 8.2.6

Solve: $x+16=-34$.

Answer available at the end of this section.

What happens when an equation has a number subtracted from the variable, as in the equation $x-5=8$? We use another

property of equations to solve equations where a number is subtracted from the variable. We want to isolate the variable, so to 'undo' the subtraction we will add the number to both sides. We use the **Addition Property of Equality**.

ADDITION PROPERTY OF EQUALITY

For any numbers *a*, *b*, and *c*,

$$\begin{aligned} &\text{If} && a = b \\ &\text{then} && a + c = b + c \end{aligned} \tag{8.2.3}$$

When you add the same quantity from both sides of an equation, you still have equality.

In Exercise 8.2.4, 37 was added to the *y* and so we subtracted 37 to 'undo' the addition. In Exercise 8.2.7, we will need to 'undo' subtraction by using the **Addition Property of Equality**.

Exercise 8.2.7

Solve: $a - 28 = -37$.

Answer available at the end of this section.

Exercise 8.2.8

Solve: $n - 61 = -75$.

Answer available at the end of this section.

Exercise 8.2.9

Solve: $p - 41 = -73$.

Answer available at the end of this section.

Exercise 8.2.10

Solve: $x - \frac{5}{8} = \frac{3}{4}$

Answer available at the end of this section.

Exercise 8.2.11

Solve: $p - \frac{2}{3} = \frac{5}{6}$.

Answer available at the end of this section.

Exercise 8.2.12

Solve: $q - \frac{1}{2} = \frac{5}{6}$.

Answer available at the end of this section.

The next example will be an equation with decimals.

Exercise 8.2.13

Solve: $n - 0.63 = -4.2$.

Answer available at the end of this section.

Exercise 8.2.14

Solve: $b - 0.47 = -2.1$.

Answer available at the end of this section.

Exercise 8.2.15

Solve: $c - 0.93 = -4.6$.

Answer available at the end of this section.

Solve Equations That Require Simplification

In the previous examples, we were able to isolate the variable with just one operation. Most of the equations we encounter in algebra will take more steps to solve. Usually, we will need to simplify one or both sides of an equation before using the Subtraction or Addition Properties of Equality.

You should always simplify as much as possible before you try to isolate the variable. Remember that to simplify an expression means to do all the operations in the expression. Simplify one side of the equation at a time. Note that **simplification** is different from the process used to solve an equation in which we apply an operation to both sides.

Exercise 8.2.16: How to Solve Equations That Require Simplification

Solve: $9x - 5 - 8x - 6 = 7$.

Answer available at the end of this section.

Exercise 8.2.17

Solve: $8y - 4 - 7y - 7 = 4$.

Answer available at the end of this section.

Exercise 8.2.18

Solve: $6z + 5 - 5z - 4 = 3$.

Answer available at the end of this section.

Exercise 8.2.19

Solve: 5(n−4)−4n=−8.

Answer available at the end of this section.

Exercise 8.2.20

Solve: $5(p-3)-4p=-10$.

Answer available at the end of this section.

Exercise 8.2.21

Solve: $4(q+2)-3q=-8$.

Answer available at the end of this section.

Exercise 8.2.22

Solve: $3(2y-1)-5y=2(y+1)-2(y+3)$.

Answer available at the end of this section.

Exercise 8.2.23

Solve: $4(2h-3)-7h=6(h-2)-6(h-1)$.

Answer available at the end of this section.

Exercise 8.2.24

Solve: $2(5x+2)-9x=3(x-2)-3(x-4)$.

Answer available at the end of this section.

Translate to an Equation and Solve

To solve applications algebraically, we will begin by translating from English sentences into equations. Our first step is to look for the word (or words) that would translate to the **equals sign**. Here are some of the words that are commonly used.

Equals =

- is
- is equal to
- is the same as
- the result is
- gives
- was
- will be

The steps we use to translate a sentence into an equation are listed below.

TRANSLATE AN ENGLISH SENTENCE TO AN ALGEBRAIC EQUATION

1. Locate the “equals” word(s). Translate to an equals sign (=).
2. Translate the words to the left of the “equals” word(s) into an algebraic expression.
3. Translate the words to the right of the “equals” word(s) into an algebraic expression.

Exercise 8.2.25

Translate and solve: Eleven more than x is equal to 54.

Answer available at the end of this section.

Exercise 8.2.26

Translate and solve: Ten more than x is equal to 41.

Answer available at the end of this section.

Exercise 8.2.27

Translate and solve: Twelve less than x is equal to 51.

Answer available at the end of this section.

Exercise 8.2.28

Translate and solve: The difference of 12t and 11t is −14.

Answer available at the end of this section.

Exercise 8.2.29

Translate and solve: The difference of 4x and 3x is 14.

Answer available at the end of this section.

Exercise 8.2.30

Translate and solve: The difference of 7a and 6a is −8.

Answer available at the end of this section.

Translate and Solve Applications

Most of the time a question that requires an algebraic solution comes out of a real life question. To begin with that question is asked in English (or the language of the person asking) and not in math symbols. Because of this, it is an important skill to be able to translate an everyday situation into algebraic language.

We will start by restating the problem in just one sentence, assign a variable, and then translate the sentence into an equation to solve. When assigning a variable, choose a letter that reminds you of what you are looking for. For example, you might use q for the number of quarters if you were solving a problem about coins.

Exercise 8.2.31: How to Solve Translate and Solve Applications

The MacIntyre family recycled newspapers for two months. The two months of newspapers weighed a total of 57 pounds. The second month, the newspapers weighed 28 pounds. How much did the newspapers weigh the first month?

Answer available at the end of this section.

Exercise 8.2.32

Translate into an algebraic equation and solve:

The Pappas family has two cats, Zeus and Athena. Together, they weigh 23 pounds. Zeus weighs 16 pounds. How much does Athena weigh?

Answer available at the end of this section.

Exercise 8.2.33

Translate into an algebraic equation and solve:

Sam and Henry are roommates. Together, they have 68 books. Sam has 26 books. How many books does Henry have?

Answer available at the end of this section.

SOLVE AN APPLICATION.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with the important information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Exercise 8.2.34

Randell paid $28,675 for his new car. This was $875 less than the sticker price. What was the sticker price of the car?

Answer available at the end of this section.

Exercise 8.2.35

Translate into an algebraic equation and solve:

Eddie paid $19875 for his new car. This was $1025 less than the sticker price. What was the sticker price of the car?

Answer available at the end of this section.

Exercise 8.2.36

Translate into an algebraic equation and solve:

The admission price for the movies during the day is $7.75. This is $3.25 less the price at night. How much does the movie cost at night?

Answer available at the end of this section.

Key Concepts

- **To Determine Whether a Number is a Solution to an Equation**
  1. **Substitute the number in for the variable in the equation.**
  2. **Simplify the expressions on both sides of the equation.**
  3. **Determine whether the resulting statement is true.**
     - If it is true, the number is a solution.
     - If it is not true, the number is not a solution.
- **Addition Property of Equality**
  - For any numbers *a*, *b*, and *c*, if a=b, then a+c=b+c.
- **Subtraction Property of Equality**
  - For any numbers *a*, *b*, and *c*, if a=b, then a−c=b−c.
- **To Translate a Sentence to an Equation**
  1. Locate the “equals” word(s). Translate to an equal sign (=).
  2. Translate the words to the left of the “equals” word(s) into an algebraic expression.
  3. Translate the words to the right of the “equals” word(s) into an algebraic expression.
- **To Solve an Application**
  1. Read the problem. Make sure all the words and ideas are understood.
  2. Identify what we are looking for.
  3. Name what we are looking for. Choose a variable to represent that quantity.
  4. Translate into an equation. It may be helpful to restate the problem in one sentence with the important information.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.

Glossary

solution of an equation
A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

Answers

Exercise 8.2.1

Since a solution to an equation is a value of the variable that makes the equation true, begin by substituting the value of the solution for the variable.

	$4x-2=2x+1$
Substitute $\frac{3}{2}$ for x.	$4\left(\frac{3}{2}\right)-2\stackrel{?}{=}2\left(\frac{3}{2}\right)+1$
Multiply.	$6-2\stackrel{?}{=}3+1$
Subtract.	$4=4\checkmark$

Since $x=\frac{3}{2}$ results in a true equation (4 is in fact equal to 4), $\frac{3}{2}$ is a solution to the equation $4x-2=2x+1$.

Exercise 8.2.2

no

Exercise 8.2.3

yes

Exercise 8.2.4

To get *y* by itself, we will undo the addition of 37 by using the Subtraction Property of Equality.

		$y+37=-13$
Subtract 37 from each side to 'undo' the addition.		$y+37-37=-13-37$
Simplify.		$y=-50$
Check:	$y+37=-13$	
Substitute $y=-50$	$-50+37=-13$	
	$-13\stackrel{?}{=}-13$ ✓	

Since y=−50 makes y+37=−13 a true statement, we have the solution to this equation.

Exercise 8.2.5

$x=-46$

Exercise 8.2.6

$x=-50$

Exercise 8.2.7

		$a-28=-37$
Add 28 to each side to 'undo' the subtraction.		$a-28+28=-37+28$
Simplify.		$a=-9$
Check:	$a-28=-37$	
Substitute $a=-9$	$-9-28=-37$	
	$-37\stackrel{?}{=}-37$ ✓	
		The solution to $a-28=-37$ is $a=-9$.

Exercise 8.2.8

$n=-14$

Exercise 8.2.9

$p=-32$

Exercise 8.2.10

		$x-\frac{5}{8}=\frac{3}{4}$
Use the Addition Property of Equality.		$x-\frac{5}{8}+\frac{5}{8}=\frac{3}{4}+\frac{5}{8}$
Find the LCD to add the fractions on the right.		$x-\frac{5}{8}+\frac{5}{8}=\frac{6}{8}+\frac{5}{8}$
Simplify.		$x=\frac{11}{8}$
Check:	$x-\frac{5}{8}=\frac{3}{4}$	
Substitute $x=\frac{11}{8}$	$\frac{11}{8}-\frac{5}{8}\stackrel{?}{=}\frac{3}{4}$	
Subtract.	$\frac{6}{8}\stackrel{?}{=}\frac{3}{4}$	

Simplify.	$\frac{3}{4}=\frac{3}{4}\checkmark$	
		The solution to $x-\frac{5}{8}=\frac{3}{4}$ is $x=\frac{11}{8}$.

Exercise 8.2.11

$p=\frac{9}{6}p=\frac{3}{2}$

Exercise 8.2.12

$q=\frac{4}{3}$

Exercise 8.2.13

		$n-0.63=-4.2$
Use the Addition Property of Equality.		$n-0.63+0.63=-4.2+0.63$
Add.		$n=-3.57$
Check:	$n=-3.57$	
Let $n=-3.57$.	$-3.57-0.63\stackrel{?}{=}-4.2$	
	$-4.2=-4.2\checkmark$	

Exercise 8.2.14

$b=-1.63$

Exercise 8.2.15

$c=-3.67$

Exercise 8.2.16**: How to Solve Equations That Require Simplification**

Step 1. Simplify the expressions on each side as much as possible.	Rearrange the terms, using the Commutative Property of Addition. Combine like terms. Notice that each side is now simplified as much as possible.	$9x-5-8x-6=7$ $9x-8x-5-6=7$ $x-11=7$
Step 2. Isolate the variable.	Now isolate x. Undo subtraction by adding 11 to both sides.	$x-11+11=7+11$
Step 3. Simplify the expressions on both sides of the equation.		$x=18$

Step 4. Check the solution.	**Check:** Substitute $x = 18$. $9x - 5 - 8x - 6 = 7$ $9(18) - 5 - 8(18) - 6 \stackrel{?}{=} 7$ $162 - 5 - 144 - 6 \stackrel{?}{=} 7$ $157 - 144 - 6 \stackrel{?}{=} 7$ $13 - 6 \stackrel{?}{=} 7$ $7 = 7$ ✓ The solution to $9x - 5 - 8x - 6 = 7$ is $x = 18$.

Exercise 8.2.17

$y = 15$

Exercise 8.2.18

$z = 2$

Exercise 8.2.19

We simplify both sides of the equation as much as possible before we try to isolate the variable.

	$5(n-4) - 4n = -8$
Distribute on the left.	$5n - 20 - 4n = -8$
Use the Commutative Property to rearrange terms.	$5n - 4n - 20 = -8$
Combine like terms.	$n - 20 = -8$
Each side is as simplified as possible. Next, isolate n.	
Undo subtraction by using the Addition Property of Equality.	$n - 20 \ +20 \ = -8 \ +20$
Add.	$n = 12$
Check. Substitute n=12. $5(n-4) - 4n = -8$ $5(12-4) - 4(12) \stackrel{?}{=} -8z$ $5(8) - 48 \stackrel{?}{=} -8$ $40 - 48 \stackrel{?}{=} -8$ $-8 = -8$ ✓	
	The solution to $5(n-4) - 4n = -8$ is $n = 12$.

Exercise 8.2.20

$p = 5$

Exercise 8.2.21

$q = -16$

Exercise 8.2.22

We simplify both sides of the equation before we isolate the variable.

	$3(2y-1) - 5y = 2(y+1) - 2(y+3)$

Distribute on both sides.	$6y-3-5y=2y+2-2y-6$
Use the Commutative Property of Addition.	$6y-5y-3=2y-2y+2-6$
Combine like terms.	$y-3=-4$
Each side is as simplified as possible. Next, isolate y.	
Undo subtraction by using the Addition Property of Equality.	$y-3+3=-4+3$
Add.	$y=-1$
Check. Let y=–1. $3(2y-1)-5y=2(y+1)-2(y+3)$ $3(2(-1)-1)-5(-1)\stackrel{?}{=}2(-1+1)-2(-1+3)$ $3(-2-1)+5\stackrel{?}{=}2(0)-2(2)$ $3(-3)+5\stackrel{?}{=}-4$ $-9+5\stackrel{?}{=}-4$ $-4=-4\checkmark$	
	The solution to $3(2y-1)-5y=2(y+1)-2(y+3)3(2y-1)-5y=2(y+1)-2(y+3)$ is $y=-1$.

Exercise 8.2.23

$h=6$

Exercise 8.2.24

$x=2$

Exercise 8.2.25

Translate.
Subtract 11 from both sides.
Simplify.
Check: Is 54 eleven more than 43? $$\begin{aligned} 43+11 &\stackrel{?}{=} 54 \\ 54 &= 54\checkmark \end{aligned} \quad (8.2.4)$$

Exercise 8.2.26

$x+10=41; x=31$

Exercise 8.2.27

y–12=51;y=63

Exercise 8.2.28

Translate.

Simplify.
Check: $$\begin{aligned} 12(-14)-11(-14) &\stackrel{?}{=} -14 \\ -168+154 &\stackrel{?}{=} -14 \\ -14 &= -14\checkmark \end{aligned} \quad (8.2.5)$$

Exercise 8.2.29

$4x-3x=14; x=14$

Exercise 8.2.30

$7a-6a=-8; a=-8$

Exercise 8.2.31**: How to Solve Translate and Solve Applications**

Step 1. Read the problem. Make sure all the words and ideas are understood.	The problem is about the weight of newspapers.	
Step 2. Identify what we are asked to find.	What are we asked to find?	"How much did the newspapers weigh the 2nd month?"
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Choose a variable.	Let w = weight of the newspapers the 1st month
Step 4. Translate into an equation. It may be helpful to restate the problem in one sentence with the important information.	Restate the problem. We know the weight of the newspapers the second month is 28 pounds. Translate into an equation, using the variable w.	Weight of newspapers the 1st month plus the weight of the newspapers the 2nd month equals 57 pounds. Weight from 1st month plus 28 equals 57. $w+28=57$
Step 5. Solve the equation using good algebra techniques.	Solve.	$w+28-28=57-28$ $w=29$
Step 6. Check the answer in the problem and make sure it makes sense.	Does 1st month's weight plus 2nd month's weight equal 57 pounds?	**Check:** Does 1st month's weight plus 2nd month's weight equal 57 pounds? $29+28\stackrel{?}{=}57$ $57=57\checkmark$

Step 7. Answer the question with a complete sentence.	Write a sentence to answer "How much did the newspapers weigh the 2nd month?"	The 2nd month the newspapers weighed 29 pounds.

Exercise 8.2.32

7 pounds

Exercise 8.2.33

42 books

Exercise 8.2.34

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	"What was the sticker price of the car?"
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let s = the sticker price of the car.
Step 4. Translate into an equation. Restate the problem in one sentence.	$28675 is $875 less than the sticker price
Step 5. Solve the equation.	$28675 is $875 less than s $28675 = s - 875$ $28675 + 875 = s - 875 + 875$ $29550 = s$
Step 6. Check the answer. Is $875 less than $29550 equal to $28675? $29550 - 875 \stackrel{?}{=} 28675$ $28675 = 28675$✓	
Step 7. Answer the question with a complete sentence.	The sticker price of the car was $29550.

Exercise 8.2.35

$20900

Exercise 8.2.36

$11.00

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8.3: Solving Equations with the Division and Multiplication Properties of Equality

Learning Objectives

By the end of this section, you will be able to:

- Solve equations using the Division and Multiplication Properties of Equality
- Solve equations that require simplification
- Translate to an equation and solve
- Translate and solve applications

Solve Equations Using the Division and Multiplication Properties of Equality

You may have noticed that all of the equations we have solved so far have been of the form $x + a = b$ or $x - a = b$. We were able to isolate the variable by adding or subtracting the constant term on the side of the equation with the variable. Now we will see how to solve equations that have a variable multiplied by a constant and so will require division to isolate the variable.

Let's look at our puzzle again with the envelopes and counters in Figure $8.3.1$.

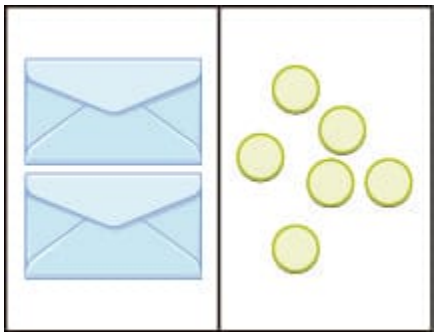


Figure $8.3.1$: The illustration shows a model of an equation with one variable multiplied by a constant. On the left side of the workspace are two instances of the unknown (envelope), while on the right side of the workspace are six counters.

In the illustration there are two identical envelopes that contain the same number of counters. Remember, the left side of the workspace must equal the right side, but the counters on the left side are "hidden" in the envelopes. So how many counters are in each envelope?

How do we determine the number? We have to separate the counters on the right side into two groups of the same size to correspond with the two envelopes on the left side. The 6 counters divided into 2 equal groups gives 3 counters in each group (since $6 \div 2 = 3$).

What equation models the situation shown in Figure $8.3.2$? There are two envelopes, and each contains xx counters. Together, the two envelopes must contain a total of 6 counters.

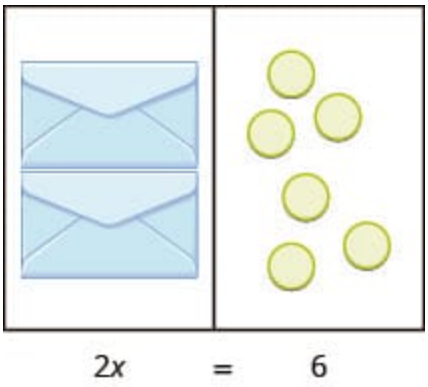


Figure 8.3.2: The illustration shows a model of the equation $2x = 6$.

	$2x = 6$
If we divide both sides of the equation by 2, as we did with the envelopes and counters,	$\frac{2x}{2} = \frac{6}{2}$
we get:	$x = 3$

We found that each envelope contains 3 counters. Does this check? We know $2 \cdot 3 = 6$, so it works! Three counters in each of two envelopes does equal six!

This example leads to the **Division Property of Equality**.

THE DIVISION PROPERTY OF EQUALITY

For any numbers *a*, *b*, and *c*, and $c \neq 0$,

$$\begin{aligned} &\text{If} \quad a = b \\ &\text{then} \quad \tfrac{a}{c} = \tfrac{b}{c} \end{aligned} \tag{8.3.1}$$

When you divide both sides of an equation by any non-zero number, you still have equality.

Note

Doing the Manipulative Mathematics activity "**Division Property of Equality**" will help you develop a better understanding of how to solve equations by using the Division Property of Equality.

The goal in solving an equation is to 'undo' the operation on the variable. In the next example, the variable is multiplied by 5, so we will divide both sides by 5 to 'undo' the multiplication.

Exercise 8.3.1

Solve: $5x = -27$.

Answer available at the end of this section.

Exercise 8.3.2

Solve: $3y = -41$.

Answer available at the end of this section.

Exercise 8.3.3

Solve: $4z = -55$.

Answer available at the end of this section.

Consider the equation $\frac{x}{4} = 3$. We want to know what number divided by 4 gives 3. So to "undo" the division, we will need to multiply by 4. The **Multiplication Property of Equality** will allow us to do this. This property says that if we start with two equal quantities and multiply both by the same number, the results are equal.

THE MULTIPLICATION PROPERTY OF EQUALITY

For any numbers *a*, *b*, and *c*,

$$\begin{array}{lrcl} \text{If} & a & = & b \\ \text{then} & ac & = & bc \end{array} \tag{8.3.2}$$

If you multiply both sides of an equation by the same number, you still have equality.

Exercise 8.3.4

Solve: $\frac{y}{-7} = -14$

Answer available at the end of this section.

Exercise 8.3.5

Solve: $\frac{a}{-7} = -42$

Answer available at the end of this section.

Exercise 8.3.6

Solve: $\frac{b}{-6} = -24$

Answer available at the end of this section.

Exercise 8.3.7

Solve: $-n = 9$

Answer available at the end of this section.

Exercise 8.3.8

Solve: $-k = 8$.

Answer available at the end of this section.

Exercise 8.3.9

Solve: $-g = 3$.

Answer available at the end of this section.

Exercise 8.3.10

Solve: $\frac{3}{4}x = 12$

Answer available at the end of this section.

Exercise 8.3.11

Solve: $\frac{2}{5}n = 14$.

Answer available at the end of this section.

Exercise 8.3.12

Solve: $\frac{5}{6}y = 15$.

Answer available at the end of this section.

In the next example, all the variable terms are on the right side of the equation. As always, our goal in solving the equation is to isolate the variable.

Exercise 8.3.13

Solve: $\frac{8}{15} = -\frac{4}{5}x$

Answer available at the end of this section.

Exercise 8.3.14

Solve: $\frac{9}{25} = -\frac{4}{5}z$

Answer available at the end of this section.

Exercise 8.3.15

$\frac{5}{6} = -\frac{8}{3}r$

Answer available at the end of this section.

Solve Equations That Require Simplification

Many equations start out more complicated than the ones we have been working with.

With these more complicated equations the first step is to simplify both sides of the equation as much as possible. This usually involves combining like terms or using the distributive property.

Exercise 8.3.16

Solve: $14 - 23 = 12y - 4y - 5y$.

Answer available at the end of this section.

Exercise 8.3.17

Solve: $18 - 27 = 15c - 9c - 3c$.

Answer available at the end of this section.

Exercise 8.3.18

Solve: $18 - 22 = 12x - x - 4x$.

Answer available at the end of this section.

Exercise 8.3.19

Solve: $-4(a - 3) - 7 = 25$.

Answer available at the end of this section.

Exercise 8.3.20

Solve: $-4(q - 2) - 8 = 24$.

Answer available at the end of this section.

Exercise 8.3.21

Solve: $-6(r - 2) - 12 = 30$.

Answer available at the end of this section.

Now we have covered all four properties of equality—subtraction, addition, division, and multiplication. We'll list them all together here for easy reference.

PROPERTIES OF EQUALITY

When you add, subtract, multiply, or divide the same quantity from both sides of an equation, you still have equality.

Subtraction Property of Equality	**Addition Property of Equality**
For any real numbers a, b, and c,	For any real numbers a, b, and c,
if $a = b$,	if $a = b$,
then $a - c = b - c$	then $a + c = b + c$
Divition Property of Equality	**Multiplication Property of Equality**
For any real numbers a, b, and c,	For any real numbers a, b, and c,
if $a = b$,	if $a = b$,
then $a - c = b - c$	then $a + c = b + c$

(8.3.3)

When you add, subtract, multiply, or divide the same quantity from both sides of an equation, you still have equality.

Translate to an Equation and Solve

In the next few examples, we will translate sentences into equations and then solve the equations. You might want to review the translation table in the previous chapter.

Exercise 8.3.22

Translate and solve: The number 143 is the product of –11 and *y*.

Answer available at the end of this section.

Exercise 8.3.23

Translate and solve: The number 132 is the product of *–12* and *y*.

Answer available at the end of this section.

Exercise 8.3.24

Translate and solve: The number 117 is the product of *–13* and *z*.

Answer available at the end of this section.

Exercise 8.3.25

Translate and solve: *n* divided by 8 is –32.

Answer available at the end of this section.

Exercise 8.3.26

Translate and solve: nn divided by 7 is equal to –21.

Answer available at the end of this section.

Exercise 8.3.27

Translate and solve: n divided by 8 is equal to −56.

Answer available at the end of this section.

Exercise 8.3.28

Translate and solve: The quotient of yy and −4 is 68.

Answer available at the end of this section.

Exercise 8.3.29

Translate and solve: The quotient of q and −8 is 72.

Answer available at the end of this section.

Exercise 8.3.30

Translate and solve: The quotient of p and −9 is 81.

Answer available at the end of this section.

Exercise 8.3.31

Translate and solve: Three-fourths of p is 18.

Answer available at the end of this section.

Exercise 8.3.32

Translate and solve: Two-fifths of f is 16.

Answer available at the end of this section.

Exercise 8.3.33

Translate and solve: Three-fourths of f is 21.

Answer available at the end of this section.

Exercise 8.3.34

Translate and solve: The sum of three-eighths and x is one-half.

Answer available at the end of this section.

Exercise 8.3.35

Translate and solve: The sum of five-eighths and x is one-fourth.

Answer available at the end of this section.

Exercise 8.3.36

Translate and solve: The sum of three-fourths and x is five-sixths.

Answer available at the end of this section.

Translate and Solve Applications

To solve applications using the Division and Multiplication Properties of Equality, we will follow the same steps we used in the last section. We will restate the problem in just one sentence, assign a variable, and then translate the sentence into an equation to solve.

Exercise 8.3.37

Denae bought 6 pounds of grapes for $10.74. What was the cost of one pound of grapes?

Answer available at the end of this section.

Exercise 8.3.38

Translate and solve:

Arianna bought a 24-pack of water bottles for $9.36. What was the cost of one water bottle?

Answer available at the end of this section.

Exercise 8.3.39

Translate and solve:

At JB's Bowling Alley, 6 people can play on one lane for $34.98. What is the cost for each person?

Answer available at the end of this section.

Exercise 8.3.40

Andreas bought a used car for $12,000. Because the car was 4-years old, its price was $\frac{3}{4}$ of the original price, when the car was new. What was the original price of the car?

Answer available at the end of this section.

Exercise 8.3.41

Translate and solve:

The annual property tax on the Mehta's house is $1,800, calculated as $\frac{15}{1000}$ of the assessed value of the house. What is the assessed value of the Mehta's house?

Answer available at the end of this section.

Exercise 8.3.42

Translate and solve:

Stella planted 14 flats of flowers in $\frac{2}{3}$ of her garden. How many flats of flowers would she need to fill the whole garden?

Answer available at the end of this section.

Key Concepts

- **The Division Property of Equality**—For any numbers *a*, *b*, and *c*, and $c \neq 0$, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$.
  When you divide both sides of an equation by any non-zero number, you still have equality.
- **The Multiplication Property of Equality**—For any numbers *a*, *b*, and *c*, if $a = b$, then $ac = bc$.
  If you multiply both sides of an equation by the same number, you still have equality.

Answers

Exercise 8.3.1

To isolate x, "undo" the multiplication by 5.		$5x = -27$
Divide to 'undo' the multiplication.		$\frac{5x}{5} = -\frac{27}{5}$
Simplify.		$x = -\frac{27}{5}$
Check:	$5x = -27$	
Substitute $-\frac{27}{5}$ for x.	$5\left(-\frac{27}{5}\right) \stackrel{?}{=} -27$	
	$-27 = -27$ ✓	
		Since this is a true statement, $x = -\frac{27}{5}$ is the solution to $5x = -27$.

Exercise 8.3.2

$y = -\frac{41}{3}$

Exercise 8.3.3

$y = -\frac{55}{4}$

Exercise 8.3.4

Here y is divided by –7. We must multiply by –7 to isolate y.

	$\frac{y}{-7} = -14$

Multiply both sides by –7.		$-7\left(\frac{y}{-7}\right)=-7(-14)$
Multiply.		$\frac{-7y}{7}=98$
Simplify.		$y=98$
Check: $\frac{y}{-7}=-14$		
Substitute y=98.	$\frac{98}{-7}\stackrel{?}{=}-14$	
Divide.	$-14=-14\checkmark$	

Exercise 8.3.5

$a=294$

Exercise 8.3.6

$b=144$

Exercise 8.3.7

		$-n=9$
Remember –n is equivalent to –1n.		$-1n=9$
Divide both sides by –1.		$\frac{-1n}{-1}=\frac{9}{-1}$
Divide.		$n=-9$
Notice that there are two other ways to solve –n=9. We can also solve this equation by multiplying both sides by –1 and also by taking the opposite of both sides.		
Check:	$-n=9$	
Substitute n=–9.	$-(-9)\stackrel{?}{=}9$	
Simplify.	$9=9\checkmark$	

Exercise 8.3.8

$k=-8$

Exercise 8.3.9

$g=-3$

Exercise 8.3.10

Since the product of a number and its reciprocal is 1, our strategy will be to isolate x by multiplying by the reciprocal of $\frac{3}{4}$.

	$\frac{3}{4}x=12$
Multiply by the reciprocal of $\frac{3}{4}$.	$\frac{4}{3}\cdot\frac{3}{4}x=\frac{4}{3}\cdot 12$
Reciprocals multiply to 1.	$1x=\frac{4}{3}\cdot\frac{12}{1}$
Multiply.	$x=16$
Notice that we could have divided both sides of the equation $\frac{3}{4}x=12$ by $\frac{3}{4}$ to isolate x. While this would work, most people would find multiplying by the reciprocal easier.	

Check:	$\frac{3}{4}x = 12$	
Substitute $x = 16$.	$\frac{3}{4} \cdot 16 \stackrel{?}{=} 12$	
	$12 = 12$ ✓	

Exercise 8.3.11

$n = 35$

Exercise 8.3.12

$y = 18$

Exercise 8.3.13

		$\frac{8}{15} = -\frac{4}{5}x$
Multiply by the reciprocal of $-\frac{4}{5}$.		$\left(-\frac{5}{4}\right)\left(\frac{8}{15}\right) = \left(-\frac{5}{4}\right)\left(-\frac{4}{5}x\right)$
Reciprocals multiply to 1.		$-\frac{5 \cdot 4 \cdot 2}{4 \cdot 3 \cdot 5} = 1x$
Multiply.		$-\frac{2}{3} = x$
Check:	$\frac{8}{15} = -\frac{4}{5}x$	
Let $x = -\frac{2}{3}$.	$\frac{8}{15} = -\frac{4}{5}\left(-\frac{2}{3}\right)$	
	$\frac{8}{15} = \frac{8}{15}$ ✓	

Exercise 8.3.14

$z = -\frac{9}{5}$

Exercise 8.3.15

$r = -\frac{5}{16}$

Exercise 8.3.16

Begin by simplifying each side of the equation.

		$14 - 23 \stackrel{?}{=} -36 + 12 + 15$
Simplify each side.		$-9 = -9$ ✓
Divide both sides by 3 to isolate y.		$14 - 23 = 12y - 4y - 5y$
Divide.		$-9 = 3y$
Check:	$14 - 23 = 12y - 4y - 5y$	
Substitute $y = -3$.	$14 - 23 \stackrel{?}{=} 12(-3) - 4(-3) - 5(-3)$	
	$14 - 23 \stackrel{?}{=} -36 + 12 + 15$	

	$-9 = -9$ ✓	

Exercise 8.3.17

$c = -3$

Exercise 8.3.18

$x = -\frac{4}{7}$

Exercise 8.3.19

Here we will simplify each side of the equation by using the distributive property first.

		$-4(a-3)-7=25$
Distribute.		$-4a+12-7=25$
Simplify.		$-4a+5=25$
Simplify.		$-4a=20$
Divide both sides by -4 to isolate a.		$\frac{-4a}{-4}=\frac{20}{-4}$
Divide.		$a=-5$
Check:	$-4(a-3)-7=25$	
Substitute $a=-5$	$-4(-5-3)-7 \stackrel{?}{=} 25$	
	$-4(-8)-7 \stackrel{?}{=} 25$	
	$32-7 \stackrel{?}{=} 25$	
	$25=25$ ✓	

Exercise 8.3.20

$q = -6$

Exercise 8.3.21

$r = -5$

Exercise 8.3.22

Translate.
Divide by –11.
Simplify.
Check: $$\begin{aligned} 143 &= -11y \\ 143 &\stackrel{?}{=} -11(-13) \\ 143 &= 143\checkmark \end{aligned} \quad (8.3.4)$$

Exercise 8.3.23

132=–12y;y=–11

Exercise 8.3.24

117=–13z;z=–9

Exercise 8.3.25

Begin by translating the sentence into an equation. Translate.		n divided by 8 is –32. $\frac{n}{8} = -32$
Multiple both sides by 8.		$8 \cdot \frac{n}{8} = 8(-32)$
Simplify.		$n = -256$
Check:	Is nn divided by 8 equal to –32?	
Let $n = -256$.	Is –256 divided by 88 equal to –32?	
Translate.	$\frac{-256}{8} \stackrel{?}{=} -32$	
Simplify.	$-32 = -32$✓	

Exercise 8.3.26

$\frac{n}{7} = -21; n = -147$

Exercise 8.3.27

$\frac{n}{8} = -56; n = -448$

Exercise 8.3.28

Begin by translating the sentence into an equation.

Translate.		The quotient of y and –4 is 68. $\frac{y}{-4} = 68$
Multiply both sides by -4.		$-4\left(\frac{y}{-4}\right) = -4(68)$
Simplify.		$y = -272$
Check:	Is the quotient of y and –4 equal to 68?	
Let y=–272.	Is the quotient of –272 and –4 equal to 68?	
Translate.	$\frac{-272}{-4} \stackrel{?}{=} 68$	
Simplify.	$68 = 68$✓	

Exercise 8.3.29

$\frac{q}{-8} = 72; q = -576$

Exercise 8.3.30

$\frac{p}{-9} = 81; p = -729$

Exercise 8.3.31

Begin by translating the sentence into an equation. Remember, “of” translates into multiplication.

Translate.	Three-fourths of p is 18. $\frac{3}{4}p = 18$
Multiply both sides by $\frac{4}{3}$.	$\frac{4}{3} \cdot \frac{3}{4}p = \frac{4}{3} \cdot 18$
Simplify.	$p = 24$

Check:	Is three-fourths of p equal to 18?	
Let p = 24.	Is three-fourths of 24 equal to 18?	
Translate.	$\frac{3}{4} \cdot 24 \stackrel{?}{=} 18$	
Simplify.	$18 = 18 \checkmark$	

Exercise 8.3.32

$\frac{2}{5}f = 16; f = 40$

Exercise 8.3.33

$\frac{3}{4}f = 21; f = 28$

Exercise 8.3.34

Begin by translating the sentence into an equation.

Translate.		The sum of three – eighths and x is $\frac{1}{2}$ $\frac{3}{8} + x = \frac{1}{2}$
Subtract $\frac{3}{8}$ from each side.		$\frac{3}{8} - \frac{3}{8} + x = \frac{1}{2} - \frac{3}{8}$
Simplify and rewrite fractions with common denominators.		$x = \frac{4}{8} - \frac{3}{8}$
Simplify.		$x = \frac{1}{8}$
Check:	Is the sum of three-eighths and x equal to one-half?	
Let $x = \frac{1}{8}$.	Is the sum of three-eighths and one-eighth equal to one-half?	
Translate.	$\frac{3}{8} + \frac{1}{8} \stackrel{?}{=} \frac{1}{2}$	
Simplify.	$\frac{4}{8} \stackrel{?}{=} \frac{1}{2}$	
Simplify.	$\frac{1}{2} = \frac{1}{2} \checkmark$	

Exercise 8.3.35

$\frac{5}{8} + x = \frac{1}{4}; x = -\frac{3}{8}$

Exercise 8.3.36

$\frac{3}{4} + x = \frac{5}{6}; x = \frac{1}{12}$

Exercise 8.3.37

What are you asked to find?	The cost of 1 pound of grapes
Assign a variable.	Let c = the cost of one pound.
Write a sentence that gives the information to find it.	The cost of 6 pounds is $10.74
Translate into an equation.	$6c = 10.74$
Solve.	$\frac{6c}{c} = \frac{10.74}{6}$ $c = 1.79$
	The grapes cost $1.79 per pound.
Check: If one pound costs $1.79, *do* 6 pounds cost $10.74?	
	$6(1.79) \stackrel{?}{=} 10.74$ $10.74 = 10.74$✓

(8.3.5)

Exercise 8.3.38

$0.39

Exercise 8.3.39

$5.83

Exercise 8.3.40

What are you asked to find?	The original price of the car
Assign a variable.	Let p = the original price.
Write a sentence that gives the information to find it.	$12000 is $\frac{3}{4}$ of the original price.
Translate into an equation.	$12000 = \frac{3}{4}p$
Solve.	$\frac{3}{4}(12000) = \frac{4}{3} \cdot \frac{3}{4}p$ $16000 = p$
	The original cost of the car was $16000.
Check: Is $\frac{3}{4}$ of $16000 equal to $12000	
	$\frac{3}{4} \cdot 16000 \stackrel{?}{=} 12000$ $12000 = 12000$✓

(8.3.6)

Exercise 8.3.41

$120000

Exercise 8.3.42

21 flats

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8.4: Solving Equations with Variables and Constants on Both Sides

Learning Objectives

By the end of this section, you will be able to:

- Solve an equation with constants on both sides
- Solve an equation with variables on both sides
- Solve an equation with variables and constants on both sides

Solve Equations with Constants on Both Sides

In all the equations we have solved so far, all the variable terms were on only one side of the equation with the constants on the other side. This does not happen all the time—so now we will learn to solve equations in which the variable terms, or constant terms, or both are on both sides of the equation.

Our strategy will involve choosing one side of the equation to be the “variable side”, and the other side of the equation to be the “constant side.” Then, we will use the Subtraction and Addition Properties of Equality to get all the variable terms together on one side of the equation and the constant terms together on the other side.

By doing this, we will transform the equation that began with variables and constants on both sides into the form $ax = b$. We already know how to solve equations of this form by using the Division or Multiplication Properties of Equality.

Exercise 8.4.1

Solve: $7x + 8 = -13$.

Answer available at the end of this section.

Exercise 8.4.2

Solve: $3x + 4 = -8$.

Answer available at the end of this section.

Exercise 8.4.3

Solve: $5a + 3 = -37$.

Answer available at the end of this section.

Exercise 8.4.4

Solve: $8y - 9 = 31$.

Answer available at the end of this section.

Exercise 8.4.5

Solve: $5y - 9 = 16$.

Answer available at the end of this section.

Exercise 8.4.6

Solve: $3m - 8 = 19$.

Answer available at the end of this section.

Solve Equations with Variables and Constants on Both Sides

The next example will be the first to have variables and constants on both sides of the equation. It may take several steps to solve this equation, so we need a clear and organized strategy.

Exercise 8.4.7

Solve: $9x = 8x - 6$.

Answer available at the end of this section.

Exercise 8.4.8

Solve: $6n = 5n - 10$.

Answer available at the end of this section.

Exercise 8.4.9

Solve: $-6c = -7c - 1$

Answer available at the end of this section.

Exercise 8.4.10

Solve: $5y - 9 = 8y$

Answer available at the end of this section.

Exercise 8.4.11

Solve: $3p - 14 = 5p$.

Answer available at the end of this section.

Exercise 8.4.12

Solve: $8m + 9 = 5m$

Answer available at the end of this section.

Exercise 8.4.13

Solve: $12x = -x + 26$

Answer available at the end of this section.

Exercise 8.4.14

Solve: $12j = -4j + 32$

Answer available at the end of this section.

Exercise 8.4.15

Solve: $8h = -4h + 12$

Answer available at the end of this section.

Solve Equations with Variables and Constants on Both Sides

The next example will be the first to have variables and constants on both sides of the equation. It may take several steps to solve this equation, so we need a clear and organized strategy.

Exercise 8.4.16: How to Solve Equations with Variables and Constants on Both Sides

Solve: $7x + 5 = 6x + 2$

Answer available at the end of this section.

Exercise 8.4.17

Solve: $12x + 8 = 6x + 2$.

Answer available at the end of this section.

Exercise 8.4.18

Solve: $9y + 4 = 7y + 12$.

Answer available at the end of this section.

We'll list the steps below so you can easily refer to them. But we'll call this the 'Beginning Strategy' because we'll be adding some

steps later in this chapter.

BEGINNING STRATEGY FOR SOLVING EQUATIONS WITH VARIABLES AND CONSTANTS ON BOTH SIDES OF THE EQUATION.

1. Choose which side will be the “variable” side—the other side will be the “constant” side.
2. Collect the variable terms to the “variable” side of the equation, using the Addition or Subtraction Property of Equality.
3. Collect all the constants to the other side of the equation, using the Addition or Subtraction Property of Equality.
4. Make the coefficient of the variable equal 1, using the Multiplication or Division Property of Equality.
5. Check the solution by substituting it into the original equation.

In Step 1, a helpful approach is to make the “variable” side the side that has the variable with the larger coefficient. This usually makes the arithmetic easier.

Exercise 8.4.19

Solve: $8n - 4 = -2n + 6$.

Answer available at the end of this section.

Exercise 8.4.20

Solve: $8q - 5 = -4q + 7$

Answer available at the end of this section.

Exercise 8.4.21

Solve: $7n - 3 = n + 3$

Answer available at the end of this section.

Exercise 8.4.22

Solve: $7a - 3 = 13a + 7$

Answer available at the end of this section.

Exercise 8.4.23

Solve: $2a - 2 = 6a + 18$

Answer available at the end of this section.

Exercise 8.4.24

Solve: $4k - 1 = 7k + 17$

Answer available at the end of this section.

In the last example, we could have made the left side the “variable” side, but it would have led to a negative coefficient on the variable term. (Try it!) While we could work with the negative, there is less chance of errors when working with positives. The strategy outlined above helps avoid the negatives!

To solve an equation with fractions, we just follow the steps of our strategy to get the solution!

Exercise 8.4.25

Solve: $\frac{5}{4}x + 6 = \frac{1}{4}x - 2$

Answer available at the end of this section.

Exercise 8.4.26

Solve: $\frac{7}{8}x - 12 = -\frac{1}{8}x - 2$

Answer available at the end of this section.

Exercise 8.4.27

Solve: $\frac{7}{6}y + 11 = \frac{1}{6}y + 8$

Answer available at the end of this section.

We will use the same strategy to find the solution for an equation with decimals.

Exercise 8.4.28

Solve: $7.8x + 4 = 5.4x - 8$.

Answer available at the end of this section.

Exercise 8.4.29

Solve: $2.8x + 12 = -1.4x - 9$

Answer available at the end of this section.

Exercise 8.4.30

Solve: $3.6y + 8 = 1.2y - 4$

Answer available at the end of this section.

Key Concepts

- **Beginning Strategy for Solving an Equation with Variables and Constants on Both Sides of the Equation**
  1. Choose which side will be the “variable” side—the other side will be the “constant” side.
  2. Collect the variable terms to the “variable” side of the equation, using the Addition or Subtraction Property of Equality.
  3. Collect all the constants to the other side of the equation, using the Addition or Subtraction Property of Equality.

4. Make the coefficient of the variable equal 1, using the Multiplication or Division Property of Equality.
5. Check the solution by substituting it into the original equation.

Answers

Exercise 8.4.1

In this equation, the variable is found only on the left side. It makes sense to call the left side the "variable" side. Therefore, the right side will be the "constant" side. We will write the labels above the equation to help us remember what goes where.

variable constant

$$7x+8=-13$$

Since the left side is the "xx", or variable side, the 8 is out of place. We must "undo" adding 8 by subtracting 8, and to keep the equality we must subtract 8 from both sides.

	variable constant $7x+8=-13$
Use the Subtraction Property of Equality.	$7x+8-8=-13-8$
Simplify.	$7x=-21$
Now all the variables are on the left and the constant on the right. The equation looks like those you learned to solve earlier.	
Use the Division Property of Equality.	$\frac{7x}{7}=\frac{-21}{7}$
Simplify.	$x=-3$
Check: $7x+8=-13$	
Let x=−3. $7(-3)+8\stackrel{?}{=}-13$	
$-21+8\stackrel{?}{=}-13$ $-13=-13$ ✓	

Exercise 8.4.2

$x=-4$

Exercise 8.4.3

$a=-8$

Exercise 8.4.4

Notice, the variable is only on the left side of the equation, so we will call this side the "variable" side, and the right side will be the "constant" side. Since the left side is the "variable" side, the 9 is out of place. It is subtracted from the 8y, so to "undo" subtraction, add 9 to both sides. Remember, whatever you do to the left, you must do to the right.

	variable constant $8y-9=31$
Add 9 to both sides.	$8y-9+9=31+9$
Simplify.	$8y=40$
	The variables are now on one side and the constants on the other. We continue from here as we did earlier.
Divide both sides by 8.	$\frac{8y}{8}=\frac{40}{8}$
Simplify.	$y=5$

Check:	$8y-9=31$	
Let y=5.	$8\cdot 5-9\stackrel{?}{=}31$	
	$40-9\stackrel{?}{=}31$	
	$31=31$ ✓	

Exercise 8.4.5

$y=5$

Exercise 8.4.6

$m=9$

Exercise 8.4.7

Here the variable is on both sides, but the constants only appear on the right side, so let's make the right side the "constant" side. Then the left side will be the "variable" side.

		variable constant $9x=8x-6$
We don't want any x's on the right, so subtract the 8x from both sides.		$9x-8x=8x-8x-6$
Simplify.		$x=-6$
We succeeded in getting the variables on one side and the constants on the other, and have obtained the solution.		
Check:	$9x=8x-6$	
Let x=−6.	$9(-6)\stackrel{?}{=}8(-6)-6$	
	$-54\stackrel{?}{=}-48-6$	
	$-54=-54$ ✓	

Exercise 8.4.8

$n=-10$

Exercise 8.4.9

$c=-1$

Exercise 8.4.10

The only constant is on the left and the y's are on both sides. Let's leave the constant on the left and get the variables to the right.

		constant variable $5y-9=8y$
Subtract 5y from both sides.		$5y-5y-9=8y-5y$
Simplify.		$-9=3y$
We have the y's on the right and the constants on the left. Divide both sides by 3.		$\frac{-9}{3}=\frac{3y}{3}$
Simplify.		$-3=y$
Check:	$5y-9=8y$	
Let $y=-3$.	$5(-3)-9\stackrel{?}{=}8(-3)$	
	$-15-9\stackrel{?}{=}-24$	

	$-24 = -24$ ✓	

Exercise 8.4.11

$p = -7$

Exercise 8.4.12

$m = -3$

Exercise 8.4.13

The only constant is on the right, so let the left side be the "variable" side.

	variable constant
	$12x = -x + 26$
Remove the –x from the right side by adding x to both sides.	$12x + x = -x + x + 26$
Simplify.	$13x = 26$
All the x's are on the left and the constants are on the right. Divide both sides by 13.	$\frac{13x}{13} = \frac{26}{13}$
Simplify.	$x = 2$

Exercise 8.4.14

$j = 2$

Exercise 8.4.15

$h = 1$

Exercise 8.4.16**: How to Solve Equations with Variables and Constants on Both Sides**

Step 1. Choose which side will be the "variable" side—the other side will be the "constant" side.	The variable terms are $7x$ and $6x$. Since 7 is greater than 6, we will make the left side the "x" side. The right side will be the "constant" side.	variable constant $7x + 5 = 6x + 2$
Step 2. Collect the variable terms to the "variable" side of the equation, using the addition or subtraction property of equality.	With the right side as the "constant" side, the $6x$ is out of place, so subtract $6x$ from both sides. Combine like terms. Now, the variable is only on the left side!	$7x - 6x + 5 = 6x - 6x + 2$ $x + 5 = 2$
Step 3. Collect all the constants to the other side of the equation, using the addition or subtraction property of equality.	The right side is the "constant" side, so the 5 is out of place. Subtract 5 from both sides. Simplify.	$x + 5 - 5 = 2 - 5$ $x = -3$

Step 4. Make the coefficient of the variable equal 1, using the multiplication or division property of equality.	The coefficient of x is one. The equation is solved.	
Step 5. Check.	Let $x = -3$ Simplify. Add.	**Check:** $7x + 6 = 6x + 2$ $(-3) + 5 = 6(-3) + 2$ $-21 + 5 = -18 + 2$ $-16 = -16$ ✓

Exercise 8.4.17

$x = -1$

Exercise 8.4.18

$y = 4$

Exercise 8.4.19

In the first step, choose the variable side by comparing the coefficients of the variables on each side.

Since $8 > -2$, make the left side the "variable" side.	variable constant $8n - 4 = -2n + 6$
We don't want variable terms on the right side—add 2n to both sides to leave only constants on the right.	$8n + 2n - 4 = -2n + 2n + 6$
Combine like terms.	$10n - 4 = 6$
We don't want any constants on the left side, so add 4 to both sides.	$10n - 4 + 4 = 6 + 4$
Simplify.	$10n = 10$
The variable term is on the left and the constant term is on the right. To get the coefficient of nn to be one, divide both sides by 10.	$\frac{10n}{10} = \frac{10}{10}$
Simplify.	$n = 1$
Check:	$8n - 4 = -2n + 6$
Let n=1.	$8 \cdot 1 - 4 \stackrel{?}{=} -2 \cdot 1 + 6$
	$8 - 4 \stackrel{?}{=} -2 + 6$
	$4 = 4$ ✓

Exercise 8.4.20

$q = 1$

Exercise 8.4.21

$n = 1$

Exercise 8.4.22

In the first step, choose the variable side by comparing the coefficients of the variables on each side.

Since 13>7, make the right side the "variable" side and the left side the "constant" side.

		constant variable $7a-3=13a+7$
Subtract 7a from both sides to remove the variable term from the left.		$7a-7a-3=13a-7a+7$
Combine like terms.		$-3=6a+7$
Subtract 7 from both sides to remove the constant from the right.		$-3-7=6a+7-7$
Simplify.		$-10=6a$
Divide both sides by 6 to make 1 the coefficient of a.		$\frac{-10}{6}=\frac{6a}{6}$
Simplify.		$-\frac{5}{3}=a$
Check:	$7a-3=13a+7$	
Let $a=-\frac{5}{3}$.	$7\left(-\frac{5}{3}\right)-3\stackrel{?}{=}13\left(-\frac{5}{3}\right)+7$	
	$-\frac{35}{3}-\frac{9}{3}\stackrel{?}{=}-\frac{65}{3}+\frac{21}{3}$	
	$-\frac{54}{3}=-\frac{54}{3}$ ✓	

Exercise 8.4.23

$a=-5$

Exercise 8.4.24

$k=-6$

Exercise 8.4.25

Since $\frac{5}{4}>\frac{1}{4}$, make the left side the “variable” side and the right side the “constant” side.

		variable constant $\frac{5}{4}x+6=\frac{1}{4}x-2$
Subtract $\frac{1}{4}x$ from both sides.		$\frac{5}{4}x-\frac{1}{4}x+6=\frac{1}{4}x-\frac{1}{4}x-2$
Combine like terms.		$x+6=-2$
Subtract 6 from both sides.		$x+6-6=-2-6$
Simplify.		$x=-8$
Check: Let $x=-8$	$\frac{5}{4}x+6 = \frac{1}{4}x-2$ $\frac{5}{4}(-8)+6 \stackrel{?}{=} \frac{1}{4}(-8)-2$ $-10+6 \stackrel{?}{=} -2-2$ $-4 = -4$✓	

Exercise 8.4.26

$x=10$

Exercise 8.4.27

$y=-3$

Exercise 8.4.28

Since $7.8 > 5.4$, make the left side the “variable” side and the right side the “constant” side.

		variable side constant side $7.8x + 4 = 5.4x - 8$
Subtract 5.4x from both sides.		$7.8x - 5.4x + 4 = 5.4x - 5.4x - 8$
Combine like terms.		$2.4x + 4 = -8$
Subtract 4 from both sides.		$2.4x + 4 - 4 = -8 - 4$
Simplify.		$2.4x = -12$
Use the Division Property of Equality.		$\frac{2.4x}{2.4} = \frac{-12}{2.4}$
Simplify.		$x = -5$
Check:	$7.8x + 4 = 5.4x - 5$	
Let $x = -5$	$7.8(-5) + 4 = 5.4(-5) - 8$	
	$-39 + 4 \stackrel{?}{=} -27 - 8$	
	$-35 = -35$ ✓	

Exercise 8.4.29

$x = -5$

Exercise 8.4.30

$y = -5$

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8.5: General Strategy to Solve Linear Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve equations using a general strategy
- Classify equations

Solve Equations Using the General Strategy

Until now we have dealt with solving one specific form of a linear equation. It is time now to lay out one overall strategy that can be used to solve any linear equation. Some equations we solve will not require all these steps to solve, but many will.

Beginning by simplifying each side of the equation makes the remaining steps easier.

Exercise 8.5.1: How to Solve Linear Equations Using the General Strategy

Solve: $-6(x+3)=24$.

Answer available at the end of this section.

Exercise 8.5.2

Solve: $5(x+3)=35$

Answer available at the end of this section.

Exercise 8.5.3

Solve: $6(y-4)=-18$

Answer available at the end of this section.

GENERAL STRATEGY FOR SOLVING LINEAR EQUATIONS.

1. **Simplify each side of the equation as much as possible.**
   Use the Distributive Property to remove any parentheses.
   Combine like terms.
2. **Collect all the variable terms on one side of the equation.**
   Use the Addition or Subtraction Property of Equality.
3. **Collect all the constant terms on the other side of the equation.**
   Use the Addition or Subtraction Property of Equality.
4. **Make the coefficient of the variable term to equal to 1.**
   Use the Multiplication or Division Property of Equality.
   State the solution to the equation.
5. **Check the solution.** Substitute the solution into the original equation to make sure the result is a true statement.

Exercise 8.5.4

Solve: $-(y+9)=8$

Answer available at the end of this section.

Exercise 8.5.5

Solve: $-(y+8)=-2$

Answer available at the end of this section.

Exercise 8.5.6

Solve: $-(z+4)=-12$

Answer available at the end of this section.

Exercise 8.5.7

Solve: $5(a-3)+5=-10$

Answer available at the end of this section.

Exercise 8.5.8

Solve: $2(m-4)+3=-1$

Answer available at the end of this section.

Exercise 8.5.9

Solve:$7(n-3)-8=-15$

Answer available at the end of this section.

Exercise 8.5.10

Solve: $\frac{2}{3}(6m-3)=8-m$

Answer available at the end of this section.

Exercise 8.5.11

Solve: $\frac{1}{3}(6u+3)=7-u$

Answer available at the end of this section.

Exercise 8.5.12

Solve: $\frac{2}{3}(9x-12)=8+2x$

Answer available at the end of this section.

Exercise 8.5.13

Solve: $8-2(3y+5)=0$

Answer available at the end of this section.

Exercise 8.5.14

Solve: $12-3(4j+3)=-17$

Answer available at the end of this section.

Exercise 8.5.15

Solve: $-6-8(k-2)=-10$

Answer available at the end of this section.

Exercise 8.5.16

Solve: $4(x-1)-2=5(2x+3)+6$

Answer available at the end of this section.

Exercise 8.5.17

Solve: $6(p-3)-7=5(4p+3)-12$

Answer available at the end of this section.

Exercise 8.5.18

Solve: $8(q+1)-5=3(2q-4)-1$

Answer available at the end of this section.

Exercise 8.5.19

Solve: $10[3-8(2s-5)]=15(40-5s)$

Answer available at the end of this section.

Exercise 8.5.20

Solve: $6[4-2(7y-1)]=8(13-8y)$.

Answer available at the end of this section.

Exercise 8.5.21

Solve: $12[1-5(4z-1)]=3(24+11z)$.

Answer available at the end of this section.

Exercise 8.5.22

Solve: $0.36(100n+5)=0.6(30n+15)$.

Answer available at the end of this section.

Exercise 8.5.23

Solve: $0.55(100n+8)=0.6(85n+14)$.

Answer available at the end of this section.

Exercise 8.5.24

Solve: $0.15(40m-120)=0.5(60m+12)$.

Answer available at the end of this section.

Classify Equations

Consider the equation we solved at the start of the last section, 7x+8=–13. The solution we found was x=–3. This means the equation 7x+8=–13 is true when we replace the variable, *x*, with the value –3. We showed this when we checked the solution x=–3 and evaluated 7x+8=–13 for x=–3.

$$7(-3)+8 \stackrel{?}{=} -13$$
$$-21+8 \stackrel{?}{=} -13$$
$$-13=-13\checkmark$$

If we evaluate 7x+8 for a different value of *x*, the left side will not be –13.

The equation 7x+8=–13 is true when we replace the variable, *x*, with the value –3, but not true when we replace *x* with any other value. Whether or not the equation 7x+8=–13 is true depends on the value of the variable. Equations like this are called conditional equations.

All the equations we have solved so far are conditional equations.

CONDITIONAL EQUATION

An equation that is true for one or more values of the variable and false for all other values of the variable is a **conditional equation**.

Now let's consider the equation 2y+6=2(y+3). Do you recognize that the left side and the right side are equivalent? Let's see what happens when we solve for *y*.

	$2y+6=2(y+3)$
Distribute.	$2y+6=2y+6$
Subtract 2y to get the y's to one side.	$2y-2y+6=2y-2y+6$
Simplify—the y's are gone!	$6=6$

But 6=6 is true.

This means that the equation 2y+6=2(y+3) is true for any value of *y*. We say the solution to the equation is all of the real numbers. An equation that is true for any value of the variable like this is called an **identity**.

IDENTITY

An equation that is true for any value of the variable is called an **identity**.

The solution of an identity is all real numbers.

What happens when we solve the equation 5z=5z–1?

	$5z=5z-1$
Subtract 5z to get the constant alone on the right.	$5z-5z=5z-5z-1$
Simplify—the z's are gone!	$0\neq -1$

But $0 \neq -1$.

Solving the equation 5z=5z–1 led to the false statement 0=–1. The equation 5z=5z–1 will not be true for any value of z. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a **contradiction**.

CONTRADICTION

An equation that is false for all values of the variable is called a **contradiction**.

A contradiction has no solution.

Exercise 8.5.25

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$6(2n-1)+3=2n-8+5(2n+1)$

Answer available at the end of this section.

Exercise 8.5.26

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$4 + 9(3x - 7) = -42x - 13 + 23(3x - 2)$

Answer available at the end of this section.

Exercise 8.5.27

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$8(1 - 3x) + 15(2x + 7) = 2(x + 50) + 4(x + 3) + 1$

Answer available at the end of this section.

Exercise 8.5.28

Classify as a conditional equation, an identity, or a contradiction. Then state the solution.

$10 + 4(p - 5) = 0$

Answer available at the end of this section.

Exercise 8.5.29

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $11(q + 3) - 5 = 19$

Answer available at the end of this section.

Exercise 8.5.30

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6 + 14(k - 8) = 95$

Answer available at the end of this section.

Exercise 8.5.31

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$5m + 3(9 + 3m) = 2(7m - 11)$

Answer available at the end of this section.

Exercise 8.5.32

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$12c + 5(5 + 3c) = 3(9c - 4)$

Answer available at the end of this section.

Exercise 8.5.33

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$4(7d+18)=13(3d-2)-11d$

Answer available at the end of this section.

Type of equation	What happens when you solve it?	Solution
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	**True** for any value of the variable	All real numbers
Contradiction	**False** for all values of the variable	No solution

Key Concepts

- **General Strategy for Solving Linear Equations**
  1. Simplify each side of the equation as much as possible.
     Use the Distributive Property to remove any parentheses.
     Combine like terms.
  2. Collect all the variable terms on one side of the equation.
     Use the Addition or Subtraction Property of Equality.
  3. Collect all the constant terms on the other side of the equation.
     Use the Addition or Subtraction Property of Equality.
  4. Make the coefficient of the variable term to equal to 1.
     Use the Multiplication or Division Property of Equality.
     State the solution to the equation.
  5. Check the solution.
     Substitute the solution into the original equation.

Answers

Exercise 8.5.1: How to Solve Linear Equations Using the General Strategy

Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is simplified as much as possible.	$-6(x+3)=24$ $-6x-18=24$
Step 2. Collect all variable terms on one side of the equation.	Nothing to do – all x's are on the left side.	
Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 18 to each side. Simplify.	$-6x-18+18=24+18$ $-6x=42$

Step 4. Make the coefficient of the variable term to equal to 1.	Divide each side by –6. Simplify.	$\frac{-6x}{-6} = \frac{42}{-6}$ $x = -7$
Step 5. Check the solution.	Let $x = -7$ Simplify. Multiply.	**Check:** $-6(x + 3) = 24$ $-6(-7 + 3) \stackrel{?}{=} 24$ $-6(-4) \stackrel{?}{=} 24$ $24 = 24$ ✓

Exercise 8.5.2

$x = 4$

Exercise 8.5.3

$y = 1$

Exercise 8.5.4

		$-(y + 9) = 8$
Simplify each side of the equation as much as possible by distributing.		$-y - 9 = 8$
The only y term is on the left side, so all variable terms are on the left side of the equation.		
Add 9 to both sides to get all constant terms on the right side of the equation.		$-y - 9 + 9 = 8 + 9$
Simplify.		$-y = 17$
Rewrite –y as –1y.		$-1y = 17$
Make the coefficient of the variable term to equal to 1 by dividing both sides by –1.		$\frac{-1y}{-1} = \frac{17}{-1}$
Simplify.		$y = -17$
Check:	$-(y + 9) = 8$	
Let y=–17.	$-(-17 + 9) \stackrel{?}{=} 8$	
	$-(-8) \stackrel{?}{=} 8$	
	$8 = 8$ ✓	

Exercise 8.5.5

$y = -6$

Exercise 8.5.6

$z = 8$

Exercise 8.5.7

	$5(a - 3) + 5 = -10$
Simplify each side of the equation as much as possible.	

Distribute.		$5a-15+5=-10$
Combine like terms.		$5a-10=-10$
The only a term is on the left side, so all variable terms are on one side of the equation.		
Add 10 to both sides to get all constant terms on the other side of the equation.		$5a-10+10=-10+10$
Simplify.		$5a=0$
Make the coefficient of the variable term to equal to 11 by dividing both sides by 55.		$\frac{5a}{5}=\frac{0}{5}$
Simplify.		$a=0$
Check:	$5(a-3)+5=-10$	
Let a=0.	$5(0-3)+5\stackrel{?}{=}-10$	
	$5(-3)+5\stackrel{?}{=}-10$	
	$-15+5\stackrel{?}{=}-10$	
	$-10=-10$ ✓	

Exercise 8.5.8

$m=2$

Exercise 8.5.9

$n=2$

Exercise 8.5.10

		$\frac{2}{3}(6m-3)=8-m$
Distribute.		$4m-2=8-m$
Add m to get the variables only to the left.		$4m+m-2=8-m+m$
Simplify.		$5m-2=8$
Add 2 to get constants only on the right.		$5m-2+2=8+2$
Simplify.		$5m=10$
Divide by 5.		$\frac{5m}{5}=\frac{10}{5}$
Simplify.		$m=2$
Check:	$\frac{2}{3}(6m-3)=8-m$	
Let m=2.	$\frac{2}{3}(6\cdot 2-3)\stackrel{?}{=}8-2$	
	$\frac{2}{3}(12-3)\stackrel{?}{=}6$	
	$\frac{2}{3}(9)\stackrel{?}{=}6$	

	$6 = 6$ ✓	

Exercise 8.5.11

$u = 2$

Exercise 8.5.12

$x = 4$

Exercise 8.5.13

	$8 - 2(3y + 5) = 0$
Simplify—use the Distributive Property.	$8 - 6y - 10 = 0$
Combine like terms.	$-6y - 2 = 0$
Add 2 to both sides to collect constants on the right.	$-6y - 2 + 2 = 0 + 2$
Simplify.	$-6y = 2$
Divide both sides by −6−6.	$\frac{-6y}{-6} = \frac{2}{-6}$
Simplify.	$y = -\frac{1}{3}$
Check: Let y=−13. $8 - 2(3y + 5) = 0$ $8 - 2\left[3\left(-\frac{1}{3}\right) + 5\right] = 0$ $8 - 2(-1 + 5) \stackrel{?}{=} 0$ $8 - 2(4) \stackrel{?}{=} 0$ $8 - 8 \stackrel{?}{=} 0$ $0 = 0$ ✓	

Exercise 8.5.14

$j = \frac{5}{3}$

Exercise 8.5.15

$k = \frac{5}{2}$

Exercise 8.5.16

	$4(x - 1) - 2 = 5(2x + 3) + 6$
Distribute.	$4x - 4 - 2 = 10x + 15 + 6$
Combine like terms.	$4x - 6 = 10x + 21$
Subtract 4x to get the variables only on the right side since $10 > 4$.	$4x - 4x - 6 = 10x - 4x + 21$

Simplify.		$-6 = 6x + 21$
Subtract 21 to get the constants on left.		$-6 - 21 = 6x + 21 - 21$
Simplify.		$-27 = 6x$
Divide by 6.		$\frac{-27}{6} = \frac{6x}{6}$
Simplify.		$-\frac{9}{2} = x$
Check:	$4(x-1)-2=5(2x+3)+6$	
Let $x = -\frac{9}{2}$.	$4\left(-\frac{9}{2}-1\right)-2 \stackrel{?}{=} 5\left[2\left(-\frac{9}{2}\right)+3\right]+6$	
	$4\left(-\frac{11}{2}\right)-2 \stackrel{?}{=} 5(-9+3)+6$	
	$-22-2 \stackrel{?}{=} 5(-6)+6$	
	$-24 \stackrel{?}{=} -30+6$	
	$-24 = -24$ ✓	

Exercise 8.5.17

$p = -2$

Exercise 8.5.18

$q = -8$

Exercise 8.5.19

		$10[3-8(2s-5)] = 15(40-5s)$
Simplify from the innermost parentheses first.		$10[3-16s+40] = 15(40-5s)$
Combine like terms in the brackets.		$10[43-16s] = 15(40-5s)$
Distribute.		$430-160s = 600-75s$
Add 160s to get the s's to the right.		$430-160s+160s = 600-75s+160s$
Simplify.		$430 = 600+85s$
Subtract 600 to get the constants to the left.		$430-600 = 600+85s-600$
Simplify.		$-170 = 85s$
Divide.		$\frac{-170}{85} = \frac{85s}{85}$
Simplify.		$-2 = s$
Check:	$10[3-8(2s-5)] = 15(40-5s)$	
Substitute s=−2.	$10[3-8(2(-2)-5)] \stackrel{?}{=} 15(40-5(-2))$	
	$10[3-8(-4-5)] \stackrel{?}{=} 15(40+10)$	

	$10[3-8(-9)] \stackrel{?}{=} 15(50)$	
	$10[3+72] \stackrel{?}{=} 750$	
	$10[75] \stackrel{?}{=} 750$	
	$750 = 750$ ✓	

Exercise 8.5.20

$y = -\frac{17}{5}$

Exercise 8.5.21

$z = 0$

Exercise 8.5.22

		$0.36(100n+5) = 0.6(30n+15)$
Distribute.		$36n + 1.8 = 18n + 9$
Subtract 18n to get the variables to the left.		$36n - 18n + 1.8 = 18n - 18n + 9$
Simplify.		$18n + 1.8 = 9$
Subtract 1.8 to get the constants to the right.		$18n + 1.8 - 1.8 = 9 - 1.8$
Simplify.		$18n = 7.2$
Divide.		$\frac{18n}{18} = \frac{7.2}{18}$
Simplify.		$n = 0.4$
Check:	$0.36(100n+5) = 0.6(30n+15)$	
Let n=0.4.	$0.36(100(0.4)+5) \stackrel{?}{=} 0.6(30(0.4)+15)$	
	$0.36(40+5) \stackrel{?}{=} 0.6(12+15)$	
	$0.36(45) \stackrel{?}{=} 0.6(27)$	
	$16.2 = 16.2$ ✓	

Exercise 8.5.23

$n = 1$

Exercise 8.5.24

$m = -1$

Exercise 8.5.25

$6(2n-1)+3 = 2n-8+5(2n+1)$	
Distribute.	$12n - 6 + 3 = 2n - 8 + 10n + 5$
Combine like terms.	$12n - 3 = 12n - 3$
Subtract 12n to get the nn's to one side.	$12n - 12n - 3 = 12n - 12n - 3$
Simplify.	$-3 = -3$
This is a true statement.	The equation is an identity. The solution is all real numbers.

Exercise 8.5.26

identity; all real numbers

Exercise 8.5.27

identity; all real numbers

Exercise 8.5.28

	$10 + 4(p - 5) = 0$
Distribute.	$10 + 4p - 20 = 0$
Combine like terms.	$4p - 10 = 0$
Add 10 to both sides.	$4p - 10 + 10 = 0 + 10$
Simplify.	$4p = 10$
Divide.	$\frac{4p}{4} = \frac{10}{4}$
Simplify.	$p = \frac{5}{2}$
The equation is true when $p = frac52$.	This is a conditional equation. The solution is $p = frac52$.

Exercise 8.5.29

conditional equation; \(q = \frac{9}{11}\

Exercise 8.5.30

conditional equation; $k = \frac{193}{14}$

Exercise 8.5.31

	$5m + 3(9 + 3m) = 2(7m - 11)$
Distribute.	$5m + 27 + 9m = 14m - 22$
Combine like terms.	$14m + 27 = 14m - 22$
Subtract 14m from both sides.	$14m + 27 - 14m = 14m - 22 - 14m$
Simplify.	$27 \neq -22$
But $27 \neq -22$.	The equation is a contradiction. It has no solution.

Exercise 8.5.32

contradiction; no solution

Exercise 8.5.33

contradiction; no solution

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8.6: Solving Equations with Fractions or Decimals

Learning Objectives

By the end of this section, you will be able to:

- Solve equations with fraction coefficients
- Solve equations with decimal coefficients

Solve Equations with Fraction Coefficients

Let's use the general strategy for solving linear equations introduced earlier to solve the equation, $\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$.

Table 8.6.1

	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
To isolate the x term, subtract $\frac{1}{2}$ from both sides.	$\frac{1}{8}x+\frac{1}{2}-\frac{1}{2}=\frac{1}{4}-\frac{1}{2}$
Simplify the left side.	$\frac{1}{8}x=\frac{1}{4}-\frac{1}{2}$
Change the constants to equivalent fractions with the LCD.	$\frac{1}{8}x=\frac{1}{4}-\frac{2}{4}$
Subtract.	$\frac{1}{8}x=-\frac{1}{4}$
Multiply both sides by the reciprocal of $\frac{1}{8}$.	$\frac{8}{1}\cdot\frac{1}{8}x=\frac{8}{1}\left(-\frac{1}{4}\right)$
Simplify.	$x=-2$

This method worked fine, but many students do not feel very confident when they see all those fractions. So, we are going to show an alternate method to solve equations with fractions. This alternate method eliminates the fractions.

We will apply the Multiplication Property of Equality and multiply both sides of an equation by the least common denominator of all the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but without fractions. This process is called "clearing" the equation of fractions.

Let's solve a similar equation, but this time use the method that eliminates the fractions.

Exercise 8.6.1: How to Solve Equations with Fraction Coefficients

Solve: $\frac{1}{6}y-\frac{1}{3}=\frac{5}{6}$

Answer available at the end of this section.

Exercise 8.6.2

Solve: $\frac{1}{4}x + \frac{1}{2} = \frac{5}{8}$

Answer available at the end of this section.

Exercise 8.6.3

Solve: $\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$

Answer available at the end of this section.

Notice in Exercise 8.6.1, once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve! We then used the General Strategy for Solving Linear Equations.

STRATEGY TO SOLVE EQUATIONS WITH FRACTION COEFFICIENTS.

1. Find the least common denominator of *all* the fractions in the equation.
2. Multiply both sides of the equation by that LCD. This clears the fractions.
3. Solve using the General Strategy for Solving Linear Equations.

Exercise 8.6.4

Solve: $6 = \frac{1}{2}v + \frac{2}{5}v - \frac{3}{4}v$

Answer available at the end of this section.

Exercise 8.6.5

Solve: $7 = \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x$

Answer available at the end of this section.

Exercise 8.6.6

Solve: $-1 = \frac{1}{2}u + \frac{1}{4}u - \frac{2}{3}u$

Answer available at the end of this section.

In the next example, we again have variables on both sides of the equation.

Exercise 8.6.7

Solve: $a + \frac{3}{4} = \frac{3}{8}a - \frac{1}{2}$

Answer available at the end of this section.

Exercise 8.6.8

Solve: $x + \frac{1}{3} = \frac{1}{6}x - \frac{1}{2}$

Answer available at the end of this section.

Exercise 8.6.9

Solve: $c + \frac{3}{4} = \frac{1}{2}c - \frac{1}{4}$

Answer available at the end of this section.

In the next example, we start by using the Distributive Property. This step clears the fractions right away.

Exercise 8.6.10

Solve: $-5 = \frac{1}{4}(8x + 4)$

Answer available at the end of this section.

Exercise 8.6.11

Solve: $-11 = \frac{1}{2}(6p + 2)$

Answer available at the end of this section.

Exercise 8.6.12

Solve: $8 = \frac{1}{3}(9q + 6)$

Answer available at the end of this section.

In the next example, even after distributing, we still have fractions to clear.

Exercise 8.6.13

Solve: $\frac{1}{2}(y - 5) = \frac{1}{4}(y - 1)$

Answer available at the end of this section.

Exercise 8.6.14

Solve: $\frac{1}{5}(n + 3) = \frac{1}{4}(n + 2)$

Answer available at the end of this section.

Exercise 8.6.15

Solve: $\frac{1}{2}(m-3)=\frac{1}{4}(m-7)$

Answer available at the end of this section.

Exercise 8.6.16

Solve: $\frac{5x-3}{4}=\frac{x}{2}$

Answer available at the end of this section.

Exercise 8.6.17

Solve: $\frac{4y-7}{3}=\frac{y}{6}$

Answer available at the end of this section.

Exercise 8.6.18

Solve: $\frac{-2z-5}{4}=\frac{z}{8}$

Answer available at the end of this section.

Exercise 8.6.19

Solve: $\frac{a}{6}+2=\frac{a}{4}+3$

Answer available at the end of this section.

Exercise 8.6.20

Solve: $\frac{b}{10}+2=\frac{b}{4}+5$

Answer available at the end of this section.

Exercise 8.6.21

Solve: $\frac{c}{6}+3=\frac{c}{3}+4$

Answer available at the end of this section.

Exercise 8.6.22

Solve: $\frac{4q+3}{2}+6=\frac{3q+5}{4}$

Answer available at the end of this section.

Exercise 8.6.23

Solve: $\frac{3r+5}{6}+1=\frac{4r+3}{3}$

Answer available at the end of this section.

Exercise 8.6.24

Solve: $\frac{2s+3}{2}+1=\frac{3s+2}{4}$

Answer available at the end of this section.

Solve Equations with Decimal Coefficients

Some equations have decimals in them. This kind of equation will occur when we solve problems dealing with money or percentages. But decimals can also be expressed as fractions. For example, $0.3=\frac{3}{10}$ and $0.17=\frac{17}{100}$. So, with an equation with decimals, we can use the same method we used to clear fractions—multiply both sides of the equation by the least common denominator.

Exercise 8.6.25

Solve: $0.06x+0.02=0.25x-1.5$

Answer available at the end of this section.

Exercise 8.6.26

Solve: $0.14h+0.12=0.35h-2.4$

Answer available at the end of this section.

Exercise 8.6.27

Solve: $0.65k-0.1=0.4k-0.35$

Answer available at the end of this section.

The next example uses an equation that is typical of the money applications in the next chapter. Notice that we distribute the decimal before we clear all the decimals.

Exercise 8.6.28

Solve: $0.25x+0.05(x+3)=2.85$

Answer available at the end of this section.

Exercise 8.6.29

Solve: $0.25n + 0.05(n + 5) = 2.95$

Answer available at the end of this section.

Exercise 8.6.30

Solve: $0.10d + 0.05(d - 5) = 2.15$

Answer available at the end of this section.

Key Concepts

- **Strategy to Solve an Equation with Fraction Coefficients**
  1. Find the least common denominator of all the fractions in the equation.
  2. Multiply both sides of the equation by that LCD. This clears the fractions.
  3. Solve using the General Strategy for Solving Linear Equations.

Answers

Exercise 8.6.1: How to Solve Equations with Fraction Coefficients

Step 1. Find the least common denominator of *all* the fractions in the equation.	What is the LCD of $\frac{1}{6}$, $\frac{1}{3}$, and $\frac{5}{6}$?	$\frac{1}{6}y - \frac{1}{3} = \frac{5}{6}$ LCD $= 6$
Step 2. Multiply both sides of the equation by that LCD. This clears the fractions.	Multiply both sides of the equation by the LCD 6. Use the Distributive Property. Simplify – and notice, no more fractions!	$6\left(\frac{1}{6}y - \frac{1}{3}\right) = 6\left(\frac{5}{6}\right)$ $6 \cdot \frac{1}{6}y - 6 \cdot \frac{1}{3} = 6 \cdot \frac{5}{6}$ $y - 2 = 5$
Step 3. Solve using the General Strategy for Solving Linear Equations.	To isolate the "y" term, add 2. Simplify.	$y - 2 + 2 = 5 + 2$ $y = 7$

Exercise 8.6.2

$x = \frac{1}{2}$

Exercise 8.6.3

$x = -2$

Exercise 8.6.4

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

Find the LCD of all fractions in the equation.	$6=\frac{1}{2}v+\frac{2}{5}v-\frac{3}{4}v$
The LCD is 20.	
Multiply both sides of the equation by 20.	$20(6)=20\cdot\left(\frac{1}{2}v+\frac{2}{5}v-\frac{3}{4}v\right)$
Distribute.	$20(6)=20\cdot\frac{1}{2}v+20\cdot\frac{2}{5}v-20\cdot\frac{3}{4}v$
Simplify—notice, no more fractions!	$120=10v+8v-15v$
Combine like terms.	$120=3v$
Divide by 3.	$\frac{120}{3}=\frac{3v}{3}$
Simplify.	$40=v$
Check:	$6=\frac{1}{2}v+\frac{2}{5}v-\frac{3}{4}v$
Let v=40.	$6\stackrel{?}{=}\frac{1}{2}(40)+\frac{2}{5}(40)-\frac{3}{4}(40)$
	$6\stackrel{?}{=}20+16-30$
	$6=6\checkmark$

Exercise 8.6.5

$x=12$

Exercise 8.6.6

$u=-12$

Exercise 8.6.7

	$a+\frac{3}{4}=\frac{3}{8}a-\frac{1}{2}$
Find the LCD of all fractions in the equation. The LCD is 8.	
Multiply both sides by the LCD.	$8\left(a+\frac{3}{4}\right)=8\left(\frac{3}{8}a-\frac{1}{2}\right)$
Distribute.	$8\cdot a+8\cdot\frac{3}{4}=8\cdot\frac{3}{8}a-8\cdot\frac{1}{2}$
Simplify—no more fractions.	$8a+6=3a-4$
Subtract 3a3a from both sides.	$8a-3a+6=3a-3a-4$
Simplify.	$5a+6=-4$
Subtract 6 from both sides.	$5a+6-6=-4-6$
Simplify.	$5a=-10$
Divide by 5.	$\frac{5a}{5}=\frac{-10}{5}$

Simplify.		$a=-2$
Check:	$a+\frac{3}{4}=\frac{3}{8}a-\frac{1}{2}$	
Let a=–2.	$-2+\frac{3}{4}\stackrel{?}{=}\frac{3}{8}(-2)-\frac{1}{2}$	
	$-\frac{8}{4}+\frac{3}{4}\stackrel{?}{=}-\frac{16}{8}-\frac{4}{8}$	
	$-\frac{5}{4}=-\frac{10}{8}$	
	$-\frac{5}{4}=-\frac{5}{4}\checkmark$	

Exercise 8.6.8

$x=-1$

Exercise 8.6.9

$c=-2$

Exercise 8.6.10

		$-5=\frac{1}{4}(8x+4)$
Distribute.		$-5=\frac{1}{4}\cdot 8x+\frac{1}{4}\cdot 4$
Simplify. Now there are no fractions.		$-5=2x+1$
Subtract 1 from both sides.		$-5-1=2x+1-1$
Simplify.		$-6=2x$
Divide by 2.		$\frac{-6}{2}=\frac{2x}{2}$
Simplify.		$-3=x$
Check:	$-5=\frac{1}{4}(8x+4)$	
Let x=–3.	$-5\stackrel{?}{=}\frac{1}{2}(4(-3)+2)$	
	$-5\stackrel{?}{=}\frac{1}{2}(-12+2)$	
	$-5\stackrel{?}{=}\frac{1}{2}(-10)$	
	$-5=-5\checkmark$	

Exercise 8.6.11

$p=-4$

Exercise 8.6.12

$q=2$

Exercise 8.6.13

	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$
Distribute.	$\frac{1}{2}\cdot y-\frac{1}{2}\cdot 5=\frac{1}{4}\cdot y-\frac{1}{4}\cdot 1$
Simplify.	$\frac{1}{2}y-\frac{5}{2}=\frac{1}{4}y-\frac{1}{4}$
Multiply by the LCD, 4.	$4\left(\frac{1}{2}y-\frac{5}{2}\right)=4\left(\frac{1}{4}y-\frac{1}{4}\right)$
Distribute.	$4\cdot\frac{1}{2}y-4\cdot\frac{5}{2}=4\cdot\frac{1}{4}y-4\cdot\frac{1}{4}$
Simplify.	$2y-10=y-1$
Collect the variables to the left.	$2y-y-10=y-y-1$
Simplify.	$y-10=-1$
Collect the constants to the right.	$y-10+10=-1+10$
Simplify.	$y=9$
Check: $\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$	
Let y=9. $\frac{1}{2}(9-5)\stackrel{?}{=}\frac{1}{4}(9-1)$	
Finish the check on your own.	

Exercise 8.6.14

$n=2$

Exercise 8.6.15

$m=-1$

Exercise 8.6.16

	$\frac{5x-3}{4}=\frac{x}{2}$
Multiply by the LCD, 4.	$4\left(\frac{5x-3}{4}\right)=4\left(\frac{x}{2}\right)$
Simplify.	$5x-3=2x$
Collect the variables to the right.	$5x-5x-3=2x-5x$
Simplify.	$-3=-3x$
Divide.	$\frac{-3}{-3}=\frac{-3x}{-3}$
Simplify.	$1=x$
Check: $\frac{5x-3}{4}=\frac{x}{2}$	

Let x=1.	$\frac{5(1)-3}{4} \stackrel{?}{=} \frac{1}{2}$	
	$\frac{2}{4} \stackrel{?}{=} \frac{1}{2}$	
	$\frac{1}{2} = \frac{1}{2}$ ✓	

Exercise 8.6.17

$y = 2$

Exercise 8.6.18

$z = -2$

Exercise 8.6.19

		$\frac{a}{6} + 2 = \frac{a}{4} + 3$
Multiply by the LCD, 12.		$12\left(\frac{a}{6} + 2\right) = 12\left(\frac{a}{4} + 3\right)$
Distribute.		$12 \cdot \frac{a}{6} + 12 \cdot 2 = 12 \cdot \frac{a}{4} + 12 \cdot 3$
Simplify.		$2a + 24 = 3a + 36$
Collect the variables to the right.		$2a - 2a + 24 = 3a - 2a + 36$
Simplify.		$24 = a + 36$
Collect the constants to the left.		$24 - 36 = a + 36 - 36$
Simplify.		$a = -12$
Check:	$\frac{a}{6} + 2 = \frac{a}{4} + 3$	
Let a=−12.	$\frac{-12}{6} + 2 \stackrel{?}{=} \frac{-12}{4} + 3$	
	$-2 + 2 \stackrel{?}{=} -3 + 3$	
	$0 = 0$ ✓	

Exercise 8.6.20

$b = -20$

Exercise 8.6.21

$c = -6$

Exercise 8.6.22

	$\frac{4q+3}{2} + 6 = \frac{3q+5}{4}$
Multiply by the LCD, 4.	$4\left(\frac{4q+3}{2} + 6\right) = 4\left(\frac{3q+5}{4}\right)$
Distribute.	$4\left(\frac{4q+3}{2}\right) + 4 \cdot 6 = 4 \cdot \left(\frac{3q+5}{4}\right)$

Simplify.	$2(4q+3)+24=3q+5$ $8q+6+24=3q+5$ $8q+30=3q+5$
Collect the variables to the left.	$8q-3q+30=3q-3q+5$
Simplify.	$5q+30=5$
Collect the constants to the right.	$5q+30-30=5-30$
Simplify.	$5q=-25$
Divide by 5.	$\frac{5q}{5}=\frac{-25}{5}$
Simplify.	$q=-5$
Check: $\frac{4q+3}{2}+6=\frac{3q+5}{4}$	
Let q=−5. $\frac{4(-5)+3}{2}+6\stackrel{?}{=}\frac{3(-5)+5}{4}$	
Finish the check on your own.	

Exercise 8.6.23

$r=1$

Exercise 8.6.24

$s=-8$

Exercise 8.6.25

Look at the decimals and think of the equivalent fractions.

$0.06=\frac{6}{100}$ $0.02=\frac{2}{100}$ $0.25=\frac{25}{100}$ $1.5=1\frac{5}{10}$

Notice, the LCD is 100.

By multiplying by the LCD, we will clear the decimals from the equation.

	$0.06x+0.02=0.25x-1.5$
Multiply both side by 100.	$100(0.06x+0.02)=100(0.25x-1.5$
Distribute.	$100(0.06x)+100(0.02)=100(0.25x)-1($
Multiply, and now we have no more decimals.	$6x+2=25x-150$
Collect the variables to the right.	$6x-6x+2=25x-6x-150$
Simplify.	$2=19x-150$
Collect the variables to the right.	$2+150=19x-150+15$
Simplify.	$152=19x$
Divide by 19.	$\frac{152}{19}=\frac{19x}{19}$
Simplify.	$8=x$

Check: Let x=8 $0.06(8)+0.02\stackrel{?}{=}0.25(8)-1.5$ $0.48+0.02\stackrel{?}{=}2.00-1.5$ $0.50=0.50\checkmark$	

Exercise 8.6.26

$h=12$

Exercise 8.6.27

$k=-1$

Exercise 8.6.28

	$0.25x+0.05(x+3)=2.85$
Distribute first.	$0.25x+0.05x+0.15=2.85$
Combine like terms.	$0.30x+0.15=2.85$
To clear decimals, multiply by 100.	$100(0.30x+0.15)=100(2.85)$
Distribute.	$30x+15=285$
Subtract 15 from both sides.	$30x+15-15=285-15$
Simplify.	$30x=270$
Divide by 30.	$\frac{30x}{30}=\frac{270}{30}$
Simplify.	$x=9$
Check it yourself by substituting x=9 into the original equation.	

Exercise 8.6.29

$n=9$

Exercise 8.6.30

$d=16$

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8.7: Solving a Formula for a Specific Variable

Learning Objectives

By the end of this section, you will be able to:

- Use the Distance, Rate, and Time formula
- Solve a formula for a specific variable

Use the Distance, Rate, and Time Formula

One formula you will use often in algebra and in everyday life is the formula for distance traveled by an object moving at a constant rate. Rate is an equivalent word for "speed." The basic idea of rate may already familiar to you. Do you know what distance you travel if you drive at a steady rate of 60 miles per hour for 2 hours? (This might happen if you use your car's cruise control while driving on the highway.) If you said 120 miles, you already know how to use this formula!

DISTANCE, RATE, AND TIME

For an object moving at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula:

$$d = rt \quad \text{where} \quad \begin{aligned} d &= \text{distance} \\ r &= \text{rate} \\ t &= \text{time} \end{aligned} \tag{8.7.1}$$

We will use the Strategy for Solving Applications that we used earlier in this chapter. When our problem requires a formula, we change Step 4. In place of writing a sentence, we write the appropriate formula. We write the revised steps here for reference.

SOLVE AN APPLICATION (WITH A FORMULA).

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. Write the appropriate formula for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

You may want to create a mini-chart to summarize the information in the problem. See the chart in this first example.

Exercise 8.7.1

Jamal rides his bike at a uniform rate of 12 miles per hour for $3\frac{1}{2}$ hours. What distance has he traveled?

Answer available at the end of this section.

Exercise 8.7.2

Lindsay drove for $5\frac{1}{2}$ hours at 60 miles per hour. How much distance did she travel?

Answer available at the end of this section.

Exercise 8.7.3

Trinh walked for $2\frac{1}{3}$ hours at 3 miles per hour. How far did she walk?

Answer available at the end of this section.

Exercise 8.7.4

Rey is planning to drive from his house in San Diego to visit his grandmother in Sacramento, a distance of 520 miles. If he can drive at a steady rate of 65 miles per hour, how many hours will the trip take?

Answer available at the end of this section.

Exercise 8.7.5

Lee wants to drive from Phoenix to his brother's apartment in San Francisco, a distance of 770 miles. If he drives at a steady rate of 70 miles per hour, how many hours will the trip take?

Answer available at the end of this section.

Exercise 8.7.6

Yesenia is 168 miles from Chicago. If she needs to be in Chicago in 3 hours, at what rate does she need to drive?

Answer available at the end of this section.

Solve a Formula for a Specific Variable

You are probably familiar with some geometry formulas. A formula is a mathematical description of the relationship between variables. Formulas are also used in the sciences, such as chemistry, physics, and biology. In medicine they are used for calculations for dispensing medicine or determining body mass index. Spreadsheet programs rely on formulas to make calculations. It is important to be familiar with formulas and be able to manipulate them easily.

In Exercise 8.7.1 and Exercise 8.7.4, we used the formula $d = rt$. This formula gives the value of d, distance, when you substitute in the values of r and t, the rate and time. But in Exercise 8.7.4, we had to find the value of t. We substituted in values of d and r and then used algebra to solve for tt. If you had to do this often, you might wonder why there is not a formula that gives the value of t when you substitute in the values of d and r. We can make a formula like this by solving the formula $d = rt$ for t.

To solve a formula for a specific variable means to isolate that variable on one side of the equals sign with a coefficient of 1. All other variables and constants are on the other side of the equals sign. To see how to solve a formula for a specific variable, we will start with the distance, rate and time formula.

Exercise 8.7.7

Solve the formula d=rt for t:

1. when d=520 and r=65
2. in general

Answer available at the end of this section.

Exercise 8.7.8

Solve the formula $d = rt$ for r:

1. when d=180 and t=4
2. in general

Answer available at the end of this section.

Exercise 8.7.9

Solve the formula $d = rt$ for r:

1. when d=780 and t=12
2. in general

Answer available at the end of this section.

Exercise 8.7.10

Solve the formula $A = \frac{1}{2}bh$ for h:

1. when $A = 90$ and $b = 15$
2. in general

Answer available at the end of this section.

Exercise 8.7.11

Solve the formula $A = \frac{1}{2}bh$ for h:

1. when $A = 170$ and $b = 17$
2. in general

Answer available at the end of this section.

Exercise 8.7.12

Solve the formula $A = \frac{1}{2}bh$ for h:

1. when $A = 62$ and $h = 31$
2. in general

Answer available at the end of this section.

The formula $I = Prt$ is used to calculate simple interest, *I*, for a principal, *P*, invested at rate, *r*, for *t* years.

Exercise 8.7.13

Solve the formula I=Prt to find the principal, P:

1. when I=$5,600, r=4% ,t=7years
2. in general

Answer available at the end of this section.

Exercise 8.7.14

Solve the formula I=Prt to find the principal, P:

1. when I=$2160, r=6% ,t=3 years
2. in general

Answer available at the end of this section.

Exercise 8.7.15

Solve the formula I=Prt to find the principal, P:

1. when I=$5400, r= 12% ,t=5 years
2. in general

Answer available at the end of this section.

Later in this class, and in future algebra classes, you'll encounter equations that relate two variables, usually *x* and *y*. You might be given an equation that is solved for *y* and need to solve it for *x*, or vice versa. In the following example, we're given an equation with both *x* and *y* on the same side and we'll solve it for *y*.

Exercise 8.7.16

Solve the formula 3x+2y=18 for *y*:

1. when x=4
2. in general

Answer available at the end of this section.

Exercise 8.7.17

Solve the formula 3x+4y=10 for *y*:

1. when $x = \frac{14}{3}$
2. in general

Answer available at the end of this section.

Exercise 8.7.18

Solve the formula 5x+2y=18 for *y*:

1. when $x = 4$
2. in general

Answer available at the end of this section.

In Exercise 8.7.7 through Exercise 8.7.18 we used the numbers in part 1 as a guide to solving in general in part 2. Now we will solve a formula in general without using numbers as a guide.

Exercise 8.7.19

Solve the formula P=a+b+c for a.

Answer available at the end of this section.

Exercise 8.7.20

Solve the formula P=a+b+c for *b*.

Answer available at the end of this section.

Exercise 8.7.21

Solve the formula P=a+b+c for *c*.

Answer available at the end of this section.

Exercise 8.7.22

Solve the formula 6x+5y=13 for *y*.

Answer available at the end of this section.

Exercise 8.7.23

Solve the formula 4x+7y=9 for *y*.

Answer available at the end of this section.

Exercise 8.7.24

Solve the formula 5x+8y=1 for *y*.

Answer available at the end of this section.

Key Concepts

- **To Solve an Application (with a formula)**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. Write the appropriate formula for the situation. Substitute in the given information.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.
- **Distance, Rate and Time**
  For an object moving at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula: d=rt where *d* = distance, *r* = rate, *t* = time.
- **To solve a formula for a specific variable** means to get that variable by itself with a coefficient of 1 on one side of the equation and all other variables and constants on the other side.

Answers

Exercise 8.7.1

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	distance traveled
Step 3. Name. Choose a variable to represent it.	Let *d* = distance.
Step 4. Translate: Write the appropriate formula.	$d = rt$
	$d = ?$ $r = 12$ mph $t = 3\frac{1}{2}$ hours
Substitute in the given information.	$d = 12 \cdot 3\frac{1}{2}$
Step 5. Solve the equation.	$d = 42$ miles
Step 6. Check	
Does 42 miles make sense?	
Jamal rides:	
12 miles in 1 hour, 24 miles in 2 hours, 36 miles in 3 hours, 42 miles in $3\frac{1}{2}$ hours is reasonable 48 miles in 4 hours.	
Step 7. Answer the question with a complete sentence.	Jamal rode 42 miles.

Exercise 8.7.2

330 miles

Exercise 8.7.3

7 miles

Exercise 8.7.4

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	How many hours (time)
Step 3. Name. Choose a variable to represent it.	Let t = time.
	$d = 520$ miles $r = 65$ mph $t = ?$ hours
Step 4. Translate. Write the appropriate formula.	$d = rt$
Substitute in the given information.	$520 = 65t$
Step 5. Solve the equation.	$t = 8$
Step 6. Check. Substitute the numbers into the formula and make sure the result is a true statement.	
$d = rt$ $520 \stackrel{?}{=} 65 \cdot 8$ $520 = 520\checkmark$	
Step 7. Answer the question with a complete sentence. Rey's trip will take 8 hours.	

Exercise 8.7.5

11 hours

Exercise 8.7.6

56 mph

Exercise 8.7.7

We will write the solutions side-by-side to demonstrate that solving a formula in general uses the same steps as when we have numbers to substitute.

1. when d=520 and r=65		2. in general	
Write the formula.	$d = rt$	Write the formula.	$d = rt$
Substitute.	$520 = 65t$		
Divide, to isolate t.	$\frac{520}{65} = \frac{65t}{65}$	Divide, to isolate tt.	$\frac{d}{r} = \frac{rt}{t}$
Simplify.	$8 = t$	Simplify.	$\frac{d}{r} = t$

We say the formula $t = \frac{d}{r}$ is solved for t.

Exercise 8.7.8

1. $r = 45$
2. $r = \frac{d}{t}$

Exercise 8.7.9

1. $r = 65$
2. $r = \frac{d}{t}$

Exercise 8.7.10

1. when $A = 90$ and $b = 15$		2. in general	
Write the formula.	$A = \frac{1}{2}bh$	Write the formula.	$A = \frac{1}{2}bh$
Substitute.	$90 = \frac{1}{2} \cdot 15 \cdot h$		

Clear the fractions.	$2 \cdot 90 = 2 \cdot \frac{1}{2} 15h$	Clear the fractions.	$2 \cdot A = 2 \cdot \frac{1}{2} bh$
Simplify.	$180 = 15h$	Simplify.	$2A = bh$
Solve for h.	$12 = h$	Solve for hh.	$\frac{2A}{b} = h$

We can now find the height of a triangle, if we know the area and the base, by using the formula $h = \frac{2A}{b}$

Exercise 8.7.11

1. $h = 20$
2. $h = \frac{2A}{b}$

Exercise 8.7.12

1. $b = 4$
2. $b = \frac{2A}{h}$

Exercise 8.7.13

1. I=$5,600, r=4% ,t=7years		2. in general	
Write the formula.	$I = Prt$	Write the formula.	$I = Prt$
Substitute.	$5600 = P(0.04)(7)$		
Simplify.	$5600 = P(0.28)$	Simplify.	$I = P(rt)$
Divide, to isolate *P*.	$\frac{5600}{0.28} = \frac{P(0.28)}{0.28}$	Divide, to isolate *P*.	$\frac{I}{rt} = \frac{P(rt)}{rt}$
Simplify.	$20,000 = P$	Simplify.	$\frac{I}{rt} = P$
The principal is	$20,000		$P = \frac{I}{rt}$

Exercise 8.7.14

1. $12000
2. $P = \frac{1}{rt}$

Exercise 8.7.15

1. $9000
2. $P = \frac{1}{rt}$

Exercise 8.7.16

1. when x=4		2. in general	
	$3x + 2y = 18$		$3x + 2y = 18$
Substitute.	$3(4) + 2y = 18$		
Subtract to isolate the y-term.	$12 - 12 + 2y = 18 - 12$	Subtract to isolate the y-term.	$3x - 3x + 2y = 18 - 3x$
Divide.	$\frac{2y}{2} = \frac{6}{2}$	Divide.	$\frac{2y}{2} = \frac{18}{2} - \frac{3x}{2}$
Simplify.	$y = 3$	Simplify.	$y = -\frac{3x}{2} + 9$

Exercise 8.7.17

1. $y = -1$
2. $y = \frac{10-3x}{4}$

Exercise 8.7.18

1. $y = -1$
2. $y = \frac{18-5x}{2}$

Exercise 8.7.19

We will isolate aa on one side of the equation.	$P = a + b + c$
Both b and c are added to a, so we subtract them from both sides of the equation.	$P - b - c = a + b + c - b - c$
Simplify.	$P - b - c = a$ $a = P - b - c$

Exercise 8.7.20

b=P−a−c

Exercise 8.7.21

c=P−a−b

Exercise 8.7.22

	$6x + 5y = 13$
Subtract 6x from both sides to isolate the term with y.	$6x - 6x + 5y = 13 - 6x$
Simplify.	$5y = 13 - 6x$
Divide by 5 to make the coefficient 1.	$\frac{5y}{5} = \frac{13-6x}{5}$
Simplify.	$y = \frac{13-6x}{5}$

The fraction is simplified. We cannot divide 13−6x by 5.

Exercise 8.7.23

$y = \frac{9-4x}{7}$

Exercise 8.7.24

$y = \frac{1-5x}{8}$

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8.8: Arithmetic Operations with Functions

Learning Objectives

- Given two functions, f and g, find their sum, $f+g$.
- Given two functions, f and g, find their difference, $f-g$.
- Given two functions, f and g, find their product, fg.
- Given two functions, f and g, find their quotient, $\frac{f}{g}$.

Introduction

You are used to adding, subtracting, multiplying, and dividing real numbers. You do these operations every day in a variety of situations. You have also learned how to perform these four basic operations on algebraic expressions. So while you may not need to calculate $30x^2+10x$ too often, you do know how to do it.

If you know how to perform the four basic operations on polynomials, then you can also add, subtract, multiply, and divide functions. The notation will look different at first, but knowing a couple of steps can help you arrive at the correct answer.

Understanding Notation

A **function** is a correspondence between two sets: the domain and the range. In addition to evaluating functions, you can do operations with functions.

Let's say you are working with the following two functions.

$$f(x)=9x-5$$
$$g(x)=4x+1$$

The sum of these functions can be written $f(x)+g(x)$ or as $(f+g)(x)$. Watch what happens when these two functions are added.

$$\begin{aligned} f(x) &= 9x-5 \\ g(x) &= 4x+1 \\ (f+g)(x) &= f(x)+g(x) \\ (f+g)(x) &= (9x-5)+(4x+1) \\ (f+g)(x) &= 9x+4x-5+1 \\ (f+g)(x) &= 13x-4 \end{aligned}$$

That's it! The sum of the two functions is the sum of the two polynomials.

Addition, subtraction, multiplication, and division will all be explained in turn. The table below shows the notation that is used for each type of arithmetic operation.

Addition	$f(x)+g(x)$	$(f+g)(x)$
Subtraction	$f(x)-g(x)$	$(f-g)(x)$
Multiplication	$f(x)\cdot g(x)$	$(f\cdot g)(x)$
Division	$\frac{f(x)}{g(x)}$	$\left(\frac{f}{g}\right)(x)$

Adding and Subtracting

You have already seen one example of adding two functions. Let's look at another one. The domain (x-values) for both functions

is all real numbers.

Example

$f(x) = 5x + 6$
$g(x) = 3x^2 - 4x + 8$

Find $(f + g)(x)$.

Solution

$(f+g)(x) = f(x) + g(x)$ $= (5x + 6) + (3x^2 - 4x + 8)$ $= 3x^2 + 5x - 4x + 6 + 8$ $= 3x^2 + x + 14$	Identify $f(x)$ and $g(x)$. Replace $f(x)$ with $5x + 6$, and $g(x)$ with $3x^2 - 4x + 8$. Then add and combine like terms.

$(f + g)(x) = 3x^2 + x + 14$

Subtracting follows the same process. As long as you remember how to subtract one polynomial from another, you can figure out how to subtract one function from another.

Example

$f(x) = 5x + 6$
$g(x) = 3x^2 - 4x + 8$

Find $(g - f)(x)$.

Solution

$(g-f)(x) = g(x) - f(x)$ $= (3x^2 - 4x + 8) - (5x + 6)$ $= 3x^2 - 4x + 8 - 5x - 6$ $= 3x^2 - 4x - 5x + 8 - 6$ $= 3x^2 - 9x + 2$	Replace $g(x)$ and $f(x)$ with their respective expressions. Then subtract and combine like terms.

$(g - f)(x) = 3x^2 - 9x + 2$

Example

$f(x) = 5x^2 + 2x - 5$
$g(x) = 7x + 8$
$h(x) = 4x^2 - 10$

Find $(f - h)(x)$.

Solution

$(f-h)(x) = f(x) - h(x)$ $= (5x^2 + 2x - 5) - (4x^2 - 10)$ $= 5x^2 + 2x - 5 - 4x^2 + 10$ $= 5x^2 - 4x^2 + 2x - 5 + 10$ $= x^2 + 2x + 5$	Notice: $(f-h)(x) = f(x) - h(x)$ You can ignore $g(x)$ since it is not required to solve this problem. Replace the function notations with their appropriate polynomials and subtract.

$(f-h)(x) = x^2 + 2x + 5$

Exercise

$f(x) = 9x^3 + 2$ and $g(x) = x^3 - 4x^2 - 3$. What is $(f-g)(x)$?

A. $10x^3 - 4x^2 - 1$
B. $8x^3 - 4x^2 - 1$
C. $8x^3 + 4x^2 + 5$
D. $-8x^3 - 4x^2 - 5$

Answer available at the end of this section.

Multiplying and Dividing

Multiplying and dividing functions is also just like multiplying and dividing polynomials. Review the following examples.

Example

$f(x) = 2x + 1$
$g(x) = 5x - 3$

Find the product of f and g.

Solution

$(f \cdot g)(x) = f(x) \cdot g(x)$	To find the product, multiply the functions.
$(f \cdot g)(x) = (2x + 1)(5x - 3)$ $= 10x^2 - 6x + 5x - 3$ $= 10x^2 - x - 3$	Replace $f(x)$ with $(2x + 1)$, and $g(x)$ with $(5x - 3)$.

$(f \cdot g)(x) = 10x^2 - x - 3$

Example

$f(x) = 12x^3 + 15x^2 - 6x$
$g(x) = 3x$

Find $\left(\frac{f}{g}\right)(x)$.

Solution

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $= \frac{12x^3 + 15x^2 - 6x}{3x}, x \neq 0$ $= \frac{3x\left(4x^2 + 5x - 2\right)}{3x}$ $= 1 \cdot \left(4x^2 + 5x - 2\right)$ $= 4x^2 + 5x - 2, x \neq 0$	To find the quotient, divide f by g. Substitute the polynomials in for $f(x)$ and $g(x)$ and divide. We add $x \neq 0$ because $x = 0$ would make the denominator $g(x) = 0$ and $\frac{f(x)}{g(x)}$ undefined. Remember to rename $\frac{3x}{3x}$ as 1.

$\left(\frac{f}{g}\right)(x) = 4x^2 + 5x - 2$

Operations with three functions work the same way. In the example below, two functions are added and then divided by a third. It is no different than what you have already done with polynomials; just continue to substitute the polynomials in for the correct functions, combine, divide, and simplify.

Example

$f(x) = 8x^3 - 3x^2$
$g(x) = 4x^3 + 9x^2$
$h(x) = 3x^2$

Find $\left(\frac{f+g}{h}\right)(x)$.

Solution

$\left(\frac{f+g}{h}\right)(x) = \frac{f(x) + g(x)}{h(x)}$ $= \frac{\left(8x^3 - 3x^2\right) + \left(4x^3 + 9x^2\right)}{3x^2}, x \neq 0$	Replace $f(x)$, $g(x)$, and $h(x)$ with the equivalent polynomials.e add $x \neq 0$ because that would make the denominator $h(x)$ of $\frac{f(x)+g(x)}{h(x)}$ zero and the fraction undefined.
$\left(\frac{f+g}{h}\right)(x) = \frac{3x^2(4x+2)}{3x^2}$ $= 1 \cdot (4x+2)$ $= 4x + 2, x \neq 0$	Add $f(x)$ and $g(x)$. Divide by $h(x)$. Pull out a factor of $3x^2$ from the numerator, and then simplify the expression, using $\frac{3x^2}{3x^2} = 1$.

$\left(\frac{f+g}{h}\right)(x) = 4x + 2, x \neq 0$

Exercise

$f(x) = 9x^2$
$g(x) = -4x$
$h(x) = -10x^3$

Find $(f \cdot h)(x)$.

A. $-90x^5$
B. $-36x^3$
C. $40x^4$
D. $360x^6$

Answer available at the end of this section.

Summary

Just like integers and algebraic expressions, functions can be added, subtracted, multiplied, and divided. To perform an arithmetic operation upon two or more functions, replace the indicated function with its respective polynomial, then combine using the regular rules of addition, subtraction, multiplication, and division.

Answers

Exercise

A. $10x^3 - 4x^2 - 1$

Incorrect. $(f + g)(x) = 10^3 - 4x^2 - 1$; this question is looking for $(f - g)(x)$. The correct answer is $8x^3 + 4x^2 + 5$.

B. $8x^3 - 4x^2 - 1$

Incorrect. It looks like you tried to calculate $(f - g)(x)$, but you subtracted incorrectly. Remember: $(f - g)(x) = (9x^3 + 2) - (x^3 - 4x^2 - 3) = 9x^3 + 2 - x^3 + 4x^2 + 3$. The correct answer is $8x^3 + 4x^2 + 5$.

C. $8x^3 + 4x^2 + 5$

Correct. To find $(f - g)(x)$, subtract $g(x)$ from $f(x)$. $(f - g)(x) = (9x^3 + 2) - (x^3 - 4x^2 - 3) = 8x^3 + 4x^2 + 5$.

D. $-8x^3 - 4x^2 - 5$

Incorrect. It looks like you tried to calculate $(g - f)(x)$. This question is looking for $(f - g)(x)$. The correct answer is $8x^3 + 4x^2 + 5$.

Exercise

A. $-90x^5$

Correct. $(f \cdot h)(x) = 9x^2 \cdot (-10x^3) = 90x^5$.

B. $-36x^3$

Incorrect. It looks like you found $(f \cdot g)(x)$; this problem is looking for $(f \cdot h)(x)$. The correct answer is $-90x^5$.

C. $40x^4$

Incorrect. It looks like you found $(g \cdot h)(x)$; this problem is looking for $(f \cdot h)(x)$. The correct answer is $-90x^5$.

D. $360x^6$

Incorrect. It looks like you found $(f \cdot g \cdot h)(x)$; this problem is looking for $(f \cdot h)(x)$. The correct answer is $-90x^5$.

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8.9: Graphing and slope

Often, to get an idea of the behavior of an equation, we will make a picture that represents the solutions to the equations called a *graph*. Let's have an introduction to graphing so that later we can construct graphs.

Definition: Graph

A ***graph*** represents a visualization of the solutions to an equation.

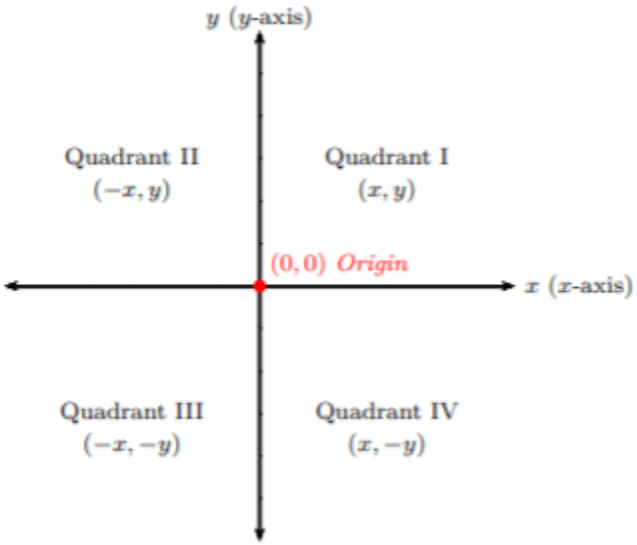


Figure 8.9.1: Coordinate plane

Coordinate Plane

- The ***coordinate plane*** is divided into four quadrants by a horizontal real number line, called the x-axis, and a vertical real number line, called the y-axis.
- The two axes meet at the ***origin***, where $x = 0$ and $y = 0$.
- The ***x-coordinate*** is the horizontal distance from the origin and the ***y-coordinate*** is the vertical distance from the origin.
- An ***ordered-pair*** is where a point in the coordinate plane is located and denoted by (x, y).
- The quadrants are labeled in counter-clockwise starting at the top right.

Note

Locations on the globe are similar to ordered-pairs. Each number is a distance from a central point, the origin, located where the prime meridian and the equator meet. This "origin" is just off the western coast of Africa.

Points and Lines

Example 8.9.1

Plot each ordered-pair and identify the quadrant in which lies the ordered pair:

$$A(-1, -5),\ B(3, -1),\ C(-2, 3),\ D(4, 2),\ E(0, 4),\ F(3, 0)$$

Solution

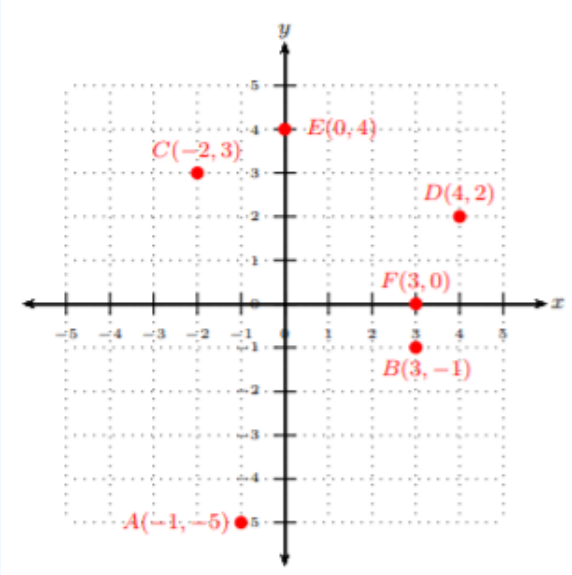


Figure 8.9.2

- For point $A(-1,-5)$, notice the x-coordinate is -1. Since the x-coordinate is the horizontal distance from the origin, then we move 1 unit to the left. Looking at the y-coordinate, -5, we see this will be the vertical distance. Hence, we will move 5 units downward from the origin. Starting at the origin, move one unit left, then 5 units down. Point A is in quadrant III.
- For point $B(3,-1)$, notice the x-coordinate is 3. Since the x-coordinate is the horizontal distance from the origin, then we move 3 units to the right. Looking at the y-coordinate, -1, we see this will be the vertical distance. Hence, we will move one unit downward from the origin. Starting at the origin, move 3 units right, then 1 unit down. Point B is in quadrant IV.
- For point $C(-2,3)$, notice the x-coordinate is -2. Since the x-coordinate is the horizontal distance from the origin, then we move 2 units to the left. Looking at the y-coordinate, 3, we see this will be the vertical distance. Hence, we will move 3 units upward from the origin. Starting at the origin, move 2 units left, then 3 units up. Point C is in quadrant III.
- For point $D(4,2)$, notice the x-coordinate is 4. Since the x-coordinate is the horizontal distance from the origin, then we move 4 units to the right. Looking at the y-coordinate, 2, we see this will be the vertical distance. Hence, we will move 2 units upward from the origin. Starting at the origin, move 4 units right, then 2 units up. Point D is in quadrant I.
- For point $E(0,4)$, notice the x-coordinate is 0. Since the x-coordinate is the horizontal distance from the origin, then we move no units horizontally from the origin. Looking at the y-coordinate, 4, we see this will be the vertical distance. Hence, we will move 4 units upward from the origin. Starting at the origin, move 4 units up. Point E is not in any quadrant as it lies on the y-axis.
- For point $F(3,0)$, notice the x-coordinate is 3. Since the x-coordinate is the horizontal distance from the origin, then we move 3 units to the right. Looking at the y-coordinate, 0, we see this will be the vertical distance. Hence, we will move no units vertically from the origin. Starting at the origin, move 3 units right. Point F is not in any quadrant as it lies on the x -axis.

Note

Notice, in points $A,\ B,\ C$, the negative coordinates didn't imply negative distance from the origin. The *negative on these coordinates implies the direction* in which we move: horizontal- we move left or right, vertical- we move up or down. If the x -coordinate is negative, then we move to the left. If the y-coordinate is negative, then we move downward.

Definition: Intercepts

From Example 8.9.1, with points E and F, we could see that these points did not lie in a quadrant, but on an axis. These are special points on graphs and are called ***intercepts***.

- The ***x-intercept*** of a graph is the point(s) where the graph crosses the x-axis, i.e., $y = 0$.
- The ***y-intercept*** of a graph is the point(s) where the graph crosses the y-axis, i.e., $x = 0$.

Example 8.9.2

Graph $y = 1 - x$ by plotting the intercepts.

Solution

To find the x and y-intercepts, we can follow the definition above and find where $y = 0$ and $x = 0$, respectively. Let's make a table.

Table 8.9.1

x	$y = 1 - x$	(x, y)
$0 = 1 - x \implies x = 1$	0	$(1, 0)$
0	$y = 1 - 0 = 1$	$(0, 1)$

We can see when $y = 0,\ \ x = 1$ since $0 = 1 - x$ only when $x = 1$. Let's plot the two intercepts from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. Draw the line to fill the grid and put arrows at the ends. It is recommended to purchase a small 6-inch ruler to make nice straight lines.

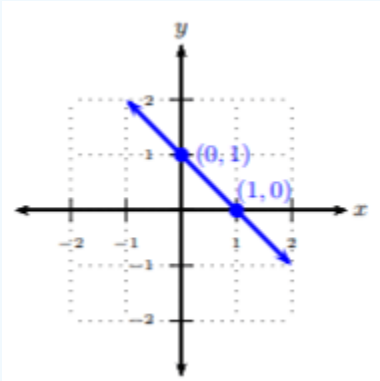


Figure 8.9.3

The main purpose of graphs is not to plot random points, but rather to give a picture of the solutions to an equation. We may have an equation such as $y = 2x - 3$ and be interested in the type of solutions that are possible for this equation. We can visualize the solution by making a graph of possible x and y combinations that makes this equation a true statement. We have to start by finding possible x and y combinations. We do this by using a table of values.

Example 8.9.3

Graph $y = 2x - 3$ by plotting points, i.e., by making a T-table.

Solution

Usually, we pick three x-coordinates, and find corresponding y-values. Each x-value being positive, negative, and zero. This is common practice, but not required.

Table 8.9.2

x	$y = 2x - 3$	(x, y)
-1	$y = 2(-1) - 3 = -2 - 3 = -5$	$(-1, -5)$
0	$y = 2(0) - 3 = 0 - 3 = -3$	$(0, -3)$
1	$y = 2(1) - 3 = 2 - 3 = -1$	$(1, -1)$

Plot the three ordered-pairs from the table. To connect the points, be sure to connect them from smallest x-value to largest x-value, i.e., left to right. Draw the line to fill the grid and put arrows at the ends. It is recommended to purchase a small 6-inch ruler to make nice straight lines.

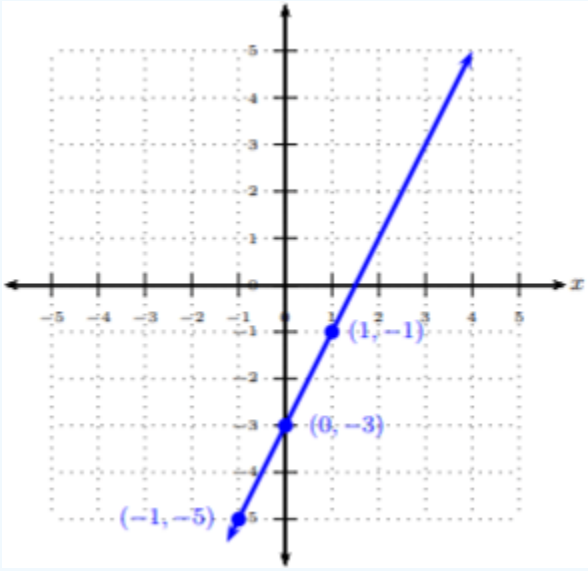


Figure 8.9.4

Example 8.9.4

Graph $2x - 3y = 6$ by plotting points, i.e., by making a T-table.

Solution

Let's begin by choosing x-values for the table. Notice this equation isn't as simple as the prior example, so we will have to do a bit of algebra to solve for the y-value. Then fill in the table.

Table 8.9.3

x	y	(x, y)
-3		
0		
3		

Let's evaluate $2x - 3y = 6$ for each of the chosen x-values:

$x = -3$:

$$\begin{aligned} 2(-3)-3y &= 6 \\ -6-3y+6 &= 6+6 \\ -\frac{1}{3}\cdot -3y &= 12\cdot -\frac{1}{3} \\ y &= -4 \end{aligned} \tag{8.9.1}$$

$x = 0$:

$$\begin{aligned} 2(0)-3y &= 6 \\ 0-3y &= 6 \\ -\frac{1}{3}\cdot -3y &= 6\cdot -\frac{1}{3} \\ y &= -2 \end{aligned} \tag{8.9.2}$$

$x = 3$:

$$\begin{aligned} 2(3)-3y &= 6 \\ 6-3y+(-6) &= 6+(-6) \\ -\frac{1}{3}\cdot -3y &= 0\cdot -\frac{1}{3} \\ y &= 0 \end{aligned} \tag{8.9.3}$$

Now, we fill in the table with y-values and ordered-pairs, and then graph $2x - 3y = 6$.

Table 8.9.4

x	y	(x, y)
-3	-4	$(-3, -4)$
0	-2	$(0, -2)$
3	0	$(3, 0)$

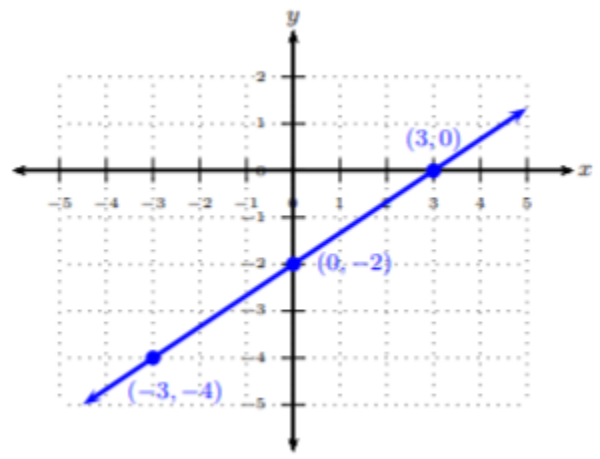


Figure 8.9.5

Obtaining the Slope of a Line from its Graph

As we graph lines, we want to identify different properties of lines. One of the most important properties of a line is its slope.

Definition: Slope

The ***slope*** of a line is the measure of the line's steepness.

- We denote slope with m. One theory from mathematicians that began working with slope was that it was called the *modular slope*.
- As $|m|$ increases, the line becomes steeper. As $|m|$ decreases, the line becomes flatter.
- A line that rises left to right has a positive slope and and a line that falls left to right has negative slope.
- m is the change of y divided by the change in x, i.e.,

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{rise}{run}$$

Example 8.9.5

Find the slope from the graph given of the line.

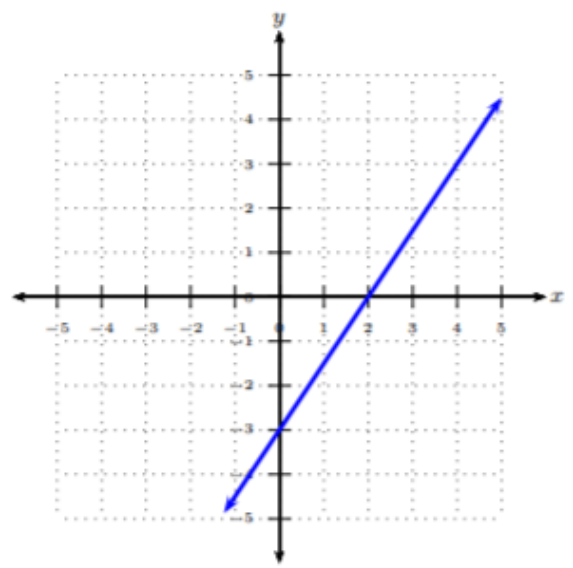


Figure 8.9.6

Solution

We start at a well-defined point, preferably a point on the y-axis, i.e., the y-intercept. Then count the number of units we rise (up/down) and run (left/right) to reach the next well-defined point. We will start at $(0, -3)$ and reach the next point, $(2, 0)$. Notice we rise upward 3 units and run to the right 2 units. Using the ratio, $m = \frac{rise}{run}$, we get the rise to be 3 and the run to be 2:

$$m = \frac{rise}{run} = \frac{3}{2}$$

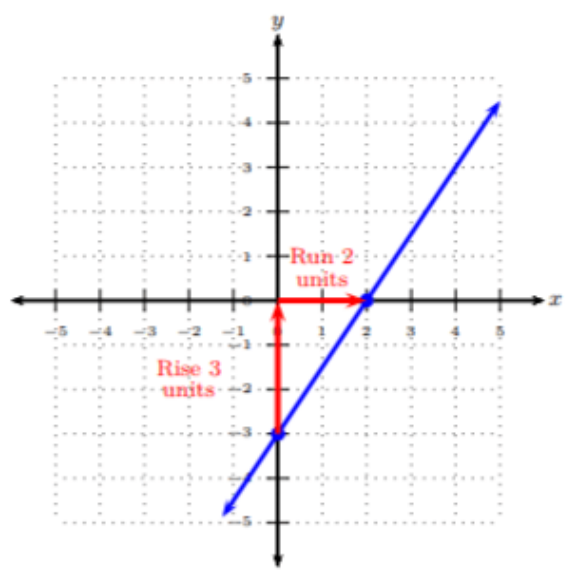


Figure 8.9.7

Thus, the slope is $\frac{3}{2}$.

Example 8.9.6

Find the slope from the graph given of the line.

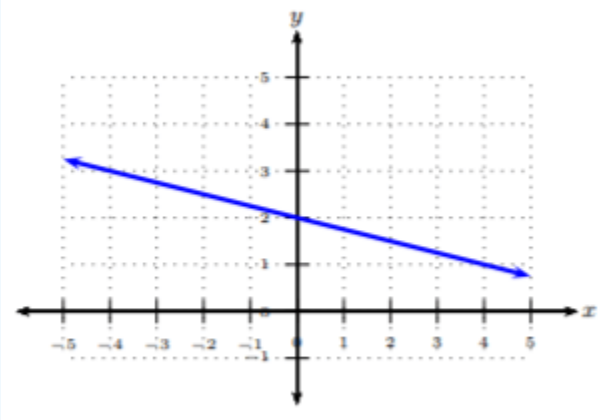


Figure 8.9.8

Solution

We start at a well-defined point, preferably a point on the y-axis, i.e., the y-intercept. Then count the number of units we rise (up/down) and run (left/right) to reach the next well-defined point. We will start at $(0, 2)$ and reach the next point, $(4, 1)$. Notice we rise downward 1 units and run to the right 4 units. Using the ratio, $m = \frac{rise}{run}$, we get the rise to be -1 and the run to be 4:

$$m = \frac{rise}{run} = \frac{-1}{4}$$

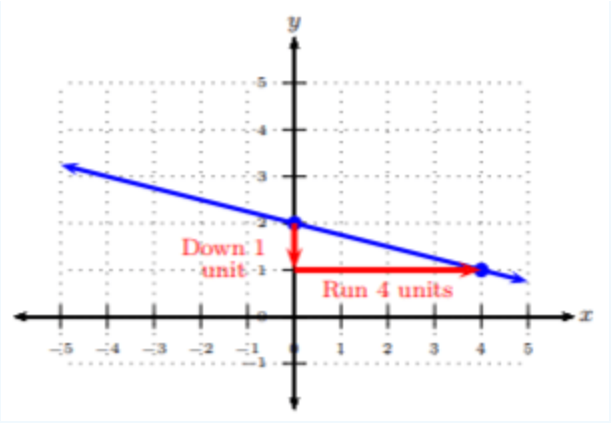


Figure 8.9.9

Thus, the slope is $-\frac{1}{4}$.

Note

Looking at Examples 8.9.5 and 8.9.6, notice the steepness. Since, in Example 8.9.5, the slope was $\frac{3}{2}$, which is larger than the slope in 8.9.6, Example 8.9.5 is a steeper line than Example 8.9.6. Also, the negative in Example 8.9.6 represents the line falling left to right; hence the negative slope.

Note

When French mathematicians *Rene Descartes* and *Pierre de Fermat* first developed the coordinate plane and the idea of graphing lines (and other functions), the y-axis was not a vertical line.

Let's look at two special cases with lines and their slope.

Example 8.9.7

Find the slope from the graph given of the line.

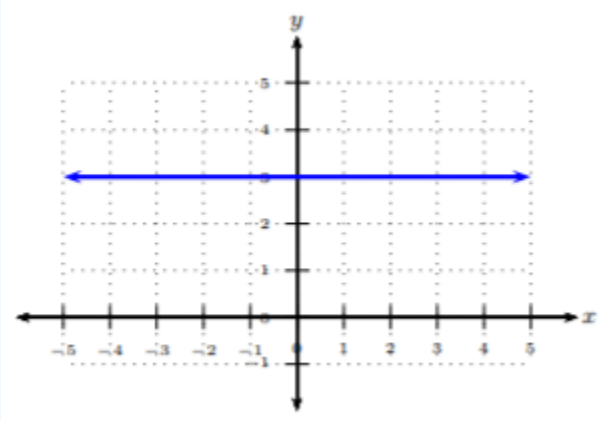


Figure 8.9.10

Solution

In this graph, there is no rise, but the run is 3 units. The slope is $\frac{0}{3} = 0$. When the slope of a line is zero, then we know the line is a horizontal line and vise versa.

Example 8.9.8

Find the slope from the graph given of the line.

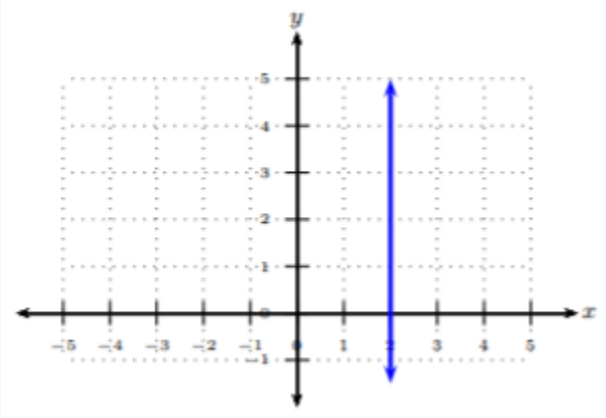


Figure 8.9.11

Solution

In this graph, there is no run, but the rise is 2 units. The slope is $\frac{2}{0} =$ undefined. When the slope of a line is undefined, then we know the line is a vertical line and vise versa.

Note

As you can see there is a big difference between having a zero slope and having undefined slope. Remember, slope is a measure of steepness. The first slope is not steep at all. In fact, it is flat. Therefore, it has a zero slope. The second slope can't get any steeper. It is so steep that there is no number large enough to express the steepness. Hence, being an undefined slope.

Obtaining the Slope of a Line from Two Points

We can find the slope of a line through two points without seeing the points on a graph. We can do this using a slope formula. If the rise is the change in y-values, we can calculate this by subtracting the y-values of a point. Similarly, if run is a change in the x -values, we can calculate this by subtracting the x-values of a point.

Definition: Slope

Slope, m, is the change of y divided by the change in x, i.e.,

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

Example 8.9.9

Find the slope between the two points $(-4, 3)$ and $(2, -9)$.

Solution

$$\begin{aligned} m &= \tfrac{y_2-y_1}{x_2-x_1} && \text{Substitute in the ordered-pairs} \\ m &= \tfrac{-9-3}{2-(-4)} && \text{Simplify} \\ m &= \tfrac{-12}{6} && \text{Reduce} \\ m &= -2 && \text{Slope} \end{aligned}$$

Since this slope is -2, then the graph of this line would be falling left to right.

Example 8.9.10

Find the slope between the two points $(4, 6)$ and $(2, -1)$.

Solution

$$\begin{aligned} m &= \tfrac{y_2-y_1}{x_2-x_1} && \text{Substitute in the ordered-pairs} \\ m &= \tfrac{-1-6}{2-4} && \text{Simplify} \\ m &= \tfrac{-7}{-2} && \text{Reduce, dividing by } -1 \\ m &= \tfrac{7}{2} && \text{Slope} \end{aligned}$$

Since this slope is $\frac{7}{2}$, then the graph of this line would be rising left to right.

Example 8.9.11

Find the slope between the two points $(-4, -1)$ and $(-4, -5)$.

Solution

$$\begin{aligned} m &= \tfrac{y_2-y_1}{x_2-x_1} && \text{Substitute in the ordered-pairs} \\ m &= \tfrac{-5-(-1)}{-4-(-4)} && \text{Simplify} \\ m &= \tfrac{-4}{0} && \text{Undefined} \\ m &= \text{ undefined} && \text{Slope} \end{aligned}$$

Since the slope is undefined, then the graph of this line is a vertical line.

Example 8.9.12

Find the slope between the two points $(3, 1)$ and $(-2, 1)$.

Solution

$$\begin{aligned} m &= \tfrac{y_2-y_1}{x_2-x_1} && \text{Substitute in the ordered-pairs} \\ m &= \tfrac{1-1}{-2-3} && \text{Simplify} \\ m &= \tfrac{0}{-5} && \text{Reduce} \\ m &= 0 && \text{Slope} \end{aligned}$$

Since the slope is zero, then the graph of this line is a horizontal line.

Example 8.9.13

Find the value of y between the points $(2, y)$ and $(5, -1)$ with slope -3.

Solution

$$
\begin{aligned}
m &= \tfrac{y_2-y_1}{x_2-x_1} && \text{We will plug values into slope formula}\\
-3 &= \tfrac{-1-y}{5-2} && \text{Simplify}\\
-3 &= \tfrac{-1-y}{3} && \text{Multiply both sides by 3}\\
3 \cdot -3 &= \tfrac{-1-y}{3} \cdot 3 && \text{Simplify}\\
-9 &= -1-y && \text{Isolate the variable term}\\
-9+1 &= -1-y+1 && \text{Simplify}\\
-8 &= -y && \text{Multiply each side by } -1\\
-1 \cdot -8 &= -y \cdot -1 &&\\
8 &= y && \text{Value of } y
\end{aligned}
$$

Example 8.9.14

Find the value of x such that the slope between the points $(-3, 2)$ and $(x, 6)$ is $\frac{2}{5}$.

Solution

$$
\begin{aligned}
m &= \tfrac{y_2-y_1}{x_2-x_1} && \text{We will plug values into slope formula}\\
\tfrac{2}{5} &= \tfrac{6-2}{x-(-3)} && \text{Simplify}\\
\tfrac{2}{5} &= \tfrac{4}{x+3} && \text{Multiply both sides by } x+3\\
(x+3) \cdot \tfrac{2}{5} &= \tfrac{4}{x+3} \cdot (x+3) && \text{Simplify}\\
\tfrac{2}{5}(x+3) &= 4 && \text{Distribute}\\
\tfrac{2}{5}x + \tfrac{6}{5} &= 4 && \text{Multiply by the LCD } = 5\\
5 \cdot \tfrac{2}{5}x + 5 \cdot \tfrac{6}{5} &= 4 \cdot 5 && \text{Simplify}\\
2x+6 &= 20 && \text{Isolate the variable term}\\
2x+6+(-6) &= 20+(-6) && \text{Simplify}\\
2x &= 14 && \text{Multiply each side by the reciprocal of 2}\\
\tfrac{1}{2} \cdot 2x &= 14 \cdot \tfrac{1}{2} && \text{Simplify}\\
x &= 7 && \text{Value of } x
\end{aligned}
$$

Graphing and Slope Homework

Exercise 8.9.1

Find the slope of the line.

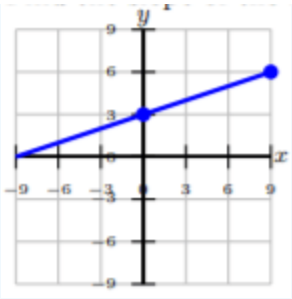


Figure 8.9.12

Exercise 8.9.2

Find the slope of the line.

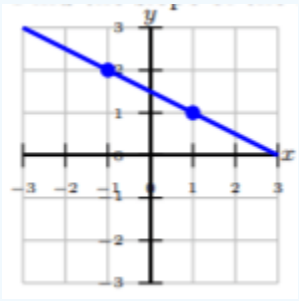


Figure 8.9.13

Exercise 8.9.3

Find the slope of the line.

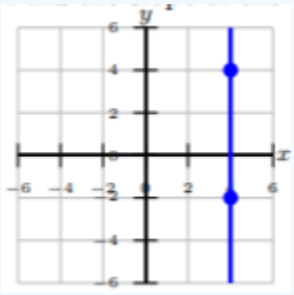


Figure 8.9.14

Exercise 8.9.4

Find the slope of the line.

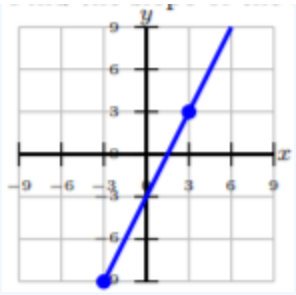


Figure 8.9.15

Exercise 8.9.5

Find the slope of the line.

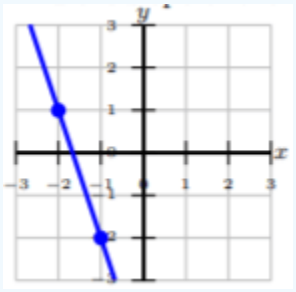


Figure 8.9.16

Exercise 8.9.6

Find the slope of the line.

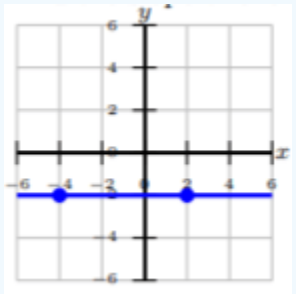


Figure 8.9.17

Find the slope of the line through each ordered-pair.

Exercise 8.9.7

$(-2, 10),\ (-2, -15)$

Exercise 8.9.8

$(-15, 10),\ (16, -7)$

Exercise 8.9.9

$(10, 18),\ (-11, -10)$

Exercise 8.9.10

$(-16, -14),\ (11, -14)$

Exercise 8.9.11

$(-4, 14),\ (-16, 8)$

Exercise 8.9.12

$(12, -19),\ (6, 14)$

Exercise 8.9.13

$(-5, -10),\ (-5, 20)$

Exercise 8.9.14

$(-17, 19),\ (10, -7)$

Exercise 8.9.15

$(7, -14),\ (-8, -9)$

Exercise 8.9.16

$(-5, 7),\ (-18, 14)$

Exercise 8.9.17

$(1, 2),\ (-6, -14)$

Exercise 8.9.18

$(13, -2),\ (7, 7)$

Exercise 8.9.19

$(-3, 6),\ (-20, 13)$

Exercise 8.9.20

$(13, 15),\ (2, 10)$

Exercise 8.9.21

$(9, -6),\ (-7, -7)$

Exercise 8.9.22

$(-16, 2),\ (15, -10)$

Exercise 8.9.23

$(8, 11),\ (-3, -13)$

Exercise 8.9.24

$(11, -2),\ (1, 17)$

Exercise 8.9.25

$(-18, -5),\ (14, -3)$

Exercise 8.9.26

$(19, 15),\ (5, 11)$

Find the value of x or y so that the line through the points has the given slope.

Exercise 8.9.27

$(2, 6)$ and $(x, 2)$; $m = \frac{4}{7}$

Exercise 8.9.28

$(-3, -2)$ and $(x, 6)$; $m = -\frac{8}{5}$

Exercise 8.9.29

$(-8, y)$ and $(-1, 1)$; $m = \frac{6}{7}$

Exercise 8.9.30

$(x, -7)$ and $(-9, -9)$; $m = \frac{2}{5}$

Exercise 8.9.31

$(x, 5)$ and $(8, 0)$; $m = -\frac{5}{6}$

Exercise 8.9.32

$(8, y)$ and $(-2, 4)$; $m = -\frac{1}{5}$

Exercise 8.9.33

$(-2, y)$ and $(2, 4)$; $m = \frac{1}{4}$

Exercise 8.9.34

$(x, -1)$ and $(-4, 6)$; $m = -\frac{7}{10}$

Exercise 8.9.35

$(2, -5)$ and $(3, y)$; $m = 6$

Exercise 8.9.36

$(6, 2)$ and $(x, 6)$; $m = -\frac{4}{5}$

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8.10: Equations of lines

In this section, we discuss applying formulas to obtain equations of lines, graphing lines, and special cases. If we can identify some properties of the line, we may be able to graph the line much quicker and easier.

The Slope-Intercept Formula

One such method is finding the slope and the y-intercept of the equation. The slope can be represented by m, and the y-intercept can be represented by $(0, b)$, where b is the y-value at which the graph crosses the y-axis. Any other point on the line can be represented by (x, y).

Example 8.10.1

Prove the slope-intercept formula by using the slope formula for the line that has slope m, y-intercept $(0, b)$, and passes through the point (x, y).

Solution

$$\begin{aligned} m &= \tfrac{y-b}{x-0} && \text{Simplify} \\ m &= \tfrac{y-b}{x} && \text{Multiply both sides by } x \\ mx &= y - b && \text{Add } b \text{ to both sides} \\ mx + b &= y \\ y &= mx + b && \text{Slope-intercept formula} \end{aligned}$$

Slope-Intercept Formula

The ***slope-intercept formula*** of a linear equation is given by $y = mx + b$, where m is the slope and $(0, b)$ is the y-intercept.

Example 8.10.2

Find the equation of the line with slope $\frac{3}{4}$ and y-intercept $(0, -3)$.

Solution

$$\begin{aligned} y &= mx + b && m \text{ is the slope, } b \text{ is the } y\text{-intercept} \\ y &= \tfrac{3}{4}x - 3 && \text{Equation of the line} \end{aligned}$$

Example 8.10.3

Find the equation of the line.

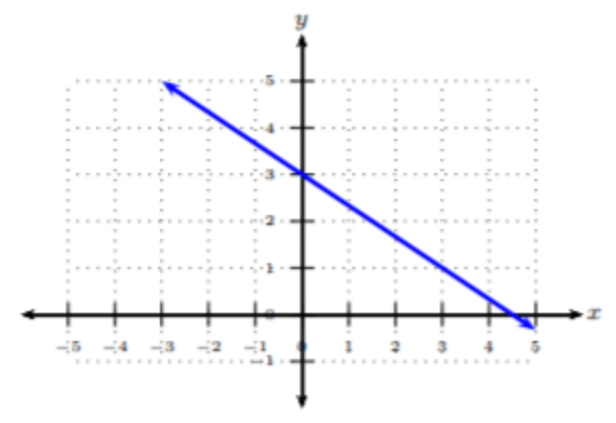


Figure 8.10.1

Solution

The y-intercept is where the graph crosses the y-axis. We can see, from the graph, that the line crosses the y-axis at $(0, 3)$. Hence, $b = 3$. To find the slope, we count the rise and run units. We can see that we rise downward 2 units and run to the right 3 units. Hence, the slope is $-\frac{2}{3}$. Now we can put the equation of the line together in slope-intercept form, where $m = -\frac{2}{3}$ and $b = 3$:

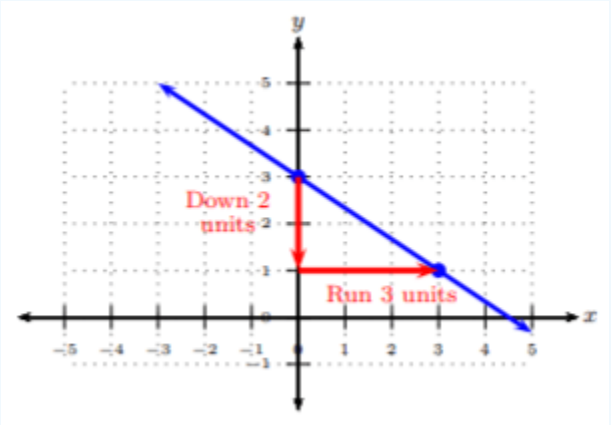


Figure 8.10.2

$$y = -\frac{2}{3}x + 3$$

Lines in Slope-Intercept Form

We can also identify the slope and y-intercept, and graph the equation from a given equation. However, we need to be sure the equation is in slope-intercept form. If it is not, we will have to rewrite the equation in slope-intercept form, i.e., solve the equation for y. Then we can easily identify the slope and the y-intercept.

Example 8.10.4

Write the equation $2x - 4y = 6$ in slope-intercept form. Find the slope and y-intercept of the line.

Solution

$$\begin{aligned} 2x-4y &= 6 && \text{Isolate the variable term } -4y \\ 2x-4y+(-2x) &= 6+(-2x) && \text{Simplify} \\ -4y &= 6-2x && \text{Multiply by the reciprocal of } -4 \\ -\tfrac{1}{4}\cdot -4y &= -\tfrac{1}{4}\cdot 6-2x\cdot -\tfrac{1}{4} && \text{Simplify} \\ y &= \tfrac{1}{2}x-\tfrac{3}{2} && \text{Slope-intercept form} \end{aligned}$$

Next, we identify the slope and the y-intercept. If we line up the general slope-intercept form with the equation we obtained, we can easily see m and b:

$$\begin{aligned} y &= mx &+b \\ y &= \tfrac{1}{2}x &-\tfrac{3}{2} \end{aligned}$$

Hence, $m=\frac{1}{2}$ and $b=-\frac{3}{2}$, i.e., the slope is $\frac{1}{2}$ and the y-intercept is $-\frac{3}{2}$.

Graphing Lines

Once we have an equation in slope-intercept form, we can graph it by first plotting the y-intercept, then applying the slope to find a second point and even a third point. We connect these points to make a line. Let's look at Example 8.10.4 and graph the line.

Example 8.10.5

Graph $y=\frac{1}{2}x-\frac{3}{2}$ by using the slope and y-intercept.

Solution

The y-intercept, or b, is where the graph crosses the y-axis. We know from Example 8.10.4 that the y-intercept is $-\frac{3}{2}$ and the line will cross the y-axis at $\left(0,-\frac{3}{2}\right)$. The slope is $\frac{1}{2}$, and, using $\frac{rise}{run}$, we need to rise upward 1 unit and run to the right 2 units to reach the next point. We continue the pattern to obtain a third point. Now we can connect the dots and create a well-defined line. Be sure to draw it to fill the grid.

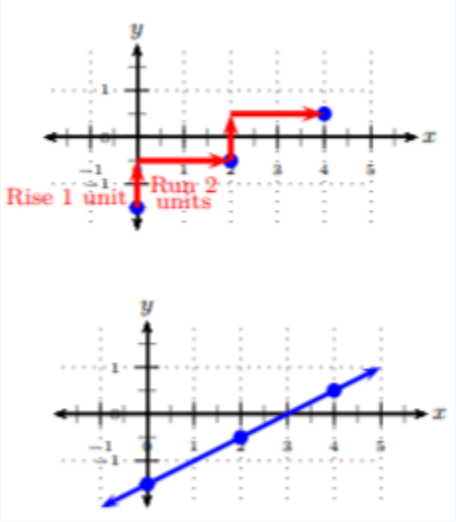


Figure 8.10.3

Note

When we graphed a line by plotting points from a table, we obtained three points. Hence, in Example 8.10.5, we obtained three points and then drew a line. Obtaining three points on a line is common practice and will help when drawing any line, even the special cases.

Note

Before our current system of graphing, French mathematician, *Nicole Oresme*, in 1323, suggested graphing lines that would look more like bar graphs with a constant slope.

Example 8.10.6

Write the equation $3x + 4y = 12$ in slope-intercept form. Find the slope and y-intercept of the line and then graph the line.

Solution

$$\begin{aligned} 3x + 4y &= 12 && \text{Isolate the variable term } 4y \\ 3x + 4y + (-3x) &= 12 + (-3x) && \text{Simplify} \\ 4y &= 12 - 3x && \text{Multiply by the reciprocal of } 4 \\ \tfrac{1}{4} \cdot 4y &= \tfrac{1}{4} \cdot 12 - 3x \cdot \tfrac{1}{4} && \text{Simplify} \\ y &= -\tfrac{3}{4}x + 3 && \text{Slope-intercept form} \end{aligned}$$

No we can graph the line. We see the y-intercept is 3 and the line will cross the y-axis at $(0, 3)$. The slope is $-\frac{3}{4}$, and, using $\frac{rise}{run}$, we need to rise downward 3 units and run to the right 4 units to reach the next point. We continue the pattern to obtain a third point. Now we can connect the dots and create a well-defined line. Be sure to draw it to fill the grid.

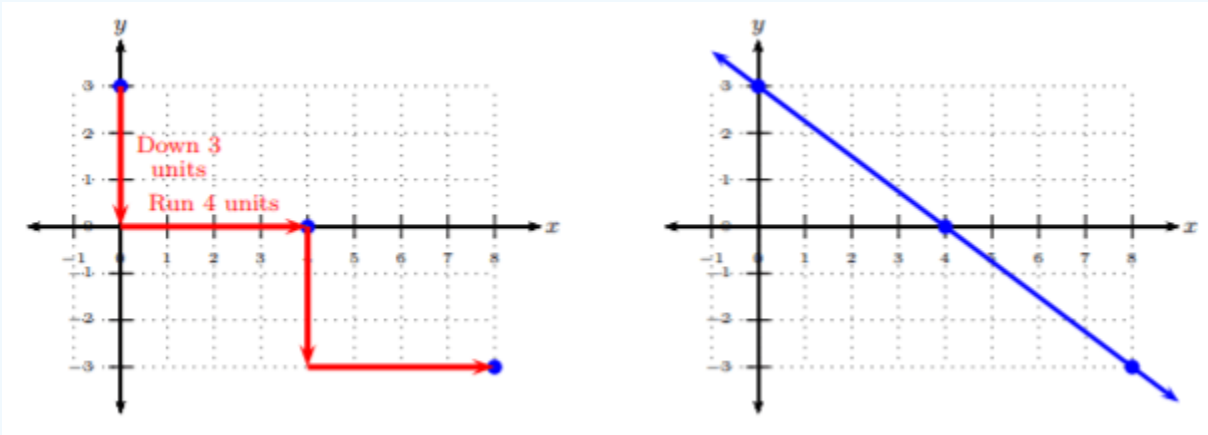


Figure 8.10.4

Vertical and Horizontal Lines

Lines with zero or undefined slope can make a problem seem very different. Zero slope, or a horizontal line, will simply have a slope of zero. So, the equation simply becomes $y = b$ or y equal to the y-coordinate of the graph. If we have undefined slope, or a vertical line, the equation can't be written in slope-intercept form because the slope is undefined. Hence, there is no y in these equations. We will simply make x equal to the x-coordinate of the graph.

Example 8.10.7

Graph the line $x = -4$.

Solution

Since $x = -4$ is a vertical line, then we know this line has no slope and the line is at its steepest. Every x-coordinate on this line is -4 and the line has no run. We can graph this line easily by plotting three points where the x-coordinate is -4. Let's plot $(-4, -1)$, $(-4, 0)$, and $(-4, 2)$; then connect the points with a well-defined line.

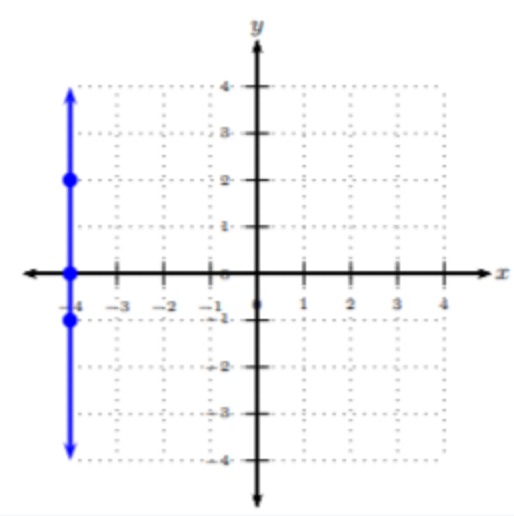


Figure 8.10.5

Example 8.10.8

Graph the line $y = 1$.

Solution

Since $y = 1$ is a horizontal line, then we know this line has zero slope and the line is at its flattest. Every y-coordinate on this line is 1 and the line has no rise. We can graph this line easily by plotting three points where the y-coordinate is 1. Let's plot $(3, 1)$, $(0, 1)$, and $(-2, 1)$; then connect the points with a well-defined line.

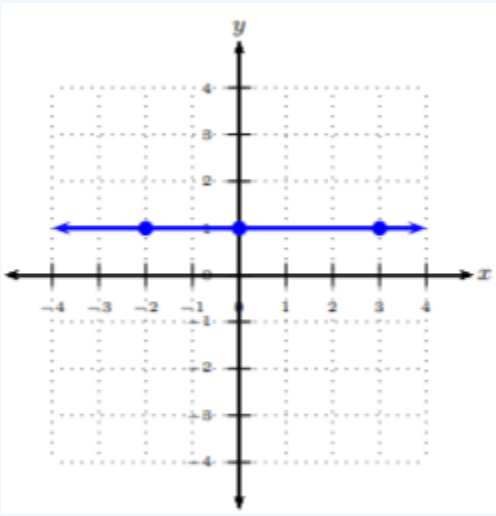


Figure 8.10.6

Point-Slope Formula

The slope-intercept form has the advantage of being simple to remember and use. However, it has one major disadvantage: we must know the y-intercept in order to graph the line. Generally, we do not know the y-intercept, but, usually, know one or more points on the line that are not the y-intercept. In these cases, we can't use the slope-intercept equation, so we will need a more general formula to assist us in graphing lines. If the slope of a line is m, and point (x_1, y_1) be a particular point on the line, and any other point on the line be (x, y), then we can use this to find this general formula.

$$
\begin{aligned}
m,\ (x_1, y_1),\ (x, y) &\quad \text{Recall slope formula} \\
\tfrac{y_2 - y_1}{x_2 - x_1} = m &\quad \text{Plug in values} \\
\tfrac{y - y_1}{x - x_1} = m &\quad \text{Multiply both sides by } (x - x_1) \\
y - y_1 = m(x - x_1) &\quad \text{New formula}
\end{aligned}
$$

Point-Slope Formula

The ***point-slope formula*** is given by

$$y - y_1 = m(x - x_1),$$

given the slope m and point (x_1, y_1) is on the line.

Example 8.10.9

Using the point-slope formula, write the equation of the line passing through the point $(3, -4)$ with a slope of $\frac{3}{5}$.

Solution

$$
\begin{aligned}
y - y_1 = m(x - x_1) &\quad \text{Plug values into point-slope formula} \\
y - (-4) = \tfrac{3}{5}(x - 3) &\quad \text{Simplify signs} \\
y + 4 = \tfrac{3}{5}(x - 3) &\quad \text{Equation in point-slope form}
\end{aligned}
$$

Note

Often, we will prefer final answers be written in slope-intercept form. If the directions prefer the equations of the line in slope-intercept form, we can distribute the slope, then solve for y.

Example 8.10.10

Let's rewrite Example 8.10.9 in slope-intercept form: $y + 4 = \frac{3}{5}(x - 3)$

Solution

$$
\begin{aligned}
y + 4 = \tfrac{3}{5}(x - 3) &\quad \text{Distribute} \\
y + 4 = \tfrac{3}{5}x - \tfrac{9}{5} &\quad \text{Isolate the variable term } y \\
y + 4 + (-4) = \tfrac{3}{5}x - \tfrac{9}{5} + (-4) &\quad \text{Simplify} \\
y = \tfrac{3}{5}x - \tfrac{11}{5} &\quad \text{Slope-intercept form}
\end{aligned}
$$

Example 8.10.11

Write the equation of the line passing through the point $(-6, 2)$ with a slope of $-\frac{2}{3}$ in slope-intercept form.

Solution

$$
\begin{aligned}
y-y_1 &= m(x-x_1) && \text{Substitute values into the point-slope formula}\\
y-2 &= -\tfrac{2}{3}(x-(-6)) && \text{Simplify}\\
y-2 &= -\tfrac{2}{3}(x+6) && \text{Distribute}\\
y-2 &= -\tfrac{2}{3}x-4 && \text{Isolate the variable term } y\\
y-2+2 &= -\tfrac{2}{3}x-4+2 && \text{Simplify}\\
y &= -\tfrac{2}{3}x-2 && \text{Slope-intercept form}
\end{aligned}
$$

Obtaining a Line Given Two Points

In order to find the equation of a line, we need to know the slope. If we aren't given the slope, but only two points on the line, then we complete some preliminary work to obtain the slope. Then we can use the point-slope formula as usual to obtain the equation of the line.

Example 8.10.12

Find the equation of the line passing through the points $(-3,4)$ and $(-1,-2)$ in slope-intercept form

Solution

Since we are given two points, we can use the slope formula to obtain the slope:

$$
\begin{aligned}
m &= \tfrac{y_2-y_1}{x_2-x_1} && \text{Substitute in the ordered-pairs}\\
m &= \tfrac{-2-4}{-1-(-3)} && \text{Simplify}\\
m &= \tfrac{-6}{2}\\
m &= -3 && \text{Slope}
\end{aligned}
$$

Now that we have the slope, we can plug-n-chug the slope and one of the points into the pointslope formula. Notice we have two points and we can choose either one; the results will be the same. Let's choose $(-3,4)$ with the slope $m=-3$.

$$
\begin{aligned}
y-y_1 &= m(x-x_1) && \text{Substitute in the point and slope}\\
y-4 &= -3(x-(-3)) && \text{Simplify}\\
y-4 &= -3(x+3) && \text{Distribute}\\
y-4 &= -3x-9 && \text{Isolate the variable term } y\\
y-4+4 &= -3x-9+4 && \text{Simplify}\\
y &= -3x-5 && \text{Slope-intercept form}
\end{aligned}
$$

Example 8.10.13

Find the equation of the line through the points $(6,-2)$ and $(-4,1)$ in slope-intercept form.

Solution

Since we are given two points, we can use the slope formula to obtain the slope:

$$
\begin{aligned}
m &= \tfrac{y_2-y_1}{x_2-x_1} && \text{Substitute in the ordered-pairs}\\
m &= \tfrac{1-(-2)}{-4-6} && \text{Simplify}\\
m &= -\tfrac{3}{10} && \text{Slope}
\end{aligned}
$$

Now that we have the slope, we can plug-n-chug the slope and one of the points into the pointslope formula. Notice we have two points and we can choose either one; the results will be the same. Let's choose $(-4,1)$ with the slope $m=-\frac{3}{10}$.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Substitute in the point and slope} \\
y - 3 &= -\tfrac{3}{10}(x - (-4)) && \text{Simplify} \\
y - 1 &= -\tfrac{3}{10}(x + 4) && \text{Distribute} \\
y - 1 &= -\tfrac{3}{10}x - \tfrac{6}{5} && \text{Isolate the variable term } y \\
y - 1 + 1 &= -\tfrac{3}{10}x - \tfrac{6}{5} + 1 && \text{Simplify} \\
y &= -\tfrac{3}{10}x - \tfrac{1}{5} && \text{Slope-intercept form}
\end{aligned}
$$

Note

The city of *Konigsberg* (now Kaliningrad, Russia) had a river that flowed through the city breaking it into several parts. There were 7 bridges that connected the parts of the city. In 1735, Leonhard Euler considered the question of whether it was possible to cross each bridge exactly once and only once. It turned out that this problem was impossible, but the work laid the foundation of what would later become graph theory.

Equations of Lines Homework

Write the equation of the line in slope-intercept form given the slope and the y-intercept.

Exercise 8.10.1

$m = 2,\ \ y - intercept = 5$

Exercise 8.10.2

$m = 1,\ \ y - intercept = -4$

Exercise 8.10.3

$m = -\frac{3}{4},\ \ y - intercept = -1$

Exercise 8.10.4

$m = \frac{1}{3},\ \ y - intercept = 1$

Exercise 8.10.5

$m = -6,\ \ y - intercept = 4$

Exercise 8.10.6

$m = -1,\ \ y - intercept = -2$

Exercise 8.10.7

$m = -\frac{1}{4},\ y - intercept = 3$

Exercise 8.10.8

$m = \frac{2}{5},\ y - intercept = 5$

Write the equation of the line in slope-intercept form given the graph or equation.

Exercise 8.10.9

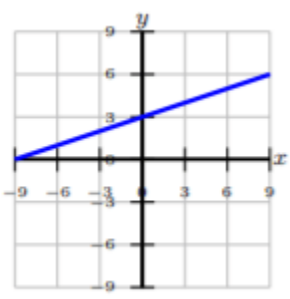


Figure 8.10.1

Exercise 8.10.10

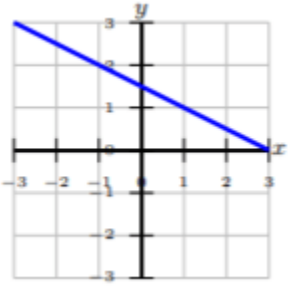


Figure 8.10.2

Exercise 8.10.11

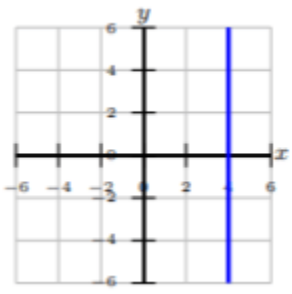


Figure 8.10.3

Exercise 8.10.12

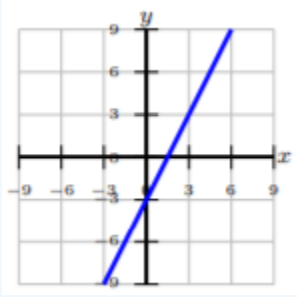


Figure 8.10.4

Exercise 8.10.13

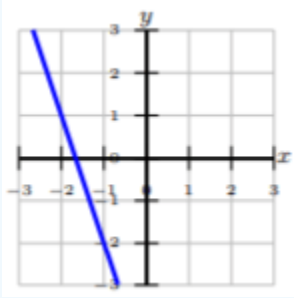


Figure 8.10.5

Exercise 8.10.14

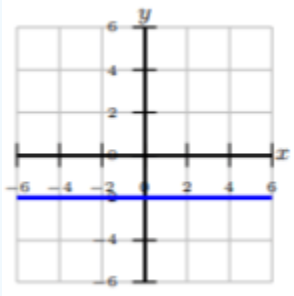


Figure 8.10.6

Exercise 8.10.15

$x + 10y = -37$

Exercise 8.10.16

$2x + y = -1$

Exercise 8.10.17

$7x - 3y = 24$

Exercise 8.10.18

$x = -8$

Exercise 8.10.19

$y - 4 = -(x + 5)$

Exercise 8.10.20

$y - 4 = 4(x - 1)$

Exercise 8.10.21

$y + 5 = -4(x - 2)$

Exercise 8.10.22

$y + 1 = -\frac{1}{2}(x - 4)$

Exercise 8.10.23

$x - 10y = 3$

Exercise 8.10.24

$6x - 11y = -70$

Exercise 8.10.25

$4x + 7y = 28$

Exercise 8.10.26

$x - 7y = -42$

Exercise 8.10.27

$y - 5 = \frac{5}{2}(x - 2)$

Exercise 8.10.28

$y - 3 = -\frac{2}{3}(x + 3)$

Exercise 8.10.29

$0 = x - 4$

Exercise 8.10.30

$y + 2 = \frac{6}{5}(x + 5)$

Sketch the graph of each line.

Exercise 8.10.31

$y = \frac{1}{3}x + 4$

Exercise 8.10.32

$y = \frac{6}{5}x - 5$

Exercise 8.10.33

$y = \frac{3}{2}x$

Exercise 8.10.34

$x - y + 3 = 0$

Exercise 8.10.35

$-y - 4 + 3x = 0$

Exercise 8.10.36

$-3y = -5x + 9$

Exercise 8.10.37

$y = -\frac{1}{5}x - 4$

Exercise 8.10.38

$y = -\frac{3}{2}x - 1$

Exercise 8.10.39

$y = -\frac{3}{4}x + 1$

Exercise 8.10.40

$4x + 5 = 5y$

Exercise 8.10.41

$-8 = 6x - 2y$

Exercise 8.10.42

$-3y = 3 - \frac{3}{2}x$

Write the equation of the line in point-slope form given a point passing through the line and its slope.

Exercise 8.10.43

$(2, 3);\ \ m =$ undefined

Exercise 8.10.44

$(2, 2);\ \ m = \frac{1}{2}$

Exercise 8.10.45

$(-1, -5);\ \ m = 9$

Exercise 8.10.46

$(-4, 1);\ \ m = \frac{3}{4}$

Exercise 8.10.47

$(0, -2);\ \ m = -3$

Exercise 8.10.48

$(0, -5);\ \ m = -\frac{1}{4}$

Exercise 8.10.49

$(-5,-3);\ m=\frac{1}{5}$

Exercise 8.10.50

$(-1,4);\ m=-\frac{5}{4}$

Exercise 8.10.51

$(1,2);\ m=0$

Exercise 8.10.52

$(2,1);\ m=-\frac{1}{2}$

Exercise 8.10.53

$(2,-2);\ m=-2$

Exercise 8.10.54

$(4,-3);\ m=-2$

Exercise 8.10.55

$(-1,1);\ m=4$

Exercise 8.10.56

$(0,2);\ m=-\frac{5}{4}$

Exercise 8.10.57

$(-1,-4);\ m=-\frac{2}{3}$

Write the equation of the line in slope-intercept form given a point passing through the line and its slope.

Exercise 8.10.58

$(-1,-5);\ m=2$

Exercise 8.10.59

$(5, -1)$; $m = -\frac{3}{5}$

Exercise 8.10.60

$(-4, 1)$; $m = \frac{1}{2}$

Exercise 8.10.61

$(4, -2)$; $m = -\frac{3}{2}$

Exercise 8.10.62

$(-5, -3)$; $m = -\frac{2}{5}$

Exercise 8.10.63

$(2, -2)$; $m = 1$

Exercise 8.10.64

$(-3, 4)$, $m =$ undefined

Exercise 8.10.65

$(-4, 2)$; $m = -\frac{1}{2}$

Exercise 8.10.66

$(2, -2)$; $m = -2$

Exercise 8.10.67

$(-2, -2)$; $m = -\frac{2}{3}$

Exercise 8.10.68

$(4, -3)$; $m = -\frac{7}{4}$

Exercise 8.10.69

$(-2, 0)$; $m = -\frac{5}{2}$

Exercise 8.10.70

$(3, 3);\ \ m = \frac{7}{3}$

Exercise 8.10.71

$(-4, -3);\ \ m = 0$

Exercise 8.10.72

$(-2, -5);\ \ m = 2$

Write the equation of the line in point-slope form given two points on the line.

Exercise 8.10.73

$(-4, 3)$ and $(-3, 1)$

Exercise 8.10.74

$(5, 1)$ and $(-3, 0)$

Exercise 8.10.75

$(-4, -2)$ and $(0, 4)$

Exercise 8.10.76

$(3, 5)$ and $(-5, 3)$

Exercise 8.10.77

$(3, -3)$ and $(-4, 5)$

Exercise 8.10.78

$(1, 3)$ and $(-3, 3)$

Exercise 8.10.79

$(-4, 5)$ and $(4, 4)$

Exercise 8.10.80

$(-4, 1)$ and $(4, 4)$

Exercise 8.10.81

$(-1, -4)$ and $(-5, 0)$

Write the equation of the line in slope-intercept form given two points on the line.

Exercise 8.10.82

$(-5, 1)$ and $(-1, -2)$

Exercise 8.10.83

$(-5, 5)$ and $(2, -3)$

Exercise 8.10.84

$(4, 1)$ and $(1, 4)$

Exercise 8.10.85

$(0, 2)$ and $(5, -3)$

Exercise 8.10.86

$(0, 3)$ and $(-1, -1)$

Exercise 8.10.87

$(-5, -1)$ and $(5, -2)$

Exercise 8.10.88

$(1, -1)$ and $(-5, -4)$

Exercise 8.10.89

$(0, 1)$ and $(-3, 0)$

Exercise 8.10.90

$(0, 2)$ and $(2, 4)$

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8.11: Parallel and perpendicular lines

In this section, we discuss parallel and perpendicular lines. The relationship between parallel lines and between perpendicular lines is unique, where the slope will be most interesting to us in this section.

The Slope of Parallel and Perpendicular Lines

Example 8.11.1

Find the slope of each line and compare. What is interesting about the slopes?

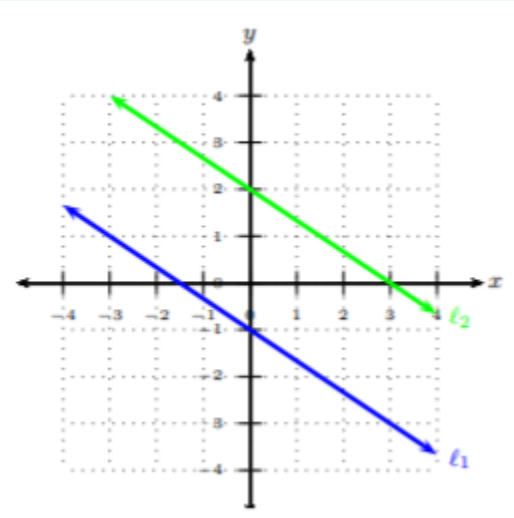


Figure 8.11.1

Solution

Looking at ℓ_1, we can start at $(-3, 1)$ and reach the next point at $(0, -1)$. We see that we will move down two units and run to the right 3 units. Hence, ℓ_1's slope is $-\frac{2}{3}$. Now let's look at ℓ_2 and obtain its slope. We will start at $(0, 2)$ and reach the next point at $(3, 0)$. We see that we will move down two units and run to the right 3 units. Hence, ℓ_2's slope is $-\frac{2}{3}$. The slopes of ℓ_1 and ℓ_2 are $-\frac{2}{3}$; they have the same exact slope but different y-intercepts.

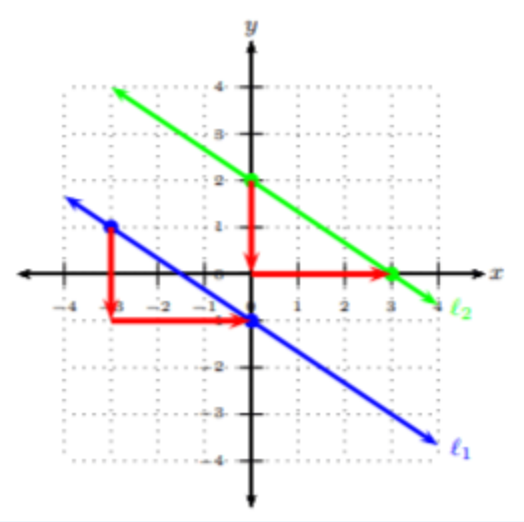


Figure 8.11.2

Definition: Parallel

Let m_1 and m_2 be slopes for lines ℓ_1 and ℓ_2, respectively. Lines ℓ_1 and ℓ_2 are ***parallel*** to each other if they have the same slope, but different y-intercepts, i.e., $m_1 = m_2$.

Example 8.11.2

Find the slope of each line and compare. What is interesting about the slopes?

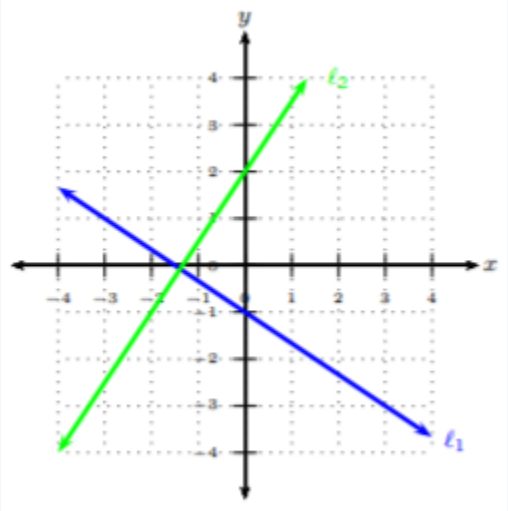


Figure 8.11.3

Solution

Looking at ℓ_1, we can start at $(-3, 1)$ and reach the next point at $(0, -1)$. We see that we will move down two units and run to the right 3 units. Hence, ℓ_1's slope is $-\frac{2}{3}$. Now let's look at ℓ_2 and obtain its slope. We will start at $(-2, -1)$ and reach the next point at $(0, 2)$. We see that we will move up three units and run to the right 2 units. Hence, ℓ_2's slope is $\frac{3}{2}$. The slopes of ℓ_1 and ℓ_2 are negative reciprocals, i.e., if one has slope m, then a line perpendicular to it will have slope $-\frac{1}{m}$. Also, note that if two lines are perpendicular, they create a right angle at the intersection.

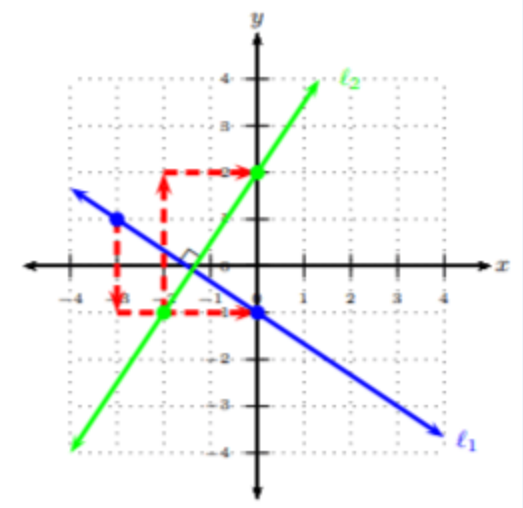


Figure 8.11.4

Definition: Perpendicular

Let m_1 and m_2 be slopes for lines ℓ_1 and ℓ_2, respectively. Lines ℓ_1 and ℓ_2 are ***perpendicular*** to each other if they have negative reciprocal slopes, i.e., ℓ_1 has slope m_1 and ℓ_2 has slope $m_2 = -\frac{1}{m_1}$.

Example 8.11.3

Find the slope of a line parallel to $5y - 2x = 7$.

Solution

We need to rewrite the equation in slope-intercept form. Then we can identify the slope and the slope for a line parallel to it.

$$\begin{aligned} 5y - 2x &= 7 && \text{Isolate the variable term } 5y \\ 5y - 2x + 2x &= 7 + 2x && \text{Simplify} \\ 5y &= 2x + 7 && \text{Multiply by the reciprocal of } 5 \\ \tfrac{1}{5} \cdot 5y &= \tfrac{1}{5} \cdot 2x + 7 \cdot \tfrac{1}{5} && \text{Simplify} \\ y &= \tfrac{2}{5}x + \tfrac{7}{5} \end{aligned}$$

We see the slope of the given line is $\frac{2}{5}$. By the definition, a line parallel will have the same slope $\frac{2}{5}$.

Example 8.11.4

Find the slope of a line perpendicular to $3x - 4y = 2$.

Solution

We need to rewrite the equation in slope-intercept form. Then we can identify the slope and the slope for a line perpendicular to it.

$$\begin{aligned} 3x-4y &= 2 && \text{Isolate the variable term } -4y \\ 3x-4y+(-3x) &= 2+(-3x) && \text{Simplify} \\ -4y &= -3x+2 && \text{Multiply by the reciprocal of } -4 \\ -\tfrac{1}{4}\cdot -4y &= -\tfrac{1}{4}\cdot -3x+2\cdot -\tfrac{1}{4} && \text{Simplify} \\ y &= \tfrac{3}{4}x-\tfrac{1}{2} \end{aligned}$$

We see the slope of the given line is $\frac{3}{4}$. By the definition, a line perpendicular will have a negative reciprocal slope $-\frac{4}{3}$.

Obtain Equations for Parallel and Perpendicular Lines

Once we have obtained the slope for a line perpendicular or parallel, it is possible to find the complete equation of the second line if we are given a point on the second line.

Example 8.11.5

Find the equation of a line passing through $(4,-5)$ and parallel to $2x-3y=6$.

Solution

First, we can rewrite the given line in slope-intercept form to obtain the slope for a line parallel to it:

$$\begin{aligned} 2x-3y &= 6 && \text{Isolate the variable term } -3y \\ 2x-3y+(-2x) &= 6+(-2x) && \text{Simplify} \\ -3y &= -2x+6 && \text{Multiply by the reciprocal of } -3 \\ -\tfrac{1}{3}\cdot -3y &= -\tfrac{1}{3}\cdot -2x+6\cdot -\tfrac{1}{3} && \text{Simplify} \\ y &= \tfrac{2}{3}x-2 \end{aligned}$$

We see the slope of the given line is $\frac{2}{3}$. By the definition, a line parallel will have the same slope $\frac{2}{3}$. Next, we can use the point-slope formula to obtain the equation of the line passing through $(4,-5)$ with slope $\frac{2}{3}$:

$$\begin{aligned} y-y_1 &= m(x-x_1) && \text{Substitute in the point and slope} \\ y-(-5) &= \tfrac{2}{3}(x-4) && \text{Simplify signs} \\ y+5 &= \tfrac{2}{3}(x-4) && \text{A line parallel to } 2x-3y=6 \text{ in point-slope form} \end{aligned}$$

Example 8.11.6

Find the equation of the line, in slope-intercept form, passing through $(6,-9)$ and perpendicular to $y=-\frac{3}{5}x+4$.

Solution

Since the given line is in slope-intercept form, we can easily observe the slope and the slope for a line perpendicular. We see the slope of the given line is $-\frac{3}{5}$. By the definition, a line perpendicular will have a negative reciprocal slope $\frac{5}{3}$. Next, we can use the point-slope formula to obtain the equation, in slope-intercept form, of the line passing through $(6,-9)$ with slope $\frac{5}{3}$:

$$\begin{aligned} y-y_1 &= m(x-x_1) && \text{Substitute in the point and slope} \\ y-(-9) &= \tfrac{5}{3}(x-6) && \text{Simplify signs} \\ y+9 &= \tfrac{5}{3}(x-6) && \text{Distribute} \\ y+9 &= \tfrac{5}{3}x-10 && \text{Isolate the variable term } y \\ y+9+(-9) &= \tfrac{5}{3}x-10+(-9) && \text{Simplify} \\ y &= \tfrac{5}{3}x-19 && \text{A line perpendicular to } y=-\tfrac{3}{5}x+4 \text{ in slope-intercept form} \end{aligned}$$

Note

Lines with zero slopes and undefined slopes may seem like opposites because a horizontal line has slope zero and a vertical line has slope that is undefined. Since a horizontal line is perpendicular to a vertical line, we can say, by definition, the slopes are negative reciprocals, i.e., $m_1 = 0$ would imply $m_2 = -\frac{1}{0}$, which is undefined.

Example 8.11.7

Find the equation of the line passing through $(3, 4)$ and perpendicular to $x = -2$.

Solution

Since $x = -2$ is a vertical line, then this line has slope that is undefined. Hence, a line perpendicular to it will have slope zero, i.e., $m = 0$. Next, we can use the point-slope formula to obtain the equation, in slope-intercept form, of the line passing through $(3, 4)$ with slope $m = 0$:

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Substitute in the point and slope} \\ y - 4 &= 0(x - 3) && \text{Distribute} \\ y - 4 &= 0 && \text{Isolate the variable term } y \\ y - 4 + 4 &= 0 + 4 && \text{Simplify} \\ y &= 4 && \text{A line perpendicular to } x = -2 \end{aligned}$$

Now, since we are aware that a line perpendicular to a vertical line is a horizontal line and we were given a point $(3, 4)$, we could have easily jumped to the equation, $y = 4$.

Parallel and Perpendicular Lines Homework

Given the line, find the slope of a line parallel.

Exercise 8.11.1

$y = 2x + 4$

Exercise 8.11.2

$y = 4x - 5$

Exercise 8.11.3

$x - y = 4$

Exercise 8.11.4

$7x + y = -2$

Exercise 8.11.5

$y = -\frac{2}{3}x + 5$

Exercise 8.11.6

$y = -\frac{10}{3}x - 5$

Exercise 8.11.7

$6x - 5y = 20$

Exercise 8.11.8

$3x + 4y = -8$

Given the line, find the slope of a line perpendicular.

Exercise 8.11.9

$x = 3$

Exercise 8.11.10

$y = -\frac{1}{3}x$

Exercise 8.11.11

$x - 3y = -6$

Exercise 8.11.12

$x + 2y = 8$

Exercise 8.11.13

$y = -\frac{1}{2}x - 1$

Exercise 8.11.14

$y = \frac{4}{5}x$

Exercise 8.11.15

$3x - y = -3$

Exercise 8.11.16

$8x - 3y = -9$

Find the equation of the line, in point-slope form, passing through the point and given the line to be parallel or perpendicular.

Exercise 8.11.17

$(2, 5)$; parallel to $x = 0$

Exercise 8.11.18

$(5, 2)$; parallel to $y = \frac{7}{5}x + 4$

Exercise 8.11.19

$(3, 4)$; parallel to $y = \frac{9}{2}x - 5$

Exercise 8.11.20

$(1, -1)$; parallel to $y = -\frac{3}{4}x + 3$

Exercise 8.11.21

$(2, 3)$; parallel to $y = \frac{7}{5}x + 4$

Exercise 8.11.22

$(-1, 3)$; parallel to $y = -3x - 1$

Exercise 8.11.23

$(4, 2)$; parallel to $x = 0$

Exercise 8.11.24

$(1, 4)$; parallel to $y = \frac{7}{5}x + 2$

Exercise 8.11.25

$(1, -5)$; perpendicular to $-x + y = 1$

Exercise 8.11.26

$(1, -2)$; perpendicular to $-x + 2y = 2$

Exercise 8.11.27

$(5, 2)$; perpendicular to $5x + y = -3$

Exercise 8.11.28

$(1, 3)$; perpendicular to $-x + y = 1$

Exercise 8.11.29

$(4, 2)$; perpendicular to $-4x + y = 0$

Exercise 8.11.30

$(-3, -5)$; perpendicular to $3x + 7y = 0$

Exercise 8.11.31

$(2, -2)$; perpendicular to $3y - x = 0$

Exercise 8.11.32

$(-2, 5)$; perpendicular to $y - 2x = 0$

Find the equation of the line, in slope-intercept form, passing through the point and given the line to be parallel or perpendicular.

Exercise 8.11.33

$(4, -3)$; parallel to $y = -2x$

Exercise 8.11.34

$(-5, 2)$; parallel to $y = \frac{3}{5}x$

Exercise 8.11.35

$(-3, 1)$; parallel to $y = -\frac{4}{3}x - 1$

Exercise 8.11.36

$(-4, 0)$; parallel to $y = -\frac{5}{4}x + 4$

Exercise 8.11.37

$(-4, -1)$; parallel to $y = -\frac{1}{2}x + 1$

Exercise 8.11.38

$(2, 3)$; parallel to $y = \frac{5}{2}x - 1$

Exercise 8.11.39

$(-2, -1)$; parallel to $y = -\frac{1}{2}x - 2$

Exercise 8.11.40

$(-5, -4)$; parallel to $y = \frac{3}{5}x - 2$

Exercise 8.11.41

$(4, 3)$; perpendicular to $x + y = -1$

Exercise 8.11.42

$(-3, -5)$; perpendicular to $x + 2y = -4$

Exercise 8.11.43

$(5, 2)$; perpendicular to $x = 0$

Exercise 8.11.44

$(5, -1)$; perpendicular to $-5x + 2y = 10$

Exercise 8.11.45

$(-2, 5)$; perpendicular to $-x + y = -2$

Exercise 8.11.46

$(2, -3)$; perpendicular to $-2x + 5y = -10$

Exercise 8.11.47

$(4, -3)$; perpendicular to $-x + 2y = -6$

Exercise 8.11.48

$(-4, 1)$; perpendicular to $4x + 3y = -9$

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8.12: System of Equations - Graphing

Three Types of Solutions to a Linear System by Graphing

Given two linear equations and after graphing the lines,

Solution 1. If the two lines intersect, then the point of intersection is the solution to the system, i.e., the solution is an ordered-pair (x, y).

Solution 2. If the two lines do not intersect and are parallel, i.e., they have the same slope and different y-intercepts, then the system has *no solution*.

Solution 3. If the two lines are the same line, then the solution is *infinitely many solutions* on that line.

Verifying Solutions

Example 4.1.1

Is the ordered-pair $(2, 1)$ a solution to the system

$$\begin{cases} 3x - y = 5 \\ x + y = 3 \end{cases} \quad ?$$

Solution

To verify whether $(2, 1)$ is the solution to the system, we plug-n-chug $(2, 1)$ into each equation and determine whether we obtain a true statement. If we obtain true statements for both equations in the system, then $(2, 1)$ will be the solution to the system.

$$\begin{aligned} 3x - y &= 5 && \text{Plug-n-chug } x = 2 \text{ and } y = 1 \\ 3(2) - (1) &\stackrel{?}{=} 5 && \text{Simplify} \\ 6 - 1 &\stackrel{?}{=} 5 && \text{Subtract} \\ 5 &= 5 && \checkmark \text{ True} \end{aligned}$$

Let's do the same for the second equation:

$$\begin{aligned} x + y &= 3 && \text{Plug-n-chug } x = 2 \text{ and } y = 1 \\ (2) + (1) &\stackrel{?}{=} 3 && \text{Add} \\ 3 &= 3 && \checkmark \text{ True} \end{aligned}$$

Since the ordered-pair $(2, 1)$ makes both statements true, then $(2, 1)$ is a solution to the system. Hence, if we were to graph these lines, they would intersect at the point $(2, 1)$.

Example 4.1.2

Is the ordered-pair $(-3, -4)$ a solution to the system

$$\begin{cases} 5x + 4y = -31 \\ 3x + 6y = -36 \end{cases} \quad ?$$

Solution

To verify whether $(-3,-4)$ is the solution to the system, we plug-n-chug $(-3,-4)$ into each equation and determine whether we obtain a true statement. If we obtain true statements for both equations in the system, then $(-3,-4)$ will be the solution to the system.

$$\begin{aligned} 5x+4y &= -31 && \text{Plug-n-chug } x=-3 \text{ and } y=-4 \\ 5(-3)+4(-4) &\stackrel{?}{=} -31 && \text{Simplify} \\ -15-16 &\stackrel{?}{=} -31 && \text{Subtract} \\ -31 &= -31 && \checkmark \text{ True} \end{aligned}$$

Let's do the same for the second equation:

$$\begin{aligned} 3x+6y &= -36 && \text{Plug-n-chug } x=-3 \text{ and } y=-4 \\ 3(-3)+6(-4) &\stackrel{?}{=} -36 && \text{Simplify} \\ -9-24 &\stackrel{?}{=} -36 && \text{Subtract} \\ -33 &\neq -36 && X \text{ False} \end{aligned}$$

Since the ordered-pair $(-3,-4)$ makes only one of the statements true, then $(-3,-4)$ is not a solution to the system. Recall, the ordered-pair must make the statement true for both equations in order to be a solution to the system.

Solve a System by Graphing

Example4.1.3

Solve the system by graphing:

$$\begin{cases} y=-\frac{1}{2}x+3 \\ y=\frac{3}{4}x-2 \end{cases}$$

Solution

We first need to decide the method in which we will graph. We learned in the previous chapter to make a table, use intercepts, or use the slope-intercept form. Notice both equations are given in slope-intercept form. Let's go ahead and use the slope-intercept form to graph the lines.

$$\begin{aligned} y &= -\frac{1}{2}x+3 \quad (1) \\ y &= \frac{3}{4}x-2 \quad (2) \end{aligned}$$

We will graph line (1) with a solid line and graph line (2) with a dashed line.

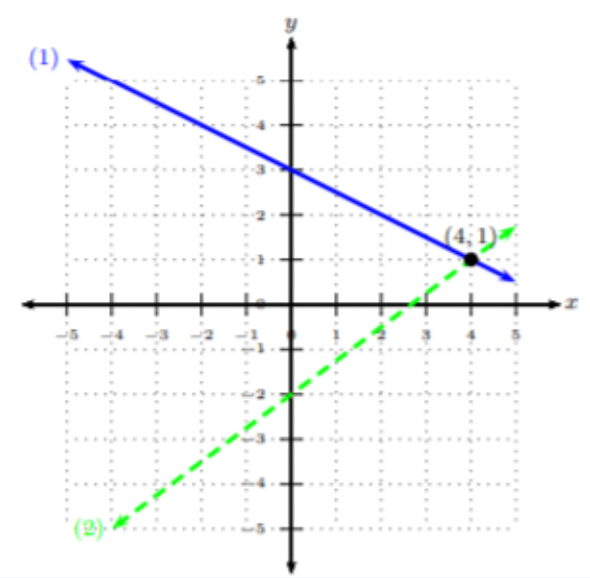


Figure 4.1.1

We can see after graphing the two lines that they intersect at the point $(4, 1)$. Hence, the solution to the system is $(4, 1)$.

Example 4.1.4

Solve the system by graphing:

$$\begin{cases} 6x - 3y = -9 \\ 2x + 2y = -6 \end{cases}$$

Solution

We first need to decide the method in which we will graph. We learned in the previous chapter to make a table, use intercepts, or use the slope-intercept form. Since both equations are not given in slope-intercept form as in Example 4.1.3 , we can rewrite them in slope-intercept form, then graph. So, let's rewrite each equation in slope-intercept form:

$$\begin{aligned} 6x - 3y &= -9 & 2x + 2y &= -6 \\ -3y &= -6x - 9 & 2y &= -2x - 6 \\ y &= \frac{-6x}{-3} - \frac{9}{-3} & y &= \frac{-2}{2}x - \frac{6}{2} \\ y &= 2x + 3 & y &= -x - 3 \end{aligned}$$

Let's go ahead and use the slope-intercept form to graph the lines.

$$\begin{aligned} y &= 2x + 3 \quad (1) \\ y &= -x - 3 \quad (2) \end{aligned}$$

We will graph line (1) with a solid line and graph line (2) with a dashed line.

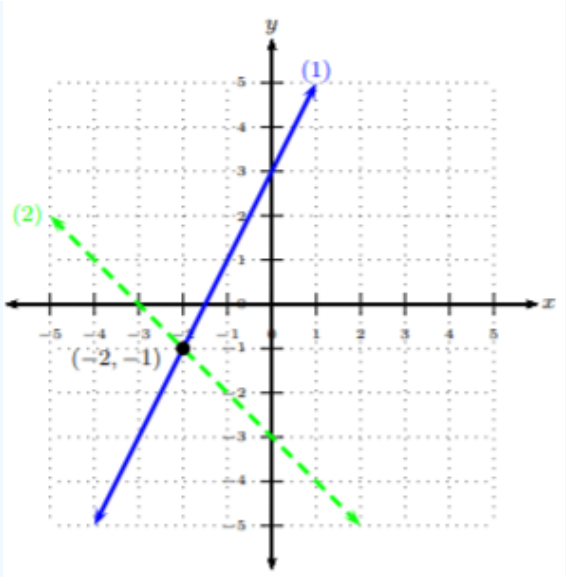


Figure 4.1.2

We can see after graphing the two lines that they intersect at the point $(-2,-1)$. Hence, the solution to the system is $(-2,-1)$.

Example 4.1.5

Solve the system by graphing:

$$\begin{cases} y=\frac{3}{2}x-4 \\ y=\frac{3}{2}x+1 \end{cases}$$

Solution

We first need to decide the method in which we will graph. We learned in the previous chapter to make a table, use intercepts, or use the slope-intercept form. Notice both equations are given in slope-intercept form. Let's go ahead and use the slope-intercept form to graph the lines.

$$y=\frac{3}{2}x-4 \quad (1)$$
$$y=\frac{3}{2}x+1 \quad (2)$$

We will graph line (1) with a solid line and graph line (2) with a dashed line.

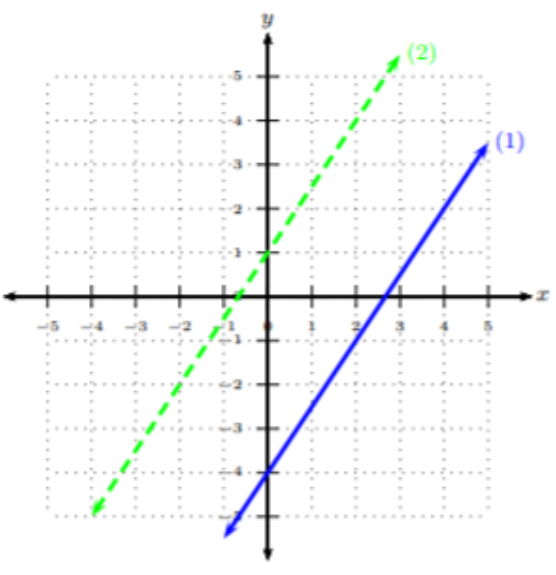


Figure 4.1.3

We can see after graphing the two lines that these two lines are parallel. Hence, there is no solution to the system (since they will never intersect). Note, we could see by the system that these lines shared the same slope, but had different y-intercepts. Without graphing, we could have seen that these lines were parallel, hence, having no solution.

Example 4.1.6

Solve the system by graphing:

$$\begin{cases} 2x - 6y = 12 \\ 3x - 9y = 18 \end{cases}$$

Solution

We first need to decide the method in which we will graph. We learned in the previous chapter to make a table, use intercepts, or use the slope-intercept form. Since neither of the equations are written in slope-intercept form, let's try graphing by making a table for each equation. Start with equation (1):

$$\begin{aligned} x = -3 \qquad 2(-3) - 6y &= 12 \\ -6 - 6y &= 12 \\ -6y &= 18 \\ y &= -3 \end{aligned}$$

$$\begin{aligned} x = 0 \qquad 2(0) - 6y &= 12 \\ -6y &= 12 \\ y &= -2 \end{aligned}$$

$$\begin{aligned} x = 3 \qquad 2(3) - 6y &= 12 \\ 6 - 6y &= 12 \\ -6y &= 6 \\ y &= -1 \end{aligned}$$

Table 4.1.1

x	y
-3	-3

x	y
0	-2
3	-1

Next, equation (2):

$$\begin{aligned} x=-3 \qquad 3(-3)-9y&=18\\ -9-9y&=18\\ -9y&=27\\ y&=-3 \end{aligned}$$

$$\begin{aligned} x=0 \qquad 3(0)-9y&=18\\ -9y&=18\\ y&=-2 \end{aligned}$$

$$\begin{aligned} x=3 \qquad 3(3)-9y&=18\\ 9-9y&=18\\ -9y&=9\\ y&=-1 \end{aligned}$$

Table 4.1.2

x	y
-3	-3
0	-2
3	-1

Now, let's graph the ordered-pairs. We will graph line (1) with a solid line and graph line (2) with a dashed line.

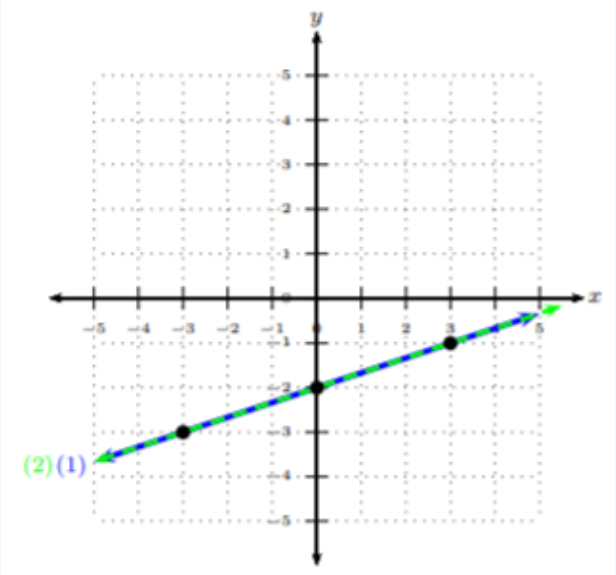


Figure 4.1.4

We can see after graphing the two lines that these two lines are the same. Hence, there are infinitely many solutions on the line $2x-6y=12$ (or the other equation) to the system (since they intersect at every point on the line). Note, we could see by the system that these lines shared the same slope and y-intercepts (after putting them in slope-intercept form). Without graphing, we could have seen that these lines were the same line, hence, having infinitely many solutions.

Note

The Babylonians were the first to work with systems of equations with two variables. However, their work with systems was quickly passed by the Greeks, around 300 AD, who would solve systems of equations with three or four variables and eventually developed methods for solving systems with any number of unknowns.

Systems of Equations: Graphing Homework

Determine whether the given ordered pair(s) is a solution to the system.

Exercise 4.1.1

$2x + 8y = 0$
$-8x + 3y = 38;\ (-4, 1)$

Exercise 4.1.2

$-5x + 6y = 11$
$4x + 2y = -4;\ (-1, 1)$

Exercise 4.1.3

$6x + 5y = 49$
$-x - 6y = -34;\ (4, 5)$

Exercise 4.1.4

$-2x + 2y = -8$
$-6x - 3y = -6;\ (2, -2)$

Solve each system by graphing.

Exercise 4.1.5

$y = -x + 1$
$y = -5x - 3$

Exercise 4.1.6

$y = -3$
$y = -x - 4$

Exercise 4.1.7

$$y = -\frac{3}{4}x + 1$$
$$y = -\frac{3}{4}x + 2$$

Exercise 4.1.8

$$y = \frac{1}{3}x + 2$$
$$y = -\frac{5}{3}x - 4$$

Exercise 4.1.9

$$y = \frac{5}{3}x + 4$$
$$y = -\frac{2}{3}x - 3$$

Exercise 4.1.10

$$x + 3y = -9$$
$$5x + 3y = 3$$

Exercise 4.1.11

$$x - y = 4$$
$$2x + y = -1$$

Exercise 4.1.12

$$2x + 3y = -6$$
$$2x + y = 2$$

Exercise 4.1.13

$$2x + y = 2$$
$$x - y = 4$$

Exercise 4.1.14

$$2x + y = -2$$
$$x + 3y = 9$$

Exercise 4.1.15

$$0 = -6x - 9y + 36$$
$$12 = 6x - 3y$$

Exercise 4.1.16

$$2x - y = -1$$
$$0 = -2x - y - 3$$

Exercise 4.1.17

$$3 + y = -x$$
$$-4 - 6x = -y$$

Exercise 4.1.18

$$-y + 7x = 4$$
$$-y - 3 + 7x = 0$$

Exercise 4.1.19

$$-12 + x = 4y$$
$$12 - 5x = 4y$$

Exercise 4.1.20

$$y = -\frac{5}{4}x - 2$$
$$y = -\frac{1}{4}x + 2$$

Exercise 4.1.21

$$y = -x - 2$$
$$y = \frac{2}{3}x + 3$$

Exercise 4.1.22

$y = 2x + 2$
$y = -x - 4$

Exercise 4.1.23

$y = 2x - 4$
$y = \frac{1}{2}x + 2$

Exercise 4.1.24

$y = \frac{1}{2}x + 4$
$y = \frac{1}{2}x + 1$

Exercise 4.1.25

$x + 4y = -12$
$2x + y = 4$

Exercise 4.1.26

$6x + y = -3$
$x + y = 2$

Exercise 4.1.27

$3x + 2y = 2$
$3x + 2y = -6$

Exercise 4.1.28

$x + 2y = 6$
$5x - 4y = 16$

Exercise 4.1.29

$x - y = 3$
$5x + 2y = 8$

Exercise 4.1.30

$$-2y + x = 4$$
$$2 = -x + \frac{1}{2}y$$

Exercise 4.1.31

$$-2y = -4 - x$$
$$-2y = -5x + 4$$

Exercise 4.1.32

$$16 = -x - 4y$$
$$-2x = -4 - 4y$$

Exercise 4.1.33

$$-4 + y = x$$
$$x + 2 = -y$$

Exercise 4.1.34

$$-5x + 1 = -y$$
$$-y + x = -3$$

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8.13: Systems of Equations - The Substitution Method

Solving a system by graphing has its limitations. First, it requires the graph to be precisely drawn. If the lines aren't drawn correctly, it is possible to obtain the incorrect point of intersection. Second, graphing is not a sufficient method to use if the point of intersection is far away from the origin, like the ordered-pair (12567, 9832), or if the answer is a decimal (or irrational number) that the graph will not display well. For these reasons, we rarely use graphing to solve systems. Instead, we use an algebraic approach. There are two approaches and the first approach is called substitution. We build the concepts of substitution through several examples and then conclude with a general four-step process to solve problems using this method.

Introduction to Substitution

Example 4.2.1

Solve the system of equations.

$$\begin{cases} x = 5 \\ y = 2x - 3 \end{cases}$$

Solution

We are given $x = 5$ in the first equation. Hence, we can plug-n-chug $x = 5$ into the second equation to find y because we know any ordered-pair with x coordinate 5 will satisfy the equation:

$$\begin{aligned} y &= 2x - 3 && \text{Plug-n-chug } x = 5 \\ y &= 2(5) - 3 && \text{Evaluate} \\ y &= 10 - 3 && \text{Subtract} \\ y &= 7 && y\text{-coordinate of the solution} \end{aligned}$$

Since it is given $x = 5$ and we obtained $y = 7$, then the solution to the system is the ordered-pair $(5, 7)$. Furthermore, if we were to graph these two lines, they would intersect at $(5, 7)$.

The Substitution Method

When we are given the solution to one of the two variables, we can easily plug-n-chug that value (or expression) in the other equation to obtain the value of the second variable. It is very important that when we substitute, we write parenthesis around the expression we are substituting; this will minimize errors with distribution.

Example 4.2.2

Solve the system by substitution.

$$\begin{cases} 2x - 3y = 7 \\ y = 3x - 7 \end{cases}$$

Solution

We can see that y is isolated in the second equation, $y = 3x - 7$, and we can substitute the right side for y in the first equation.

$$\begin{aligned} 2x-3y&=7 && \text{Plug-n-chug } y=3x-7 && \text{into the first equation}\\ 2x-3(3x-7)&=7 && \text{Distribute}\\ 2x-9x+21&=7 && \text{Combine like terms}\\ -7x+21&=7 && \text{Isolate the variable term}\\ -7x&=-14 && \text{Multiply by the reciprocal of } -7\\ x&=2 && x\text{-coordinate of the solution} \end{aligned}$$

Since $x=2$, then we can plug-n-chug $x=2$ into one of the equations to obtain y:

$$\begin{aligned} y&=3x-7 && \text{Plug-n-chug } x=2\\ y&=3(2)-7 && \text{Evaluate}\\ y&=6-7 && \text{Subtract}\\ y&=-1 && y\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(2,-1)$. Furthermore, if we were to graph these two lines, we know they would intersect at $(2,-1)$. When we obtain a solution to a system, we call this system a *consistent system*. When we obtain one ordered-pair as the solution to the system, we call this solution an *independent solution*.

Definition: Systems

Given a system of equations,

- If there exists at least one solution to the system, then this is a ***consistent system***.
  - If the consistent system has one unique solution, i.e., the solution is an ordered-pair, then this is an ***independent system***.
  - If the consistent system has more than one solution, i.e., there are infinitely many solutions, then this is a ***dependent system***.
- If a solution to the system doesn't exist, i.e., there is no solution, then this is an ***inconsistent system***.

Solve for a Variable

Example 4.2.3

Solve the system by substitution:

$$\begin{cases} 3x+2y=1\\ x-5y=6 \end{cases}$$

Solution

Notice neither of the equations have y or x isolated. Hence, we will have to pick an equation and variable, and solve for that variable in that equation. We always want to work smarter, not harder, so let's be clever in choosing the equation and variable. Looking at the first equation, there is a coefficient in front of each variable. Looking at the second equation, we see only the y has a coefficient other than 1, and x's coefficient is one. Let's pick this because solving for x is a one-step equation while the other is a two-step equation.

$$\begin{aligned} x-5y&=6\\ x&=6+5y \end{aligned} \tag{8.13.1}$$

Now, we can substitute x into the first equation:

$$\begin{aligned} 3x+2y &= 1 && \text{Plug-n-chug } x=6+5y \\ 3(6+5y)+2y &= 1 && \text{Distribute} \\ 18+15y+2y &= 1 && \text{Combine like terms} \\ 18+17y &= 1 && \text{Isolate the variable term} \\ 17y &= -17 && \text{Multiply by the reciprocal of 17} \\ y &= -1 && y\text{-coordinate of the solution} \end{aligned}$$

Since $y=-1$, then we can plug-n-chug $y=-1$ into one of the equations to obtain x:

$$\begin{aligned} x &= 6+5y && \text{Plug-n-chug } y=-1 \\ x &= 6+5(-1) && \text{Evaluate} \\ x &= 6-5 && \text{Subtract} \\ x &= 1 && x\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(1,-1)$. Furthermore, if we were to graph these two lines, they would intersect at $(1,-1)$. Also, we know this system is a consistent system that is independent.

Note

When choosing the equation and variable to use in substitution, *we want to choose the one-step equation*, i.e., one equation with one of the variables with a coefficient of one. Even though we are used to always solving for y when graphing lines, with this method, it doesn't matter. We want to work smarter, not harder, and pick the equation that will give us the solution the quickest.

The Substitution Method

Given a system of two linear equations in two variables, we can use the following steps to solve by substitution.

- **Step 1.** Choose an equation and then solve for x or y. *(Choose the one-step equation when possible.)*
- **Step 2.** Substitute the expression for x or y in the other equation.
- **Step 3.** Solve the equation.
- **Step 4.** Substitute the value into one of the original equations to find the remaining variable.

It is common practice to write your answer as an ordered-pair of the form (x,y) since this is the point of intersection. Be sure to verify the solution.

Example 4.2.4

Solve the system by substitution:

$$\begin{cases} 4x-2y=2 \\ 2x+y=-5 \end{cases}$$

Solution

Notice none of the equations have y or x isolated. Hence, we will have to pick an equation and variable, and solve for that variable in that equation.

Step 1. We want to choose the one-step equation, if there is one. Looking at the second equation, we see only the y has a coefficient of one, and x's coefficient is two. Let's pick this because solving for y is a one-step equation while the other is a two-step equation.

$$\begin{aligned} 2x + y &= -5 \\ y &= -2x - 5 \end{aligned} \tag{8.13.2}$$

Step 2. Now, we can substitute y into the first equation:

$$\begin{aligned} 4x - 2y &= 2 \quad \text{Plug-n-chug } y = -2x - 5 \\ 4x - 2(-2x - 5) &= 2 \end{aligned}$$

Step 3. Solve for x:

$$\begin{aligned} 4x - 2(-2x - 5) &= 2 \quad \text{Distribute} \\ 4x + 4x + 10 &= 2 \quad \text{Combine like terms} \\ 8x + 10 &= 2 \quad \text{Isolate the variable term} \\ 8x &= -8 \quad \text{Multiply by the reciprocal of 8} \\ x &= -1 \quad x\text{-coordinate of the solution} \end{aligned}$$

Step 4. Since $x = -1$, then we can plug-n-chug $x = -1$ into one of the equations to obtain y:

$$\begin{aligned} y &= -2x - 5 \quad \text{Plug-n-chug } x = -1 \\ y &= -2(-1) - 5 \quad \text{Evaluate} \\ y &= 2 - 5 \quad \text{Subtract} \\ y &= -3 \quad y\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(-1, -3)$. Furthermore, if we were to graph these two lines, they would intersect at $(-1, -3)$. Also, we know this system is a consistent system that is independent.

Substitution: Special Cases

Example 4.2.5

Solve the system by substitution:

$$\begin{cases} y + 4 = 3x \\ 2y - 6x = -8 \end{cases}$$

Solution

Notice none of the equations have y or x isolated. Hence, we will have to pick an equation and variable, and solve for that variable in that equation.

Step 1. We want to choose the one-step equation, if there is one. Looking at the first equation, we see only the y has a coefficient of one, and x's coefficient is three. Let's pick this because solving for y is a one-step equation while the other is a two-step equation.

$$\begin{aligned} y + 4 &= 3x \\ y &= 3x - 4 \end{aligned} \tag{8.13.3}$$

Step 2. Now, we can substitute y into the second equation:

$$\begin{aligned} 2y - 6x &= -8 \quad \text{Plug-n-chug } y = 3x - 4 \\ 2(3x - 4) - 6x &= -8 \end{aligned}$$

Step 3. Solve for x:

$$\begin{aligned} 2(3x - 4) - 6x &= -8 \quad \text{Distribute} \\ 6x - 8 - 6x &= -8 \quad \text{Combine like terms} \\ -8 &= -8 \end{aligned}$$

Since all the variables cancel and we are left with a statement without variables, we ask,“ Is this statement true?”

$$-8 \stackrel{?}{=} -8 \quad \text{Is this true?}$$
$$-8 = -8 \quad \checkmark \text{ True}$$

Since this statement is true, then there are infinitely many solutions on the line $y + 4 = 3x$. Furthermore, if we were to graph these two lines, we know they would be the same line and intersect at every point on the line. Also, we know this system is a consistent system that is dependent.

Example 4.2.6

Solve the system by substitution:

$$\begin{cases} 6x - 3y = -9 \\ -2x + y = 5 \end{cases}$$

Solution

Notice none of the equations have y or x isolated. Hence, we will have to pick an equation and variable, and solve for that variable in that equation.

Step 1. We want to choose the one-step equation, if there is one. Looking at the second equation, we see only the y has a coefficient of one, and x's coefficient is -2. Let's pick this because solving for y is a one-step equation while the other is a two-step equation.

$$\begin{aligned} -2x + y &= 5 \\ y &= 2x + 5 \end{aligned} \tag{8.13.4}$$

Step 2. Now, we can substitute y into the first equation:

$$\begin{aligned} 6x - 3y &= -9 \quad \text{Plug-n-chug } y = 2x + 5 \\ 6x - 3(2x + 5) &= -9 \end{aligned}$$

Step 3. Solve for x:

$$\begin{aligned} 6x - 3(2x + 5) &= -9 \quad \text{Distribute} \\ 6x - 6x - 15 &= -9 \quad \text{Combine like terms} \\ -15 &= -9 \end{aligned}$$

Since all the variables cancel and we are left with a statement without variables, we ask," Is this statement true?"

$$-15 \stackrel{?}{=} -9 \quad \text{Is this true?}$$
$$-15 \neq -9 \quad X \text{ False}$$

Since this statement is false, then there is no solution to this system. Furthermore, if we were to graph these two lines, we know they would be parallel. Hence, this system is an inconsistent system.

Note

French mathematician *Rene Descartes* wrote a book which included an appendix on geometry. It was in this book that he suggested using letters from the end of the alphabet for unknown values. This is the reason we are often solving for the variables x, y, and z.

Example 4.2.7

Solve the system by substitution:

$$\begin{cases} 5x - 6y = -14 \\ -2x + 4y = 12 \end{cases}$$

Solution

Notice none of the equations have y or x isolated. Hence, we will have to pick an equation and variable, and solve for that variable in that equation.

Step 1. We want to choose the one-step equation, if there is one. Looking at both equations, we see none of these are one-step equations. Hence, we can just pick one equation and solve for a variable. Notice in the second equation, all coefficients are divisible by 2. Let's pick this because solving for x would avoid fractions.

$$\begin{aligned} -2x + 4y &= 12 \\ -2x &= -4y + 12 \\ x &= 2y - 6 \end{aligned} \tag{8.13.5}$$

Step 2. Now, we can substitute x into the first equation:

$$\begin{aligned} 5x - 6y &= -14 \quad \text{Plug-n-chug } x = 2y - 6 \\ 5(2y - 6) - 6y &= -14 \end{aligned}$$

Step 3. Solve for y:

$$\begin{aligned} 5(2y - 6) - 6y &= -14 && \text{Distribute} \\ 10y - 20 - 6y &= -14 && \text{Combine like terms} \\ 4y - 30 &= -14 && \text{Isolate the variable term} \\ 4y &= 16 && \text{Multiply by the reciprocal of 4} \\ y &= 4 && y\text{-coordinate of the solution} \end{aligned}$$

Step 4. Since $y = 4$, then we can plug-n-chug $y = 4$ into one of the equations to obtain x:

$$\begin{aligned} x &= 2y - 6 && \text{Plug-n-chug } y = 4 \\ x &= 2(4) - 6 && \text{Evaluate} \\ x &= 8 - 6 && \text{Subtract} \\ x &= 2 && x\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(2, 4)$. Furthermore, if we were to graph these two lines, we know they would intersect at $(2, 4)$. Also, we know this system is a consistent system that is independent.

Systems of Equations: The Substitution Method Homework

Solve each system by substitution. Determine if each system is consistent, independent or dependent, or inconsistent.

Exercise 4.2.1

$y = -3x$
$y = 6x - 9$

Exercise 4.2.2

$y = -2x - 9$
$y = 2x - 1$

Exercise 4.2.3

$y = 6x + 4$
$y = -3x - 5$

Exercise 4.2.4

$y = 3x + 2$
$y = -3x + 8$

Exercise 4.2.5

$y = 2x - 3$
$y = -2x + 9$

Exercise 4.2.6

$y = 6x - 6$
$-3x - 3y = -24$

Exercise 4.2.7

$y = -6$
$3x - 6y = 30$

Exercise 4.2.8

$y = -5$
$3x + 4y = -17$

Exercise 4.2.9

$-2x + 2y = 18$
$y = 7x + 15$

Exercise 4.2.10

$y = -8x + 19$
$-x + 6y = 16$

Exercise 4.2.11

$7x - 2y = -7$
$y = 7$

Exercise 4.2.12

$x - 5y = 7$
$2x + 7y = -20$

Exercise 4.2.13

$-2x - y = -5$
$x - 8y = -23$

Exercise 4.2.14

$-6x + y = 20$
$-3x - 3y = -18$

Exercise 4.2.15

$3x + y = 9$
$2x + 8y = -16$

Exercise 4.2.16

$y = x + 5$
$y = -2x - 4$

Exercise 4.2.17

$y = -6x + 3$
$y = 6x + 3$

Exercise 4.2.18

$y = 3x + 13$
$y = -2x - 22$

Exercise 4.2.19

$y = -2x - 9$
$y = -5x - 21$

Exercise 4.2.20

$y = 7x - 24$
$y = -3x + 16$

Exercise 4.2.21

$-x + 3y = 12$
$y = 6x + 21$

Exercise 4.2.22

$6x - 4y = -8$
$y = -6x + 2$

Exercise 4.2.23

$7x + 2y = -7$
$y = 5x + 5$

Exercise 4.2.24

$y = x + 4$
$3x - 4y = -19$

Exercise 4.2.25

$y = -2x + 8$
$-7x - 6y = -8$

Exercise 4.2.26

$x - 2y = -13$
$4x + 2y = 18$

Exercise 4.2.27

$3x - 4y = 15$
$7x + y = 4$

Exercise 4.2.28

$6x + 4y = 16$
$-2x + y = -3$

Exercise 4.2.29

$7x + 5y = -13$
$x - 4y = -16$

Exercise 4.2.30

$-5x - 5y = -20$
$-2x + y = 7$

Exercise 4.2.31

$2x + y = 2$
$3x + 7y = 14$

Exercise 4.2.32

$x + 5y = 15$
$-3x + 2y = 6$

Exercise 4.2.33

$-2x + 4y = -16$
$y = -2$

Exercise 4.2.34

$-6x + 6y = -12$
$8x - 3y = 16$

Exercise 4.2.35

$$2x + 3y = 16$$
$$-7x - y = 20$$

Exercise 4.2.36

$$2x + y = -7$$
$$5x + 3y = -21$$

Exercise 4.2.37

$$2x + 3y = -10$$
$$7x + y = 3$$

Exercise 4.2.38

$$-2x + 2y = -22$$
$$-5x - 7y = -19$$

Exercise 4.2.39

$$-8x + 2y = -6$$
$$-2x + 3y = 11$$

Exercise 4.2.40

$$-x - 4y = -14$$
$$-6x + 8y = 12$$

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8.14: System of Equations - The Addition Method

We first found that graphing isn't a sophisticated way for generally solving systems. We then considered a second method known as substitution. The substitution method is often used for solving systems in various areas of algebra. However, substitution can get quite involved, especially if there are fractions because this only allows more room for error. Hence, we need an even more sophisticated way for solving systems in general. We call this method the **addition method**, also called the elimination method. We will build the concept in the following examples, then define a four-step process we can use to solve by elimination.

The Addition Method

Example 4.3.1

Solve the system by addition (elimination).

$$\begin{cases} 3x - 4y = 8 \\ 5x + 4y = -24 \end{cases}$$

Solution

We solve the system by addition because we do just that- add. We want to add the two equations together to obtain an equation of one variable. Hence, we cannot just add right away; we need to make sure that when we add, we will eliminate one of the variables. Looking at the y variable terms, we can see that the coefficients of y are the same but opposite signs. We can foresee that when we add these two equations, the y variable terms will cancel:

$$\begin{array}{rl} 3x - 4y = 8 & \\ \underline{+5x + 4y = -24} & \text{Add the equations} \\ 8x = -16 & \end{array}$$

Notice the y variable terms canceled and we are left with one equation in one variable. This is always the goal. Now, we can easily solve as usual:

$$\begin{array}{rl} 8x = -16 & \text{Multiply by the reciprocal of } 8 \\ x = -2 & x\text{-coordinate of the solution} \end{array}$$

Since $x = -2$, then we can plug-n-chug $x = -2$ into one of the equations to obtain y:

$$\begin{array}{rl} 3x - 4y = 8 & \text{Plug-n-chug } x = -2 \\ 3(-2) - 4y = 8 & \text{Evaluate} \\ -6 - 4y = 8 & \text{Isolate the variable term} \\ -4y = 14 & \text{Multiply by the reciprocal of } -4 \\ y = -\dfrac{14}{4} & \text{Reduce the fraction} \\ y = -\dfrac{7}{2} & y\text{-coordinate of the solution} \end{array}$$

The solution to the system is the ordered-pair $\left(-2, -\frac{7}{2}\right)$. Furthermore, if we were to graph these two lines, we know they would intersect at $\left(-2, -\frac{7}{2}\right)$. Also, we know this system is a consistent system with an independent solution.

The Addition Method with Multiplication

In Example 4.3.1 , one of the variable terms had the same coefficient, but opposite signs, and adding these together eliminated

the variable terms completely, which allowed us to solve for the other variable. This is the idea behind the addition method. However, generally, we aren't given variable terms that have the same coefficient with opposite signs. So, we will manipulate the equations by multiplying one or both equations by the LCM of the coefficients for one of the variables. We want to work smarter, not harder, so we should be clever in which variable we choose.

Example 4.3.2

Solve the system by addition (elimination).

$$\begin{cases} -6x + 5y = 22 \\ 2x + 3y = 2 \end{cases}$$

Solution

Since none of the variable terms have the same coefficient with opposite signs, we need to choose a variable and rewrite the equations so we can cancel the variable. Recall, the goal is to obtain an equation in one variable after adding. Looking at the x variable terms, we can see that the coefficients of x have opposite signs. So let's choose to eliminate x and we multiply the second equation by 3 to obtain the $\text{LCM}(2, 6) = 6$:

$$\begin{aligned} 3 \cdot (2x + 3y) &= (2) \cdot 3 \quad \text{Distribute} \\ 6x + 9y &= 6 \end{aligned}$$

Notice the x variable terms have the same coefficients with opposite signs. Now we can add and eliminate x:

$$\begin{array}{rl} -6x + 5y = 22 & \\ + \quad 6x + 9y = 6 & \quad \text{Add the equations} \\ \hline 14y = 28 & \end{array}$$

Notice the x variable terms canceled and we are left with one equation in one variable. This is always the goal. Now, we can easily solve as usual:

$$\begin{aligned} 14y &= 28 \quad \text{Multiply by the reciprocal of 14} \\ y &= 2 \quad y\text{-coordinate of the solution} \end{aligned}$$

Since $y = 2$, then we can plug-n-chug $y = 2$ into one of the equations to obtain x:

$$\begin{aligned} 2x + 3y &= 2 \quad \text{Plug-n-chug } y = 2 \\ 2x + 3(2) &= 2 \quad \text{Evaluate} \\ 2x + 6 &= 2 \quad \text{Isolate the variable term} \\ 2x &= -4 \quad \text{Multiply by the reciprocal of 2} \\ x &= -2 \quad x\text{-coordinate of the solution} \end{aligned}$$

The solution to the system is the ordered-pair $(-2, 2)$. Furthermore, if we were to graph these two lines, we know they would intersect at $(-2, 2)$. Also, we know this system is a consistent system with an independent solution.

Multiplying Two Equations

Example 4.3.3

Solve the system by addition (elimination).

$$\begin{cases} 3x + 6y = -9 \\ 2x + 9y = -26 \end{cases}$$

Solution

Since none of the variable terms have the same coefficient with opposite signs, we need to choose a variable and rewrite the equations so we can cancel the variable. Recall, the goal is to obtain an equation in one variable after adding. Looking at the x and y variable terms, we can see that none of the coefficients are the same or with opposite signs. So we can choose any variable to eliminate. Let's choose to eliminate y and we multiply both equations by a factor to obtain the $\text{LCM}(9,6)=18$ with opposite signs:

$$\begin{aligned} -3\cdot(3x+6y) &= (-9)\cdot -3 \\ 2\cdot(2x+9y) &= (-26)\cdot 2 \end{aligned} \tag{8.14.1}$$

Notice the y variable terms have the same coefficients with opposite signs

$$\begin{aligned} -9x-18y &= 27 \\ 4x+18y &= -52 \end{aligned}$$

Now we can add and eliminate y:

$$\begin{array}{rl} -9x-18y=27 & \\ +\quad 4x+18y=-52 & \text{Add the equations} \\ \hline -5x=-25 & \end{array}$$

Notice the y variable terms canceled and we are left with one equation in one variable. This is always the goal. Now, we can easily solve as usual:

$$\begin{array}{rl} -5x=-25 & \text{Multiply by the reciprocal of } -5 \\ x=5 & x\text{-coordinate of the solution} \end{array}$$

Since $x=5$, then we can plug-n-chug $x=5$ into one of the equations to obtain y:

$$\begin{array}{rl} 2x+9y=-26 & \text{Plug-n-chug } x=5 \\ 2(5)+9y=-26 & \text{Evaluate} \\ 10+9y=-26 & \text{Isolate the variable term} \\ 9y=-36 & \text{Multiply by the reciprocal of 9} \\ y=-4 & y\text{-coordinate of the solution} \end{array}$$

The solution to the system is the ordered-pair $(5,-4)$. Furthermore, if we were to graph these two lines, we know they would intersect at $(5,-4)$. Also, we know this system is a consistent system with an independent solution.

The Addition Method

Given a system of two linear equations in two variables, we can use the following steps to solve by addition (elimination).

- **Step 1.** Choose a variable to eliminate. *(Choose the variable with the variable terms with opposite signs, same coefficient, or both, if possible.)*
- **Step 2.** Multiply one or both equations so that the coefficients of this variable are the LCM of the coefficients with opposite signs.
- **Step 3.** Add the equations together, then solve.
- **Step 4.** Substitute the value into one of the original equations to find the remaining variable.

It is common practice to write your answer as an ordered-pair of the form (x,y) since this is the point of intersection. Be sure to verify the solution.

Example 4.3.4

Solve the system by addition (elimination):

$$\begin{cases} 2x - 5y = -13 \\ 5x - 3y = -4 \end{cases}$$

Solution

Since none of the variable terms have the same coefficient, opposite signs, or both, we need to choose a variable and rewrite the equations so we can cancel the variable. We can choose any variable to eliminate.

Step 1. Let's choose to eliminate x and we multiply both equations by a factor to obtain the $\text{LCM}(2,5) = 10$ with opposite signs.

Step 2. We can mulitply the first equation by a factor of 5 and the second equation by a factor of -2 so that we obtain variable terms with the same coefficient, 10, with opposite signs:

$$\begin{aligned} 5 \cdot (2x - 5y) &= (-13) \cdot 5 \\ -2 \cdot (5x - 3y) &= (-4) \cdot -2 \end{aligned} \tag{8.14.2}$$

Notice the x variable terms have the same coefficients with opposite signs:

$$\begin{aligned} 10x - 25y &= -65 \\ -10x + 6y &= 8 \end{aligned} \tag{8.14.3}$$

Step 3. Now we can add and eliminate x:

$$\begin{array}{rl} 10x - 25y = -65 & \\ + \quad -10x + 6y = 8 & \text{Add the equations} \\ \hline -19y = -57 & \end{array}$$

Notice the x variable terms canceled and we are left with one equation in one variable. This is always the goal. Now, we can easily solve as usual:

$$\begin{array}{rl} -19y = -57 & \text{Multiply by the reciprocal of } -19 \\ y = 3 & y\text{-coordinate of the solution} \end{array}$$

Step 4. Since $y = 3$, then we can plug-n-chug $y = 3$ into one of the equations to obtain x:

$$\begin{array}{rl} 5x - 3y = -4 & \text{Plug-n-chug } y = 3 \\ 5x - 3(3) = -4 & \text{Evaluate} \\ 5x - 9 = -4 & \text{Isolate the variable term} \\ 5x = 5 & \text{Multiply by the reciprocal of 5} \\ x = 1 & x\text{-coordinate of the solution} \end{array}$$

The solution to the system is the ordered-pair $(1,3)$. Furthermore, if we were to graph these two lines, we know they would intersect at $(1,3)$. Also, we know this system is a consistent system with an independent solution.

Note

The famous mathematical text, *The Nine Chapters on the Mathematical Art*, which was printed around 179 AD in China, describes a process very similar to Gaussian elimination which is very similar to the addition method.

Addition: Special Cases

Just as with graphing and substitution, it is possible to have no solution or infinite solutions with elimination. Just as with substitution, if the variables eliminate, a true statement will indicate infinitely many solutions and a false statement will indicate

there is no solution.

Example 4.3.5

Solve the system by addition (elimination):

$$\begin{cases} 2x - 5y = 3 \\ -6x + 15y = -9 \end{cases}$$

Solution

Since none of the variable terms have the same coefficient, but both have opposite signs, we can choose any variable to eliminate.

Step 1. Let's choose to eliminate y and we multiply both equations by a factor to obtain the $\text{LCM}(5, 15) = 15$ with opposite signs.

Step 2. We can mulitply the first equation by a factor of 3 and leave the second equation alone so that we obtain variable terms with the same coefficient, 15, with opposite signs:

$$\begin{aligned} 3 \cdot (2x - 5y) &= (3) \cdot 3 \quad \text{Distribute} \\ 6x - 15y &= 9 \end{aligned}$$

Notice the y variable terms have the same coefficients with opposite signs

$$\begin{aligned} 6x - 5y &= 9 \\ -6x + 15y &= -9 \end{aligned}$$

Step 3. Now we can add and eliminate x:

$$\begin{aligned} 6x - 15y &= 9 \\ + \quad -6x + 15y &= -9 \qquad \text{Add the equations} \\ \hline 0 &= 0 \end{aligned}$$

Since all the variables cancel and we are left with a statement without variables, we ask," Is this statement true?"

$$\begin{aligned} 0 &\stackrel{?}{=} 0 \quad \text{Is this true?} \\ 0 &= 0 \quad \checkmark \text{ True} \end{aligned}$$

Since this statement is true, then there are infinitely many solutions to this system. Furthermore, if we were to graph these two lines, we know they would be the same line. Hence, this system is a consistent system with a dependent solution.

Example 4.3.6

Solve the system by addition (elimination):

$$\begin{cases} 4x - 6y = 8 \\ 6x - 9y = 15 \end{cases}$$

Solution

Since none of the variable terms have the same coefficient, opposite signs, or both, we need to choose a variable and rewrite the equations so we can cancel the variable. We can choose any variable to eliminate.

Step 1. Let's choose to eliminate x and we multiply both equations by a factor to obtain the $\text{LCM}(4, 6) = 12$ with opposite signs.

Step 2. We can multiply the first equation by a factor of 3 and the second equation by -2 so that we obtain variable terms with the same coefficient, 12, with opposite signs:

$$\begin{aligned} 3 \cdot (4x - 6y) &= (8) \cdot 3 \\ -2 \cdot (6x - 9y) &= (15) \cdot -2 \end{aligned} \tag{8.14.4}$$

Notice the y variable terms have the same coefficients with opposite signs:

$$\begin{aligned} 12x - 18y &= 24 \\ -12x + 18y &= -20 \end{aligned}$$

Step 3. Now we can add and eliminate x:

$$\begin{array}{rl} & 12x - 18y = 24 \\ + & -12x + 18y = -20 \quad \text{Add the equations} \\ \hline & 0 = 4 \end{array}$$

Since all the variables cancel and we are left with a statement without variables, we ask," Is this statement true?"

$$\begin{aligned} 0 &\stackrel{?}{=} 4 \quad \text{Is this true?} \\ 0 &\neq 4 \quad X \text{ False} \end{aligned}$$

Since this statement is false, then there is no solution to this system. Furthermore, if we were to graph these two lines, we know they would be parallel. Hence, this system is an inconsistent system.

We discussed three different methods that can be used to solve a system of two equations in two variables. While all three can be used to solve any system, graphing works great for small integer solutions. Substitution works great when we have a given variable term with a coefficient of one, and addition works great for all other cases. As each method has its own strengths, it is important to be familiar with all three methods. Next, we use these methods to solve application problems.

System of Equations: The Addition Method Homework

Solve each system by addition (elimination). Determine if each system is consistent, independent or dependent, or inconsistent.

Exercise 4.3.1

$$\begin{aligned} 4x + 2y &= 0 \\ -4x - 9y &= -28 \end{aligned}$$

Exercise 4.3.2

$$\begin{aligned} -9x + 5y &= -22 \\ 9x - 5y &= 13 \end{aligned}$$

Exercise 4.3.3

$$\begin{aligned} -6x + 9y &= 3 \\ 6x - 9y &= -9 \end{aligned}$$

Exercise 4.3.4

$$4x - 6y = -10$$
$$4x - 6y = -14$$

Exercise 4.3.5

$$-x - 5y = 28$$
$$-x + 4y = -17$$

Exercise 4.3.6

$$2x - y = 5$$
$$5x + 2y = -28$$

Exercise 4.3.7

$$10x + 6y = 24$$
$$-6x + y = 4$$

Exercise 4.3.8

$$2x + 4y = 24$$
$$4x - 12y = 8$$

Exercise 4.3.9

$$-7x + 4y = -4$$
$$10x - 8y = -8$$

Exercise 4.3.10

$$5x + 10y = 20$$
$$-6x - 5y = -3$$

Exercise 4.3.11

$$-7x - 3y = 12$$
$$-6x - 5y = 20$$

Exercise 4.3.12

$$9x - 2y = -18$$
$$5x - 7y = -10$$

Exercise 4.3.13

$$9x + 6y = -21$$
$$-10x - 9y = 28$$

Exercise 4.3.14

$$-7x + 5y = -8$$
$$-3x - 3y = 12$$

Exercise 4.3.15

$$-8x - 8y = -8$$
$$10x + 9y = 1$$

Exercise 4.3.16

$$9y = 7 - x$$
$$-18y + 4x = -26$$

Exercise 4.3.17

$$0 = 9x + 5y$$
$$y = \frac{2}{7}x$$

Exercise 4.3.18

$$-7x + y = -10$$
$$-9x - y = -22$$

Exercise 4.3.19

$$-x - 2y = -7$$
$$x + 2y = 7$$

Exercise 4.3.20

$5x - 5y = -15$
$5x - 5y = -15$

Exercise 4.3.21

$-3x + 3y = -12$
$-3x + 9y = -24$

Exercise 4.3.22

$-10x - 5y = 0$
$-10x - 10y = -30$

Exercise 4.3.23

$-5x + 6y = -17$
$x - 2y = 5$

Exercise 4.3.24

$x + 3y = -1$
$10x + 6y = -10$

Exercise 4.3.25

$-6x + 4y = 12$
$12x + 6y = 18$

Exercise 4.3.26

$-6x + 4y = 4$
$-3x - y = 26$

Exercise 4.3.27

$-9x - 5y = -19$
$3x - 7y = -11$

Exercise 4.3.28

$$-5x + 4y = 4$$
$$-7x - 10y = -10$$

Exercise 4.3.29

$$3x + 7y = -8$$
$$4x + 6y = -4$$

Exercise 4.3.30

$$-4x - 5y = 12$$
$$-10x + 6y = 30$$

Exercise 4.3.31

$$8x + 7y = -24$$
$$6x + 3y = -18$$

Exercise 4.3.32

$$-7x + 10y = 13$$
$$4x + 9y = 22$$

Exercise 4.3.33

$$0 = -9x - 21 + 12y$$
$$1 + \frac{4}{3}y + \frac{7}{3}x = 0$$

Exercise 4.3.34

$$-6 - 42y = -12x$$
$$x - \frac{1}{2} - \frac{7}{2}y = 0$$

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8.15: Applications with systems of equations

We saw these types of examples in a previous chapter, but with one variable. In this section, we review the same types of applications, but solving in a more sophisticated way using systems of equations. Once we set up the system, we can solve using any method we choose. However, setting up the system may be the challenge, but as long as we follow the method we used before, we will be fine. We use tables to organize the parameters.

Value & Interest Problems

Example 4.4.1

There were 41 tickets sold for an event. Tickets for children cost $\$1.50$ and tickets for adults cost $\$2.00$. Total receipts for the event were $\$73.50$. How many of each type of ticket were sold?

Solution

First, we can make a table to organize the given information and then create an equation. Let c represent the number of children tickets sold and a represent the number of adult tickets sold.

Table 4.4.1

	Amount	Value (in $)	Total Value
Adult tickets	a	$\$2$	$2a$
Children tickets	c	$\$1.50$	$1.5c$
Total	41		$\$73.50$

Now let's set up the system. The total number of tickets sold was 41 and the total revenue from the tickets was $\$73.50$, and so we obtain the system

$$\begin{aligned} a+c &= 41 \\ 2a+1.5c &= 73.50 \end{aligned} \tag{8.15.1}$$

At this point, we can solve using any method we choose. Since the coefficient of a and c in the first equation are both one, then let's use the method of substitution. We will solve for a in the first equation:

$$\begin{aligned} a+c &= 41 \\ a &= 41-c \end{aligned} \tag{8.15.2}$$

Now, we can substitute a into the second equation and solve:

$$\begin{aligned} 2a+1.5c &= 73.50 && \text{Plug-n-chug } a=41-c \\ 2(41-c)+1.5c &= 73.50 && \text{Distribute} \\ 82-2c+1.5c &= 73.50 && \text{Combine like terms} \\ 82-0.5c &= 73.50 && \text{Isolate the variable term} \\ -0.5c &= -8.50 && \text{Multiply by the reciprocal of } -0.5 \\ c &= 17 && \text{Number of children tickets} \end{aligned}$$

Since $c=17$, then we can plug-n-chug $c=17$ into one of the equations to obtain a:

$$\begin{aligned} a &= 41-c && \text{Plug-n-chug } c=17 \\ a &= 41-(17) && \text{Evaluate} \\ a &= 24 && \text{Number of adult tickets} \end{aligned}$$

Thus, there were 17 children tickets and 24 adult tickets sold.

Example 4.4.2

Aaron invests $\$9,700$ in two different accounts. The first account paid 7%, the second account paid 11% in interest. At the end of the first year he had earned $\$863$ in interest. How much was in each account?

Solution

First, we can make a table to organize the given information and then create an equation. Let x represent the amount of investment in the first account and y represent the amount of investment in the second account.

Table 4.4.2

	Principal	rate	Total Interest
Account 1	x	0.07	$0.07x$
Account 2	y	0.11	$0.11y$
Total	$\$9,700$		$\$863$

Now let's set up the system. The total interest made in the one year was $863 and the total invested was $9,700, so we obtain the system

$$\begin{aligned} x + y &= 9700 \\ 0.07x + 0.11y &= 863 \end{aligned} \tag{8.15.3}$$

At this point, we can solve using any method we choose. Since the coefficient of x and y in the first equation are both one, then let's use the method of substitution. We will solve for y in the first equation:

$$\begin{aligned} x + y &= 9700 \\ y &= 9700 - x \end{aligned} \tag{8.15.4}$$

Now, we can substitute y into the second equation and solve:

$$\begin{aligned} 0.07x + 0.11(9700 - x) &= 863 && \text{Distribute} \\ 0.07x + 1067 - 0.11x &= 863 && \text{Combine like terms} \\ -0.04x + 1067 &= 863 && \text{Isolate the variable term} \\ -0.04x &= -204 && \text{Multiply by the reciprocal of } -0.04 \\ x &= 5100 && \text{Investment ammount for Account 1} \end{aligned}$$

Since the investment amount for Account 1 was $\$5,100$, then the investment amount for Account 2 was $\$4,600$ $(9700 - 5100 = 4600)$. Thus, the investment amounts for Account 1 and Account 2 was $\$5,100$ and $\$4,600$, respectively.

Mixture Problems

Example 4.4.3

A farmer has two types of milk, one that is 24% butterfat and another which is 18% butterfat. How much of each should he use to end up with 42 gallons of 20% butterfat?

Solution

First, we can make a table to organize the given information and then create a system. Let x represent the number of gallons of the 24% butterfat milk and y represent the number of gallons of the 18% butterfat milk.

Table 4.4.3

	Amount	Concentration	Total Butterfat
24% butterfat	x	0.24	$0.24x$
18% butterfat	y	0.18	$0.18y$
20% butterfat	42	0.20	$0.20(42)$

Now let's set up the system:

$$\begin{aligned} x + y &= 42 \\ 0.24x + 0.18y &= 8.4 \end{aligned} \tag{8.15.5}$$

At this point, we can solve using any method we choose. Let's solve using elimination. We can choose to eliminate x and we will multiply the first equation by -0.24:

$$\begin{aligned} -0.24 \cdot (x + y) &= (42) \cdot -0.24 \quad \text{Distribute} \\ -0.24x - 0.24y &= -10.08 \end{aligned}$$

Notice the x variable terms have the same coefficients with opposite signs. Now we can add and eliminate x:

$$\begin{aligned} -0.24x - 0.24y &= -10.08 \\ +\quad 0.24x + 0.18y &= 8.4 \qquad \text{Add the equations} \\ \hline -0.06y &= -1.68 \end{aligned}$$

Now, we can easily solve as usual:

$$\begin{aligned} -0.06y &= -1.68 \quad \text{Multiply by the reciprocal of } -0.06 \\ y &= 28 \quad \text{Number of gallons from the 18} \end{aligned}$$

Since the number of gallons from the 18% butterfat milk was 28, then the number of gallons from the 24% butterfat milk was $14 (42 - 28 = 14)$. Thus, the farmer will need 14 gallons of the 24% butterfat milk and 28 gallons of the 18% butterfat milk to make 42 gallons of a 20% butterfat milk.

Example 4.4.4

A solution of pure antifreeze is mixed with water to make a 65% antifreeze solution. How much of each should be used to make 70 liters?

Solution

Mixture problems with a pure solution or water contains no other chemicals. For pure solutions, the percentage is 100% (or 1 in the table) and for water, the percentage is 0%. First, we can make a table to organize the given information and then create a system. Let a represent the number of liters of antifreeze and w represent the number of liters of water.

Table 4.4.4

	Amount	Concentration	Total Butterfat
Antifreeze	a	1	$1a$
Water	w	0	0
65% Solution	70	0.65	$0.65(70)$

Now let's set up the system:

$$\begin{aligned} a + w &= 70 \\ 1a &= 45.5 \end{aligned} \tag{8.15.6}$$

At this point, we can solve using any method we choose. Since we see from the system that $a = 45.5$, then let's solve by substitution. We can put the second equation in for a in the first equation:

$$\begin{aligned} a + w &= 70 && \text{Plug-n-chug } a = 45.5 \\ 45.5 + w &= 70 && \text{Isolate the variable term} \\ w &= 24.5 && \text{The number of liters in water} \end{aligned}$$

Thus, the number of liters of water needed is 24.5 liters and antifreeze needed is 45.5 liters.

Example 4.4.5

In a candy shop, chocolate, which sells for $\$4$ per pound, is mixed with nuts, which are sold for $\$2.50$ per pound, to form a chocolate-nut candy which sells for $\$3.50$ a pound. How many pounds of each are used to make 30 pounds of the mixture?

Solution

First, we can make a table to organize the given information and then create an equation. Let c represent the number of pounds of chocolate and n represent the number of pounds of nuts.

Table 4.4.5

	Amount	Cost	Total Cost
Chocolate	c	$\$4$	$4c$
Nuts	n	$\$2.50$	$2.5n$
Mix	30	$\$3.50$	$3.5(30)$

Now let's set up the system:

$$\begin{aligned} c + n &= 30 \\ 4c + 2.5n &= 105 \end{aligned} \tag{8.15.7}$$

At this point, we can solve using any method we choose. Let's solve using elimination. We can choose to eliminate c and we will multiply the first equation by -4:

$$\begin{aligned} -4 \cdot (c + n) &= (3) \cdot -4 && \text{Distribute} \\ -4c - 4n &= -120 \end{aligned}$$

Notice the c variable terms have the same coefficients with opposite signs. Now we can add and eliminate c:

$$\begin{aligned} -4c - 4n &= -120 \\ +\quad 4c + 2.5n &= 105 && \text{Add the equations} \\ \hline -1.5n &= -15 \end{aligned}$$

Now, we can easily solve as usual:

$$\begin{aligned} -1.5n &= -15 && \text{Multiply by the reciprocal of } -1.5 \\ n &= 10 && \text{Number of pounds of nuts} \end{aligned}$$

Since we need 10 pounds of nuts, then this implies that we need 20 pounds of chocolate $(30 - 10 = 20)$.

Uniform Motion with Unknown Rates

When we looked at uniform motion in a prior chapter, we always were given something about the rate. However, now we discuss uniform motion with a wind force and a water current rates, where we know very little and will have to use a system to solve for the rates.

Example 4.4.6

Turkey the Pigeon travels the same distance of 72 miles in 4 hours against the wind as it does traveling 3 hours with the wind in local skies. What is the rate of Turkey the Pigeon in still air and the rate of the wind?

Solution

First, we can make a table to organize the given information and then create an equation. Let r represent the rate of Turkey in still air and w represent the rate of the wind. If Turkey travels with the wind, then Turkey is getting a little push from the wind, meaning traveling a little faster. If Turkey travels against the wind, then Turkey is getting a little push back from the wind, meaning traveling a little slower.

Table 4.4.6

	rate	time	distance
with the wind	$r+w$	3	$3(r+w)$
against the wind	$r-w$	4	$4(r-w)$

Since Turkey is traveling a distance of 72 miles, then this is the distance for both routes. Now let's set up the system:

$$\begin{aligned} 3(r+w) &= 72 \\ 4(r-w) &= 72 \end{aligned} \tag{8.15.8}$$

Since 3 is a factor of 72 and 4 is a factor of 72, let's divide each side of each equation:

$$\begin{aligned} \frac{3(r+w)}{3} &= \frac{72}{3} \\ r+w &= 24 \end{aligned}$$

$$\begin{aligned} \frac{4(r-w)}{4} &= \frac{72}{4} \\ r-w &= 18 \end{aligned}$$

Notice the w variable terms have the same coefficients with opposite signs. Now we can add and eliminate w:

$$\begin{aligned} r+w &= 24 \\ +\quad r-w &= 18 \qquad \text{Add the equations} \\ \hline 2r &= 42 \end{aligned}$$

Now, we can easily solve as usual:

$$\begin{aligned} 2r &= 42 \quad \text{Multiply by the reciprocal of 2} \\ r &= 21 \quad \text{Rate of Turkey in still air} \end{aligned}$$

Since Turkey's rate is 21 miles per hour in still air, then this implies that the rate of the wind is 3 miles per hour $(24-21=3)$.

Example 4.4.7

A boat travels upstream for 156 miles in 3 hours and returns in 2 hours traveling downstream in a local stream of water. What is the rate of the boat in still water and the rate of the current?

Solution

First, we can make a table to organize the given information and then create an equation. Let r represent the rate of boat in still water and c represent the rate of the current. If the boat travels with the current, then the boat is getting a little push from the current, meaning traveling a little faster. If the boat travels against the current, then the boat is getting a little push back from the current, meaning traveling a little slower.

Table 4.4.7

	rate	time	distance
with the current	$r+c$	2	$2(r+c)$
against the current	$r-c$	3	$3(r-c)$

Since the boat is traveled a distance of 156 miles, then this is the distance for both routes. Now let's set up the system:

$$\begin{aligned} 2(r+c) &= 156 \\ 3(r-c) &= 156 \end{aligned} \tag{8.15.9}$$

Since 2 is a factor of 156 and 3 is a factor of 156, let's divide each side of each equation:

$$\begin{aligned} \frac{2(r+c)}{2} &= \frac{156}{2} \\ r+c &= 78 \end{aligned}$$

$$\begin{aligned} \frac{3(r-c)}{3} &= \frac{156}{3} \\ r-c &= 32 \end{aligned}$$

Notice the c variable terms have the same coefficients with opposite signs. Now we can add and eliminate c:

$$\begin{aligned} r+c &= 78 \\ +\quad r-c &= 52 \qquad \text{Add the equations} \\ 2r &= 130 \end{aligned}$$

Now, we can easily solve as usual:

$$\begin{aligned} 2r &= 130 \quad \text{Multiply by the reciprocal of 2} \\ r &= 65 \quad \text{Rate of the boat in still water} \end{aligned}$$

Since the rate of the boat in still water is 65 miles per hour, then this implies that the rate of the current is 13 miles per hour $(65-52=13)$.

Applications with Systems of Equations Homework

Exercise 4.4.1

The attendance at a school concert was 578. Admission was $\$2.00$ for adults and $\$1.50$ for children. The total receipts were $\$985.00$. How many adults and how many children attended?

Exercise 4.4.2

There were 429 people at a play. Admission was $\$1$ each for adults and 75 cents each for children. The receipts were $\$372.50$. How many children and how many adults attended?

Exercise 4.4.3

There were 200 tickets sold for a women's basketball game. Tickets forstudents were 50 cents each and for adults 75 cents each. The total amount of money collected was $\$132.50$. How many of each type of ticket was sold?

Exercise 4.4.4

There were 203 tickets sold for a volleyball game. For activity-card holders, the price was $\$1.25$ each and for non-card holders the price was $\$2$ each. The total amount of money collected was $\$310$. How many of each type of ticket was sold?

Exercise 4.4.5

At a local ball game the hot dogs sold for $\$2.50$ each and the hamburgers sold for $\$2.75$ each. There were 131 total hamburgers and hot dogs sold for a total value of $\$342$. How many of each was sold?

Exercise 4.4.6

At a recent Vikings game, there was $\$445$ in admission tickets. The cost of a student ticket was $\$1.50$ and the cost of a non-student ticket was $\$2.50$. A total of 232 tickets were sold. How many students and how many non-students attended the game?

Exercise 4.4.7

A total of $\$27,000$ is invested, part of it at 12% and the rest at 13%. The total interest after one year is $\$3,385$. How much was invested at each rate?

Exercise 4.4.8

A total of $\$50,000$ is invested, part of it at 5% and the rest at 7.5%. The total interest after one year is $\$3,250$. How much was invested at each rate?

Exercise 4.4.9

A total of $\$9,000$ is invested, part of it at 10% and the rest at 12%. The total interest after one year is $\$1,030$. How much was invested at each rate?

Exercise 4.4.10

A total of $\$18,000$ is invested, part of it at 6% and the rest at 9%. The total interest after one year is $\$1,248$. How much was invested at each rate?

Exercise 4.4.11

An inheritance of $\$10,000$ is invested in 2 ways, part at 9.5 and the remainder at 11%. The combined annual interest was $\$1,038.50$. How much was invested at each rate?

Exercise 4.4.12

Kerry earned a total of $\$900$ last year on his investments. If $\$7,000$ was invested at a certain rate of return and $\$9,000$ was invested in a fund with a rate that was 2% higher, find the two rates of interest.

Exercise 4.4.13

Jason earned $\$256$ interest last year on his investments. If $\$1,600$ was invested at a certain rate of return and $\$2,400$ was invested in a fund with a rate that was double the rate of the first fund, find the two rates of interest.

Exercise 4.4.14

Millicent earned $\$435$ last year in interest. If $\$3,000$ was invested at a certain rate of return and $\$4,500$ was invested in a fund with a rate that was 2% lower, find the two rates of interest.

Exercise 4.4.15

A total of $\$8,500$ is invested, part of it at 6% and the rest at 3.5%. The total interest after one year is $\$385$. How much was invested at each rate?

Exercise 4.4.16

A total of $\$12,000$ was invested, part of it at 9% and the rest at 7.5%. The total interest after one year is $\$1,005$. How much was invested at each rate?

Exercise 4.4.17

A total of $\$15,000$ is invested, part of it at 8% and the rest at 11%. The total interest after one year is $\$1,455$. How much was invested at each rate?

Exercise 4.4.18

A total of $\$17,500$ is invested, part of it at 7.25% and the rest at 6.5%. The total interest after one year is $\$1,227.50$. How much was invested at each rate?

Exercise 4.4.19

A total of $\$6,000$ is invested, part of it at 4.25% and the rest at 5.75%. The total interest after one year is $\$300$. How much was invested at each rate?

Exercise 4.4.20

A total of $\$14,000$ is invested, part of it at 5.5% and the rest at 9%. The total interest after one year is $\$910$. How much was invested at each rate?

Exercise 4.4.21

A total of $\$11,000$ is invested, part of it at 6.8% and the rest at 8.2%. The total interest after one year is $\$797$. How much was invested at each rate?

Exercise 4.4.22

An investment portfolio earned $\$2,010$ in interest last year. If $\$3,000$ was invested at a certain rate of return and $\$24,000$ was invested in a fund with a rate that was 4% lower, find the two rates of interest.

Exercise 4.4.23

Samantha earned $\$1,480$ in interest last year on her investments. If $\$5,000$ was invested at a certain rate of return and $\$11,000$ was invested in a fund with a rate that was two-thirds the rate of the first fund, find the two rates of interest.

Exercise 4.4.24

Solution A is 50% acid and solution B is 80% acid. How much of each should be used to make 100 cc. of a solution that is 68 % acid?

Exercise 4.4.25

A certain grade of milk contains 10% butterfat and a certain grade of cream 60% butterfat. How many quarts of each must be taken so as to obtain a mixture of 100 quarts that will be 45% butterfat?

Exercise 4.4.26

A farmer has some cream which is 21% butterfat and some which is 15% butterfat. How many gallons of each must be mixed to produce 60 gallons of cream which is 19% butterfat?

Exercise 4.4.27

A syrup manufacturer has some pure maple syrup and some which is 85% maple syrup. How many liters of each should be mixed to make 150 liters which is 96% maple syrup?

Exercise 4.4.28

A chemist wants to make 50 mL of a 16% acid solution by mixing a 13% acid solution and an 18% acid solution. How many milliliters of each solution should the chemist use?

Exercise 4.4.29

A hair dye is made by blending 7% hydrogen peroxide solution and a 4% hydrogen peroxide solution. How many milliliters of each are used to make a 300 mL solution that is 5% hydrogen peroxide?

Exercise 4.4.30

A paint that contains 21% green dye is mixed with a paint that contains 15% green dye. How many gallons of each must be used to make 60 gallons of paint that is 19% green dye?

Exercise 4.4.31

A candy mix sells for $\$2.20$ per kilogram. It contains chocolates worth $\$1.80$ per kilogram and other candy worth $\$3.00$ per kilogram. How much of each are in 15 kilograms of the mixture?

Exercise 4.4.32

To make a weed and feed mixture, the Green Thumb Garden Shop mixes fertilizer worth $\$4$ per pound with a weed killer worth $\$8$ per pound. The mixture will cost $\$6.00$ per pound. How much of each should be used to prepare 500 pounds of the mixture?

Exercise 4.4.33

A grocer is mixing a 40-cent per pound coffee with a 60-cent per pound coffee to make a mixture worth $54¢$ per pound How much of each kind of coffee should be used to make 70 pounds of the mixture?

Exercise 4.4.34

A grocer wishes to mix sugar at 9 cents per pound with sugar at 6 cents per pound to make 60 pounds at 7 cents per pound. What quantity of each must he take?

Exercise 4.4.35

A high-protein diet supplement that costs $\$6.75$ per pound is mixed with a vitamin supplement that costs $\$3.25$ per pound. How many pounds of each should be used to make 5 pounds of a mixture that costs $\$4.65$ per pound?

Exercise 4.4.36

A goldsmith combined an alloy that costs $\$4.30$ per ounce with an alloy that costs $\$1.80$ per ounce. How many ounces of each were used to make a mixture of 200 ounces costing $\$2.50$ per ounce?

Exercise 4.4.37

A grocery store offers a cheese and fruit sampler that combines cheddar cheese that costs $\$8$ per kilogram with kiwis that cost $\$3$ per kilogram. How many kilograms of each were used to make a 5 kilogram mixture that costs $\$4.50$ per kilogram?

Exercise 4.4.38

A caterer made an ice cream punch by combining fruit juice that cost $\$2.25$ per gallon with ice cream that costs $\$3.25$ per gallon. How many gallons of each were used to make 100 gallons of punch costing $\$2.50$ per pound?

Exercise 4.4.39

A clothing manufacturer has some pure silk thread and some thread that is 85% silk. How many kilograms of each must be woven together to make 75 kilograms of cloth that is 96% silk?

Exercise 4.4.40

A carpet manufacturer blends two fibers, one 20% wool and the second 50% wool. How many pounds of each fiber should be woven together to produce 600 pounds of a fabric that is 28% wool?

Exercise 4.4.41

The manager of a specialty food store combined almonds that cost $\$4.50$ per pound with walnuts that cost $\$2.50$ per pound. How many pounds of each were used to make a 100-pound mixture that cost $\$3.24$ per pound?

Exercise 4.4.42

A tea that is 20% jasmine is blended with a tea that is 15% jasmine. How many pounds of each tea are used to make 5 pounds of tea that is 18% jasmine?

Exercise 4.4.43

How many milliliters of pure chocolate must be added to 150 mL of chocolate topping that is 50% chocolate to make a topping that is 75% chocolate?

Exercise 4.4.44

How many ounces of pure bran flakes must be added to 50 ounces of cereal that is 40% bran flakes to produce a mixture that is 50% bran flakes?

Exercise 4.4.45

A ground meat mixture is formed by combining meat that costs $\$2.20$ per pound with meat that costs $\$4.20$ per pound. How many pounds of each were used to make a 50-pound mixture that costs $\$3$ per pound?

Exercise 4.4.46

How many grams of pure water must be added to 50 grams of pure acid to make a solution that is 40% acid?

Exercise 4.4.47

A lumber company combined oak wood chips that cost $\$3.10$ per pound with pine wood chips that cost $\$2.50$ per pound. How many pounds of each were used to make an 80-pound mixture costing $\$2.65$ per pound?

Exercise 4.4.48

How many ounces of pure water must be added to 50 ounces of a 15% saline solution to make a saline solution that is 10% salt?

Exercise 4.4.49

A boat travels upstream for 216 miles in 4 hours and returns in 3 hours traveling downstream in a local stream of water. What is the rate of the boat in still water and the rate of the current?

Exercise 4.4.50

A boat travels upstream for 12 miles in 3 hours and returns in 2 hours traveling downstream in a local stream of water. What is the rate of the boat in still water and the rate of the current?

Exercise 4.4.51

A boat travels upstream for 336 miles in 4 hours and returns in 3 hours traveling downstream in a local stream of water. What is the rate of the boat in still water and the rate of the current?

Exercise 4.4.52

Turkey the Pigeon travels the same distance of 280 miles in 5 hours against the wind as it does traveling 4 hours with the wind in local skies. What is the rate of Turkey the Pigeon in still air and the rate of the wind?

Exercise 4.4.53

Turkey the Pigeon travels the same distance of 24 miles in 4 hours against the wind as it does traveling 3 hours with the wind in local skies. What is the rate of Turkey the Pigeon in still air and the rate of the wind?

Exercise 4.4.54

Turkey the Pigeon travels the same distance of 120 miles in 4 hours against the wind as it does traveling 3 hours with the wind in local skies. What is the rate of Turkey the Pigeon in still air and the rate of the wind?

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9: PROBABILITY TOPICS

Probability theory is concerned with probability, the analysis of random phenomena. The central objects of probability theory are random variables, stochastic processes, and events: mathematical abstractions of non-deterministic events or measured quantities that may either be single occurrences or evolve over time in an apparently random fashion.

9.1: Introduction

9.2: Terminology

9.3: Independent and Mutually Exclusive Events

9.4: Two Basic Rules of Probability

9.5: Contingency Tables

9.6: Tree and Venn Diagrams

9.7: Probability Topics (Worksheet)

9.E: Probability Topics (Exericses)

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9.1: Introduction

This is a photo taken of the night sky. A meteor and its tail are shown entering the earth's atmosphere.Figure 9.1.1. Meteor showers are rare, but the probability of them occurring can be calculated. (credit: Navicore/flickr)

CHAPTER OBJECTIVES

By the end of this chapter, the student should be able to:

- Understand and use the terminology of probability.
- Determine whether two events are mutually exclusive and whether two events are independent.
- Calculate probabilities using the Addition Rules and Multiplication Rules.
- Construct and interpret Contingency Tables.
- Construct and interpret Venn Diagrams.
- Construct and interpret Tree Diagrams.

It is often necessary to "guess" about the outcome of an event in order to make a decision. Politicians study polls to guess their likelihood of winning an election. Teachers choose a particular course of study based on what they think students can comprehend. Doctors choose the treatments needed for various diseases based on their assessment of likely results. You may have visited a casino where people play games chosen because of the belief that the likelihood of winning is good. You may have chosen your course of study based on the probable availability of jobs.

You have, more than likely, used probability. In fact, you probably have an intuitive sense of probability. Probability deals with the chance of an event occurring. Whenever you weigh the odds of whether or not to do your homework or to study for an exam, you are using probability. In this chapter, you will learn how to solve probability problems using a systematic approach.

Collaborative Exercise

Your instructor will survey your class. Count the number of students in the class today.

- Raise your hand if you have any change in your pocket or purse. Record the number of raised hands.
- Raise your hand if you rode a bus within the past month. Record the number of raised hands.
- Raise your hand if you answered "yes" to BOTH of the first two questions. Record the number of raised hands.

Use the class data as estimates of the following probabilities. $P(\text{change})$ means the probability that a randomly chosen person in your class has change in his/her pocket or purse. $P(\text{bus})$ means the probability that a randomly chosen person in your class rode a bus within the last month and so on. Discuss your answers.

- Find $P(\text{change})$.
- Find $P(\text{bus})$.
- Find $P(\text{change AND bus})$. Find the probability that a randomly chosen student in your class has change in his/her pocket or purse and rode a bus within the last month.
- Find $P(\text{change}|\text{bus})$. Find the probability that a randomly chosen student has change given that he or she rode a bus within the last month. Count all the students that rode a bus. From the group of students who rode a bus, count those who have change. The probability is equal to those who have change and rode a bus divided by those who rode a bus.

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9.2: Terminology

Probability is a measure that is associated with how certain we are of outcomes of a particular experiment or activity. An **experiment** is a planned operation carried out under controlled conditions. If the result is not predetermined, then the experiment is said to be a **chance** experiment. Flipping one fair coin twice is an example of an experiment.

A result of an experiment is called an **outcome**. The **sample space** of an experiment is the set of all possible outcomes. Three ways to represent a sample space are: to list the possible outcomes, to create a tree diagram, or to create a Venn diagram. The uppercase letter S is used to denote the sample space. For example, if you flip one fair coin, $S = \{\text{H}, \text{T}\}$ where $\text{H} =$ heads and $\text{T} =$ tails are the outcomes.

An **event** is any combination of outcomes. Upper case letters like A and B represent events. For example, if the experiment is to flip one fair coin, event A might be getting at most one head. The probability of an event A is written $P(\text{A})$.

Definition: Probability

The *probability* of any outcome is the long-term relative frequency of that outcome. Probabilities are between zero and one, inclusive (that is, zero and one and all numbers between these values).

- $P(\text{A}) = 0$ means the event A can never happen.
- $P(\text{A}) = 1$ means the event A always happens.
- $P(\text{A}) = 0.5$ means the event A is equally likely to occur or not to occur. For example, if you flip one fair coin repeatedly (from 20 to 2,000 to 20,000 times) the relative frequency of heads approaches 0.5 (the probability of heads).

Equally likely means that each outcome of an experiment occurs with equal probability. For example, if you toss a **fair**, six-sided die, each face (1, 2, 3, 4, 5, or 6) is as likely to occur as any other face. If you toss a fair coin, a Head (H) and a Tail (T) are equally likely to occur. If you randomly guess the answer to a true/false question on an exam, you are equally likely to select a correct answer or an incorrect answer.

To calculate the probability of an event *A* when all outcomes in the sample space are equally likely, count the number of outcomes for event A and divide by the total number of outcomes in the sample space. For example, if you toss a fair dime and a fair nickel, the sample space is $\{\text{HH}, \text{TH}, \text{HT}, \text{TT}\}$ where $\text{T} =$ tails and $\text{H} =$ heads. The sample space has four outcomes. $\text{A} =$ getting one head. There are two outcomes that meet this condition $\{\text{HT}, \text{TH}\}$, so $P(\text{A}) = \frac{2}{4} = 0.5$.

Suppose you roll one fair six-sided die, with the numbers {1, 2, 3, 4, 5, 6} on its faces. Let event $\text{E} =$ rolling a number that is at least five. There are two outcomes {5, 6}. $P(\text{E}) = \frac{2}{6}$. If you were to roll the die only a few times, you would not be surprised if your observed results did not match the probability. If you were to roll the die a very large number of times, you would expect that, overall, $\frac{2}{6}$ of the rolls would result in an outcome of "at least five". You would not expect exactly $\frac{2}{6}$. The long-term relative frequency of obtaining this result would approach the theoretical probability of $\frac{2}{6}$ as the number of repetitions grows larger and larger.

Definition: Law of Large Numbers

This important characteristic of probability experiments is known as the law of large numbers which states that as the number of repetitions of an experiment is increased, the relative frequency obtained in the experiment tends to become closer and closer to the theoretical probability. Even though the outcomes do not happen according to any set pattern or order, overall, the long-term observed relative frequency will approach the theoretical probability. (The word **empirical** is often used instead of the word observed.)

It is important to realize that in many situations, the outcomes are not equally likely. A coin or die may be **unfair**, or **biased**. Two math professors in Europe had their statistics students test the Belgian one Euro coin and discovered that in 250 trials, a head was obtained 56% of the time and a tail was obtained 44% of the time. The data seem to show that the coin is not a fair coin; more repetitions would be helpful to draw a more accurate conclusion about such bias. Some dice may be biased. Look at the

dice in a game you have at home; the spots on each face are usually small holes carved out and then painted to make the spots visible. Your dice may or may not be biased; it is possible that the outcomes may be affected by the slight weight differences due to the different numbers of holes in the faces. Gambling casinos make a lot of money depending on outcomes from rolling dice, so casino dice are made differently to eliminate bias. Casino dice have flat faces; the holes are completely filled with paint having the same density as the material that the dice are made out of so that each face is equally likely to occur. Later we will learn techniques to use to work with probabilities for events that are not equally likely.

The "OR" Event

An outcome is in the event $\mathrm{A\ OR\ B}$ if the outcome is in A or is in B or is in both A and B. For example, let $\mathrm{A} = \{1, 2, 3, 4, 5\}$ and $\mathrm{B} = \{4, 5, 6, 7, 8\}$. $\mathrm{A\ OR\ B} = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Notice that 4 and 5 are NOT listed twice.

The "AND" Event

An outcome is in the event $\mathrm{A\ AND\ B}$ if the outcome is in both A and B at the same time. For example, let A and B be {1, 2, 3, 4, 5} and {4, 5, 6, 7, 8}, respectively. Then $\mathrm{A\ AND\ B} = 4, 5$.

The **complement** of event A is denoted A' (read "A prime"). A' consists of all outcomes that are **NOT** in A. Notice that

$$P(\mathrm{A}) + P(\mathrm{A}') = 1.$$

For example, let $\mathrm{S} = \{1, 2, 3, 4, 5, 6\}$ and let $\mathrm{A} = 1, 2, 3, 4$. Then, $\mathrm{A}' = 5, 6$ and $P(A) = \frac{4}{6}$, $P(\mathrm{A}') = \frac{2}{6}$, and

$$P(\mathrm{A}) + P(\mathrm{A}') = \frac{4}{6} + \frac{2}{6} = 1.$$

The conditional probability of A given B is written $P(\mathrm{A}|\mathrm{B})$. $P(\mathrm{A}|\mathrm{B})$ is the probability that event A will occur given that the event B has already occurred. **A conditional reduces the sample space**. We calculate the probability of A from the reduced sample space B. The formula to calculate $P(\mathrm{A}|\mathrm{B})$ is

$$P(\mathrm{A}|\mathrm{B}) = \frac{\mathrm{P(A\ AND\ B)}}{\mathrm{P(B)}}$$

where $P(\mathrm{B})$ is greater than zero.

For example, suppose we toss one fair, six-sided die. The sample space $\mathrm{S} = \{1, 2, 3, 4, 5, 6\}$. Let A = face is 2 or 3 and B = face is even (2, 4, 6). To calculate $P(\mathrm{A}|\mathrm{B})$, we count the number of outcomes 2 or 3 in the sample space $\mathrm{B} = \{2, 4, 6\}$. Then we divide that by the number of outcomes B (rather than S).

We get the same result by using the formula. Remember that S has six outcomes.

$$\begin{aligned} P(\mathrm{A}|\mathrm{B}) &= \frac{\mathrm{P(A\ AND\ B)}}{P(\mathrm{B})} \\ &= \frac{\frac{\text{the number of outcomes that are 2 or 3 and even in S}}{6}}{\frac{\text{the number of outcomes that are even in S}}{6}} \\ &= \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3} \end{aligned}$$

Understanding Terminology and Symbols

It is important to read each problem carefully to think about and understand what the events are. Understanding the wording is the first very important step in solving probability problems. Reread the problem several times if necessary. Clearly identify the event of interest. Determine whether there is a condition stated in the wording that would indicate that the probability is

conditional; carefully identify the condition, if any.

Example 9.2.1

The sample space S is the whole numbers starting at one and less than 20.

a. $S =$ _____

Let event $A =$ the even numbers and event $B =$ numbers greater than 13.

b. $A =$ _____, $B =$ _____
c. $P(\text{A}) =$ _____, $P(\text{B}) =$ _____
d. A AND B = _____, A OR B = _____
e. $P(\text{A AND B}) =$ _____, $P(\text{A OR B}) =$ _____
f. $\text{A}' =$ _____, $P(\text{A}') =$ _____
g. $P(\text{A}) + P(\text{A}') =$ _____
h. $P(\text{A}|\text{B}) =$ _____, $P(\text{B}|\text{A}) =$ _____; are the probabilities equal?

Answer

a. $\text{S} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$
b. $\text{A} = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}, \text{B} = \{14, 15, 16, 17, 18, 19\}$
c. $P(\text{A}) = \frac{9}{19}$, $P(\text{B}) = \frac{6}{19}$
d. A AND B $= \{14, 16, 18\}$, A OR B $= \{2, 4, 6, 8, 10, 12, 14, 15, 16, 17, 18, 19\}$
e. $P(\text{A AND B}) = \frac{3}{19}$, $P(\text{A OR B}) = \frac{12}{19}$
f. $\text{A}' = 1, 3, 5, 7, 9, 11, 13, 15, 17, 19$; $P(\text{A}') = \frac{10}{19}$
g. $P(\text{A}) + P(\text{A}') = 1 \left(\left(\frac{9}{19} + \frac{10}{19} = 1\right)\right.$
h. $P(\text{A}|\text{B}) = \frac{\text{P(A AND B)}}{\text{P(B)}} = \frac{3}{6}, P(\text{B}|\text{A}) = \frac{\text{P(A AND B)}}{\text{P(A)}} = \frac{3}{9}$, No

Exercise 9.2.1

The sample space S is the ordered pairs of two whole numbers, the first from one to three and the second from one to four (Example: (1, 4)).

a. $S =$ _____
Let event $A =$ the sum is even and event $B =$ the first number is prime.
b. $A =$ _____, $B =$ _____
c. $P(\text{A}) =$ _____, $P(\text{B}) =$ _____
d. A AND B = _____, A OR B = _____
e. $P(\text{A AND B}) =$ _____, $P(\text{A OR B}) =$ _____
f. $\text{B}' =$ _____, $P(\text{B}') =$ _____
g. $P(\text{A}) + P(\text{A}') =$ _____
h. $P(\text{A}|\text{B}) =$ _____, $P(\text{B}|\text{A}) =$ _____; are the probabilities equal?

Answer

a. $\text{S} = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4)\}$
b. $\text{A} = \{(1,1), (1,3), (2,2), (2,4), (3,1), (3,3)\}$
$\text{B} = \{(2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4)\}$
c. $P(\text{A}) = \frac{1}{2}$, $P(\text{B}) = \frac{2}{3}$
d. A AND B $= \{(2,2), (2,4), (3,1), (3,3)\}$
A OR B $= \{(1,1), (1,3), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4)\}$
e. $P(\text{A AND B}) = \frac{1}{3}, P(\text{A OR B}) = \frac{5}{6}$
f. $\text{B}' = \{(1,1), (1,2), (1,3), (1,4)\}, P(\text{B}') = \frac{1}{3}$
g. $P(\text{B}) + P(\text{B}') = 1$

h. $P(\text{A}|\text{B}) = \frac{P(\text{A AND B})}{P(\text{B})} = \frac{1}{2}, P(\text{B}|\text{A}) = \frac{P(\text{A AND B})}{P(\text{B})} = \frac{2}{3}$, No.

Example $9.2.2A$

A fair, six-sided die is rolled. Describe the sample space S, identify each of the following events with a subset of S and compute its probability (an outcome is the number of dots that show up).

a. Event $\text{T} =$ the outcome is two.
b. Event $\text{A} =$ the outcome is an even number.
c. Event $\text{B} =$ the outcome is less than four.
d. The complement of A.
e. A GIVEN B
f. B GIVEN A
g. A AND B
h. A OR B
i. $\text{A OR B}'$
j. Event $\text{N} =$ the outcome is a prime number.
k. Event $\text{I} =$ the outcome is seven.

Solution

a. $\text{T} = \{2\}$, $P(\text{T}) = \frac{1}{6}$
b. $A = \{2, 4, 6\}$, $P(\text{A}) = \frac{1}{2}$
c. $\text{B} = \{1, 2, 3\}$, $P(\text{B}) = \frac{1}{2}$
d. $\text{A}' = \{1, 3, 5\}, P(\text{A}') = \frac{1}{2}$
e. $\text{A}|\text{B} = \{2\}$, $P(\text{A}|\text{B}) = \frac{1}{3}$
f. $\text{B}|\text{A} = \{2\}$, $P(\text{B}|\text{A}) = \frac{1}{3}$
g. $\text{A AND B} = 2, P(\text{A AND B}) = \frac{1}{6}$
h. $\text{A OR B} = \{1, 2, 3, 4, 6\}$, $P(\text{A OR B}) = \frac{5}{6}$
i. $\text{A OR B}' = \{2, 4, 5, 6\}$, $P(\text{A OR B}') = \frac{2}{3}$
j. $\text{N} = \{2, 3, 5\}$, $P(\text{N}) = \frac{1}{2}$
k. A six-sided die does not have seven dots. $P(7) = 0$.

Example $9.2.2B$

Table describes the distribution of a random sample S of 100 individuals, organized by gender and whether they are right- or left-handed.

	Right-handed	Left-handed
Males	43	9
Females	44	4

Let's denote the events $M =$ the subject is male, $F =$ the subject is female, $R =$ the subject is right-handed, $L =$ the subject is left-handed. Compute the following probabilities:

a. $P(\text{M})$
b. $P(\text{F})$
c. $P(\text{R})$
d. $P(\text{L})$
e. $P(\text{M AND R})$
f. $P(\text{F AND L})$
g. $P(\text{M OR F})$
h. $P(\text{M OR R})$

i. $P(\text{F OR L})$
j. $P(\text{M'})$
k. $P(\text{R}|\text{M})$
l. $P(\text{F}|\text{L})$
m. $P(\text{L}|\text{F})$

Answer

a. $P(\text{M}) = 0.52$
b. $P(\text{F}) = 0.48$
c. $P(\text{R}) = 0.87$
d. $P(\text{L}) = 0.13$
e. $P(\text{M AND R}) = 0.43$
f. $P(\text{F AND L}) = 0.04$
g. $P(\text{M OR F}) = 1$
h. $P(\text{M OR R}) = 0.96$
i. $P(\text{F OR L}) = 0.57$
j. $P(\text{M'}) = 0.48$
k. $P(\text{R}|\text{M}) = 0.8269$ (rounded to four decimal places)
l. $P(\text{F}|\text{L}) = 0.3077$ (rounded to four decimal places)
m. $P(\text{L}|\text{F}) = 0.0833$

References

1. "Countries List by Continent." Worldatlas, 2013. Available online at http://www.worldatlas.com/cntycont.htm (accessed May 2, 2013).

Review

In this module we learned the basic terminology of probability. The set of all possible outcomes of an experiment is called the sample space. Events are subsets of the sample space, and they are assigned a probability that is a number between zero and one, inclusive.

Formula Review

A and B are events

$P(\text{S}) = 1$ where S is the sample space

$0 \leq P(\text{A}) \leq 1$

$P(\text{A}|\text{B}) = \frac{\text{P(A AND B)}}{\text{P(B)}}$

Glossary

Conditional Probability
the likelihood that an event will occur given that another event has already occurred

Equally Likely
Each outcome of an experiment has the same probability.

Event
a subset of the set of all outcomes of an experiment; the set of all outcomes of an experiment is called a **sample space** and is usually denoted by S. An event is an arbitrary subset in S. It can contain one outcome, two outcomes, no outcomes (empty subset), the entire sample space, and the like. Standard notations for events are capital letters

such as A, B, C, and so on.

Experiment
a planned activity carried out under controlled conditions

Outcome
a particular result of an experiment

Probability
a number between zero and one, inclusive, that gives the likelihood that a specific event will occur; the foundation of statistics is given by the following 3 axioms (by A.N. Kolmogorov, 1930's): Let S denote the sample space and A and B are two events in S. Then:

- $0 \leq P(\mathrm{A}) \leq 1$
- If A and B are any two mutually exclusive events, then $\mathrm{P(A\ OR\ B)} = P(\mathrm{A}) + P(\mathrm{B})$.
- $P(\mathrm{S}) = 1$

Sample Space
the set of all possible outcomes of an experiment

The AND Event
An outcome is in the event $\mathrm{A\ AND\ B}$ if the outcome is in both $\mathrm{A\ AND\ B}$ at the same time.

The Complement Event
The complement of event A consists of all outcomes that are NOT in A.

The Conditional Probability of *A* GIVEN *B*
$P(\mathrm{A|B})$ is the probability that event A will occur given that the event B has already occurred.

The Or Event
An outcome is in the event $\mathrm{A\ OR\ B}$ if the outcome is in A or is in B or is in both A and B.

Exercise 3.2.2

In a particular college class, there are male and female students. Some students have long hair and some students have short hair. Write the **symbols** for the probabilities of the events for parts a through j. (Note that you cannot find numerical answers here. You were not given enough information to find any probability values yet; concentrate on understanding the symbols.)

- Let F be the event that a student is female.
- Let M be the event that a student is male.
- Let S be the event that a student has short hair.
- Let L be the event that a student has long hair.

a. The probability that a student does not have long hair.
b. The probability that a student is male or has short hair.
c. The probability that a student is a female and has long hair.
d. The probability that a student is male, given that the student has long hair.
e. The probability that a student has long hair, given that the student is male.
f. Of all the female students, the probability that a student has short hair.
g. Of all students with long hair, the probability that a student is female.
h. The probability that a student is female or has long hair.
i. The probability that a randomly selected student is a male student with short hair.
j. The probability that a student is female.

Answer

a. $P(\mathrm{L}') = P(\mathrm{S})$
b. $P(\mathrm{M\ OR\ S})$

c. $P(\text{F AND L})$
d. $P(\text{M}|\text{L})$
e. $P(\text{L}|\text{M})$
f. $P(\text{S}|\text{F})$
g. $P(\text{F}|\text{L})$
h. $P(\text{F OR L})$
i. $P(\text{M AND S})$
j. $P(\text{F})$

Use the following information to answer the next four exercises. A box is filled with several party favors. It contains 12 hats, 15 noisemakers, ten finger traps, and five bags of confetti.

Let $H =$ the event of getting a hat.

Let $N =$ the event of getting a noisemaker.

Let $F =$ the event of getting a finger trap.

Let $C =$ the event of getting a bag of confetti.

Exercise 3.2.3

Find $P(\text{H})$.

Exercise 3.2.4

Find $P(\text{N})$.

Answer

$P(\text{N}) = \frac{15}{42} = \frac{5}{14} = 0.36$

Exercise 3.2.5

Find $P(\text{F})$.

Exercise 3.2.6

Find $P(\text{C})$.

Answer

$P(\text{C}) = \frac{5}{42} = 0.12$

Use the following information to answer the next six exercises. A jar of 150 jelly beans contains 22 red jelly beans, 38 yellow, 20 green, 28 purple, 26 blue, and the rest are orange.

Let $B =$ the event of getting a blue jelly bean

Let $G =$ the event of getting a green jelly bean.

Let $O =$ the event of getting an orange jelly bean.

Let $P =$ the event of getting a purple jelly bean.

Let $R =$ the event of getting a red jelly bean.

Let $Y =$ the event of getting a yellow jelly bean.

Exercise 3.2.7

Find $P(\mathrm{B})$.

Exercise 3.2.8

Find $P(\mathrm{G})$.

Answer

$P(\mathrm{G}) = \frac{20}{150} = \frac{2}{15} = 0.13$

Exercise 3.2.9

Find $P(\mathrm{P})$.

Exercise 3.2.10

Find $P(\mathrm{R})$.

Answer

$P(\mathrm{R}) = \frac{22}{150} = \frac{11}{75} = 0.15$

Exercise 3.2.11

Find $P(\mathrm{Y})$.

Exercise 3.2.12

Find $P(\mathrm{O})$.

Answer

$P(textO) = \frac{150-22-38-20-28-26}{150} = \frac{16}{150} = \frac{8}{75} = 0.11$

Use the following information to answer the next six exercises. There are 23 countries in North America, 12 countries in South America, 47 countries in Europe, 44 countries in Asia, 54 countries in Africa, and 14 in Oceania (Pacific Ocean region).

Let $\mathrm{A} =$ the event that a country is in Asia.

Let $\mathrm{E} =$ the event that a country is in Europe.

Let $\mathrm{F} =$ the event that a country is in Africa.

Let N = the event that a country is in North America.

Let O = the event that a country is in Oceania.

Let S = the event that a country is in South America.

Exercise 3.2.13

Find $P(\mathrm{A})$.

Exercise 3.2.14

Find $P(\mathrm{E})$.

Answer

$P(\mathrm{E}) = \frac{47}{194} = 0.24$

Exercise 3.2.15

Find $P(\mathrm{F})$.

Exercise 3.2.16

Find $P(\mathrm{N})$.

Answer

$P(\mathrm{N}) = \frac{23}{194} = 0.12$

Exercise 3.2.17

Find $P(\mathrm{O})$.

Exercise 3.2.18

Find $P(\mathrm{S})$.

Answer

$P(\mathrm{S}) = \frac{12}{194} = \frac{6}{97} = 0.06$

Exercise 3.2.19

What is the probability of drawing a red card in a standard deck of 52 cards?

Exercise 3.2.20

What is the probability of drawing a club in a standard deck of 52 cards?

Answer

$\frac{13}{52} = \frac{1}{4} = 0.25$

Exercise 3.2.21

What is the probability of rolling an even number of dots with a fair, six-sided die numbered one through six?

Exercise 3.2.22

What is the probability of rolling a prime number of dots with a fair, six-sided die numbered one through six?

Answer

$\frac{3}{6} = \frac{1}{2} = 0.5$

Use the following information to answer the next two exercises. You see a game at a local fair. You have to throw a dart at a color wheel. Each section on the color wheel is equal in area.

CNX_Stats_C03_M01_021.jpg Figure $9.2.1$.

Let B = the event of landing on blue.

Let R = the event of landing on red.

Let G = the event of landing on green.

Let Y = the event of landing on yellow.

Exercise 3.2.23

If you land on Y, you get the biggest prize. Find $P(\mathrm{Y})$.

Exercise 3.2.24

If you land on red, you don't get a prize. What is $P(\mathrm{R})$?

Answer

$\mathrm{P}(R) = \frac{4}{8} = 0.5$

Use the following information to answer the next ten exercises. On a baseball team, there are infielders and outfielders. Some players are great hitters, and some players are not great hitters.

Let I = the event that a player in an infielder.

Let O = the event that a player is an outfielder.

Let $\mathrm{H} =$ the event that a player is a great hitter.

Let $\mathrm{N} =$ the event that a player is not a great hitter.

Exercise 3.2.25

Write the symbols for the probability that a player is not an outfielder.

Exercise 3.2.26

Write the symbols for the probability that a player is an outfielder or is a great hitter.

Answer

$P(\mathrm{O\ OR\ H})$

Exercise 3.2.27

Write the symbols for the probability that a player is an infielder and is not a great hitter.

Exercise 3.2.28

Write the symbols for the probability that a player is a great hitter, given that the player is an infielder.

Answer

$P(\mathrm{H}|\mathrm{I})$

Exercise 3.2.29

Write the symbols for the probability that a player is an infielder, given that the player is a great hitter.

Exercise 3.2.30

Write the symbols for the probability that of all the outfielders, a player is not a great hitter.

Answer

$P(\mathrm{N}|\mathrm{O})$

Exercise 3.2.31

Write the symbols for the probability that of all the great hitters, a player is an outfielder.

Exercise 3.2.32

Write the symbols for the probability that a player is an infielder or is not a great hitter.

Answer

$P(\text{I OR N})$

Exercise 3.2.33

Write the symbols for the probability that a player is an outfielder and is a great hitter.

Exercise 3.2.34

Write the symbols for the probability that a player is an infielder.

Answer

$P(\text{I})$

Exercise 3.2.35

What is the word for the set of all possible outcomes?

Exercise 3.2.36

What is conditional probability?

Answer

The likelihood that an event will occur given that another event has already occurred.

Exercise 3.2.37

A shelf holds 12 books. Eight are fiction and the rest are nonfiction. Each is a different book with a unique title. The fiction books are numbered one to eight. The nonfiction books are numbered one to four. Randomly select one book

Let $\text{F} =$ event that book is fiction

Let $\text{N} =$ event that book is nonfiction

What is the sample space?

Exercise 3.2.38

What is the sum of the probabilities of an event and its complement?

Answer

1

Use the following information to answer the next two exercises. You are rolling a fair, six-sided number cube. Let $\text{E} =$ the event that it lands on an even number. Let $\text{M} =$ the event that it lands on a multiple of three.

Exercise 3.2.39

What does $P(\mathrm{E}|\mathrm{M})$ mean in words?

Exercise 3.2.40

What does $P(\mathrm{E}\ \mathrm{OR}\ \mathrm{M})$ mean in words?

Answer

the probability of landing on an even number or a multiple of three

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9.3: Independent and Mutually Exclusive Events

Independent and *mutually exclusive* do not mean the same thing.

Independent Events

Two events are independent if the following are true:

- $P(\mathrm{A}|\mathrm{B}) = P(\mathrm{A})$
- $P(\mathrm{B}|\mathrm{A}) = P(\mathrm{B})$
- $P(\mathrm{A\ AND\ B}) = P(\mathrm{A})P(\mathrm{B})$

Two events A and B are independent if the knowledge that one occurred does not affect the chance the other occurs. For example, the outcomes of two roles of a fair die are independent events. The outcome of the first roll does not change the probability for the outcome of the second roll. To show two events are independent, you must show only one of the above conditions. If two events are NOT independent, then we say that they are dependent.

Sampling a population

Sampling may be done with replacement or without replacement (Figure $9.3.1$):

- **With replacement:** If each member of a population is replaced after it is picked, then that member has the possibility of being chosen more than once. When sampling is done with replacement, then events are considered to be *independent*, meaning the result of the first pick will not change the probabilities for the second pick.
- **Without replacement:** When sampling is done without replacement, each member of a population may be chosen only once. In this case, the probabilities for the second pick are affected by the result of the first pick. The events are considered to be *dependent* or *not independent*.

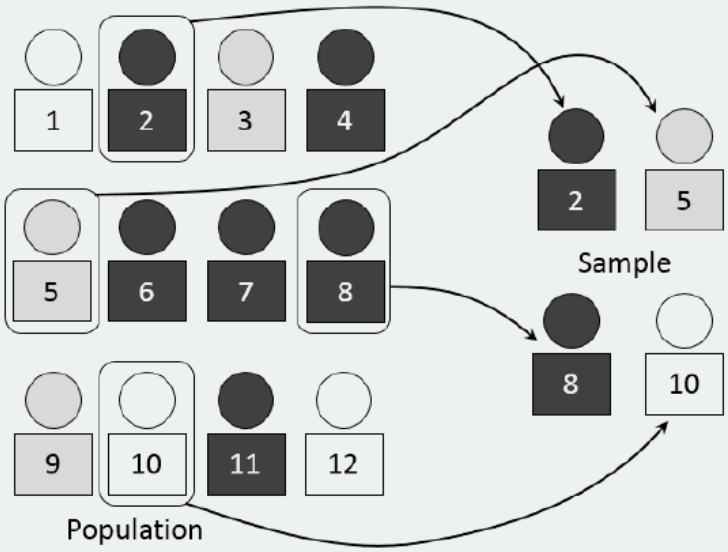


Figure $9.3.1$: A visual representation of the sampling process. If the sample items are replaced after each sampling event, then this is "sampling with replacement" if not, then it is "sampling without replacement". (CC BY-SA 4.0; Dan Kernler).

If it is not known whether A and B are independent or dependent, assume they are dependent until you can show otherwise.

Example 9.3.1: Sampling with and without replacement

You have a fair, well-shuffled deck of 52 cards. It consists of four suits. The suits are clubs, diamonds, hearts and spades. There are 13 cards in each suit consisting of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king) of that suit.

a. Sampling with replacement:

Suppose you pick three cards with replacement. The first card you pick out of the 52 cards is the Q of spades. You put this card back, reshuffle the cards and pick a second card from the 52-card deck. It is the ten of clubs. You put this card back, reshuffle the cards and pick a third card from the 52-card deck. This time, the card is the Q of spades again. Your picks are { Q of spades, ten of clubs, Q of spades}. You have picked the Q of spades twice. You pick each card from the 52-card deck.

b. Sampling without replacement:

Suppose you pick three cards without replacement. The first card you pick out of the 52 cards is the K of hearts. You put this card aside and pick the second card from the 51 cards remaining in the deck. It is the three of diamonds. You put this card aside and pick the third card from the remaining 50 cards in the deck. The third card is the J of spades. Your picks are {K of hearts, three of diamonds, J of spades}. Because you have picked the cards without replacement, you cannot pick the same card twice.

Exercise 9.3.1

You have a fair, well-shuffled deck of 52 cards. It consists of four suits. The suits are clubs, diamonds, hearts and spades. There are 13 cards in each suit consisting of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king) of that suit. Three cards are picked at random.

a. Suppose you know that the picked cards are Q of spades, K of hearts and Qof spades. Can you decide if the sampling was with or without replacement?
b. Suppose you know that the picked cards are Q of spades, K of hearts, and Jof spades. Can you decide if the sampling was with or without replacement?

Answer a available at the end of this section.

Example 9.3.2

You have a fair, well-shuffled deck of 52 cards. It consists of four suits. The suits are clubs, diamonds, hearts, and spades. There are 13 cards in each suit consisting of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), and K (king) of that suit. S = spades, H = Hearts, D = Diamonds, C = Clubs.

a. Suppose you pick four cards, but do not put any cards back into the deck. Your cards are QS, 1D, 1C, QD.
b. Suppose you pick four cards and put each card back before you pick the next card. Your cards are KH, 7D, 6D, KH.

Which of a. or b. did you sample with replacement and which did you sample without replacement?

Answer a available at the end of this section.

Exercise 9.3.2

You have a fair, well-shuffled deck of 52 cards. It consists of four suits. The suits are clubs, diamonds, hearts, and spades. There are 13 cards in each suit consisting of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), and K (king) of that suit. S = spades, H = Hearts, D = Diamonds, C = Clubs. Suppose that you sample four cards without replacement. Which of the following outcomes are possible? Answer the same question for sampling with replacement.

a. QS, 1D, 1C, QD

b. KH, 7D, 6D, KH
c. QS, 7D, 6D, KS

Answer - without replacement available at the end of this section.

Mutually Exclusive Events

A and B are mutually exclusive events if they **cannot** occur at the same time. This means that A and B do not share any outcomes and $P(\text{A AND B}) = 0$.

For example, suppose the sample space

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

Let $\text{A} = \{1, 2, 3, 4, 5\}$, $\text{B} = \{4, 5, 6, 7, 8\}$, and $\text{C} = \{7, 9\}$. $\text{A AND B} = \{4, 5\}$.

$$P(\text{A AND B}) = \frac{2}{10}$$

and is not equal to zero. Therefore, A and B are not mutually exclusive. A and C do not have any numbers in common so $P(\text{A AND C}) = 0$. Therefore, A and C are mutually exclusive.

If it is not known whether A and B are mutually exclusive, **assume they are not until you can show otherwise**. The following examples illustrate these definitions and terms.

Example 9.3.3

Flip two fair coins.

The sample space is $\{HH, HT, TH, TT\}$ where T = tails and H = heads. The outcomes are HH, HT, TH, and TT. The outcomes HT and TH are different. The HT means that the first coin showed heads and the second coin showed tails. The TH means that the first coin showed tails and the second coin showed heads.

- Let A = the event of getting **at most one tail**. (At most one tail means zero or one tail.) Then A can be written as $\{HH, HT, TH\}$. The outcome HH shows zero tails. HT and THeach show one tail.
- Let B = the event of getting all tails. B can be written as $\{TT\}$. B is the **complement** of A, so $\text{B} = \text{A}'$. Also, $P(\text{A}) + P(\text{B}) = P(\text{A}) + P(\text{A}') = 1$.
- The probabilities for A and for B are $P(\text{A}) = \frac{3}{4}$ and $P(\text{B}) = \frac{1}{4}$.
- Let C = the event of getting all heads. $\text{C} = \{HH\}$. Since $\text{B} = \{TT\}$, $P(\text{B AND C}) = 0$. B and Care mutually exclusive. B and C have no members in common because you cannot have all tails and all heads at the same time.)
- Let D = event of getting **more than one** tail. $\text{D} = \{TT\}$. $P(\text{D}) = \frac{1}{4}$
- Let E = event of getting a head on the first roll. (This implies you can get either a head or tail on the second roll.) $\text{E} = \{HT, HH\}$. $P(\text{E}) = \frac{2}{4}$
- Find the probability of getting **at least one** (one or two) tail in two flips. Let F = event of getting at least one tail in two flips. $\text{F} = \{HT, TH, TT\}$. $P(\text{F}) = \frac{3}{4}$

Exercise 9.3.3

Draw two cards from a standard 52-card deck with replacement. Find the probability of getting at least one black card.

Answer available at the end of this section.

Example 9.3.4

Flip two fair coins. Find the probabilities of the events.

a. Let F = the event of getting at most one tail (zero or one tail).
b. Let G = the event of getting two faces that are the same.
c. Let H = the event of getting a head on the first flip followed by a head or tail on the second flip.
d. Are F and G mutually exclusive?
e. Let J = the event of getting all tails. Are J and H mutually exclusive?

Solution

Look at the sample space in Example 9.3.3.

1. Zero (0) or one (1) tails occur when the outcomes HH, TH, HT show up. $P(\text{F}) = \frac{3}{4}$
2. Two faces are the same if HH or TT show up. $P(\text{G}) = \frac{2}{4}$
3. A head on the first flip followed by a head or tail on the second flip occurs when HH or HT show up. $P(\text{H}) = \frac{2}{4}$
4. F and G share HH so $P(\text{F AND G})$ is not equal to zero (0). F and G are not mutually exclusive.
5. Getting all tails occurs when tails shows up on both coins (TT). H's outcomes are HH and HT.

J and H have nothing in common so $P(\text{J AND H}) = 0$. J and H are mutually exclusive.

Exercise 9.3.4

A box has two balls, one white and one red. We select one ball, put it back in the box, and select a second ball (sampling with replacement). Find the probability of the following events:

a. Let F = the event of getting the white ball twice.
b. Let G = the event of getting two balls of different colors.
c. Let H = the event of getting white on the first pick.
d. Are F and G mutually exclusive?
e. Are G and H mutually exclusive?

Answer

a. $P(\text{F}) = \frac{1}{4}$
b. $P(\text{G}) = \frac{1}{2}$
c. $P(\text{H}) = \frac{1}{2}$
d. Yes
e. No

Example 9.3.5

Roll one fair, six-sided die. The sample space is {1, 2, 3, 4, 5, 6}. Let event A = a face is odd. Then $\text{A} = \{1, 3, 5\}$. Let event B = a face is even. Then $\text{B} = \{2, 4, 6\}$.

- Find the complement of A, A′. The complement of A, A′, is B because A and B together make up the sample space. $P(\text{A}) + P(\text{B}) = P(\text{A}) + P(\text{A}') = 1$. Also, $P(\text{A}) = \frac{3}{6}$ and $P(\text{B}) = \frac{3}{6}$.
- Let event C = odd faces larger than two. Then $\text{C} = \{3, 5\}$. Let event D = all even faces smaller than five. Then $\text{D} = \{2, 4\}$. $P(\text{C AND D}) = 0$ because you cannot have an odd and even face at the same time. Therefore, C and D are mutually exclusive events.

- Let event E = all faces less than five. $\mathrm{E} = \{1, 2, 3, 4\}$.

Are C and E mutually exclusive events? (Answer yes or no.) Why or why not?

Answer

No. $\mathrm{C} = \{3, 5\}$ and $\mathrm{E} = \{1, 2, 3, 4\}$. $P(\mathrm{C\ AND\ E}) = \frac{1}{6}$. To be mutually exclusive, $P(\mathrm{C\ AND\ E})$ must be zero.

- Find $P(\mathrm{C|A})$. This is a conditional probability. Recall that the event C is {3, 5} and event A is {1, 3, 5}. To find $P(\mathrm{C|A})$, find the probability of C using the sample space A. You have reduced the sample space from the original sample space {1, 2, 3, 4, 5, 6} to {1, 3, 5}. So, $P(\mathrm{C|A}) = \frac{2}{3}$.

Exercise 9.3.5

Let event A = learning Spanish. Let event B = learning German. Then $\mathrm{A\ AND\ B}$ = learning Spanish and German. Suppose $P(\mathrm{A}) = 0.4$ and $P(\mathrm{B}) = 0.2$. $P(\mathrm{A\ AND\ B}) = 0.08$. Are events A and B independent? Hint: You must show ONE of the following:

- $P(\mathrm{A|B}) = P(\mathrm{A})$
- $P(\mathrm{B|A})$
- $P(\mathrm{A\ AND\ B}) = P(\mathrm{A})P(\mathrm{B})$

Answer

$$P(\mathrm{A|B}) = \frac{\mathrm{P(A\ AND\ B)}}{P(\mathrm{B})} = \frac{0.08}{0.2} = 0.4 = P(\mathrm{A}) \tag{9.3.1}$$

The events are independent because $P(\mathrm{A|B}) = P(\mathrm{A})$.

Example 9.3.6

Let event G = taking a math class. Let event H = taking a science class. Then, $\mathrm{G\ AND\ H}$ = taking a math class and a science class. Suppose $P(\mathrm{G}) = 0.6$, $P(\mathrm{H}) = 0.5$, and $P(\mathrm{G\ AND\ H}) = 0.3$. Are G and H independent?

If G and H are independent, then you must show **ONE** of the following:

- $P(\mathrm{G|H}) = P(\mathrm{G})$
- $P(\mathrm{H|G}) = P(\mathrm{H})$
- $P(\mathrm{G\ AND\ H}) = P(\mathrm{G})P(\mathrm{H})$

The choice you make depends on the information you have. You could choose any of the methods here because you have the necessary information.

a. a. Show that $P(\mathrm{G|H}) = P(\mathrm{G})$.
b. b. Show $P(\mathrm{G\ AND\ H}) = P(\mathrm{G})P(\mathrm{H})$.

Solution

a. $P(\mathrm{G|H}) = \frac{P(\mathrm{G\ AND\ H})}{P(\mathrm{H})} = \frac{0.3}{0.5} = 0.6 = P(\mathrm{G})$
b. $P(\mathrm{G})P(\mathrm{H}) = (0.6)(0.5) = 0.3 = P(\mathrm{G\ AND\ H})$

Since G and H are independent, knowing that a person is taking a science class does not change the chance that he or she is taking a math class. If the two events had not been independent (that is, they are dependent) then knowing that a person is taking a science class would change the chance he or she is taking math. For practice, show that $P(\mathrm{H|G}) = P(\mathrm{H})$ to show that G and H are independent events.

Exercise 9.3.6

In a bag, there are six red marbles and four green marbles. The red marbles are marked with the numbers 1, 2, 3, 4, 5, and 6. The green marbles are marked with the numbers 1, 2, 3, and 4.

- $\mathrm{R} =$ a red marble
- $\mathrm{G} =$ a green marble
- $\mathrm{O} =$ an odd-numbered marble
- The sample space is $\mathrm{S} = \{R1, R2, R3, R4, R5, R6, G1, G2, G3, G4\}$.

S has ten outcomes. What is $P(\mathrm{G\ AND\ O})$?

Answer

Event G and $\mathrm{O} = \{G1, G3\}$

$P(\mathrm{G\ and\ O}) = \frac{2}{10} = 0.2$

Example 9.3.7

Let event $\mathrm{C} =$ taking an English class. Let event $\mathrm{D} =$ taking a speech class.

Suppose $P(\mathrm{C}) = 0.75$, $P(\mathrm{D}) = 0.3$, $P(\mathrm{C}|\mathrm{D}) = 0.75$ and $P(\mathrm{C\ AND\ D}) = 0.225$.

Justify your answers to the following questions numerically.

a. Are C and D independent?
b. Are C and D mutually exclusive?
c. What is $P(\mathrm{D}|\mathrm{C})$?

Solution

a. Yes, because $P(\mathrm{C}|\mathrm{D}) = P(\mathrm{C})$.
b. No, because $P(\mathrm{C\ AND\ D})$ is not equal to zero.
c. $P(\mathrm{D}|\mathrm{C}) = \frac{P(\mathrm{C\ AND\ D})}{P(\mathrm{C})} = \frac{0.225}{0.75} = 0.3$

Exercise 9.3.7

A student goes to the library. Let events $\mathrm{B} =$ the student checks out a book and $\mathrm{D} =$ the student checks out a DVD. Suppose that $P(\mathrm{B}) = 0.40$, $P(\mathrm{D}) = 0.30$ and $P(\mathrm{B\ AND\ D}) = 0.20$.

a. Find $P(\mathrm{B}|\mathrm{D})$.
b. Find $P(\mathrm{D}|\mathrm{B})$.
c. Are B and D independent?
d. Are B and D mutually exclusive?

Answer

a. $P(\mathrm{B}|\mathrm{D}) = 0.6667$
b. $P(\mathrm{D}|\mathrm{B}) = 0.5$
c. No
d. No

Example 9.3.8

In a box there are three red cards and five blue cards. The red cards are marked with the numbers 1, 2, and 3, and the blue cards are marked with the numbers 1, 2, 3, 4, and 5. The cards are well-shuffled. You reach into the box (you cannot see into it) and draw one card.

Let

- R = red card is drawn,
- B = blue card is drawn,
- E = even-numbered card is drawn.

The sample space $S = R1, R2, R3, B1, B2, B3, B4, B5$.

S has eight outcomes.

- $P(\mathrm{R}) = \frac{3}{8}$. $P(\mathrm{B}) = \frac{5}{8}$. $P(\mathrm{R\ AND\ B}) = 0$. (You cannot draw one card that is both red and blue.)
- $P(\mathrm{E}) = \frac{3}{8}$. (There are three even-numbered cards, $R2, B2$, and $B4$.)
- $P(\mathrm{E|B}) = \frac{2}{5}$. (There are five blue cards: $B1, B2, B3, B4$, and $B5$. Out of the blue cards, there are two even cards; $B2$ and $B4$.)
- $P(\mathrm{B|E}) = \frac{2}{3}$. (There are three even-numbered cards: $R2, B2$, and $B4$. Out of the even-numbered cards, to are blue; $B2$ and $B4$.)
- The events R and B are mutually exclusive because $P(\mathrm{R\ AND\ B}) = 0$.
- Let G = card with a number greater than 3. $\mathrm{G} = \{B4, B5\}$. $P(\mathrm{G}) = \frac{2}{8}$. Let H = blue card numbered between one and four, inclusive. $\mathrm{H} = \{B1, B2, B3, B4\}$. $P(\mathrm{G|H}) = \frac{1}{4}$. (The only card in H that has a number greater than three is B4.) Since $\frac{2}{8} = \frac{1}{4}$, $P(\mathrm{G}) = P(\mathrm{G|H})$, which means that G and H are independent.

Exercise 9.3.8

In a basketball arena,

- 70% of the fans are rooting for the home team.
- 25% of the fans are wearing blue.
- 20% of the fans are wearing blue and are rooting for the away team.
- Of the fans rooting for the away team, 67% are wearing blue.

Let A be the event that a fan is rooting for the away team.

Let B be the event that a fan is wearing blue.

Are the events of rooting for the away team and wearing blue independent? Are they mutually exclusive?

Answer

- $P(\mathrm{B|A}) = 0.67$
- $P(\mathrm{B}) = 0.25$

So $P(\mathrm{B})$ does not equal $P(\mathrm{B|A})$ which means that $\mathrm{B} and \mathrm{A}$ are not independent (wearing blue and rooting for the away team are not independent). They are also not mutually exclusive, because $P(\mathrm{B\ AND\ A}) = 0.20$, not 0.

Example 9.3.9

In a particular college class, 60% of the students are female. Fifty percent of all students in the class have long hair. Forty-five percent of the students are female and have long hair. Of the female students, 75% have long hair. Let F be the event that a student is female. Let L be the event that a student has long hair. One student is picked randomly. Are the events of being female and having long hair independent?

- The following probabilities are given in this example:
- $P(\text{F}) = 0.60$; $P(\text{L}) = 0.50$
- $P(\text{F AND L}) = 0.45$
- $P(\text{L}|\text{F}) = 0.75$

The choice you make depends on the information you have. You could use the first or last condition on the list for this example. You do not know $P(\text{F}|\text{L})$ yet, so you cannot use the second condition.

Solution 1

Check whether $P(\text{F AND L}) = P(\text{F})P(\text{L})$. We are given that $P(\text{F AND L}) = 0.45$, but $P(\text{F})P(\text{L}) = (0.60)(0.50) = 0.30$. The events of being female and having long hair are not independent because $P(\text{F AND L})$ does not equal $P(\text{F})P(\text{L})$.

Solution 2

Check whether $P(\text{L}|\text{F})$ equals $P(\text{L})$. We are given that $P(\text{L}|\text{F}) = 0.75$, but $P(\text{L}) = 0.50$; they are not equal. The events of being female and having long hair are not independent.

Interpretation of Results

The events of being female and having long hair are not independent; knowing that a student is female changes the probability that a student has long hair.

Exercise 9.3.9

Mark is deciding which route to take to work. His choices are $\text{I} = \text{the Interstate}$ and $\text{F} = \text{Fifth Street}$

- $P(\text{I}) = 0.44$ and $P(\text{F}) = 0.55$
- $P(\text{I AND F}) = 0$ because Mark will take only one route to work.

What is the probability of $P(\text{I OR F})$?

Answer

Because $P(\text{I AND F}) = 0$,

$P(\text{I OR F}) = P(\text{I}) + P(\text{F}) - P(\text{I AND F}) = 0.44 + 0.56 - 0 = 1$

Example 9.3.10

a. Toss one fair coin (the coin has two sides, H and T). The outcomes are _____. Count the outcomes. There are ___ outcomes.
b. Toss one fair, six-sided die (the die has 1, 2, 3, 4, 5 or 6 dots on a side). The outcomes are _____. Count the outcomes. There are ___ outcomes.
c. Multiply the two numbers of outcomes. The answer is _____.
d. If you flip one fair coin and follow it with the toss of one fair, six-sided die, the answer in three is the number of outcomes (size of the sample space). What are the outcomes? (Hint: Two of the outcomes are $H1$ and $T6$.)

e. Event A = heads (H) on the coin followed by an even number (2, 4, 6) on the die. A = {_____}. Find $P(\mathrm{A})$.
f. Event B = heads on the coin followed by a three on the die. B = {_____}. Find $P(\mathrm{B})$.
g. Are A and B mutually exclusive? (Hint: What is $P(\mathrm{A\ AND\ B})$? If $P(\mathrm{A\ AND\ B}) = 0$, then A and B are mutually exclusive.)
h. Are A and B independent? (Hint: Is $P(\mathrm{A\ AND\ B}) = P(\mathrm{A})P(\mathrm{B})$? If $P(\mathrm{A\ AND\ B}) = P(\mathrm{A})P(\mathrm{B})$, then A and B are independent. If not, then they are dependent).

Solution

a. H and T; 2
b. 1, 2, 3, 4, 5, 6; 6
c. 2(6) = 12
d. $T1, T2, T3, T4, T5, T6, H1, H2, H3, H4, H5, H6$
e. $\mathrm{A} = \{H2, H4, H6\}$; $P(\mathrm{A}) = \frac{3}{12}$
f. $\mathrm{B} = \{H3\}$; $P(\mathrm{B}) = \frac{1}{12}$
g. Yes, because $P(\mathrm{A\ AND\ B}) = 0$
h. $P(\mathrm{A\ AND\ B}) = 0$. $P(\mathrm{A})P(\mathrm{B}) = \left(\frac{3}{12}\right)\left(\frac{1}{12}\right)$. $P(\mathrm{A\ AND\ B})$ does not equal $P(\mathrm{A})P(\mathrm{B})$, so A and B are dependent.

Exercise 9.3.10

A box has two balls, one white and one red. We select one ball, put it back in the box, and select a second ball (sampling with replacement). Let T be the event of getting the white ball twice, F the event of picking the white ball first, S the event of picking the white ball in the second drawing.

a. Compute $P(\mathrm{T})$.
b. Compute $P(\mathrm{T}|\mathrm{F})$.
c. Are T and F independent?.
d. Are F and S mutually exclusive?
e. Are F and S independent?

Answer

a. $P(\mathrm{T}) = \frac{1}{4}$
b. $P(\mathrm{T}|\mathrm{F}) = \frac{1}{2}$
c. No
d. No
e. Yes

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Review

Two events A and B are independent if the knowledge that one occurred does not affect the chance the other occurs. If two events are not independent, then we say that they are dependent.

In sampling with replacement, each member of a population is replaced after it is picked, so that member has the possibility of being chosen more than once, and the events are considered to be independent. In sampling without replacement, each member of a population may be chosen only once, and the events are considered not to be independent. When events do not share

outcomes, they are mutually exclusive of each other.

Formula Review

- If A and B are independent, $P(\text{A AND B}) = P(\text{A})P(\text{B})$, $P(\text{A}|\text{B}) = P(\text{A})$ and $P(\text{B}|\text{A}) = P(\text{B})$.
- If A and B are mutually exclusive, $P(\text{A OR B}) = P(\text{A}) + P(\text{B}) and P(\text{A AND B}) = 0$.

Exercise 9.3.11

E and F are mutually exclusive events. $P(\text{E}) = 0.4$; $P(\text{F}) = 0.5$. Find $P(\text{E}|\text{F})$.

Exercise 9.3.12

J and K are independent events. $P(\text{J}|\text{K}) = 0.3$. Find $P(\text{J})$.

Answer

$P(\text{J}) = 0.3$

Exercise 9.3.13

U and V are mutually exclusive events. $P(\text{U}) = 0.26$; $P(\text{V}) = 0.37$. Find:

a. $P(\text{U AND V}) =$
b. $P(\text{U}|\text{V}) =$
c. $P(\text{U OR V}) =$

Exercise 9.3.14

Q and R are independent events. $P(\text{Q}) = 0.4$ and $P(\text{Q AND R}) = 0.1$. Find $P(\text{R})$.

Answer

$P(\text{Q AND R}) = P(\text{Q})P(\text{R})$

$0.1 = (0.4)P(\text{R})$

$P(\text{R}) = 0.25$

Bringing It Together

Exercise 9.3.16

A previous year, the weights of the members of the **San Francisco 49ers** and the **Dallas Cowboys** were published in the *San Jose Mercury News*. The factual data are compiled into Table.

Shirt#	≤ 210	211–250	251–290	290≤
1–33	21	5	0	0
34–66	6	18	7	4

Shirt#	≤ 210	211–250	251–290	290≤
66–99	6	12	22	5

For the following, suppose that you randomly select one player from the 49ers or Cowboys.

If having a shirt number from one to 33 and weighing at most 210 pounds were independent events, then what should be true about $P(\text{Shirt\#1–33} | \leq 210 \text{ pounds})$?

Exercise 9.3.17

The probability that a male develops some form of cancer in his lifetime is 0.4567. The probability that a male has at least one false positive test result (meaning the test comes back for cancer when the man does not have it) is 0.51. Some of the following questions do not have enough information for you to answer them. Write "not enough information" for those answers. Let C = a man develops cancer in his lifetime and P = man has at least one false positive.

a. $P(\text{C}) =$ _____
b. $P(\text{P}|\text{C}) =$ _____
c. $P(\text{P}|\text{C'}) =$ _____
d. If a test comes up positive, based upon numerical values, can you assume that man has cancer? Justify numerically and explain why or why not.

Answer

a. $P(\text{C}) = 0.4567$
b. not enough information
c. not enough information
d. No, because over half (0.51) of men have at least one false positive text

Exercise 9.3.18

Given events G and H : $P(\text{G}) = 0.43$; $P(\text{H}) = 0.26$; $P(\text{H AND G}) = 0.14$

a. Find $P(\text{H OR G})$.
b. Find the probability of the complement of event (H AND G).
c. Find the probability of the complement of event (H OR G).

Exercise 9.3.19

Given events J and K : $P(\text{J}) = 0.18$; $P(\text{K}) = 0.37$; $P(\text{J OR K}) = 0.45$

a. Find $P(\text{J AND K})$.
b. Find the probability of the complement of event (J AND K).
c. Find the probability of the complement of event (J AND K).

Answer

a. $P(\text{J OR K}) = P(\text{J}) + P(\text{K}) - P(\text{J AND K}); 0.45 = 0.18 + 0.37 - P(\text{J AND K})$; solve to find $P(\text{J AND K}) = 0.10$
b. $P(\text{NOT (J AND K)}) = 1 - P(\text{J AND K}) = 1 - 0.10 = 0.90$
c. $P(\text{NOT (J OR K)}) = 1 - P(\text{J OR K}) = 1 - 0.45 = 0.55$

Glossary

Dependent Events
If two events are NOT independent, then we say that they are dependent.

Sampling with Replacement
If each member of a population is replaced after it is picked, then that member has the possibility of being chosen more than once.

Sampling without Replacement
When sampling is done without replacement, each member of a population may be chosen only once.

The Conditional Probability of One Event Given Another Event
P(*A*|*B*) is the probability that event *A* will occur given that the event *B* has already occurred.

The OR of Two Events
An outcome is in the event *A* OR *B* if the outcome is in *A*, is in *B*, or is in both *A* and *B*.

Answers

Exercise 9.3.1
With replacement
No

Example 9.3.2
Without replacement
With replacement

Exercise 9.3.2
a. Possible; b. Impossible, c. Possible
a. Possible; c. Possible, c. Possible

Exercise 9.3.3
The sample space of drawing two cards with replacement from a standard 52-card deck with respect to color is $\{BB, BR, RB, RR\}$.
Event A = Getting at least one black card $= \{BB, BR, RB\}$
$P(\text{A}) = \frac{3}{4} = 0.75$

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9.4: Two Basic Rules of Probability

When calculating probability, there are two rules to consider when determining if two events are independent or dependent and if they are mutually exclusive or not.

The Multiplication Rule

If A and B are two events defined on a sample space, then:

$$P(A \text{ AND } B) = P(B)P(A|B) \tag{9.4.1}$$

This rule may also be written as:

$$P(A|B) = \frac{P(A \text{ AND } B)}{P(B)}$$

(The probability of A given B equals the probability of A and B divided by the probability of B.)

If A and B are *independent*, then

$$P(A|B) = P(A).$$

and Equation 9.4.1 becomes

$$P(A \text{ AND } B) = P(A)P(B).$$

The Addition Rule

If A and B are defined on a sample space, then:

$$P(A \text{ OR } B) = P(A) + P(B) - P(A \text{ AND } B) \tag{9.4.2}$$

If A and B are **mutually exclusive**, then

$$P(A \text{ AND } B) = 0.$$

and Equation 9.4.2 becomes

$$P(A \text{ OR } B) = P(A) + P(B).$$

Example 9.4.1

Klaus is trying to choose where to go on vacation. His two choices are: $\mathrm{A} = \text{New Zealand}$ and $\mathrm{B} = \text{Alaska}$.

- Klaus can only afford one vacation. The probability that he chooses A is $P(\mathrm{A}) = 0.6$ and the probability that he chooses B is $P(\mathrm{B}) = 0.35$.
- $P(\mathrm{A \text{ AND } B}) = 0$ because Klaus can only afford to take one vacation
- Therefore, the probability that he chooses either New Zealand or Alaska is $P(\mathrm{A \text{ OR } B}) = P(\mathrm{A}) + P(\mathrm{B}) = 0.6 + 0.35 = 0.95$. Note that the probability that he does not choose to go anywhere on vacation must be 0.05.

Carlos plays college soccer. He makes a goal 65% of the time he shoots. Carlos is going to attempt two goals in a row in the next game. A = the event Carlos is successful on his first attempt. $P(\mathrm{A}) = 0.65$. B = the event Carlos is successful on his second attempt. $P(\mathrm{B}) = 0.65$. Carlos tends to shoot in streaks. The probability that he makes the second goal **GIVEN** that he made the first goal is 0.90.

a. What is the probability that he makes both goals?
b. What is the probability that Carlos makes either the first goal or the second goal?
c. Are A and B independent?
d. Are A and B mutually exclusive?

Solutions

a. The problem is asking you to find $P(\text{A AND B}) = P(\text{B AND A})$. Since $P(\text{B}|\text{A}) = 0.90$: $P(\text{B AND A}) = P(\text{B}|\text{A})P(\text{A}) = (0.90)(0.65) = 0.585$

Carlos makes the first and second goals with probability 0.585.

b. The problem is asking you to find $P(\text{A OR B})$.

$$P(\text{A OR B}) = P(\text{A}) + P(\text{B}) - P(\text{A AND B}) = 0.65 + 0.65 - 0.585 = 0.715 \tag{9.4.3}$$

Carlos makes either the first goal or the second goal with probability 0.715.

c. No, they are not, because $P(\text{B AND A}) = 0.585$.

$$P(\text{B})P(\text{A}) = (0.65)(0.65) = 0.423 \tag{9.4.4}$$

$$0.423 \neq 0.585 = P(\text{B AND A}) \tag{9.4.5}$$

So, $P(\text{B AND A})$ is **not** equal to $P(\text{B})P(\text{A})$.

d. No, they are not because $P(\text{A and B}) = 0.585$.

To be mutually exclusive, $P(\text{A AND B})$ must equal zero.

Exercise 9.4.1

Helen plays basketball. For free throws, she makes the shot 75% of the time. Helen must now attempt two free throws. C = the event that Helen makes the first shot. $P(\text{C}) = 0.75$. D = the event Helen makes the second shot. $P(\text{D}) = 0.75$. The probability that Helen makes the second free throw given that she made the first is 0.85. What is the probability that Helen makes both free throws?

Answer

$$P(\text{D}|\text{C}) = 0.85 \tag{9.4.6}$$

$$P(\text{C AND D}) = P(\text{D AND C}) \tag{9.4.7}$$

$$P(\text{D AND C}) = P(\text{D}|\text{C})P(\text{C}) = (0.85)(0.75) = 0.6375 \tag{9.4.8}$$

Helen makes the first and second free throws with probability 0.6375.

Example 9.4.2

A community swim team has **150** members. **Seventy-five** of the members are advanced swimmers. **Forty-seven** of the members are intermediate swimmers. The remainder are novice swimmers. **Forty** of the advanced swimmers practice four times a week. **Thirty** of the intermediate swimmers practice four times a week. **Ten** of the novice swimmers practice four times a week. Suppose one member of the swim team is chosen randomly.

a. What is the probability that the member is a novice swimmer?
b. What is the probability that the member practices four times a week?
c. What is the probability that the member is an advanced swimmer and practices four times a week?

d. What is the probability that a member is an advanced swimmer and an intermediate swimmer? Are being an advanced swimmer and an intermediate swimmer mutually exclusive? Why or why not?
e. Are being a novice swimmer and practicing four times a week independent events? Why or why not?

Answer

a. $\frac{28}{150}$
b. $\frac{80}{150}$
c. $\frac{40}{150}$
d. $P(\text{advanced AND intermediate}) = 0$, so these are mutually exclusive events. A swimmer cannot be an advanced swimmer and an intermediate swimmer at the same time.
e. No, these are not independent events.

$$P(\text{novice AND practices four times per week}) = 0.0667 \tag{9.4.9}$$

$$P(\text{novice})P(\text{practices four times per week}) = 0.0996 \tag{9.4.10}$$

$$0.0667 \neq 0.0996 \tag{9.4.11}$$

Exercise 9.4.2

A school has 200 seniors of whom 140 will be going to college next year. Forty will be going directly to work. The remainder are taking a gap year. Fifty of the seniors going to college play sports. Thirty of the seniors going directly to work play sports. Five of the seniors taking a gap year play sports. What is the probability that a senior is taking a gap year?

Answer

$$P = \frac{200 - 140 - 40}{200} = \frac{20}{200} = 0.1 \tag{9.4.12}$$

Example 9.4.3

Felicity attends Modesto JC in Modesto, CA. The probability that Felicity enrolls in a math class is 0.2 and the probability that she enrolls in a speech class is 0.65. The probability that she enrolls in a math class GIVEN that she enrolls in speech class is 0.25.

Let: M = math class, S = speech class, $\mathrm{M|S}$ = math given speech

a. What is the probability that Felicity enrolls in math and speech?
Find $P(\mathrm{M\ AND\ S}) = P(\mathrm{M|S})P(\mathrm{S})$.
b. What is the probability that Felicity enrolls in math or speech classes?
Find $P(\mathrm{M\ OR\ S}) = P(\mathrm{M}) + P(\mathrm{S}) - P(\mathrm{M\ AND\ S})$.
c. Are M and S independent? Is $P(\mathrm{M|S}) = P(\mathrm{M})$?
d. Are M and S mutually exclusive? Is $P(\mathrm{M\ AND\ S}) = 0$?

Answer

a. 0.1625, b. 0.6875, c. No, d. No

Exercise 9.4.3

A student goes to the library. Let events B = the student checks out a book and D = the student check out a DVD. Suppose that $P(\mathrm{B}) = 0.40, P(\mathrm{D}) = 0.30$ and $P(\mathrm{D|B}) = 0.5$.

a. Find $P(\mathrm{B\ AND\ D})$.

b. Find $P(\text{B OR D})$.

Answer

a. $P(\text{B AND D}) = P(\text{D}|\text{B})P(\text{B}) = (0.5)(0.4) = 0.20$.
b. $P(\text{B OR D}) = P(\text{B}) + P(\text{D}) - P(\text{B AND D}) = 0.40 + 0.30 - 0.20 = 0.50$

Example 9.4.4

Studies show that about one woman in seven (approximately 14.3%) who live to be 90 will develop breast cancer. Suppose that of those women who develop breast cancer, a test is negative 2% of the time. Also suppose that in the general population of women, the test for breast cancer is negative about 85% of the time. Let $\text{B} =$ woman develops breast cancer and let $\text{N} =$ tests negative. Suppose one woman is selected at random.

a. What is the probability that the woman develops breast cancer? What is the probability that woman tests negative?
b. Given that the woman has breast cancer, what is the probability that she tests negative?
c. What is the probability that the woman has breast cancer AND tests negative?
d. What is the probability that the woman has breast cancer or tests negative?
e. Are having breast cancer and testing negative independent events?
f. Are having breast cancer and testing negative mutually exclusive?

Answers

a. $P(\text{B}) = 0.143; P(\text{N}) = 0.85$
b. $P(\text{N}|\text{B}) = 0.02$
c. $P(\text{B AND N}) = P(\text{B})P(\text{N}|\text{B}) = (0.143)(0.02) = 0.0029$
d. $P(\text{B OR N}) = P(\text{B}) + P(\text{N}) - P(\text{B AND N}) = 0.143 + 0.85 - 0.0029 = 0.9901$
e. No. $P(\text{N}) = 0.85; P(\text{N}|\text{B}) = 0.02$. So, $P(\text{N}|\text{B})$ does not equal $P(\text{N})$.
f. No. $P(\text{B AND N}) = 0.0029$. For B and N to be mutually exclusive, $P(\text{B AND N})$ must be zero

Exercise 9.4.4

A school has 200 seniors of whom 140 will be going to college next year. Forty will be going directly to work. The remainder are taking a gap year. Fifty of the seniors going to college play sports. Thirty of the seniors going directly to work play sports. Five of the seniors taking a gap year play sports. What is the probability that a senior is going to college and plays sports?

Answer

Let $\text{A} =$ student is a senior going to college.

Let $\text{B} =$ student plays sports.

$$P(\text{B}) = \frac{140}{200}$$

$$P(\text{B}|\text{A}) = \frac{50}{140}$$

$$P(\text{A AND B}) = P(\text{B}|\text{A})P(\text{A})$$

$$P(\text{A AND B}) = (\frac{140}{200})(\frac{50}{140}) = \frac{1}{4}$$

Example 9.4.5

Refer to the information in Example 9.4.4. $\text{P} =$ tests positive.

a. Given that a woman develops breast cancer, what is the probability that she tests positive. Find $P(\mathrm{P}|\mathrm{B}) = 1 - P(\mathrm{N}|\mathrm{B})$.
b. What is the probability that a woman develops breast cancer and tests positive. Find $P(\mathrm{B\ AND\ P}) = P(\mathrm{P}|\mathrm{B})P(\mathrm{B})$.
c. What is the probability that a woman does not develop breast cancer. Find $P(\mathrm{B}') = 1 - P(\mathrm{B})$.
d. What is the probability that a woman tests positive for breast cancer. Find $P(\mathrm{P}) = 1 - P(\mathrm{N})$.

Answer

a. 0.98; b. 0.1401; c. 0.857; d. 0.15

Exercise 9.4.5

A student goes to the library. Let events B = the student checks out a book and D = the student checks out a DVD. Suppose that $P(\mathrm{B}) = 0.40, P(\mathrm{D}) = 0.30$ and $P(\mathrm{D}|\mathrm{B}) = 0.5$.

a. Find $P(\mathrm{B}')$.
b. Find $P(\mathrm{D\ AND\ B})$.
c. Find $P(\mathrm{B}|\mathrm{D})$.
d. Find $P(\mathrm{D\ AND\ B}')$.
e. Find $P(\mathrm{D}|\mathrm{B}')$.

Answer

a. $P(\mathrm{B}') = 0.60$
b. $P(\mathrm{D\ AND\ B}) = P(\mathrm{D}|\mathrm{B})P(\mathrm{B}) = 0.20$
c. $P(\mathrm{B}|\mathrm{D}) = \dfrac{P(\mathrm{B\ AND\ D})}{P(\mathrm{D})} = \dfrac{(0.20)}{(0.30)} = 0.66$
d. $P(\mathrm{D\ AND\ B}') = P(\mathrm{D}) - P(\mathrm{D\ AND\ B}) = 0.30 - 0.20 = 0.10$
e. $P(\mathrm{D}|\mathrm{B}') = P(\mathrm{D\ AND\ B}')P(\mathrm{B}') = (P(\mathrm{D}) - P(\mathrm{D\ AND\ B}))(0.60) = (0.10)(0.60) = 0.06$

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Review

The multiplication rule and the addition rule are used for computing the probability of A and B, as well as the probability of A or B for two given events A, B defined on the sample space. In sampling with replacement each member of a population is replaced after it is picked, so that member has the possibility of being chosen more than once, and the events are considered to be independent. In sampling without replacement, each member of a population may be chosen only once, and the events are considered to be not independent. The events A and B are mutually exclusive events when they do not have any outcomes in

common.

Formula Review

The multiplication rule: $P(\text{A AND B}) = P(\text{A}|\text{B})P(\text{B})$

The addition rule: $P(\text{A OR B}) = P(\text{A}) + P(\text{B}) - P(\text{A AND B})$

Use the following information to answer the next ten exercises. Forty-eight percent of all Californians registered voters prefer life in prison without parole over the death penalty for a person convicted of first degree murder. Among Latino California registered voters, 55% prefer life in prison without parole over the death penalty for a person convicted of first degree murder. 37.6% of all Californians are Latino.

In this problem, let:

- $\text{C} =$ Californians (registered voters) preferring life in prison without parole over the death penalty for a person convicted of first degree murder.
- $\text{L} =$ Latino Californians

Suppose that one Californian is randomly selected.

Exercise 9.4.5

Find $P(\text{C})$.

Exercise 9.4.6

Find $P(\text{L})$.

Answer

0.376

Exercise 9.4.7

Find $P(\text{C}|\text{L})$.

Exercise 9.4.8

In words, what is $\text{C}|\text{L}$?

Answer

$\text{C}|\text{L}$ means, given the person chosen is a Latino Californian, the person is a registered voter who prefers life in prison without parole for a person convicted of first degree murder.

Exercise 9.4.9

Find $P(\text{L AND C})$

Exercise 9.4.10

In words, what is L AND C?

Answer

L AND C is the event that the person chosen is a Latino California registered voter who prefers life without parole over the death penalty for a person convicted of first degree murder.

Exercise 9.4.11

Are L and C independent events? Show why or why not.

Exercise 9.4.12

Find $P(\mathrm{L}$ OR $\mathrm{C})$.

Answer

0.6492

Exercise 9.4.13

In words, what is L OR C?

Exercise 9.4.14

Are L and C mutually exclusive events? Show why or why not.

Answer

No, because $P(\mathrm{L}$ AND $\mathrm{C})$ does not equal 0.

Glossary

Independent Events

The occurrence of one event has no effect on the probability of the occurrence of another event. Events A and B are independent if one of the following is true:

1. $P(\mathrm{A}|\mathrm{B}) = P(\mathrm{A})$
2. $P(\mathrm{B}|\mathrm{A}) = P(\mathrm{B})$
3. $P(\mathrm{A}$ AND $\mathrm{B}) = P(\mathrm{A})P(\mathrm{B})$

Mutually Exclusive

Two events are mutually exclusive if the probability that they both happen at the same time is zero. If events A and B are mutually exclusive, then $P(\mathrm{A}$ AND $\mathrm{B}) = 0$.

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9.5: Contingency Tables

A *contingency table* provides a way of portraying data that can facilitate calculating probabilities. The table helps in determining conditional probabilities quite easily. The table displays sample values in relation to two different variables that may be dependent or contingent on one another. Later on, we will use contingency tables again, but in another manner.

Example 9.5.1

Suppose a study of speeding violations and drivers who use cell phones produced the following fictional data:

	Speeding violation in the last year	No speeding violation in the last year	Total
Cell phone user	25	280	305
Not a cell phone user	45	405	450
Total	70	685	755

The total number of people in the sample is 755. The row totals are 305 and 450. The column totals are 70 and 685. Notice that 305 + 450 = 755 and 70 + 685 = 755.

Calculate the following probabilities using the table.

a. Find $P(\text{Person is a cell phone user})$.
b. Find $P(\text{person had no violation in the last year})$.
c. Find $P(\text{Person had no violation in the last year AND was a cell phone user})$.
d. Find $P(\text{Person is a cell phone user OR person had no violation in the last year})$.
e. Find $P(\text{Person is a cell phone user GIVEN person had a violation in the last year})$.
f. Find $P(\text{Person had no violation last year GIVEN person was not a cell phone user})$

Answer

a. $\frac{\text{number of cell phone users}}{\text{total number in study}} = \frac{305}{755}$
b. $\frac{\text{number that had no violation}}{\text{total number in study}} = \frac{685}{755}$
c. $\frac{280}{755}$
d. $\left(\frac{305}{755} + \frac{685}{755}\right) - \frac{280}{755} = \frac{710}{755}$
e. $\frac{25}{70}$ (The sample space is reduced to the number of persons who had a violation.)
f. $\frac{405}{450}$ (The sample space is reduced to the number of persons who were not cell phone users.)

Exercise 9.5.1

Table shows the number of athletes who stretch before exercising and how many had injuries within the past year.

	Injury in last year	No injury in last year	Total
Stretches	55	295	350
Does not stretch	231	219	450
Total	286	514	800

a. What is $P(\text{athlete stretches before exercising})$?
b. What is $P(\text{athlete stretches before exercising}|\text{no injury in the last year})$?

Answer

a. $P(\text{athlete stretches before exercising}) = \frac{350}{800} = 0.4375$
b. $P(\text{athlete stretches before exercising}|\text{no injury in the last year}) = \frac{295}{514} = 0.5739$

Example 9.5.2

Table shows a random sample of 100 hikers and the areas of hiking they prefer.

Hiking Area Preference

Sex	The Coastline	Near Lakes and Streams	On Mountain Peaks	Total
Female	18	16	___	45
Male	___	___	14	55
Total	___	41	___	___

a. Complete the table.
b. Are the events "being female" and "preferring the coastline" independent events? Let *F* = being female and let *C* = preferring the coastline.
   1. Find *P*(*F* AND *C*).
   2. Find *P*(*F*)*P*(*C*)
   3. Are these two numbers the same? If they are, then *F* and *C* are independent. If they are not, then *F* and *C* are not independent.
c. Find the probability that a person is male given that the person prefers hiking near lakes and streams. Let $\mathrm{M} =$ being male, and let $\mathrm{L} =$ prefers hiking near lakes and streams.
   1. What word tells you this is a conditional?
   2. Fill in the blanks and calculate the probability: P(___|___) = ___.
   3. Is the sample space for this problem all 100 hikers? If not, what is it?
d. Find the probability that a person is female or prefers hiking on mountain peaks. Let $\mathrm{F} =$ being female, and let $\mathrm{P} =$ prefers mountain peaks.
   1. Find $P(\mathrm{F})$.
   2. Find $P(\mathrm{P})$.
   3. Find $P(\mathrm{F\ AND\ P})$.
   4. Find $P(\mathrm{F\ OR\ P})$.

Answers

a.

Hiking Area Preference

Sex	The Coastline	Near Lakes and Streams	On Mountain Peaks	Total
Female	18	16	**11**	45
Male	**16**	**25**	14	55
Total	**34**	41	**25**	**100**

b.

$$P(\text{F AND C}) = \frac{18}{100} = 0.18$$

$$P(\text{F})P(\text{C}) = \left(\frac{45}{100}\right)\left(\frac{34}{100}\right) = (0.45)(0.34) = 0.153$$

$P(\text{F AND C}) \neq P(\text{F})P(\text{C})$, so the events F and C are not independent.

c.

1. The word 'given' tells you that this is a conditional.
2. $P(\text{M}|\text{L}) = \frac{25}{41}$
3. No, the sample space for this problem is the 41 hikers who prefer lakes and streams.

d.

a. Find $P(\text{F})$.
b. Find $P(\text{P})$.
c. Find $P(\text{F AND P})$.
d. Find $P(\text{F OR P})$.

d.

1. $P(\text{F}) = \frac{45}{100}$
2. $P(\text{P}) = \frac{25}{100}$
3. $P(\text{F AND P}) = \frac{11}{100}$
4. $P(\text{F OR P}) = \frac{45}{100} + \frac{25}{100} - \frac{11}{100} = \frac{59}{100}$

Exercise 9.5.2

Table shows a random sample of 200 cyclists and the routes they prefer. Let M = males and H = hilly path.

Gender	Lake Path	Hilly Path	Wooded Path	Total
Female	45	38	27	110
Male	26	52	12	90
Total	71	90	39	200

a. Out of the males, what is the probability that the cyclist prefers a hilly path?
b. Are the events “being male” and “preferring the hilly path” independent events?

Answer

a. $P(H|M) = \frac{52}{90} = 0.5778$
b. For M and H to be independent, show $P(H|M) = P(H)$
$P(H|M) = 0.5778$, $P(H) = \frac{90}{200} = 0.45$
$P(H|M)$ does not equal $P(H)$ so M and H are NOT independent.

Example 9.5.3

Muddy Mouse lives in a cage with three doors. If Muddy goes out the first door, the probability that he gets caught by Alissa the cat is $\frac{1}{5}$ and the probability he is not caught is $\frac{4}{5}$. If he goes out the second door, the probability he gets caught by Alissa is $\frac{1}{4}$ and the probability he is not caught is $\frac{3}{4}$. The probability that Alissa catches Muddy coming out of the third door is $\frac{1}{2}$ and the probability she does not catch Muddy is $\frac{1}{2}$. It is equally likely that Muddy will choose any of the three doors so the probability of choosing each door is $\frac{1}{3}$.

Door Choice

Caught or Not	Door One	Door Two	Door Three	Total
Caught	$\frac{1}{15}$	$\frac{1}{12}$	$\frac{1}{6}$	____
Not Caught	$\frac{4}{15}$	$\frac{3}{12}$	$\frac{1}{6}$	____
Total	____	____	____	1

- The first entry $\frac{1}{15} = \left(\frac{1}{5}\right)\left(\frac{1}{3}\right)$ is $P(\text{Door One AND Caught})$
- The entry $\frac{4}{15} = \left(\frac{4}{5}\right)\left(\frac{1}{3}\right)$ is $P(\text{Door One AND Not Caught})$

Verify the remaining entries.

a. Complete the probability contingency table. Calculate the entries for the totals. Verify that the lower-right corner entry is 1.
b. What is the probability that Alissa does not catch Muddy?
c. What is the probability that Muddy chooses Door One OR Door Two given that Muddy is caught by Alissa?

Solution

Door Choice

Caught or Not	Door One	Door Two	Door Three	Total
Caught	$\frac{1}{15}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{19}{60}$
Not Caught	$\frac{4}{15}$	$\frac{3}{12}$	$\frac{1}{6}$	$\frac{41}{60}$
Total	$\frac{5}{15}$	$\frac{4}{12}$	$\frac{2}{6}$	1

b. $\frac{41}{60}$

c. $\frac{9}{19}$

Example 9.5.4

Table contains the number of crimes per 100,000 inhabitants from 2008 to 2011 in the U.S.

United States Crime Index Rates Per 100,000 Inhabitants 2008–2011

Year	Robbery	Burglary	Rape	Vehicle	Total
2008	145.7	732.1	29.7	314.7	
2009	133.1	717.7	29.1	259.2	
2010	119.3	701	27.7	239.1	
2011	113.7	702.2	26.8	229.6	
Total					

TOTAL each column and each row. Total data = 4,520.7

a. Find $P(2009 \text{ AND Robbery})$.
b. Find $P(2010 \text{ AND Burglary})$.
c. Find $P(2010 \text{ OR Burglary})$.
d. Find $P(2011|\text{Rape})$.
e. Find $P(\text{Vehicle}|2008)$.

Answer

a. 0.0294, b. 0.1551, c. 0.7165, d. 0.2365, e. 0.2575

Exercise 9.5.3

Table relates the weights and heights of a group of individuals participating in an observational study.

Weight/Height	Tall	Medium	Short	Totals
Obese	18	28	14	
Normal	20	51	28	
Underweight	12	25	9	
Totals				

a. Find the total for each row and column
b. Find the probability that a randomly chosen individual from this group is Tall.
c. Find the probability that a randomly chosen individual from this group is Obese and Tall.
d. Find the probability that a randomly chosen individual from this group is Tall given that the idividual is Obese.
e. Find the probability that a randomly chosen individual from this group is Obese given that the individual is Tall.
f. Find the probability a randomly chosen individual from this group is Tall and Underweight.
g. Are the events Obese and Tall independent?

Answer

Weight/Height	Tall	Medium	Short	Totals
Obese	18	28	14	60
Normal	20	51	28	99
Underweight	12	25	9	46
Totals	50	104	51	205

a. Row Totals: 60, 99, 46. Column totals: 50, 104, 51.

b. $P(\text{Tall}) = \frac{50}{205} = 0.244$
c. $P(\text{Obese AND Tall}) = \frac{18}{205} = 0.088$
d. $P(\text{Tall}|\text{Obese}) = \frac{18}{60} = 0.3$
e. $P(\text{Obese}|\text{Tall}) = \frac{18}{50} = 0.36$
f. $P(\text{Tall AND Underweight}) = \frac{12}{205} = 0.0585$
g. No. $P(\text{Tall})$ does not equal $P(\text{Tall}|\text{Obese})$.

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Review

There are several tools you can use to help organize and sort data when calculating probabilities. Contingency tables help display data and are particularly useful when calculating probabilities that have multiple dependent variables.

Use the following information to answer the next four exercises. Table shows a random sample of musicians and how they learned to play their instruments.

Gender	Self-taught	Studied in School	Private Instruction	Total
Female	12	38	22	72
Male	19	24	15	58
Total	31	62	37	130

Exercise 3.5.4

Find *P*(musician is a female).

Exercise 3.5.5

Find $P(\text{musician is a male AND had private instruction})$.

Answer

$P(\text{musician is a male AND had private instruction}) = \frac{15}{130} = \frac{3}{26} = 0.12$

Exercise 3.5.6

Find *P*(musician is a female OR is self taught).

Exercise 3.5.7

Are the events “being a female musician” and “learning music in school” mutually exclusive events?

Answer

The events are not mutually exclusive. It is possible to be a female musician who learned music in school.

Bringing it Together

Use the following information to answer the next seven exercises. An article in the *New England Journal of Medicine*, reported about a study of smokers in California and Hawaii. In one part of the report, the self-reported ethnicity and smoking levels per day were given. Of the people smoking at most ten cigarettes per day, there were 9,886 African Americans, 2,745 Native Hawaiians, 12,831 Latinos, 8,378 Japanese Americans, and 7,650 Whites. Of the people smoking 11 to 20 cigarettes per day, there were 6,514 African Americans, 3,062 Native Hawaiians, 4,932 Latinos, 10,680 Japanese Americans, and 9,877 Whites. Of the people smoking 21 to 30 cigarettes per day, there were 1,671 African Americans, 1,419 Native Hawaiians, 1,406 Latinos, 4,715 Japanese Americans, and 6,062 Whites. Of the people smoking at least 31 cigarettes per day, there were 759 African Americans, 788 Native Hawaiians, 800 Latinos, 2,305 Japanese Americans, and 3,970 Whites.

Exercise 3.5.8

Complete the table using the data provided. Suppose that one person from the study is randomly selected. Find the probability that person smoked 11 to 20 cigarettes per day.

Smoking Levels by Ethnicity

Smoking Level	African American	Native Hawaiian	Latino	Japanese Americans	White	TOTALS
1–10						
11–20						
21–30						
31+						
TOTALS						

Exercise 3.5.9

Suppose that one person from the study is randomly selected. Find the probability that person smoked 11 to 20 cigarettes per day.

Answer

$$\frac{35,065}{100,450}$$

Exercise 3.5.10

Find the probability that the person was Latino.

Exercise 3.5.11

In words, explain what it means to pick one person from the study who is "Japanese American **AND** smokes 21 to 30 cigarettes per day." Also, find the probability.

Answer

To pick one person from the study who is Japanese American AND smokes 21 to 30 cigarettes per day means that the person has to meet both criteria: both Japanese American and smokes 21 to 30 cigarettes. The sample space should include everyone in the study. The probability is $\frac{4,715}{100,450}$.

Exercise 3.5.12

In words, explain what it means to pick one person from the study who is "Japanese American **OR** smokes 21 to 30 cigarettes per day." Also, find the probability.

Exercise 3.5.13

In words, explain what it means to pick one person from the study who is "Japanese American **GIVEN** that person smokes 21 to 30 cigarettes per day." Also, find the probability.

Answer

To pick one person from the study who is Japanese American given that person smokes 21-30 cigarettes per day, means that the person must fulfill both criteria and the sample space is reduced to those who smoke 21-30 cigarettes per day. The probability is $\frac{4,715}{15,273}$.

Exercise 3.5.14

Prove that smoking level/day and ethnicity are dependent events.

Glossary

contingency table
: the method of displaying a frequency distribution as a table with rows and columns to show how two variables may be dependent (contingent) upon each other; the table provides an easy way to calculate conditional probabilities.

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9.6: Tree and Venn Diagrams

Sometimes, when the probability problems are complex, it can be helpful to graph the situation. Tree diagrams and Venn diagrams are two tools that can be used to visualize and solve conditional probabilities.

Tree Diagrams

A *tree diagram* is a special type of graph used to determine the outcomes of an experiment. It consists of "branches" that are labeled with either frequencies or probabilities. Tree diagrams can make some probability problems easier to visualize and solve. The following example illustrates how to use a tree diagram.

Example 9.6.1: Probabilities from Sampling with replacement

In an urn, there are 11 balls. Three balls are red (R) and eight balls are blue (B). Draw two balls, one at a time, with replacement (remember that "with replacement" means that you put the first ball back in the urn before you select the second ball). The tree diagram using frequencies that show all the possible outcomes follows.

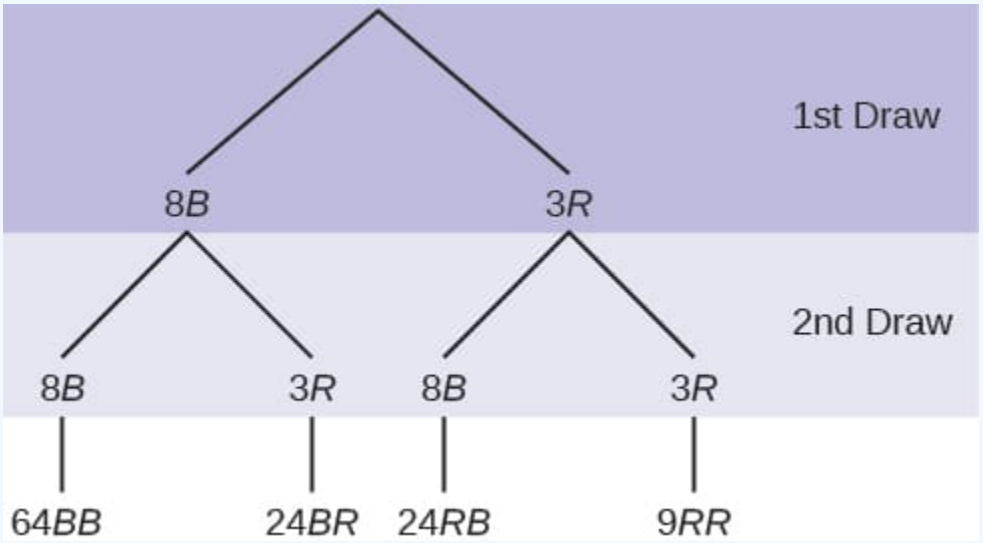


Figure 9.6.1: Total = 64 + 24 + 24 + 9 = 121

The first set of branches represents the first draw. The second set of branches represents the second draw. Each of the outcomes is distinct. In fact, we can list each red ball as *R*1, *R*2, and *R*3 and each blue ball as *B*1, *B*2, *B*3, *B*4, *B*5, *B*6, *B*7, and *B*8. Then the nine *RR* outcomes can be written as:

*R*1*R*1 *R*1*R*2 *R*1*R*3 *R*2*R*1 *R*2*R*2 *R*2*R*3 *R*3*R*1 *R*3*R*2 *R*3*R*3

The other outcomes are similar.

There are a total of 11 balls in the urn. Draw two balls, one at a time, with replacement. There are 11(11) = 121 outcomes, the size of the sample space.

Exercise 9.6.1

In a standard deck, there are 52 cards. 12 cards are face cards (event F) and 40 cards are not face cards (event N). Draw two cards, one at a time, with replacement. All possible outcomes are shown in the tree diagram as frequencies. Using the tree diagram, calculate $P(\text{FF})$.

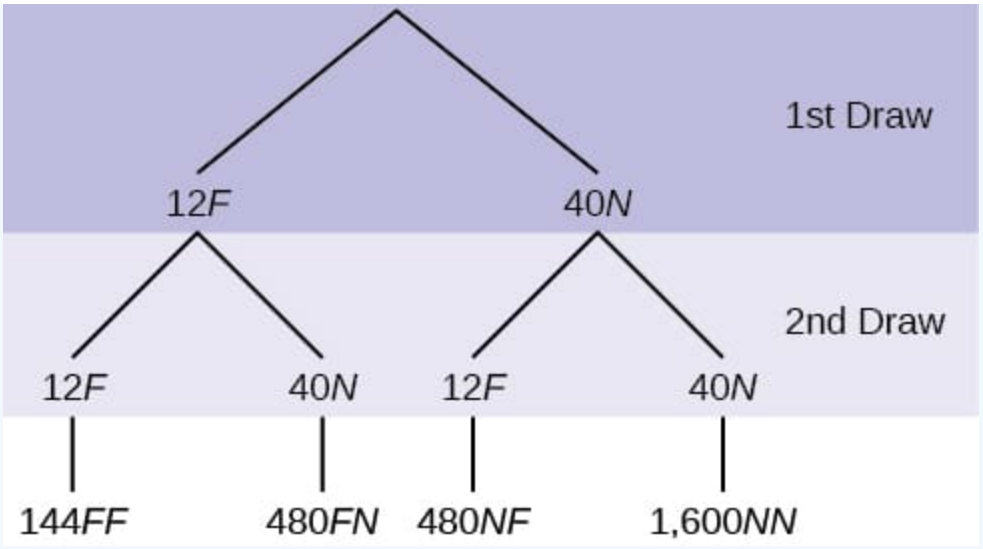


Figure 9.6.2:

Answer

Total number of outcomes is 144 + 480 + 480 + 1600 = 2,704.

$$P(\text{FF}) = \frac{144}{144 + 480 + 480 + 1,600} = \frac{144}{2,704} = \frac{9}{169} \tag{9.6.1}$$

1. a. List the 24 *BR* outcomes: *B*1*R*1, *B*1*R*2, *B*1*R*3, ...
2. b. Using the tree diagram, calculate *P*(*RR*).
3. c. Using the tree diagram, calculate $P(\text{RB OR BR})$.
4. d. Using the tree diagram, calculate $P(\text{R on 1st draw AND B on 2nd draw})$.
5. e. Using the tree diagram, calculate *P*(*R* on 2nd draw GIVEN *B* on 1st draw).
6. Using the tree diagram, calculate $P(\text{BB})$.
7. g. Using the tree diagram, calculate $P(\text{B on the 2nd draw given R on the first draw})$.

Solution

a. *B*1*R*1; *B*1*R*2; *B*1*R*3; *B*2*R*1; *B*2*R*2; *B*2*R*3; *B*3*R*1; *B*3*R*2; *B*3*R*3; *B*4*R*1; *B*4*R*2; *B*4*R*3; *B*5*R*1; *B*5*R*2; *B*5*R*3; *B*6*R*1; *B*6*R*2; *B*6*R*3 *B*7*R*1; *B*7*R*2; *B*7*R*3; *B*8*R*1; *B*8*R*2; *B*8*R*3

b. $P(\text{RR}) = \left(\frac{3}{11}\right)\left(\frac{3}{11}\right) = \frac{9}{121}$

c. $P(\text{RB OR BR}) = \left(\frac{3}{11}\right)\left(\frac{8}{11}\right) + \left(\frac{8}{11}\right)\left(\frac{3}{11}\right) = \frac{48}{121}$

d. $P(\text{R on 1st draw AND B on 2nd draw}) = P(\text{RB}) = \left(\frac{3}{11}\right)\left(\frac{8}{11}\right) = \frac{24}{121}$

e. *P*(*R* on 2nd draw GIVEN *B* on 1st draw) = *P*(*R* on 2nd|*B* on 1st) = $\frac{24}{88} = \frac{3}{11}$ This problem is a conditional one. The sample space has been reduced to those outcomes that already have a blue on the first draw. There are 24 + 64 = 88 possible outcomes (24 *BR* and 64 *BB*). Twenty-four of the 88 possible outcomes are *BR*. $\frac{24}{88} = \frac{3}{11}$

f. $P(\text{BB}) = \frac{64}{121}$

g. $P(\text{B on 2nd draw}|\text{R on 1st draw}) = \frac{8}{11}$. There are 9 + 24 outcomes that have R on the first draw (9 *RR* and 24 *RB*). The sample space is then 9 + 24 = 33. 24 of the 33 outcomes have B on the second draw. The probability is then $\frac{24}{33}$.

Example 9.6.2: Probabilities from Sampling without replacement

An urn has three red marbles and eight blue marbles in it. Draw two marbles, one at a time, this time without replacement, from the urn. (remember that "without replacement" means that you do not put the first ball back before you select the second marble). Following is a tree diagram for this situation. The branches are labeled with probabilities instead of frequencies. The numbers at the ends of the branches are calculated by multiplying the numbers on the two corresponding branches, for example, $\left(\frac{3}{11}\right)\left(\frac{2}{10}\right) = \frac{6}{110}$.

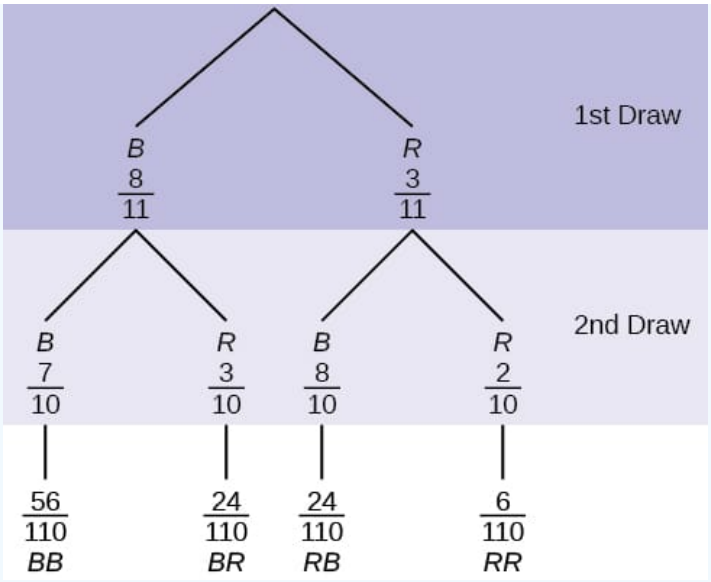


Figure 9.6.3: Total $= \frac{56+24+24+6}{110} = \frac{110}{110} = 1$

If you draw a red on the first draw from the three red possibilities, there are two red marbles left to draw on the second draw. You do not put back or replace the first marble after you have drawn it. You draw ***without replacement****, so that on the second draw there are ten marbles left in the urn.*

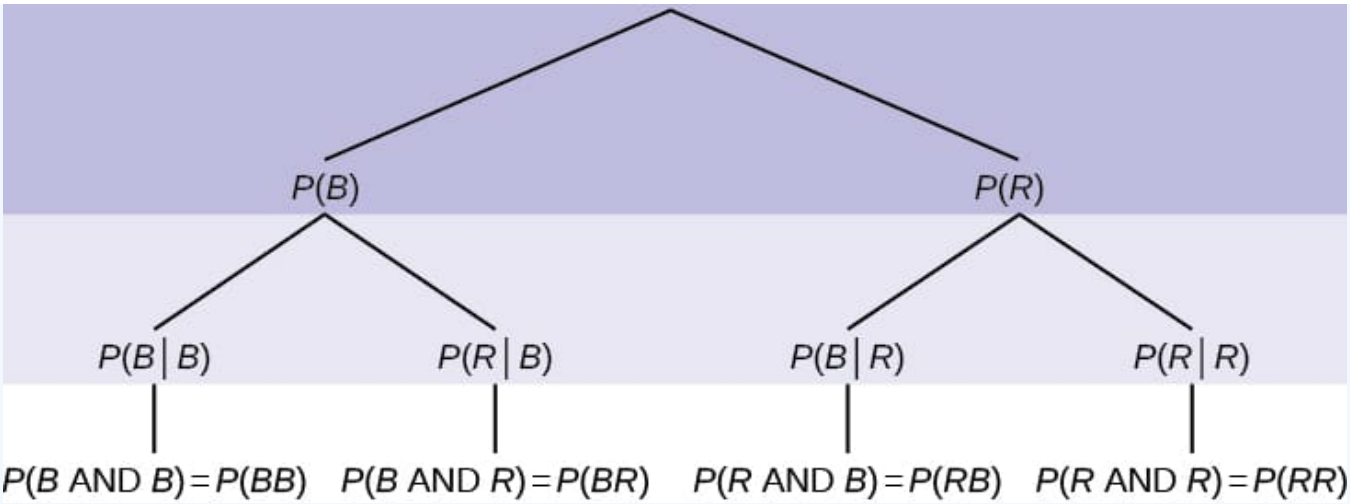
Calculate the following probabilities using the tree diagram.

a. $P(\text{RR}) =$ _____
b. Fill in the blanks: $P(\text{RB OR BR}) = \left(\frac{3}{11}\right)\left(\frac{8}{10}\right) +$ (___)(___) $= \frac{48}{110}$
c. $P(\text{R on 2nd}|\text{B on 1st}) =$
d. Fill in the blanks: $P(\text{R on 1st AND B on 2nd}) = P(\text{RB}) =$ (___)(___) $= \frac{24}{110}$
e. Find $P(\text{BB})$.
f. Find $P(\text{B on 2nd}|\text{R on 1st})$.

Answers

a. $P(\text{RR}) = \left(\frac{3}{11}\right)\left(\frac{2}{10}\right) = \frac{6}{110}$
b. $P(\text{RB OR BR}) = \left(\frac{3}{11}\right)\left(\frac{8}{10}\right) + \left(\frac{8}{11}\right)\left(\frac{3}{10}\right) = \frac{48}{110}$
c. $P(\text{R on 2nd}|\text{B on 1st}) = \frac{3}{10}$
d. $P(\text{R on 1st AND B on 2nd}) = P(\text{RB}) = \left(\frac{3}{11}\right)\left(\frac{8}{10}\right) = \frac{24}{110}$
e. $P(\text{BB}) = \left(\frac{8}{11}\right)\left(\frac{7}{10}\right)$
f. Using the tree diagram, $P(\text{B on 2nd}|\text{R on 1st}) = P(\text{R}|\text{B}) = \frac{8}{10}$.

If we are using probabilities, we can label the tree in the following general way.



- $P(\mathrm{R}|\mathrm{R})$ here means $P(\mathrm{R}\text{ on 2nd}|\mathrm{R}\text{ on 1st})$
- $P(\mathrm{B}|\mathrm{R})$ here means $P(\mathrm{B}\text{ on 2nd}|\mathrm{R}\text{ on 1st})$
- $P(\mathrm{R}|\mathrm{B})$ here means $P(\mathrm{R}\text{ on 2nd}|\mathrm{B}\text{ on 1st})$
- $P(\mathrm{B}|\mathrm{B})$ here means $P(\mathrm{B}\text{ on 2nd}|\mathrm{B}\text{ on 1st})$

Exercise 9.6.2

In a standard deck, there are 52 cards. Twelve cards are face cards (F) and 40 cards are not face cards (N). Draw two cards, one at a time, without replacement. The tree diagram is labeled with all possible probabilities.

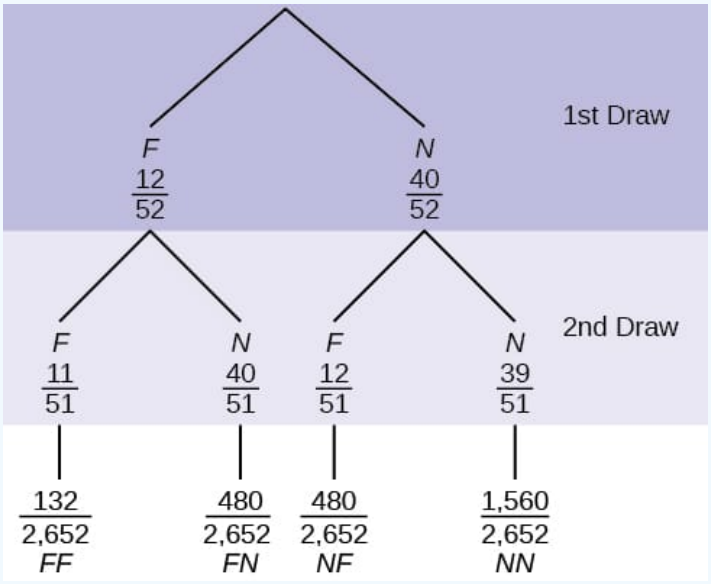


Figure 9.6.4:

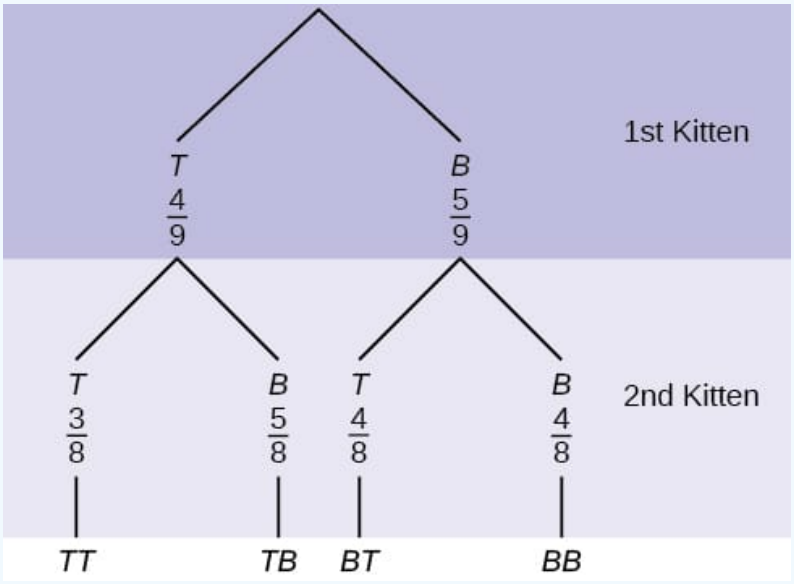
a. Find $P(\mathrm{FN}\text{ OR }\mathrm{NF})$.
b. Find $P(\mathrm{N}|\mathrm{F})$.
c. Find $P(\text{at most one face card})$.
   Hint: "At most one face card" means zero or one face card.
d. Find $P(\text{at least on face card})$.
   Hint: "At least one face card" means one or two face cards.

Answer

a. $P(\text{FN OR NF}) = \frac{480}{2,652} + \frac{480}{2,652} = \frac{960}{2,652} = \frac{80}{221}$
b. $P(\text{N|F}) = \frac{40}{51}$
c. $P(\text{at most one face card}) = \frac{(480+480+1,560)}{2,652} = \frac{2,520}{2,652}$
d. $P(\text{at least one face card}) = \frac{(132+480+480)}{2,652} = \frac{1,092}{2,652}$

Example 9.6.3

A litter of kittens available for adoption at the Humane Society has four tabby kittens and five black kittens. A family comes in and randomly selects two kittens (without replacement) for adoption.



a. What is the probability that both kittens are tabby?
   a. $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$ b. $\left(\frac{4}{9}\right)\left(\frac{4}{9}\right)$ c. $\left(\frac{4}{9}\right)\left(\frac{3}{8}\right)$ d. $\left(\frac{4}{9}\right)\left(\frac{5}{9}\right)$
b. What is the probability that one kitten of each coloring is selected?
   a. $\left(\frac{4}{9}\right)\left(\frac{5}{9}\right)$ b. $\left(\frac{4}{9}\right)\left(\frac{5}{8}\right)$ c. $\left(\frac{4}{9}\right)\left(\frac{5}{9}\right) + \left(\frac{5}{9}\right)\left(\frac{4}{9}\right)$ d. $\left(\frac{4}{9}\right)\left(\frac{5}{8}\right) + \left(\frac{5}{9}\right)\left(\frac{4}{8}\right)$
c. What is the probability that a tabby is chosen as the second kitten when a black kitten was chosen as the first?
d. What is the probability of choosing two kittens of the same color?

Answer

a. c, b. d, c. $\frac{4}{8}$, d. $\frac{32}{72}$

Exercise 9.6.3

Suppose there are four red balls and three yellow balls in a box. Three balls are drawn from the box without replacement. What is the probability that one ball of each coloring is selected?

Answer

$\left(\frac{4}{7}\right)\left(\frac{3}{6}\right) + \left(\frac{3}{7}\right)\left(\frac{4}{6}\right)$

Venn Diagram

A Venn diagram is a picture that represents the outcomes of an experiment. It generally consists of a box that represents the sample space S together with circles or ovals. The circles or ovals represent events.

Example 9.6.4

Suppose an experiment has the outcomes 1, 2, 3, ... , 12 where each outcome has an equal chance of occurring. Let event A = {1, 2, 3, 4, 5, 6} and event B = {6, 7, 8, 9}. Then A AND B = {6} and A OR B = {1, 2, 3, 4, 5, 6, 7, 8, 9}. The Venn diagram is as follows:

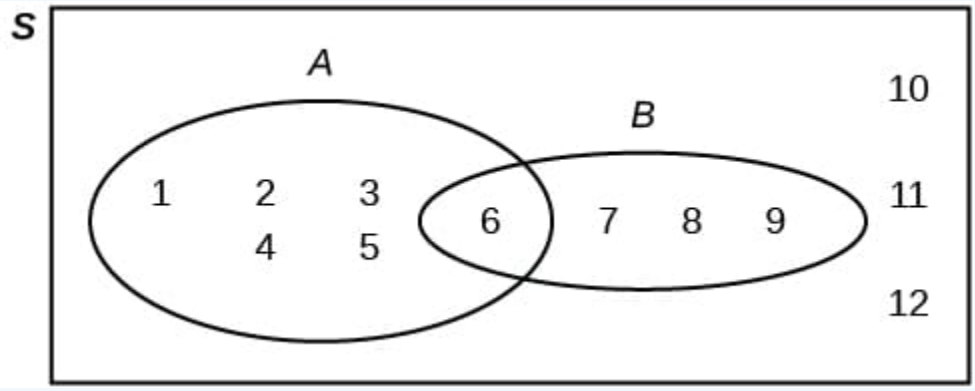


Figure 9.6.5:

Exercise 9.6.4

Suppose an experiment has outcomes black, white, red, orange, yellow, green, blue, and purple, where each outcome has an equal chance of occurring. Let event C = {green, blue, purple} and event P = {red, yellow, blue}. Then C AND P = {blue} and C OR P = {green, blue, purple, red, yellow}. Draw a Venn diagram representing this situation.

Answer

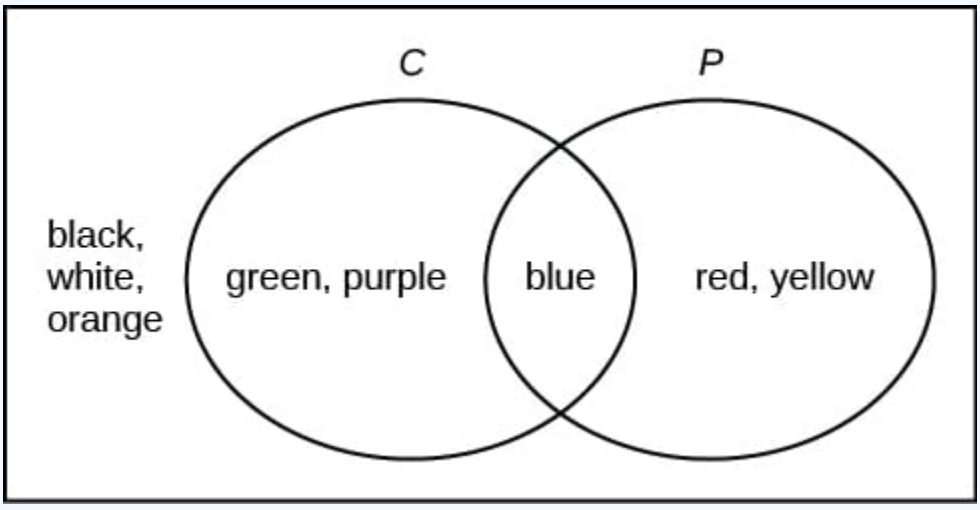


Figure 9.6.6:

Example 9.6.5

Flip two fair coins. Let A = tails on the first coin. Let B = tails on the second coin. Then A = {*TT*,*TH*} and B = {*TT*, *HT*}. Therefore, A AND B = {*TT*}. A OR B = {*TH*, *TT*, *HT*}.

The sample space when you flip two fair coins is X = {*HH*, *HT*, *TH*, *TT*}. The outcome *HH* is in NEITHER A NOR B. The Venn diagram is as follows:

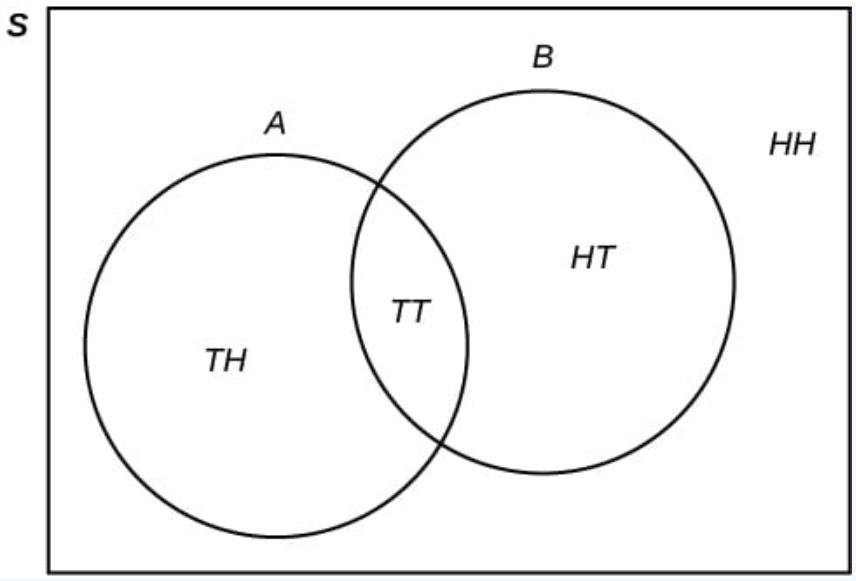
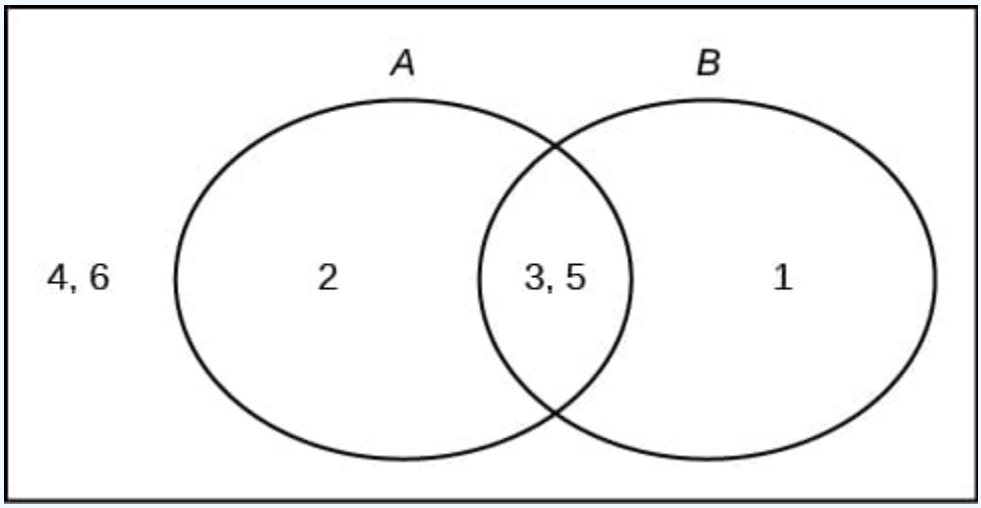


Figure 9.6.7:

Exercise 9.6.5

Roll a fair, six-sided die. Let A = a prime number of dots is rolled. Let B = an odd number of dots is rolled. Then A = {2, 3, 5} and B = {1, 3, 5}. Therefore, A AND B = {3, 5}. A OR B = {1, 2, 3, 5}. The sample space for rolling a fair die is S = {1, 2, 3, 4, 5, 6}. Draw a Venn diagram representing this situation.

Answer



Example 9.6.6: Probability and Venn Diagrams

Forty percent of the students at a local college belong to a club and 50% work part time. Five percent of the students work part time and belong to a club. Draw a Venn diagram showing the relationships. Let C = student belongs to a club and PT = student works part time.

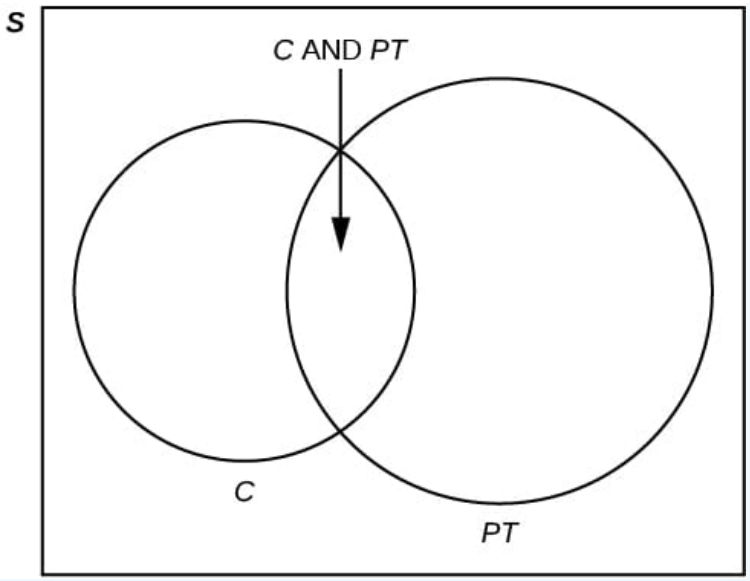


Figure 9.6.8:

If a student is selected at random, find

- the probability that the student belongs to a club. $P(\mathrm{C}) = 0.40$
- the probability that the student works part time. $P(\mathrm{PT}) = 0.50$
- the probability that the student belongs to a club AND works part time. $P(\mathrm{C\ AND\ PT}) = 0.05$
- the probability that the student belongs to a club **given** that the student works part time. $P(\mathrm{C|PT}) = \frac{P(\mathrm{C\ AND\ PT})}{P(\mathrm{PT})} = \frac{0.05}{0.50} = 0.1$
- the probability that the student belongs to a club **OR** works part time. $P(\mathrm{C\ OR\ PT}) = P(\mathrm{C}) + P(\mathrm{PT}) - P(\mathrm{C\ AND\ PT}) = 0.40 + 0.50 - 0.05 = 0.85$

Exercise 9.6.6

Fifty percent of the workers at a factory work a second job, 25% have a spouse who also works, 5% work a second job and have a spouse who also works. Draw a Venn diagram showing the relationships. Let W = works a second job and S = spouse also works.

Answer

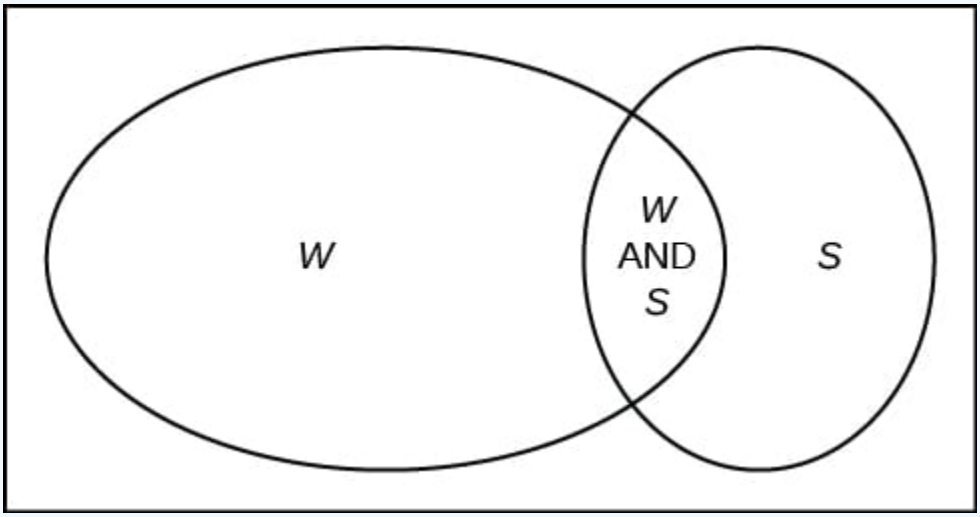


Figure 9.6.9:

Example 9.6.7

A person with type O blood and a negative Rh factor (Rh-) can donate blood to any person with any blood type. Four percent of African Americans have type O blood and a negative RH factor, 5–10% of African Americans have the Rh- factor, and 51% have type O blood.

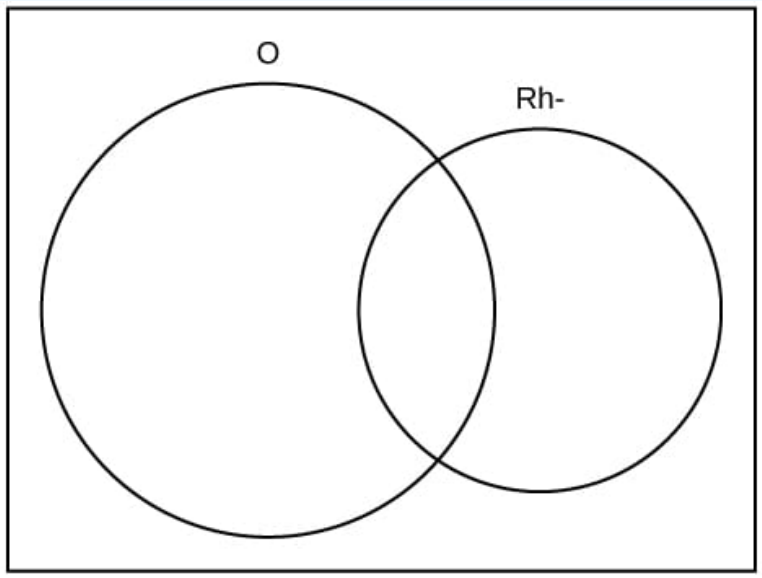


Figure 9.6.10:

The "O" circle represents the African Americans with type O blood. The "Rh-" oval represents the African Americans with the Rh- factor.

We will take the average of 5% and 10% and use 7.5% as the percent of African Americans who have the Rh- factor. Let $\mathrm{O} =$ African American with Type O blood and $\mathrm{R} =$ African American with Rh- factor.

a. $P(\mathrm{O}) =$ _____
b. $P(\mathrm{R}) =$ _____
c. $P(\mathrm{O\ AND\ R}) =$ _____
d. $P(\mathrm{O\ OR\ R}) =$ _____
e. In the Venn Diagram, describe the overlapping area using a complete sentence.
f. In the Venn Diagram, describe the area in the rectangle but outside both the circle and the oval using a complete sentence.

Answer

a. 0.51; b. 0.075; c. 0.04; d. 0.545; e. The area represents the African Americans that have type O blood and the Rh- factor. f. The area represents the African Americans that have neither type O blood nor the Rh- factor.

Exercise 9.6.7

In a bookstore, the probability that the customer buys a novel is 0.6, and the probability that the customer buys a non-fiction book is 0.4. Suppose that the probability that the customer buys both is 0.2.

a. Draw a Venn diagram representing the situation.
b. Find the probability that the customer buys either a novel or anon-fiction book.
c. In the Venn diagram, describe the overlapping area using a complete sentence.
d. Suppose that some customers buy only compact disks. Draw an oval in your Venn diagram representing this event.

Answer

a. and d. In the following Venn diagram below, the blue oval represent customers buying a novel, the red oval represents customer buying non-fiction, and the yellow oval customer who buy compact disks.

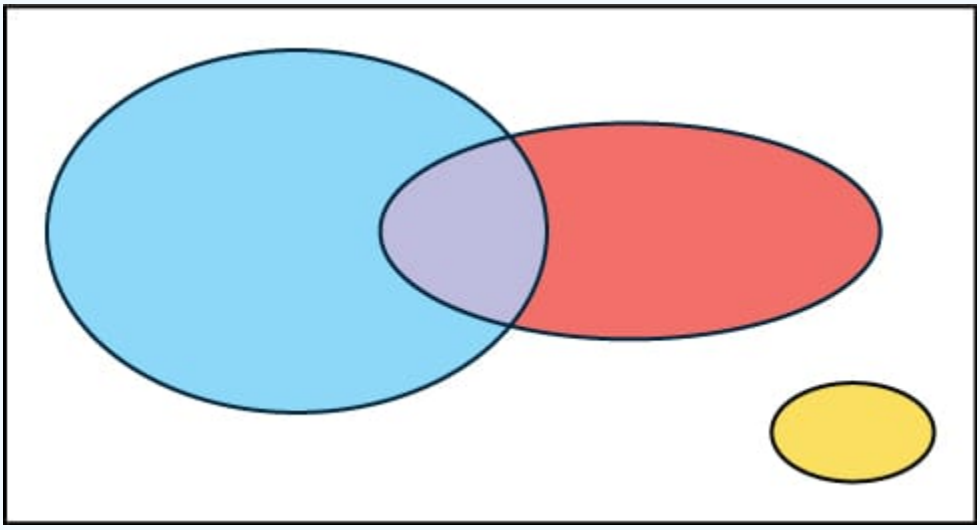


Figure 9.6.11:

b. $P(\text{novel or non-fiction}) = P(\text{Blue OR Red}) = P(\text{Blue}) + P(\text{Red}) - P(\text{Blue AND Red}) = 0.6 + 0.4 - 0.2 = 0.8$.

c. The overlapping area of the blue oval and red oval represents the customers buying both a novel and a nonfiction book.

References

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3. Data from The Data and Story Library, 1996. Available online at http://lib.stat.cmu.edu/DASL/ (accessed May 2, 2013).
4. Data from the Federal Highway Administration, part of the United States Department of Transportation.
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Review

A tree diagram use branches to show the different outcomes of experiments and makes complex probability questions easy to visualize. A Venn diagram is a picture that represents the outcomes of an experiment. It generally consists of a box that represents the sample space S together with circles or ovals. The circles or ovals represent events. A Venn diagram is especially helpful for visualizing the OR event, the AND event, and the complement of an event and for understanding conditional probabilities.

Glossary

Tree Diagram
: the useful visual representation of a sample space and events in the form of a "tree" with branches marked by possible outcomes together with associated probabilities (frequencies, relative frequencies)

Venn Diagram
: the visual representation of a sample space and events in the form of circles or ovals showing their intersections

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9.7: Probability Topics (Worksheet)

Name: _____

Section: _____

Student ID#:_____

Work in groups on these problems. You should try to answer the questions without referring to your textbook. If you get stuck, try asking another group for help.

Student Learning Outcomes

- The student will use theoretical and empirical methods to estimate probabilities.
- The student will appraise the differences between the two estimates.
- The student will demonstrate an understanding of long-term relative frequencies.

Do the Experiment

Count out 40 mixed-color M&Ms® which is approximately one small bag's worth. Record the number of each color in Table. Use the information from this table to complete Table. Next, put the M&Ms in a cup. The experiment is to pick two M&Ms, one at a time. Do **not** look at them as you pick them. The first time through, replace the first M&M before picking the second one. Record the results in the "With Replacement" column of Table. Do this 24 times. The second time through, after picking the first M&M, do **not** replace it before picking the second one. Then, pick the second one. Record the results in the "Without Replacement" column section of Table. After you record the pick, put **both** M&Ms back. Do this a total of 24 times, also. Use the data from Table to calculate the empirical probability questions. Leave your answers in unreduced fractional form. Do **not** multiply out any fractions.

Population

Color	Quantity
Yellow (Y)	
Green (G)	
Blue (BL)	
Brown (B)	
Orange (O)	
Red (R)	

Theoretical Probabilities

	With Replacement	Without Replacement
P(2 reds)		
$P(R_1B_2$ OR $B_1R_2)$		
$P(R_1$ AND $G_2)$		
$P(G_2\|R_1)$		
P(no yellows)		
P(doubles)		
P(no doubles)		

G_2 = green on second pick; R_1 = red on first pick; B_1 = brown on first pick; B_2 = brown on second pick; doubles = both picks are the same colour.

Empirical Results

With Replacement	Without Replacement
(__ , __) (__ , __)	(__ , __) (__ , __)
(__ , __) (__ , __)	(__ , __) (__ , __)
(__ , __) (__ , __)	(__ , __) (__ , __)
(__ , __) (__ , __)	(__ , __) (__ , __)
(__ , __) (__ , __)	(__ , __) (__ , __)
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(__ , __) (__ , __)	(__ , __) (__ , __)
(__ , __) (__ , __)	(__ , __) (__ , __)
(__ , __) (__ , __)	(__ , __) (__ , __)

Empirical Probabilities

	With Replacement	Without Replacement
P(2 reds)		
$P(R_1B_2$ OR $B_1R_2)$		
$P(R_1$ AND $G_2)$		
$P(G_2\|R_1)$		
P(no yellows)		
P(doubles)		
P(no doubles)		

Discussion Questions

a. Why are the “With Replacement” and “Without Replacement” probabilities different?
b. Convert P(no yellows) to decimal format for both Theoretical “With Replacement” and for Empirical “With Replacement”. Round to four decimal places.
    1. Theoretical “With Replacement”: P(no yellows) = _____
    2. Empirical “With Replacement”: P(no yellows) = _____
    3. Are the decimal values “close”? Did you expect them to be closer together or farther apart? Why?
c. If you increased the number of times you picked two M&Ms to 240 times, why would empirical probability values change?
d. Would this change (see part 3) cause the empirical probabilities and theoretical probabilities to be closer together or farther apart? How do you know?
e. Explain the differences in what $P(G_1$ AND $R_2)$ and $P(R_1|G_2)$ represent. Hint: Think about the sample space for each probability.

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9.E: Probability Topics (Exericses)

These are homework exercises to accompany the Textmap created for "Introductory Statistics" by OpenStax.

3.1: Introduction

3.2: Terminology

Q 3.2.1

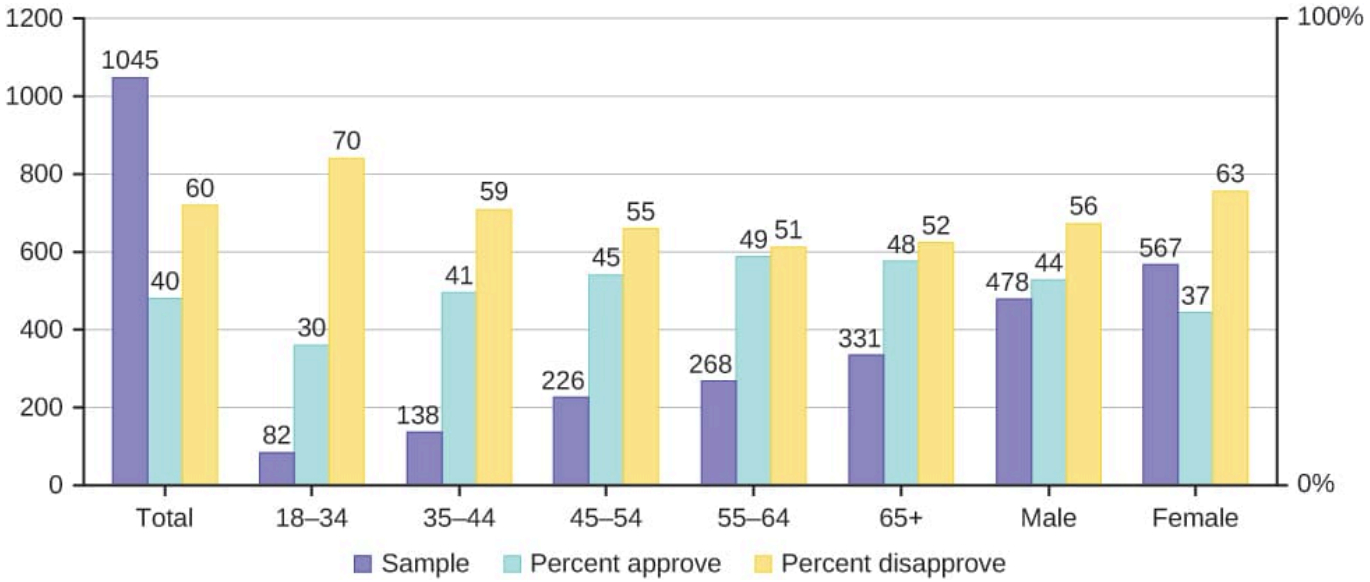


Figure 3.2.3.2.11.

The graph in Figure 3.2.1 displays the sample sizes and percentages of people in different age and gender groups who were polled concerning their approval of Mayor Ford's actions in office. The total number in the sample of all the age groups is 1,045.

a. Define three events in the graph.
b. Describe in words what the entry 40 means.
c. Describe in words the complement of the entry in question 2.
d. Describe in words what the entry 30 means.
e. Out of the males and females, what percent are males?
f. Out of the females, what percent disapprove of Mayor Ford?
g. Out of all the age groups, what percent approve of Mayor Ford?
h. Find $P(\text{Approve}|\text{Male})$.
i. Out of the age groups, what percent are more than 44 years old?
j. Find $P(\text{Approve}|\text{Age} < 35)$.

Q 3.2.2

Explain what is wrong with the following statements. Use complete sentences.

a. If there is a 60% chance of rain on Saturday and a 70% chance of rain on Sunday, then there is a 130% chance of rain over the weekend.
b. The probability that a baseball player hits a home run is greater than the probability that he gets a successful hit.

S 3.2.2

a. You can't calculate the joint probability knowing the probability of both events occurring, which is not in the information given; the probabilities should be multiplied, not added; and probability is never greater than 100%
b. A home run by definition is a successful hit, so he has to have at least as many successful hits as home runs.

3.3: Independent and Mutually Exclusive Events

Use the following information to answer the next 12 exercises. The graph shown is based on more than 170,000 interviews done by Gallup that took place from January through December 2012. The sample consists of employed Americans 18 years of age or older. The Emotional Health Index Scores are the sample space. We randomly sample one Emotional Health Index Score.

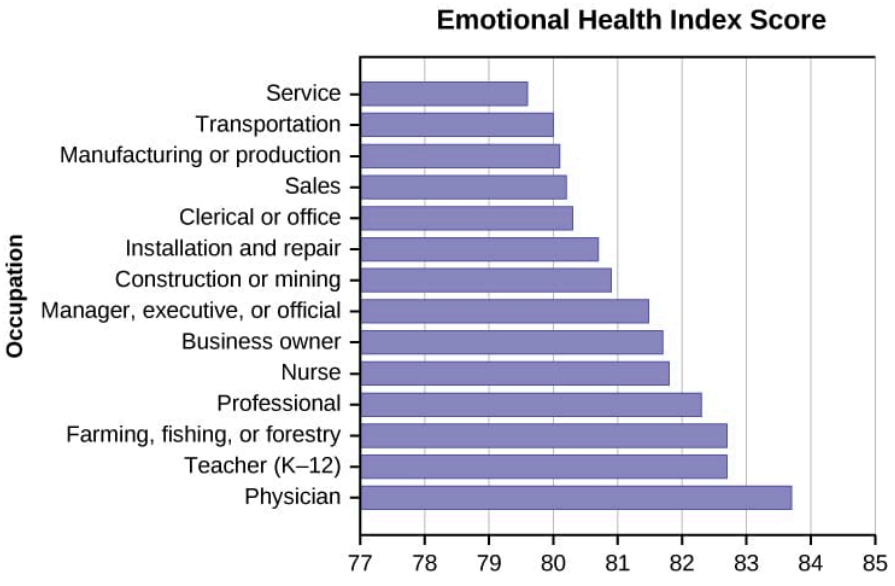


Figure 3.3.1.

Q 3.3.1

Find the probability that an Emotional Health Index Score is 82.7.

Q 3.3.2

Find the probability that an Emotional Health Index Score is 81.0.

S 3.3.2

0

Q 3.3.3

Find the probability that an Emotional Health Index Score is more than 81?

Q 3.3.4

Find the probability that an Emotional Health Index Score is between 80.5 and 82?

S 3.3.4

0.3571

Q 3.3.5

If we know an Emotional Health Index Score is 81.5 or more, what is the probability that it is 82.7?

Q 3.3.6

What is the probability that an Emotional Health Index Score is 80.7 or 82.7?

S 3.3.6

0.2142

Q 3.3.7

What is the probability that an Emotional Health Index Score is less than 80.2 given that it is already less than 81.

Q 3.3.8

What occupation has the highest emotional index score?

S 3.3.8

Physician (83.7)

Q 3.3.9

What occupation has the lowest emotional index score?

Q 3.3.10

What is the range of the data?

S 3.3.10

83.7 – 79.6 = 4.1

Q 3.3.11

Compute the average EHIS.

Q 3.3.12

If all occupations are equally likely for a certain individual, what is the probability that he or she will have an occupation with lower than average EHIS?

S 3.3.12

$P(\text{Occupation} < 81.3) = 0.5$

3.4: Two Basic Rules of Probability

Q 3.4.1

On February 28, 2013, a Field Poll Survey reported that 61% of California registered voters approved of allowing two people of the same gender to marry and have regular marriage laws apply to them. Among 18 to 39 year olds (California registered voters), the approval rating was 78%. Six in ten California registered voters said that the upcoming Supreme Court's ruling about the constitutionality of California's Proposition 8 was either very or somewhat important to them. Out of those CA registered voters who support same-sex marriage, 75% say the ruling is important to them.

In this problem, let:

- C = California registered voters who support same-sex marriage.
- B = California registered voters who say the Supreme Court's ruling about the constitutionality of California's Proposition 8 is very or somewhat important to them
- A = California registered voters who are 18 to 39 years old.

a. Find $P(\mathrm{C})$.
b. Find $P(\mathrm{B})$.
c. Find $P(\mathrm{C}|\mathrm{A})$.
d. Find $P(\mathrm{B}|\mathrm{C})$.
e. In words, what is $\mathrm{C}|\mathrm{A}$?
f. In words, what is $\mathrm{B}|\mathrm{C}$?
g. Find $P(\mathrm{C}\ \mathrm{AND}\ \mathrm{B})$.
h. In words, what is $\mathrm{C}\ \mathrm{AND}\ \mathrm{B}$?
i. Find $P(\mathrm{C}\ \mathrm{OR}\ \mathrm{B})$.
j. Are C and B mutually exclusive events? Show why or why not.

Q 3.4.2

After Rob Ford, the mayor of Toronto, announced his plans to cut budget costs in late 2011, the Forum Research polled 1,046 people to measure the mayor's popularity. Everyone polled expressed either approval or disapproval. These are the results their poll produced:

- In early 2011, 60 percent of the population approved of Mayor Ford's actions in office.
- In mid-2011, 57 percent of the population approved of his actions.
- In late 2011, the percentage of popular approval was measured at 42 percent.

a. What is the sample size for this study?
b. What proportion in the poll disapproved of Mayor Ford, according to the results from late 2011?
c. How many people polled responded that they approved of Mayor Ford in late 2011?
d. What is the probability that a person supported Mayor Ford, based on the data collected in mid-2011?
e. What is the probability that a person supported Mayor Ford, based on the data collected in early 2011?

S 3.4.2

a. The Forum Research surveyed 1,046 Torontonians.
b. 58%
c. 42% of 1,046 = 439 (rounding to the nearest integer)
d. 0.57
e. 0.60.

Use the following information to answer the next three exercises. The casino game, roulette, allows the gambler to bet on the probability of a ball, which spins in the roulette wheel, landing on a particular color, number, or range of numbers. The table used to place bets contains of 38 numbers, and each number is assigned to a color and a range.

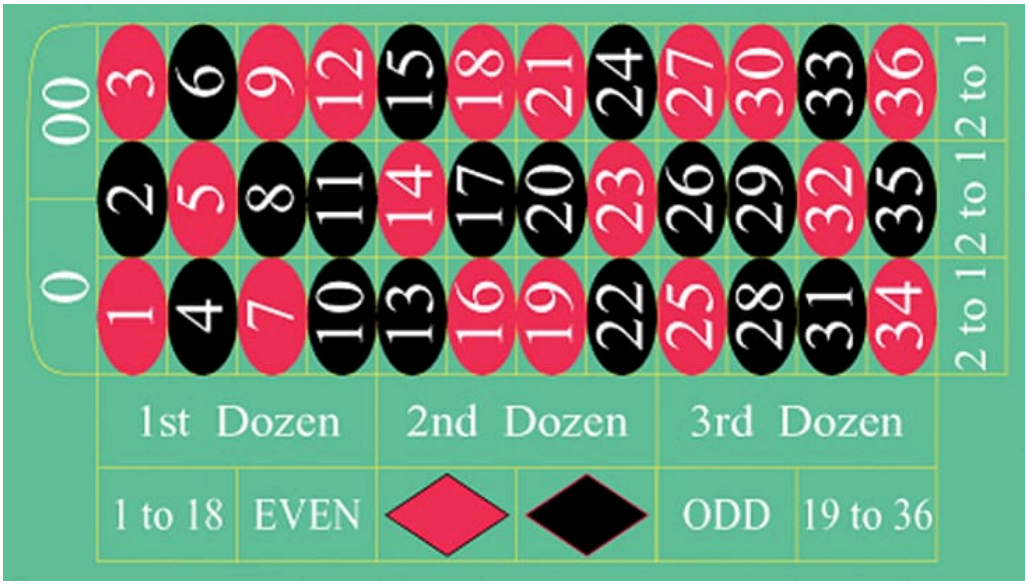


Figure 3.4.1

Q 3.4.3

a. List the sample space of the 38 possible outcomes in roulette.
b. You bet on red. Find $P(\text{red})$.
c. You bet on -1st 12- (1st Dozen). Find $P(\text{-1st 12-})$.
d. You bet on an even number. Find $P(\text{even number})$.
e. Is getting an odd number the complement of getting an even number? Why?
f. Find two mutually exclusive events.
g. Are the events Even and 1st Dozen independent?

Q 3.4.4

Compute the probability of winning the following types of bets:

a. Betting on two lines that touch each other on the table as in 1-2-3-4-5-6
b. Betting on three numbers in a line, as in 1-2-3
c. Betting on one number
d. Betting on four numbers that touch each other to form a square, as in 10-11-13-14
e. Betting on two numbers that touch each other on the table, as in 10-11 or 10-13
f. Betting on 0-00-1-2-3
g. Betting on 0-1-2; or 0-00-2; or 00-2-3

S 3.4.4

1. $P(\text{Betting on two line that touch each other on the table}) = \frac{6}{38}$
2. $P(\text{Betting on three numbers in a line}) = \frac{3}{38}$
3. $P(\text{Betting on one number}) = \frac{1}{38}$
4. $P(\text{Betting on four number that touch each other to form a square}) = \frac{4}{38}$
5. $P(\text{Betting on two number that touch each other on the table}) = \frac{2}{38}$
6. $P(\text{Betting on 0-00-1-2-3}) = \frac{5}{38}$
7. $P(\text{Betting on 0-1-2; or 0-00-2; or 00-2-3}) = \frac{3}{38}$

Q 3.4.5

Compute the probability of winning the following types of bets:

a. Betting on a color
b. Betting on one of the dozen groups
c. Betting on the range of numbers from 1 to 18
d. Betting on the range of numbers 19–36
e. Betting on one of the columns
f. Betting on an even or odd number (excluding zero)

Q 3.4.6

Suppose that you have eight cards. Five are green and three are yellow. The five green cards are numbered 1, 2, 3, 4, and 5. The three yellow cards are numbered 1, 2, and 3. The cards are well shuffled. You randomly draw one card.

- $\mathrm{G} =$ card drawn is green
- $\mathrm{E} =$ card drawn is even-numbered
  a. List the sample space.
  b. $P(\mathrm{G}) =$ _____
  c. $P(\mathrm{G}|\mathrm{E}) =$ _____
  d. $P(\mathrm{G\ AND\ E}) =$ _____
  e. $P(\mathrm{G\ OR\ E}) =$ _____
  f. Are G and E mutually exclusive? Justify your answer numerically.

S 3.4.6

1. $\{G1, G2, G3, G4, G5, Y1, Y2, Y3\}$
2. $\frac{5}{8}$
3. $\frac{2}{3}$
4. $\frac{2}{8}$
5. $\frac{1}{6}8$
6. No, because $P(\mathrm{G\ AND\ E})$ does not equal 0.

Q 3.4.7

Roll two fair dice. Each die has six faces.

a. List the sample space.
b. Let A be the event that either a three or four is rolled first, followed by an even number. Find $P(\mathrm{A})$.
c. Let *B* be the event that the sum of the two rolls is at most seven. Find $P(\mathrm{B})$.
d. In words, explain what "$P(\mathrm{A}|\mathrm{B})$" represents. Find $P(\mathrm{A}|\mathrm{B})$.
e. Are A and B mutually exclusive events? Explain your answer in one to three complete sentences, including numerical justification.
f. Are A and B independent events? Explain your answer in one to three complete sentences, including numerical justification.

Q 3.4.8

A special deck of cards has ten cards. Four are green, three are blue, and three are red. When a card is picked, its color of it is recorded. An experiment consists of first picking a card and then tossing a coin.

a. List the sample space.
b. Let A be the event that a blue card is picked first, followed by landing a head on the coin toss. Find $P(\mathrm{A})$.
c. Let B be the event that a red or green is picked, followed by landing a head on the coin toss. Are the events A and B mutually exclusive? Explain your answer in one to three complete sentences, including numerical justification.
d. Let C be the event that a red or blue is picked, followed by landing a head on the coin toss. Are the events A and C mutually

exclusive? Explain your answer in one to three complete sentences, including numerical justification.

S 3.4.9

The coin toss is independent of the card picked first.

a. $\{(G,H)(G,T)(B,H)(B,T)(R,H)(R,T)\}$
b. $P(\text{A}) = P(\text{blue})P(\text{head}) = \left(\frac{3}{10}\right)\left(\frac{1}{2}\right) = \frac{3}{20}$
c. Yes, A and B are mutually exclusive because they cannot happen at the same time; you cannot pick a card that is both blue and also (red or green). $P(\text{A AND B}) = 0$
d. No, A and C are not mutually exclusive because they can occur at the same time. In fact, C includes all of the outcomes of A ; if the card chosen is blue it is also (red or blue). $P(\text{A AND C}) = P(\text{A}) = 320$

Q 3.4.10

An experiment consists of first rolling a die and then tossing a coin.

a. List the sample space.
b. Let A be the event that either a three or a four is rolled first, followed by landing a head on the coin toss. Find $P(\text{A})$.
c. Let B be the event that the first and second tosses land on heads. Are the events $\text{A}$and B mutually exclusive? Explain your answer in one to three complete sentences, including numerical justification.

Q 3.4.11

An experiment consists of tossing a nickel, a dime, and a quarter. Of interest is the side the coin lands on.

a. List the sample space.
b. Let A be the event that there are at least two tails. Find $P(\text{A})$.
c. Let B be the event that the first and second tosses land on heads. Are the events A and B mutually exclusive? Explain your answer in one to three complete sentences, including justification.

S 3.4.12

a. $S = (\text{HHH}), (\text{HHT}), (\text{HTH}), (\text{HTT}), (\text{THH}), (\text{THT}), (\text{TTH}), (\text{TTT})$
b. $\frac{4}{8}$
c. Yes, because if A has occurred, it is impossible to obtain two tails. In other words, $P(\text{A AND B}) = 0$.

Q 3.4.13

Consider the following scenario:

Let $P(\text{C}) = 0.4$.

Let $P(\text{D}) = 0.5$.

Let $P(\text{C}|\text{D}) = 0.6$.

a. Find $P(CANDD)$.
b. Are C and D mutually exclusive? Why or why not?
c. Are C and D independent events? Why or why not?
d. Find $P(\text{C OR D})$.
e. Find $P(\text{D}|\text{C})$.

Q 3.4.14

Y and Z are independent events.

a. Rewrite the basic Addition Rule $P(\mathrm{Y\ OR\ Z}) = P(\mathrm{Y}) + P(\mathrm{Z}) - P(\mathrm{Y\ AND\ Z})$ using the information that Y and Z are independent events.
b. Use the rewritten rule to find $P(\mathrm{Z})$ if $P(\mathrm{Y\ OR\ Z}) = 0.71$ and $P(\mathrm{Y}) = 0.42$.

S 3.4.14

a. If Y and Z are independent, then $P(\mathrm{Y\ AND\ Z}) = P(\mathrm{Y})P(\mathrm{Z})$, so $P(\mathrm{Y\ OR\ Z}) = P(\mathrm{Y}) + P(\mathrm{Z}) - P(\mathrm{Y})P(\mathrm{Z})$.
b. 0.5

Q 3.4.15

G and H are mutually exclusive events. $P(\mathrm{G}) = 0.5 P(\mathrm{H}) = 0.3$

a. Explain why the following statement MUST be false: $P(\mathrm{H}|\mathrm{G}) = 0.4$.
b. Find $P(\mathrm{H\ OR\ G})$.
c. Are G and H independent or dependent events? Explain in a complete sentence.

Q 3.4.16

Approximately 281,000,000 people over age five live in the United States. Of these people, 55,000,000 speak a language other than English at home. Of those who speak another language at home, 62.3% speak Spanish.

Let: E = speaks English at home; E' = speaks another language at home; S = speaks Spanish;

Finish each probability statement by matching the correct answer.

Probability Statements	Answers
a. $P(\mathrm{E}') =$	i. 0.8043
b. $P(\mathrm{E}) =$	ii. 0.623
c. $P(\mathrm{S\ and\ E}') =$	iii. 0.1957
d. $P(\mathrm{S}|\mathrm{E}') =$	iv. 0.1219

S 3.4.16

a. iii
b. i
c. iv
d. ii

Q 3.4.17

1994, the U.S. government held a lottery to issue 55,000 Green Cards (permits for non-citizens to work legally in the U.S.). Renate Deutsch, from Germany, was one of approximately 6.5 million people who entered this lottery. Let G = won green card.

a. What was Renate's chance of winning a Green Card? Write your answer as a probability statement.
b. In the summer of 1994, Renate received a letter stating she was one of 110,000 finalists chosen. Once the finalists were chosen, assuming that each finalist had an equal chance to win, what was Renate's chance of winning a Green Card? Write your answer as a conditional probability statement. Let F = was a finalist.
c. Are G and F independent or dependent events? Justify your answer numerically and also explain why.

d. Are G and F mutually exclusive events? Justify your answer numerically and explain why.

Q 3.4.18

Three professors at George Washington University did an experiment to determine if economists are more selfish than other people. They dropped 64 stamped, addressed envelopes with $10 cash in different classrooms on the George Washington campus. 44% were returned overall. From the economics classes 56% of the envelopes were returned. From the business, psychology, and history classes 31% were returned.

Let: R = money returned; E = economics classes; O = other classes

a. Write a probability statement for the overall percent of money returned.
b. Write a probability statement for the percent of money returned out of the economics classes.
c. Write a probability statement for the percent of money returned out of the other classes.
d. Is money being returned independent of the class? Justify your answer numerically and explain it.
e. Based upon this study, do you think that economists are more selfish than other people? Explain why or why not. Include numbers to justify your answer.

S 3.4.18

a. $P(\mathrm{R}) = 0.44$
b. $P(\mathrm{R}|\mathrm{E}) = 0.56$
c. $P(\mathrm{R}|\mathrm{O}) = 0.31$
d. No, whether the money is returned is not independent of which class the money was placed in. There are several ways to justify this mathematically, but one is that the money placed in economics classes is not returned at the same overall rate; $P(\mathrm{R}|\mathrm{E}) \neq P(\mathrm{R})$.
e. No, this study definitely does not support that notion; *in fact*, it suggests the opposite. The money placed in the economics classrooms was returned at a higher rate than the money place in all classes collectively; $P(\mathrm{R}|\mathrm{E}) > P(\mathrm{R})$.

Q 3.4.19

The following table of data obtained from www.baseball-almanac.com shows hit information for four players. Suppose that one hit from the table is randomly selected.

Name	Single	Double	Triple	Home Run	Total Hits
Babe Ruth	1,517	506	136	714	2,873
Jackie Robinson	1,054	273	54	137	1,518
Ty Cobb	3,603	174	295	114	4,189
Hank Aaron	2,294	624	98	755	3,771
Total	8,471	1,577	583	1,720	12,351

Are "the hit being made by Hank Aaron" and "the hit being a double" independent events?

a. Yes, because $P(\text{hit by Hank Aaron}|\text{hit is a double}) = P(\text{hit by Hank Aaron})$
b. No, because $P(\text{hit by Hank Aaron}|\text{hit is a double}) \neq P(\text{hit is a double})$
c. No, because $P(\text{hit is by Hank Aaron}|\text{hit is a double}) \neq P(\text{hit by Hank Aaron})$
d. Yes, because $P(\text{hit is by Hank Aaron}|\text{hit is a double}) = P(\text{hit is a double})$

Q 3.4.29

United Blood Services is a blood bank that serves more than 500 hospitals in 18 states. According to their website, a person with type O blood and a negative Rh factor (Rh-) can donate blood to any person with any blood type. Their data show that 43% of people have type O blood and 15% of people have Rh- factor; 52% of people have type O or Rh- factor.

a. Find the probability that a person has both type O blood and the Rh- factor.
b. Find the probability that a person does NOT have both type O blood and the Rh- factor.

S 3.4.30

a. $P(\text{type O OR Rh-}) = P(\text{type O}) + P(\text{Rh-}) - P(\text{type O AND Rh-})$

   $0.52 = 0.43 + 0.15 - P(\text{type O AND Rh-})$; solve to find $P(\text{type O AND Rh-}) = 0.06$

   6% of people have type O, Rh- blood

b. $P(\text{NOT}(\text{type O AND Rh-})) = 1 - P(\text{type O AND Rh-}) = 1 - 0.06 = 0.94$

   94% of people do not have type O, Rh- blood

Q 3.4.31

At a college, 72% of courses have final exams and 46% of courses require research papers. Suppose that 32% of courses have a research paper and a final exam. Let *F* be the event that a course has a final exam. Let *R* be the event that a course requires a research paper.

1. Find the probability that a course has a final exam or a research project.
2. Find the probability that a course has NEITHER of these two requirements.

Q 3.4.32

In a box of assorted cookies, 36% contain chocolate and 12% contain nuts. Of those, 8% contain both chocolate and nuts. Sean is allergic to both chocolate and nuts.

1. Find the probability that a cookie contains chocolate or nuts (he can't eat it).
2. Find the probability that a cookie does not contain chocolate or nuts (he can eat it).

S 3.4.32

a. Let $C =$ be the event that the cookie contains chocolate. Let $N =$ the event that the cookie contains nuts.
b. $P(\text{C OR N}) = P(\text{C}) + P(\text{N}) - P(\text{C AND N}) = 0.36 + 0.12 - 0.08 = 0.40$
c. $P(\text{NEITHER chocolate NOR nuts}) = 1 - P(\text{C OR N}) = 1 - 0.40 = 0.60$

Q 3.4.33

A college finds that 10% of students have taken a distance learning class and that 40% of students are part time students. Of the part time students, 20% have taken a distance learning class. Let $\text{D} =$ event that a student takes a distance learning class and $\text{E} =$ event that a student is a part time student

a. Find $P(\text{D AND E})$.
b. Find $P(\text{E}|\text{D})$.
c. Find $P(\text{D OR E})$.
d. Using an appropriate test, show whether D and E are independent.
e. Using an appropriate test, show whether D and E are mutually exclusive.

3.5: Contingency Tables

Use the information in the Table to answer the next eight exercises. The table shows the political party affiliation of each of 67 members of the US Senate in June 2012, and when they are up for reelection.

Up for reelection:	Democratic Party	Republican Party	Other	Total
November 2014	20	13	0	
November 2016	10	24	0	
Total				

Q 3.5.1

What is the probability that a randomly selected senator has an “Other” affiliation?

S 3.5.1

0

Q 3.5.2

What is the probability that a randomly selected senator is up for reelection in November 2016?

Q 3.5.3

What is the probability that a randomly selected senator is a Democrat and up for reelection in November 2016?

S 3.5.3

$\frac{10}{67}$

Q 3.5.4

What is the probability that a randomly selected senator is a Republican or is up for reelection in November 2014?

Q 3.5.5

Suppose that a member of the US Senate is randomly selected. Given that the randomly selected senator is up for reelection in November 2016, what is the probability that this senator is a Democrat?

S 3.5.5

$\frac{10}{34}$

Q 3.5.6

Suppose that a member of the US Senate is randomly selected. What is the probability that the senator is up for reelection in November 2014, knowing that this senator is a Republican?

Q 3.5.7

The events “Republican” and “Up for reelection in 2016” are _____

a. mutually exclusive.
b. independent.
c. both mutually exclusive and independent.

d. neither mutually exclusive nor independent.

S 3.5.7

d

Q 3.5.8

The events “Other” and “Up for reelection in November 2016” are _____

a. mutually exclusive.
b. independent.
c. both mutually exclusive and independent.
d. neither mutually exclusive nor independent.

Q 3.5.9

This table gives the number of participants in the recent National Health Interview Survey who had been treated for cancer in the previous 12 months. The results are sorted by age, race (black or white), and sex. We are interested in possible relationships between age, race, and sex.

Race and Sex	15-24	25-40	41-65	over 65	TOTALS
white, male	1,165	2,036	3,703		8,395
white, female	1,076	2,242	4,060		9,129
black, male	142	194	384		824
black, female	131	290	486		1,061
all others					
TOTALS	2,792	5,279	9,354		21,081

Do not include "all others" for parts f and g.

a. Fill in the column for cancer treatment for individuals over age 65.
b. Fill in the row for all other races.
c. Find the probability that a randomly selected individual was a white male.
d. Find the probability that a randomly selected individual was a black female.
e. Find the probability that a randomly selected individual was black
f. Find the probability that a randomly selected individual was a black or white male.
g. Out of the individuals over age 65, find the probability that a randomly selected individual was a black or white male.

S 3.5.9

a.

Race and Sex	1–14	15–24	25–64	over 64	TOTALS
white, male	210	3,360	13,610	4,870	22,050
white, female	80	580	3,380	890	4,930
black, male	10	460	1,060	140	1,670
black, female	0	40	270	20	330
all others				100	
TOTALS	310	4,650	18,780	6,020	29,760

b.

Race and Sex	1–14	15–24	25–64	over 64	TOTALS
white, male	210	3,360	13,610	4,870	22,050
white, female	80	580	3,380	890	4,930
black, male	10	460	1,060	140	1,670
black, female	0	40	270	20	330
all others	10	210	460	100	780
TOTALS	310	4,650	18,780	6,020	29,760

c. $\frac{22,050}{29,760}$
d. $\frac{330}{29,760}$
e. $\frac{2,000}{29,760}$
f. $\frac{23,720}{29,760}$
g. $\frac{5,010}{6,020}$

Use the following information to answer the next two exercises. The table of data obtained from*www.baseball-almanac.com* shows hit information for four well known baseball players. Suppose that one hit from the table is randomly selected.

NAME	Single	Double	Triple	Home Run	TOTAL HITS
Babe Ruth	1,517	506	136	714	2,873
Jackie Robinson	1,054	273	54	137	1,518
Ty Cobb	3,603	174	295	114	4,189
Hank Aaron	2,294	624	98	755	3,771
TOTAL	8,471	1,577	583	1,720	12,351

Q 3.5.10

Find $P(\text{hit was made by Babe Ruth})$.

a. $\frac{1518}{2873}$
b. $\frac{2873}{12351}$
c. $\frac{583}{12351}$
d. $\frac{4189}{12351}$

Q 3.5.11

Find $P(\text{hit was made by Ty Cobb}|\text{The hit was a Home Run})$.

a. $\frac{4189}{12351}$
b. $\frac{114}{1720}$
c. $\frac{1720}{4189}$
d. $\frac{114}{12351}$

S 3.5.11

b

Q 3.5.12

Table identifies a group of children by one of four hair colors, and by type of hair.

Hair Type	Brown	Blond	Black	Red	Totals
Wavy	20		15	3	43
Straight	80	15		12	
Totals		20			215

a. Complete the table.
b. What is the probability that a randomly selected child will have wavy hair?
c. What is the probability that a randomly selected child will have either brown or blond hair?
d. What is the probability that a randomly selected child will have wavy brown hair?
e. What is the probability that a randomly selected child will have red hair, given that he or she has straight hair?
f. If B is the event of a child having brown hair, find the probability of the complement of B.
g. In words, what does the complement of B represent?

Q 3.5.13

In a previous year, the weights of the members of the **San Francisco 49ers** and the **Dallas Cowboys** were published in the *San Jose Mercury News*. The factual data were compiled into the following table.

Shirt#	≤ 210	211–250	251–290	> 290
1–33	21	5	0	0
34–66	6	18	7	4
66–99	6	12	22	5

For the following, suppose that you randomly select one player from the 49ers or Cowboys.

a. Find the probability that his shirt number is from 1 to 33.
b. Find the probability that he weighs at most 210 pounds.
c. Find the probability that his shirt number is from 1 to 33 AND he weighs at most 210 pounds.
d. Find the probability that his shirt number is from 1 to 33 OR he weighs at most 210 pounds.
e. Find the probability that his shirt number is from 1 to 33 GIVEN that he weighs at most 210 pounds.

S 3.5.13

a. $\frac{26}{106}$
b. $\frac{33}{106}$
c. $\frac{21}{106}$
d. $\left(\frac{26}{106}\right)+\left(\frac{33}{106}\right)-\left(\frac{21}{106}\right)=\left(\frac{38}{106}\right)$
e. $\frac{21}{33}$

3.6: Tree and Venn Diagrams

Exercise 3.6.8

The probability that a man develops some form of cancer in his lifetime is 0.4567. The probability that a man has at least one false positive test result (meaning the test comes back for cancer when the man does not have it) is 0.51. Let: $\mathrm{C}=$ a man develops cancer in his lifetime; $\mathrm{P}=$ man has at least one false positive. Construct a tree diagram of the situation.

Answer

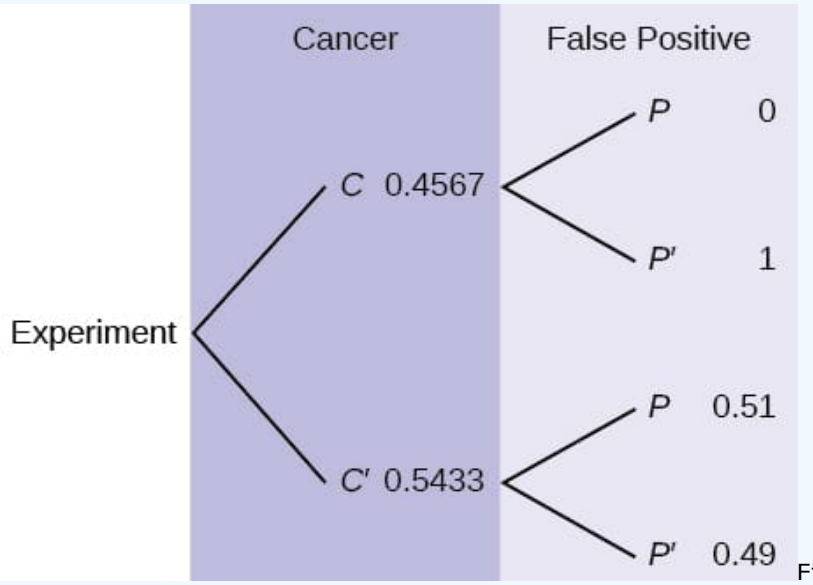


Figure 3.6.12

Bring It Together

Use the following information to answer the next two exercises. Suppose that you have eight cards. Five are green and three are yellow. The cards are well shuffled.

Exercise 3.6.9

Suppose that you randomly draw two cards, one at a time, **with replacement**.

Let $G_1 =$ first card is green
Let $G_2 =$ second card is green

a. Draw a tree diagram of the situation.
b. Find $P(G_1 \text{AND} G_2)$.
c. Find $P(\text{at least one green})$.
d. Find $P(G_2|G_1)$.
e. Are G_1 and G_2 independent events? Explain why or why not.

Answer

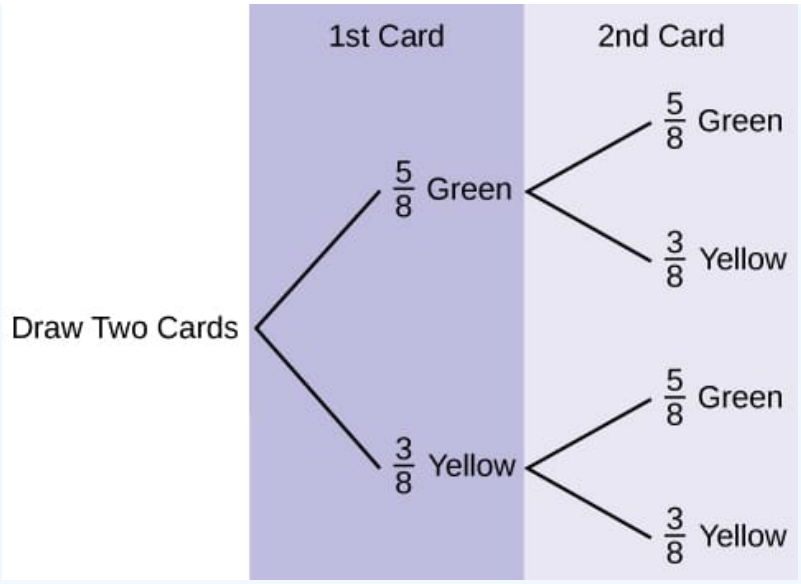
a. 

Figure 3.6.14

b. $P(\text{GG}) = \left(\frac{5}{8}\right)\left(\frac{5}{8}\right) = \frac{25}{64}$
c. $P(\text{at least one green}) = P(\text{GG}) + P(\text{GY}) + P(\text{YG}) = \frac{25}{64} + \frac{15}{64} + \frac{15}{64} = \frac{55}{64}$
d. $P(\text{G}|\text{G}) = \frac{5}{8}$
e. Yes, they are independent because the first card is placed back in the bag before the second card is drawn; the composition of cards in the bag remains the same from draw one to draw two.

Exercise 3.6.10

Suppose that you randomly draw two cards, one at a time, **without replacement**.

G_1 = first card is green
G_2 = second card is green

a. Draw a tree diagram of the situation.
b. Find $P(\text{G}_1 \text{AND } \text{G}_2)$.
c. Find $P(\text{at least one green})$.
d. Find $P(\text{G}_2|\text{G}_1)$.
e. Are G_2 and G_1 independent events? Explain why or why not.

Use the following information to answer the next two exercises. The percent of licensed U.S. drivers (from a recent year) that are female is 48.60. Of the females, 5.03% are age 19 and under; 81.36% are age 20–64; 13.61% are age 65 or over. Of the licensed U.S. male drivers, 5.04% are age 19 and under; 81.43% are age 20–64; 13.53% are age 65 or over.

Exercise 3.6.11

Complete the following.

a. Construct a table or a tree diagram of the situation.
b. Find $P(\text{driver is female})$.
c. Find $P(\text{driver is age 65 or over}|\text{driver is female})$.
d. Find $P(\text{driver is age 65 or over AND female})$.
e. In words, explain the difference between the probabilities in part c and part d.
f. Find $P(\text{driver is age 65 or over})$.
g. Are being age 65 or over and being female mutually exclusive events? How do you know?

Answer

a.

	<20	20–64	>64	Totals
Female	0.0244	0.3954	0.0661	0.486
Male	0.0259	0.4186	0.0695	0.514
Totals	0.0503	0.8140	0.1356	1

b. $P(\text{F}) = 0.486$
c. $P(>64|\text{F}) = 0.1361$
d. $P(>64 \text{ and } \text{F}) = P(\text{F})P(>64|\text{F}) = (0.486)(0.1361) = 0.0661$
e. $P(>64|\text{F})$ is the percentage of female drivers who are 65 or older and $P(>64 \text{ and } \text{F})$ is the percentage of drivers who are female and 65 or older.
f. $P(>64) = P(>64 \text{ and } \text{F}) + P(>64 \text{ and } \text{M}) = 0.1356$
g. No, being female and 65 or older are not mutually exclusive because they can occur at the same time $P(>64 \text{ and } \text{F}) = 0.0661$.

Exercise 3.6.12

Suppose that 10,000 U.S. licensed drivers are randomly selected.

a. How many would you expect to be male?
b. Using the table or tree diagram, construct a contingency table of gender versus age group.
c. Using the contingency table, find the probability that out of the age 20–64 group, a randomly selected driver is female.

Exercise 3.6.13

Approximately 86.5% of Americans commute to work by car, truck, or van. Out of that group, 84.6% drive alone and 15.4% drive in a carpool. Approximately 3.9% walk to work and approximately 5.3% take public transportation.

a. Construct a table or a tree diagram of the situation. Include a branch for all other modes of transportation to work.
b. Assuming that the walkers walk alone, what percent of all commuters travel alone to work?
c. Suppose that 1,000 workers are randomly selected. How many would you expect to travel alone to work?
d. Suppose that 1,000 workers are randomly selected. How many would you expect to drive in a carpool?

Answer

a.

	Car, Truck or Van	Walk	Public Transportation	Other	Totals
Alone	0.7318				
Not Alone	0.1332				
Totals	0.8650	0.0390	0.0530	0.0430	1

b. If we assume that all walkers are alone and that none from the other two groups travel alone (which is a big assumption) we have: $P(\text{Alone}) = 0.7318 + 0.0390 = 0.7708$.
c. Make the same assumptions as in (b) we have: (0.7708)(1,000) = 771
d. (0.1332)(1,000) = 133

Exercise 3.6.14

When the Euro coin was introduced in 2002, two math professors had their statistics students test whether the Belgian one Euro coin was a fair coin. They spun the coin rather than tossing it and found that out of 250 spins, 140 showed a head (event H) while 110 showed a tail (event T). On that basis, they claimed that it is not a fair coin.

a. Based on the given data, find $P(\mathrm{H})$ and $P(\mathrm{T})$.
b. Use a tree to find the probabilities of each possible outcome for the experiment of tossing the coin twice.
c. Use the tree to find the probability of obtaining exactly one head in two tosses of the coin.
d. Use the tree to find the probability of obtaining at least one head.

Exercise 3.6.15

Use the following information to answer the next two exercises. The following are real data from Santa Clara County, CA. As of a certain time, there had been a total of 3,059 documented cases of AIDS in the county. They were grouped into the following categories:

* *includes homosexual/bisexual IV drug users*

	Homosexual/Bisexual	IV Drug User*	Heterosexual Contact	Other	Totals
Female	0	70	136	49	___
Male	2,146	463	60	135	___
Totals	___	___	___	___	___

Suppose a person with AIDS in Santa Clara County is randomly selected.

a. Find $P(\text{Person is female})$.
b. Find $P(\text{Person has a risk factor heterosexual contact})$.
c. Find $P(\text{Person is female OR has a risk factor of IV drug user})$.
d. Find $P(\text{Person is female AND has a risk factor of homosexual/bisexual})$.
e. Find $P(\text{Person is male AND has a risk factor of IV drug user})$.
f. Find $P(\text{Person is female GIVEN person got the disease from heterosexual contact})$.
g. Construct a Venn diagram. Make one group females and the other group heterosexual contact.

Answer

The completed contingency table is as follows:

* *includes homosexual/bisexual IV drug users*

	Homosexual/Bisexual	IV Drug User*	Heterosexual Contact	Other	Totals
Female	0	70	136	49	**255**
Male	2,146	463	60	135	**2,804**
Totals	**2,146**	**533**	**196**	**184**	**3,059**

a. $\frac{255}{2059}$
b. $\frac{196}{3059}$
c. $\frac{718}{3059}$
d. 0
e. $\frac{463}{3059}$
f. $\frac{136}{196}$

g.

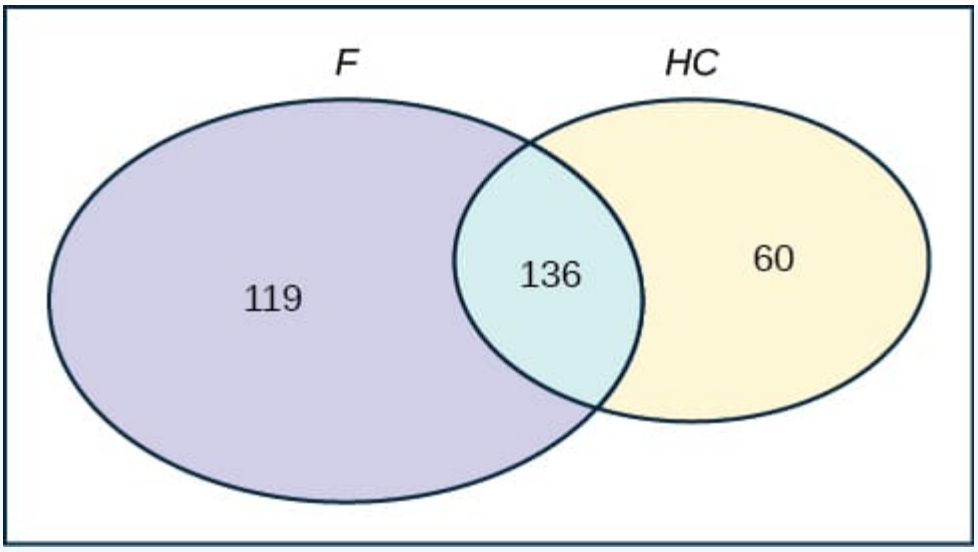


Figure 3.6.15

Exercise 3.6.16

Answer these questions using probability rules. Do NOT use the contingency table. Three thousand fifty-nine cases of AIDS had been reported in Santa Clara County, CA, through a certain date. Those cases will be our population. Of those cases, 6.4% obtained the disease through heterosexual contact and 7.4% are female. Out of the females with the disease, 53.3% got the disease from heterosexual contact.

a. Find $P(\text{Person is female})$.
b. Find $P(\text{Person obtained the disease through heterosexual contact})$.
c. Find $P(\text{Person is female GIVEN person got the disease from heterosexual contact})$
d. Construct a Venn diagram representing this situation. Make one group females and the other group heterosexual contact. Fill in all values as probabilities.

Use the following information to answer the next two exercises. This tree diagram shows the tossing of an unfair coin followed by drawing one bead from a cup containing three red (R), four yellow (Y) and five blue (B) beads. For the coin, $P(\text{H}) = \frac{2}{3}$ and $P(\text{T}) = \frac{1}{3}$ where H is heads and T is tails.

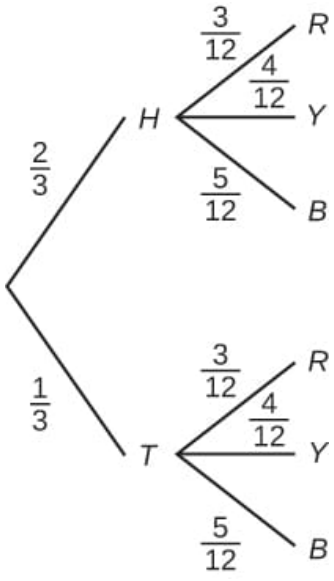


Figure 3.6.1.

Q 3.6.1

Find $P(\text{tossing a Head on the coin AND a Red bead})$

a. $\frac{2}{3}$
b. $\frac{5}{15}$
c. $\frac{6}{36}$
d. $\frac{5}{36}$

Q 3.6.2

Find $P(\text{Blue bead})$.

a. $\frac{15}{36}$
b. $\frac{10}{36}$
c. $\frac{10}{12}$
d. $\frac{6}{36}$

S 3.6.2

a

Q 3.6.3

A box of cookies contains three chocolate and seven butter cookies. Miguel randomly selects a cookie and eats it. Then he randomly selects another cookie and eats it. (How many cookies did he take?)

a. Draw the tree that represents the possibilities for the cookie selections. Write the probabilities along each branch of the tree.
b. Are the probabilities for the flavor of the SECOND cookie that Miguel selects independent of his first selection? Explain.
c. For each complete path through the tree, write the event it represents and find the probabilities.
d. Let S be the event that both cookies selected were the same flavor. Find $P(\mathrm{S})$.
e. Let T be the event that the cookies selected were different flavors. Find $P(\mathrm{T})$ by two different methods: by using the complement rule and by using the branches of the tree. Your answers should be the same with both methods.
f. Let U be the event that the second cookie selected is a butter cookie. Find $P(\mathrm{U})$.

3.7: Probability Topics

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10: DESCRIPTIVE STATISTICS

In this chapter, you will study numerical and graphical ways to describe and display your data. This area of statistics is called "Descriptive Statistics." You will learn how to calculate, and even more importantly, how to interpret these measurements and graphs.

10.1: Prelude to Descriptive Statistics

10.2: Stem-and-Leaf Graphs (Stemplots), Line Graphs, and Bar Graphs

10.3: Histograms, Frequency Polygons, and Time Series Graphs

10.4: Measures of the Location of the Data

10.4E: Measures of the Location of the Data (Exercises)

10.5: Box Plots

10.6: Measures of the Center of the Data

10.7: Skewness and the Mean, Median, and Mode

10.8: Measures of the Spread of the Data

10.9: Descriptive Statistics (Worksheet)

10.E: Descriptive Statistics (Exercises)

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10.1: Prelude to Descriptive Statistics

Learning Objectives

By the end of this chapter, the student should be able to:

- Display data graphically and interpret graphs: stemplots, histograms, and box plots.
- Recognize, describe, and calculate the measures of location of data: quartiles and percentiles.
- Recognize, describe, and calculate the measures of the center of data: mean, median, and mode.
- Recognize, describe, and calculate the measures of the spread of data: variance, standard deviation, and range.

Once you have collected data, what will you do with it? Data can be described and presented in many different formats. For example, suppose you are interested in buying a house in a particular area. You may have no clue about the house prices, so you might ask your real estate agent to give you a sample data set of prices. Looking at all the prices in the sample often is overwhelming. A better way might be to look at the median price and the variation of prices. The median and variation are just two ways that you will learn to describe data. Your agent might also provide you with a graph of the data.


Figure 10.1.1: When you have large amounts of data, you will need to organize it in a way that makes sense. These ballots from an election are rolled together with similar ballots to keep them organized. (credit: William Greeson)

In this chapter, you will study numerical and graphical ways to describe and display your data. This area of statistics is called **"Descriptive Statistics."** You will learn how to calculate, and even more importantly, how to interpret these measurements and graphs.

A statistical graph is a tool that helps you learn about the shape or distribution of a sample or a population. A graph can be a more effective way of presenting data than a mass of numbers because we can see where data clusters and where there are only a few data values. Newspapers and the Internet use graphs to show trends and to enable readers to compare facts and figures quickly. Statisticians often graph data first to get a picture of the data. Then, more formal tools may be applied.

Some of the types of graphs that are used to summarize and organize data are the dot plot, the bar graph, the histogram, the stem-and-leaf plot, the frequency polygon (a type of broken line graph), the pie chart, and the box plot. In this chapter, we will briefly look at stem-and-leaf plots, line graphs, and bar graphs, as well as frequency polygons, and time series graphs. Our emphasis will be on histograms and box plots.

This book contains instructions for constructing a histogram and a box plot for the TI-83+ and TI-84 calculators. The Texas Instruments (TI) website provides additional instructions for using these calculators.

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10.2: Stem-and-Leaf Graphs (Stemplots), Line Graphs, and Bar Graphs

One simple graph, the *stem-and-leaf graph* or *stemplot*, comes from the field of exploratory data analysis. It is a good choice when the data sets are small. To create the plot, divide each observation of data into a stem and a leaf. The leaf consists of a final significant digit. For example, 23 has stem two and leaf three. The number 432 has stem 43 and leaf two. Likewise, the number 5,432 has stem 543 and leaf two. The decimal 9.3 has stem nine and leaf three. Write the stems in a vertical line from smallest to largest. Draw a vertical line to the right of the stems. Then write the leaves in increasing order next to their corresponding stem.

Example 10.2.1

For Susan Dean's spring pre-calculus class, scores for the first exam were as follows (smallest to largest):

33; 42; 49; 49; 53; 55; 55; 61; 63; 67; 68; 68; 69; 69; 72; 73; 74; 78; 80; 83; 88; 88; 88; 90; 92; 94; 94; 94; 94; 96; 100

Stem-and-Leaf Graph

Stem	Leaf
3	3
4	2 9 9
5	3 5 5
6	1 3 7 8 8 9 9
7	2 3 4 8
8	0 3 8 8 8
9	0 2 4 4 4 4 6
10	0

The stemplot shows that most scores fell in the 60s, 70s, 80s, and 90s. Eight out of the 31 scores or approximately 26% $\left(\frac{8}{31}\right)$ were in the 90s or 100, a fairly high number of As.

Exercise 10.2.2

For the Park City basketball team, scores for the last 30 games were as follows (smallest to largest):

32; 32; 33; 34; 38; 40; 42; 42; 43; 44; 46; 47; 47; 48; 48; 48; 49; 50; 50; 51; 52; 52; 52; 53; 54; 56; 57; 57; 60; 61

Construct a stem plot for the data.

Answer available at the end of this section.

The stemplot is a quick way to graph data and gives an exact picture of the data. You want to look for an overall pattern and any outliers. An outlier is an observation of data that does not fit the rest of the data. It is sometimes called an **extreme value.** When you graph an outlier, it will appear not to fit the pattern of the graph. Some outliers are due to mistakes (for example, writing down 50 instead of 500) while others may indicate that something unusual is happening. It takes some background information to explain outliers, so we will cover them in more detail later.

Example 10.2.3

The data are the distances (in kilometers) from a home to local supermarkets. Create a stemplot using the data:

1.1; 1.5; 2.3; 2.5; 2.7; 3.2; 3.3; 3.3; 3.5; 3.8; 4.0; 4.2; 4.5; 4.5; 4.7; 4.8; 5.5; 5.6; 6.5; 6.7; 12.3

Do the data seem to have any concentration of values?

HINT: The leaves are to the right of the decimal.

Answer

The value 12.3 may be an outlier. Values appear to concentrate at three and four kilometers.

Stem	Leaf
1	1 5
2	3 5 7
3	2 3 3 5 8
4	0 2 5 5 7 8
5	5 6
6	5 7
7	
8	
9	
10	
11	
12	3

Exercise 10.2.4

The following data show the distances (in miles) from the homes of off-campus statistics students to the college. Create a stem plot using the data and identify any outliers:

0.5; 0.7; 1.1; 1.2; 1.2; 1.3; 1.3; 1.5; 1.5; 1.7; 1.7; 1.8; 1.9; 2.0; 2.2; 2.5; 2.6; 2.8; 2.8; 2.8; 3.5; 3.8; 4.4; 4.8; 4.9; 5.2; 5.5; 5.7; 5.8; 8.0

Answer available at the end of this section.

Example 10.2.5: Side-by-Side Stem-and-Leaf plot

A side-by-side stem-and-leaf plot allows a comparison of the two data sets in two columns. In a side-by-side stem-and-leaf plot, two sets of leaves share the same stem. The leaves are to the left and the right of the stems. Tables 10.2.1 and 10.2.2 show the ages of presidents at their inauguration and at their death. Construct a side-by-side stem-and-leaf plot using this data.

Table 10.2.1: Presidential Ages at Inauguration

President	Ageat Inauguration	President	Age	President	Age
Pierce	48	Harding	55	Obama	47

President	Ageat Inauguration	President	Age	President	Age
Polk	49	T. Roosevelt	42	G.H.W. Bush	64
Fillmore	50	Wilson	56	G. W. Bush	54
Tyler	51	McKinley	54	Reagan	69
Van Buren	54	B. Harrison	55	Ford	61
Washington	57	Lincoln	52	Hoover	54
Jefferson	57	Grant	46	Truman	60
Madison	57	Hayes	54	Eisenhower	62
J. Q. Adams	57	Arthur	51	L. Johnson	55
Monroe	58	Garfield	49	Kennedy	43
J. Adams	61	A. Johnson	56	F. Roosevelt	51
Jackson	61	Cleveland	47	Nixon	56
Taylor	64	Taft	51	Clinton	47
Buchanan	65	Coolidge	51	Trump	70
W. H. Harrison	68	Cleveland	55	Carter	52

10.2.2 *Presidential Age at Death*

President	Age	President	Age	President	Age
Washington	67	Lincoln	56	Hoover	90
J. Adams	90	A. Johnson	66	F. Roosevelt	63
Jefferson	83	Grant	63	Truman	88
Madison	85	Hayes	70	Eisenhower	78
Monroe	73	Garfield	49	Kennedy	46
J. Q. Adams	80	Arthur	56	L. Johnson	64
Jackson	78	Cleveland	71	Nixon	81
Van Buren	79	B. Harrison	67	Ford	93
W. H. Harrison	68	Cleveland	71	Reagan	93
Tyler	71	McKinley	58		
Polk	53	T. Roosevelt	60		
Taylor	65	Taft	72		
Fillmore	74	Wilson	67		
Pierce	64	Harding	57		
Buchanan	77	Coolidge	60		

Answer

Ages at Inauguration		Ages at Death
9 9 8 7 7 7 6 3 2	4	6 9
8 7 7 7 7 6 6 6 5 5 5 5 4 4 4 4 4 2 1 1 1 1 1 0	5	3 6 6 7 7 8

Ages at Inauguration		Ages at Death
9 5 4 4 2 1 1 1 0	6	0 0 3 3 4 4 5 6 7 7 7 8
	7	0 0 1 1 1 3 4 7 8 8 9
	8	0 1 3 5 8
	9	0 0 3 3

Exercise 10.2.6

The table shows the number of wins and losses the Atlanta Hawks have had in 42 seasons. Create a side-by-side stem-and-leaf plot of these wins and losses.

Losses	Wins	Year	Losses	Wins	Year
34	48	1968–1969	41	41	1989–1990
34	48	1969–1970	39	43	1990–1991
46	36	1970–1971	44	38	1991–1992
46	36	1971–1972	39	43	1992–1993
36	46	1972–1973	25	57	1993–1994
47	35	1973–1974	40	42	1994–1995
51	31	1974–1975	36	46	1995–1996
53	29	1975–1976	26	56	1996–1997
51	31	1976–1977	32	50	1997–1998
41	41	1977–1978	19	31	1998–1999
36	46	1978–1979	54	28	1999–2000
32	50	1979–1980	57	25	2000–2001
51	31	1980–1981	49	33	2001–2002
40	42	1981–1982	47	35	2002–2003
39	43	1982–1983	54	28	2003–2004
42	40	1983–1984	69	13	2004–2005
48	34	1984–1985	56	26	2005–2006
32	50	1985–1986	52	30	2006–2007
25	57	1986–1987	45	37	2007–2008
32	50	1987–1988	35	47	2008–2009
30	52	1988–1989	29	53	2009–2010

Answer available at the end of this section.

Another type of graph that is useful for specific data values is a **line graph**. In the particular line graph shown in Example, the ***x*-axis** (horizontal axis) consists of **data values** and the ***y*-axis** (vertical axis) consists of **frequency points**. The frequency points are connected using line segments.

Example 10.2.7

In a survey, 40 mothers were asked how many times per week a teenager must be reminded to do his or her chores. The results are shown in Table and in Figure.

Number of times teenager is reminded	Frequency
0	2
1	5
2	8
3	14
4	7
5	4

Answer

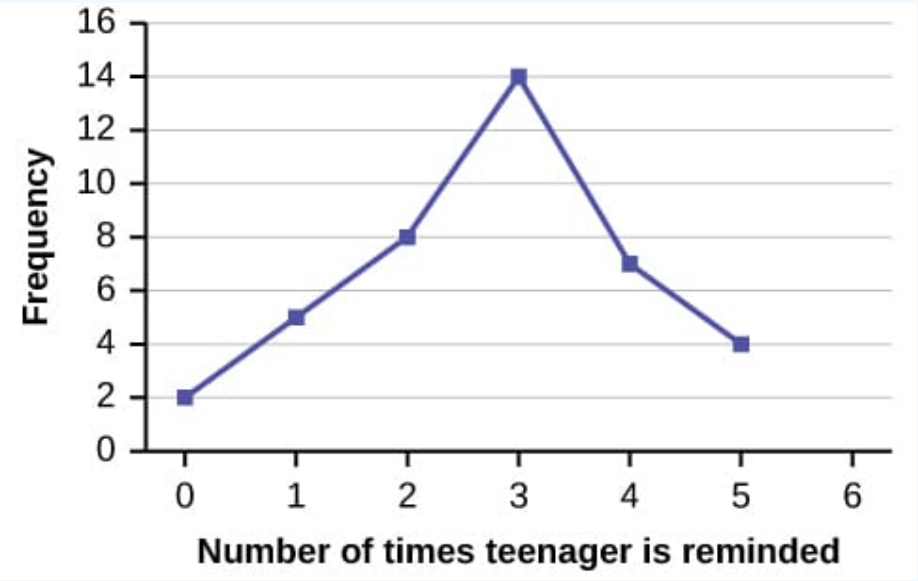


Figure 10.2.1: A line graph showing the number of times a teenager needs to be reminded to do chores on the x-axis and frequency on the y-axis.

Exercise 10.2.8

In a survey, 40 people were asked how many times per year they had their car in the shop for repairs. The results are shown in Table. Construct a line graph.

Number of times in shop	Frequency
0	7
1	10
2	14
3	9

Answer



Figure 10.2.2: A line graph showing the number of times a car is in the shop on the x-axis and frequency on the y-axis.

Bar graphs consist of bars that are separated from each other. The bars can be rectangles or they can be rectangular boxes (used in three-dimensional plots), and they can be vertical or horizontal. The **bar graph** shown in Example 10.2.9 has age groups represented on the ***x*-axis** and proportions on the ***y*-axis**.

Example 10.2.9

By the end of 2011, Facebook had over 146 million users in the United States. Table shows three age groups, the number of users in each age group, and the proportion (%) of users in each age group. Construct a bar graph using this data.

Age groups	Number of Facebook users	Proportion (%) of Facebook users
13–25	65,082,280	45%
26–44	53,300,200	36%
45–64	27,885,100	19%

Answer

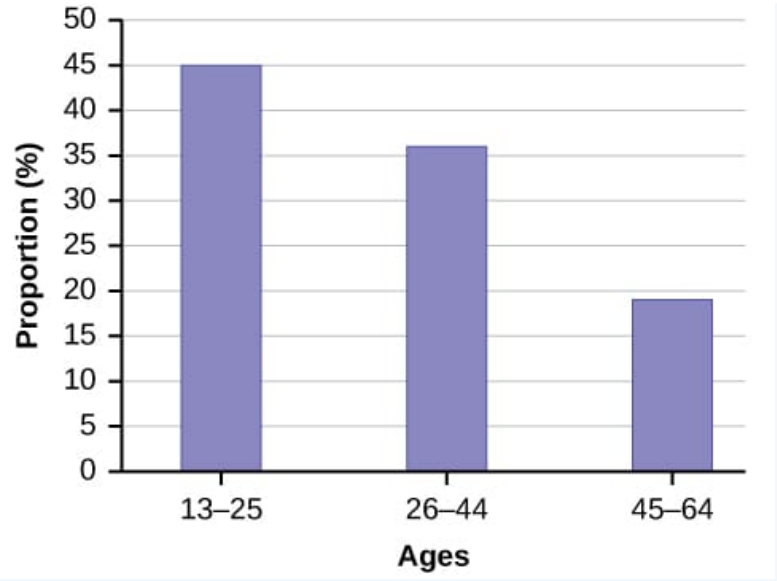


Figure 10.2.3: This is a bar graph that matches the supplied data. The x-axis shows age groups and the y-axis show the percentages of Facebook users

Exercise 10.2.10

The population in Park City is made up of children, working-age adults, and retirees. Table shows the three age groups, the number of people in the town from each age group, and the proportion (%) of people in each age group. Construct a bar graph showing the proportions.

Age groups	Number of people	Proportion of population
Children	67,059	19%
Working-age adults	152,198	43%
Retirees	131,662	38%

Answer

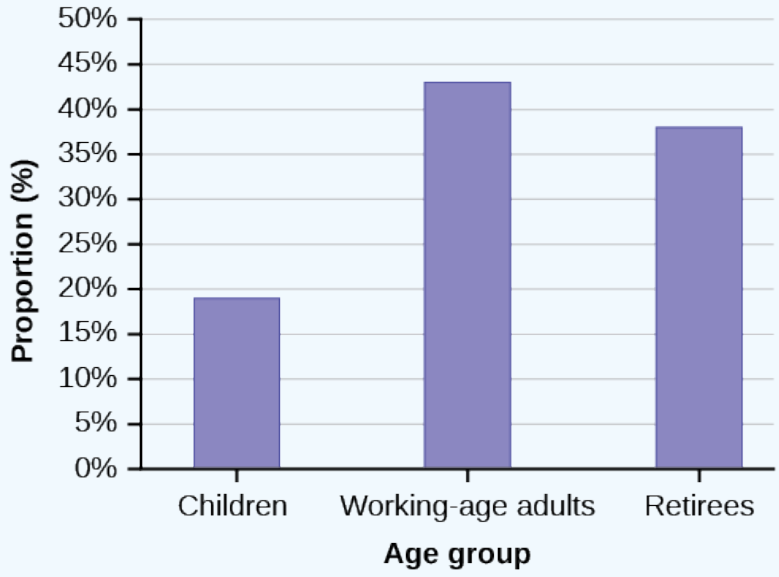


Figure 10.2.4: This is a bar graph that matches the supplied data. The x-axis shows age groups, and the y-axis shows the percentages of Park City's population.

Example 10.2.11

The columns in Table contain: the race or ethnicity of students in U.S. Public Schools for the class of 2011, percentages for the Advanced Placement examine population for that class, and percentages for the overall student population. Create a bar graph with the student race or ethnicity (qualitative data) on the *x*-axis, and the Advanced Placement examinee population percentages on the *y*-axis.

Race/Ethnicity	AP Examinee Population	Overall Student Population
1 = Asian, Asian American or Pacific Islander	10.3%	5.7%
2 = Black or African American	9.0%	14.7%
3 = Hispanic or Latino	17.0%	17.6%
4 = American Indian or Alaska Native	0.6%	1.1%
5 = White	57.1%	59.2%
6 = Not reported/other	6.0%	1.7%

Solution

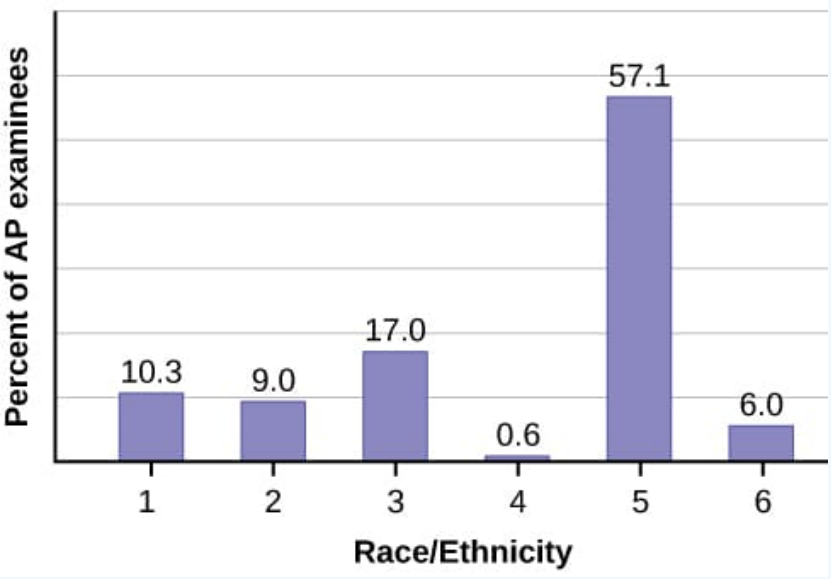


Figure 10.2.5: This is a bar graph that matches the supplied data. The x-axis shows race and ethnicity, and the y-axis shows the percentages of AP examinees.

Exercise 10.2.12

Park city is broken down into six voting districts. The table shows the percent of the total registered voter population that lives in each district as well as the percent total of the entire population that lives in each district. Construct a bar graph that shows the registered voter population by district.

District	Registered voter population	Overall city population
1	15.5%	19.4%
2	12.2%	15.6%
3	9.8%	9.0%
4	17.4%	18.5%

District	Registered voter population	Overall city population
5	22.8%	20.7%
6	22.3%	16.8%

Answer



Figure 10.2.6: This is a bar graph that matches the supplied data. The x-axis shows Park City voting districts, and the y-axis shows the percentages of the registered voter population.

Summary

A **stem-and-leaf plot** is a way to plot data and look at the distribution. In a stem-and-leaf plot, all data values within a class are visible. The advantage in a stem-and-leaf plot is that all values are listed, unlike a histogram, which gives classes of data values. A **line graph** is often used to represent a set of data values in which a quantity varies with time. These graphs are useful for finding trends. That is, finding a general pattern in data sets including temperature, sales, employment, company profit or cost over a period of time. A **bar graph** is a chart that uses either horizontal or vertical bars to show comparisons among categories. One axis of the chart shows the specific categories being compared, and the other axis represents a discrete value. Some bar graphs present bars clustered in groups of more than one (grouped bar graphs), and others show the bars divided into subparts to show cumulative effect (stacked bar graphs). Bar graphs are especially useful when categorical data is being used.

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1. Burbary, Ken. *Facebook Demographics Revisited – 2001 Statistics,* 2011. Available online at www.kenburbary.com/2011/03/fa...-statistics-2/ (accessed August 21, 2013).
2. "9th Annual AP Report to the Nation." CollegeBoard, 2013. Available online at http://apreport.collegeboard.org/goa...omoting-equity (accessed September 13, 2013).
3. "Overweight and Obesity: Adult Obesity Facts." Centers for Disease Control and Prevention. Available online at http://www.cdc.gov/obesity/data/adult.html (accessed September 13, 2013).

Answers

Exercise 10.2.2

Stem	Leaf
3	2 2 3 4 8

Stem	Leaf
4	0 2 2 3 4 6 7 7 8 8 8 9
5	0 0 1 2 2 2 3 4 6 7 7
6	0 1

Exercise 10.2.4

Stem	Leaf
0	5 7
1	1 2 2 3 3 5 5 7 7 8 9
2	0 2 5 6 8 8 8
3	5 8
4	4 8 9
5	2 5 7 8
6	
7	
8	0

The value 8.0 may be an outlier. Values appear to concentrate at one and two miles.

Exercise 10.2.6

Table 10.2.3*: Atlanta Hawks Wins and Losses*

Number of Wins		Number of Losses
3	1	9
9 8 8 6 5	2	5 5 9
8 7 6 6 5 5 4 3 1 1 1 1 0	3	0 2 2 2 2 4 4 5 6 6 6 9 9 9
8 8 7 6 6 6 3 3 3 2 2 1 1 0	4	0 0 1 1 2 4 5 6 6 7 7 8 9
7 7 6 3 2 0 0 0 0	5	1 1 1 2 3 4 4 6 7
	6	9

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10.3: Histograms, Frequency Polygons, and Time Series Graphs

For most of the work you do in this book, you will use a histogram to display the data. One advantage of a histogram is that it can readily display large data sets. A rule of thumb is to use a histogram when the data set consists of 100 values or more.

A histogram consists of contiguous (adjoining) boxes. It has both a horizontal axis and a vertical axis. The horizontal axis is labeled with what the data represents (for instance, distance from your home to school). The vertical axis is labeled either frequency or relative frequency (or percent frequency or probability). The graph will have the same shape with either label. The histogram (like the stemplot) can give you the shape of the data, the center, and the spread of the data.

The relative frequency is equal to the frequency for an observed value of the data divided by the total number of data values in the sample.(Remember, frequency is defined as the number of times an answer occurs.) If:

- f is frequency
- n is total number of data values (or the sum of the individual frequencies), and
- RF is relative frequency,

then:

$$RF = \frac{f}{n} \tag{10.3.1}$$

For example, if three students in Mr. Ahab's English class of 40 students received from 90% to 100%, then, f = 3, n = 40, and RF = fn = 340 = 0.075. 7.5% of the students received 90–100%. 90–100% are quantitative measures.

To construct a histogram, first decide how many bars or intervals, also called classes, represent the data. Many histograms consist of five to 15 bars or classes for clarity. The number of bars needs to be chosen. Choose a starting point for the first interval to be less than the smallest data value. A convenient starting point is a lower value carried out to one more decimal place than the value with the most decimal places. For example, if the value with the most decimal places is 6.1 and this is the smallest value, a convenient starting point is $6.05 (6.1–0.05 = 6.05)$. We say that 6.05 has more precision. If the value with the most decimal places is 2.23 and the lowest value is 1.5, a convenient starting point is $1.495 (1.5–0.005 = 1.495)$. If the value with the most decimal places is 3.234 and the lowest value is 1.0, a convenient starting point is $0.9995 (1.0–0.0005 = 0.9995)$. If all the data happen to be integers and the smallest value is two, then a convenient starting point is $1.5 (2 – 0.5 = 1.5)$. Also, when the starting point and other boundaries are carried to one additional decimal place, no data value will fall on a boundary. The next two examples go into detail about how to construct a histogram using continuous data and how to create a histogram using discrete data.

Example 10.3.1

The following data are the heights (in inches to the nearest half inch) of 100 male semiprofessional soccer players. The heights are **continuous** data, since height is measured.

60; 60.5; 61; 61; 61.5

63.5; 63.5; 63.5

64; 64; 64; 64; 64; 64; 64; 64.5; 64.5; 64.5; 64.5; 64.5; 64.5; 64.5

66; 66; 66; 66; 66; 66; 66; 66; 66; 66; 66.5; 66.5; 66.5; 66.5; 66.5; 66.5; 66.5; 66.5; 66.5; 66.5; 67; 67; 67; 67; 67; 67; 67; 67; 67; 67; 67; 67.5; 67.5; 67.5; 67.5; 67.5; 67.5; 67.5

68; 68; 69; 69; 69; 69; 69; 69; 69; 69; 69; 69; 69.5; 69.5; 69.5; 69.5; 69.5

70; 70; 70; 70; 70; 70; 70.5; 70.5; 70.5; 71; 71; 71

72; 72; 72; 72.5; 72.5; 73; 73.5

74

The smallest data value is 60. Since the data with the most decimal places has one decimal (for instance, 61.5), we want our starting point to have two decimal places. Since the numbers 0.5, 0.05, 0.005, etc. are convenient numbers, use 0.05 and subtract it from 60, the smallest value, for the convenient starting point.

60 – 0.05 = 59.95 which is more precise than, say, 61.5 by one decimal place. The starting point is, then, 59.95.

The largest value is 74, so 74 + 0.05 = 74.05 is the ending value.

Next, calculate the width of each bar or class interval. To calculate this width, subtract the starting point from the ending value and divide by the number of bars (you must choose the number of bars you desire). Suppose you choose eight bars.

$$\frac{74.05 - 59.95}{8} = 1.76 \tag{10.3.2}$$

We will round up to two and make each bar or class interval two units wide. Rounding up to two is one way to prevent a value from falling on a boundary. Rounding to the next number is often necessary even if it goes against the standard rules of rounding. For this example, using 1.76 as the width would also work. A guideline that is followed by some for the width of a bar or class interval is to take the square root of the number of data values and then round to the nearest whole number, if necessary. For example, if there are 150 values of data, take the square root of 150 and round to 12 bars or intervals.

The boundaries are:

- 59.95
- 59.95 + 2 = 61.95
- 61.95 + 2 = 63.95
- 63.95 + 2 = 65.95
- 65.95 + 2 = 67.95
- 67.95 + 2 = 69.95
- 69.95 + 2 = 71.95
- 71.95 + 2 = 73.95
- 73.95 + 2 = 75.95

The heights 60 through 61.5 inches are in the interval 59.95–61.95. The heights that are 63.5 are in the interval 61.95–63.95. The heights that are 64 through 64.5 are in the interval 63.95–65.95. The heights 66 through 67.5 are in the interval 65.95–67.95. The heights 68 through 69.5 are in the interval 67.95–69.95. The heights 70 through 71 are in the interval 69.95–71.95. The heights 72 through 73.5 are in the interval 71.95–73.95. The height 74 is in the interval 73.95–75.95.

The following histogram displays the heights on the *x*-axis and relative frequency on the *y*-axis.

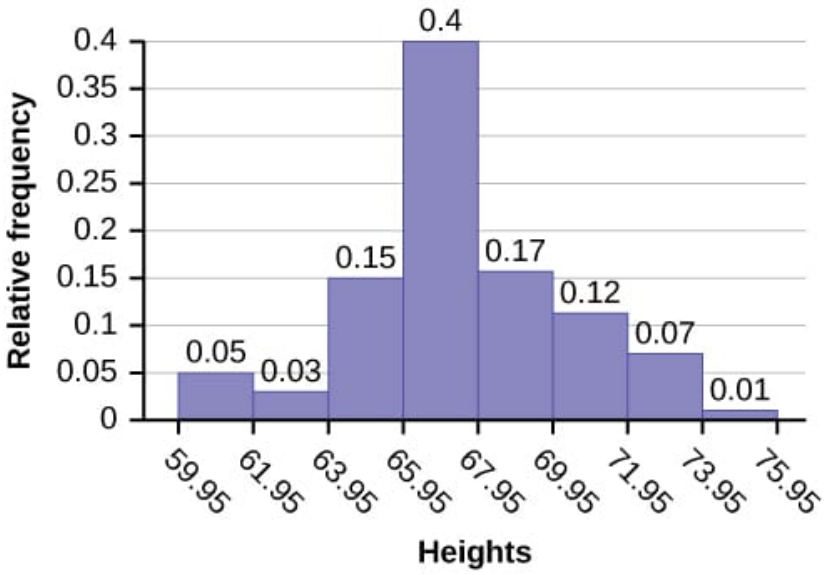


Figure 10.3.1: Histogram of something

Exercise 10.3.1

The following data are the shoe sizes of 50 male students. The sizes are discrete data since shoe size is measured in whole and half units only. Construct a histogram and calculate the width of each bar or class interval. Suppose you choose six bars.

9; 9; 9.5; 9.5; 10; 10; 10; 10; 10; 10; 10.5; 10.5; 10.5; 10.5; 10.5; 10.5; 10.5; 10.5
11; 11; 11; 11; 11; 11; 11; 11; 11; 11; 11; 11; 11.5; 11.5; 11.5; 11.5; 11.5; 11.5
12; 12; 12; 12; 12; 12; 12; 12.5; 12.5; 12.5; 12.5; 14

Answer

Smallest value: 9

Largest value: 14

Convenient starting value: 9 – 0.05 = 8.95

Convenient ending value: 14 + 0.05 = 14.05

$\frac{14.05-8.95}{6}$ = 0.85

The calculations suggests using 0.85 as the width of each bar or class interval. You can also use an interval with a width equal to one.

Example 10.3.2

The following data are the number of books bought by 50 part-time college students at ABC College. The number of books is **discrete data**, since books are counted.

1; 1; 1; 1; 1; 1; 1; 1; 1; 1
2; 2; 2; 2; 2; 2; 2; 2; 2; 2
3; 3; 3; 3; 3; 3; 3; 3; 3; 3; 3; 3; 3; 3; 3
4; 4; 4; 4; 4; 4
5; 5; 5; 5; 5
6; 6

Eleven students buy one book. Ten students buy two books. Sixteen students buy three books. Six students buy four books. Five students buy five books. Two students buy six books.

Because the data are integers, subtract 0.5 from 1, the smallest data value and add 0.5 to 6, the largest data value. Then the starting point is 0.5 and the ending value is 6.5.

Next, calculate the width of each bar or class interval. If the data are discrete and there are not too many different values, a width that places the data values in the middle of the bar or class interval is the most convenient. Since the data consist of the numbers 1, 2, 3, 4, 5, 6, and the starting point is 0.5, a width of one places the 1 in the middle of the interval from 0.5 to 1.5, the 2 in the middle of the interval from 1.5 to 2.5, the 3 in the middle of the interval from 2.5 to 3.5, the 4 in the middle of the interval from _____ to _____, the 5 in the middle of the interval from _____ to _____, and the _____ in the middle of the interval from _____ to _____ .

Answer

Calculate the number of bars as follows:

$$\frac{6.5 - 0.5}{\text{number of bars}} = 1$$

where 1 is the width of a bar. Therefore, bars = 6.

The following histogram displays the number of books on the *x*-axis and the frequency on the *y*-axis.

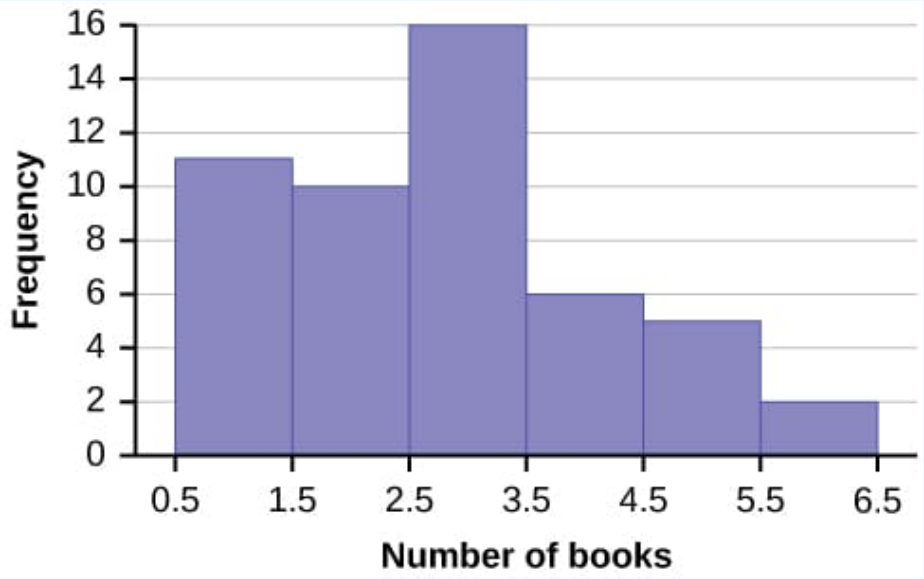


Figure 10.3.2: Histogram consists of 6 bars with the y-axis in increments of 2 from 0-16 and the x-axis in intervals of 1 from 0.5-6.5.

Note

Go to [link]. There are calculator instructions for entering data and for creating a customized histogram. Create the histogram for Example.

- Press Y=. Press CLEAR to delete any equations.
- Press STAT 1:EDIT. If L1 has data in it, arrow up into the name L1, press CLEAR and then arrow down. If necessary, do the same for L2.
- Into L1, enter 1, 2, 3, 4, 5, 6.
- Into L2, enter 11, 10, 16, 6, 5, 2.
- Press WINDOW. Set Xmin = .5, Xscl = (6.5 – .5)/6, Ymin = –1, Ymax = 20, Yscl = 1, Xres = 1.
- Press 2nd Y=. Start by pressing 4:Plotsoff ENTER.
- Press 2nd Y=. Press 1:Plot1. Press ENTER. Arrow down to TYPE. Arrow to the 3^{rd}picture (histogram). Press ENTER.

- Arrow down to Xlist: Enter L1 (2nd 1). Arrow down to Freq. Enter L2 (2nd 2).
- Press GRAPH.
- Use the TRACE key and the arrow keys to examine the histogram.

Exercise 10.3.2

The following data are the number of sports played by 50 student athletes. The number of sports is discrete data since sports are counted.

1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1

2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2
3; 3; 3; 3; 3; 3; 3; 3

20 student athletes play one sport. 22 student athletes play two sports. Eight student athletes play three sports.

Fill in the blanks for the following sentence. Since the data consist of the numbers 1, 2, 3, and the starting point is 0.5, a width of one places the 1 in the middle of the interval 0.5 to _____, the 2 in the middle of the interval from _____ to _____, and the 3 in the middle of the interval from _____ to _____.

Answer

1.5

1.5 to 2.5

2.5 to 3.5

Example 10.3.3

Using this data set, construct a histogram.

Number of Hours My Classmates Spent Playing Video Games on Weekends

9.95	10	2.25	16.75	0
19.5	22.5	7.5	15	12.75
5.5	11	10	20.75	17.5
23	21.9	24	23.75	18
20	15	22.9	18.8	20.5

Answer

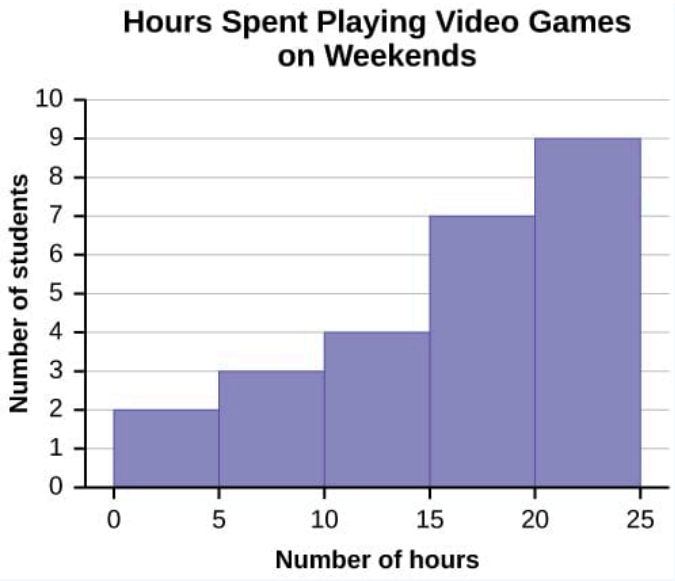


Figure 10.3.3: This is a histogram that matches the supplied data. The x-axis consists of 5 bars in intervals of 5 from 0 to 25. The y-axis is marked in increments of 1 from 0 to 10. The x-axis shows the number of hours spent playing video games on the weekends, and the y-axis shows the number of students.

Some values in this data set fall on boundaries for the class intervals. A value is counted in a class interval if it falls on the left boundary, but not if it falls on the right boundary. Different researchers may set up histograms for the same data in different ways. There is more than one correct way to set up a histogram.

Exercise 10.3.3

The following data represent the number of employees at various restaurants in New York City. Using this data, create a histogram.

22; 35; 15; 26; 40; 28; 18; 20; 25; 34; 39; 42; 24; 22; 19; 27; 22; 34; 40; 20; 38 and 28

Use 10–19 as the first interval.

Count the money (bills and change) in your pocket or purse. Your instructor will record the amounts. As a class, construct a histogram displaying the data. Discuss how many intervals you think is appropriate. You may want to experiment with the number of intervals.

Frequency Polygons

Frequency polygons are analogous to line graphs, and just as line graphs make continuous data visually easy to interpret, so too do frequency polygons. To construct a frequency polygon, first examine the data and decide on the number of intervals, or class intervals, to use on the *x*-axis and *y*-axis. After choosing the appropriate ranges, begin plotting the data points. After all the points are plotted, draw line segments to connect them.

Example 10.3.4

A frequency polygon was constructed from the frequency table below.

Frequency Distribution for Calculus Final Test Scores

Lower Bound	Upper Bound	Frequency	Cumulative Frequency
49.5	59.5	5	5

Lower Bound	Upper Bound	Frequency	Cumulative Frequency
59.5	69.5	10	15
69.5	79.5	30	45
79.5	89.5	40	85
89.5	99.5	15	100

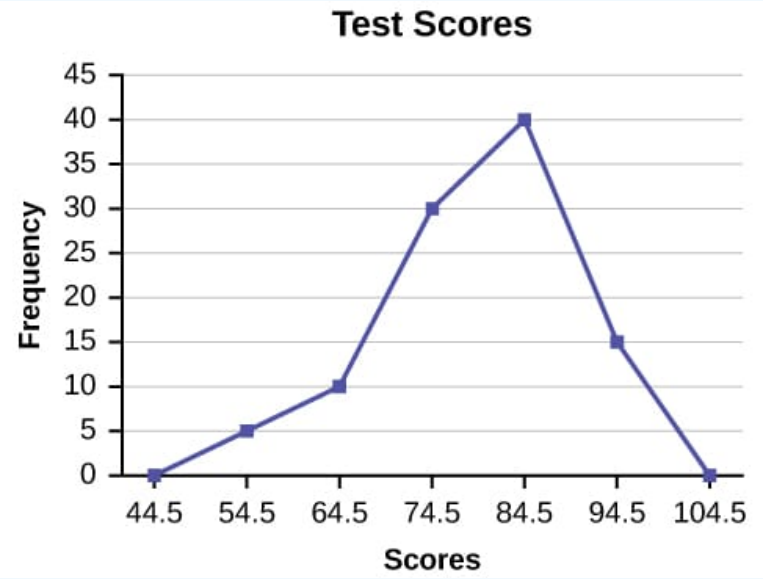


Figure 10.3.4: A frequency polygon was constructed from the frequency table above.

The first label on the *x*-axis is 44.5. This represents an interval extending from 39.5 to 49.5. Since the lowest test score is 54.5, this interval is used only to allow the graph to touch the *x*-axis. The point labeled 54.5 represents the next interval, or the first "real" interval from the table, and contains five scores. This reasoning is followed for each of the remaining intervals with the point 104.5 representing the interval from 99.5 to 109.5. Again, this interval contains no data and is only used so that the graph will touch the *x*-axis. Looking at the graph, we say that this distribution is skewed because one side of the graph does not mirror the other side.

Exercise 10.3.4

Construct a frequency polygon of U.S. Presidents' ages at inauguration shown in the Table.

Age at Inauguration	Frequency
41.5–46.5	4
46.5–51.5	11
51.5–56.5	14
56.5–61.5	9
61.5–66.5	4
66.5–71.5	2

Answer

The first label on the *x*-axis is 39. This represents an interval extending from 36.5 to 41.5. Since there are no ages less than 41.5, this interval is used only to allow the graph to touch the *x*-axis. The point labeled 44 represents the next interval, or the first "real" interval from the table, and contains four scores. This reasoning is followed for each of the remaining intervals with the point 74 representing the interval from 71.5 to 76.5. Again, this interval contains no data and is only used so that the graph will touch the *x*-axis. Looking at the graph, we say that this distribution is skewed because one side of the graph does not mirror the other side.

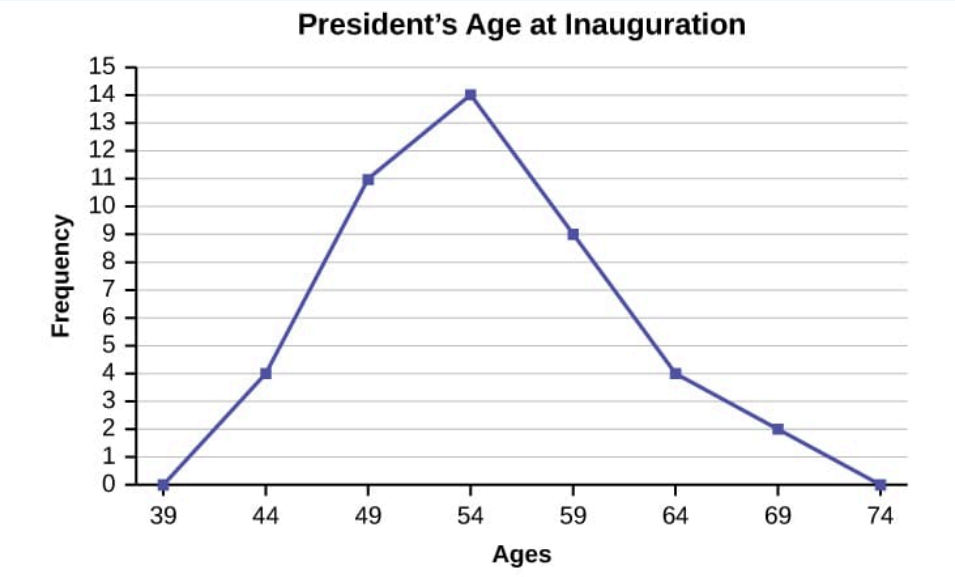


Figure 10.3.5: This figure shows a graph entitled, 'President's Age at Inauguration.' The x-axis is labeled 'Ages' and is marked off at 39, 44, 49, 54, 59, 64, 69 and 74. The y-axis is labeled, 'Frequency,' and is marked off in intervals of 1 from 0 to 15. The following points are plotted and a line connects one to the other to create the frequency polygon: (39, 0), (44, 4), (49, 11), (54, 14), (59, 9), (64, 4), (69, 2), (74, 0).

Frequency polygons are useful for comparing distributions. This is achieved by overlaying the frequency polygons drawn for different data sets.

Example 10.3.5

We will construct an overlay frequency polygon comparing the scores from Example with the students' final numeric grade.

Frequency Distribution for Calculus Final Test Scores

Lower Bound	Upper Bound	Frequency	Cumulative Frequency
49.5	59.5	5	5
59.5	69.5	10	15
69.5	79.5	30	45
79.5	89.5	40	85
89.5	99.5	15	100

Frequency Distribution for Calculus Final Grades

Lower Bound	Upper Bound	Frequency	Cumulative Frequency
49.5	59.5	10	10
59.5	69.5	10	20
69.5	79.5	30	50

Lower Bound	Upper Bound	Frequency	Cumulative Frequency
79.5	89.5	45	95
89.5	99.5	5	100

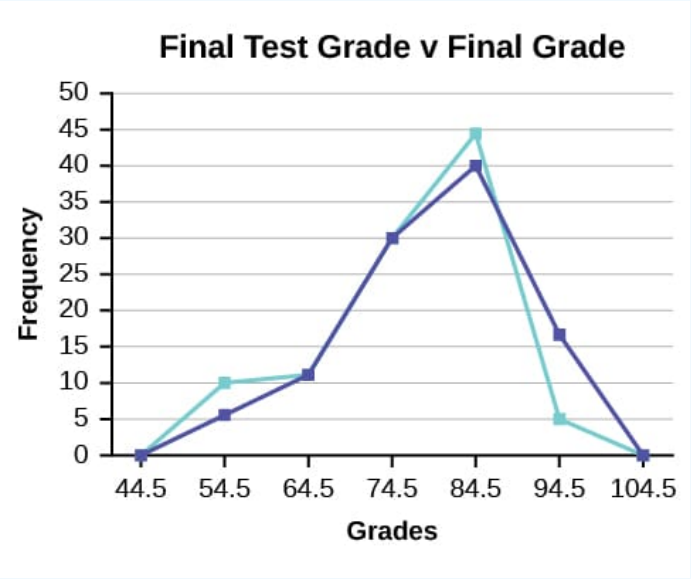


Figure 10.3.6: This is an overlay frequency polygon that matches the supplied data. The x-axis shows the grades, and the y-axis shows the frequency.

Suppose that we want to study the temperature range of a region for an entire month. Every day at noon we note the temperature and write this down in a log. A variety of statistical studies could be done with this data. We could find the mean or the median temperature for the month. We could construct a histogram displaying the number of days that temperatures reach a certain range of values. However, all of these methods ignore a portion of the data that we have collected.

One feature of the data that we may want to consider is that of time. Since each date is paired with the temperature reading for the day, we don't have to think of the data as being random. We can instead use the times given to impose a chronological order on the data. A graph that recognizes this ordering and displays the changing temperature as the month progresses is called a time series graph.

Constructing a Time Series Graph

To construct a time series graph, we must look at both pieces of our **paired data set**. We start with a standard Cartesian coordinate system. The horizontal axis is used to plot the date or time increments, and the vertical axis is used to plot the values of the variable that we are measuring. By doing this, we make each point on the graph correspond to a date and a measured quantity. The points on the graph are typically connected by straight lines in the order in which they occur.

Example 10.3.6

The following data shows the Annual Consumer Price Index, each month, for ten years. Construct a time series graph for the Annual Consumer Price Index data only.

Year	Jan	Feb	Mar	Apr	May	Jun	Jul
2003	181.7	183.1	184.2	183.8	183.5	183.7	183.9
2004	185.2	186.2	187.4	188.0	189.1	189.7	189.4
2005	190.7	191.8	193.3	194.6	194.4	194.5	195.4
2006	198.3	198.7	199.8	201.5	202.5	202.9	203.5

Year	Jan	Feb	Mar	Apr	May	Jun	Jul
2007	202.416	203.499	205.352	206.686	207.949	208.352	208.299
2008	211.080	211.693	213.528	214.823	216.632	218.815	219.964
2009	211.143	212.193	212.709	213.240	213.856	215.693	215.351
2010	216.687	216.741	217.631	218.009	218.178	217.965	218.011
2011	220.223	221.309	223.467	224.906	225.964	225.722	225.922
2012	226.665	227.663	229.392	230.085	229.815	229.478	229.104

Year	Aug	Sep	Oct	Nov	Dec	Annual
2003	184.6	185.2	185.0	184.5	184.3	184.0
2004	189.5	189.9	190.9	191.0	190.3	188.9
2005	196.4	198.8	199.2	197.6	196.8	195.3
2006	203.9	202.9	201.8	201.5	201.8	201.6
2007	207.917	208.490	208.936	210.177	210.036	207.342
2008	219.086	218.783	216.573	212.425	210.228	215.303
2009	215.834	215.969	216.177	216.330	215.949	214.537
2010	218.312	218.439	218.711	218.803	219.179	218.056
2011	226.545	226.889	226.421	226.230	225.672	224.939
2012	230.379	231.407	231.317	230.221	229.601	229.594

Answer

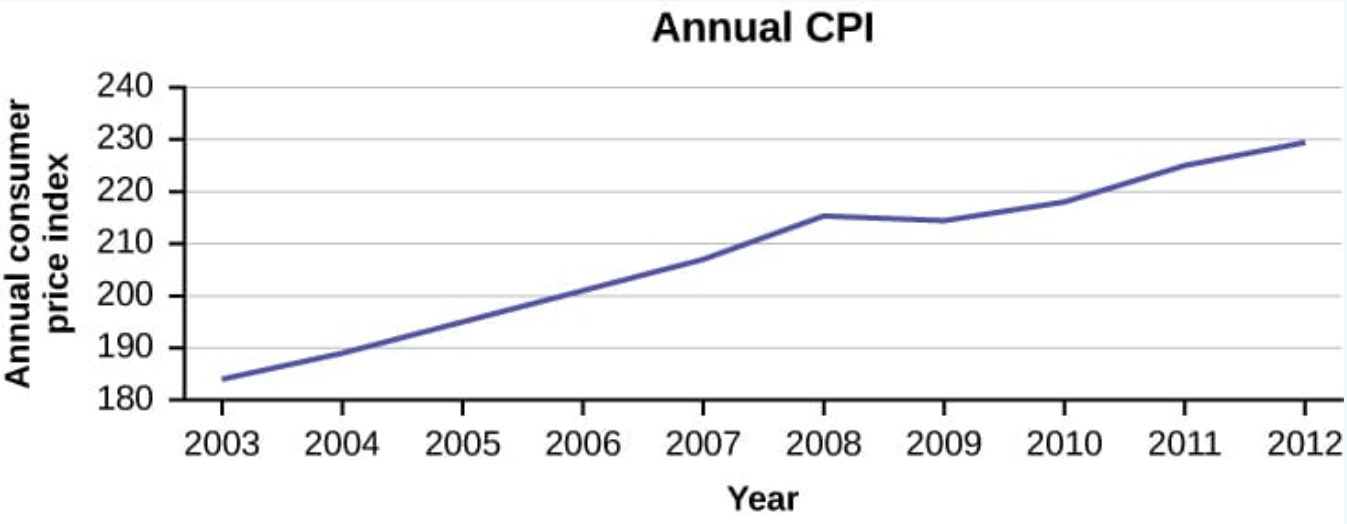


Figure 10.3.7: This is a times series graph that matches the supplied data. The x-axis shows years from 2003 to 2012, and the y-axis shows the annual CPI.

Exercise 10.3.5

The following table is a portion of a data set from www.worldbank.org. Use the table to construct a time series graph for CO_2 emissions for the United States.

CO_2 Emissions

	Ukraine	United Kingdom	United States
2003	352,259	540,640	5,681,664
2004	343,121	540,409	5,790,761
2005	339,029	541,990	5,826,394
2006	327,797	542,045	5,737,615
2007	328,357	528,631	5,828,697
2008	323,657	522,247	5,656,839
2009	272,176	474,579	5,299,563

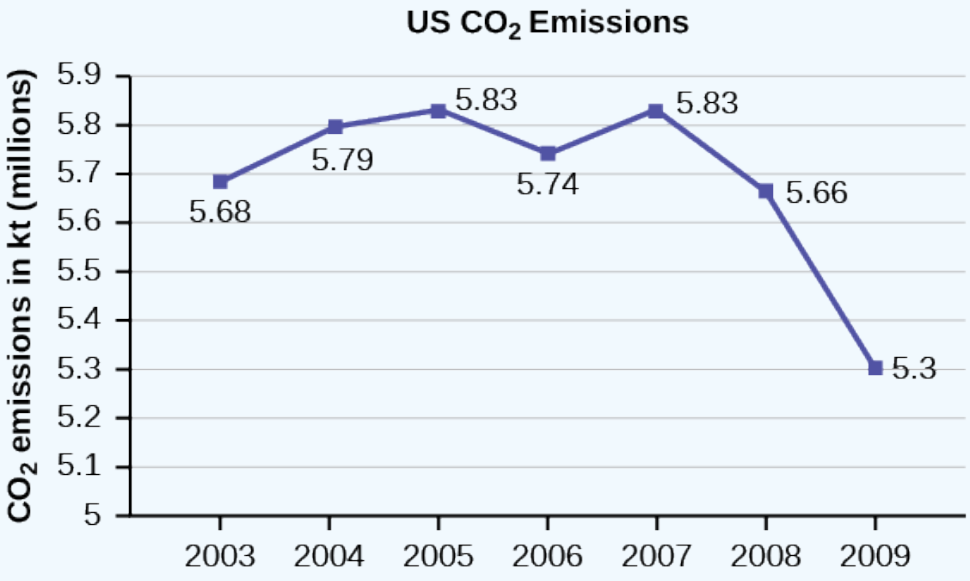


Figure 10.3.8: This is a times series graph that matches the supplied data. The x-axis shows years from 2003 to 2012, and the y-axis shows the annual CPI.

Uses of a Time Series Graph

Time series graphs are important tools in various applications of statistics. When recording values of the same variable over an extended period of time, sometimes it is difficult to discern any trend or pattern. However, once the same data points are displayed graphically, some features jump out. Time series graphs make trends easy to spot.

Review

A **histogram** is a graphic version of a frequency distribution. The graph consists of bars of equal width drawn adjacent to each other. The horizontal scale represents classes of quantitative data values and the vertical scale represents frequencies. The heights of the bars correspond to frequency values. Histograms are typically used for large, continuous, quantitative data sets. A frequency polygon can also be used when graphing large data sets with data points that repeat. The data usually goes on *y*-axis with the frequency being graphed on the *x*-axis. Time series graphs can be helpful when looking at large amounts of data for one variable over a period of time.Glossary

References

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10. “Overweight and Obesity: Adult Obesity Facts.” Centers for Disease Control and Prevention. Available online at http://www.cdc.gov/obesity/data/adult.html (accessed September 13, 2013).

Frequency
the number of times a value of the data occurs

Histogram
a graphical representation in $x - y$ form of the distribution of data in a data set; x represents the data and y represents the frequency, or relative frequency. The graph consists of contiguous rectangles.

Relative Frequency
the ratio of the number of times a value of the data occurs in the set of all outcomes to the number of all outcomes

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10.4: Measures of the Location of the Data

The common measures of location are **quartiles** and **percentiles.** Quartiles are special percentiles. The first quartile, Q_1, is the same as the 25th percentile, and the third quartile, Q_3, is the same as the 75th percentile. The median, *M*, is called both the second quartile and the 50th percentile.

To calculate quartiles and percentiles, the data must be ordered from smallest to largest. Quartiles divide ordered data into quarters. Percentiles divide ordered data into hundredths. To score in the 90th percentile of an exam does not mean, necessarily, that you received 90% on a test. It means that 90% of test scores are the same or less than your score and 10% of the test scores are the same or greater than your test score.

Percentiles are useful for comparing values. For this reason, universities and colleges use percentiles extensively. One instance in which colleges and universities use percentiles is when SAT results are used to determine a minimum testing score that will be used as an acceptance factor. For example, suppose Duke accepts SAT scores at or above the 75th percentile. That translates into a score of at least 1220.

Percentiles are mostly used with very large populations. Therefore, if you were to say that 90% of the test scores are less (and not the same or less) than your score, it would be acceptable because removing one particular data value is not significant.

The **median** is a number that measures the "center" of the data. You can think of the median as the "middle value," but it does not actually have to be one of the observed values. It is a number that separates ordered data into halves. Half the values are the same number or smaller than the median, and half the values are the same number or larger. For example, consider the following data.

1; 11.5; 6; 7.2; 4; 8; 9; 10; 6.8; 8.3; 2; 2; 10; 1

Ordered from smallest to largest:

1; 1; 2; 2; 4; 6; 6.8; 7.2; 8; 8.3; 9; 10; 10; 11.5

Since there are 14 observations, the median is between the seventh value, 6.8, and the eighth value, 7.2. To find the median, add the two values together and divide by two.

$$\frac{6.8 + 7.2}{2} = 7 \tag{10.4.1}$$

The median is seven. Half of the values are smaller than seven and half of the values are larger than seven.

Quartiles are numbers that separate the data into quarters. Quartiles may or may not be part of the data. To find the quartiles, first find the median or second quartile. The first quartile, Q_1, is the middle value of the lower half of the data, and the third quartile, Q_3, is the middle value, or median, of the upper half of the data. To get the idea, consider the same data set:

1; 1; 2; 2; 4; 6; 6.8; 7.2; 8; 8.3; 9; 10; 10; 11.5

The median or **second quartile** is seven. The lower half of the data are 1, 1, 2, 2, 4, 6, 6.8. The middle value of the lower half is two.

1; 1; 2; 2; 4; 6; 6.8

The number two, which is part of the data, is the **first quartile**. One-fourth of the entire sets of values are the same as or less than two and three-fourths of the values are more than two.

The upper half of the data is 7.2, 8, 8.3, 9, 10, 10, 11.5. The middle value of the upper half is nine.

The **third quartile**, Q3, is nine. Three-fourths (75%) of the ordered data set are less than nine. One-fourth (25%) of the ordered data set are greater than nine. The third quartile is part of the data set in this example.

The **interquartile range** is a number that indicates the spread of the middle half or the middle 50% of the data. It is the difference between the third quartile (Q_3) and the first quartile (Q_1).

$$IQR = Q_3 – Q_1 \tag{2.4.1}$$

The *IQR* can help to determine potential **outliers**. **A value is suspected to be a potential outlier if it is less than (1.5)(*IQR*) below the first quartile or more than (1.5)(*IQR*) above the third quartile**. Potential outliers always require further investigation.

Definition: Outliers

A potential outlier is a data point that is significantly different from the other data points. These special data points may be errors or some kind of abnormality or they may be a key to understanding the data.

Example 2.4.1

For the following 13 real estate prices, calculate the *IQR* and determine if any prices are potential outliers. Prices are in dollars.

389,950; 230,500; 158,000; 479,000; 639,000; 114,950; 5,500,000; 387,000; 659,000; 529,000; 575,000; 488,800; 1,095,000

Answer

Order the data from smallest to largest.

114,950; 158,000; 230,500; 387,000; 389,950; 479,000; 488,800; 529,000; 575,000; 639,000; 659,000; 1,095,000; 5,500,000

$$M = 488,800$$

$$Q_1 = \frac{230,500 + 387,000}{2} = 308,750$$

$$Q_3 = \frac{639,000 + 659,000}{2} = 649,000$$

$$IQR = 649,000 – 308,750 = 340,250$$

$$(1.5)(IQR) = (1.5)(340,250) = 510,375$$

$$Q_1 – (1.5)(IQR) = 308,750 – 510,375 = –201,625$$

$$Q_3 + (1.5)(IQR) = 649,000 + 510,375 = 1,159,375$$

No house price is less than –201,625. However, 5,500,000 is more than 1,159,375. Therefore, 5,500,000 is a potential **outlier**.

Exercise 10.4.1

For the following 11 salaries, calculate the *IQR* and determine if any salaries are outliers. The salaries are in dollars.

$33,000; $64,500; $28,000; $54,000; $72,000; $68,500; $69,000; $42,000; $54,000; $120,000; $40,500

Answer

Order the data from smallest to largest.

$28,000; $33,000; $40,500; $42,000; $54,000; $54,000; $64,500; $68,500; $69,000; $72,000; $120,000

Median = $54,000

$$Q_1 = \$40,500$$

$$Q_3 = \$69,000$$

$$IQR = \$69,000 - \$40,500 = \$28,500$$

$$(1.5)(IQR) = (1.5)(\$28,500) = \$42,750$$

$$Q_1 - (1.5)(IQR) = \$40,500 - \$42,750 = -\$2,250$$

$$Q_3 + (1.5)(IQR) = \$69,000 + \$42,750 = \$111,750$$

No salary is less than –$2,250. However, $120,000 is more than $11,750, so $120,000 is a potential outlier.

Example 2.4.2

For the two data sets in the test scores example, find the following:

a. The interquartile range. Compare the two interquartile ranges.
b. Any outliers in either set.

Answer

The five number summary for the day and night classes is

	Minimum	Q_1	Median	Q_3	Maximum
Day	32	56	74.5	82.5	99
Night	25.5	78	81	89	98

a. The *IQR* for the day group is $Q_3 - Q_1 = 82.5 - 56 = 26.5$

The *IQR* for the night group is $Q_3 - Q_1 = 89 - 78 = 11$

The interquartile range (the spread or variability) for the day class is larger than the night class *IQR*. This suggests more variation will be found in the day class's class test scores.

b. Day class outliers are found using the IQR times 1.5 rule. So,
   - $Q_1 - IQR(1.5) = 56–26.5(1.5) = 16.25$
   - $Q_3 + IQR(1.5) = 82.5 + 26.5(1.5) = 122.25$

   Since the minimum and maximum values for the day class are greater than 16.25 and less than 122.25, there are no outliers.

   Night class outliers are calculated as:

   - $Q_1 - IQR(1.5) = 78–11(1.5) = 61.5$
   - $Q_3 + IQR(1.5) = 89 + 11(1.5) = 105.5$

   For this class, any test score less than 61.5 is an outlier. Therefore, the scores of 45 and 25.5 are outliers. Since no test score is greater than 105.5, there is no upper end outlier.

Exercise 10.4.2

Find the interquartile range for the following two data sets and compare them.

Test Scores for Class *A*

69; 96; 81; 79; 65; 76; 83; 99; 89; 67; 90; 77; 85; 98; 66; 91; 77; 69; 80; 94

Test Scores for Class *B*

90; 72; 80; 92; 90; 97; 92; 75; 79; 68; 70; 80; 99; 95; 78; 73; 71; 68; 95; 100

Answer

Class *A*

Order the data from smallest to largest.

65; 66; 67; 69; 69; 76; 77; 77; 79; 80; 81; 83; 85; 89; 90; 91; 94; 96; 98; 99

$Median = \frac{80 + 81}{2}$ = 80.5

$Q_1 = \frac{69 + 76}{2} = 72.5$

$Q_3 = \frac{90 + 91}{2} = 90.5$

$IQR = 90.5 - 72.5 = 18$

Class *B*

Order the data from smallest to largest.

68; 68; 70; 71; 72; 73; 75; 78; 79; 80; 80; 90; 90; 92; 92; 95; 95; 97; 99; 100

$Median = \frac{80 + 80}{2} = 80$

$Q_1 = \frac{72 + 73}{2} = 72.5$

$Q_3 = \frac{92 + 95}{2} = 93.5$

$IQR = 93.5 - 72.5 = 21$

The data for Class *B* has a larger *IQR*, so the scores between Q_3 and Q_1 (middle 50%) for the data for Class *B* are more spread out and not clustered about the median.

Example 10.4.3

Fifty statistics students were asked how much sleep they get per school night (rounded to the nearest hour). The results were:

AMOUNT OF SLEEP PER SCHOOL NIGHT (HOURS)	FREQUENCY	RELATIVE FREQUENCY	CUMULATIVE RELATIVE FREQUENCY
4	2	0.04	0.04

AMOUNT OF SLEEP PER SCHOOL NIGHT (HOURS)	FREQUENCY	RELATIVE FREQUENCY	CUMULATIVE RELATIVE FREQUENCY
5	5	0.10	0.14
6	7	0.14	0.28
7	12	0.24	0.52
8	14	0.28	0.80
9	7	0.14	0.94
10	3	0.06	1.00

Find the 28th percentile. Notice the 0.28 in the "cumulative relative frequency" column. Twenty-eight percent of 50 data values is 14 values. There are 14 values less than the 28th percentile. They include the two 4s, the five 5s, and the seven 6s. The 28th percentile is between the last six and the first seven. **The 28th percentile is 6.5.**

Find the median. Look again at the "cumulative relative frequency" column and find 0.52. The median is the 50th percentile or the second quartile. 50% of 50 is 25. There are 25 values less than the median. They include the two 4s, the five 5s, the seven 6s, and eleven of the 7s. The median or 50th percentile is between the 25th, or seven, and 26th, or seven, values. **The median is seven.**

Find the third quartile. The third quartile is the same as the 75th percentile. You can "eyeball" this answer. If you look at the "cumulative relative frequency" column, you find 0.52 and 0.80. When you have all the fours, fives, sixes and sevens, you have 52% of the data. When you include all the 8s, you have 80% of the data. **The 75th percentile, then, must be an eight**. Another way to look at the problem is to find 75% of 50, which is 37.5, and round up to 38. The third quartile, Q_3, is the 38th value, which is an eight. You can check this answer by counting the values. (There are 37 values below the third quartile and 12 values above.)

Exercise 10.4.3

Forty bus drivers were asked how many hours they spend each day running their routes (rounded to the nearest hour). Find the 65th percentile.

Amount of time spent on route (hours)	Frequency	Relative Frequency	Cumulative Relative Frequency
2	12	0.30	0.30
3	14	0.35	0.65
4	10	0.25	0.90
5	4	0.10	1.00

Answer

The 65th percentile is between the last three and the first four.

The 65th percentile is 3.5.

Example 2.4.4

Using the table above in Example 10.4.3

a. Find the 80th percentile.
b. Find the 90th percentile.
c. Find the first quartile. What is another name for the first quartile?

Solution

Using the data from the frequency table, we have:

a. The 80th percentile is between the last eight and the first nine in the table (between the 40th and 41st values). Therefore, we need to take the mean of the 40th an 41st values. The 80th percentile $= \frac{8+9}{2} = 8.5$
b. The 90th percentile will be the 45th data value (location is $0.90(50) = 45$) and the 45th data value is nine.
c. Q_1 is also the 25th percentile. The 25th percentile location calculation: $P_{25} = 0.25(50) = 12.5 \approx 13$ the 13th data value. Thus, the 25th percentile is six.

Exercise 10.4.4

Refer to the table above in Exercise 10.4.3. Find the third quartile. What is another name for the third quartile?

Answer

The third quartile is the 75th percentile, which is four. The 65th percentile is between three and four, and the 90th percentile is between four and 5.75. The third quartile is between 65 and 90, so it must be four.

COLLABORATIVE STATISTICS

Your instructor or a member of the class will ask everyone in class how many sweaters they own. Answer the following questions:

a. How many students were surveyed?
b. What kind of sampling did you do?
c. Construct two different histograms. For each, starting value = _____ ending value = ____.
d. Find the median, first quartile, and third quartile.
e. Construct a table of the data to find the following:
   i. the 10th percentile
   ii. the 70th percentile
   iii. the percent of students who own less than four sweaters

A Formula for Finding the *k*th Percentile

If you were to do a little research, you would find several formulas for calculating the kth percentile. Here is one of them.

- $k =$ the kth percentile. It may or may not be part of the data.
- $i =$ the index (ranking or position of a data value)
- $n =$ the total number of data

Order the data from smallest to largest.

Calculate $i = \frac{k}{100}(n+1)$

If i is an integer, then the kth percentile is the data value in the ith position in the ordered set of data.

If i is not an integer, then round i up and round i down to the nearest integers. Average the two data values in these two positions in the ordered data set. This is easier to understand in an example.

Example 2.4.5

Listed are 29 ages for Academy Award winning best actors *in order from smallest to largest.*

18; 21; 22; 25; 26; 27; 29; 30; 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77

a. Find the 70th percentile.
b. Find the 83rd percentile.

Solution

a.
- $k = 70$
- i = the index
- $n = 29$

$i = \frac{k}{100}(n+1) = \frac{70}{100}(29+1) = 21$. Twenty-one is an integer, and the data value in the 21st position in the ordered data set is 64. The 70th percentile is 64 years.

b.
- k = 83rd percentile
- i= the index
- $n = 29$

$i = \frac{k}{100}(n+1) = (\frac{83}{100})(29+1) = 24.9$, which is NOT an integer. Round it down to 24 and up to 25. The age in the 24th position is 71 and the age in the 25th position is 72. Average 71 and 72. The 83rd percentile is 71.5 years.

Exercise 10.4.5

Listed are 29 ages for Academy Award winning best actors *in order from smallest to largest.*

18; 21; 22; 25; 26; 27; 29; 30; 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77

Calculate the 20th percentile and the 55th percentile.

Answer

$k = 20$. Index $= i = \frac{k}{100}(n+1) = \frac{20}{100}(29+1) = 6$. The age in the sixth position is 27. The 20th percentile is 27 years.

$k = 55$. Index $= i = \frac{k}{100}(n+1) = \frac{55}{100}(29+1) = 16.5$. Round down to 16 and up to 17. The age in the 16th position is 52 and the age in the 17th position is 55. The average of 52 and 55 is 53.5. The 55th percentile is 53.5 years.

Note 2.4.2

You can calculate percentiles using calculators and computers. There are a variety of online calculators.

A Formula for Finding the Percentile of a Value in a Data Set

- Order the data from smallest to largest.
- $x =$ the number of data values counting from the bottom of the data list up to but not including the data value for which you want to find the percentile.
- $y =$ the number of data values equal to the data value for which you want to find the percentile.

- $n =$ the total number of data.
- Calculate $\frac{x + 0.5y}{n}(100)$. Then round to the nearest integer.

Example 2.4.6

Listed are 29 ages for Academy Award winning best actors *in order from smallest to largest.*

18; 21; 22; 25; 26; 27; 29; 30; 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77

a. Find the percentile for 58.
b. Find the percentile for 25.

Solution

a. Counting from the bottom of the list, there are 18 data values less than 58. There is one value of 58.

$x = 18$ and $y = 1$. $\frac{x + 0.5y}{n}(100) = \frac{18 + 0.5(1)}{29}(100) = 63.80$. 58 is the 64th percentile.

b. Counting from the bottom of the list, there are three data values less than 25. There is one value of 25.

$x = 3$ and $y = 1$. $\frac{x + 0.5y}{n}(100) = \frac{3 + 0.5(1)}{29}(100) = 12.07$. Twenty-five is the 12thpercentile.

Exercise 10.4.6

Listed are 30 ages for Academy Award winning best actors in order from smallest to largest.

18; 21; 22; 25; 26; 27; 29; 30; 31, 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77

Find the percentiles for 47 and 31.

Answer

Percentile for 47: Counting from the bottom of the list, there are 15 data values less than 47. There is one value of 47.

$x = 15$ and $y = 1$. $\frac{x + 0.5y}{n}(100) = \frac{15 + 0.5(1)}{30}(100) = 51.67$. 47 is the 52nd percentile.

Percentile for 31: Counting from the bottom of the list, there are eight data values less than 31. There are two values of 31.

$x = 8$ and $y = 2$. $\frac{x + 0.5y}{n}(100) = \frac{8 + 0.5(2)}{30}(100) = 30$. 31 is the 30th percentile.

Interpreting Percentiles, Quartiles, and Median

A percentile indicates the relative standing of a data value when data are sorted into numerical order from smallest to largest. Percentages of data values are less than or equal to the pth percentile. For example, 15% of data values are less than or equal to the 15th percentile.

- Low percentiles always correspond to lower data values.
- High percentiles always correspond to higher data values.

A percentile may or may not correspond to a value judgment about whether it is "good" or "bad." The interpretation of whether a certain percentile is "good" or "bad" depends on the context of the situation to which the data applies. In some situations, a low percentile would be considered "good;" in other contexts a high percentile might be considered "good". In many situations, there

is no value judgment that applies.

Understanding how to interpret percentiles properly is important not only when describing data, but also when calculating probabilities in later chapters of this text.

GUIDELINE

When writing the interpretation of a percentile in the context of the given data, the sentence should contain the following information.

- information about the context of the situation being considered
- the data value (value of the variable) that represents the percentile
- the percent of individuals or items with data values below the percentile
- the percent of individuals or items with data values above the percentile.

On a timed math test, the first quartile for time it took to finish the exam was 35 minutes. Interpret the first quartile in the context of this situation.

Answer

- Twenty-five percent of students finished the exam in 35 minutes or less.
- Seventy-five percent of students finished the exam in 35 minutes or more.
- A low percentile could be considered good, as finishing more quickly on a timed exam is desirable. (If you take too long, you might not be able to finish.)

Exercise 10.4.7

For the 100-meter dash, the third quartile for times for finishing the race was 11.5 seconds. Interpret the third quartile in the context of the situation.

Answer

Twenty-five percent of runners finished the race in 11.5 seconds or more. Seventy-five percent of runners finished the race in 11.5 seconds or less. A lower percentile is good because finishing a race more quickly is desirable.

On a 20 question math test, the 70th percentile for number of correct answers was 16. Interpret the 70th percentile in the context of this situation.

Answer

- Seventy percent of students answered 16 or fewer questions correctly.
- Thirty percent of students answered 16 or more questions correctly.
- A higher percentile could be considered good, as answering more questions correctly is desirable.

Exercise 10.4.8

On a 60 point written assignment, the 80th percentile for the number of points earned was 49. Interpret the 80th percentile in the context of this situation.

Answer

Eighty percent of students earned 49 points or fewer. Twenty percent of students earned 49 or more points. A higher percentile is good because getting more points on an assignment is desirable.

Example 2.4.9

At a community college, it was found that the 30th percentile of credit units that students are enrolled for is seven units. Interpret the 30th percentile in the context of this situation.

Answer

- Thirty percent of students are enrolled in seven or fewer credit units.
- Seventy percent of students are enrolled in seven or more credit units.
- In this example, there is no "good" or "bad" value judgment associated with a higher or lower percentile. Students attend community college for varied reasons and needs, and their course load varies according to their needs.

Exercise 10.4.9

During a season, the 40th percentile for points scored per player in a game is eight. Interpret the 40th percentile in the context of this situation.

Answer

Forty percent of players scored eight points or fewer. Sixty percent of players scored eight points or more. A higher percentile is good because getting more points in a basketball game is desirable.

Example 2.4.10

Sharpe Middle School is applying for a grant that will be used to add fitness equipment to the gym. The principal surveyed 15 anonymous students to determine how many minutes a day the students spend exercising. The results from the 15 anonymous students are shown.

0 minutes; 40 minutes; 60 minutes; 30 minutes; 60 minutes

10 minutes; 45 minutes; 30 minutes; 300 minutes; 90 minutes;

30 minutes; 120 minutes; 60 minutes; 0 minutes; 20 minutes

Determine the following five values.

- Min = 0
- Q_1 = 20
- Med = 40
- Q_3 = 60
- Max = 300

If you were the principal, would you be justified in purchasing new fitness equipment? Since 75% of the students exercise for 60 minutes or less daily, and since the *IQR* is 40 minutes (60 – 20 = 40), we know that half of the students surveyed exercise between 20 minutes and 60 minutes daily. This seems a reasonable amount of time spent exercising, so the principal would be justified in purchasing the new equipment.

However, the principal needs to be careful. The value 300 appears to be a potential outlier.

$$Q_3 + 1.5(IQR) = 60 + (1.5)(40) = 120 \tag{10.4.2}$$

.

The value 300 is greater than 120 so it is a potential outlier. If we delete it and calculate the five values, we get the following values:

- Min = 0
- Q_1 = 20
- Q_3 = 60
- Max = 120

We still have 75% of the students exercising for 60 minutes or less daily and half of the students exercising between 20 and 60 minutes a day. However, 15 students is a small sample and the principal should survey more students to be sure of his survey results.

References

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2. Data from the United States Department of Commerce: United States Census Bureau. Available online at http://www.census.gov/ (accessed April 3, 2013).
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4. Data from *San Jose Mercury News*.
5. Data from *Time Magazine*; survey by Yankelovich Partners, Inc.

Review

The values that divide a rank-ordered set of data into 100 equal parts are called percentiles. Percentiles are used to compare and interpret data. For example, an observation at the 50th percentile would be greater than 50 percent of the other obeservations in the set. Quartiles divide data into quarters. The first quartile (Q_1) is the 25th percentile,the second quartile (Q_2 or median) is 50th percentile, and the third quartile (Q_3) is the the 75th percentile. The interquartile range, or *IQR*, is the range of the middle 50 percent of the data values. The *IQR* is found by subtracting Q_1 from Q_3, and can help determine outliers by using the following two expressions.

- $Q_3 + IQR(1.5)$
- $Q_1 - IQR(1.5)$

Formula Review

$$i = \frac{k}{100}(n+1)$$

where i = the ranking or position of a data value,

- k = the kth percentile,
- n = total number of data.

Expression for finding the percentile of a data value: $\left(\frac{x+0.5y}{n}\right)(100)$

where $x =$ the number of values counting from the bottom of the data list up to but not including the data value for which you want to find the percentile,

$y =$ the number of data values equal to the data value for which you want to find the percentile,

n = total number of data

Glossary

Interquartile Range
: or *IQR*, is the range of the middle 50 percent of the data values; the *IQR* is found by subtracting the first quartile from the third quartile.

Outlier
: an observation that does not fit the rest of the data

Percentile
: a number that divides ordered data into hundredths; percentiles may or may not be part of the data. The median of the data is the second quartile and the 50th percentile. The first and third quartiles are the 25th and the 75th percentiles, respectively.

Quartiles
: the numbers that separate the data into quarters; quartiles may or may not be part of the data. The second quartile is the median of the data.

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10.4E: Measures of the Location of the Data (Exercises)

Exercise 10.4E.10

Listed are 29 ages for Academy Award winning best actors *in order from smallest to largest.*

18; 21; 22; 25; 26; 27; 29; 30; 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77

a. Find the 40th percentile.
b. Find the 78th percentile.

Answer

a. The 40th percentile is 37 years.
b. The 78th percentile is 70 years.

Exercise 10.4E.11

Listed are 32 ages for Academy Award winning best actors *in order from smallest to largest.*

18; 18; 21; 22; 25; 26; 27; 29; 30; 31; 31; 33; 36; 37; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77

a. Find the percentile of 37.
b. Find the percentile of 72.

Exercise 10.4E.12

Jesse was ranked 37th in his graduating class of 180 students. At what percentile is Jesse's ranking?

Answer

Jesse graduated 37th out of a class of 180 students. There are 180 – 37 = 143 students ranked below Jesse. There is one rank of 37.

$x = 143$ and $y = 1$. $\frac{x + 0.5y}{n}(100) = \frac{143 + 0.5(1)}{180}(100) = 79.72$. Jesse's rank of 37 puts him at the 80th percentile.

Exercise 10.4E.13

a. For runners in a race, a low time means a faster run. The winners in a race have the shortest running times. Is it more desirable to have a finish time with a high or a low percentile when running a race?
b. The 20th percentile of run times in a particular race is 5.2 minutes. Write a sentence interpreting the 20th percentile in the context of the situation.
c. A bicyclist in the 90th percentile of a bicycle race completed the race in 1 hour and 12 minutes. Is he among the fastest or slowest cyclists in the race? Write a sentence interpreting the 90th percentile in the context of the situation.

Exercise 10.4E.14

a. For runners in a race, a higher speed means a faster run. Is it more desirable to have a speed with a high or a low percentile when running a race?
b. The 40th percentile of speeds in a particular race is 7.5 miles per hour. Write a sentence interpreting the 40th percentile in the context of the situation.

Answer

a. For runners in a race it is more desirable to have a high percentile for speed. A high percentile means a higher speed which is faster.
b. 40% of runners ran at speeds of 7.5 miles per hour or less (slower). 60% of runners ran at speeds of 7.5 miles per hour or more (faster).

Exercise 10.4E.15

On an exam, would it be more desirable to earn a grade with a high or low percentile? Explain.

Exercise 10.4E.16

Mina is waiting in line at the Department of Motor Vehicles (DMV). Her wait time of 32 minutes is the 85th percentile of wait times. Is that good or bad? Write a sentence interpreting the 85th percentile in the context of this situation.

Answer

When waiting in line at the DMV, the 85th percentile would be a long wait time compared to the other people waiting. 85% of people had shorter wait times than Mina. In this context, Mina would prefer a wait time corresponding to a lower percentile. 85% of people at the DMV waited 32 minutes or less. 15% of people at the DMV waited 32 minutes or longer.

Exercise 10.4E.17

In a survey collecting data about the salaries earned by recent college graduates, Li found that her salary was in the 78th percentile. Should Li be pleased or upset by this result? Explain.

Exercise 10.4E.18

In a study collecting data about the repair costs of damage to automobiles in a certain type of crash tests, a certain model of car had $1,700 in damage and was in the 90th percentile. Should the manufacturer and the consumer be pleased or upset by this result? Explain and write a sentence that interprets the 90th percentile in the context of this problem.

Answer

The manufacturer and the consumer would be upset. This is a large repair cost for the damages, compared to the other cars in the sample. INTERPRETATION: 90% of the crash tested cars had damage repair costs of $1700 or less; only 10% had damage repair costs of $1700 or more.

Exercise 10.4E.19

The University of California has two criteria used to set admission standards for freshman to be admitted to a college in the UC system:

a. Students' GPAs and scores on standardized tests (SATs and ACTs) are entered into a formula that calculates an "admissions index" score. The admissions index score is used to set eligibility standards intended to meet the goal of admitting the top 12% of high school students in the state. In this context, what percentile does the top 12% represent?
b. Students whose GPAs are at or above the 96th percentile of all students at their high school are eligible (called eligible in the local context), even if they are not in the top 12% of all students in the state. What percentage of students from each high school are "eligible in the local context"?

Exercise 10.4E.20

Suppose that you are buying a house. You and your realtor have determined that the most expensive house you can afford is the 34th percentile. The 34th percentile of housing prices is $240,000 in the town you want to move to. In this town, can you afford 34% of the houses or 66% of the houses?

Answer

You can afford 34% of houses. 66% of the houses are too expensive for your budget. INTERPRETATION: 34% of houses cost $240,000 or less. 66% of houses cost $240,000 or more.

Use Exercise to calculate the following values:

Exercise 10.4E.21

First quartile = _____

Exercise 10.4E.22

Second quartile = median = 50th percentile = _____

Answer

4

Exercise 10.4E.23

Third quartile = _____

Exercise 10.4E.24

Interquartile range (*IQR*) = _____ – _____ = _____

Answer

$6 - 4 = 2$

Exercise $10.4E.25$

10th percentile = _____

Exercise $10.4E.26$

70th percentile = _____

Answer

6

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10.5: Box Plots

Box plots (also called *box-and-whisker plots* or *box-whisker plots*) give a good graphical image of the concentration of the data. They also show how far the extreme values are from most of the data. A box plot is constructed from five values: the minimum value, the first quartile, the median, the third quartile, and the maximum value. We use these values to compare how close other data values are to them.

To construct a box plot, use a horizontal or vertical number line and a rectangular box. The smallest and largest data values label the endpoints of the axis. The first quartile marks one end of the box and the third quartile marks the other end of the box. Approximately *the middle 50 percent of the data fall inside the box*. The "whiskers" extend from the ends of the box to the smallest and largest data values. The median or second quartile can be between the first and third quartiles, or it can be one, or the other, or both. The box plot gives a good, quick picture of the data.

> You may encounter box-and-whisker plots that have dots marking outlier values. In those cases, the whiskers are not extending to the minimum and maximum values.

Consider, again, this dataset.

1; 1; 2; 2; 4; 6; 6;.8; 7.2; 8; 8.3; 9; 10; 10; 11.5

The first quartile is two, the median is seven, and the third quartile is nine. The smallest value is one, and the largest value is 11.5. The following image shows the constructed box plot.

> See the calculator instructions on the TI web site or in the appendix.

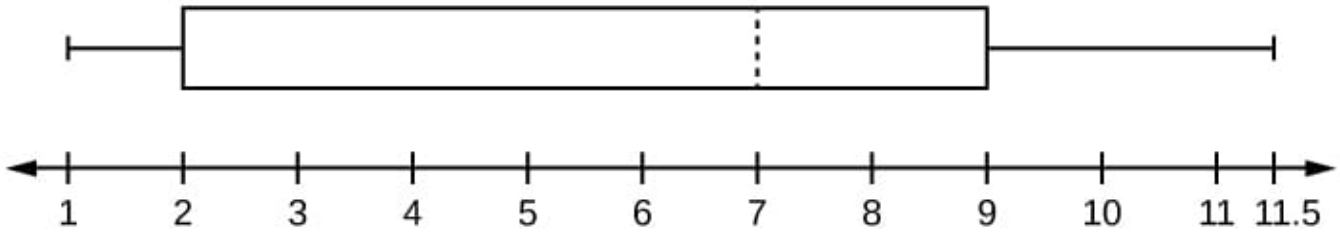


Figure 10.5.1

The two whiskers extend from the first quartile to the smallest value and from the third quartile to the largest value. The median is shown with a dashed line.

> It is important to start a box plot with a **scaled number line**. Otherwise the box plot may not be useful.

Example 10.5.1

The following data are the heights of 40 students in a statistics class.

59; 60; 61; 62; 62; 63; 63; 64; 64; 64; 65; 65; 65; 65; 65; 65; 65; 65; 66; 66; 67; 67; 68; 68; 69; 70; 70; 70; 70; 70; 71; 71; 72; 72; 73; 74; 74; 75; 77

Construct a box plot with the following properties; the calculator instructions for the minimum and maximum values as well as the quartiles follow the example.

- Minimum value = 59
- Maximum value = 77
- Q1: First quartile = 64.5

- Q2: Second quartile or median= 66
- Q3: Third quartile = 70

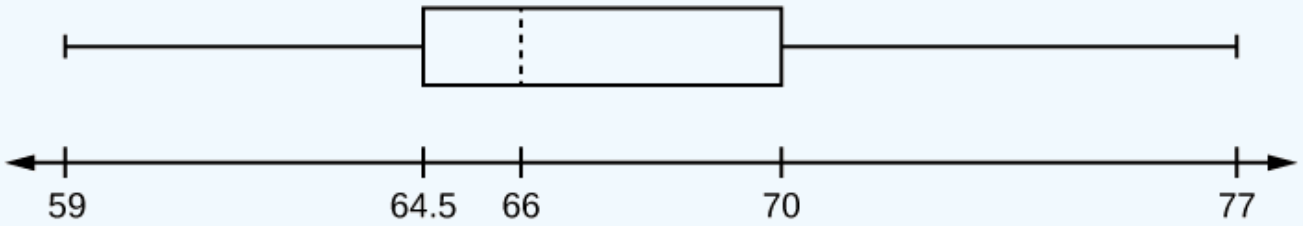


Figure 10.5.2

a. Each quarter has approximately 25% of the data.
b. The spreads of the four quarters are 64.5 – 59 = 5.5 (first quarter), 66 – 64.5 = 1.5 (second quarter), 70 – 66 = 4 (third quarter), and 77 – 70 = 7 (fourth quarter). So, the second quarter has the smallest spread and the fourth quarter has the largest spread.
c. $\text{Range} = \text{maximum value} - \text{the minimum value} = 77 - 59 = 18$
d. Interquartile Range: $IQR = Q_3 – Q_1 = 70 - 64.5 = 5.5$.
e. The interval 59–65 has more than 25% of the data so it has more data in it than the interval 66 through 70 which has 25% of the data.
f. The middle 50% (middle half) of the data has a range of 5.5 inches.

Calculator

To find the minimum, maximum, and quartiles:

Enter data into the list editor (Pres STAT 1:EDIT). If you need to clear the list, arrow up to the name L1, press CLEAR, and then arrow down.

Put the data values into the list L1.

Press STAT and arrow to CALC. Press 1:1-VarStats. Enter L1.

Press ENTER.

Use the down and up arrow keys to scroll.

Smallest value = 59.

Largest value = 77.

Q_1: First quartile = 64.5.

Q_2: Second quartile or median = 66.

Q_3: Third quartile = 70.

To construct the box plot:

Press 4: Plotsoff. Press ENTER.

Arrow down and then use the right arrow key to go to the fifth picture, which is the box plot. Press ENTER.

Arrow down to Xlist: Press 2nd 1 for L1

Arrow down to Freq: Press ALPHA. Press 1.

Press Zoom. Press 9: ZoomStat.

Press TRACE, and use the arrow keys to examine the box plot.

Exercise 10.5.1

The following data are the number of pages in 40 books on a shelf. Construct a box plot using a graphing calculator, and state the interquartile range.

136; 140; 178; 190; 205; 215; 217; 218; 232; 234; 240; 255; 270; 275; 290; 301; 303; 315; 317; 318; 326; 333; 343; 349; 360; 369; 377; 388; 391; 392; 398; 400; 402; 405; 408; 422; 429; 450; 475; 512

Answer

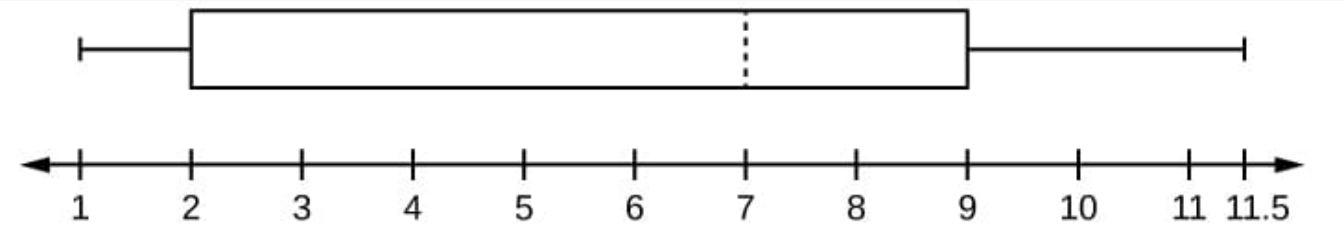


Figure 10.5.3

$IQR = 158$

For some sets of data, some of the largest value, smallest value, first quartile, median, and third quartile may be the same. For instance, you might have a data set in which the median and the third quartile are the same. In this case, the diagram would not have a dotted line inside the box displaying the median. The right side of the box would display both the third quartile and the median. For example, if the smallest value and the first quartile were both one, the median and the third quartile were both five, and the largest value was seven, the box plot would look like:



Figure 10.5.4

In this case, at least 25% of the values are equal to one. Twenty-five percent of the values are between one and five, inclusive. At least 25% of the values are equal to five. The top 25% of the values fall between five and seven, inclusive.

Example 10.5.2

Test scores for a college statistics class held during the day are:

99; 56; 78; 55.5; 32; 90; 80; 81; 56; 59; 45; 77; 84.5; 84; 70; 72; 68; 32; 79; 90

Test scores for a college statistics class held during the evening are:

98; 78; 68; 83; 81; 89; 88; 76; 65; 45; 98; 90; 80; 84.5; 85; 79; 78; 98; 90; 79; 81; 25.5

a. Find the smallest and largest values, the median, and the first and third quartile for the day class.

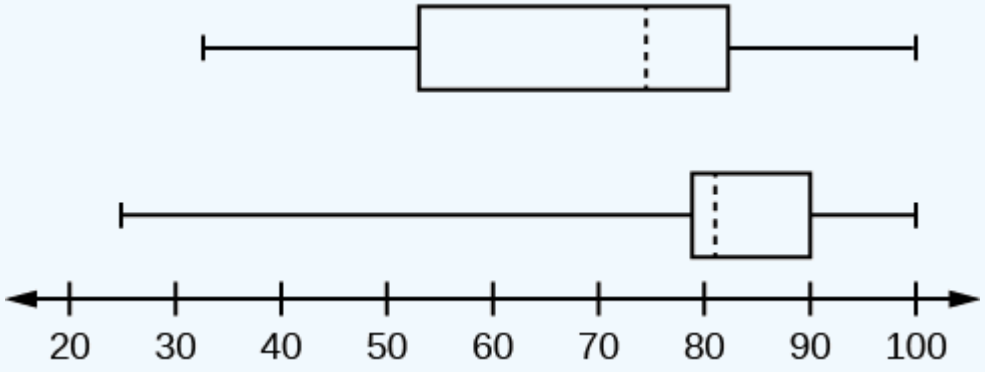
b. Find the smallest and largest values, the median, and the first and third quartile for the night class.
c. For each data set, what percentage of the data is between the smallest value and the first quartile? the first quartile and the median? the median and the third quartile? the third quartile and the largest value? What percentage of the data is between the first quartile and the largest value?
d. Create a box plot for each set of data. Use one number line for both box plots.
e. Which box plot has the widest spread for the middle 50% of the data (the data between the first and third quartiles)? What does this mean for that set of data in comparison to the other set of data?

Answer

a.
- Min = 32
- $Q_1 = 56$
- $M = 74.5$
- $Q_3 = 82.5$
- Max = 99

b.
- Min = 25.5
- $Q_1 = 78$
- $M = 81$
- $Q_3 = 89$
- Max = 98

c. Day class: There are six data values ranging from 32 to 56: 30%. There are six data values ranging from 56 to 74.5: 30%. There are five data values ranging from 74.5 to 82.5: 25%. There are five data values ranging from 82.5 to 99: 25%. There are 16 data values between the first quartile, 56, and the largest value, 99: 75%. Night class:

d.


e. The first data set has the wider spread for the middle 50% of the data. The *IQR*for the first data set is greater than the *IQR* for the second set. This means that there is more variability in the middle 50% of the first data set.

Exercise 10.5.2

The following data set shows the heights in inches for the boys in a class of 40 students.

66; 66; 67; 67; 68; 68; 68; 68; 68; 69; 69; 69; 70; 71; 72; 72; 72; 73; 73; 74

The following data set shows the heights in inches for the girls in a class of 40 students.

61; 61; 62; 62; 63; 63; 63; 65; 65; 65; 66; 66; 66; 67; 68; 68; 68; 69; 69; 69

Construct a box plot using a graphing calculator for each data set, and state which box plot has the wider spread for the middle 50% of the data.

Answer

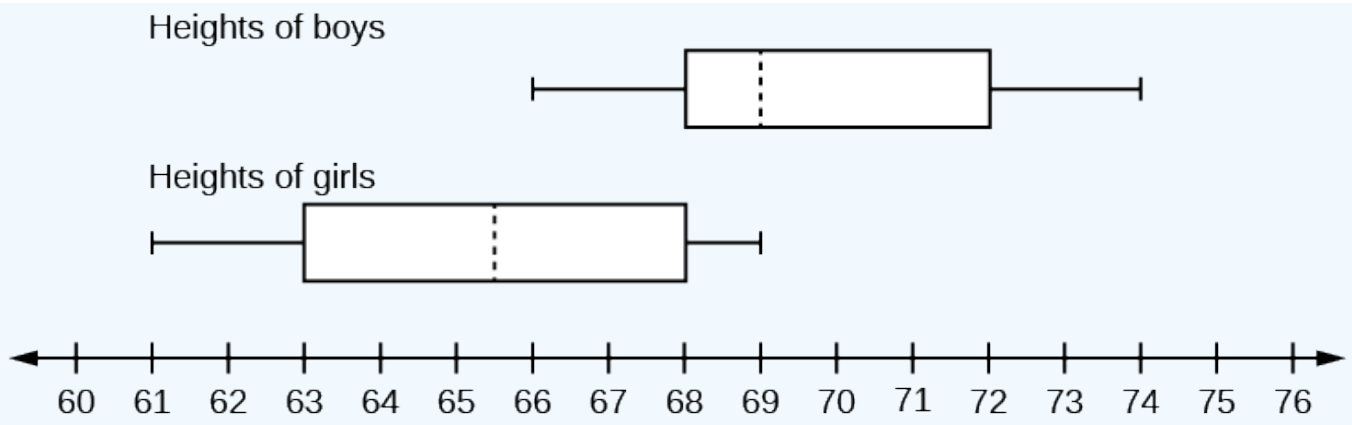


Figure 10.5.6

IQR for the boys = 4

IQR for the girls = 5

The box plot for the heights of the girls has the wider spread for the middle 50% of the data.

Example 10.5.3

Graph a box-and-whisker plot for the data values shown.

10; 10; 10; 15; 35; 75; 90; 95; 100; 175; 420; 490; 515; 515; 790

The five numbers used to create a box-and-whisker plot are:

- Min: 10
- Q_1: 15
- Med: 95
- Q_3: 490
- Max: 790

The following graph shows the box-and-whisker plot.

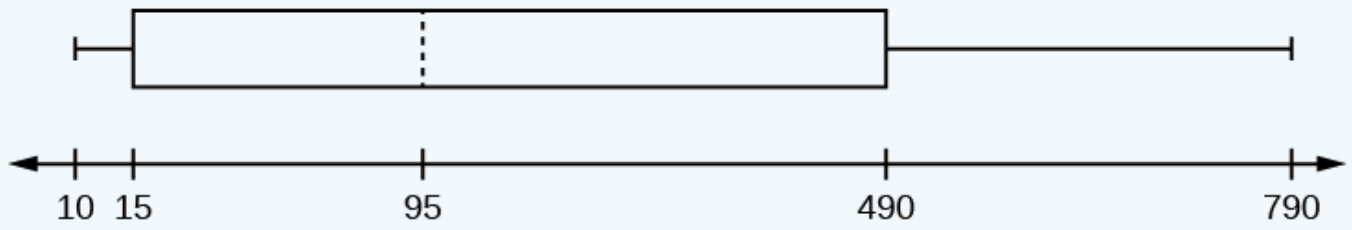


Figure 10.5.7

Exercise 10.5.3

Follow the steps you used to graph a box-and-whisker plot for the data values shown.

0; 5; 5; 15; 30; 30; 45; 50; 50; 60; 75; 110; 140; 240; 330

Answer

The data are in order from least to greatest. There are 15 values, so the eighth number in order is the median: 50. There are seven data values written to the left of the median and 7 values to the right. The five values that are used to create the boxplot are:

- Min: 0
- Q_1: 15
- Med: 50
- Q_3: 110
- Max: 330

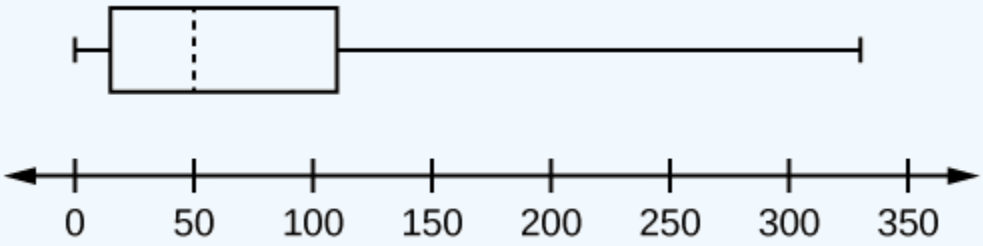


Figure 10.5.8

References

1. Data from *West Magazine*.

Review

Box plots are a type of graph that can help visually organize data. To graph a box plot the following data points must be calculated: the minimum value, the first quartile, the median, the third quartile, and the maximum value. Once the box plot is graphed, you can display and compare distributions of data.

Sixty-five randomly selected car salespersons were asked the number of cars they generally sell in one week. Fourteen people answered that they generally sell three cars; nineteen generally sell four cars; twelve generally sell five cars; nine generally sell six cars; eleven generally sell seven cars.

Exercise 2.5.4

Construct a box plot below. Use a ruler to measure and scale accurately.

Exercise 2.5.5

Looking at your box plot, does it appear that the data are concentrated together, spread out evenly, or concentrated in some areas, but not in others? How can you tell?

Answer

More than 25% of salespersons sell four cars in a typical week. You can see this concentration in the box plot because the first quartile is equal to the median. The top 25% and the bottom 25% are spread out evenly; the whiskers have the same length.

Bringing It Together

Exercise 2.5.6

Santa Clara County, CA, has approximately 27,873 Japanese-Americans. Their ages are as follows:

Age Group	Percent of Community
0–17	18.9
18–24	8.0
25–34	22.8
35–44	15.0
45–54	13.1
55–64	11.9
65+	10.3

a. Construct a histogram of the Japanese-American community in Santa Clara County, CA. The bars will **not** be the same width for this example. Why not? What impact does this have on the reliability of the graph?
b. What percentage of the community is under age 35?
c. Which box plot most resembles the information above?

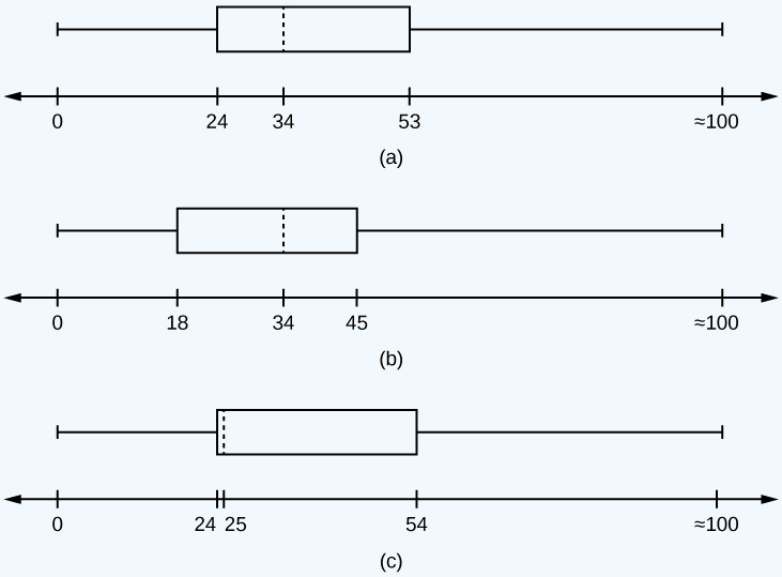


Figure 10.5.9.

Answer

a. For graph, check student's solution.
b. 49.7% of the community is under the age of 35.
c. Based on the information in the table, graph (a) most closely represents the data.

Glossary

Box plot
a graph that gives a quick picture of the middle 50% of the data

First Quartile
the value that is the median of the of the lower half of the ordered data set

Frequency Polygon
looks like a line graph but uses intervals to display ranges of large amounts of data

Interval
also called a class interval; an interval represents a range of data and is used when displaying large data sets

Paired Data Set
two data sets that have a one to one relationship so that:

- both data sets are the same size, and
- each data point in one data set is matched with exactly one point from the other set.

Skewed
used to describe data that is not symmetrical; when the right side of a graph looks “chopped off” compared the left side, we say it is “skewed to the left.” When the left side of the graph looks “chopped off” compared to the right side, we say the data is “skewed to the right.” Alternatively: when the lower values of the data are more spread out, we say the data are skewed to the left. When the greater values are more spread out, the data are skewed to the right.

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10.6: Measures of the Center of the Data

The "center" of a data set is also a way of describing location. The two most widely used measures of the "center" of the data are the **mean** (average) and the median. To calculate the **mean weight** of 50 people, add the 50 weights together and divide by 50. To find the **median weight** of the 50 people, order the data and find the number that splits the data into two equal parts. The median is generally a better measure of the center when there are extreme values or outliers because it is not affected by the precise numerical values of the outliers. The mean is the most common measure of the center.

Note

The words “mean” and “average” are often used interchangeably. The substitution of one word for the other is common practice. The technical term is “arithmetic mean” and “average” is technically a center location. However, in practice among non-statisticians, “average" is commonly accepted for “arithmetic mean.”

When each value in the data set is not unique, the mean can be calculated by multiplying each distinct value by its frequency and then dividing the sum by the total number of data values. The letter used to represent the sample mean is an x with a bar over it (pronounced “x bar”): $\overline{x}$.

The Greek letter μ (pronounced "mew") represents the **population mean**. One of the requirements for the **sample mean** to be a good estimate of the **population mean** is for the sample taken to be truly random.

To see that both ways of calculating the mean are the same, consider the sample:

1; 1; 1; 2; 2; 3; 4; 4; 4; 4; 4

$$\bar{x} = \frac{1+1+1+2+2+3+4+4+4+4+4}{11} = 2.7 \tag{10.6.1}$$

$$\bar{x} = \frac{3(1)+2(2)+1(3)+5(4)}{11} = 2.7 \tag{10.6.2}$$

In the second calculation, the frequencies are 3, 2, 1, and 5.

You can quickly find the location of the median by using the expression

$$\frac{n+1}{2} \tag{10.6.3}$$

The letter n is the total number of data values in the sample. If n is an odd number, the median is the middle value of the ordered data (ordered smallest to largest). If n is an even number, the median is equal to the two middle values added together and divided by two after the data has been ordered. For example, if the total number of data values is 97, then

$$\frac{n+1}{2} = \frac{97+1}{2} = 49. \tag{10.6.4}$$

The median is the 49th value in the ordered data. If the total number of data values is 100, then

$$\frac{n+1}{2} = \frac{100+1}{2} = 50.5. \tag{10.6.5}$$

The median occurs midway between the 50th and 51st values. The location of the median and the value of the median are **not** the same. The upper case letter M is often used to represent the median. The next example illustrates the location of the median and the value of the median.

Example 10.6.1

AIDS data indicating the number of months a patient with AIDS lives after taking a new antibody drug are as follows (smallest to largest):

3; 4; 8; 8; 10; 11; 12; 13; 14; 15; 15; 16; 16; 17; 17; 18; 21; 22; 22; 24; 24; 25; 26; 26; 27; 27; 29; 29; 31; 32; 33; 33; 34; 34; 35; 37; 40; 44; 44; 47

Calculate the mean and the median.

Answer

The calculation for the mean is:

$$\bar{x} = \frac{[3 + 4 + (8)(2) + 10 + 11 + 12 + 13 + 14 + (15)(2) + (16)(2) + ... + 35 + 37 + 40 + (44)(2) + 47]}{40} = 23.6 \quad (10.6.6)$$

To find the median, M, first use the formula for the location. The location is:

$$\frac{n+1}{2} = \frac{40+1}{2} = 20.5 \quad (10.6.7)$$

Starting at the smallest value, the median is located between the 20th and 21st values (the two 24s):

3; 4; 8; 8; 10; 11; 12; 13; 14; 15; 15; 16; 16; 17; 17; 18; 21; 22; 22; 24; 24; 25; 26; 26; 27; 27; 29; 29; 31; 32; 33; 33; 34; 34; 35; 37; 40; 44; 44; 47

$$M = \frac{24 + 24}{2} = 24 \quad (10.6.8)$$

Calculator

To find the mean and the median:

Clear list L1. Pres STAT 4:ClrList. Enter 2nd 1 for list L1. Press ENTER.

Enter data into the list editor. Press STAT 1:EDIT.

Put the data values into list L1.

Press STAT and arrow to CALC. Press 1:1-VarStats. Press 2nd 1 for L1 and then ENTER.

Press the down and up arrow keys to scroll.

$\bar{x}$ = 23.6, M = 24

Exercise 10.6.1

The following data show the number of months patients typically wait on a transplant list before getting surgery. The data are ordered from smallest to largest. Calculate the mean and median.

3; 4; 5; 7; 7; 7; 7; 8; 8; 9; 9; 10; 10; 10; 10; 10; 11; 12; 12; 13; 14; 14; 15; 15; 17; 17; 18; 19; 19; 19; 21; 21; 22; 22; 23; 24; 24; 24; 24

Answer

Mean: $3 + 4 + 5 + 7 + 7 + 7 + 7 + 8 + 8 + 9 + 9 + 10 + 10 + 10 + 10 + 10 + 11 + 12 + 12 + 13 + 14 + 14 + 15 + 15 + 17 + 17 + 18 + 19 + 19 + 19 + 21 + 21 + 22 + 22 + 23 + 24 + 24 + 24 = 544$

$$\frac{544}{39} = 13.95 \tag{10.6.9}$$

Median: Starting at the smallest value, the median is the 20th term, which is 13.

Example 10.6.2

Suppose that in a small town of 50 people, one person earns $5,000,000 per year and the other 49 each earn $30,000. Which is the better measure of the "center": the mean or the median?

Solution

$$\bar{x} = \frac{5,000,000 + 49(30,000)}{50} = 129,400 \tag{10.6.10}$$

$M = 30,000$

(There are 49 people who earn $30,000 and one person who earns $5,000,000.)

The median is a better measure of the "center" than the mean because 49 of the values are 30,000 and one is 5,000,000. The 5,000,000 is an outlier. The 30,000 gives us a better sense of the middle of the data.

Exercise 10.6.2

In a sample of 60 households, one house is worth $2,500,000. Half of the rest are worth $280,000, and all the others are worth $315,000. Which is the better measure of the “center”: the mean or the median?

Answer

The median is the better measure of the “center” than the mean because 59 of the values are $280,000 and one is $2,500,000. The $2,500,000 is an outlier. Either $280,000 or $315,000 gives us a better sense of the middle of the data.

Another measure of the center is the mode. The **mode** is the most frequent value. There can be more than one mode in a data set as long as those values have the same frequency and that frequency is the highest. A data set with two modes is called bimodal.

Example 10.6.3

Statistics exam scores for 20 students are as follows:

50; 53; 59; 59; 63; 63; 72; 72; 72; 72; 72; 76; 78; 81; 83; 84; 84; 84; 90; 93

Find the mode.

Answer

The most frequent score is 72, which occurs five times. Mode = 72.

Exercise 10.6.3

The number of books checked out from the library from 25 students are as follows:

0; 0; 0; 1; 2; 3; 3; 4; 4; 5; 5; 7; 7; 7; 7; 8; 8; 8; 9; 10; 10; 11; 11; 12; 12

Find the mode.

Answer

The most frequent number of books is 7, which occurs four times. Mode = 7.

Example 10.6.4

Five real estate exam scores are 430, 430, 480, 480, 495. The data set is bimodal because the scores 430 and 480 each occur twice.

When is the mode the best measure of the "center"? Consider a weight loss program that advertises a mean weight loss of six pounds the first week of the program. The mode might indicate that most people lose two pounds the first week, making the program less appealing.

> The mode can be calculated for qualitative data as well as for quantitative data. For example, if the data set is: red, red, red, green, green, yellow, purple, black, blue, the mode is red.

Statistical software will easily calculate the mean, the median, and the mode. Some graphing calculators can also make these calculations. In the real world, people make these calculations using software.

Exercise 10.6.4

Five credit scores are 680, 680, 700, 720, 720. The data set is bimodal because the scores 680 and 720 each occur twice. Consider the annual earnings of workers at a factory. The mode is $25,000 and occurs 150 times out of 301. The median is $50,000 and the mean is $47,500. What would be the best measure of the “center”?

Answer

Because $25,000 occurs nearly half the time, the mode would be the best measure of the center because the median and mean don’t represent what most people make at the factory.

The Law of Large Numbers and the Mean

The Law of Large Numbers says that if you take samples of larger and larger size from any population, then the mean $\bar{x}$ of the sample is very likely to get closer and closer to μ. This is discussed in more detail later in the text.

Sampling Distributions and Statistic of a Sampling Distribution

You can think of a sampling distribution as a **relative frequency distribution** with a great many samples. (See **Sampling and Data** for a review of relative frequency). Suppose thirty randomly selected students were asked the number of movies they watched the previous week. The results are in the **relative frequency table** shown below.

# of movies	Relative Frequency
0	$\frac{5}{30}$
1	$\frac{15}{30}$

# of movies	Relative Frequency
2	$\frac{6}{30}$
3	$\frac{3}{30}$
4	$\frac{1}{30}$

If you let the number of samples get very large (say, 300 million or more), the relative frequency table becomes a relative frequency distribution.

A statistic is a number calculated from a sample. Statistic examples include the mean, the median and the mode as well as others. The sample mean $\bar{x}$ is an example of a statistic which estimates the population mean μ.

Calculating the Mean of Grouped Frequency Tables

When only grouped data is available, you do not know the individual data values (we only know intervals and interval frequencies); therefore, you cannot compute an exact mean for the data set. What we must do is estimate the actual mean by calculating the mean of a frequency table. A frequency table is a data representation in which grouped data is displayed along with the corresponding frequencies. To calculate the mean from a grouped frequency table we can apply the basic definition of mean:

$$mean = \frac{\text{data sum}}{\text{number of data values}}. \tag{10.6.11}$$

We simply need to modify the definition to fit within the restrictions of a frequency table.

Since we do not know the individual data values we can instead find the midpoint of each interval. The midpoint is

$$\frac{\text{lower boundary}+\text{upper boundary}}{2}. \tag{10.6.12}$$

We can now modify the mean definition to be

$$\text{Mean of Frequency Table} = \frac{\sum fm}{\sum f} \tag{10.6.13}$$

where f is the frequency of the interval and m is the midpoint of the interval.

Example 10.6.5

A frequency table displaying professor Blount's last statistic test is shown. Find the best estimate of the class mean.

Grade Interval	Number of Students
50–56.5	1
56.5–62.5	0
62.5–68.5	4
68.5–74.5	4
74.5–80.5	2

Grade Interval	Number of Students
80.5–86.5	3
86.5–92.5	4
92.5–98.5	1

Solution

- Find the midpoints for all intervals

Grade Interval	Midpoint
50–56.5	53.25
56.5–62.5	59.5
62.5–68.5	65.5
68.5–74.5	71.5
74.5–80.5	77.5
80.5–86.5	83.5
86.5–92.5	89.5
92.5–98.5	95.5

- Calculate the sum of the product of each interval frequency and midpoint. $53.25(1) + 59.5(0) + 65.5(4) + 71.5(4) + 77.5(2) + 83.5(3) + 89.5(4) + 95.5(1) = 1460.25$
- $\mu = \frac{\sum fm}{\sum f} = \frac{1460.25}{19} = 76.86$

Exercise 10.6.5

Maris conducted a study on the effect that playing video games has on memory recall. As part of her study, she compiled the following data:

Hours Teenagers Spend on Video Games	Number of Teenagers
0–3.5	3
3.5–7.5	7
7.5–11.5	12
11.5–15.5	7
15.5–19.5	9

What is the best estimate for the mean number of hours spent playing video games?

Answer

Find the midpoint of each interval, multiply by the corresponding number of teenagers, add the results and then divide by the total number of teenagers

The midpoints are 1.75, 5.5, 9.5, 13.5,17.5.

$$Mean = (1.75)(3) + (5.5)(7) + (9.5)(12) + (13.5)(7) + (17.5)(9) = 409.75 \tag{10.6.14}$$

References

1. Data from The World Bank, available online at http://www.worldbank.org (accessed April 3, 2013).
2. “Demographics: Obesity – adult prevalence rate.” Indexmundi. Available online at http://www.indexmundi.com/g/r.aspx?t=50&v=2228&l=en (accessed April 3, 2013).

Review

The mean and the median can be calculated to help you find the "center" of a data set. The mean is the best estimate for the actual data set, but the median is the best measurement when a data set contains several outliers or extreme values. The mode will tell you the most frequently occuring datum (or data) in your data set. The mean, median, and mode are extremely helpful when you need to analyze your data, but if your data set consists of ranges which lack specific values, the mean may seem impossible to calculate. However, the mean can be approximated if you add the lower boundary with the upper boundary and divide by two to find the midpoint of each interval. Multiply each midpoint by the number of values found in the corresponding range. Divide the sum of these values by the total number of data values in the set.

Formula Review

$$\mu = \frac{\sum fm}{\sum f} \tag{10.6.15}$$

where f = interval frequencies and m = interval midpoints.

Exercise 2.6.6

Find the mean for the following frequency tables.

a.

Grade	Frequency
49.5–59.5	2
59.5–69.5	3
69.5–79.5	8
79.5–89.5	12
89.5–99.5	5

b.

Daily Low Temperature	Frequency
49.5–59.5	53
59.5–69.5	32
69.5–79.5	15
79.5–89.5	1
89.5–99.5	0

c.

Points per Game	Frequency
49.5–59.5	14
59.5–69.5	32
69.5–79.5	15
79.5–89.5	23

Points per Game	Frequency
89.5–99.5	2

Use the following information to answer the next three exercises: The following data show the lengths of boats moored in a marina. The data are ordered from smallest to largest: 16; 17; 19; 20; 20; 21; 23; 24; 25; 25; 25; 26; 26; 27; 27; 27; 28; 29; 30; 32; 33; 33; 34; 35; 37; 39; 40

Exercise 2.6.7

Calculate the mean.

Answer

Mean: $16 + 17 + 19 + 20 + 20 + 21 + 23 + 24 + 25 + 25 + 25 + 26 + 26 + 27 + 27 + 27 + 28 + 29 + 30 + 32 + 33 + 33 + 34 + 35 + 37 + 39 + 40 = 738$;

$\frac{738}{27} = 27.33$

Exercise 2.6.8

Identify the median.

Exercise 2.6.9

Identify the mode.

Answer

The most frequent lengths are 25 and 27, which occur three times. Mode = 25, 27

Use the following information to answer the next three exercises: Sixty-five randomly selected car salespersons were asked the number of cars they generally sell in one week. Fourteen people answered that they generally sell three cars; nineteen generally sell four cars; twelve generally sell five cars; nine generally sell six cars; eleven generally sell seven cars. Calculate the following:

Exercise 2.6.10

sample mean = $\bar{x}$ = _____

Exercise 2.6.11

median = _____

Answer

4

Bringing It Together

Exercise 2.6.12

Javier and Ercilia are supervisors at a shopping mall. Each was given the task of estimating the mean distance that shoppers live from the mall. They each randomly surveyed 100 shoppers. The samples yielded the following information.

	Javier	Ercilia
$\bar{x}$	6.0 miles	6.0 miles
s	4.0 miles	7.0 miles

a. How can you determine which survey was correct ?
b. Explain what the difference in the results of the surveys implies about the data.
c. If the two histograms depict the distribution of values for each supervisor, which one depicts Ercilia's sample? How do you know?

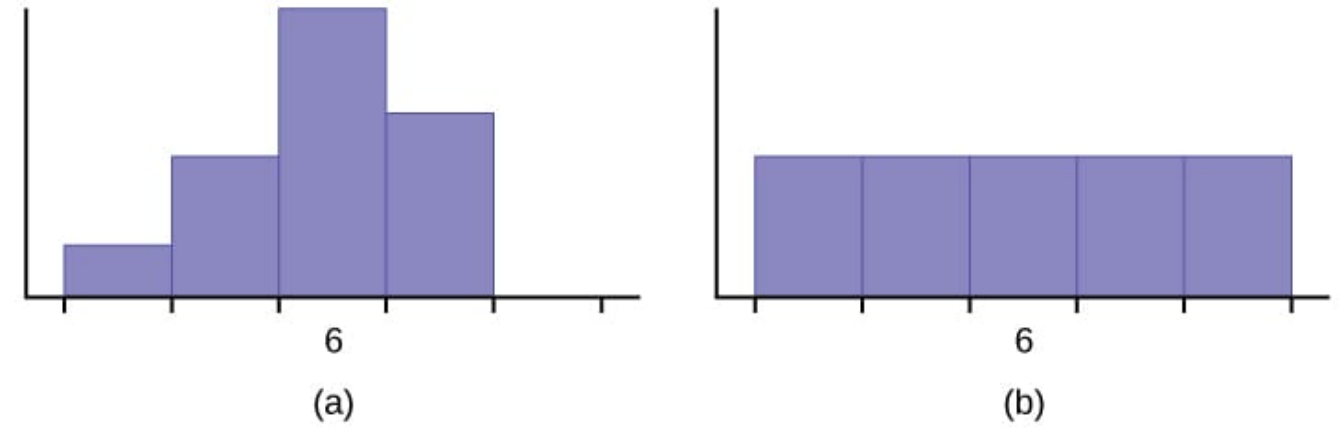


Figure 10.6.1: This shows two histograms. The first histogram shows a fairly symmetrical distribution with a mode of 6. The second histogram shows a uniform distribution.

1. If the two box plots depict the distribution of values for each supervisor, which one depicts Ercilia's sample? How do you know?

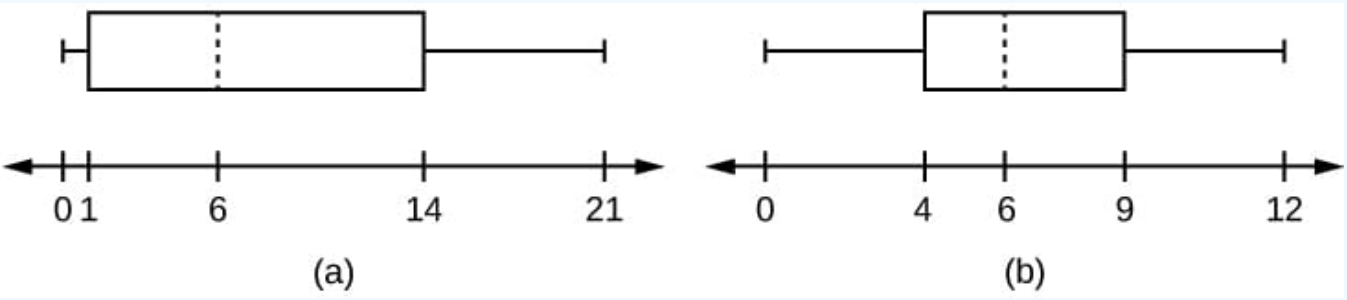


Figure 10.6.2: This shows two horizontal boxplots. The first boxplot is graphed over a number line from 0 to 21. The first whisker extends from 0 to 1. The box begins at the first quartile, 1, and ends at the third quartile, 14. A vertical, dashed line marks the median at 6. The second whisker extends from the third quartile to the largest value, 21. The second boxplot is graphed over a number line from 0 to 12. The first whisker extends from 0 to 4. The box begins at the first quartile, 4, and ends at the third quartile, 9. A vertical, dashed line marks the median at 6. The second whisker extends from the third quartile to the largest value, 12.

Use the following information to answer the next three exercises: We are interested in the number of years students in a particular elementary statistics class have lived in California. The information in the following table is from the entire section.

Number of years	Frequency	Number of years	Frequency
7	1	22	1
14	3	23	1
15	1	26	1
18	1	40	2
19	4	42	2
20	3		
			Total = 20

Exercise 2.6.13

What is the *IQR*?

a. 8
b. 11
c. 15
d. 35

Answer

a

Exercise 2.6.14

What is the mode?

a. 19
b. 19.5
c. 14 and 20
d. 22.65

Exercise 2.6.15

Is this a sample or the entire population?

a. sample
b. entire population
c. neither

Answer

b

Glossary

Frequency Table
a data representation in which grouped data is displayed along with the corresponding frequencies

Mean
a number that measures the central tendency of the data; a common name for mean is 'average.' The term 'mean' is a shortened form of 'arithmetic mean.' By definition, the mean for a sample (denoted by $\bar{x}$) is $\bar{x}$

$= \frac{\text{Sum of all values in the sample}}{\text{Number of values in the sample}}$, and the mean for a population (denoted by μ) is $\mu = \frac{\text{Sum of all values in the population}}{\text{Number of values in the population}}$.

Median

a number that separates ordered data into halves; half the values are the same number or smaller than the median and half the values are the same number or larger than the median. The median may or may not be part of the data.

Midpoint

the mean of an interval in a frequency table

Mode

the value that appears most frequently in a set of data

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10.7: Skewness and the Mean, Median, and Mode

Consider the following data set.

4; 5; 6; 6; 6; 7; 7; 7; 7; 7; 7; 8; 8; 8; 9; 10

This data set can be represented by following histogram. Each interval has width one, and each value is located in the middle of an interval.

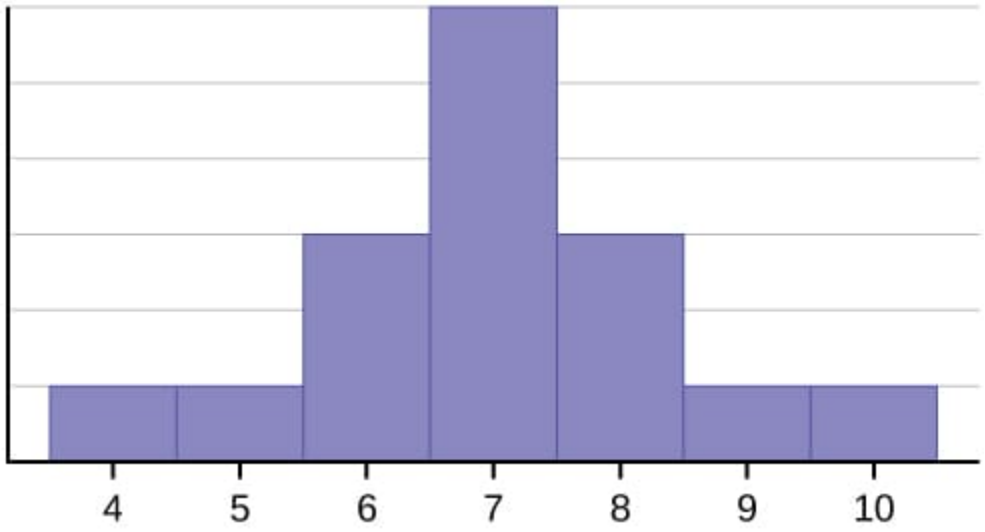


Figure 10.7.1

The histogram displays a **symmetrical** distribution of data. A distribution is symmetrical if a vertical line can be drawn at some point in the histogram such that the shape to the left and the right of the vertical line are mirror images of each other. The mean, the median, and the mode are each seven for these data. **In a perfectly symmetrical distribution, the mean and the median are the same.** This example has one mode (unimodal), and the mode is the same as the mean and median. In a symmetrical distribution that has two modes (bimodal), the two modes would be different from the mean and median.

The histogram for the data: 4; 5; 6; 6; 6; 7; 7; 7; 7; 8 is not symmetrical. The right-hand side seems "chopped off" compared to the left side. A distribution of this type is called **skewed to the left** because it is pulled out to the left.

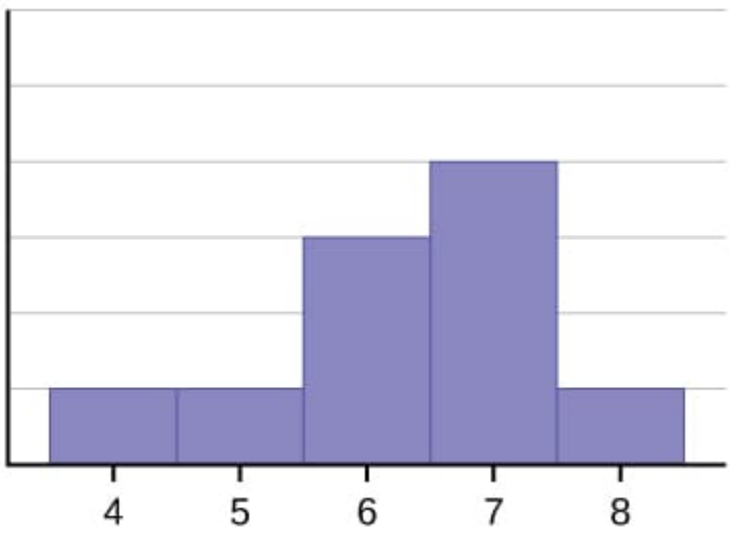


Figure 10.7.2

The mean is 6.3, the median is 6.5, and the mode is seven. **Notice that the mean is less than the median, and they are both**

less than the mode. The mean and the median both reflect the skewing, but the mean reflects it more so.

The histogram for the data: 6; 7; 7; 7; 7; 8; 8; 8; 9; 10, is also not symmetrical. It is **skewed to the right**.

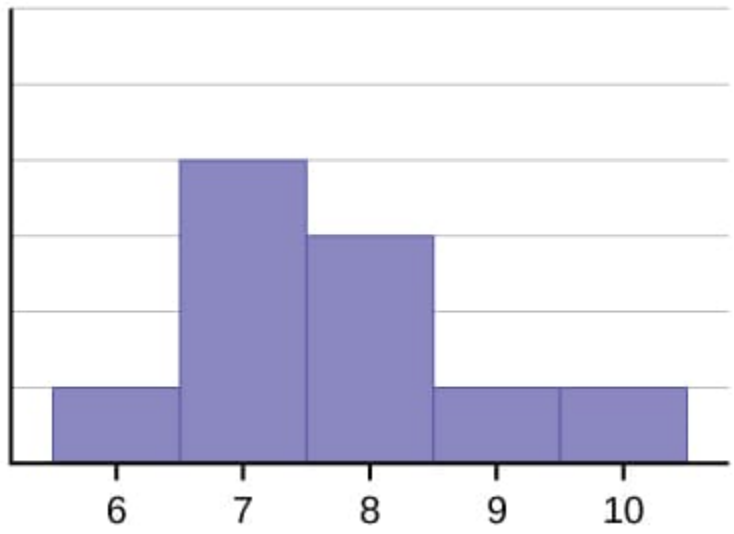


Figure 10.7.3

The mean is 7.7, the median is 7.5, and the mode is seven. Of the three statistics, **the mean is the largest, while the mode is the smallest**. Again, the mean reflects the skewing the most.

> Generally, if the distribution of data is skewed to the left, the mean is less than the median, which is often less than the mode. If the distribution of data is skewed to the right, the mode is often less than the median, which is less than the mean.

Skewness and symmetry become important when we discuss probability distributions in later chapters.

Example 10.7.1

Statistics are used to compare and sometimes identify authors. The following lists shows a simple random sample that compares the letter counts for three authors.

- Terry: 7; 9; 3; 3; 3; 4; 1; 3; 2; 2
- Davis: 3; 3; 3; 4; 1; 4; 3; 2; 3; 1
- Maris: 2; 3; 4; 4; 4; 6; 6; 6; 8; 3

a. Make a dot plot for the three authors and compare the shapes.
b. Calculate the mean for each.
c. Calculate the median for each.
d. Describe any pattern you notice between the shape and the measures of center.

Solution

a.

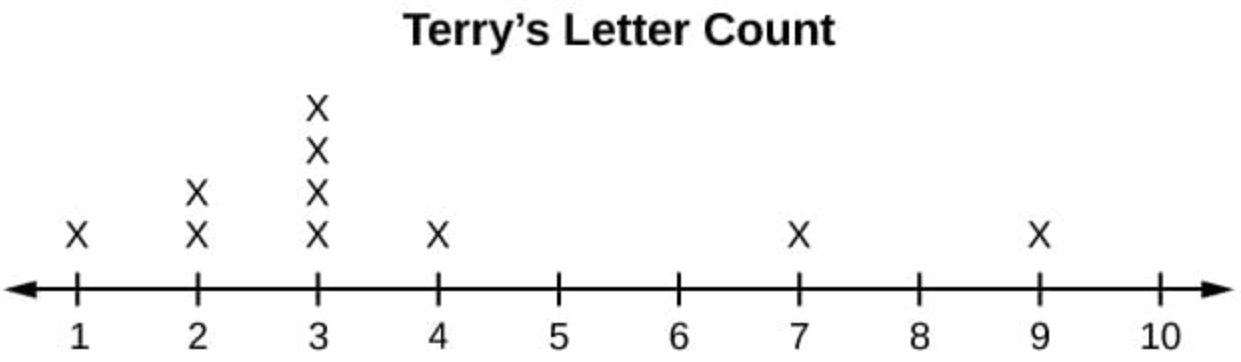


Figure 10.7.4 : This dot plot matches the supplied data for Terry. The plot uses a number line from 1 to 10. It shows one x over 1, two x's over 2, four x's over 3, one x over 4, one x over 7, and one x over 9. There are no x's over the numbers 5, 6, 8, and 10.

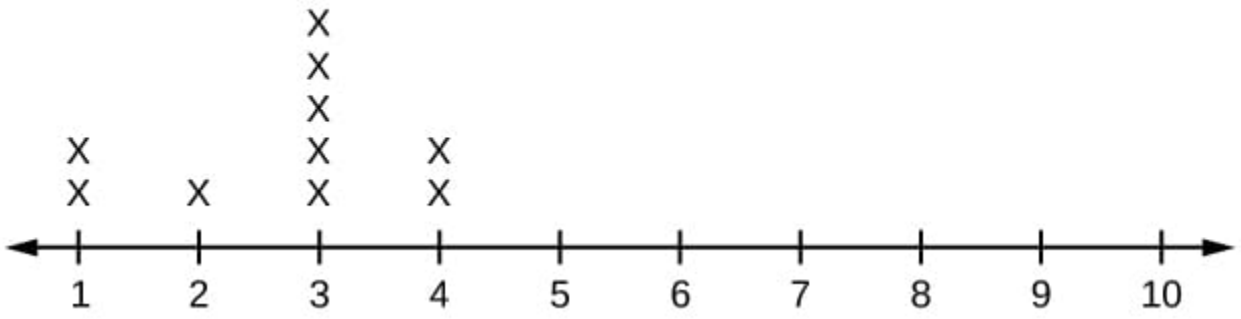


Figure 10.7.5 : Copy and Paste Caption here. (Copyright; author via source)

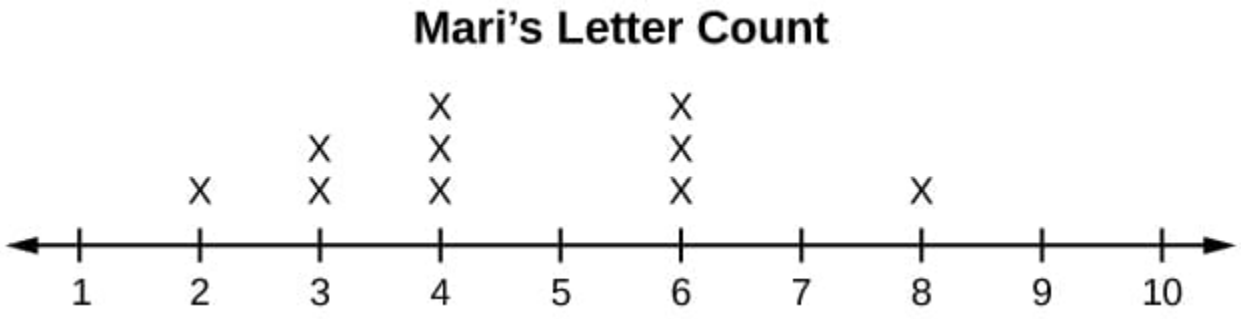


Figure 10.7.6 : Copy and Paste Caption here. (Copyright; author via source)

- Terry's mean is 3.7, Davis' mean is 2.7, Maris' mean is 4.6.
- Terry's median is three, Davis' median is three. Maris' median is four.
- It appears that the median is always closest to the high point (the mode), while the mean tends to be farther out on the tail. In a symmetrical distribution, the mean and the median are both centrally located close to the high point of the distribution.

Exercise 10.7.1

Discuss the mean, median, and mode for each of the following problems. Is there a pattern between the shape and measure of the center?

a.



Figure 10.7.7: This dot plot matches the supplied data. The plot uses a number line from 0 to 14. It shows two x's over 0, four x's over 1, three x's over 2, one x over 3, two x's over the number 4, 5, 6, and 9, and 1 x each over 10 and 14. There are no x's over the numbers 7, 8, 11, 12, and 13.

b.

The Ages Former U.S Presidents Died	
4	6 9
5	3 6 7 7 7 8
6	0 0 3 3 4 4 5 6 7 7 7 8
7	0 1 1 2 3 4 7 8 8 9
8	0 1 3 5 8
9	0 0 3 3
Key: 8|0 means 80.	

c.



Figure 10.7.8: This is a histogram titled Hours Spent Playing Video Games on Weekends. The x-axis shows the number of hours spent playing video games with bars showing values at intervals of 5. The y-axis shows the number of students. The first bar for 0 - 4.99 hours has a height of 2. The second bar from 5 - 9.99 has a height of 3. The third bar from 10 - 14.99 has a height of 4. The fourth bar from 15 - 19.99 has a height of 7. The fifth bar from 20 - 24.99 has a height of 9.

Review

Looking at the distribution of data can reveal a lot about the relationship between the mean, the median, and the mode. There are three types of distributions. A **left (or negative) skewed** distribution has a shape like Figure 10.7.2. A **right (or positive) skewed** distribution has a shape like Figure 10.7.3. A **symmetrical** distribution looks like Figure 10.7.1.

Use the following information to answer the next three exercises: State whether the data are symmetrical, skewed to the left, or skewed to the right.

Exercise 2.7.2

1; 1; 1; 2; 2; 2; 2; 3; 3; 3; 3; 3; 3; 3; 3; 4; 4; 4; 5; 5

Answer

The data are symmetrical. The median is 3 and the mean is 2.85. They are close, and the mode lies close to the middle of the data, so the data are symmetrical.

Exercise 2.7.3

16; 17; 19; 22; 22; 22; 22; 22; 23

Exercise 2.7.4

87; 87; 87; 87; 87; 88; 89; 89; 90; 91

Answer

The data are skewed right. The median is 87.5 and the mean is 88.2. Even though they are close, the mode lies to the left of the middle of the data, and there are many more instances of 87 than any other number, so the data are skewed right.

Exercise 2.7.5

When the data are skewed left, what is the typical relationship between the mean and median?

Exercise 2.7.6

When the data are symmetrical, what is the typical relationship between the mean and median?

Answer

When the data are symmetrical, the mean and median are close or the same.

Exercise 2.7.7

What word describes a distribution that has two modes?

Exercise 2.7.8

Describe the shape of this distribution.

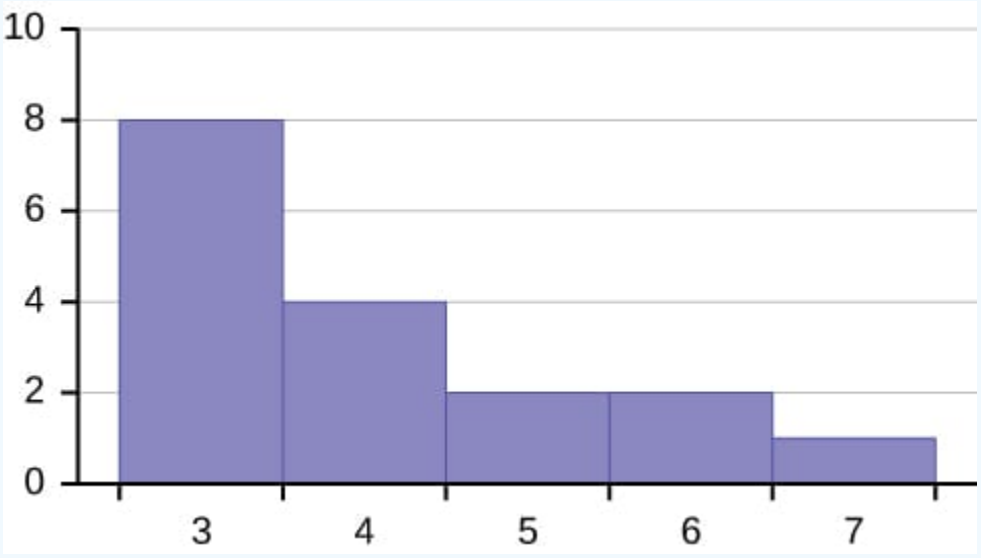


Figure 10.7.9: This is a historgram which consists of 5 adjacent bars with the x-axis split into intervals of 1 from 3 to 7. The bar heights peak at the first bar and taper lower to the right.

Answer

The distribution is skewed right because it looks pulled out to the right.

Exercise 2.7.9

Describe the relationship between the mode and the median of this distribution.

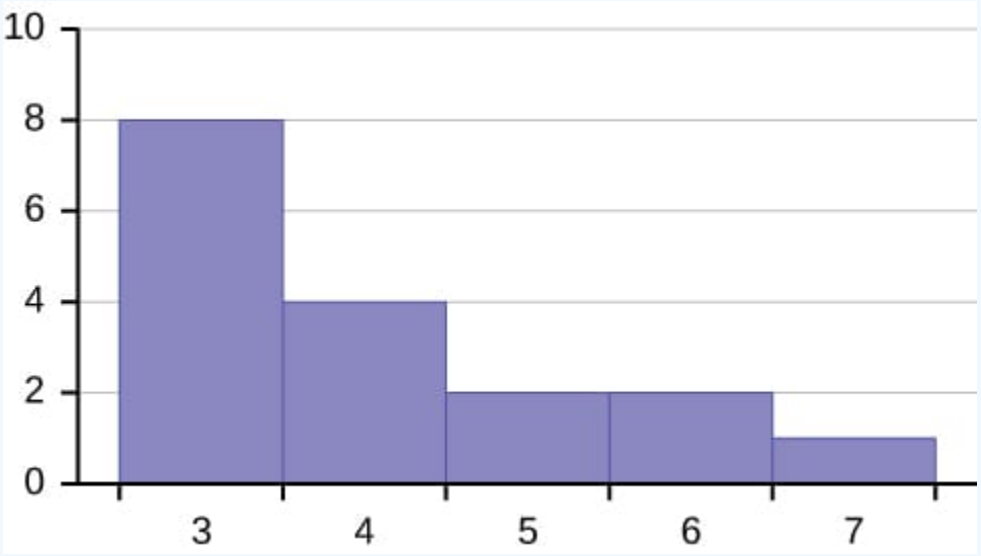


Figure 10.7.10: This is a histogram which consists of 5 adjacent bars with the x-axis split into intervals of 1 from 3 to 7. The bar heights peak at the first bar and taper lower to the right. The bar heighs from left to right are: 8, 4, 2, 2, 1.

Exercise 2.7.10

Describe the relationship between the mean and the median of this distribution.

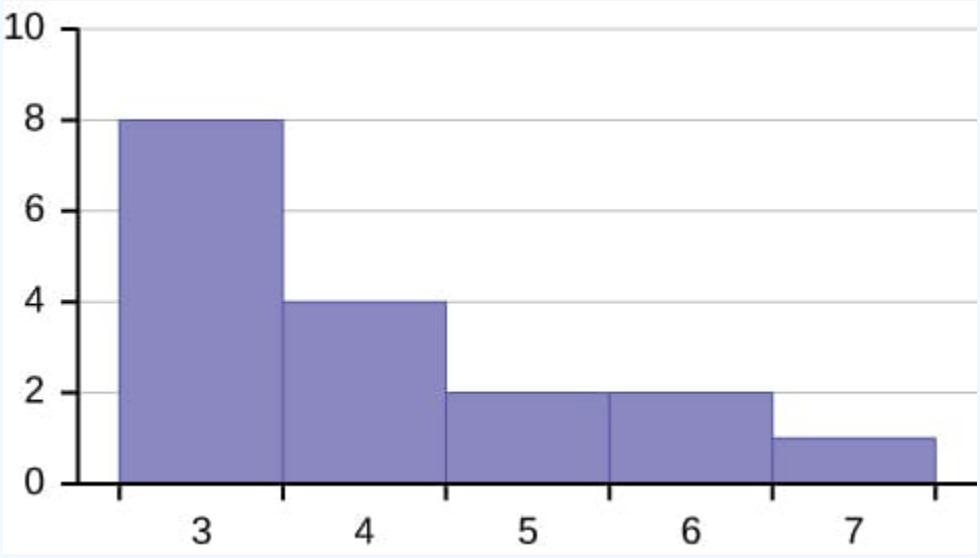


Figure 10.7.11: This is a histogram which consists of 5 adjacent bars with the x-axis split into intervals of 1 from 3 to 7. The bar heights peak at the first bar and taper lower to the right. The bar heights from left to right are: 8, 4, 2, 2, 1.

Answer

The mean is 4.1 and is slightly greater than the median, which is four.

Exercise 2.7.11

Describe the shape of this distribution.

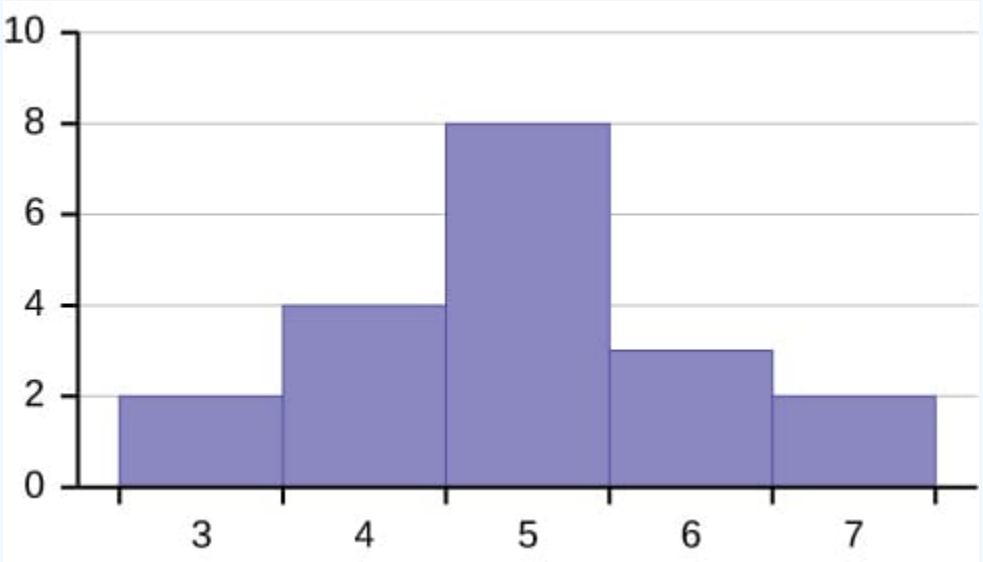


Figure 10.7.12

Exercise 2.7.12

Describe the relationship between the mode and the median of this distribution.

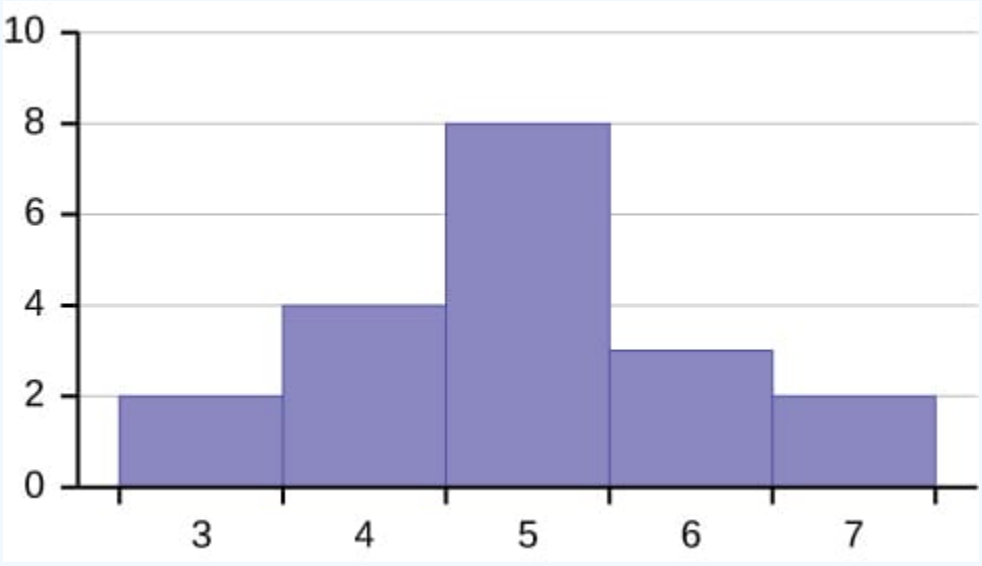


Figure 10.7.13

Answer

The mode and the median are the same. In this case, they are both five.

Exercise 2.7.13

Are the mean and the median the exact same in this distribution? Why or why not?

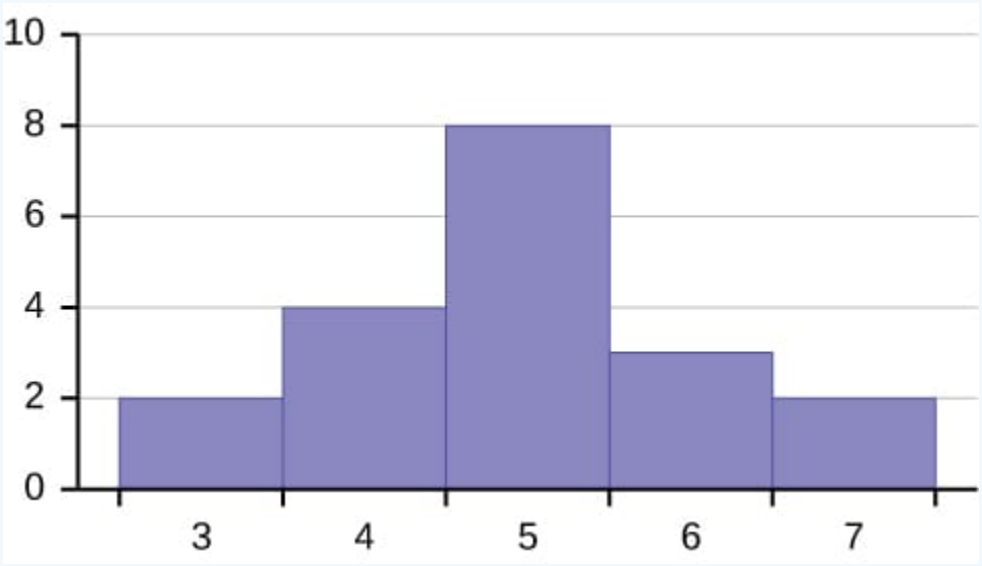


Figure 10.7.14

Exercise 2.7.14

Describe the shape of this distribution.

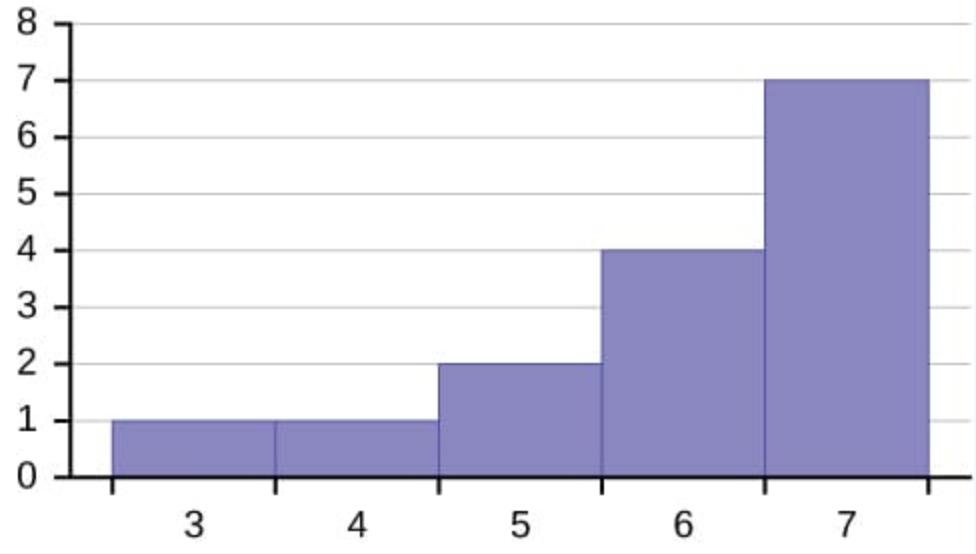


Figure 10.7.15

Answer

The distribution is skewed left because it looks pulled out to the left.

Exercise 2.7.15

Describe the relationship between the mode and the median of this distribution.

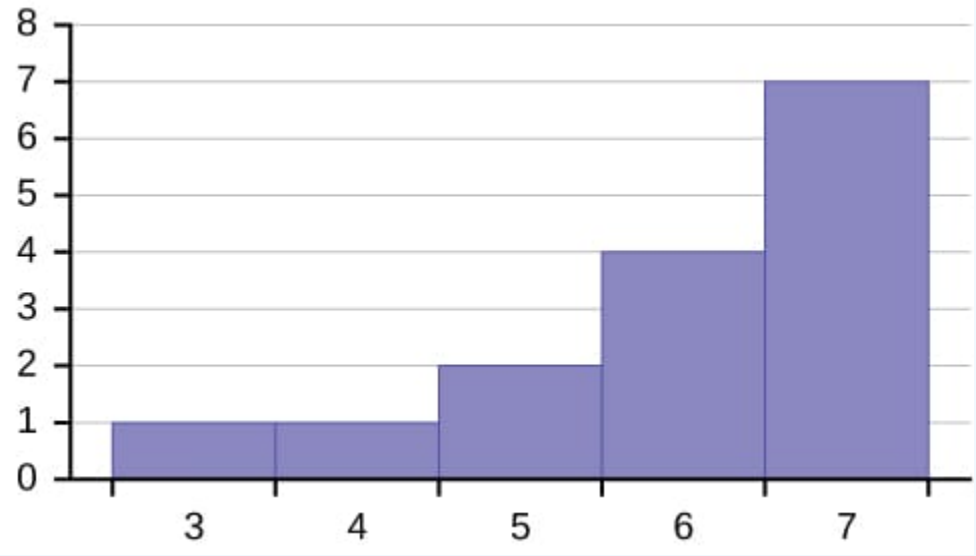


Figure 10.7.16: Copy and Paste Caption here. (Copyright; author via source)

Exercise 2.7.16

Describe the relationship between the mean and the median of this distribution.

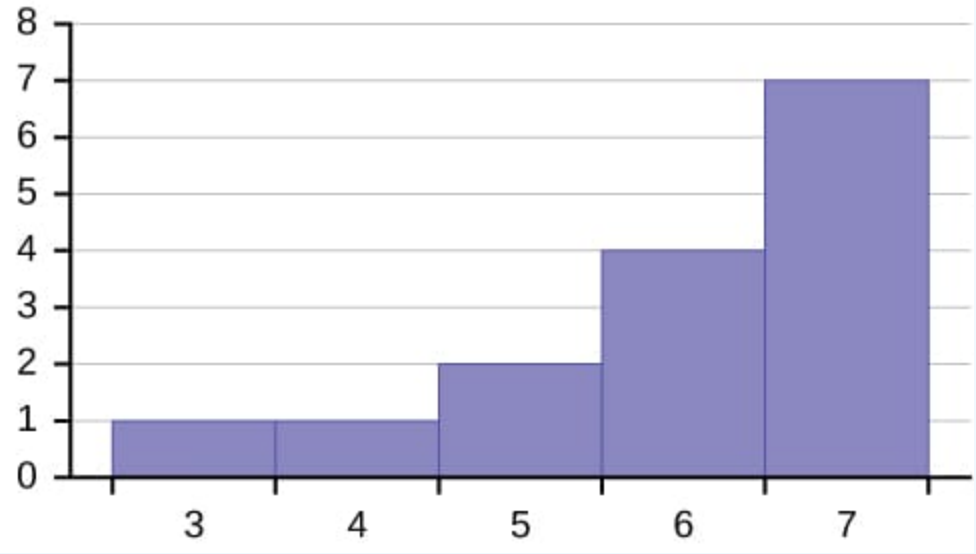


Figure 10.7.17

Answer

The mean and the median are both six.

Exercise 2.7.17

The mean and median for the data are the same.

3; 4; 5; 5; 6; 6; 6; 6; 7; 7; 7; 7; 7; 7; 7

Is the data perfectly symmetrical? Why or why not?

Exercise 2.7.18

Which is the greatest, the mean, the mode, or the median of the data set?

11; 11; 12; 12; 12; 12; 13; 15; 17; 22; 22; 22

Answer

The mode is 12, the median is 12.5, and the mean is 15.1. The mean is the largest.

Exercise 2.7.19

Which is the least, the mean, the mode, and the median of the data set?

56; 56; 56; 58; 59; 60; 62; 64; 64; 65; 67

Exercise 2.7.20

Of the three measures, which tends to reflect skewing the most, the mean, the mode, or the median? Why?

Answer

The mean tends to reflect skewing the most because it is affected the most by outliers.

Exercise 2.7.21

In a perfectly symmetrical distribution, when would the mode be different from the mean and median?

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10.8: Measures of the Spread of the Data

An important characteristic of any set of data is the variation in the data. In some data sets, the data values are concentrated closely near the mean; in other data sets, the data values are more widely spread out from the mean. The most common measure of variation, or spread, is the standard deviation. The standard deviation is a number that measures how far data values are from their mean.

The standard deviation

- provides a numerical measure of the overall amount of variation in a data set, and
- can be used to determine whether a particular data value is close to or far from the mean.

The standard deviation provides a measure of the overall variation in a data set

The standard deviation is always positive or zero. The standard deviation is small when the data are all concentrated close to the mean, exhibiting little variation or spread. The standard deviation is larger when the data values are more spread out from the mean, exhibiting more variation.

Suppose that we are studying the amount of time customers wait in line at the checkout at supermarket *A* and supermarket *B*. the average wait time at both supermarkets is five minutes. At supermarket *A*, the standard deviation for the wait time is two minutes; at supermarket *B* the standard deviation for the wait time is four minutes.

Because supermarket *B* has a higher standard deviation, we know that there is more variation in the wait times at supermarket *B*. Overall, wait times at supermarket *B* are more spread out from the average; wait times at supermarket *A* are more concentrated near the average.

The standard deviation can be used to determine whether a data value is close to or far from the mean.

Suppose that Rosa and Binh both shop at supermarket *A*. Rosa waits at the checkout counter for seven minutes and Binh waits for one minute. At supermarket *A*, the mean waiting time is five minutes and the standard deviation is two minutes. The standard deviation can be used to determine whether a data value is close to or far from the mean.

Rosa waits for seven minutes:

- Seven is two minutes longer than the average of five; two minutes is equal to one standard deviation.
- Rosa's wait time of seven minutes is **two minutes longer than the average** of five minutes.
- Rosa's wait time of seven minutes is **one standard deviation above the average** of five minutes.

Binh waits for one minute.

- One is four minutes less than the average of five; four minutes is equal to two standard deviations.
- Binh's wait time of one minute is **four minutes less than the average** of five minutes.
- Binh's wait time of one minute is **two standard deviations below the average** of five minutes.
- A data value that is two standard deviations from the average is just on the borderline for what many statisticians would consider to be far from the average. Considering data to be far from the mean if it is more than two standard deviations away is more of an approximate "rule of thumb" than a rigid rule. In general, the shape of the distribution of the data affects how much of the data is further away than two standard deviations. (You will learn more about this in later chapters.)

The number line may help you understand standard deviation. If we were to put five and seven on a number line, seven is to the right of five. We say, then, that seven is **one** standard deviation to the **right** of five because $5 + (1)(2) = 7$.

If one were also part of the data set, then one is **two** standard deviations to the **left** of five because $5 + (-2)(2) = 1$.



Figure 10.8.1

- In general, a **value = mean + (#ofSTDEV)(standard deviation)**
- where #ofSTDEVs = the number of standard deviations
- #ofSTDEV does not need to be an integer
- One is **two standard deviations less than the mean** of five because: $1 = 5 + (-2)(2)$.

The equation value = mean + (#ofSTDEVs)(standard deviation) can be expressed for a sample and for a population.

- sample: $x = x + (\#\text{ of STDEV })(s)$
- Population: $x = \mu + (\#\text{ ofSTDEV })(\sigma)$

The lower case letter s represents the sample standard deviation and the Greek letter σ (sigma, lower case) represents the population standard deviation.

The symbol $\bar{x}$ is the sample mean and the Greek symbol μ is the population mean.

Calculating the Standard Deviation

If x is a number, then the difference "x – mean" is called its **deviation**. In a data set, there are as many deviations as there are items in the data set. The deviations are used to calculate the standard deviation. If the numbers belong to a population, in symbols a deviation is $x - \mu$. For sample data, in symbols a deviation is $x - \bar{x}$.

The procedure to calculate the standard deviation depends on whether the numbers are the entire population or are data from a sample. The calculations are similar, but not identical. Therefore the symbol used to represent the standard deviation depends on whether it is calculated from a population or a sample. The lower case letter s represents the sample standard deviation and the Greek letter σ (sigma, lower case) represents the population standard deviation. If the sample has the same characteristics as the population, then s should be a good estimate of σ.

To calculate the standard deviation, we need to calculate the variance first. The variance is the **average of the squares of the deviations** (the $x - \bar{x}$ values for a sample, or the $x - \mu$ values for a population). The symbol σ^2 represents the population variance; the population standard deviation σ is the square root of the population variance. The symbol s^2 represents the sample variance; the sample standard deviation s is the square root of the sample variance. You can think of the standard deviation as a special average of the deviations.

If the numbers come from a census of the entire **population** and not a sample, when we calculate the average of the squared deviations to find the variance, we divide by N, the number of items in the population. If the data are from a **sample** rather than a population, when we calculate the average of the squared deviations, we divide by ***n* – 1**, one less than the number of items in the sample.

Formulas for the Sample Standard Deviation

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}} \tag{10.8.1}$$

or

$$s = \sqrt{\frac{\sum f(x - \bar{x})^2}{n - 1}} \tag{10.8.2}$$

For the sample standard deviation, the denominator is $n - 1$, that is the sample size MINUS 1.

Formulas for the Population Standard Deviation

$$\sigma = \sqrt{\frac{\sum (x-\mu)^2}{N}} \tag{10.8.3}$$

or

$$\sigma = \sqrt{\frac{\sum f(x-\mu)^2}{N}} \tag{10.8.4}$$

For the population standard deviation, the denominator is N, the number of items in the population.

In Equations 10.8.2 and 10.8.4, f represents the frequency with which a value appears. For example, if a value appears once, f is one. If a value appears three times in the data set or population, f is three.

Sampling Variability of a Statistic

The statistic of a sampling distribution was discussed in Section 2.6. How much the statistic varies from one sample to another is known as the sampling variability of a statistic. You typically measure the **sampling variability of a statistic** by its standard error.

The **standard error of the mean** is an example of a standard error. It is a special standard deviation and is known as the standard deviation of the sampling distribution of the mean. You will cover the standard error of the mean in Chapter 7. The notation for the standard error of the mean is $\frac{\sigma}{\sqrt{n}}$ where σ is the standard deviation of the population and n is the size of the sample.

In practice, USE A CALCULATOR OR COMPUTER SOFTWARE TO CALCULATE THE STANDARD DEVIATION. If you are using a TI-83, 83+, 84+ calculator, you need to select the appropriate standard deviation σ_x or s_x from the summary statistics. We will concentrate on using and interpreting the information that the standard deviation gives us. However you should study the following step-by-step example to help you understand how the standard deviation measures variation from the mean. (The calculator instructions appear at the end of this example.)

Example 10.8.1

In a fifth grade class, the teacher was interested in the average age and the sample standard deviation of the ages of her students. The following data are the ages for a SAMPLE of *n* = 20 fifth grade students. The ages are rounded to the nearest half year:

9; 9.5; 9.5; 10; 10; 10; 10; 10.5; 10.5; 10.5; 10.5; 11; 11; 11; 11; 11; 11; 11.5; 11.5; 11.5;

$$\bar{x} = \frac{9 + 9.5(2) + 10(4) + 10.5(4) + 11(6) + 11.5(3)}{20} = 10.525$$

The average age is 10.53 years, rounded to two places.

The variance may be calculated by using a table. Then the standard deviation is calculated by taking the square root of the variance. We will explain the parts of the table after calculating s.

Data	Freq.	Deviations	*Deviations*^{2}	(Freq.)(*Deviations*^{2})
x	*f*	(x – $\bar{x}$)	(x – $\bar{x}$)2	(f)(x – $\bar{x}$)2
9	1	9 – 10.525 = –1.525	(–1.525)2 = 2.325625	1 × 2.325625 = 2.325625
9.5	2	9.5 – 10.525 = –1.025	(–1.025)^2 = 1.050625	2 × 1.050625 = 2.101250

Data	Freq.	Deviations	$(Deviations)^2$	(Freq.)$(Deviations^2)$
10	4	10 – 10.525 = –0.525	$(-0.525)^2 = 0.275625$	4 × 0.275625 = 1.1025
10.5	4	10.5 – 10.525 = –0.025	$(-0.025)^2 = 0.000625$	4 × 0.000625 = 0.0025
11	6	11 – 10.525 = 0.475	$(0.475)^2 = 0.225625$	6 × 0.225625 = 1.35375
11.5	3	11.5 – 10.525 = 0.975	$(0.975)^2 = 0.950625$	3 × 0.950625 = 2.851875
				The total is 9.7375

The sample variance, s^2, is equal to the sum of the last column (9.7375) divided by the total number of data values minus one (20 – 1):

$$s^2 = \frac{9.7375}{20-1} = 0.5125$$

The **sample standard deviation** s is equal to the square root of the sample variance:

$$s = \sqrt{0.5125} = 0.715891$$

and this is rounded to two decimal places, $s = 0.72$.

Typically, you do the calculation for the standard deviation on your calculator or computer. The intermediate results are not rounded. This is done for accuracy.

- For the following problems, recall that **value = mean + (#ofSTDEVs)(standard deviation)**. Verify the mean and standard deviation or a calculator or computer.
- For a sample: $x = \bar{x}$ + (#ofSTDEVs)(s)
- For a population: $x = \mu$ + (#ofSTDEVs)σ
- For this example, use $x = \bar{x}$ + (#ofSTDEVs)(s) because the data is from a sample

a. Verify the mean and standard deviation on your calculator or computer.
b. Find the value that is one standard deviation above the mean. Find ($\bar{x}$ + 1s).
c. Find the value that is two standard deviations below the mean. Find ($\bar{x}$ – 2s).
d. Find the values that are 1.5 standard deviations **from** (below and above) the mean.

Solution

a.
- Clear lists L1 and L2. Press STAT 4:ClrList. Enter 2nd 1 for L1, the comma (,), and 2nd 2 for L2.
- Enter data into the list editor. Press STAT 1:EDIT. If necessary, clear the lists by arrowing up into the name. Press CLEAR and arrow down.
- Put the data values (9, 9.5, 10, 10.5, 11, 11.5) into list L1 and the frequencies (1, 2, 4, 4, 6, 3) into list L2. Use the arrow keys to move around.
- Press STAT and arrow to CALC. Press 1:1-VarStats and enter L1 (2nd 1), L2 (2nd 2). Do not forget the comma. Press ENTER.
- $\bar{x}$ = 10.525
- Use Sx because this is sample data (not a population): Sx=0.715891

b. $(\bar{x} + 1s) = 10.53 + (1)(0.72) = 11.25$
c. $(\bar{x} - 2s) = 10.53–(2)(0.72) = 9.09$
d.
- $(\bar{x} - 1.5s) = 10.53–(1.5)(0.72) = 9.45$
- $(\bar{x} + 1.5s) = 10.53 + (1.5)(0.72) = 11.61$

Exercise 2.8.1

On a baseball team, the ages of each of the players are as follows:

21; 21; 22; 23; 24; 24; 25; 25; 28; 29; 29; 31; 32; 33; 33; 34; 35; 36; 36; 36; 36; 38; 38; 38; 40

Use your calculator or computer to find the mean and standard deviation. Then find the value that is two standard deviations above the mean.

Answer

μ = 30.68

$s = 6.09$

$(\bar{x} + 2s = 30.68 + (2)(6.09) = 42.86.$

Explanation of the standard deviation calculation shown in the table

The deviations show how spread out the data are about the mean. The data value 11.5 is farther from the mean than is the data value 11 which is indicated by the deviations 0.97 and 0.47. A positive deviation occurs when the data value is greater than the mean, whereas a negative deviation occurs when the data value is less than the mean. The deviation is –1.525 for the data value nine. **If you add the deviations, the sum is always zero**. (For Example $10.8.1$, there are $n = 20$ deviations.) So you cannot simply add the deviations to get the spread of the data. By squaring the deviations, you make them positive numbers, and the sum will also be positive. The variance, then, is the average squared deviation.

The variance is a squared measure and does not have the same units as the data. Taking the square root solves the problem. The standard deviation measures the spread in the same units as the data.

Notice that instead of dividing by $n = 20$, the calculation divided by $n - 1 = 20 - 1 = 19$ because the data is a sample. For the **sample** variance, we divide by the sample size minus one $(n - 1)$. Why not divide by n? The answer has to do with the population variance. **The sample variance is an estimate of the population variance.** Based on the theoretical mathematics that lies behind these calculations, dividing by $(n - 1)$ gives a better estimate of the population variance.

Your concentration should be on what the standard deviation tells us about the data. The standard deviation is a number which measures how far the data are spread from the mean. Let a calculator or computer do the arithmetic.

The standard deviation, s or σ, is either zero or larger than zero. When the standard deviation is zero, there is no spread; that is, all the data values are equal to each other. The standard deviation is small when the data are all concentrated close to the mean, and is larger when the data values show more variation from the mean. When the standard deviation is a lot larger than zero, the data values are very spread out about the mean; outliers can make s or σ very large.

The standard deviation, when first presented, can seem unclear. By graphing your data, you can get a better "feel" for the deviations and the standard deviation. You will find that in symmetrical distributions, the standard deviation can be very helpful but in skewed distributions, the standard deviation may not be much help. The reason is that the two sides of a skewed distribution have different spreads. In a skewed distribution, it is better to look at the first quartile, the median, the third quartile, the smallest value, and the largest value. Because numbers can be confusing, **always graph your data**. Display your data in a histogram or a box plot.

Example $10.8.2$

Use the following data (first exam scores) from Susan Dean's spring pre-calculus class:

33; 42; 49; 49; 53; 55; 55; 61; 63; 67; 68; 68; 69; 69; 72; 73; 74; 78; 80; 83; 88; 88; 88; 90; 92; 94; 94; 94; 94; 96; 100

a. Create a chart containing the data, frequencies, relative frequencies, and cumulative relative frequencies to three decimal places.
b. Calculate the following to one decimal place using a TI-83+ or TI-84 calculator:
    i. The sample mean

ii. The sample standard deviation
iii. The median
iv. The first quartile
v. The third quartile
vi. *IQR*

c. Construct a box plot and a histogram on the same set of axes. Make comments about the box plot, the histogram, and the chart.

Answer

a. See Table
b. i. The sample mean = 73.5
ii. The sample standard deviation = 17.9
iii. The median = 73
iv. The first quartile = 61
v. The third quartile = 90
vi. *IQR* = 90 – 61 = 29
c. The x-axis goes from 32.5 to 100.5; y-axis goes from -2.4 to 15 for the histogram. The number of intervals is five, so the width of an interval is $(100.5 - 32.5)$ divided by five, is equal to 13.6. Endpoints of the intervals are as follows: the starting point is 32.5, $32.5 + 13.6 = 46.1$, $46.1 + 13.6 = 59.7$, $59.7 + 13.6 = 73.3$, $73.3 + 13.6 = 86.9$, $86.9 + 13.6 = 100.5 =$ the ending value; No data values fall on an interval boundary.

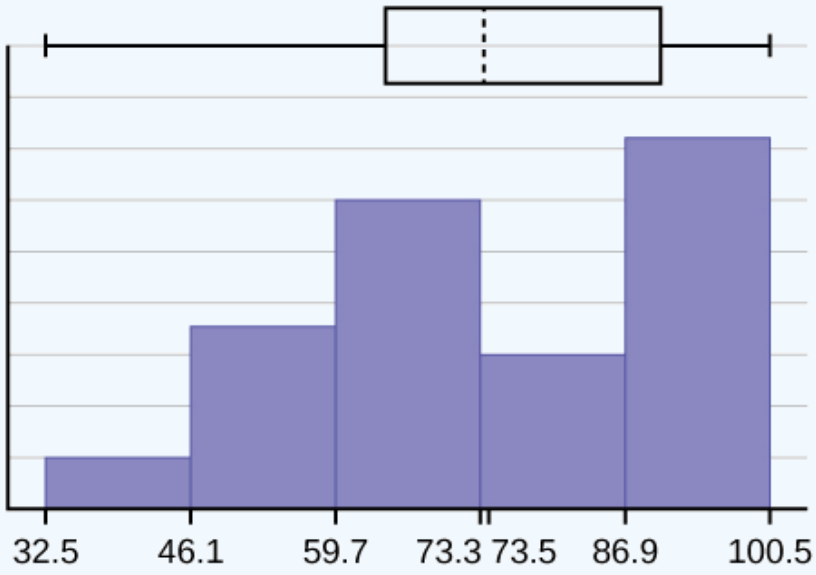


Figure 10.8.2.

The long left whisker in the box plot is reflected in the left side of the histogram. The spread of the exam scores in the lower 50% is greater $(73 - 33 = 40)$ than the spread in the upper 50% $(100 - 73 = 27)$. The histogram, box plot, and chart all reflect this. There are a substantial number of A and B grades (80s, 90s, and 100). The histogram clearly shows this. The box plot shows us that the middle 50% of the exam scores (*IQR* = 29) are Ds, Cs, and Bs. The box plot also shows us that the lower 25% of the exam scores are Ds and Fs.

Data	Frequency	Relative Frequency	Cumulative Relative Frequency
33	1	0.032	0.032
42	1	0.032	0.064
49	2	0.065	0.129
53	1	0.032	0.161
55	2	0.065	0.226

Data	Frequency	Relative Frequency	Cumulative Relative Frequency
61	1	0.032	0.258
63	1	0.032	0.29
67	1	0.032	0.322
68	2	0.065	0.387
69	2	0.065	0.452
72	1	0.032	0.484
73	1	0.032	0.516
74	1	0.032	0.548
78	1	0.032	0.580
80	1	0.032	0.612
83	1	0.032	0.644
88	3	0.097	0.741
90	1	0.032	0.773
92	1	0.032	0.805
94	4	0.129	0.934
96	1	0.032	0.966
100	1	0.032	0.998 (Why isn't this value 1?)

Exercise 10.8.2

The following data show the different types of pet food stores in the area carry.

6; 6; 6; 6; 7; 7; 7; 7; 7; 8; 9; 9; 9; 9; 10; 10; 10; 10; 10; 11; 11; 11; 11; 12; 12; 12; 12; 12; 12;

Calculate the sample mean and the sample standard deviation to one decimal place using a TI-83+ or TI-84 calculator.

Answer

$\mu = 9.3$ and $s = 2.2$

Standard deviation of Grouped Frequency Tables

Recall that for grouped data we do not know individual data values, so we cannot describe the typical value of the data with precision. In other words, we cannot find the exact mean, median, or mode. We can, however, determine the best estimate of the measures of center by finding the mean of the grouped data with the formula:

$$\text{Mean of Frequency Table} = \frac{\sum fm}{\sum f} \tag{10.8.5}$$

where f interval frequencies and $m =$ interval midpoints.

Just as we could not find the exact mean, neither can we find the exact standard deviation. Remember that standard deviation describes numerically the expected deviation a data value has from the mean. In simple English, the standard deviation allows us to compare how “unusual” individual data is compared to the mean.

Example 10.8.3

Find the standard deviation for the data in Table 10.8.3.

Table 10.8.3

Class	Frequency, *f*	Midpoint, *m*	m^2	$\bar{x}$	fm^2	Standard Deviation
0–2	1	1	1	7.58	1	3.5
3–5	6	4	16	7.58	96	3.5
6–8	10	7	49	7.58	490	3.5
9–11	7	10	100	7.58	700	3.5
12–14	0	13	169	7.58	0	3.5
15–17	2	16	256	7.58	512	3.5

For this data set, we have the mean, $\bar{x}$ = 7.58 and the standard deviation, s_x = 3.5. This means that a randomly selected data value would be expected to be 3.5 units from the mean. If we look at the first class, we see that the class midpoint is equal to one. This is almost two full standard deviations from the mean since 7.58 – 3.5 – 3.5 = 0.58. While the formula for calculating the standard deviation is not complicated, $s_x = \sqrt{\frac{f(m-\bar{x})^2}{n-1}}$ where s_x = sample standard deviation, $\bar{x}$ = sample mean, the calculations are tedious. It is usually best to use technology when performing the calculations.

Find the standard deviation for the data from the previous example

Class	0-2	3-5	6-8	9–11	12–14	15–17
Frequency, *f*	1	6	10	7	0	2

First, press the **STAT** key and select **1:Edit**



Figure 10.8.3

Input the midpoint values into **L1** and the frequencies into **L2**

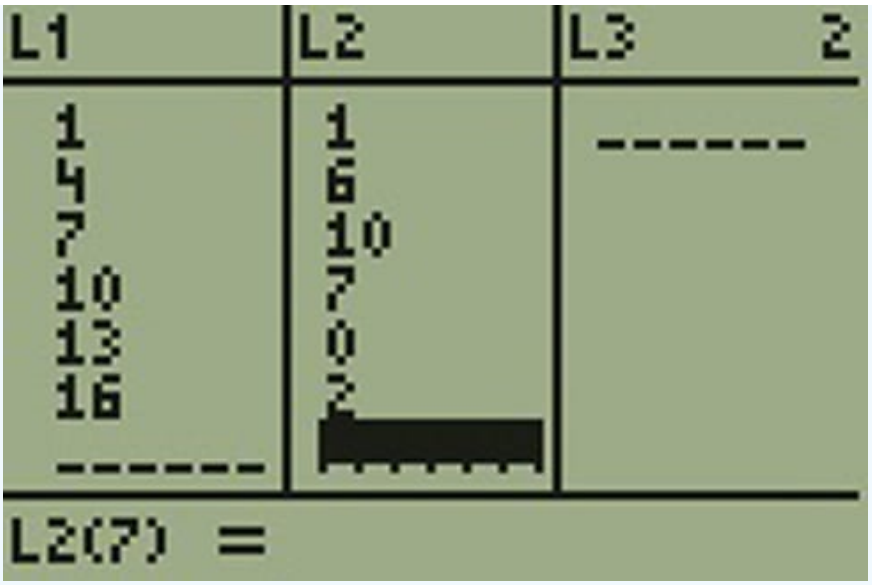


Figure 10.8.4

Select **STAT**, **CALC**, and **1: 1-Var Stats**

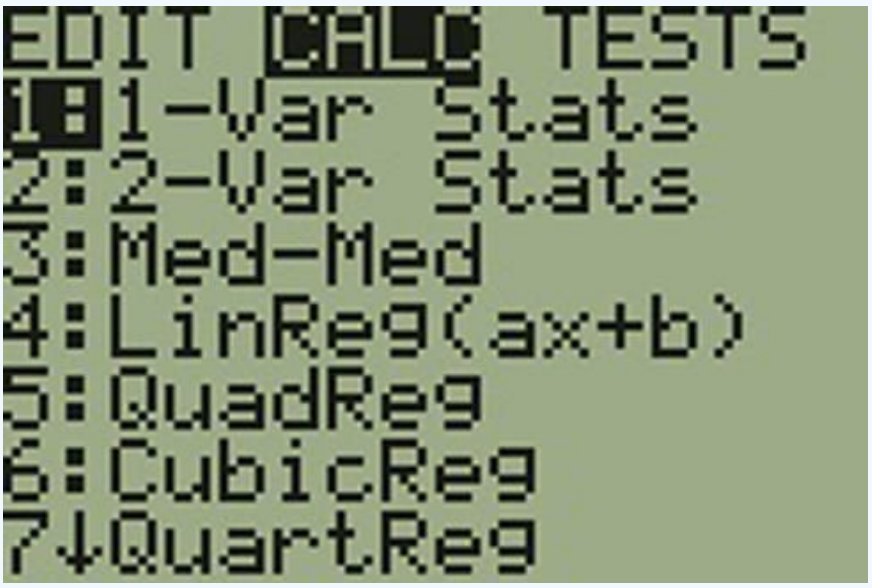


Figure 10.8.5

Select **2nd** then **1** then , **2nd** then **2 Enter**

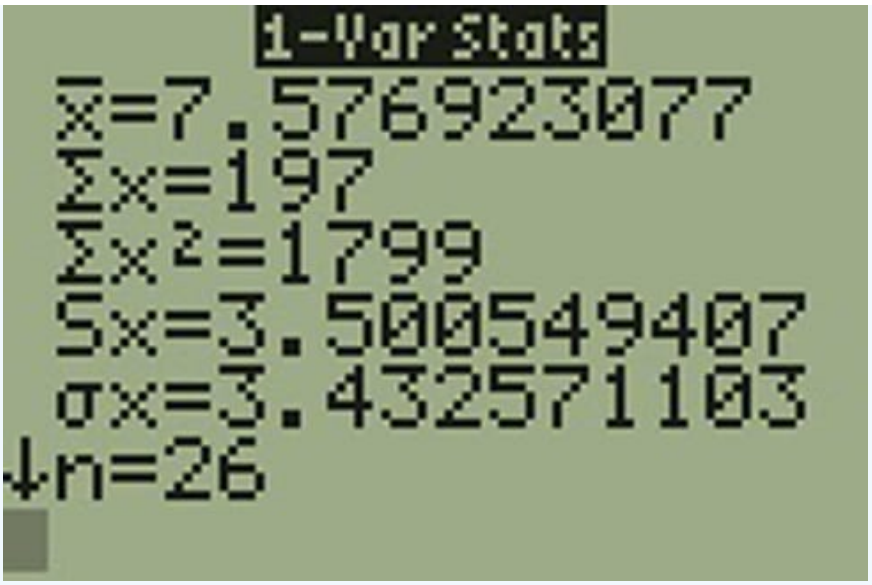


Figure 10.8.6

You will see displayed both a population standard deviation, σ_x, and the sample standard deviation, s_x.

Comparing Values from Different Data Sets

The standard deviation is useful when comparing data values that come from different data sets. If the data sets have different means and standard deviations, then comparing the data values directly can be misleading.

- For each data value, calculate how many standard deviations away from its mean the value is.
- Use the formula: value = mean + (#ofSTDEVs)(standard deviation); solve for #ofSTDEVs.
- Misplaced '#'
- Compare the results of this calculation.

#ofSTDEVs is often called a "z-score"; we can use the symbol z. In symbols, the formulas become:

Sample	$x = \bar{x} + zs$	$z = \frac{x - \bar{x}}{s}$
Population	$x = \mu + z\sigma$	$z = \frac{x - \mu}{\sigma}$

Example 10.8.4

Two students, John and Ali, from different high schools, wanted to find out who had the highest GPA when compared to his school. Which student had the highest GPA when compared to his school?

Student	GPA	School Mean GPA	School Standard Deviation
John	2.85	3.0	0.7
Ali	77	80	10

Answer

For each student, determine how many standard deviations (#ofSTDEVs) his GPA is away from the average, for his school. Pay careful attention to signs when comparing and interpreting the answer.

Misplaced '#'

For John,

Misplaced '#'

For Ali,

Misplaced '#'

John has the better GPA when compared to his school because his GPA is 0.21 standard deviations **below** his school's mean while Ali's GPA is 0.3 standard deviations **below** his school's mean.

John's *z*-score of –0.21 is higher than Ali's *z*-score of –0.3. For GPA, higher values are better, so we conclude that John has the better GPA when compared to his school.

Exercise 10.8.4

Two swimmers, Angie and Beth, from different teams, wanted to find out who had the fastest time for the 50 meter freestyle when compared to her team. Which swimmer had the fastest time when compared to her team?

Swimmer	Time (seconds)	Team Mean Time	Team Standard Deviation
Angie	26.2	27.2	0.8
Beth	27.3	30.1	1.4

Answer

For Angie:

$$z = \left(\frac{26.2 - 27.2}{0.8}\right) = -1.25$$

For Beth:

$$z = \left(\frac{27.3 - 30.1}{1.4}\right) = -2$$

The following lists give a few facts that provide a little more insight into what the standard deviation tells us about the distribution of the data.

For ANY data set, no matter what the distribution of the data is:

- At least 75% of the data is within two standard deviations of the mean.
- At least 89% of the data is within three standard deviations of the mean.
- At least 95% of the data is within 4.5 standard deviations of the mean.
- This is known as Chebyshev's Rule.

For data having a distribution that is BELL-SHAPED and SYMMETRIC:

- Approximately 68% of the data is within one standard deviation of the mean.
- Approximately 95% of the data is within two standard deviations of the mean.
- More than 99% of the data is within three standard deviations of the mean.
- This is known as the Empirical Rule.
- It is important to note that this rule only applies when the shape of the distribution of the data is bell-shaped and symmetric. We will learn more about this when studying the "Normal" or "Gaussian" probability distribution in later chapters.

References

1. Data from Microsoft Bookshelf.
2. King, Bill.“Graphically Speaking.” Institutional Research, Lake Tahoe Community College. Available online at www.ltcc.edu/web/about/institutional-research (accessed April 3, 2013).

Review

The standard deviation can help you calculate the spread of data. There are different equations to use if are calculating the standard deviation of a sample or of a population.

- The Standard Deviation allows us to compare individual data or classes to the data set mean numerically.
- $s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$ or $s = \sqrt{\frac{\sum f(x - \bar{x})^2}{n - 1}}$ is the formula for calculating the standard deviation of a sample. To calculate the standard deviation of a population, we would use the population mean, μ, and the formula $\sigma = \sqrt{\frac{\sum (x - \mu)^2}{N}}$ or $\sigma = \sqrt{\frac{\sum f(x - \mu)^2}{N}}$.

Formula Review

$$s_x = \sqrt{\frac{\sum fm^2}{n} - \bar{x}^2} \tag{10.8.6}$$

where $s_x = \text{sample standard deviation}$ and $\bar{x} = \text{sample mean}$

Use the following information to answer the next two exercises: The following data are the distances between 20 retail stores and a large distribution center. The distances are in miles.

29; 37; 38; 40; 58; 67; 68; 69; 76; 86; 87; 95; 96; 96; 99; 106; 112; 127; 145; 150

Exercise 2.8.4

Use a graphing calculator or computer to find the standard deviation and round to the nearest tenth.

Answer

s = 34.5

Exercise 2.8.5

Find the value that is one standard deviation below the mean.

Exercise 2.8.6

Two baseball players, Fredo and Karl, on different teams wanted to find out who had the higher batting average when compared to his team. Which baseball player had the higher batting average when compared to his team?

Baseball Player	Batting Average	Team Batting Average	Team Standard Deviation
Fredo	0.158	0.166	0.012
Karl	0.177	0.189	0.015

Answer

For Fredo:

$$z = \frac{0.158 - 0.166}{0.012} = -0.67$$

For Karl:

$$z = \frac{0.177 - 0.189}{0.015} = -0.8$$

Fredo's z-score of –0.67 is higher than Karl's z-score of –0.8. For batting average, higher values are better, so Fredo has a better batting average compared to his team.

Exercise 2.8.7

Use Table to find the value that is three standard deviations:

- above the mean
- below the mean

Find the standard deviation for the following frequency tables using the formula. Check the calculations with the TI 83/84.

Exercise 2.8.5

Find the standard deviation for the following frequency tables using the formula. Check the calculations with the TI 83/84.

a.

Grade	Frequency
49.5–59.5	2
59.5–69.5	3
69.5–79.5	8
79.5–89.5	12
89.5–99.5	5

b.

Daily Low Temperature	Frequency
49.5–59.5	53
59.5–69.5	32
69.5–79.5	15
79.5–89.5	1
89.5–99.5	0

c.

Points per Game	Frequency
49.5–59.5	14
59.5–69.5	32
69.5–79.5	15
79.5–89.5	23

Points per Game	Frequency
89.5–99.5	2

Answer

a. $s_x = \sqrt{\frac{\sum fm^2}{n} - \bar{x}^2} = \sqrt{\frac{193157.45}{30} - 79.5^2} = 10.88$
b. $s_x = \sqrt{\frac{\sum fm^2}{n} - \bar{x}^2} = \sqrt{\frac{380945.3}{101} - 60.94^2} = 7.62$
c. $s_x = \sqrt{\frac{\sum fm^2}{n} - \bar{x}^2} = \sqrt{\frac{440051.5}{86} - 70.66^2} = 11.14$

Bringing It Together

Exercise 2.8.7

Twenty-five randomly selected students were asked the number of movies they watched the previous week. The results are as follows:

# of movies	Frequency
0	5
1	9
2	6
3	4
4	1

a. Find the sample mean $\bar{x}$.
b. Find the approximate sample standard deviation, s.

Answer

a. 1.48
b. 1.12

Exercise 2.8.8

Forty randomly selected students were asked the number of pairs of sneakers they owned. Let $X =$ the number of pairs of sneakers owned. The results are as follows:

X	Frequency
1	2
2	5
3	8
4	12
5	12
6	0

X	Frequency
7	1

a. Find the sample mean $\bar{x}$
b. Find the sample standard deviation, s
c. Construct a histogram of the data.
d. Complete the columns of the chart.
e. Find the first quartile.
f. Find the median.
g. Find the third quartile.
h. Construct a box plot of the data.
i. What percent of the students owned at least five pairs?
j. Find the 40th percentile.
k. Find the 90th percentile.
l. Construct a line graph of the data
m. Construct a stemplot of the data

Exercise 2.8.9

Following are the published weights (in pounds) of all of the team members of the San Francisco 49ers from a previous year.

177; 205; 210; 210; 232; 205; 185; 185; 178; 210; 206; 212; 184; 174; 185; 242; 188; 212; 215; 247; 241; 223; 220; 260; 245; 259; 278; 270; 280; 295; 275; 285; 290; 272; 273; 280; 285; 286; 200; 215; 185; 230; 250; 241; 190; 260; 250; 302; 265; 290; 276; 228; 265

a. Organize the data from smallest to largest value.
b. Find the median.
c. Find the first quartile.
d. Find the third quartile.
e. Construct a box plot of the data.
f. The middle 50% of the weights are from _____ to _____.
g. If our population were all professional football players, would the above data be a sample of weights or the population of weights? Why?
h. If our population included every team member who ever played for the San Francisco 49ers, would the above data be a sample of weights or the population of weights? Why?
i. Assume the population was the San Francisco 49ers. Find:
   i. the population mean, μ.
   ii. the population standard deviation, σ.
   iii. the weight that is two standard deviations below the mean.
   iv. When Steve Young, quarterback, played football, he weighed 205 pounds. How many standard deviations above or below the mean was he?
j. That same year, the mean weight for the Dallas Cowboys was 240.08 pounds with a standard deviation of 44.38 pounds. Emmit Smith weighed in at 209 pounds. With respect to his team, who was lighter, Smith or Young? How did you determine your answer?

Answer

a. 174; 177; 178; 184; 185; 185; 185; 185; 188; 190; 200; 205; 205; 206; 210; 210; 210; 212; 212; 215; 215; 220; 223; 228; 230; 232; 241; 241; 242; 245; 247; 250; 250; 259; 260; 260; 265; 265; 270; 272; 273; 275; 276; 278; 280; 280; 285; 285; 286; 290; 290; 295; 302
b. 241
c. 205.5
d. 272.5
e. A box plot with a whisker between 174 and 205.5, a solid line at 205.5, a dashed line at 241, a solid line at 272.5, and a whisker between 272.5 and 302.

f. 205.5, 272.5
g. sample
h. population
i. i. 236.34
ii. 37.50
iii. 161.34
iv. 0.84 std. dev. below the mean
j. Young

Exercise 2.8.10

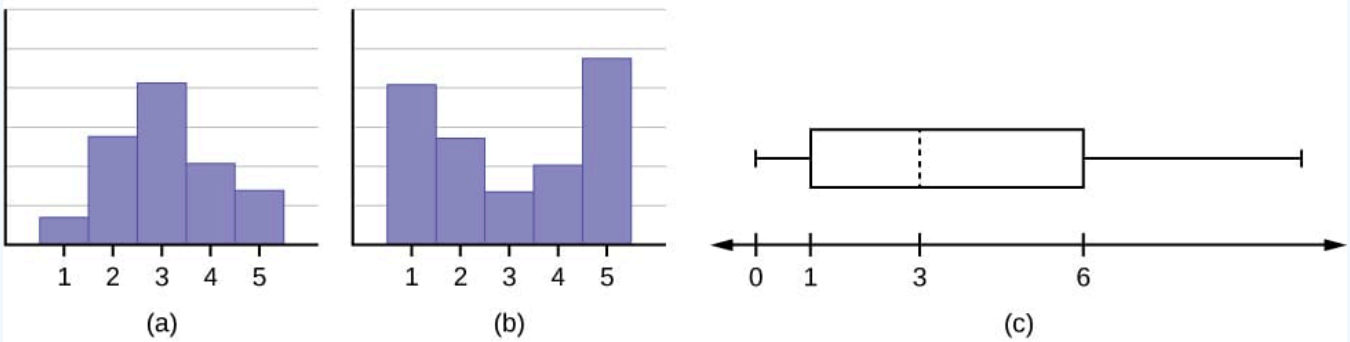
One hundred teachers attended a seminar on mathematical problem solving. The attitudes of a representative sample of 12 of the teachers were measured before and after the seminar. A positive number for change in attitude indicates that a teacher's attitude toward math became more positive. The 12 change scores are as follows:

3; 8; –1; 2; 0; 5; –3; 1; –1; 6; 5; –2

a. What is the mean change score?
b. What is the standard deviation for this population?
c. What is the median change score?
d. Find the change score that is 2.2 standard deviations below the mean.

Exercise 2.8.11

Refer to Figure determine which of the following are true and which are false. Explain your solution to each part in complete sentences.



Figure

.

a. The medians for all three graphs are the same.
b. We cannot determine if any of the means for the three graphs is different.
c. The standard deviation for graph b is larger than the standard deviation for graph a.
d. We cannot determine if any of the third quartiles for the three graphs is different.

Answer

a. True
b. True

c. True
d. False

Exercise 2.8.12

In a recent issue of the *IEEE Spectrum*, 84 engineering conferences were announced. Four conferences lasted two days. Thirty-six lasted three days. Eighteen lasted four days. Nineteen lasted five days. Four lasted six days. One lasted seven days. One lasted eight days. One lasted nine days. Let X = the length (in days) of an engineering conference.

a. Organize the data in a chart.
b. Find the median, the first quartile, and the third quartile.
c. Find the 65th percentile.
d. Find the 10th percentile.
e. Construct a box plot of the data.
f. The middle 50% of the conferences last from _____ days to _____ days.
g. Calculate the sample mean of days of engineering conferences.
h. Calculate the sample standard deviation of days of engineering conferences.
i. Find the mode.
j. If you were planning an engineering conference, which would you choose as the length of the conference: mean; median; or mode? Explain why you made that choice.
k. Give two reasons why you think that three to five days seem to be popular lengths of engineering conferences.

Exercise 2.8.13

A survey of enrollment at 35 community colleges across the United States yielded the following figures:

6414; 1550; 2109; 9350; 21828; 4300; 5944; 5722; 2825; 2044; 5481; 5200; 5853; 2750; 10012; 6357; 27000; 9414; 7681; 3200; 17500; 9200; 7380; 18314; 6557; 13713; 17768; 7493; 2771; 2861; 1263; 7285; 28165; 5080; 11622

a. Organize the data into a chart with five intervals of equal width. Label the two columns "Enrollment" and "Frequency."
b. Construct a histogram of the data.
c. If you were to build a new community college, which piece of information would be more valuable: the mode or the mean?
d. Calculate the sample mean.
e. Calculate the sample standard deviation.
f. A school with an enrollment of 8000 would be how many standard deviations away from the mean?

Answer

a.

Enrollment	Frequency
1000-5000	10
5000-10000	16
10000-15000	3
15000-20000	3
20000-25000	1
25000-30000	2

b. Check student's solution.
c. mode
d. 8628.74
e. 6943.88

f. –0.09

Use the following information to answer the next two exercises. $X =$ the number of days per week that 100 clients use a particular exercise facility.

x	Frequency
0	3
1	12
2	33
3	28
4	11
5	9
6	4

Exercise 2.8.14

The 80th percentile is _____

a. 5
b. 80
c. 3
d. 4

Exercise 2.8.15

The number that is 1.5 standard deviations BELOW the mean is approximately _____

a. 0.7
b. 4.8
c. –2.8
d. Cannot be determined

Answer

a

Exercise 2.8.16

Suppose that a publisher conducted a survey asking adult consumers the number of fiction paperback books they had purchased in the previous month. The results are summarized in the Table.

# of books	Freq.	Rel. Freq.
0	18	
1	24	
2	24	
3	22	

# of books	Freq.	Rel. Freq.
4	15	
5	10	
7	5	
9	1	

a. Are there any outliers in the data? Use an appropriate numerical test involving the *IQR* to identify outliers, if any, and clearly state your conclusion.
b. If a data value is identified as an outlier, what should be done about it?
c. Are any data values further than two standard deviations away from the mean? In some situations, statisticians may use this criteria to identify data values that are unusual, compared to the other data values. (Note that this criteria is most appropriate to use for data that is mound-shaped and symmetric, rather than for skewed data.)
d. Do parts a and c of this problem give the same answer?
e. Examine the shape of the data. Which part, a or c, of this question gives a more appropriate result for this data?
f. Based on the shape of the data which is the most appropriate measure of center for this data: mean, median or mode?

Glossary

Standard Deviation
a number that is equal to the square root of the variance and measures how far data values are from their mean; notation: s for sample standard deviation and σ for population standard deviation.

Contributors and Attributions

Variance
mean of the squared deviations from the mean, or the square of the standard deviation; for a set of data, a deviation can be represented as $x - \bar{x}$ where x is a value of the data and $\bar{x}$ is the sample mean. The sample variance is equal to the sum of the squares of the deviations divided by the difference of the sample size and one.

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10.9: Descriptive Statistics (Worksheet)

Name: _____

Section: _____

Student ID#:_____

Work in groups on these problems. You should try to answer the questions without referring to your textbook. If you get stuck, try asking another group for help.

Student Learning Outcomes

- The student will construct a histogram and a box plot.
- The student will calculate univariate statistics.
- The student will examine the graphs to interpret what the data implies.

Collect the Data

Record the number of pairs of shoes you own.

1. Randomly survey 30 classmates about the number of pairs of shoes they own. Record their values.

   Survey Results

_____	_____	_____	_____	_____
_____	_____	_____	_____	_____
_____	_____	_____	_____	_____
_____	_____	_____	_____	_____
_____	_____	_____	_____	_____
_____	_____	_____	_____	_____

2. Construct a histogram. Make five to six intervals. Sketch the graph using a ruler and pencil and scale the axes.

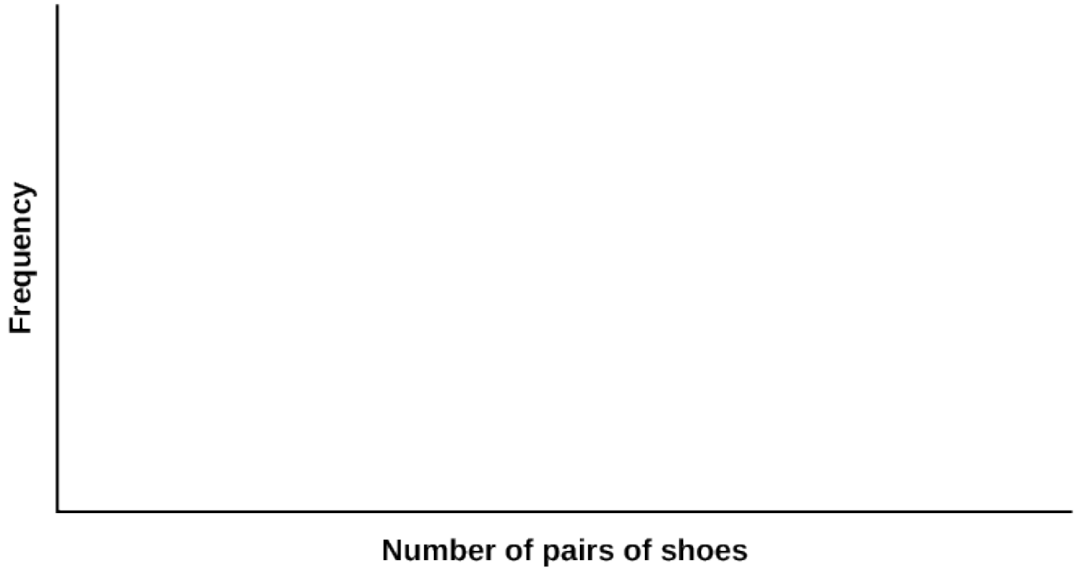


Figure 10.9.1

3. Calculate the following values.
   a. $\bar{x}$ = _____
   b. s = _____
4. Are the data discrete or continuous? How do you know?
5. In complete sentences, describe the shape of the histogram.
6. Are there any potential outliers? List the value(s) that could be outliers. Use a formula to check the end values to determine if they are potential outliers.

Analyze the Data

1. Determine the following values.
   a. Min = _____
   b. M = _____
   c. Max = _____
   d. Q_1 = _____
   e. Q_3 = _____
   f. IQR = _____
2. Construct a box plot of data
3. What does the shape of the box plot imply about the concentration of data? Use complete sentences.
4. Using the box plot, how can you determine if there are potential outliers?
5. How does the standard deviation help you to determine concentration of the data and whether or not there are potential outliers?
6. What does the IQR represent in this problem?
7. Show your work to find the value that is 1.5 standard deviations:
   a. above the mean.
   b. below the mean.

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10.E: Descriptive Statistics (Exercises)

2.2: Stem-and-Leaf Graphs (Stemplots), Line Graphs, and Bar Graphs

Q 2.2.1

Student grades on a chemistry exam were: 77, 78, 76, 81, 86, 51, 79, 82, 84, 99

a. Construct a stem-and-leaf plot of the data.
b. Are there any potential outliers? If so, which scores are they? Why do you consider them outliers?

Q 2.2.2

The table below contains the 2010 obesity rates in U.S. states and Washington, DC.

State	Percent (%)	State	Percent (%)	State	Percent (%)
Alabama	32.2	Kentucky	31.3	North Dakota	27.2
Alaska	24.5	Louisiana	31.0	Ohio	29.2
Arizona	24.3	Maine	26.8	Oklahoma	30.4
Arkansas	30.1	Maryland	27.1	Oregon	26.8
California	24.0	Massachusetts	23.0	Pennsylvania	28.6
Colorado	21.0	Michigan	30.9	Rhode Island	25.5
Connecticut	22.5	Minnesota	24.8	South Carolina	31.5
Delaware	28.0	Mississippi	34.0	South Dakota	27.3
Washington, DC	22.2	Missouri	30.5	Tennessee	30.8
Florida	26.6	Montana	23.0	Texas	31.0
Georgia	29.6	Nebraska	26.9	Utah	22.5
Hawaii	22.7	Nevada	22.4	Vermont	23.2
Idaho	26.5	New Hampshire	25.0	Virginia	26.0
Illinois	28.2	New Jersey	23.8	Washington	25.5
Indiana	29.6	New Mexico	25.1	West Virginia	32.5
Iowa	28.4	New York	23.9	Wisconsin	26.3
Kansas	29.4	North Carolina	27.8	Wyoming	25.1

a. Use a random number generator to randomly pick eight states. Construct a bar graph of the obesity rates of those eight states.
b. Construct a bar graph for all the states beginning with the letter "A."
c. Construct a bar graph for all the states beginning with the letter "M."

S 2.2.2

a. Example solution for using the random number generator for the TI-84+ to generate a simple random sample of 8 states. Instructions are as follows.

- Number the entries in the table 1–51 (Includes Washington, DC; Numbered vertically)
- Press MATH
- Arrow over to PRB
- Press 5:randInt(
- Enter 51,1,8)

Eight numbers are generated (use the right arrow key to scroll through the numbers). The numbers correspond to the numbered states (for this example: {47 21 9 23 51 13 25 4}. If any numbers are repeated, generate a different number by using 5:randInt(51,1)). Here, the states (and Washington DC) are {Arkansas, Washington DC, Idaho, Maryland, Michigan, Mississippi, Virginia, Wyoming}.

Corresponding percents are {30.1, 22.2, 26.5, 27.1, 30.9, 34.0, 26.0, 25.1}.

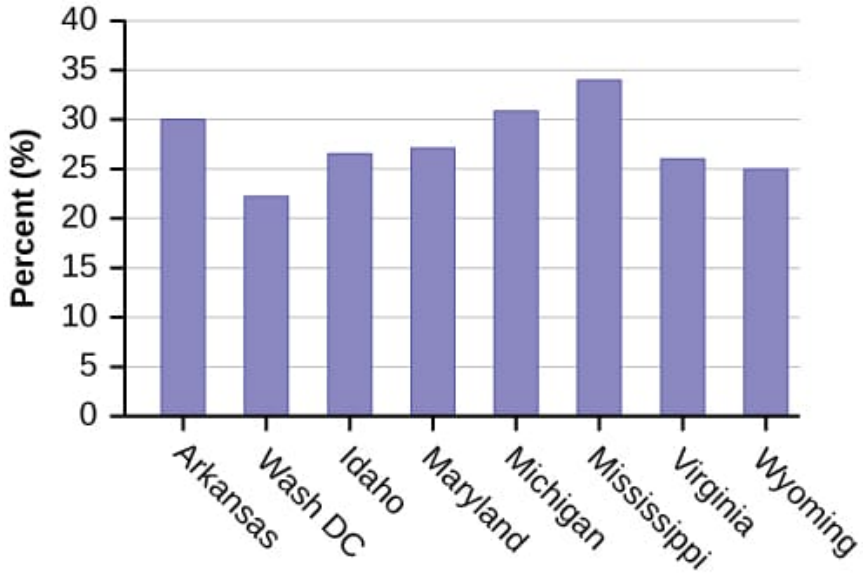


Figure $10.E.1$: (a)

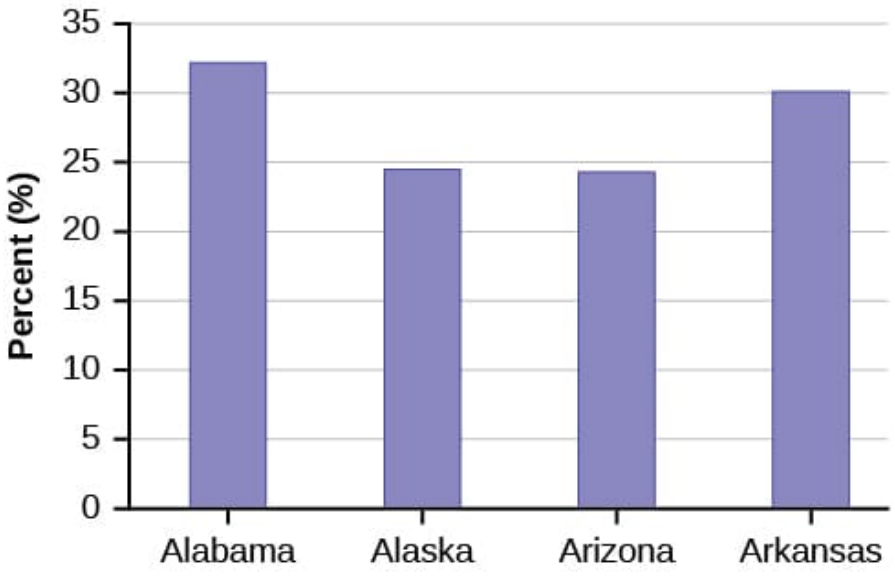


Figure $10.E.1$: (b)

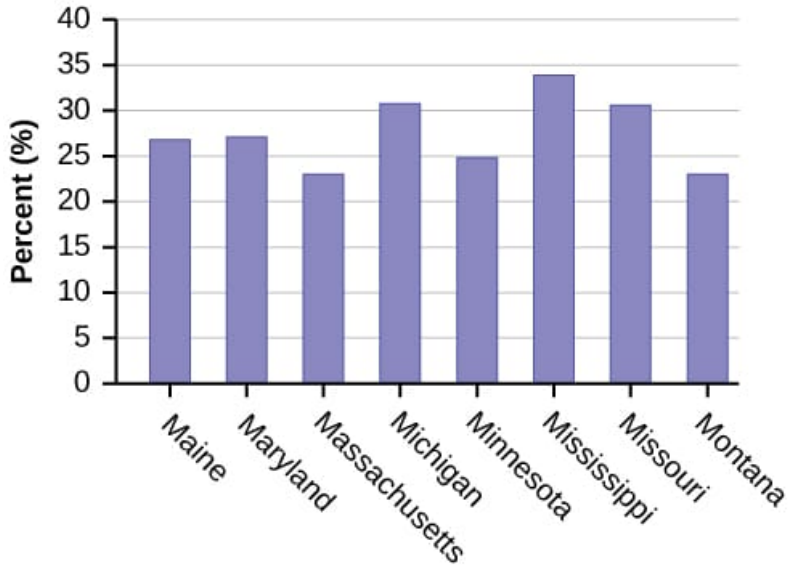


Figure $10.E.1$: (c)

For each of the following data sets, create a stem plot and identify any outliers.

Exercise 2.2.7

The miles per gallon rating for 30 cars are shown below (lowest to highest).

19, 19, 19, 20, 21, 21, 25, 25, 25, 26, 26, 28, 29, 31, 31, 32, 32, 33, 34, 35, 36, 37, 37, 38, 38, 38, 38, 41, 43, 43

Answer

Stem	Leaf
1	9 9 9
2	0 1 1 5 5 5 6 6 8 9
3	1 1 2 2 3 4 5 6 7 7 8 8 8 8
4	1 3 3

The height in feet of 25 trees is shown below (lowest to highest).

25, 27, 33, 34, 34, 34, 35, 37, 37, 38, 39, 39, 39, 40, 41, 45, 46, 47, 49, 50, 50, 53, 53, 54, 54

The data are the prices of different laptops at an electronics store. Round each value to the nearest ten.

249, 249, 260, 265, 265, 280, 299, 299, 309, 319, 325, 326, 350, 350, 350, 365, 369, 389, 409, 459, 489, 559, 569, 570, 610

Answer

Stem	Leaf
2	5 5 6 7 7 8

Stem	Leaf
3	0 0 1 2 3 3 5 5 5 7 7 9
4	1 6 9
5	6 7 7
6	1

The data are daily high temperatures in a town for one month.

61, 61, 62, 64, 66, 67, 67, 67, 68, 69, 70, 70, 70, 71, 71, 72, 74, 74, 74, 75, 75, 75, 76, 76, 77, 78, 78, 79, 79, 95

For the next three exercises, use the data to construct a line graph.

Exercise 2.2.8

In a survey, 40 people were asked how many times they visited a store before making a major purchase. The results are shown in the Table below.

Number of times in store	Frequency
1	4
2	10
3	16
4	6
5	4

Answer



Figure $10.E.2$:

Exercise 2.2.9

In a survey, several people were asked how many years it has been since they purchased a mattress. The results are shown in Table.

Years since last purchase	Frequency
0	2
1	8
2	13
3	22
4	16
5	9

Exercise 2.2.10

Several children were asked how many TV shows they watch each day. The results of the survey are shown in the Table below.

Number of TV Shows	Frequency
0	12
1	18
2	36
3	7
4	2

Answer

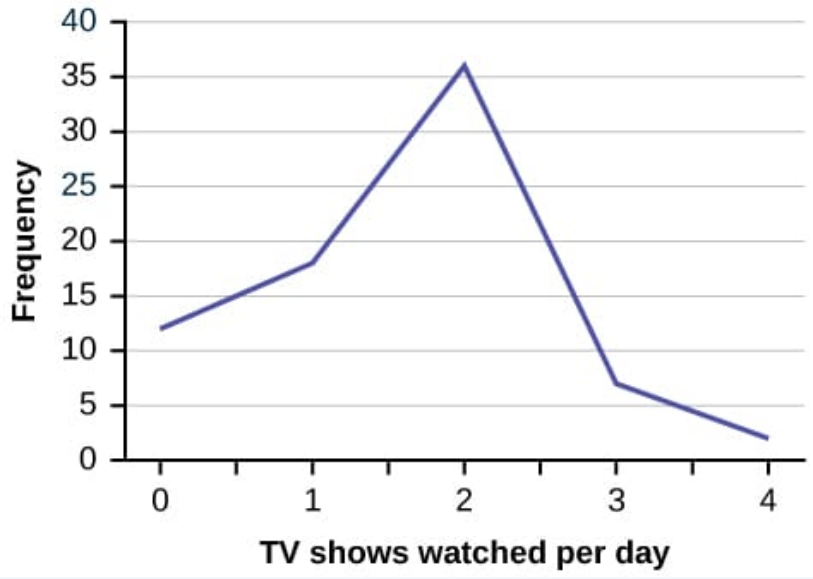


Figure $10.E.3$:

Exercise 2.2.11

The students in Ms. Ramirez's math class have birthdays in each of the four seasons. Table shows the four seasons, the number of students who have birthdays in each season, and the percentage (%) of students in each group. Construct a bar graph showing the number of students.

Seasons	Number of students	Proportion of population
Spring	8	24%
Summer	9	26%
Autumn	11	32%
Winter	6	18%

Using the data from Mrs. Ramirez's math class supplied in the table above, construct a bar graph showing the percentages.

Answer

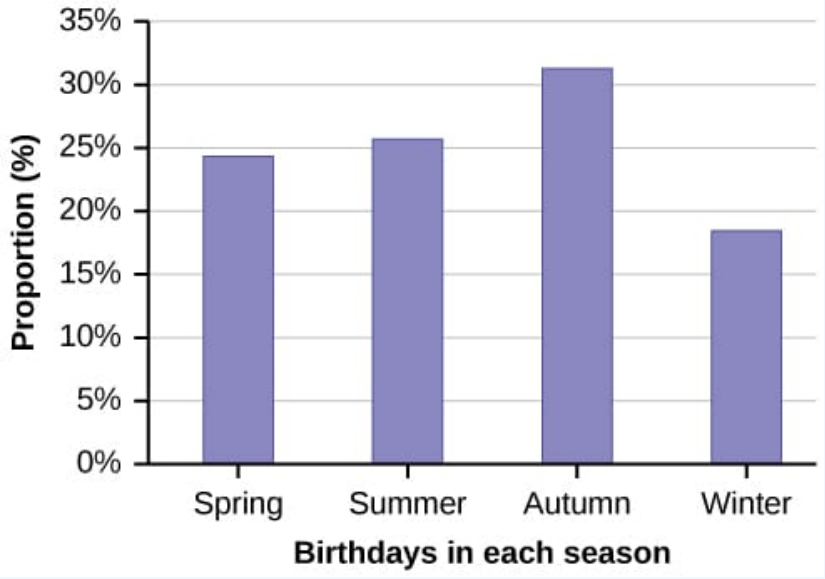


Figure $10.E.4$:

Exercise 2.2.12

David County has six high schools. Each school sent students to participate in a county-wide science competition. Table shows the percentage breakdown of competitors from each school, and the percentage of the entire student population of the county that goes to each school. Construct a bar graph that shows the population percentage of competitors from each school.

High School	Science competition population	Overall student population
Alabaster	28.9%	8.6%
Concordia	7.6%	23.2%
Genoa	12.1%	15.0%
Mocksville	18.5%	14.3%
Tynneson	24.2%	10.1%

High School	Science competition population	Overall student population
West End	8.7%	28.8%

Use the data from the David County science competition supplied in Exercise. Construct a bar graph that shows the county-wide population percentage of students at each school.

Answer

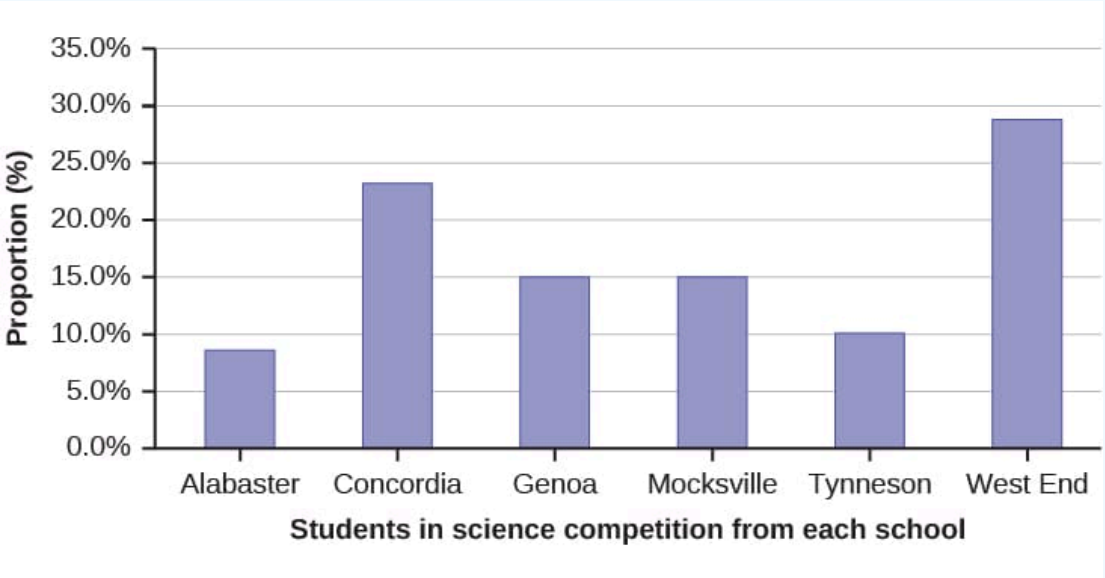


Figure $10.E.5$:

2.3: Histograms, Frequency, Polygons, and Time Series Graphs

Q 2.3.1

Suppose that three book publishers were interested in the number of fiction paperbacks adult consumers purchase per month. Each publisher conducted a survey. In the survey, adult consumers were asked the number of fiction paperbacks they had purchased the previous month. The results are as follows:

Publisher A

# of books	Freq.	Rel. Freq.
0	10	
1	12	
2	16	
3	12	
4	8	
5	6	
6	2	
8	2	

Publisher B

# of books	Freq.	Rel. Freq.
0	18	
1	24	
2	24	
3	22	
4	15	
5	10	
7	5	
9	1	

Publisher C

# of books	Freq.	Rel. Freq.
0–1	20	
2–3	35	
4–5	12	
6–7	2	
8–9	1	

a. Find the relative frequencies for each survey. Write them in the charts.
b. Using either a graphing calculator, computer, or by hand, use the frequency column to construct a histogram for each publisher's survey. For Publishers A and B, make bar widths of one. For Publisher C, make bar widths of two.
c. In complete sentences, give two reasons why the graphs for Publishers A and B are not identical.
d. Would you have expected the graph for Publisher C to look like the other two graphs? Why or why not?
e. Make new histograms for Publisher A and Publisher B. This time, make bar widths of two.
f. Now, compare the graph for Publisher C to the new graphs for Publishers A and B. Are the graphs more similar or more different? Explain your answer.

Q 2.3.2

Often, cruise ships conduct all on-board transactions, with the exception of gambling, on a cashless basis. At the end of the cruise, guests pay one bill that covers all onboard transactions. Suppose that 60 single travelers and 70 couples were surveyed as to their on-board bills for a seven-day cruise from Los Angeles to the Mexican Riviera. Following is a summary of the bills for each group.

Singles

Amount($)	Frequency	Rel. Frequency
51–100	5	
101–150	10	
151–200	15	
201–250	15	
251–300	10	
301–350	5	

Couples

Amount($)	Frequency	Rel. Frequency
100–150	5	
201–250	5	
251–300	5	
301–350	5	
351–400	10	
401–450	10	
451–500	10	
501–550	10	
551–600	5	
601–650	5	

a. Fill in the relative frequency for each group.
b. Construct a histogram for the singles group. Scale the *x*-axis by $50 widths. Use relative frequency on the *y*-axis.
c. Construct a histogram for the couples group. Scale the *x*-axis by $50 widths. Use relative frequency on the *y*-axis.
d. Compare the two graphs:
    i. List two similarities between the graphs.
    ii. List two differences between the graphs.
    iii. Overall, are the graphs more similar or different?
e. Construct a new graph for the couples by hand. Since each couple is paying for two individuals, instead of scaling the *x*-axis by $50, scale it by $100. Use relative frequency on the *y*-axis.
f. Compare the graph for the singles with the new graph for the couples:
    i. List two similarities between the graphs.
    ii. Overall, are the graphs more similar or different?
g. How did scaling the couples graph differently change the way you compared it to the singles graph?
h. Based on the graphs, do you think that individuals spend the same amount, more or less, as singles as they do person by person as a couple? Explain why in one or two complete sentences.

S 2.3.2

Singles

Amount($)	Frequency	Relative Frequency
51–100	5	0.08
101–150	10	0.17
151–200	15	0.25
201–250	15	0.25
251–300	10	0.17
301–350	5	0.08

Couples

Amount($)	Frequency	Relative Frequency
100–150	5	0.07
201–250	5	0.07

Amount($)	Frequency	Relative Frequency
251–300	5	0.07
301–350	5	0.07
351–400	10	0.14
401–450	10	0.14
451–500	10	0.14
501–550	10	0.14
551–600	5	0.07
601–650	5	0.07

a. See the tables above
b. In the following histogram data values that fall on the right boundary are counted in the class interval, while values that fall on the left boundary are not counted (with the exception of the first interval where both boundary values are included).

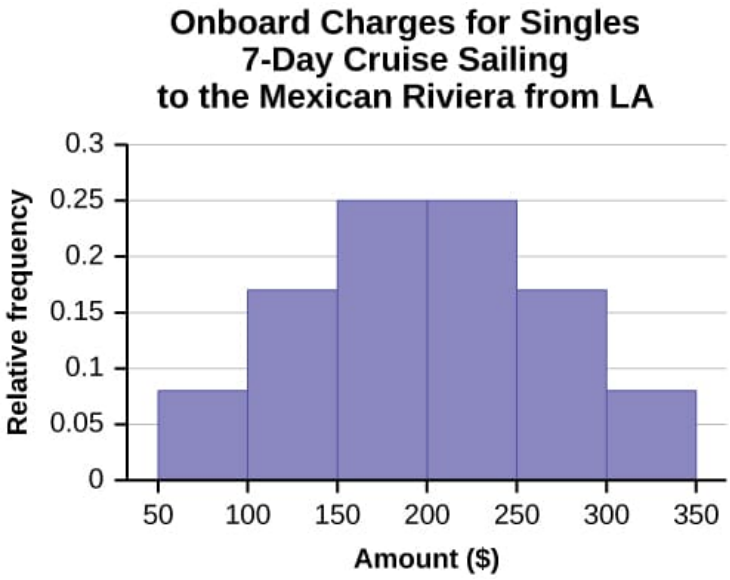


Figure $10.E.6$:

c. In the following histogram, the data values that fall on the right boundary are counted in the class interval, while values that fall on the left boundary are not counted (with the exception of the first interval where values on both boundaries are included).

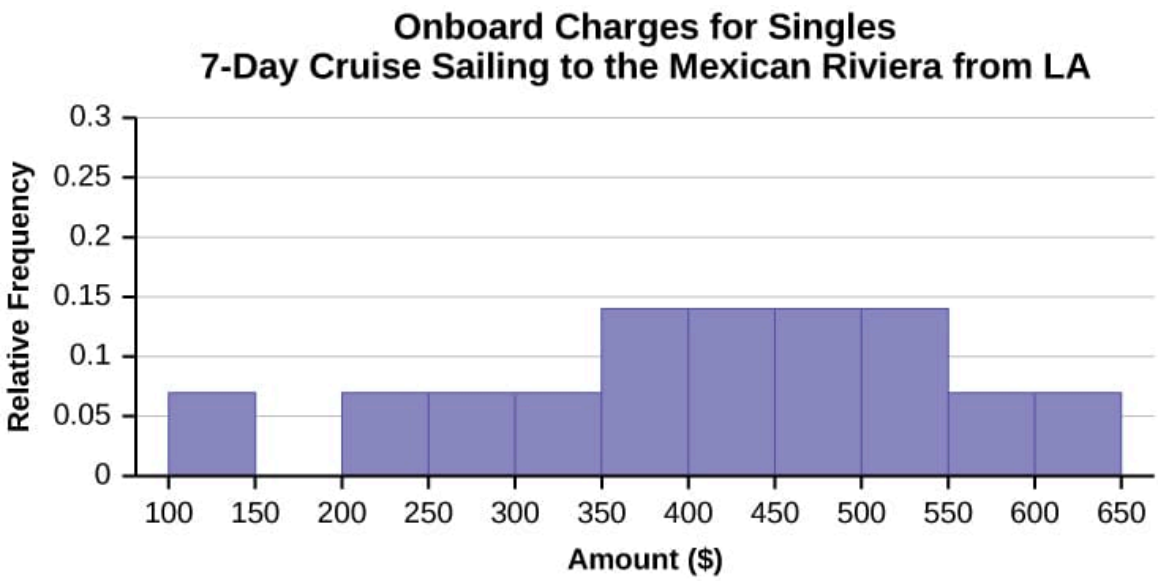


Figure $10.E.7$:

d. Compare the two graphs:
   i. Answers may vary. Possible answers include:
      - Both graphs have a single peak.
      - Both graphs use class intervals with width equal to $50.
   ii. Answers may vary. Possible answers include:
      - The couples graph has a class interval with no values.
      - It takes almost twice as many class intervals to display the data for couples.
   iii. Answers may vary. Possible answers include: The graphs are more similar than different because the overall patterns for the graphs are the same.
e. Check student's solution.
f. Compare the graph for the Singles with the new graph for the Couples:
   i.
      - Both graphs have a single peak.
      - Both graphs display 6 class intervals.
      - Both graphs show the same general pattern.
   ii. Answers may vary. Possible answers include: Although the width of the class intervals for couples is double that of the class intervals for singles, the graphs are more similar than they are different.
g. Answers may vary. Possible answers include: You are able to compare the graphs interval by interval. It is easier to compare the overall patterns with the new scale on the Couples graph. Because a couple represents two individuals, the new scale leads to a more accurate comparison.
h. Answers may vary. Possible answers include: Based on the histograms, it seems that spending does not vary much from singles to individuals who are part of a couple. The overall patterns are the same. The range of spending for couples is approximately double the range for individuals.

Q 2.3.3

Twenty-five randomly selected students were asked the number of movies they watched the previous week. The results are as follows.

# of movies	Frequency	Relative Frequency	Cumulative Relative Frequency
0	5		
1	9		
2	6		
3	4		

# of movies	Frequency	Relative Frequency	Cumulative Relative Frequency
4	1		

a. Construct a histogram of the data.
b. Complete the columns of the chart.

Use the following information to answer the next two exercises: Suppose one hundred eleven people who shopped in a special t-shirt store were asked the number of t-shirts they own costing more than $19 each.

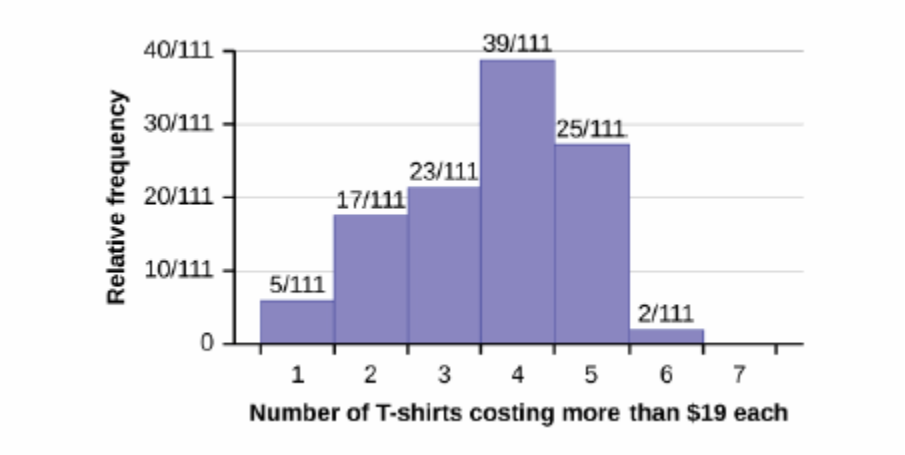


Figure $10.E.8$:

Q 2.3.4

The percentage of people who own at most three t-shirts costing more than $19 each is approximately:

a. 21
b. 59
c. 41
d. Cannot be determined

S 2.3.4

c

Q 2.3.5

If the data were collected by asking the first 111 people who entered the store, then the type of sampling is:

a. cluster
b. simple random
c. stratified
d. convenience

Q 2.3.6

Following are the 2010 obesity rates by U.S. states and Washington, DC.

State	Percent (%)	State	Percent (%)	State	Percent (%)
Alabama	32.2	Kentucky	31.3	North Dakota	27.2
Alaska	24.5	Louisiana	31.0	Ohio	29.2
Arizona	24.3	Maine	26.8	Oklahoma	30.4
Arkansas	30.1	Maryland	27.1	Oregon	26.8
California	24.0	Massachusetts	23.0	Pennsylvania	28.6
Colorado	21.0	Michigan	30.9	Rhode Island	25.5
Connecticut	22.5	Minnesota	24.8	South Carolina	31.5
Delaware	28.0	Mississippi	34.0	South Dakota	27.3
Washington, DC	22.2	Missouri	30.5	Tennessee	30.8
Florida	26.6	Montana	23.0	Texas	31.0
Georgia	29.6	Nebraska	26.9	Utah	22.5
Hawaii	22.7	Nevada	22.4	Vermont	23.2
Idaho	26.5	New Hampshire	25.0	Virginia	26.0
Illinois	28.2	New Jersey	23.8	Washington	25.5
Indiana	29.6	New Mexico	25.1	West Virginia	32.5
Iowa	28.4	New York	23.9	Wisconsin	26.3
Kansas	29.4	North Carolina	27.8	Wyoming	25.1

Construct a bar graph of obesity rates of your state and the four states closest to your state. Hint: Label the x-axis with the states.

S 2.3.7

Answers will vary.

Exercise 2.3.6

Sixty-five randomly selected car salespersons were asked the number of cars they generally sell in one week. Fourteen people answered that they generally sell three cars; nineteen generally sell four cars; twelve generally sell five cars; nine generally sell six cars; eleven generally sell seven cars. Complete the table.

Data Value (# cars)	Frequency	Relative Frequency	Cumulative Relative Frequency

Exercise 2.3.7

What does the frequency column in the Table above sum to? Why?

Answer

65

Exercise 2.3.8

What does the relative frequency column in in the Table above sum to? Why?

Exercise 2.3.9

What is the difference between relative frequency and frequency for each data value in in the Table above ?

Answer

The relative frequency shows the *proportion* of data points that have each value. The frequency tells the *number* of data points that have each value.

Exercise 2.3.10

What is the difference between cumulative relative frequency and relative frequency for each data value?

Exercise 2.3.11

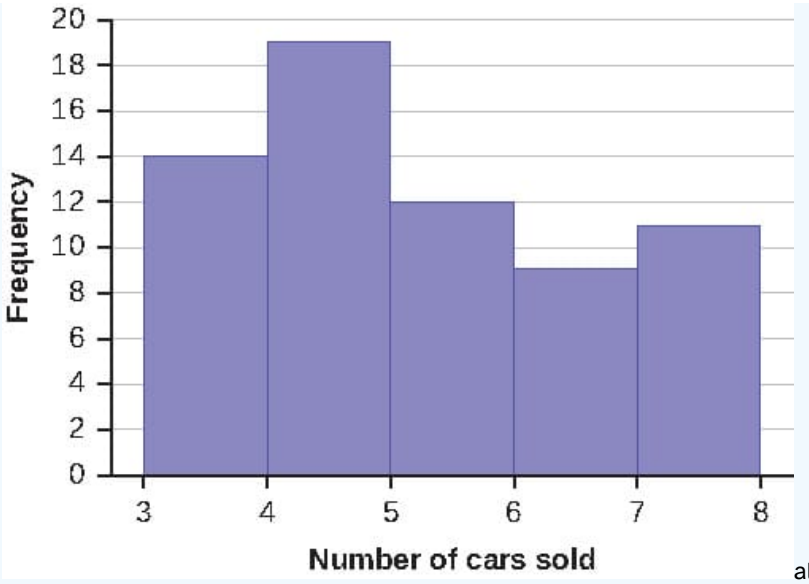
To construct the histogram for the data in in the Table above , determine appropriate minimum and maximum *x* and *y* values and the scaling. Sketch the histogram. Label the horizontal and vertical axes with words. Include numerical scaling.



Figure $10.E.9$:

Answer

Answers will vary. One possible histogram is shown:



alt Figure $10.E.10$:

Exercise 2.3.12

Construct a frequency polygon for the following:

a.

Pulse Rates for Women	Frequency
60–69	12
70–79	14
80–89	11
90–99	1
100–109	1
110–119	0
120–129	1

b.

Actual Speed in a 30 MPH Zone	Frequency
42–45	25
46–49	14
50–53	7
54–57	3
58–61	1

c.

Tar (mg) in Nonfiltered Cigarettes	Frequency
10–13	1
14–17	0
18–21	15
22–25	7
26–29	2

Exercise 2.3.13

Construct a frequency polygon from the frequency distribution for the 50 highest ranked countries for depth of hunger.

Depth of Hunger	Frequency
230–259	21
260–289	13
290–319	5
320–349	7
350–379	1
380–409	1
410–439	1

Answer

Find the midpoint for each class. These will be graphed on the *x*-axis. The frequency values will be graphed on the *y*-axis values.

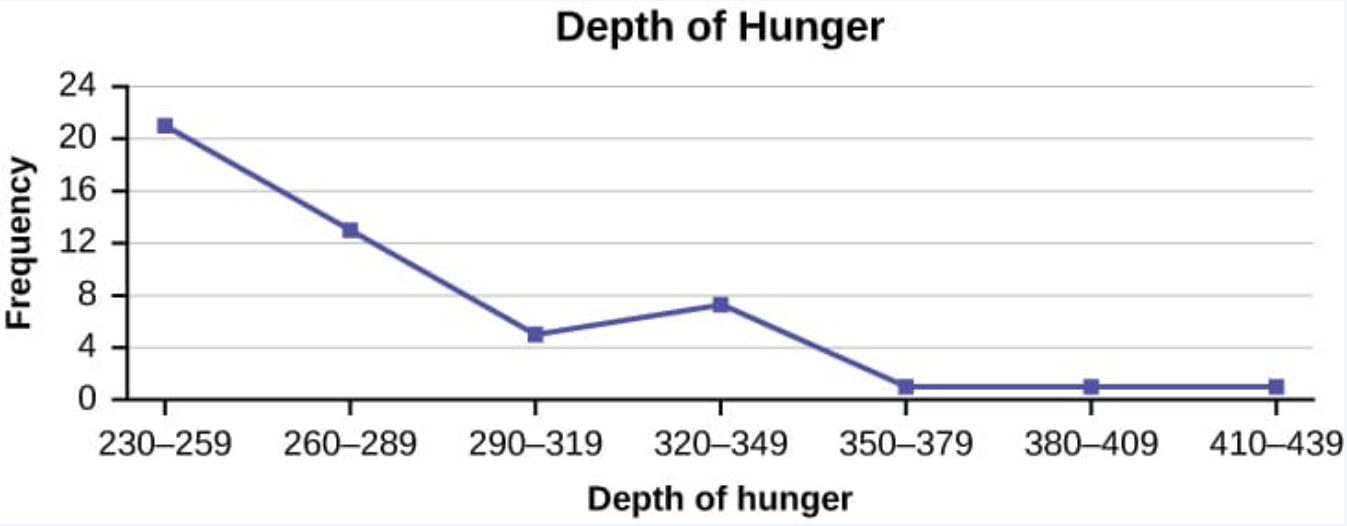


Figure 10.*E*.11:

Exercise 2.3.14

Use the two frequency tables to compare the life expectancy of men and women from 20 randomly selected countries. Include an overlayed frequency polygon and discuss the shapes of the distributions, the center, the spread, and any outliers. What can we conclude about the life expectancy of women compared to men?

Life Expectancy at Birth – Women	Frequency
49–55	3
56–62	3
63–69	1
70–76	3
77–83	8
84–90	2

Life Expectancy at Birth – Men	Frequency
49–55	3
56–62	3
63–69	1
70–76	1
77–83	7
84–90	5

Exercise 2.3.15

Construct a times series graph for (a) the number of male births, (b) the number of female births, and (c) the total number of births.

Sex/Year	1855	1856	1857	1858	1859	1860	1861
Female	45,545	49,582	50,257	50,324	51,915	51,220	52,403
Male	47,804	52,239	53,158	53,694	54,628	54,409	54,606
Total	93,349	101,821	103,415	104,018	106,543	105,629	107,009

Sex/Year	1862	1863	1864	1865	1866	1867	1868	1869
Female	51,812	53,115	54,959	54,850	55,307	55,527	56,292	55,033
Male	55,257	56,226	57,374	58,220	58,360	58,517	59,222	58,321
Total	107,069	109,341	112,333	113,070	113,667	114,044	115,514	113,354

Sex/Year	1871	1870	1872	1871	1872	1827	1874	1875
Female	56,099	56,431	57,472	56,099	57,472	58,233	60,109	60,146
Male	60,029	58,959	61,293	60,029	61,293	61,467	63,602	63,432
Total	116,128	115,390	118,765	116,128	118,765	119,700	123,711	123,578

Answer

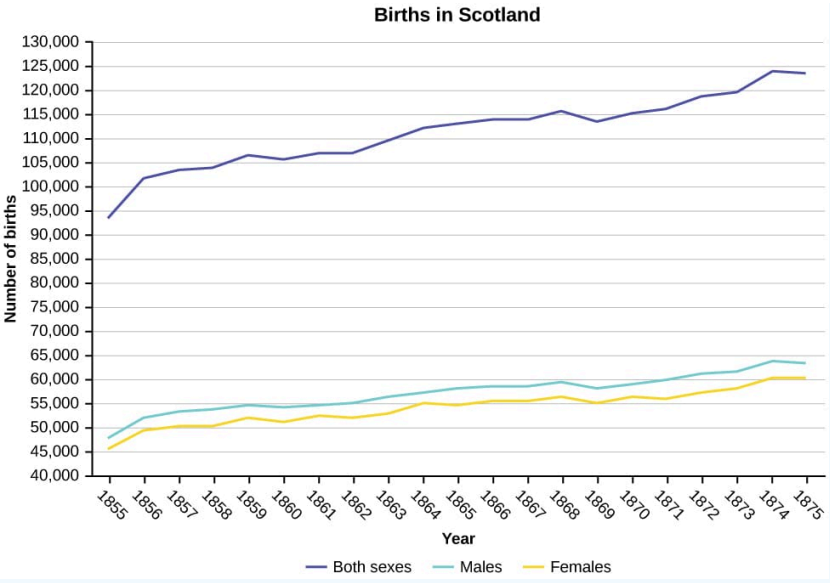


Figure $10.E.12$:

Exercise 2.3.16

The following data sets list full time police per 100,000 citizens along with homicides per 100,000 citizens for the city of Detroit, Michigan during the period from 1961 to 1973.

Year	1961	1962	1963	1964	1965	1966	1967
Police	260.35	269.8	272.04	272.96	272.51	261.34	268.89
Homicides	8.6	8.9	8.52	8.89	13.07	14.57	21.36

Year	1968	1969	1970	1971	1972	1973
Police	295.99	319.87	341.43	356.59	376.69	390.19
Homicides	28.03	31.49	37.39	46.26	47.24	52.33

a. Construct a double time series graph using a common *x*-axis for both sets of data.
b. Which variable increased the fastest? Explain.
c. Did Detroit's increase in police officers have an impact on the murder rate? Explain.

2.4: Measures of the Location of the Data

Q 2.4.1

The median age for U.S. blacks currently is 30.9 years; for U.S. whites it is 42.3 years.

a. Based upon this information, give two reasons why the black median age could be lower than the white median age.
b. Does the lower median age for blacks necessarily mean that blacks die younger than whites? Why or why not?
c. How might it be possible for blacks and whites to die at approximately the same age, but for the median age for whites to be higher?

Q 2.4.2

Six hundred adult Americans were asked by telephone poll, "What do you think constitutes a middle-class income?" The results are in the Table below. Also, include left endpoint, but not the right endpoint.

Salary ($)	Relative Frequency
< 20,000	0.02
20,000–25,000	0.09
25,000–30,000	0.19
30,000–40,000	0.26
40,000–50,000	0.18
50,000–75,000	0.17
75,000–99,999	0.02
100,000+	0.01

a. What percentage of the survey answered "not sure"?
b. What percentage think that middle-class is from $25,000 to $50,000?
c. Construct a histogram of the data.
   i. Should all bars have the same width, based on the data? Why or why not?
   ii. How should the <20,000 and the 100,000+ intervals be handled? Why?
d. Find the 40th and 80th percentiles
e. Construct a bar graph of the data

S 2.4.2

a. $1 - (0.02 + 0.09 + 0.19 + 0.26 + 0.18 + 0.17 + 0.02 + 0.01) = 0.06$
b. $0.19 + 0.26 + 0.18 = 0.63$
c. Check student's solution.
d. 40th percentile will fall between 30,000 and 40,000

   80th percentile will fall between 50,000 and 75,000

e. Check student's solution.

Q 2.4.3

Given the following box plot:

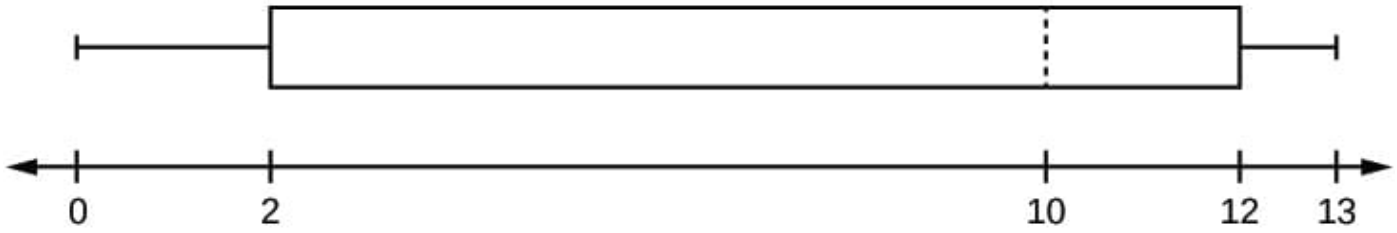


Figure $10.E.13$:

a. which quarter has the smallest spread of data? What is that spread?
b. which quarter has the largest spread of data? What is that spread?
c. find the interquartile range (*IQR*).
d. are there more data in the interval 5–10 or in the interval 10–13? How do you know this?
e. which interval has the fewest data in it? How do you know this?
   i. 0–2
   ii. 2–4
   iii. 10–12
   iv. 12–13

v. need more information

Q 2.4.4

The following box plot shows the U.S. population for 1990, the latest available year.

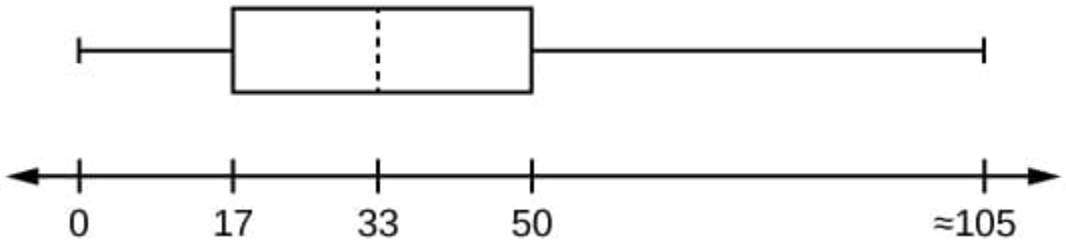


Figure $10.E.14$:

a. Are there fewer or more children (age 17 and under) than senior citizens (age 65 and over)? How do you know?
b. 12.6% are age 65 and over. Approximately what percentage of the population are working age adults (above age 17 to age 65)?

S 2.4.4

a. more children; the left whisker shows that 25% of the population are children 17 and younger. The right whisker shows that 25% of the population are adults 50 and older, so adults 65 and over represent less than 25%.
b. 62.4%

2.5: Box Plots

Q 2.5.1

In a survey of 20-year-olds in China, Germany, and the United States, people were asked the number of foreign countries they had visited in their lifetime. The following box plots display the results.

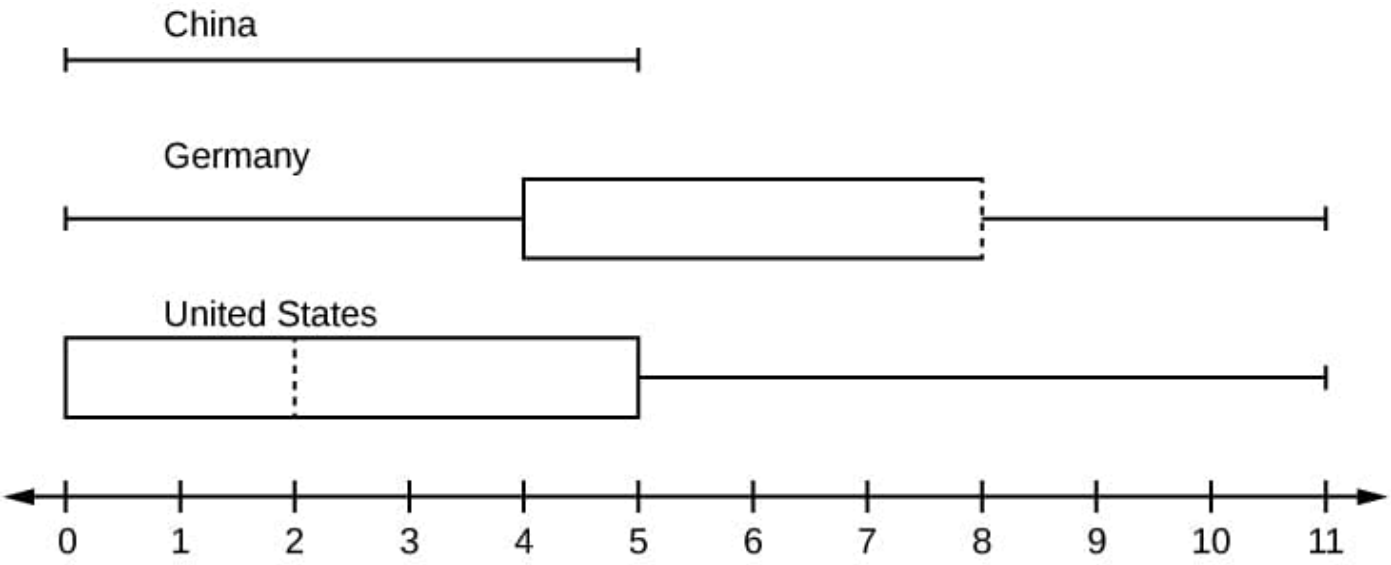


Figure $10.E.15$:

a. In complete sentences, describe what the shape of each box plot implies about the distribution of the data collected.
b. Have more Americans or more Germans surveyed been to over eight foreign countries?
c. Compare the three box plots. What do they imply about the foreign travel of 20-year-old residents of the three countries

when compared to each other?

Q 2.5.2

Given the following box plot, answer the questions.

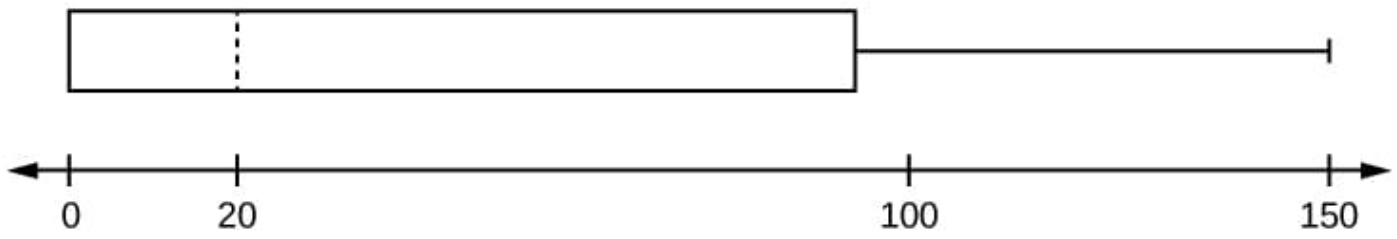


Figure $10.E.16$:

a. Think of an example (in words) where the data might fit into the above box plot. In 2–5 sentences, write down the example.
b. What does it mean to have the first and second quartiles so close together, while the second to third quartiles are far apart?

S 2.5.2

a. Answers will vary. Possible answer: State University conducted a survey to see how involved its students are in community service. The box plot shows the number of community service hours logged by participants over the past year.
b. Because the first and second quartiles are close, the data in this quarter is very similar. There is not much variation in the values. The data in the third quarter is much more variable, or spread out. This is clear because the second quartile is so far away from the third quartile.

Q 2.5.3

Given the following box plots, answer the questions.

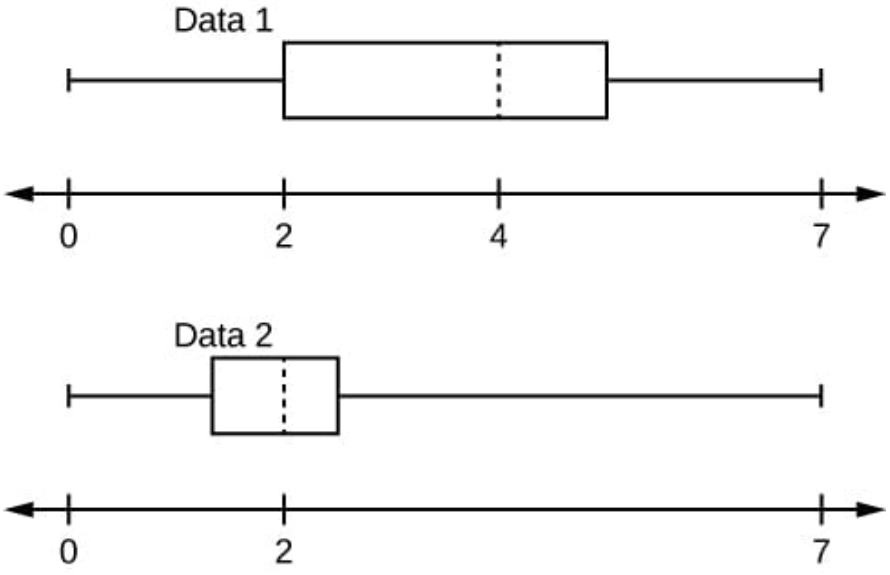


Figure $10.E.17$:

a. In complete sentences, explain why each statement is false.
   i. **Data 1** has more data values above two than **Data 2** has above two.
   ii. The data sets cannot have the same mode.
   iii. For **Data 1**, there are more data values below four than there are above four.
b. For which group, Data 1 or Data 2, is the value of “7” more likely to be an outlier? Explain why in complete sentences.

Q 2.5.4

A survey was conducted of 130 purchasers of new BMW 3 series cars, 130 purchasers of new BMW 5 series cars, and 130 purchasers of new BMW 7 series cars. In it, people were asked the age they were when they purchased their car. The following box plots display the results.

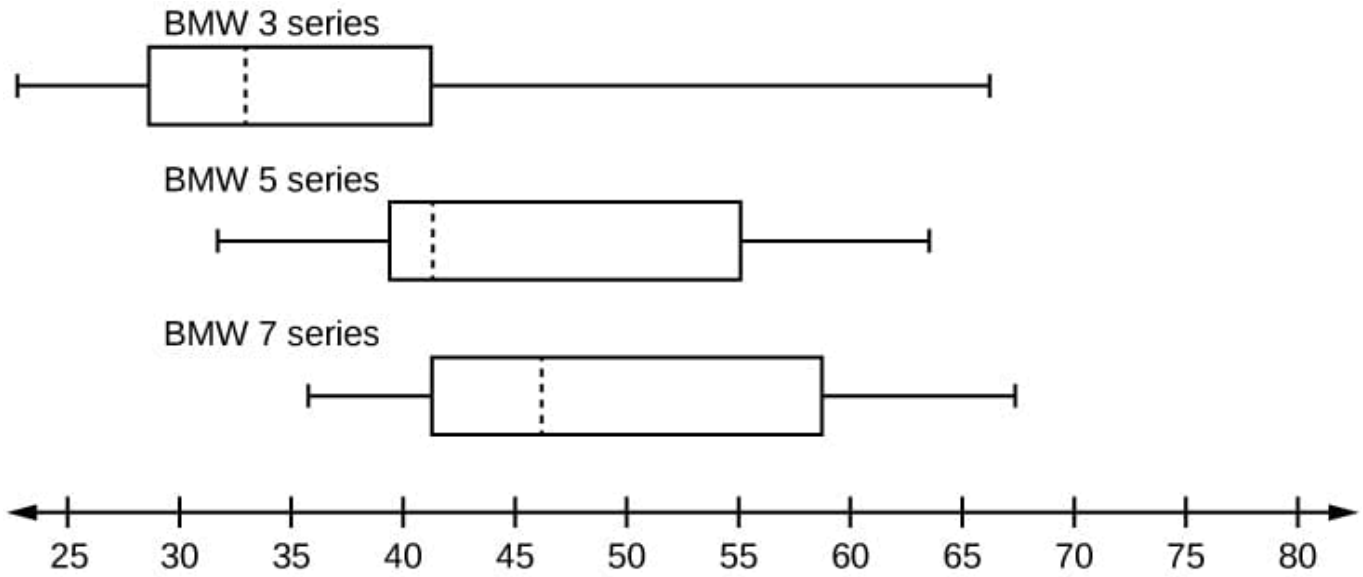


Figure 10.*E*.18:

a. In complete sentences, describe what the shape of each box plot implies about the distribution of the data collected for that car series.
b. Which group is most likely to have an outlier? Explain how you determined that.
c. Compare the three box plots. What do they imply about the age of purchasing a BMW from the series when compared to each other?
d. Look at the BMW 5 series. Which quarter has the smallest spread of data? What is the spread?
e. Look at the BMW 5 series. Which quarter has the largest spread of data? What is the spread?
f. Look at the BMW 5 series. Estimate the interquartile range (IQR).
g. Look at the BMW 5 series. Are there more data in the interval 31 to 38 or in the interval 45 to 55? How do you know this?
h. Look at the BMW 5 series. Which interval has the fewest data in it? How do you know this?
    i. 31–35
    ii. 38–41
    iii. 41–64

S 2.5.4

a. Each box plot is spread out more in the greater values. Each plot is skewed to the right, so the ages of the top 50% of buyers are more variable than the ages of the lower 50%.
b. The BMW 3 series is most likely to have an outlier. It has the longest whisker.
c. Comparing the median ages, younger people tend to buy the BMW 3 series, while older people tend to buy the BMW 7 series. However, this is not a rule, because there is so much variability in each data set.
d. The second quarter has the smallest spread. There seems to be only a three-year difference between the first quartile and the median.
e. The third quarter has the largest spread. There seems to be approximately a 14-year difference between the median and the third quartile.
f. *IQR* ~ 17 years
g. There is not enough information to tell. Each interval lies within a quarter, so we cannot tell exactly where the data in that quarter is concentrated.
h. The interval from 31 to 35 years has the fewest data values. Twenty-five percent of the values fall in the interval 38 to 41, and 25% fall between 41 and 64. Since 25% of values fall between 31 and 38, we know that fewer than 25% fall between 31 and

35.

Q 2.5.5

Twenty-five randomly selected students were asked the number of movies they watched the previous week. The results are as follows:

# of movies	Frequency
0	5
1	9
2	6
3	4
4	1

Construct a box plot of the data.

2.6: Measures of the Center of the Data

Q 2.6.1

The most obese countries in the world have obesity rates that range from 11.4% to 74.6%. This data is summarized in the following table.

Percent of Population Obese	Number of Countries
11.4–20.45	29
20.45–29.45	13
29.45–38.45	4
38.45–47.45	0
47.45–56.45	2
56.45–65.45	1
65.45–74.45	0
74.45–83.45	1

a. What is the best estimate of the average obesity percentage for these countries?
b. The United States has an average obesity rate of 33.9%. Is this rate above average or below?
c. How does the United States compare to other countries?

Q 2.6.2

The table below gives the percent of children under five considered to be underweight. What is the best estimate for the mean percentage of underweight children?

Percent of Underweight Children	Number of Countries
16–21.45	23
21.45–26.9	4
26.9–32.35	9

Percent of Underweight Children	Number of Countries
32.35–37.8	7
37.8–43.25	6
43.25–48.7	1

S 2.6.2

The mean percentage, $\bar{x} = \frac{1328.65}{50} = 26.75$

2.7: Skewness and the Mean, Median, and Mode

Q 2.7.1

The median age of the U.S. population in 1980 was 30.0 years. In 1991, the median age was 33.1 years.

a. What does it mean for the median age to rise?
b. Give two reasons why the median age could rise.
c. For the median age to rise, is the actual number of children less in 1991 than it was in 1980? Why or why not?

2.8: Measures of the Spread of the Data

Use the following information to answer the next nine exercises: The population parameters below describe the full-time equivalent number of students (FTES) each year at Lake Tahoe Community College from 1976–1977 through 2004–2005.

- $\mu = 1000$ FTES
- median = 1,014 FTES
- $\sigma = 474$ FTES
- first quartile = 528.5 FTES
- third quartile = 1,447.5 FTES
- $n = 29$ years

Q 2.8.1

A sample of 11 years is taken. About how many are expected to have a FTES of 1014 or above? Explain how you determined your answer.

S 2.8.1

The median value is the middle value in the ordered list of data values. The median value of a set of 11 will be the 6th number in order. Six years will have totals at or below the median.

Q 2.8.2

75% of all years have an FTES:

a. at or below: _____
b. at or above: _____

Q 2.8.3

The population standard deviation = _____

S 2.8.3

474 FTES

Q 2.8.4

What percent of the FTES were from 528.5 to 1447.5? How do you know?

Q 2.8.5

What is the *IQR*? What does the *IQR* represent?

S 2.8.5

919

Q 2.8.6

How many standard deviations away from the mean is the median?

Additional Information: The population FTES for 2005–2006 through 2010–2011 was given in an updated report. The data are reported here.

Year	2005–06	2006–07	2007–08	2008–09	2009–10	2010–11
Total FTES	1,585	1,690	1,735	1,935	2,021	1,890

Q 2.8.7

Calculate the mean, median, standard deviation, the first quartile, the third quartile and the*IQR*. Round to one decimal place.

S 2.8.7

- mean = 1,809.3
- median = 1,812.5
- standard deviation = 151.2
- first quartile = 1,690
- third quartile = 1,935
- *IQR* = 245

Q 2.8.8

Construct a box plot for the FTES for 2005–2006 through 2010–2011 and a box plot for the FTES for 1976–1977 through 2004–2005.

Q 2.8.9

Compare the *IQR* for the FTES for 1976–77 through 2004–2005 with the *IQR* for the FTES for 2005-2006 through 2010–2011. Why do you suppose the *IQR*s are so different?

S 2.8.10

Hint: Think about the number of years covered by each time period and what happened to higher education during those periods.

Q 2.8.11

Three students were applying to the same graduate school. They came from schools with different grading systems. Which student had the best GPA when compared to other students at his school? Explain how you determined your answer.

Student	GPA	School Average GPA	School Standard Deviation
Thuy	2.7	3.2	0.8
Vichet	87	75	20
Kamala	8.6	8	0.4

Q 2.8.12

A music school has budgeted to purchase three musical instruments. They plan to purchase a piano costing $3,000, a guitar costing $550, and a drum set costing $600. The mean cost for a piano is $4,000 with a standard deviation of $2,500. The mean cost for a guitar is $500 with a standard deviation of $200. The mean cost for drums is $700 with a standard deviation of $100. Which cost is the lowest, when compared to other instruments of the same type? Which cost is the highest when compared to other instruments of the same type. Justify your answer.

S 2.8.12

For pianos, the cost of the piano is 0.4 standard deviations BELOW the mean. For guitars, the cost of the guitar is 0.25 standard deviations ABOVE the mean. For drums, the cost of the drum set is 1.0 standard deviations BELOW the mean. Of the three, the drums cost the lowest in comparison to the cost of other instruments of the same type. The guitar costs the most in comparison to the cost of other instruments of the same type.

Q 2.8.13

An elementary school class ran one mile with a mean of 11 minutes and a standard deviation of three minutes. Rachel, a student in the class, ran one mile in eight minutes. A junior high school class ran one mile with a mean of nine minutes and a standard deviation of two minutes. Kenji, a student in the class, ran 1 mile in 8.5 minutes. A high school class ran one mile with a mean of seven minutes and a standard deviation of four minutes. Nedda, a student in the class, ran one mile in eight minutes.

a. Why is Kenji considered a better runner than Nedda, even though Nedda ran faster than he?
b. Who is the fastest runner with respect to his or her class? Explain why.

Q 2.8.14

The most obese countries in the world have obesity rates that range from 11.4% to 74.6%. This data is summarized in the table belo2

Percent of Population Obese	Number of Countries
11.4–20.45	29
20.45–29.45	13
29.45–38.45	4
38.45–47.45	0
47.45–56.45	2
56.45–65.45	1
65.45–74.45	0
74.45–83.45	1

What is the best estimate of the average obesity percentage for these countries? What is the standard deviation for the listed obesity rates? The United States has an average obesity rate of 33.9%. Is this rate above average or below? How "unusual" is the United States' obesity rate compared to the average rate? Explain.

S 2.8.14

- $\bar{x} = 23.32$
- Using the TI 83/84, we obtain a standard deviation of: $s_x = 12.95$.
- The obesity rate of the United States is 10.58% higher than the average obesity rate.
- Since the standard deviation is 12.95, we see that $23.32 + 12.95 = 36.27$ is the obesity percentage that is one standard deviation from the mean. The United States obesity rate is slightly less than one standard deviation from the mean. Therefore, we can assume that the United States, while 34% obese, does not have an unusually high percentage of obese people.

Q 2.8.15

The Table below gives the percent of children under five considered to be underweight.

Percent of Underweight Children	Number of Countries
16–21.45	23
21.45–26.9	4
26.9–32.35	9
32.35–37.8	7
37.8–43.25	6
43.25–48.7	1

What is the best estimate for the mean percentage of underweight children? What is the standard deviation? Which interval(s) could be considered unusual? Explain.

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11: GEOMETRY

> Geometry is the art of good reasoning from bad drawings. – Henri Poincaré

11.1: Introduction

11.2: Tangrams

11.3: Triangles and Quadrilaterals

11.4: Polygons

11.5: Platonic Solids

11.6: Painted Cubes

11.7: Symmetry

11.8: Geometry in Art and Science

11.9: Problem Bank

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11.1: Introduction

The word “geometry” comes from the ancient Greek words “geo” meaning Earth and “metron” meaning measurement. It is probably the oldest field of mathematics, because of its usefulness in calculating lengths, areas, and volumes of everyday objects.

The study of geometry has evolved a great deal during the last 3,000 years or so. Like all of mathematics, what’s really important in geometry is reasoning, making sense of problems, and justifying your solutions.

The mathematician Henri Poincaré said that

> *Geometry is the art of good reasoning from bad drawings.*

This insight should guide your study in this chapter. You should never trust a drawing. You might find that one line segment appears to be longer than another, or an angle looks like it might be 90 degrees. But “appears to be” and “looks like” are simply not good enough. You have to reason through the situation and figure out what you know for sure and why you know it.

Think / Pair / Share

Reflect on your learning of geometry in the past. What is geometry really about? Also think about these questions:

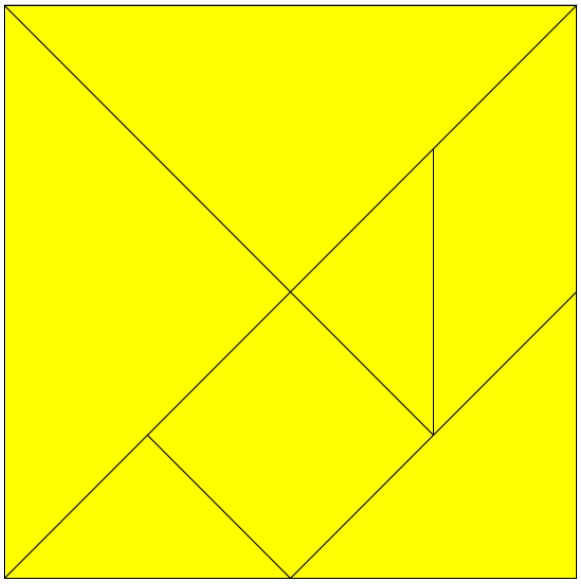
- What is a point?
- What is a line? A segment? A ray?
- What is a plane?
- What is a circle?
- What is an angle?
- Which of these basic objects can be measured? How are they measured? What kinds of tools are useful?

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11.2: Tangrams

Tangrams are a seven-piece geometric puzzle that dates back at least to the Song Dynasty in China (about 1100 AD). Below[1] you will find the seven puzzle pieces. Make a careful copy (a photocopy or printout is best), cut out the puzzle pieces, and then use them to solve the problems in this section.

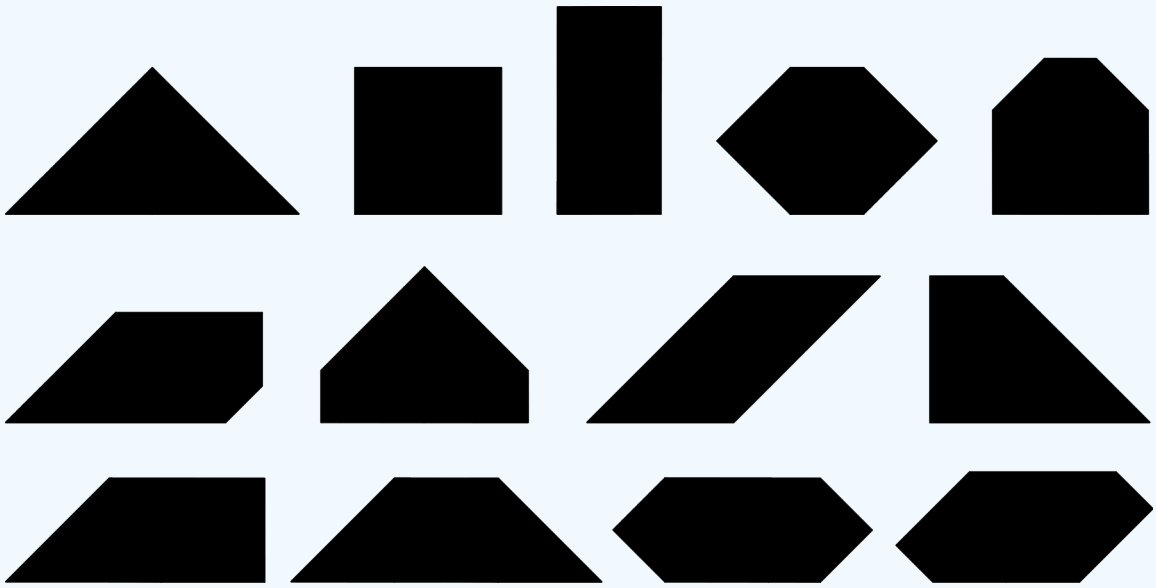


Whenever you solve a tangram puzzle, your job is to **use all seven pieces to form the shape**. They should fit together like puzzle pieces, sitting flat on the table; **no overlapping** of the pieces is allowed.

You can trace around your solutions to remember what you have done and to have a record of your work.

Problem 1

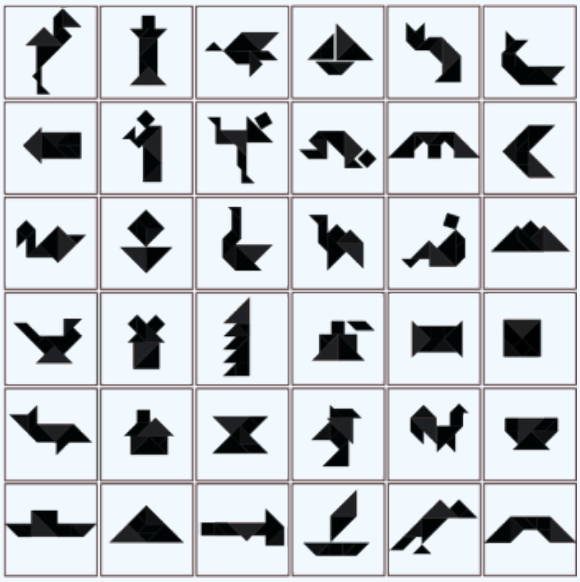
Use your tangram pieces to build the following designs[2]. How many can you make?



(These are all separate challenges. Each one requires all seven pieces. Once you solve one, trace your solution. Then try to solve another one.)

Problem 2

Use your tangram pieces to build the following designs[3]. How many can you make?



(These are all separate challenges. Each one requires all seven pieces. Once you solve one, trace your solution. Then try to solve another one.)

Think / Pair / Share

- Which tangram problems were easier and which were harder: making “real life” objects like cats and people, or purely mathematical objects like the rectangle?
- What do you think made one kind of problem easier or harder?

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2. Tangram puzzles from Wikimedia Commons, public domain. ↵
3. Tangram puzzles from pixababy.com,CC0 Creative Commons. ↵

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11.3: Triangles and Quadrilaterals

Think / Pair / Share

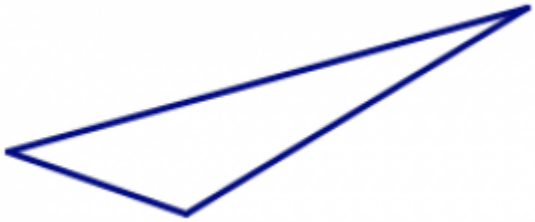
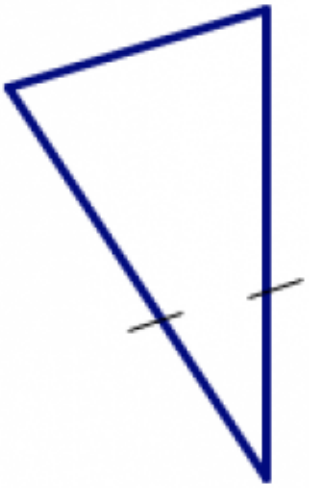
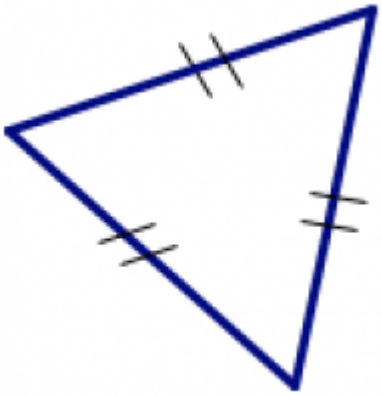
Follow these directions on your own:

- Draw any triangle on your paper.
- Draw a second triangle that is different in some way from your first one. Write down a sentence or two to say how it is different.
- Draw a third triangle that is different from both of your other two. Describe how it is different.
- Draw two more triangles, different from all the ones that came before.

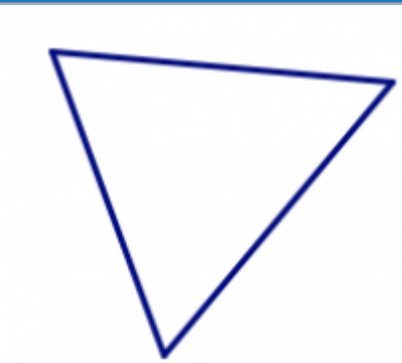
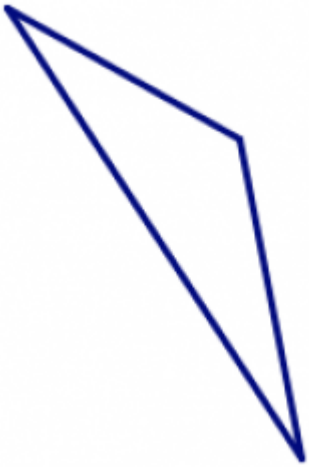
Compare your triangles and descriptions with a partner. To make "different" triangles, you have to change some feature of the triangle. Make a list of the features that you or your partner changed.

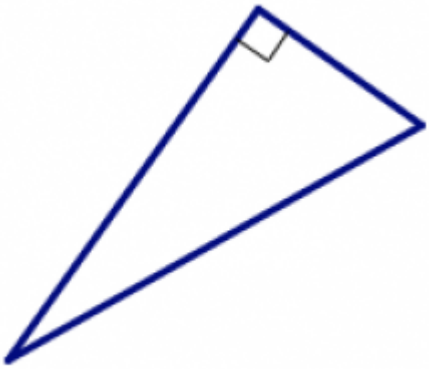
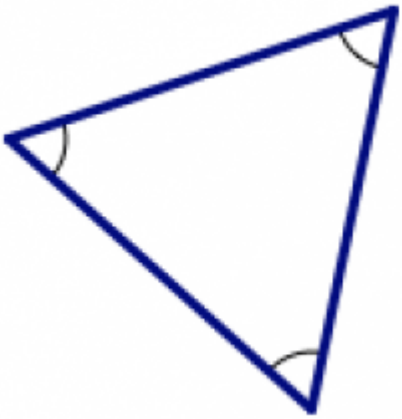
Triangles are classified according to different properties. The point of learning geometry is not to learn a lot of vocabulary, but it's useful to use the correct terms for objects, so that we can communicate clearly. Here's a quick dictionary of some types of triangles.

Classification by sides

scalene	isosceles	equilateral
		
all sides have different lengths	*two sides have the same length*	*all three sides have the same length*

Classification by angles

acute	obtuse
	
all interior angles measure less than 90°	*one interior angle measures more than 90°*

right	equiangular
	
one interior angle measures exactly 90°	*all interior angles have the same measure*

Remember that “geometry is the art of good reasoning from bad drawings.” That means you can’t always trust your eyes. If you look at a picture of a triangle and one side looks like it’s longer than another, that may just mean the drawing was done a bit sloppily.

Notation: Tick marks

Mathematicians either write down measurements or use tick marks to indicate when sides and angles are supposed to be equal.

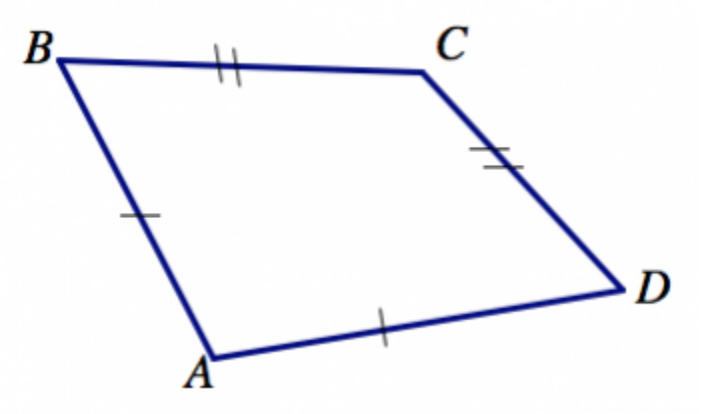
If two sides have the same measurement or the same number of tick marks, you must believe they are equal and work out the problem accordingly, *even if it doesn't look that way to your eyes*.

You can see examples of these in some of the pictures above. Another example is the little square used to indicate a right angle in the picture of the right triangle.

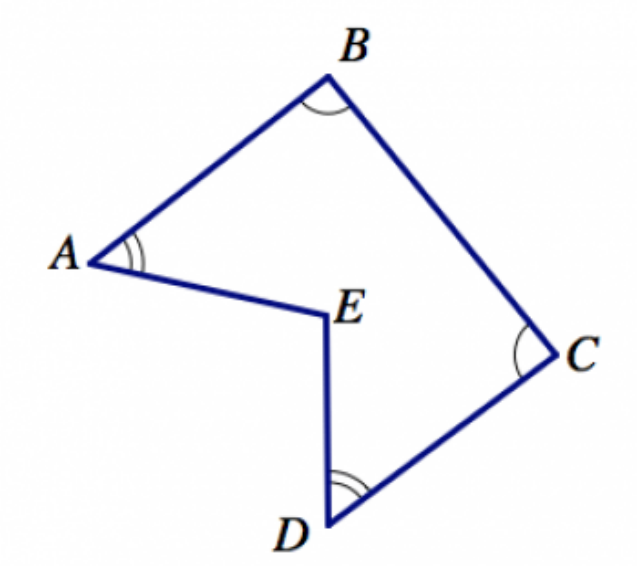
On Your Own

Work on the following exercises on your own or with a partner.

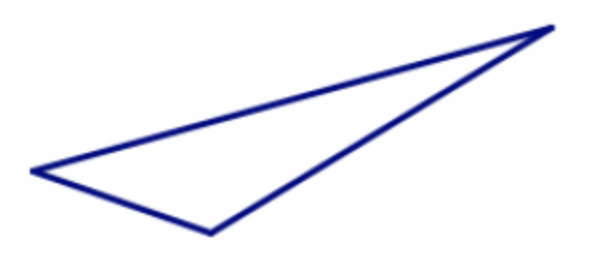
1. In the picture below, which sides are understood to have the same length (even if it doesn't look that way in the drawing)?



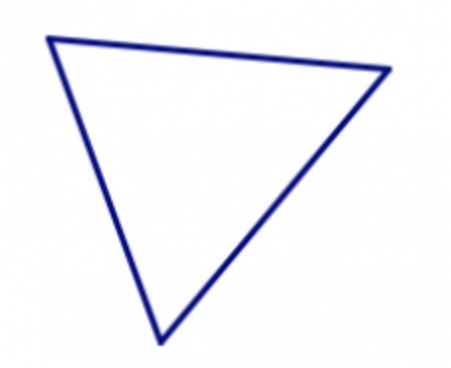
2. In the picture below, which angles are understood to have the same measure (even if if doesn't look that way in the drawing)?



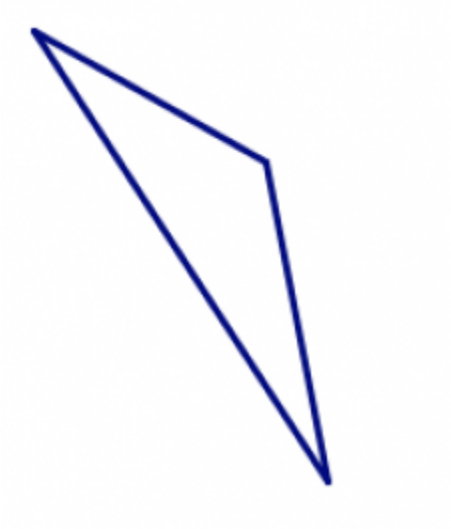
3. Here is a scalene triangle. Sketch two more scalene triangles, each of which is different from the one shown here in some way.



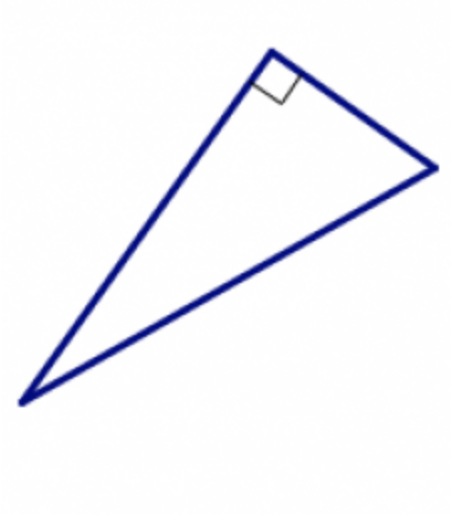
4. Here is an acute triangle. Sketch two more acute triangles, each of which is different from the one shown here in some way.



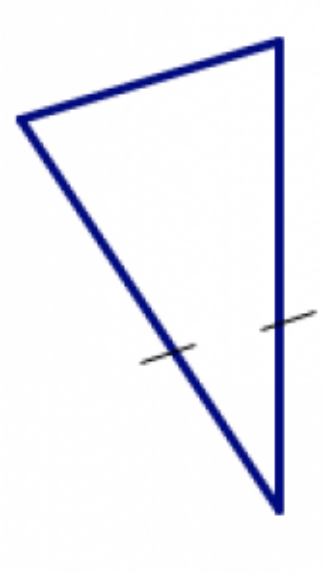
5. Here is an obtuse triangle. Sketch two more obtuse triangles, each of which is different from the one shown here in some way.



6. Here is a right triangle. Sketch two more right triangles, each of which is different from the one shown here in some way. Be sure to indicate which angle is 90°.



7. Here is an isosceles triangle. Sketch two more isosceles triangles, each of which is different from the one shown here in some way. Use tick marks to indicate which sides are equal.

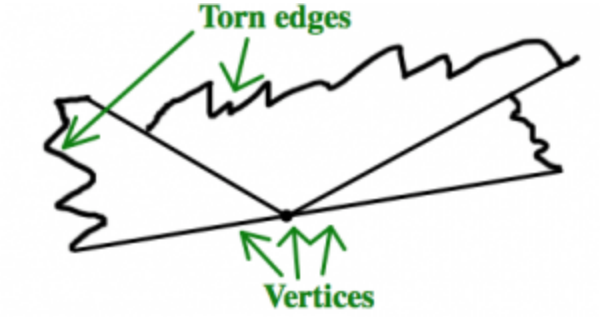


Angle Sum

Think / Pair / Share

By now, you have drawn several different triangles on your paper. Choose one of your triangles, and follow these directions:

- Using scissors, cut the triangle out.
- Tear (do not cut) off the corners, and place the three vertices together. Your should have something that looks a bit like this picture:



What do you notice? What does this suggest about the angles in a triangle?

You may remember learning that the sum of the angles in any triangle is 180°. In your class, you now have lots of examples of triangles where the sum of the angles seems to be 180°. But remember, our drawings are not exact. How can we be sure that our eyes are not deceiving us? How can we be sure that the sum of the angles in a triangle isn't 181° or 178°, but is really 180° on the nose in every case?

Think / Pair / Share

What would convince you beyond all doubt that the sum of the angles in any triangle is 180°? Would testing lots of cases be enough? How many is enough? Could you ever test every possible triangle?

History: Euclid's axioms

Often high school geometry teachers prove the sum of the angles in a triangle is 180°, usually using some facts about parallel lines. But (maybe surprisingly?) it's just as good to take this as an *axiom*, as a given fact about how geometry works, and go from there. Perhaps this is less satisfying than proving it from some other statement, and if you're curious you can certainly find a proof or your instructor can share one with you.

Note

In about 300BC, Euclid[1] was the first mathematician (as far as we know) who tried to write down careful axioms and then build from those axioms rigorous proofs of mathematical truths.



Euclid

Euclid had five axioms for geometry, the first four of which seemed pretty obvious to mathematicians. People felt they were reasonable assumptions from which to build up geometric truths:

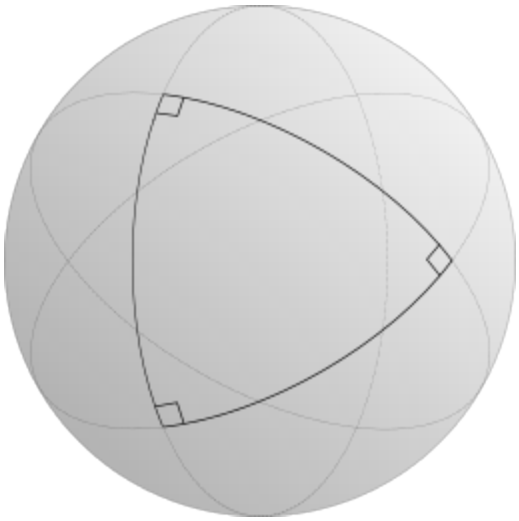
1. Given two points, you can connect them with a straight line segment.
2. Given a line segment, you can extend it as far as you like in either direction, making a line.
3. Given a line segment, you can draw a circle having that segment as a radius.
4. All right angles are congruent.

The fifth postulate bothered people a bit more. It was originally stated in more flowery language, but it was equivalent to this statement:

5. The sum of the angles in a triangle is 180°.

It's easy to see why this fifth axiom caused such a ruckus in mathematics. It seemed much less obvious than the other four, and mathematicians felt like they were somehow cheating if they just assumed it rather than proving it had to be true. Many mathematicians spent many, many years trying to prove this fifth axiom from the other axioms, but they couldn't do it. And with good reason: There are other kinds of geometries where the first four axioms are true, but the fifth one is not!

For example, if you do geometry on a sphere — like a basketball or more importantly on the surface of the Earth — rather than on a flat plane, the first four axioms are true. But triangles are a little strange on the surface of the earth. Every triangle you can draw on the surface of the earth has an angle sum strictly *greater than* 180°. In fact, you can draw a triangle on the Earth that has three right angles[2], making an angle sum of 270°.



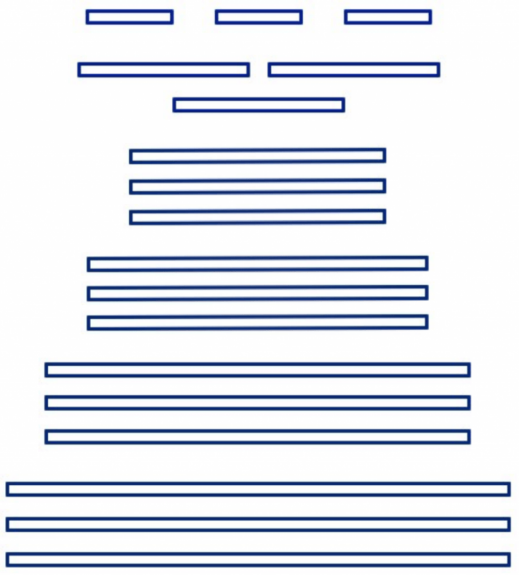
Triangle with three right angles on a sphere.

On a sphere like the Earth, the angle sum is not constant among all triangles. Bigger triangles have bigger angle sums, and smaller triangles have smaller angle sums, but even tiny triangles have angle sums that are greater than 180°.

The geometry you study in school is called *Euclidean geometry*; it is the geometry of a flat plane, of a flat world. It's a pretty good approximation for the little piece of the Earth that we see at any given time, but it's not the only geometry out there!

Triangle Inequality

Make a copy of these strips of paper and cut them out. They have lengths from 1 unit to 6 units. You may want to color the strips, write numbers on them, or do something that makes it easy to keep track of the different lengths.



Problem 3

Repeat the following process several times (at least 10) and keep track of the results (a table has been started for you).

- Pick three strips of paper. (The lengths do not have to be all different from each other; that's why you have multiple copies of each length.)
- Try to make a triangle with those three strips, and decide if you think it is possible or not. (Don't overlap the strips, cut them, or fold them. The length of the strips should be the length of the sides of the triangle.)

Length 1	Length 2	Length 3	Triangle?
4	3	2	yes
4	2	1	no
4	2	2	??

Your goal is to come up with a **rule that describes when three lengths will make a triangle** and when they will not. Write down the rule in your own words.

Think / Pair / Share

Compare your rule with other students. Then use your rule to answer the following questions. Keep in mind the goal is not to try to build the triangle, but to **predict the outcome** based on your rule.

- Suppose you were asked to make a triangle with sides 40 units, 40 units, and 100 units long. Do you think you could do it? Explain your answer.
- Suppose you were asked to make a triangle with sides 2.5 units, 2.6 units, and 5 units long. Do you think you could do it? Explain your answer.

You probably came up with some version of this statement:

Theorem: Triangle Inequality

The sum of the lengths of two sides in a triangle is greater than the length of the third side.

Of course, we know that in geometry we should not believe our eyes. You need to look for an *explanation*. Why does your statement make sense?

Remember that "geometry is the art of good reasoning from bad drawings." Our materials weren't very precise, so how can we be sure this rule we've come up with is is correct?

Well in this case, the rule is really just the same as the saying "the shortest distance between two points is a straight line." In fact, this is exactly what we mean by the words *straight line* in geometry.

SSS Congruence

We say that two triangles (or any two geometric objects) are *congruent* if they are exactly the same shape and the same size. That means that if you could pick one of them up and move it to put down on the other, they would exactly overlap.

Problem 4

Repeat the following process several times and keep track of the results.

- Pick three strips of paper that will definitely form a triangle.
- Try to make two *different* (non-congruent) triangles with the same three strips of paper. Record if you were able to do so.

Problem 5

Repeat the following process several times and keep track of the results.

- Pick four strips of paper and form a quadrilateral with them. (If your four strips do not form a quadrilateral, pick another four strips.)
- Try to make two *different* (non-congruent) quadrilaterals with the same four strips of paper. Record if you were able to do so.

Think / Pair / Share

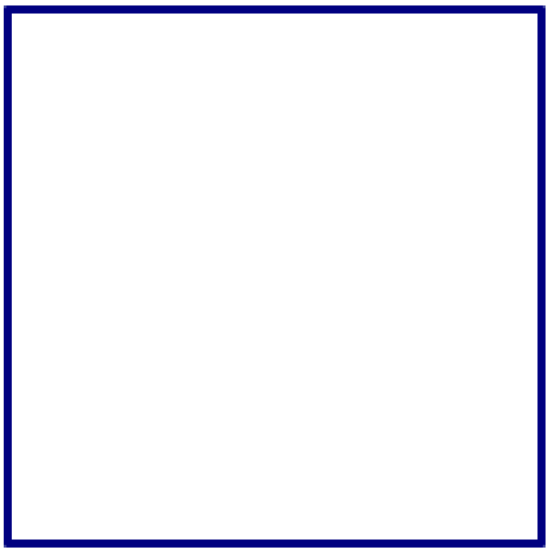
What do you notice from Problems 4 and 5? Can you make a general statement to describe what's going on? Can you explain why your statement makes sense?

You probably came up with some version of this statement:

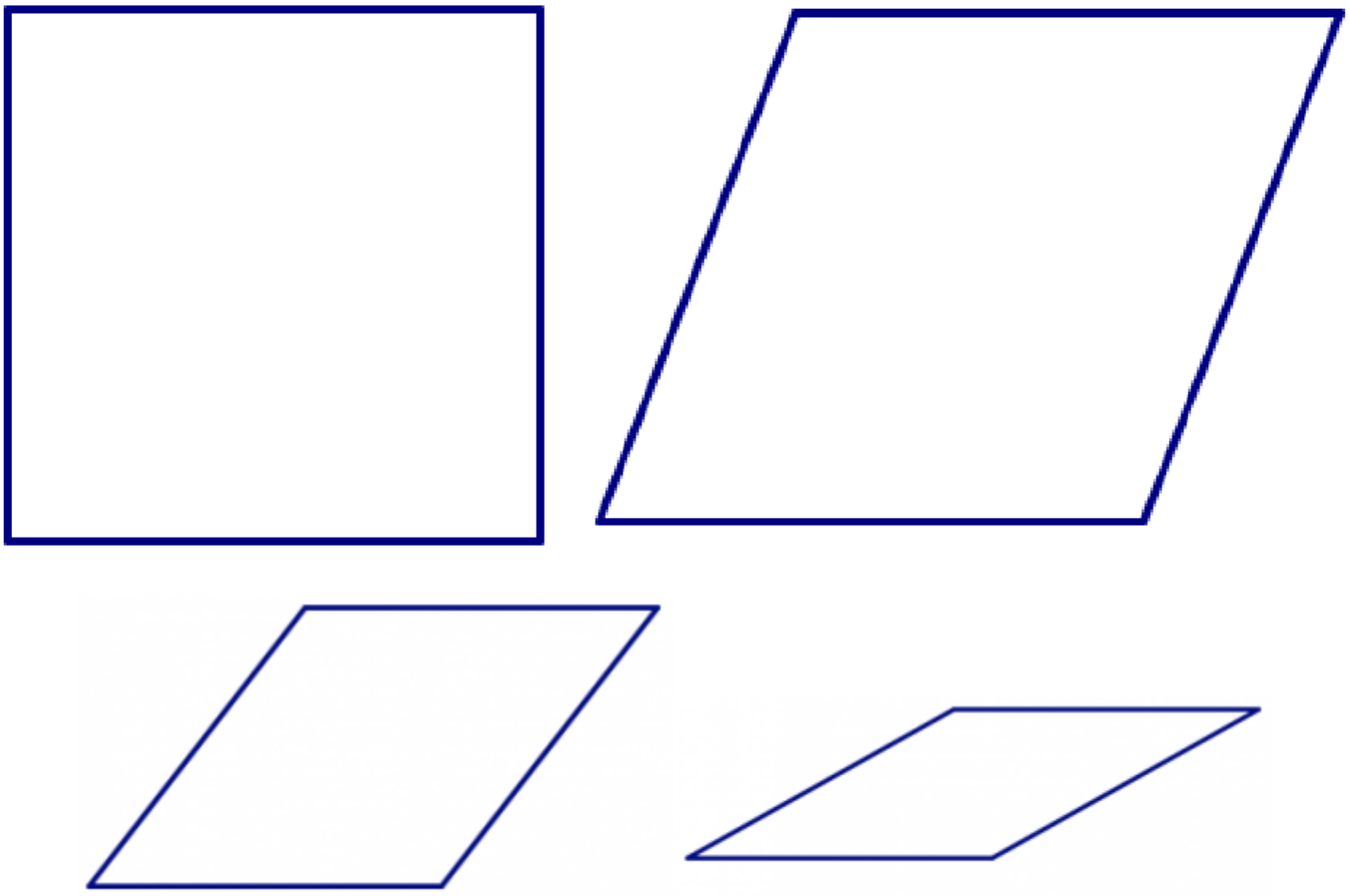
Theorem: SSS (side-side-side) Congruence

If two triangles have the same side lengths, then the triangles are congruent.

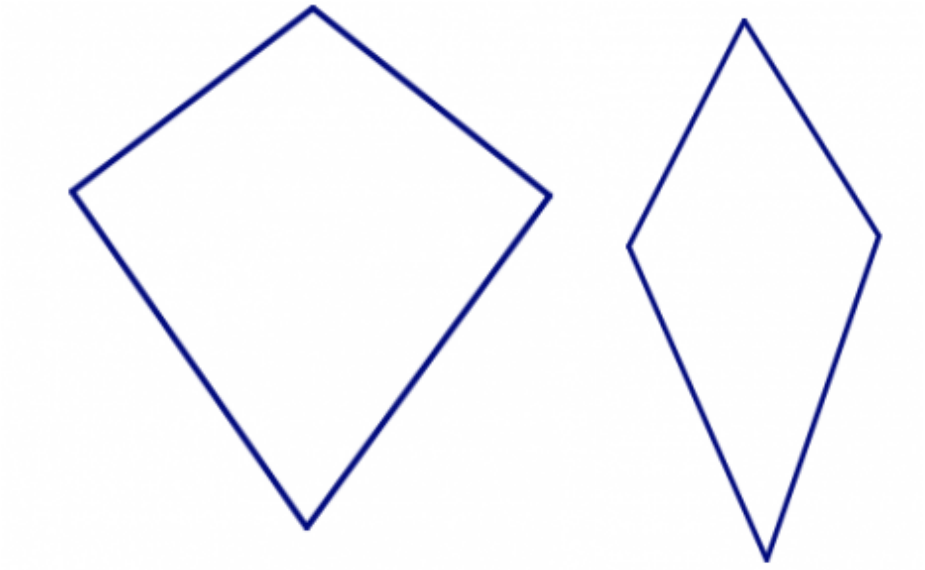
This most certainly is not true for quadrilaterals. For example, if you choose four strips that are all the same length, you can make a square:



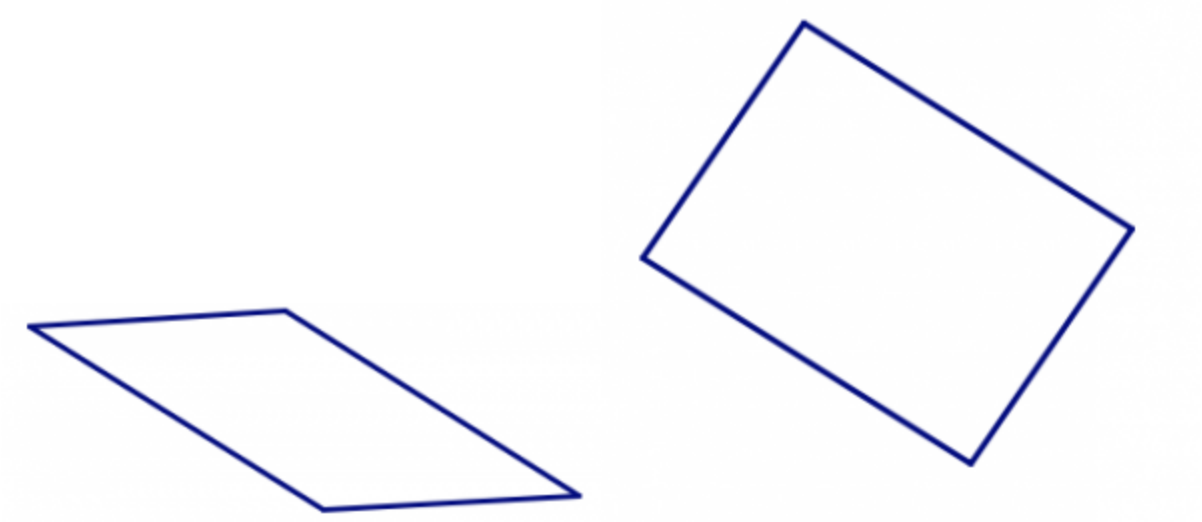
But you can also squish that square into a non-square rhombus. (Try it!)



If you don't choose four lengths that are all the same, in addition to "squishing" the shape, you can rearrange the sides to make different (non-congruent) shapes. (Try it!)



These two quadrilaterals have the same four side lengths in the same order.



These two quadrilaterals have the same four side lengths as the two above, but the sides are in a different order.

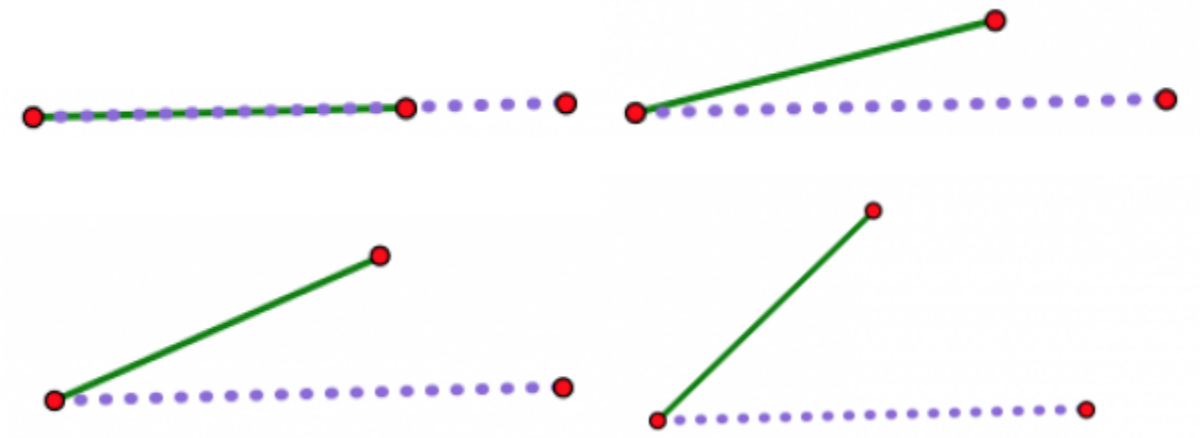
But this can't happen with triangles. Why not? Well, certainly you can't rearrange the three sides. That would be just the same as rotating the triangle or flipping it over, but not making a new shape.

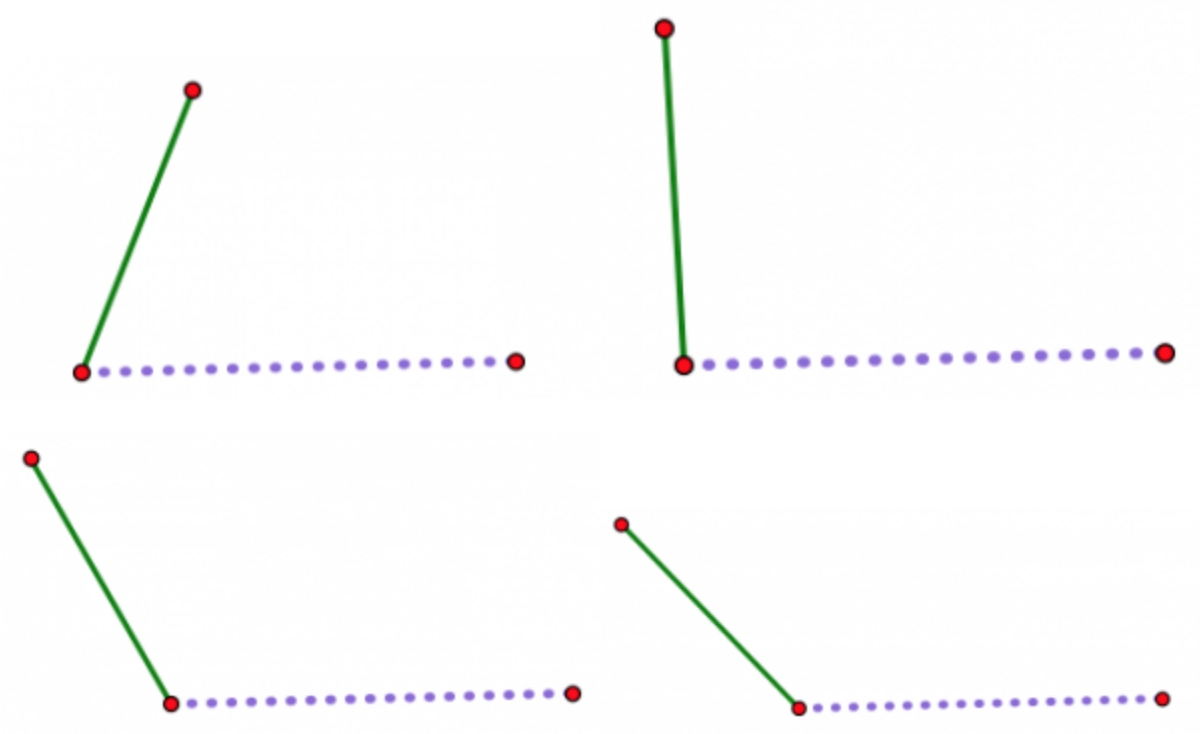
Why can't the triangles "squish" the way a quadrilateral (and other shapes) can? Here's one way to understand it. Imagine you pick two of your three lengths and lay them on top of each other, hinged at one corner.



This shows a longer purple dashed segment and a shorter green segment. The two segments are hinged at the red dot on the left.

Now imagine opening up the hinge a little at a time.





As the hinge opens up, the two non-hinged endpoints get farther and farther apart. Whatever your third length is (assuming you are actually able to make a triangle with your three lengths), there is **exactly one position** of the hinge where it will just exactly fit to close off the triangle. No other position will work.

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1. Portrait of Euclid from Wikimedia Commons, licensed under the Creative Commons Attribution 4.0 International license. ↵
2. Image by Coyau / Wikimedia Commons, via Wikimedia Commons, licensed under Creative Commons Attribution-Share Alike 3.0 Unported. ↵

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11.4: Polygons

It can seem like the study of geometry in elementary school is nothing more than learning a bunch of definitions and then classifying objects. In this part, you'll explore some problem solving and reasoning activities that are based in geometry. But definitions are still important! So let's start with this one.

Definition

A **polygon** is:

1. a plane figure
2. that is bounded by a finite number of straight line segments
3. in which each segment meets exactly two others, one at each of its endpoints.

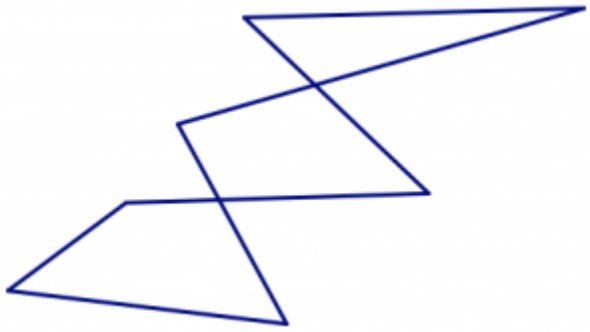
Think / Pair / Share

Just as the first step in problem solving is to understand the problem, the first step in reading a mathematical definition is to understand the definition.

- Use the definition above to draw several examples of figures that are definitely polygons. (You should be able to say why your example fits the definition.)
- Draw several non-examples as well: shapes that are definitely not polygons. (You should be able to say which part of the definition fails for your non-examples.)

A few comments about polygons:

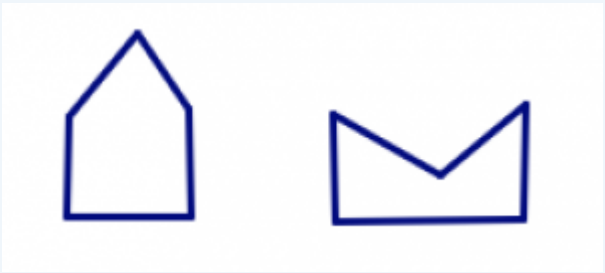
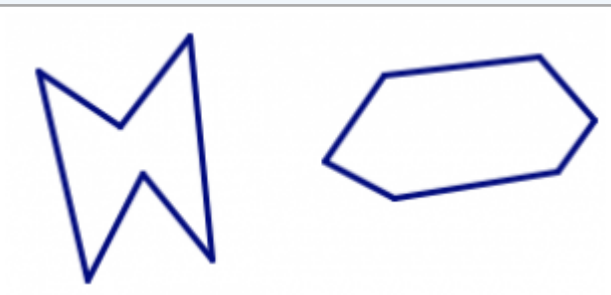
- The line segments that make up a polygon are called its **edges** and the points where they meet are called its **vertices** (singular: **vertex**).
- Because of properties (2) and (3) in the definition, the boundaries of polygons are not self-intersecting.



Not a polygon.

- Polygons are named based on the number of sides they have.

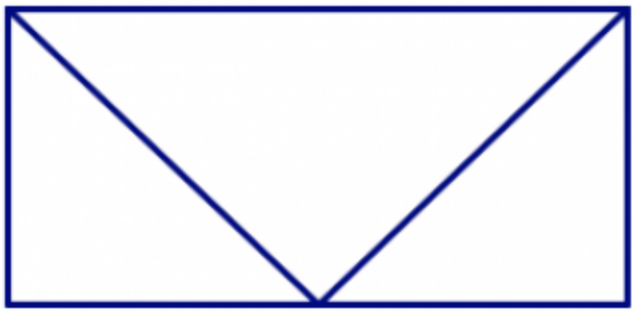
name	# of sides	examples

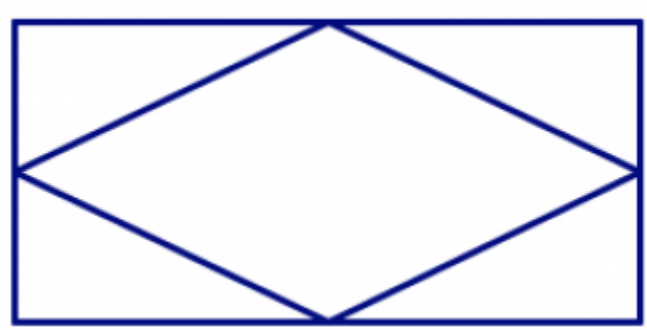
triangle	3	
quadrilateral	4	
pentagon	5	
hexagon	6	
heptagon	7	
octagon	8	
nonagon	9	
decagon	10	

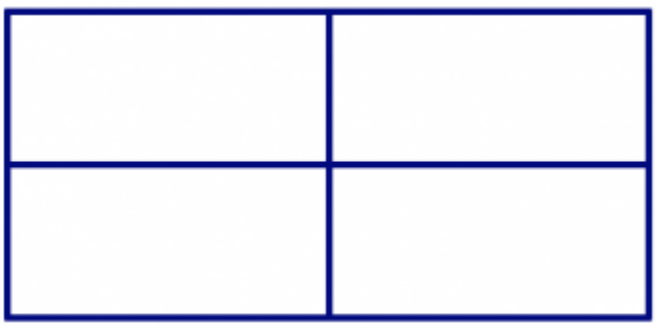
- In general, we call a polygon with n sides an *n-gon*.

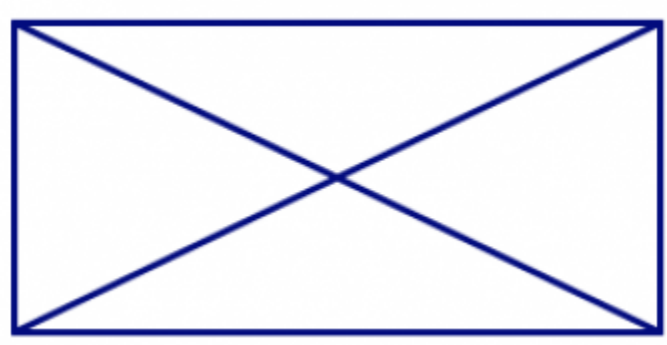
Problem 6

In the pictures below, there are polygons hidden in the design. In each design, find all of the triangles, quadrilaterals, pentagons, and hexagons. How can you be sure you've found them all and haven't counted any twice?









Angle Sum

You know that the sum of the interior angles in any triangle is 180°. Can you say anything about the angles in other polygons?

You probably know that rectangles have four 90° angles. So *if if all quadrilaterals have the same interior angle sum*, it must be

360° (since 4 × 90° = 360°).

But notice: We don't necessarily have any reason to believe this constant sum would be true. Remember that SSS congruence is true for triangles, but not for any other polygons. Triangles are special, and we shouldn't assume that true statements about triangles will hold true for other shapes.

Think / Pair / Share

Any quadrilateral can be split into two triangles, where the vertices of the triangles all coincide with the vertices of the quadrilateral:



Use the pictures above to carefully explain why all quadrilaterals do, indeed, have an angle sum of 360°.

On Your Own

Work on the following exercises on your own or with a partner.

1. Draw several different pentagons on your paper. Show that each of them can be split into exactly three triangles in such a way that the vertices of the triangles all coincide with the vertices of the pentagon.
2. Use the fact that every pentagon can be split into three triangles in this way to find the sum of the angles in any pentagon.
3. Draw several different hexagons on your paper. Show that each of them can be split into exactly four triangles so that the vertices of the triangles all coincide with the vertices of the hexagon.
4. Use the fact that every hexagon can be split into four triangles in this way to find the sum of the angles in any hexagon.

Problem 7

Use your work on the exercises above to complete this general statement:

Angle Sum in Polygons

The sum of the interior angles in an n-gon (a polygon with n sides) is

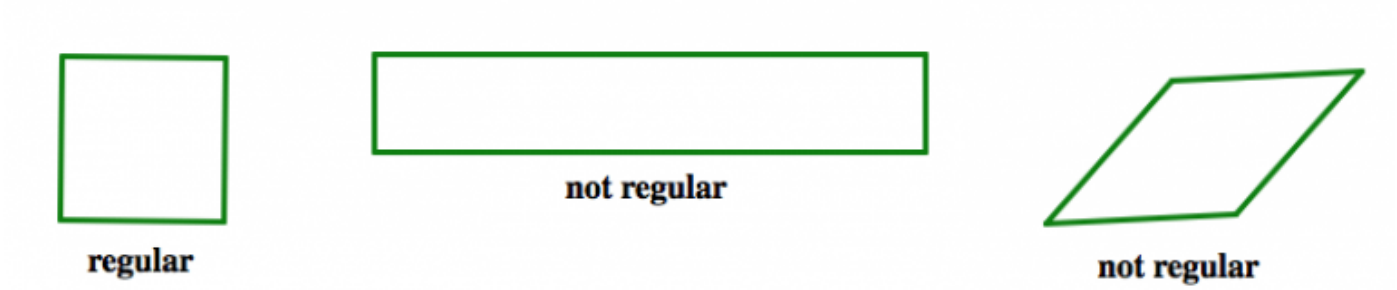
_____.

Explain how you know your statement is true.

Definition

A **regular polygon** has all sides the same length and all angles the same measure.

For example, squares are regular quadrilaterals — all four sides are the same length, and all four angles measure 90°. But a non-square rectangle is *not regular*. Even though all of the angles are 90°, the sides are not all the same length. Similarly, a non-square rhombus is *not regular*. Even though the sides of a rhombus are all the same length, the angles can be different.



Problem 8

Since a square is a regular quadrilateral, you know that every angle in a regular quadrilateral measure 90°. What about angles in other regular polygons?

1. What is the measure of each angle in a regular triangle? Explain how you know you are right.
2. What is the measure of each angle in a regular pentagon? Explain how you know you are right.
3. What is the measure of each angle in a regular hexagon? Explain how you know you are right.
4. What is the measure of each angle in a regular *n-gon*? Explain how you know you are right.

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11.5: Platonic Solids

Of course, we live in a three-dimensional world (at least!), so only studying flat geometry doesn't make a lot of sense. Why not think about some three-dimensional objects as well?

Definition

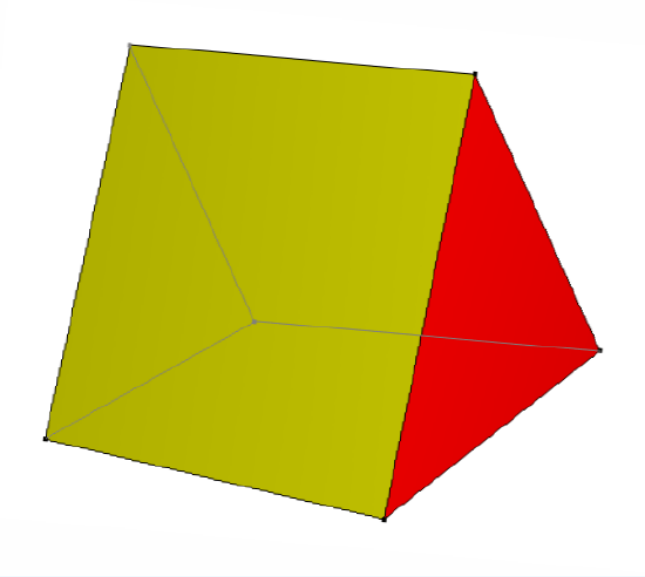
A **polyhedron** is a solid (3-dimensional) figure bounded by polygons. A polyhedron has **faces** that are flat polygons, straight **edges** where the faces meet in pairs, and **vertices** where three or more edges meet.

The plural of polyhedron is **polyhedra.**

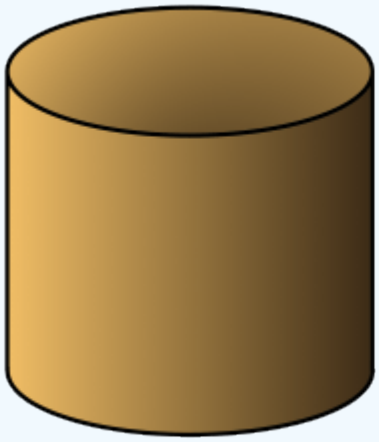
Think / Pair / Share

Look at the pictures of solids below, and decide which are polyhedra and which are not. You should be able to say why each figure does or does not fit the definition.

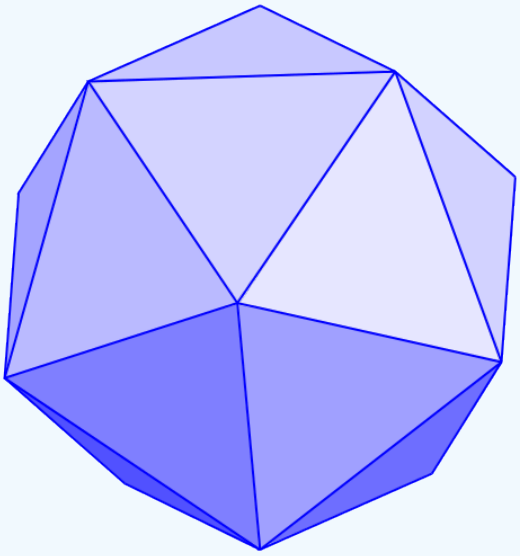
a. [1]



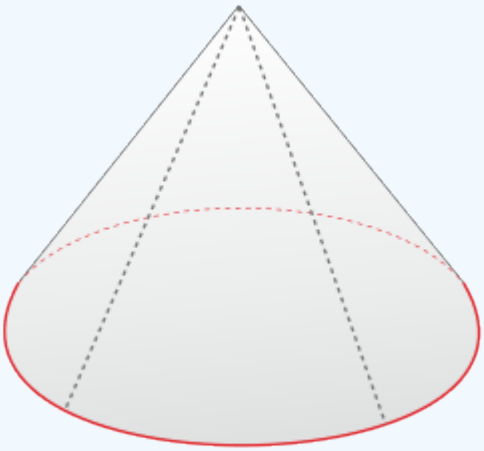
b. [2]



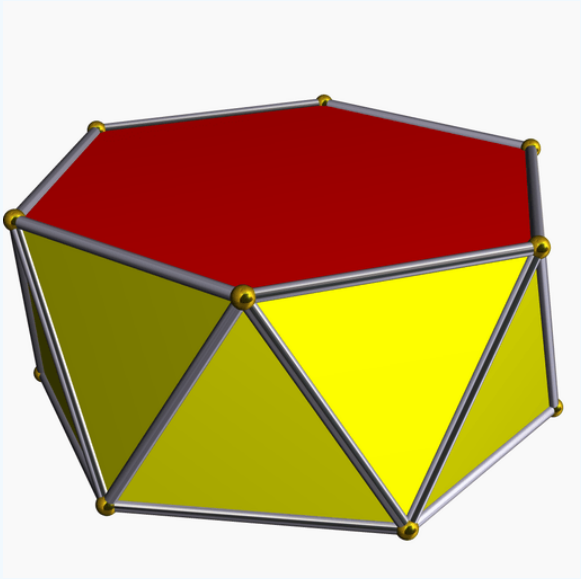
c. [3]



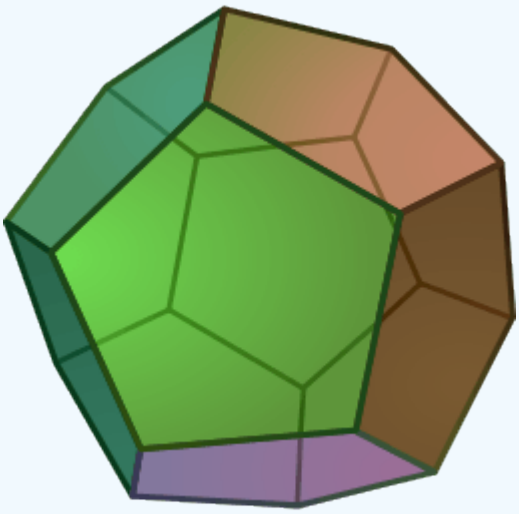
d. [4]



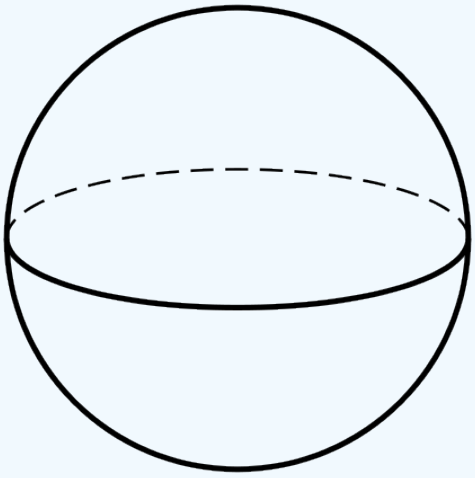
e. [5]



f. [6]



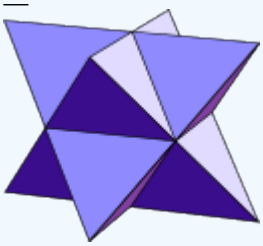
g. [7]



h. [8]



i. [9]



Remember that a *regular polygon* has all sides the same length and all angles the same measure. There is a similar (if slightly more complicated) notion of *regular* for solid figures.

Definition

A **regular polyhedron** has faces that are all *identical (congruent) regular polygons*. All vertices are also identical (the same number of faces meet at each vertex).

Regular polyhedra are also called **Platonic solids** (named for Plato).

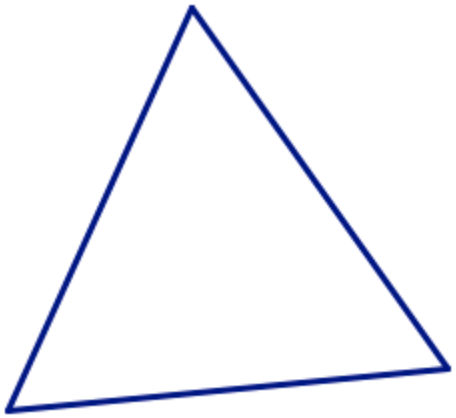
If you fix the number of sides and their length, there is one and only one regular polygon with that number of sides. That is, every regular quadrilateral is a square, but there can be different sized squares. Every regular octagon looks like a stop sign, but it may be scaled up or down. Your job in this section is to figure out what we can say about regular polyhedra.

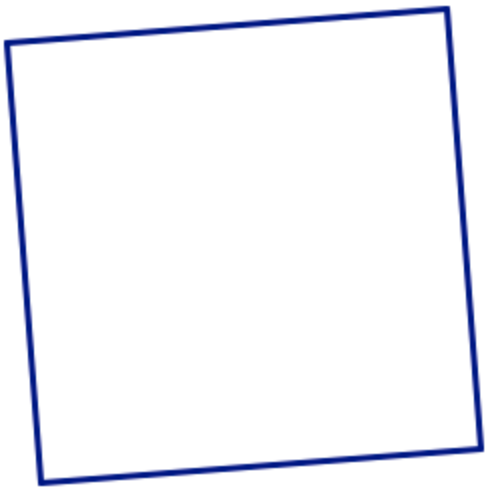
On Your Own

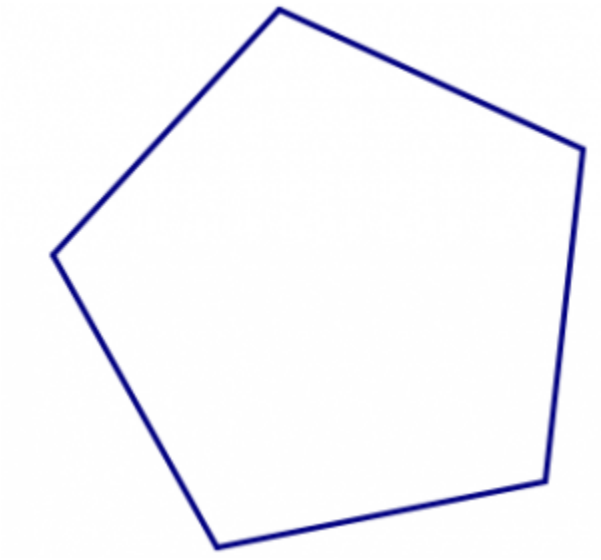
Work on the following exercises on your own or with a partner. You will need to make lots of copies of the regular polygons below. Copy and cut out at least:

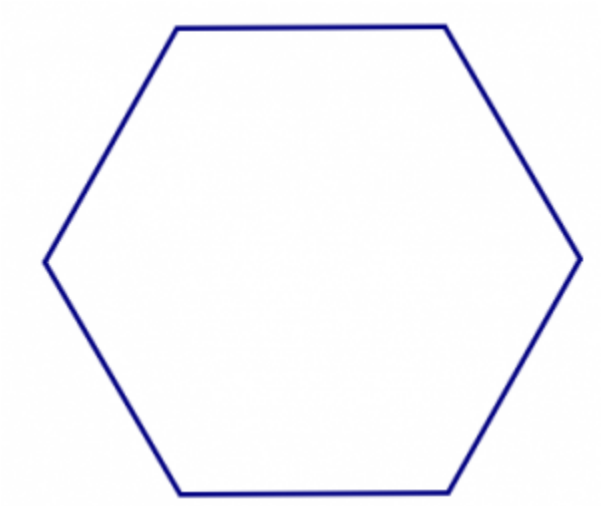
- 40 copies of the equilateral triangle,
- 15 copies of the square,
- 20 copies of the regular pentagon, and
- 10 copies each of the hexagon, heptagon, and octagon.

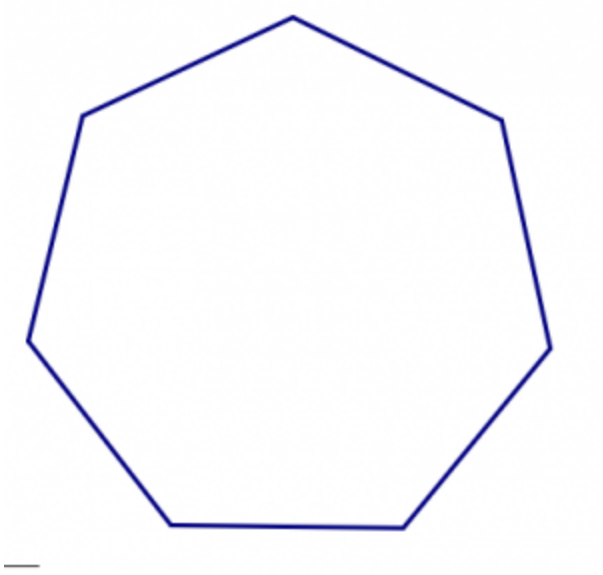
You will also need some tape.

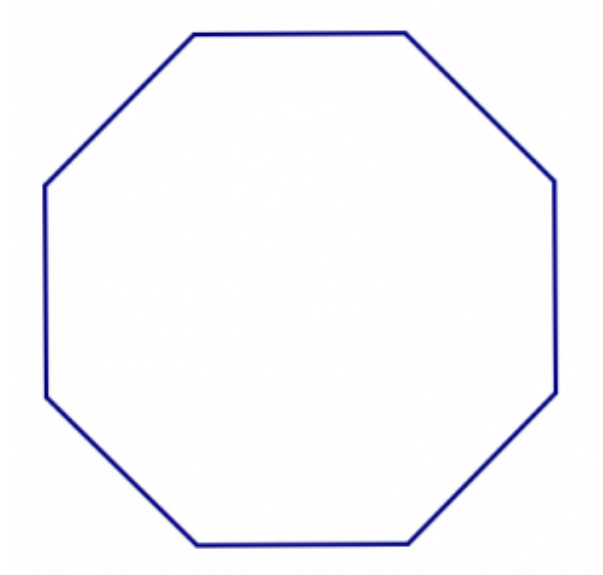












1. In any polyhedron, at least three polygons meet at each vertex. Start with the equilateral triangles: Put **three** of them together meeting at a vertex and tape them together. Then close them up so they form a solid shape. Can you complete this shape into a platonic solid? Be sure to check that at every vertex you have **exactly three** triangles meeting.
2. Now repeat this process, but start with **four** equilateral triangles around a single vertex. Then close them up so they form a solid shape. Can you complete this into a platonic solid? Be sure to check that at every vertex you have **exactly four** triangles meeting.
3. Repeat this process with **five** equilateral triangles, then six, then seven, and so on. Keep going until you are convinced you understand what's happening with Platonic solids that have triangular faces.
4. When you are done with triangular faces, move on to square faces. Work systematically: Try to build a Platonic solid with three squares at each vertex, then four, then five, etc. Keep going until you can make a definitive statement about Platonic solids with square faces.
5. Repeat this process with the other regular polygons you cut out: pentagons, hexagons, heptagons, and octagons.

You must have noticed that the situation for Platonic solids is quite different from the situation for regular polygons. There are *infinitely many* regular polygons (even if you don't account for size). There is a regular polygon with n sides for every value of n bigger than 2. But for solids, we have the following (perhaps surprising) result.

Theorem

There are exactly five Platonic solids.

The key fact is that for a three-dimensional solid to close up and form a polyhedron, there must be less than 360° around each vertex. Otherwise, it either lies flat (if there is exactly 360°) or folds over on itself (if there is more than 360°).

Problem 9

Based on your work in the exercises, you should be able to write a convincing justification of the Theorem above. Here's a sketch, and you should fill in the explanations.

1. If a Platonic solid has faces that are equilateral triangles, then fewer than 6 faces must meet at each vertex. Why?
2. If a Platonic solid has square faces, then three faces can meet at each vertex, but not more than that. Why?
3. If a Platonic solid has faces that are regular pentagons, then three faces can meet at each vertex, but not more than that. Why?
4. Regular hexagons cannot be used as the faces for a Platonic solid. Why?
5. Similarly, regular n-gons for n bigger than 6 cannot be used as the faces for a Platonic solid. Why?

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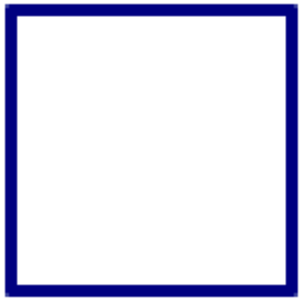
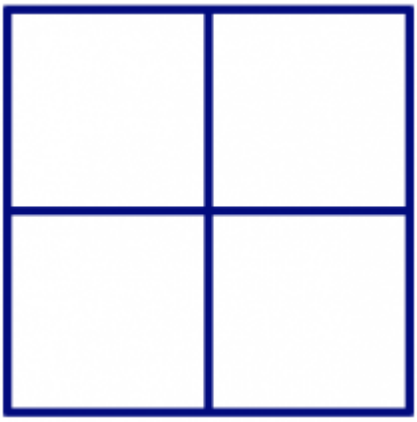
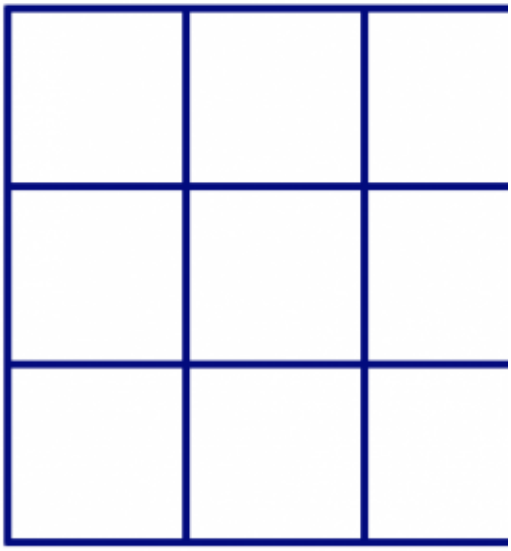
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6. Image DTR CC-BY-SA-3.0], via Wikimedia Commons ↵
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8. Imgae via Wikimedia Commons [Public domain]. ↵
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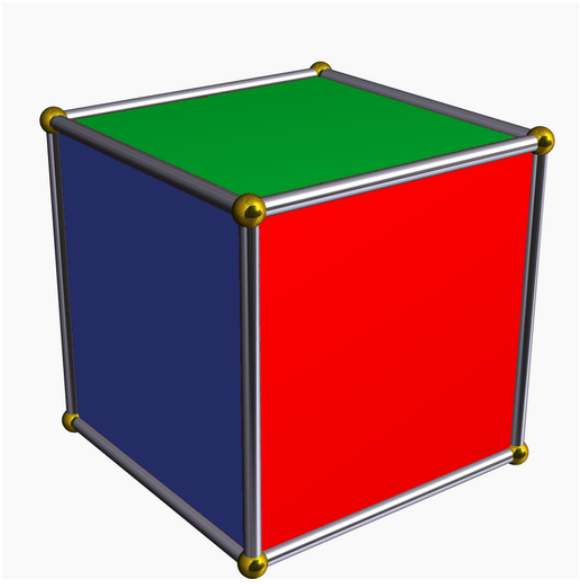
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11.6: Painted Cubes

You can build up squares from smaller squares:

		
1 × 1 square	2 × 2 square	3 × 3 square

In a similar way, you can build up cubes from smaller cubes:

		
1 × 1 × 1 cube[1]	2 × 2 × 2 cube[2]	

Think / Pair / Share

We call a 1 × 1 × 1 cube a **unit cube**.

- How many unit cubes are in a 2 × 2 × 2 cube?
- How many unit cubes are in a 3 × 3 × 3 cube?
- How many unit cubes are in a $n \times n \times n$ cube?

Explain your answers.

Problem 10

Imagine you build a 3 × 3 × 3 cube from 27 small white unit cubes. Then you take your cube and dip it into a bucket of bright blue paint. After the cube dries, you take it apart, separating the small unit cubes.

1. After you take the cube apart, some of the unit cubes are still all white (no blue paint). How many? How do you know you are right?
2. After you take the cube apart, some of the unit cubes have blue paint on just one face. How many? How do you know you are right?
3. After you take the cube apart, some of the unit cubes have blue paint on two faces. How many? How do you know you are right?
4. After you take the cube apart, some of the unit cubes have blue paint on three faces. How many? How do you know you are right?
5. After you take the cube apart, do any of the unit cubes have blue paint on more than three faces? How many? How do you know you are right?

Problem 11

Generalize your work on Problem 10. What if you started with a 2 × 2 × 2 cube? Answer the same questions. What about a 4 × 4 × 4 cube? How about an $n \times n \times n$ cube? Be sure to justify what you say.

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1. Image by Robert Webb's Stella software: http://www.software3d.com/Stella.php, via Wikimedia Commons. ↵
2. Image by Mike Gonzalez (TheCoffee) (Work by Mike Gonzalez (TheCoffee)) [CC BY-SA 3.0], via Wikimedia Commons. ↵
3. Image by Mike Gonzalez (TheCoffee) (Work by Mike Gonzalez (TheCoffee)) [CC BY-SA 3.0], via Wikimedia Commons. ↵

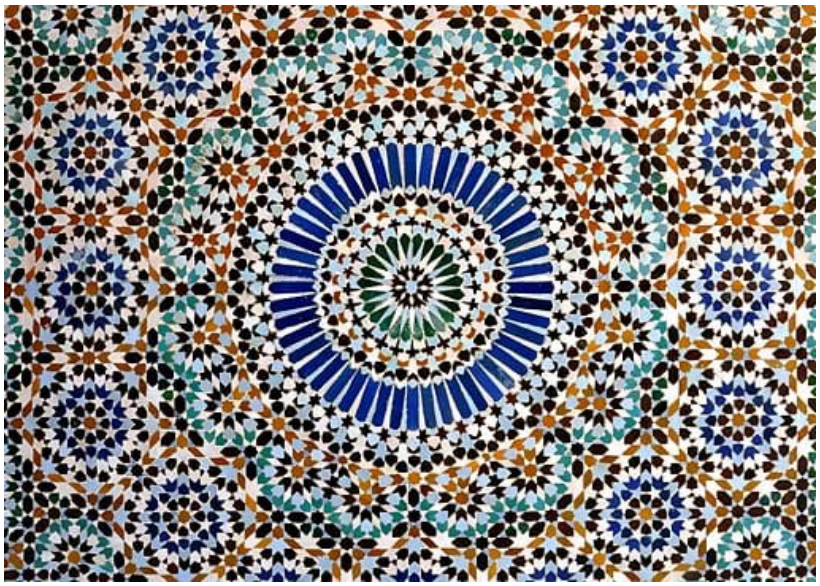
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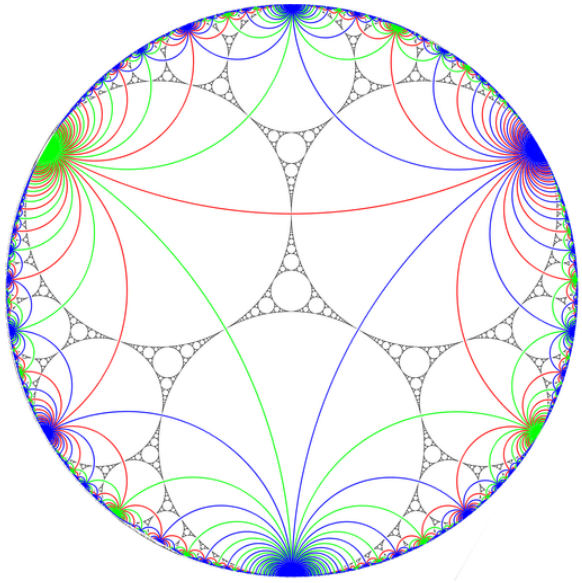
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11.7: Symmetry

Mathematicians use symmetry in all kinds of situations. There can be symmetry in calculations, for example. But the most recognizable kinds of symmetry are those in geometric designs.

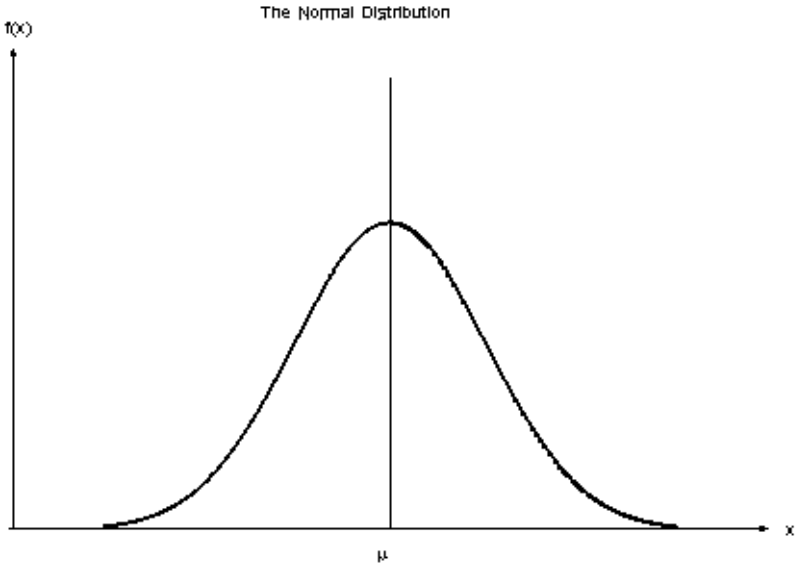
Geometric and real-world objects can have different kinds of symmetries[1].











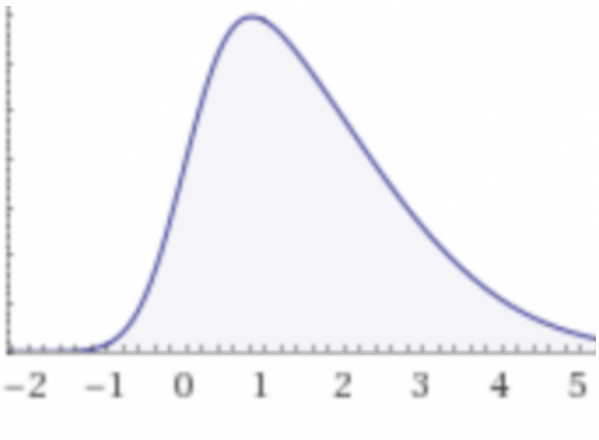


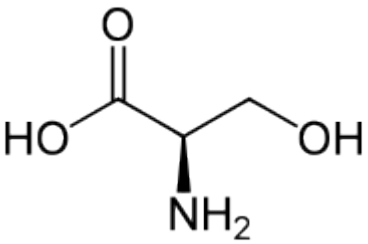
Or they might have no symmetry[2] at all.

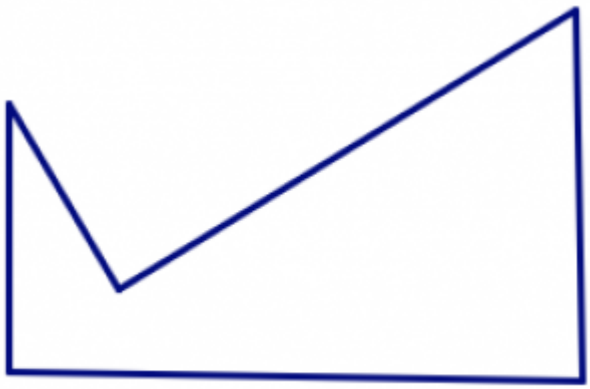












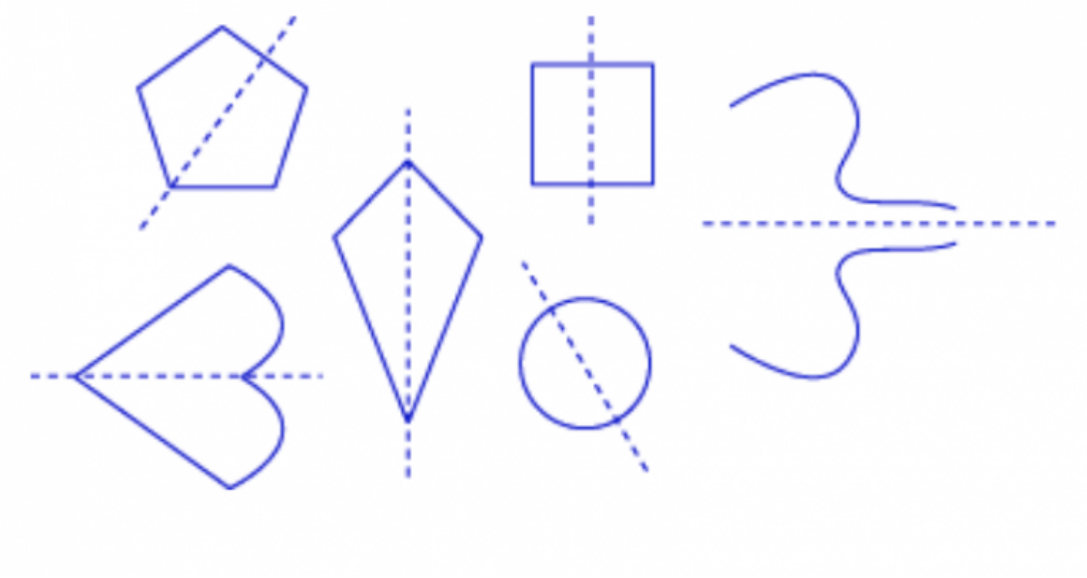
Think / Pair / Share

- What do you already know about the idea of *symmetry*? What does it mean to say a design is *symmetric*?

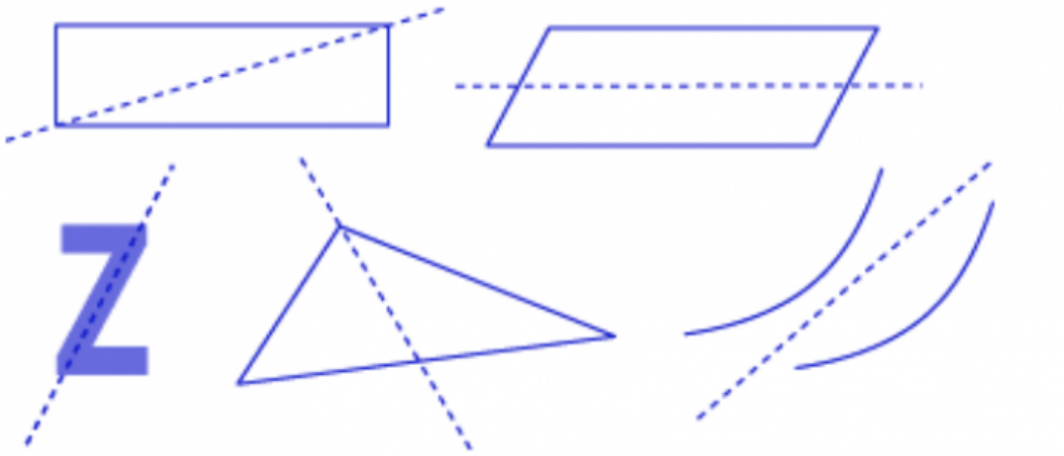
- Do you know about different types of symmetry? What types?
- Can you give examples of real-world objects that are symmetric? What about objects that are not symmetric?

Line Symmetry

If you can flip a figure over a line — this is called *reflecting* the figure — and then it appears unchanged, then the figure has **reflection symmetry** or **line symmetry**. A **line of symmetry** divides an object into two mirror-image halves. The dashed lines below are lines of symmetry:



Compare with the dashed lines below. Though they do cut the figures in half, they don't create mirror-image halves. These are **not** lines of symmetry:



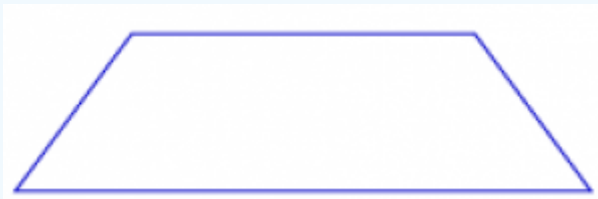
Think / Pair / Share

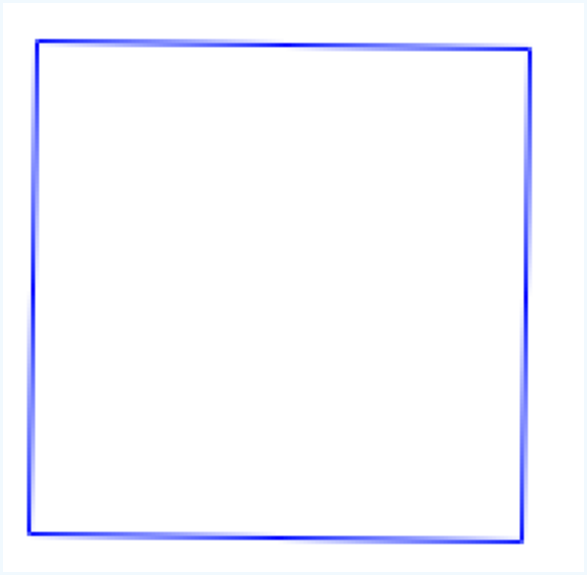
Look at the first set of pictures at the start of this chapter. Do any of them have lines of symmetry? How can you tell?

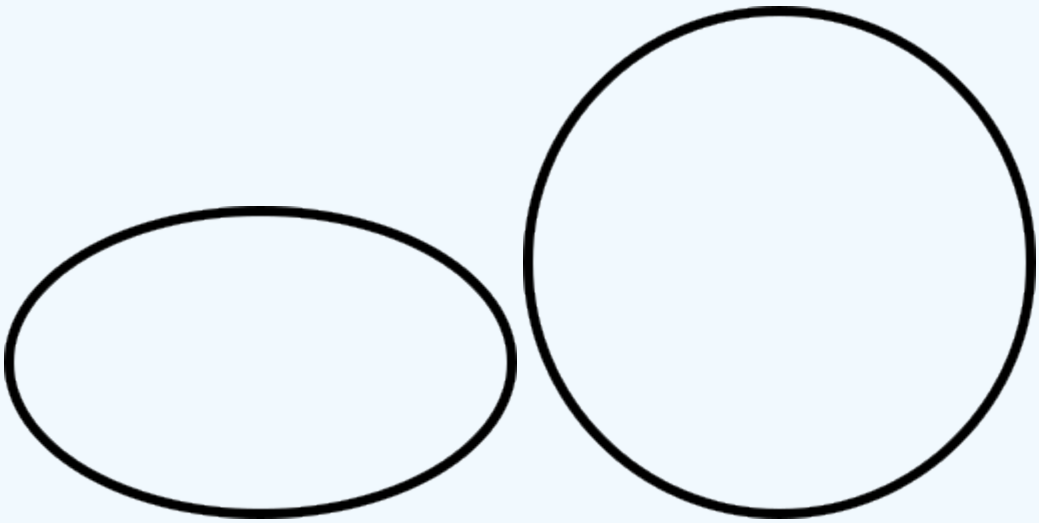
Problem 12

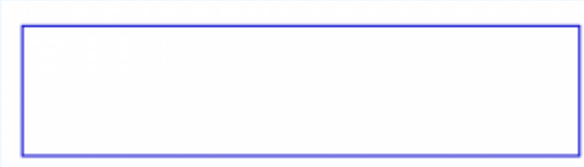
For each of the figures[3] below:

1. Decide if it has any lines of symmetry. If not, how do you know?
2. If it does have one or more lines of symmetry, find / describe all of them. Explain how you did it.







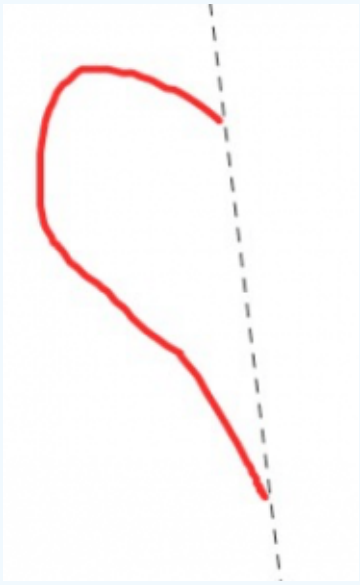


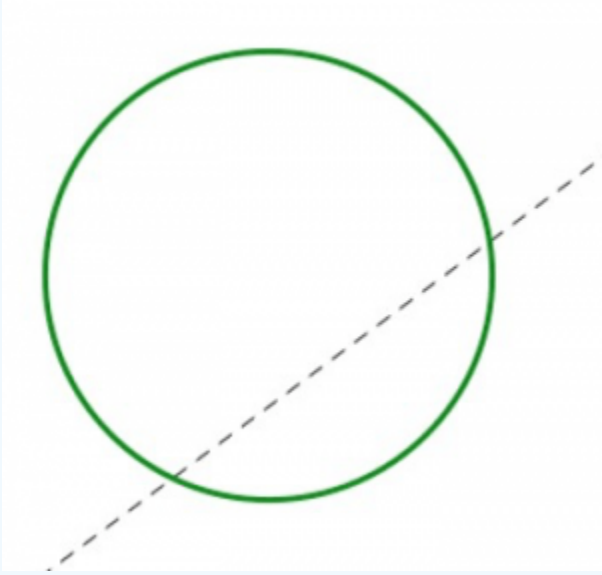




Problem 13

Each picture below shows **half** of a design with line symmetry. The line of symmetry (dashed) is shown. Can you complete the design? Explain how you did it.

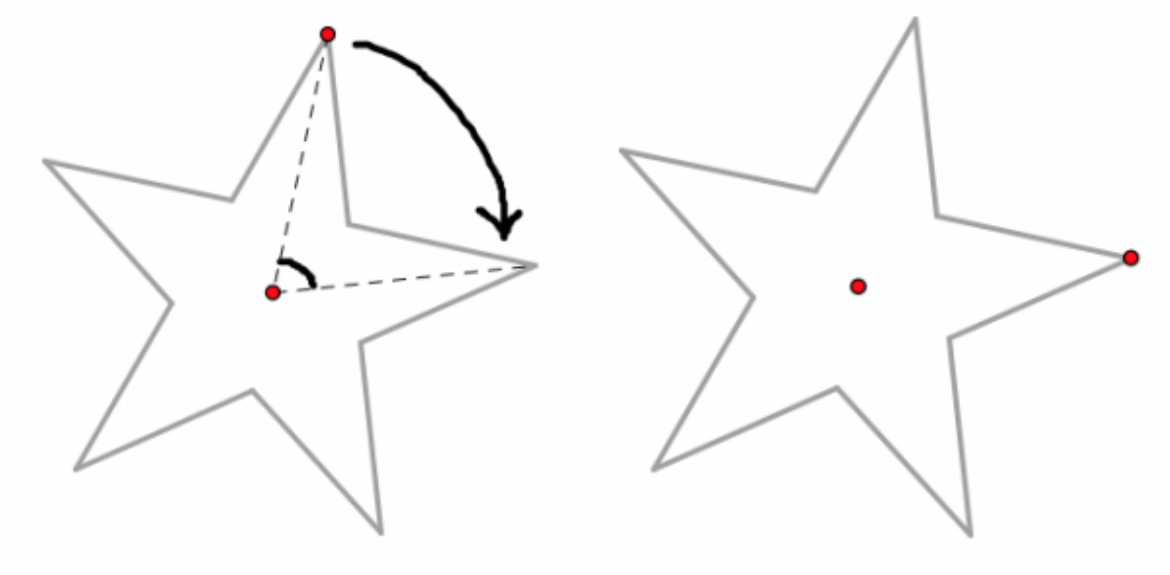




Rotational Symmetry

If you can turn a figure around a center point less than a full circle — this is called a *rotation* — and the figure appears unchanged, then the figure has **rotational symmetry**. The point around which you rotate is called the center of rotation, and the smallest angle you need to turn is called the angle of rotation.

This star has rotational symmetry of 72°, and the center of rotation is the center of the star. One point is marked to help you visualize the rotation.



Think / Pair / Share

- How can you be certain that the angle of rotation for the star is exactly 72°?
- Look at the first set of pictures at the start of this chapter. Do any of them have rotational symmetry? How can you tell?

Problem 14

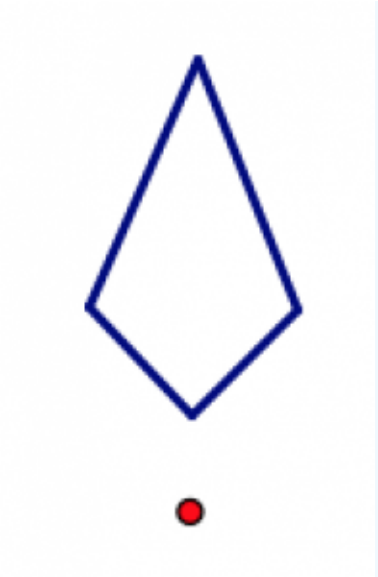
Each of the figures below has rotational symmetry. Find the center of rotation and the angle of rotation. Explain your thinking.

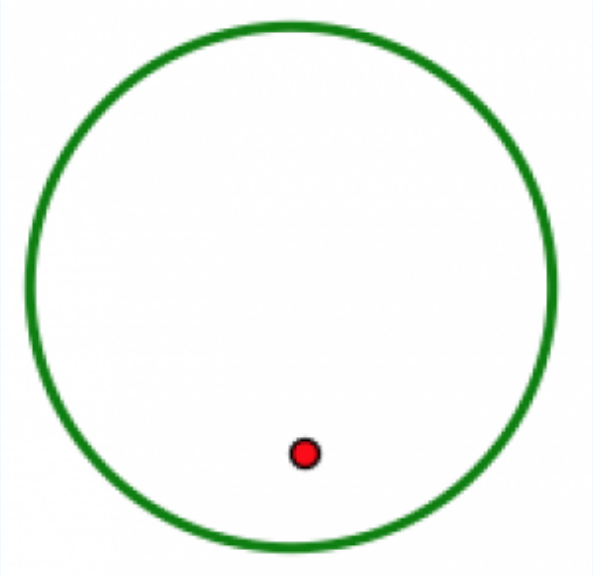




Problem 15

Each picture below shows part of a design with a marked center of rotation and an angle of rotation given. Can you complete the design so that it has the correct rotational symmetry? Explain how you did it.

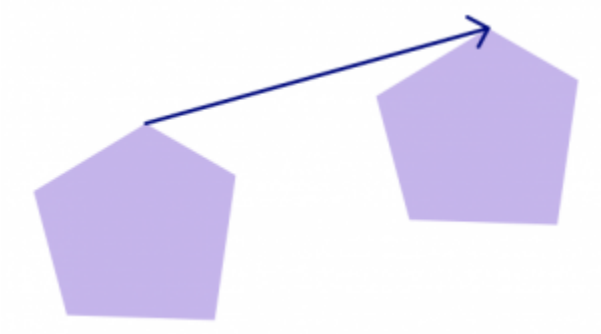




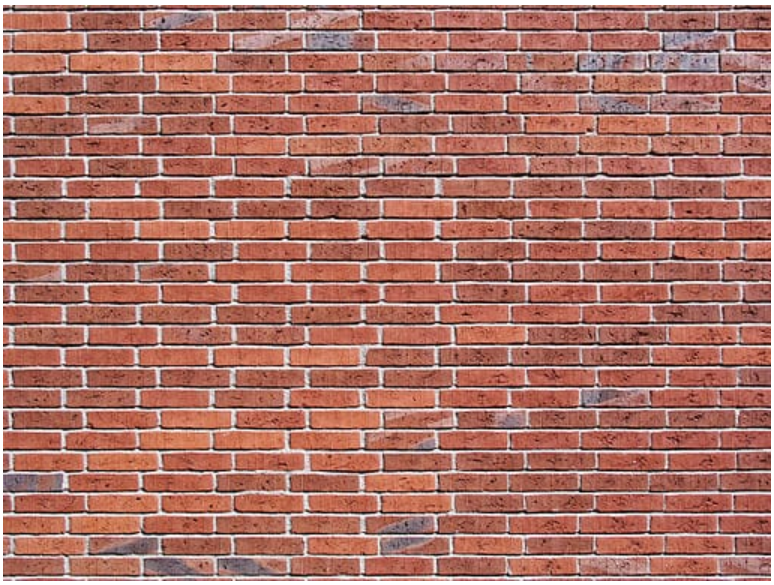
90° 45°

Translational Symmetry

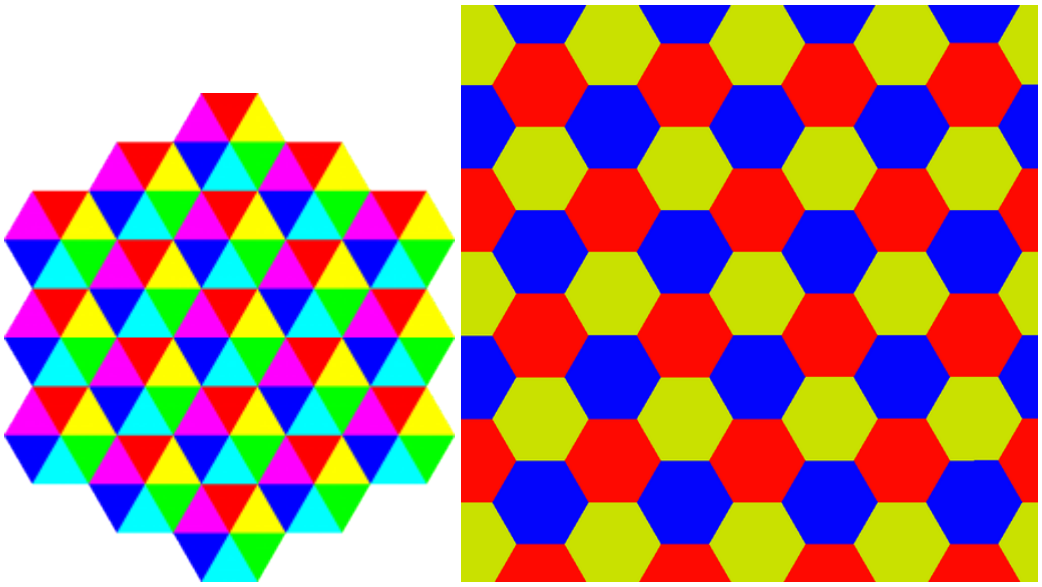
A **translation** (also called a slide) involves moving a figure in a specific direction for a specific distance. A **vector** (a line segment with an arrow on one end) can be used to describe a translation, because the vector communicates both a distance (the length of the segment) and a direction (the direction the arrow points).

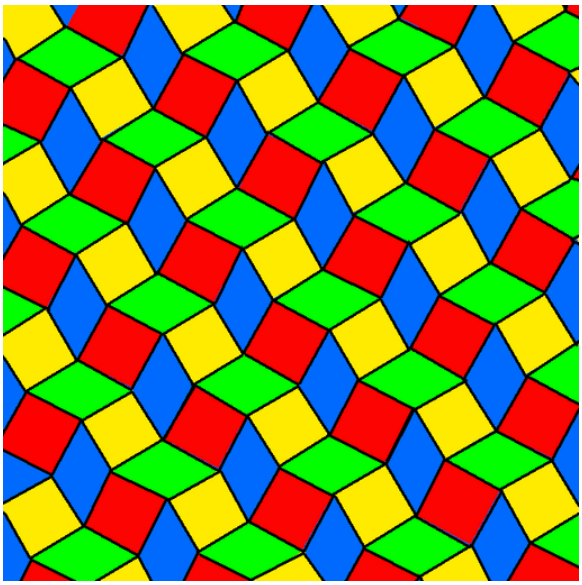


A design has **translational symmetry** if you can perform a translation on it and the figure appears unchanged. A brick wall[4] has translational symmetry in lots of directions!

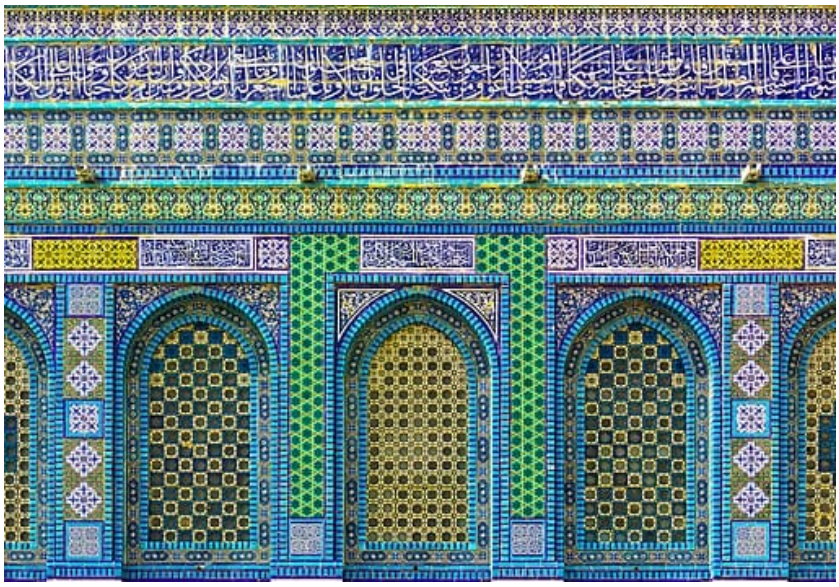


The brick wall is one example of a *tessellation*[5], which you'll learn more about in the next chapter.





You can see translation symmetry in lots of places. It's in architecture and design[6].






It's in art, most famously that by M.C. Escher. (You might want to visit http://www.mcescher.com/gallery/symmetry/ and browse the "Symmetry" gallery.)

And it appears in traditional Hawaiian and other Polynesian tattoo[7] designs.





Think / Pair / Share

- On each of the pictures with translational symmetry above, sketch a vector to indicate the direction and distance of the translational symmetry.
- Create your own design with translational symmetry. Explain how you did it.

---

1. Mosaic image by MarcCooperUK (Flickr: Paris central mosque) [CC BY 2.0], via Wikimedia Commons. Apollonian Circle Packing by Tomruen (Own work) [CC BY-SA 3.0], via Wikimedia Commons. Butterfly by Bernard DUPONT from FRANCE (Swallowtail Butterfly (Papilio oribazus)) [CC BY-SA 2.0], via Wikimedia Commons. Starfish by Paul Shaffner [CC BY 2.0], via Wikimedia Commons. Normal distribution from Wikimedia Commons [Public domain]. Water drop from pixababy.com [CC0 Creative Commons]. ↵

2. Pillar coral, wave, and molecule from Wikimedia commons [Public domain]. Head of a woman by Pablo Picasso, image from Gandalf's Gallery on flickr [CC-BY-NC-SA 2.0] ↵
3. Circle and ellipse by Paris 16 (Own work) [CC BY-SA 4.0], via Wikimedia Commons ↵
4. Image by I, Xauxa [CC-BY-SA-3.0], via Wikimedia Commons ↵
5. Triangular tessellation from pixababy [CC0]. Hexagonal and rhombic tessellations from Wikimedia Commons [Public domain]. ↵
6. Tile at Jerusalem temple by Andrew Shiva / Wikipedia, via Wikimedia Commons [CC BY-SA 4.0]. Mosque by Hisham Binsuwaif via flickr [CC BY-SA 2.0]. British Museum great court by Andrew Dunn, http://www.andrewdunnphoto.com/ (Own work) [CC BY-SA 2.0], via Wikimedia Commons ↵
7. Royal Hawaiian officer via Wikimedia Commons [Public domain]. Shoulder and arm tattoos by Micael Faccio on flicker [CC BY-2.0]. ↵

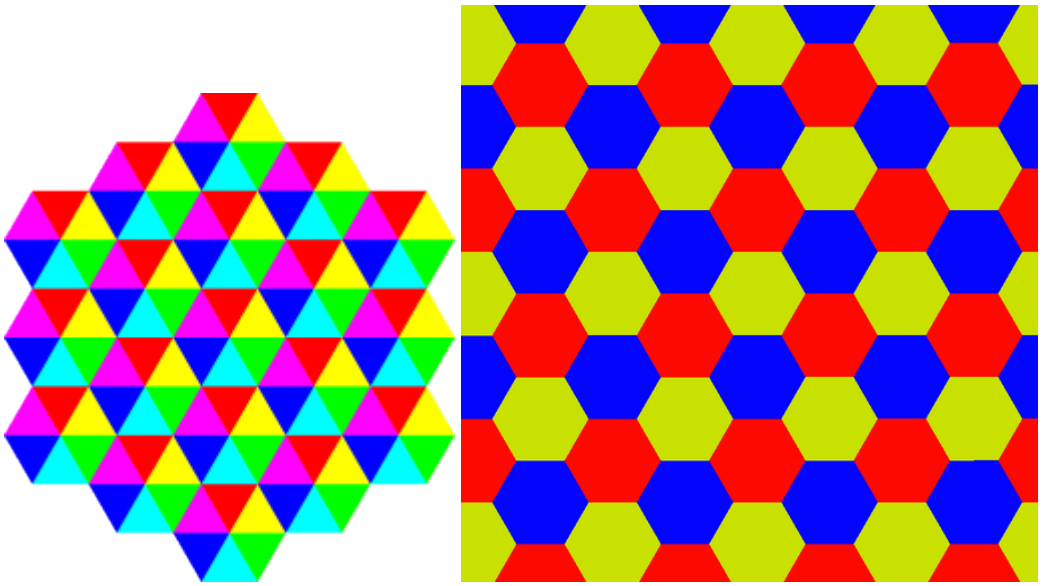
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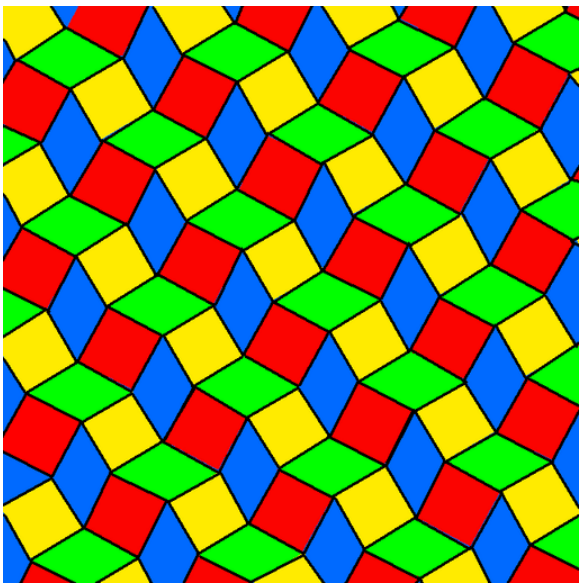
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11.8: Geometry in Art and Science

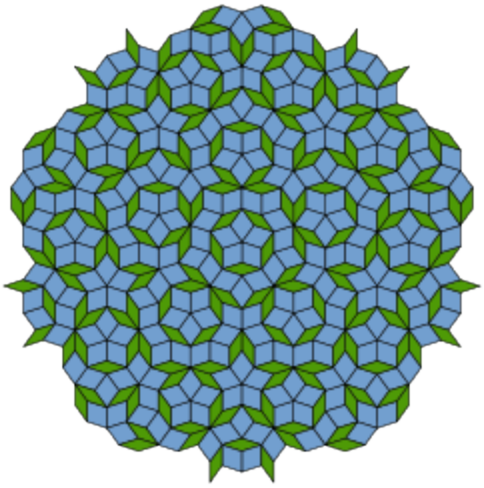
Tessellations

A *tessellation*[1] is a design using one ore more geometric shapes with no overlaps and no gaps. The idea is that the design could be continued infinitely far to cover the whole plane (though of course we can only draw a small portion of it).





Many tessellations have translational symmetry, but it's not strictly necessary. The *Penrose tiling* shown below[2] does not have any translational symmetry.

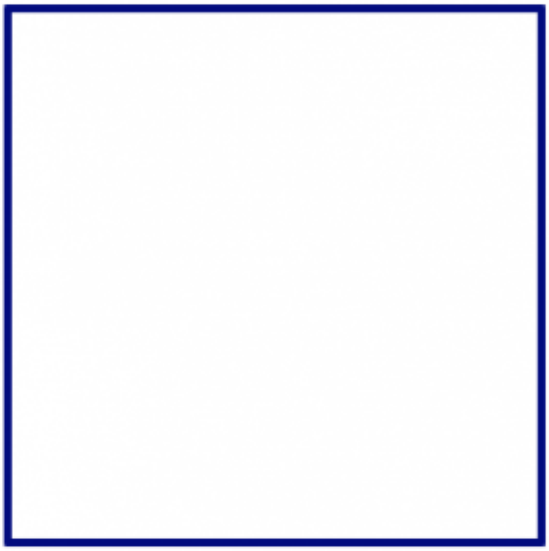


It's actually much harder to come up with these "aperiodic" tessellations than to come up with ones that have translational symmetry. So we'll focus on how to make symmetric tessellations.

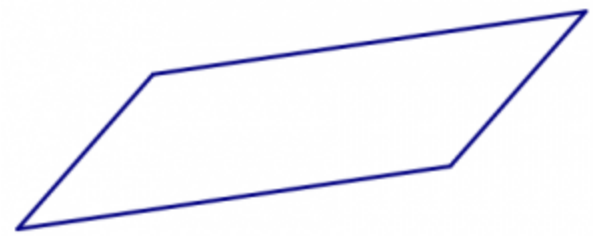
The first two tessellations above were made with a single geometric shape (called a *tile)* designed so that they can fit together without gaps or overlaps. The third design uses two basic tiles. Tessellations are often called *tilings*, and that's what you should think about: If I had tiles made in this shape, could I use them to tile my kitchen floor? Or would it be impossible?

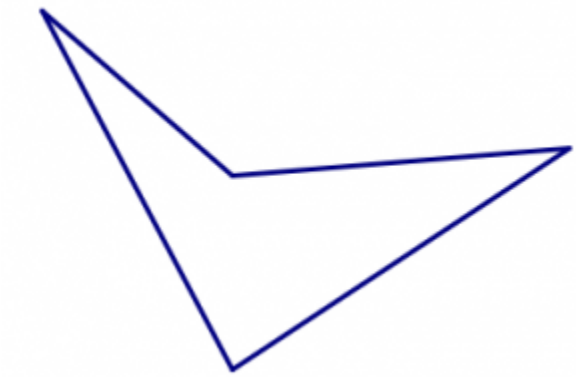
On Your Own

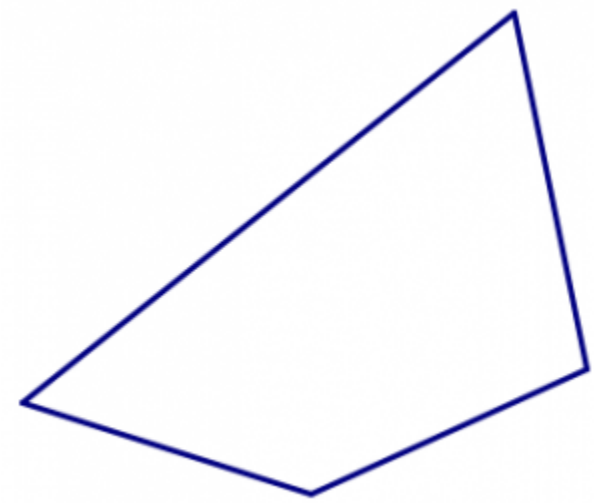
Work on these exercises on your own or with a partner. You will need lots of copies (maybe 10–15 each) of each shape below. In each problem, focus on just a single tile for making your tessellation.

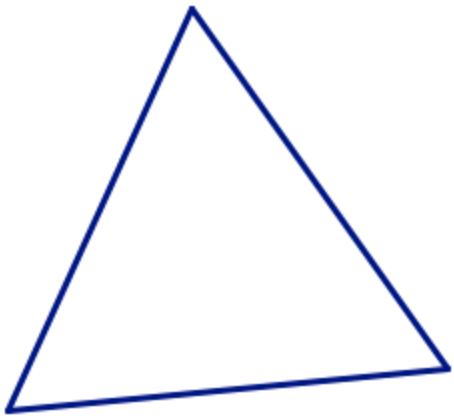


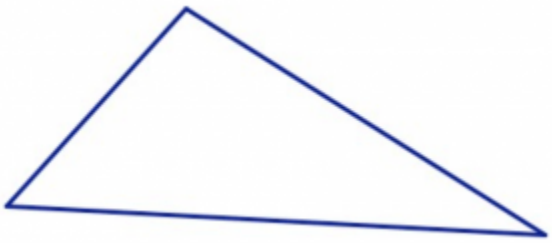




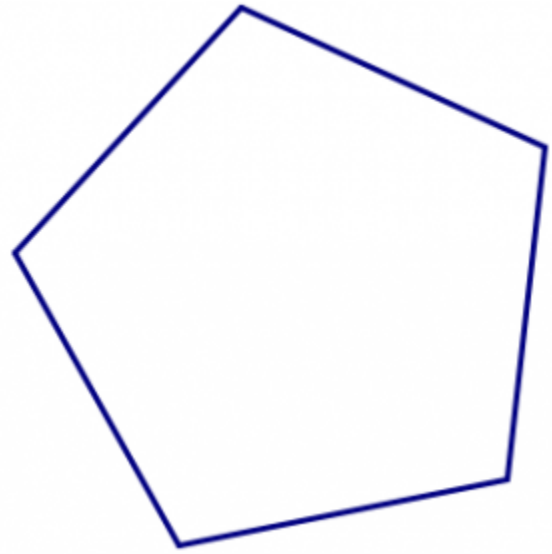


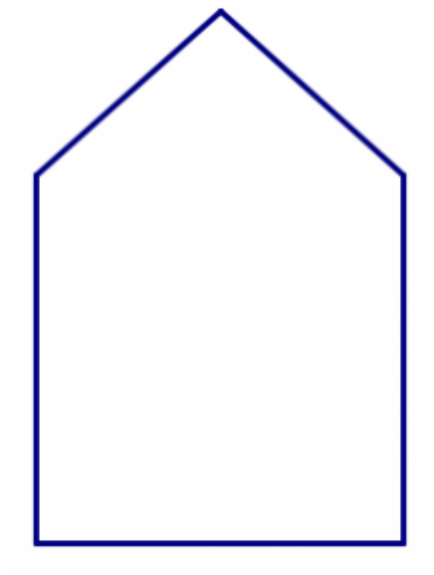


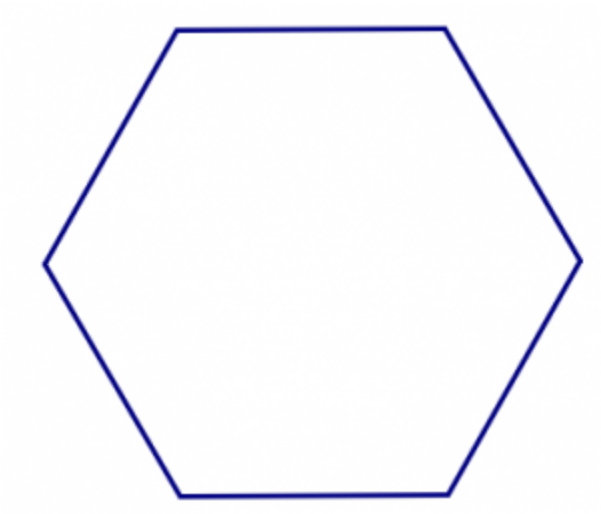


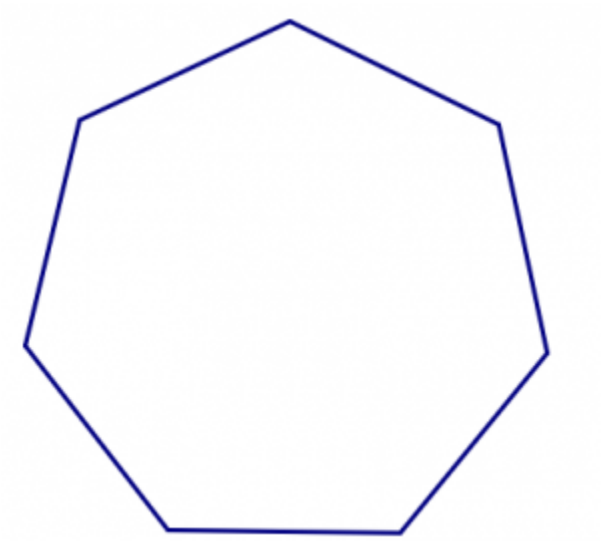


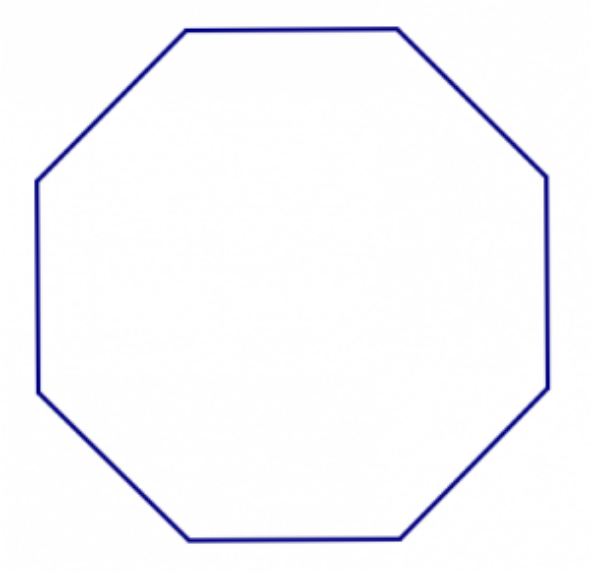












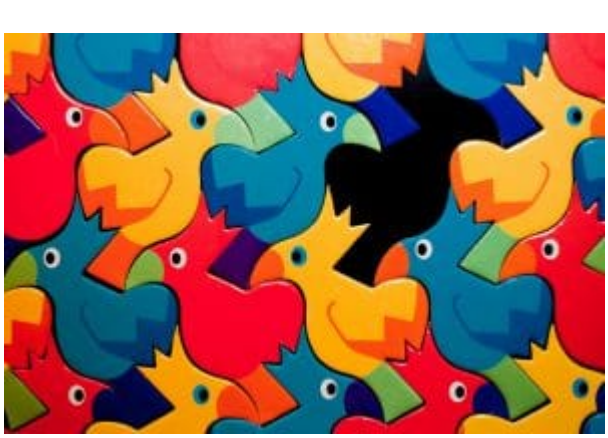
1. Start with the square tile. Can you fit the squares together in a pattern that could be continued forever, with no gaps and no overlaps? Can you do it in more than one way?
2. Now try one of the triangular tiles. Can you use many copies of a single triangle to tessellate the plane?
3. Repeat this process with each of the other tiles. Keep track of your findings.

Think / Pair / Share

- Which of the tiles given above tessellate, and which do not?
- Do you have any conjectures based on this experience, about which shapes will tessellate and which will not?

Escher Drawings

The artist M.C. Escher created many works of art inspired by mathematics, including some very beautiful tessellations. Below you will see some images[3] inspired by his work. You can view the real thing at http://www.mcescher.com/ in the “Symmetry” gallery.





You can make your own Escher-like drawings using some facts that you learned while studying tessellations.

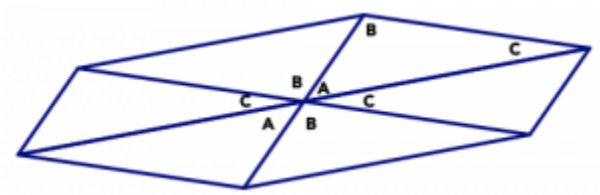
Theorem: Tessellations

Any triangle will tessellate. So will any quadrilateral.

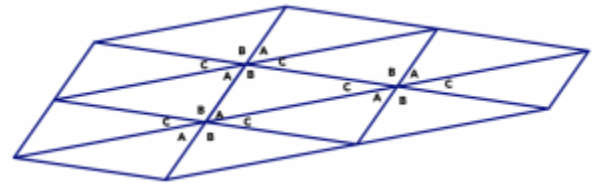
The explanation for this comes down to what you know about the sums of angles. The sum of the angles in a triangle is 180°.



So if you make six copies of a single triangle and put them together at a point so that each angle appears twice, there will be a total of 360° around the point, meaning the triangles fit together perfectly with no gaps and no overlaps.



You can then repeat this at every vertex, using more and more copies of the same triangles.



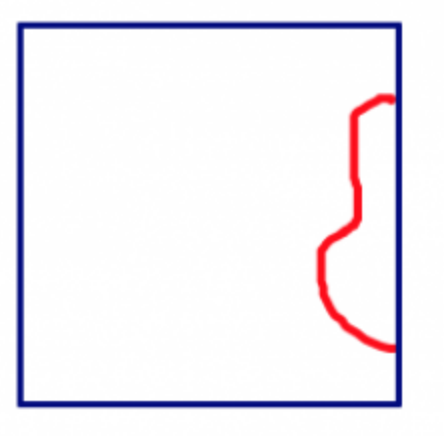
Think / Pair / Share

- Use the fact that the sum of the angles in any quadrilateral is 360° to explain why every quadrilateral will tessellate.
- Use angles to explain why regular hexagons will tessellate.
- Explain why regular pentagons will not tessellate.

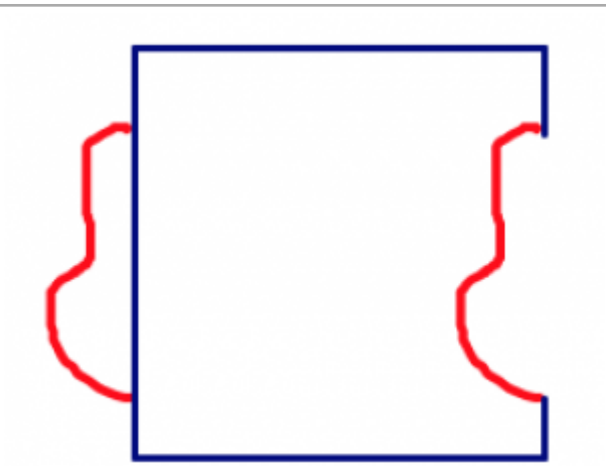
On Your Own

Work on the following exercises on your own or with a partner. Here's how you can create your own Escher-like drawings.

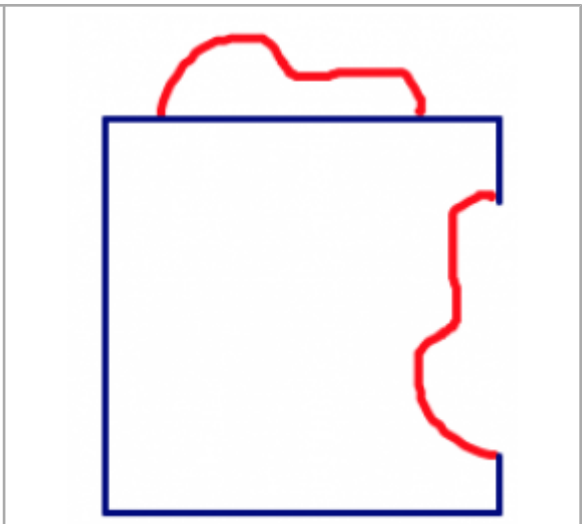
1. Select your basic tile. The first time you do this, it's easiest to start with a simple shape that you know will tessellate, like an equilateral triangle, a square, or a regular hexagon.
2. Draw a "squiggle" on one side of your basic tile.



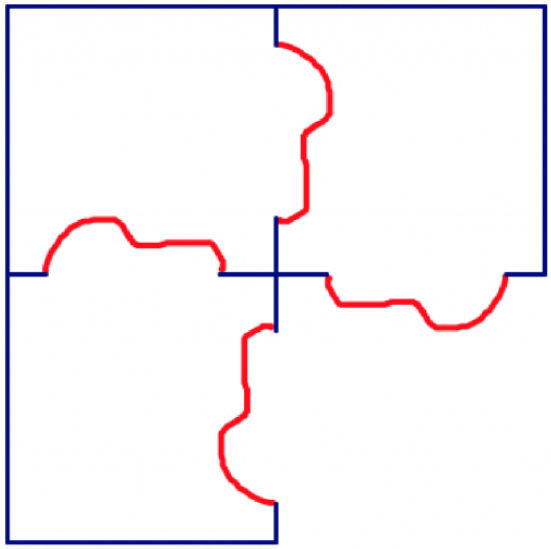
3. Cut out the squiggle, and move it to another side of your shape. You can either translate it straight across or rotate it.



or



4. It's important that the cut-out lines up along the new edge in the same place that it appeared on its original edge.
5. Tape the squiggle into its new location. This is your basic tile. On a large piece of paper, trace around your tile. Then move it the same way you moved the squiggle (translate or rotate) so that the squiggle fits in exactly where you cut it out.



6. The shape will still tessellate, so go ahead and fill up your paper.
7. Now get creative. Color in your basic shape to look like something — an animal? a flower? a colorful blob? Add color and design throughout the tessellation to transform it into your own Escher-like drawing.
8. If you want to try a more complicated version, cut two different squiggles out of two different sides, and move them both.

Building Towers

For this activity, you will need some construction materials:

- You'll need lots of toothpicks.
- You'll also need something to connect the toothpicks together. The best material for this is mini marshmallows; you can stick the ends of the toothpicks into the marshmallows to connect them. You can also use pieces of clay, bits of gummy candies, or other similar (sticky) material.

Problem 16

Try this as a warm-up activity. Grab exactly six toothpicks. Your job is to make four triangles using all six toothpicks. You cannot break any of the toothpicks or add any other materials besides the marshmallow connectors.

Problem 17

Now comes the main challenge. You have ten minutes to build the *tallest free-standing structure* that you can make. "Free-standing" means that it will stand up on its own. You can't hold it up or lean it against something. When the ten minutes are up, back away from your tower and measure its height.

Think / Pair / Share

Look at your own tower and at other students' towers. Talk about these questions:

- What design choices led to taller free-stranding structures? Why do you think that is?
- If you had another ten minutes to try this activity again, what would you do differently and why?

---

1. Triangular tessellation from pixababy [CC0]. Hexagonal and rhombic tessellations from Wikimedia Commons [Public

domain]. ↵
2. Image via Wikimedia Commons [Public donmain]. ↵
3. Images from flickr [CC BY-NC-SA 2.0]. Birds by Sharon Drummond. Lizard tiles by Ben Lawson. ↵

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11.9: Problem Bank

Problem 18

In the Tangrams chapter, you first saw all 7 tangram pieces arranged into a square.

1. If the large square you made with all seven pieces is one whole, assign a (fractional) value to each of the seven tangram pieces. Justify your answers.
2. The tangram puzzle contains a small square. If the small square (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.
3. The tangram set contains two large triangles. If a large triangle (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.
4. The tangram set contains one medium triangle. If the medium triangle (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.
5. The tangram set contains two small triangles. If a small triangle (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.

Problem 19

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. A right triangle that is scalene.
2. A right triangle that is isosceles.
3. A right triangle that is equilateral.

Problem 20

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. An acute triangle that is scalene.
2. An acute triangle that is isosceles.
3. An acute triangle that is equilateral.

Problem 21

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. An obtuse triangle that is scalene.
2. An obtuse triangle that is isosceles.
3. An obtuse triangle that is equilateral.

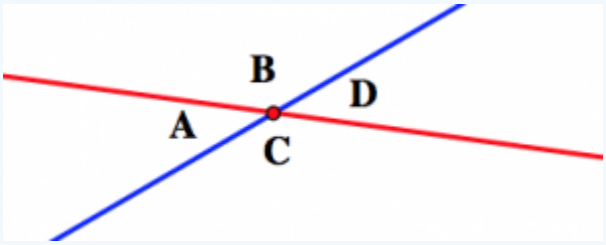
Problem 22

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. An equiangular triangle that is scalene.
2. An equiangular triangle that is isosceles.
3. An equiangular triangle that is equilateral.

Problem 23

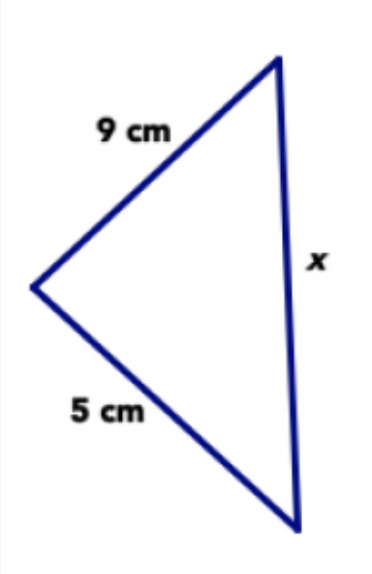
Look at the picture below, which shows two lines intersecting. Angles A and D are called “vertical angles,” and so are angles B and C.



Use this drawing to explain why vertical angles must have the same measure. (Hint: what is the sum of the measures of angle A angle B? How do you know?)

Problem 24

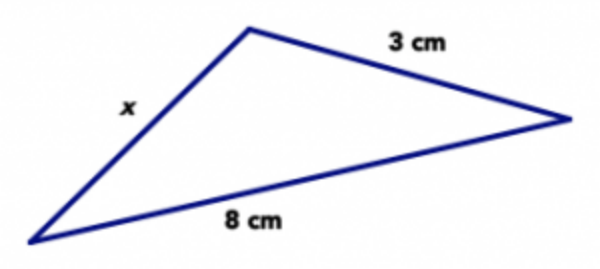
Answer the following questions about the triangle below. Be sure to focus on what you *know for sure* and not what the picture *looks like*.



1. Could it be true that $x = 4$ cm? Explain your answer.
2. Could it be true that $x = 20$ cm? Explain your answer.
3. Give three possible values of x, based on the information in the picture.

Problem 25

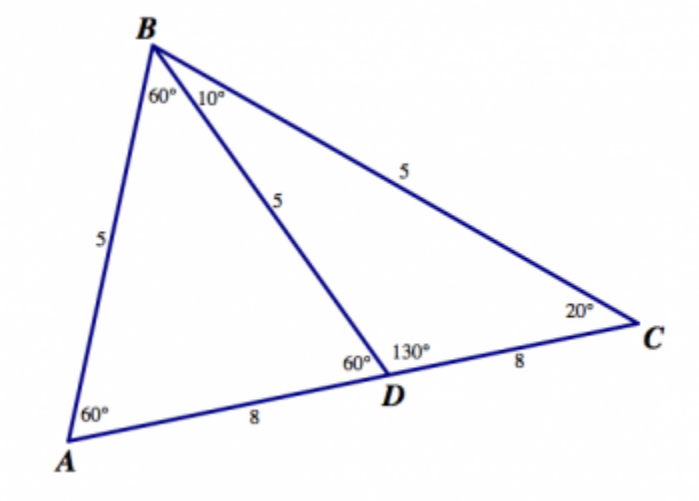
Answer the following questions about the triangle below. Be sure to focus on what you know for sure and not what the picture looks like.



1. If $x = 3$ cm, the triangle is isosceles. Is this possible? Explain your answer.
2. If $x = 8$ cm, the triangle is isosceles. Is this possible? Explain your answer.
3. Give three *impossible* values of x, based on the information in the picture.

Problem 26

Prof. Faber drew this picture on the board, saying it showed three triangles: △ABC, △ABD, and △CBD. Side lengths and angle measurements are shown for each of the triangles.



There are **lots of mistakes** in this picture. Use what you know about side lengths and angles in triangles to find all the mistakes you can. For each mistake, say what is wrong with the picture, and why it's a mistake. Explain your thinking as clearly as you can.

Problem 27

Because of SSS congruence, triangles are exceptionally sturdy. This means they are used frequently in architecture and design to provide supports for buildings, bridges, and other man-made objects. Take your camera with you, and find several places in your neighborhood or near your campus that use triangular supports. Snap a picture, and describe what the structure is and where you see the triangles.

Problem 28

It is possible to create designs that have multiple symmetries. See if you can find images (or create your own!) that have both:

1. reflection symmetry and rotational symmetry,

2. reflection symmetry and translational symmetry, and
3. rotational symmetry and translational symmetry.

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12: CONGRUENCE AND SIMILARITY WITH CONSTRUCTIONS

12.1: Similar Triangles

12.1.1: Proportions

12.1.2: Similar Triangles

12.1.3: Transversals to Three Parallel Lines

12.1.4: Pythagorean Theorem

12.1.5: Special Right Triangles

12.1.6: Distance from a Point to a Line

Congruence and Similarity with Constructions

- 12-1 Congruence Through Construction
- 12-2 Additional Congruence Theorem
- 12-3 Additional Construction
- 12-4 Similar Triangles and Other Similar Figures

Links to sources we want to aggregate for sections 12.1-3:

- 1. https://www.ck12.org/book/ck-12-geometry-second-edition/section/4.2/
  2. https://www.ck12.org/book/ck-12-geometry-second-edition/section/4.3/
  3. https://www.ck12.org/book/ck-12-geometry-second-edition/section/4.4/
  4. https://www.ck12.org/book/ck-12-geometry-second-edition/section/4.5/

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12.1: SIMILAR TRIANGLES

12.1.1: Proportions

12.1.2: Similar Triangles

12.1.3: Transversals to Three Parallel Lines

12.1.4: Pythagorean Theorem

12.1.5: Special Right Triangles

12.1.6: Distance from a Point to a Line

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- 4: Similar Triangles by Henry Africk is licensed CC BY-NC-SA 4.0. Original source: https://academicworks.cuny.edu/ny_oers/44.

12.1.1: Proportions

In our discussion of similar triangles the idea of a proportion will play an important role. In this section we will review the important properties of proportions.

A **proportion** is an equation which states that two fractions are equal. For example, $\frac{2}{6}=\frac{4}{12}$ is a **proportion**. We sometimes say "2 is to 6 as 4 is to 12." This is also written $2:6=4:12$. The **extremes** of this proportion are the numbers 2 and 12 and the **means** are the numbers 6 and 4. Notice that the product of the means $6\times 4=24$ is the same as the product of the extremes $2\times 12=24$.

Theorem 12.1.1.1

If $\frac{a}{b}=\frac{c}{d}$ then $ad=be$. Conversely, if $ad=bc$ then $\frac{a}{b}=\frac{c}{d}$. (The product of the means is equal to the product of the extremes).

EXAMPLES:

- $\frac{2}{6}=\frac{4}{12}$ and $2\times 12=6\times 4$ are both true.
- $\frac{2}{3}=\frac{6}{9}$ and $2\times 9=3\times 6$ are both true.
- $\frac{1}{4}=\frac{4}{12}$ and $1\times 12=4\times 4$ are both false.

Proof of Theorem 1: If $\frac{a}{b}=\frac{c}{d}$, multiply both sides of the equation by bd:

$$\frac{a}{\cancel{b}}(\cancel{b}d)=\frac{c}{\cancel{d}}(b\cancel{d}) \tag{12.1.1.1}$$

We obtain $ad=bc$.

Conversely, if $ad=bc$, divide both sides of the equation by bd:

$$\frac{d\cancel{d}}{b\cancel{d}}=\frac{\cancel{b}c}{\cancel{b}d} \tag{12.1.1.2}$$

The result is $\frac{a}{b}=\frac{c}{d}$.

The following theorem shows that we can interchange the means or the extremes or both of them simultaneously and still have a valid proportion:

Theorem 12.1.1.2

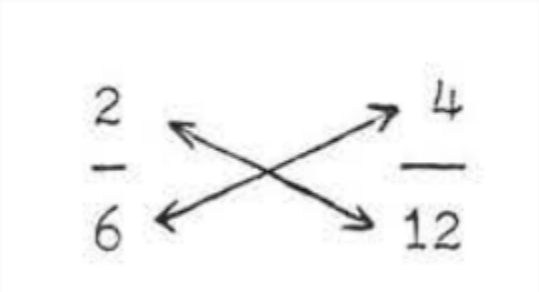
If one of the following is true then they are all true:

1. $\frac{a}{b}=\frac{c}{d}$
2. $\frac{a}{c}=\frac{b}{d}$
3. $\frac{d}{b}=\frac{c}{a}$
4. $\frac{d}{c}=\frac{b}{a}$

Proof available at the end of this section.

EXAMPLE: $\frac{2}{6}=\frac{4}{12}, \frac{2}{4}=\frac{6}{12}, \frac{12}{6}=\frac{4}{2}, \frac{12}{4}=\frac{6}{2}$ are all true because $2 \times 12=6 \times 4$.

The process of converting a proportion $\frac{2}{6}=\frac{4}{12}$ to the equivalent equation $2 \times 12=6 \times 4$ is sometimes called **cross multiplication**. The idea is conveyed by the following notation:



Example 12.1.1.1

Find x : $\frac{3}{x}=\frac{4}{20}$

Solution

By "cross multiplication,"

$$\begin{aligned} 3(20) &= x(4) \\ 60 &= 4x \\ 15 &= x \end{aligned} \tag{12.1.1.3}$$

Check:

$\frac{3}{x}=\frac{3}{15}=\frac{1}{5}$. $\frac{4}{20}=\frac{1}{5}$.

Answer: $x=15$.

Example 12.1.1.2

Find x: $\frac{x-1}{x-3}=\frac{2x+2}{x+1}$

Solution

$$\begin{aligned} (x-1)(x+1) &= (x-3)(2x+2) \\ x^2-1 &= 2x^2-4x-6 \\ 0 &= x^2-4x-5 \\ 0 &= (x-5)(x+1) \\ 0 &= x-5 \qquad 0=x+1 \\ 5 &= x \qquad\quad -1=x \end{aligned} \tag{12.1.1.4}$$

Check, $x=5$:

$\frac{x-1}{x-3}=\frac{5-1}{5-3}=\frac{4}{2}=2$. $\frac{2x+2}{x+1}=\frac{2(5)+2}{5+1}=\frac{12}{6}=2$.

Check, $x=-1$:

$\frac{x-1}{x-3}=\frac{-1-1}{-1-3}=\frac{-2}{-4}=\frac{1}{2}$. $\frac{2x+2}{x+1}=\frac{2(-1)+2}{-1+1}=\frac{-2+2}{0}=\frac{0}{0}$.

Since $\frac{0}{0}$ is undefined, we reject this answer.

Answer: $x=5$.

Problems

1 - 12. Find x:

1. $\frac{6}{x}=\frac{18}{3}$
2. $\frac{4}{x}=\frac{2}{6}$
3. $\frac{7}{1}=\frac{x}{3}$
4. $\frac{x}{8}=\frac{9}{6}$
5. $\frac{7}{1}=\frac{x}{3}$
6. $\frac{10}{2}=\frac{25}{x}$
7. $\frac{x+5}{x}=\frac{5}{4}$
8. $\frac{x-6}{4}=\frac{5}{10}$
9. $\frac{3+x}{x}=\frac{3}{2}$
10. $\frac{x}{x+3}=\frac{4}{x}$
11. $\frac{3x-3}{2x+6}=\frac{x-1}{x}$
12. $\frac{3x-6}{x-2}=\frac{2x+2}{x-1}$

Answers

Theorem 12.1.1.2

If any one of these proportions is true when $ad=bc$ by Theorem 12.1.1.1. The remaining proportions can then be obtained from $ad=bc$ by division, as in Theorem 12.1.1.1.

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12.1.2: Similar Triangles

Two triangles are said to be similar if they have equal sets of angles. In Figure 12.1.2.1, $\triangle ABC$ is similar to $\triangle DEF$. The angles which are equal are called **corresponding angles**. In Figure 12.1.2.1, $\angle A$ corresponds to $\angle D$, $\angle B$ corresponds to $\angle E$, and $\angle C$ corresponds to $\angle F$. The sides joining corresponding vertices are called **corresponding sides**. In Figure 12.1.2.1, AB corresponds to DE, BC corresponds to EF, and AC corresponds to DF. The symbol for similar is $\sim$. The similarity statement $\triangle ABC \sim \triangle DEF$ will always be written so that corresponding vertices appear in the same order.

For the triangles in Figure 12.1.2.1, we could also write $\triangle BAC \sim \triangle BDF$ or $\triangle ACB \sim \triangle DFE$ but never $\triangle ABC \sim \triangle EDF$ nor $\triangle ACB \sim \triangle DEF$.

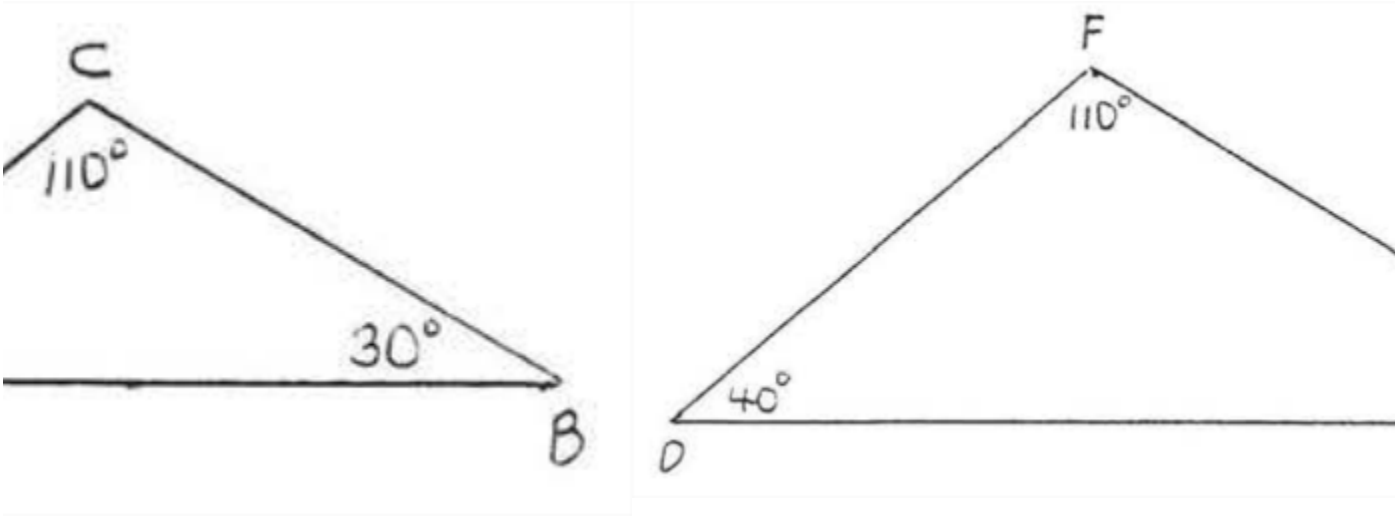
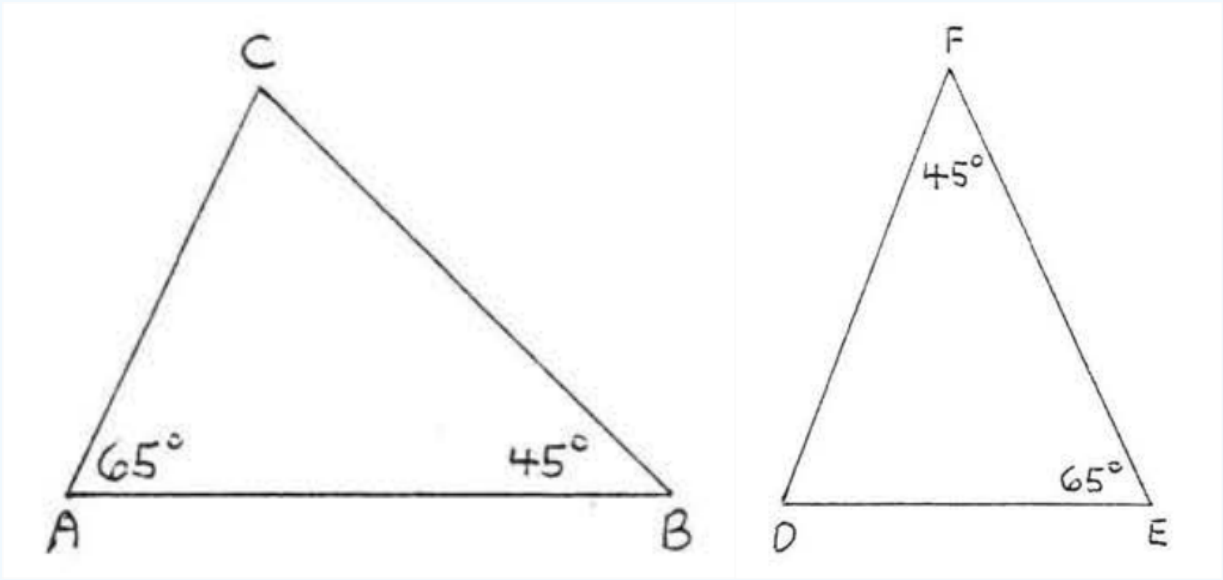


Figure 12.1.2.1: $\triangle ABC$ is similar to $\triangle DEF$.

We can tell which sides correspond from the similarity statement. For example, if $\triangle ABC \sim \triangle DEF$, then side AB corresponds to side DE because both are the first two letters. BC corresponds to EF because both are the last two letters, AC corresponds to DF because both consist of the first and last letters.

Example 12.1.2.1

Determine if the triangles are similar, and if so, write the similarity statement:



Solution

$$\angle C = 180^\circ - (65^\circ + 45^\circ) = 180^\circ - 110^\circ = 70^\circ$$

$$\angle D = 180^\circ - (65^\circ + 45^\circ) = 180^\circ - 110^\circ = 70^\circ$$

Therefore both triangles have the same angles and $\triangle ABC \sim \triangle EFD$.

Answer: $\triangle ABC \sim \triangle EFD$.

Example A suggests that to prove similarity it is only necessary to know that **two** of the corresponding angles are equal:

Theorem 12.1.2.1

Two triangles are similar if two angles of one equal two angles of the other $(AA = AA)$.

In Figure 12.1.2.2, $\triangle ABC \sim \triangle DEF$ because $\angle A = \angle D$ and $\angle B = \angle E$.

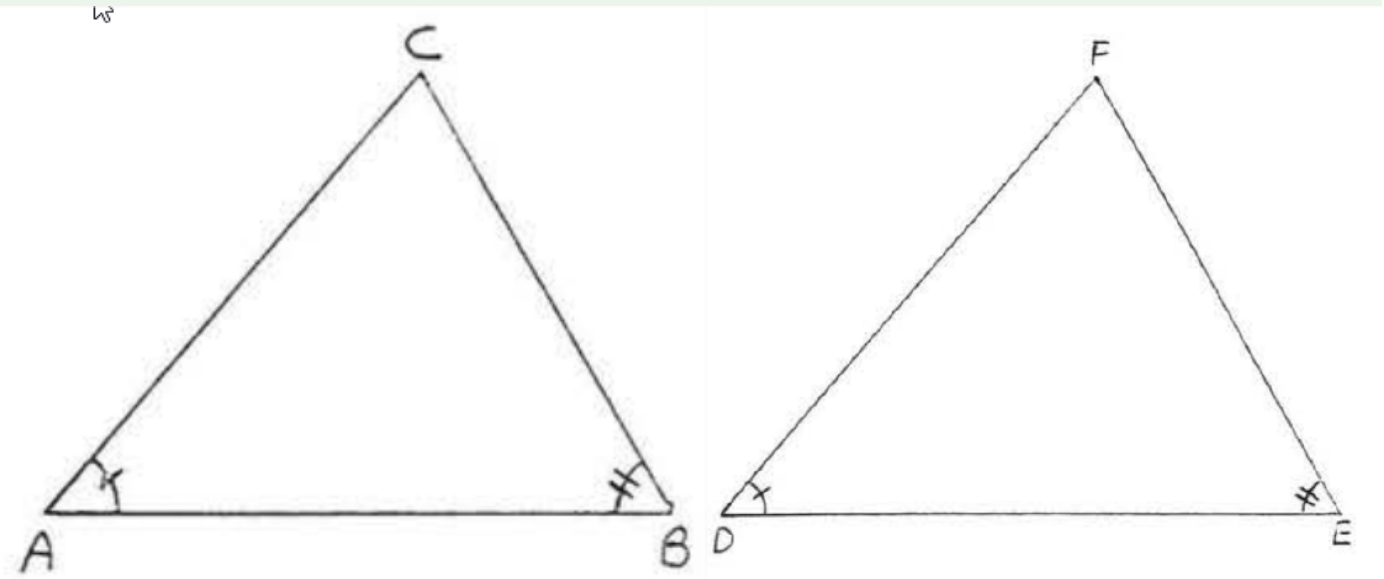
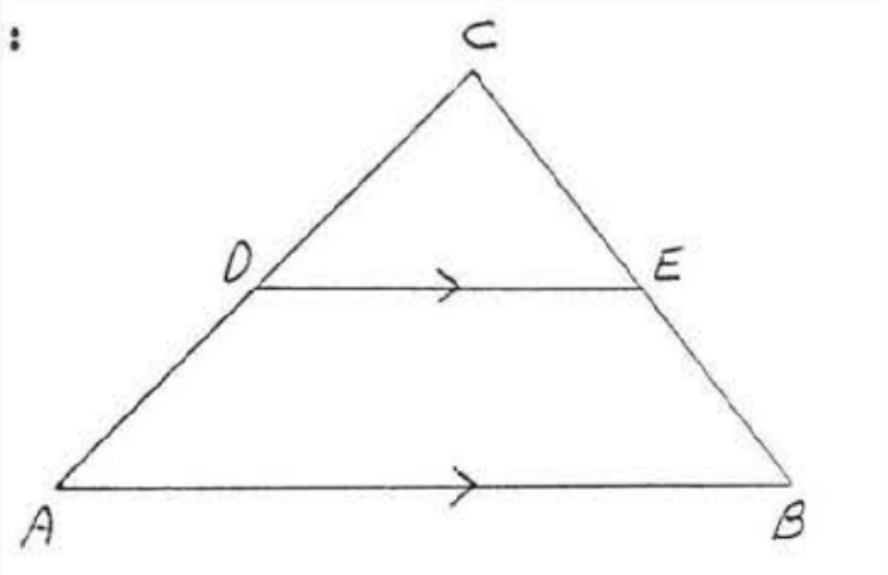


Figure 12.1.2.2. $\triangle ABC \sim \triangle DEF$ because $AA = AA$.

Proof available at the end of this section.

Example 12.1.2.2

Determine which triangles are similar and write a similarity statement:



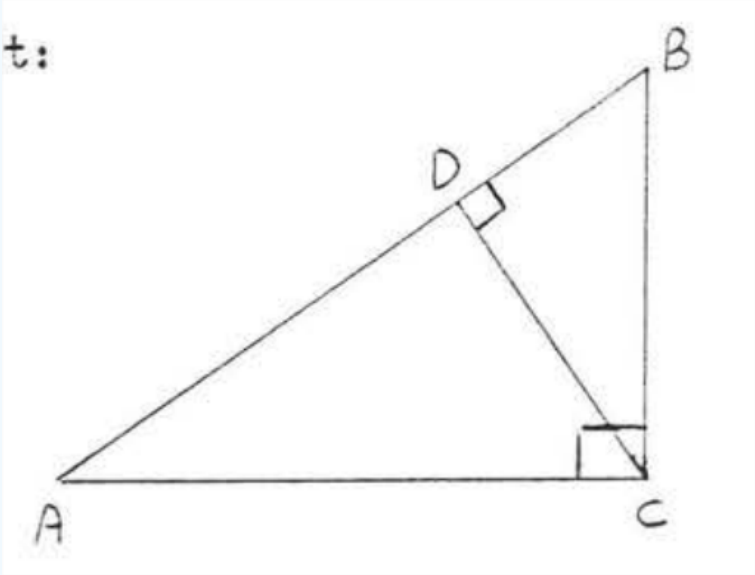
Solution

$\angle A = \angle CDE$ because they are corresponding angles of parallel lines. $\angle C = \angle C$ because of identity. Therefore $\triangle ABC \sim \triangle DEC$ by $AA = AA$.

Answer: $\triangle ABC \sim \triangle DEC$.

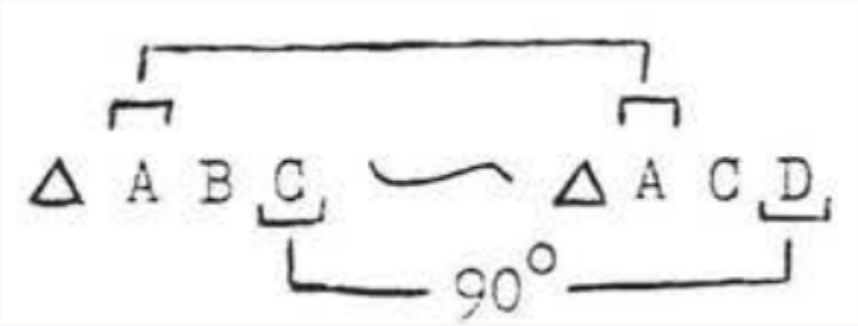
Example 12.1.2.3

Determine which triangles are similar and write a similarity statement:

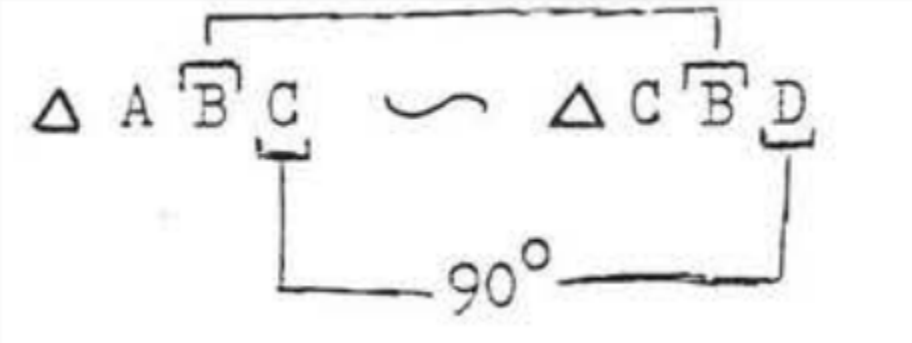


Solution

$\angle A = \angle A$ identity. $\angle ACB = \angle ADC = 90°$. Therefore



Also $\angle B = \angle B$, identity, $\angle BDC = \angle BCA = 90°$. Therefore



Answer: $\triangle ABC \sim \triangle ACD \sim \triangle CBD$.

Similar triangIes are important because of the following theorem:

Theorem 12.1.2.2

The corresponding sides of similar triangles are proportional. This means that if $\triangle ABC \sim \triangle DEF$ then

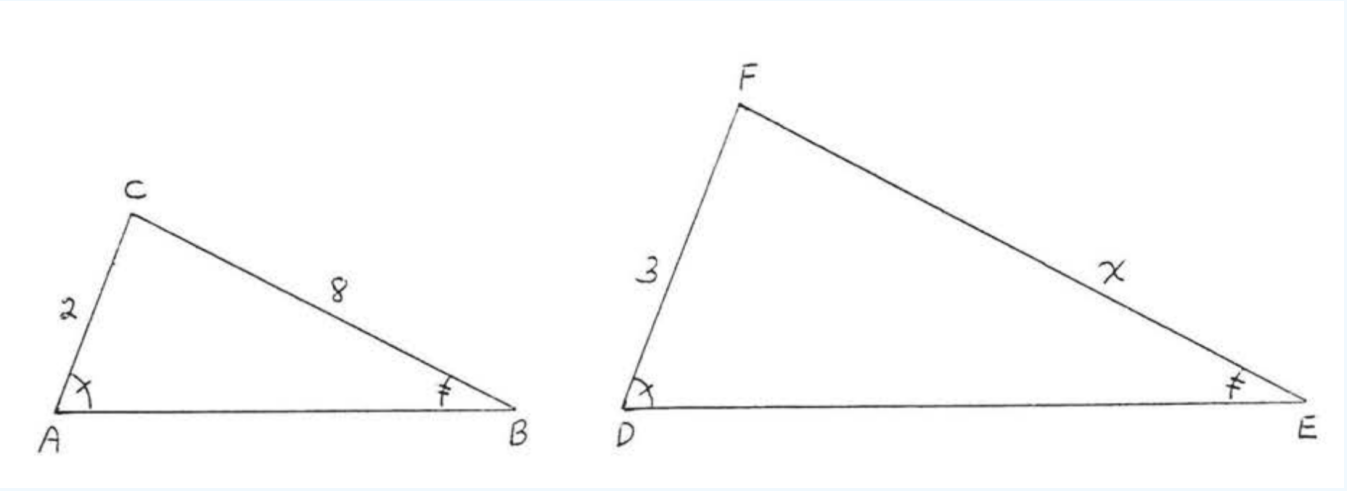
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

That is, the first two letters of $\triangle ABC$ are to the first two letters of $\triangle DEF$ as the last two letters of $\triangle ABC$ are to the last two letters of $\triangle DEF$ as the first and last letters of $\triangle ABC$ are to the first and last letters of $\triangle DEF$.

Before attempting to prove Theorem 12.1.2.2, we will give several examples of how it is used:

Example 12.1.2.4

Find x:



Solution

$\angle A = \angle D$ and $\angle B = \angle E$ so $\triangle ABC \sim \triangle DEF$. By Theorem 12.1.2.2,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

We will ignore $\frac{AB}{DE}$ here since we do not know and do not have to find either AB or DE.

$$\begin{aligned} \frac{BC}{EF} &= \frac{AC}{DF} \\ \frac{8}{x} &= \frac{2}{3} \\ 24 &= 2x \\ 12 &= x \end{aligned} \tag{12.1.2.1}$$

Check:

$$\frac{BC}{EF} = \frac{AC}{DF}$$

$$\frac{8}{x} \quad \frac{2}{3}$$

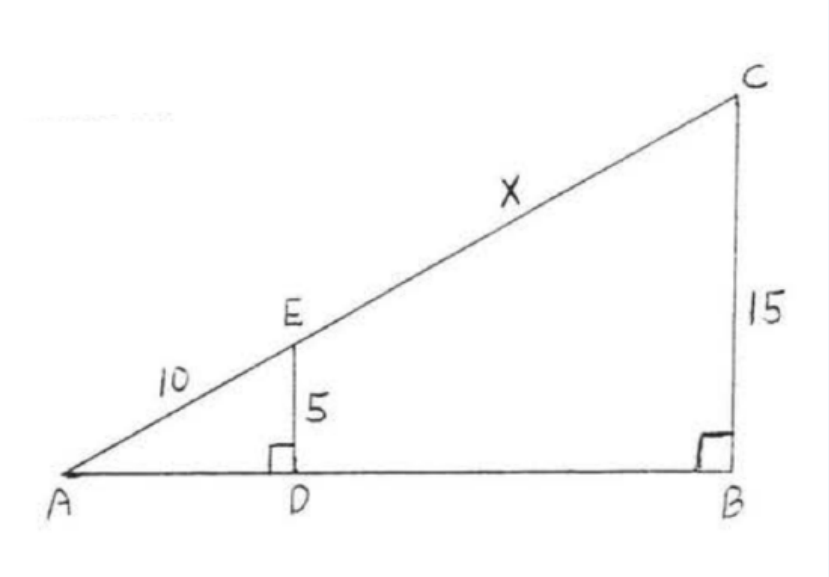
$$\frac{8}{12}$$

$$\frac{2}{3}$$

Answer: $x = 12$.

Example 12.1.2.5

Find x:



Solution

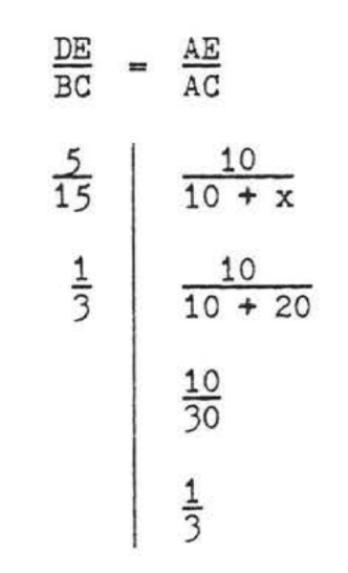
$\angle A = \angle A, \angle ADE = \angle ABC$, so $\triangle ADE \sim \triangle ABC$ by $AA = AA$.

$$\frac{AD}{AB} = \frac{DE}{BC} = \frac{AE}{AC}.$$

We ignore $\frac{AD}{AB}$.

$$\begin{aligned} \frac{DE}{BC} &= \frac{AE}{AC} \\ \frac{5}{15} &= \frac{10}{10+x} \\ 5(10+x) &= 15(10) \\ 50+5x &= 150 \\ 5x &= 150-50 \\ 5x &= 100 \\ x &= 20 \end{aligned} \tag{12.1.2.2}$$

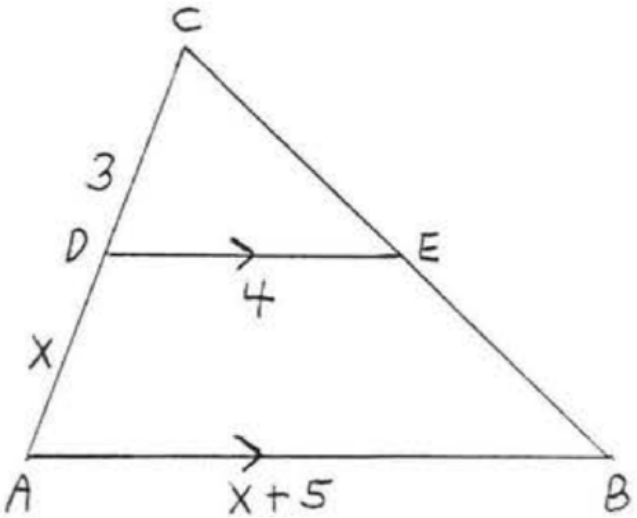
Check:



Answer: $x = 20$.

Example 12.1.2.6

Find x:



Solution

$\angle A = \angle CDE$ because they are corresponding angles of parallel lines. $\angle C = \angle C$ because of identity. Therefore $\triangle ABC \sim \triangle DEC$ by $AA = AA$.

$$\frac{AB}{DE} = \frac{BC}{EC} = \frac{AC}{DC}$$

We ignore $\frac{BC}{EC}$:

$$\begin{aligned} \frac{AB}{DE} &= \frac{AC}{DC} \\ \frac{x+5}{4} &= \frac{x+3}{3} \\ (x+5)(3) &= (4)(x+3) \\ 3x+15 &= 4x+12 \\ 15-12 &= 4x-3x \\ 3 &= x \end{aligned} \tag{12.1.2.3}$$

Check:

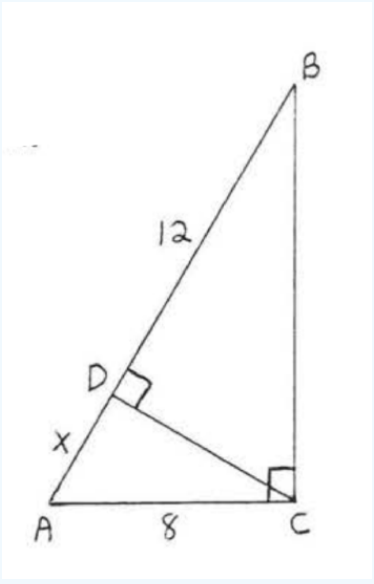
$$\frac{AB}{DE} = \frac{AC}{DC}$$

$\frac{x+5}{4}$	$\frac{x+3}{3}$
$\frac{3+5}{4}$	$\frac{3+3}{3}$
$\frac{8}{4}$	$\frac{6}{3}$
2	2

Answer: $x = 3$.

Example 12.1.2.7

Find x:



Solution

$\angle A = \angle A$, $\angle ACB = \angle ADC = 90°$, $\triangle ABC \sim \triangle ACD$.

$$\begin{aligned} \frac{AB}{AC} &= \frac{AC}{AD} \\ \frac{x+12}{8} &= \frac{8}{x} \\ (x+12)(x) &= (8)(8) \\ x^2+12x &= 64 \\ x^2+12x-64 &= 0 \\ (x-4)(x+16) &= 0 \\ x=4 \qquad x &= -16 \end{aligned} \tag{12.1.2.4}$$

We reject the answer $x=-16$ because $AD=x$ cannot be negative.

Check, $x=4$

$$\frac{AB}{AC} = \frac{AC}{AD}$$

$$\frac{x+12}{8} \Big| \frac{8}{x}$$

$$\frac{4+12}{8} \Big| \frac{8}{4}$$

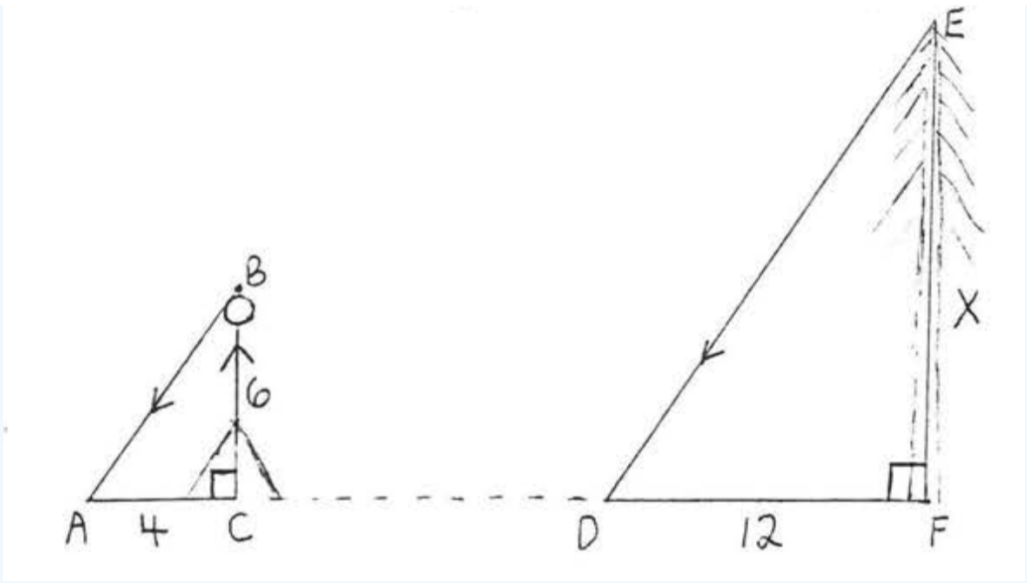
$$\frac{16}{8} \Big| 2$$

$$2 \Big|$$

Answer: $x=4$.

Example 12.1.2.8

A tree casts a shadow 12 feet long at the same time a 6 foot man casts a shadow 4 feet long. What is the height of the tree?



Solution

In the diagram AB and DE are parallel rays of the sun. Therefore $\angle A = \angle D$ because they are corresponding angles of parallel lines with respect to the transversal AF. Since also $\angle C = \angle F = 90°$, we have $\triangle ABC \sim \triangle DEF$ by $AA = AA$.

$$\begin{aligned} \frac{AC}{DF} &= \frac{BC}{EF} \\ \frac{4}{12} &= \frac{6}{x} \\ 4x &= 72 \\ x &= 18 \end{aligned} \tag{12.1.2.5}$$

Answer: $x = 18$ feet.

Proof of Theorem 12.1.2.2 ("The corresponding sides of similar triangles are proportional"):

We illustrate the proof using the triangles of Example 12.1.2.4 (Figure 12.1.2.3). The proof for other similar triangles follows the same pattern. Here we will prove that $x = 12$ so that $\frac{2}{3} = \frac{8}{x}$.

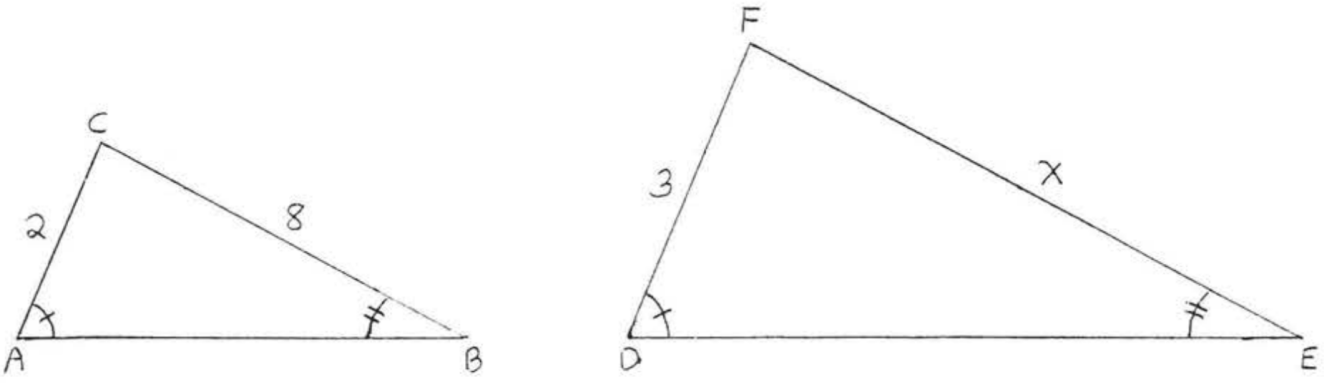


Figure 12.1.2.3. The triangles of Example 12.1.2.4.

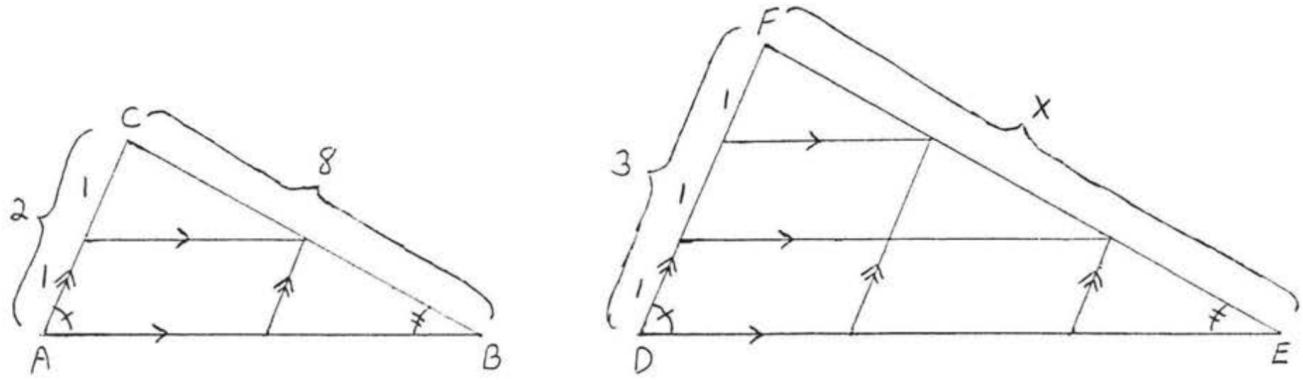


Figure 12.1.2.4: Draw lines parallel to the sides of $\triangle ABC$ and $\triangle DEF$.

First draw lines parallel to the sides of $\triangle ABC$ and $\triangle DEF$ as shown in Figure 12.1.2.4. The corresponding angles of these parallel lines are equal and each of the parallelograms with a side equal to 1 has its opposite side equal to 1 as well, Therefore all of the small triangles with a side equal to 1 are congruent by $AAS = AAS$. The corresponding sides of these triangles form side $BC = 8$ of $\triangle ABC$ (see Figure 12.1.2.5). Therefore each of these sides must equal 4 and $x = EF = 4 + 4 + 4 = 12$ (Figure 12.1.2.6).

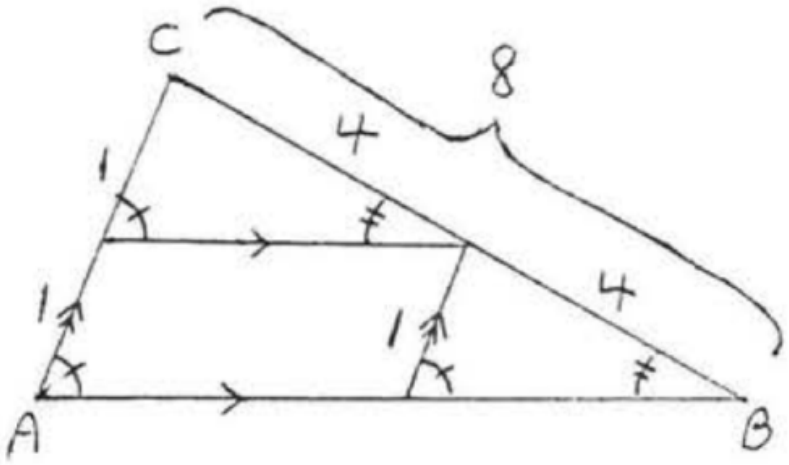


Figure 12.1.2.5. The small triangles are congruent hence the corresponding sides lying on BC must each be equal to 4.

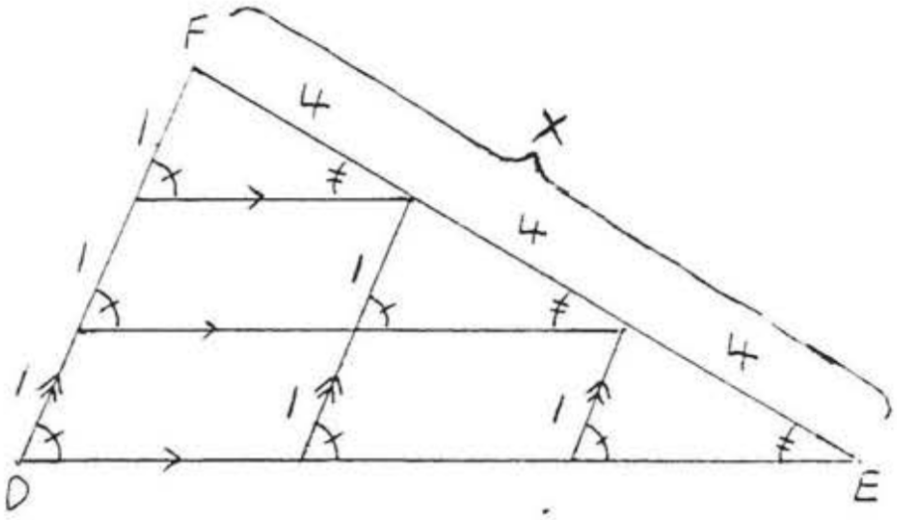


Figure 12.1.2.6. The small triangles of $\triangle DEF$ are congruent to the small triangles of $\triangle ABC$ hence $x = EF = 4 + 4 + 4 = 12$.

(Note to instructor: This proof can be carried out whenever the lengths of the sides of the triangles are rational numbers. However, since irrational numbers can be approximated as closely as necessary by rationals, the proof extends to that case as well.)

Historical Note

Thales (c. 600 B.C.) used the proportionality of sides of similar triangles to measure the heights of the pyramids in Egypt. His method was much like the one we used in Example 12.1.2.8 to measure the height of trees.

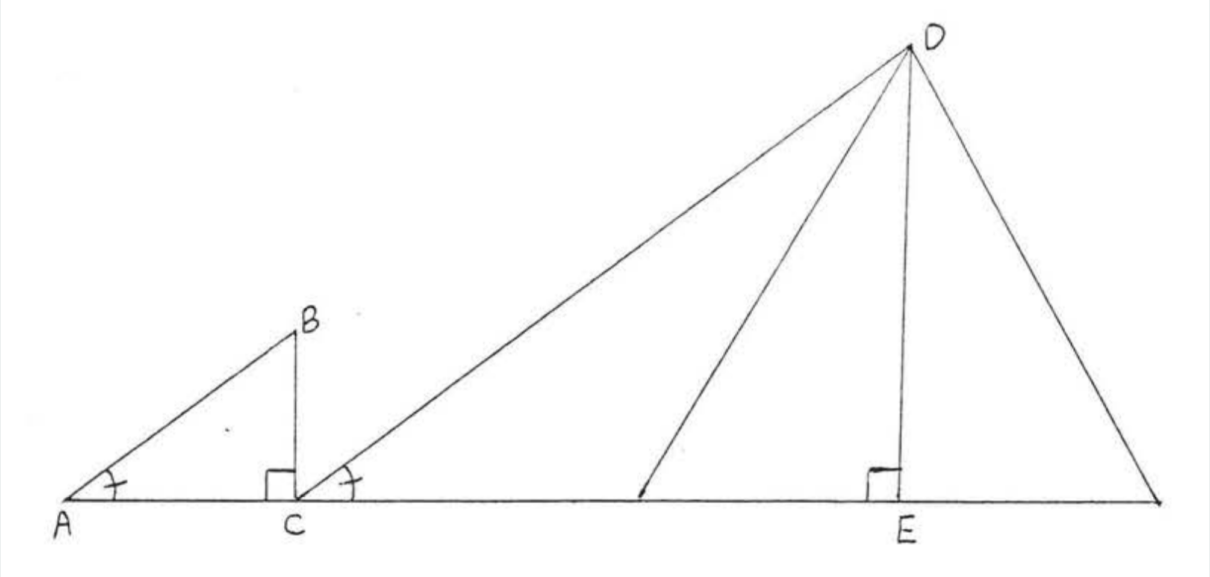


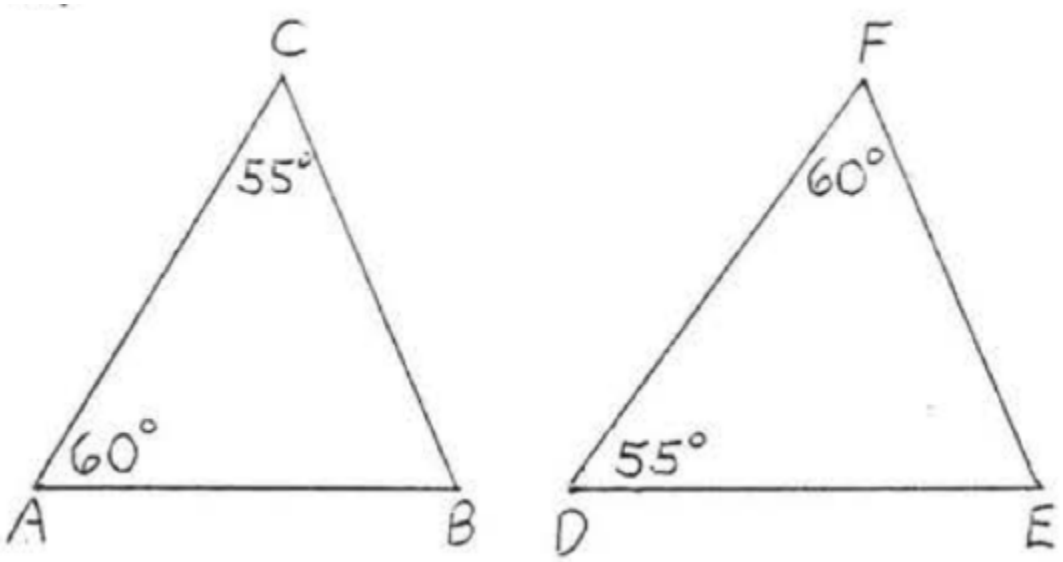
Figure 12.1.2.7. Using similar triangles to measure the height of a pyramid.

In Figure 12.1.2.7, DE represent the height of the pyramid and CE is the length of its shadow. BC represents a vertical stick and AC is the length of its shadow. We have $\triangle ABC \sim \triangle CDE$. Thales was able to measure directly the lengths AC, BC, and CE. Substituting these values in the proportion $\frac{BC}{DE} = \frac{AC}{CE}$, he was able to find the height DE.

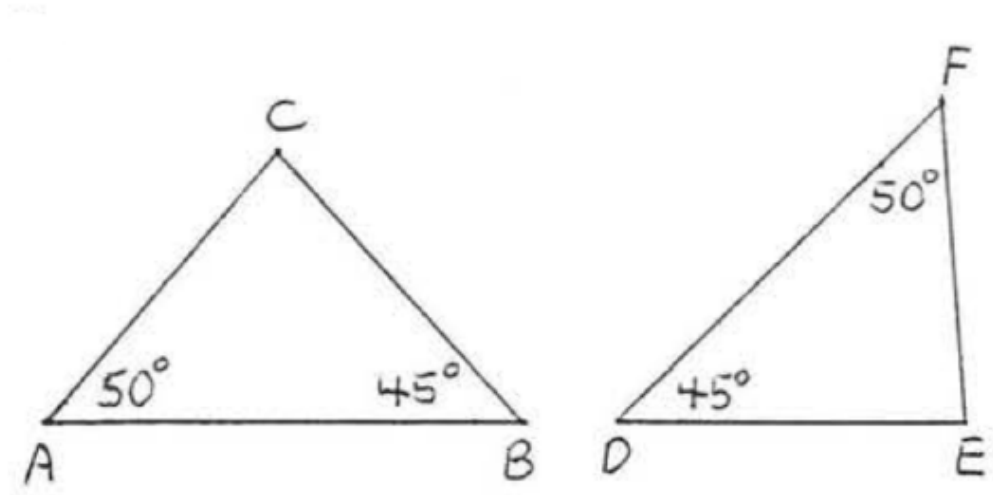
Problems

1 - 6. Determine which triangles are similar and write the similarity statement:

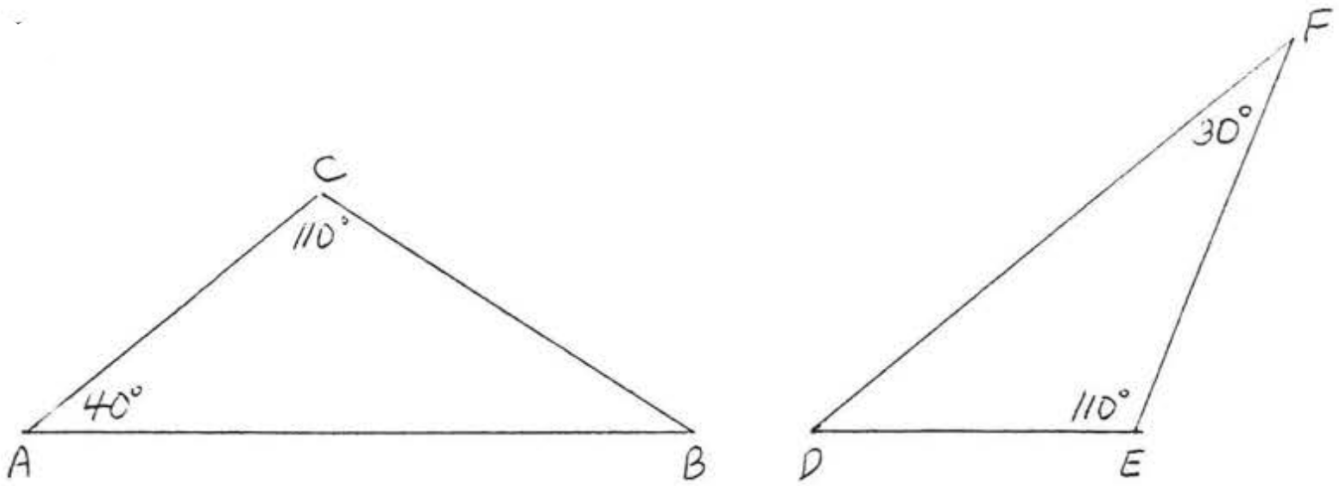
1.



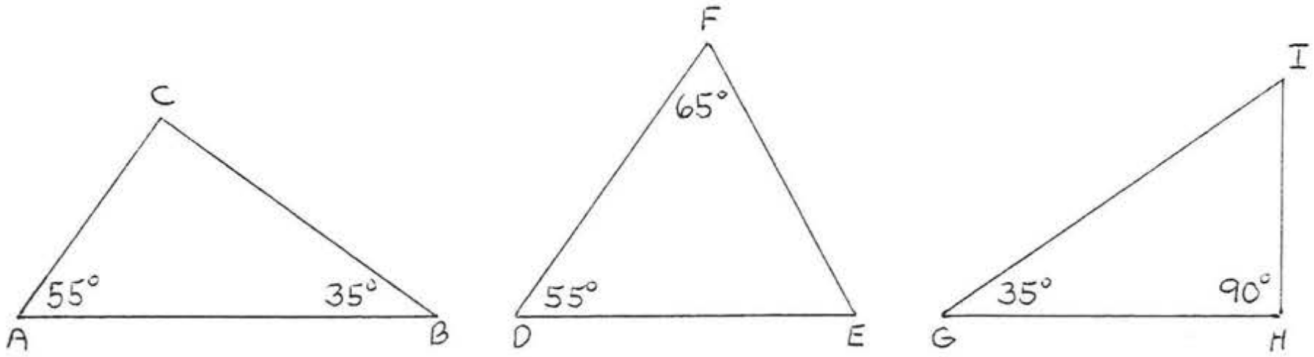
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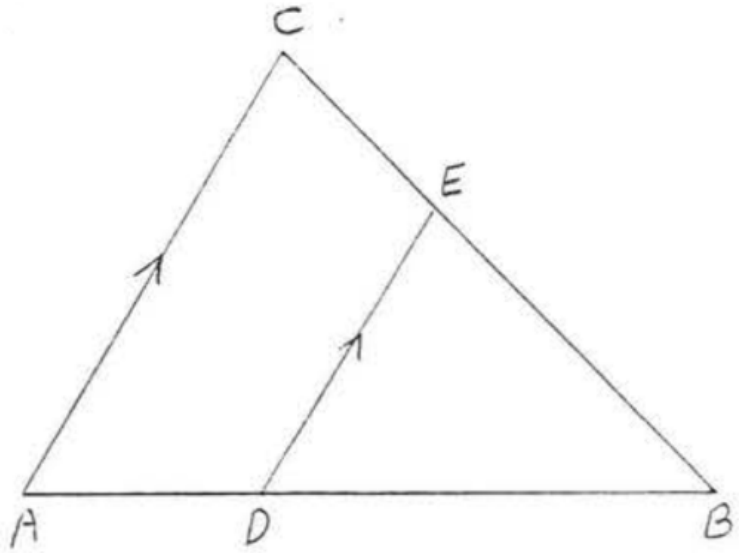
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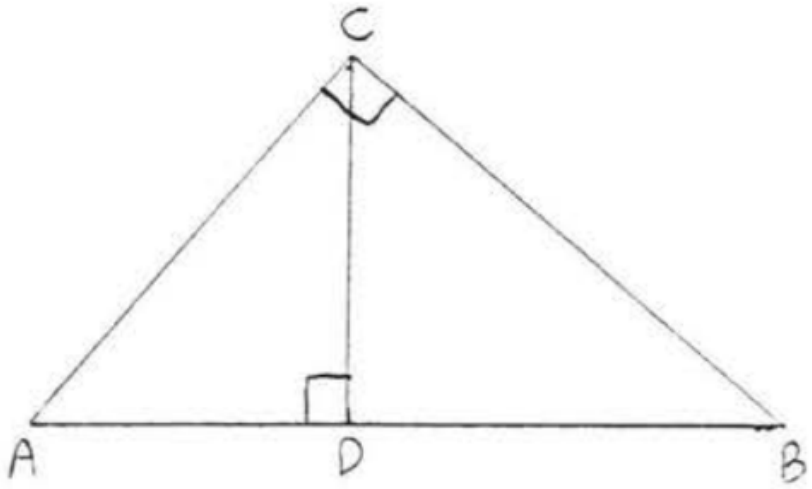
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5.



6.



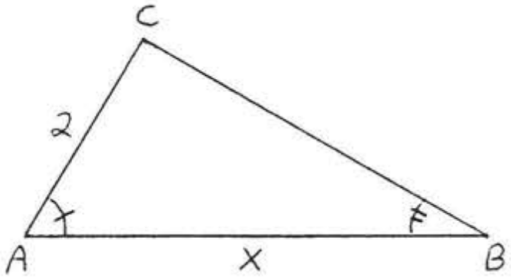
7 - 22. For each of the following

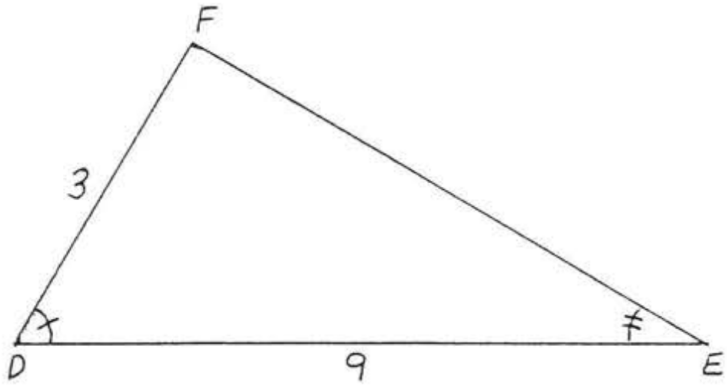
(1) write the similarity statement

(2) write the proportion between the corresponding sides

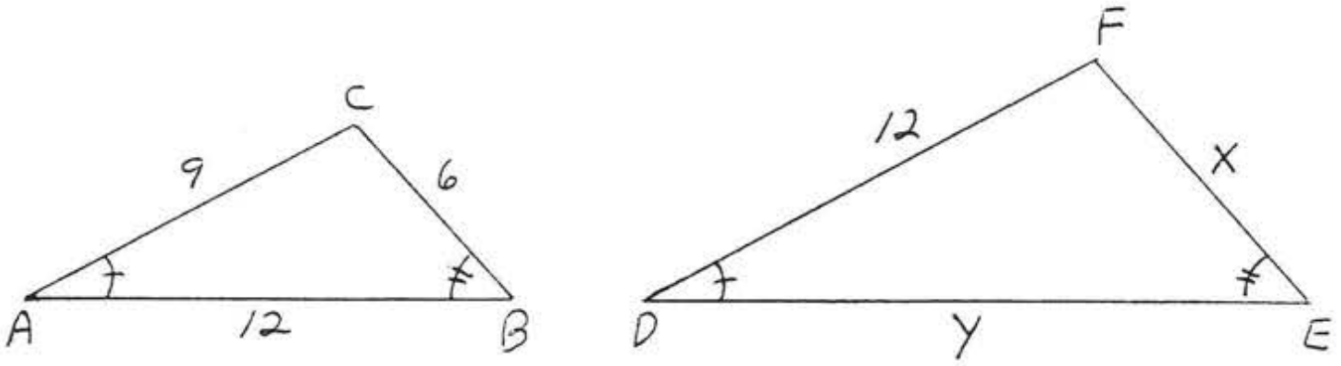
(3) solve for x or x and y.

7.

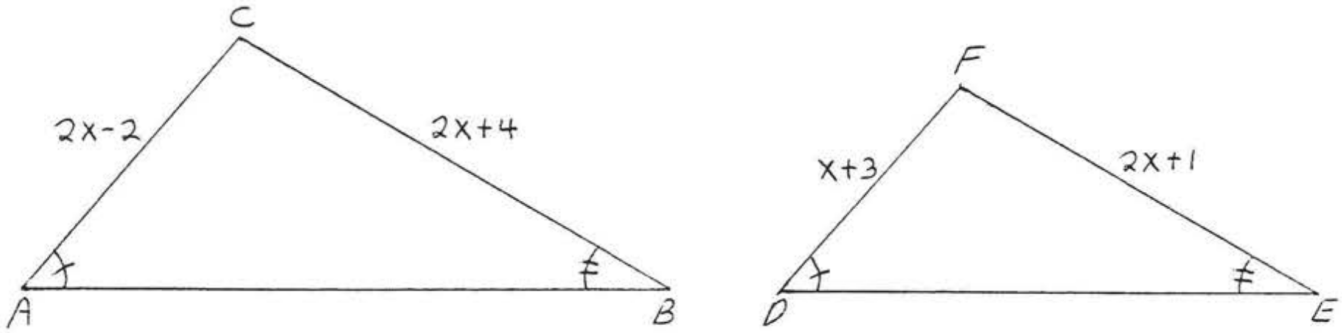




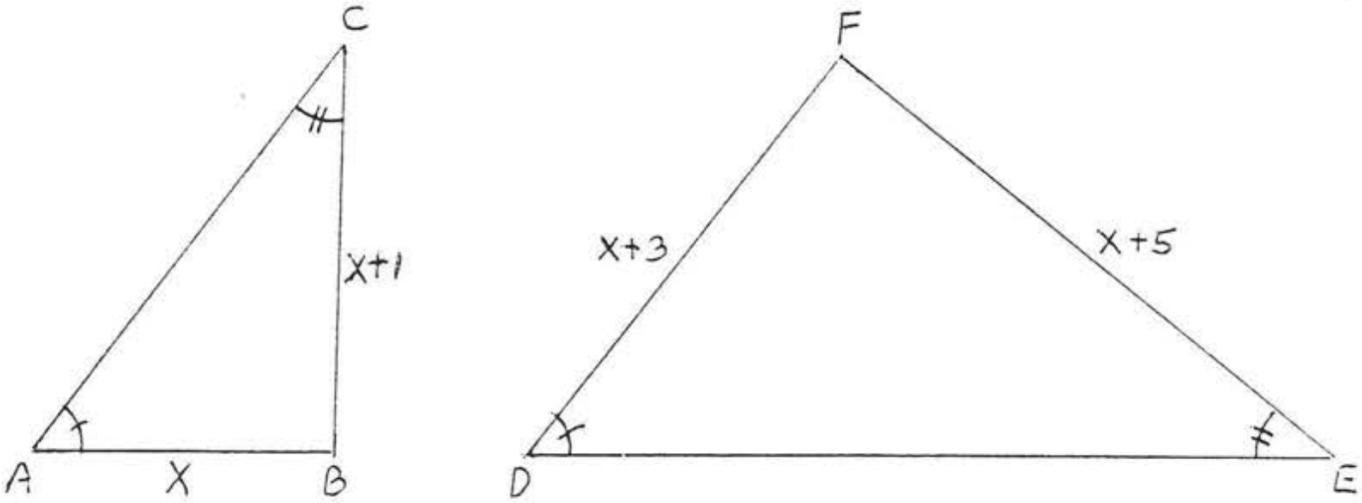
8.



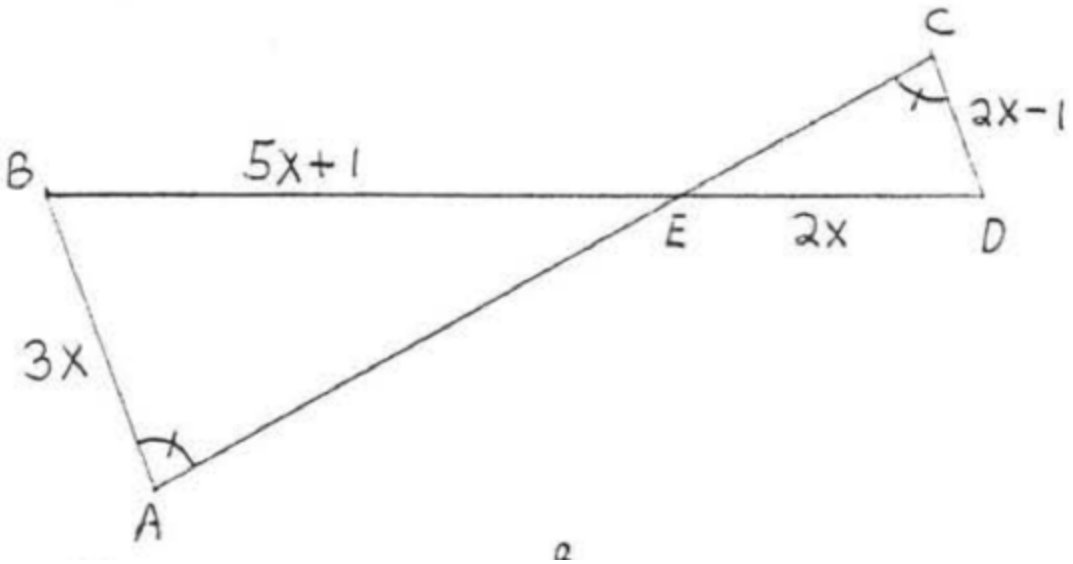
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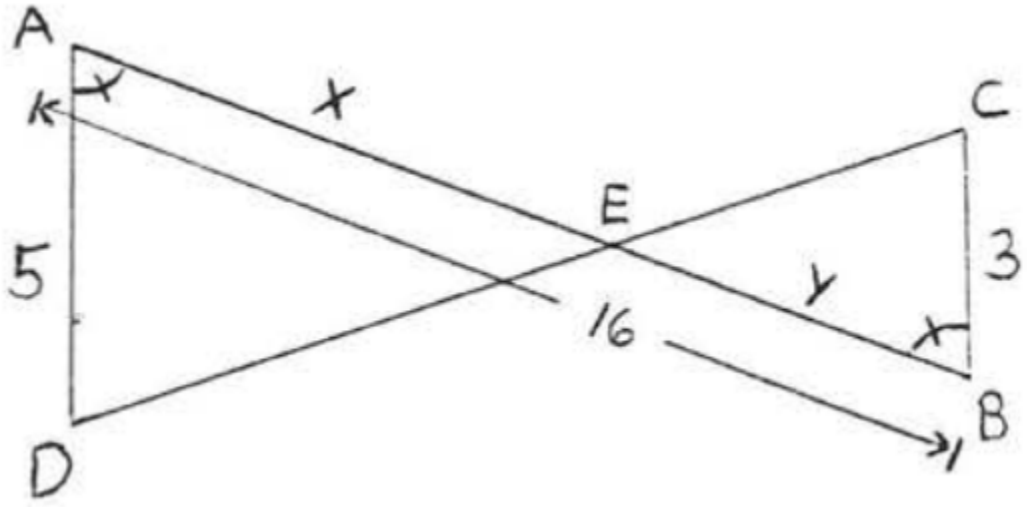
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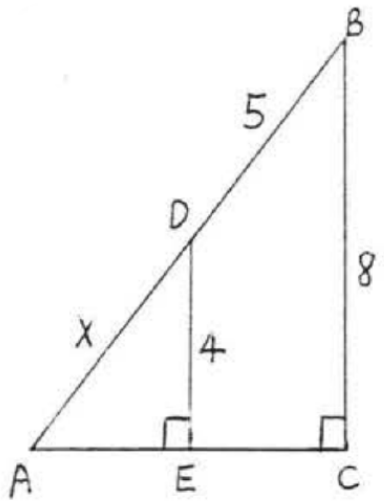
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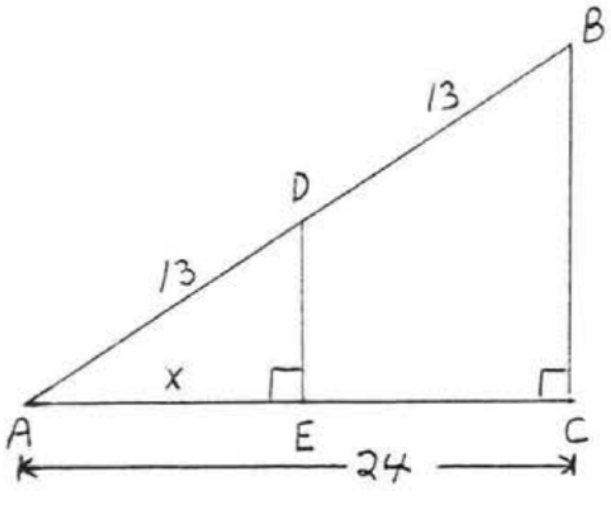
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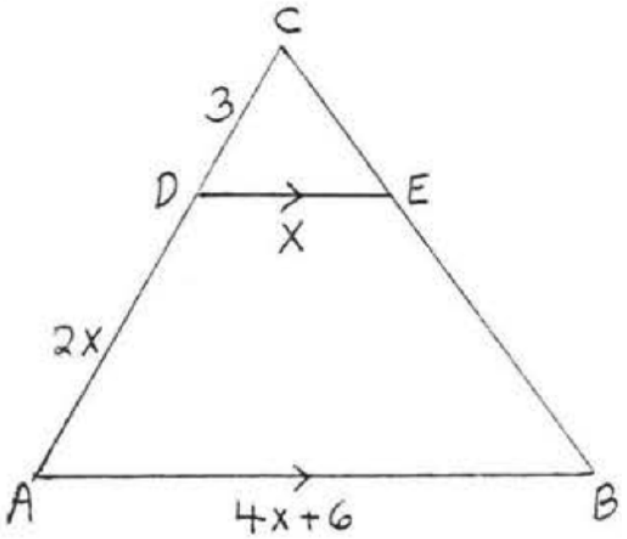
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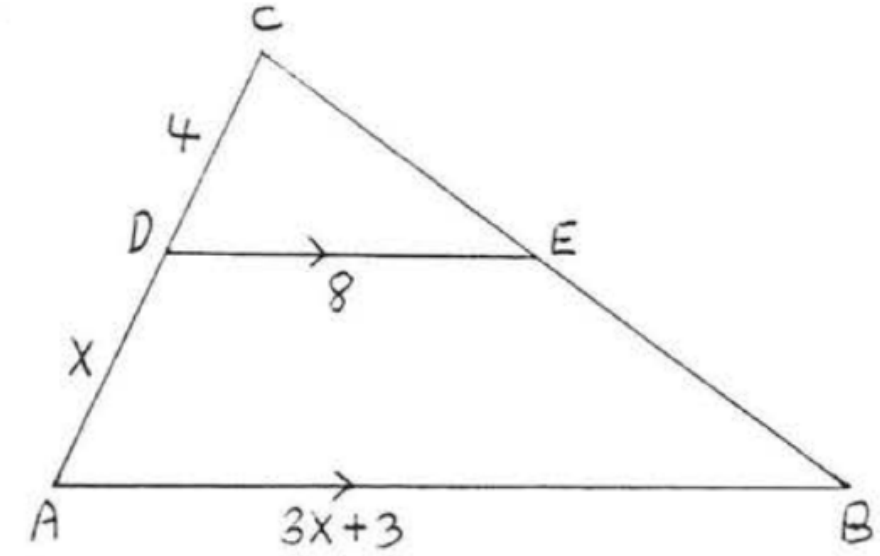
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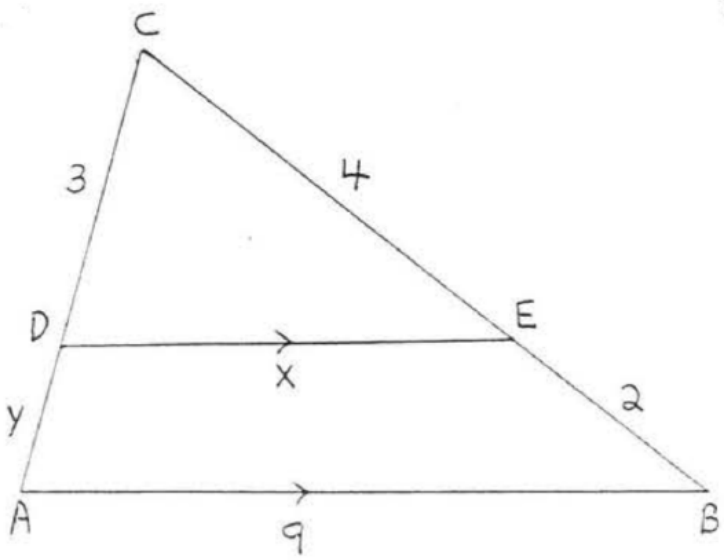
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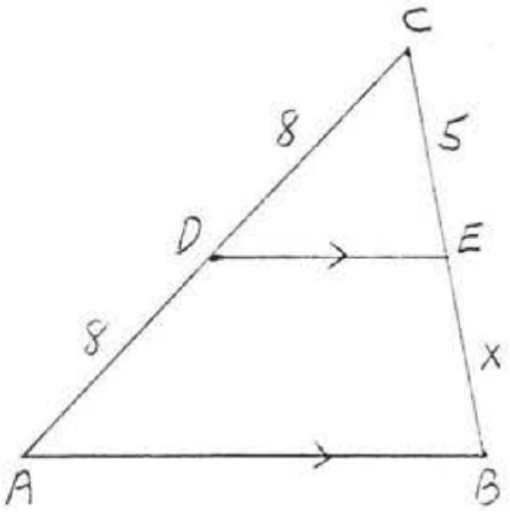
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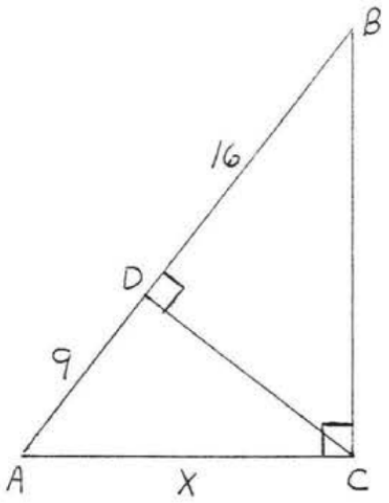
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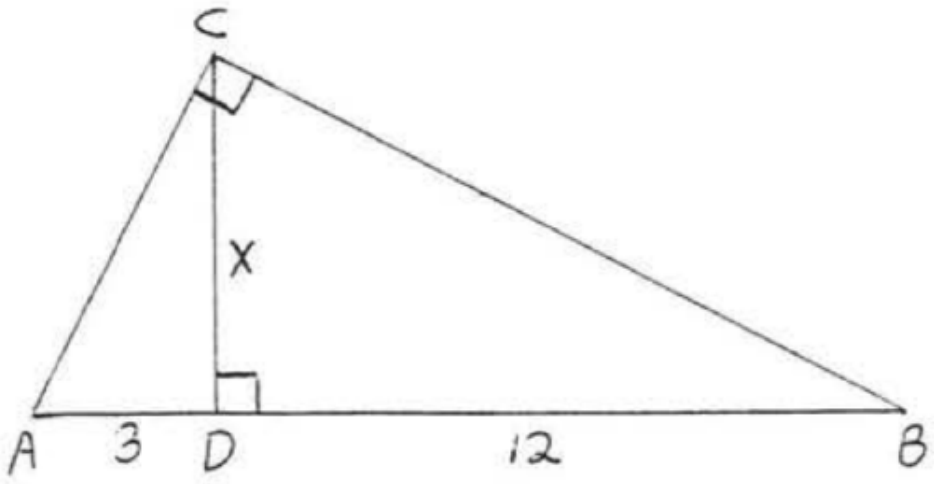
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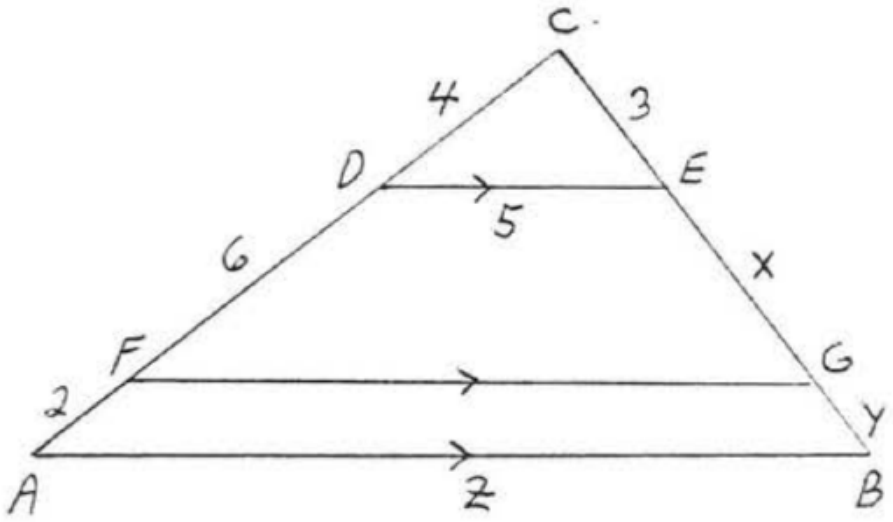
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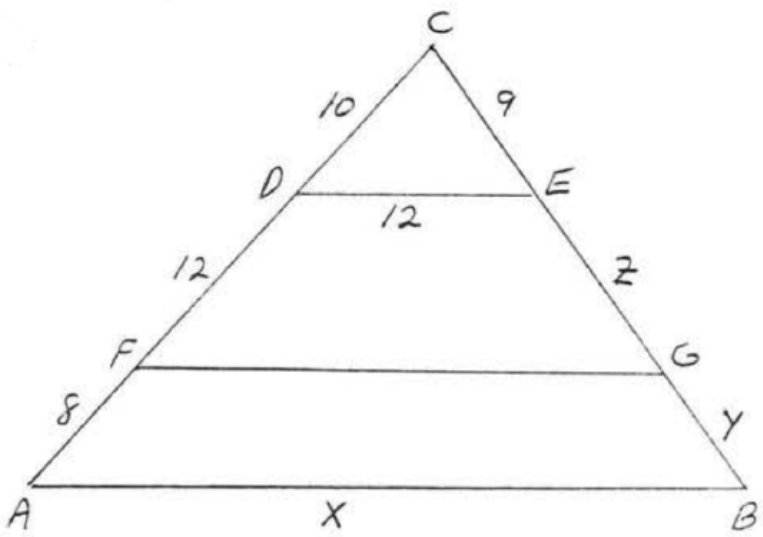
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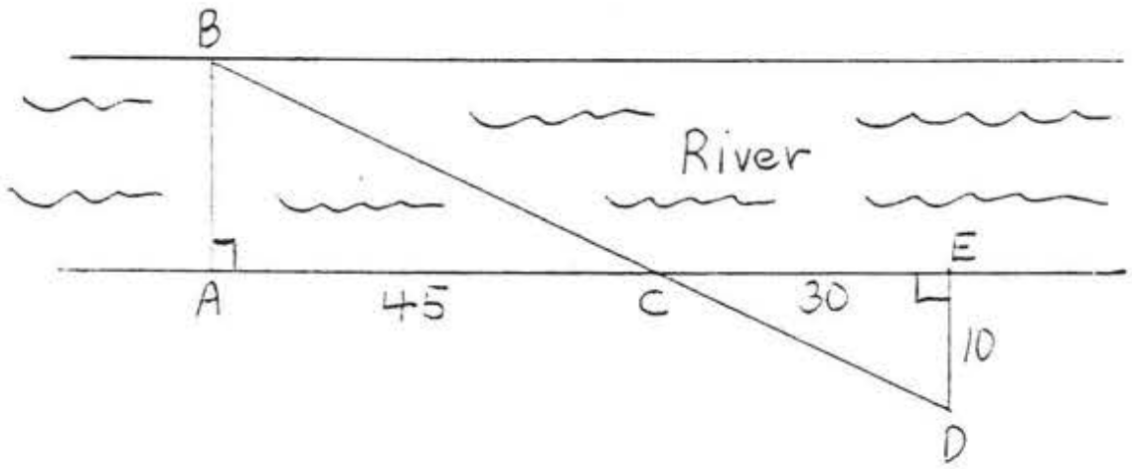


22.



23. A flagpole casts a shadow 80 feet long at the same time a 5 foot boy casts a shadow 4 feet long. How tall is the flagpole?

24. Find the width AB of the river:



Answers

Theorem 12.1.2.1

$\triangle C = 180^\circ - (\angle A + \angle B) = 180^\circ - (\angle D + \angle E) = \angle F.$

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12.1.3: Transversals to Three Parallel Lines

In Chapter 1 we defined a transversal to be a line which intersects two other lines, We will now extend the definition to a line which intersects three other lines. In Figure 12.1.3.1, AB is a transversal to three lines.

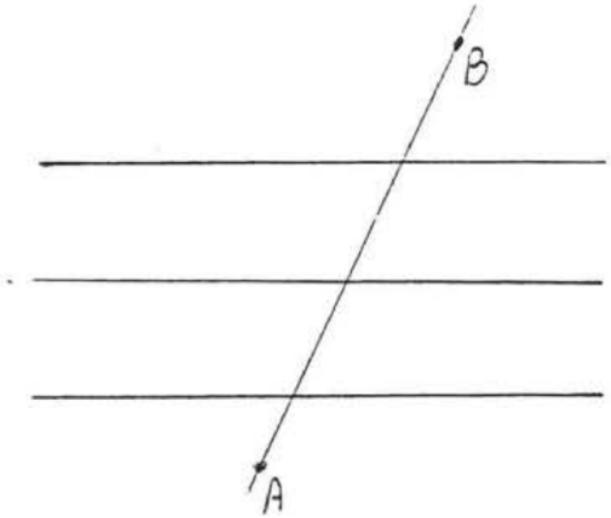


Figure 12.1.3.1: AB is a transversal to three lines.

If the three lines are parallel and we have two such transversals we may state the following theorem:

Theorem 12.1.3.1

The line segments formed by two transversals crossing three parallel lines are proportional.

In Figure 12.1.3.2, $\frac{a}{b} = \frac{c}{d}$.

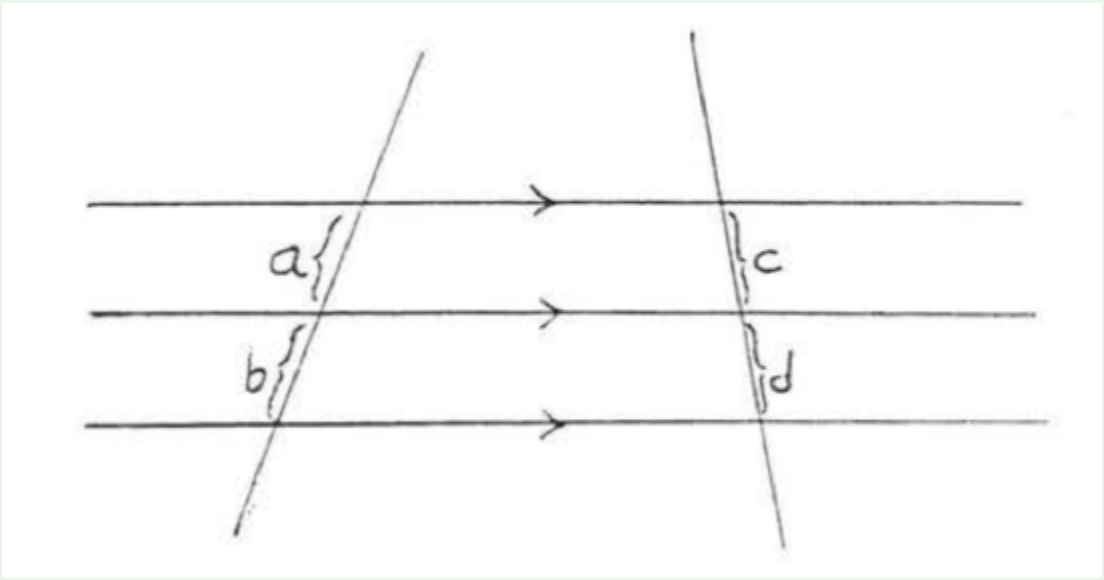
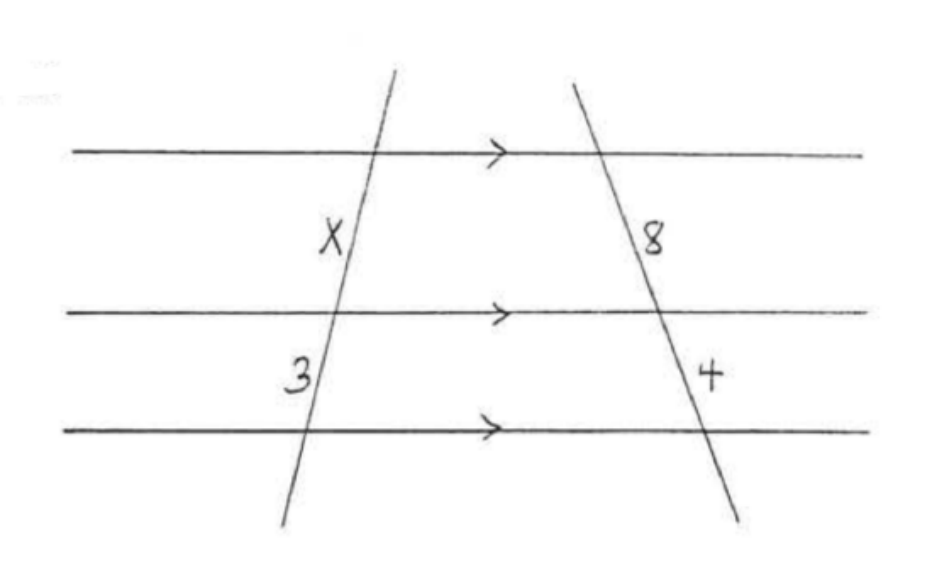


Figure 12.1.3.2: The line segments formed by the transversals are proportional.

Example 12.1.3.1

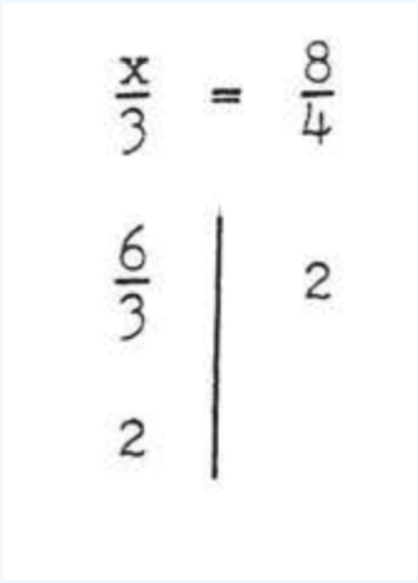
Find x:



Solution

$$\begin{aligned} \frac{x}{3} &= \frac{8}{4} \\ 4x &= 24 \\ x &= 6 \end{aligned} \tag{12.1.3.1}$$

Check:



Answer: $x = 6$.

Proof of Theorem 12.1.3.1

Draw GB and HC parallel to DF (Figure 12.1.3.3). The corresponding angles of the parallel lines are equal and so $\triangle BCH \sim \triangle ABG$. Therefore

$$\frac{BC}{AB} = \frac{CH}{BG}. \tag{12.1.3.2}$$

Now $CH = FE = c$ and $BG = ED = d$ because they are the opposite sides of a parallelogram. Substituting, we obtain

$$\frac{a}{b} = \frac{c}{d}. \tag{12.1.3.3}$$

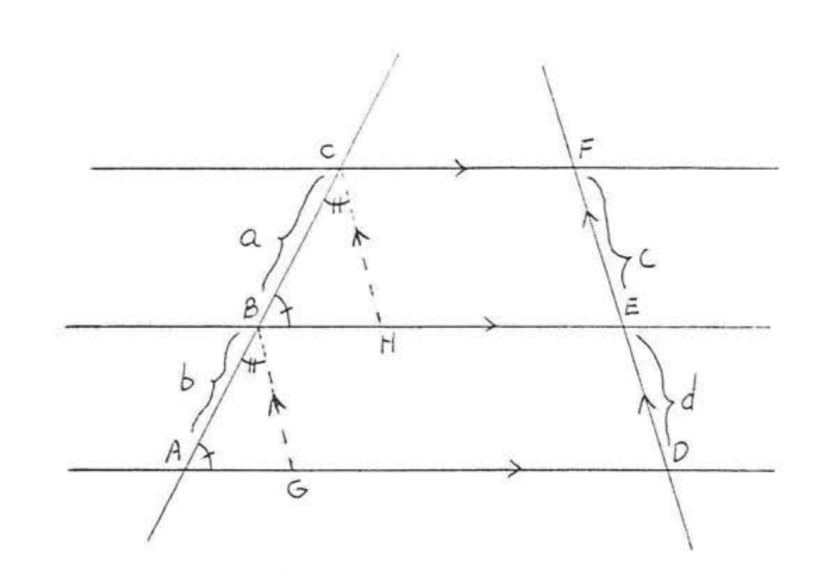
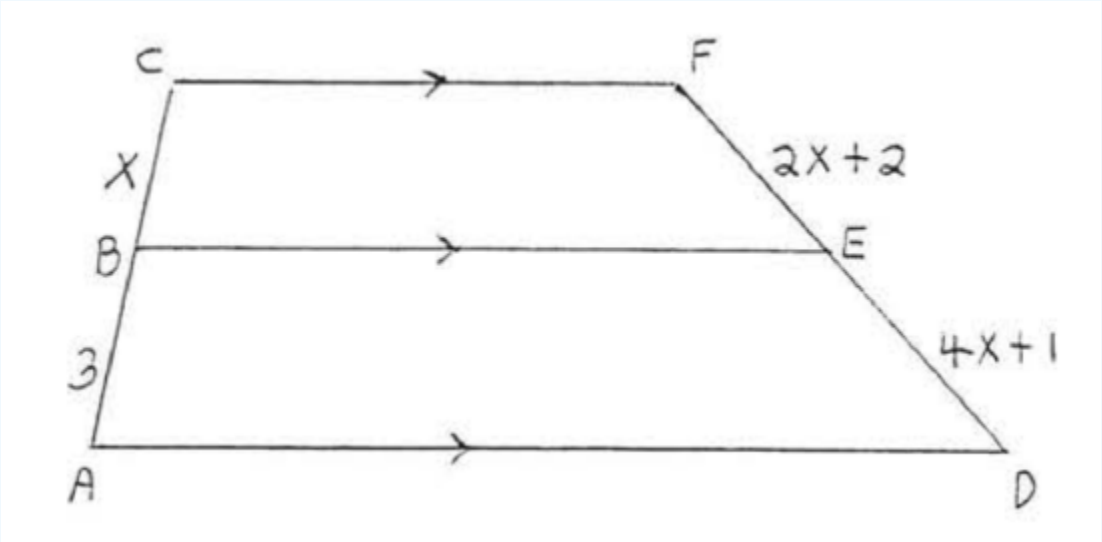


Figure 12.1.3.3: Draw GB and HC parallel to DF.

Example 12.1.3.2

Find x:



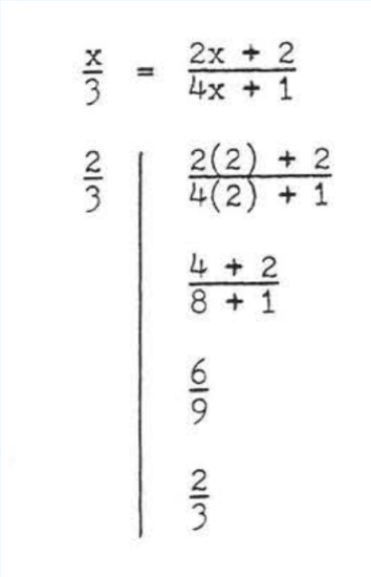
Solution

$$\begin{aligned} \frac{x}{3} &= \frac{2x+2}{4x+1} \\ (x)(4x+1) &= (3)(2x+2) \\ 4x^2+x &= 6x+6 \\ 4x^2-5x-6 &= 0 \\ (x-2)(4x+3) &= 0 \end{aligned}$$

$$\begin{aligned} x-2 &= 0 \\ x &= -2 \end{aligned} \quad \text{or} \quad \begin{aligned} 4x+3 &= 0 \\ 4x &= -3 \\ x &= -\frac{3}{4} \end{aligned}$$

We reject $x = -\frac{3}{4}$ because $BC = x$ cannot be negative.

Check, $x = 2$:

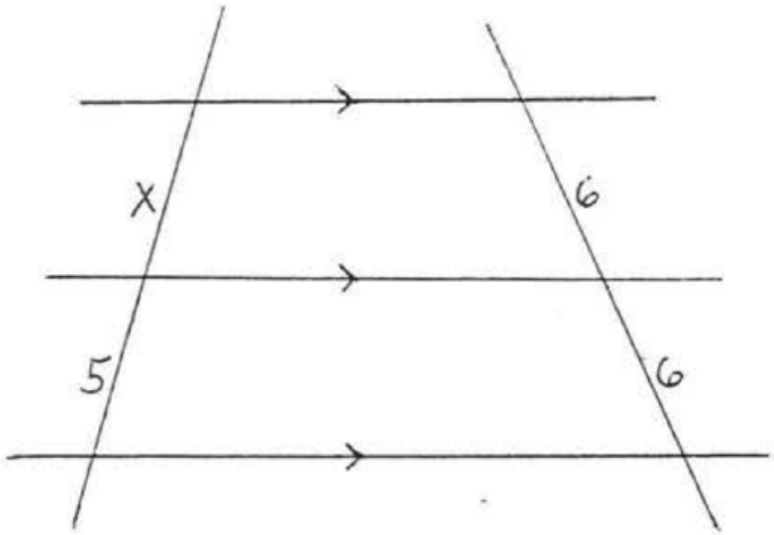


Answer: $x = 2$.

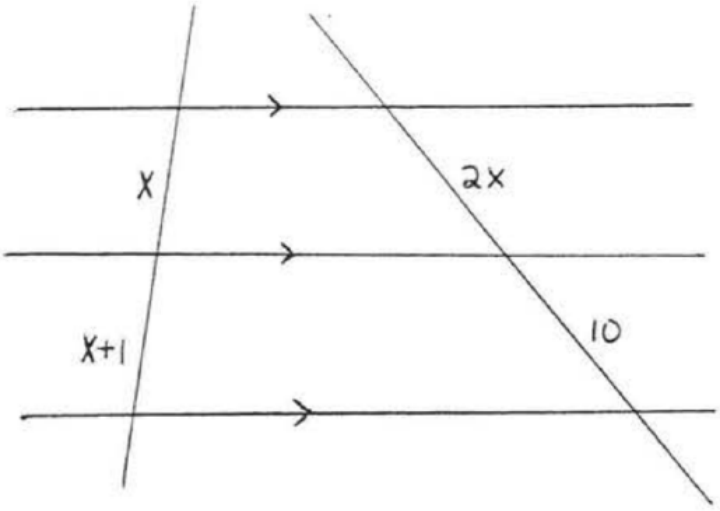
Problems

1 - 6. Find x:

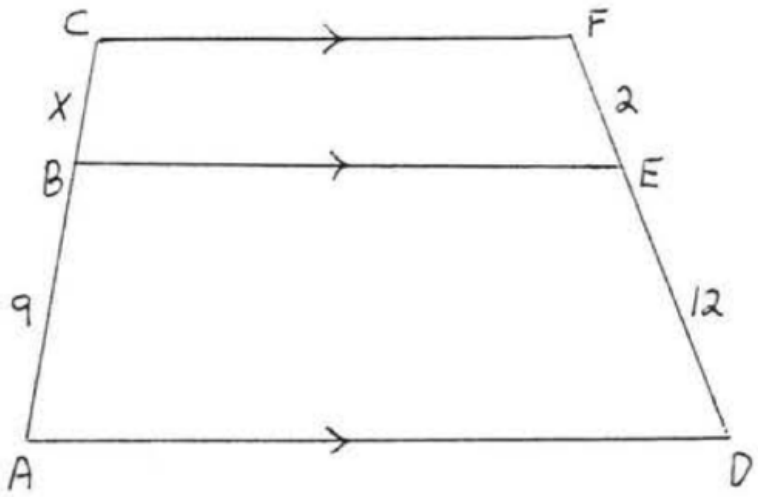
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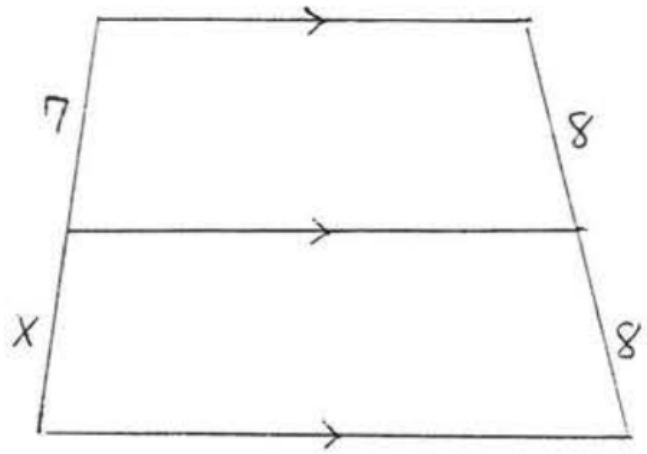
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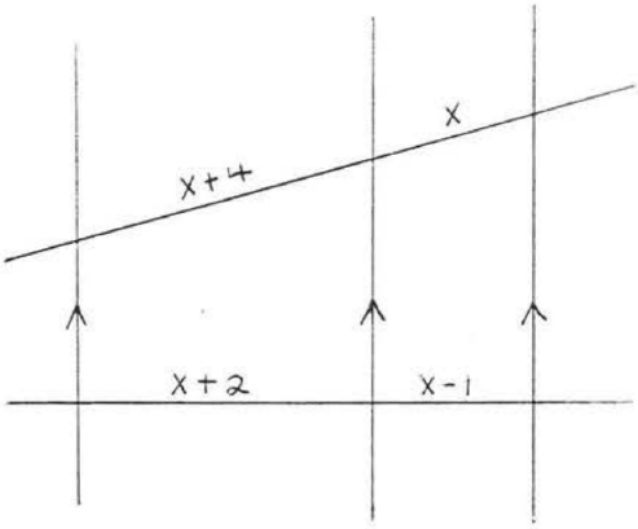
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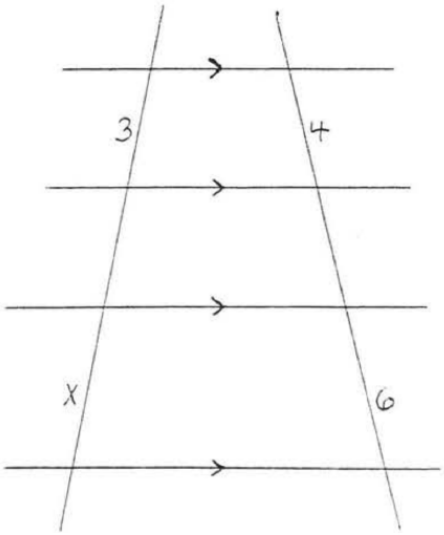
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5.



6.



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12.1.4: Pythagorean Theorem

In a right triangle, the sides of the right angle are called the **legs** of the triangle and the remaining side is called the **hypotenuse**. In Figure 12.1.4.1, side AC and BC are the legs and side AB is the hypotenuse.

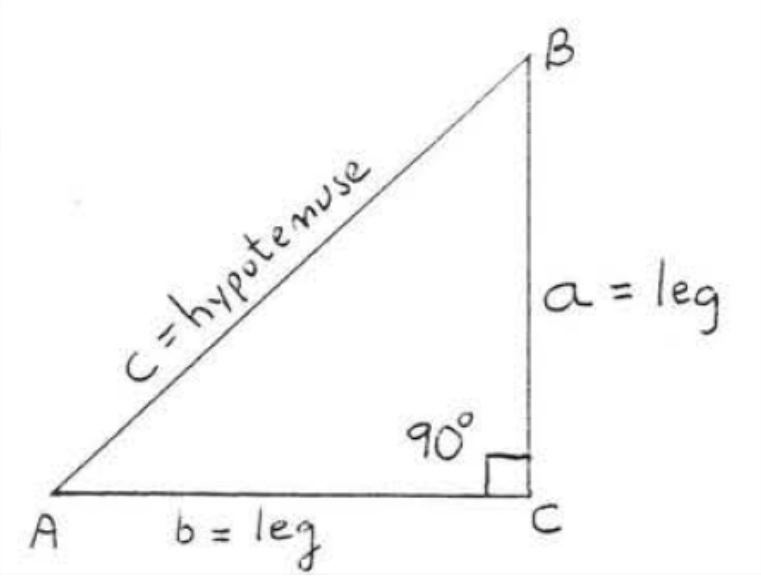


Figure 12.1.4.1: A right triangle.

The following is one of the most famous theorems in mathematics.

Theorem 12.1.4.1: Pythagorean Theorem

In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs. That is,

$$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2 \tag{12.1.4.1}$$

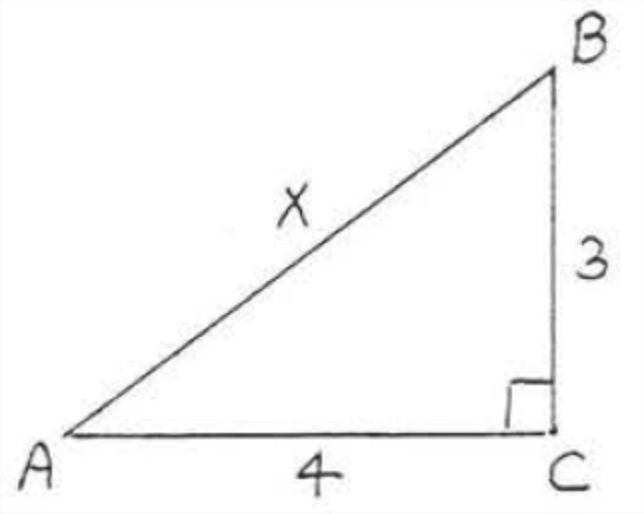
Thus, for the sides of the triangle in Figure 12.1.4.1,

$$a^2 + b^2 = c^2$$

Before we prove Theorem 12.1.4.1, we will give several examples.

Example 12.1.4.1

Find x



Solution

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 3^2 + 4^2 &= x^2 \\ 9 + 16 &= x^2 \\ 25 &= x^2 \\ 5 &= x \end{aligned}$$

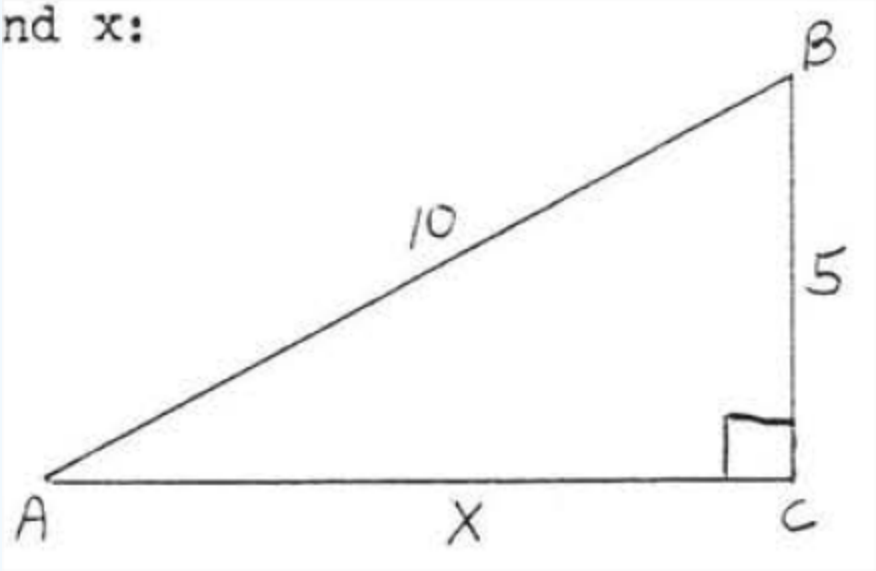
Check:

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 3^2 + 4^2 &\mid x^2 \\ 9 + 16 &\mid 5^2 \\ 25 &\mid 25 \end{aligned}$$

Answer: $x = 5$.

Example 12.1.4.2

Find x:



Solution

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 5^2 + x^2 &= 10^2 \\ 25 + x^2 &= 100 \\ x^2 &= 75 \\ x &= \sqrt{75} = \sqrt{25}\sqrt{3} = 5\sqrt{3} \end{aligned}$$

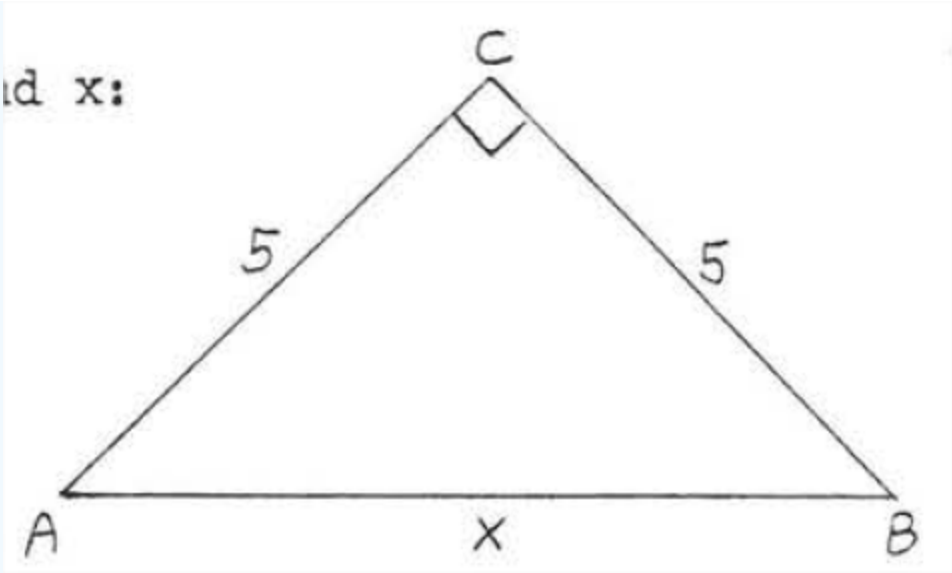
Check:

$$\begin{array}{rcl} \text{leg}^2 + \text{leg}^2 & = & \text{hyp}^2 \\ 5^2 + x^2 & & 10^2 \\ 25 + (5\sqrt{3})^2 & & 100 \\ 25 + 25\sqrt{9} & & \\ 25 + 25(3) & & \\ 25 + 75 & & \\ 100 & & \end{array}$$

Answer: $x = 5\sqrt{3}$.

Example 12.1.4.3

Find x:



Solution

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 5^2 + 5^2 &= x^2 \\ 25 + 25 &= x^2 \\ 50 &= x^2 \\ x &= \sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2} \end{aligned}$$

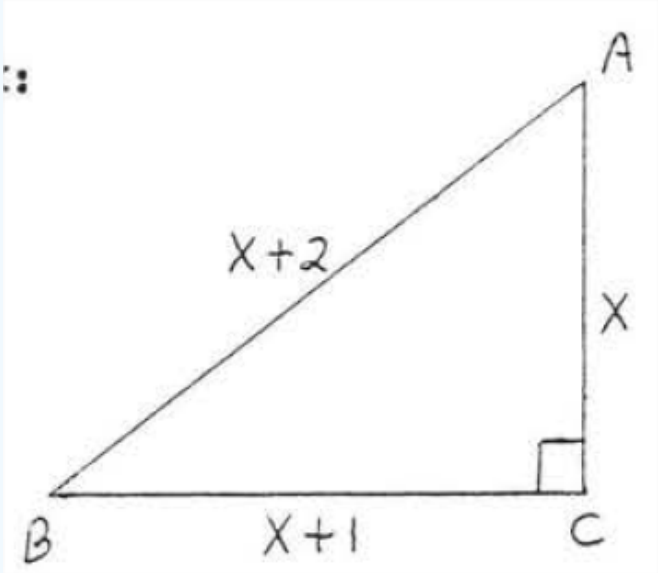
Check:

$$\begin{array}{rcl} \text{leg}^2 + \text{leg}^2 & = & \text{hyp}^2 \\ 5^2 + 5^2 & & x^2 \\ 25 + 25 & & (5\sqrt{2})^2 \\ 50 & & 25\sqrt{4} \\ & & 25(2) \\ & & 50 \end{array}$$

Answer: $x = 5\sqrt{2}$.

Example 12.1.4.4

Find x



Solution

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ x^2 + (x+1)^2 &= (x+2)^2 \\ x^2 + x^2 + 2x + 1 &= x^2 + 4x + 4 \\ x^2 + x^2 + 2x + 1 - x^2 - 4x - 4 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \end{aligned}$$

$$\begin{aligned} x - 3 &= 0 & x + 1 &= 0 \\ x &= 3 & x &= -1 \end{aligned}$$

We reject $x = -1$ because $AC = x$ cannot be negative.

Check, $x = 3$:

$$\begin{array}{r|l} \text{leg}^2 + \text{leg}^2 = & \text{hyp}^2 \\ x^2 + (x+1)^2 & (x+2)^2 \\ 3^2 + (3+1)^2 & (3+2)^2 \\ 9 + 4^2 & 5^2 \\ 9 + 16 & 25 \\ 25 & \end{array}$$

Answer: $x = 3$.

We will now restate and prove Theorem 12.1.4.1:

Theorem 12.1.4.1 Pythagorean Theorem

In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs. That is,

$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$.

In Figure 12.1.4.1,

$a^2 + b^2 = c^2$.

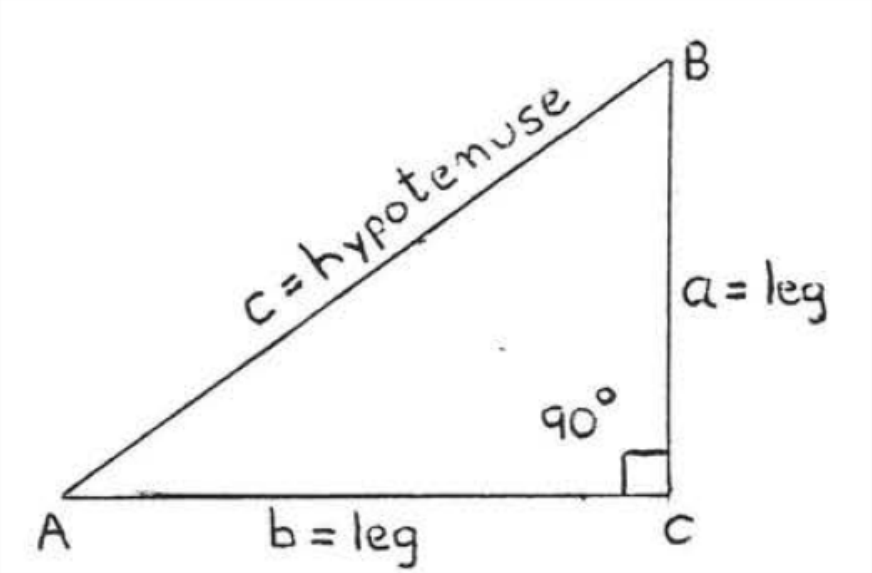


Figure 12.1.4.1. A right triangle.

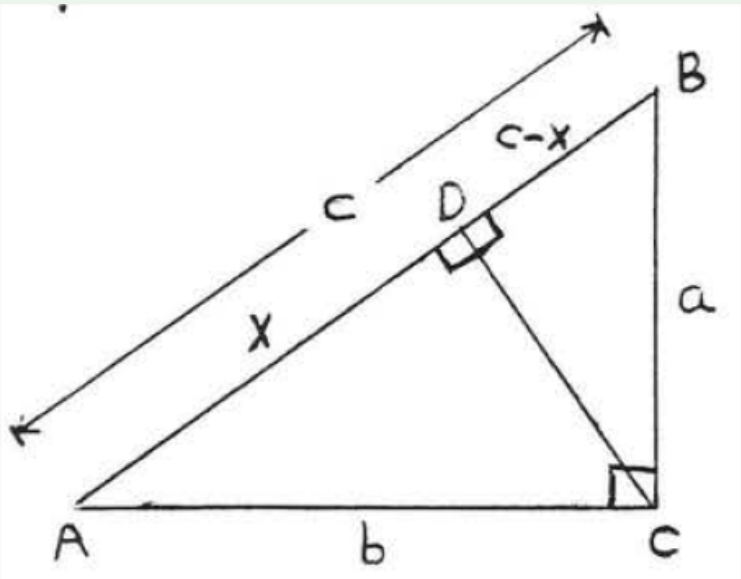


Figure 12.1.4.2. Draw CD perpendicular to AB.

Proof available at the end of this section.

The converse of the Pythagorean Theorem also holds:

Theorem 12.1.4.2 (converse of the Pythagorean Theorem).

In a triangle, if the square of one side is equal to the sun of the squares of the other two sides then the triangle is a right triangle.

In Figure 12.1.4.3, if $c^2 = a^2 + b^2$ then $\triangle ABC$ is a right triangle with $\angle C = 90^\circ$.

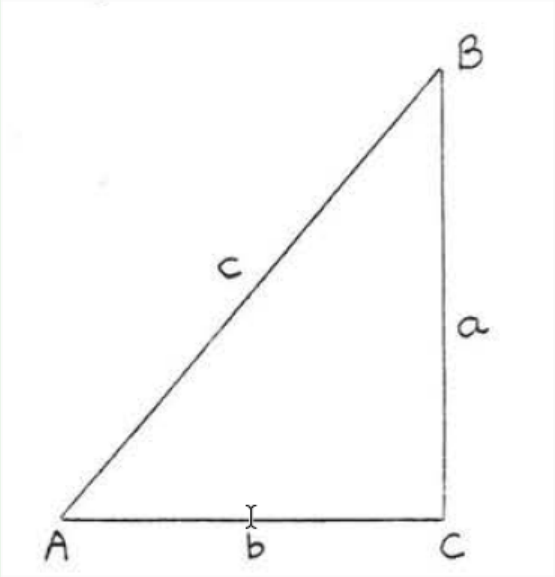
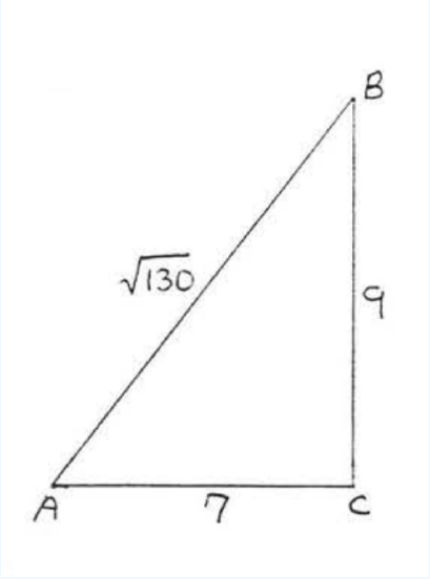


Figure 12.1.4.3: If $c^2 = a^2 + b^2$ then $\angle C = 90^\circ$.

Proof available at the end of this section.

Example 12.1.4.5

Is $\triangle ABC$ a right triangle?



Solution

$AC^2 = 7^2 = 49$

$BC^2 = 9^2 = 81$

$AB^2 = (\sqrt{130})^2 = 130$

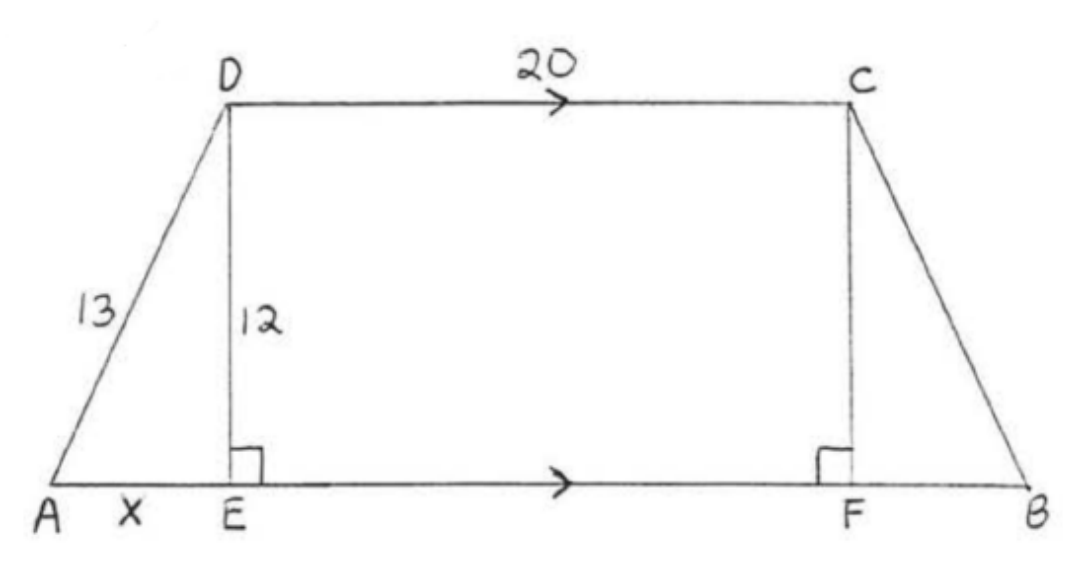
$49 + 81 = 130$.

so by Theorem 12.1.4.2, $\triangle ABC$ is a right triangle.

Answer: yes.

Example 12.1.4.6

Find x and AB:



Solution

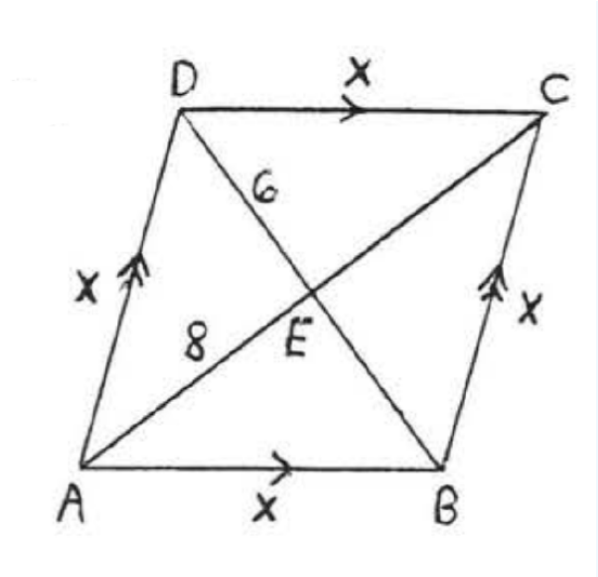
$$\begin{aligned} x^2 + 12^2 &= 13^2 \\ x^2 + 144 &= 169 \\ x^2 &= 169 - 144 \\ x^2 &= 25 \\ x &= 5 \end{aligned}$$

$CDEF$ is a rectangle so $EF = CD = 20$ and $CF = DE = 12$. Therefore $FB = 5$ and $AB = AE + EF + FB = 5 + 20 + 5 = 30$.

Answer: $x = 5$, $AB = 30$.

Example 12.1.4.7

Find x, AC and BD:



Solution

$ABCD$ is a rhombus. The diagonals of a rhombus are perpendicular and bisect each other.

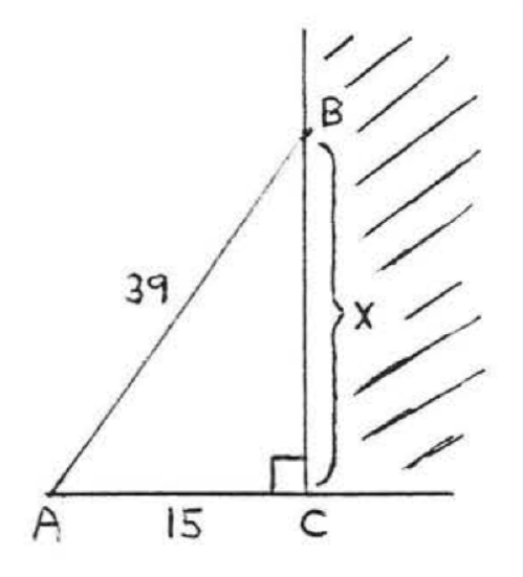
$$\begin{aligned} 6^2+8^2 &= x^2 \\ 36+64 &= x^2 \\ 100 &= x^2 \\ 10 &= x \end{aligned}$$

$AC = 8+8 = 16, BD = 6+6 = 12.$

Answer: $x = 10, AC = 16, BD = 12.$

Example 12.1.4.8

A ladder 39 feet long leans against a building, How far up the side of the building does the ladder reach if the foot of the ladder is 15 feet from the building?



Solution

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ x^2 + 15^2 &= 39^2 \\ x^2 + 225 &= 1521 \\ x^2 &= 1521 - 225 \\ x^2 &= 1296 \\ x &= \sqrt{1296} = 36 \end{aligned}$$

Answer: 36 feet.

Historical Note

Pythagoras (c. 582 - 507 B.C.) was not the first to discover the theorem which bears his name. It was known long before his time by the Chinese, the Babylonians, and perhaps also the Egyptians and the Hindus, According to tradition, Pythagoras was the first to give a nroof of the theorem, His proof probably made use of areas, like the one suggested. In Figure $12.1.4.5$ below, (each square contains four congruent right triangles with sides of lengths a, b, and c, In addition the square on the left contains a square with side a and a square with side b while the one on the right contains a square with side c.)

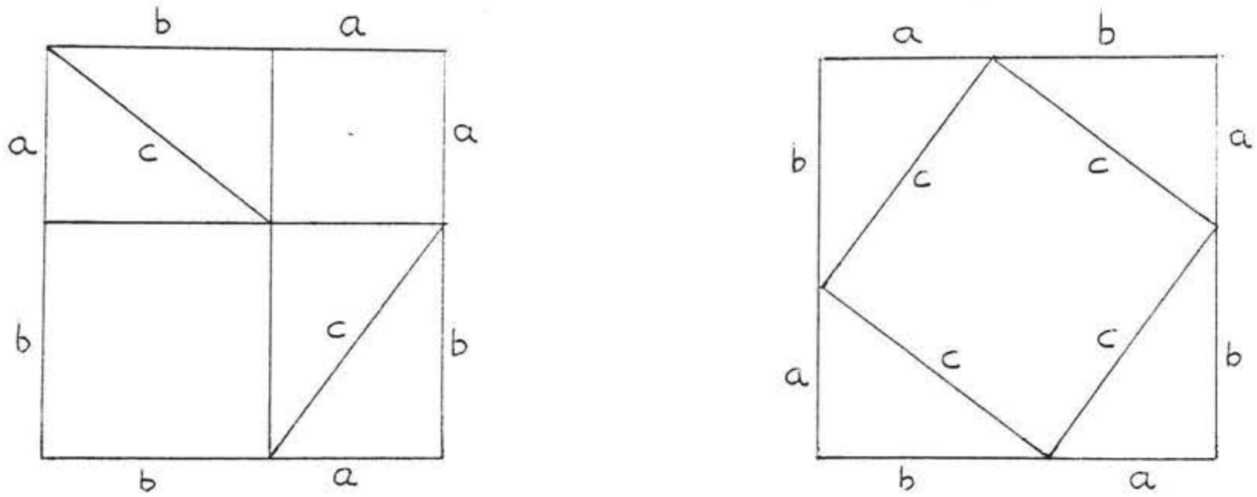


Figure 12.1.4.5: Pythagoras may have proved $a^2 + b^2 = c^2$ in this way.

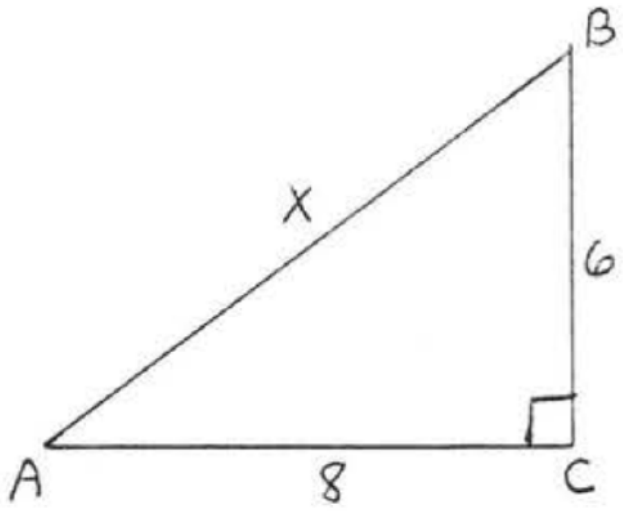
Since the time of Pythagoras, at least several hundred different proofs of the Pythagorean Theorem have been proposed, Pythagoras was the founder of the Pythagorean school, a secret religious society devoted to the study of philosophy, mathematics, and science. Its membership was a select group, which tended to keep the discoveries and practices of the society secret from outsiders. The Pythagoreans believed that numbers were the ultimate components of the universe and that all physical relationships could be expressed with whole numbers, This belief was prompted in part by their discovery that the notes of the musical scale were related by numerical ratios. The Pythagoreans made important contributions to medicine, physics, and astronomy, In geometry, they are credited with the angle s

um theorem for triangles, the properties of parallel lines, and the theory of similar triangles and proportions.

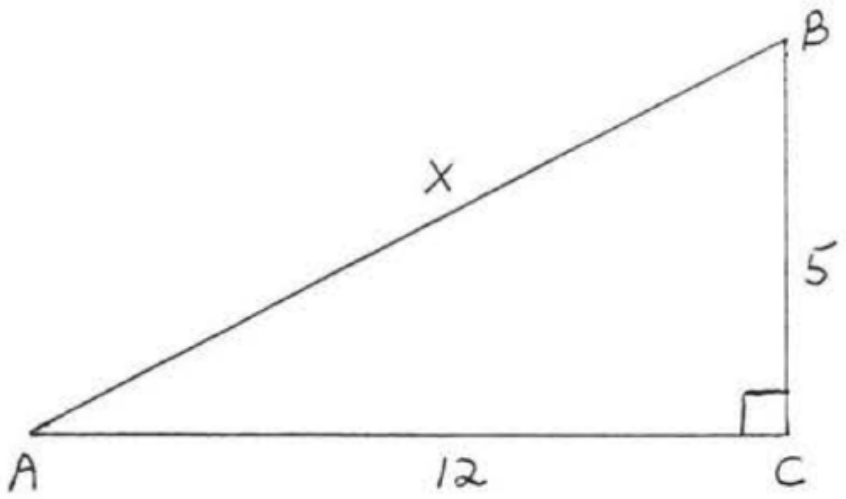
Problems

1 - 10. Find x. Leave answers in simplest radical form.

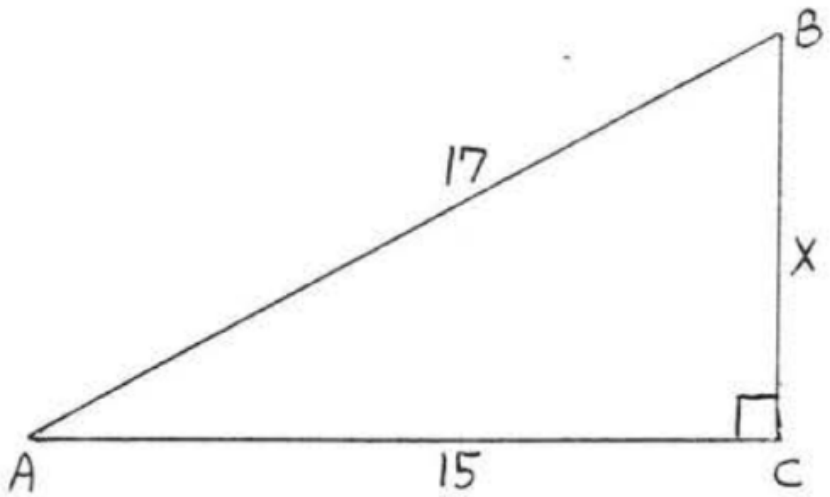
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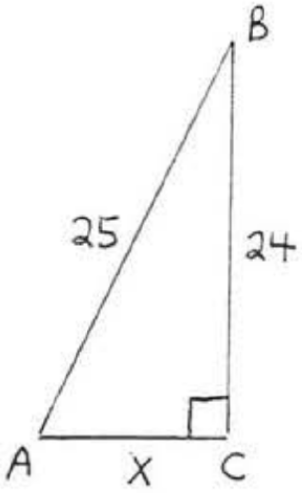
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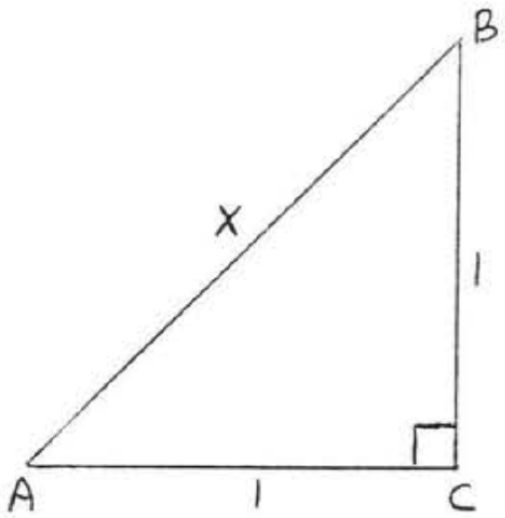
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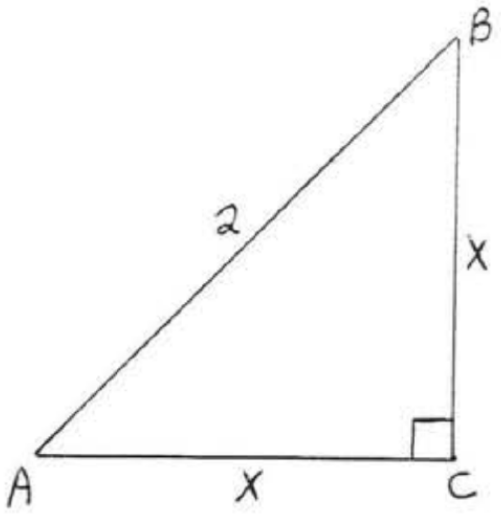
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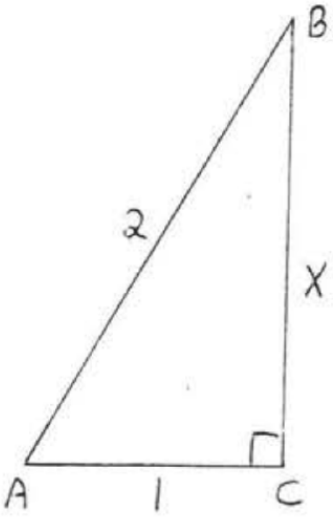
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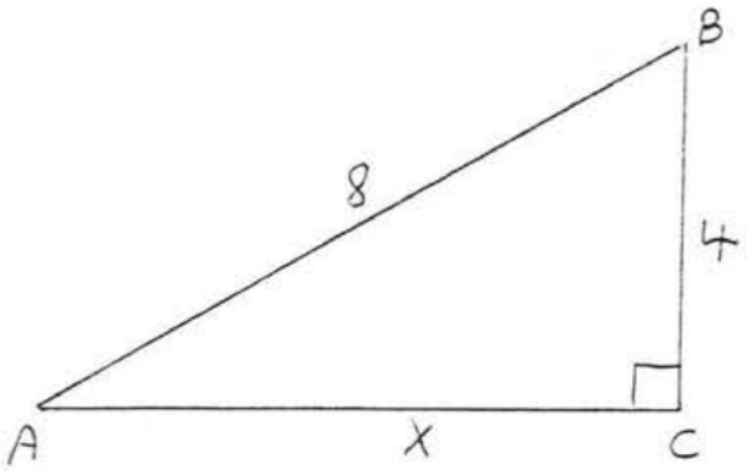
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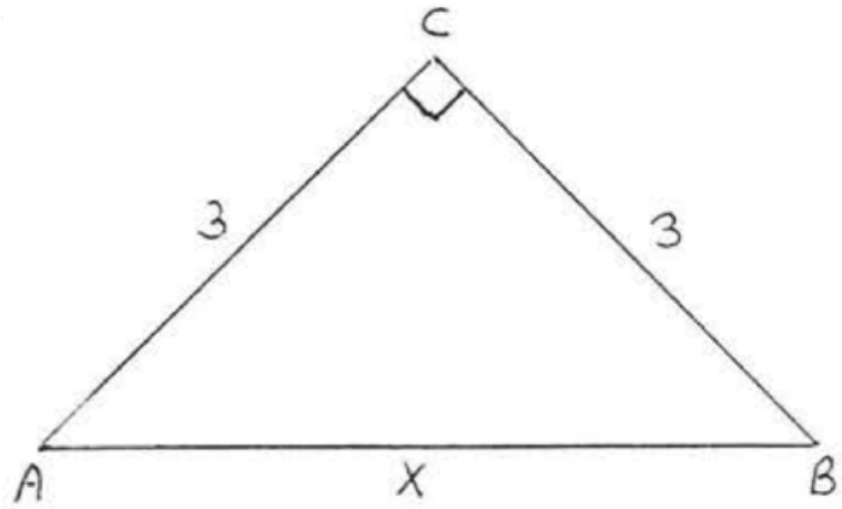
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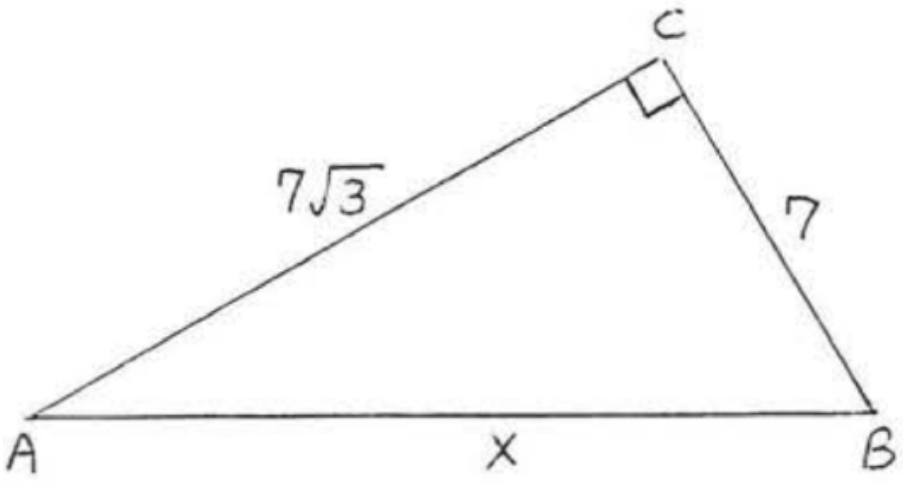
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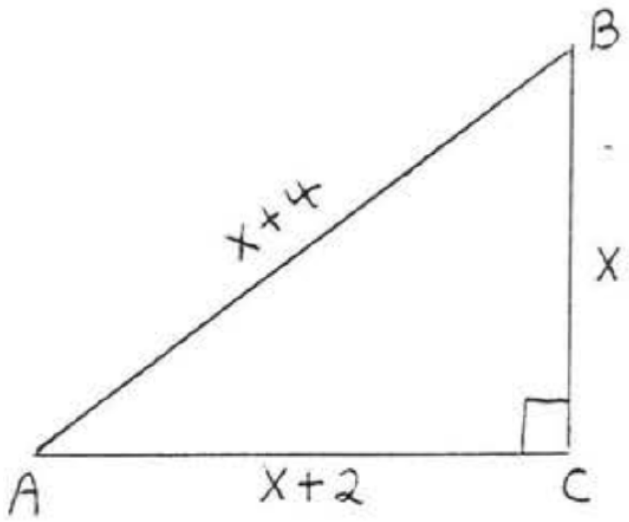


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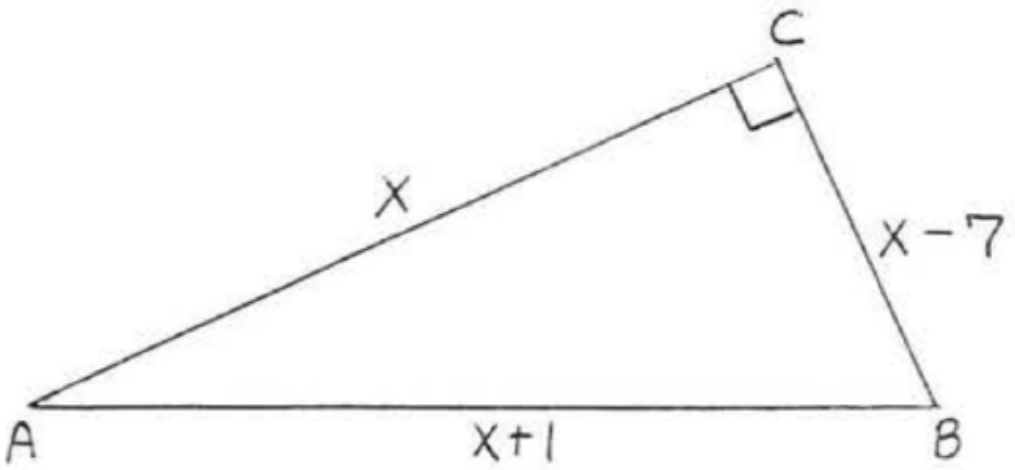


11 - 14. Find x and all sides of the triangle:

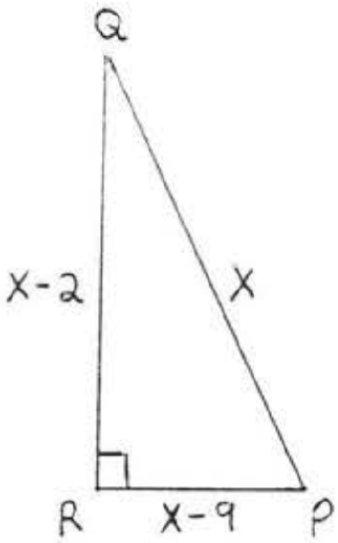
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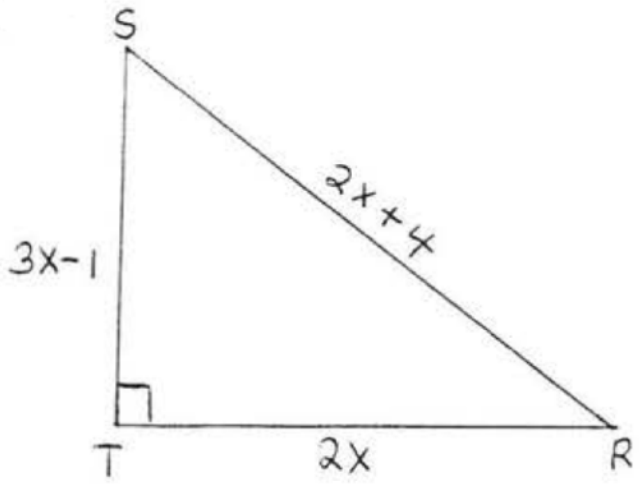
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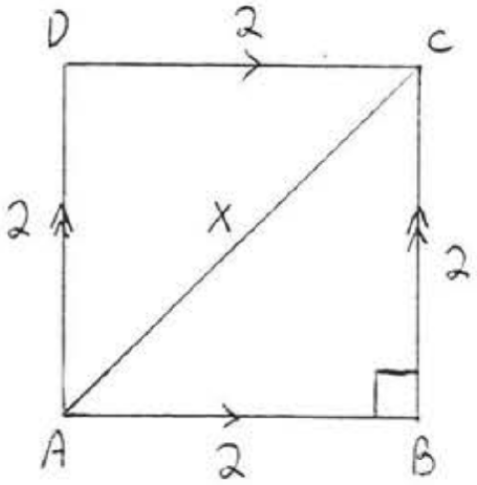


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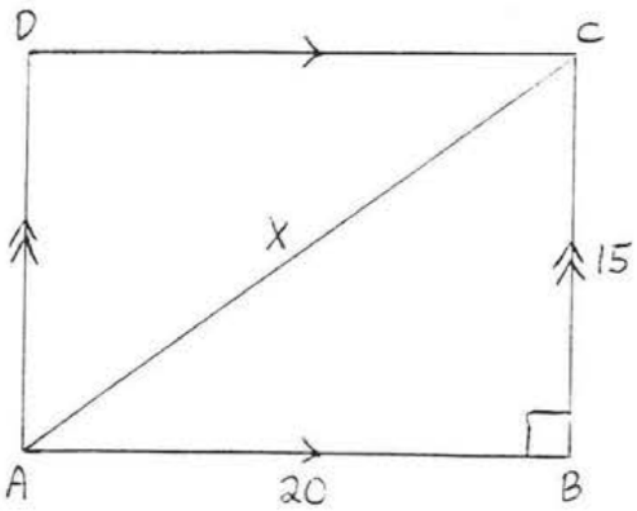


15 - 16. Find x:

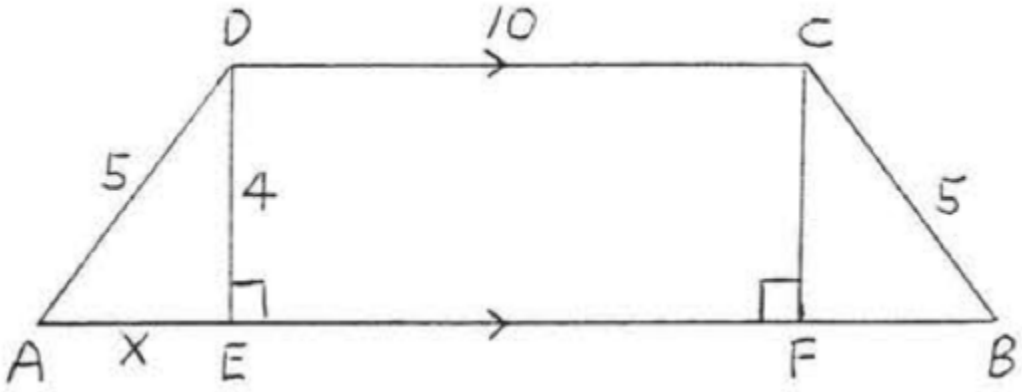
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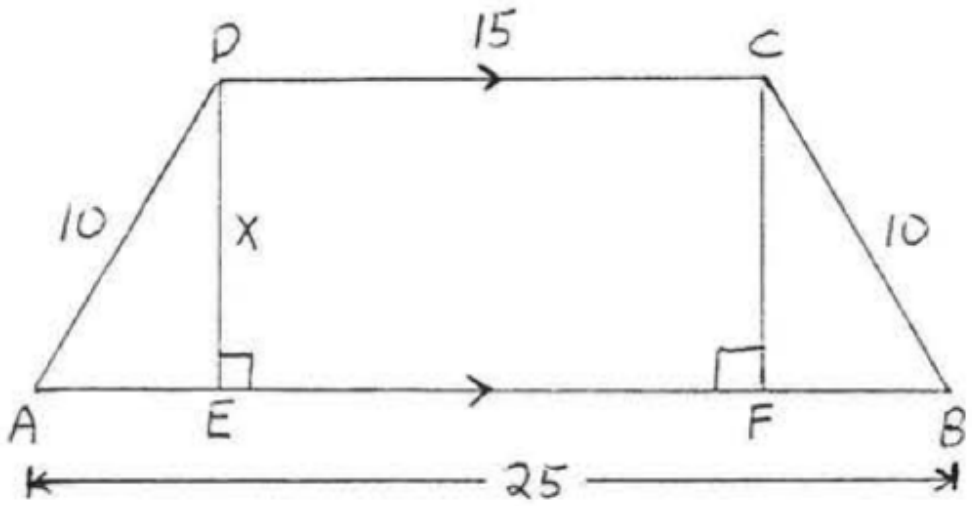
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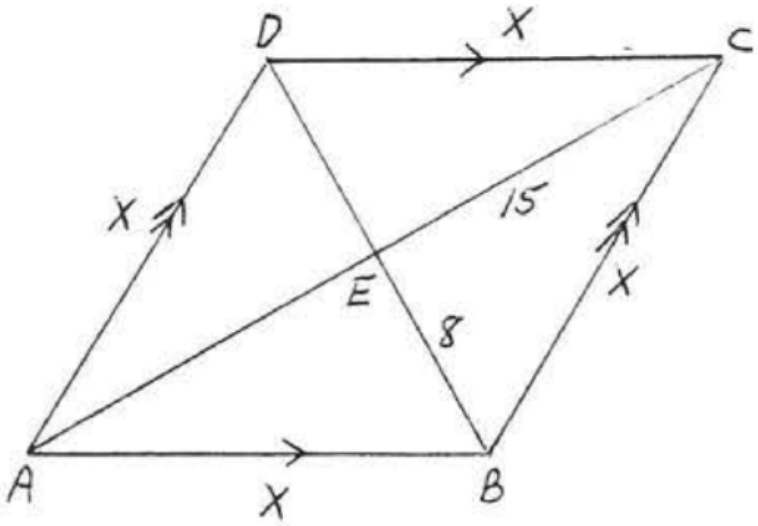
17. Find x and AB.



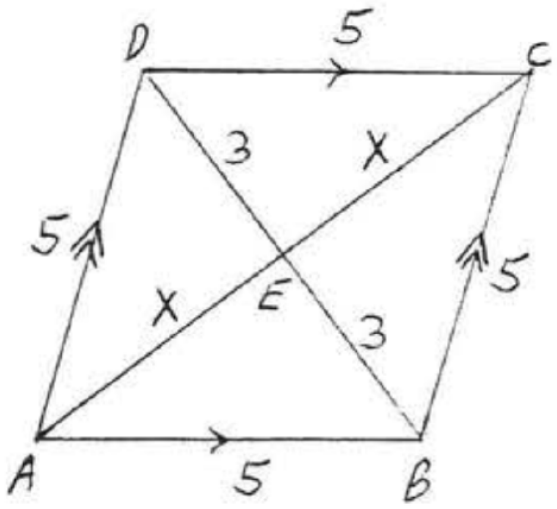
18. Find x:



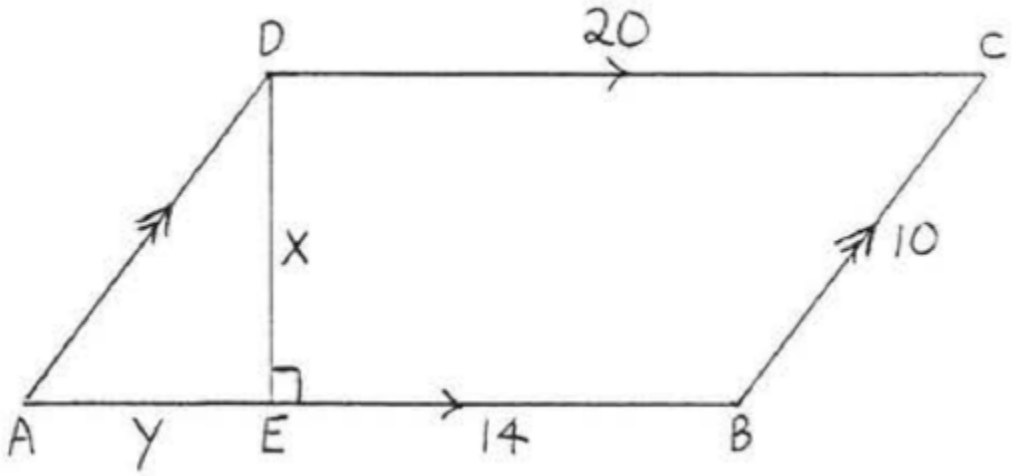
19. Find x, AC and BD:



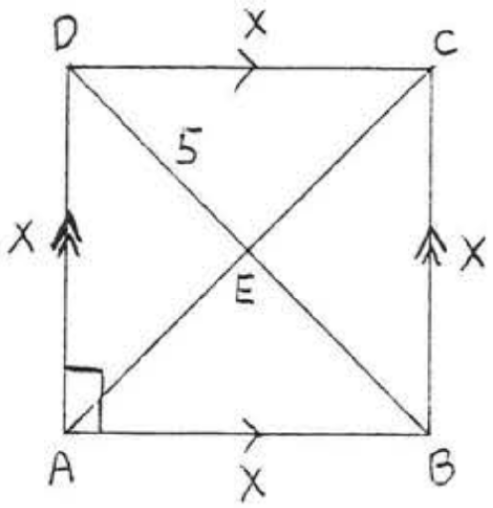
20. Find x, AC and BD:



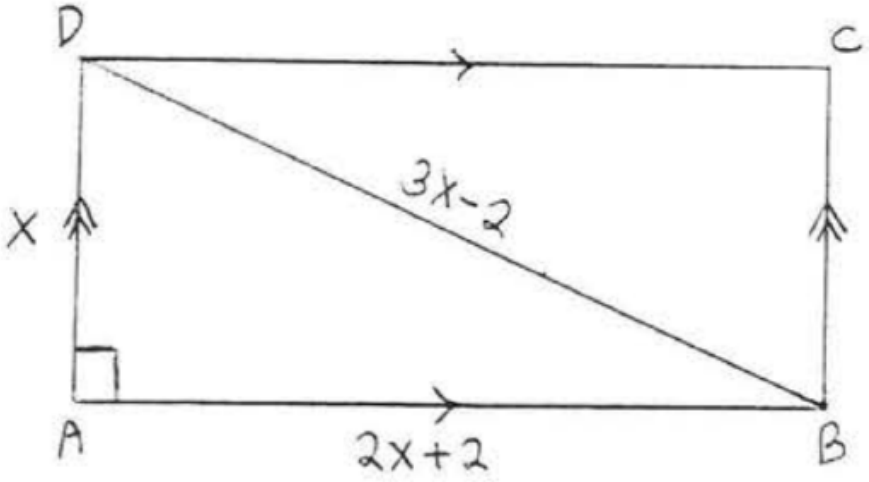
21. Find x and y:



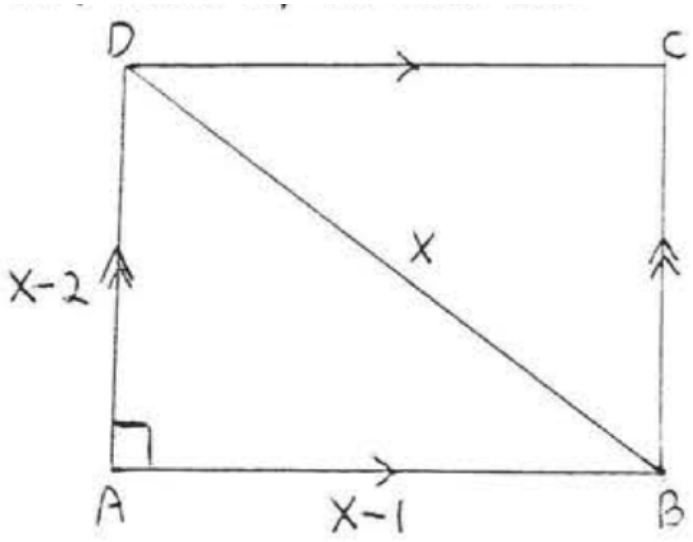
22. Find x, AC and BD:



23. Find x, AB and BD:

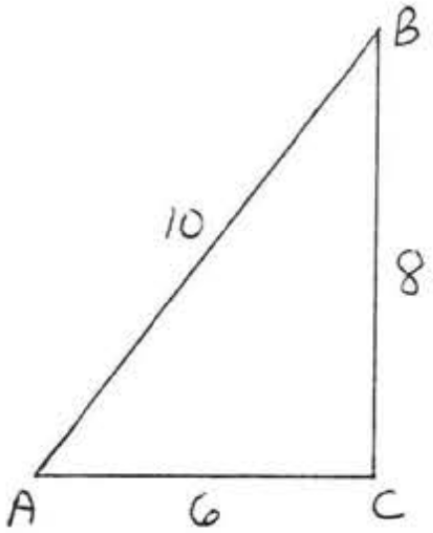


24. Find x, AB and AD:

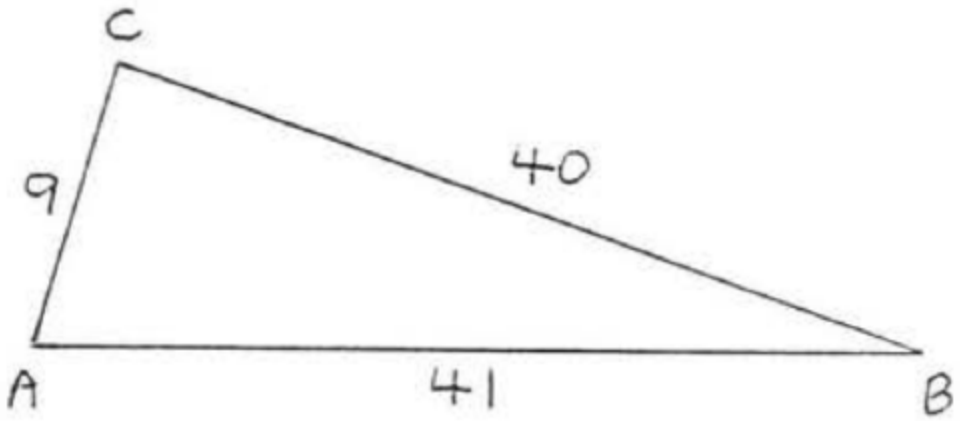


25 - 30. Is $\triangle ABC$ a right triangle?

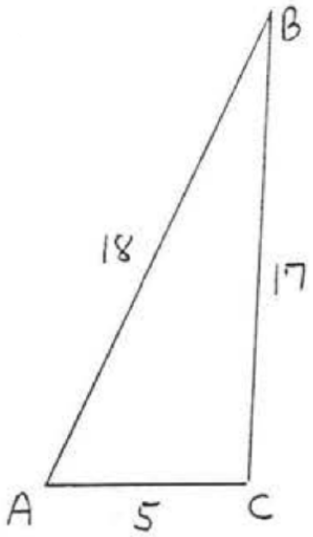
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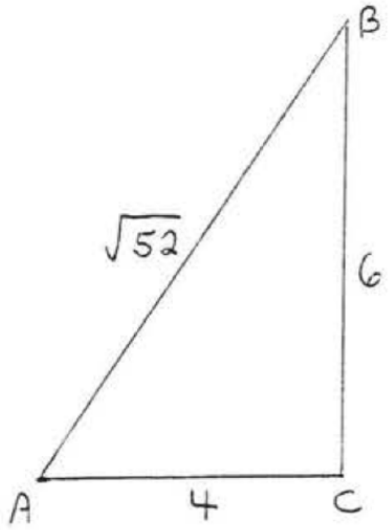
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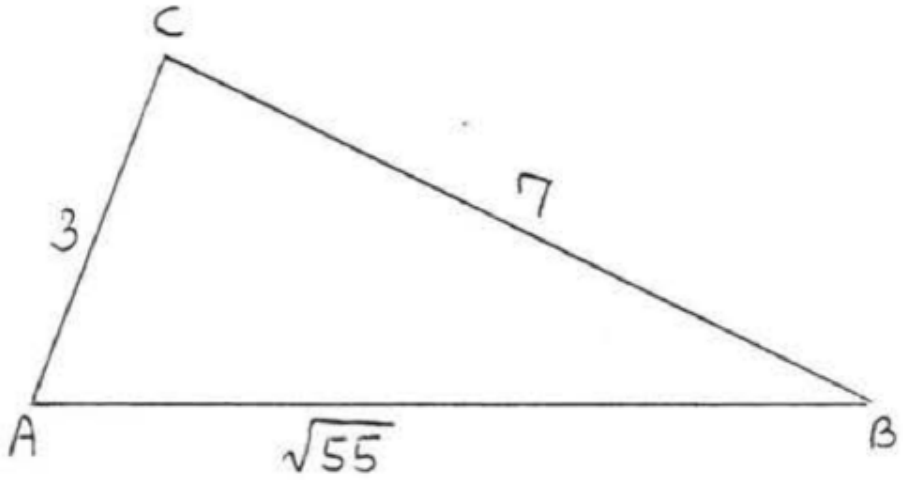
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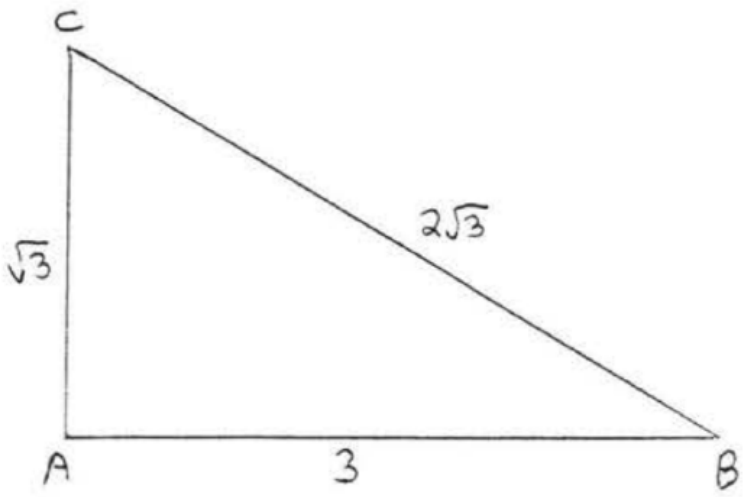
28.



29.



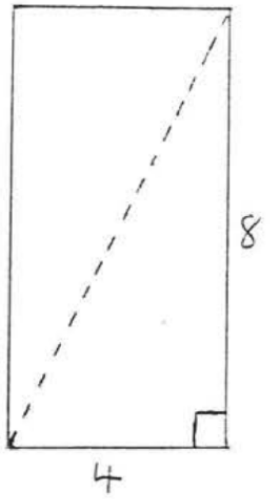
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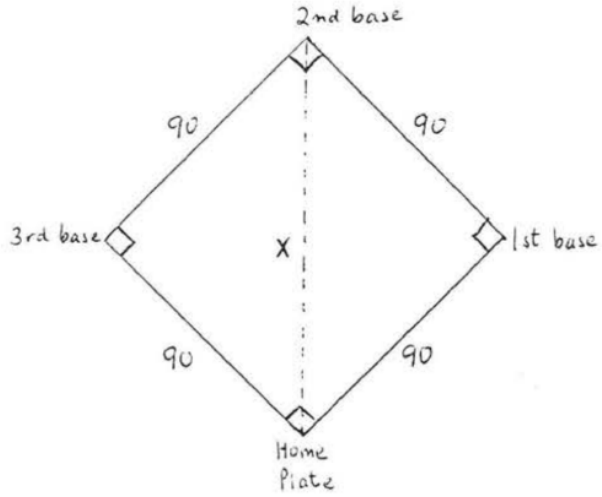
31. A ladder 25 feet long leans against a building, How far up the side of the building does the ladder reach if the foot of the ladder is 7 feet from the building?

32. A man travels 24 miles east and then 10 miles north. At the end of his journey how far is he from his starting point?

33. Can a table 9 feet wide (with its legs folded) fit through a rectangular doorway 4 feet by 8 feet?



34. A baseball diamond is a square 90 feet on each side, Find the distance from home plate to second base (leave answer in simplest radical form).



Answers

Theorem 12.1.4.1 **Pythagorean Theorem**

In Figure 12.1.4.1, draw CD perpendicular to AB. Let $x = AD$. Then $BD = c - x$ (Figure 12.1.4.2). As in Example 12.1.4.3, section 4.2, $\triangle ABC \sim \triangle ACD$ and $\triangle ABC \sim \triangle CBD$. From these two similarities we obtain two proportions:

$$\triangle ABC \sim \triangle ACD \qquad \triangle ABC \sim \triangle CBD$$

$$\frac{AB}{AC} = \frac{AC}{AD} \qquad \frac{AB}{CB} = \frac{BC}{BD}$$

$$\frac{c}{b} = \frac{b}{x} \qquad \frac{c}{a} = \frac{a}{c - x}$$

$$cx = b^2 \qquad c(c - x) = a^2$$

$$c^2 - cx = a^2$$

$$c^2 - b^2 = a^2$$

$$c^2 = a^2 + b^2$$

Theorem 12.1.4.2 **(converse of the Pythagorean Theorem).**

Draw a new triangle, $\triangle DEF$, so that $\angle F = 90^\circ$, $d = a$, and $e = b$ (Figure 12.1.4.4). $\triangle DEF$ is a right triangle, so by Theorem 12.1.4.1, $f^2 = d^2 + e^2$. We have $f^2 = d^2 + e^2 = a^2 + b^2 = c^2$ and therefore $f = c$. Therefore $\triangle ABC \cong \triangle DEF$ because $SSS = SSS$. Therefore, $\angle C + \angle F = 90^\circ$.

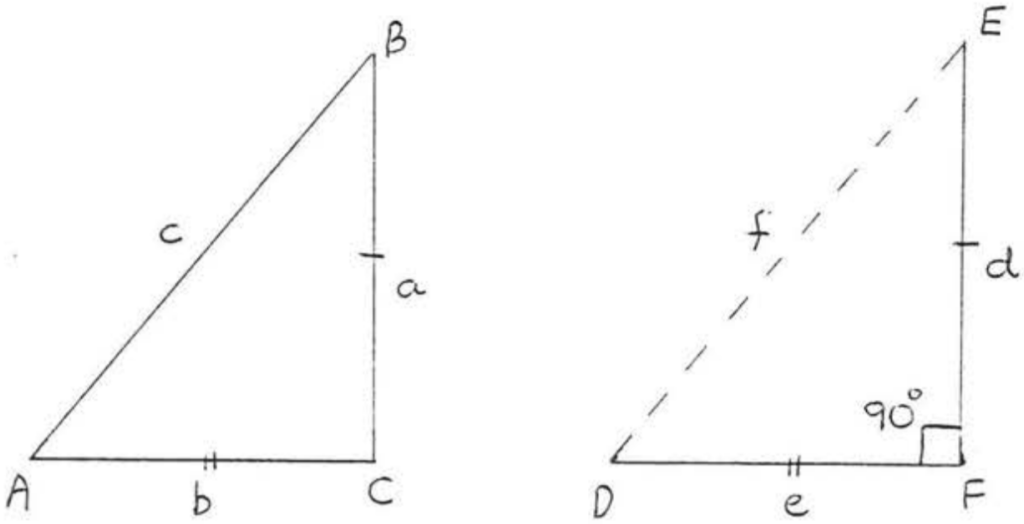


Figure 12.1.4.4: Given $\triangle ABC$, draw $\triangle DEF$ so that $\angle F = 90^\circ$, $d = a$ and $e = b$.

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12.1.5: Special Right Triangles

There are two kinds of right triangle which deserve special attention: the $30^\circ - 60^\circ - 90^\circ$ right triangle and the $45^\circ - 45^\circ - 90^\circ$ right triangle.

30°–60°–90° Right Triangles

A triangle whose angles are $30^\circ, 60^\circ$, and 90° is called a $30^\circ - 60^\circ - 90^\circ$ triangle. $\triangle ABC$ in Figure $12.1.5.1$ is a $30^\circ - 60^\circ - 90^\circ$ triangle with side $AC = 1$.

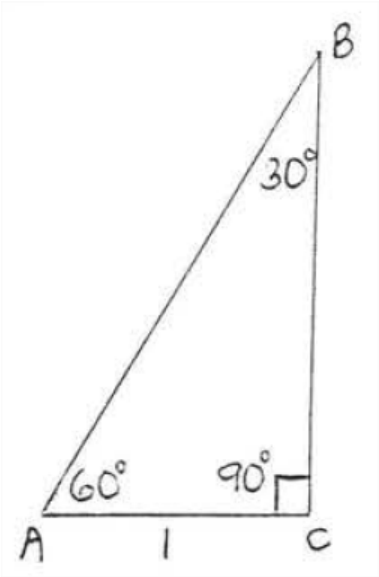


Figure $12.1.5.1$: A $30^\circ - 60^\circ - 90^\circ$ triangIe.

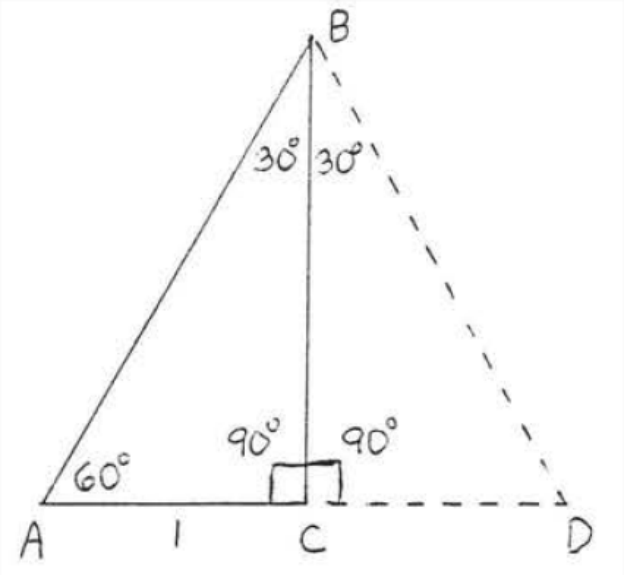


Figure $12.1.5.2$: Draw BD and CD.

To learn more about this triangle let us draw lines BD and CD as in Figure $12.1.5.2$. $\triangle ABC \cong \triangle DBC$ by $ASA = ASA$ so $AC = DC = 1$. $\triangle ABD$ is an equiangular triangle so all the sides must be equal to 2.. Therefore $AB = 2$ (Figure $12.1.5.3$).

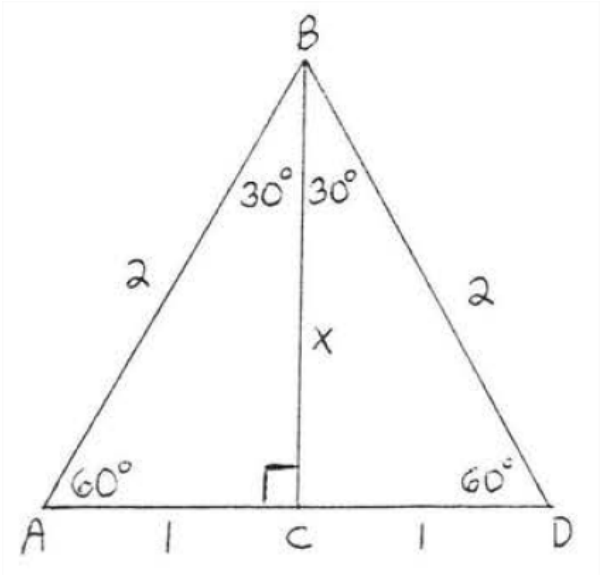


Figure 12.1.5.3: $\triangle ABD$ is equiangular with all sides equal to 2 .

Let $x = BC$. Let us find x. Applying the Pythagorean Theorem to $\triangle ABC$,

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 1^2 + x^2 &= x^2 \\ 1 + x^2 &= 4 \\ x^2 &= 3 \\ x &= \sqrt{3} \end{aligned}$$

Now suppose we are given another $30^\circ - 60^\circ - 90^\circ$ triangle $\triangle DEF$, with side $DF = 8$ (Figure 12.1.5.4). $\triangle DEF$ is similar to $\triangle ABC$ of Figure 12.1.5.3 Therefore

$$\begin{aligned} \frac{DF}{AC} &= \frac{DE}{AB} \\ \frac{8}{1} &= \frac{DE}{2} \\ 16 &= DE \end{aligned} \quad \text{and} \quad \begin{aligned} \frac{DF}{AC} &= \frac{EF}{BC} \\ \frac{8}{1} &= \frac{EF}{\sqrt{3}} \\ 8\sqrt{3} &= EF \end{aligned}$$

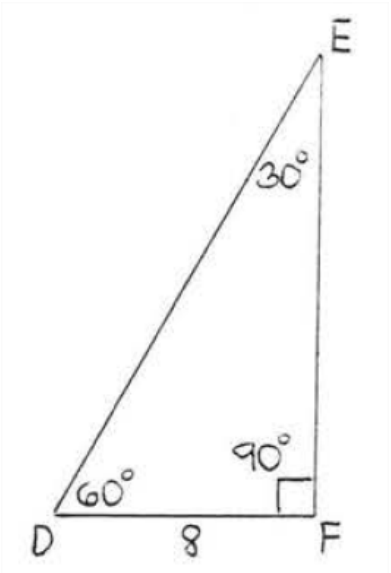


Figure 12.1.5.3:

Our conclusions about triangles ABC and DEF suggest the following theorem:

Theorem 12.1.5.1

In the $30^\circ - 60^\circ - 90^\circ$ triangle the hypotenuse is always twice as large as the leg opposite the 30° angle (the shorter leg). The leg opposite the 60° angle (the longer leg) is always equal to the shorter leg times $\sqrt{3}$.

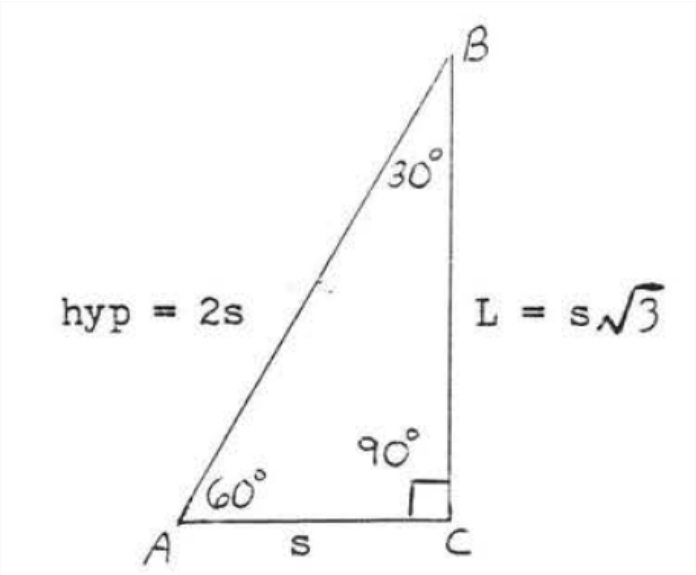


Figure 12.1.5.5: The hypotenuse is twice the shorter leg and the longer leg is equal to the shorter leg times the $\sqrt{3}$.

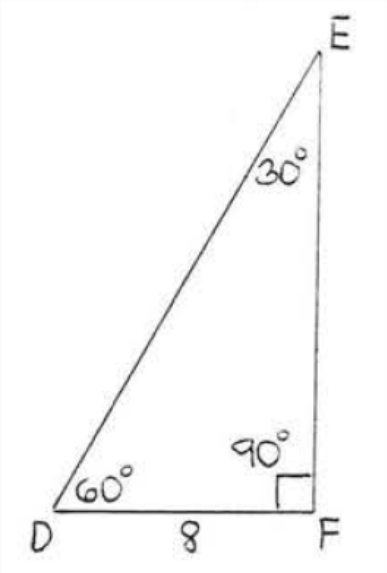
In Figure 12.1.5.5, $s =$ shorter leg, $L =$ longer leg, and hyp = hypotenuse. Theorem 12.1.5.1 says that

Erroneous nesting of equation structures

Note that the longer leg is always the leg opposite (furthest away from) the 60° angle and the shorter leg is always the leg opposite (furthest away from) the 30° angle.

Example 12.1.5.1

Find x and y:



Solution

$\angle B = 180^\circ - (60^\circ + 90^\circ) = 180^\circ - 150^\circ = 30^\circ$, so $\triangle ABC$ is a $30^\circ - 60^\circ - 90^\circ$ triangle. By theorem 12.1.5.1,

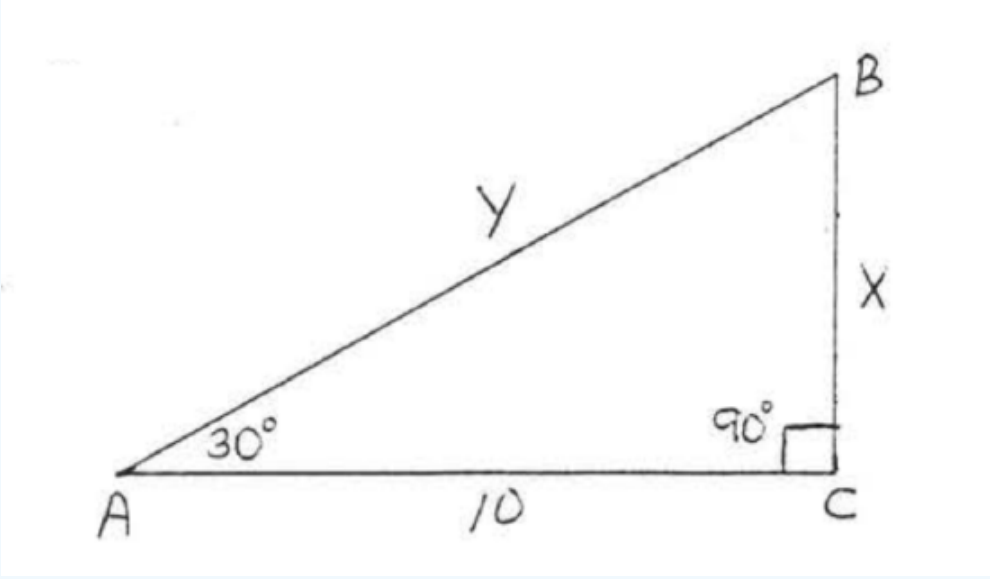
$$\begin{aligned} \text{hyp} &= 2s & L &= s\sqrt{3} \\ y &= 2(7) = 14. & x &= 7\sqrt{3} \end{aligned}$$

Answer

$x = 7\sqrt{3}$, $y = 14$.

Example 12.1.5.2

Find x and y:



Solution

$\angle B = 60^\circ$ so $\triangle ABC$ is a $30^\circ - 60^\circ - 90^\circ$ triangle. By Theorem 12.1.5.1,

$$\begin{aligned} L &= s\sqrt{3} \\ 10 &= x\sqrt{3} \\ \frac{10}{\sqrt{3}} &= \frac{x\cancel{\sqrt{3}}}{\cancel{\sqrt{3}}} \\ x &= \frac{10}{\sqrt{3}} = \frac{10}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{10\sqrt{3}}{3}. \\ \text{hyp} &= 2s \\ y &= 2x = 2(\frac{10\sqrt{3}}{3}) = \frac{20\sqrt{3}}{3} \end{aligned}$$

Answer

$x = \frac{10\sqrt{3}}{3}, y = \frac{20\sqrt{3}}{3}$.

45°–45°–90° Right Triangles

The second special triangle we will consider is the $45^\circ - 45^\circ - 90^\circ$ triangle. A triangle whose angles are 45°, 45°, and 90° is called a $45^\circ - 45^\circ - 90^\circ$ triangle or an isosceles right triangle. $\triangle ABC$ in Figure 12.1.5.6 is a $45^\circ - 45^\circ - 90^\circ$ triangle with side $AC = 1$.

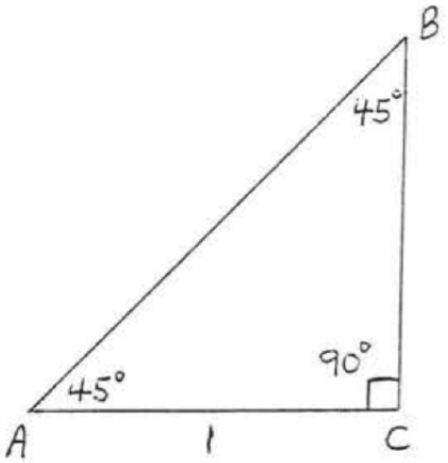


Figure 12.1.5.6: A $45^\circ - 45^\circ - 90^\circ$ triangle.

Since $\angle A = \angle B = 45^\circ$, the sides opposite these angles must be equal (Theorem 2.5.2, Section 2.5). Therefore $AC = BC = 1$.

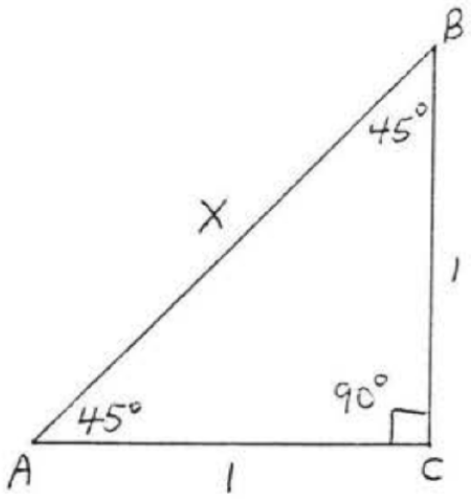


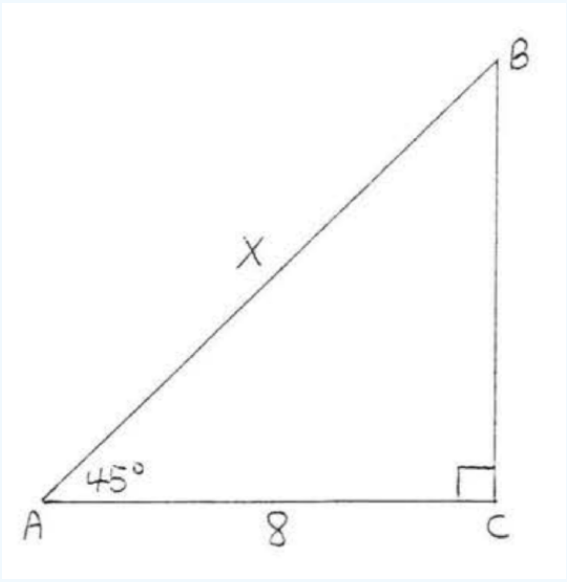
Figure 12.1.5.7: The legs of the $45^\circ - 45^\circ - 90^\circ$ triangle are equal.

Let $x = AB$ (Figure 12.1.5.7). By the Pythagorean Theorem,

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 1^2 + 1^2 &= x^2 \\ 1 + 1 &= x^2 \\ 2 &= x^2 \\ \sqrt{2} &= x \end{aligned}$$

Example 12.1.5.3

Find x:



Solution

$\angle B = 180^\circ - (45^\circ + 90^\circ) = 180^\circ - 135^\circ = 45^\circ$. So $\triangle ABC$ is a $45^\circ - 45^\circ - 90^\circ$ triangle. $AC = BC = 8$ because these sides are opposite equal angles. By the Pythagorean Theorem,

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 8^2 + 8^2 &= x^2 \\ 64 + 64 &= x^2 \\ 128 &= x^2 \\ x &= \sqrt{128} = \sqrt{64}\sqrt{2} = 8\sqrt{2} \end{aligned}$$

Answer

$x = 8\sqrt{2}$.

The triangles of Figure 12.1.5.6 and Example 12.1.5.3 suggest the following theorem:

Theorem 12.1.5.2

In the $45^\circ - 45^\circ - 90^\circ$ triangle the legs are equal and the hypotenuse is equal to either leg times $\sqrt{2}$.

In Figure 12.1.5.8, hyp is the hypotenuse and L is the length of each leg. Theorem 12.1.5.2 says that

$$\boxed{\text{hyp} = \sqrt{2}L} \tag{12.1.5.1}$$

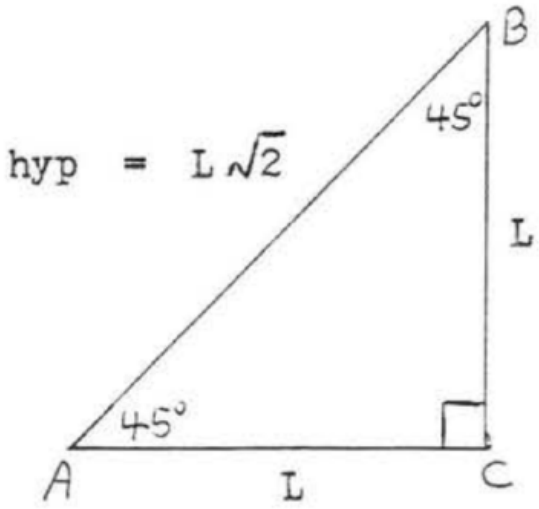
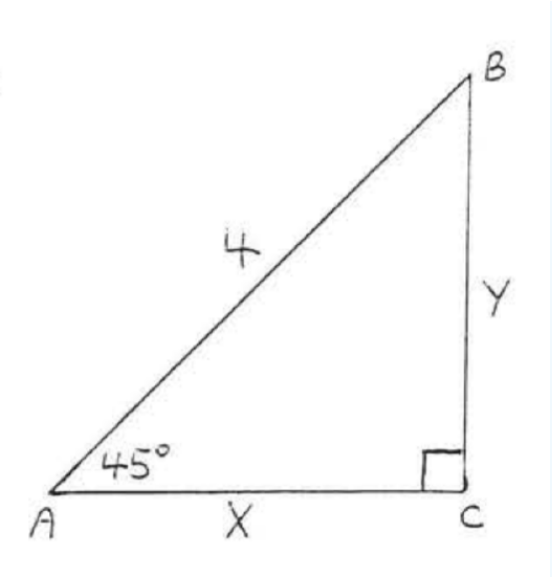


Figure 12.1.5.8: The legs are equal and the hypotenuse is equal to either leg times $\sqrt{2}$.

Example 12.1.5.4

Find x and y:



Solution

$\angle B = 45^\circ$. So $\triangle ABC$ is an isosceles right triangle and $x = y$.

$$\begin{aligned} x^2 + y^2 &= 4^2 \\ x^2 + x^2 &= 16 \\ 2x^2 &= 16 \\ x^2 &= 8 \\ x &= \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2} \end{aligned}$$

Answer: $x = y = 2\sqrt{2}$.

Another method:

$\triangle ABC$ is a $45^\circ - 45^\circ - 90^\circ$ triangle. Hence by Theorem 12.1.5.2,

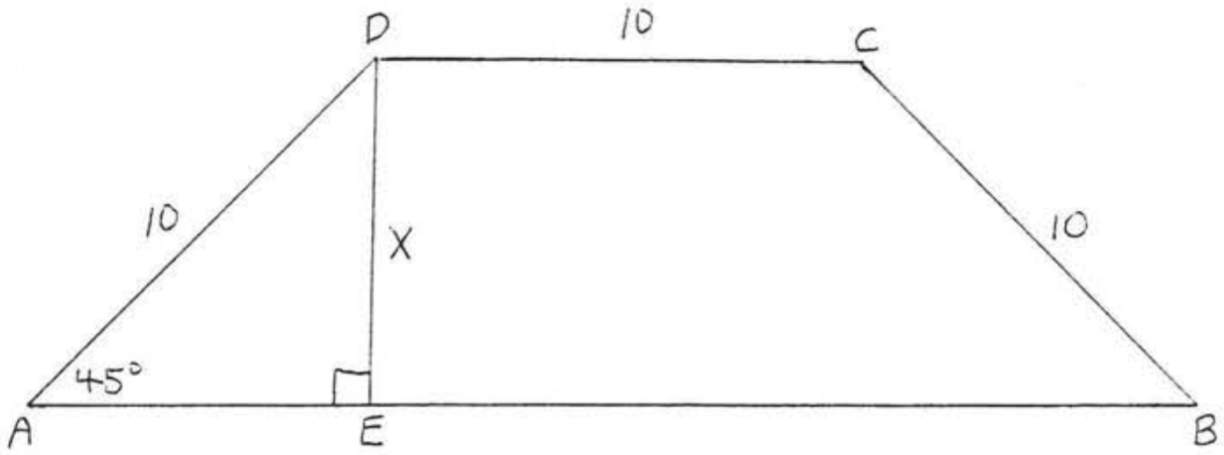
$$\begin{aligned} \text{hyp} &= L\sqrt{2} \\ 4 &= x\sqrt{2} \\ \frac{4}{\sqrt{2}} &= \frac{x\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} \\ x &= \frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2} \end{aligned}$$

Answer

$x = y = 2\sqrt{2}$.

Example 12.1.5.5

Find AB:



Solution

$\triangle ADE$ is a $45^\circ - 45^\circ - 90^\circ$ triangle. Hence

$$\begin{aligned} \text{hyp} &= L\sqrt{2} \\ 10 &= x\sqrt{2} \\ \frac{10}{\sqrt{2}} &= \frac{x\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} \\ x &= \frac{10}{\sqrt{2}} = \frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2} \\ AE = x &= 5\sqrt{2} \end{aligned}$$

Now draw CF perpendicular to AB (Figure 12.1.5.9). $\angle B = 45^\circ$ since $ABCD$ is an isosceles trapezoid (Theorem 3.2.4, Section 3.2).

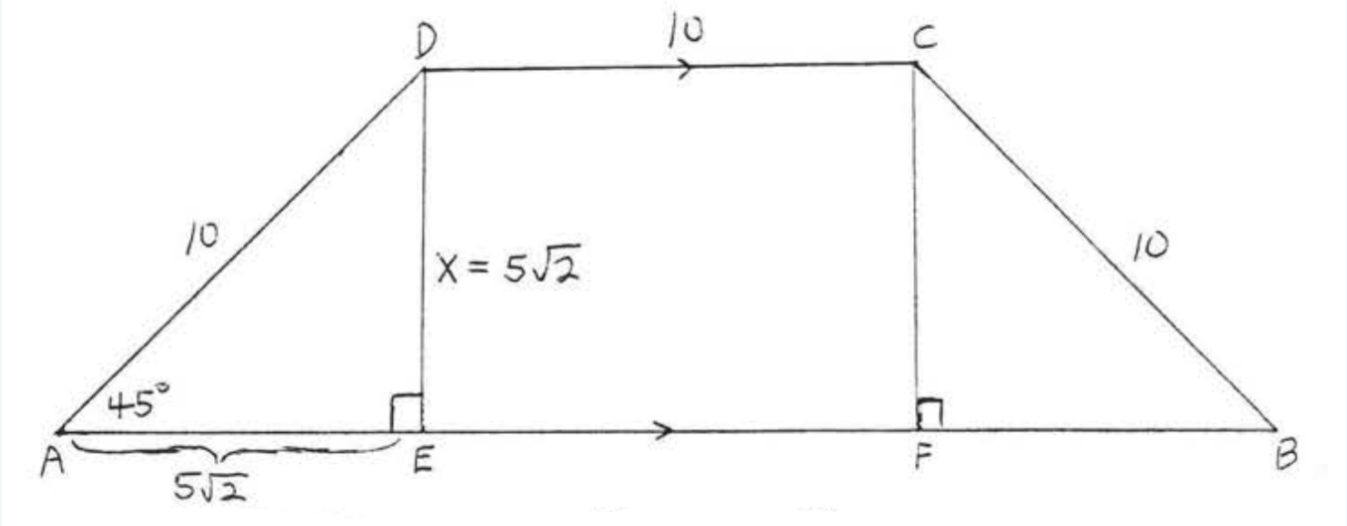


Figure 12.1.5.9: Draw CF perpendicular to AB.

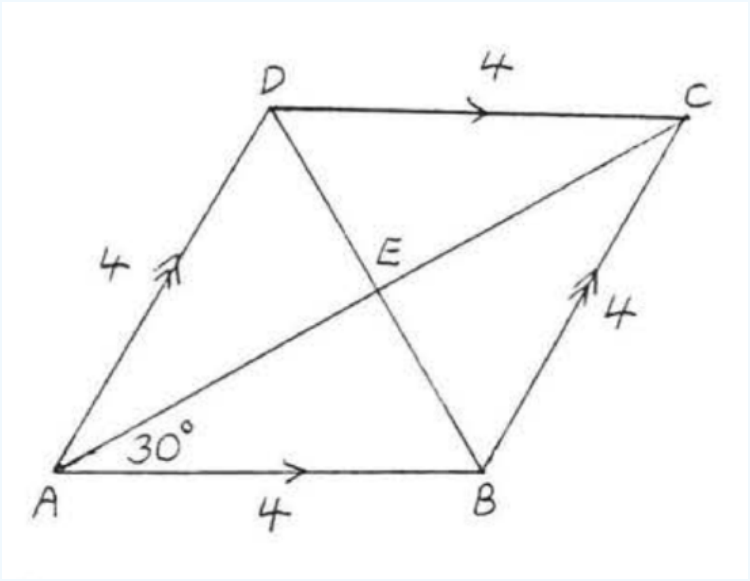
So $\triangle BCF$ is a $45^\circ - 45^\circ - 90^\circ$ triangle congruent to $\triangle ADE$ and therefore $BF = 5\sqrt{2}$. $CDEF$ is a rectangle and therefore $EF = 10$. We have $AB = AE + EF + FB = 5\sqrt{2} + 10 + 5\sqrt{2} = 10\sqrt{2} + 10$.

Answer

$AB = 10\sqrt{2} + 10.$

Example 12.1.5.6

Find AC and BD:



Solution

$ABCD$ is a rhombus. The diagonals AC and BD are perpendicular and bisect each other. $\angle AEB = 90^\circ$ and $\angle ABE = 180^\circ - (90^\circ - 30^\circ) = 60^\circ$. So $\triangle AEB$ is a $30^\circ - 60^\circ - 90^\circ$ triangle.

$$\begin{aligned} \text{hyp} &= 2s & L &= s\sqrt{3} \\ 4 &= 2(BE) & AE &= 2\sqrt{3} \\ 2 &= BE & AC &= 2\sqrt{3} + 2\sqrt{3} \\ bd &= 2 + 2 = 4 & AC &= 4\sqrt{3} \end{aligned}$$

Answer

$AC = 4\sqrt{3}, BD = 4.$

Historical Note: Irrational Numbers

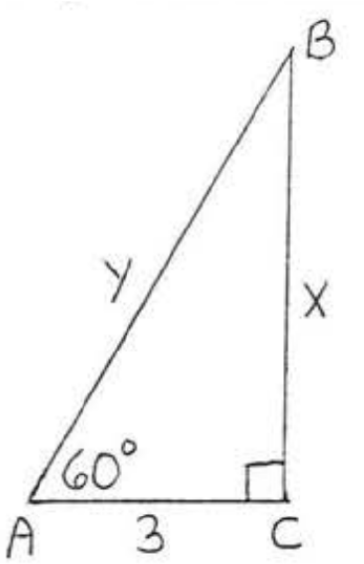
The Pythagoreans believed that all physical relation ships could be expressed with whole numbers. However the sides of the special triangles described in this section are related by **irrational** numbers, $\sqrt{2}$ and $\sqrt{3}$. An irrational number is a number which can be approximated, but not expressed exactly, by a ratio of whole numbers. For example $\sqrt{2}$ can be approximated with increasing accuracy by such ratios as $1.4 = \frac{14}{10}$, $1.41 = \frac{141}{100}$, $1.414 = \frac{1414}{1000}$, etc., but there is no fraction of whole numbers which is **exactly** equal to $\sqrt{2}$. (For more details and a proof, see the book by Richardson listed in the References). The Pythagoreans discovered that $\sqrt{2}$ was irrational in about the 5th century B.C. It was a tremendous shock to them that not all triangles could be measured "exactly." They may have even tried to keep this discovery a secret for fear of the damage it would do to their philosophical credibility.

The inability of the Pythagoreans to accept irrational numbers had unfortunate consequences for the development of mathematics. Later Greek mathematicians avoided giving numerical values to lengths of line segments. Problems whose algebraic solutions might be irrational numbers, such as those involving quadratic equations, were instead stated and solved geometrically. The result was that geometry flourished at the expense of algebra. It was left for the Hindus and the Arabs to resurrect the study of algebra in the Middle Ages. And it was not until the 19th century that irrational numbers were placed in the kind of logical framework that the Greeks had given to geometry 2000 years before.

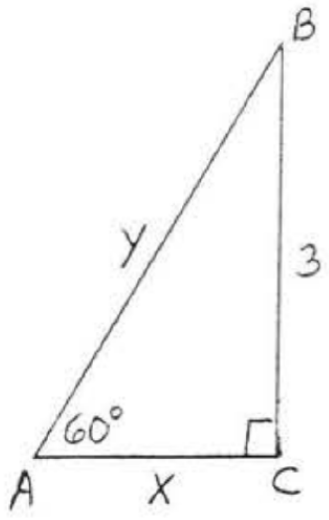
Problems

1 - 10. Find x and y:

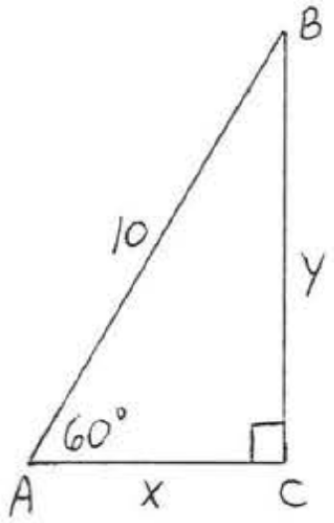
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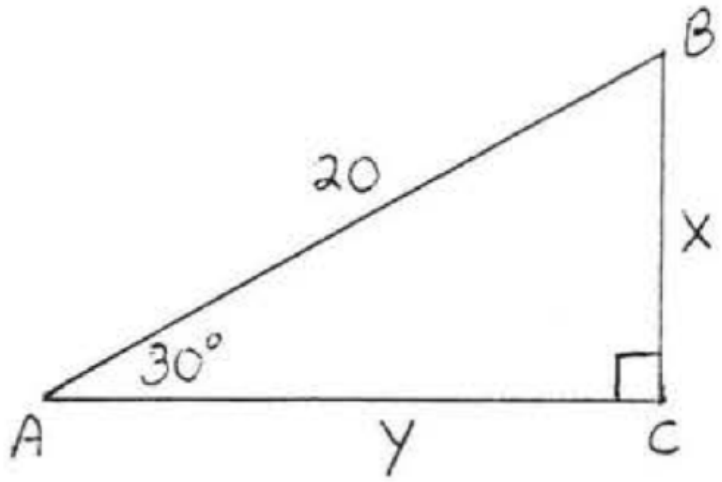
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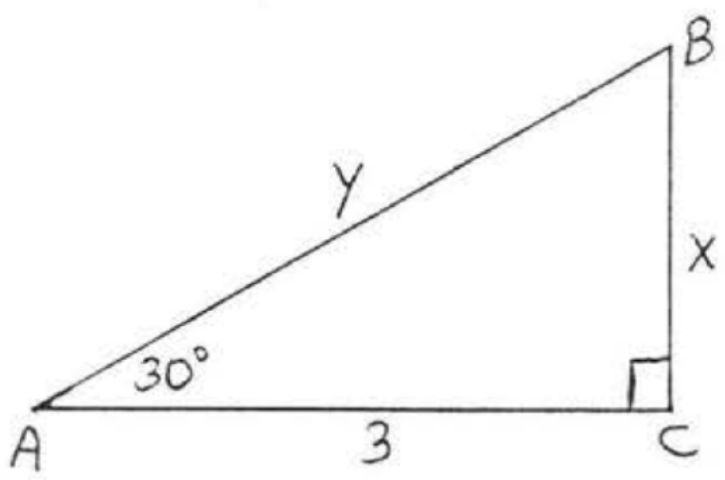
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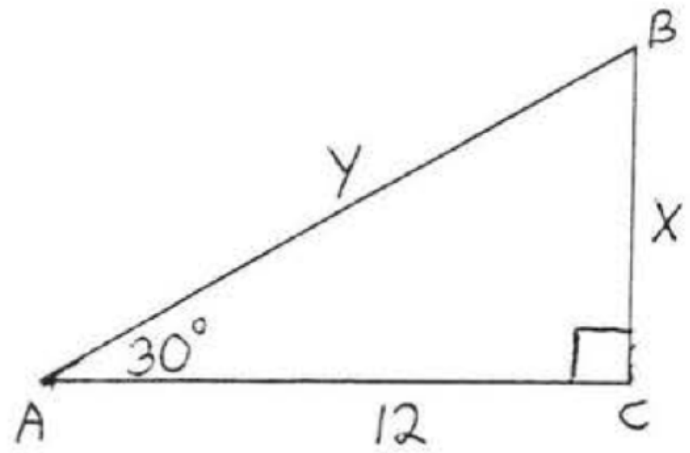
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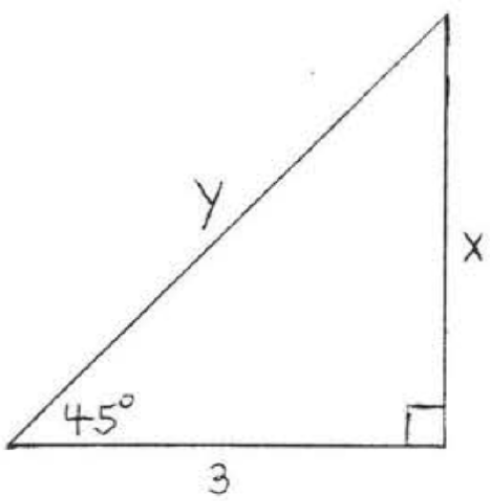
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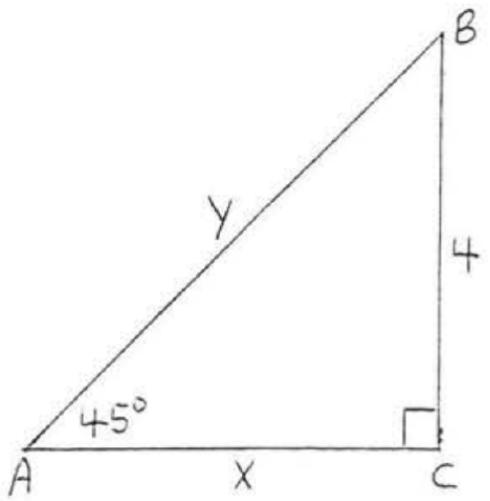
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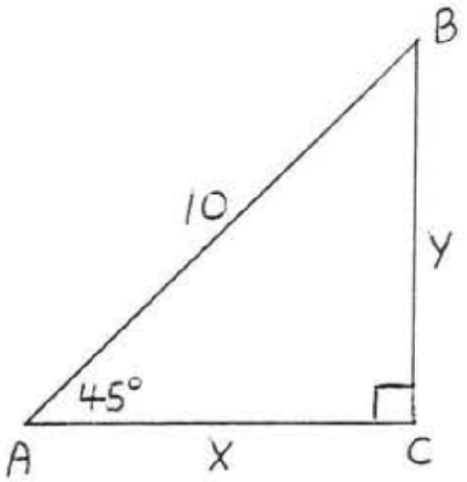
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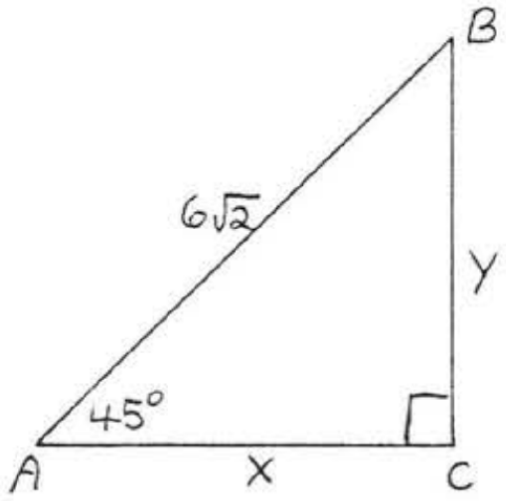
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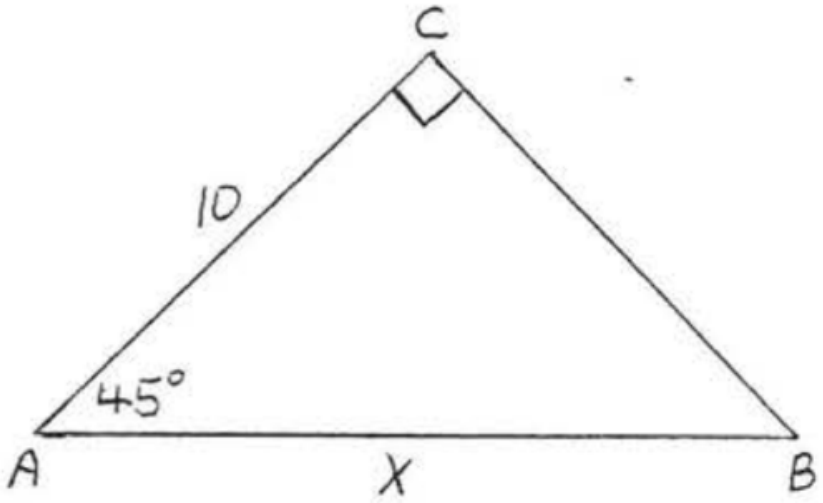


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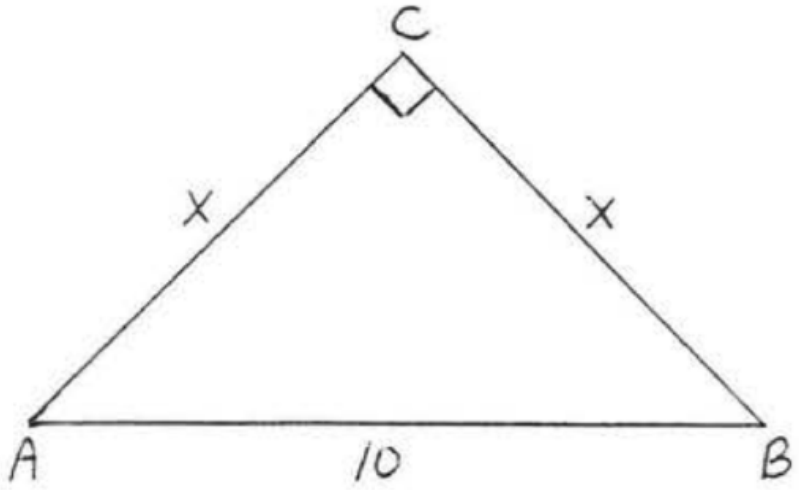


11 - 14. Find x:

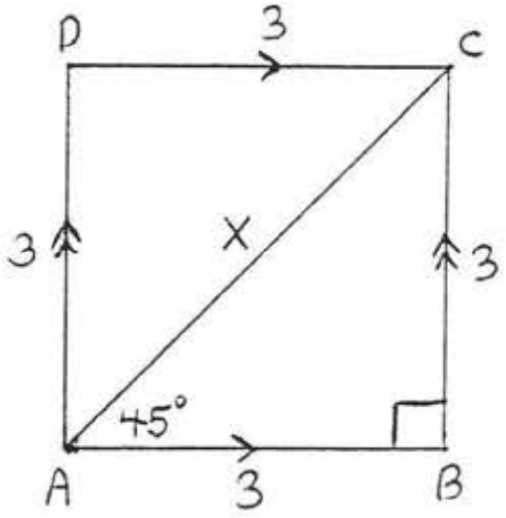
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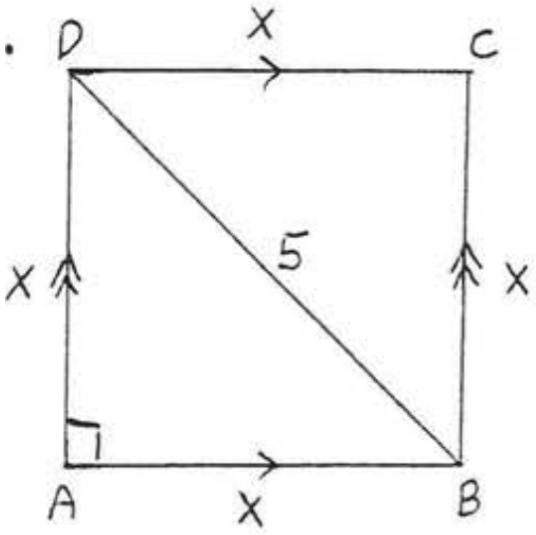
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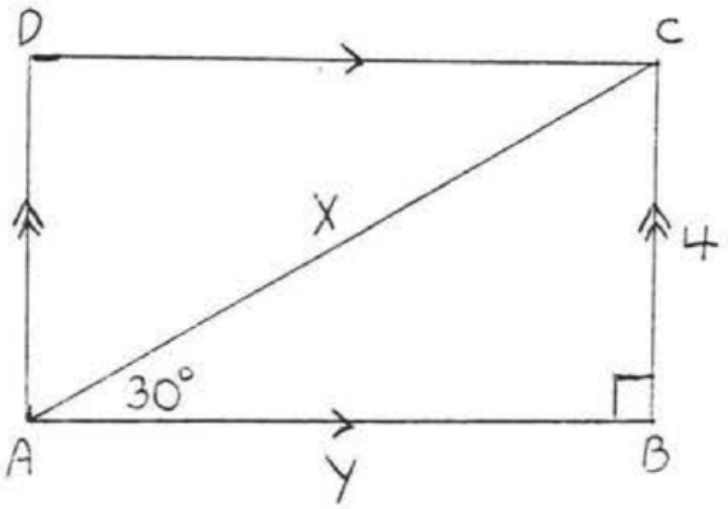


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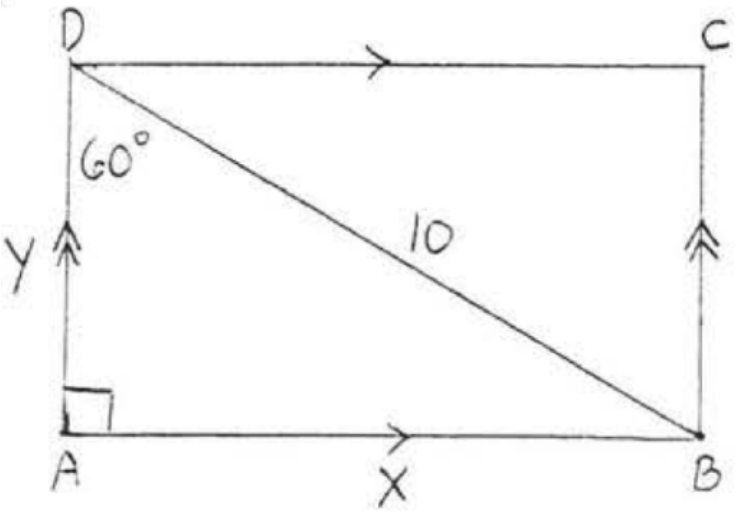


15 - 20. Find x and y:

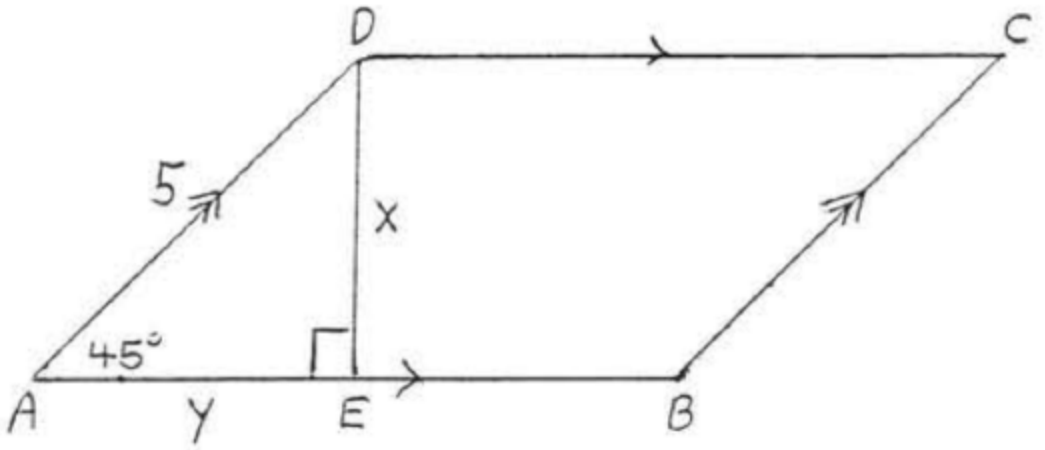
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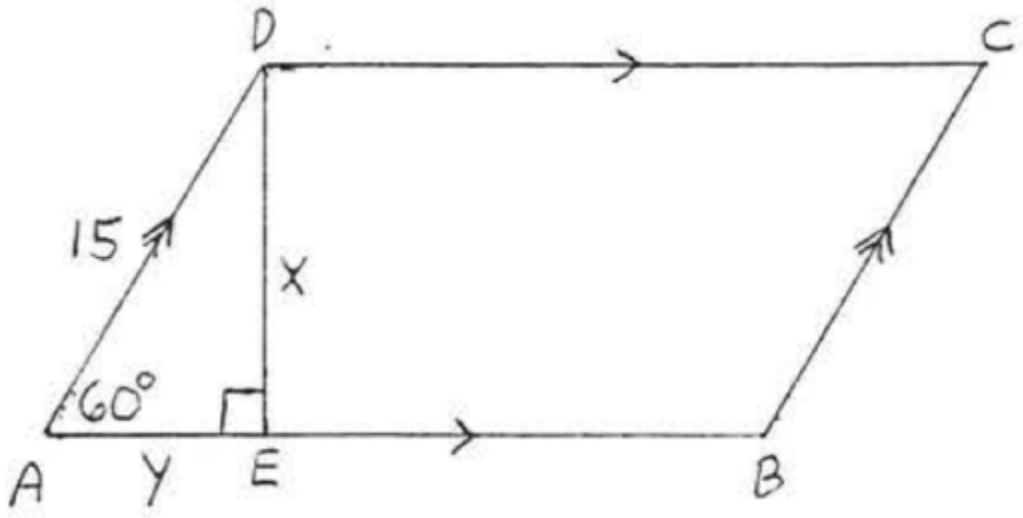
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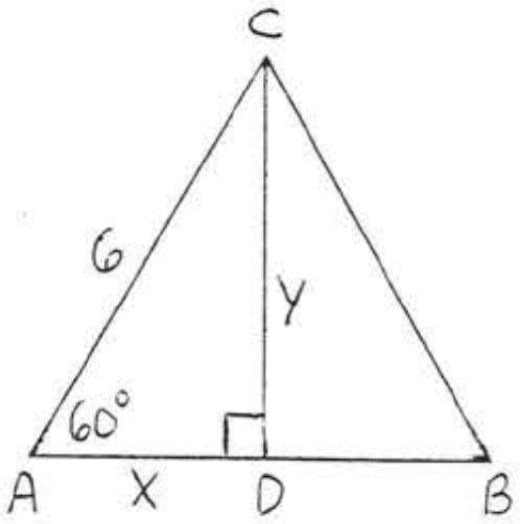
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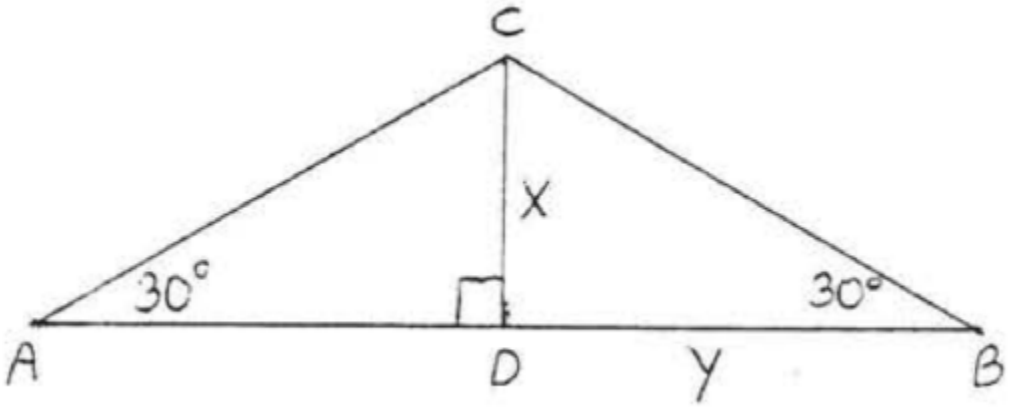
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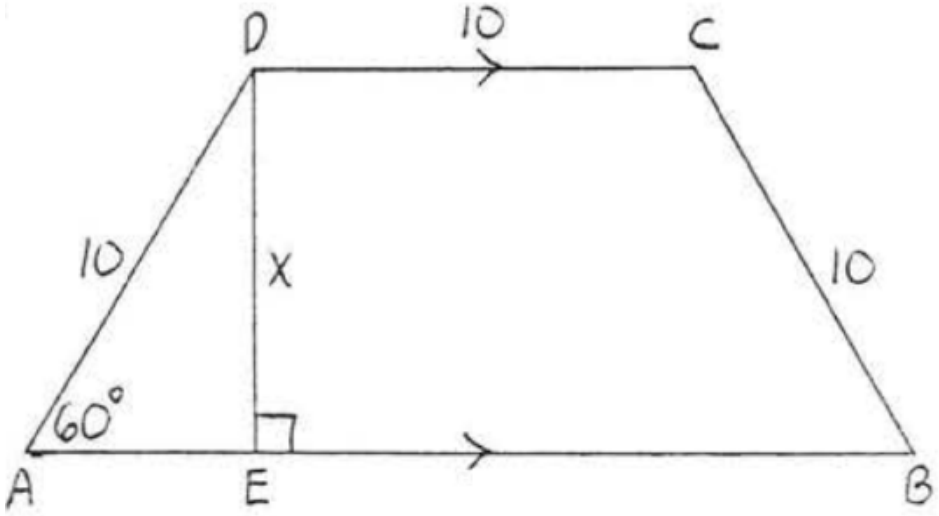


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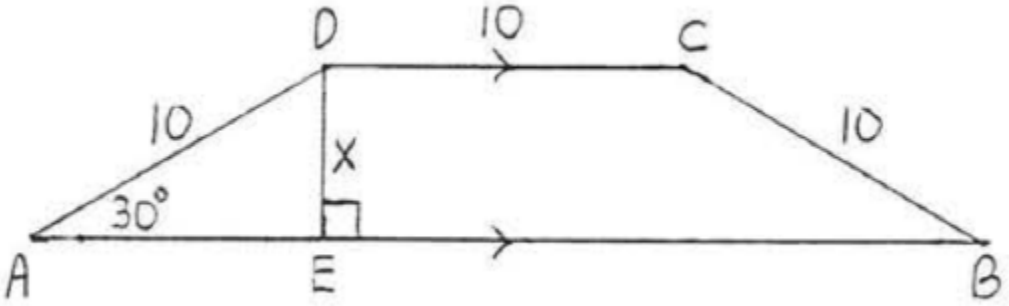


21 - 22. Find x and AB:

21.

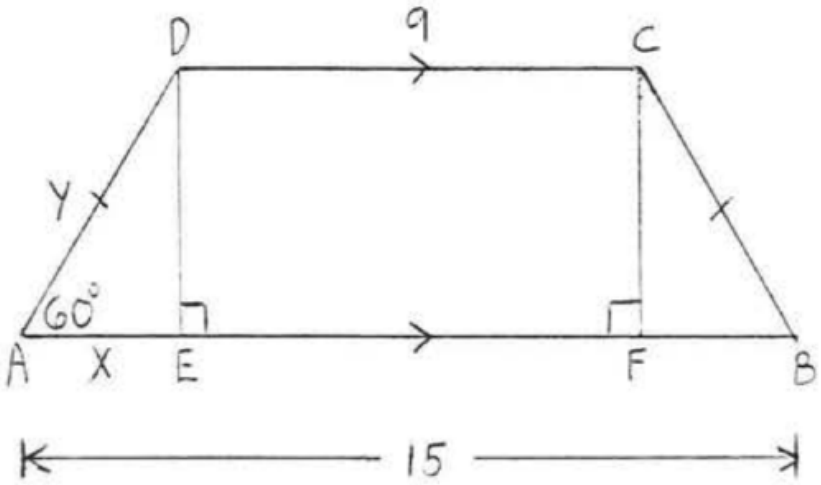


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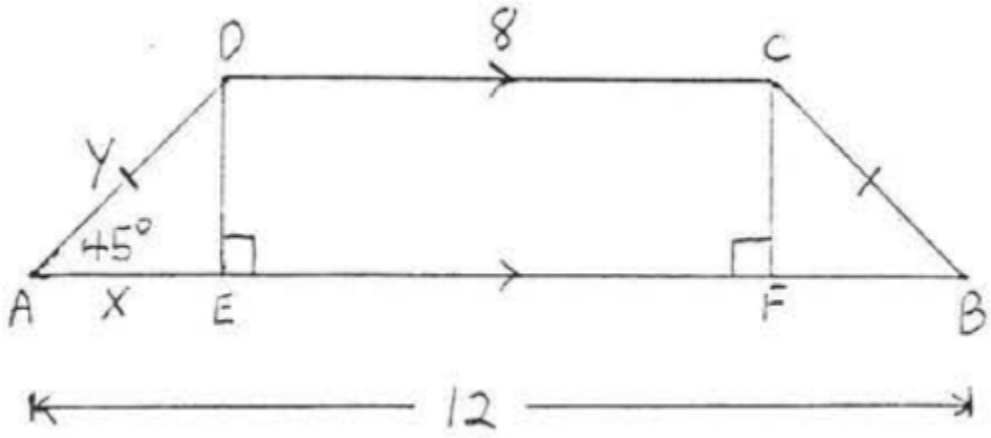


23 - 24. Find x and y:

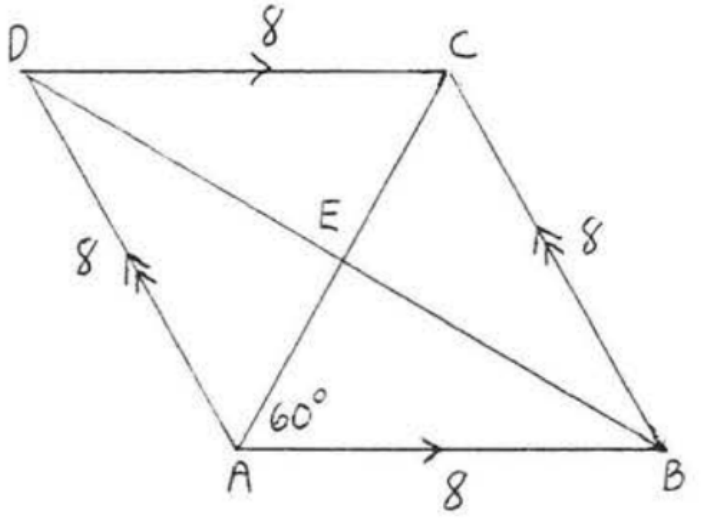
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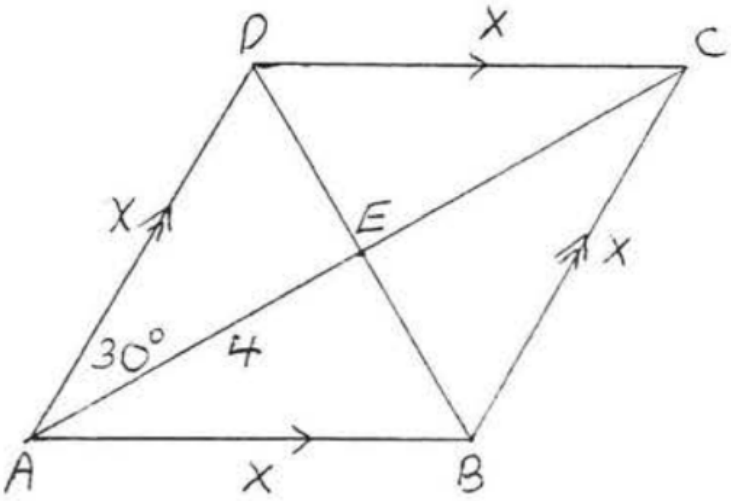
24.



25. Find AC and BD:



26. Find x, AC and BD:



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12.1.6: Distance from a Point to a Line

Suppose we are given a point P and a line $\overleftrightarrow{AB}$ as in Figure 12.1.6.1. We would like to find the shortest line segment that can be drawn from P to $\overleftrightarrow{AB}$.

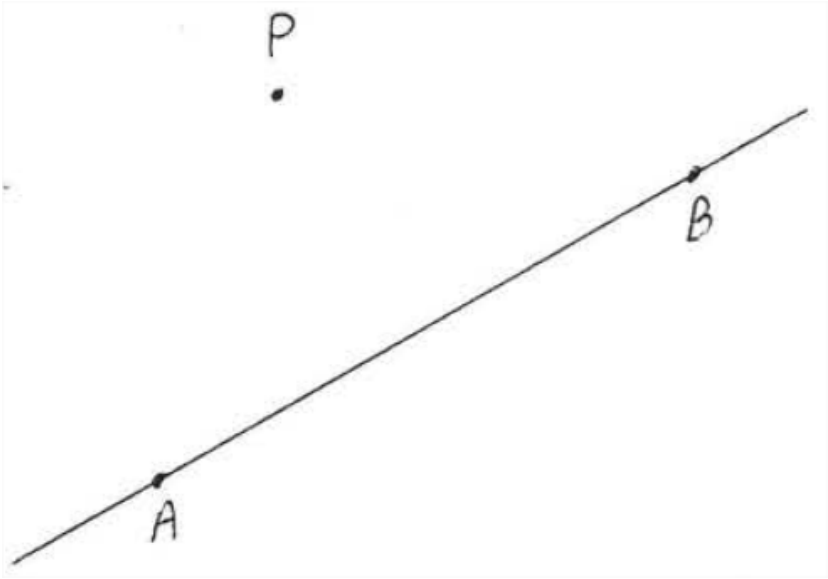


Figure 12.1.6.1: Point P and Line $\overleftrightarrow{AB}$.

First we will prove a theorem:

Theorem 12.1.6.1

In a right triangle, the hypotenuse is larger than either leg. In Figure 12.1.6.1, $c > a$ and $c > b$. (The symbol ">" means "is greater than.")

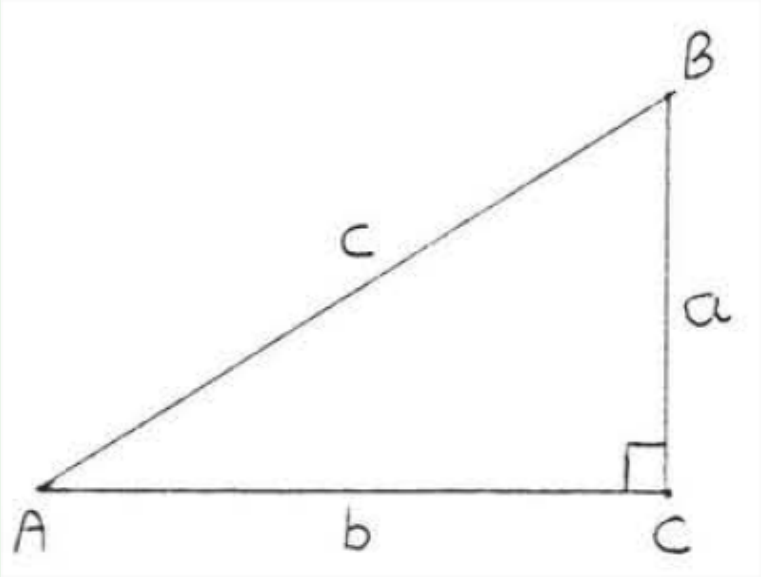


Figure 12.1.6.2: c is larger than either a or b.

Proof available at the end of this section.

Now we can give the answer to our question:

Theorem 12.1.6.2

The perpendicular is the shortest line segment that can be drawn from a point to a straight line.

In Figure 12.1.6.3 the shortest line segment from P to $\overleftrightarrow{AB}$ is PD. Any other line segment, such as PC, must be longer.

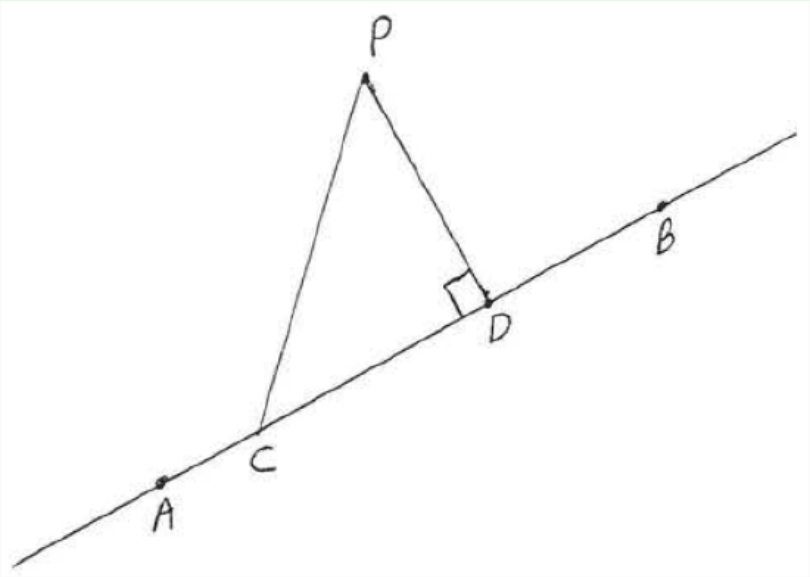


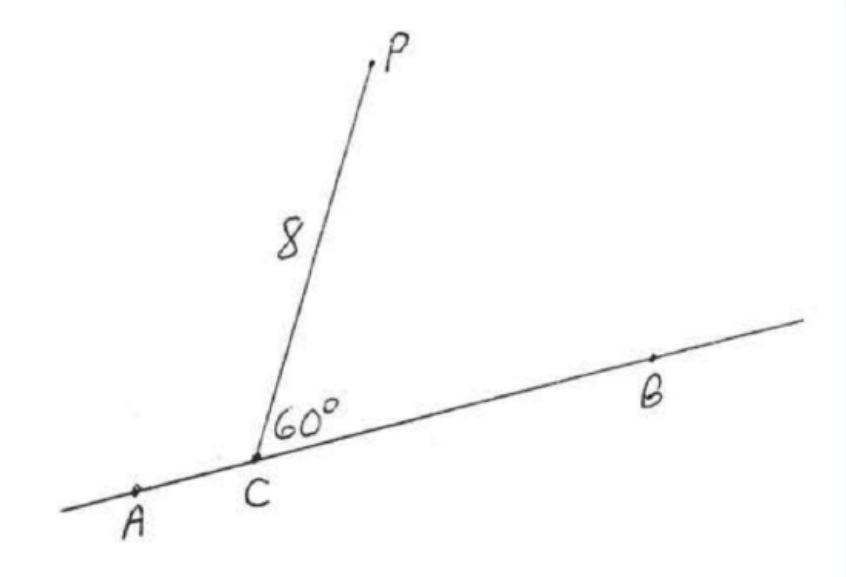
Figure 12.1.6.3: PD is the shortest line segment from P to $\overleftrightarrow{AB}$.

Proof available at the end of this section.

We define the distance from a point to a line to be the length of the perpendicular.

Example 12.1.6.1

Find the distance from P to $\overleftrightarrow{AB}$:



Solution

Draw PD perpendicular to $\overleftrightarrow{AB}$ (Figure 12.1.6.4). $\triangle PCD$ is a $30^\circ - 60^\circ - 90^\circ$ triangle.

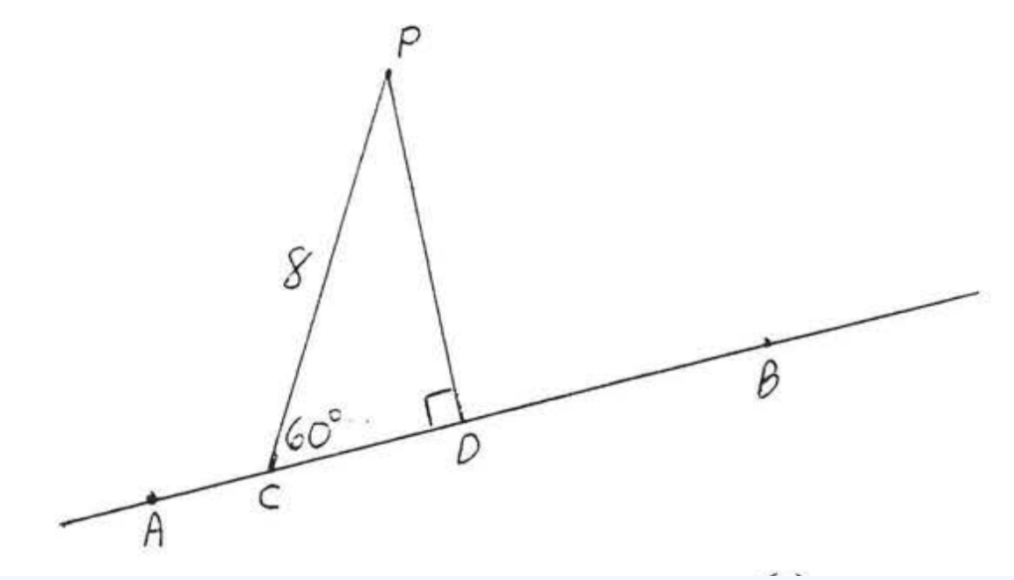


Figure 12.1.6.4: Draw PD perpendicular to $\overleftrightarrow{AB}$.

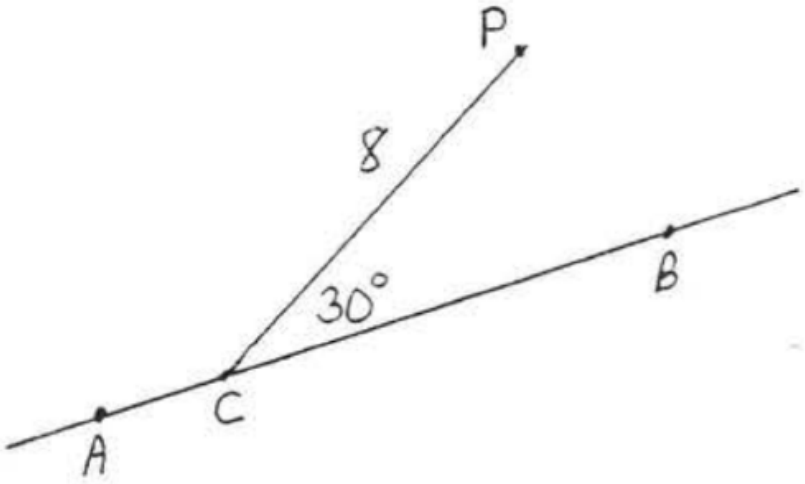
$$\begin{aligned} \text{hyp} &= 2s \\ 8 &= 2(CD) \\ 4 &= CD \\ L &= s\sqrt{3} \\ PD &= 4\sqrt{3} \end{aligned}$$

Answer: $4\sqrt{3}$

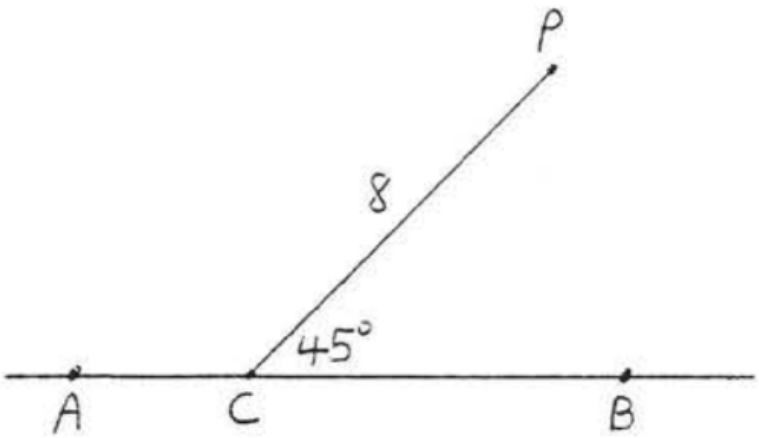
Problems

1 - 6. Find the distance from P to $\overleftrightarrow{AB}$:

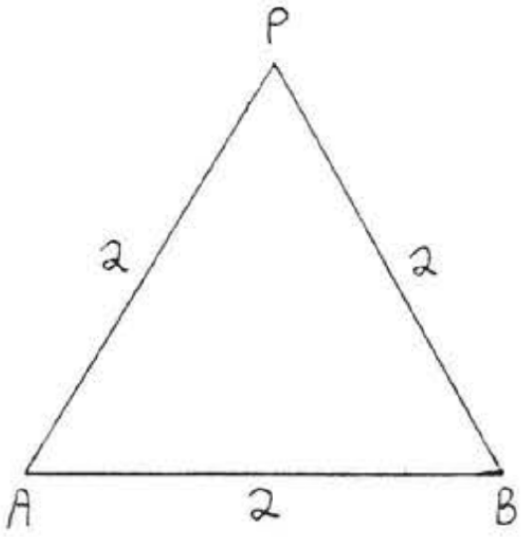
1.



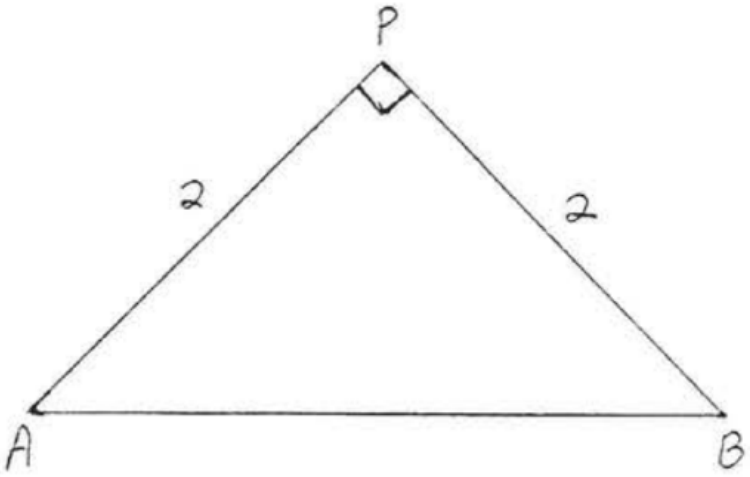
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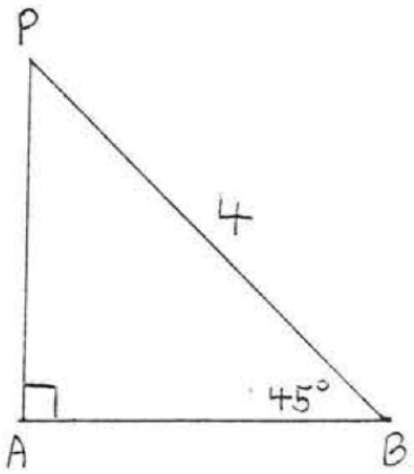
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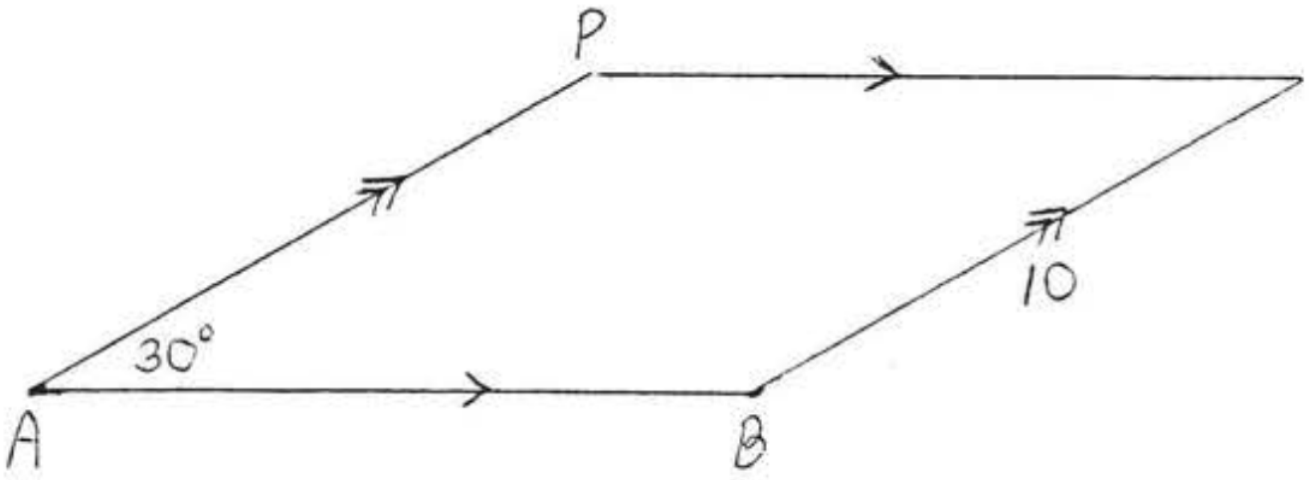
4.



5.



6.



Answers

Theorem 12.1.6.1

By the Pythagorean Theorem,

$$c = \sqrt{a^2 + b^2} > \sqrt{a^2} = a.$$

$c = \sqrt{a^2 + b^2} > \sqrt{b^2} = b.$

Theorem 12.1.6.2

PC is the hypotenuse of right triangle PCD. Therefore by Theorem 12.1.6.1, $PC > PD$.

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13: AREA, PYTHAGOREAN THEOREM, AND VOLUME

Area, Pythagorean Theorem, and Volume

- 13-1 Linear Measure
- 13-2 Areas of Polygons and Circles
- 13-3 The Pythagorean Theorem, Distance Formula, and Equation of a Circle
- 13-4 Surface Areas
- 13-5 Volume, Mass, and Temperature
- Chapter 13 Review 890
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13.1: Lines

13.2: Parallelograms

13.3: Other Quadrilaterals

13.4: Regular Polygons

13.5: Circles

13.6: Metric Units of Measurement

13.7: Temperature Scales

13.8: Temperature and Time

13.8.1: Exercises

13.9: Dilations on a Square Grid

13.10: More Dilations

Links we'd like to aggregate:

- 13-3 The Pythagorean Theorem, Distance Formula, and Equation of a Circle
  1. https://www.ck12.org/book/ck-12-geometry-second-edition/section/8.1/
  2. https://www.ck12.org/book/ck-12-geometry-second-edition/section/8.2/
  3. https://www.ck12.org/book/ck-12-geometry-second-edition/section/8.3/
  4. https://www.ck12.org/book/ck-12-geometry-second-edition/section/8.4/
- 13-4 Surface Areas
  1. https://www.ck12.org/book/ck-12-geometry-second-edition/section/11.1/
  2. https://www.ck12.org/book/ck-12-geometry-second-edition/section/11.2/
  3. https://www.ck12.org/book/ck-12-geometry-second-edition/section/11.3/
     - https://www.ck12.org/book/ck-12-geometry-second-edition/section/11.6/
- 13-5 Volume, Mass, and Temperature
  1. https://www.ck12.org/book/ck-12-geometry-second-edition/section/11.4/
  2. https://www.ck12.org/book/ck-12-geometry-second-edition/section/11.5/

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13.1: Lines

Geometry (from Greek words meaning earth-measure) originally developed as a means of surveying land areas, In its simplest form, it is a study of figures that can be drawn on a perfectly smooth flat surface, or **plane**. It is this **plane geometry** which we will study in this bock and which serves as a foundation for trigonometry, solid and analytic geometry, and calculus.

The simplest figures that can be drawn on a plane are the point and the line. By a line we will always mean a **straight line**. Through two distinct points one and only one (straight) line can be drawn. The line through points A and B will be denoted by $\overleftrightarrow{AB}$ (Figure 13.1.1). The arrows indicate that the line extends indefinitely in each direction, The **line segment** from A to B consists of A, B and that part of $\overleftrightarrow{AB}$ between A and B, It is denoted by AB (some textbooks use the notation $\overline{AB}$ for line segment). The ray $\overrightarrow{AB}$ is the part of $\overleftrightarrow{AB}$ which begins at A and extends indefinitely in the direction of B.

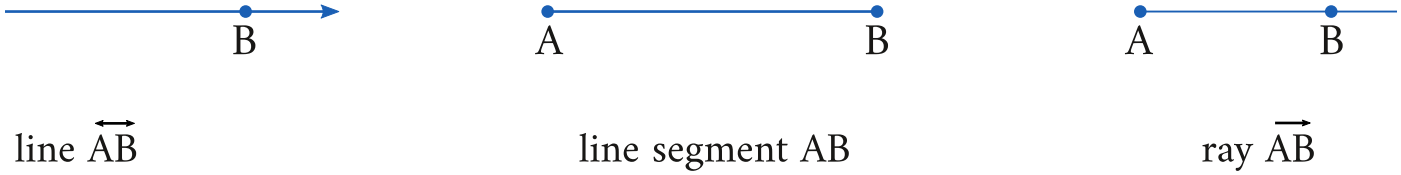


Figure 13.1.1: Line $\overleftrightarrow{AB}$, line segment $\overline{AB}$, and ray $\overrightarrow{AB}$. (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

We assume everyone is familiar with the notion of length of a line segment and how it can be measured in inches, or feet, or meters, etc, The distance between two points A and B is the same as the length of AB.

Two line segments are equal if they have the same length, e.g., in Figure 13.1.2, $AB = CD$,



Figure 13.1.2: $AB = CD$. (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

We often indicate two line segments are equal by marking them in the same way, e.g., in Figure 13.1.3, $AB = CD$ and $EF = GH$.



Figure 13.1.3: $AB = CD$ and $EF = GH$. (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

Example 13.1.1

Find x if $AB = CD$:



Figure 13.1.E1: (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

Solution

$$\begin{aligned} AB &= CD \\ 3x-6 &= x \\ 3x-x &= 6 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

Check:

$$\begin{array}{rcl|l} \text{AB} & = & \text{CD} \\ 3\text{x} - 6 & & \text{x} \\ 3(3) - 6 & & 3 \\ 9 - 6 & & \\ 3 & & \end{array}$$

Answer: $x = 3$.

Notice that in Example 13.1.1 we have not indicated the unit of measurement. Strictly speaking, we should specify that $AB = 3x - 6$ inches (or feet or meters) and that $BC = x$ inches. However since the answer would still be $x = 3$ we will usually omit this information to save space.

We say that B is the midpoint of AC if B is A point on AC and $AB = BC$ (Figure 13.1.4).

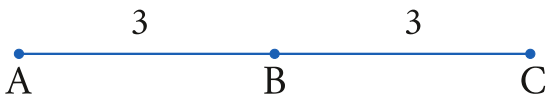


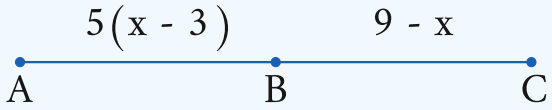
Figure 13.1.4: B is the midpoint of AC. (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

Example 13.1.2

Find x and AC if B is the midpoint of AC and $AB = 5(x - 3)$ and $BC = 9 - x$,

Solution

We first draw a picture to help visualize the given information:



Since B is a midpoint,

$$\begin{aligned} AB &= BC \\ 5(x-3) &= 9-x \\ 5x-15 &= 9-x \\ 5x+x &= 9+15 \\ 6x &= 24 \\ x &= 4 \end{aligned}$$

Check:

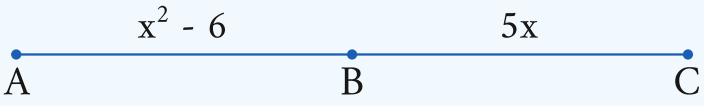
AB	=	BC
5(x - 3)		9 - x
5(4 - 3)		9 - 4
5(1)		5
5		

We obtain $AC = AB + BC = 5 + 5 = 10$.

Answer: $x = 4$, $AC = 10$.

Example 13.1.3

Find AB if B is the midpoint of AC:



Solution

$$\begin{aligned} AB &= BC \\ x^2-6 &= 5x \\ x^2-5x-6 &= 0 \\ (x-6)(x+1) &= 0 \end{aligned}$$

$$\begin{aligned} x-6 &= 0 \\ x &= 6 \end{aligned} \qquad \begin{aligned} x+1 &= 0 \\ x &= -1 \end{aligned}$$

If $x = 6$ then $AB = x^2 - 6 = 6^2 - 6 = 36 - 6 = 30$.

If $x = -1$ then $AB = (-1)^2 - 6 = 1 - 6 = -5$.

We reject the answer $x = -1$ and $AB = -5$ because the length of a line segment is always positive. Therefore $x = 6$ and $AB = 30$.

Check:

AB	=	BC
x^2 - 6		5x
6^2 - 6		5(6)
36 - 6		30
30		

Answer: $AB = 30$.

Three points are collinear if they lie on the same line.



Figure 13.1.5: A, B, and C are collinear $AB = 5$, $BC = 3$, and $AC = 8$ (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

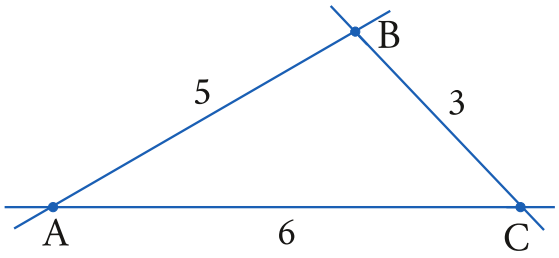


Figure 13.1.6: A, B, and C are not collinear. $AB = 5$, $BC = 3$, $AC = 6$. (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

A, B, and C are collinear if and only if $AB + BC = AC$.

Example 13.1.4

If A, B, and C are collinear and $AC = 7$, find x:



Solution

$$\begin{aligned} AB + BC &= AC \\ 8 - 2x + x + 1 &= 7 \\ 9 - x &= 7 \\ 2 &= x \end{aligned}$$

Check:

AB + BC	=	AC
8 - 2x + x + 1		7
8 - 2(2) + 2 + 1		
8 - 4 + 3		
4 + 3		
7		

Answer: $x = 2$

Historical Note

Geometry originated in the solution of practical problems, The architectural remains of Babylon, Egypt, and other ancient civilizations show a knowledge of simple geometric relationships, The digging of canals, erection of buildings, and the laying out of cities required computations of lengths, areas, and volumes, Surveying is said to have developed in Egypt so that tracts of land could be relocated after the annual overflow of the Nile, Geometry was also utilized by ancient civilizations in their astronomical observations and the construction of their calendars.

The Greeks transformed the practical geometry of the Babylonians and Egyptians into an organized body of knowledge. Thales (c, 636 - c. 546 B,C.), one of the "seven wise men" of antiquity, is credited with being the first to obtain geometrical results by logical reasoning, instead of just by intuition and experiment. Pythagoras (c. 582 - c. 507 B,C.) continued the work of Thales, He founded the Pythagorean school, a mystical society devoted to the unified study of philosophy, mathematics, and science, About 300 B,C., Euclid, a Greek teacher of mathematics at the university at Alexandria, wrote a systematic exposition of elementary geometry called the **Elements**, In his **Elements**, Euclid used a few simple principles, called *axioms* or *postulates*, to derive most of the mathematics known at the time, For over 2000 years, Euclid's Elements has been accepted as the standard textbook of geometry and is the basis for most other elementary texts, including this one.

Problems

1. Find x if $AB = CD$:



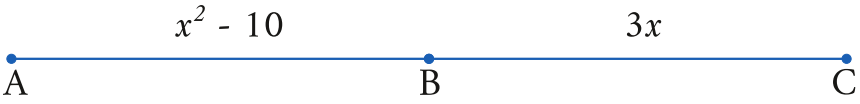
2. Find x if $AB = CD$:



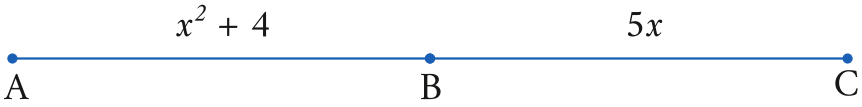
3. Find x and AC if B is the midpoint of AC and $AB = 3(x - 5)$ and $BC = x + 3$.

4. Find x and AC if B is the midpoint of AC and $AB = 2x + 9$ and $BC = 5(x - 9)$,

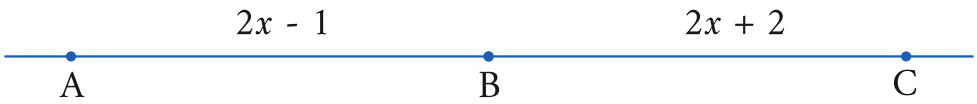
5. Find AB if B is the midpoint of AC:



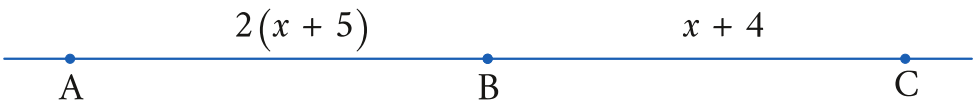
6, Find AB if B is the midpoint of AC:



7. If A, B, and C are collinear and $AC = 13$ find x:



8. If A, B, and C are collinear and $AC = 26$ find x:



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13.2: Parallelograms

A **polygon** is a figure formed by line segments which bound a portion of the plane (Figure 13.2.1), The bounding line segments are called the **sides** of the polygon, The angles formed by the sides are the **angles** of the polygon and the **vertices** of these angles are the vertices of the polygon, The simplest polygon is the triangle, which has 3 sides, In this chapter we will study the **quadrilateral**, the polygon with 4 sides (Figure 13.2.2). Other polygons are the **pentagon** (5 sides), the **hexagon** (6 sides), the **octagon** (8 sides), and the **decagon** (10 sides).

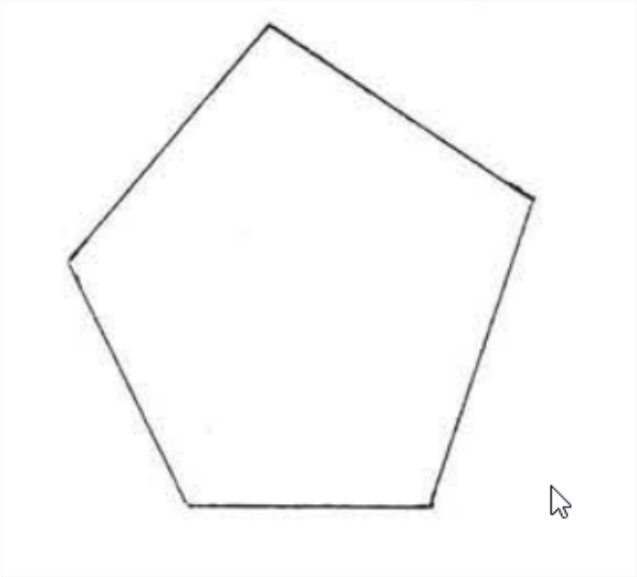


Figure 13.2.1: A polygon

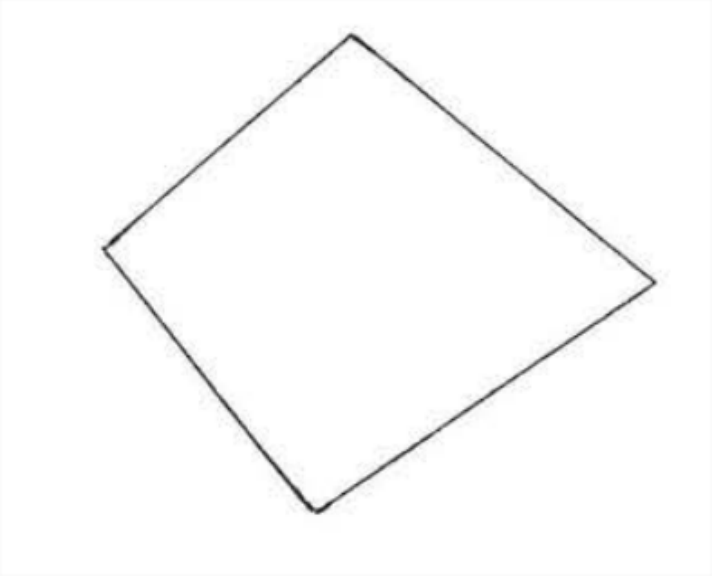


Figure 13.2.2: A quadrilateral

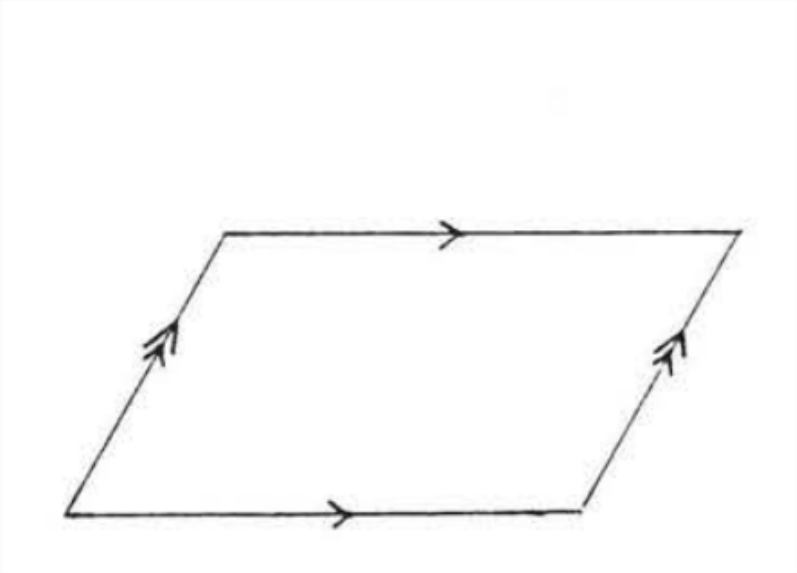


Figure 13.2.3: A parallelogram.

A **parallelogram** is a quadrilateral in which the opposite sides are parallel (Figure 13.2.3). To discover its properties, we will draw a **diagonal**, a line connecting the opposite vertices of the parallelogram. In Figure 4, AC is a diagonal of parallelogram $ABCD$. We will now prove $\Delta ABC \cong \Delta CDA$.

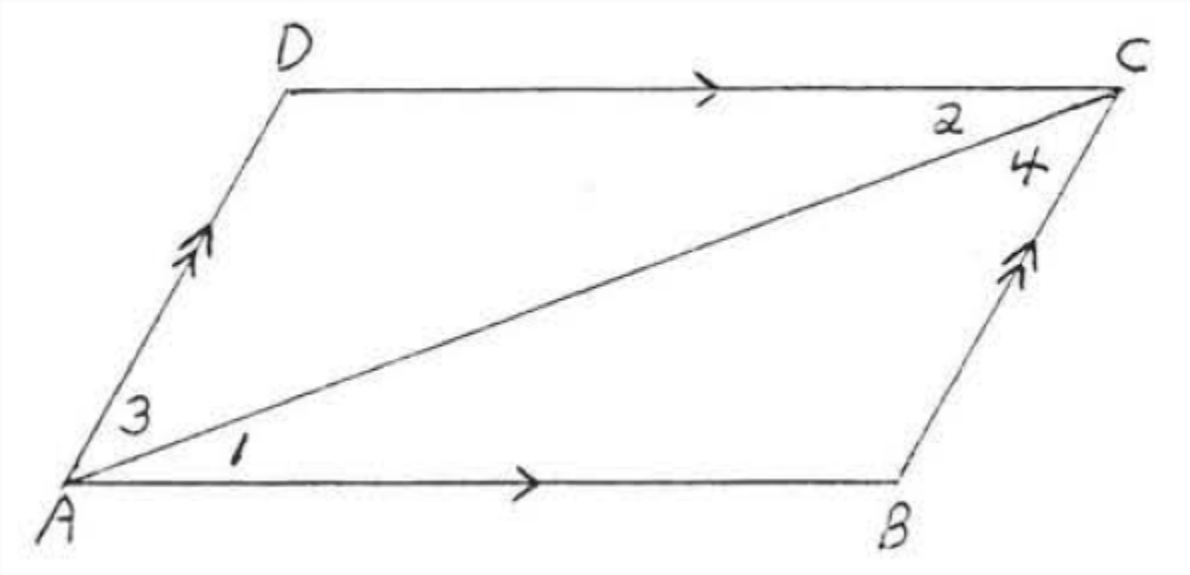


Figure 13.2.4: Diagonal AC divides parallelogram $ABCD$ into two congruent triangles.

Statements	Reasons
1. $\angle 1 = \angle 2$.	1. The alternate interior angles of parallel lines AB and CD are equal.
2. $\angle 3 = \angle 4$.	2. The alternate interior angles of parallel lines BC and AD are equal.
3. $AC = AC$.	3. Identity.
4. $\triangle ABC \cong \triangle CDA$.	4. $ASA = ASA$.
5. $AB = CD$, $BC = DA$.	5. The corresponding sides of congruent triangles are equal.
6. $\angle B = \angle D$.	6. The corresponding angles of congruent triangles are equal.
7. $\angle A = \angle C$.	7. $\angle A = \angle 1 + \angle 3 = \angle 2 + \angle 4 = \angle C$ (Add statements 1 and 2).

We have proved the following theorem:

Theorem 13.2.1

The opposite sides and opposite angles of a parallelogram are equal.

In parallelogram $ABCD$ of Figure 13.2.5, $AB = CD$, $AD = BC$, $\angle A = \angle C$, and $\angle B = \angle D$.

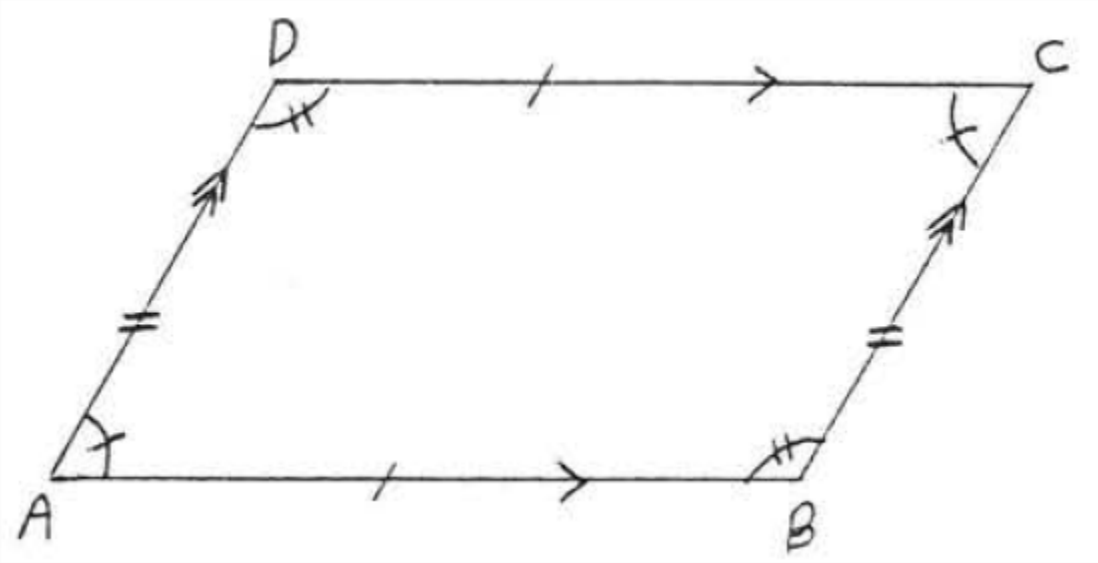
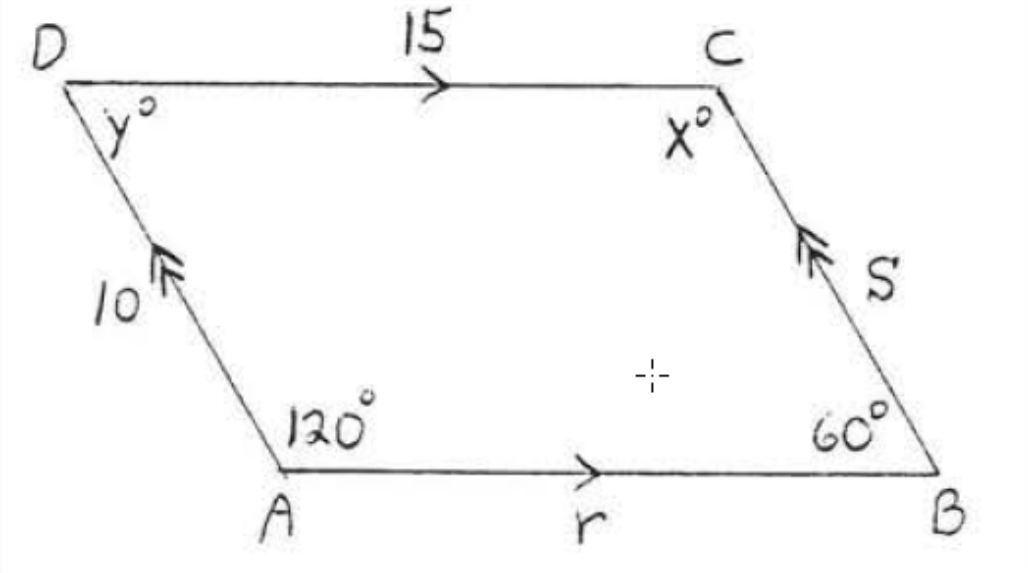


Figure 13.2.5: The opposite sides and opposite angles of a parallelogram are equal.

Example 13.2.1

Find x, y, r and s:



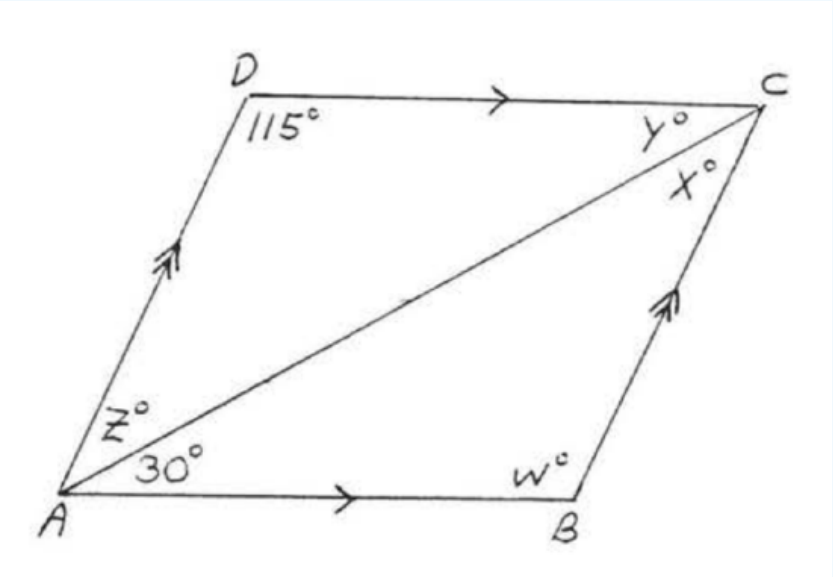
Solution

By Theorem 13.2.1, the opposite sides and opposite angles are equal. Hence $x^\circ = 120^\circ$, $y^\circ = 60^\circ$, $r = 15$, and $s = 10$.

Answer: $x = 120, y = 60, r = 15, s = 10$.

Example 13.2.2

Find x, y, x and z :



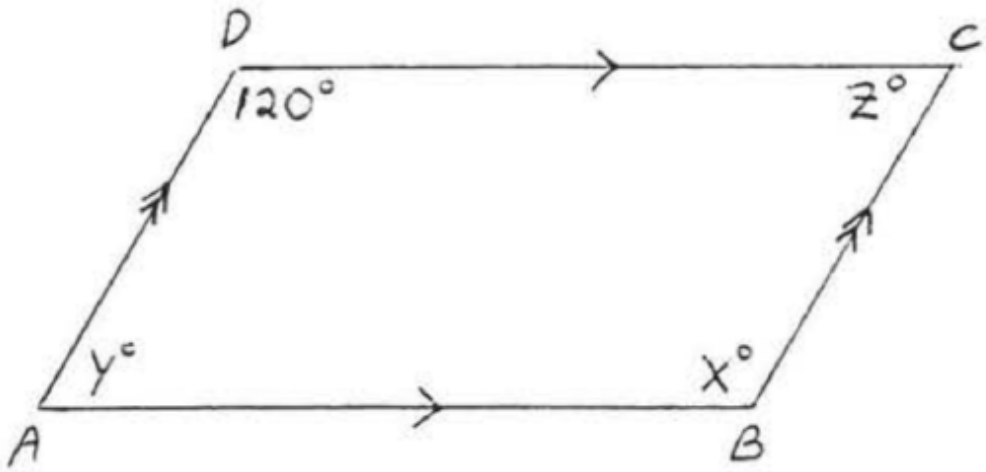
Solution

$w^\circ = 115^\circ$ since the opposite angles of a parallelogram are equal. $x^\circ = 180^\circ - (w^\circ + 30^\circ) = 180^\circ - (115^\circ + 30^\circ) = 180^\circ - 145^\circ = 35^\circ$, because the sum of the angles of $\triangle ABC$ is 180°, $y^\circ = 30^\circ$ and $x^\circ = x^\circ = 35^\circ$ because they are alternate interior angles of parallel lines.

Answer: $w = 115$, $x = z = 35$, $y = 30$.

Example 13.2.3

Find x, y, and z:



Solution

$x = 120$ and $y = z$ because the opposite angles are equal, $\angle A$ and $\angle D$ are supplementary J because they are interior angles on the same side of the transversal of parallel lines (they form the letter "C." Theorem $13.2.3$, section 1.4).

Answer: $x = 120, y = z = 60$.

In Example $13.2.3$, $\angle A$ and $\angle B$, $\angle B$ and $\angle C$, $\angle C$ and $\angle D$, and $\angle D$ and $\angle A$ are called the successive angles of parallelogram $ABCD$. Example $13.2.3$ suggests the following theorem:

Theorem 13.2.2

The successive angles of a parallelogram are supplementary.

In Figure 6,$\angle A + \angle B = \angle B + \angle C = \angle C + \angle D = \angle D + \angle A = 180^\circ$.

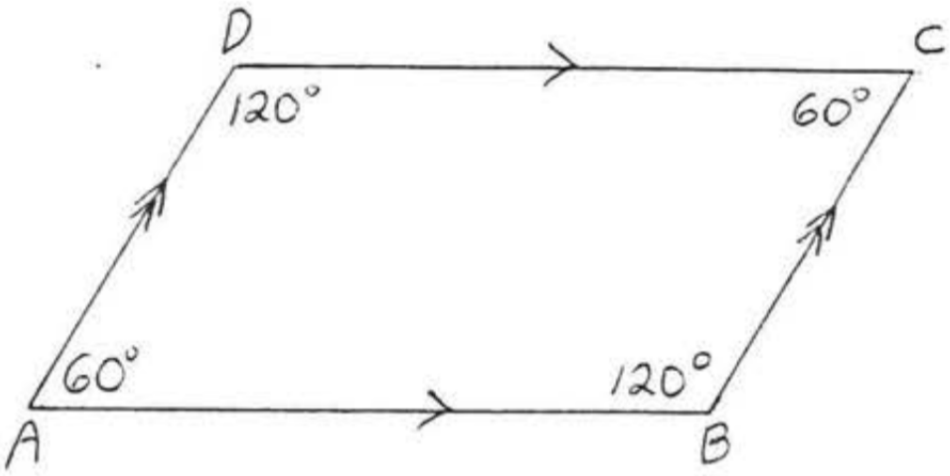
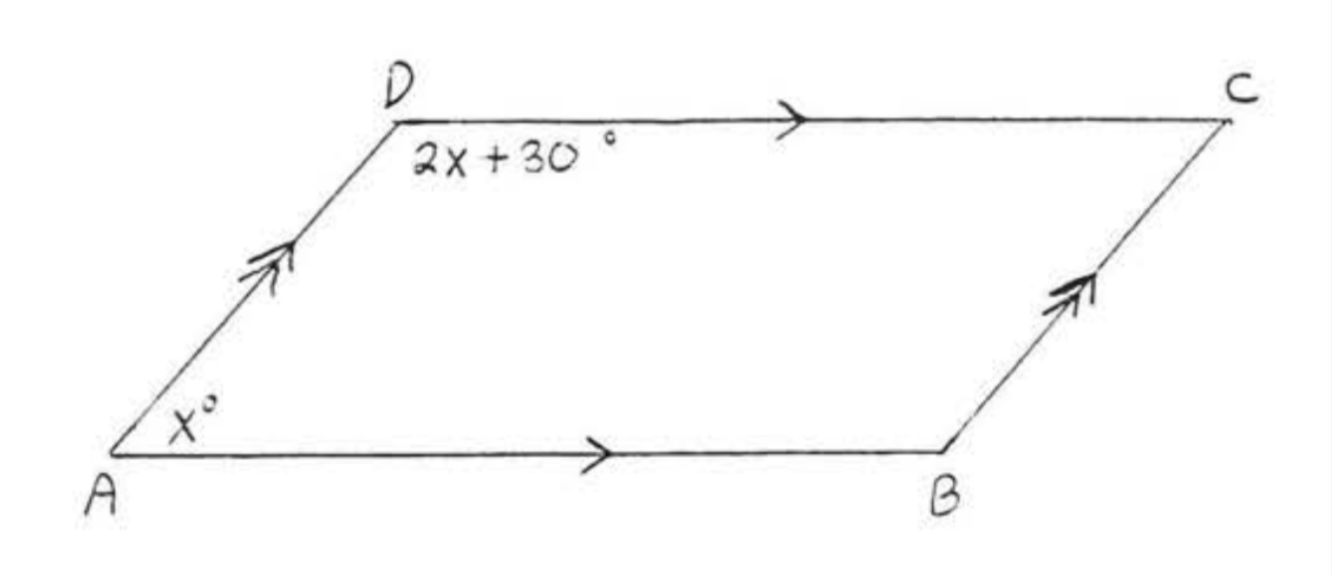


Figure $13.2.6$, The successive angles of parallelogram $ABCD$ are supplementary.

Example 13.2.4

Find x, $\angle A$, $\angle B$, $\angle C$, and $\angle D$.



Solution

$\angle A$ and $\angle D$ are supplementary by Theorem 13.2.2.

$$\begin{aligned} \angle A + \angle D &= 180^\circ \\ x + 2x + 30 &= 180 \\ 3x + 30 &= 180 \\ 3x &= 180 - 30 \\ 3x &= 150 \\ x &= 50 \end{aligned} \tag{13.2.1}$$

$\angle A = x^\circ = 50^\circ$

$\angle C = \angle A = 50^\circ$

$\angle D = 2x + 30^\circ = 2(50) + 30^\circ = 100 + 30^\circ = 130^\circ$.

$\angle B = \angle D = 130^\circ$.

Check:

$$\angle A + \angle D = 180^\circ$$

$$x + 2x + 30$$

$$50 + 2(50) + 30$$

$$50 + 130$$

$$180$$

Answer: $x = 50$, $A = 50^\circ$, $B = 130^\circ$, $C = 50^\circ$, $D = 130^\circ$.

Suppose now that both diagonals of parallelogram are drawn (Figure 13.2.7):

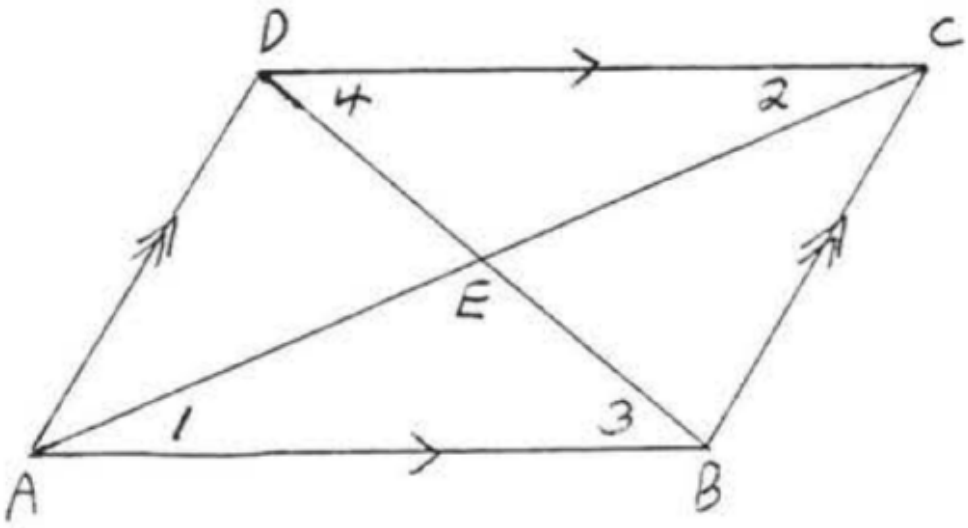


Figure 13.2.7. Parallelogram $ABCD$ with diagonals AC and BD.

We have $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$ (both pairs of angles are alternate interior angles of parallel lines AB and CD. Also $AB = CD$ from Theorem 13.2.1. Therefore $\triangle ABE \cong \triangle CDE$ by $ASA = ASA$. Since corresponding sides of congruent triangles are equal, $AE = CE$ and $DE = BE$. We have proven:

Theorem 13.2.3

The diagonals of a parallelogram bisect each other (cut each other in half).

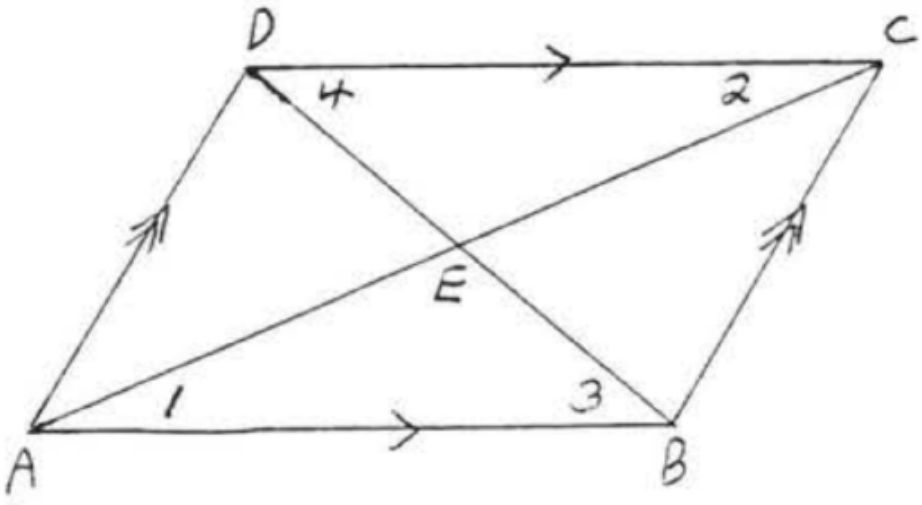
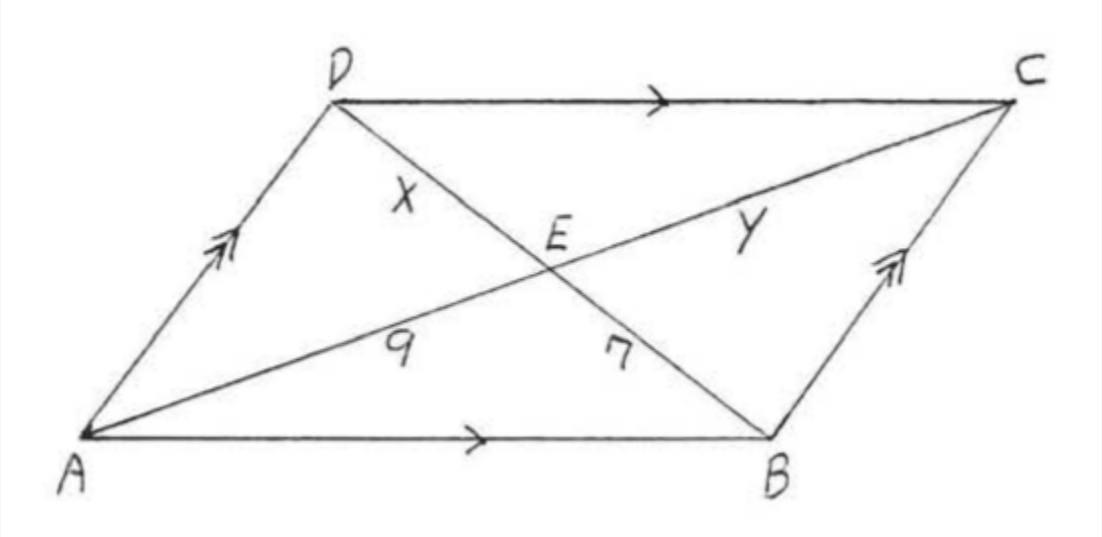


Figure 13.2.8. The diagonals of parallelogram $ABCD$ bisect each other.

Example 13.2.5

Find x, y, AC, and BD:



Solution

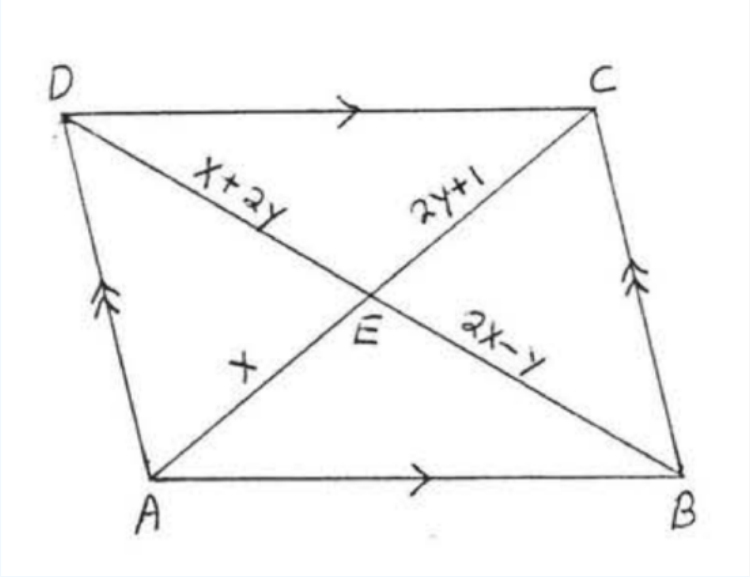
By Theorem 13.2.3 the diagonals bisect each other.

$$\begin{aligned} x &= 7 \\ y &= 9 \\ AC &= 9+9=18 \\ BD &= 7+7=14 \end{aligned} \tag{13.2.2}$$

Answer: $x=7, y=9, AC=18, BD=14$.

Example 13.2.6

Find x, y, AC, and BD:



Solution

By Theorem 13.2.3 the diagonals bisect each other.

$$\begin{aligned} AE &= CE \\ x &= 2y+1 \\ x-2y &= 1 \end{aligned} \qquad \begin{aligned} BE &= DE \\ 2x-y &= x+2y \\ 2x-y-x-2y &= 0 \\ x-3y &= 0 \end{aligned}$$

$$\begin{aligned} x - 2y &= 1 \longrightarrow & x - 2y &= 1 \\ x - 3y &= 0 \xrightarrow{-1} & -x + 3y &= 0 \\ & & y &= 1 \end{aligned}$$

$$x = 2y + 1 = 2(1) + 1 = 3$$

Check:

AE	=	CE
x		2y + 1
3		2(1) + 1
		3

AC = AE + CE = 3 + 3 = 6.

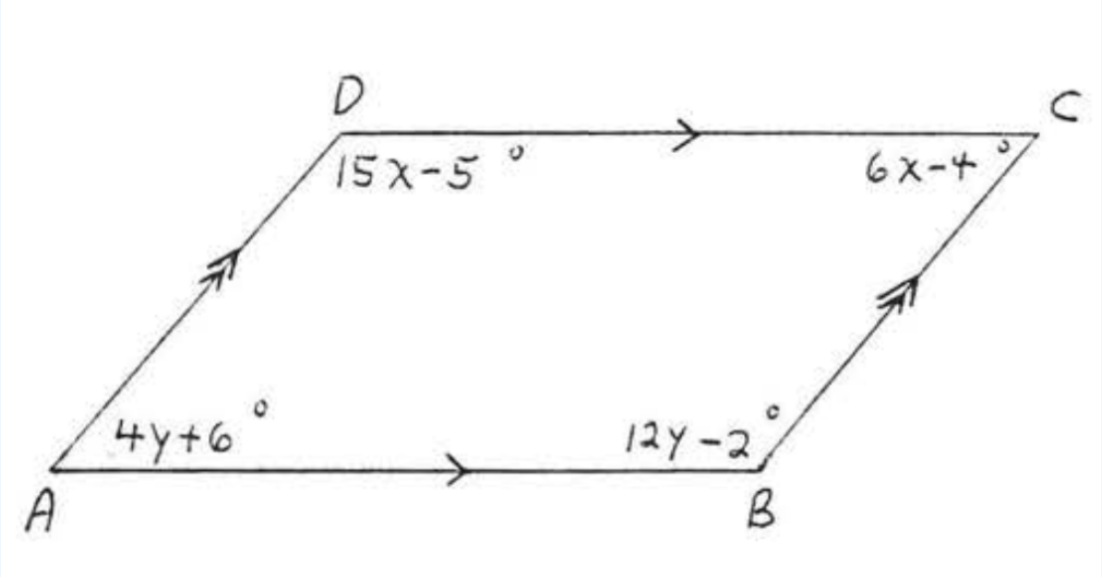
BE	=	DE
2x - y		x + 2y
2(3) - 1		3 + 2(1)
5		5

BD = BE + DE = 5 + 5 = 10.

Answer: $x = 3, y = 1, AC = 6, BD = 10$.

Example 13.2.7

Find $x, y, \angle A, \angle B, \angle C$, and $\angle D$:



Solution

By Theorem 13.2.2:

$$\begin{aligned} \angle A + \angle B &= 180^\circ \\ 4y + 6 + 12y - 2 &= 180 \\ 16y + 4 &= 180 \\ 16y &= 180 - 4 \\ 16y &= 176 \\ y &= 11 \end{aligned}$$

and

$$\begin{aligned} \angle C + \angle D &= 180^\circ \\ 6x - 4 + 15x - 5 &= 180 \\ 21x - 9 &= 180 \\ 21x &= 180 + 9 \\ 21x &= 189 \\ x &= 9 \end{aligned}$$

Check:

$$\angle A + \angle B = 180^\circ$$

$$4y + 6 + 12y - 2$$

$$4(11) + 6 + 12(11) - 2$$

$$50^\circ + 130^\circ$$

$$180^\circ$$

$$\angle C + \angle D = 180^\circ$$

$$6x - 4 + 15x - 5$$

$$6(9) - 4 + 15(9) - 5$$

$$50^\circ + 130^\circ$$

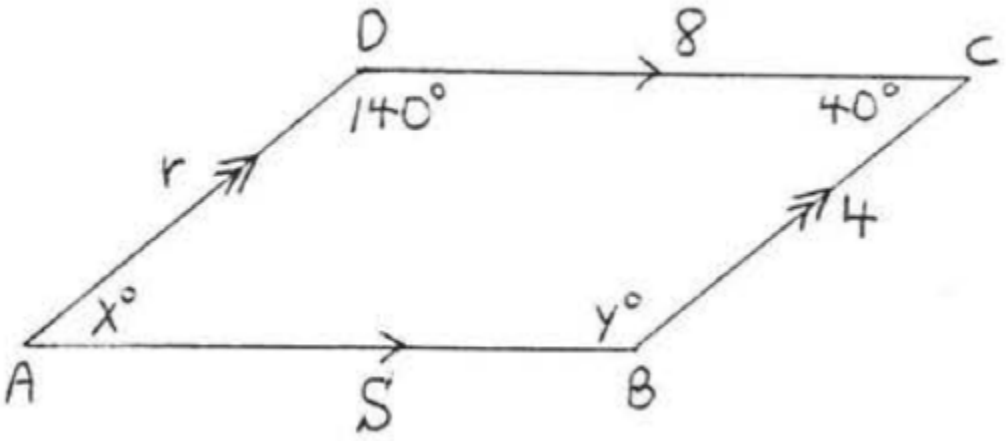
$$180^\circ$$

Answer: $x = 9, y = 11, \angle A = \angle C = 50^\circ, \angle B = \angle D = 130^\circ$.

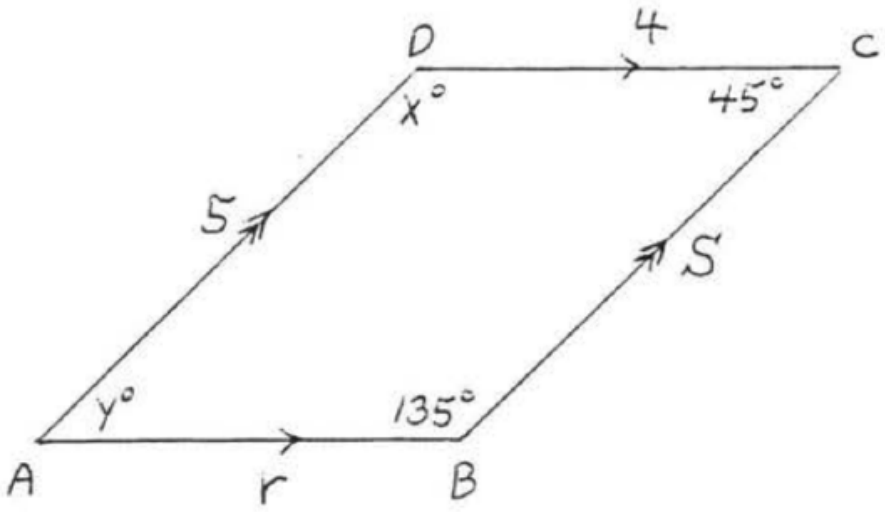
Problems

For each of the following state any theorem used in obtaining your answer(s):

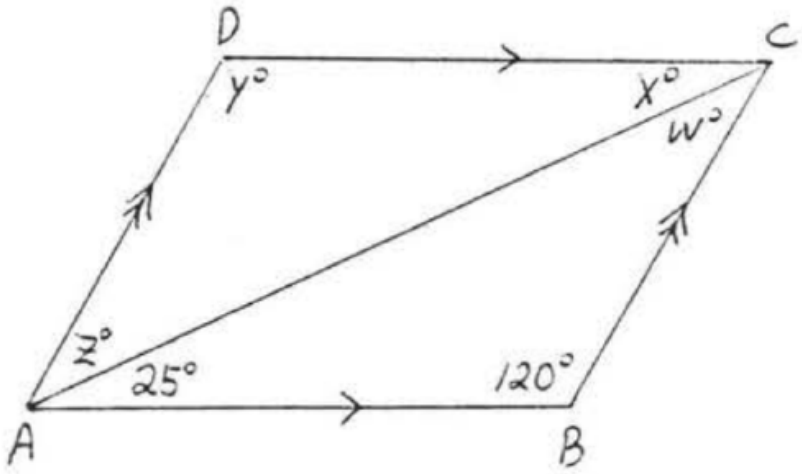
1. Find x, y, r, and s:



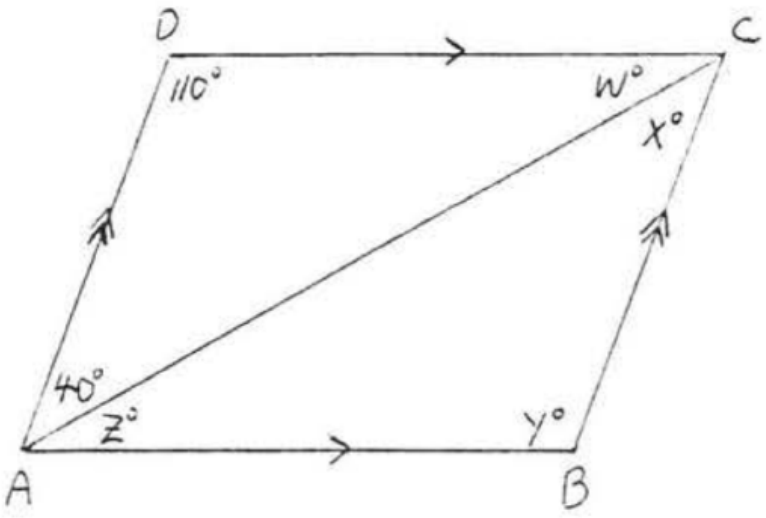
2. Find x, y, r, and s:



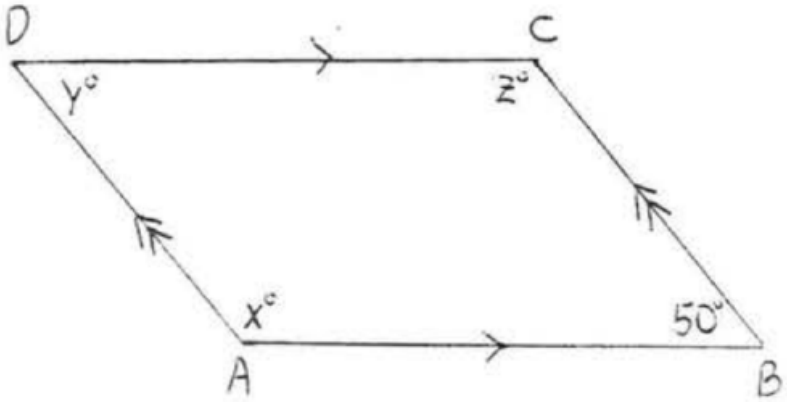
3. Find w, x, y, and z:



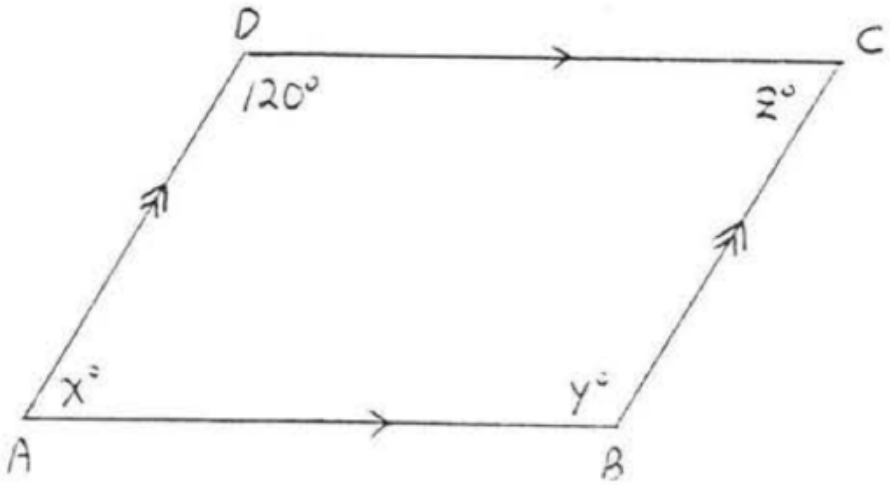
4. Find w, x, y, and z:



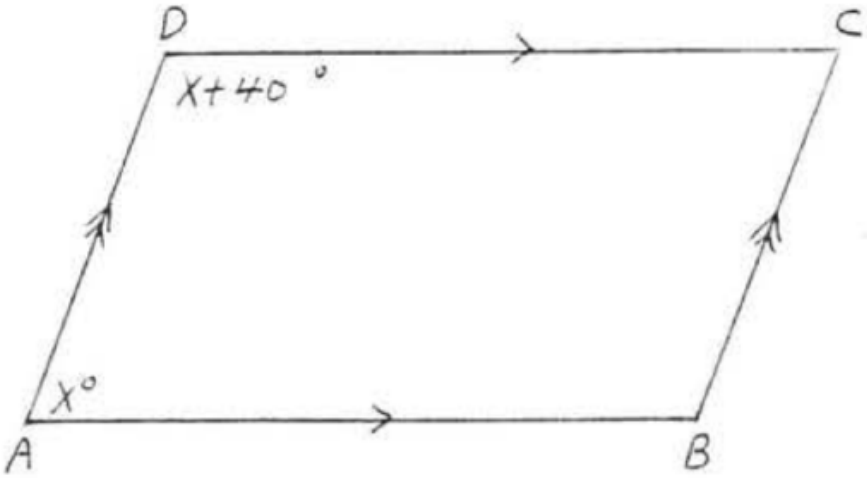
5. Find x, y, and z:



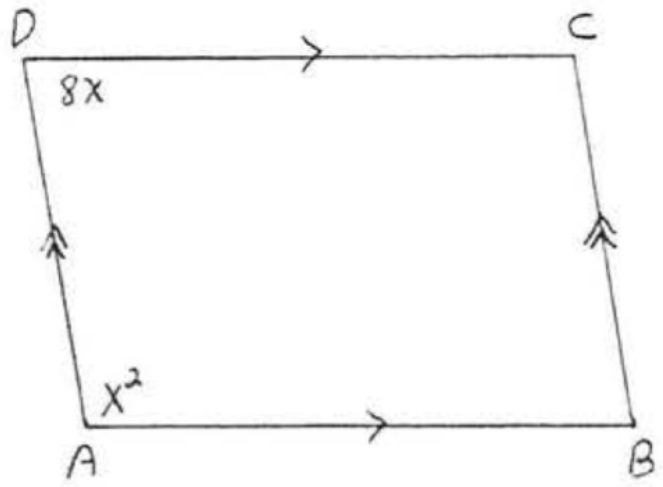
6. Find x, y, and z:



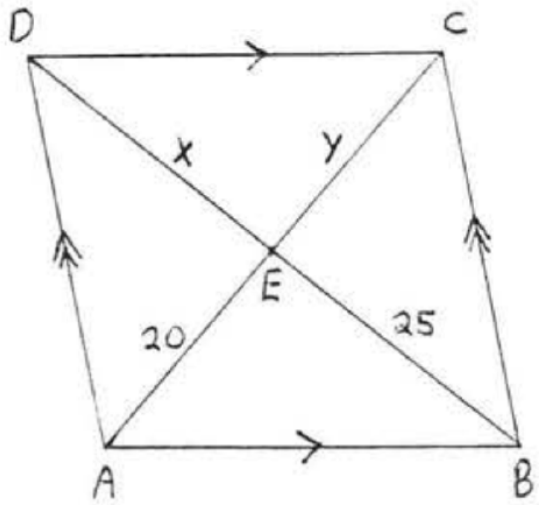
7. Find $x, \angle A, \angle B, \angle C$, and $\angle D$:



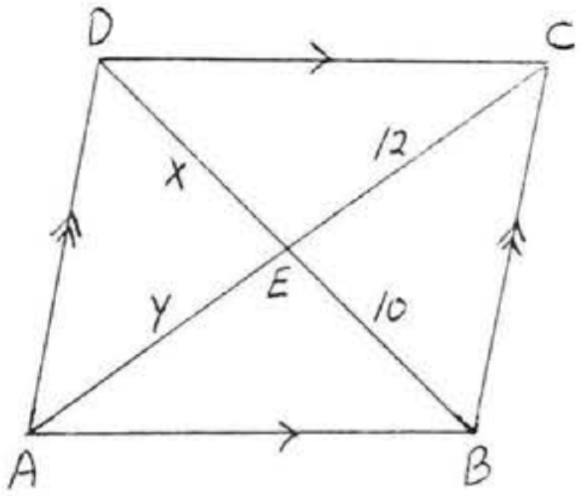
8. Find $x, \angle A, \angle B, \angle C$, and $\angle D$:



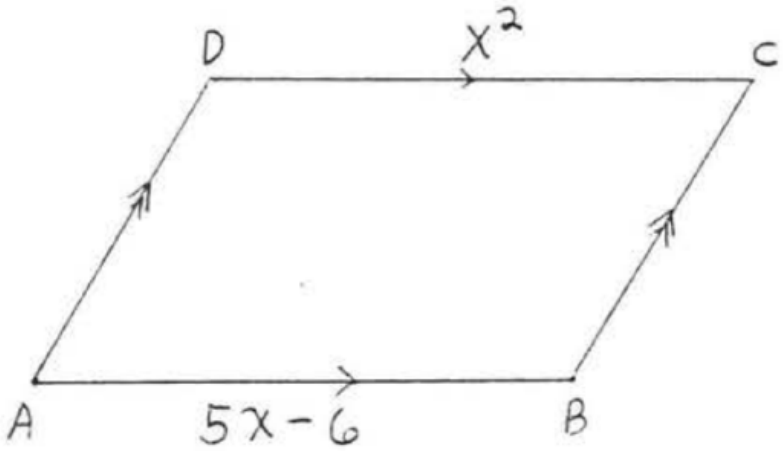
9. Find x, y, AC, and BD:



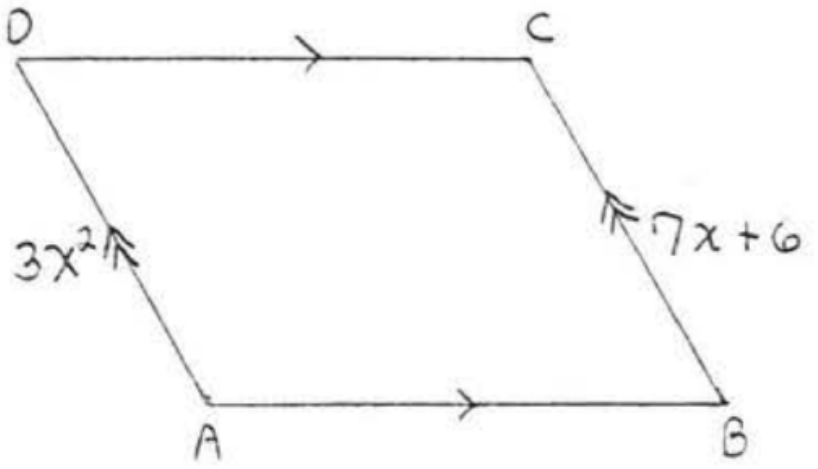
10. Find x, y, AC, and BD:



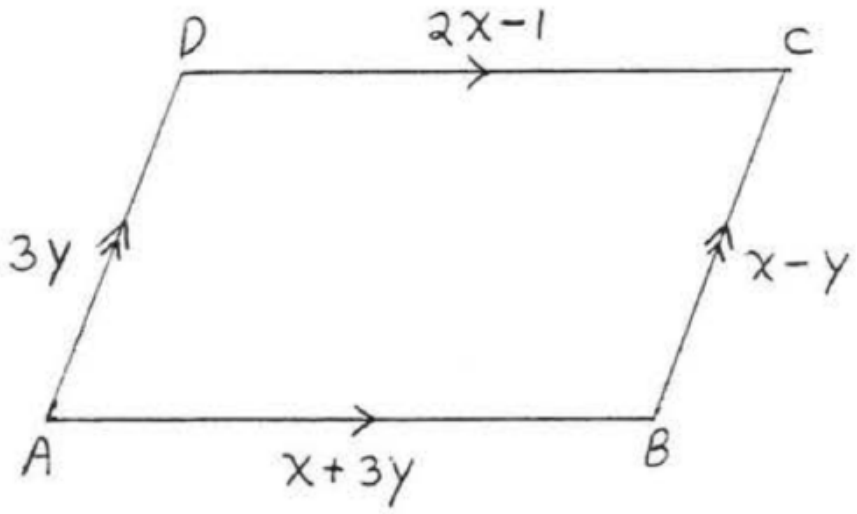
11. Find x, AB, and CD:



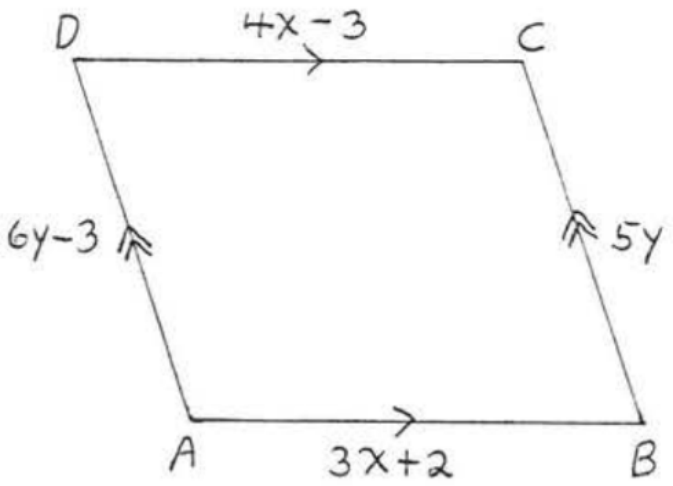
12. Find x, AD, and BC:



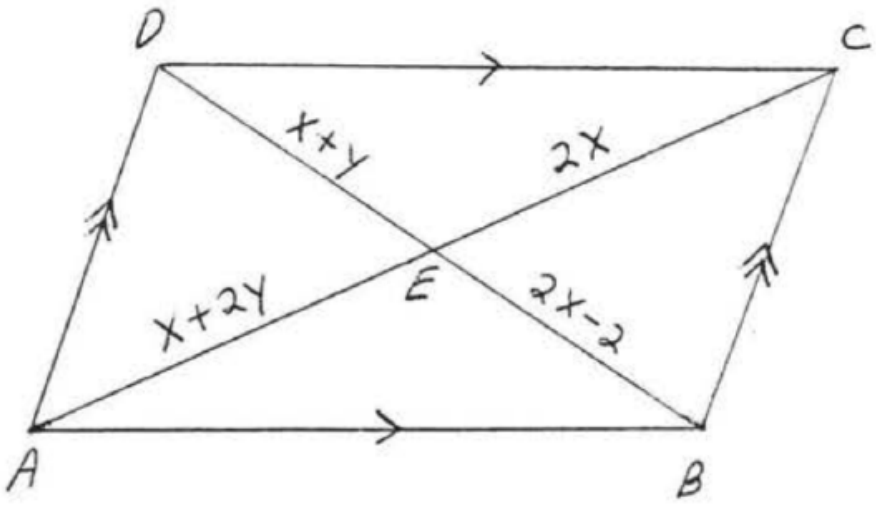
13. Find x, y, AB, BC, CD, and AD:



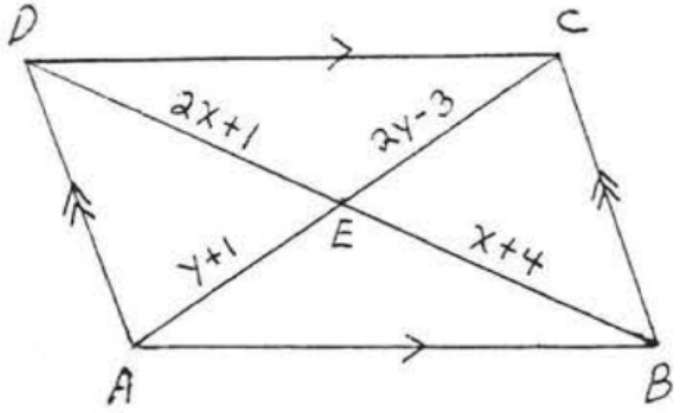
14. Find x, y, AB, BC, CD, and AD:



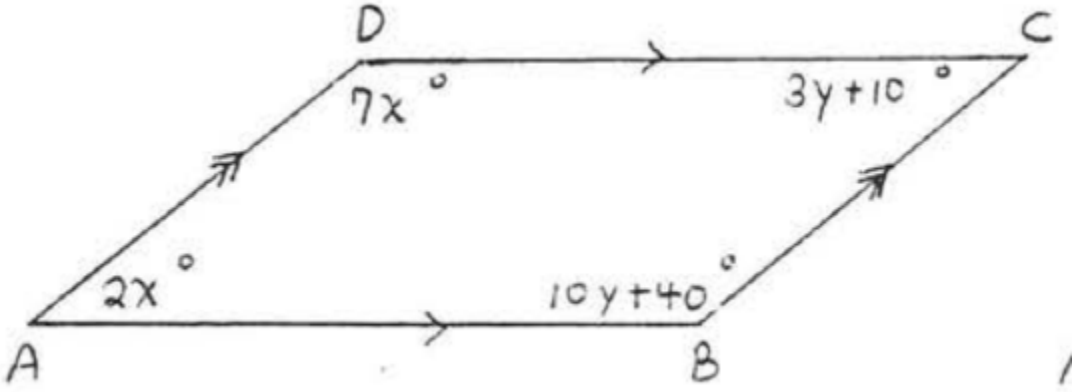
15. Find x, y, AC, and BD:



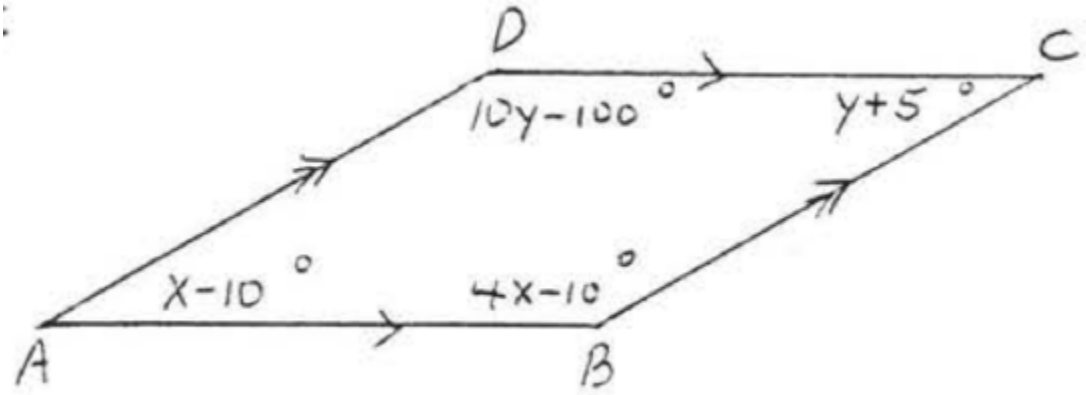
16. Find x, y, AC, and BD:



17. Find $x, y, \angle A, \angle B, \angle C$, and $\angle D$:



18. Find $x, y, \angle A, \angle B, \angle C$, and $\angle D$:



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13.3: Other Quadrilaterals

In this section we will consider other quadrilaterals with special properties: the rhombus, the rectangle, the square, and the trapezoid.

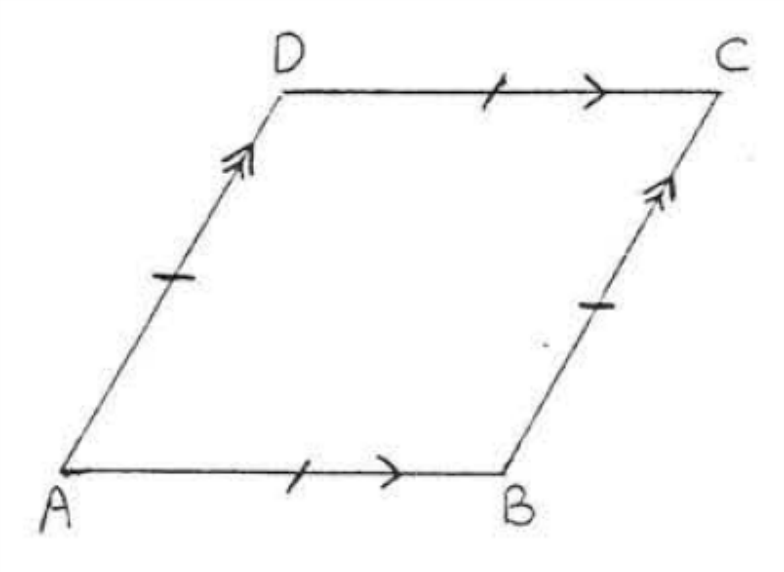


Figure 13.3.1: A rhombus.

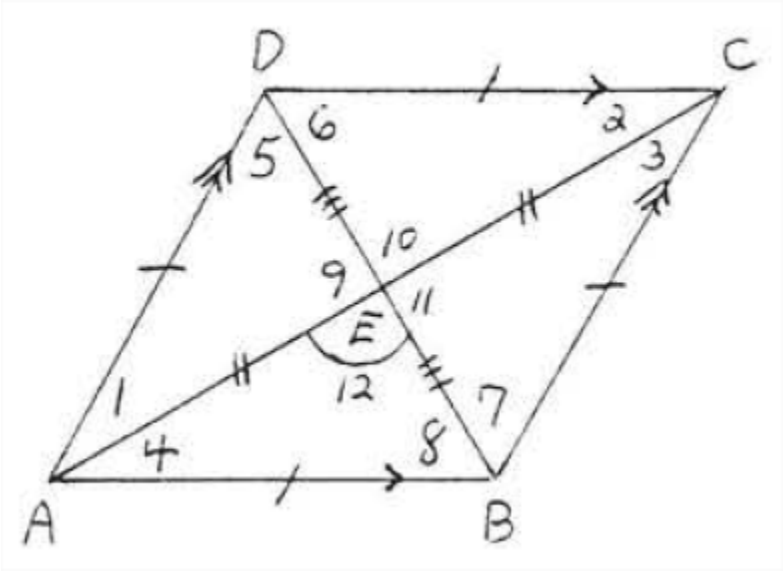


Figure 13.3.2: A rhombus with diagonals.

A **rhombus** is a parallelogram in which all sides are equal (Figure 13.3.1). It has all the properties of a parallelogram plus some additional ones as well. Let us draw the diagonals AC and BD (Figure 13.3.2). By Theorem 13.3.3 of section 3.1 the diagonals bisect each other. Hence

$$\triangle ADE \cong \triangle CDE \cong \triangle CBE \cong \triangle ABE$$

by $SSS = SSS$. The corresponding angles of the congruent triangles are equal:

$$\angle 1 = \angle 2 = \angle 3 = \angle 4,$$

$$\angle 5 = \angle 6 = \angle 7 = \angle 8$$

and

$$\angle 9 = \angle 10 = \angle 11 = \angle 12.$$

$\angle 9$ and $\angle 10$ are supplementary in addition to being equal, hence $\angle 9 = \angle 10 = \angle 11 = \angle 12 = 90^\circ$. We have proven the following theorem:

Theorem 13.3.1

The diagonals of a rhombus are perpendicular and bisect the angles. See Figure 13.3.3.

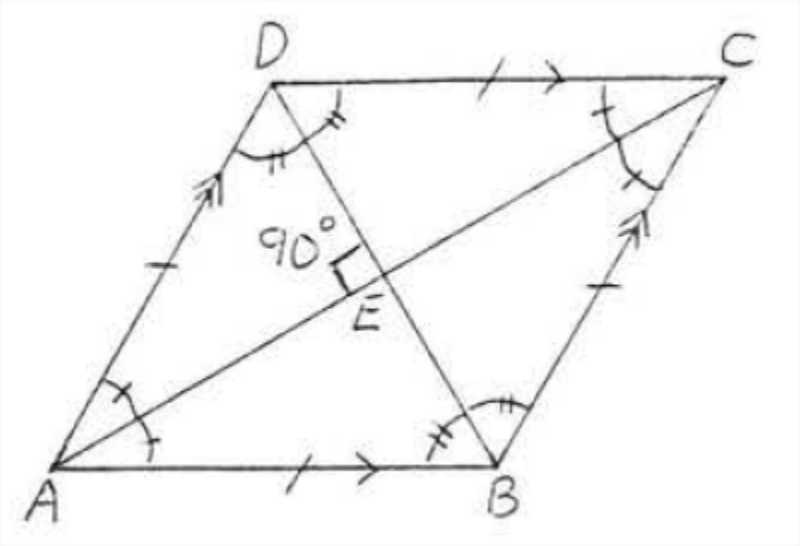
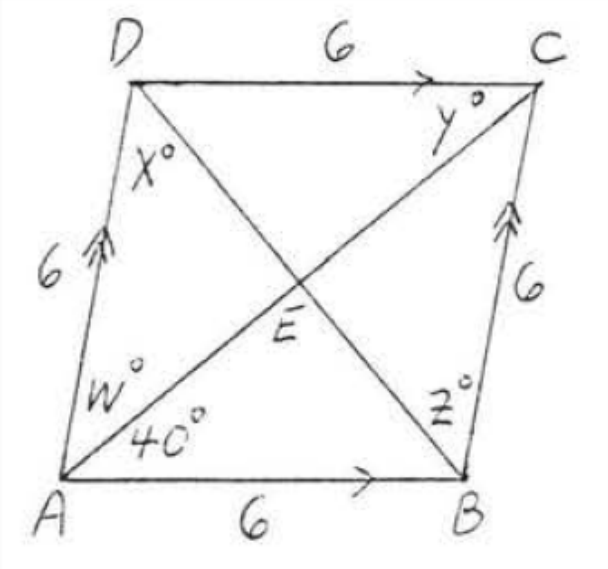


Figure 13.3.3: The diagonals of a rhombus are perpendicular and bisect the angles.

Example 13.3.1

Find w, x, y, and z:



Solution

$ABCD$ is a rhombus since it is a parallelogram all of whose sides equal 6. According to Theorem $13.3.1$, the diagonals are perpendicular and bisect the angles. Therefore $w^\circ = 40^\circ$ since AC bisects $\angle BAD$. $\angle AED = 90^\circ$ so $x^\circ = 180^\circ - (90^\circ + 40^\circ) = 180^\circ - 130^\circ = 50^\circ$ (the sum of the angles of $\angle AED$ is 180°). Finally $y^\circ = w^\circ = 40^\circ$ (compare with Figure $13.3.3$) and $z^\circ = x^\circ = 50^\circ$.

Answer

$w = 40, x = 50, y = 40, z = 50$.

Figure $13.3.4$ shows rhombus $ABCD$ of Example $13.3.1$ with all its angles identified.

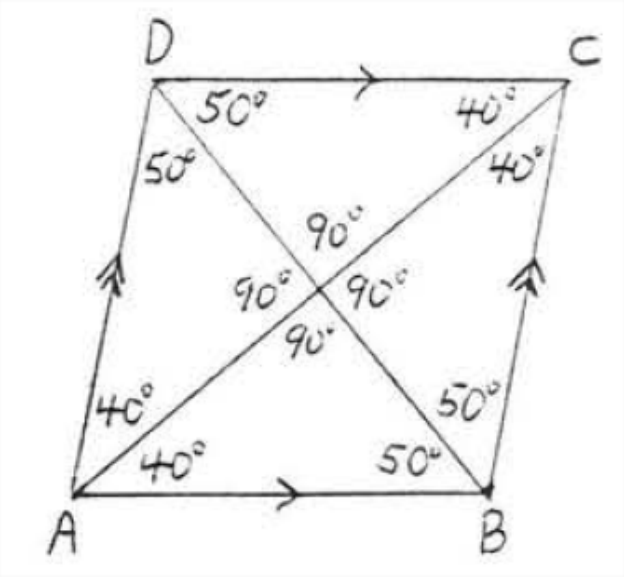


Figure $13.3.4$: The rhombus of Example $13.3.1$ with all angles identified.

A **rectangle** is a parallelogram in which all the angles are right angles (Figure $13.3.5$). It has all the properties of a parallelogram plus some additional ones as well. It is not actually necessary to be told that all the angles are right angles:

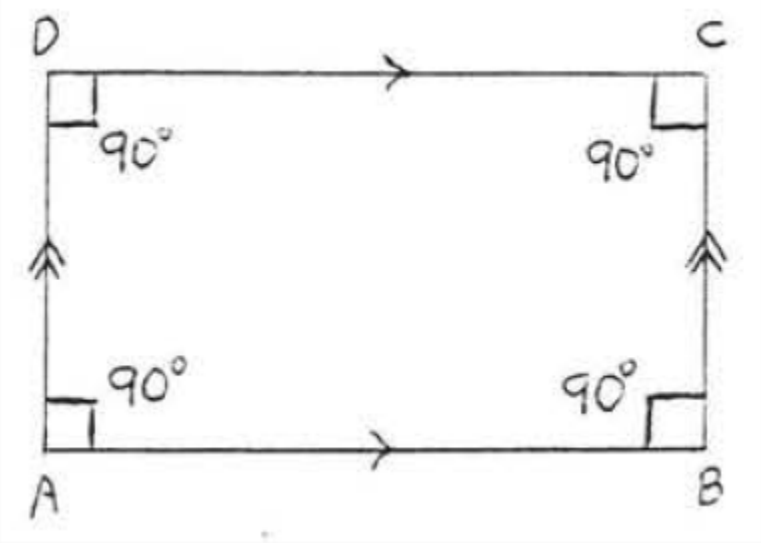


Figure $13.3.5$: A rectangle.

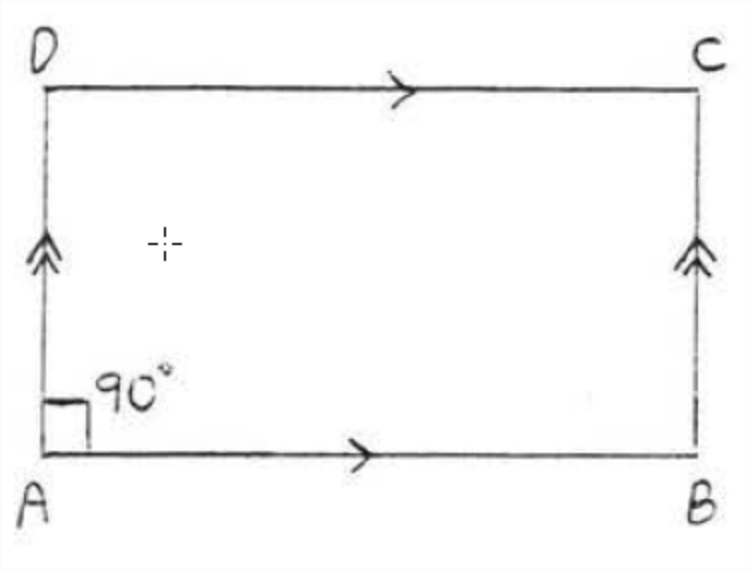


Figure 13.3.6: A parallelogram with just one right angle must also be a rectangle.

Theorem 13.3.2

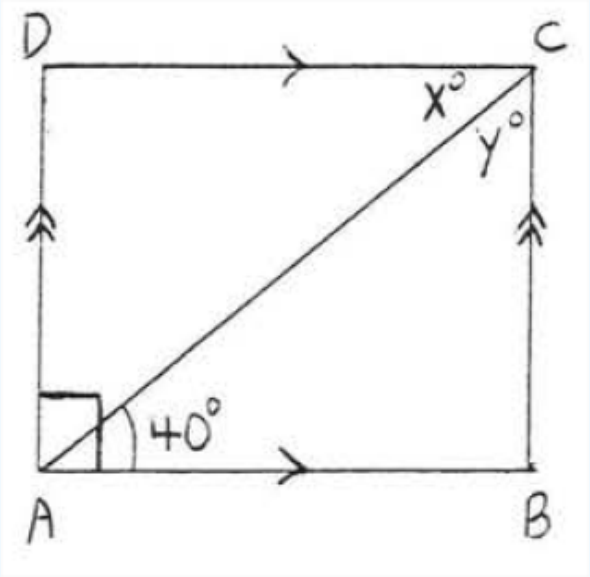
A parallelogram with just one right angle must be a rectangle.

In Figure 13.3.6 if $\angle A$ is a right angle then all the other angles must be right angles too.

Proof available at the end of this section.

Example 13.3.2

Find x and y:



Solution

Theorem 13.3.2, $ABCD$ is a rectangle. $x^\circ = 40^\circ$ because alternate interior angles of parallel lines AB and CD must be equal. Since the figure is a rectangle $\angle BCD = 90^\circ$ and $y^\circ = 90^\circ - x^\circ = 90^\circ - 40^\circ = 50^\circ$.

Answer: $x = 40, y = 50$

Let us draw the diagonals of rectangle $ABCD$ (Figure 13.3.7).

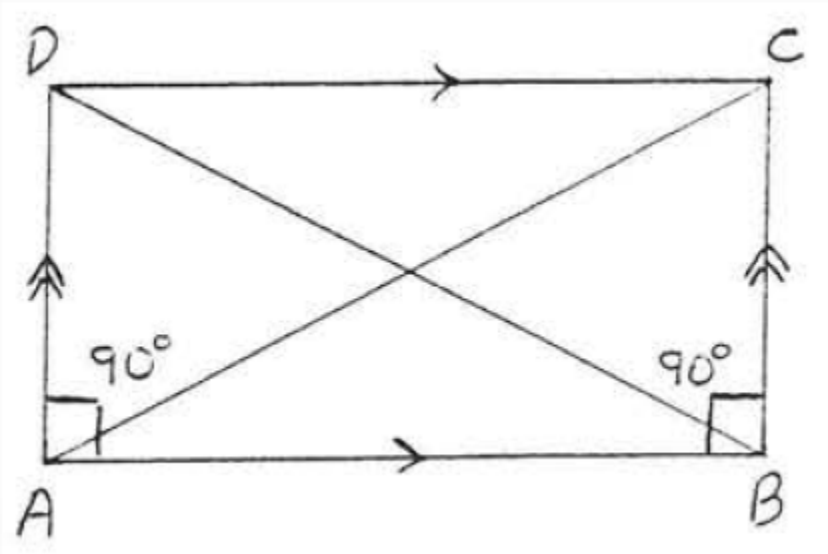


Figure 13.3.7: Rectangle wIth diagonals drawn.

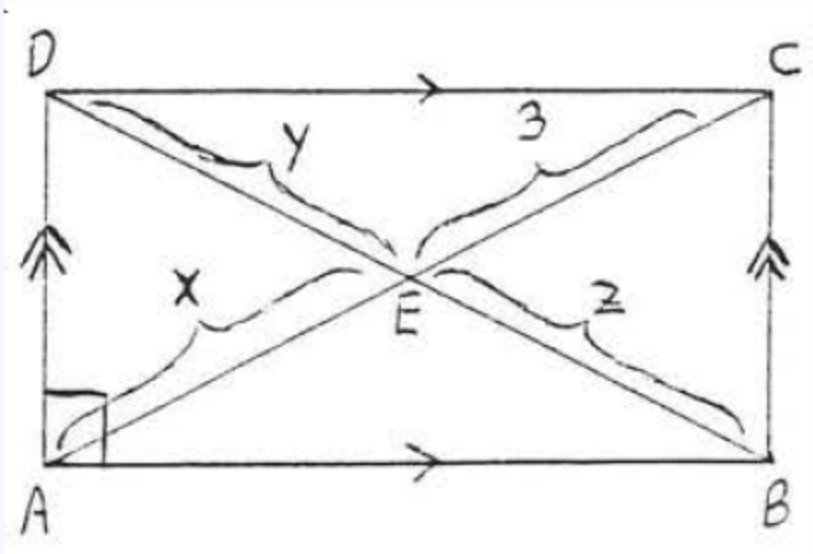
We will show $\triangle ABC \cong \triangle BAD$. $AB = BA$ because of identity. $\angle A = \angle B = 90^\circ$. $BC = AD$ because the opposite sides of a parallelogram are equal. Then $\triangle ABC \cong \triangle BAD$ by $SAS = SAS$. Therefore diagonal AC = diagonal BD because they are corresponding sides of congruent triangles. We have proven:

Theorem 13.3.3

The diagonals of a rectangle are equal. In Figure 13.3.7, $AC = BD$.

Example 13.3.3

Find w, x, y, z, AC and BD:



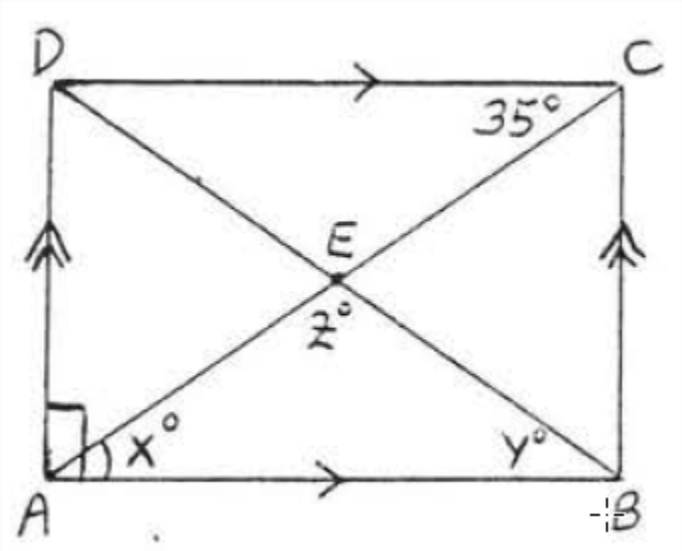
Solution

$x = 3$ because the diagonals of a parallelogram bisect each other. So $AC = 3 + 3 = 6$. $BD = AC = 6$ since the diagonals of a rectangle are equal (Theorem 13.3.3). Therefore $y = z = 3$ since diagonal BD is bisected by diagonal AC.

Answer: $x = y = z = 3$ and $AC = BD = 6$.

Example 13.3.4

Find x, y, and z:



Solution

$x^\circ = 35^\circ$, because alternate interior angles of parallel lines are equal. $y^\circ = x^\circ = 35^\circ$ because they are base angles of isosceles triangle ABE so (AE= BE\) because the diagonals of a rectangle are equal and bisect each other). $z^\circ = 180^\circ - (x^\circ + y^\circ) = 180^\circ - (35^\circ + 35^\circ) = 180^\circ - 70^\circ = 110^\circ$. Figure 13.3.8 shows rectangle $ABCD$ with all the angles identified.

Answer: $x = y = z = 3, AC = BD = 6$.

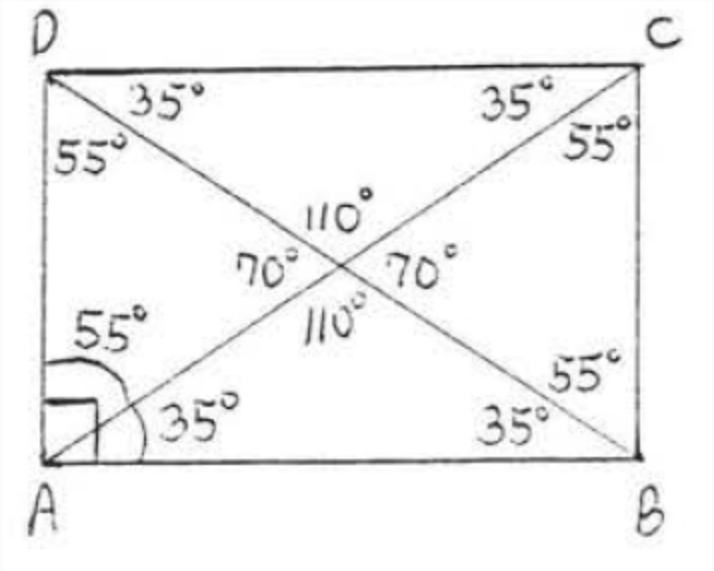


Figure 13.3.8: The rectangle of Example 13.3.4 with all the angles identified.

The Square

A **square** is a rectangle with all its sides equal. It is therefore also a rhombus. So it has all the properties of the rectangle and all the properties of the rhombus.

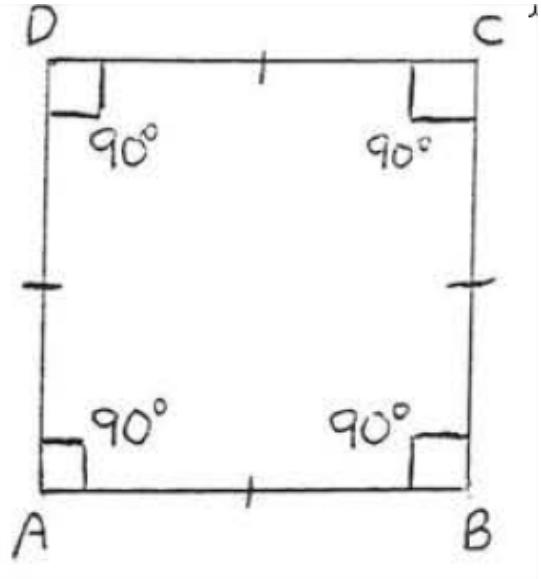


Figure 13.3.9: A square.

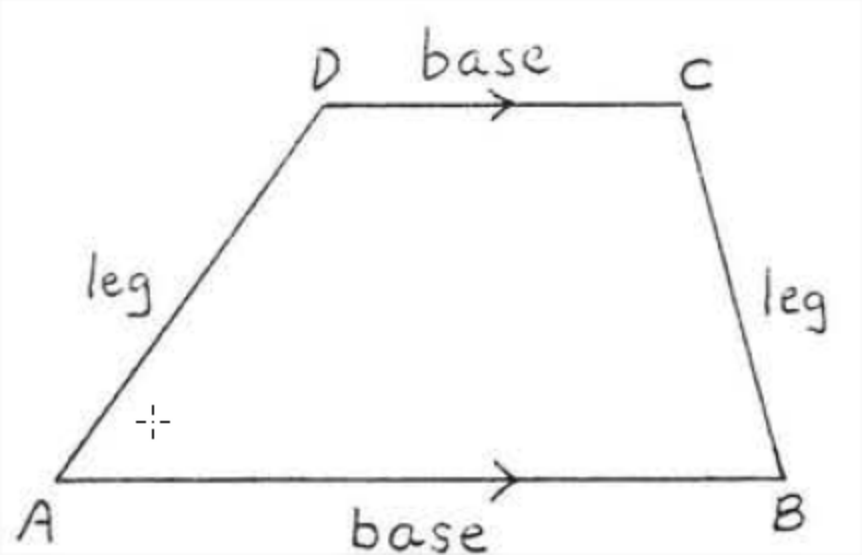


Figure 13.3.10: A trapezoid.

A **trapezoid** is a quadrilateral with two and only two sides parallel. The parallel sides are called **bases** and the other two sides are called **legs**. In Figure 13.3.8 AB and CD are the bases and AD and BC are the legs. $\angle A$ and $\angle B$ are a pair of **base angles**. $\angle C$ and $\angle D$ are another pair of base angles.

An **isosceles trapezoid** is a trapezoid in which the legs are equal. In Figure 13.3.8, $ABCD$ is an isosceles trapezoid with $AD = BC$. An isosceles trapezoid has the following property:

Theorem 13.3.4

The base angles of an isosceles trapezoid are equal. In Figure 13.3.11, $\angle A = \angle B$ and $\angle C = \angle D$

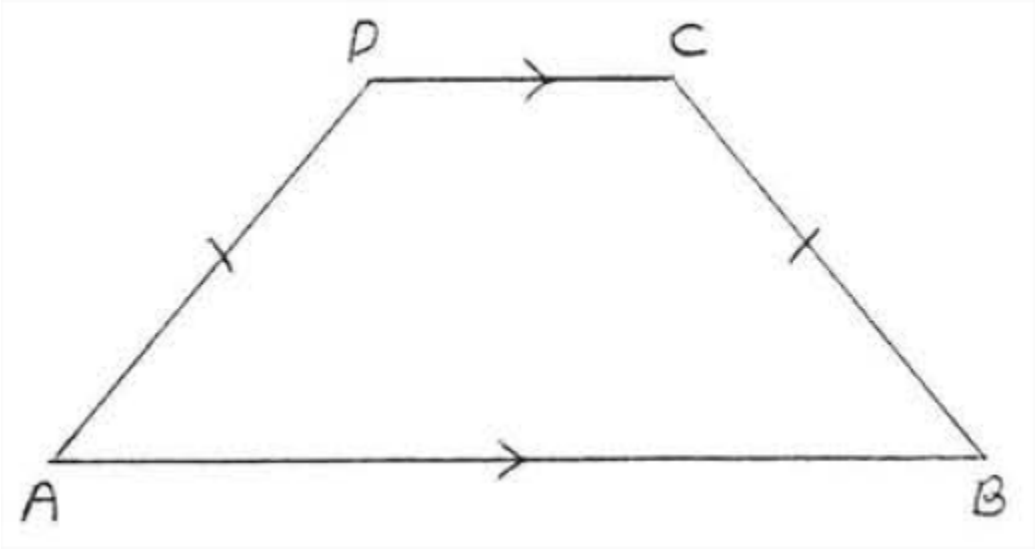
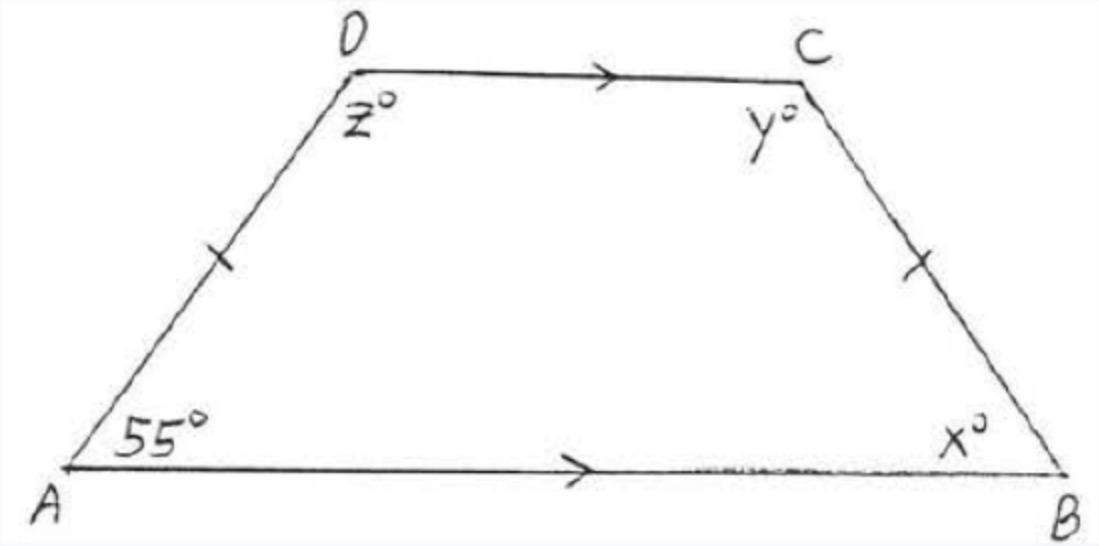


Figure 13.3.11: An isosceles trapezoid

Example 13.3.5

Find x, y, and z :



Solution

$x^\circ = 55^\circ$ because $\angle A$ and $\angle B$, the base angles of isosceles trapezoid $ABCD$, are equal. Now the interior angles of parallel lines on the same side of the transversal are supplementary (Theorem 3 section 1.4). Therefore $y^\circ = 180^\circ - x^\circ = 180^\circ - 55^\circ = 125^\circ$ and $z^\circ = 180^\circ - 55^\circ = 125^\circ$.

Answer: $x = 22$, $y = z = 125$.

Proof of Theorem $13.3.4$: Draw DE parallel to CB as in Figure $13.3.12$. $\angle 1 = \angle B$ because corresponding angles of parallel lines are equal, $DE = BC$ because they are the opposite sides of parallelogram $BCDE$. Therefore $AD = DE$. So $\triangle ADE$ is isosceles and its base angles, $\angle A$ and $\angle 1$, are equal. We have proven $A = \angle 1 = \angle B$. To prove $\angle C = \angle D$, observe that they are both supplements of $\angle A = \angle B$ (Theorem $13.3.3$, section 1.4).

The isosceles trapezoid has one additional property:

Theorem $13.3.5$

The diagonals of an isosecles trapezoid are equal.

In Figure $13.3.13$, $AC = BD$

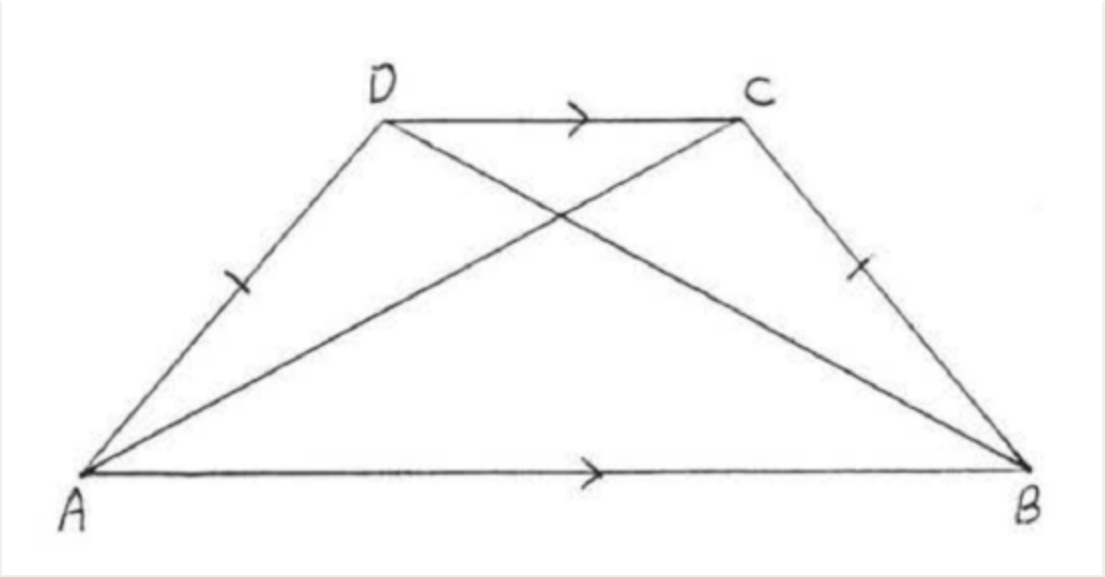
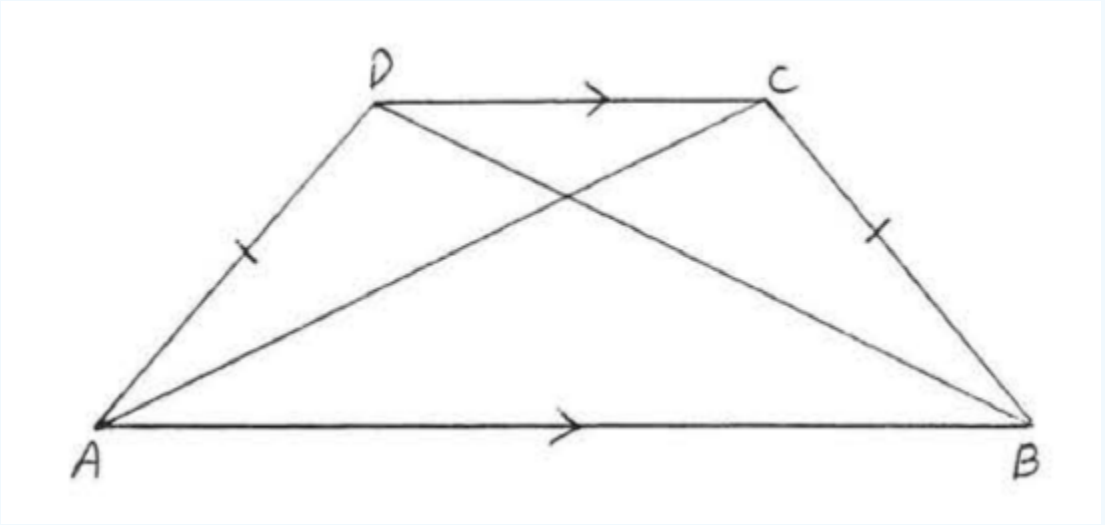


Figure 13.3.13. The diagonals AC and BD are equal.

Proof available at the end of this section.

Example 13.3.6

Find x if $AC = \dfrac{2}{x}$ and $BD = 3 - x$:



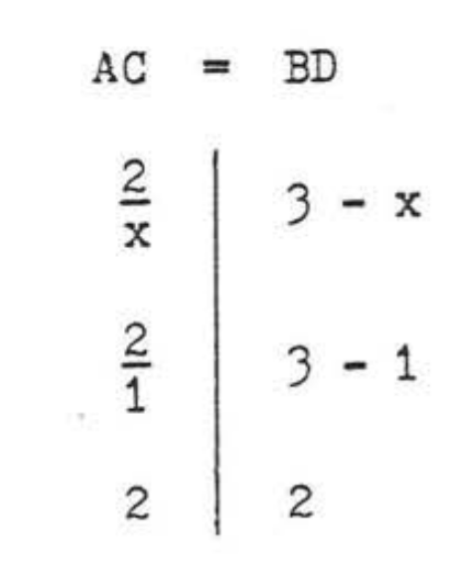
Solution

By Theorem 13.3.5,

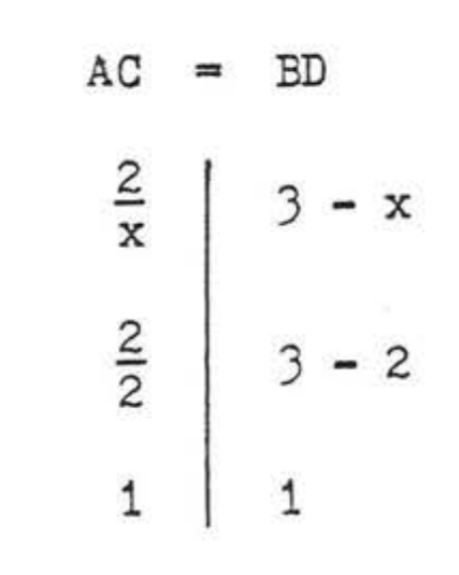
$$\begin{aligned} AC &= BD \\ \frac{2}{x} &= 3-x \\ (x)\frac{2}{x} &= (3-x)(x) \\ 2 &= 3x-x^2 \\ x^2-3x+2 &= 0 \\ (x-1)(x-2) &= 0 \end{aligned}$$

$$\begin{aligned} x-1 &= 0 \\ x &= 1 \end{aligned} \qquad \begin{aligned} x-2 &= 0 \\ x &= 2 \end{aligned}$$

Check, $x = 1$:

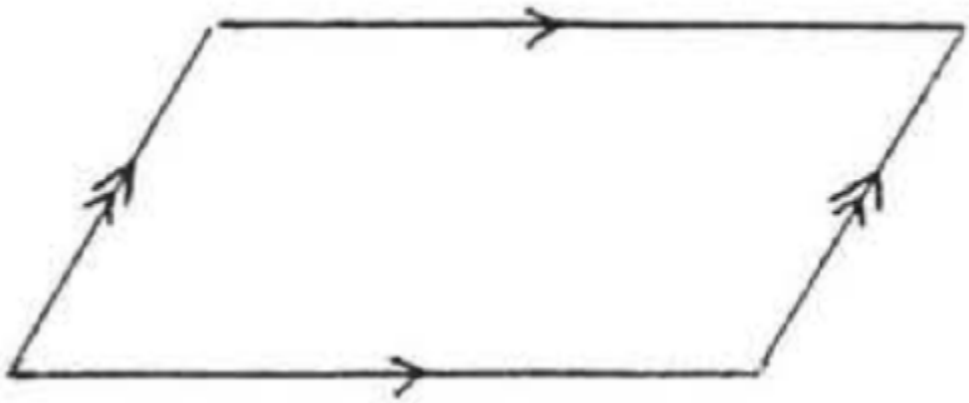


Check, $x = 2$:



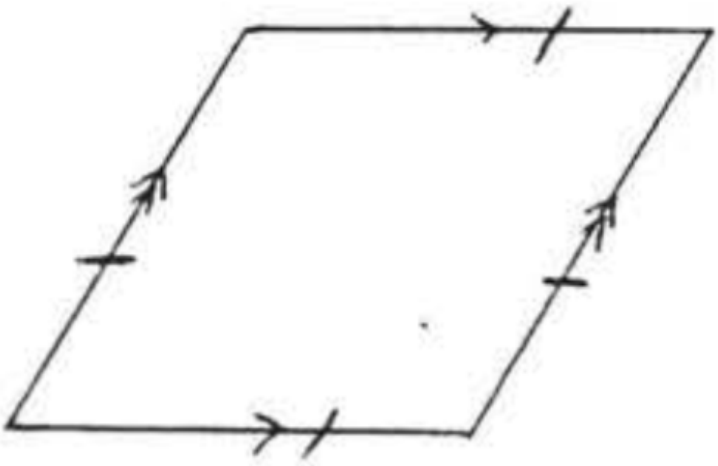
Answer: $x = 1$ or $x = 2$.

SUMMARY



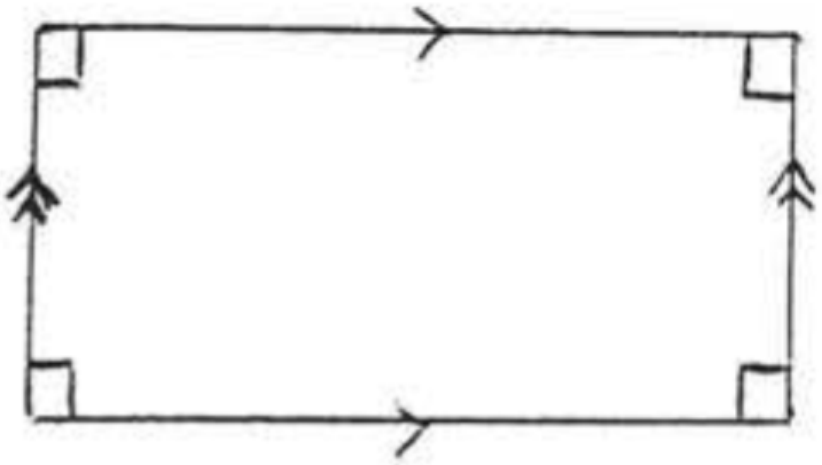
THE PARALLELOGRAM

A quadrilateral in which the opposite sides are parallel.



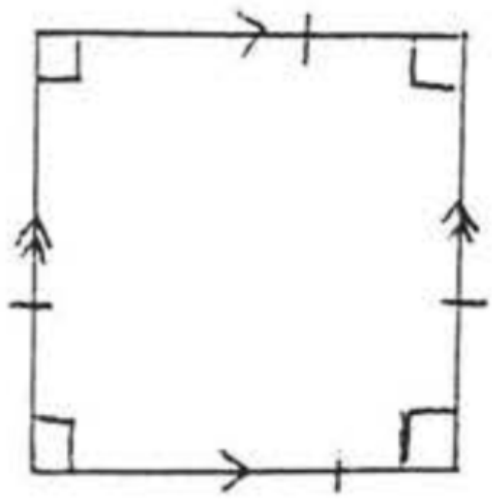
THE RHOMBUS

A parallelogram in which all of the sides are equal.



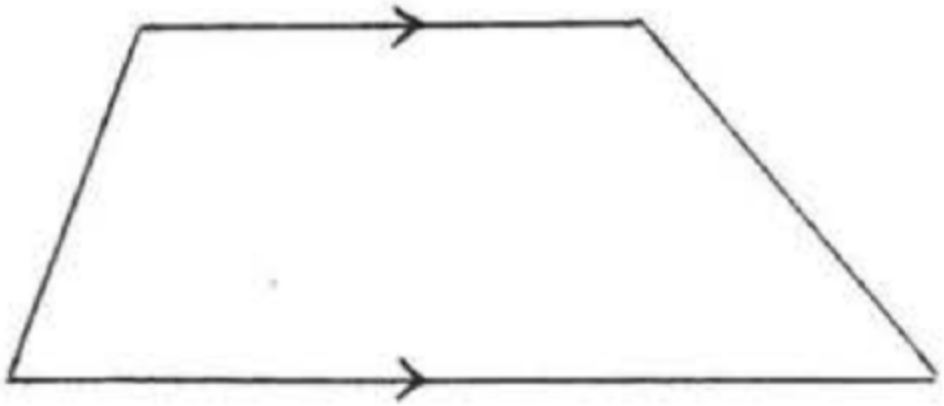
THE RECTANGLE

A parallelogram in which all of the angles are equal to 90°.



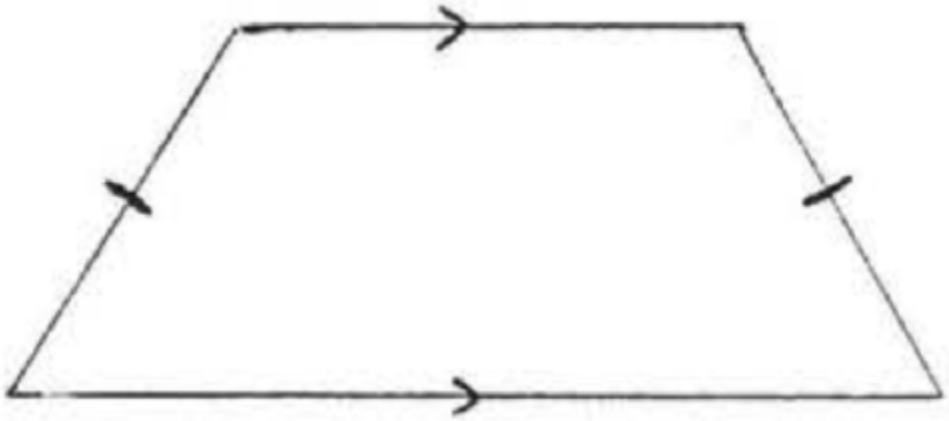
THE SQUARE

A parallelogram which is both a rhombus and a rectangle.



THE TRAPEZOID

A quadrilateral with just one pair of parallel sides.



THE ISOSCELES TRAPEZOID

A trapezoid in which the non-parallel sides are equal.

PROPERTIES OF QUADRILATERALS

	Opposite sides are parallel	Opposite sides are equal	Opposite angles	Diagonals bisect each other	Diagonals are equal	Diagonals are perpendicular	Diagonals bisect the angles	All sides are equal	All angles are equal
Parallelogram	YES	YES	YES	YES	-	-	-	-	-
Rhombus	YES	YES	YES	YES	-	YES	YES	YES	-
Rectangle	YES	YES	YES	YES	YES	-	-	-	YES

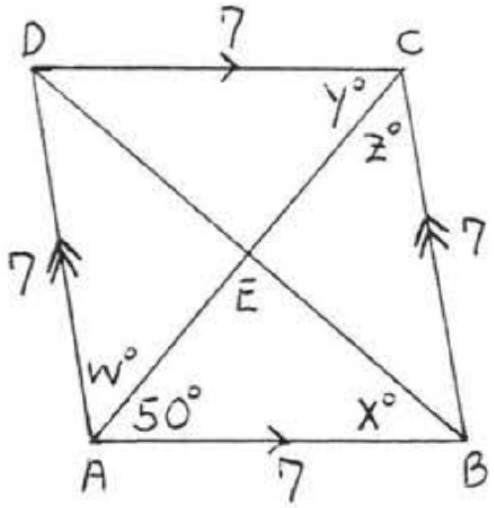
	Opposite sides are parallel	Opposite sides are equal	Opposite angles	Diagonals bisect each other	Diagonals are equal	Diagonals are perpendicular	Diagonals bisect the angles	All sides are equal	All angles are equal
Trapezoid	*	-	-	-	-	-	-	-	-
Isosceles Trapezoid	*	*	-	-	YES	-	-	-	-

*One pair only.

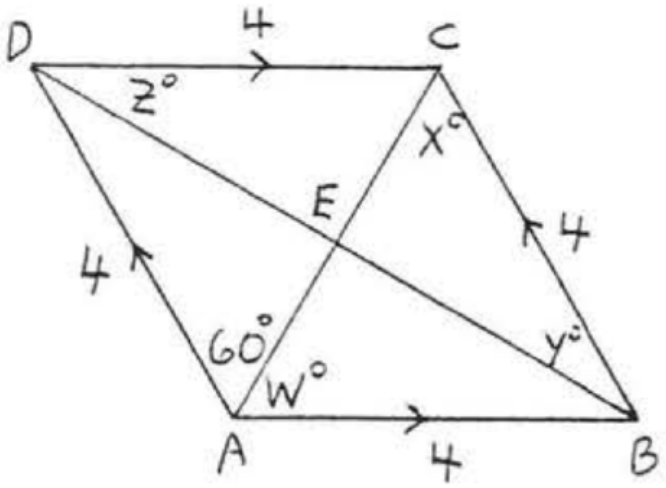
Problems

For each of the following state any theorems used in obtaining your answer.

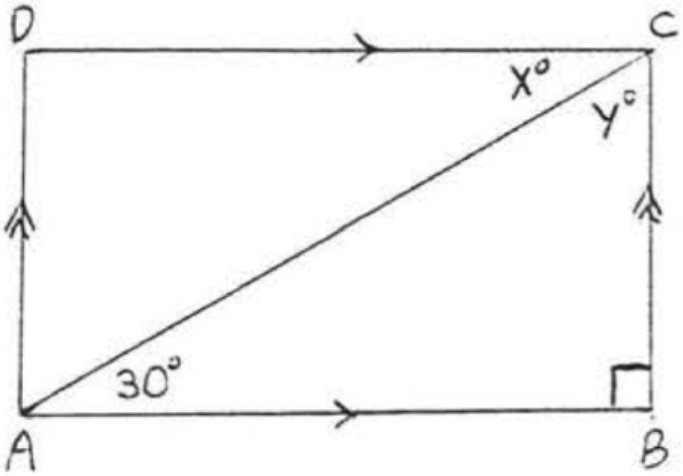
1. Find w, x, y, and z:



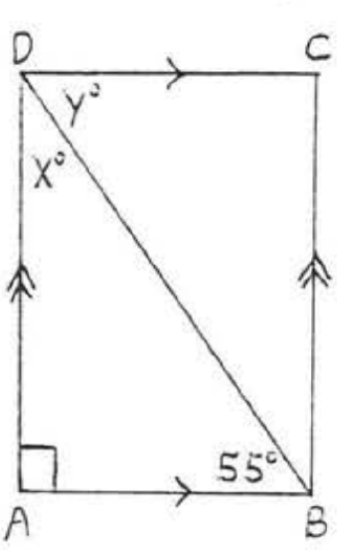
2. Find w, x, y, and z:



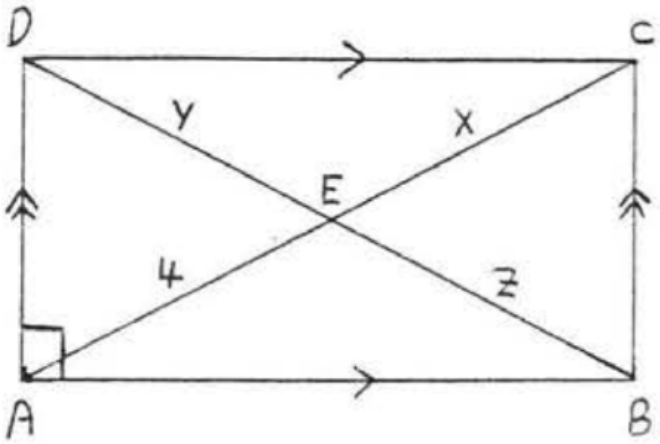
3. Find x and y:



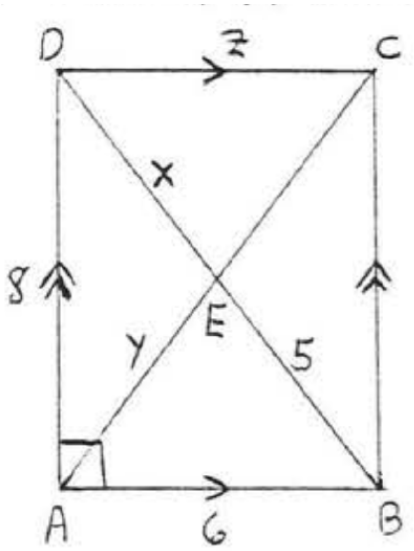
4. Find x and y:



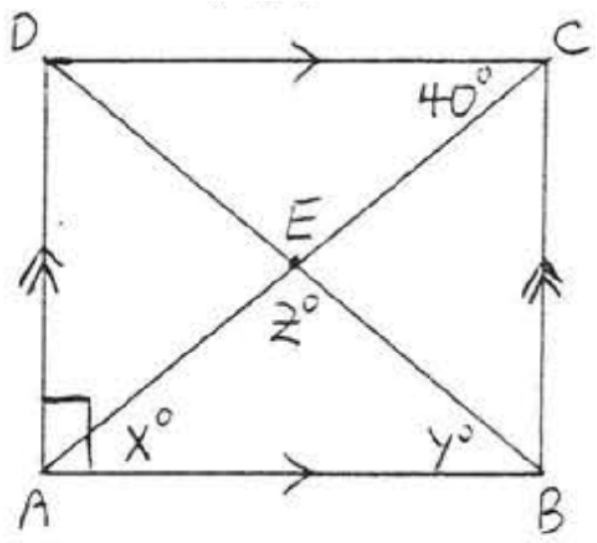
5. Find x, y, z, AC and BD:



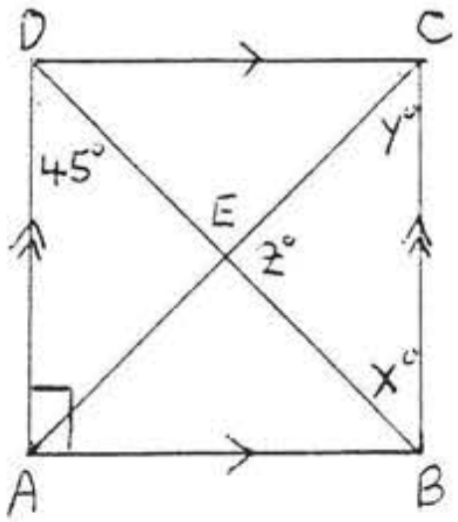
6. Find x, y, and z:



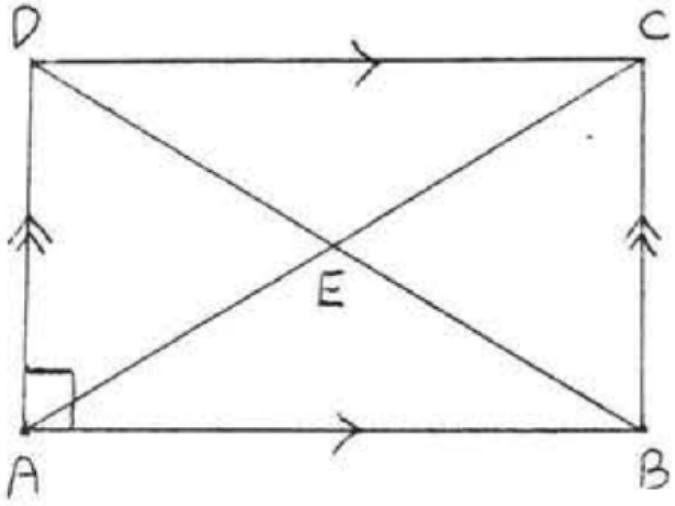
7. Find x, y, and z:



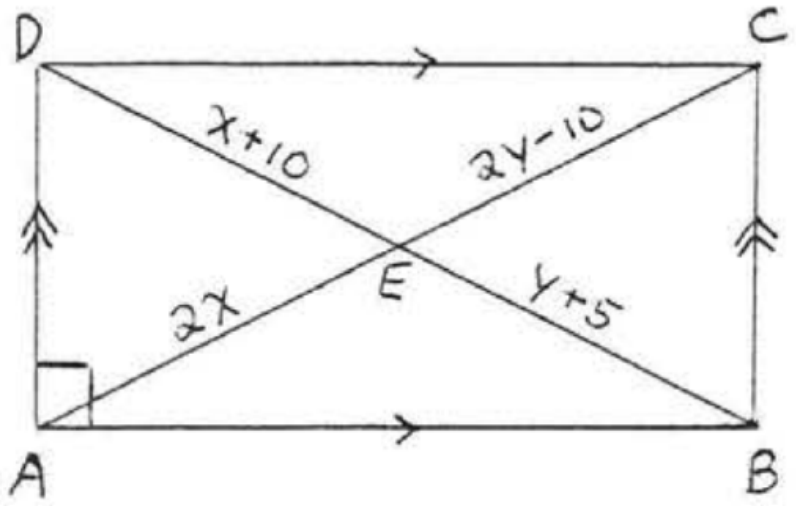
8. Find x, y, and z:



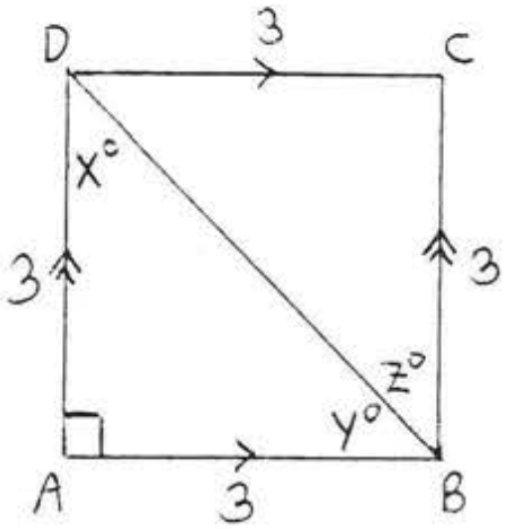
9. Find x if $AC = \dfrac{3}{x}$ and $BD = 4x - 1$:



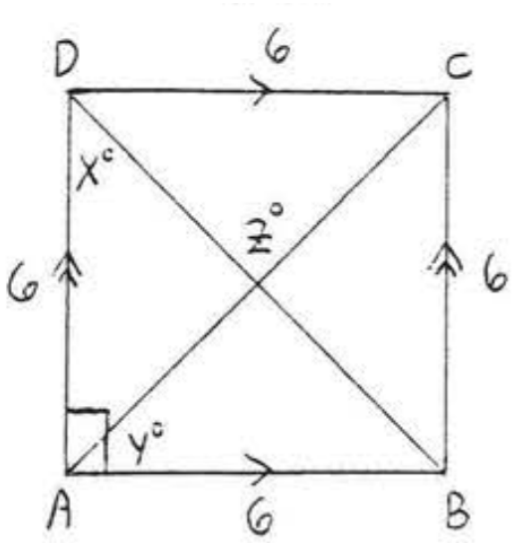
10. Find x and y:



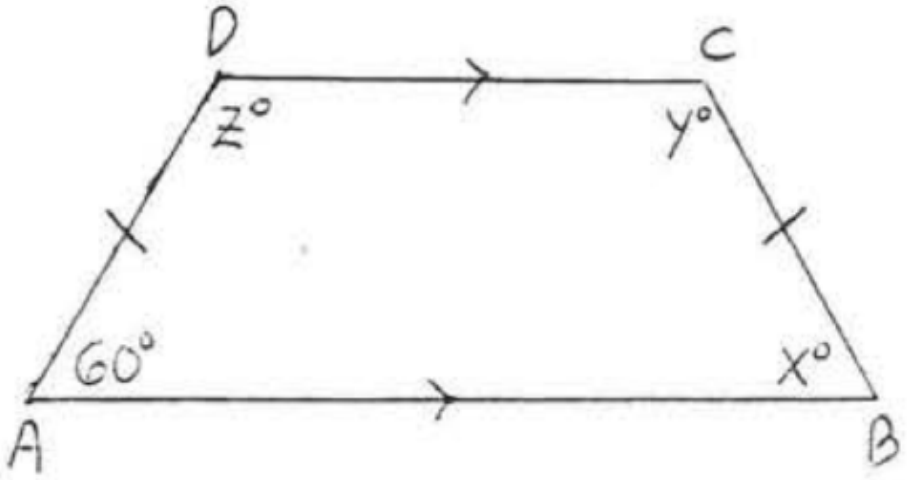
11. Find x, y, and z:



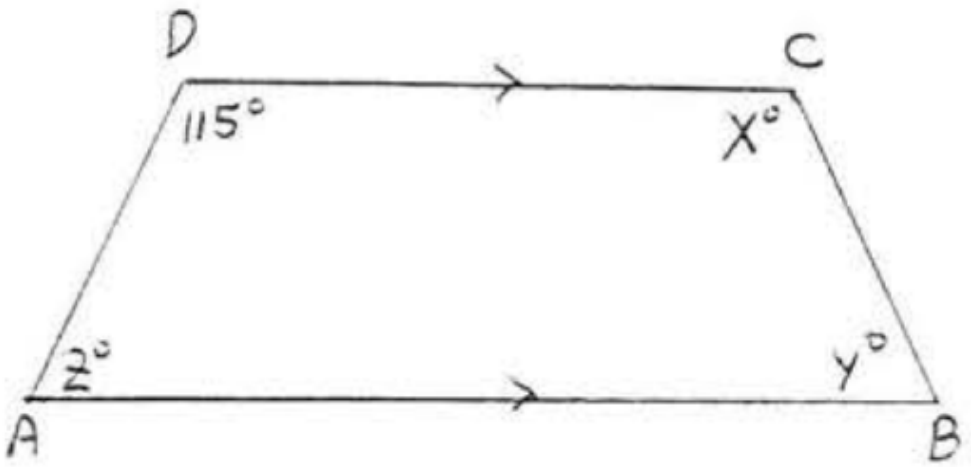
12. Find x, y, and z:



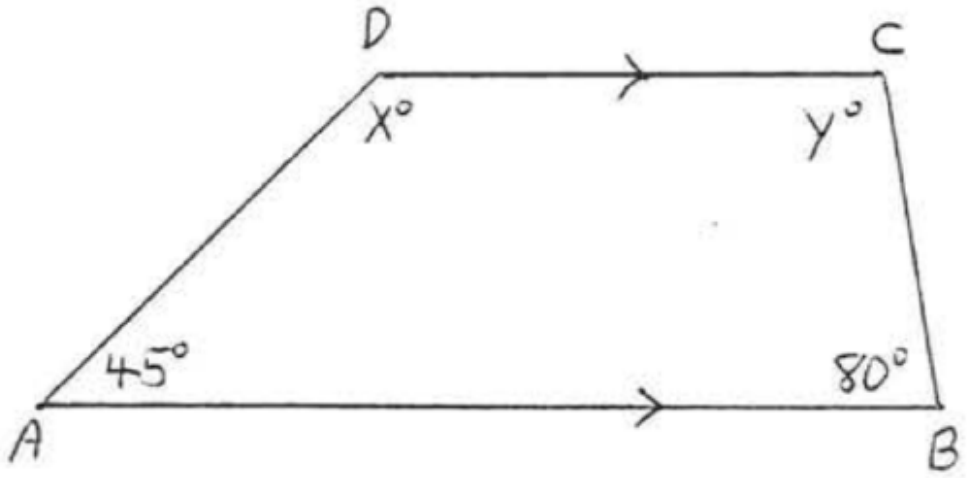
13. Find x, y, and z:



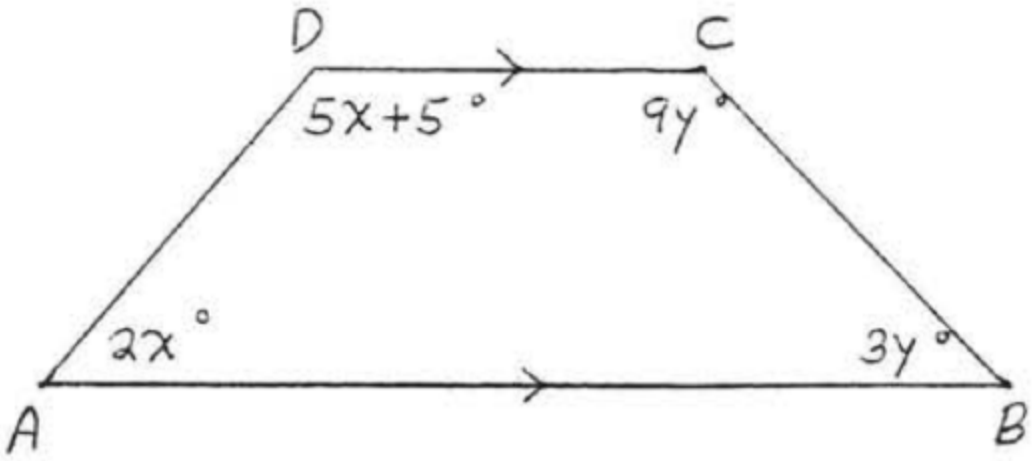
14. Find x, y, and z:



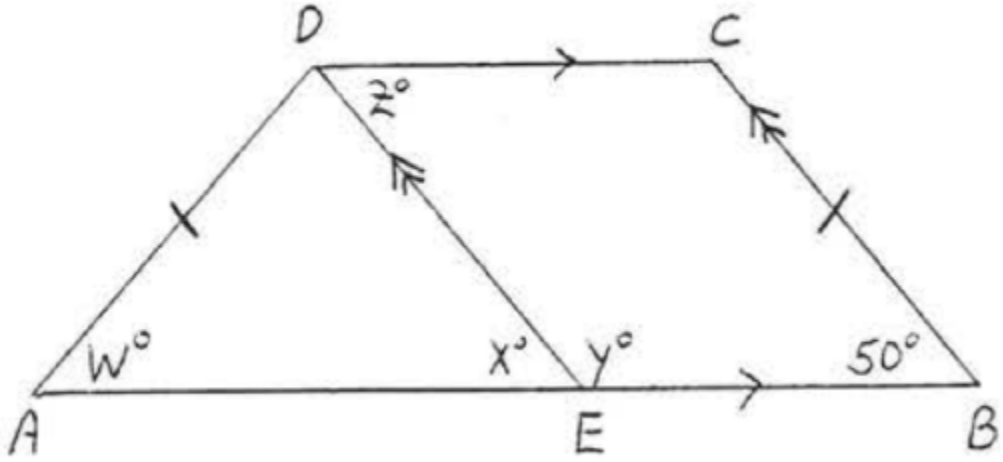
15. Find x and y:



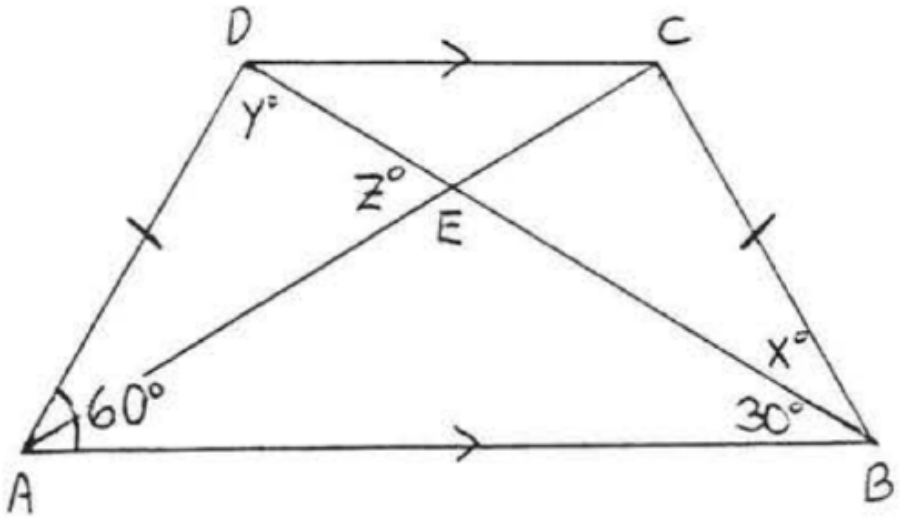
16. Find $x, y, \angle A, \angle B, \angle C$, and $\angle D$:



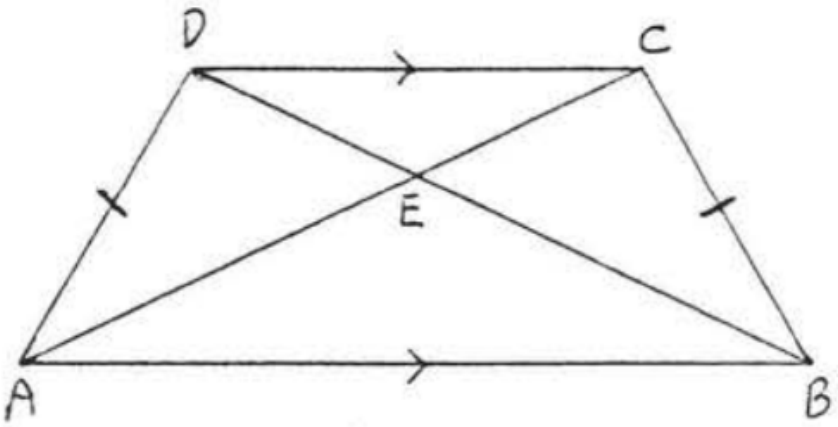
17. Find w, x, y, and z:



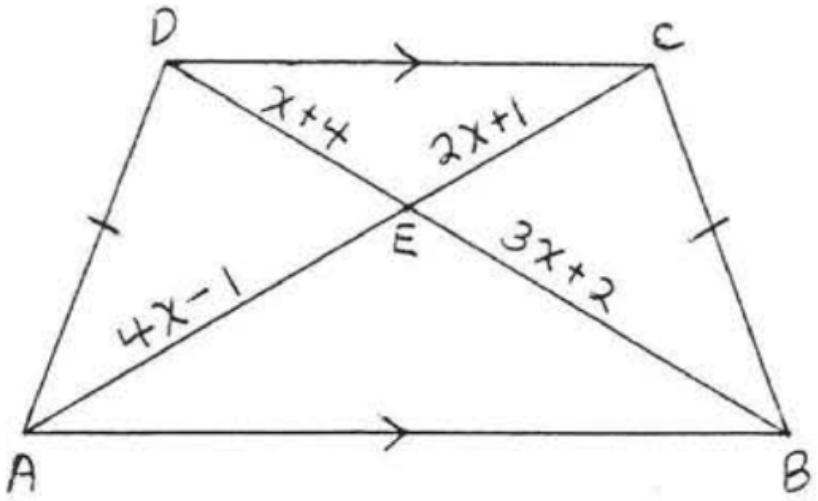
18. Find x, y, and z:



19. Find x if $AC = x^2 - 13$ and $BD = 2x + 2$:



20. Find x, AC and BD:



Answers

Theorem 13.3.2

In Figure 13.3.6, $\angle C = \angle A = 90^\circ$ because the opposite angles of a parallelogram are equal (Theorem 13.3.1, section 3.1). $\angle B = 90^\circ$ and $\angle D = 90^\circ$ because the successive angles of a parallelogram are supplementary (Theorem 13.3.2, section 3.2).

Theorem 13.3.5

$BC = AD$, given, $\angle ABC = \angle BAD$ because they are the base angles of isosceles trapezoid $ABCD$ (Theorem 13.3.4). $AB = BA$, identity. Therefore $\triangle ABC \cong \triangle BAD$ by $SAS = SAS$. So $AC = BD$ because they are corresponding sides of the congruent triangles.

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13.4: Regular Polygons

A **regular polygon** is a polygon in which all sides are equal and all angles are equal, Examples of a regular polygon are the equilateral triangle (3 sides), the square (4 sides), the regular pentagon (5 sides), and the regular hexagon (6 sides). The angles of a regular polygon can easily be found using the methods of section 1.5.

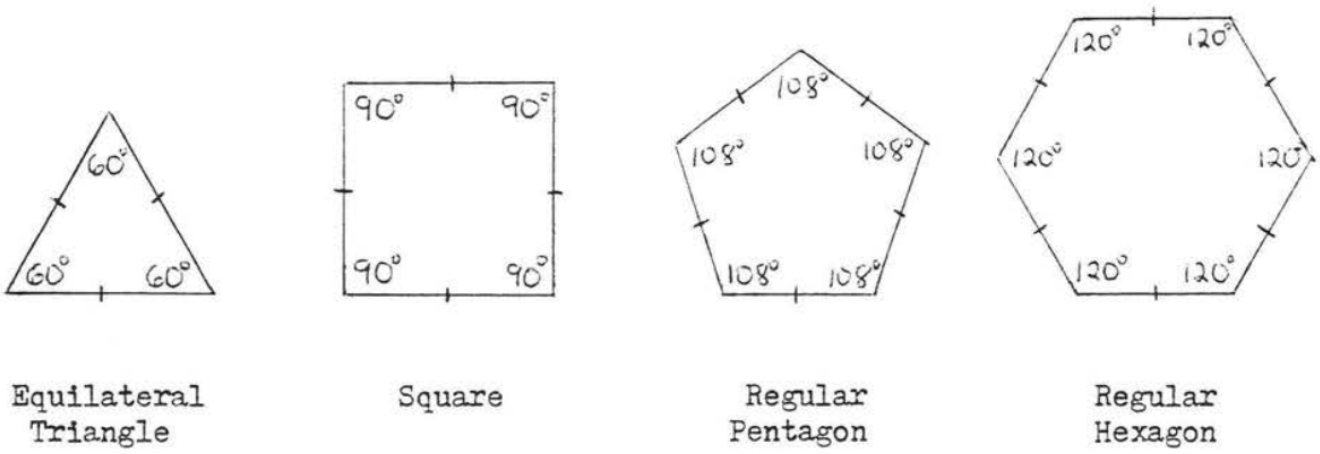


Figure 13.4.1: Examples of regular polygons.

Suppose we draw the angle bisector of each angle of a regular polygon, We will find these angle bisectors all meet at the same point (Figure 13.4.2).

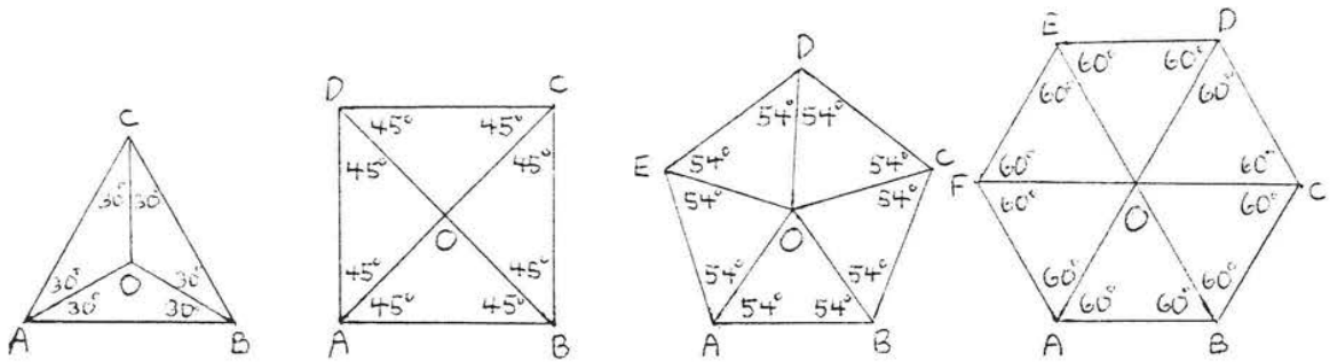


Figure 13.4.2: The angle bisectors of a regular polygon meet at the same point, O. O is called the center of the regular polygon.

Theorem 13.4.1

The angle bisectors of each angle of a regular polygon meet at the same point. This point is called the **center** of the regular polygon.

In Figure 13.4.2. O is the center of each regular polygon. The segment of each angle bisector from the center to the vertex is called a **radius**. For example, OA, OB, OC, OD, and OE are the five radii of regular pentagon $ABCDE$.

Theorem 13.4.2

The radii of a regular polygon divide the polygon into congruent isosceles triangles. All the radii are equal.

In Figure 13.4.3, radii OA, OB, OC, OD, and OE divide the regular pentagon into five isosceles triangles with $OA = OB = OC = OD = OE$.

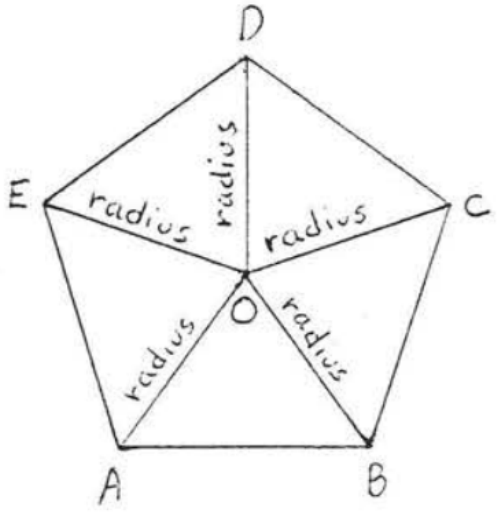
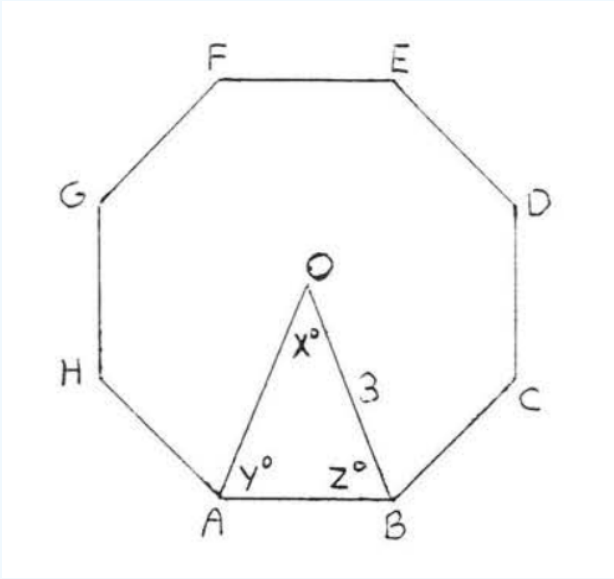


Figure 13.4.3: The five radii of a regular pentagon.

Example 13.4.1

Find the radius CA, and the angles x°, y°, and z° in the regular octagon (eight-sided figure):



Solution

The radii divide the octagon into 8 congruent isosceles triangles. Therefore $OA = OB = 3$.

$x^\circ = \frac{1}{8}(360^\circ) = 45^\circ$.

$y^\circ = z^\circ = \frac{1}{2}(180^\circ - 45^\circ) = \frac{1}{2}(135^\circ) = 67\frac{1}{2}^\circ$.

Answer: $OA = 3, x^\circ = 45^\circ, y^\circ = z^\circ = 67\frac{1}{2}^\circ$.

Theorem $13.4.1$ and Theorem $13.4.2$ appear to be true intuitively, but we verify them with a formal proof:

Proof of Theorem $13.4.1$ and Theorem $13.4.2$: We will prove these theorems for the regular pentagon. The proof for other regular polygons is similar.

Draw the angle bisectors of $\angle A$ and $\angle B$ as in Figure $13.4.4$ and call their point of intersection O. We will show OC, OD, and OE are the angle bisectors of $\angle C$, $\angle D$, and $\angle E$ respectively.

$\angle EAB = \angle ABC$ since the angles of a regular pentagon are equal. $\angle 1 = \angle 2 = \frac{1}{2}$ of $\angle EAB = \frac{1}{2}$ of $\angle ABC = \angle 3 = \angle 4$ since OA and OB are angle bisectors.

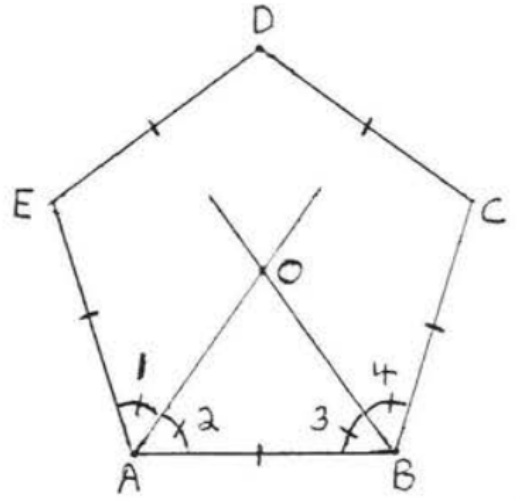


Figure $13.4.4$: Draw the angle bisectors of $\angle A$ and $\angle B$ and call their point of intersection O.

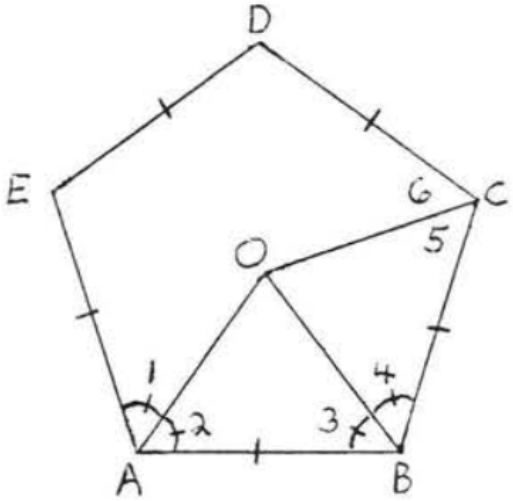


Figure 13.4.5. Draw OC.

Draw OC (Figure 13.4.5). $AB = BC$ since the sides of a regular pentagon are equal. Therefore $\triangle AOB \cong \triangle COB$ by $SAS = SAS$. Therefore $\angle 5 = \angle 2 = \frac{1}{2}$ of $\angle EAB = \frac{1}{2}$ of $\angle BCD$. So OC is the angle bisector of $\angle BCD$.

Similarly we can show $\triangle BOC \cong \triangle DOC$, $\triangle COD \cong \triangle EOD$, $\triangle DOE \cong \triangle AOE$ and that OD and OE are angle bisectors. The triangles are all isosceles because their base angles are equal. This completes the proof.

A line segment drawn from the center perpendicular to the sides of a regular polygon is called an apothem (see Figure $PageIndex6$).

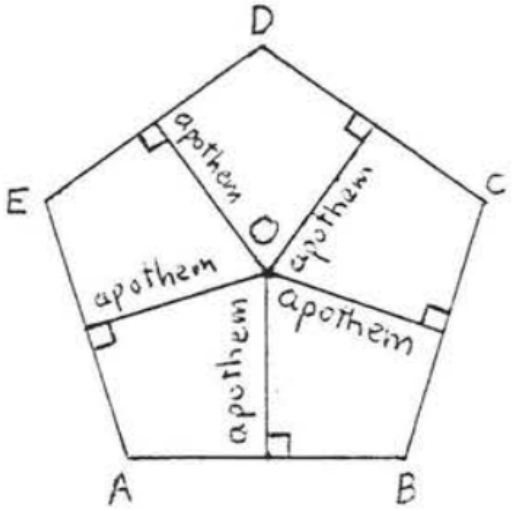


Figure 13.4.6: The apothems of a regular pentagon.

Theorem 13.4.1

The apothems of a regular polygon are all equal, They bisect the sides of the regular polygon.

Proof available at the end of this section.

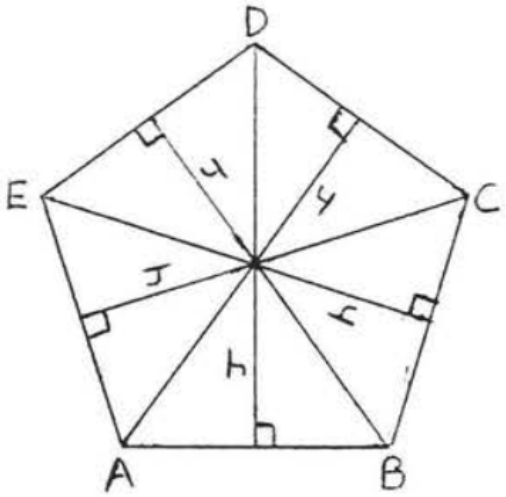


Figure 13.4.7: The apothems are the heights of the congruent isosceles triangles formed by the radii.

Example 13.4.2

Find the apothem of a regular pentagon with side 20, to the nearest tenth.

Solution

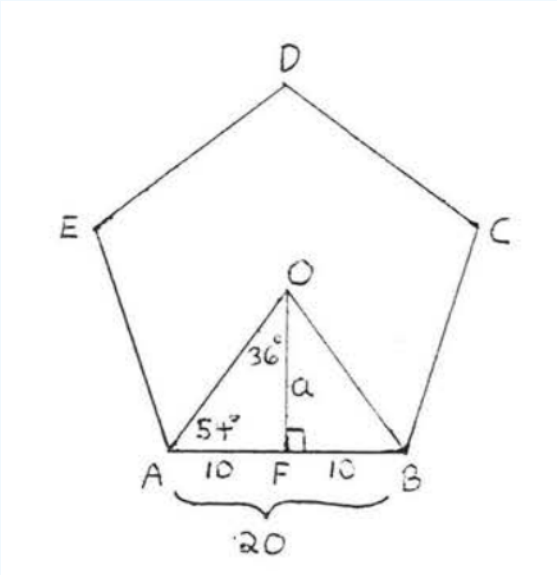


Figure 13.4.8: A regular pentagon with side 20.

In Figure 13.4.8,

$\angle AOB = \frac{1}{5}(360^\circ) = 72^\circ,$

$\angle AOF = \frac{1}{2}\angle AOB = \frac{1}{2}(72^\circ) = 36^\circ,$

and $\angle OAF = 90^\circ - 36^\circ = 54^\circ.$

$$\begin{aligned} \tan 54^\circ &= \frac{a}{10} \\ (10)1.3764 &= \frac{a}{10}(10) \\ 13.764 &= a \\ 13.8 &= a \end{aligned}$$

Answer: 13.8

The apothem of a regular polygon is important because it is used to find the area:

Theorem 13.4.4

The area of a regular polygon is one-half the product of the apothem and the perimeter.

$$A = \frac{1}{2}aP \tag{13.4.1}$$

Proof available at the end of this section.

Example 13.4.3

Find the area of a regular pentagon with side 20, to the nearest tenth.

Solution

From Example 13.4.2 we know $a = 13.764$. The perimeter $P = (5)(20) = 100$. Therefore $A = \frac{1}{2}aP = \frac{1}{2}(13.764)(100) = \frac{1}{2}(1376.4) = 688.2$.

Answer: 688.2

To find the perimeter of a regular polygon, all we have to do is to multiply the length of a side by the number of sides. For example, the pentagon of Figure 13.4.8 has perimeter $P = 5(20) = 100$. However it is also useful to have a formula for the perimeter when only the **radius** is known:

Theorem 13.4.5

The perimeter of a regular polygon of n sides with radius r is given by the formula

$$P = 2rn \sin \frac{180^\circ}{n} \tag{13.4.2}$$

Proof available at the end of this section.

Example 13.4.4

Find the perimeter of a regular pentagon with radius 10, to the nearest tenth.

Solution

A pentagon has $n = 5$ sides. Using the formula of Theorem 13.4.5, $P = 2rn \sin \frac{180^\circ}{n} = 2(10)(5) \sin \frac{180^\circ}{5} = 100 \sin 36^\circ = 100(.5878) = 58.78 = 58.8$.

Answer: 58.8.

He can also give explicit formulas for the various regular polygons, as in the following table:

Table 13.4.1

Regular Figure	n	$n \sin \frac{180^\circ}{n}$	$P = 2rn \sin \frac{180^\circ}{n}$
Triangle	3	$3 \sin 60^\circ = 2.5980$	$5.1960\ r$
Square	4	$4 \sin 45^\circ = 2.8284$	$5.6568\ r$
Pentagon	5	$5 \sin 36^\circ = 2.9390$	$5.8780\ r$
Hexagon	6	$6 \sin 30^\circ = 3.0000$	$6.0000\ r$
Decagon	10	$10 \sin 18^\circ = 3.090$	$6.180\ r$
45-sided figure	45	$45 \sin 4^\circ = 3.139$	$6.278\ r$
90-sided figure	90	$90 \sin 2^\circ = 3.141$	$6.282\ r$
1000-sided figure	1000	$1000 \sin 180^\circ = 3.1416$	$6.283\ r$

From Table 13.4.1 we can see that as the number of sides increases, the perimeter of a regular polygon becomes approximately 6.28 times the radius. You may also recognize that the value of $n \sin \frac{180^\circ}{n}$ comes close to the number π. We will return to this point when we discuss the circumference of a circle in section 7.5.

Example 13.4.4 (repeated)

Find the perimeter of a regular pentagon Hith radius 10, to the nearest tenth.

Solution

From the table

$$\begin{aligned} P &= 5.8780\ r \\ &= 5.8780(10) \\ &= 58.78 \\ &= 58.8 \end{aligned}.$$

Answer: 58.8.

Example 13.4.5

Find the apothem and area of a regular pentagon with radius 10, to the nearest tenth.

Solution

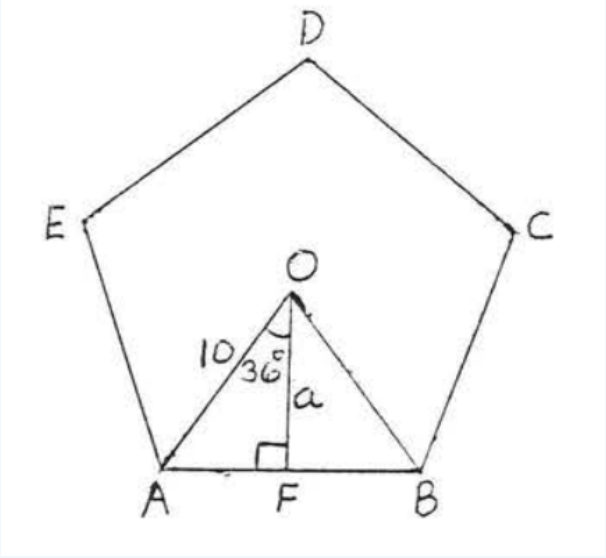


Figure 13.4.11: A regular pentagon with radius 10.

In Figure 13.4.11

$$\angle AOB = \frac{1}{5}(360^\circ) = 72^\circ$$

and

$$\angle AOF = \frac{1}{2}\angle AOB = \frac{1}{2}(72^\circ) = 36^\circ.$$

Applying trigonometry to right triangle AOF,

$$\begin{aligned} \cos 36^\circ &= \frac{a}{10} \\ (10).8090 &= \frac{a}{10}(10) \\ 8.090 &= a \end{aligned}$$

From Example 13.4.4, $P = 58.78$. Therefore, by Theorem 13.4.4,

$$A = \frac{1}{2}aP = \frac{1}{2}(8.09)(58.78) = \frac{1}{2}(475.5302) = 237.7651 = 237.8.$$

Answer: $a = 8.1, P = 237.8$.

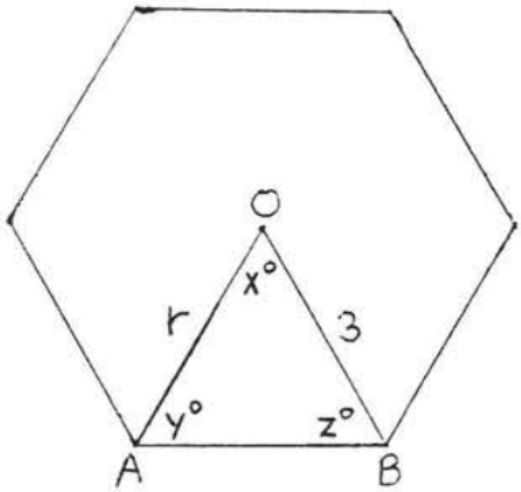
Historical Note

In 1936 archeologists unearthed a group of ancient Babylonian tables containing formulas for the areas of regular polygons of three, four, five, six and seven sides, There is evidence that regular polygons were commonly used in the architecture and designs of other ancient civilizations as well, A classical problem of Greek mathematics was to construct a regular polygon using just a ruler and compass, Regular polygons were usually studied in relationship to circles. As we shall see later in this chapter, the formulas for the area and perimeter of a circle can be derived from the corresponding formulas for regular polygons.

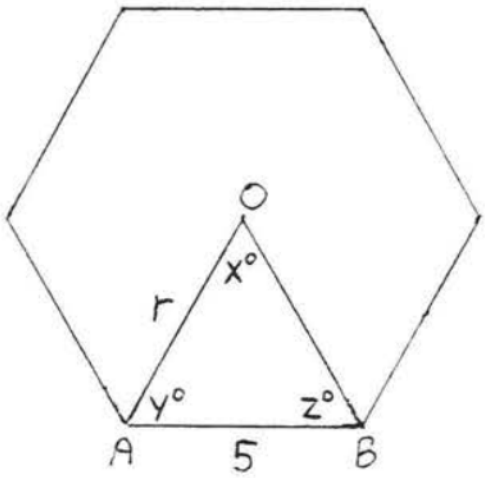
Problems

1 - 6. Find the angles $x^\circ, y^\circ, z^\circ$ and radius r of the regular polygons:

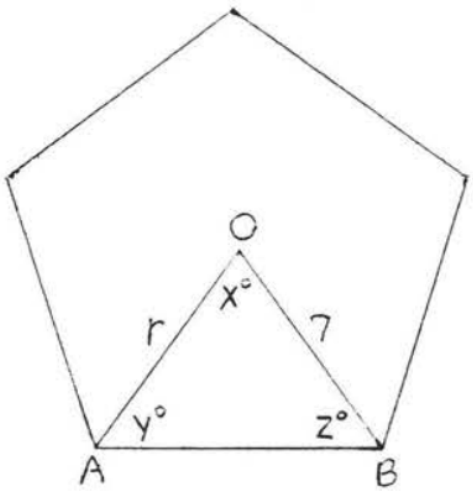
1.



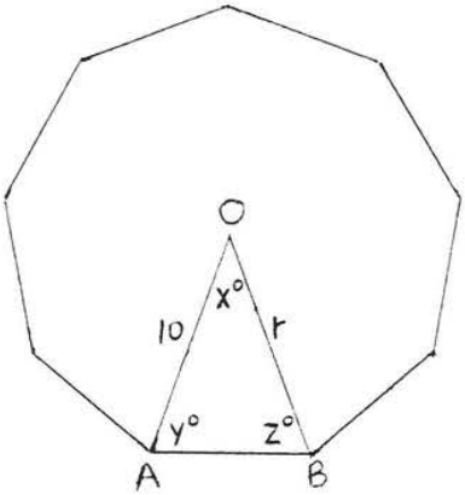
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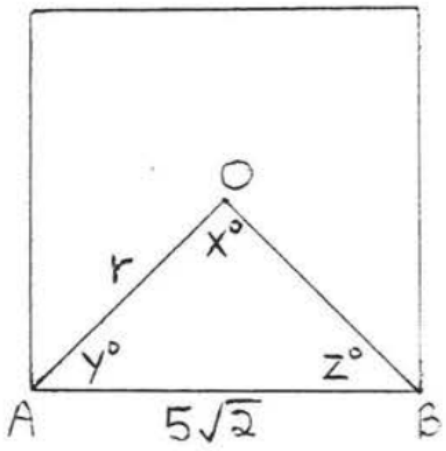
3.



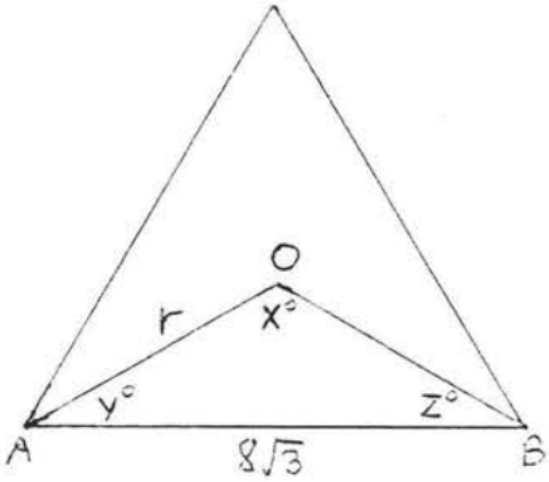
4.



5.



6.



7 - 18. Find the apothem, perimeter and area to the nearest tenth:

7. regular pentagon with side 40.

8. regular pentagon with side 16.

9. regular hexagon with side 20.

10. regular hexagon with side 16.

11. regular decagon (ten-sided figure) with side 20.

12. regular nonagon (nine-sided figure) with side 20.

13. regular pentagon with radius 20.

14. regular pentagon with radius 5.

15. regular hexagon with radius 10.

16. regular hexagon with radius 20.

17. regular decagon with radius 10.

18. regular nonagon with radius 20.

Answers

Theorem 13.4.1

The apothems are all equal because they are the heights of the congruent isosceles triangles formed by the radii (see Theorem 13.4.2), Each apothem divides the isosceles triangle into two congruent right triangles, Therefore each apothem bisects a side of the polygon, which is what we wanted to prove.

Theorem 13.4.4

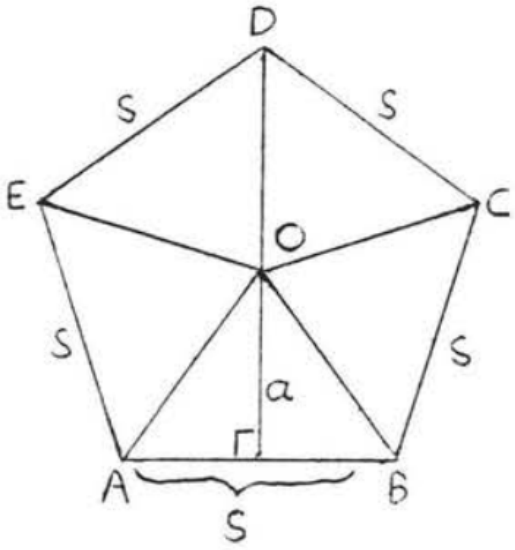


Figure $PageIndex9$. The area of $\triangle AOB$ is $\frac{1}{2}$ as, where s is the side of the pentagon.

We prove the Theorem for the regular pentagon. The proof for other regular polygons is similar.

The radii of a regular pentagon divide the regular pentagon into five congruent triangles. The area of each triangle is $\frac{1}{2}$ as, where s is the side of the pentagon (Figure $PageIndex9$). Therefore, area of the pentagon = 5(\dfrac{1}{2} as) = \dfrac{1}{2} a (5s) = \dfrac{1}{2}aP\), which is the formula we wanted to prove.

Theorem 13.4.5

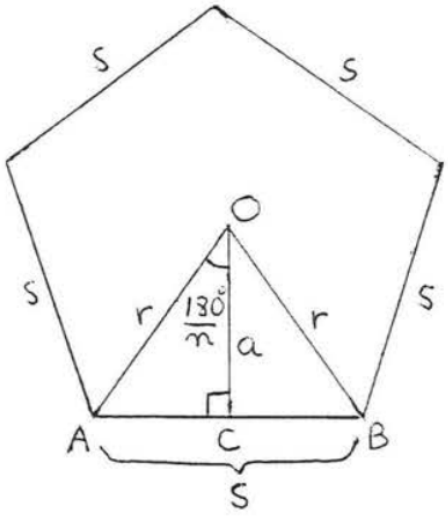


Figure 13.4.10: A regular polygon with radius r and side s.

Let us label the regular polygon as in Figure $PageIndex10$. Since the radii of the regular polygon divide the polygon into n congruent triangles (Theorem $13.4.2$), we have

$$\angle AOB = \frac{1}{n}(360^\circ) = \frac{360^\circ}{n}.$$

By Theorem $13.4.3$ apothem OC divides AOB into two congruent right triangles, so

$$\angle AOC = \frac{1}{2}\angle AOB = \frac{1}{2}(\frac{360^\circ}{n}) = \frac{180^\circ}{n}.$$

Applying trigonometry to right triangle AOC we have

$$\begin{aligned} \sin\frac{180^\circ}{n} &= \frac{AC}{r} \\ (r)\sin\frac{180^\circ}{n} &= \frac{AC}{r}(r) \\ r\sin\frac{180^\circ}{n} &= AC \end{aligned}$$

Since OC bisects AB,

$$s = 2(AC) = 2r\sin\frac{180^\circ}{n}$$

and therefore

$$P = ns = n(2r\sin\frac{180^\circ}{n}) = 2rn\sin\frac{180^\circ}{n}$$

which is the formula that we wish to prove.

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13.5: Circles

The circle is one of the most frequently encountered geometric figures. Wheels, rings, phonograph records, clocks, coins are just a few examples of common objects with circular shape. The circle has many applications in the construction of machinery and in architectural and ornamental design.

To draw a circle we use an instrument called a **compass** (Figure $13.5.1$). The compass consists of two arms, one ending in a sharp metal point and the other attached to a pencil. We draw the circle by rotating the pencil while the metal point is held so that it does not move, The position of the metal point is called the center of the circle. The distance between the **center** and the tip of the pencil is called the radius of the circle, the radius remains the same as the circle is drawn.

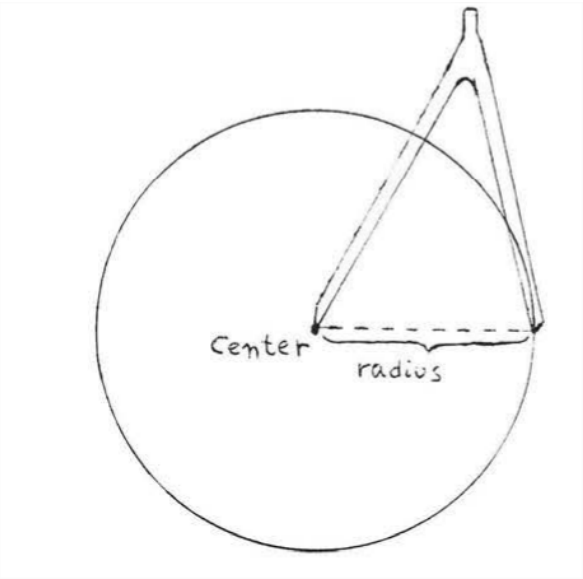


Figure $13.5.1$: Using a compass to draw a circle.

The method of constructing a circle suggests the following definition:

Definition: Circle

A **circle** is a figure consisting of all points which are a given distance from a fixed point called the center. For example the circle in Figure 2 consists of all points which are a distance of 3 from the center $\mathbb{0}$. The radius is the **distance** of any point on the circle from the center.

The circle in Figure $13.5.2$ has radius 2. The term radius is also used to denote any of the line segments from a point on the circle to the center. In Figure $13.5.2$, each of the line segments OA, OB, and OC is a radius. It follows from the definition of circle that all radii of a circle are equal. So in Figure $13.5.2$ the three radii OA, OB, and OC are all equal to 3.

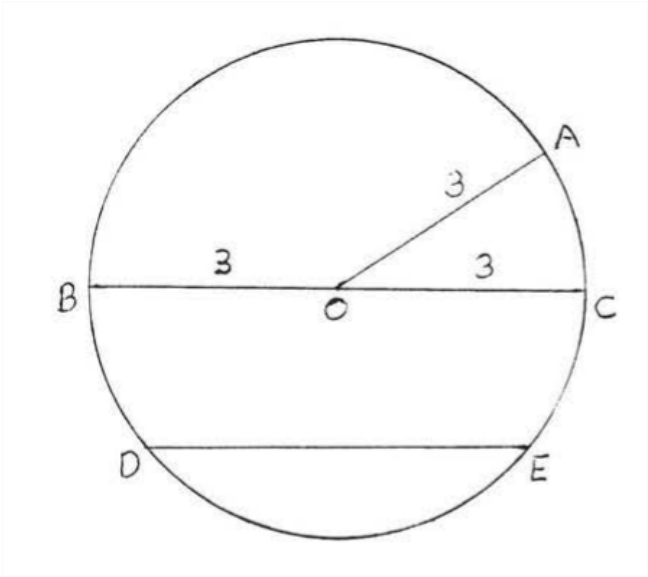


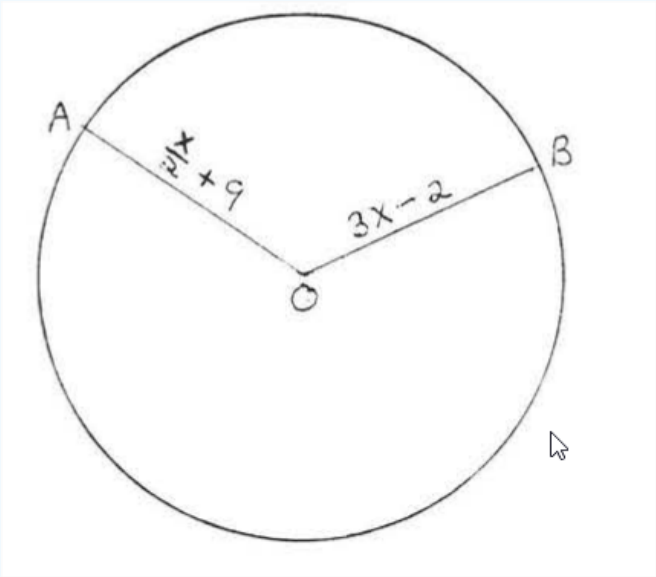
Figure 13.5.2: A circle with radius 3. (Copyright; author via source)

A circle is usually named for its center. The circle in Figure 13.5.2 is called **circle** O.

A **chord** is a line segment joining two points on a circle. In Figure 13.5.2, DE is a chord. A **diameter** is a chord which passes through the center. \(BC is a diameter. A diameter is always twice the length of a radius since it consists of two radii. Any diameter of circle 0 is equal to 6. All diameters of a circle are equal.

Example 13.5.1

Find the radius and diameter and diameter.



Solution

All radii are equal so

\[\begin{aligned}

O A &=O B \

\frac{x}{2}+9 &=3 x-2 \

(2)\left(\frac{x}{2}+9\right) &=(3 x-2)(2) \

x+18 &=6 x-4 \

22 &=5 x \

x &=\frac{22}{5}=4.4

\end{aligned}\]

Check:

$$OA = OB$$

\[\begin{array}{r|l}

\frac{x}{2}+9 & 3 x-2 \

\frac{4.4}{2}+9 & 3(4.4)-2 \

2.2+9 & 13.2-2 \

11.2 & 11.2

\end{array}\]

Therefore the radius = $OA = OB$ = 11.2 and the diameter = 2(11.2) =22.4.

Answer: radius =11.2, diameter =22.4.

The following three theorems show that a diameter of a circle and the perpendicular bisector of a chord in a circle are actually the same thing.

Theorem 13.5.1

A diameter perpendicular to a chord bisects the chord.

In Figure 13.5.3, if $AB \perp CD$ then $AE = EB$.

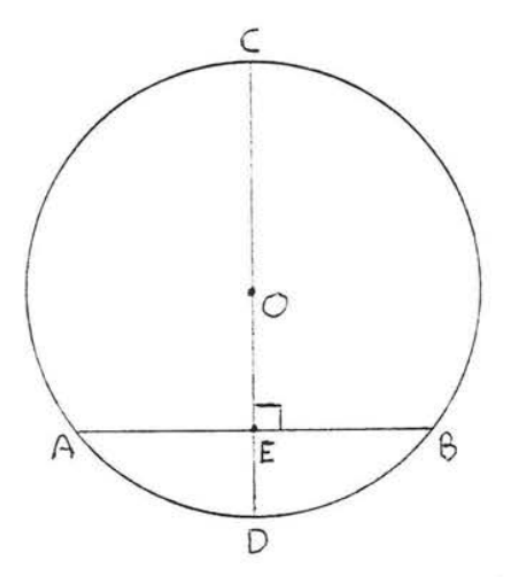
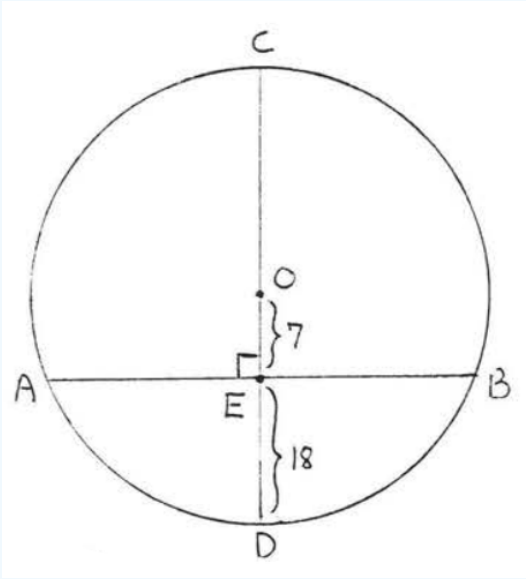


Figure 13.5.3: The diameter CD is perpendicular to chord AB.

Proof available at the end of this section.

Example 13.5.2

Find AB:



Solution

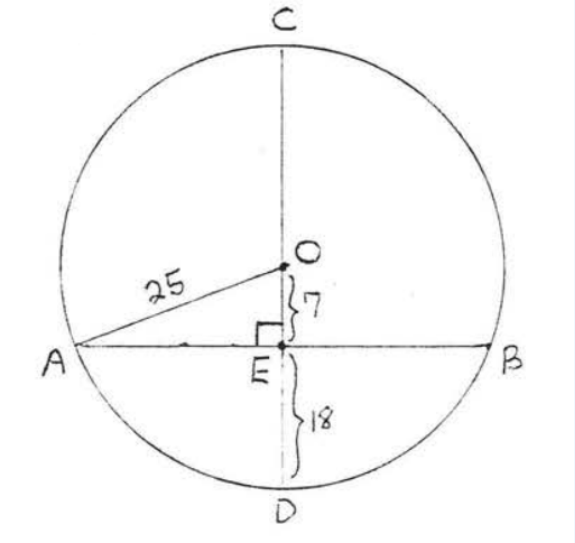


Figure 13.5.5: Draw OA.

Draw OA (Figure 13.5.5). $OA = \text{radius} = OD = 18 + 7 = 25$. $\triangle AOE$ is a right triangle and therefore we can use the Pythagorean theorem to find AE:

$$\begin{aligned} \mathrm{AE}^2 + \mathrm{CE}^2 &= \mathrm{CA}^2 \\ \mathrm{AE}^2 + 7^2 &= 25^2 \\ \mathrm{AE}^2 + 49 &= 625 \\ \mathrm{AE}^2 &= 576 \\ \mathrm{AE} &= 24 \end{aligned}$$

By Theorem 13.5.1, $EB = AE = 24$ so $AB = AE + EB = 24 + 24 = 48$.

Answer: $AB = 48$.

Theorem 13.5.2

A diameter that bisects a chord which is not a diameter is perpendicular to it.

In Figure 13.5.6, if $AE = EB$ then $AB \perp CD$.

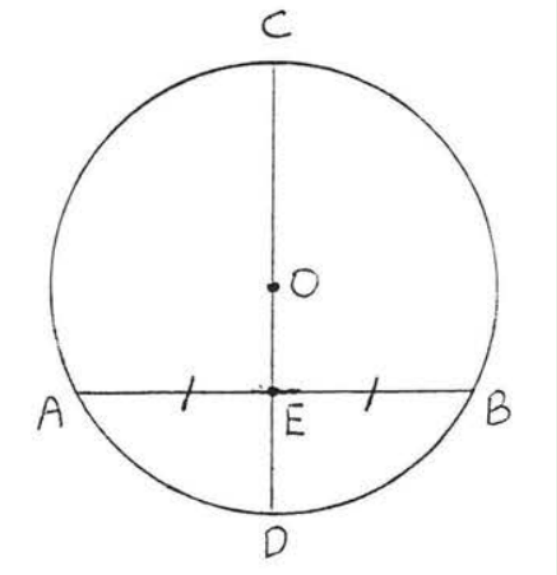
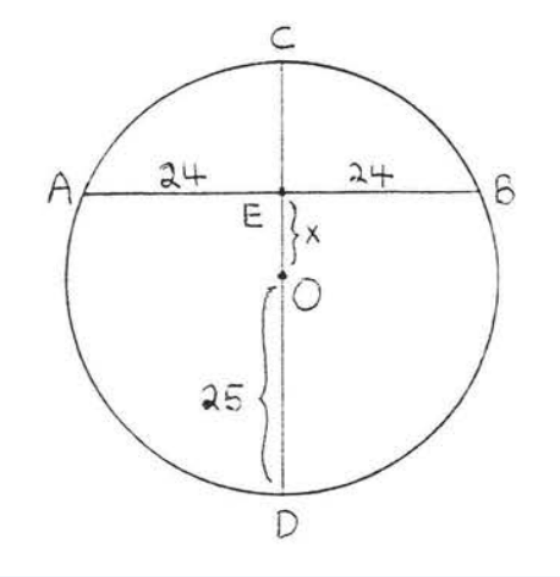


Figure 13.5.6: Diameter CD bisects chord AB.

Proof available at the end of this section.

Example 13.5.3

Find x:



Solution

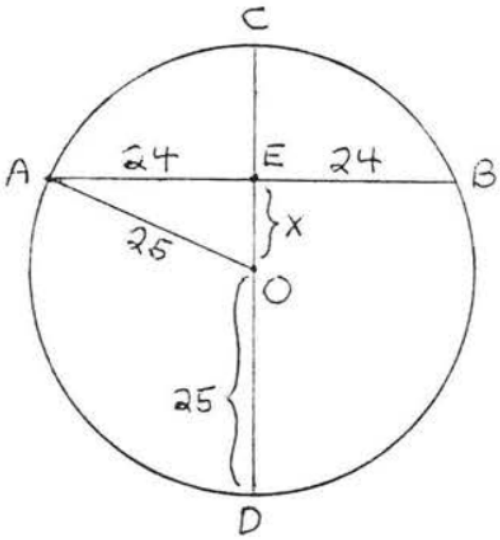


Figure 13.5.8: Draw OA.

Draw OA (Figure $PageIndex8$). $OA = \text{radius} = OD = 25$. According to Theorem 13.5.2, $AB \perp CD$. Therefore $\triangle AOE$ is a right triangle, and we can use the Pythagorean theorem to find x:

$$\begin{aligned} \mathrm{OE}^2 + \mathrm{AE}^2 &= \mathrm{OA}^2 \\ x^2 + 24^2 &= 25^2 \\ x^2 + 576 &= 625 \\ x^2 &= 49 \\ x &= 7 \end{aligned} \tag{13.5.1}$$

Answer: $x = 7$.

Theorem 13.5.3

The perpendicular bisector of a chord must pass through the center of the circle (that is, it is a diameter).

In Figure 13.5.9, if $CD \perp AB$ and $AE = EB$ then O must lie on CD.

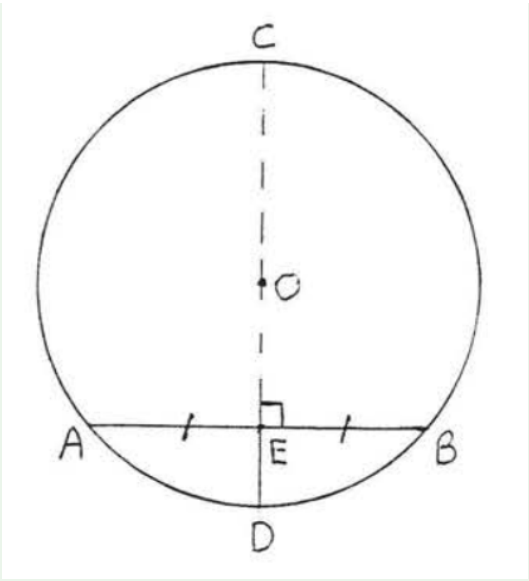
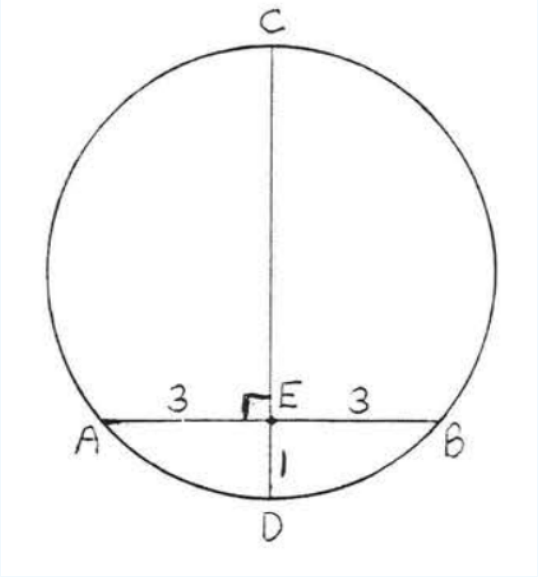


Figure 13.5.9: If CD is a perpendicular bisector of AB then CD must pass through O.

Proof available at the end of this section.

Example 13.5.4

Find the radius of the circle:



Solution

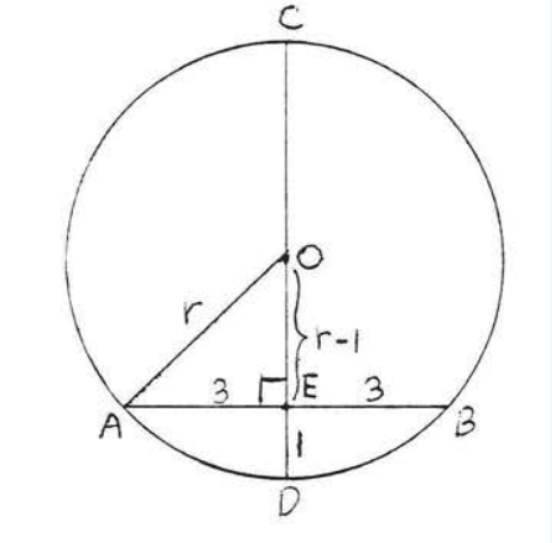


Figure 13.5.11: Draw OA and let r be the radius.

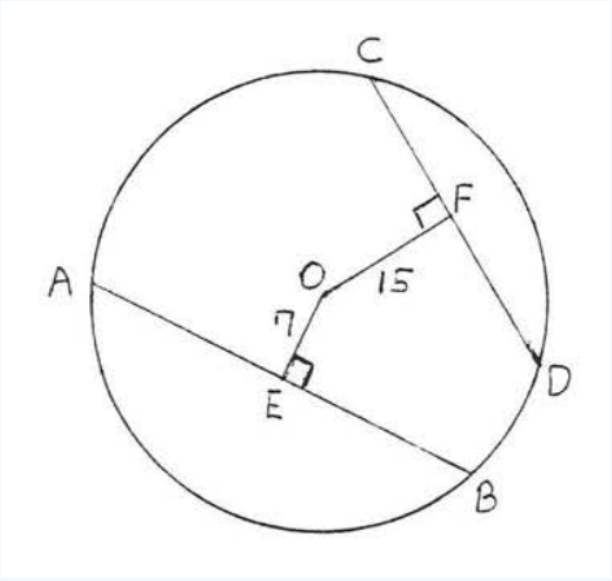
According to Theorem 13.5.3, O must lie on CD. Draw OA (Figure 13.5.11). Let r be the radius. Then $OA = OD = r$ and $OE = r - 1$. To find r we apply the Pythagorean theorem to right triangle AOE:

$$\begin{aligned} \mathrm{AE}^2 + \mathrm{OE}^2 &= \mathrm{OA}^2 \\ 3^2 + (r-1)^2 &= r^2 \\ 9 + r^2 - 2r + 1 &= r^2 \\ 10 &= 2r \\ 5 &= r \end{aligned} \tag{13.5.2}$$

Answer: $r = 5$.

Example 13.5.5

Find which chord, AB or CD, is larger if the radius of the circle is 25:



Solution

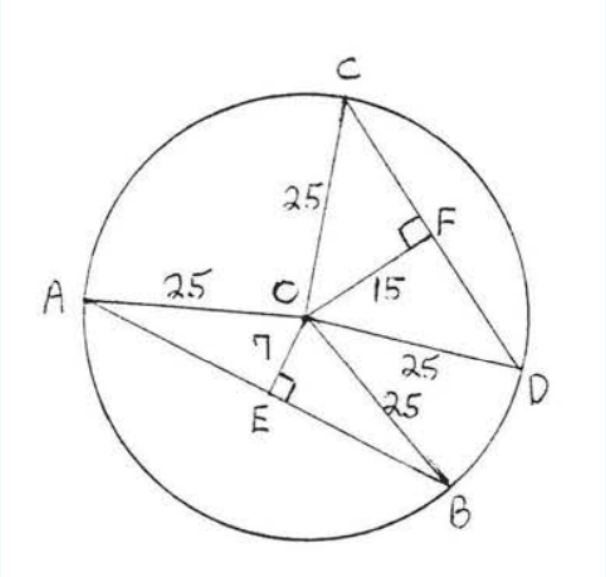


Figure 13.5.12: Draw OA, OB, OC and OD.

Draw OA, OB, OC and OD (Figure 13.5.12). Each is a radius and equal to 25. We use the Pythagorean theorem, applied to right triangle AOE, to find AE:

$$\begin{aligned} \mathrm{AE}^2 + \mathrm{OE}^2 &= \mathrm{OA}^2 \\ \mathrm{AE}^2 + 7^2 &= 25^2 \\ \mathrm{AE}^2 + 49 &= 625 \\ \mathrm{AE}^2 &= 576 \\ \mathrm{AE} &= 24 \end{aligned} \tag{13.5.3}$$

Since perpendicular OE bisects AB (Theorem 13.5.1) $BE = AE = 24$ and so $AB = AE + BE = 24 + 24 = 48$.

Similarly, to find CF, we apply the Pythagorean theorem to right triangle COF:

$$\begin{aligned} \mathrm{CF}^2 + \mathrm{OF}^2 &= \mathrm{OC}^2 \\ \mathrm{CF}^2 + 15^2 &= 25^2 \\ \mathrm{CF}^2 + 225 &= 625 \\ \mathrm{CF}^2 &= 400 \\ \mathrm{CF} &= 20 \end{aligned} \tag{13.5.4}$$

Again, from Theorem 13.5.1, we know OF bisects CD, hence $DF = CF = 20$ and $CD = 40$.

Answer: $AB = 48$, $CD = 40$, AB is larger than CD.

Example 13.5.5 suggests the following Theorem (which we state without proof):

Theorem 13.5.4

The length of a chord is determined by its distance from the center of the circle; the closer to the center, the larger the chord.

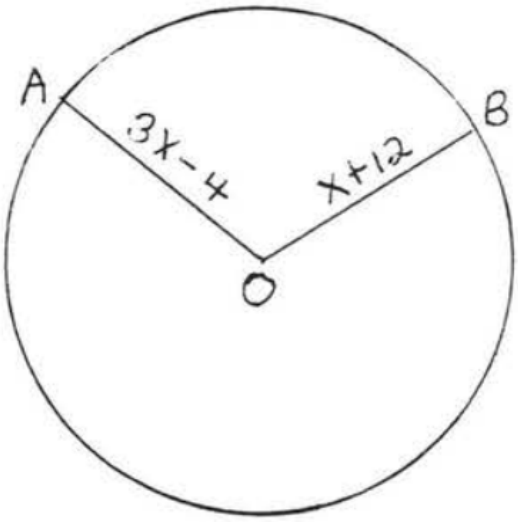
Historical Note

The definition of a circle and essentially all of the theorems of this and the next two sections can be found in Book III of Euclid's Elements.

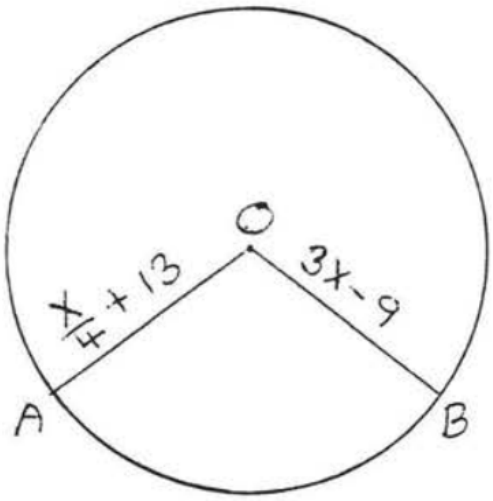
Problems

1 - 2. Find the radius and diameter:

1.

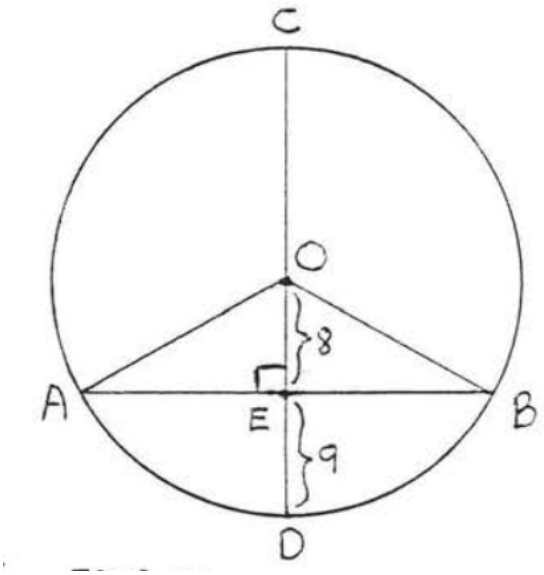


2.

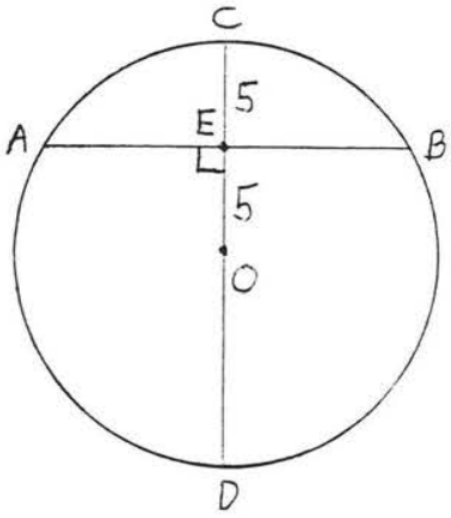


3 - 4. Find AB:

3.

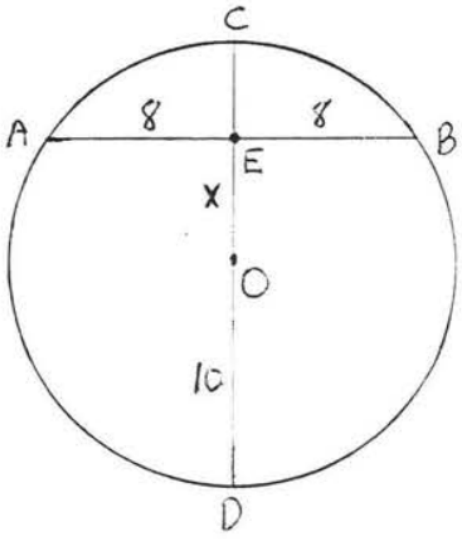


4.

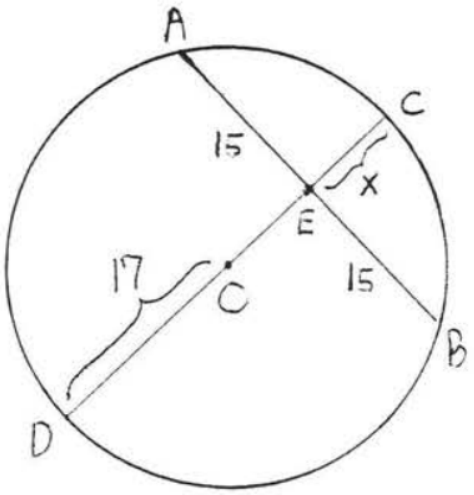


5 - 6. Find x:

5.

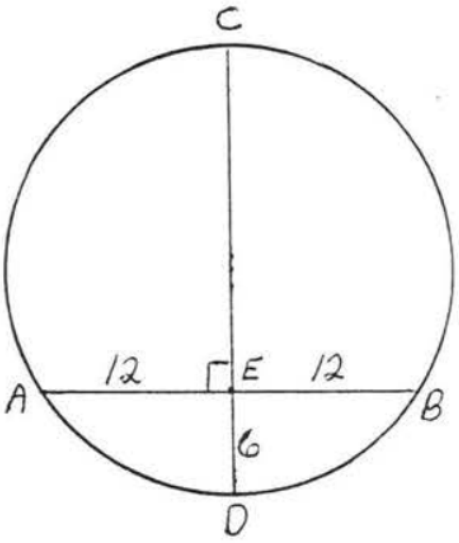


6.

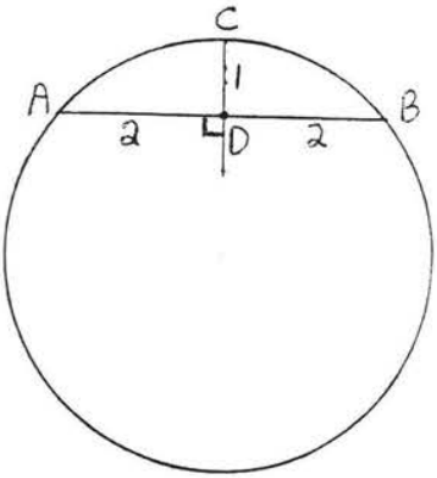


7 - 10. Find the radius and diameter:

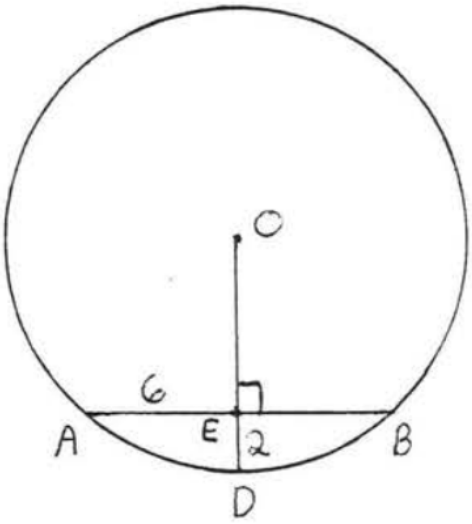
7.



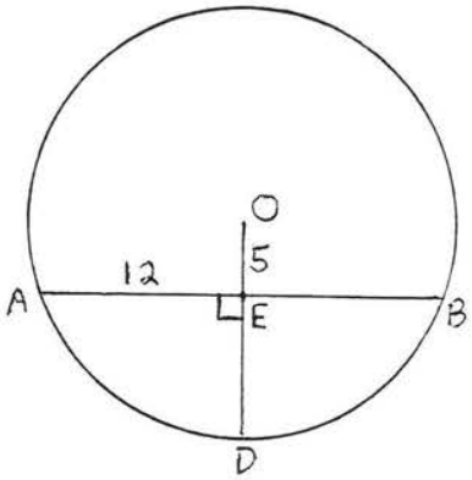
8.



9.

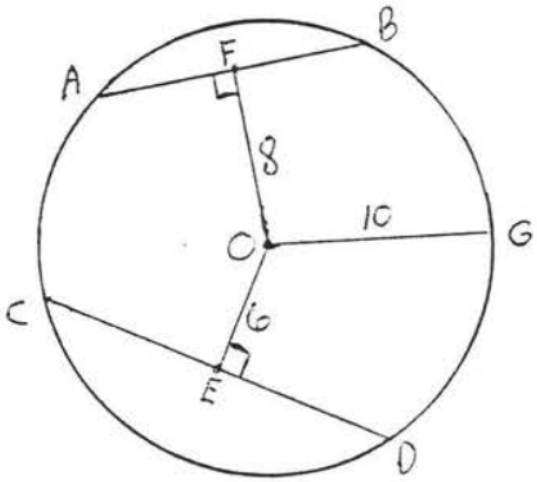


10.

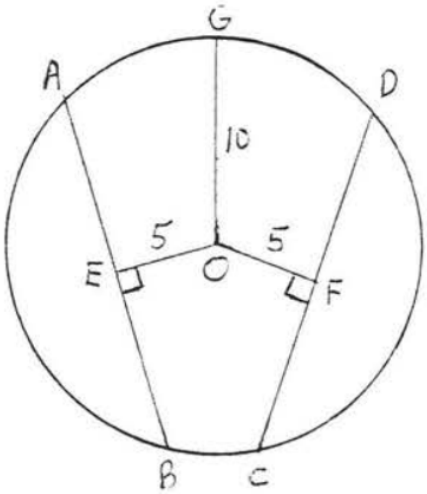


11 - 12. Find the lengths of AB and CD:

11.



12.



Answers

Theorem 13.5.1

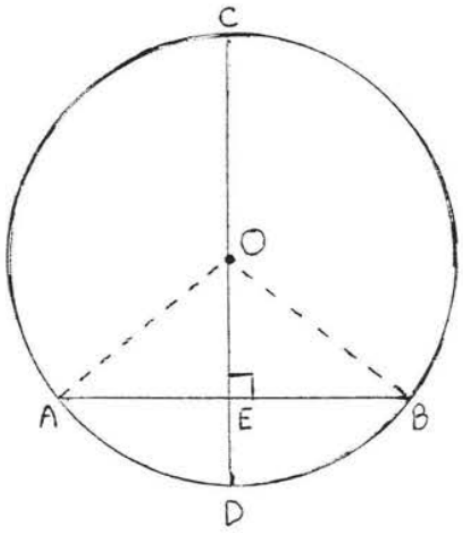


Figure 13.5.4: Draw OA and OB.

Draw OA and OB (Figure 13.5.4). $OA = OB$ because all radii of a circle are equal. $OE = OE$ because of identity. Therefore $\triangle ACE \cong \triangle BOE$ by Hyp-Leg = Hyp-Leg. Hence $AE = BE$ because they are corresponding sides of congruent triangles.

Theorem 13.5.2

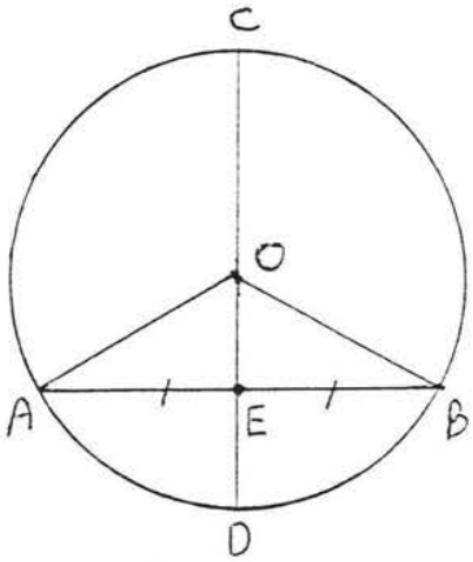


Figure 13.5.7: Draw OA and OB.

Draw OA and OB (Figure 13.5.7). $OA = OB$ because all radii are equal, $OE = OE$ (identity) and $AE = EB$ (given). Therefore $\triangle AOE \cong \triangle BOE$ by $SSS = SSS$. Therefore $\triangle AEO = \triangle BEO$. Since $\angle AEO$ and $\angle BEO$ are also supplementary we must also have $\angle AEO = \angle BEO = 90^\circ$, which is what we had to prove.

Theorem 13.5.3

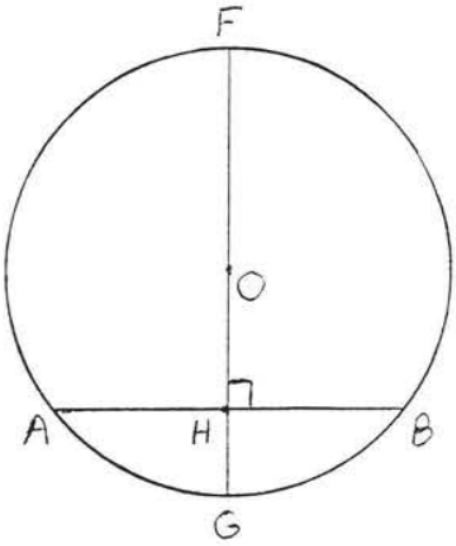


Figure 13.5.10: Draw FG through O perpendicular to AB.

Draw a diameter FG through O perpendicular to AB at H (Figure 13.5.10). Then according to Theorem 13.5.1 H must bisect AB. Hence H and E are the same point and FG and CD are the same line. So O lies on CD. This completes the proof.

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13.6: Metric Units of Measurement

Learning Objectives

1. Describe the general relationship between the U.S. customary units and metric units of length, weight/mass, and volume.
2. Define the metric prefixes and use them to perform basic conversions among metric units.

Introduction

In the United States, both the **U.S. customary measurement system** and the **metric system** are used, especially in medical, scientific, and technical fields. In most other countries, the metric system is the primary system of measurement. If you travel to other countries, you will see that road signs list distances in kilometers and milk is sold in liters. People in many countries use words like “kilometer,” “liter,” and “milligram” to measure the length, volume, and weight of different objects. These measurement units are part of the metric system.

Unlike the U.S. customary system of measurement, the metric system is based on 10s. For example, a liter is 10 times larger than a deciliter, and a centigram is 10 times larger than a milligram. This idea of “10” is not present in the U.S. customary system—there are 12 inches in a foot, and 3 feet in a yard...and 5,280 feet in a mile!

So, what if you have to find out how many milligrams are in a decigram? Or, what if you want to convert meters to kilometers? Understanding how the metric system works is a good start.

What is Metric?

The metric system uses units such as meter, liter, and gram to measure length, liquid volume, and mass, just as the U.S. customary system uses feet, quarts, and ounces to measure these.

In addition to the difference in the basic units, the metric system is based on factors of 10, and different measures for length include kilometer, meter, decimeter, centimeter, and millimeter. Notice that the word “meter” is part of all of these units.

The metric system also applies the idea that units within the system get larger or smaller by a power of 10. This means that a meter is 100 times larger than a centimeter, and a kilogram is 1,000 times heavier than a gram. You will explore this idea a bit later. For now, notice how this idea of “getting bigger or smaller by 10” is very different than the relationship between units in the U.S. customary system, where 3 feet equals 1 yard, and 16 ounces equals 1 pound.

Length, Mass, and Volume

The table below shows the basic units of the metric system. Note that the names of all metric units follow from these three basic units.

Length	Mass	Volume
basic units		
meter	gram	liter
other units you may see		
kilometer	kilogram	dekaliter
centimeter	centigram	centiliter
millimeter	milligram	milliliter

In the metric system, the basic unit of length is the meter. A meter is slightly larger than a yardstick, or just over three feet.

The basic metric unit of mass is the gram. A regular-sized paperclip has a mass of about 1 gram.

Among scientists, one gram is defined as the mass of water that would fill a 1-centimeter cube. You may notice that the word “mass” is used here instead of “weight.” In the sciences and technical fields, a distinction is made between weight and mass. Weight is a measure of the pull of gravity on an object. For this reason, an object’s weight would be different if it was weighed on Earth or on the moon because of the difference in the gravitational forces. However, the object’s mass would remain the same in both places because mass measures the amount of substance in an object. As long as you are planning on only measuring objects on Earth, you can use mass/weight fairly interchangeably—but it is worth noting that there is a difference!

Finally, the basic metric unit of volume is the liter. A liter is slightly larger than a quart.

		
The handle of a shovel is about 1 meter.	A paperclip weighs about 1 gram.	A medium-si container of m about 1 lite

Though it is rarely necessary to convert between the customary and metric systems, sometimes it helps to have a mental image of how large or small some units are. The table below shows the relationship between some common units in both systems.

	Common Measurements in Customary and Metric Systems
Length	1 centimeter is a little less than half an inch.
	1.6 kilometers is about 1 mile.
	1 meter is about 3 inches longer than 1 yard.
Mass	1 kilogram is a little more than 2 pounds
	28 grams is about the same as 1 ounce.
Volume	1 liter is a little more than 1 quart.
	4 liters is a little more than 1 gallon.

Prefixes in the Metric System

The metric system is a base-10 system. This means that each successive unit is 10 times larger than the previous one.

The names of metric units are formed by adding a prefix to the basic unit of measurement. To tell how large or small a unit is, you look at the prefix. To tell whether the unit is measuring length, mass, or volume, you look at the base.

Prefixes in the Metric System						
kilo-	*hecto-*	*deka-*	*meter/ gram/ liter*	*deci-*	*centi-*	*milli-*

1,000 times **larger** than base unit	100 times **larger** than base unit	10 times **larger** than base unit	base units	10 times **smaller** than base unit	100 times **smaller** than base unit	1000 times **smaller** than base unit

Using this table as a reference, you can see the following:

- A kilogram is 1,000 times larger than one gram (so 1 kilogram = 1,000 grams).
- A centimeter is 100 times smaller than one meter (so 1 meter = 100 centimeters).
- A dekaliter is 10 times larger than one liter (so 1 dekaliter = 10 liters).

Here is a similar table that just shows the metric units of measurement for mass, along with their size relative to 1 gram (the base unit). The common abbreviations for these metric units have been included as well.

Prefixes in the Metric System						
kilogram (kg)	hectogram (hg)	dekagram (dag)	gram (g)	decigram (dg)	centigram (cg)	milligram (mg)
1,000 gram	100 gram	10 gram	gram	0.1 gram	0.01 gram	0.001 gram

Since the prefixes remain constant through the metric system, you could create similar charts for length and volume. The prefixes have the same meanings whether they are attached to the units of length (meter), mass (gram), or volume (liter).

Try It Now 1

Which of the following sets of three units are all metric measurements of length?

A) inch, foot, yard

B) kilometer, centimeter, millimeter

C) kilogram, gram, centigram

D) kilometer, foot, decimeter

Converting Units Up and Down the Metric Scale

Converting between metric units of measure requires knowledge of the metric prefixes and an understanding of the decimal system—that's about it.

For instance, you can figure out how many centigrams are in one dekagram by using the table above. One dekagram is larger than one centigram, so you expect that one dekagram will equal many centigrams.

In the table, each unit is 10 times larger than the one to its immediate right. This means that 1 dekagram = 10 grams; 10 grams = 100 decigrams; and 100 decigrams = 1,000 centigrams. So, 1 dekagram = 1,000 centigrams.

Example 13.6.1

How many milligrams are in one decigram?

Solution

Identify locations of milligrams and decigrams.

kg	hg	dag	g	**dg**	cg	**mg**

Decigrams (dg) are larger than milligrams (mg), so you expect there to be many mg in one dg.

Dg is 10 times larger than a cg, and a cg is 10 times larger than a milligram.

x10 x10

kg	hg	dag	g	**dg**	cg	**mg**

→ → → → → → → → → → → → → → →

Since you are going from a larger unit to a smaller unit, multiply.

Multiply: $1 \cdot 10 \cdot 10$, to find the number of milligrams in one decigram.

$$1 \text{ dg} \cdot 10 \cdot 10 = 100 \text{ mg}$$

Answer: There are 100 milligrams (mg) in 1 decigram (dg).

Example 13.6.2

Convert 1 centimeter to kilometers.

Solution

Identify locations of kilometers and centimeters.

km	hm	dam	m	dm	**cm**	mm

Kilometers (km) are larger than centimeters (cm), so you expect there to be less than one km in a cm.

Cm is 10 times smaller than a dm; a dm is 10 times smaller than a m, etc.

÷10 ÷10 ÷10 ÷10 ÷10

km	hm	dam	m	dm	**cm**	mm

← ←

Since you are going from a smaller unit to a larger unit, divide.

Divide: $1 \div 10 \div 10 \div 10 \div 10 \div 10$, to find the number of kilometers in one centimeter.

$$1 \text{ cm} \div 10 \div 10 \div 10 \div 10 \div 10 = 0.00001 \text{ km}$$

Answer: 1 centimeter (cm) = 0.00001 kilometers (km).

Once you begin to understand the metric system, you can use a shortcut to convert among different metric units. The size of metric units increases tenfold as you go up the metric scale. The decimal system works the same way: a tenth is 10 times larger than a hundredth; a hundredth is 10 times larger than a thousandth, etc. By applying what you know about decimals to the metric system, converting among units is as simple as moving decimal points.

Here is the first problem from above: How many milligrams are in one decigram? You can recreate the order of the metric units as shown below:

kg hg dag g dg cg mg

This question asks you to start with 1 decigram and convert that to milligrams. As shown above, milligrams is two places to the right of decigrams. You can just move the decimal point two places to the right to convert decigrams to milligrams: $1.\ \mathrm{dg} = 100.\ \mathrm{mg}$ (Note the location of the decimal points).

The same method works when you are converting from a smaller to a larger unit, as in the problem: Convert 1 centimeter to kilometers.

km hm dam m dm cm mm

Note that instead of moving to the right, you are now moving to the left—so the decimal point must do the same: $1.\ \mathrm{cm} = 0.00001\ \mathrm{km}$.

Try It Now 2

How many milliliters are in 1 liter?

Summary

The metric system is an alternative system of measurement used in most countries, as well as in the United States. The metric system is based on joining one of a series of prefixes, including kilo- , hecto-, deka-, deci-, centi-, and milli-, with a base unit of measurement, such as meter, liter, or gram. Units in the metric system are all related by a power of 10, which means that each successive unit is 10 times larger than the previous one. This makes converting one metric measurement to another a straightforward process, and is often as simple as moving a decimal point. It is always important, though, to consider the direction of the conversion. If you are converting a smaller unit to a larger unit, then the decimal point has to move to the left (making your number smaller); if you are converting a larger unit to a smaller unit, then the decimal point has to move to the right (making your number larger).

Try It Now Answers

1. B) kilometer, centimeter, millimeter; all of these measurements are from the metric system. They are measurements of length because they all contain the word “meter.”

2. 1,000; there are 10 milliliters in a centiliter, 10 centiliters in a deciliter, and 10 deciliters in a liter. Multiply: $10 \cdot 10 \cdot 10$, to find the number of milliliters in a liter, 1,000.

Converting within the Metric System

Learning Objective

1. Perform arithmetic calculations on metric units of length, mass, and volume.

Introduction

While knowing the different units used in the metric system is important, the real purpose behind learning the metric system is for you to be able to use these measurement units to calculate the size, mass, or volume of different objects. In practice, it is often necessary to convert one metric measurement to another unit—this happens frequently in the medical, scientific, and technical fields, where the metric system is commonly used.

If you have a prescription for 5,000 mg of medicine, and upon getting it filled, the dosage reads 5 g of medicine, did the pharmacist make a mistake?

For a moment, imagine that you are a pharmacist. You receive three prescriptions for liquid amoxicillin: one calls for 2.5 centiliters,

one calls for 0.3 deciliters, and one calls for 450 milliliters. Amoxicillin is stored in the refrigerator in 1 liter, 1 deciliter, and 1 centiliter containers. Which container should you use to ensure you are not wasting any of the unused drug?

To solve this problem, you need to know how to convert from one measurement to another as well as how to add different quantities together. Let's take a look at how to do this.

Converting from Larger to Smaller Units

Converting between measurements in the metric system is simply a matter of identifying the unit that you have, the unit that you want to convert to, and then counting the number of units between them. A basic example of this is shown below.

Example 13.6.3

Convert 1 kilometer to decimeters

Solution

Identify locations of kilometers and decimeters.

km	hm	dam	m	**dm**	cm	mm

Kilometers (km) are larger than decimeters (dm), so you expect there to be more than one dm in a km.

x10 x10 x10 x10

km	hm	dam	m	**dm**	cm	mm

→ →

Count the intermediate units, multiplying by 10 as you go. (Since you are going from a larger unit to a smaller unit, you multiply.)

Multiply to find the number of decimeters in one kilometer.

$$1\ \text{km} \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10000\ \text{dm}$$

Answer: 1 kilometer = 10,000 decimeters

This problem is straightforward because you are converting 1 kilometer to another unit. The example below shows how you would solve this problem if you were asked to convert 8.2 kilometers to decimeters. Notice that most steps are the same; the critical difference is that you multiply by 8.2 in the final step.

Example 13.6.4

Convert 8.2 kilometers to decimeters

Solution

Identify locations of kilometers and decimeters.

km	hm	dam	m	**dm**	cm	mm

Kilometers (km) are larger than decimeters (dm), so you expect there to be more than one dm in a km.

x10 x10 x10 x10

km	hm	dam	m	**dm**	cm	mm

→ →

Count the intermediate units, multiplying by 10 as you go. Since you are going from a larger unit to a smaller unit, multiply.

Multiply to find the number of decimeters in 8.2 kilometers.

$$8.2 \text{ km} \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10000 \text{ dm}$$

Answer: 8.2 kilometer = 82,000 decimeters

You can also apply the rules of base 10 to use the "move the decimal" shortcut method in this example. Notice how decimeters (dm) is four places to the right of kilometers (km); similarly, you move the decimal point four places to the right when converting 8.2 kilometers to decimeters.

km hm dam m dm cm mm

8.2000 km = 82000. dm

8.2 km = 82000 dm

Example $13.6.5$

Convert 0.55 liters to centiliters.

Solution

Count two places from liters to centiliters.

kl hl dal l dl cl ml

In 0.55 *l*, move the decimal point two places to the right.

0.55 *l* = 0.55.*cl*

0.55 *l* = 55 *cl*

Answer: 0.55 liters = 55 centiliters

Try It Now 1

How many dekaliters are in 0.5 deciliters?

Converting from Smaller to Larger Units

You can use similar processes when converting from smaller to larger units. When converting a larger unit to a smaller one, you multiply; when you convert a smaller unit to a larger one, you divide. Here is an example.

Example 13.6.6

Convert 739 centigrams to grams.

Solution

Identify locations of centigrams and grams.

kg	hg	dag	**g**	dg	**cg**	mg

Centigrams (cg) are smaller than grams (g), so you expect there to be less than 739 g in 739 cg.

Count the intermediate units, dividing by 10 as you go. Since you are going from a smaller unit to a larger unit, divide.

÷10 ÷10

kg	hg	dag	**g**	dg	**cg**	mg

← ← ← ← ← ← ← ← ← ← ← ← ← ← ←

Since you are going from a smaller unit to a larger unit, divide.

Divide to find the number of grams in 739 centigrams.

$$739 \div 10 \div 10 = 7.39 \text{ g}$$

Answer: 739 centigrams = 7.39 grams

Notice that the shortcut method of counting prefixes and moving the decimal the same number of places also works here. Just make sure you are moving the decimal point in the correct direction for the conversion.

Example 13.6.7

Convert 205.5 milliliters to kiloliters.

Solution

Count six places from milliliters to kiloliters.

kl hl dal l dl cl ml

Milliliter is smaller than kiloliter, so you expect the number 205.5 to get smaller as you move up the metric chart.

0.000205.5

In 205.5 ml, move the decimal point six places to the left.

205.5 *ml* = 0.0002055 *kl*

Answer: 205.5 milliliters = 0.0002055 kiloliters

Try It Now 2

Convert 3,085 milligrams to grams

Factor Label Method

There is yet another method that you can use to convert metric measurements—the **factor label method**. You used this method when you were converting measurement units within the U.S. customary system.

The factor label method works the same in the metric system; it relies on the use of unit fractions and the reducing of intermediate units. The table below shows some of the **unit equivalents** and **unit fractions** for length in the metric system. (You should notice that all of the unit fractions contain a factor of 10. Remember that the metric system is based on the notion that each unit is 10 times larger than the one that came before it.)

Also, notice that two new prefixes have been added here: mega- (which is very big) and micro- (which is very small).

Unit Equivalents	Conversion Factors	
1 meter = 1,000,000 micrometers	$\frac{1\text{ m}}{1000000\text{ µm}}$	$\frac{1000000\text{ µm}}{1\text{ m}}$
1 meter = 1,000 millimeters	$\frac{1\text{ m}}{1000\text{ mm}}$	$\frac{1000\text{ mm}}{1\text{ m}}$
1 meter = 100 centimeters	$\frac{1\text{ m}}{100\text{ cm}}$	$\frac{100\text{ cm}}{1\text{ m}}$
1 meter = 10 decimeters	$\frac{1\text{ m}}{10\text{ dm}}$	$\frac{10\text{ dm}}{1\text{ m}}$
1 dekameter = 10 meters	$\frac{1\text{ dam}}{10\text{ m}}$	$\frac{10\text{ m}}{1\text{ dam}}$
1 hectometer = 100 meters	$\frac{1\text{ hm}}{100\text{ m}}$	$\frac{100\text{ m}}{1\text{ hm}}$
1 kilometer = 1,000 meters	$\frac{1\text{ km}}{1000\text{ m}}$	$\frac{1000\text{ m}}{1\text{ km}}$
1 megameter = 1,000,000 meters	$\frac{1\text{ Mm}}{1000000\text{ m}}$	$\frac{1000000\text{ m}}{1\text{ Mm}}$

When applying the factor label method in the metric system, be sure to check that you are not skipping over any intermediate units of measurement.

Example 13.6.8

Convert 7,225 centimeters to meters.

Solution

Meters is larger than centimeters, so you expect your answer to be less than 7,225.

$$7225\text{ cm} = \_\_\_\_\_\text{m}$$

Using the factor label method, write 7,225 cm as a fraction and use unit fractions to convert it to m.

$$\frac{7225\text{ cm}}{1}\cdot\frac{1\text{ m}}{100\text{ cm}} = \_\_\_\_\_\text{m}$$

Reduce similar units, multiply, and simplify.

$$\frac{7225}{1}\cdot\frac{1\text{ m}}{100} = \frac{7225}{100}\text{ m}$$

$$\frac{7225\text{ m}}{100} = 72.25\text{ m}$$

Answer: 7,225 centimeters = 72.25 meters

Try It Now 3

Using whichever method you prefer, convert 32.5 kilometers to meters.

Now that you have seen how to convert among metric measurements in multiple ways, let's revisit the problem posed earlier.

Example 13.6.9

If you have a prescription for 5,000 mg of medicine, and upon getting it filled, the dosage reads 5 g of medicine, did the pharmacist make a mistake?

Solution

Need to convert mg to g.

$$5000 \text{ mg} = \_\_\_\_\_ \text{g?}$$

$$\frac{5000 \text{ mg}}{1} \cdot \frac{1 \text{ g}}{1000 \text{ mg}} = \_\_\_\_\_ \text{g}$$

$$\frac{5000}{1} \cdot \frac{1 \text{ g}}{1000} = \frac{7225}{100} \text{ m}$$

$$\frac{5000 \cdot 1 \text{ g}}{1 \cdot 1000} = \frac{5000 \text{ g}}{1000}$$

$$\frac{5000 \text{ g}}{1000} = 5 \text{ m}$$

Answer: 5 g = 5,000 mg, so the pharmacist did not make a mistake.

Summary

To convert among units in the metric system, identify the unit that you have, the unit that you want to convert to, and then count the number of units between them. If you are going from a larger unit to a smaller unit, you multiply by 10 successively. If you are going from a smaller unit to a larger unit, you divide by 10 successively. The factor label method can also be applied to conversions within the metric system. To use the factor label method, you multiply the original measurement by unit fractions; this allows you to represent the original measurement in a different measurement unit.

Try It Now Answers

1. 0.005 dekaliters; one deciliter is 100 times smaller than a dekaliter, so you move the decimal point two places to the left to convert 0.5 deciliters to 0.005 dekaliters.
2. 3.085 grams; one gram is 1,000 times larger than a milligram, so you can move the decimal point in 3,085 three places to the left.
3. 32,500 meters; to find the number of m in 32.5 km, you can set up the following equation: $\frac{32.5 \text{ km}}{1} \cdot \frac{1000 \text{ m}}{1 \text{ km}} = \frac{32500 \text{ m}}{1}$. The km units reduce, leaving the answer in meters.

Using Metric Conversions to Solve Problems

Learning Objectives

1. Solve application problems involving metric units of length, mass, and volume.

Introduction

Learning how to solve real-world problems using metric conversions is as important as learning how to do the conversions themselves. Mathematicians, scientists, nurses, and even athletes are often confronted with situations where they are presented with information using metric measurements, and must then make informed decisions based on that data.

To solve these problems effectively, you need to understand the context of a problem, perform conversions, and then check the reasonableness of your answer. Do all three of these steps and you will succeed in whatever measurement system you find yourself using.

Understanding Context and Performing Conversions

The first step in solving any real-world problem is to understand its context. This will help you figure out what kinds of solutions are reasonable (and the problem itself may give you clues about what types of conversions are necessary). Here is an example.

Example 13.6.10

In the Summer Olympic Games, athletes compete in races of the following lengths: 100 meters, 200 meters, 400 meters, 800 meters, 1500 meters, 5000 meters and 10,000 meters. If a runner were to run in all these races, how many kilometers would he run?

Solution

To figure out how many kilometers he would run, you need to first add all of the lengths of the races together and *then* convert that measurement to kilometers.

$$10000 + 5000 + 1500 + 800 + 400 + 200 + 100 = 18000$$

Use the factor label method and unit fractions to convert from meters to kilometers.

$$\frac{18000 \text{ m}}{1} \cdot \frac{1 \text{ km}}{1000 \text{ m}} = \_\_\_\_\_ \text{ km}$$

Reduce, multiply, and solve.

$$\frac{18000}{1} \cdot \frac{1 \text{ km}}{1000} = \frac{18000 \text{ km}}{1000}$$

$$\frac{18000 \text{ km}}{1000} = 18 \text{ km}$$

Answer: The runner would run 18 kilometers.

This may not be likely to happen (a runner would have to be quite an athlete to compete in all of these races) but it is an interesting question to consider. The problem required you to find the total distance that the runner would run (in kilometers). The example showed how to add the distances, in meters, and then convert that number to kilometers.

An example with a different context, but still requiring conversions, is shown below.

Example 13.6.11

One bottle holds 295 dl while another one holds 28,000 ml. What is the difference in capacity between the two bottles?

Solution

The two measurements are in different units. You can convert both units to liters and then compare them

$$295 \text{ dl} = \_\_\_\_\_ \text{l}$$

$$28000 \text{ ml} = \_\_\_\_\_ \text{l}$$

Convert dl to liters.

$$\frac{295 \text{ dl}}{1} \cdot \frac{1 \text{ l}}{10 \text{ dl}} = \_\_\_\_\_ \text{l}$$

Reduce common units and multiply.

$$\frac{295}{1} \cdot \frac{1 \text{ l}}{10} = \_\_\_\_\_ \text{l}$$

295 dl = 29.5 liters.

$$\frac{295 \text{ l}}{10} = 29.5 \text{ l}$$

Convert ml to liters.

$$\frac{28000 \text{ ml}}{1} \cdot \frac{1 \text{ l}}{1000 \text{ ml}} = \_\_\_\_\_ \text{l}$$

$$\frac{28000}{1} \cdot \frac{1 \text{ l}}{1000} = \frac{28000 \text{ l}}{1000}$$

28,000 ml = 28 liters

$$\frac{28000 \text{ l}}{1000} = 28 \text{ l}$$

The question asks for “difference in capacity” between the bottles.

$$29.5 \text{ liters} – 28 \text{ liters} = 1.5 \text{ liters}$$

Answer: There is a difference in capacity of 1.5 liters between the two bottles.

This problem asked for the difference between two quantities. The easiest way to find this is to convert one quantity so that both quantities are measured in the same unit, and then subtract one from the other.

Try It Now 1

One boxer weighs in at 85 kg. He is 80 dag heavier than his opponent. How much does his opponent weigh?

Checking your Conversions

Sometimes it is a good idea to check your conversions using a second method. This usually helps you catch any errors that you may make, such as using the wrong unit fractions or moving the decimal point the wrong way.

Example 13.6.12

A two-liter bottle contains 87 centiliters of oil and 4.1 deciliters of water. How much more liquid is needed to fill the bottle?

Solution

You are looking for the amount of liquid needed to fill the bottle. Convert both measurements to liters and then solve the problem.

$$87 \text{ cl} + 4.1 \text{ dl} + \_\_\_\_\_ = 2 \text{ l}$$

Convert 87 cl to liters.

$$87 \text{ cl} = \_\_\_\_\_\text{l}$$

$$\frac{87 \text{ cl}}{1} \cdot \frac{1 \text{ l}}{100 \text{ cl}} = \_\_\_\_\_\text{l}$$

$$\frac{87}{1} \cdot \frac{1 \text{ l}}{100} = \frac{87 \text{ l}}{100}$$

$$\frac{87 \text{ l}}{100} = 0.87 \text{ l}$$

Convert 4.1 dl to liters

$$4.1 \text{ dl} = \_\_\_\_\_\text{l}$$

$$\frac{4.1 \text{ dl}}{1} \cdot \frac{1 \text{ l}}{10 \text{ dl}} = \_\_\_\_\_\text{l}$$

$$\frac{4.1}{1} \cdot \frac{1 \text{ l}}{10} = \frac{4.1 \text{ l}}{10}$$

$$\frac{4.1 \text{ l}}{10} = 0.41 \text{ l}$$

Subtract to find how much more liquid is needed to fill the bottle.

$$87 \text{ cl} + 4.1 \text{ dl} + \_\_\_\_\_ = 2 \text{ l}$$

$$0.87 \text{ liter} + 0.41 \text{ liter} + \_\_\_\_\_ = 2 \text{ liters}$$

$$2 \text{ liters} – 0.87 \text{ liter} – 0.41 \text{ liter} = 0.72 \text{ liter}$$

Answer: The amount of liquid needed to fill the bottle is 0.72 liter.

Having come up with the answer, you could also check your conversions using the quicker “move the decimal” method, shown below.

Example 13.6.13

A two-liter bottle contains 87 centiliters of oil and 4.1 deciliters of water. How much more liquid is needed to fill the bottle?

Solution

You are looking for the amount of liquid needed to fill the bottle. Convert both measurements to liters and then solve the problem.

$$87 \text{ cl} + 4.1 \text{ dl} + \_\_\_\_\_ = 2 \text{ l}$$

Convert 87 cl to liters.

$$87 \text{ cl} = \_\_\_\_\_ \text{l}$$

On the chart, l is two places to the left of cl.

kl hl dal ***l*** *dl* ***cl*** *ml*

Move the decimal point two places to the left in 87 cl.

.87. *cl*

87 *cl* = 0.87 *l*

Convert 4.1 dl to liters.

$$4.1 \text{ dl} = \_\_\_\_\_ \text{l}$$

On the chart, l is one place to the left of dl.

kl hl dal ***l*** ***dl*** *cl ml*

Move the decimal point one place to the left in 4.1 dl.

.4.1 *dl*

4.1 *dl* = 0.41 *l*

Subtract to find how much more liquid is needed to fill the bottle.

$$87 \text{ cl} + 4.1 \text{ dl} + \_\_\_\_\_ = 2 \text{ l}$$

$$0.87 \text{ liter} + 0.41 \text{ liter} + \_\_\_\_\_ = 2 \text{ liters}$$

$$2 \text{ liters} – 0.87 \text{ liter} – 0.41 \text{ liter} = 0.72 \text{ liter}$$

Answer: The amount of liquid needed to fill the bottle is 0.72 liter.

The initial answer checks out—0.72 liter of liquid is needed to fill the bottle. Checking one conversion with another method is a good practice for catching any errors in scale.

Summary

Understanding the context of real-life application problems is important. Look for words within the problem that help you identify what operations are needed, and then apply the correct unit conversions. Checking your final answer by using another conversion method (such as the “move the decimal” method, if you have used the factor label method to solve the problem) can cut down on errors in your calculations.

Try It Now Answer

1. 84.2 kg; 80 dag = 0.8 kg, and 85 – 0.8 = 84.2

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13.7: Temperature Scales

Learning Objectives

1. State the freezing and boiling points of water on the Celsius and Fahrenheit temperature scales.
2. Convert from one temperature scale to the other, using conversion formulas

Introduction

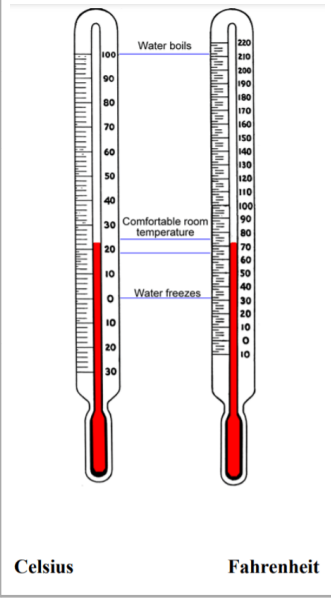
Turn on the television any morning and you will see meteorologists talking about the day's weather forecast. In addition to telling you what the weather conditions will be like (sunny, cloudy, rainy, muggy), they also tell you the day's forecast for high and low temperatures. A hot summer day may reach 100° in Philadelphia, while a cool spring day may have a low of 40° in Seattle.

If you have been to other countries, though, you may notice that meteorologists measure heat and cold differently outside of the United States. For example, a TV weatherman in San Diego may forecast a high of 89°, but a similar forecaster in Tijuana, Mexico—which is only 20 miles south— may look at the same weather pattern and say that the day's high temperature is going to be 32°. What's going on here?

The difference is that the two countries use different temperature scales. In the United States, temperatures are usually measured using the **Fahrenheit** scale, while most countries that use the metric system use the **Celsius** scale to record temperatures. Learning about the different scales— including how to convert between them—will help you figure out what the weather is going to be like, no matter which country you find yourself in.

Measuring Temperature on Two Scales

Fahrenheit and Celsius are two different scales for measuring temperature.

A thermometer measuring a temperature of 22° Celsius is shown here.		A thermometer measuring a temperature of 72° Fahrenheit is shown here.
On the Celsius scale, water freezes at 0° and boils at 100°.		On the Fahrenheit scale, water freezes at 32° and boils at 212°.
If the United States were to adopt the Celsius scale, forecast temperatures would rarely go below -30° or above 45°. (A temperature of -18° may be forecast for a cold winter day in Michigan, while a temperature of 43° may be predicted for a hot summer day in Arizona.)		In the United States, forecast temperatures measured in Fahrenheit rarely go below - 20° or above 120°. (A temperature of 0° may be forecast for a cold winter day in Michigan, while a temperature of 110° may be predicted for a hot summer day in Arizona.)
Most office buildings maintain an indoor temperature between 18°C and 24°C to keep employees comfortable.		Most office buildings maintain an indoor temperature between 65°F and 75°F to keep employees comfortable.

Try It Now 1

A cook puts a thermometer into a pot of water to see how hot it is. The thermometer reads 132°, but the water is not boiling yet. Which temperature scale is the thermometer measuring?

Converting Between the Scales

By looking at the two thermometers shown, you can make some general comparisons between the scales. For example, many people tend to be comfortable in outdoor temperatures between 50°F and 80°F (or between 10°C and 25°C). If a meteorologist predicts an average temperature of 0°C (or 32°F), then it is a safe bet that you will need a winter jacket.

Sometimes, it is necessary to convert a Celsius measurement to its exact Fahrenheit measurement or vice versa. For example, what if you want to know the temperature of your child in Fahrenheit, and the only thermometer you have measures temperature in Celsius measurement? Converting temperature between the systems is a straightforward process as long as you use the formulas provided below.

Temperature Conversion Formulas

To convert a Fahrenheit measurement to a Celsius measurement, use this formula.

$$C = \frac{5}{9}(F - 32) \tag{13.7.1}$$

To convert a Celsius measurement to a Fahrenheit measurement, use this formula.

\[F = \dfrac{9}{5}C+32 \label{CtoF \]

How were these formulas developed? They came from comparing the two scales. Since the freezing point is 0° in the Celsius scale and 32° on the Fahrenheit scale, we subtract 32 when converting from Fahrenheit to Celsius, and add 32 when converting from Celsius to Fahrenheit.

There is a reason for the fractions $\frac{5}{9}$ and $\frac{9}{5}$, also. There are 100 degrees between the freezing (0°) and boiling points (100°) of water on the Celsius scale and 180 degrees between the similar points (32° and 212°) on the Fahrenheit scale. Writing these two scales as a ratio, $\frac{F°}{C°}$, gives $\frac{180°}{100°} = \frac{180°÷20}{100°÷20} = \frac{9}{5}$. If you flip the ratio to be $\frac{C°}{F°}$, you get $\frac{100°}{180°} = \frac{100°÷20}{180°÷20} = \frac{5}{9}$. Notice how these fractions are used in the conversion formulas.

The example below illustrates the conversion of Celsius temperature to Fahrenheit temperature, using the boiling point of water, which is 100° C.

Example 13.7.1

The boiling point of water is 100°C. What temperature does water boil at in the Fahrenheit scale?

Solution

A Celsius temperature is given. To convert it to the Fahrenheit scale, use Equation ???.

$$F = \frac{9}{5}C + 32$$

Substitute 100 for C and multiply.

$$F = \frac{9}{5}(100) + 32$$

$$F = \frac{900}{5} + 32$$

Simplify $\frac{900}{5}$ by dividing numerator and denominator by 5.

$$F = \frac{900 \div 5}{5 \div 5} + 32$$

$$F = \frac{180}{1} + 32$$

Add 180 + 32.

$$F = 212$$

Answer: The boiling point of water is 212°F.

Example 13.7.2

Water freezes at 32°F. On the Celsius scale, what temperature is this?

Solution

A Fahrenheit temperature is given. To convert it to the Celsius scale, use Equation 13.7.1.

$$C = \frac{5}{9}(F - 32)$$

Substitute 32 for F and subtract.

$$C = \frac{5}{9}(32 - 32)$$

Any number multiplied by 0 is 0

$$C = \frac{5}{9}(0)$$

$$C = 0$$

Answer: The freezing point of water is 0°C.

The two previous problems used the conversion formulas to verify some temperature conversions that were discussed earlier—the boiling and freezing points of water. The next example shows how these formulas can be used to solve a real-world problem using different temperature scales.

Example 13.7.3

Two scientists are doing an experiment designed to identify the boiling point of an unknown liquid. One scientist gets a result of 120°C; the other gets a result of 250°F. Which temperature is higher and by how much?

Solution

One temperature is given in °C, and the other is given in °F. To find the difference between them, we need to measure them on the same scale.

What is the difference between 120°C and 250°F?

Use the conversion formula to convert 120°C to °F. (You could convert 250°F to °C instead; this is explained in the text after this example.)

$$F = \frac{9}{5}C + 32$$

Substitute 120 for C.

$$F = \frac{9}{5}(120) + 32$$

Multiply.

$$F = \frac{1080}{5} + 32$$

Simplify $\frac{1080}{5}$ by dividing numerator and denominator by 5.

$$F = \frac{1080 \div 5}{5 \div 5} + 32$$

Add 216 + 32.

$$F = \frac{216}{1} + 32$$

You have found that 120°C = 248°F.

$$F = 248$$

To find the difference between 248°F and 250°F, subtract.

$$250\ °F – 248\ °F = 2\ °F$$

Answer: 250°F is the higher temperature by 2°F

You could have converted 250°F to °C instead, and then found the difference in the two measurements. (Had you done it this way, you would have found that 250°F = 121.1°C, and that 121.1°C is 1.1°C higher than 120°C.) Whichever way you choose, it is important to compare the temperature measurements within the same scale, and to apply the conversion formulas accurately.

Try It Now 2

Tatiana is researching vacation destinations, and she sees that the average summer temperature in Barcelona, Spain is around 26°C. What is the average temperature in degrees Fahrenheit?

Summary

Temperature is often measured in one of two scales: the Celsius scale and the Fahrenheit scale. A Celsius thermometer will measure the boiling point of water at 100° and its freezing point at 0°; a Fahrenheit thermometer will measure the same events at 212° for the boiling point of water and 32° as its freezing point. You can use conversion formulas to convert a measurement made in one scale to the other scale.

Try It Now Answers

1. Fahrenheit; water boils at 212° on the Fahrenheit scale, so a measurement of 132° on a Fahrenheit scale is legitimate for hot (but non-boiling) water.

2. 79°F; Tatiana can find the Fahrenheit equivalent by solving the equation $F = \frac{9}{5}(26) + 32$. The result is 78.8°F, which rounds to 79°F.

Unit Recap

5.1: Length

The four basic units of measurement that are used in the U.S. customary measurement system are: inch, foot, yard, and mile. Typically, people use yards, miles, and sometimes feet to describe long distances. Measurement in inches is common for shorter objects or lengths. You need to convert from one unit of measure to another if you are solving problems that include measurements involving more than one type of measurement. Each of the units can be converted to one of the other units using the table of equivalents, the conversion factors, and/or the factor label method shown in this topic.

5.1: Weight

In the U.S. customary system of measurement, weight is measured in three units: ounces, pounds, and tons. A pound is equivalent to 16 ounces, and a ton is equivalent to 2,000 pounds. While an object's weight can be described using any of these units, it is typical to describe very heavy objects using tons and very light objects using an ounce. Pounds are used to describe the weight of many objects and people. Often, in order to compare the weights of two objects or people or to solve problems involving weight, you must convert from one unit of measurement to another unit of measurement. Using conversion factors with the factor label method is an effective strategy for converting units and solving problems.

5.1: Capacity

There are five basic units for measuring capacity in the U.S. customary measurement system. These are the fluid ounce, cup, pint, quart, and gallon. These measurement units are related to one another, and capacity can be described using any of the units. Typically, people use gallons to describe larger quantities and fluid ounces, cups, pints, or quarts to describe smaller quantities. Often, in order to compare or to solve problems involving the amount of liquid in a container, you need to convert from one unit of measurement to another.

5.2: The Metric System

The metric system is an alternative system of measurement used in most countries, as well as in the United States. The metric system is based on joining one of a series of prefixes, including kilo- , hecto-, deka-, deci-, centi-, and milli-, with a base unit of measurement, such as meter, liter, or gram. Units in the metric system are all related by a power of 10, which means that each successive unit is 10 times larger than the previous one. This makes converting one metric measurement to another a straightforward process, and is often as simple as moving a decimal point. It is always important, though, to consider the direction of the conversion. If you are converting a smaller unit to a larger unit, then the decimal point has to move to the left (making your number smaller); if you are converting a larger unit to a smaller unit, then the decimal point has to move to the right (making your number larger).

5.2: Converting within the Metric System

To convert among units in the metric system, identify the unit that you have, the unit that you want to convert to, and then count the number of units between them. If you are going from a larger unit to a smaller unit, you multiply by 10 successively. If you are going from a smaller unit to a larger unit, you divide by 10 successively. The factor label method can also be applied to conversions within the metric system. To use the factor label method, you multiply the original measurement by unit fractions; this allows you to represent the original measurement in a different measurement unit.

5.2: Using Metric Conversion to Solve Problems

Understanding the context of real-life application problems is important. Look for words within the problem that help you identify what operations are needed, and then apply the correct unit conversions. Checking your final answer by using another conversion method (such as the "move the decimal" method, if you have used the factor label method to solve the problem) can cut down on errors in your calculations.

5.3: Temperature Scales

Temperature is often measured in one of two scales: the Celsius scale and the Fahrenheit scale. A Celsius thermometer will measure the boiling point of water at 100° and its freezing point at 0°; a Fahrenheit thermometer will measure the same events at 212° for the boiling point of water and 32° as its freezing point. You can use conversion formulas to convert a measurement made in one scale to the other scale.

Glossary:

capacity	The amount of liquid (or other pourable substance) that an object can hold when it's full.
Celsius	A measure of temperature commonly used in countries that use the metric system. On the Celsius scale, water freezes at 0° and boils at 100°.
cup	A unit of capacity equal to 8 fluid ounces.
factor label method	One method of converting a measurement from one unit of measurement to another unit of measurement. In this method, you multiply the original measurement by unit fractions containing different units of measurement to obtain the new unit of measurement.
Fahrenheit	A measure of temperature commonly used in the United States. On the Fahrenheit scale, water freezes at 32° F and boils at 212° F.
fluid ounce	A unit of capacity equal to $\frac{1}{8}$ of a cup. One fluid ounce of water at 62°F weighs about one ounce.
foot	A unit for measuring length in the U.S. customary measurement system. 1 foot = 12 inches
gallon	A unit equal to 4 quarts, or 128 fluid ounces.
gram	The base unit of mass in the Metric system.
inch	A unit for measuring length in the U.S. customary measurement system. 1 foot = 12 inches
length	The distance from one end to the other or the distance from one point to another.
liter	The base unit of volume in the Metric system.
measurement	The use of standard units to find out the size or quantity of items such as length, width, height, mass, weight, volume, temperature or time.
meter	The base unit of length in the Metric system.
metric system	A widely-used system of measurement that is based on the decimal system and multiples of 10.
mile	A unit for measuring length in the U.S. customary measurement system. 1 mile = 5,280 feet or 1,760 yards.
ounce	A unit for measuring weight in the U.S. customary measurement system. 16 ounces = 1 pound.
pint	A unit of capacity equal to 16 fluid ounces, or 2 cups.
pound	A unit for measuring weight in the U.S. customary measurement system. 16 ounces = 1 pound.
prefix	A short set of letters that denote the size of measurement units in the Metric System. Metric prefixes include centi-, milli-, kilo-, and hecto-.
quart	A unit of capacity equal to 32 fluid ounces, or 4 cups.
ton	A unit for measuring the weight of heavier items in the U.S. customary measurement system. 1 ton = 2,000 pounds.
U.S. customary measurement system	The most common system of measurement used in the United States. It is based on English measurement systems of the 18th century.
unit equivalents	Statements of equivalence between measurement units within a system or in comparison to another system of units. For example, 1 foot = 12 inches or 1 inch = 2.54 centimeters are both examples of unit equivalents.

unit fractions	A fraction where the numerator and denominator are equal amounts, as in $\frac{1\text{ kg}}{1000\text{ g}}$ or $\frac{12\text{ inches}}{1\text{ foot}}$. Unit fractions serve to help with conversions in the Factor Label method.
unit of measurement	A standard amount or quantity. For example, an inch is a unit of measurement.
weight	A mathematical description of how heavy an object is.
yard	A unit for measuring length in the U.S. customary measurement system. 1 yard = 3 feet or 36 inches.

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13.8: Temperature and Time

6.1.4 Learning Objectives

- Perform arithmetic calculations on temperature and time
- Convert from one unit of measure to another unit of measure.
- Solve application problems using each form of measurement.

Definition: Time, Temperature

Temperature is used to describe the level of heat or cold of an object or environment.

Time is used to describe the amount of existence between events.

Some of the common units (along with their abbreviations) for temperature and time are listed in the following table.

Unit Conversion Table	
Temperature	Fahrenheit $(^\circ F) = \frac{9}{5}$°C$+32$ Kelvin (K)$=$°C$+273.15$ Celsius (°C) $= \frac{5}{9}$(°F-32) Celsius (°C)$=$K-273.15
Time	1 minute (min) = 60 seconds (sec) 1 hour (hr) = 60 minutes 1 day (da) = 24 hours 1 week (wk) = 7 days 1 year (yr)= 52 weeks

Example 1

Convert 73°F to Celsius.

Solution

To convert from Fahrenheit to Celsius, we need the equation (°C) $= \frac{5}{9}$ °F-32). Substitute in the given value for Fahrenheit to find the answer. Round to the nearest degree.

(°C) $= \frac{5}{9}$°F-32(=

\dfrac{5}{9}$73^\circ F$- 32

=\dfrac{5}{9} \times 41=\dfrac{41 \times 5}{9}=23.7\approx 24^\circ F

Thus, 73°F ≈ 24°C\)

Example 2

Convert 3763 K to Fahrenheit.

Solution

First we need to covert to Celsius, then we can convert to Fahrenheit. Round to two decimal places.

$(\ ^\circ\text{C}) = \text{K} - 273.15 = 3763 + 273.15 = 3489.85\ ^\circ\text{C}$

$(\ ^\circ\text{F}) = \frac{9}{5}\ ^\circ\text{C} + 32 = \frac{9}{5}3489.85 + 32 = \frac{9 \times 3489.85}{5} + 32 = 6281.73 + 32 = 6313.73^\circ\text{F}$

Thus, 3763 K $= 6313.73^\circ$F

Example 3

Convert 2,016 hr to weeks.

Solution

Looking in the unit conversion table under *time*, we see that $1\ \text{wk} = 7\ \text{da}$ and that $1\ \text{da} = 24\ \text{hr}$. To convert from hours to weeks, we must first convert from hours to days and then from days to weeks. We need two unit fractions.

The unit fraction needed for converting from hours to days is $\frac{1\ \text{da}}{24\ \text{hr}}$. The unit fraction needed for converting from days to weeks is $\frac{1\ \text{wk}}{7\ \text{da}}$.

$$\begin{aligned} 2{,}016\ \text{hr} &= \frac{2{,}016\ \text{hr}}{1} \cdot \frac{1\ \text{da}}{24\ \text{hr}} \cdot \frac{1\ \text{wk}}{7\ \text{da}} && \text{Divide out common units.} \\ &= \frac{2{,}016\ \cancel{\text{hr}}}{1} \cdot \frac{1\ \cancel{\text{da}}}{24\ \cancel{\text{hr}}} \cdot \frac{1\ \text{wk}}{7\ \cancel{\text{da}}} \\ &= \frac{2{,}016 \cdot 1\ \text{wk}}{24 \cdot 7} && \text{Simplify} \\ &= 12\ \text{wk} \end{aligned}$$

Thus, $2{,}016\ \text{hr} = 12\ \text{wk}$.

Example 4

Your employer tracks your accumulated vacation time in hours. You want to take a two week vacation and have accumulated 412 hours of paid time off. Do you have enough hours for your vacation?

Solution

To solve this we need to covert 412 hours to weeks.

The unit fraction needed for converting from hours to days is $\frac{1\ \text{da}}{24\ \text{hr}}$. The unit fraction needed for converting from days to weeks is $\frac{1\ \text{wk}}{7\ \text{da}}$.

$$\begin{aligned} 412\ \text{hr} &= \frac{412\ \text{hr}}{1} \cdot \frac{1\ \text{da}}{24\ \text{hr}} \cdot \frac{1\ \text{wk}}{7\ \text{da}} && \text{Divide out common units.} \\ &= \frac{412\ \cancel{\text{hr}}}{1} \cdot \frac{1\ \cancel{\text{da}}}{24\ \cancel{\text{hr}}} \cdot \frac{1\ \text{wk}}{7\ \cancel{\text{da}}} \\ &= \frac{412 \cdot 1\ \text{wk}}{24 \cdot 7} && \text{Simplify} \\ &= 2.5\ \text{wk} \end{aligned}$$

Thus, $412 \text{ hr} = 2.5 \text{ wk}$.

Yes, you have enough time for a 2.5 week vacation.

Try It Now 1

Make the following conversions. If a fraction occurs, convert it to a decimal rounded to two decimal places.

1. Convert 32°C to °F
2. Convert 1732°F to K
3. Convert 52 min to hours.

Answer available at the end of this section.

Answers

Try It Now 1

1. 89.6°F
2. 1217.59 K
3. 0.87 hr

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13.8.1: Exercises

1. What unit of measurement do these abbreviations belong to?
   a. K
   b. da
   c. wk
   d. F
   e. C
   f. yr
   g. sec
   h. min
   i. hr
2. Convert the following:
   a. 21°C to °F
   b. 1736 K to °C
   c. 81°F to °C
   d. 789°C to K
   e. 235,000 sec to yr
   f. 67 hr to da
   g. 19 wk to hr
   h. 4 da to min
3. Australia uses Celsius to measure temperature. Your travel agent said it will be between 23° C and 27° C during your visit. What is the temperature range in Fahrenheit?
4. You want to binge watch a TV show that has 11 episodes that are 45 minutes each. How many hours do you need to set aside this weekend to watch the entire series?
5. A friend from Switzerland is coming to visit the U.S. in December. You tell them it will be about 35° C when they're here. You forgot to tell them that the measurement was in degrees Fahrenheit. They are used to Celsius and assume that is the temperature you told them. Are they planning for cold or hot weather? What Celsius temperature, should you have told them?
6. Your aunt always tells you her new child's age in weeks. The last time you saw them she said her child was 93 weeks old. How much is that in years?
7. To tell your colleague in Europe the temperature you are currently experiencing, you need to tell it to them in Celsius. If it is currently 64°F, what would you tell them?
8. The photosphere of the Sun is approximately 5778 K. How much is that in Celsius?

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13.9: Dilations on a Square Grid

Lesson

Let's dilate figures on a square grid.

Exercise 13.9.1: Estimating a Scale Factor

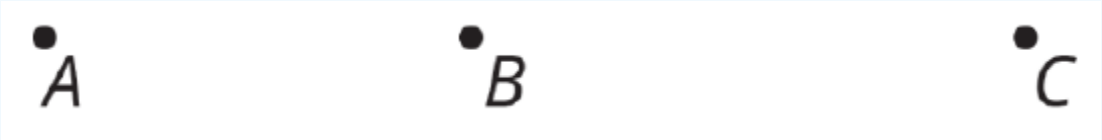


Figure 13.9.1

Point C is the dilation of point B with center of dilation and scale factor . Estimate . Be prepared to explain your reasoning.

Exercise 13.9.2: Dilations on a Grid

1. Find the dilation of quadrilateral $ABCD$ with center P and scale factor 2.

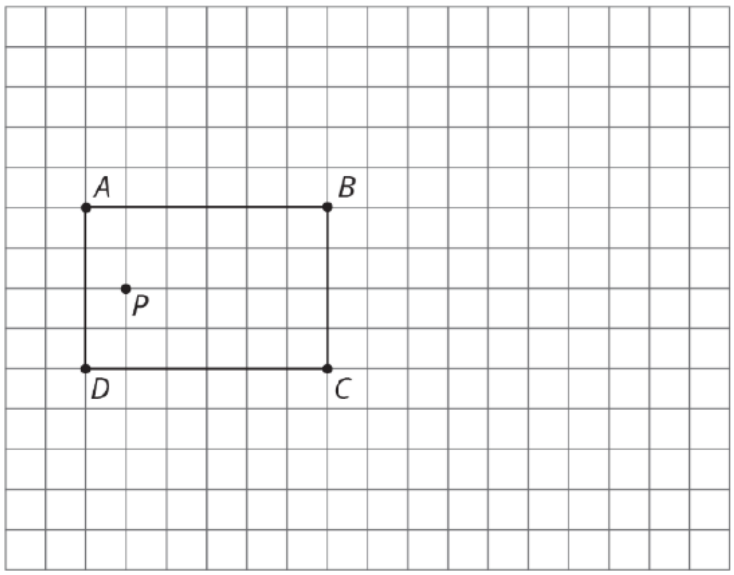


Figure 13.9.2

2. Find the dilation of triangle QRS with center T and scale factor 2.

3. Find the dilation of triangle QRS with center T and scale factor $\frac{1}{2}$.

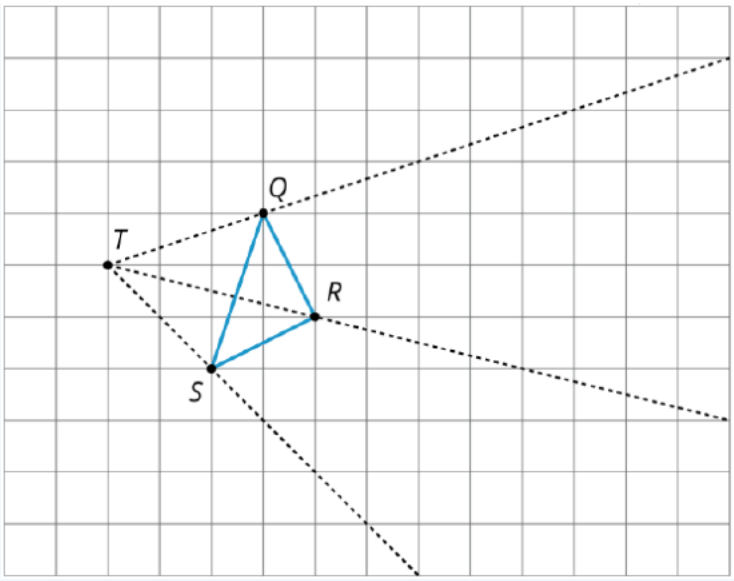


Figure 13.9.3: Point T, triangle Q R S and three projection rays on a square grid. Let the lower left corner be (0 comma 0). Then triangle Q R S is Q(5 comma 7), R(6 comma 5), S(4 comma 4) and point T is T(2 comma 6). The dashed projection rays are T Q, T R and T S.

Exercise 13.9.3: Card Sort: Matching Dilations on a Coordinate Grid

Your teacher will give you some cards. Each of Cards 1 through 6 shows a figure in the coordinate plane and describes a dilation.

Each of Cards A through E describes the image of the dilation for one of the numbered cards.

Match number cards with letter cards. One of the number cards will not have a match. For this card, you'll need to draw an image.

Are you ready for more?

The image of a circle under dilation is a circle when the center of the dilation is the center of the circle. What happens if the center of dilation is a point on the circle? Using center of dilation $(0, 0)$ and scale factor 1.5, dilate the circle shown on the diagram. This diagram shows some points to try dilating.

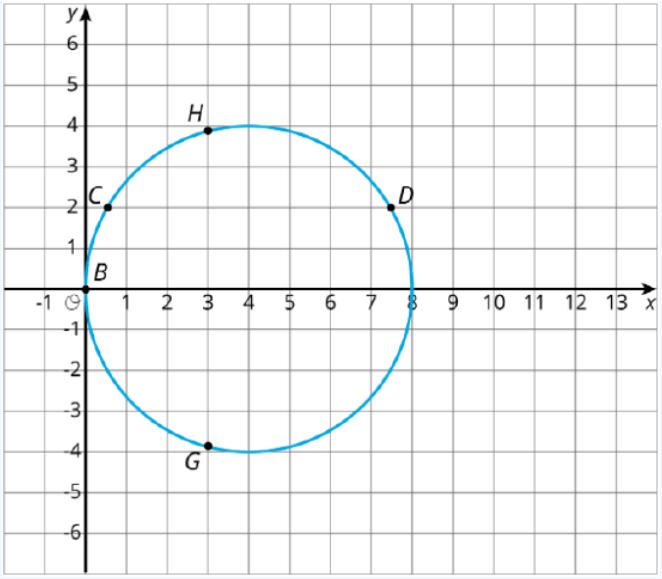


Figure 13.9.4: Five points on a circle on a coordinate plane, origin O. Horizontal axis scale negative 1 to 13 by 1's. Vertical axis scale negative 6 to 6 by 1's. Circle has radius 4 and is centered at (4 comma 0). The five points on the circle are B(0 comma 0), C(point 5 comma 2), H(3 comma 4), D(7 point 5 comma 2) and G(3 comma negative 4).

Summary

Square grids can be useful for showing dilations. The grid is helpful especially when the center of dilation and the point(s) being dilated lie at grid points. Rather than using a ruler to measure the distance between the points, we can count grid units.

For example, suppose we want to dilate point Q with center of dilation P and scale factor $\frac{3}{2}$. Since Q is 4 grid squares to the left and 2 grid squares down from P, the dilation will be 6 grid squares to the left and 3 grid squares down from P' (can you see why?). The dilated image is marked as Q' in the picture.

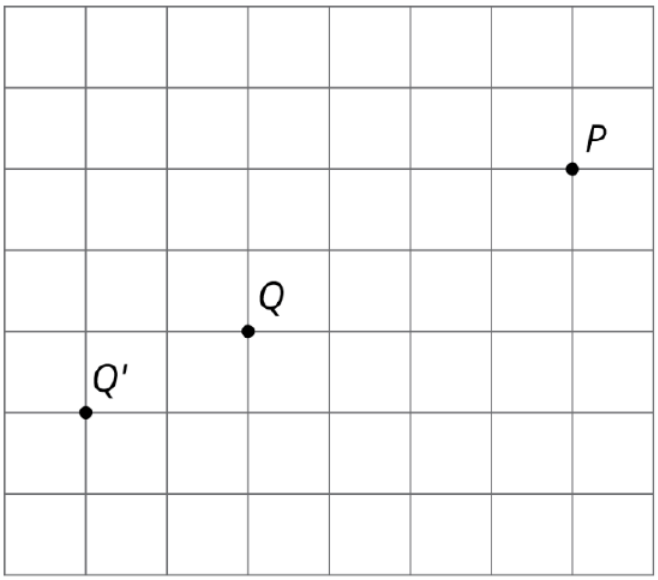


Figure 13.9.5

Sometimes the square grid comes with coordinates. The coordinate grid gives us a convenient way to *name* points, and sometimes the coordinates of the image can be found with just arithmetic.

For example, to make a dilation with center $(0, 0)$ and scale factor 2 of the triangle with coordinates $(-1, -2)$, $(3, 1)$, and $(2, -1)$, we can just double the coordinates to get $(-2, -4)$, $(6, 2)$, and $(4, -2)$.

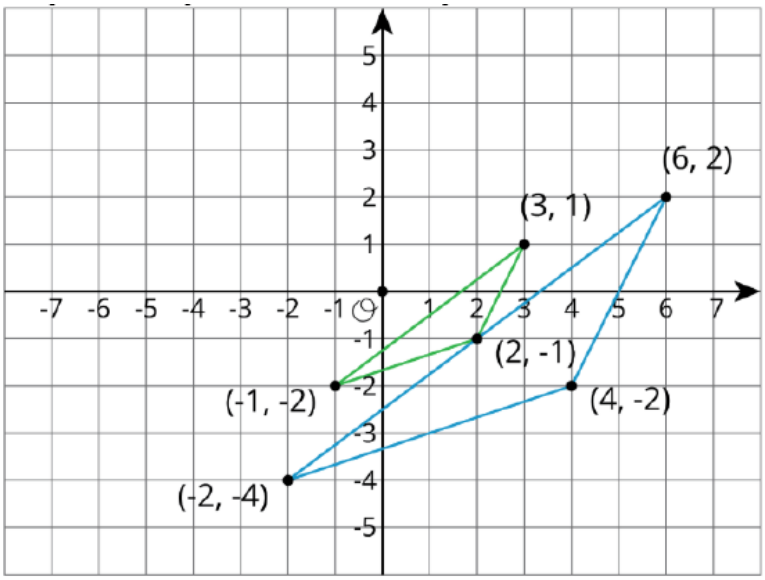


Figure 13.9.6: Two triangles on a coordinate plane, origin O. Horizontal axis scale negative 7 to 7 by 1's. Vertical axis scale negative 5 to 5 by 1's. The coordinates of the triangle are (negative 1 comma negative 2), (3 comma 1), (2 comma negative 1). The coordinates of the image are(negative 2 comma negative 4), (6 comma 2), (4 comma negative 2).

Glossary

Definition: Center of a Dilation

The center of a dilation is a fixed point on a plane. It is the starting point from which we measure distances in a dilation.

In this diagram, point P is the center of the dilation.

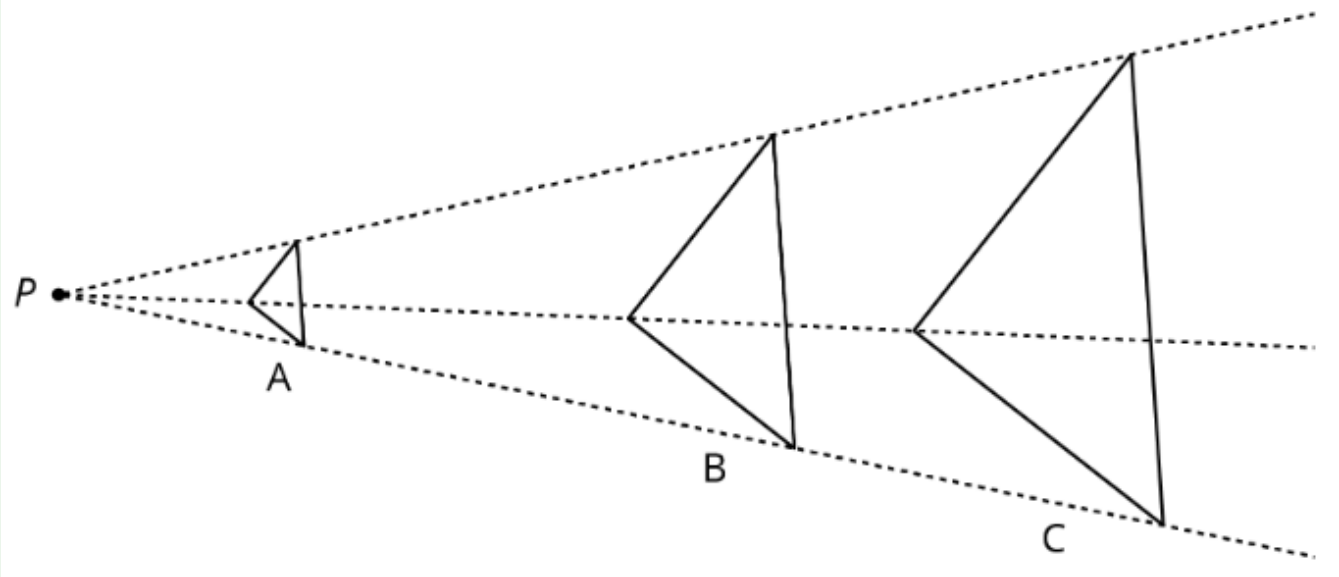


Figure 13.9.4

Definition: Dilation

A dilation is a transformation in which each point on a figure moves along a line and changes its distance from a fixed point. The fixed point is the center of the dilation. All of the original distances are multiplied by the same scale factor.

For example, triangle DEF is a dilation of triangle ABC. The center of dilation is O and the scale factor is 3.

This means that every point of triangle DEF is 3 times as far from O as every corresponding point of triangle ABC.

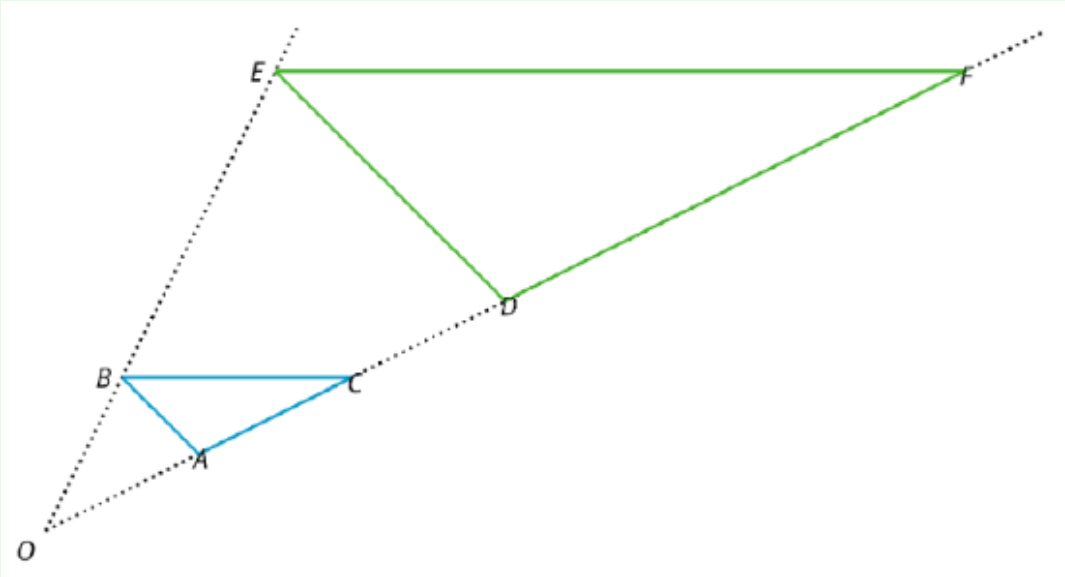


Figure 13.9.5

Definition: Scale Factor

To create a scaled copy, we multiply all the lengths in the original figure by the same number. This number is called the scale factor.

In this example, the scale factor is 1.5, because $4 \cdot (1.5) = 6$, $5 \cdot (1.5) = 7.5$, and $6 \cdot (1.5) = 9$.

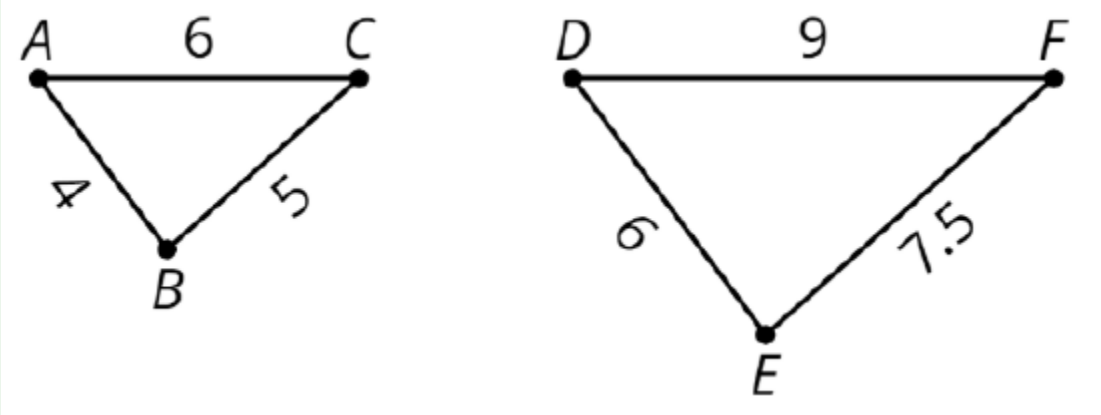


Figure 13.9.6

Practice

Exercise 13.9.4

Triangle ABC is dilated using D as the center of dilation with scale factor 2.

The image is triangle $A'B'C'$. Clare says the two triangles are congruent, because their angle measures are the same. Do you agree? Explain how you know.

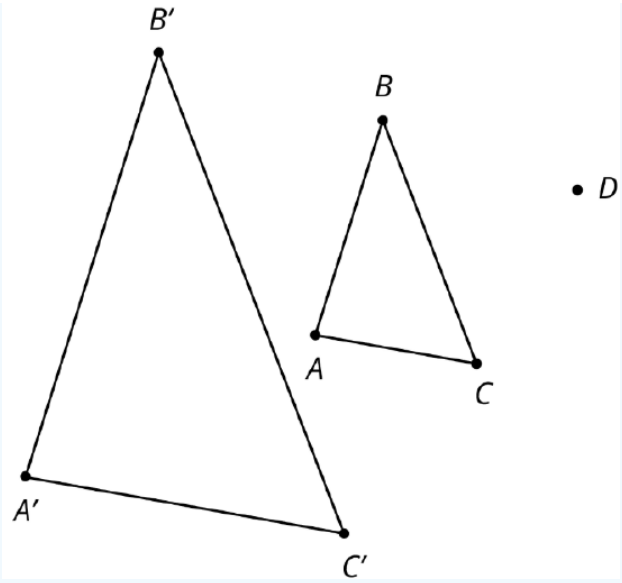


Figure 13.9.7: Triangle A B C, it's image after dilation triangle A prime B prime C prime and point D. Point D is on the right, smaller triangle A B C is in the middle and larger image triangle A prime, B prime and C prime is on the right.

Exercise 13.9.5

On graph paper, sketch the image of quadrilateral PQRS under the following dilations:

1. The dilation centered at R with scale factor 2.
2. The dilation centered at O with scale factor $\frac{1}{2}$.
3. The dilation centered at S with scale factor $\frac{1}{2}$.

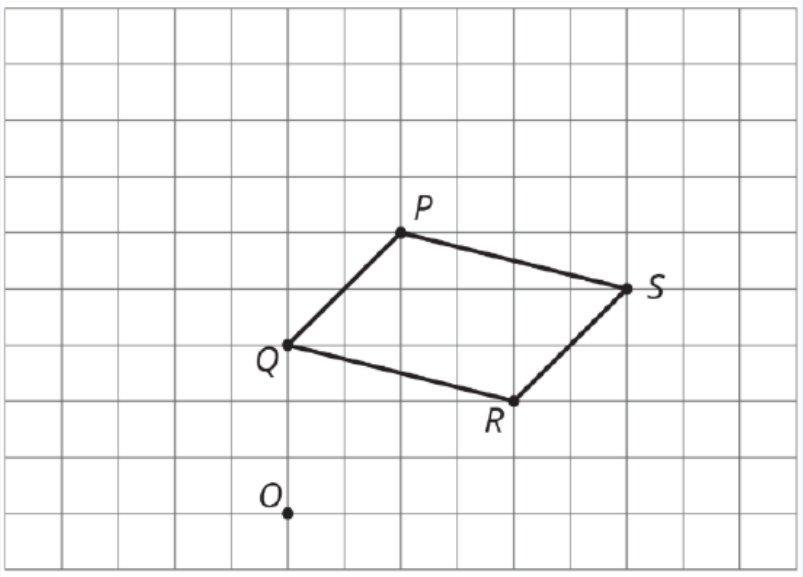


Figure 13.9.8: Quadrilateral P Q R S and point O on a square grid. Let the lower left corner be (0 comma 0). Then the coordinates of P Q R S are P(7 comma 6), Q(5 comma 4), R(9 comma 3), S(11 comma 5). The coordinates of point O are O(5 comma 1).

Exercise 13.9.6

The diagram shows three lines with some marked angle measures.

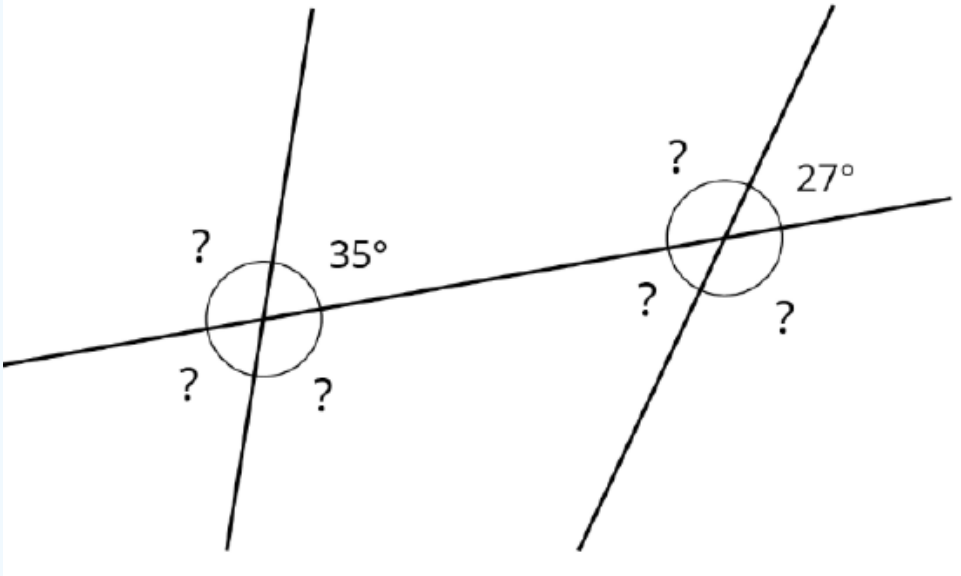


Figure 13.9.9:

A line slanting up and to the right. Two lines intersect this line, both slanting up and to the right, but do not intersect. At the first intersection, the top right angle is labeled 35 degrees. The rest are labeled ?. At the second intersection, the top right angle is labeled 27 degrees. The rest are labeled ?.

Find the missing angle measures marked with question marks.

(From Unit 1.4.4)

Exercise 13.9.7

Describe a sequence of translations, rotations, and reflections that takes Polygon P to Polygon Q.

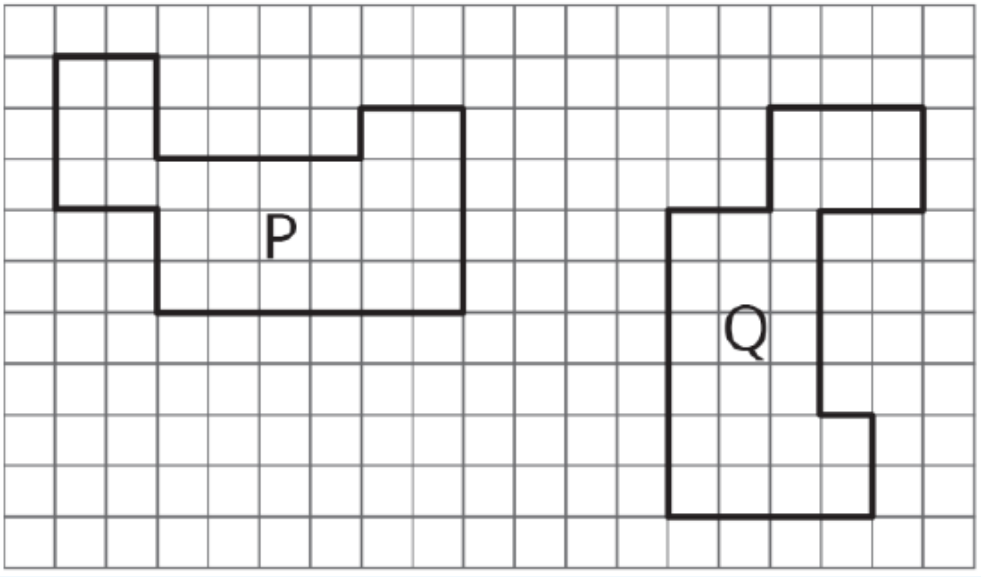


Figure 13.9.10

(From Unit 1.1.4)

Exercise 13.9.8

Point B has coordinates $(-2, -5)$. After a translation 4 units down, a reflection across the -axis, and a translation 6 units up, what are the coordinates of the image?

(From Unit 1.1.6)

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13.10: More Dilations

Lesson

Let's look at dilations in the coordinate plane.

Exercise 13.10.1: Many Dilations of a Triangle

Explore the applet and observe the dilation of triangle ABC. The dilation always uses center P, but you can change the scale factor. What connections can you make between the scale factor and the dilated triangle?

Exercise 13.10.2: Info Gap: Dilations

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.
4. Continue to ask questions until you have enough information to solve the problem.
5. Share the *problem card* and solve the problem independently.
6. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner *"What specific information do you need?"* and wait for them to *ask* for information.

   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.

3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Are you ready for more?

Triangle EFG was created by dilating triangle ABC using a scale factor of 2 and center D. Triangle HIJ was created by dilating triangle ABC using a scale factor of $\frac{1}{2}$ and center D.

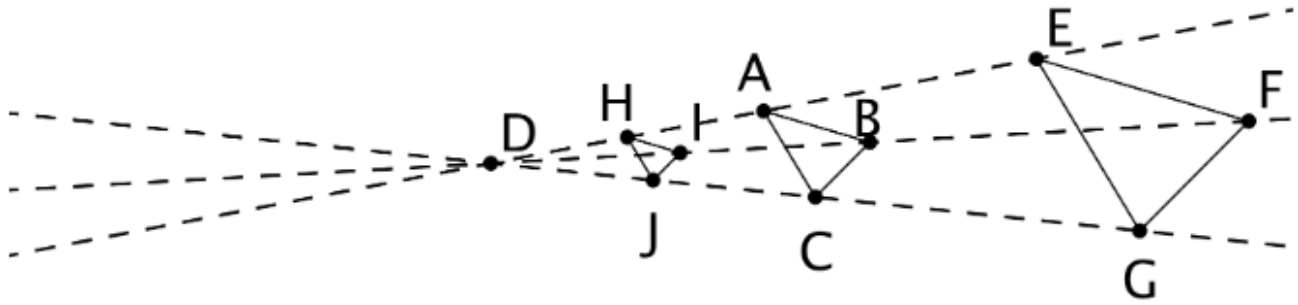


Figure 13.10.1: A triangle A B C, two images after dilation, point D and three dashed projection lines. The dashed lines meet near the middle at point D, then to right are the increasing in size triangles, H I J, A B C and E F G. The upper line goes through points H, A and E. The nearly horizontal middle line goes through points I, B and F. The lower line goes through points J, C and G.

1. What would the image of triangle ABC look like under a dilation with scale factor 0?
2. What would the image of the triangle look like under dilation with a scale factor of -1? If possible, draw it and label the vertices A', B', and C'. If it's not possible, explain why not.
3. If possible, describe what happens to a shape if it is dilated with a negative scale factor. If dilating with a negative scale factor is not possible, explain why not.

Summary

One important use of coordinates is to communicate geometric information precisely. Let's consider a quadrilateral $ABCD$ in the coordinate plane. Performing a dilation of $ABCD$ requires three vital pieces of information:

1. The coordinates of A, B, C, and D
2. The coordinates of the center of dilation, P
3. The scale factor of the dilation

With this information, we can dilate the vertices A, B, C, and D and then draw the corresponding segments to find the dilation of $ABCD$. Without coordinates, describing the location of the new points would likely require sharing a picture of the polygon and the center of dilation.

Glossary Entries

Definition: Center of a Dilation

The center of a dilation is a fixed point on a plane. It is the starting point from which we measure distances in a dilation.

In this diagram, point P is the center of the dilation.

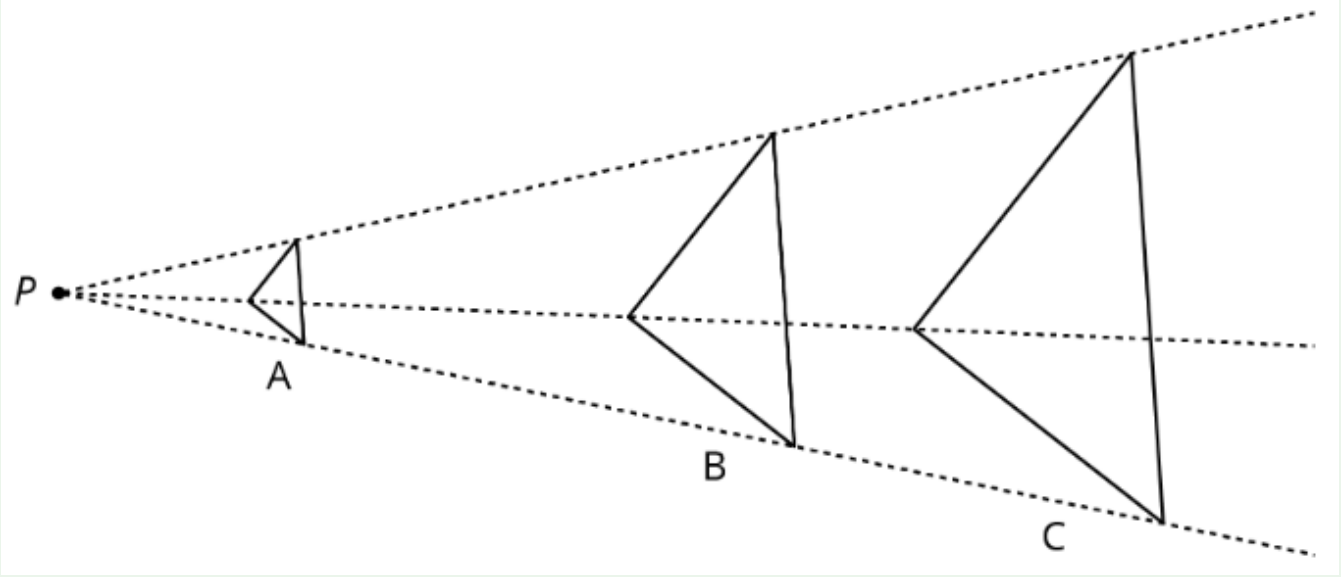


Figure 13.10.2

Definition: Dilation

A dilation is a transformation in which each point on a figure moves along a line and changes its distance from a fixed point. The fixed point is the center of the dilation. All of the original distances are multiplied by the same scale factor.

For example, triangle DEF is a dilation of triangle ABC. The center of dilation is O and the scale factor is 3.

This means that every point of triangle DEF is 3 times as far from O as every corresponding point of triangle ABC.

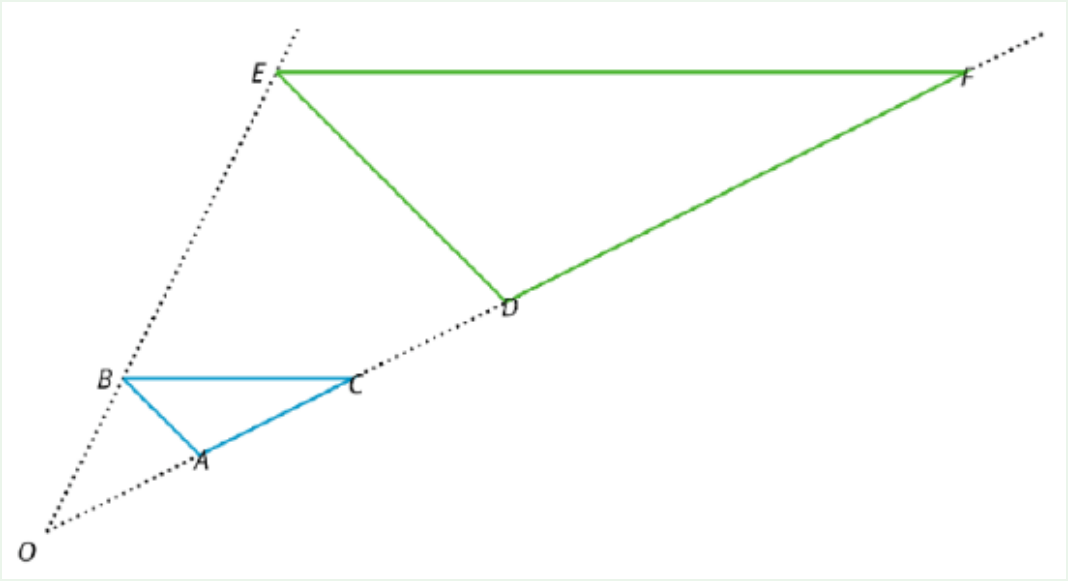


Figure 13.10.3

Definition: Scale Factor

To create a scaled copy, we multiply all the lengths in the original figure by the same number. This number is called the scale factor.

In this example, the scale factor is 1.5, because $4 \cdot (1.5) = 6$, $5 \cdot (1.5) = 7.5$, and $6 \cdot (1.5) = 9$.

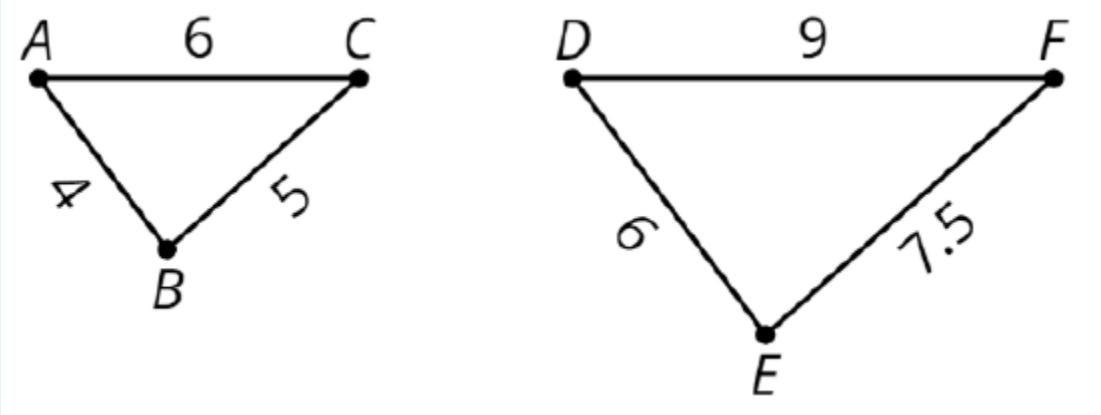


Figure 13.10.4

Practice

Exercise 13.10.3

Quadrilateral $ABCD$ is dilated with center $(0, 0)$ taking B to B'. Draw $A'B'C'D'$.

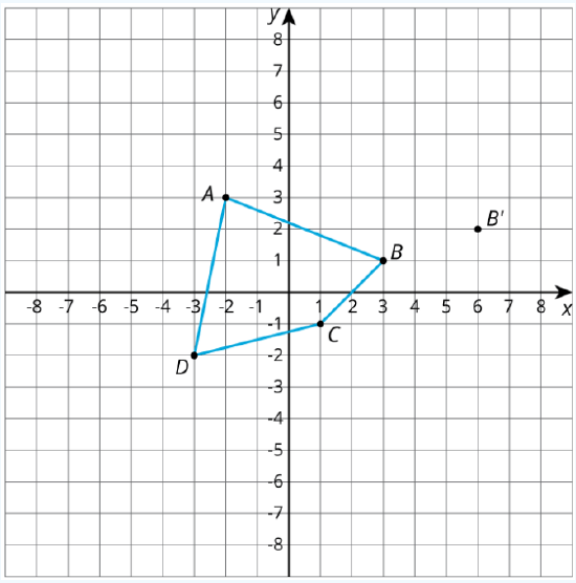


Figure 13.10.5

Exercise 13.10.4

Triangles B and C have been built by dilating Triangle A.

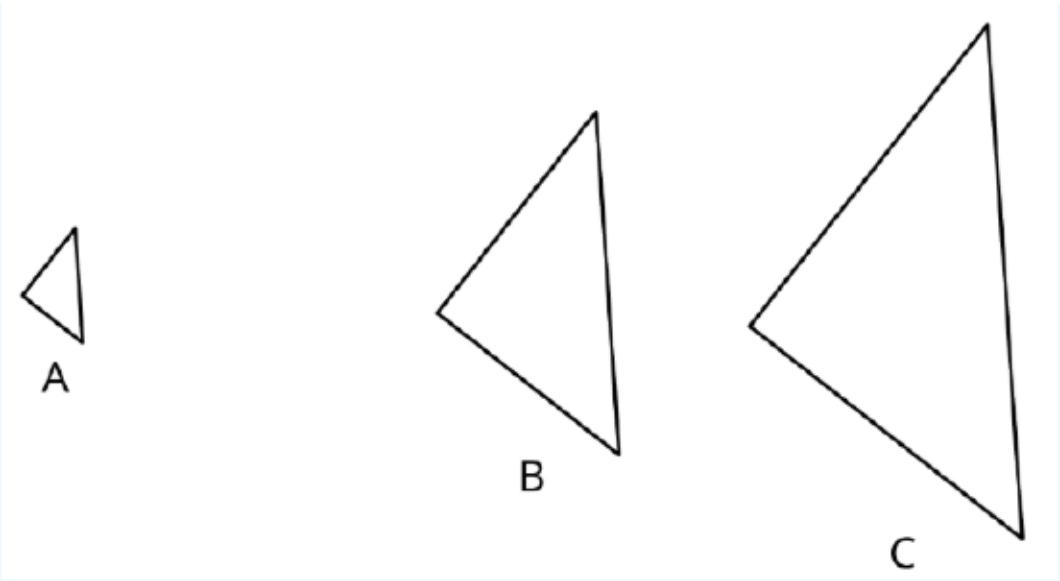


Figure 13.10.6

1. Find the center of dilation.
2. Triangle B is a dilation of A with approximately what scale factor?
3. Triangle A is a dilation of B with approximately what scale factor?
4. Triangle B is a dilation of C with approximately what scale factor?

Exercise 13.10.5

Here is a triangle.

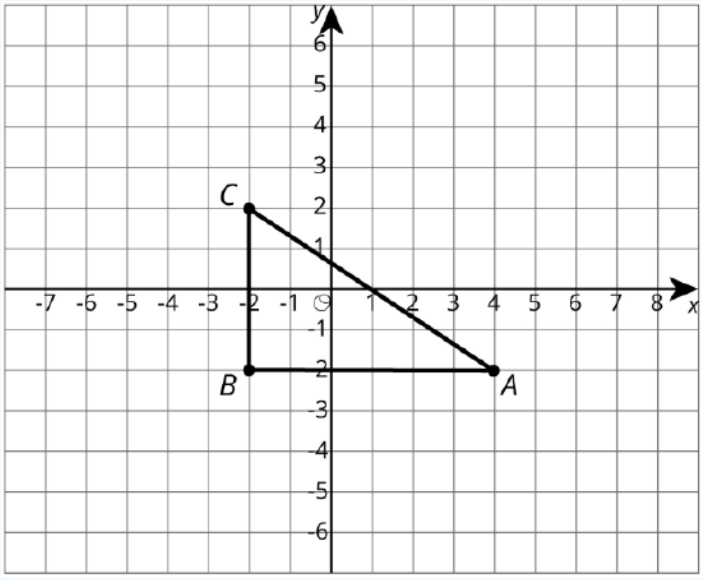


Figure 13.10.7

1. Draw the dilation of triangle ABC, with center $(0,0)$, and scale factor 2. Label this triangle $A'B'C'$.
2. Draw the dilation of triangle ABC, with center $(0,0)$, and scale factor $\frac{1}{2}$. Label this triangle $A''B''C''$.
3. Is $A''B''C''$ a dilation of triangle $A'B'C'$? If yes, what are the center of dilation and the scale factor?

Exercise 13.10.1

Triangle DEF is a right triangle, and the measure of angle D is 28°. What are the measures of the other two angles?

(From Unit 1.4.2)

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14: Transformations

Transformations

- 14-1 Translations, Rotations, and Tessellations
- 14-2 Reflections and Glide Reflections
- 14-3 Dilations 935
- Chapter 14 Review

Links we'd like to aggregate:

Transformations

- 14-1 Translations, Rotations, and Tessellations
  1. https://www.ck12.org/book/ck-12-geometry-second-edition/section/12.1/
  2. https://www.ck12.org/book/ck-12-geometry-second-edition/section/12.2/
  3. https://www.ck12.org/book/ck-12-geometry-second-edition/section/12.4/
  4. https://www.ck12.org/book/ck-12-geometry-second-edition/section/12.6/
- 14-2 Reflections and Glide Reflections
  1. https://www.ck12.org/book/ck-12-geometry-second-edition/section/12.3/

https://www.ck12.org/book/ck-12-geometry-second-edition/section/12.5/

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