SWC BUSINESS CALCULUS



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Business Calculus

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Licensing

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CHAPTER OVERVIEW

Unit 1: Pre-Calculus and Systems

1.1: Review of Functions

1.1E: Exercises for Section 1.1

1.2: Linear Functions

1.3: Systems of Linear Equations - Two Variables

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1.1: Review of Functions

Learning Objectives

- Use functional notation to evaluate a function.
- Determine the domain and range of a function.
- Draw the graph of a function.
- Find the zeros of a function.
- Recognize a function from a table of values.
- Make new functions from two or more given functions.
- Describe the symmetry properties of a function.

In this section, we provide a formal definition of a function and examine several ways in which functions are represented—namely, through tables, formulas, and graphs. We study formal notation and terms related to functions. We also define composition of functions and symmetry properties. Most of this material will be a review for you, but it serves as a handy reference to remind you of some of the algebraic techniques useful for working with functions.

1.1.1 Functions

Given two sets A and B a set with elements that are ordered pairs (x, y) where x is an element of A and y is an element of B, is a relation from A to B. A relation from A to B defines a relationship between those two sets. A function is a special type of relation in which each element of the first set is related to exactly one element of the second set. The element of the first set is called the ***input***; the element of the second set is called the ***output***. Functions are used all the time in mathematics to describe relationships between two sets. For any function, when we know the input, the output is determined, so we say that the output is a function of the input. For example, the area of a square is determined by its side length, so we say that the area (the output) is a function of its side length (the input). The velocity of a ball thrown in the air can be described as a function of the amount of time the ball is in the air. The cost of mailing a package is a function of the weight of the package. Since functions have so many uses, it is important to have precise definitions and terminology to study them.



Figure 1.1.1: A function can be visualized as an input/output device

Definition: Functions

A *function* f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the *domain* of the function. The set of outputs is called the *range* of the *function*.

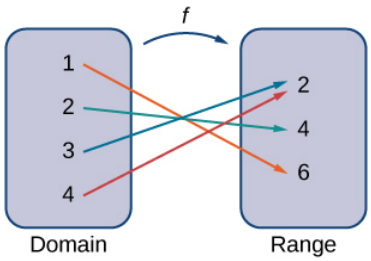


Figure 1.1.2: A function maps every element in the domain to exactly one element in the range. Although each input can be sent to only one output, two different inputs can be sent to the same output.

For example, consider the function f, where the domain is the set of all real numbers and the rule is to square the input. Then, the input $x = 3$ is assigned to the output $3^2 = 9$.

Since every nonnegative real number has a real-value square root, every nonnegative number is an element of the range of this function. Since there is no real number with a square that is negative, the negative real numbers are not elements of the range. We

conclude that the range is the set of nonnegative real numbers.

For a general function f with domain D, we often use x to denote the input and y to denote the output associated with x. When doing so, we refer to x as the **independent variable** and y as the **dependent variable**, because it depends on x. Using function notation, we write $y = f(x)$, and we read this equation as "y equals f of x." For the squaring function described earlier, we write $f(x) = x^2$.

The concept of a function can be visualized using Figures 1.1.1 - 1.1.3.

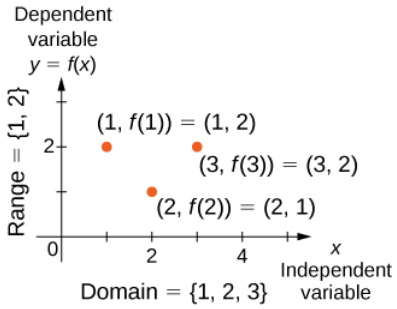


Figure 1.1.3: In this case, a graph of a function f has a domain of $\{1,2,3\}$ and a range of $\{1,2\}$. The independent variable is x and the dependent variable is y.

We can also visualize a function by plotting points (x, y) in the coordinate plane where $y = f(x)$. The graph of a function is the set of all these points. For example, consider the function f, where the domain is the set $D = \{1, 2, 3\}$ and the rule is $f(x) = 3 - x$. In Figure 1.1.4, we plot a graph of this function.

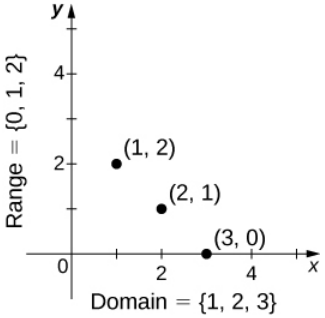


Figure 1.1.4: Here we see a graph of the function f with domain $\{1,2,3\}$ and rule $f(x) = 3 - x$. The graph consists of the points $(x, f(x))$ for all x in the domain.

Every function has a domain. However, sometimes a function is described by an equation, as in $f(x) = x^2$, with no specific domain given. In this case, the domain is taken to be the set of all real numbers x for which $f(x)$ is a real number. For example, since any real number can be squared, if no other domain is specified, we consider the domain of $f(x) = x^2$ to be the set of all real numbers. On the other hand, the square root function $f(x) = \sqrt{x}$ only gives a real output if x is nonnegative. Therefore, the domain of the function $f(x) = \sqrt{x}$ is the set of nonnegative real numbers, sometimes called the natural domain.

For the functions $f(x) = x^2$ and $f(x) = \sqrt{x}$, the domains are sets with an infinite number of elements. Clearly we cannot list all these elements. When describing a set with an infinite number of elements, it is often helpful to use set-builder or interval notation. When using set-builder notation to describe a subset of all real numbers, denoted R, we write

$$\{x \mid x \, has \, some \, property\}.$$

We read this as the set of real numbers x such that x has some property. For example, if we were interested in the set of real numbers that are greater than one but less than five, we could denote this set using set-builder notation by writing

$$\{x \mid 1 < x < 5\}.$$

A set such as this, which contains all numbers greater than a and less than b, can also be denoted using the *interval notation* (a, b). Therefore,

$$(1, 5) = \{x \mid 1 < x < 5\}.$$

The numbers 1 and 5 are called the endpoints of this set. If we want to consider the set that includes the endpoints, we would denote this set by writing

$$[1,5]=\{x\mid 1\le x\le 5\}.$$

We can use similar notation if we want to include one of the endpoints, but not the other. To denote the set of nonnegative real numbers, we would use the set-builder notation

$$\{x\mid x\ge 0\}.$$

The smallest number in this set is zero, but this set does not have a largest number. Using interval notation, we would use the symbol ∞, which refers to positive infinity, and we would write the set as

$$[0,\infty)=\{x\mid x\ge 0\}.$$

It is important to note that ∞ is not a real number. It is used symbolically here to indicate that this set includes all real numbers greater than or equal to zero. Similarly, if we wanted to describe the set of all nonpositive numbers, we could write

$$(-\infty,0]=\{x\mid x\le 0\}.$$

Here, the notation $-\infty$ refers to negative infinity, and it indicates that we are including all numbers less than or equal to zero, no matter how small. The set

$$(-\infty,\infty)=\{x\mid x\ is\ any\ real\ number\}$$

refers to the set of all real numbers. Some functions are defined using different equations for different parts of their domain. These types of functions are known as piecewise-defined functions. For example, suppose we want to define a function f with a domain that is the set of all real numbers such that $f(x)=3x+1$ for $x\ge 2$ and $f(x)=x^2$ for $x<2$. We denote this function by writing

$$f(x)=\begin{cases}3x+1, & \text{if } x\ge 2\\ x^2, & \text{if } x<2\end{cases}$$

When evaluating this function for an input x, the equation to use depends on whether $x\ge 2$ or $x<2$. For example, since $5>2$, we use the fact that $f(x)=3x+1$ for $x\ge 2$ and see that $f(5)=3(5)+1=16$. On the other hand, for $x=-1$, we use the fact that $f(x)=x^2$ for $x<2$ and see that $f(-1)=1$.

Example 1.1.1: Evaluating Functions

For the function $f(x)=3x^2+2x-1$, evaluate:

a. $f(-2)$
b. $f(\sqrt{2})$
c. $f(a+h)$

Solution

Substitute the given value for x in the formula for $f(x)$.

a. $f(-2)=3(-2)^2+2(-2)-1=12-4-1=7$
b. $f(\sqrt{2})=3(\sqrt{2})^2+2\sqrt{2}-1=6+2\sqrt{2}-1=5+2\sqrt{2}$
c. $f(a+h)=3(a+h)^2+2(a+h)-1=3(a^2+2ah+h^2)+2a+2h-1=3a^2+6ah+3h^2+2a+2h-1$

Exercise 1.1.1

For $f(x)=x^2-3x+5$, evaluate $f(1)$ and $f(a+h)$.

Hint

Substitute 1 and $a+h$ for x in the formula for $f(x)$.

Answer

$f(1)=3$ and $f(a+h)=a^2+2ah+h^2-3a-3h+5$

Example 1.1.2: Finding Domain and Range

For each of the following functions, determine the i. domain and ii. range.

a. $f(x)=(x-4)^2+5$
b. $f(x)=\sqrt{3x+2}-1$
c. $f(x)=\dfrac{3}{x-2}$

Solution

a. Consider $f(x)=(x-4)^2+5$.

1.Since $f(x)=(x-4)^2+5$ is a real number for any real number x, the domain of f is the interval $(-\infty,\infty)$.

2. Since $(x-4)^2\geq 0$, we know $f(x)=(x-4)^2+5\geq 5$. Therefore, the range must be a subset of $\{y\,|\,y\geq 5\}$. To show that every element in this set is in the range, we need to show that for a given y in that set, there is a real number x such that $f(x)=(x-4)^2+5=y$. Solving this equation for x, we see that we need x such that

$$(x-4)^2=y-5.$$

This equation is satisfied as long as there exists a real number x such that

$$x-4=\pm\sqrt{y-5}$$

Since $y\geq 5$, the square root is well-defined. We conclude that for $x=4\pm\sqrt{y-5}$, $f(x)=y$, and therefore the range is $\{y\,|\,y\geq 5\}$.

b. Consider $f(x)=\sqrt{3x+2}-1$.

1.To find the domain of f, we need the expression $3x+2\geq 0$. Solving this inequality, we conclude that the domain is $\{x\,|\,x\geq -2/3\}$.

2.To find the range of f, we note that since $\sqrt{3x+2}\geq 0$, $f(x)=\sqrt{3x+2}-1\geq -1$. Therefore, the range of f must be a subset of the set $\{y\,|\,y\geq -1\}$. To show that every element in this set is in the range of f, we need to show that for all y in this set, there exists a real number x in the domain such that $f(x)=y$. Let $y\geq -1$. Then, $f(x)=y$ if and only if

$$\sqrt{3x+2}-1=y.$$

Solving this equation for x, we see that x must solve the equation

$$\sqrt{3x+2}=y+1.$$

Since $y\geq -1$, such an x could exist. Squaring both sides of this equation, we have $3x+2=(y+1)^2$.

Therefore, we need

$$3x=(y+1)^2-2,$$

which implies

$$x=\tfrac{1}{3}(y+1)^2-\tfrac{2}{3}.$$

We just need to verify that x is in the domain of f. Since the domain of f consists of all real numbers greater than or equal to $\frac{-2}{3}$, and

$$\tfrac{1}{3}(y+1)^2-\tfrac{2}{3}\geq -\tfrac{2}{3},$$

there does exist an x in the domain of f. We conclude that the range of f is $\{y\,|\,y\geq -1\}$.

c. Consider $f(x)=\dfrac{3}{x-2}$.

1.Since $3/(x-2)$ is defined when the denominator is nonzero, the domain is $\{x\,|\,x\neq 2\}$.

2.To find the range of f, we need to find the values of y such that there exists a real number x in the domain with the property that

$$\frac{3}{x-2} = y.$$

Solving this equation for x, we find that

$$x = \frac{3}{y} + 2.$$

Therefore, as long as $y \neq 0$, there exists a real number x in the domain such that $f(x) = y$. Thus, the range is $\{y \mid y \neq 0\}$.

? Exercise 1.1.2

Find the domain and range for $f(x) = \sqrt{4-2x} + 5$.

Hint

Use $4 - 2x \geq 0$.

Answer

Domain = $\{x \mid x \leq 2\}$ and range = $\{y \mid y \geq 5\}$

1.1.1.1 Representing Functions

Typically, a function is represented using one or more of the following tools:

- A table
- A graph
- A formula

We can identify a function in each form, but we can also use them together. For instance, we can plot on a graph the values from a table or create a table from a formula.

1.1.1.2 Tables

Functions described using a **table of values** arise frequently in real-world applications. Consider the following simple example. We can describe temperature on a given day as a function of time of day. Suppose we record the temperature every hour for a 24-hour period starting at midnight. We let our input variable x be the time after midnight, measured in hours, and the output variable y be the temperature x hours after midnight, measured in degrees Fahrenheit. We record our data in Table 1.1.1.

Table 1.1.1: Temperature as a Function of Time of Day

Hour After Midnight	Temperature(°F)	Hour After Midnight	Temperature(°F)
0	58	12	84
1	54	13	85
2	53	14	85
3	52	15	83
4	52	16	82
5	55	17	80
6	60	18	77
7	64	19	74
8	72	20	69
9	75	21	65
10	78	22	60

Hour After Midnight	Temperature(°F)	Hour After Midnight	Temperature(°F)
11	80	23	58

We can see from the table that temperature is a function of time, and the temperature decreases, then increases, and then decreases again. However, we cannot get a clear picture of the behavior of the function without graphing it.

1.1.1.3 Graphs

Given a function f described by a table, we can provide a visual picture of the function in the form of a graph. Graphing the temperatures listed in Table 1.1.1 can give us a better idea of their fluctuation throughout the day. Figure 1.1.5 shows the plot of the temperature function.

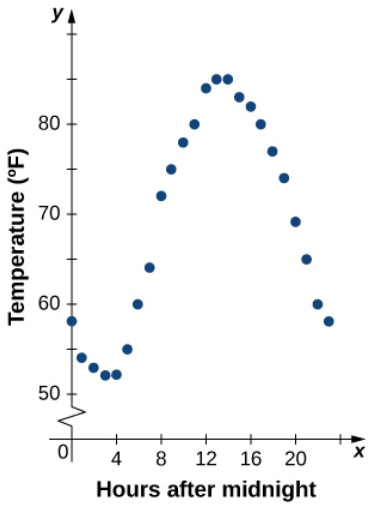


Figure 1.1.5: The graph of the data from Table 1.1.1 shows temperature as a function of time.

From the points plotted on the graph in Figure 1.1.5, we can visualize the general shape of the graph. It is often useful to connect the dots in the graph, which represent the data from the table. In this example, although we cannot make any definitive conclusion regarding what the temperature was at any time for which the temperature was not recorded, given the number of data points collected and the pattern in these points, it is reasonable to suspect that the temperatures at other times followed a similar pattern, as we can see in Figure 1.1.6.

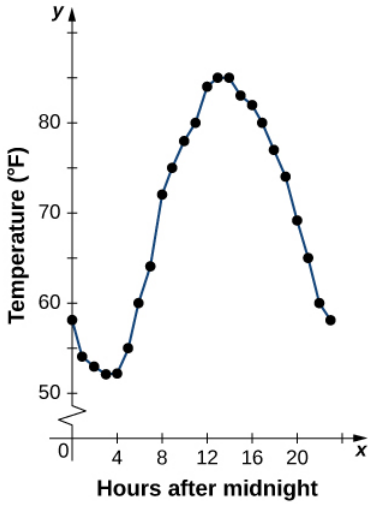


Figure 1.1.6: Connecting the dots in Figure 1.1.5 shows the general pattern of the data.

1.1.2 Algebraic Formulas

Sometimes we are not given the values of a function in table form, rather we are given the values in an explicit formula. Formulas arise in many applications. For example, the area of a circle of radius r is given by the formula $A(r) = \pi r^2$. When an object is

thrown upward from the ground with an initial velocity v_0 ft/s, its height above the ground from the time it is thrown until it hits the ground is given by the formula $s(t) = -16t^2 + v_0 t$. When P dollars are invested in an account at an annual interest rate r compounded continuously, the amount of money after t years is given by the formula $A(t) = Pe^{rt}$. Algebraic formulas are important tools to calculate function values. Often we also represent these functions visually in graph form.

Given an algebraic formula for a function f, the graph of f is the set of points $(x, f(x))$, where x is in the domain of f and $f(x)$ is in the range. To graph a function given by a formula, it is helpful to begin by using the formula to create a table of inputs and outputs. If the domain of f consists of an infinite number of values, we cannot list all of them, but because listing some of the inputs and outputs can be very useful, it is often a good way to begin.

When creating a table of inputs and outputs, we typically check to determine whether zero is an output. Those values of x where $f(x) = 0$ are called the zeros of a function. For example, the zeros of $f(x) = x^2 - 4$ are $x = \pm 2$. The zeros determine where the graph of f intersects the x-axis, which gives us more information about the shape of the graph of the function. The graph of a function may never intersect the x-axis, or it may intersect multiple (or even infinitely many) times.

Another point of interest is the y -intercept, if it exists. The y-intercept is given by $(0, f(0))$.

Since a function has exactly one output for each input, the graph of a function can have, at most, one y-intercept. If $x = 0$ is in the domain of a function f, then f has exactly one y-intercept. If $x = 0$ is not in the domain of f, then f has no y-intercept. Similarly, for any real number c, if c is in the domain of f, there is exactly one output $f(c)$, and the line $x = c$ intersects the graph of f exactly once. On the other hand, if c is not in the domain of f, $f(c)$ is not defined and the line $x = c$ does not intersect the graph of f. This property is summarized in the vertical line test.

Vertical Line Test

Given a function f, every vertical line that may be drawn intersects the graph of f no more than once. If any vertical line intersects a set of points more than once, the set of points does not represent a function.

We can use this test to determine whether a set of plotted points represents the graph of a function (Figure 1.1.7).

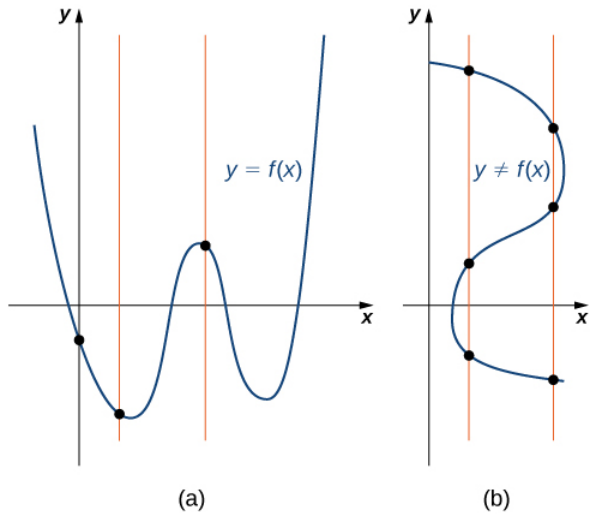


Figure 1.1.7: (a) The set of plotted points represents the graph of a function because every vertical line intersects the set of points, at most, once. (b) The set of plotted points does not represent the graph of a function because some vertical lines intersect the set of points more than once.

Example 1.1.3: Finding Zeros and y-Intercepts of a Function

Consider the function $f(x) = -4x + 2$.

a. Find all zeros of f.
b. Find the y-intercept (if any).
c. Sketch a graph of f.

Solution

1.To find the zeros, solve $f(x) = -4x + 2 = 0$. We discover that f has one zero at $x = 1/2$.

2. The y-intercept is given by $(0, f(0)) = (0, 2)$.

3. Given that f is a linear function of the form $f(x) = mx + b$ that passes through the points $(1/2, 0)$ and $(0, 2)$, we can sketch the graph of f (Figure 1.1.8).

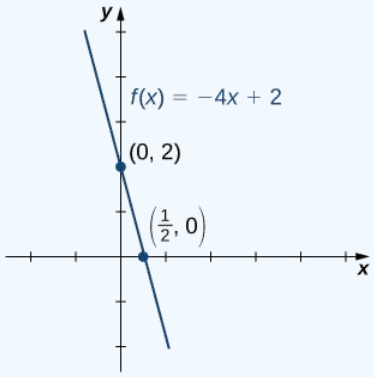


Figure 1.1.8: The function $f(x) = -4x + 2$ is a line with x-intercept $(1/2, 0)$ and y-intercept $(0, 2)$.

Example 1.1.4: Using Zeros and y-Intercepts to Sketch a Graph

Consider the function $f(x) = \sqrt{x+3} + 1$.

a. Find all zeros of f.
b. Find the y-intercept (if any).
c. Sketch a graph of f.

Solution

1.To find the zeros, solve $\sqrt{x+3} + 1 = 0$. This equation implies $\sqrt{x+3} = -1$. Since $\sqrt{x+3} \geq 0$ for all x, this equation has no solutions, and therefore f has no zeros.

2.The y-intercept is given by $(0, f(0)) = (0, \sqrt{3}+1)$.

3.To graph this function, we make a table of values. Since we need $x + 3 \geq 0$, we need to choose values of $x \geq -3$. We choose values that make the square-root function easy to evaluate.

x	-3	-2	1
$f(x)$	1	2	3

Making use of the table and knowing that, since the function is a square root, the graph of f should be similar to the graph of $y = \sqrt{x}$, we sketch the graph (Figure 1.1.9).

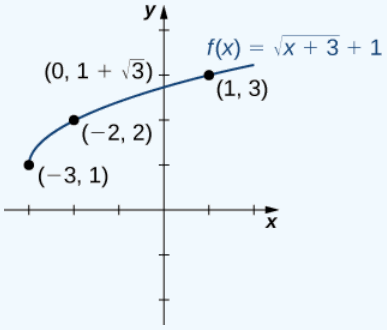


Figure 1.1.9: The graph of $f(x) = \sqrt{x+3} + 1$ has a y-intercept but no x-intercepts.

Exercise 1.1.4

Find the zeros of $f(x) = x^3 - 5x^2 + 6x$.

Hint

Factor the polynomial.

Answer

$x = 0, 2, 3$

Example 1.1.5: Finding the Height of a Free-Falling Object

If a ball is dropped from a height of 100 ft, its height s at time t is given by the function $s(t) = -16t^2 + 100$, where s is measured in feet and t is measured in seconds. The domain is restricted to the interval $[0, c]$, where $t = 0$ is the time when the ball is dropped and $t = c$ is the time when the ball hits the ground.

a. Create a table showing the height s(t) when $t = 0,\ 0.5,\ 1,\ 1.5,\ 2$,and 2.5. Using the data from the table, determine the domain for this function. That is, find the time c when the ball hits the ground.
b. Sketch a graph of s.

Solution

t	0	0.5	1	1.5	2	2.5
$s(t)$	100	96	84	64	36	0

Since the ball hits the ground when $t = 2.5$, the domain of this function is the interval $[0, 2.5]$.

2.

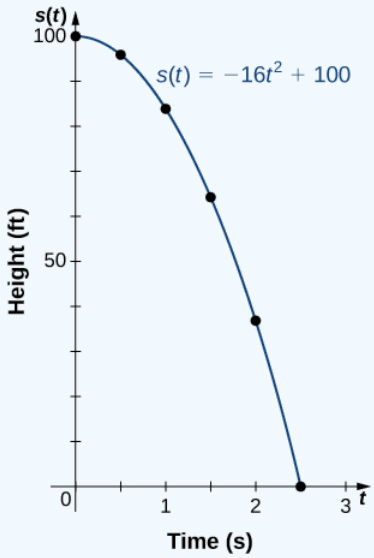


Figure 1.1.8, the values of $f(x)$ are getting smaller as x is getting larger. A function with this property is said to be decreasing. On the other hand, for the function $f(x) = \sqrt{x+3} + 1$ graphed in Figure 1.1.9, the values of $f(x)$ are getting larger as the values of x are getting larger. A function with this property is said to be increasing. It is important to note, however, that a function can be increasing on some interval or intervals and decreasing over a different interval or intervals. For example, using our temperature function plotted above, we can see that the function is decreasing on the interval $(0, 4)$, increasing on the interval $(4, 14)$, and then decreasing on the interval $(14, 23)$. We make the idea of a function increasing or decreasing over a particular interval more precise in the next definition.

Definition: Increasing and Decreasing on an Interval

We say that a function f is ***increasing on the interval I*** if for all $x_1,\ x_2 \in I$,

$$f(x_1) \leq f(x_2) \text{ when } x_1 < x_2.$$

We say f is strictly increasing on the interval I if for all $x_1, x_2 \in I$,

$$f(x_1) < f(x_2) \text{ when } x_1 < x_2.$$

We say that a function f is ***decreasing on the interval I*** if for all $x_1, x_2 \in I$,

$$f(x_1) \geq f(x_2) \text{ if } x_1 < x_2.$$

We say that a function f is strictly decreasing on the interval I if for all $x_1, x_2 \in I$,

$$f(x_1) > f(x_2) \text{ if } x_1 < x_2.$$

For example, the function $f(x) = 3x$ is increasing on the interval $(-\infty, \infty)$ because $3x_1 < 3x_2$ whenever $x_1 < x_2$. On the other hand, the function $f(x) = -x^3$ is decreasing on the interval $(-\infty, \infty)$ because $-x_1^3 > -x_2^3$ whenever $x_1 < x_2$ (Figure 1.1.10).

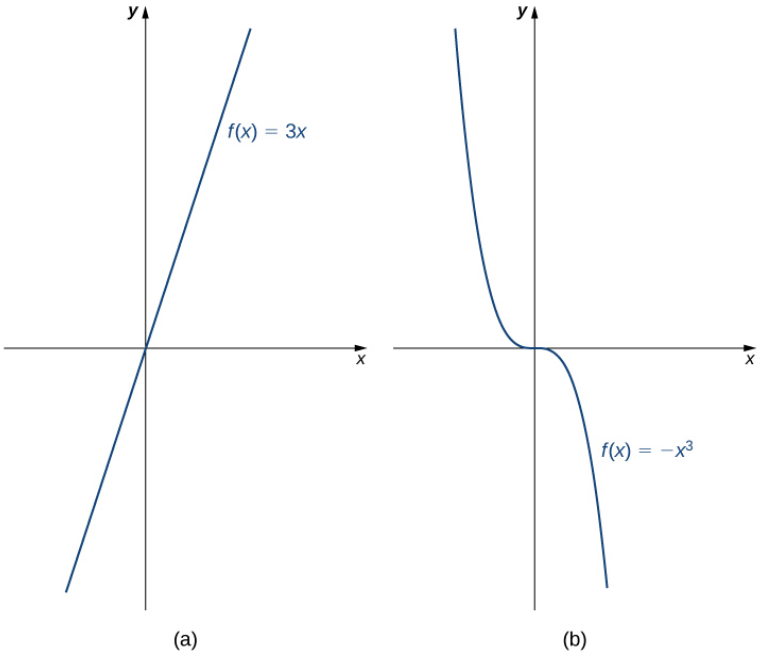


Figure 1.1.10: (a) The function $f(x) = 3x$ is increasing on the interval $(-\infty, \infty)$. (b) The function $f(x) = -x^3$ is decreasing on the interval $(-\infty, \infty)$.

1.1.3 Combining Functions

Now that we have reviewed the basic characteristics of functions, we can see what happens to these properties when we combine functions in different ways, using basic mathematical operations to create new functions. For example, if the cost for a company to manufacture x items is described by the function $C(x)$ and the revenue created by the sale of x items is described by the function $R(x)$, then the profit on the manufacture and sale of x items is defined as $P(x) = R(x) - C(x)$. Using the difference between two functions, we created a new function.

Alternatively, we can create a new function by composing two functions. For example, given the functions $f(x) = x^2$ and $g(x) = 3x + 1$, the composite function $f \circ g$ is defined such that

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (3x + 1)^2.$$

The composite function $g \circ f$ is defined such that

$$(g \circ f)(x) = g(f(x)) = 3f(x) + 1 = 3x^2 + 1.$$

Note that these two new functions are different from each other.

1.1.4 Combining Functions with Mathematical Operators

To combine functions using mathematical operators, we simply write the functions with the operator and simplify. Given two functions f and g, we can define four new functions:

$(f+g)(x)=f(x)+g(x)$	Sum
$(f-g)(x)=f(x)-g(x)$	Difference
$(f\cdot g)(x)=f(x)g(x)$	Product
$(\frac{f}{g})(x)=\frac{f(x)}{g(x)}$ for $g(x)\neq 0$	Quotient

Example 1.1.6: Combining Functions Using Mathematical Operations

Given the functions $f(x)=2x-3$ and $g(x)=x^2-1$, find each of the following functions and state its domain.

a. $(f+g)(x)$
b. $(f-g)(x)$
c. $(f\cdot g)(x)$
d. $\left(\frac{f}{g}\right)(x)$

Solution

1. $(f+g)(x)=(2x-3)+(x^2-1)=x^2+2x-4.$

The domain of this function is the interval $(-\infty,\infty)$.

2.$(f-g)(x)=(2x-3)-(x^2-1)=-x^2+2x-2.$

The domain of this function is the interval $(-\infty,\infty)$.

3. $(f\cdot g)(x)=(2x-3)(x^2-1)=2x^3-3x^2-2x+3.$

The domain of this function is the interval $(-\infty,\infty)$.

4. $\left(\frac{f}{g}\right)(x)=\frac{2x-3}{x^2-1}.$

The domain of this function is $\{x\mid x\neq\pm 1\}$.

Exercise 1.1.6

For $f(x)=x^2+3$ and $g(x)=2x-5$, find $(f/g)(x)$ and state its domain.

Hint

The new function $(f/g)(x)$ is a quotient of two functions. For what values of x is the denominator zero?

Answer

$\left(\frac{f}{g}\right)(x)=\frac{x^2+3}{2x-5}$. The domain is $\{x\mid x\neq\frac{5}{2}\}$.

1.1.5 Function Composition

When we compose functions, we take a function of a function. For example, suppose the temperature T on a given day is described as a function of time t (measured in hours after midnight) as in Table 1.1.1. Suppose the cost C, to heat or cool a building for 1 hour, can be described as a function of the temperature T. Combining these two functions, we can describe the cost of heating or cooling a building as a function of time by evaluating $C(T(t))$. We have defined a new function, denoted $C\circ T$, which is defined such that $(C\circ T)(t)=C(T(t))$ for all t in the domain of T. This new function is called a composite function. We note that since

cost is a function of temperature and temperature is a function of time, it makes sense to define this new function $(C \circ T)(t)$. It does not make sense to consider $(T \circ C)(t)$, because temperature is not a function of cost.

Definition: Composite Functions

Consider the function f with domain A and range B, and the function g with domain D and range E. If B is a subset of D, then the composite function $(g \circ f)(x)$ is the function with domain A such that

$$(g \circ f)(x) = g(f(x))$$

A composite function $g \circ f$ can be viewed in two steps. First, the function f maps each input x in the domain of f to its output $f(x)$ in the range of f. Second, since the range of f is a subset of the domain of g, the output $f(x)$ is an element in the domain of g, and therefore it is mapped to an output $g(f(x))$ in the range of g. In Figure 1.1.11, we see a visual image of a composite function.

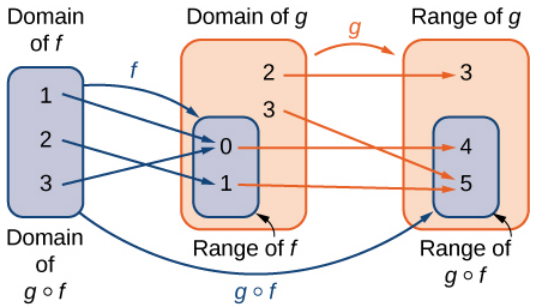


Figure 1.1.11: For the composite function $g \circ f$, we have $(g \circ f)(1) = 4$, $(g \circ f)(2) = 5$, and $(g \circ f)(3) = 4$.

Example 1.1.7: Compositions of Functions Defined by Formulas

Consider the functions $f(x) = x^2 + 1$ and $g(x) = 1/x$.

a. Find $(g \circ f)(x)$ and state its domain and range.
b. Evaluate $(g \circ f)(4)$, $(g \circ f)(-1/2)$.
c. Find $(f \circ g)(x)$ and state its domain and range.
d. Evaluate $(f \circ g)(4)$, $(f \circ g)(-1/2)$.

Solution

1. We can find the formula for $(g \circ f)(x)$ in two different ways. We could write

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = \frac{1}{x^2 + 1} \quad .$$

Alternatively, we could write

$$(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)} = \frac{1}{x^2 + 1}.$$

Since $x^2 + 1 \neq 0$ for all real numbers x, the domain of $(g \circ f)(x)$ is the set of all real numbers. Since $0 < 1/(x^2 + 1) \leq 1$, the range is, at most, the interval $(0, 1]$. To show that the range is this entire interval, we let $y = 1/(x^2 + 1)$ and solve this equation for x to show that for all y in the interval $(0, 1]$, there exists a real number x such that $y = 1/(x^2 + 1)$. Solving this equation for x, we see that $x^2 + 1 = 1/y$, which implies that

$$x = \pm\sqrt{\tfrac{1}{y} - 1}$$

If y is in the interval $(0, 1]$, the expression under the radical is nonnegative, and therefore there exists a real number x such that $1/(x^2 + 1) = y$. We conclude that the range of $g \circ f$ is the interval $(0, 1]$.

2. $(g \circ f)(4) = g(f(4)) = g(4^2 + 1) = g(17) = \frac{1}{17}$

$(g \circ f)(-\frac{1}{2}) = g(f(-\frac{1}{2})) = g((-\frac{1}{2})^2 + 1) = g(\frac{5}{4}) = \frac{4}{5}$

3. We can find a formula for $(f \circ g)(x)$ in two ways. First, we could write

$$(f\circ g)(x)=f(g(x))=f(\tfrac{1}{x})=(\tfrac{1}{x})^2+1.$$

Alternatively, we could write

$$(f\circ g)(x)=f(g(x))=(g(x))^2+1=(\tfrac{1}{x})^2+1.$$

The domain of $f\circ g$ is the set of all real numbers x such that $x\neq 0$. To find the range of f, we need to find all values y for which there exists a real number $x\neq 0$ such that

$$\left(\frac{1}{x}\right)^2+1=y.$$

Solving this equation for x, we see that we need x to satisfy

$$\left(\frac{1}{x}\right)^2=y-1,$$

which simplifies to

$$\frac{1}{x}=\pm\sqrt{y-1}$$

Finally, we obtain

$$x=\pm\frac{1}{\sqrt{y-1}}.$$

Since $1/\sqrt{y-1}$ is a real number if and only if $y>1$, the range of f is the set $\{y\,|\,y\geq 1\}$.

4.$(f\circ g)(4)=f(g(4))=f(\tfrac{1}{4})=(\tfrac{1}{4})^2+1=\tfrac{17}{16}$

$(f\circ g)(-\tfrac{1}{2})=f(g(-\tfrac{1}{2}))=f(-2)=(-2)^2+1=5$

In Example 1.1.7, we can see that $(f\circ g)(x)\neq(g\circ f)(x)$. This tells us, in general terms, that the order in which we compose functions matters.

? Exercise 1.1.7

Let $f(x)=2-5x$. Let $g(x)=\sqrt{x}$. Find $(f\circ g)(x)$.

Solution

$(f\circ g)(x)=2-5\sqrt{x}$.

✔ Example 1.1.8: Composition of Functions Defined by Tables

Consider the functions f and g described by

x	-3	-2	-1	0	1	2	3	4
$f(x)$	0	4	2	4	-2	0	-2	4

x	-4	-2	0	2	4
$g(x)$	1	0	3	0	5

a. Evaluate $(g\circ f)(3)$,$(g\circ f)(0)$.
b. State the domain and range of $(g\circ f)(x)$.
c. Evaluate $(f\circ f)(3)$,$(f\circ f)(1)$.
d. State the domain and range of $(f\circ f)(x)$.

Solution:

1. $(g\circ f)(3)=g(f(3))=g(-2)=0$

$(g \circ f)(0) = g(4) = 5$

2.The domain of $g \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$.Since the range of f is the set $\{-2, 0, 2, 4\}$,the range of $g \circ f$ is the set $\{0, 3, 5\}$.

3. $(f \circ f)(3) = f(f(3)) = f(-2) = 4$

$(f \circ f)(1) = f(f(1)) = f(-2) = 4$

4.The domain of $f \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$.Since the range of f is the set $\{-2, 0, 2, 4\}$,the range of $f \circ f$ is the set $\{0, 4\}$.

Example 1.1.9: Application Involving a Composite Function

A store is advertising a sale of 20% off all merchandise. Caroline has a coupon that entitles her to an additional 15% off any item, including sale merchandise. If Caroline decides to purchase an item with an original price of x dollars, how much will she end up paying if she applies her coupon to the sale price? Solve this problem by using a composite function.

Solution

Since the sale price is 20% off the original price, if an item is x dollars, its sale price is given by $f(x) = 0.80x$. Since the coupon entitles an individual to 15% off the price of any item, if an item is y dollars, the price, after applying the coupon, is given by g(y)=0.85y. Therefore, if the price is originally x dollars, its sale price will be $f(x) = 0.80x$ and then its final price after the coupon will be $g(f(x)) = 0.85(0.80x) = 0.68x$

Exercise 1.1.9

If items are on sale for 10% off their original price, and a customer has a coupon for an additional 30% off, what will be the final price for an item that is originally x dollars, after applying the coupon to the sale price?

Hint

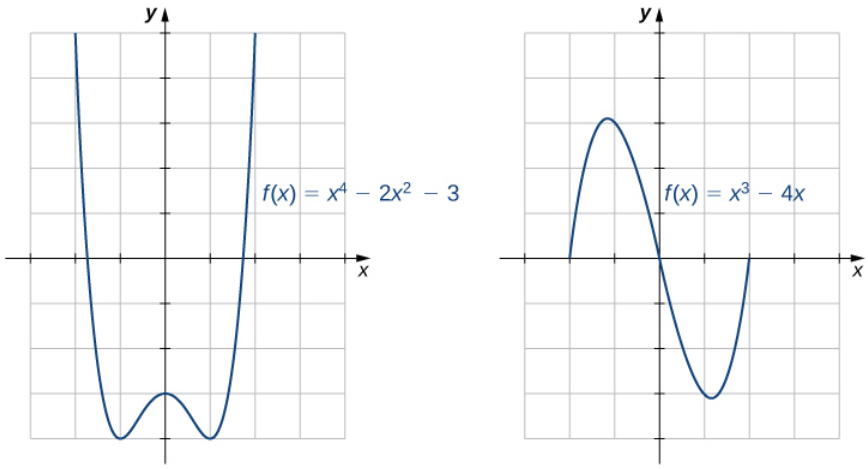
The sale price of an item with an original price of x dollars is $f(x) = 0.90x$. The coupon price for an item that is y dollars is $g(y) = 0.70y$.

Solution

$(g \circ f)(x) = 0.63x$

1.1.6 Symmetry of Functions

The graphs of certain functions have symmetry properties that help us understand the function and the shape of its graph. For example, consider the function $f(x) = x^4 - 2x^2 - 3$ shown in Figure $1.1.12a$. If we take the part of the curve that lies to the right of the y-axis and flip it over the y-axis, it lays exactly on top of the curve to the left of the y-axis. In this case, we say the function has **symmetry about the y-axis**. On the other hand, consider the function $f(x) = x^3 - 4x$ shown in Figure $1.1.12b$ If we take the graph and rotate it 180° about the origin, the new graph will look exactly the same. In this case, we say the function has **symmetry about the origin**.



(a) Symmetry about the y-axis (b) Symmetry about the origin

Figure 1.1.12: (a) A graph that is symmetric about the y-axis. (b) A graph that is symmetric about the origin.

If we are given the graph of a function, it is easy to see whether the graph has one of these symmetry properties. But without a graph, how can we determine algebraically whether a function f has symmetry? Looking at Figure $1.1.12a$ again, we see that since f is symmetric about the y-axis, if the point (x, y) is on the graph, the point $(-x, y)$ is on the graph. In other words, $f(-x) = f(x)$. If a function f has this property, we say f is an even function, which has symmetry about the y-axis. For example, $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x).$$

In contrast, looking at Figure $1.1.12b$ again, if a function f is symmetric about the origin, then whenever the point (x, y) is on the graph, the point $(-x, -y)$ is also on the graph. In other words, $f(-x) = -f(x)$. If f has this property, we say f is an odd function, which has symmetry about the origin. For example, $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

Definition: Even and Odd Functions

- If $f(x) = f(-x)$ for all x in the domain of f, then f is an *even* function. An even function is symmetric about the y-axis.
- If $f(-x) = -f(x)$ for all x in the domain of f, then f is an *odd* function. An odd function is symmetric about the origin.

Example 1.1.10: Even and Odd Functions

Determine whether each of the following functions is even, odd, or neither.

a. $f(x) = -5x^4 + 7x^2 - 2$
b. $f(x) = 2x^5 - 4x + 5$
c. $f(x) = \frac{3x}{x^2+1}$

Solution

To determine whether a function is even or odd, we evaluate $f(-x)$ and compare it to $f(x)$ and $-f(x)$.

1. $f(-x) = -5(-x)^4 + 7(-x)^2 - 2 = -5x^4 + 7x^2 - 2 = f(x)$. Therefore, f is even.

2. $f(-x) = 2(-x)^5 - 4(-x) + 5 = -2x^5 + 4x + 5$. Now, $f(-x) \neq f(x)$. Furthermore, noting that $-f(x) = -2x^5 + 4x - 5$, we see that $f(-x) \neq -f(x)$. Therefore, f is neither even nor odd.

3. $f(-x) = 3(-x)/((-x)2 + 1) = -3x/(x^2 + 1) = -[3x/(x^2 + 1)] = -f(x)$. Therefore, f is odd.

? Exercise 1.1.10

Determine whether $f(x) = 4x^3 - 5x$ is even, odd, or neither.

Hint

Compare $f(-x)$ with $f(x)$ and $-f(x)$.

Answer

$f(x)$ is odd.

One symmetric function that arises frequently is the **absolute value function**, written as $|x|$. The absolute value function is defined as

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Some students describe this function by stating that it "makes everything positive." By the definition of the absolute value function, we see that if $x < 0$, then $|x| = -x > 0$, and if $x > 0$, then $|x| = x > 0$. However, for $x = 0$, $|x| = 0$. Therefore, it is more accurate to say that for all nonzero inputs, the output is positive, but if $x = 0$, the output $|x| = 0$. We conclude that the range of the absolute value function is $\{y \mid y \geq 0\}$. In Figure 1.1.13, we see that the absolute value function is symmetric about the y-axis and is therefore an even function.

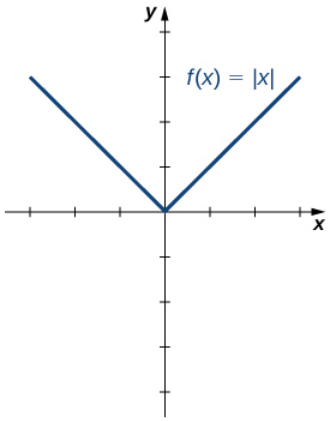


Figure 1.1.13: The graph of $f(x) = |x|$ is symmetric about the y-axis.

✓ Example 1.1.11: Working with the Absolute Value Function

Find the domain and range of the function $f(x) = 2|x - 3| + 4$.

Solution

Since the absolute value function is defined for all real numbers, the domain of this function is $(-\infty, \infty)$. Since $|x - 3| \geq 0$ for all x, the function $f(x) = 2|x - 3| + 4 \geq 4$. Therefore, the range is, at most, the set $\{y \mid y \geq 4\}$. To see that the range is, in fact, this whole set, we need to show that for $y \geq 4$ there exists a real number x such that

$$2|x - 3| + 4 = y$$

A real number x satisfies this equation as long as

$$|x - 3| = \tfrac{1}{2}(y - 4)$$

Since $y \geq 4$, we know $y - 4 \geq 0$, and thus the right-hand side of the equation is nonnegative, so it is possible that there is a solution. Furthermore,

$$|x - 3| = \begin{cases} -(x - 3), & \text{if } x < 3 \\ x - 3, & \text{if } x \geq 3 \end{cases}$$

Therefore, we see there are two solutions:

$$x = \pm\tfrac{1}{2}(y-4)+3\ .$$

The range of this function is $\{y \mid y \geq 4\}$.

? Exercise 1.1.11: Domain and Range

For the function $f(x) = |x+2| - 4$, find the domain and range.

Hint

$|x+2| \geq 0$ for all real numbers x.

Answer

Domain = $(-\infty, \infty)$, range = $\{y \mid y \geq -4\}$.

1.1.7 Key Concepts

- A function is a mapping from a set of inputs to a set of outputs with exactly one output for each input.
- If no domain is stated for a function $y = f(x)$, the domain is considered to be the set of all real numbers x for which the function is defined.
- When sketching the graph of a function f, each vertical line may intersect the graph, at most, once.
- A function may have any number of zeros, but it has, at most, one y-intercept.
- To define the composition $g \circ f$, the range of f must be contained in the domain of g.
- Even functions are symmetric about the y-axis whereas odd functions are symmetric about the origin.

1.1.8 Key Equations

- **Composition of two functions**

  $(g \circ f)(x) = g\big(f(x)\big)$

- **Absolute value function**

  $$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

1.1.9 Glossary

absolute value function

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

composite function

given two functions f and g, a new function, denoted $g \circ f$, such that $(g \circ f)(x) = g(f(x))$

decreasing on the interval I

a function decreasing on the interval I if, for all $x_1,\ x_2 \in I,\ f(x_1) \geq f(x_2)$ if $x_1 < x_2$

dependent variable

the output variable for a function

domain

the set of inputs for a function

even function

a function is even if $f(-x) = f(x)$ for all x in the domain of f

function

a set of inputs, a set of outputs, and a rule for mapping each input to exactly one output

graph of a function
the set of points (x, y) such that x is in the domain of f and $y = f(x)$

increasing on the interval I
a function increasing on the interval I if for all $x_1,\ x_2 \in I,\ f(x_1) \leq f(x_2)$ if $x_1 < x_2$

independent variable
the input variable for a function

odd function
a function is odd if $f(-x) = -f(x)$ for all x in the domain of f

range
the set of outputs for a function

symmetry about the origin
the graph of a function f is symmetric about the origin if $(-x, -y)$ is on the graph of f whenever (x, y) is on the graph

symmetry about the y-axis
the graph of a function f is symmetric about the y-axis if $(-x, y)$ is on the graph of f whenever (x, y) is on the graph

table of values
a table containing a list of inputs and their corresponding outputs

vertical line test
given the graph of a function, every vertical line intersects the graph, at most, once

zeros of a function
when a real number x is a zero of a function $f,\ f(x) = 0$

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1.1E: Exercises for Section 1.1

For exercises 1 - 6, (a) determine the domain and the range of each relation, and (b) state whether the relation is a function.

1)

x	y	x	y
-3	9	1	1
-2	4	2	4
-1	1	3	9
0	0		

Answer

a. Domain = $\{-3, -2, -1, 0, 1, 2, 3\}$, Range = $\{0, 1, 4, 9\}$

b. Yes, a function

2)

x	y	x	y
-3	-2	1	1
-2	-8	2	8
-1	-1	3	-2
0	0		

3)

x	y	x	y
1	-3	1	1
2	-2	2	2
3	-1	3	3
0	0		

Answer

a. Domain = $\{0, 1, 2, 3\}$, Range = $\{-3, -2, -1, 0, 1, 2, 3\}$

b. No, not a function

4)

x	y	x	y
1	1	5	1
2	1	6	1
3	1	7	1
4	1		

5)

x	y	x	y
3	3	15	1
5	2	21	2
8	1	33	3
10	0		

Answer

a. Domain = $\{3, 5, 8, 10, 15, 21, 33\}$ Range = $\{0, 1, 2, 3\}$

b. Yes, a function

6)

x	y	x	y
-7	11	1	-2
-2	5	3	4
-2	1	6	11
0	-1		

For exercises 7 - 13, find the values for each function, if they exist, then simplify.

a. $f(0)$ b. $f(1)$ c. $f(3)$ d. $f(-x)$ e. $f(a)$ f. $f(a+h)$

7) $f(x) = 5x - 2$

Answer

a. -2 b. 3 c. 13 d. $-5x - 2$ e. $5a - 2$ f. $5a + 5h - 2$

8) $f(x) = 4x^2 - 3x + 1$

9) $f(x) = \dfrac{2}{x}$

Answer

a. Undefined b. 2 c. $\frac{2}{3}$ d. $-\dfrac{2}{x}$ e. $\dfrac{2}{a}$ f. $\dfrac{2}{a+h}$

10) $f(x) = |x - 7| + 8$

11) $f(x) = \sqrt{6x + 5}$

Answer

a. $\sqrt{5}$ b. $\sqrt{11}$ c. $\sqrt{23}$ d. $\sqrt{-6x + 5}$ e. $\sqrt{6a + 5}$ f. $\sqrt{6a + 6h + 5}$

12) $f(x) = \dfrac{x - 2}{3x + 7}$

13) $f(x) = 9$

Answer

a. 9 b. 9 c. 9 d. 9 e. 9 f. 9

For exercises 14 - 21, find the domain, range, and all zeros/intercepts, if any, of the functions.

14) $f(x) = \dfrac{x}{x^2 - 16}$

15) $g(x) = \sqrt{8x - 1}$

Answer

$x \geq \frac{1}{8}; \quad y \geq 0; \quad x = \frac{1}{8}$; no y-intercept

16) $h(x) = \dfrac{3}{x^2 + 4}$

17) $f(x) = -1 + \sqrt{x + 2}$

Answer

$x \geq -2; \quad y \geq -1; \quad x = -1; \quad y = -1 + \sqrt{2}$

18) $f(x) = 1x - \sqrt{9}$

19) $g(x) = \dfrac{3}{x - 4}$

Answer

$x \neq 4; \quad y \neq 0$; no x-intercept; $y = -\frac{3}{4}$

20) $f(x) = 4|x + 5|$

21) $g(x) = \sqrt{\dfrac{7}{x - 5}}$

Answer

$x > 5; \quad y > 0$; no intercepts

For exercises 22 - 27, set up a table to sketch the graph of each function using the following values: $x = -3, -2, -1, 0, 1, 2, 3.$

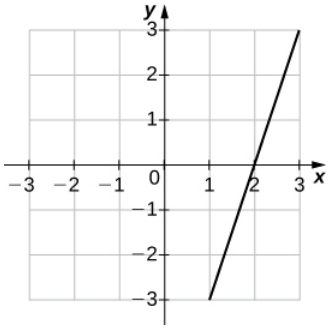
22) $f(x) = x^2 + 1$

x	y	x	y
-3	10	1	2
-2	5	2	5
-1	2	3	10
0	1		

23) $f(x) = 3x - 6$

x	y	x	y
-3	-15	1	-3
-2	-12	2	0
-1	-9	3	3
0	-6		

Answer



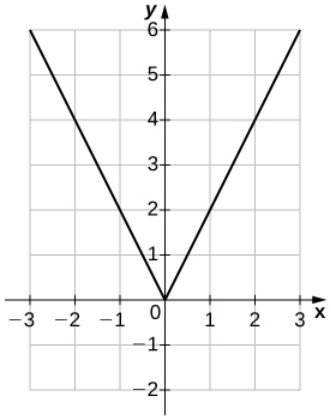
24) $f(x) = \frac{1}{2}x + 1$

x	y	x	y
-3	$-\frac{1}{2}$	1	$\frac{3}{2}$
-2	0	2	2
-1	$\frac{1}{2}$	3	$\frac{5}{2}$
0	1		

25) $f(x) = 2|x|$

x	y	x	y
-3	6	1	2
-2	4	2	4
-1	2	3	6
0	0		

Answer



26) $f(x) = -x^2$

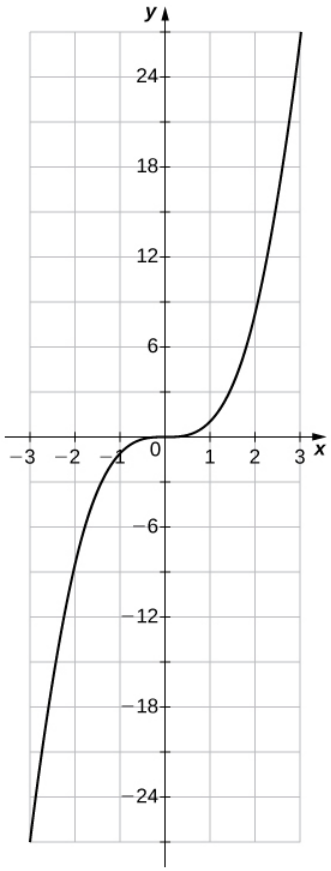
x	y	x	y
-3	-9	1	-1
-2	-4	2	-4
-1	-1	3	-9

x	y	x	y
0	0		

27) $f(x) = x^3$

x	y	x	y
-3	-27	1	1
-2	-8	2	8
-1	-1	3	27
0	0		

Answer



For exercises 28 - 35, use the vertical line test to determine whether each of the given graphs represents a function. Assume that a graph continues at both ends if it extends beyond the given grid. If the graph represents a function, then determine the following for each graph:

a. Domain and range

b. x -intercept, if any (estimate where necessary)

c. y-Intercept, if any (estimate where necessary)

d. The intervals for which the function is increasing

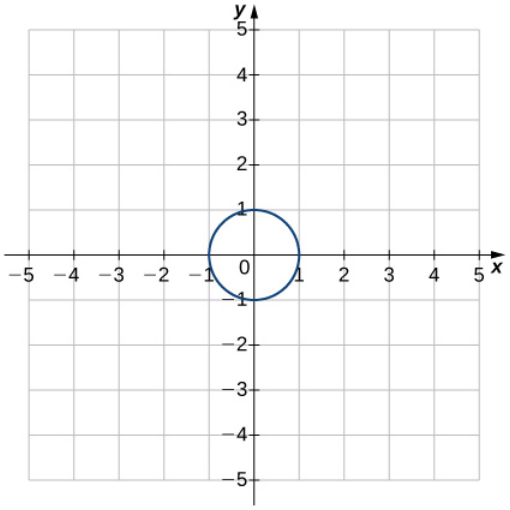
e. The intervals for which the function is decreasing

f. The intervals for which the function is constant

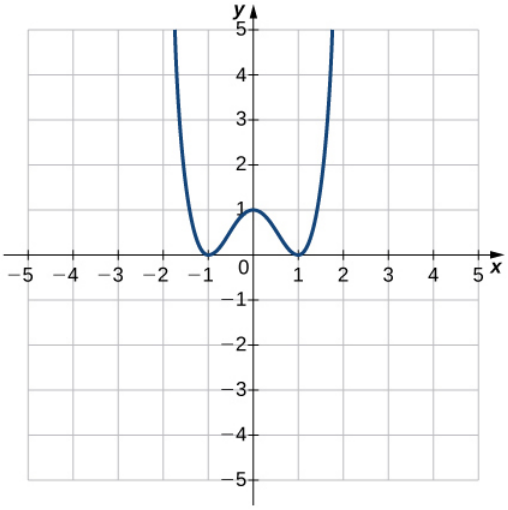
g. Symmetry about any axis and/or the origin

h. Whether the function is even, odd, or neither

28)



29)



Answer

Function;

a. Domain: all real numbers, range: $y \geq 0$

b. $x = \pm 1$

c. $y = 1$

d. $-1 < x < 0$ and $1 < x < \infty$

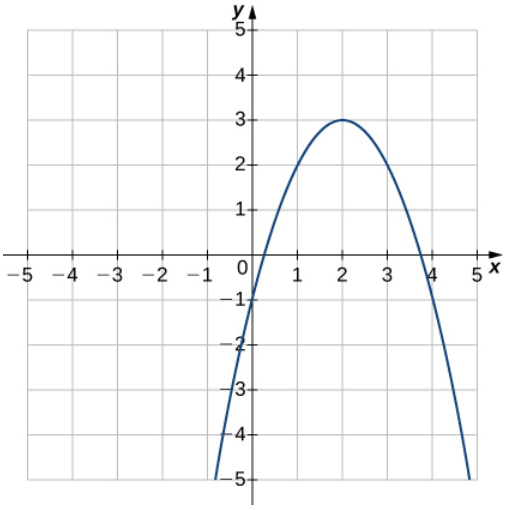
e. $-\infty < x < -1$ and $0 < x < 1$

f. Not constant

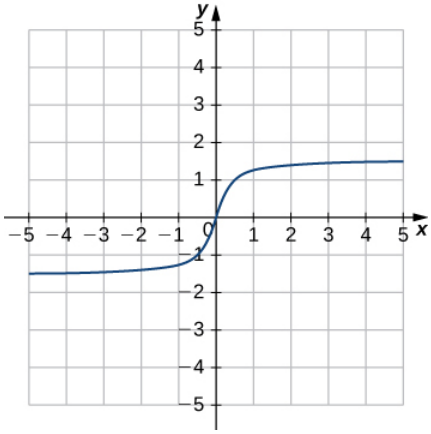
g. y-axis

h. Even

30)



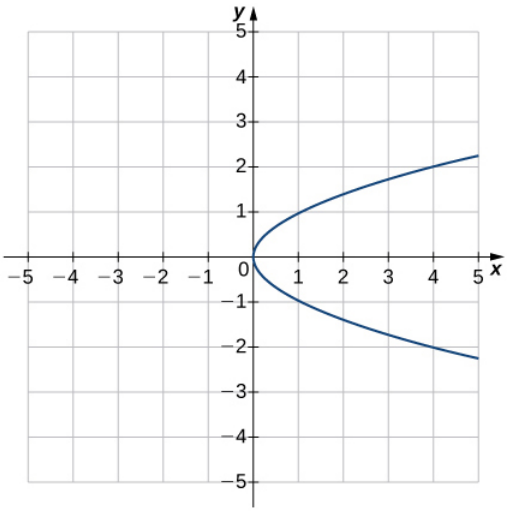
31)



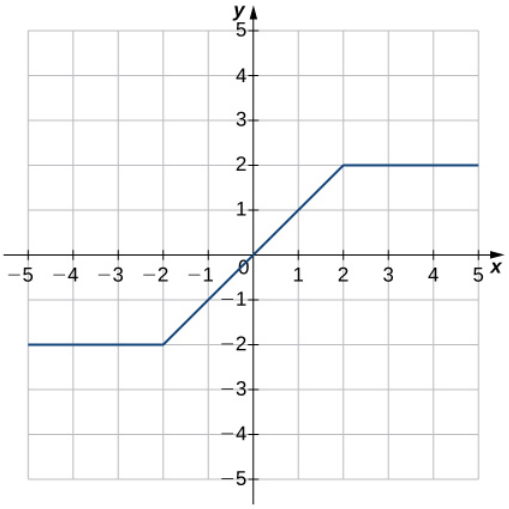
Answer

Function;
a. Domain: all real numbers, range: $-1.5 \leq y \leq 1.5$
b. $x = 0$
c. $y = 0$
d. all real numbers
e. None
f. Not constant
g. Origin
h. Odd

32)



33)



Answer

Function;

a. Domain: $-\infty < x < \infty$, range: $-2 \leq y \leq 2$

b. $x = 0$

c. $y = 0$

d. $-2 < x < 2$

e. Not decreasing

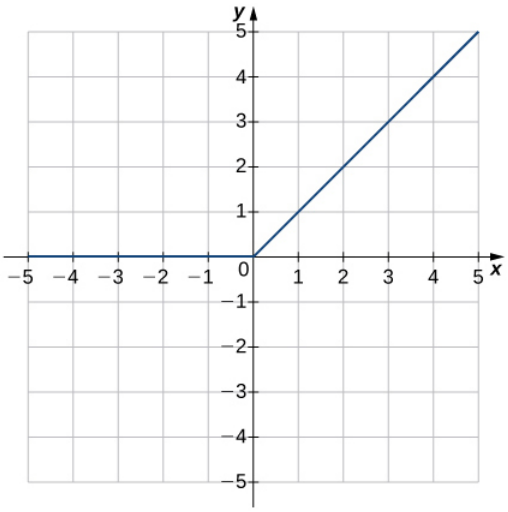
f. $-\infty < x < -2$ and $2 < x < \infty$

g. Origin

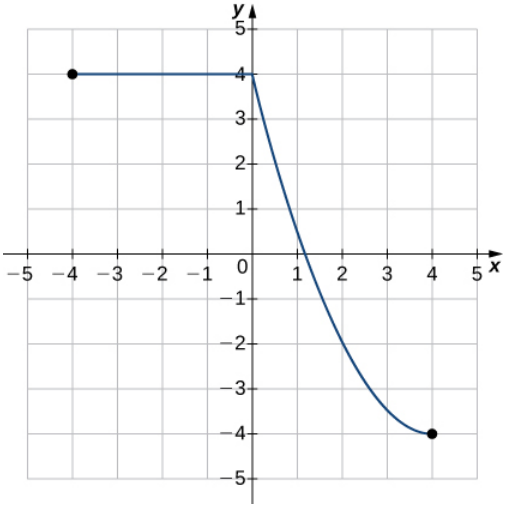
h. Odd

34)

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35)



Answer

Function;
a. Domain: $-4 \leq x \leq 4$, range: $-4 \leq y \leq 4$
b. $x = 1.2$
c. $y = 4$
d. Not increasing
e. $0 < x < 4$
f. $-4 < x < 0$
g. No Symmetry
h. Neither

For exercises 36 - 41, for each pair of functions, find a. $f+g$ b. $f-g$ c. $f \cdot g$ d. f/g. Determine the domain of each of these new functions.

36) $f(x) = 3x + 4, \quad g(x) = x - 2$

37) $f(x) = x - 8, \quad g(x) = 5x^2$

Answer

a. $5x^2 + x - 8$; all real numbers
b. $-5x^2 + x - 8$; all real numbers

c. $5x^3 - 40x^2$; all real numbers

d. $\dfrac{x-8}{5x^2}; \quad x \neq 0$

38) $f(x) = 3x^2 + 4x + 1, \quad g(x) = x + 1$

39) $f(x) = 9 - x^2, \quad g(x) = x^2 - 2x - 3$

Answer

a. $-2x + 6$; all real numbers

b. $-2x^2 + 2x + 12$; all real numbers

c. $-x^4 + 2x^3 + 12x^2 - 18x - 27$; all real numbers

d. $-\dfrac{x+3}{x+1}; \quad x \neq -1, 3$

40) $f(x) = \sqrt{x}, \quad g(x) = x - 2$

41) $f(x) = 6 + \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$

Answer

a. $6 + \dfrac{2}{x}; \quad x \neq 0$

b. $6; \quad x \neq 0$

c. $6x + \dfrac{1}{x^2}; \quad x \neq 0$

d. $6x + 1; \quad x \neq 0$

For exercises 42 - 48, for each pair of functions, find a. $(f \circ g)(x)$ and b. $(g \circ f)(x)$ Simplify the results. Find the domain of each of the results.

42) $f(x) = 3x, \quad g(x) = x + 5$

43) $f(x) = x + 4, \quad g(x) = 4x - 1$

Answer

a. $4x + 3$; all real numbers

b. $4x + 15$; all real numbers

44) $f(x) = 2x + 4, \quad g(x) = x^2 - 2$

45) $f(x) = x^2 + 7, \quad g(x) = x^2 - 3$

Answer

a. $x^4 - 6x^2 + 16$; all real numbers

b. $x^4 + 14x^2 + 46$; all real numbers

46) $f(x) = \sqrt{x}, \quad g(x) = x + 9$

47) $f(x) = \dfrac{3}{2x+1}, \quad g(x) = \dfrac{2}{x}$

Answer

a. $\dfrac{3x}{4+x}; \quad x \neq 0, -4$

b. $\dfrac{4x+2}{3}; \quad x \neq -\frac{1}{2}$

48) $f(x) = |x + 1|, \quad g(x) = x^2 + x - 4$

49) The table below lists the NBA championship winners for the years 2001 to 2012.

Year	Winner
2001	LA Lakers
2002	LA Lakers
2003	Sam Antonio Spurs
2004	Detroit Pistons
2005	Sam Antonio Spurs
2006	Miami Heat
2007	Sam Antonio Spurs
2008	Boston Celtics
2009	LA Lakers
2010	LA Lakers
2011	Dallas Mavericks
2012	Miami Heat

a. Consider the relation in which the domain values are the years 2001 to 2012 and the range is the corresponding winner. Is this relation a function? Explain why or why not.

b. Consider the relation where the domain values are the winners and the range is the corresponding years. Is this relation a function? Explain why or why not.

Answer

a. Yes, because there is only one winner for each year.
b. No, because there are three teams that won more than once during the years 2001 to 2012.

50) [T] The area A of a square depends on the length of the side s.

a. Write a function $A(s)$ for the area of a square.

b. Find and interpret $A(6.5)$.

c. Find the exact and the two-significant-digit approximation to the length of the sides of a square with area 56 square units.

51) [T] The volume of a cube depends on the length of the sides s.

a. Write a function $V(s)$ for the area of a square.

b. Find and interpret $V(11.8)$.

Answer

a. $V(s) = s^3$
b. $V(11.8) \approx 1643$; a cube of side length 11.8 each has a volume of approximately 1643 cubic units.

52) **[T]** A rental car company rents cars for a flat fee of $20 and an hourly charge of $10.25. Therefore, the total cost C to rent a car is a function of the hours t the car is rented plus the flat fee.

a. Write the formula for the function that models this situation.

b. Find the total cost to rent a car for 2 days and 7 hours.

c. Determine how long the car was rented if the bill is $432.73.

53) [T] A vehicle has a 20-gal tank and gets 15 mpg. The number of miles N that can be driven depends on the amount of gas x in the tank.

a. Write a formula that models this situation.

b. Determine the number of miles the vehicle can travel on (i) a full tank of gas and (ii) 3/4 of a tank of gas.

c. Determine the domain and range of the function.

d. Determine how many times the driver had to stop for gas if she has driven a total of 578 mi.

Answer

a. $N(x) = 15x$
b. i. $N(20) = 15(20) = 300$; therefore, the vehicle can travel 300 mi on a full tank of gas.
ii. $N(15) = 225$; therefore, the vehicle can travel 225 mi on 3/4 of a tank of gas.
c. Domain: $0 \le x \le 20$; range: $[0, 300]$
d. The driver had to stop at least once, given that it takes approximately 39 gal of gas to drive a total of 578 mi.

54) [T] The volume V of a sphere depends on the length of its radius as $V = (4/3)\pi r^3$. Because Earth is not a perfect sphere, we can use the mean radius when measuring from the center to its surface. The mean radius is the average distance from the physical center to the surface, based on a large number of samples. Find the volume of Earth with mean radius 6.371×106 m.

55) [T] A certain bacterium grows in culture in a circular region. The radius of the circle, measured in centimeters, is given by $r(t) = 6 - \dfrac{5}{t^2+1}$, where t is time measured in hours since a circle of a 1-cm radius of the bacterium was put into the culture.

a. Express the area of the bacteria as a function of time.

b. Find the exact and approximate area of the bacterial culture in 3 hours.

c. Express the circumference of the bacteria as a function of time.

d. Find the exact and approximate circumference of the bacteria in 3 hours.

Answer

a. $A(t) = A(r(t)) = \pi \cdot (6 - \frac{5}{t^2+1})^2$
b. Exact: $\frac{121\pi}{4}$; approximately 95 cm^2
c. $C(t) = C(r(t)) = 2\pi(6 - \frac{5}{t^2+1})$
d. Exact: 11π; approximately 35 cm

56) [T] An American tourist visits Paris and must convert U.S. dollars to Euros, which can be done using the function $E(x) = 0.79x$, where x is the number of U.S. dollars and $E(x)$ is the equivalent number of Euros. Since conversion rates fluctuate, when the tourist returns to the United States 2 weeks later, the conversion from Euros to U.S. dollars is $D(x) = 1.245x$, where x is the number of Euros and $D(x)$ is the equivalent number of U.S. dollars.

a. Find the composite function that converts directly from U.S. dollars to U.S. dollars via Euros. Did this tourist lose value in the conversion process?

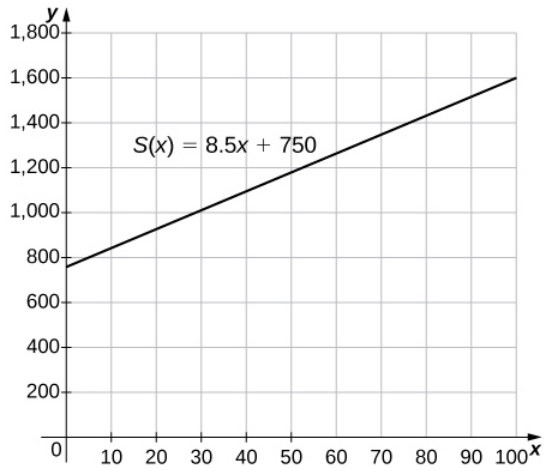
b. Use (a) to determine how many U.S. dollars the tourist would get back at the end of her trip if she converted an extra $200 when she arrived in Paris.

57) [T] The manager at a skateboard shop pays his workers a monthly salary S of $750 plus a commission of $8.50 for each skateboard they sell.

a. Write a function $y = S(x)$ that models a worker's monthly salary based on the number of skateboards x he or she sells.

b. Find the approximate monthly salary when a worker sells 25, 40, or 55 skateboards.

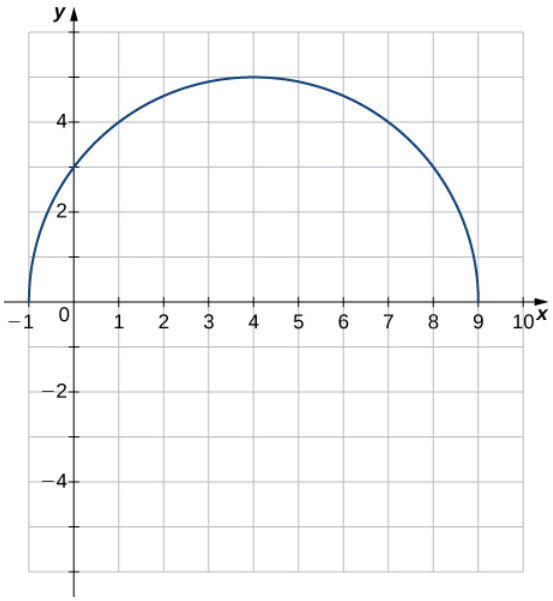
c. Use the INTERSECT feature on a graphing calculator to determine the number of skateboards that must be sold for a worker to earn a monthly income of $1400. (Hint: Find the intersection of the function and the line $y = 1400$.)



Answer

a. $S(x) = 8.5x + 750$ b. \$962.50, \$1090, \$1217.50 c. 77 skateboards

58) [T] Use a graphing calculator to graph the half-circle $y = \sqrt{25 - (x-4)^2}$. Then, use the INTERCEPT feature to find the value of both the x- and y-intercepts.



1.1E.1 Contributors

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1.2: Linear Functions

Learning Objectives

- Represent a linear function.
- Determine whether a linear function is increasing, decreasing, or constant.
- Interpret slope as a rate of change.
- Write and interpret an equation for a linear function.
- Graph linear functions.
- Determine whether lines are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.

Just as with the growth of a bamboo plant, there are many situations that involve constant change over time. Consider, for example, the first commercial maglev train in the world, the Shanghai MagLev Train (maglev train in the world, the Shanghai MagLev Train (Figure 1.2.1). It carries passengers comfortably for a 30-kilometer trip from the airport to the subway station in only eight minutes.



Figure 1.2.1: Shanghai MagLev Train (credit: “kanegen”/Flickr)

Suppose a maglev train were to travel a long distance, and that the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train’s distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train’s distance from the station at a given point in time.maglev train were to travel a long distance, and that the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train’s distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train’s distance from the station at a given point in time.

1.2.1 Representing Linear Functions

The function describing the train’s motion is a **linear function**, which is defined as a function with a constant rate of change, that is, a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train’s motion as a function using each method.

1.2.1.1 Representing a Linear Function in Word Form

Let’s begin by describing the linear function in words. For the train problem we just considered, the following word sentence may be used to describe the function relationship.

- *The train’s distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at constant speed.*

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

1.2.1.2 Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the form known as the slope-intercept form of a line, where x is the input value, m is the rate of change, and b is the initial value of the dependent variable.

$$\begin{aligned} &\text{Equation form} && y = mx + b \\ &\text{Equation notation} && f(x) = mx + b \end{aligned}$$

In the example of the train, we might use the notation $D(t)$ in which the total distance D is a function of the time t. The rate, m, is 83 meters per second. The initial value of the dependent variable b is the original distance from the station, 250 meters. We can write a generalized equation to represent the motion of the train.

$$D(t) = 83t + 250 \tag{1.2.1}$$

1.2.1.3 Representing a Linear Function in Tabular Form

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in Figure 1.2.2. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.

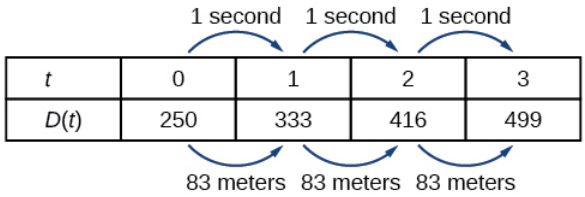


Figure 1.2.2: Tabular representation of the function D showing selected input and output values

Q/A? Can the input in the previous example be any real number?

No. The input represents time, so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

1.2.1.4 Representing a Linear Function in Graphical Form

Another way to represent linear functions is visually, using a graph. We can use the function relationship from above, $D(t) = 83t + 250$, to draw a graph, represented in Figure 1.2.3. Notice the graph is a line. When we plot a linear function, the graph is always a line.

The rate of change, which is constant, determines the slant, or **slope** of the line. The point at which the input value is zero is the vertical intercept, or **y-intercept**, of the line. We can see from the graph in Figure 1.2.3 that the y-intercept in the train example we just saw is $(0, 250)$ and represents the distance of the train from the station when it began moving at a constant speed.

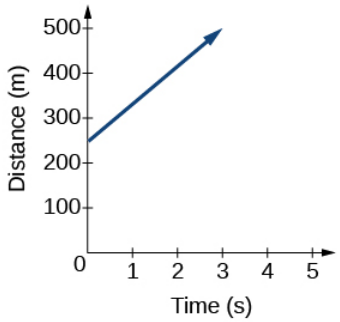


Figure 1.2.3: The graph of $D(t) = 83t + 250$. Graphs of linear functions are lines because the rate of change is constant.

Notice that the graph of the train example is restricted, but this is not always the case. Consider the graph of the line $f(x) = 2x + 1$. Ask yourself what numbers can be input to the function, that is, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled, and then have one added to the product.

Definition: Linear Function

A **linear function** is a function whose graph is a line. Linear functions can be written in the slope-intercept form of a line

$$f(x) = mx + b \tag{1.2.2}$$

where b is the initial or starting value of the function (when input, $x = 0$), and m is the constant rate of change, or **slope** of the function. The **y-intercept** is at $(0, b)$.

Example 1.2.1: Using a Linear Function to Find the Pressure on a Diver

The pressure, P, in pounds per square inch (PSI) on the diver in Figure 1.2.4 depends upon her depth below the water surface, d, in feet. This relationship may be modeled by the equation, $P(d) = 0.434d + 14.696$. Restate this function in words.



Figure 1.2.4: (credit: Ilse Reijs and Jan-Noud Hutten)Hutten)

To restate the function in words, we need to describe each part of the equation. The pressure as a function of depth equals four hundred thirty-four thousandths times depth plus fourteen and six hundred ninety-six thousandths.

Analysis

The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.

1.2.2 Determining whether a Linear Function Is Increasing, Decreasing, or Constant

The linear functions we used in the two previous examples increased over time, but not every linear function does. A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an **increasing function** has a positive slope. A line with a positive slope slants upward from left to right as in Figure 1.2.5(a). For a **decreasing function**, the slope is negative. The output values decrease as the input values increase. A line with a negative slope slants downward from left to right as in Figure 1.2.5(b). If the function is constant, the output values are the same for all input values so the slope is zero. A line with a slope of zero is horizontal as in Figure 1.2.5(c).

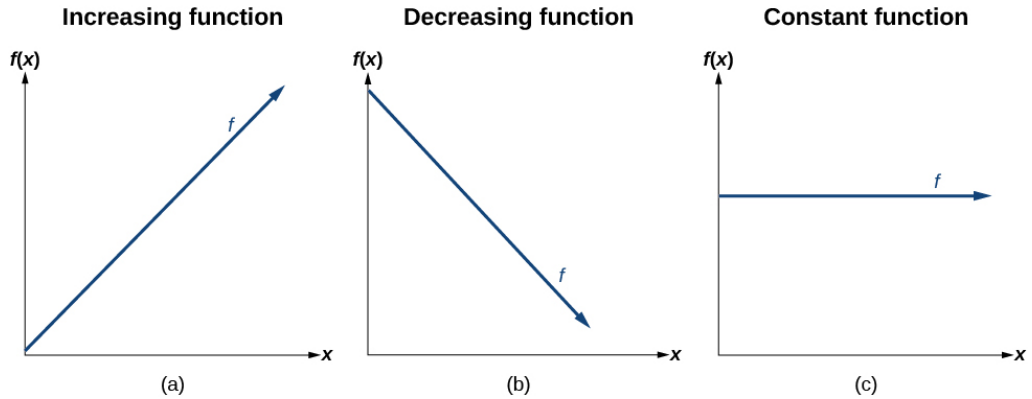


Figure 1.2.5: Three graphs depicting an increasing function, a decreasing function, and a constant function.]
Increasing and Decreasing Functions

Increasing and Decreasing Functions

The slope determines if the function is an **increasing linear function**, a **decreasing linear function**, or a constant function.

- $f(x) = mx + b$ is an increasing function if $m > 0$.
- $f(x) = mx + b$ is an decreasing function if $m < 0$.
- $f(x) = mx + b$ is a constant function if $m = 0$.

Example 1.2.2: decide whether a Function is Increasing, Decreasing, or Constant

Some recent studies suggest that a teenager sends an average of 60 texts per day. For each of the following scenarios, find the linear function that describes the relationship between the input value and the output value. Then, determine whether the graph of the function is increasing, decreasing, or constant.

a. The total number of texts a teen sends is considered a function of time in days. The input is the number of days, and output is the total number of texts sent.
b. A teen has a limit of 500 texts per month in his or her data plan. The input is the number of days, and output is the total number of texts remaining for the month.
c. A teen has an unlimited number of texts in his or her data plan for a cost of $50 per month. The input is the number of days, and output is the total cost of texting each month.

Solution

Analyze each function.

a. The function can be represented as $f(x) = 60x$ where x is the number of days. The slope, 60, is positive so the function is increasing. This makes sense because the total number of texts increases with each day.
b. The function can be represented as $f(x) = 500 - 60x$ where x is the number of days. In this case, the slope is negative so the function is decreasing. This makes sense because the number of texts remaining decreases each day and this function represents the number of texts remaining in the data plan after x days.
c. The cost function can be represented as $f(x) = 50$ because the number of days does not affect the total cost. The slope is 0 so the function is constant.

1.2.3 Calculating and Interpreting Slope

In the examples we have seen so far, we have had the slope provided for us. However, we often need to calculate the slope given input and output values. Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2—which can be represented by a set of points, (x_1, y_1) and (x_2, y_2)—we can calculate the slope m, as follows

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}}$$

$$= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where Δy is the vertical displacement and Δx is the horizontal displacement. Note in function notation two corresponding values for the output y_1 and y_2 for the function f, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Figure 1.2.6 indicates how the slope of the line between the points, (x_1, y_1) and (x_2, y_2), is calculated. Recall that the slope measures steepness. The greater the absolute value of the slope, the steeper the line is.

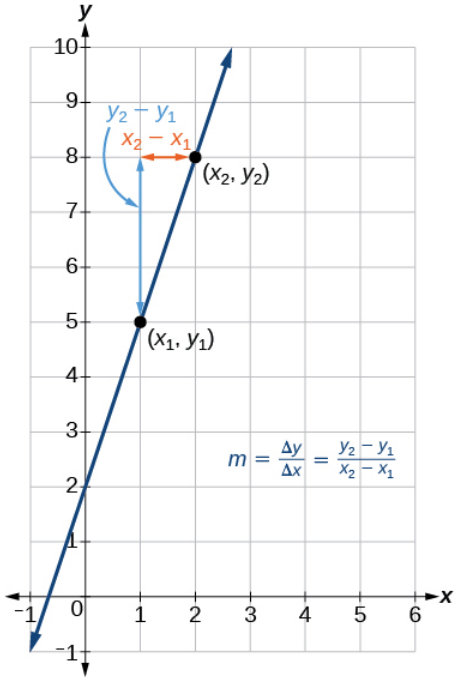


Figure 1.2.6: The slope of a function is calculated by the change in y divided by the change in x. It does not matter which coordinate is used as the (x_2, y_2) and which is the (x_1, y_1),as long as each calculation is started with the elements from the same coordinate pair.

Q & A

Are the units for slope always $\frac{\text{units for the output}}{\text{units for the input}}$?

Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

Note: Calculate Slope

The slope or rate of change of a function, m, can be calculated according to the following:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.2.3}$$

where x_1 and x_2 are input values, y_1 and y_2 are output values.

How to: Given two points from a linear function, calculate and interpret the slope.

1. Determine the units for output and input values.
2. Calculate the change of output values and change of input values.
3. Interpret the slope as the change in output values per unit of the input value.

Example 1.2.3: Finding the Slope of a Linear Function

If $f(x)$ is a linear function, and $(3,-2)$ and $(8,1)$ are points on the line, find the slope. Is this function increasing or decreasing?

Solution

The coordinate pairs are $(3,-2)$ and $(8,1)$. To find the rate of change, we divide the change in output by the change in input.

$$m=\frac{\text{change in output (rise)}}{\text{change in input (run)}}=\frac{1-(-2)}{8-3}=\frac{3}{5} \tag{1.2.4}$$

We could also write the slope as $m=0.6$. The function is increasing because $m>0$.

Analysis

As noted earlier, the order in which we write the points does not matter when we compute the slope of the line as long as the first output value, or *y*-coordinate, used corresponds with the first input value, or *x*-coordinate, used.

Try it! 1.2.1

If $f(x)$ is a linear function, and $(2,3)$ and $(0,4)$ are points on the line, find the slope. Is this function increasing or decreasing?

Answer

$m=\frac{4-3}{0-2}=\frac{1}{-2}=-\frac{1}{2}$; decreasing because $m<0$.

Example 1.2.4: Finding the Population Change from a Linear Function

The population of a city increased from 23,400 to 27,800 between 2008 and 2012. Find the change of population per year if we assume the change was constant from 2008 to 2012.

The rate of change relates the change in population to the change in time. The population increased by $27,800-23,400=4400$ people over the four-year time interval. To find the rate of change, divide the change in the number of people by the number of years.

$$\frac{4,400 \text{ people}}{4 \text{ years}}=1,100\frac{\text{people}}{\text{year}} \tag{1.2.5}$$

So the population increased by 1,100 people per year.

Analysis

Because we are told that the population increased, we would expect the slope to be positive. This positive slope we calculated is therefore reasonable.

Try it! 1.2.2

The population of a small town increased from 1,442 to 1,868 between 2009 and 2012. Find the change of population per year if we assume the change was constant from 2009 to 2012.

Answer

$m = \frac{1{,}868-1{,}442}{2{,}012-2{,}009} = \frac{426}{3} = 142$ people per year

1.2.4 Writing the Point-Slope Form of a Linear Equation

Up until now, we have been using the slope-intercept form of a linear equation to describe linear functions. Here, we will learn another way to write a linear function, the **point-slope form**.

$$y - y_1 = m(x - x_1) \tag{1.2.6}$$

The point-slope form is derived from the slope formula.

$$\begin{aligned} m &= \frac{y-y_1}{x-x_1} && \text{assuming } x \neq x_1 \\ m(x-x_1) &= \frac{y-y_1}{x-x_1}(x-x_1) && \text{Multiply both sides by } (x-x_1). \\ m(x-x_1) &= y-y_1 && \text{Simplify} \\ y-y_1 &= m(x-x_1) && \text{Rearrange} \end{aligned}$$

Keep in mind that the slope-intercept form and the point-slope form can be used to describe the same function. We can move from one form to another using basic algebra. For example, suppose we are given an equation in point-slope form, $y-4 = -\frac{1}{2}(x-6)$. We can convert it to the slope-intercept form as shown.

$$\begin{aligned} y-4 &= -\frac{1}{2}(x-6) \\ y-4 &= -\frac{1}{2}x+3 && \text{Distribute the } -\frac{1}{2}. \\ y &= -\frac{1}{2}x+7 && \text{Add 4 to each side.} \end{aligned}$$

Therefore, the same line can be described in slope-intercept form as $y = \frac{1}{2}x + 7$.

Note: Point-Slope Form of a Linear Equation

The **point-slope** form of a linear equation takes the form

$$y - y_1 = m(x - x_1) \tag{1.2.7}$$

where m is the slope, x_1 and y_1 are the x and y coordinates of a specific point through which the line passes.

1.2.4.1 Writing the Equation of a Line Using a Point and the Slope

The point-slope form is particularly useful if we know one point and the slope of a line. Suppose, for example, we are told that a line has a slope of 2 and passes through the point $(4, 1)$. We know that $m = 2$ and that $x_1 = 4$ and $y_1 = 1$. We can substitute these values into the general point-slope equation.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-1 &= 2(x-4) \end{aligned}$$

If we wanted to then rewrite the equation in slope-intercept form, we apply algebraic techniques.

$$\begin{aligned} y-1 &= 2(x-4) \\ y-1 &= 2x-8 && \text{Distribute the 2.} \\ y &= 2x-7 && \text{Add 1 to each side.} \end{aligned}$$

Both equations, $y-1=2(x-4)$ and $y=2x-7$, describe the same line. See Figure 1.2.7.

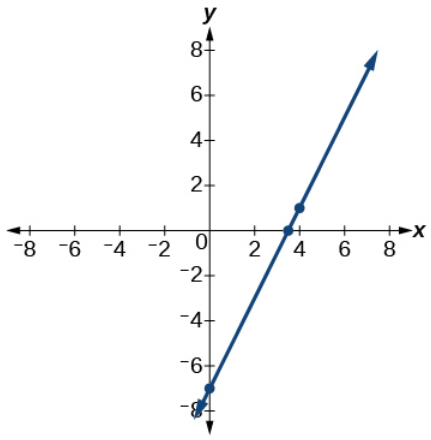


Figure 1.2.7

Example 1.2.5: Writing Linear Equations Using a Point and the Slope

Write the point-slope form of an equation of a line with a slope of 3 that passes through the point $(6,-1)$. Then rewrite it in the slope-intercept form.

Solution

Let's figure out what we know from the given information. The slope is 3, so $m=3$. We also know one point, so we know $x_1=6$ and $y_1=-1$. Now we can substitute these values into the general point-slope equation.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-(-1) &= 3(x-6) && \text{Substitute known values.} \\ y+1 &= 3(x-6) && \text{Distribute } -1 \text{ to find point-slope form.} \end{aligned}$$

Then we use algebra to find the slope-intercept form.

$$\begin{aligned} y+1 &= 3(x-6) \\ y+1 &= 3x-18 && \text{Distribute 3.} \\ y &= 3x-19 && \text{Simplify to slope-intercept form.} \end{aligned}$$

? Try it! 1.2.3

Write the point-slope form of an equation of a line with a slope of –2 that passes through the point $(-2,2)$. Then rewrite it in the slope-intercept form.

Answer

$y-2=-2(x+2)$; $y=-2x-2$

1.2.4.2 Writing the Equation of a Line Using Two Points

The point-slope form of an equation is also useful if we know any two points through which a line passes. Suppose, for example, we know that a line passes through the points $(0,1)$ and $(3,2)$. We can use the coordinates of the two points to find the slope.

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{2-1}{3-0} \\ &= \frac{1}{3} \end{aligned}$$

Now we can use the slope we found and the coordinates of one of the points to find the equation for the line. Let use $(0,1)$ for our point.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-1 &= \frac{1}{3}(x-0) \end{aligned}$$

As before, we can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y-1 &= \frac{1}{3}(x-0) && \\ y-1 &= \frac{1}{3}x && \text{Distribute the } \frac{1}{3}. \\ y &= \frac{1}{3}x+1 && \text{Add 1 to each side.} \end{aligned}$$

Both equations describe the line shown in Figure 1.2.8.

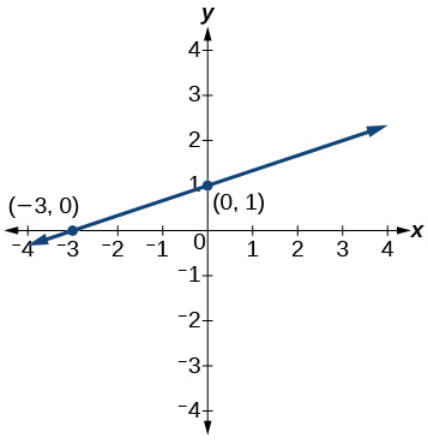


Figure 1.2.8

Example 1.2.6: Writing Linear Equations Using Two Points

Write the point-slope form of an equation of a line that passes through the points $(5,1)$ and $(8,7)$. Then rewrite it in the slope-intercept form.

Let's begin by finding the slope.

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{7-1}{8-5} \\ &= \frac{6}{3} \\ &= 2 \end{aligned}$$

So $m=2$. Next, we substitute the slope and the coordinates for one of the points into the general point-slope equation. We can choose either point, but we will use $(5,1)$.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-1 &= 2(x-5) \end{aligned}$$

The point-slope equation of the line is $y_2-1=2(x_2-5)$. To rewrite the equation in slope-intercept form, we use algebra.

$$\begin{aligned} y-1 &= 2(x-5) \\ y-1 &= 2x-10 \\ y &= 2x-9 \end{aligned}$$

The slope-intercept equation of the line is $y = 2x - 9$.

Try It! 1.2.4

Write the point-slope form of an equation of a line that passes through the points $(-1, 3)$ and $(0, 0)$. Then rewrite it in the slope-intercept form.

Answer

$y - 0 = -3(x - 0)$; $y = -3x$.

1.2.5 Writing and Interpreting an Equation for a Linear Function

Now that we have written equations for linear functions in both the slope-intercept form and the point-slope form, we can choose which method to use based on the information we are given. That information may be provided in the form of a graph, a point and a slope, two points, and so on. Look at the graph of the function f in Figure 1.2.9.

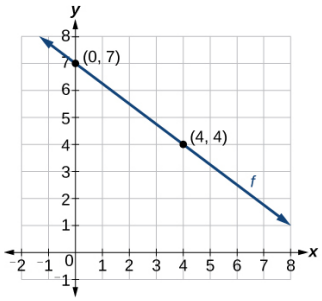


Figure 1.2.9: Graph depicting how to calculate the slope of a line.

We are not given the slope of the line, but we can choose any two points on the line to find the slope. Let's choose $(0, 7)$ and $(4, 4)$. We can use these points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 7}{4 - 0} \\ &= -\frac{3}{4} \end{aligned}$$

Now we can substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= -\frac{3}{4}(x - 4) \end{aligned}$$

If we want to rewrite the equation in the slope-intercept form, we would find

$$\begin{aligned} y - 4 &= -\frac{3}{4}(x - 4) \\ y - 4 &= -\frac{3}{4}x + 3 \\ y &= -\frac{3}{4}x + 7 \end{aligned}$$

If we wanted to find the slope-intercept form without first writing the point-slope form, we could have recognized that the line crosses the y-axis when the output value is 7. Therefore, $b = 7$. We now have the initial value b and the slope m so we can substitute m and b into the slope-intercept form of a line.

$$f(x) = mx + b$$

$$f(x) = -\frac{3}{4}x + 7$$

Figure 1.2.10

So the function is $f(x) = -\frac{3}{4}x + 7$, and the linear equation would be $y = -\frac{3}{4}x + 7$.

How to: Given the graph of a linear function, write an equation to represent the function.

1. Identify two points on the line.
2. Use the two points to calculate the slope.
3. Determine where the line crosses the y-axis to identify the y-intercept by visual inspection.
4. Substitute the slope and y-intercept into the slope-intercept form of a line equation.

Example 1.2.7: Writing an Equation for a Linear Function

Write an equation for a linear function given a graph of f shown in Figure 1.2.11.

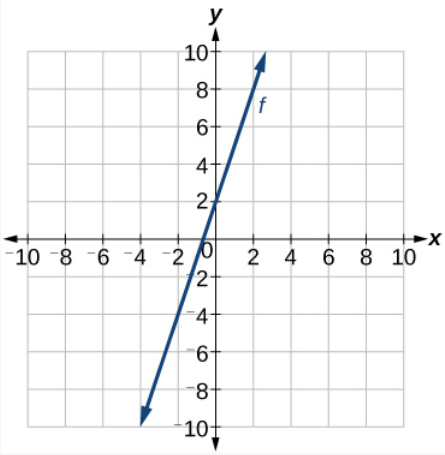


Figure 1.2.11

Solution

Identify two points on the line, such as $(0, 2)$ and $(-2, -4)$. Use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 2}{-2 - 0} \\ &= \frac{-6}{-2} \\ &= 3 \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-4) &= 3(x - (-2)) \\ y + 4 &= 3(x + 2) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y+4 &= 3(x+2) \\ y+4 &= 3x+6 \\ y &= 3x+2 \end{aligned}$$

Analysis

This makes sense because we can see from Figure 1.2.12 that the line crosses the y-axis at the point $(0, 2)$, which is the y-intercept, so $b = 2$.

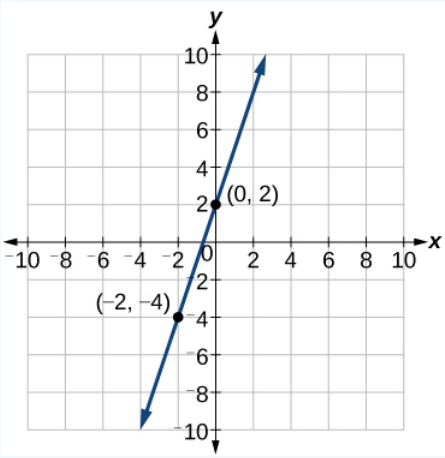


Figure 1.2.12: Graph of an increasing line with points at $(0, 2)$ and $(-2, -4)$.

Example 1.2.8: Writing an Equation for a Linear Cost Function

Suppose Ben starts a company in which he incurs a fixed cost of $1,250 per month for the overhead, which includes his office rent. His production costs are $37.50 per item. Write a linear function C where $C(x)$ is the cost for x items produced in a given month.

Solution

The fixed cost is present every month, $1,250. The costs that can vary include the cost to produce each item, which is $37.50 for Ben. The variable cost, called the marginal cost, is represented by 37.5. The cost Ben incurs is the sum of these two costs, represented by $C(x) = 1250 + 37.5x$.

Analysis

If Ben produces 100 items in a month, his monthly cost is represented by

$$\begin{aligned} C(100) &= 1250 + 37.5(100) \\ &= 5000 \end{aligned}$$

So his monthly cost would be $5,000.

Example 1.2.9: Writing an Equation for a Linear Function Given Two Points

If f is a linear function, with $f(3) = -2$, and $f(8) = 1$, find an equation for the function in slope-intercept form.

Solution

We can write the given points using coordinates.

$$\begin{aligned} f(3) = -2 &\rightarrow (3, 2) \\ f(8) = 1 &\rightarrow (8, 1) \end{aligned}$$

We can then use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1-(-2)}{8-3} \\ &= \frac{3}{5} \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y-(-2) &= \frac{3}{5}(x-3) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y+2 &= \frac{3}{5}(x-3) \\ y+2 &= \frac{3}{5}x - \frac{9}{5} \\ y &= \frac{3}{5}x - \frac{19}{5} \end{aligned}$$

? Try It! 1.2.5

If $f(x)$ is a linear function, with $f(2) = -11$, and $f(4) = -25$, find an equation for the function in slope-intercept form.

Answer

$y = -7x + 3$

1.2.6 Modeling Real-World Problems with Linear Functions

In the real world, problems are not always explicitly stated in terms of a function or represented with a graph. Fortunately, we can analyze the problem by first representing it as a linear function and then interpreting the components of the function. As long as we know, or can figure out, the initial value and the rate of change of a linear function, we can solve many different kinds of real-world problems.

How To: Given a linear function f and the initial value and rate of change, evaluate $f(c)$.

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into $f(x) = mx + b$.
3. Evaluate the function at $x = c$.

Example 1.2.10: Using a Linear Function to Determine the Number of Songs in a Music Collection

Marcus currently has 200 songs in his music collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his collection as a function of time, t, the number of months. How many songs will he own in a year?

Solution

The initial value for this function is 200 because he currently owns 200 songs, so $N(0) = 200$, which means that $b = 200$.

The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. Therefore we know that $m = 15$. We can substitute the initial value and the rate of change into the slope-intercept form of a line.

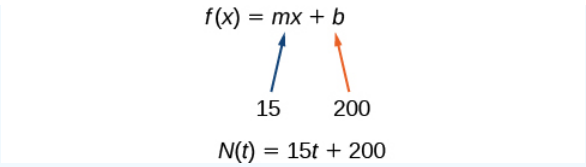


Figure 1.2.13

We can write the formula $N(t) = 15t + 200$.

With this formula, we can then predict how many songs Marcus will have in 1 year (12 months). In other words, we can evaluate the function at $t = 12$.

$$\begin{aligned} N(12) &= 15(12) + 200 \\ &= 180 + 200 \\ &= 380 \end{aligned}$$

Marcus will have 380 songs in 12 months.

Analysis

Notice that N is an increasing linear function. As the input (the number of months) increases, the output (number of songs) increases as well.

Example 1.2.11: Using a Linear Function to Calculate Salary Plus Commission

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya's weekly income, I,depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies and earned \$920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

Solution

The given information gives us two input-output pairs: $(3, 760)$ and $(5, 920)$. We start by finding the rate of change.

$$\begin{aligned} m &= \frac{920 - 760}{5 - 3} \\ &= \frac{\$160}{2 \text{ policies}} \\ &= \$80 \text{ per policy} \end{aligned}$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy. Therefore, Ilya earns a commission of \$80 for each policy sold during the week.

We can then solve for the initial value.

$$\begin{aligned} I(n) &= 80n + b \\ 760 &= 80(3) + b \text{ when } n = 3, I(3) = 760 \\ 760 - 80(3) &= b \\ 520 &= b \end{aligned}$$

The value of b is the starting value for the function and represents Ilya's income when $n = 0$, or when no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

We can now write the final equation.

$$I(n) = 80n + 520$$

Our final interpretation is that Ilya's base salary is \$520 per week and he earns an additional \$80 commission for each policy sold.

Example 1.2.12: Using Tabular Form to Write an Equation for a Linear Function

Table 1.2.1 relates the number of rats in a population to time, in weeks. Use the table to write a linear equation.

Table 1.2.1

w, number of weeks	0	2	4	6
P(w), number of rats	1000	1080	1160	1240

Solution

We can see from the table that the initial value for the number of rats is 1000, so $b = 1000$.

Rather than solving for m, we can tell from looking at the table that the population increases by 80 for every 2 weeks that pass. This means that the rate of change is 80 rats per 2 weeks, which can be simplified to 40 rats per week.

$$P(w) = 40w + 1000$$

If we did not notice the rate of change from the table we could still solve for the slope using any two points from the table. For example, using $(2, 1080)$ and $(6, 1240)$

$$\begin{aligned} m &= \frac{1240 - 1080}{6 - 2} \\ &= \frac{160}{4} \\ &= 40 \end{aligned}$$

Is the initial value always provided in a table of values?

No. Sometimes the initial value is provided in a table of values, but sometimes it is not. If you see an input of 0, then the initial value would be the corresponding output. If the initial value is not provided because there is no value of input on the table equal to 0, find the slope, substitute one coordinate pair and the slope into $f(x) = mx + b$, and solve for b.

Try It 1.2.6

A new plant food was introduced to a young tree to test its effect on the height of the tree. Table 1.2.2 shows the height of the tree, in feet, x months since the measurements began. Write a linear function, $H(x)$, where x is the number of months since the start of the experiment.

Table 1.2.2

x	0	2	4	8	12
$H(x)$	12.5	13.5	14.5	16.5	18.5

Answer

$H(x) = 0.5x + 12.5$

1.2.7 Key Equations

- slope-intercept form of a line: $f(x) = mx + b$

- slope: $m = \dfrac{\text{change in output (rise)}}{\text{change in input (run)}} = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$
- point-slope form of a line: $y - y_1 = m(x - x_1)$

1.2.8 Key Concepts

- The ordered pairs given by a linear function represent points on a line.
- Linear functions can be represented in words, function notation, tabular form, and graphical form.
- The rate of change of a linear function is also known as the slope.
- An equation in the slope-intercept form of a line includes the slope and the initial value of the function.
- The initial value, or y-intercept, is the output value when the input of a linear function is zero. It is the y-value of the point at which the line crosses the y-axis.
- An increasing linear function results in a graph that slants upward from left to right and has a positive slope.
- A decreasing linear function results in a graph that slants downward from left to right and has a negative slope.
- A constant linear function results in a graph that is a horizontal line.
- Analyzing the slope within the context of a problem indicates whether a linear function is increasing, decreasing, or constant.
- The slope of a linear function can be calculated by dividing the difference between y-values by the difference in corresponding x-values of any two points on the line.
- The slope and initial value can be determined given a graph or any two points on the line.
- One type of function notation is the slope-intercept form of an equation.
- The point-slope form is useful for finding a linear equation when given the slope of a line and one point.
- The point-slope form is also convenient for finding a linear equation when given two points through which a line passes.
- The equation for a linear function can be written if the slope m and initial value b are known.
- A linear function can be used to solve real-world problems.
- A linear function can be written from tabular form.

1.2.9 Footnotes

1 www.chinahighlights.com/shang...glev-train.htm
2 www.cbsnews.com/8301-501465_1...ay-study-says/

Glossary

decreasing linear function

a function with a negative slope: If $f(x) = mx + b$, then $m < 0$.

increasing linear function

a function with a positive slope: If $f(x) = mx + b$, then $m > 0$.

linear function

a function with a constant rate of change that is a polynomial of degree 1, and whose graph is a straight line

point-slope form

the equation for a line that represents a linear function of the form \(y−y_1=m(x−x_1)

slope

the ratio of the change in output values to the change in input values; a measure of the steepness of a line

slope-intercept form

the equation for a line that represents a linear function in the form $f(x) = mx + b$

y-intercept

the value of a function when the input value is zero; also known as initial value

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1.3: Systems of Linear Equations - Two Variables

Learning Objectives

- Solve systems of equations by graphing.
- Solve systems of equations by substitution.
- Solve systems of equations by addition.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables.

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.



Figure 1.3.1

1.3.0.0.1 Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. **A system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

$$\begin{aligned} 2x+y &= 15 \\ 3x-y &= 5 \end{aligned}$$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair $(4,7)$ is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

$$\begin{aligned} 2(4)+(7) &= 15 \text{ True} \\ 3(4)-(7) &= 5 \text{ True} \end{aligned}$$

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A consistent system of equations has at least one solution. A consistent system is considered to be an **independent system** if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same y-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different y-intercepts. There are no points common to both lines; hence, there is no solution to the system.

TYPES OF LINEAR SYSTEMS

There are three types of systems of linear equations in two variables, and three types of solutions.

- An independent system has exactly one solution pair (x, y). The point where the two lines intersect is the only solution.
- An inconsistent system has no solution. Notice that the two lines are parallel and will never intersect.
- A dependent system has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

Figure $1.3.2$ compares graphical representations of each type of system.

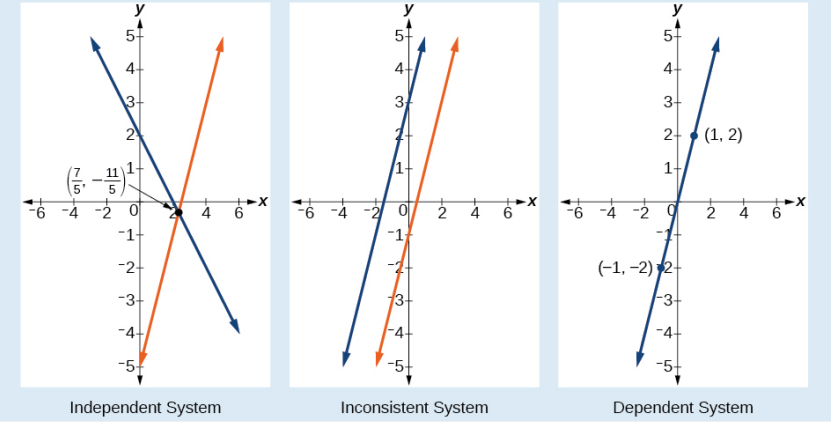


Figure 1.3.2

Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

Example 1.3.1: Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5, 1)$ is a solution to the given system of equations.

$$\begin{aligned} x + 3y &= 8 \\ 2x - 9 &= y \end{aligned}$$

Solution

Substitute the ordered pair $(5, 1)$ into both equations.

$$\begin{aligned} (5) + 3(1) &= 8 \\ 8 &= 8 \text{ True} \\ 2(5) - 9 &= (1) \\ 1 &= 1 \text{ True} \end{aligned}$$

The ordered pair $(5, 1)$ satisfies both equations, so it is the solution to the system.

Analysis

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See Figure $1.3.3$.

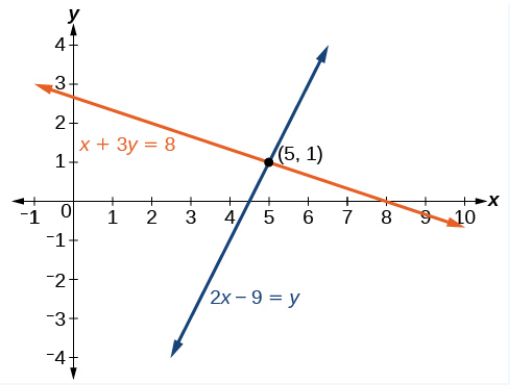


Figure 1.3.3

? Exercise 1.3.1

Determine whether the ordered pair $(8, 5)$ is a solution to the following system.

$$\begin{aligned} 5x - 4y &= 20 \\ 2x + 1 &= 3y \end{aligned}$$

Answer

Not a solution.

1.3.0.0.1 Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

✓ Example 1.3.2: Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.

$$\begin{aligned} 2x + y &= -8 \\ x - y &= -1 \end{aligned}$$

Solution

Solve the first equation for y.

$$\begin{aligned} 2x + y &= -8 \\ y &= -2x - 8 \end{aligned}$$

Solve the second equation for y.

$$\begin{aligned} x - y &= -1 \\ y &= x + 1 \end{aligned}$$

Graph both equations on the same set of axes as in Figure 1.3.4.

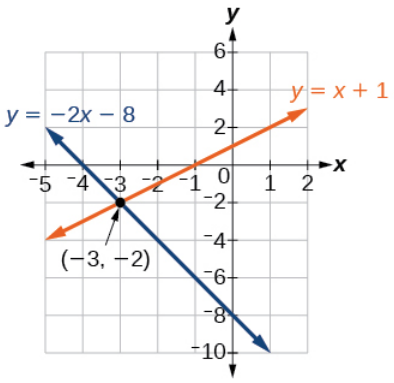


Figure 1.3.4

The lines appear to intersect at the point $(-3,-2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

$$\begin{aligned} 2(-3)+(-2) &= -8 \\ -8 &= -8 \text{ True} \\ (-3)-(-2) &= -1 \\ -1 &= -1 \text{ True} \end{aligned}$$

The solution to the system is the ordered pair $(-3,-2)$,so the system is independent.

Exercise 1.3.2

Solve the following system of equations by graphing.

$$\begin{aligned} 2x-5y &= -25 \\ -4x+5y &= 35 \end{aligned}$$

Answer

The solution to the system is the ordered pair $(-5,3)$.

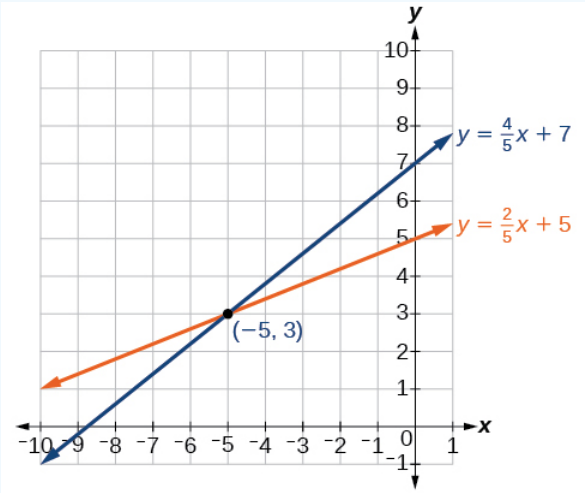


Figure 1.3.5

Q&A

Can graphing be used if the system is inconsistent or dependent?

Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent

system.

1.3.0.0.1 Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a **system of linear equations** that are more precise than graphing. One such method is solving a system of equations by the **substitution method**, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

How to: Given a system of two equations in two variables, solve using the substitution method.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

Example 1.3.3: Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.

$$\begin{aligned} -x+y &= -5 \\ 2x-5y &= 1 \end{aligned}$$

Solution

First, we will solve the first equation for y.

$$\begin{aligned} -x+y &= -5 \\ y &= x-5 \end{aligned}$$

Now we can substitute the expression $x-5$ for y in the second equation.

$$\begin{aligned} 2x-5y &= 1 \\ 2x-5(x-5) &= 1 \\ 2x-5x+25 &= 1 \\ -3x &= -24 \\ x &= 8 \end{aligned}$$

Now, we substitute $x=8$ into the first equation and solve for y.

$$\begin{aligned} -(8)+y &= -5 \\ y &= 3 \end{aligned}$$

Our solution is $(8,3)$.

Check the solution by substituting $(8,3)$ into both equations.

$$\begin{aligned} -x+y &= -5 \\ -(8)+(3) &= -5 \text{ True} \\ 2x-5y &= 1 \\ 2(8)-5(3) &= 1 \text{ True} \end{aligned}$$

Exercise 1.3.3

Solve the following system of equations by substitution.

$$x = y + 3$$
$$4 = 3x - 2y$$

Answer

$(-2, -5)$

Q&A

Can the substitution method be used to solve any linear system in two variables?

Yes, but the method works best if one of the equations contains a coefficient of 1 *or* -1 *so that we do not have to deal with fractions.*

1.3.0.0.1 Solving Systems of Equations in Two Variables by the Addition Method

A third method of solving systems of linear equations is the addition method. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

How to: Given a system of equations, solve using the addition method.

1. Write both equations with *x*- and *y*-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

Example 1.3.4: Solving a System by the Addition Method

Solve the given system of equations by addition.

$$x + 2y = -1$$
$$-x + y = 3$$

Solution

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, -1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.

$$\begin{aligned} x + 2y &= -1 \\ \underline{-x + y} &= \underline{3} \\ 3y &= 2 \end{aligned}$$

Now that we have eliminated x, we can solve the resulting equation for y.

$$3y = 2$$
$$y = \frac{2}{3}$$

Then, we substitute this value for y into one of the original equations and solve for x.

$$\begin{aligned} -x+y &= 3 \\ -x+\frac{2}{3} &= 3 \\ -x &= 3-\frac{2}{3} \\ -x &= \frac{7}{3} \\ x &= -\frac{7}{3} \end{aligned}$$

The solution to this system is $\left(-\frac{7}{3}, \frac{2}{3}\right)$.

Check the solution in the first equation.

$$\begin{aligned} x+2y &= -1 \\ \left(-\frac{7}{3}\right)+2\left(\frac{2}{3}\right) &= \\ -\frac{7}{3}+\frac{4}{3} &= -\frac{3}{3} \\ -1 &= -1 \qquad \text{True} \end{aligned}$$

Analysis

We gain an important perspective on systems of equations by looking at the graphical representation. See Figure 1.3.6 to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.

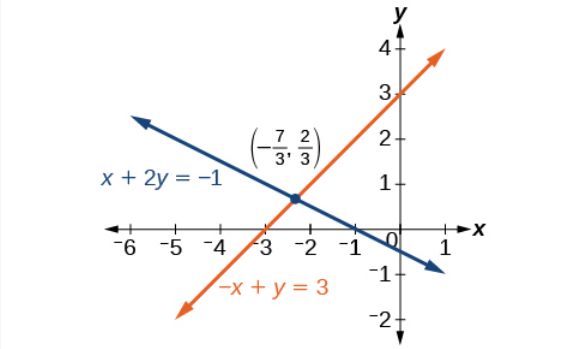


Figure 1.3.6

Example 1.3.5: Using the Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the addition method.

$$\begin{aligned} 3x+5y &= -11 \\ x-2y &= 11 \end{aligned}$$

Solution

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So if we multiply the second equation by -3,the x-terms will add to zero.

$$\begin{aligned} x-2y &= 11 \\ -3(x-2y) &= -3(11) \qquad \text{Multiply both sides by } -3. \\ -3x+6y &= -33 \qquad \text{Use the distributive property.} \end{aligned}$$

Now, let's add them.

$$\begin{aligned} 3x+5y &= -11 \\ \underline{-3x+6y} &= \underline{-33} \\ 11y &= -44 \\ y &= -4 \end{aligned}$$

For the last step, we substitute $y=-4$ into one of the original equations and solve for x.

$$\begin{aligned} 3x+5y &= -11 \\ 3x+5(-4) &= -11 \\ 3x-20 &= -11 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Our solution is the ordered pair $(3,-4)$. See Figure 1.3.7. Check the solution in the original second equation.

$$\begin{aligned} x-2y &= 11 \\ (3)-2(-4) &= 3+8 \\ &= 11 \qquad \text{True} \end{aligned}$$

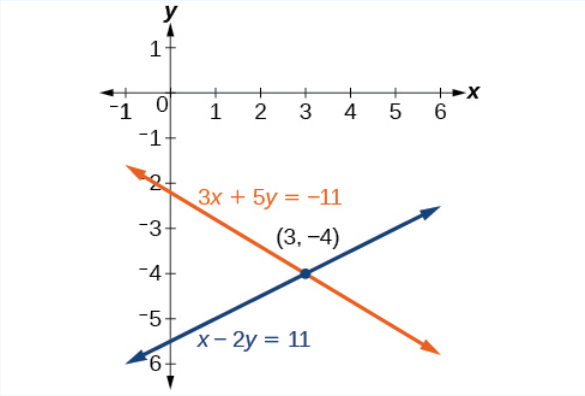


Figure 1.3.7

Exercise 1.3.4

Solve the system of equations by addition.

$$\begin{aligned} 2x-7y &= 2 \\ 3x+y &= -20 \end{aligned}$$

Answer

$(-6,-2)$

Example 1.3.6: Using the Addition Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables by addition.

$$\begin{aligned} 2x+3y &= -16 \\ 5x-10y &= 30 \end{aligned}$$

Solution

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate x by multiplying the first equation by -5 and the second equation by 2.

$$\begin{aligned} -5(2x+3y) &= -5(-16) \\ -10x-15y &= 80 \\ 2(5x-10y) &= 2(30) \\ 10x-20y &= 60 \end{aligned}$$

Then, we add the two equations together.

$$\begin{aligned} -10x-15y &= 80 \\ \underline{10x-20y} &= \underline{60} \\ -35y &= 140 \\ y &= -4 \end{aligned}$$

Substitute $y=-4$ into the original first equation.

$$\begin{aligned} 2x+3(-4) &= -16 \\ 2x-12 &= -16 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

The solution is $(-2,-4)$. Check it in the other equation.

$$\begin{aligned} 5x-10y &= 30 \\ 5(-2)-10(-4) &= 30 \\ -10+40 &= 30 \\ 30 &= 30 \end{aligned}$$

See Figure 1.3.8.

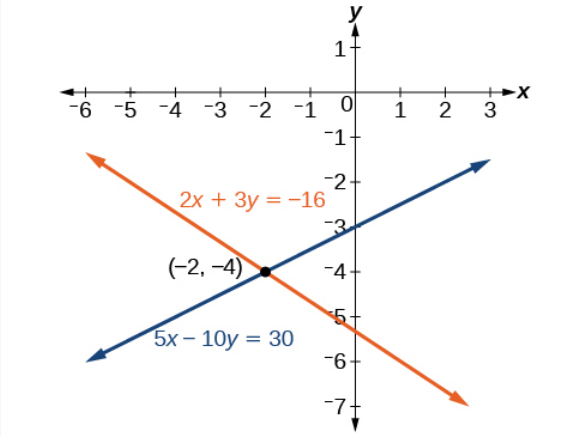


Figure 1.3.8

Example 1.3.7: Using the Addition Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables by addition.

$$\begin{aligned} \frac{x}{3}+\frac{y}{6} &= 3 \\ \frac{x}{2}-\frac{y}{4} &= 1 \end{aligned}$$

Solution

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.

$$6\left(\frac{x}{3}+\frac{y}{6}\right)=6(3)$$
$$2x+y=18$$
$$4\left(\frac{x}{2}-\frac{y}{4}\right)=4(1)$$
$$2x-y=4$$

Now multiply the second equation by -1 so that we can eliminate the x-variable.

$$-1(2x-y)=-1(4)$$
$$-2x+y=-4$$

Add the two equations to eliminate the x-variable and solve the resulting equation.

$$2x+y=18$$
$$-2x+y=-4$$
$$2y=14$$
$$y=7$$

Substitute $y=7$ into the first equation.

$$2x+(7)=18$$
$$2x=11$$
$$x=\frac{11}{2}$$
$$=7.5$$

The solution is $\left(\frac{11}{2},7\right)$. Check it in the other equation.

$$\frac{x}{2}-\frac{y}{4}=1$$
$$\frac{\frac{11}{2}}{2}-\frac{7}{4}=1$$
$$\frac{11}{4}-\frac{7}{4}=1$$
$$\frac{4}{4}=1$$

? Exercise 1.3.5

Solve the system of equations by addition.

$$2x+3y=8$$
$$3x+5y=10$$

Answer

$(10,-4)$

1.3.0.0.1 Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12=0$.

Example 1.3.8: Solving an Inconsistent System of Equations

Solve the following system of equations.

$$\begin{aligned} x &= 9-2y \\ x+2y &= 13 \end{aligned}$$

Solution

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.

$$\begin{aligned} x+2y &= 13 \\ (9-2y)+2y &= 13 \\ 9+0y &= 13 \\ 9 &= 13 \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

$$\begin{aligned} x &= 9-2y \\ 2y &= -x+9 \\ y &= -\frac{1}{2}x+\frac{9}{2} \end{aligned}$$

We then convert the second equation expressed to slope-intercept form.

$$\begin{aligned} x+2y &= 13 \\ 2y &= -x+13 \\ y &= -\frac{1}{2}x+\frac{13}{2} \end{aligned}$$

Comparing the equations, we see that they have the same slope but different y-intercepts. Therefore, the lines are parallel and do not intersect.

$$\begin{aligned} y &= -\frac{1}{2}x+\frac{9}{2} \\ y &= -\frac{1}{2}x+\frac{13}{2} \end{aligned}$$

Analysis

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in Figure 1.3.9.

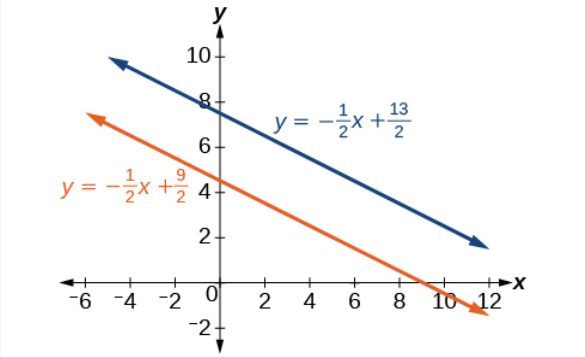


Figure 1.3.9

https://math.libretexts.org/@go/page/149850

? Exercise 1.3.6

Solve the following system of equations in two variables.

$$\begin{aligned} 2y-2x &= 2 \\ 2y-2x &= 6 \end{aligned}$$

Answer

No solution. It is an inconsistent system.

1.3.0.0.1 Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as $0=0$.

✓ Example 1.3.9: Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the **addition method**.

$$\begin{aligned} x+3y &= 2 \\ 3x+9y &= 6 \end{aligned}$$

Solution

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x-variable.

$$\begin{aligned} x+3y &= 2 \\ (-3)(x+3y) &= (-3)(2) \\ -3x-9y &= -6 \end{aligned}$$

Now add the equations.

$$\begin{aligned} -3x-9y &= -6 \\ \underline{+\,3x+9y} &= \underline{6} \\ 0 &= 0 \end{aligned}$$

We can see that there will be an infinite number of solutions that satisfy both equations.

Analysis

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let's look at what happens when we convert the system to slope-intercept form.

$$\begin{aligned} x+3y &= 2 \\ 3y &= -x+2 \\ y &= -\frac{1}{3}x+\frac{2}{3} \\ 3x+9y &= 6 \\ 9y &= -3x+6 \\ y &= -\frac{3}{9}x+\frac{6}{9} \\ y &= -\frac{1}{3}x+\frac{2}{3} \end{aligned}$$

See Figure 1.3.10. Notice the results are the same. The general solution to the system is $\left(x,-\frac{1}{3}x+\frac{2}{3}\right)$.

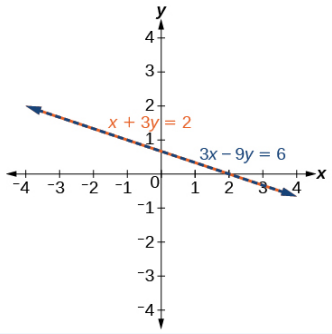


Figure 1.3.10

? Exercise 1.3.7

Solve the following system of equations in two variables.

$$\begin{aligned} y-2x &= 5 \\ -3y+6x &= -15 \end{aligned}$$

Answer

The system is dependent so there are infinite solutions of the form $(x, 2x+5)$.

1.3.1 Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's revenue function is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R = xp$, where x=quantity and p=price. The revenue function is shown in orange in Figure 1.3.11.

The cost function is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in Figure 1.3.11. The x-axis represents quantity in hundreds of units. The y-axis represents either cost or revenue in hundreds of dollars.

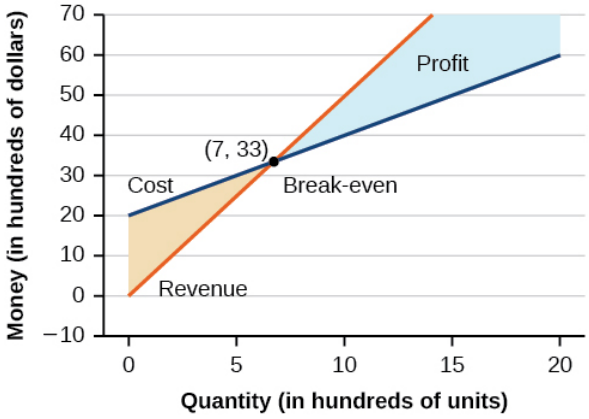


Figure 1.3.11

The point at which the two lines intersect is called the break-even point. We can see from the graph that if 700 units are produced, the cost is $\$3,300$ and the revenue is also $\$3,300$. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The profit function is the revenue function minus the cost function, written as $P(x) = R(x) - C(x)$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

Example 1.3.10: Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C(x) = 0.85x + 35,000$ and the revenue function $R(x) = 1.55x$, find the break-even point and the profit function.

Solution

Write the system of equations using y to replace function notation.

$$\begin{aligned} y &= 0.85x + 35,000 \\ y &= 1.55x \end{aligned}$$

Substitute the expression $0.85x + 35,000$ from the first equation into the second equation and solve for x.

$$\begin{aligned} 0.85x + 35,000 &= 1.55x \\ 35,000 &= 0.7x \\ 50,000 &= x \end{aligned}$$

Then, we substitute $x = 50,000$ into either the cost function or the revenue function.

$$1.55(50,000) = 77,500$$

The break-even point is $(50,000, 77,500)$.

The profit function is found using the formula $P(x) = R(x) - C(x)$.

$$\begin{aligned} P(x) &= 1.55x - (0.85x + 35,000) \\ &= 0.7x - 35,000 \end{aligned}$$

The profit function is $P(x) = 0.7x - 35,000$.

Analysis

The cost to produce $50,000$ units is $\$77,500$, and the revenue from the sales of $50,000$ units is also $\$77,500$. To make a profit, the business must produce and sell more than $50,000$ units. See Figure 1.3.12.

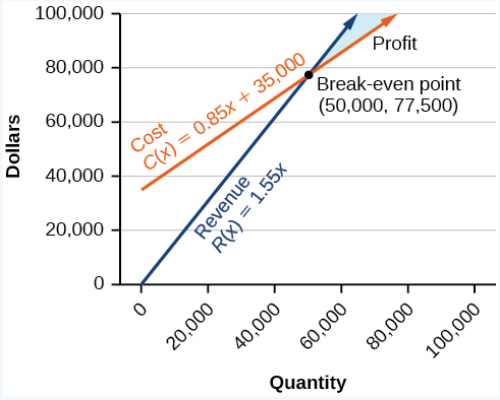


Figure 1.3.12

We see from the graph in Figure 1.3.13 that the profit function has a negative value until $x = 50,000$, when the graph crosses the x-axis. Then, the graph emerges into positive y-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.

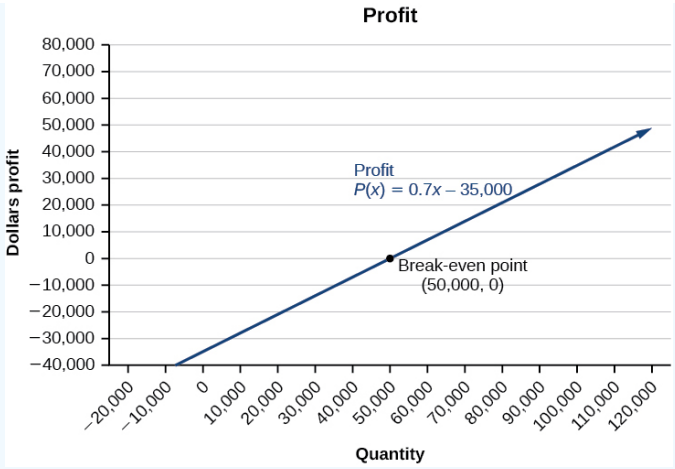


Figure 1.3.13

Example 1.3.11: Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $\$25.00$ for children and $\$50.00$ for adults. On a certain day, attendance at the circus is $2,000$ and the total gate revenue is $\$70,000$. How many children and how many adults bought tickets?

Solution

Let c = the number of children and a = the number of adults in attendance.

The total number of people is $2,000$. We can use this to write an equation for the number of people at the circus that day.

$$c+a=2,000$$

The revenue from all children can be found by multiplying $\$25.00$ by the number of children, $25c$. The revenue from all adults can be found by multiplying $\$50.00$ by the number of adults, $50a$. The total revenue is $\$70,000$. We can use this to write an equation for the revenue.

$$25c+50a=70,000$$

We now have a system of linear equations in two variables.

$$c+a=2,000$$
$$25c+50a=70,000$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c or a. We will solve for a.

$$\begin{aligned} c+a &= 2,000 \\ a &= 2,000-c \end{aligned}$$

Substitute the expression $2,000-c$ in the second equation fora a and solve for c.

$$\begin{aligned} 25c+50(2,000-c) &= 70,000 \\ 25c+100,000-50c &= 70,000 \\ -25c &= -30,000 \\ c &= 1,200 \end{aligned}$$

Substitute $c=1,200$ into the first equation to solve for a.

$$\begin{aligned} 1,200+a &= 2,000 \\ a &= 800 \end{aligned}$$

We find that $1,200$ children and 800 adults bought tickets to the circus that day.

https://math.libretexts.org/@go/page/149850

Exercise 1.3.8

Meal tickets at the circus cost $\$4.00$ for children and $\$12.00$ for adults. If $1,650$ meal tickets were bought for a total of, $\$14,200$, how many children and how many adults bought meal tickets?

Answer

700 children, 950 adults

Media

Access these online resources for additional instruction and practice with systems of linear equations.

- Solving Systems of Equations Using Substitution
- Solving Systems of Equations Using Elimination
- Applications of Systems of Equations

1.3.1.0.1 Key Concepts

- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. See Example 1.3.1.
- Systems of equations are classified as independent with one solution, dependent with an infinite number of solutions, or inconsistent with no solution.
- One method of solving a system of linear equations in two variables is by graphing. In this method, we graph the equations on the same set of axes. See Example 1.3.2.
- Another method of solving a system of linear equations is by substitution. In this method, we solve for one variable in one equation and substitute the result into the second equation. See Example 1.3.3.
- A third method of solving a system of linear equations is by addition, in which we can eliminate a variable by adding opposite coefficients of corresponding variables. See Example 1.3.4.
- It is often necessary to multiply one or both equations by a constant to facilitate elimination of a variable when adding the two equations together. See Example 1.3.5, Example 1.3.6, and Example 1.3.7.
- Either method of solving a system of equations results in a false statement for inconsistent systems because they are made up of parallel lines that never intersect. See Example 1.3.8.
- The solution to a system of dependent equations will always be true because both equations describe the same line. See Example 1.3.9.
- Systems of equations can be used to solve real-world problems that involve more than one variable, such as those relating to revenue, cost, and profit. See Example 1.3.10 and Example 1.3.11.

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1.4: Systems of Linear Equations with Three Variables

Learning Objectives

- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables.

John received an inheritance of $\$12,000$ that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. John invested $\$4,000$ more in municipal funds than in municipal bonds. He earned $\$670$ in interest the first year. How much did John invest in each type of fund?



Figure 1.4.1: (credit: "Elembis," Wikimedia Commons)

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

1.4.0.0.1 Solving Systems of Three Equations in Three Variables

In order to solve systems of equations in three variables, known as three-by-three systems, the primary tool we will be using is called Gaussian elimination, named after the prolific German mathematician Karl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve upper triangular form, the ideal form for a three-by-three system because it allows for straightforward back-substitution to find a solution (x, y, z), which we call an ordered triple. A system in upper triangular form looks like the following:

$$\begin{aligned} Ax + By + Cz &= D \\ Ey + Fz &= G \\ Hz &= K \end{aligned}$$

The third equation can be solved for z,and then we back-substitute to find y and x. To write the system in upper triangular form, we can perform the following operations:

1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

The **solution set** to a three-by-three system is an ordered triple (x, y, z). Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

NUMBER OF POSSIBLE SolutionS

Figure 1.4.2 and Figure 1.4.3 illustrate possible solution scenarios for three-by-three systems.

- Systems that have a single solution are those which, after elimination, result in a solution set consisting of an ordered triple (x, y, z). Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0 = 0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3 = 0$. Graphically, a system with no solution is represented by three planes with no point in common.

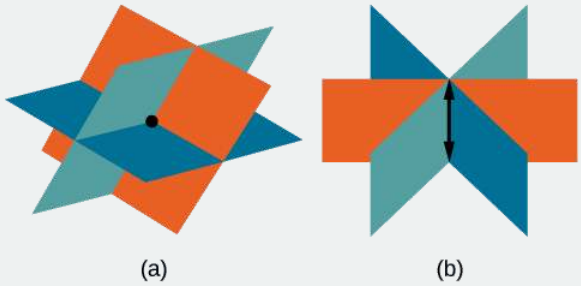


Figure 1.4.2: (a)Three planes intersect at a single point, representing a three-by-three system with a single solution. (b) Three planes intersect in a line, representing a three-by-three system with infinite solutions.

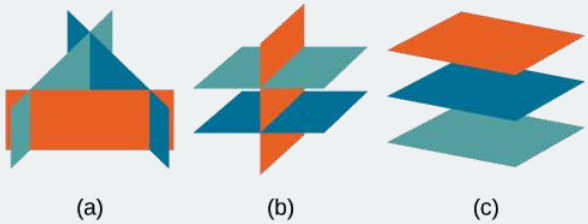


Figure 1.4.3: All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other, but not at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection.

Example 1.4.1: Determining Whether an Ordered Triple Is a Solution to a System

Determine whether the ordered triple $(3, -2, 1)$ is a solution to the system.

$$\begin{aligned} x + y + z &= 2 \\ 6x - 4y + 5z &= 31 \\ 5x + 2y + 2z &= 13 \end{aligned}$$

Solution

We will check each equation by substituting in the values of the ordered triple for x, y, and z.

$x + y + z = 2$	$6x - 4y + 5z = 31$	$5x + 2y + 2z = 13$
$(3) + (-2) + (1) = 2$	$6(3) - 4(-2) + 5(1) = 31$	$5(3) + 2(-2) + 2(1) = 13$
True	$18 + 8 + 5 = 31$	$15 - 4 + 2 = 13$
	True	True

The ordered triple $(3, -2, 1)$ is indeed a solution to the system.

How to: Given a linear system of three equations, solve for three unknowns

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.

4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

Example 1.4.2: Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:

$$\begin{aligned} x-2y+3z &= 9 \quad (1)\\ -x+3y-z &= -6 \quad (2)\\ 2x-5y+5z &= 17 \quad (3) \end{aligned}$$

Solution

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate x by adding equations (1) and (2).

$$\begin{aligned} x-2y+3z &= 9 \quad (1)\\ \underline{-x+3y-z} &\underline{= -6 \quad (2)}\\ y+2z &= 3 \quad (4) \end{aligned}$$

The second step is multiplying equation (1) by -2 and adding the result to equation (3). These two steps will eliminate the variable x.

$$\begin{aligned} -2x+4y-6z &= -18 \quad (1) \quad \text{multiplied by } -2\\ \underline{2x-5y+5z} &\underline{= 17 \quad (3)}\\ -y-z &= -1 \quad (5) \end{aligned}$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for z by adding the two equations.

$$\begin{aligned} y+2z &= 3 \quad (4)\\ \underline{-y-z} &\underline{= -1 \quad (5)}\\ z &= 2 \quad (6) \end{aligned}$$

Choosing one equation from each new system, we obtain the upper triangular form:

$$\begin{aligned} x-2y+3z &= 9 \quad (1)\\ y+2z &= 3 \quad (4)\\ z &= 2 \quad (6) \end{aligned}$$

Next, we back-substitute $z=2$ into equation (4) and solve for y.

$$\begin{aligned} y+2(2) &= 3\\ y+4 &= 3\\ y &= -1 \end{aligned}$$

Finally, we can back-substitute $z=2$ and $y=-1$ into equation (1). This will yield the solution for x.

$$\begin{aligned} x-2(-1)+3(2) &= 9\\ x+2+6 &= 9\\ x &= 1 \end{aligned}$$

The solution is the ordered triple $(1,-1,2)$. See Figure 1.4.4.

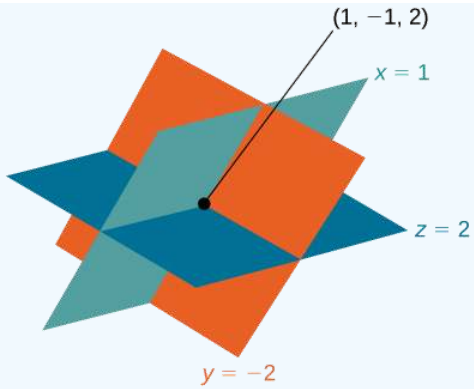


Figure 1.4.4

Example 1.4.3: Solving a Real-World Problem Using a System of Three Equations in Three Variables

In the problem posed at the beginning of the section, John invested his inheritance of $\$12,000$ in three different funds: part in a money-market fund paying 3% interest annually; part in municipal bonds paying 4% annually; and the rest in mutual funds paying 7% annually. John invested $\$4,000$ more in mutual funds than he invested in municipal bonds. The total interest earned in one year was $\$670$. How much did he invest in each type of fund?

Solution

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

$$x = \text{amount invested in money-market fund}$$
$$y = \text{amount invested in municipal bonds}$$
$$z = \text{amount invested in mutual funds}$$

The first equation indicates that the sum of the three principal amounts is $\$12,000$.

$$x + y + z = 12,000$$

We form the second equation according to the information that John invested $\$4,000$ more in mutual funds than he invested in municipal bonds.

$$z = y + 4,000$$

The third equation shows that the total amount of interest earned from each fund equals $\$670$.

$$0.03x + 0.04y + 0.07z = 670$$

Then, we write the three equations as a system.

$$\begin{aligned} x + y + z &= 12,000 \\ -y + z &= 4,000 \\ 0.03x + 0.04y + 0.07z &= 670 \end{aligned}$$

To make the calculations simpler, we can multiply the third equation by 100. Thus,

$$\begin{aligned} x + y + z &= 12,000 \quad (1) \\ -y + z &= 4,000 \quad (2) \\ 3x + 4y + 7z &= 67,000 \quad (3) \end{aligned}$$

Step 1. Interchange equation (2) and equation (3) so that the two equations with three variables will line up.

$$\begin{aligned} x+y+z &= 12,000 \\ 3x+4y+7z &= 67,000 \\ -y+z &= 4,000 \end{aligned}$$

Step 2. Multiply equation (1) by -3 and add to equation (2). Write the result as row 2.

$$\begin{aligned} x+y+z &= 12,000 \\ y+4z &= 31,000 \\ -y+z &= 4,000 \end{aligned}$$

Step 3. Add equation (2) to equation (3) and write the result as equation (3).

$$\begin{aligned} x+y+z &= 12,000 \\ y+4z &= 31,000 \\ 5z &= 35,000 \end{aligned}$$

Step 4. Solve for z in equation (3). Back-substitute that value in equation (2) and solve for y. Then, back-substitute the values for z and y into equation (1) and solve for x.

$$\begin{aligned} 5z &= 35,000 \\ z &= 7,000 \end{aligned}$$

$$\begin{aligned} y+4(7,000) &= 31,000 \\ y &= 3,000 \end{aligned}$$

$$\begin{aligned} x+3,000+7,000 &= 12,000 \\ x &= 2,000 \end{aligned}$$

John invested $\$2,000$ in a money-market fund, $\$3,000$ in municipal bonds, and $\$7,000$ in mutual funds.

? Exercise 1.4.1

Solve the system of equations in three variables.

$$\begin{aligned} 2x+y-2z &= -1 \\ 3x-3y-z &= 5 \\ x-2y+3z &= 6 \end{aligned}$$

Answer

$(1,-1,1)$

1.4.0.0.1 Identifying Inconsistent Systems of Equations Containing Three Variables

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as $3=7$ or some other contradiction.

✓ Example 1.4.4: Solving an Inconsistent System of Three Equations in Three Variables

Solve the following system.

$$x-3y+z=4 \quad (1.4.1)$$
$$-x+2y-5z=3 \quad (1.4.2)$$
$$5x-13y+13z=8 \quad (1.4.3)$$

Solution

Looking at the coefficients of x, we can see that we can eliminate x by adding Equation 1.4.1 to Equation 1.4.2.

$$\begin{aligned} x-3y+z&=4\,(1)\\ -x+2y-5z&=3\,(2)\\ \hline -y-4z&=7\,(4)\end{aligned}$$

Next, we multiply equation (1) by -5 and add it to equation (3).

$$\begin{aligned} -5x+15y-5z&=-20\,(1) \quad \text{multiplied by } -5\\ 5x-13y+13z&=8\,(3)\\ \hline 2y+8z&=-12\,(5)\end{aligned}$$

Then, we multiply equation (4) by 2 and add it to equation (5).

$$\begin{aligned} -2y-8z&=14\,(4) \quad \text{multiplied by } 2\\ 2y+8z&=-12\,(5)\\ \hline 0&=2\end{aligned}$$

The final equation $0=2$ is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

Analysis

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

? Exercise 1.4.2

Solve the system of three equations in three variables.

$$x+y+z=2$$
$$y-3z=1$$
$$2x+y+5z=0$$

Answer

No solution.

1.4.0.0.1 Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

✓ Example 1.4.5: Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.

$$\begin{aligned} 2x+y-3z &= 0 \quad (1) \\ 4x+2y-6z &= 0 \quad (2) \\ x-y+z &= 0 \quad (3) \end{aligned}$$

Solution

First, we can multiply equation (1) by -2 and add it to equation (2).

$$\begin{aligned} -4x-2y+6z &= 0 \ (1) \quad \text{multiplied by } -2 \\ \underline{4x+2y-6z} &= \underline{0 \ (2)} \\ 0 &= 0 \end{aligned}$$

We do not need to proceed any further. The result we get is an identity, $0=0$, which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation (3) by -2, and adding it to equation (1). We then perform the same steps as above and find the same result, $0=0$.

When a system is dependent, we can find general expressions for the solutions. Adding equations (1) and (3), we have

$$\begin{aligned} 2x+y-3z &= 0 \\ x-y+z &= 0 \\ 3x-2z &= 0 \end{aligned}$$

We then solve the resulting equation for z.

$$\begin{aligned} 3x-2z &= 0 \\ z &= \frac{3}{2}x \end{aligned}$$

We back-substitute the expression for z into one of the equations and solve for y.

$$\begin{aligned} 2x+y-3\left(\frac{3}{2}x\right) &= 0 \\ 2x+y-\frac{9}{2}x &= 0 \\ y &= \frac{9}{2}x-2x \\ y &= \frac{5}{2}x \end{aligned}$$

So the general solution is $\left(x, \frac{5}{2}x, \frac{3}{2}x\right)$. In this solution, x can be any real number. The values of y and z are dependent on the value selected for x.

Analysis

As shown in Figure 1.4.5, two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.

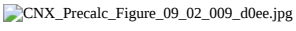


Figure 1.4.5

Q&A: Does the generic solution to a dependent system always have to be written in terms of x?

No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of x and if needed x and y.

Exercise 1.4.3:

Solve the following system.

$$\begin{aligned} x+y+z &= 7 \\ 3x-2y-z &= 4 \\ x+6y+5z &= 24 \end{aligned}$$

Answer

Infinite number of solutions of the form $(x, 4x-11, -5x+18)$.

Media

Access these online resources for additional instruction and practice with systems of equations in three variables.

- Ex 1: System of Three Equations with Three Unknowns Using Elimination
- Ex. 2: System of Three Equations with Three Unknowns Using Elimination

1.4.1 Key Concepts

- A solution set is an ordered triple {(x,y,z)} that represents the intersection of three planes in space. See Example 1.4.1.
- A system of three equations in three variables can be solved by using a series of steps that forces a variable to be eliminated. The steps include interchanging the order of equations, multiplying both sides of an equation by a nonzero constant, and adding a nonzero multiple of one equation to another equation. See Example 1.4.2.
- Systems of three equations in three variables are useful for solving many different types of real-world problems. See Example 1.4.3.
- A system of equations in three variables is inconsistent if no solution exists. After performing elimination operations, the result is a contradiction. See Example 1.4.4.
- Systems of equations in three variables that are inconsistent could result from three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
- A system of equations in three variables is dependent if it has an infinite number of solutions. After performing elimination operations, the result is an identity. See Example 1.4.5.
- Systems of equations in three variables that are dependent could result from three identical planes, three planes intersecting at a line, or two identical planes that intersect the third on a line.

1.4.1.0.1 Contributors

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1.5: Matrices and Matrix Operations

Learning Objectives

- Find the sum and difference of two matrices.
- Find scalar multiples of a matrix.
- Find the product of two matrices.

Two club soccer teams, the Wildcats and the Mud Cats, are hoping to obtain new equipment for an upcoming season. Table 1.5.1 shows the needs of both teams.

Table 1.5.1

	Wildcats	Mud Cats
Goals	6	10
Balls	30	24
Jerseys	14	20

A goal costs $300; a ball costs $10; and a jersey costs $30. How can we find the total cost for the equipment needed for each team? In this section, we discover a method in which the data in the soccer equipment table can be displayed and used for calculating other information. Then, we will be able to calculate the cost of the equipment.



Figure 1.5.1

1.5.0.0.1 Finding the Sum and Difference of Two Matrices

To solve a problem like the one described for the soccer teams, we can use a matrix, which is a rectangular array of numbers. A row in a matrix is a set of numbers that are aligned horizontally. A column in a matrix is a set of numbers that are aligned vertically. Each number is an entry, sometimes called an element, of the matrix. Matrices (plural) are enclosed in [ ] or ( ), and are usually named with capital letters. For example, three matrices named A, B, and C are shown below.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -5 & 6 \\ 7 & 8 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 3 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}$$

A matrix is often referred to by its size or dimensions: $m \times n$ indicating m rows and n columns. Matrix entries are defined first by row and then by column. For example, to locate the entry in matrix A identified as a_{ij},we look for the entry in row i,column j. In matrix A, shown below, the entry in row 2, column 3 is a_{23}.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- A **square matrix** is a matrix with dimensions $n \times n$, meaning that it has the same number of rows as columns. The 3×3 matrix above is an example of a square matrix.
- A **row matrix** is a matrix consisting of one row with dimensions $1 \times n$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$$

- A **column matrix** is a matrix consisting of one column with dimensions $m \times 1$.

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

A matrix may be used to represent a system of equations. In these cases, the numbers represent the coefficients of the variables in the system. Matrices often make solving systems of equations easier because they are not encumbered with variables. We will investigate this idea further in the next section, but first we will look at basic **matrix operations**.

Definition: MATRICES

A *matrix* is a rectangular array of numbers that is usually named by a capital letter: A, B, C,and so on. Each entry in a matrix is referred to as a_{ij},such that i represents the row and j represents the column. Matrices are often referred to by their dimensions: $m \times n$ indicating m rows and n columns.

Example 1.5.1: Finding the Dimensions of the Given Matrix and Locating Entries

Given matrix A:

a. What are the dimensions of matrix A?
b. What are the entries at a_{31} and a_{22}?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 4 & 7 \\ 3 & 1 & -2 \end{bmatrix}$$

Solution

a. The dimensions are 3×3 because there are three rows and three columns.
b. Entry a_{31} is the number at row 3, column 1, which is 3. The entry a_{22} is the number at row 2, column 2, which is 4. Remember, the row comes first, then the column.

Adding and Subtracting Matrices

We use matrices to list data or to represent systems. Because the entries are numbers, we can perform operations on matrices. We add or subtract matrices by adding or subtracting corresponding entries. To do this, the entries must correspond. Therefore, *addition and subtraction of matrices is only possible when the matrices have the same dimensions*. We can add or subtract a 3×3 matrix and another 3×3 matrix, but we cannot add or subtract a 2×3 matrix and a 3×3 matrix because some entries in one matrix will not have a corresponding entry in the other matrix.

ADDING AND SUBTRACTING MATRICES

Given matrices A and B of like dimensions, addition and subtraction of A and B will produce matrix C or matrix D of the same dimension.

$$A+B=C \tag{1.5.1}$$

such that $a_{ij}+b_{ij}=c_{ij}$

$$A-B=D \tag{1.5.2}$$

such that $a_{ij}-b_{ij}=d_{ij}$

Matrix addition is *commutative*.

$$A+B=B+A \tag{1.5.3}$$

It is also *associative*.

$$(A+B)+C=A+(B+C) \tag{1.5.4}$$

Example 1.5.2A: Finding the Sum of Matrices

Find the sum of A and B, given

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$B=\begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Solution

Add corresponding entries.

$$\begin{aligned} A+B &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}+\begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ &= \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \end{aligned}$$

Example 1.5.2B: Adding Matrix A and Matrix B

Find the sum of A and B.

$$A=\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

and

$$B=\begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix}$$

Solution

Add corresponding entries. Add the entry in row 1, column 1, a_{11}, of matrix A to the entry in row 1, column 1, b_{11}, of B. Continue the pattern until all entries have been added.

$$\begin{aligned} A+B &= \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}+\begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 4+5 & 1+9 \\ 3+0 & 2+7 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 10 \\ 3 & 9 \end{bmatrix} \end{aligned}$$

Example 1.5.2C: Finding the Difference of Two Matrices

Find the difference of A and B.

$$A=\begin{bmatrix}-2 & 3\\ 0 & 1\end{bmatrix} \text{ and } B=\begin{bmatrix}8 & 1\\ 5 & 4\end{bmatrix}$$

Solution

We subtract the corresponding entries of each matrix.

$$\begin{aligned} A-B &= \begin{bmatrix}-2 & 3\\ 0 & 1\end{bmatrix}-\begin{bmatrix}8 & 1\\ 5 & 4\end{bmatrix}\\ &= \begin{bmatrix}-2-8 & 3-1\\ 0-5 & 1-4\end{bmatrix}\\ &= \begin{bmatrix}-10 & 2\\ -5 & -3\end{bmatrix}\end{aligned}$$

Example 1.5.2D: Finding the Sum and Difference of Two 3 x 3 Matrices

Given A and B:

a. Find the sum.
b. Find the difference.

$$A=\begin{bmatrix}2 & -10 & -2\\ 14 & 12 & 10\\ 4 & -2 & 2\end{bmatrix}$$

and

$$B=\begin{bmatrix}6 & 10 & -2\\ 0 & -12 & -4\\ -5 & 2 & -2\end{bmatrix}$$

Solution

a. Add the corresponding entries.

$$\begin{aligned} A+B &= \begin{bmatrix}2 & -10 & -2\\ 14 & 12 & 10\\ 4 & -2 & 2\end{bmatrix}+\begin{bmatrix}6 & 10 & -2\\ 0 & -12 & -4\\ -5 & 2 & -2\end{bmatrix}\\ &= \begin{bmatrix}2+6 & -10+10 & -2-2\\ 14+0 & 12-12 & 10-4\\ 4-5 & -2+2 & 2-2\end{bmatrix}\\ &= \begin{bmatrix}8 & 0 & -4\\ 14 & 0 & 6\\ -1 & 0 & 0\end{bmatrix}\end{aligned}$$

b. Subtract the corresponding entries.

$$A-B=\begin{bmatrix}2&-10&-2\\14&12&10\\4&-2&2\end{bmatrix}-\begin{bmatrix}6&10&-2\\0&-12&-4\\-5&2&-2\end{bmatrix}$$
$$=\begin{bmatrix}2-6&-10-10&-2+2\\14-0&12+12&10+4\\4+5&-2-2&2+2\end{bmatrix}$$
$$=\begin{bmatrix}-4&-20&0\\14&24&14\\9&-4&4\end{bmatrix}$$

? Exercise 1.5.1

Add matrix A and matrix B.

$$A=\begin{bmatrix}2&6\\1&0\\1&-3\end{bmatrix}$$

and

$$B=\begin{bmatrix}3&-2\\1&5\\-4&3\end{bmatrix}$$

Answer

$$A+B=\begin{bmatrix}2&6\\1&0\\1&-3\end{bmatrix}+\begin{bmatrix}3&-2\\1&5\\-4&3\end{bmatrix}$$
$$=\begin{bmatrix}2+3&6+(-2)\\1+1&0+5\\1+(-4)&-3+3\end{bmatrix}$$
$$=\begin{bmatrix}5&4\\2&5\\-3&0\end{bmatrix}$$

1.5.1 Finding Scalar Multiples of a Matrix

Besides adding and subtracting whole matrices, there are many situations in which we need to multiply a matrix by a constant called a scalar. Recall that a scalar is a real number quantity that has magnitude, but not direction. For example, time, temperature, and distance are scalar quantities. The process of scalar multiplication involves multiplying each entry in a matrix by a scalar. A scalar multiple is any entry of a matrix that results from scalar multiplication.

Consider a real-world scenario in which a university needs to add to its inventory of computers, computer tables, and chairs in two of the campus labs due to increased enrollment. They estimate that 15 more equipment is needed in both labs. The school's current inventory is displayed in Table 1.5.2.

Table 1.5.2

	Lab A	Lab B
Computers	15	27
Computer Tables	16	34
Chairs	16	34

Converting the data to a matrix, we have

$$C_{2013} = \begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix}$$

To calculate how much computer equipment will be needed, we multiply all entries in matrix C by 0.15.

$$(0.15)C_{2013} = \begin{bmatrix} (0.15)15 & (0.15)27 \\ (0.15)16 & (0.15)34 \\ (0.15)16 & (0.15)34 \end{bmatrix} = \begin{bmatrix} 2.25 & 4.05 \\ 2.4 & 5.1 \\ 2.4 & 5.1 \end{bmatrix}$$

We must round up to the next integer, so the amount of new equipment needed is

$$\begin{bmatrix} 3 & 5 \\ 3 & 6 \\ 3 & 6 \end{bmatrix}$$

Adding the two matrices as shown below, we see the new inventory amounts.

$$\begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 3 & 6 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{bmatrix}$$

This means

$$C_{2014} = \begin{bmatrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{bmatrix}$$

Thus, Lab A will have 18 computers, 19 computer tables, and 19 chairs; Lab B will have 32 computers, 40 computer tables, and 40 chairs.

SCALAR MULTIPLICATION

Scalar multiplication involves finding the product of a constant by each entry in the matrix. Given

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the scalar multiple cA is

$$cA = c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
$$= \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

Scalar multiplication is distributive. For the matrices A, B,and C with scalars a and b,

$$a(A+B) = aA + aB \tag{1.5.5}$$
$$(a+b)A = aA + bA \tag{1.5.6}$$

Example 1.5.3: Multiplying the Matrix by a Scalar

Multiply matrix A by the scalar 3.

$$A = \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix}$$

Solution

Multiply each entry in A by the scalar 3.

$$\begin{aligned} 3A &= 3\begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3\cdot 8 & 3\cdot 1 \\ 3\cdot 5 & 3\cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 24 & 3 \\ 15 & 12 \end{bmatrix} \end{aligned}$$

? Exercise 1.5.2

Given matrix B, find $-2B$ where

$$B = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

Answer

$$-2B = \begin{bmatrix} -8 & -2 \\ -6 & -4 \end{bmatrix}$$

✓ Example 1.5.4: Finding the Sum of Scalar Multiples

Find the sum $3A + 2B$.

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 2 \\ 4 & 3 & -6 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 1 & -4 \end{bmatrix}$$

Solution

First, find $3A$, then $2B$.

$$\begin{aligned} 3A &= \begin{bmatrix} 3\cdot 1 & 3(-2) & 3\cdot 0 \\ 3\cdot 0 & 3(-1) & 3\cdot 2 \\ 3\cdot 4 & 3\cdot 3 & 3(-6) \end{bmatrix} \\ &= \begin{bmatrix} 3 & -6 & 0 \\ 0 & -3 & 6 \\ 12 & 9 & -18 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} 2B &= \begin{bmatrix} 2(-1) & 2\cdot 2 & 2\cdot 1 \\ 2\cdot 0 & 2(-3) & 2\cdot 2 \\ 2\cdot 0 & 2\cdot 1 & 2(-4) \end{bmatrix} \\ &= \begin{bmatrix} -2 & 4 & 2 \\ 0 & -6 & 4 \\ 0 & 2 & -8 \end{bmatrix} \end{aligned}$$

Now, add $3A + 2B$.

$$3A+2B=\begin{bmatrix}3&-6&0\\0&-3&6\\12&9&-18\end{bmatrix}+\begin{bmatrix}-2&4&2\\0&-6&4\\0&2&-8\end{bmatrix}$$
$$=\begin{bmatrix}3-2&-6+4&0+2\\0+0&-3-6&6+4\\12+0&9+2&-18-8\end{bmatrix}$$
$$=\begin{bmatrix}1&-2&2\\0&-9&10\\12&11&-26\end{bmatrix}$$

1.5.1.0.1 Finding the Product of Two Matrices

In addition to multiplying a matrix by a scalar, we can multiply two matrices. Finding the product of two matrices is only possible when the inner dimensions are the same, meaning that the number of columns of the first matrix is equal to the number of rows of the second matrix. If A is an $m\times r$ matrix and B is an $r\times n$ matrix, then the product matrix AB is an $m\times n$ matrix. For example, the product AB is possible because the number of columns in A is the same as the number of rows in B. If the inner dimensions do not match, the product is not defined.

$$\underset{2\times 3}{A}\quad\cdot\quad\underbrace{\underset{3\times 3}{B}}_{\text{same}}$$

We multiply entries of A with entries of B according to a specific pattern as outlined below. The process of **matrix multiplication** becomes clearer when working a problem with real numbers.

To obtain the entries in row i of AB, we multiply the entries in row i of A by column j in B and add. For example, given matrices A and B, where the dimensions of A are 2×3 and the dimensions of B are 3×3 ,the product of AB will be a 2×3 matrix.

$$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}$$

and

$$B=\begin{bmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{bmatrix}$$

Multiply and add as follows to obtain the first entry of the product matrix AB.

1. To obtain the entry in row 1, column 1 of AB,multiply the first row in A by the first column in B,and add.

$$\begin{bmatrix}a_{11}&a_{12}&a_{13}\end{bmatrix}\cdot\begin{bmatrix}b_{11}\\b_{21}\\b_{31}\end{bmatrix}=a_{11}\cdot b_{11}+a_{12}\cdot b_{21}+a_{13}\cdot b_{31}$$

2. To obtain the entry in row 1, column 2 of AB,multiply the first row of A by the second column in B,and add.

$$\begin{bmatrix}a_{11}&a_{12}&a_{13}\end{bmatrix}\cdot\begin{bmatrix}b_{12}\\b_{22}\\b_{32}\end{bmatrix}=a_{11}\cdot b_{12}+a_{12}\cdot b_{22}+a_{13}\cdot b_{32}$$

3. To obtain the entry in row 1, column 3 of AB,multiply the first row of A by the third column in B,and add.

$$\begin{bmatrix}a_{11}&a_{12}&a_{13}\end{bmatrix}\cdot\begin{bmatrix}b_{13}\\b_{23}\\b_{33}\end{bmatrix}=a_{11}\cdot b_{13}+a_{12}\cdot b_{23}+a_{13}\cdot b_{33}$$

We proceed the same way to obtain the second row of AB. In other words, row 2 of A times column 1 of B; row 2 of A times column 2 of B; row 2 of A times column 3 of B. When complete, the product matrix will be

$$AB = \begin{bmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \end{bmatrix}$$

PROPERTIES OF MATRIX MULTIPLICATION

For the matrice A, B,and C the following properties hold.

- Matrix multiplication is *associative*:

$$(AB)C = A(BC). \tag{1.5.7}$$

- Matrix multiplication is *distributive*:

$$C(A+B) = CA + CB \tag{1.5.8}$$

$$(A+B)C = AC + BC. \tag{1.5.9}$$

Note that matrix multiplication is not commutative.

Example 1.5.5A: Multiplying Two Matrices

Multiply matrix A and matrix B.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Solution

First, we check the dimensions of the matrices. Matrix A has dimensions 2×2 and matrix B has dimensions 2×2. The inner dimensions are the same so we can perform the multiplication. The product will have the dimensions 2×2.

We perform the operations outlined previously.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1(5) + 2(7) & 1(6) + 2(8) \\ 3(5) + 4(7) & 3(6) + 4(8) \end{bmatrix} \\ &= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \end{aligned}$$

Example 1.5.5B: Multiplying Two Matrices

Given A and B:

a. Find AB.
b. Find BA.

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix}$$

Solution

a. As the dimensions of A are 2×3 and the dimensions of B are 3×2,these matrices can be multiplied together because the number of columns in A matches the number of rows in B. The resulting product will be a 2×2 matrix, the number of rows in A by the number of columns in B.

$$\begin{aligned} AB &= \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -1(5)+2(-4)+3(2) & -1(-1)+2(0)+3(3) \\ 4(5)+0(-4)+5(2) & 4(-1)+0(0)+5(3) \end{bmatrix} \\ &= \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix} \end{aligned}$$

b. The dimensions of B are 3×2 and the dimensions of A are 2×3. The inner dimensions match so the product is defined and will be a 3×3 matrix.

$$\begin{aligned} BA &= \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 5(-1)+-1(4) & 5(2)+-1(0) & 5(3)+-1(5) \\ -4(-1)+0(4) & -4(2)+0(0) & -4(3)+0(5) \\ 2(-1)+3(4) & 2(2)+3(0) & 2(3)+3(5) \end{bmatrix} \\ &= \begin{bmatrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{bmatrix} \end{aligned}$$

Analysis

Notice that the products AB and BA are not equal.

$$AB = \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix} \neq \begin{bmatrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{bmatrix} = BA$$

This illustrates the fact that matrix multiplication is **not** commutative.

Q&A: Is it possible for AB to be defined but not BA?

Yes, consider a matrix A with dimension 3×4 and matrix B with dimension 4×2. For the product AB the inner dimensions are 4 and the product is defined, but for the product BA the inner dimensions are 2 and 3 so the product is undefined.

Example 1.5.6: Using Matrices in Real-World Problems

Let's return to the problem presented at the opening of this section. We have Table 1.5.3, representing the equipment needs of two soccer teams.

Table 1.5.3

	Wildcats	Mud Cats
Goals	6	10

	Wildcats	Mud Cats
Balls	30	24
Jerseys	14	20

We are also given the prices of the equipment, as shown in Table 1.5.4.

Table 1.5.4

Goal	$300
Ball	$10
Jersey	$30

We will convert the data to matrices. Thus, the equipment need matrix is written as

$$E = \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix}$$

The cost matrix is written as

$$C = \begin{bmatrix} 300 & 10 & 30 \end{bmatrix}$$

We perform matrix multiplication to obtain costs for the equipment.

$$\begin{aligned} CE &= \begin{bmatrix} 300 & 10 & 30 \end{bmatrix} \cdot \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix} \\ &= \begin{bmatrix} 300(6)+10(30)+30(14) & 300(10)+10(24)+30(20) \end{bmatrix} \\ &= \begin{bmatrix} 2,520 & 3,840 \end{bmatrix} \end{aligned}$$

The total cost for equipment for the Wildcats is $\$2,520$, and the total cost for equipment for the Mud Cats is $\$3,840$.

How to: Given a matrix operation, evaluate using a calculator

1. Save each matrix as a matrix variable $[A], [B], [C], \ldots$
2. Enter the operation into the calculator, calling up each matrix variable as needed.
3. If the operation is defined, the calculator will present the solution matrix; if the operation is undefined, it will display an error message.

Example 1.5.7: Using a Calculator to Perform Matrix Operations

Find $AB - C$ given

$$A = \begin{bmatrix} -15 & 25 & 32 \\ 41 & -7 & -28 \\ 10 & 34 & -2 \end{bmatrix}, B = \begin{bmatrix} 45 & 21 & -37 \\ -24 & 52 & 19 \\ 6 & -48 & -31 \end{bmatrix}, \text{ and } C = \begin{bmatrix} -100 & -89 & -98 \\ 25 & -56 & 74 \\ -67 & 42 & -75 \end{bmatrix}$$

Solution

On the matrix page of the calculator, we enter matrix A above as the matrix variable $[A]$, matrix B above as the matrix variable $[B]$, and matrix C above as the matrix variable $[C]$.

On the home screen of the calculator, we type in the problem and call up each matrix variable as needed.

$$[A] \times [B] - [C]$$

The calculator gives us the following matrix.

$$\begin{bmatrix} -983 & -462 & 136 \\ 1,820 & 1,897 & -856 \\ -311 & 2,032 & 413 \end{bmatrix}$$

Media

Access these online resources for additional instruction and practice with matrices and matrix operations.

- Dimensions of a Matrix
- Matrix Addition and Subtraction
- Matrix Operations
- Matrix Multiplication

1.5.2 Key Concepts

- A matrix is a rectangular array of numbers. Entries are arranged in rows and columns.
- The dimensions of a matrix refer to the number of rows and the number of columns. A 3×2 matrix has three rows and two columns. See Example $1.5.1$.
- We add and subtract matrices of equal dimensions by adding and subtracting corresponding entries of each matrix. See Example $1.5.2$, Example $1.5.3$, Example $1.5.4$, and Example $1.5.5$.
- Scalar multiplication involves multiplying each entry in a matrix by a constant. See Example $1.5.6$.
- Scalar multiplication is often required before addition or subtraction can occur. See Example $1.5.7$.
- Multiplying matrices is possible when inner dimensions are the same—the number of columns in the first matrix must match the number of rows in the second.
- The product of two matrices, A and B,is obtained by multiplying each entry in row 1 of A by each entry in column 1 of B; then multiply each entry of row 1 of A by each entry in columns 2 of B,and so on. See Example $1.5.8$ and Example $1.5.9$.
- Many real-world problems can often be solved using matrices. See Example $1.5.10$.
- We can use a calculator to perform matrix operations after saving each matrix as a matrix variable. See Example $1.5.11$.

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1.6: Solving Systems with Gaussian Elimination

Learning Objectives

- Write the augmented matrix of a system of equations.
- Write the system of equations from an augmented matrix.
- Perform row operations on a matrix.
- Solve a system of linear equations using matrices.

Carl Friedrich Gauss lived during the late 18th century and early 19th century, but he is still considered one of the most prolific mathematicians in history. His contributions to the science of mathematics and physics span fields such as algebra, number theory, analysis, differential geometry, astronomy, and optics, among others. His discoveries regarding matrix theory changed the way mathematicians have worked for the last two centuries.



Figure 1.6.1: German mathematician Carl Friedrich Gauss (1777–1855).

We first encountered Gaussian elimination in Systems of Linear Equations: Two Variables. In this section, we will revisit this technique for solving systems, this time using matrices.

1.6.0.0.1 Writing the Augmented Matrix of a System of Equations

A matrix can serve as a device for representing and solving a system of equations. To express a system in matrix form, we extract the coefficients of the variables and the constants, and these become the entries of the matrix. We use a vertical line to separate the coefficient entries from the constants, essentially replacing the equal signs. When a system is written in this form, we call it an **augmented matrix**.

For example, consider the following 2×2 system of equations.

$$\begin{aligned} 3x+4y &= 7 \\ 4x-2y &= 5 \end{aligned}$$

We can write this system as an augmented matrix:

$$\left[\begin{array}{cc|c} 3 & 4 & 7 \\ 4 & -2 & 5 \end{array}\right]$$

We can also write a matrix containing just the coefficients. This is called the **coefficient matrix.**

$$\begin{bmatrix} 3 & 4 \\ 4 & -2 \end{bmatrix}$$

A three-by-three **system of equations** such as

$$\begin{aligned} 3x-y-z &= 0 \\ x+y &= 5 \\ 2x-3z &= 2 \end{aligned}$$

has a coefficient matrix

$$\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & -3 \end{bmatrix}$$

and is represented by the augmented matrix

$$\left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 1 & 1 & 0 & 5 \\ 2 & 0 & -3 & 2 \end{array}\right]$$

Notice that the matrix is written so that the variables line up in their own columns: x-terms go in the first column, y-terms in the second column, and z-terms in the third column. It is very important that each equation is written in standard form $ax+by+cz=d$ so that the variables line up. When there is a missing variable term in an equation, the coefficient is 0.

How to: Given a system of equations, write an augmented matrix

1. Write the coefficients of the x-terms as the numbers down the first column.
2. Write the coefficients of the y-terms as the numbers down the second column.
3. If there are z-terms, write the coefficients as the numbers down the third column.
4. Draw a vertical line and write the constants to the right of the line.

Example 1.6.1: Writing the Augmented Matrix for a System of Equations

Write the augmented matrix for the given system of equations.

$$\begin{aligned} x+2y-z &= 3 \\ 2x-y+2z &= 6 \\ x-3y+3z &= 4 \end{aligned}$$

Solution

The augmented matrix displays the coefficients of the variables, and an additional column for the constants.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -1 & 2 & 6 \\ 1 & -3 & 3 & 4 \end{array}\right]$$

Exercise 1.6.1

Write the augmented matrix of the given system of equations.

$$\begin{aligned} 4x-3y &= 11 \\ 3x+2y &= 4 \end{aligned}$$

Answer

$$\left[\begin{array}{cc|c} 4 & -3 & 11 \\ 3 & 2 & 4 \end{array}\right]$$

1.6.0.0.1 Writing a System of Equations from an Augmented Matrix

We can use augmented matrices to help us solve systems of equations because they simplify operations when the systems are not encumbered by the variables. However, it is important to understand how to move back and forth between formats in order to make finding solutions smoother and more intuitive. Here, we will use the information in an augmented matrix to write the system of equations in standard form.

Example 1.6.2: Writing a System of Equations from an Augmented Matrix Form

Find the system of equations from the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ -3 & 5 & 4 & 6 \end{array}\right]$$

Solution

When the columns represent the variables x, y, and z,

$$\left[\begin{array}{ccc|c} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ -3 & 5 & 4 & 6 \end{array}\right] \rightarrow \begin{array}{r} x-3y-5z=-2 \\ 2x-5y-4z=5 \\ -3x+5y+4z=6 \end{array}$$

Exercise 1.6.2

Write the system of equations from the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 2 & -1 & 3 & 1 \\ 0 & 1 & 1 & -9 \end{array}\right]$$

Answer

$$\begin{array}{r} x-y+z=5 \\ 2x-y+3z=1 \\ y+z=-9 \end{array}$$

1.6.1 Performing Row Operations on a Matrix

Now that we can write systems of equations in augmented matrix form, we will examine the various **row operations** that can be performed on a matrix, such as addition, multiplication by a constant, and interchanging rows.

Performing row operations on a matrix is the method we use for solving a system of equations. In order to solve the system of equations, we want to convert the matrix to **row-echelon** form, in which there are ones down the **main diagonal** from the upper left corner to the lower right corner, and zeros in every position below the main diagonal as shown.

$$\text{Row-echelon form } \begin{bmatrix} 1 & a & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}$$

We use row operations corresponding to equation operations to obtain a new matrix that is **row-equivalent** in a simpler form. Here are the guidelines to obtaining row-echelon form.

1. In any nonzero row, the first nonzero number is a 1. It is called a *leading* 1.
2. Any all-zero rows are placed at the bottom on the matrix.
3. Any leading 1 is below and to the right of a previous leading 1.
4. Any column containing a leading 1 has zeros in all other positions in the column.

To solve a system of equations we can perform the following row operations to convert the **coefficient matrix** to row-echelon form and do back-substitution to find the solution.

1. Interchange rows. (Notation: $R_i \leftrightarrow R_j$)
2. Multiply a row by a constant. (Notation: cR_i)
3. Add the product of a row multiplied by a constant to another row. (Notation: $R_i + cR_j$)

Each of the row operations corresponds to the operations we have already learned to solve systems of equations in three variables. With these operations, there are some key moves that will quickly achieve the goal of writing a matrix in row-echelon form. To

obtain a matrix in row-echelon form for finding solutions, we use Gaussian elimination, a method that uses row operations to obtain a 1 as the first entry so that row 1 can be used to convert the remaining rows.

GAUSSIAN ELIMINATION

The Gaussian elimination method refers to a strategy used to obtain the row-echelon form of a matrix. The goal is to write matrix A with the number 1 as the entry down the main diagonal and have all zeros below.

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{After\ Gaussian\ elimination} A=\begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

The first step of the Gaussian strategy includes obtaining a 1 as the first entry, so that row 1 may be used to alter the rows below.

How to: Given an augmented matrix, perform row operations to achieve row-echelon form

1. The first equation should have a leading coefficient of 1. Interchange rows or multiply by a constant, if necessary.
2. Use row operations to obtain zeros down the first column below the first entry of 1.
3. Use row operations to obtain a 1 in row 2, column 2.
4. Use row operations to obtain zeros down column 2, below the entry of 1.
5. Use row operations to obtain a 1 in row 3, column 3.
6. Continue this process for all rows until there is a \(1 in every entry down the main diagonal and there are only zeros below.
7. If any rows contain all zeros, place them at the bottom.

Example 1.6.3: Solving a 2×2 System by Gaussian Elimination

Solve the given system by Gaussian elimination.

$$\begin{aligned} 2x+3y &= 6 \\ x-y &= \frac{1}{2} \end{aligned}$$

Solution

First, we write this as an augmented matrix.

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 1 & -1 & \frac{1}{2} \end{array}\right]$$

We want a 1 in row 1, column 1. This can be accomplished by interchanging row 1 and row 2.

$$R_1 \leftrightarrow R_2 \rightarrow \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 2 & 3 & 6 \end{array}\right]$$

We now have a 1 as the first entry in row 1, column 1. Now let's obtain a 0 in row 2, column 1. This can be accomplished by multiplying row 1 by -2, and then adding the result to row 2.

$$-2R_1+R_2=R_2 \rightarrow \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 5 & 5 \end{array}\right]$$

We only have one more step, to multiply row 2 by $\frac{1}{5}$.

$$\frac{1}{5}R_2=R_2 \rightarrow \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 1 & 1 \end{array}\right]$$

Use back-substitution. The second row of the matrix represents $y = 1$. Back-substitute $y = 1$ into the first equation.

$$x - (1) = \frac{1}{2}$$
$$x = \frac{3}{2}$$

The solution is the point $\left(\frac{3}{2}, 1\right)$.

Exercise 1.6.3

Solve the given system by Gaussian elimination.

$$4x + 3y = 11$$
$$x - 3y = -1$$

Answer

$(2, 1)$

Example 1.6.4: Using Gaussian Elimination to Solve a System of Equations

Use **Gaussian elimination** to solve the given 2×2 **system of equations**.

$$2x + y = 1$$
$$4x + 2y = 6$$

Solution

Write the system as an **augmented matrix**.

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Obtain a 1 in row 1, column 1. This can be accomplished by multiplying the first row by $\frac{1}{2}$.

$$\frac{1}{2}R_1 = R_1 \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 4 & 2 & 6 \end{array}\right]$$

Next, we want a 0 in row 2, column 1. Multiply row 1 by -4 and add row 1 to row 2.

$$-4R_1 + R_2 = R_2 \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 4 \end{array}\right]$$

The second row represents the equation $0 = 4$. Therefore, the system is inconsistent and has no solution.

Example 1.6.5: Solving a Dependent System

Solve the system of equations.

$$3x + 4y = 12$$
$$6x + 8y = 24$$

Solution

Perform **row operations** on the augmented matrix to try and achieve **row-echelon form**.

$$A = \left[\begin{array}{cc|c} 3 & 4 & 12 \\ 6 & 8 & 24 \end{array}\right]$$

$$-\frac{1}{2}R_2+R_1=R_1\rightarrow\left[\begin{array}{cc|c}0&0&0\\6&8&24\end{array}\right]$$

$$R_1\leftrightarrow R_2=\left[\begin{array}{cc|c}6&8&24\\0&0&0\end{array}\right]$$

The matrix ends up with all zeros in the last row: $0y=0$. Thus, there are an infinite number of solutions and the system is classified as dependent. To find the generic solution, return to one of the original equations and solve for y.

$$\begin{aligned}3x+4y&=12\\4y&=12-3x\\y&=3-\frac{3}{4}x\end{aligned}$$

So the solution to this system is $\left(x,3-\frac{3}{4}x\right)$.

Example 1.6.6: Performing Row Operations on a 3×3 Augmented Matrix to Obtain Row-Echelon Form

Perform row operations on the given matrix to obtain row-echelon form.

$$\left[\begin{array}{ccc|c}1&-3&4&3\\2&-5&6&6\\-3&3&4&6\end{array}\right]$$

Solution

The first row already has a 1 in row 1, column 1. The next step is to multiply row 1 by -2 and add it to row 2. Then replace row 2 with the result.

$$-2R_1+R_2=R_2\left[\begin{array}{ccc|c}1&-3&4&3\\0&1&-2&0\\-3&3&4&6\end{array}\right]$$

Next, obtain a zero in row 3, column 1.

$$3R_1+R_3=R_3\left[\begin{array}{ccc|c}1&-3&4&3\\0&1&-2&0\\0&-6&16&15\end{array}\right]$$

Next, obtain a zero in row 3, column 2.

$$6R_2+R_3=R_3\left[\begin{array}{ccc|c}1&-3&4&3\\0&1&-2&0\\0&0&4&15\end{array}\right]$$

The last step is to obtain a 1 in row 3, column 3.

$$\frac{1}{4}R_3=R_3\left[\begin{array}{ccc|c}1&-3&4&3\\0&1&-2&0\\0&0&1&\frac{15}{4}\end{array}\right]$$

Exercise 1.6.4

Write the system of equations in row-echelon form.

$$\begin{aligned} x-2y+3z &= 9 \\ -x+3y &= -4 \\ 2x-5y+5z &= 17 \end{aligned}$$

Answer

$$\left[\begin{array}{ccc|c} 1 & -\frac{5}{2} & \frac{5}{2} & \frac{17}{2} \\ 0 & 1 & 5 & 9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

1.6.1.0.1 Solving a System of Linear Equations Using Matrices

We have seen how to write a **system of equations** with an **augmented matrix**, and then how to use row operations and back-substitution to obtain row-echelon form. Now, we will take **row-echelon form** a step farther to solve a 3 by 3 system of linear equations. The general idea is to eliminate all but one variable using row operations and then back-substitute to solve for the other variables.

Example 1.6.7: Solving a System of Linear Equations Using Matrices

Solve the system of linear equations using matrices.

$$\begin{aligned} x-y+z &= 8 \\ 2x+3y-z &= -2 \\ 3x-2y-9z &= 9 \end{aligned}$$

Solution

First, we write the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{array}\right]$$

Next, we perform row operations to obtain row-echelon form.

$$-2R_1+R_2=R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 3 & -2 & -9 & 9 \end{array}\right]$$

$$-3R_1+R_3=R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 0 & 1 & -12 & -15 \end{array}\right]$$

The easiest way to obtain a 1 in row 2 of column 1 is to interchange R_2 and R_3.

$$Interchange\ R_2\ and\ R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 5 & -3 & -18 \end{array}\right]$$

Then

$$-5R_2+R_3=R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 57 & 57 \end{array}\right]$$

$$-\frac{1}{57}R_3=R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The last matrix represents the equivalent system.

$$\begin{aligned} x-y+z &= 8 \\ y-12z &= -15 \\ z &= 1 \end{aligned}$$

Using back-substitution, we obtain the solution as $(4,-3,1)$.

Example 1.6.8: Solving a Dependent System of Linear Equations Using Matrices

Solve the following system of linear equations using matrices.

$$\begin{aligned} -x-2y+z &= -1 \\ 2x+3y &= 2 \\ y-2z &= 0 \end{aligned}$$

Solution

Write the augmented matrix.

$$\left[\begin{array}{ccc|c} -1 & -2 & 1 & -1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

First, multiply row 1 by -1 to get a 1 in row 1, column 1. Then, perform **row operations** to obtain row-echelon form.

$$-R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 2 & 3 & 0 & 2 \end{array}\right]$$

$$-2R_1+R_3=R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$R_2+R_3=R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix represents the following system.

$$\begin{aligned} x+2y-z &= 1 \\ y-2z &= 0 \\ 0 &= 0 \end{aligned}$$

We see by the identity $0=0$ that this is a dependent system with an infinite number of solutions. We then find the generic solution. By solving the second equation for y and substituting it into the first equation we can solve for z in terms of x.

$$\begin{aligned} x+2y-z &= 1 \\ y &= 2z \\ x+2(2z)-z &= 1 \\ x+3z &= 1 \\ z &= \frac{1-x}{3} \end{aligned}$$

Now we substitute the expression for z into the second equation to solve for y in terms of x.

$$\begin{aligned} y-2z &= 0 \\ z &= \frac{1-x}{3} \\ y-2\left(\frac{1-x}{3}\right) &= 0 \\ y &= \frac{2-2x}{3} \end{aligned}$$

The generic solution is $\left(x, \frac{2-2x}{3}, \frac{1-x}{3}\right)$.

? Exercise 1.6.5

Solve the system using matrices.

$$\begin{aligned} x+4y-z &= 4 \\ 2x+5y+8z &= 1 \\ 5x+3y-3z &= 1 \end{aligned}$$

Answer

$(1,1,1)$

Q&A: Can any system of linear equations be solved by Gaussian elimination?

Yes, a system of linear equations of any size can be solved by Gaussian elimination.

How to: Given a system of equations, solve with matrices using a calculator

1. Save the augmented matrix as a matrix variable $[A], [B], [C], \ldots$.
2. Use the **ref(** function in the calculator, calling up each matrix variable as needed.

Example 1.6.9A: Solving Systems of Equations with Matrices Using a Calculator

Solve the system of equations.

$$\begin{aligned} 5x+3y+9z &= -1 \\ -2x+3y-z &= -2 \\ -x-4y+5z &= 1 \end{aligned}$$

Solution

Write the augmented matrix for the system of equations.

$$\left[\begin{array}{ccc|c} 5 & 3 & 9 & -1 \\ -2 & 3 & -1 & -2 \\ -1 & -4 & 5 & 1 \end{array}\right]$$

On the matrix page of the calculator, enter the augmented matrix above as the matrix variable $[A]$.

$$[A] = \left[\begin{array}{ccc|c} 5 & 3 & 9 & -1 \\ -2 & 3 & -1 & -2 \\ -1 & -4 & 5 & 1 \end{array}\right]$$

Use the **ref(** function in the calculator, calling up the matrix variable $[A]$.

ref([A])

Evaluate

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{5} & \frac{9}{5} & \frac{1}{5} \\ 0 & 1 & \frac{13}{21} & -\frac{4}{7} \\ 0 & 0 & 1 & -\frac{24}{187} \end{array}\right] \rightarrow \begin{aligned} x+\frac{3}{5}y+\frac{9}{5}z &= -\frac{1}{5} \\ y+\frac{13}{21}z &= -\frac{4}{7} \\ z &= -\frac{24}{187} \end{aligned}$$

Using back-substitution, the solution is $\left(\frac{61}{187}, -\frac{92}{187}, -\frac{24}{187}\right)$.

Example 1.6.9B: Applying 2×2 Matrices to Finance

Carolyn invests a total of $\$12,000$ in two municipal bonds, one paying 10.5 interest and the other paying 12 interest. The annual interest earned on the two investments last year was $\$1,335$. How much was invested at each rate?

Solution

We have a system of two equations in two variables. Let $x =$ the amount invested at 10.5 interest, and $y =$ the amount invested at 12 interest.

$$\begin{aligned} x+y &= 12,000 \\ 0.105x+0.12y &= 1,335 \end{aligned}$$

As a matrix, we have

$$\left[\begin{array}{cc|c} 1 & 1 & 12,000 \\ 0.105 & 0.12 & 1,335 \end{array}\right]$$

Multiply row 1 by -0.105 and add the result to row 2.

$$\left[\begin{array}{cc|c} 1 & 1 & 12,000 \\ 0 & 0.015 & 75 \end{array}\right]$$

Then,

$$\begin{aligned} 0.015y &= 75 \\ y &= 5,000 \end{aligned}$$

So $12,000-5,000=7,000$.

Thus, $\$5,000$ was invested at 12 interest and $\$7,000$ at 10.5 interest.

Example 1.6.10: Applying 3×3 Matrices to Finance

Ava invests a total of $\$10,000$ in three accounts, one paying 5 interest, another paying 8 interest, and the third paying 9 interest. The annual interest earned on the three investments last year was $\$770$. The amount invested at 9 was twice the amount invested at 5. How much was invested at each rate?

Solution

We have a system of three equations in three variables. Let x be the amount invested at 5 interest, let y be the amount invested at 8 interest, and let z be the amount invested at 9 interest. Thus,

$$\begin{aligned} x+y+z &= 10,000 \\ 0.05x+0.08y+0.09z &= 770 \\ 2x-z &= 0 \end{aligned}$$

As a matrix, we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10,000 \\ 0.05 & 0.08 & 0.09 & 770 \\ 2 & 0 & -1 & 0 \end{array}\right]$$

Now, we perform Gaussian elimination to achieve row-echelon form.

$$-0.05R_1 + R_2 = R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10,000 \\ 0 & 0.03 & 0.04 & 270 \\ 2 & 0 & -1 & 0 \end{array}\right]$$

$$-2R_1 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10,000 \\ 0 & 0.03 & 0.04 & 270 \\ 0 & -2 & -3 & -20,000 \end{array}\right]$$

$$\frac{1}{0.03}R_2 = R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10,000 \\ 0 & 1 & \frac{4}{3} & 9,000 \\ 0 & -2 & -3 & -20,000 \end{array}\right]$$

$$2R_2 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10,000 \\ 0 & 1 & \frac{4}{3} & 9,000 \\ 0 & 0 & -\frac{1}{3} & -2,000 \end{array}\right]$$

The third row tells us $-\frac{1}{3}z = -2,000$; thus $z = 6,000$.

The second row tells us $y + \frac{4}{3}z = 9,000$. Substituting $z = 6,000$,we get

$$\begin{aligned} y + \frac{4}{3}(6,000) &= 9,000 \\ y + 8,000 &= 9,000 \\ y &= 1,000 \end{aligned}$$

The first row tells us $x + y + z = 10,000$. Substituting $y = 1,000$ and $z = 6,000$,we get

$$\begin{aligned} x + 1,000 + 6,000 &= 10,000 \\ x &= 3,000 \end{aligned}$$

The answer is $\$3,000$invested at 5 interest, $\$1,000$invested at 8, and $\$6,000$invested at 9 interest.

Exercise 1.6.6

A small shoe company took out a loan of $\$1,500,000$to expand their inventory. Part of the money was borrowed at 7, part was borrowed at 8, and part was borrowed at 10. The amount borrowed at 10 was four times the amount borrowed at 7, and the annual interest on all three loans was $\$130,500$ Use matrices to find the amount borrowed at each rate.

Answer

$\$150,000$at 7, $\$750,000$at 8, $\$600,000$at 10

Media

Access these online resources for additional instruction and practice with solving systems of linear equations using Gaussian elimination.

- Solve a System of Two Equations Using an Augmented Matrix
- Solve a System of Three Equations Using an Augmented Matrix

- Augmented Matrices on the Calculator

1.6.2 Key Concepts

- An augmented matrix is one that contains the coefficients and constants of a system of equations. See Example 1.6.1.
- A matrix augmented with the constant column can be represented as the original system of equations. See Example 1.6.2.
- Row operations include multiplying a row by a constant, adding one row to another row, and interchanging rows.
- We can use Gaussian elimination to solve a system of equations. See Example 1.6.3, Example 1.6.4, and Example 1.6.5.
- Row operations are performed on matrices to obtain row-echelon form. See Example 1.6.6.
- To solve a system of equations, write it in augmented matrix form. Perform row operations to obtain row-echelon form. Back-substitute to find the solutions. See Example 1.6.7 and Example 1.6.8.
- A calculator can be used to solve systems of equations using matrices. See Example 1.6.9.
- Many real-world problems can be solved using augmented matrices. See Example 1.6.10 and Example 1.6.11.

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1.7: Solving Systems with Inverses

Learning Objectives

- Find the inverse of a matrix.
- Solve a system of linear equations using an inverse matrix

Nancy plans to invest $\$10,500$ into two different bonds to spread out her risk. The first bond has an annual return of 10, and the second bond has an annual return of 6. To receive an 8.5 return from the two bonds, how much should Nancy invest in each bond? What is the best method to solve this problem? There are several ways we can solve this problem. As we have seen in previous sections, systems of equations and matrices are useful in solving real-world problems involving finance. After studying this section, we will have the tools to solve the bond problem using the inverse of a matrix.

1.7.1 Finding the Inverse of a Matrix

We know that the multiplicative inverse of a real number a is a^{-1}, so

$$aa^{-1} = a^{-1}a = \left(\frac{1}{a}\right)a = 1 \tag{1.7.1}$$

For example, consider the scalar multiplication situation

$$2^{-1} = \frac{1}{2}$$

therefore from Equation 1.7.1

$$\left(\frac{1}{2}\right)2 = 1.$$

The **multiplicative inverse of a matrix** is similar in concept, except that the product of matrix A and its inverse A^{-1} equals the **identity matrix.** The identity matrix is a square matrix containing ones down the main diagonal and zeros everywhere else. We identify identity matrices by I_n where n represents the dimension of the matrix. Equations 1.7.2 and 1.7.3 are the identity matrices for a 2×2 matrix and a 3×3 matrix, respectively:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.7.2}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.7.3}$$

The identity matrix acts as a 1 in matrix algebra. For example,

$$AI = IA = A$$

A matrix that has a multiplicative inverse has the properties

$$AA^{-1} = I \tag{1.7.4}$$

$$A^{-1}A = I \tag{1.7.5}$$

A matrix that has a multiplicative inverse is called an **invertible matrix**. Only a square matrix may have a multiplicative inverse, as the reversibility,

$$AA^{-1} = A^{-1}A = I \tag{1.7.6}$$

is a requirement. Not all square matrices have an inverse, but if A is invertible, then A^{-1} is unique. We will look at two methods for finding the inverse of a 2×2 matrix and a third method that can be used on both 2×2 and 3×3 matrices.

Definitions: THE IDENTITY MATRIX AND MULTIPLICATIVE INVERSE

The *identity matrix*, I_n, is a square matrix containing ones down the main diagonal and zeros everywhere else.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

as for the 2×2 identity matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as for the 3×3 identity matrix

If A is an $n \times n$ matrix and B is an $n \times n$ matrix such that $AB = BA = I_n$, then $B = A - 1$, the **multiplicative inverse of a matrix A.**

Example 1.7.1: Showing That the Identity Matrix Acts as a 1

Given matrix A, show that $AI = IA = A$.

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

Solution

Use matrix multiplication to show that the product of A and the identity matric is equal to the product of the identity matrix and A.

$$\begin{aligned} AI &= \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \cdot 1 + 4 \cdot 0 & 3 \cdot 0 + 4 \cdot 1 \\ -2 \cdot 1 + 5 \cdot 0 & -2 \cdot 0 + 5 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} AI &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 3 + 0 \cdot (-2) & 1 \cdot 4 + 0 \cdot 5 \\ 0 \cdot 3 + 1 \cdot (-2) & 0 \cdot 4 + 1 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \end{aligned}$$

How to: Given two matrices, show that one is the multiplicative inverse of the other

- Given matrix A of order $n \times n$ and matrix B of order $n \times n$ multiply AB.
- If $AB = I$, then find the product BA. If $BA = I$, then $B = A^{-1}$ and $A = B^{-1}$.

Example 1.7.2: Showing That Matrix A Is the Multiplicative Inverse of Matrix B

Show that the given matrices are multiplicative inverses of each other.

$$A = \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix}$$

Solution

Multiply AB and BA. If both products equal the identity, then the two matrices are inverses of each other.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} \cdot \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(-9)+5(2) & 1(-5)+5(1) \\ -2(-9)-9(2) & -2(-5)-9(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} BA &= \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} \\ &= \begin{bmatrix} -9(1)-5(-2) & -9(5)-5(-9) \\ 2(1)+1(-2) & 2(-5)+1(-9) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

A and B are inverses of each other.

? Exercise 1.7.1

Show that the following two matrices are inverses of each other.

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

Answer

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(-3)+4(1) & 1(-4)+4(1) \\ -1(-3)+-3(1) & -1(-4)+-3(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -3(1)+-4(-1) & -3(4)+-4(-3) \\ 1(1)+1(-1) & 1(4)+1(-3) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

1.7.2 Finding the Multiplicative Inverse Using Matrix Multiplication

We can now determine whether two matrices are inverses, but how would we find the inverse of a given matrix? Since we know that the product of a matrix and its inverse is the identity matrix, we can find the inverse of a matrix by setting up an equation using **matrix multiplication**.

Example 1.7.3: Finding the Multiplicative Inverse Using Matrix Multiplication

Use matrix multiplication to find the inverse of the given matrix.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

Solution

For this method, we multiply A by a matrix containing unknown constants and set it equal to the identity.

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Find the product of the two matrices on the left side of the equal sign.

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1a-2c & 1b-2d \\ 2a-3c & 2b-3d \end{bmatrix}$$

Next, set up a system of equations with the entry in row 1, column 1 of the new matrix equal to the first entry of the identity, 1. Set the entry in row 2, column 1 of the new matrix equal to the corresponding entry of the identity, which is 0.

$$1a-2c=1 \; R_1$$

$$2a-3c=0 \; R_2$$

Using row operations, multiply and add as follows: $(-2)R_1+R_2 \rightarrow R_2$. Add the equations, and solve for c.

$$\begin{aligned} 1a-2c &= 1 \\ 0+1c &= -2 \\ c &= -2 \end{aligned}$$

Back-substitute to solve for a.

$$\begin{aligned} a-2(-2) &= 1 \\ a+4 &= 1 \\ a &= -3 \end{aligned}$$

Write another system of equations setting the entry in row 1, column 2 of the new matrix equal to the corresponding entry of the identity, 0. Set the entry in row 2, column 2 equal to the corresponding entry of the identity.

$$1b-2d=0 \; R_1$$

$$2b-3d=1 \; R_2$$

Using row operations, multiply and add as follows: $(-2)R_1+R_2=R_2$. Add the two equations and solve for d.

$$\begin{aligned} 1b-2d &= 0 \\ 0+1d &= 1 \\ d &= 1 \end{aligned}$$

Once more, back-substitute and solve for b.

$$b-2(1)=0$$
$$b-2=0$$
$$b=2$$
$$A^{-1}=\begin{bmatrix}-3 & 2\\ -2 & 1\end{bmatrix}$$

1.7.3 Finding the Multiplicative Inverse by Augmenting with the Identity

Another way to find the **multiplicative inverse** is by augmenting with the identity. When matrix A is transformed into I, the augmented matrix I transforms into A^{-1}.

For example, given

$$A=\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}$$

augment A with the identity

$$\left[\begin{array}{cc|cc}2 & 1 & 1 & 0\\ 5 & 3 & 0 & 1\end{array}\right]$$

Perform row operations with the goal of turning A into the identity.

1. Switch row 1 and row 2.

$$\left[\begin{array}{cc|cc}5 & 3 & 0 & 1\\ 2 & 1 & 1 & 0\end{array}\right]$$

2. Multiply row 2 by −2 and add to row 1.

$$\left[\begin{array}{cc|cc}1 & 1 & -2 & 1\\ 2 & 1 & 1 & 0\end{array}\right]$$

3. Multiply row 1 by −2 and add to row 2.

$$\left[\begin{array}{cc|cc}1 & 1 & -2 & 1\\ 0 & -1 & 5 & -2\end{array}\right]$$

4. Add row 2 to row 1.

$$\left[\begin{array}{cc|cc}1 & 0 & 3 & -1\\ 0 & -1 & 5 & -2\end{array}\right]$$

5. Multiply row 2 by−1. −1.

$$\left[\begin{array}{cc|cc}1 & 0 & 3 & -1\\ 0 & 1 & -5 & 2\end{array}\right]$$

The matrix we have found is A^{-1}.

$$A^{-1}=\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}$$

1.7.4 Finding the Multiplicative Inverse of 2×2 Matrices Using a Formula

When we need to find the **multiplicative inverse** of a 2×2 matrix, we can use a special formula instead of using matrix multiplication or augmenting with the identity.

If A is a 2×2 matrix, such as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the multiplicative inverse of A is given by the formula

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $ad-bc \neq 0$. If $ad-bc=0$, then A has no inverse.

Example 1.7.4: Using the Formula to Find the Multiplicative Inverse of Matrix A

Use the formula to find the multiplicative inverse of

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

Solution

We can check that our formula works by using one of the other methods to calculate the inverse. Let's augment A with the identity.

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right]$$

Perform row operations with the goal of turning A into the identity.

1. Multiply row 1 by -2 and add to row 2.

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

2. Multiply row 1 by 2 and add to row 1.

$$\left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

So, we have verified our original solution.

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

Exercise 1.7.2

Use the formula to find the inverse of matrix A. Verify your answer by augmenting with the identity matrix.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

Answer

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

Example 1.7.5: Finding the Inverse of the Matrix, If It Exists

Find the inverse, if it exists, of the given matrix.

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

Solution

We will use the method of augmenting with the identity.

$$\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

1. Switch row 1 and row 2.

$$\left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 3 & 6 & 1 & 0 \end{array}\right]$$

2. Multiply row 1 by -3 and add it to row 2.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right]$$

3. There is nothing further we can do. The zeros in row 2 indicate that this matrix has no inverse.

Finding the Multiplicative Inverse of 3×3 Matrices

Unfortunately, we do not have a formula similar to the one for a 2×2 matrix to find the inverse of a 3×3 matrix. Instead, we will augment the original matrix with the identity matrix and use row operations to obtain the inverse.

Given a 3×3 matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

augment A with the identity matrix

$$A \left| I \right. = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

To begin, we write the **augmented matrix** with the identity on the right and A on the left. Performing elementary **row operations** so that the **identity matrix** appears on the left, we will obtain the **inverse matrix** on the right. We will find the inverse of this matrix in the next example.

How to: Given a 3×3 matrix, find the inverse

1. Write the original matrix augmented with the identity matrix on the right.
2. Use elementary row operations so that the identity appears on the left.
3. What is obtained on the right is the inverse of the original matrix.
4. Use matrix multiplication to show that $AA^{-1} = I$ and $A^{-1}A = I$.

Example 1.7.6: Finding the Inverse of a 3×3 Matrix

Given the 3×3 matrix A, find the inverse.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

Solution

Augment A with the identity matrix, and then begin row operations until the identity matrix replaces A. The matrix on the right will be the inverse of A.

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Interchange } R_2 \text{ and } R_1} \left[\begin{array}{ccc|ccc} 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$-R_2 + R_1 = R_1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$-R_2 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 2 & 3 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$-2R_1 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & 1 & 3 & -2 & 0 \end{array}\right]$$

$$-3R_2 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$$

Thus,

$$A^{-1} = B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

Analysis

To prove that $B = A^{-1}$, let's multiply the two matrices together to see if the product equals the identity, if $AA^{-1} = I$ and $A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-1)+3(-1)+1(6) & 2(1)+3(0)+1(-2) & 2(0)+3(1)+1(-3) \\ 3(-1)+3(-1)+1(6) & 3(1)+3(0)+1(-2) & 3(0)+3(1)+1(-3) \\ 2(-1)+4(-1)+1(6) & 2(1)+4(0)+1(-2) & 2(0)+4(1)+1(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & & & \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} & 2 & 31 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(2)+1(3)+0(2) & -1(3)+1(3)+0(4) & -1(1)+1(1)+0(1) \\ -1(2)+0(3)+1(2) & -1(3)+0(3)+1(4) & -1(1)+0(1)+1(1) \\ 6(2)+-2(3)+-3(2) & 6(3)+-2(3)+-3(4) & 6(1)+-2(1)+-3(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

? Exercise 1.7.3

Find the inverse of the 3×3 matrix.

$$A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$$

Answer

$$A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$$

1.7.5 Solving a System of Linear Equations Using the Inverse of a Matrix

Solving a system of linear equations using the inverse of a matrix requires the definition of two new matrices: X is the matrix representing the variables of the system, and B is the matrix representing the constants. Using **matrix multiplication**, we may define a system of equations with the same number of equations as variables as

$$AX = B$$

To solve a system of linear equations using an inverse matrix, let A be the coefficient matrix, let X be the variable matrix, and let B be the constant matrix. Thus, we want to solve a system $AX = B$. For example, look at the following system of equations.

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

From this system, the coefficient matrix is

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

The variable matrix is

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

And the constant matrix is

$$B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Then $AX = B$ looks like

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Recall the discussion earlier in this section regarding multiplying a real number by its inverse, $(2^{-1})2 = \left(\frac{1}{2}\right) 2 = 1$. To solve a single linear equation $ax = b$ for x, we would simply multiply both sides of the equation by the multiplicative inverse (reciprocal) of a. Thus,

$$\begin{aligned} ax &= b \\ \left(\frac{1}{a}\right) ax &= \left(\frac{1}{a}\right) b \\ \left(a^{-1}\right) ax &= \left(a^{-1}\right) b \\ \left[\left(a^{-1}\right) a\right] x &= \left(a^{-1}\right) b \\ 1x &= \left(a^{-1}\right) b \\ x &= \left(a^{-1}\right) b \end{aligned}$$

The only difference between a solving a linear equation and a system of equations written in matrix form is that finding the inverse of a matrix is more complicated, and matrix multiplication is a longer process. However, the goal is the same—to isolate the variable.

We will investigate this idea in detail, but it is helpful to begin with a 2×2 system and then move on to a 3×3 system.

SOLVING A SYSTEM OF EQUATIONS USING THE INVERSE OF A MATRIX

Given a system of equations, write the coefficient matrix A, the variable matrix X, and the constant matrix B. Then

$$AX = B$$

Multiply both sides by the inverse of A to obtain the solution.

$$\begin{aligned} \left(A^{-1}\right) AX &= \left(A^{-1}\right) B \\ \left[\left(A^{-1}\right) A\right] X &= \left(A^{-1}\right) B \\ IX &= \left(A^{-1}\right) B \\ X &= \left(A^{-1}\right) B \end{aligned}$$

Q&A: If the coefficient matrix does not have an inverse, does that mean the system has no solution?

No, if the coefficient matrix is not invertible, the system could be inconsistent and have no solution, or be dependent and have infinitely many solutions.

Example 1.7.7: Solving a 2×2 System Using the Inverse of a Matrix

Solve the given system of equations using the inverse of a matrix.

$$\begin{aligned} 3x+8y &= 5 \\ 4x+11y &= 7 \end{aligned}$$

Solution

Write the system in terms of a coefficient matrix, a variable matrix, and a constant matrix.

$$A=\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}, B=\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Then

$$\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

First, we need to calculate A^{-1}. Using the formula to calculate the inverse of a 2 by 2 matrix, we have:

$$\begin{aligned} A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{3(11)-8(4)}\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \\ &= \frac{1}{1}\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \end{aligned}$$

So,

$$A^{-1}=\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$$

Now we are ready to solve. Multiply both sides of the equation by A^{-1}.

$$\begin{aligned} \left(A^{-1}\right)AX &= \left(A^{-1}\right)B \\ \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 11(5)+(-8)7 \\ -4(5)+3(7) \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

The solution is $(-1, 1)$.

Q&A: Can we solve for X by finding the product BA^{-1}?

No, recall that matrix multiplication is not commutative, so $A^{-1}B \neq BA^{-1}$. Consider our steps for solving the matrix equation.

$$\begin{aligned} \left(A^{-1}\right)AX &= \left(A^{-1}\right)B \\ \left[\left(A^{-1}\right)A\right]X &= \left(A^{-1}\right)B \\ IX &= \left(A^{-1}\right)B \\ X &= \left(A^{-1}\right)B \end{aligned}$$

Notice in the first step we multiplied both sides of the equation by A^{-1}, but the A^{-1} was to the left of A on the left side and to the left of B on the right side. Because matrix multiplication is not commutative, order matters.

Example 1.7.8: Solving a 3 × 3 System Using the Inverse of a Matrix

Solve the following system using the inverse of a matrix.

$$\begin{aligned} 5x+15y+56z &= 35 \\ -4x-11y-41z &= -26 \\ -x-3y-11z &= -7 \end{aligned}$$

Solution

Write the equation $AX = B$.

$$\begin{bmatrix} 5 & 15 & 56 \\ -4 & -11 & -41 \\ -1 & -3 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35 \\ -26 \\ -7 \end{bmatrix}$$

First, we will find the inverse of A by augmenting with the identity.

$$\left[\begin{array}{ccc|ccc} 5 & 15 & 56 & 1 & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Multiply row 1 by $\frac{1}{5}$.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Multiply row 1 by 4 and add to row 2.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Add row 1 to row 3.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 1 \end{array}\right]$$

Multiply row 2 by -3 and add to row 1.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{5} & -\frac{11}{5} & -3 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 1 \end{array}\right]$$

Multiply row 3 by 5.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{5} & -\frac{11}{5} & -3 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right]$$

Multiply row 3 by $\frac{1}{5}$ and add to row 1.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right]$$

Multiply row 3 by $-\frac{19}{5}$ and add to row 2.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & 1 & -19 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right]$$

So,

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix}$$

Multiply both sides of the equation by A^{-1}. We want $A^{-1}AX = A^{-1}B$:

$$\begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 15 & 56 \\ -4 & -11 & -41 \\ -1 & -3 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 35 \\ -26 \\ -7 \end{bmatrix}$$

Thus,

$$A^{-1}B = \begin{bmatrix} -70+78-7 \\ -105-26+133 \\ 35+0-35 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

The solution is $(1, 2, 0)$.

? Exercise 1.7.4

Solve the system using the inverse of the coefficient matrix.

$$\begin{aligned} 2x - 17y + 11z &= 0 \\ -x + 11y - 7z &= 8 \\ 3y - 2z &= -2 \end{aligned}$$

Answer

$$X = \begin{bmatrix} 4 \\ 38 \\ 58 \end{bmatrix}$$

How to: Given a system of equations, solve with matrix inverses using a calculator

1. Save the coefficient matrix and the constant matrix as matrix variables $[A]$ and $[B]$.
2. Enter the multiplication into the calculator, calling up each matrix variable as needed.
3. If the coefficient matrix is invertible, the calculator will present the solution matrix; if the coefficient matrix is not invertible, the calculator will present an error message.

Example 1.7.9: Using a Calculator to Solve a System of Equations with Matrix Inverses

Solve the system of equations with matrix inverses using a calculator

$$\begin{aligned} 2x + 3y + z &= 32 \\ 3x + 3y + z &= -27 \\ 2x + 4y + z &= -2 \end{aligned}$$

Solution

On the matrix page of the calculator, enter the coefficient matrix as the matrix variable $[A]$, and enter the constant matrix as the matrix variable $[B]$.

$$[A] = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}, [B] = \begin{bmatrix} 32 \\ -27 \\ -2 \end{bmatrix}$$

On the home screen of the calculator, type in the multiplication to solve for X, calling up each matrix variable as needed.

$$[A]^{-1} \times [B]$$

Evaluate the expression.

$$\begin{bmatrix} -59 \\ -34 \\ 252 \end{bmatrix}$$

Media

Access these online resources for additional instruction and practice with solving systems with inverses.

- The Identity Matrix
- Determining Inverse Matrices
- Using a Matrix Equation to Solve a System of Equations

1.7.6 Key Equations

Identity matrix for a 2×2 matrix	$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Identity matrix for a 3×3 matrix	$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Multiplicative inverse of a 2×2 matrix	$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$

1.7.7 Key Concepts

- An identity matrix has the property $AI = IA = A$. See Example 1.7.1.
- An invertible matrix has the property $AA^{-1} = A^{-1}A = I$. See Example 1.7.2.
- Use matrix multiplication and the identity to find the inverse of a 2×2 matrix. See Example 1.7.3.
- The multiplicative inverse can be found using a formula. See Example 1.7.4.
- Another method of finding the inverse is by augmenting with the identity. See Example 1.7.5.
- We can augment a 3×3 matrix with the identity on the right and use row operations to turn the original matrix into the identity, and the matrix on the right becomes the inverse. See Example 1.7.6.
- Write the system of equations as $AX = B$, and multiply both sides by the inverse of A: $A^{-1}AX = A^{-1}B$. See Example 1.7.7 and Example 1.7.8.
- We can also use a calculator to solve a system of equations with matrix inverses. See Example 1.7.9.

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CHAPTER OVERVIEW

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  - 2.1: 2.1(b)- Factoring Polynomials
  - 2.1: 2.1(b) - Simplify Rational Expressions
- 2.2: Basic Classes of Functions
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2.1: 2.1(a)- Polynomials

Learning Objectives

In this section students will:

- Identify the degree and leading coefficient of polynomials.
- Add and subtract polynomials.
- Multiply polynomials.
- Use FOIL to multiply binomials.
- Perform operations with polynomials of several variables.

Earl is building a doghouse, whose front is in the shape of a square topped with a triangle. There will be a rectangular door through which the dog can enter and exit the house. Earl wants to find the area of the front of the doghouse so that he can purchase the correct amount of paint. Using the measurements of the front of the house, shown in Figure 2.1.1, we can create an expression that combines several variable terms, allowing us to solve this problem and others like it.

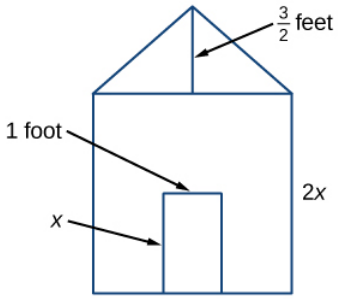


Figure 2.1.1

- First find the area of the square in square feet.

$$\begin{aligned} A &= s^2 \\ &= (2x)^2 \\ &= 4x^2 \end{aligned}$$

- Then find the area of the triangle in square feet.

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(2x)\left(\frac{3}{2}\right) \\ &= \frac{3}{2}x \end{aligned}$$

- Next find the area of the rectangular door in square feet.

$$\begin{aligned} A &= lw \\ &= x \times 1 \\ &= x \end{aligned}$$

The area of the front of the doghouse can be found by adding the areas of the square and the triangle, and then subtracting the area of the rectangle. When we do this, we get

$$4x^2 + \frac{3}{2}x - x \ \ ft^2$$

or

$$4x^2 + \frac{1}{2}x \ \ ft^2$$

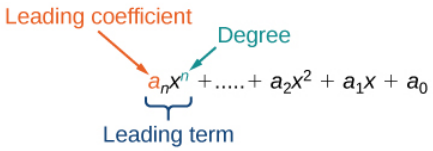
In this section, we will examine expressions such as this one, which combine several variable terms.

2.1.1 Identifying the Degree and Leading Coefficient of Polynomials

The formula just found is an example of a **polynomial**, which is a sum of or difference of terms, each consisting of a variable raised to a nonnegative integer power. A number multiplied by a variable raised to an exponent, such as 384π, is known as a **coefficient**. Coefficients can be positive, negative, or zero, and can be whole numbers, decimals, or fractions. Each product $a_i x^i$, such as $384\pi w$, is a **term of a polynomial**. If a term does not contain a variable, it is called a constant.

A polynomial containing only one term, such as $5x^4$, is called a **monomial**. A polynomial containing two terms, such as $2x-9$, is called a **binomial**. A polynomial containing three terms, such as $-3x^2+8x-7$, is called a **trinomial**.

We can find the **degree** of a polynomial by identifying the highest power of the variable that occurs in the polynomial. The term with the highest degree is called the **leading term** because it is usually written first. The coefficient of the leading term is called the **leading coefficient**. When a polynomial is written so that the powers are descending, we say that it is in standard form.



Polynomials

A polynomial is an expression that can be written in the form

$$a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 \tag{2.1.1}$$

Each real number ai is called a **coefficient**. The number a_0 that is not multiplied by a variable is called aconstant. Each product $a_i x^i$ is a **term of a polynomial**. The highest power of the variable that occurs in the polynomial is called the **degree** of a polynomial. The **leading term** is the term with the highest power, and its coefficient is called the **leading coefficient.**

How to: Given a polynomial expression, identify the degree and leading coefficient.

1. Find the highest power of x to determine the degree.
2. Identify the term containing the highest power of x to find the leading term.
3. Identify the coefficient of the leading term.

Example 2.1.1: Identifying the Degree and Leading Coefficient of a Polynomial

For the following polynomials, identify the degree, the leading term, and the leading coefficient.

a. $3+2x^2-4x^3$
b. $5t^5-2t^3+7t$
c. $6p-p^3-2$

Solution

a. The highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.
b. The highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.
c. The highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$, The leading coefficient is the coefficient of that term, -1.

Exercise 2.1.1

Identify the degree, leading term, and leading coefficient of the polynomial $4x^2-x^6+2x-6$.

Answer

The degree is 6, the leading term is $-x^6$, and the leading coefficient is -1.

2.1.2 Adding and Subtracting Polynomials

We can add and subtract polynomials by combining like terms, which are terms that contain the same variables raised to the same exponents. For example, $5x^2$ and $-2x^2$ are like terms, and can be added to get $3x^2$, but $3x$ and $3x^2$ are not like terms, and therefore cannot be added.

How to: Given multiple polynomials, add or subtract them to simplify the expressions.

1. Combine like terms.
2. Simplify and write in standard form.

Example 2.1.2: Adding Polynomials

Find the sum.

$$(12x^2+9x-21)+(4x^3+8x^2-5x+20)$$

Solution

$$\begin{aligned} &4x^3+(12x^2+8x^2)+(9x-5x)+(-21+20) && \text{Combine like terms} \\ &4x^3+20x^2+4x-1 && \text{Simplify} \end{aligned}$$

Analysis

We can check our answers to these types of problems using a graphing calculator. To check, graph the problem as given along with the simplified answer. The two graphs should be equivalent. Be sure to use the same window to compare the graphs. Using different windows can make the expressions seem equivalent when they are not.

Exercise 2.1.2

Find the sum.

$$(2x^3+5x^2-x+1)+(2x^2-3x-4)$$

Answer

$2x^3+7x^2-4x-3$

Example 2.1.3: Subtracting Polynomials

Find the difference.

$$(7x^4-x^2+6x+1)-(5x^3-2x^2+3x+2)$$

Solution

$7x^4-5x^3+(-x^2+2x^2)+(6x-3x)+(1-2)$ Combine like terms

$7x^4-5x^3+x^2+3x-1$ Simplify

Analysis

Note that finding the difference between two polynomials is the same as adding the opposite of the second polynomial to the first.

Exercise 2.1.3

Find the difference.

$$(-7x^3-7x^2+6x-2)-(4x^3-6x^2-x+7)$$

Answer

$-11x^3 - x^2 + 7x - 9$

2.1.3 Multiplying Polynomials

Multiplying polynomials is a bit more challenging than adding and subtracting polynomials. We must use the distributive property to multiply each term in the first polynomial by each term in the second polynomial. We then combine like terms. We can also use a shortcut called the FOIL method when multiplying binomials. Certain special products follow patterns that we can memorize and use instead of multiplying the polynomials by hand each time. We will look at a variety of ways to multiply polynomials.

2.1.3.1 Multiplying Polynomials Using the Distributive Property

To multiply a number by a polynomial, we use the distributive property. The number must be distributed to each term of the polynomial. We can distribute the 2 in $2(x+7)$ to obtain the equivalent expression $2x+14$. When multiplying polynomials, the distributive property allows us to multiply each term of the first polynomial by each term of the second. We then add the products together and combine like terms to simplify.

How to: Given the multiplication of two polynomials, use the distributive property to simplify the expression.

1. Multiply each term of the first polynomial by each term of the second.
2. Combine like terms.
3. Simplify.

Example 2.1.4: Multiplying Polynomials Using the Distributive Property

Find the product.

$$(2x+1)(3x^2 - x + 4)$$

Solution

$$\begin{aligned} &2x(3x^2 - x + 4) + 1(3x^2 - x + 4) && \text{Use the distributive property} \\ &(6x^3 - 2x^2 + 8x) + (3x^2 - x + 4) && \text{Multiply} \\ &6x^3 + (-2x^2 + 3x^2) + (8x - x) + 4 && \text{Combine like terms} \\ &6x^3 + x^2 + 7x + 4 && \text{Simplify} \end{aligned}$$

Analysis

We can use a table to keep track of our work, as shown in Table 2.1.1. Write one polynomial across the top and the other down the side. For each box in the table, multiply the term for that row by the term for that column. Then add all of the terms together, combine like terms, and simplify.

Table 2.1.1

	$3x^2$	$-x$	$+4$
$2x$	$6x^3$	$-2x^2$	$8x$
$+1$	$3x^2$	$-x$	4

Exercise 2.1.4

Find the product.

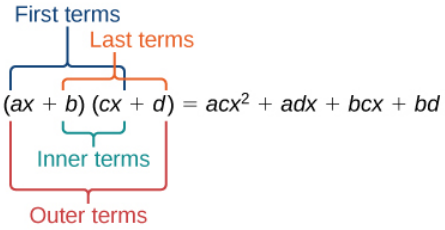
$(3x+2)(x^3 - 4x^2 + 7)$

Answer

$3x^4 - 10x^3 - 8x^2 + 21x + 14$

2.1.3.2 Using FOIL to Multiply Binomials

A shortcut called FOIL is sometimes used to find the product of two binomials. It is called FOIL because we multiply the first terms, the outer terms, the inner terms, and then the last terms of each binomial.



The FOIL method arises out of the distributive property. We are simply multiplying each term of the first binomial by each term of the second binomial, and then combining like terms.

FOIL to simplify expression

Given two binomials, use FOIL to simplify the expression.

1. Multiply the outer terms of the binomials.
2. Multiply the last terms of each binomial.
3. Combine like terms and simplify.

Example 2.1.5: Using FOIL to Multiply Binomials

Use FOIL to find the product.

$$(2x-10)(3x+3)$$

Solution

Find the product of the first terms.

$2x - 18 \quad 3x + 3 \qquad 2x \cdot 3x = 6x^2$

Find the product of the outer terms.

$2x - 18 \quad 3x + 3 \qquad 2x \cdot 3 = 6x$

Find the product of the inner terms.

$2x - 18 \quad 3x + 3 \qquad -18 \cdot 3x = -54x$

Find the product of the last terms.

$2x - 18 \quad 3x + 3 \qquad -18 \cdot 3 = -54$

$$\begin{aligned} 6x^2+6x-54x-54 &\quad \text{Add the products} \\ 6x^2+(6x-54x)-54 &\quad \text{Combine like terms} \\ 6x^2-48x-54 &\quad \text{Simplify} \end{aligned}$$

Exercise 2.1.5

Use FOIL to find the product.

$$(x+7)(3x-5)$$

Answer

$3x^2+16x-35$

2.1.3.3 Perfect Square Trinomials

Certain binomial products have special forms. When a binomial is squared, the result is called a perfect square trinomial. We can find the square by multiplying the binomial by itself. However, there is a special form that each of these perfect square trinomials takes, and memorizing the form makes squaring binomials much easier and faster. Let's look at a few perfect square trinomials to familiarize ourselves with the form.

$$(x+5)^2 = x^2+10x+25$$
$$(x-3)^2 = x^2-6x+9$$
$$(4x-1)^2 = 16x^2-8x+1$$

Notice that the first term of each trinomial is the square of the first term of the binomial and, similarly, the last term of each trinomial is the square of the last term of the binomial. The middle term is double the product of the two terms. Lastly, we see that the first sign of the trinomial is the same as the sign of the binomial.

Perfect Square Trinomials

When a binomial is squared, the result is the first term squared added to double the product of both terms and the last term squared.

$$(x+a)^2 = (x+a)(x+a) = x^2+2ax+a^2 \tag{2.1.2}$$

How to: Given a binomial, square it using the formula for perfect square trinomials.

1. Square the first term of the binomial.
2. Square the last term of the binomial.
3. For the middle term of the trinomial, double the product of the two terms.
4. Add and simplify.

Example 2.1.6: Expanding Perfect Squares

Expand $(3x-8)^2$.

Solution

Begin by squaring the first term and the last term. For the middle term of the trinomial, double the product of the two terms.

$$(3x)^2-2(3x)(8)+(-8)^2$$
$$9x^2-48x+64 \qquad \text{Simplify}$$

Exercise 2.1.6

Expand $(4x-1)^2$.

Answer

$16x^2-8x+1$

2.1.3.4 Difference of Squares

Another special product is called the difference of squares, which occurs when we multiply a binomial by another binomial with the same terms but the opposite sign. Let's see what happens when we multiply $(x+1)(x-1)$ using the FOIL method.

$$\begin{aligned}(x+1)(x-1) &= x^2 - x + x - 1 \\ &= x^2 - 1\end{aligned}$$

The middle term drops out, resulting in a difference of squares. Just as we did with the perfect squares, let's look at a few examples.

$$(x+5)(x-5) = x^2 - 25$$
$$(x+11)(x-11) = x^2 - 121$$
$$(2x+3)(2x-3) = 4x^2 - 9$$

Because the sign changes in the second binomial, the outer and inner terms cancel each other out, and we are left only with the square of the first term minus the square of the last term.

Q&A

Is there a special form for the sum of squares?

No. The difference of squares occurs because the opposite signs of the binomials cause the middle terms to disappear. There are no two binomials that multiply to equal a sum of squares.

Difference of Squares

When a binomial is multiplied by a binomial with the same terms separated by the opposite sign, the result is the square of the first term minus the square of the last term.

$$(a+b)(a-b) = a^2 - b^2 \tag{2.1.3}$$

Howto: Given a binomial multiplied by a binomial with the same terms but the opposite sign, find the difference of squares.

1. Square the first term of the binomials.
2. Square the last term of the binomials.
3. Subtract the square of the last term from the square of the first term.

Example 2.1.7: Multiplying Binomials Resulting in a Difference of Squares

Multiply $(9x+4)(9x-4)$.

Solution

Square the first term to get $(9x)^2 = 81x^2$. Square the last term to get $4^2 = 16$. Subtract the square of the last term from the square of the first term to find the product of $81x^2 - 16$.

Exercise 2.1.7

Multiply $(2x+7)(2x-7)$.

Answer

$4x^2 - 49$

2.1.4 Performing Operations with Polynomials of Several Variables

We have looked at polynomials containing only one variable. However, a polynomial can contain several variables. All of the same rules apply when working with polynomials containing several variables. Consider an example:

$(a+2b)(4a-b-c)a(4a-b-c)+2b(4a-b-c)$	Use the distributive property
$4a^2-ab-ac+8ab-2b^2-2bc$	Multiply
$4a^2+(-ab+8ab)-ac-2b^2-2bc$	Combine like terms
$4a^2+7ab-ac-2bc-2b^2$	Simplify

Example 2.1.8: Multiplying Polynomials Containing Several Variables

Multiply $(x+4)(3x-2y+5)$.

Solution

$x(3x-2y+5)+4(3x-2y+5)$	Use the distributive property
$3x^2-2xy+5x+12x-8y+20$	Multiply
$3x^2-2xy+(5x+12x)-8y+20$	Combine like terms
$3x^2-2xy+17x-8y+20$	Simplify

Exercise 2.1.8

Multiply $(3x-1)(2x+7y-9)$.

Answer

$6x^2+21xy-29x-7y+9$

Media

Access these online resources for additional instruction and practice with polynomials.

1. Adding and Subtracting Polynomials
2. Multiplying Polynomials
3. Special Products of Polynomials

2.1.5 Key Equations

perfect square trinomial	$(x+a)^2=(x+a)(x+a)=x^2+2ax+a^2$
difference of squares	$(a+b)(a-b)=a^2-b^2$

2.1.6 Key Concepts

- A polynomial is a sum of terms each consisting of a variable raised to a non-negative integer power. The degree is the highest power of the variable that occurs in the polynomial. The leading term is the term containing the highest degree, and the leading coefficient is the coefficient of that term. See Example.
- We can add and subtract polynomials by combining like terms. See Example and Example.
- To multiply polynomials, use the distributive property to multiply each term in the first polynomial by each term in the second. Then add the products. See Example.
- FOIL (First, Outer, Inner, Last) is a shortcut that can be used to multiply binomials. See Example.
- Perfect square trinomials and difference of squares are special products. See Example and Example.
- Follow the same rules to work with polynomials containing several variables. See Example.

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2.1: 2.1(b)- Factoring Polynomials

Learning Objectives

In this section students will:

- Factor the greatest common factor of a polynomial.
- Factor a trinomial.
- Factor by grouping.
- Factor a perfect square trinomial.
- Factor a difference of squares.
- Factor the sum and difference of cubes.
- Factor expressions using fractional or negative exponents.

Imagine that we are trying to find the area of a lawn so that we can determine how much grass seed to purchase. The lawn is the green portion in Figure 2.1.1.

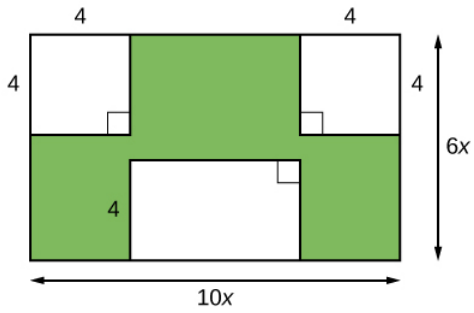


Figure 2.1.1

The area of the entire region can be found using the formula for the area of a rectangle.

$$\begin{aligned} A &= lw \\ &= 10x \times 6x \\ &= 60x^2 \ units^2 \end{aligned}$$

The areas of the portions that do not require grass seed need to be subtracted from the area of the entire region. The two square regions each have an area of $A = s^2 = 4^2 = 16 \ units^2$. The other rectangular region has one side of length $10x - 8$ and one side of length 4, giving an area of

$$A = lw = 4(10x - 8) = 40x - 32 \text{ units}^2.$$

So the region that must be subtracted has an area of

$$2(16) + 40x - 32 = 40x \text{ units}^2.$$

The area of the region that requires grass seed is found by subtracting $60x^2 - 40x \text{ units}^2$. This area can also be expressed in factored form as $20x(3x - 2) \text{ units}^2$. We can confirm that this is an equivalent expression by multiplying.

Many polynomial expressions can be written in simpler forms by factoring. In this section, we will look at a variety of methods that can be used to factor polynomial expressions.

2.1.1 Factoring the Greatest Common Factor of a Polynomial

When we study fractions, we learn that the greatest common factor (GCF) of two numbers is the largest number that divides evenly into both numbers. For instance, 4 is the GCF of 16 and 20 because it is the largest number that divides evenly into both 16 and 20 The GCF of polynomials works the same way: $4x$ is the GCF of $16x$ and $20x^2$ because it is the largest polynomial that divides evenly into both $16x$ and $20x^2$.

When factoring a polynomial expression, our first step should be to check for a GCF. Look for the GCF of the coefficients, and then look for the GCF of the variables.

Definition: Greatest Common Factor

The *greatest common factor* (GCF) of polynomials is the largest polynomial that divides evenly into the polynomials.

Howto: Given a polynomial expression, factor out the greatest common factor

1. Identify the GCF of the coefficients.
2. Identify the GCF of the variables.
3. Combine to find the GCF of the expression.
4. Determine what the GCF needs to be multiplied by to obtain each term in the expression.
5. Write the factored expression as the product of the GCF and the sum of the terms we need to multiply by.

Example 2.1.1: Factoring the Greatest Common Factor

Factor $6x^3y^3+45x^2y^2+21xy$.

Solution

First, find the GCF of the expression. The GCF of 6, 45, and 21 is 3. The GCF of x^3,x^2, and x is x. (Note that the GCF of a set of expressions in the form x^n will always be the exponent of lowest degree.) And the GCF of y^3,y^2, and y is y. Combine these to find the GCF of the polynomial, $3xy$.

Next, determine what the GCF needs to be multiplied by to obtain each term of the polynomial. We find that

- $3xy(2x^2y^2)=6x^3y^3$,
- $3xy(15xy)=45x^2y^2$, and
- $3xy(7)=21xy$.

Finally, write the factored expression as the product of the GCF and the sum of the terms we needed to multiply by.

$$(3xy)(2x^2y^2+15xy+7)$$

Analysis

After factoring, we can check our work by multiplying. Use the distributive property to confirm that

$$(3xy)(2x^2y^2+15xy+7)=6x^3y^3+45x^2y^2+21xy$$

Exercise 2.1.1

Factor $x(b^2-a)+6(b^2-a)$ by pulling out the GCF.

Answer

$(b^2-a)(x+6)$

2.1.2 Factoring a Trinomial with Leading Coefficient 1

Although we should always begin by looking for a GCF, pulling out the GCF is not the only way that polynomial expressions can be factored. The polynomial x^2+5x+6 has a GCF of 1, but it can be written as the product of the factors $(x+2)$ and $(x+3)$.

Trinomials of the form x^2+bx+c can be factored by finding two numbers with a product of c and a sum of b. The trinomial $x^2+10x+16$, for example, can be factored using the numbers 2 and 8 because the product of those numbers is 16 and their sum is 10. The trinomial can be rewritten as the product of $(x+2)$ and $(x+8)$.

FACTORING A TRINOMIAL WITH LEADING COEFFICIENT 1

A trinomial of the form x^2+bx+c can be written in factored form as $(x+p)(x+q)$ where $pq=c$ and $p+q=b$.

Q&A: Can every trinomial be factored as a product of binomials?

No. Some polynomials cannot be factored. These polynomials are said to be *prime*.

Howto: Given a trinomial in the form $x^2 + bx + c$, factor it

1. List factors of c.
2. Find p and q, a pair of factors of c with a sum of b.
3. Write the factored expression $(x+p)(x+q)$.

Example 2.1.2: Factoring a Trinomial with Leading Coefficient 1

Factor $x^2+2x-15$.

Solution

We have a trinomial with leading coefficient 1, $b=2$, and $c=-15$. We need to find two numbers with a product of -15 and a sum of 2. In Table 2.1.1, we list factors until we find a pair with the desired sum.

Table 2.1.1

Factors of −15	Sum of Factors
1,−15	−14
−1,15	14
3,−5	−2
−3,5	

Now that we have identified p and q as -3 and 5, write the factored form as $(x-3)(x+5)$.

Analysis

We can check our work by multiplying. Use FOIL to confirm that $(x-3)(x+5)=x^2+2x-15$.

Q&A: Does the order of the factors matter?

No. Multiplication is commutative, so the order of the factors does not matter.

Exercise 2.1.2

Factor x^2-7x+6 .

Answer

$(x-6)(x-1)$

2.1.3 Factoring by Grouping

Trinomials with leading coefficients other than 1 are slightly more complicated to factor. For these trinomials, we can **factor by grouping** by dividing the x term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression. The trinomial $2x^2+5x+3$ can be rewritten as $(2x+3)(x+1)$ using this process. We begin by rewriting the original expression as $2x^2+2x+3x+3$ and then factor each portion of the expression to obtain $2x(x+1)+3(x+1)$. We then pull out the GCF of $(x+1)$ to find the factored expression.

Factor by Grouping

To factor a trinomial in the form ax^2+bx+c by grouping, we find two numbers with a product of ac and a sum of b. We use these numbers to divide the x term into the sum of two terms and factor each portion of the expression separately, then factor out the GCF of the entire expression.

Howto: Given a trinomial in the form ax^2+bx+c, factor by grouping.

1. List factors of ac.
2. Find p and q, a pair of factors of ac with a sum of b.
3. Rewrite the original expression as $ax^2+px+qx+c$.
4. Pull out the GCF of ax^2+px.
5. Pull out the GCF of $qx+c$.
6. Factor out the GCF of the expression.

Example 2.1.3: Factoring a Trinomial by Grouping

Factor $5x^2+7x-6$ by grouping.

Solution

We have a trinomial with $a=5, b=7$, and $c=-6$. First, determine $ac=-30$. We need to find two numbers with a product of -30 and a sum of 7. In the table below, we list factors until we find a pair with the desired sum.

Table 2.1.2

Factors of −30	Sum of Factors
1,−30	−29
−1,30	29
2,−15	−13
−2,15	13
3,−10	−7
−3,10	7

So $p=-3$ and $q=10$.

$5x^2-3x+10x-6$ Rewrite the original expression as $ax^2+px+qx+c$.

$x(5x-3)+2(5x-3)$ Factor out the GCF of each part

$(5x-3)(x+2)$ Factor out the GCF of the expression.

Analysis

We can check our work by multiplying. Use FOIL to confirm that $(5x-3)(x+2)=5x^2+7x-6$.

Exercise 2.1.3

Factor:

a. $2x^2+9x+9$
b. $6x^2+x-1$

Answer a

$(2x+3)(x+3)$

Answer b

$(3x-1)(2x+1)$

2.1.4 Factoring a Perfect Square Trinomial

A perfect square trinomial is a trinomial that can be written as the square of a binomial. Recall that when a binomial is squared, the result is the square of the first term added to twice the product of the two terms and the square of the last term.

$$a^2+2ab+b^2=(a+b)^2 \tag{2.1.1}$$

and

$$a^2-2ab+b^2=(a-b)^2 \tag{2.1.2}$$

We can use this equation to factor any perfect square trinomial.

Perfect Square Trinomials

A perfect square trinomial can be written as the square of a binomial:

$$a^2+2ab+b^2=(a+b)^2 \tag{2.1.3}$$

Howto: Given a perfect square trinomial, factor it into the square of a binomial

1. Confirm that the first and last term are perfect squares.
2. Confirm that the middle term is twice the product of ab.
3. Write the factored form as $(a+b)^2$.

Example 2.1.4: Factoring a Perfect Square Trinomial

Factor $25x^2+20x+4$.

Solution

Notice that $25x^2$ and 4 are perfect squares because $25x^2=(5x)^2$ and $4=2^2$. Then check to see if the middle term is twice the product of $5x$ and 2. The middle term is, indeed, twice the product: $2(5x)(2)=20x$. Therefore, the trinomial is a perfect square trinomial and can be written as $(5x+2)^2$.

Exercise 2.1.4

Factor $49x^2-14x+1$.

Answer

$(7x-1)^2$

2.1.5 Factoring a Difference of Squares

A difference of squares is a perfect square subtracted from a perfect square. Recall that a difference of squares can be rewritten as factors containing the same terms but opposite signs because the middle terms cancel each other out when the two factors are multiplied.

$$a^2-b^2=(a+b)(a-b) \tag{2.1.4}$$

We can use this equation to factor any differences of squares.

Differences of Squares

A difference of squares can be rewritten as two factors containing the same terms but opposite signs.

$$a^2 - b^2 = (a+b)(a-b) \tag{2.1.5}$$

Howto: Given a difference of squares, factor it into binomials

1. Confirm that the first and last term are perfect squares.
2. Write the factored form as $(a+b)(a-b)$.

Example 2.1.5: Factoring a Difference of Squares

Factor $9x^2 - 25$.

Solution

Notice that $9x^2$ and 25 are perfect squares because $9x^2 = (3x)^2$ and $25 = 5^2$. The polynomial represents a difference of squares and can be rewritten as $(3x+5)(3x-5)$.

Exercise 2.1.5

Factor $81y^2 - 100$.

Answer

$(9y+10)(9y-10)$

Q&A: Is there a formula to factor the sum of squares?

No. A sum of squares cannot be factored.

2.1.6 Factoring the Sum and Difference of Cubes

Now, we will look at two new special products: the sum and difference of cubes. Although the sum of squares cannot be factored, the sum of cubes can be factored into a binomial and a trinomial.

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) \tag{2.1.6}$$

Similarly, the sum of cubes can be factored into a binomial and a trinomial, but with different signs.

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) \tag{2.1.7}$$

We can use the acronym SOAP to remember the signs when factoring the sum or difference of cubes. The first letter of each word relates to the signs: Same Opposite Always Positive. For example, consider the following example.

$$x^3 - 2^3 = (x-2)(x^2 + 2x + 4) \tag{2.1.8}$$

The sign of the first 2 is the same as the sign between $x^3 - 2^3$. The sign of the $2x$ term is opposite the sign between $x^3 - 2^3$. And the sign of the last term, 4, is always positive.

Sum and Difference of Cubes

We can factor the sum of two cubes as

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) \tag{2.1.9}$$

We can factor the difference of two cubes as

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) \tag{2.1.10}$$

Howto: Given a sum of cubes or difference of cubes, factor it

1. Confirm that the first and last term are cubes, a^3+b^3 or a^3-b^3.
2. For a sum of cubes, write the factored form as $(a+b)(a^2-ab+b^2)$. For a difference of cubes, write the factored form as $(a-b)(a^2+ab+b^2)$.

Example 2.1.6: Factoring a Sum of Cubes

Factor x^3+512.

Solution

Notice that x^3 and 512 are cubes because $8^3=512$. Rewrite the sum of cubes as $(x+8)(x^2-8x+64)$.

Analysis

After writing the sum of cubes this way, we might think we should check to see if the trinomial portion can be factored further. However, the trinomial portion cannot be factored, so we do not need to check.

Exercise 2.1.6

Factor the sum of cubes: $216a^3+b^3$.

Answer

$(6a+b)(36a^2-6ab+b^2)$

Example 2.1.7: Factoring a Difference of Cubes

Factor $8x^3-125$.

Solution

Notice that $8x^3$ and 125 are cubes because $8x^3=(2x)^3$ and $125=5^3$. Write the difference of cubes as $(2x-5)(4x^2+10x+25)$.

Analysis

Just as with the sum of cubes, we will not be able to further factor the trinomial portion.

Exercise 2.1.7

Factor the difference of cubes: $1000x^3-1$

Answer

$(10x-1)(100x^2+10x+1)$

2.1.7 Factoring Expressions with Fractional or Negative Exponents

Expressions with fractional or negative exponents can be factored by pulling out a GCF. Look for the variable or exponent that is common to each term of the expression and pull out that variable or exponent raised to the lowest power. These expressions follow the same factoring rules as those with integer exponents. For instance, $2x^{\frac{1}{4}}+5x^{\frac{3}{4}}$ can be factored by pulling out $x^{\frac{1}{4}}$ and being rewritten as $x^{\frac{1}{4}}(2+5x^{\frac{1}{2}})$.

Example 2.1.8: Factoring an Expression with Fractional or Negative Exponents

Factor $3x(x+2)^{-\frac{1}{3}}+4(x+2)^{\frac{2}{3}}$.

Solution

Factor out the term with the lowest value of the exponent. In this case, that would be $(x+2)^{-\frac{1}{3}}$.

$$(x+2)^{-\frac{1}{3}}(3x+4(x+2)) \qquad \text{Factor out the GCF}$$
$$(x+2)^{-\frac{1}{3}}(3x+4x+8) \qquad \text{Simplify}$$
$$(x+2)^{-\frac{1}{3}}(7x+8)$$

Exercise 2.1.8

Factor $2(5a-1)^{\frac{3}{4}}+7a(5a-1)^{-\frac{1}{4}}$.

Answer

$$(5a-1)^{-\frac{1}{4}}(17a-2)$$

Media

Access these online resources for additional instruction and practice with factoring polynomials.

1. Identify GCF
2. Factor Trinomials when a Equals 1
3. Factor Trinomials when a is not equal to 1
4. Factor Sum or Difference of Cubes

2.1.8 Key Equations

difference of squares	$a^2-b^2=(a+b)(a-b)$
perfect square trinomial	$a^2+2ab+b^2=(a+b)^2$
sum of cubes	$a^3+b^3=(a+b)(a^2-ab+b^2)$
difference of cubes	$a^3-b^3=(a-b)(a^2+ab+b^2)$

- The greatest common factor, or GCF, can be factored out of a polynomial. Checking for a GCF should be the first step in any factoring problem. See Example.
- Trinomials with leading coefficient 1 can be factored by finding numbers that have a product of the third term and a sum of the second term. See Example.
- Trinomials can be factored using a process called factoring by grouping. See Example.
- Perfect square trinomials and the difference of squares are special products and can be factored using equations. See Example and Example.
- The sum of cubes and the difference of cubes can be factored using equations. See Example and Example.
- Polynomials containing fractional and negative exponents can be factored by pulling out a GCF. See Example.

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2.1: 2.1(b) - Simplify Rational Expressions

Learning Objectives

In this section students will:

- Simplify rational expressions

A pastry shop has fixed costs of $\$280$ per week and variable costs of $\$9$ per box of pastries. The shop's costs per week in terms of x, the number of boxes made, is $280+9x$. We can divide the costs per week by the number of boxes made to determine the cost per box of pastries.

$$\frac{280+9x}{x}$$

Notice that the result is a polynomial expression divided by a second polynomial expression. In this section, we will explore quotients of polynomial expressions.

Simplifying Rational Expressions

The quotient of two polynomial expressions is called a rational expression. We can apply the properties of fractions to rational expressions, such as simplifying the expressions by canceling common factors from the numerator and the denominator. To do this, we first need to factor both the numerator and denominator. Let's start with the rational expression shown.

$$\frac{x^2+8x+16}{x^2+11x+28}$$

We can factor the numerator and denominator to rewrite the expression.

$$\frac{(x+4)^2}{(x+4)(x+7)}$$

Then we can simplify that expression by canceling the common factor $(x+4)$.

$$\frac{x+4}{x+7}$$

Howto: Given a rational expression, simplify it

1. Factor the numerator and denominator.
2. Cancel any common factors.

Example 2.1.1: Simplifying Rational Expressions

Simplify $\dfrac{x^2-9}{x^2+4x+3}$

Solution

$$\frac{(x+3)(x-3)}{(x+3)(x+1)} \qquad \text{Factor the numerator and the denominator}$$

$$\frac{x-3}{x+1} \qquad \text{Cancel common factor } (x+3)$$

Analysis

We can cancel the common factor because any expression divided by itself is equal to 1.

Q&A

Can the x^2 term be cancelled in the last example?

No. A factor is an expression that is multiplied by another expression. The x^2 term is not a factor of the numerator or the denominator.

Exercise 2.1.1

Simplify $\dfrac{x-6}{x^2-36}$

Answer

$$\frac{1}{x+6}$$

Key Concepts

- Rational expressions can be simplified by cancelling common factors in the numerator and denominator. See Example.

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2.2: Basic Classes of Functions

Learning Objectives

- Calculate the slope of a linear function and interpret its meaning.
- Recognize the degree of a polynomial.
- Find the roots of a quadratic polynomial.
- Describe the graphs of basic odd and even polynomial functions.
- Identify a rational function.
- Describe the graphs of power and root functions.
- Graph a piecewise-defined function.
- Sketch the graph of a function that has been shifted, stretched, or reflected from its initial graph position.

We have studied the general characteristics of functions, so now let's examine some specific classes of functions. We begin by reviewing the basic properties of linear and quadratic functions, and then generalize to include higher-degree polynomials. By combining root functions with polynomials, we can define general algebraic functions and distinguish them from the transcendental functions we examine later in this chapter. We finish the section with examples of piecewise-defined functions and take a look at how to sketch the graph of a function that has been shifted, stretched, or reflected from its initial form.

2.2.1 Linear Functions and Slope

The easiest type of function to consider is a **linear function**. Linear functions have the form $f(x) = ax + b$, where a and b are constants. In Figure 2.2.1, we see examples of linear functions when a is positive, negative, and zero. Note that if $a > 0$, the graph of the line rises as x increases. In other words, $f(x) = ax + b$ is increasing on $(-\infty, \infty)$. If $a < 0$, the graph of the line falls as x increases. In this case, $f(x) = ax + b$ is decreasing on $(-\infty, \infty)$. If $a = 0$, the line is horizontal.

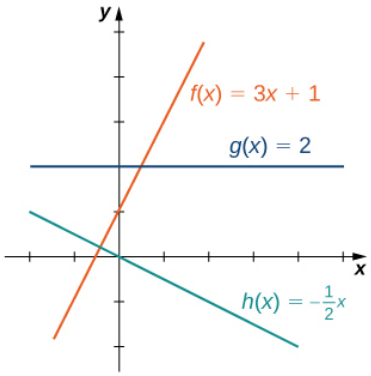


Figure 2.2.1: These linear functions are increasing or decreasing on (∞, ∞) and one function is a horizontal line.

As suggested by Figure 2.2.1, the graph of any linear function is a line. One of the distinguishing features of a line is its slope. The **slope** is the change in y for each unit change in x. The slope measures both the steepness and the direction of a line. If the slope is positive, the line points upward when moving from left to right. If the slope is negative, the line points downward when moving from left to right. If the slope is zero, the line is horizontal. To calculate the slope of a line, we need to determine the ratio of the change in y versus the change in x. To do so, we choose any two points (x_1, y_1) and (x_2, y_2) on the line and calculate $\dfrac{y_2 - y_1}{x_2 - x_1}$. In Figure 2.2.2, we see this ratio is independent of the points chosen.

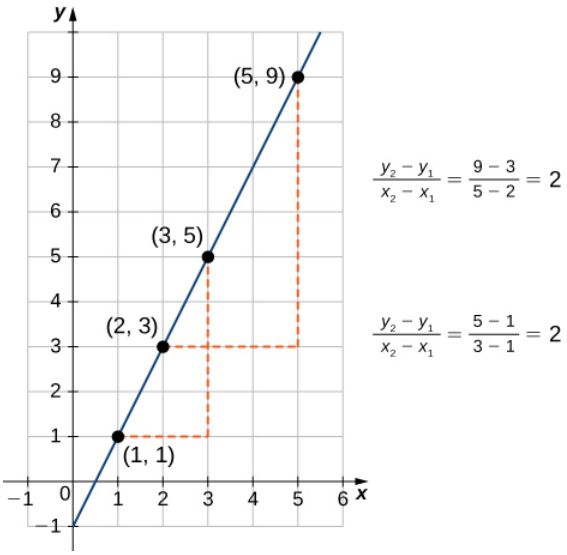


Figure 2.2.2: For any linear function, the slope $(y_2 - y_1)/(x_2 - x_1)$ is independent of the choice of points (x_1, y_1) and (x_2, y_2) on the line.

Definition: Slope of a Linear Function

Consider line L passing through points (x_1, y_1) and (x_2, y_2). Let $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$ denote the changes in y and x,respectively. The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

We now examine the relationship between slope and the formula for a linear function. Consider the linear function given by the formula $f(x) = ax + b$. As discussed earlier, we know the graph of a linear function is given by a line. We can use our definition of slope to calculate the slope of this line. As shown, we can determine the slope by calculating $(y_2 - y_1)/(x_2 - x_1)$ for any points (x_1, y_1) and (x_2, y_2) on the line. Evaluating the function f at $x = 0$, we see that $(0, b)$ is a point on this line. Evaluating this function at $x = 1$, we see that $(1, a + b)$ is also a point on this line. Therefore, the slope of this line is

$$\frac{(a+b) - b}{1 - 0} = a.$$

We have shown that the coefficient a is the slope of the line. We can conclude that the formula $f(x) = ax + b$ describes a line with slope a. Furthermore, because this line intersects the y-axis at the point $(0, b)$, we see that the y-intercept for this linear function is $(0, b)$. We conclude that the formula $f(x) = ax + b$ tells us the slope, a, and the y-intercept, $(0, b)$, for this line. Since we often use the symbol m to denote the slope of a line, we can write

$$\underbrace{f(x) = mx + b}_{\text{slope-intercept form}}$$

to denote the **slope-intercept form** of a linear function.

Sometimes it is convenient to express a linear function in different ways. For example, suppose the graph of a linear function passes through the point (x_1, y_1) and the slope of the line is m. Since any other point $(x, f(x))$ on the graph of f must satisfy the equation

$$m = \frac{f(x) - y_1}{x - x_1},$$

this linear function can be expressed by writing

$$\underbrace{f(x) - y_1 = m(x - x_1)}_{\text{point-slope equation}}.$$

We call this equation the **point-slope equation** for that linear function.

Since every nonvertical line is the graph of a linear function, the points on a nonvertical line can be described using the slope-intercept or point-slope equations. However, a vertical line does not represent the graph of a function and cannot be expressed in either of these forms. Instead, a vertical line is described by the equation $x = k$ for some constant k. Since neither the slope-intercept form nor the point-slope form allows for vertical lines, we use the notation

$$\underbrace{ax + by = c}_{\text{standard form}},$$

where a, b are both not zero, to denote the **standard form of a line.**

Definition: Point-Slope Equation, and the Slope-Intercept Form and Standard Form of the Equation of a Line

Consider a line passing through the point (x_1, y_1) with slope m. The equation

$$y - y_1 = m(x - x_1)$$

is the *point-slope equation* for that line.

Consider a line with slope m and y-intercept $(0, b)$. The equation

$$y = mx + b$$

is an equation for that line in *slope-intercept form.*

The *standard form of a line* is given by the equation

$$ax + by = c,$$

where a and b are both not zero. This form is more general because it allows for a vertical line, $x = k$.

Example 2.2.1: Finding the Slope and Equations of Lines

Consider the line passing through the points $(11, -4)$ and $(-4, 5)$, as shown in Figure 2.2.3.

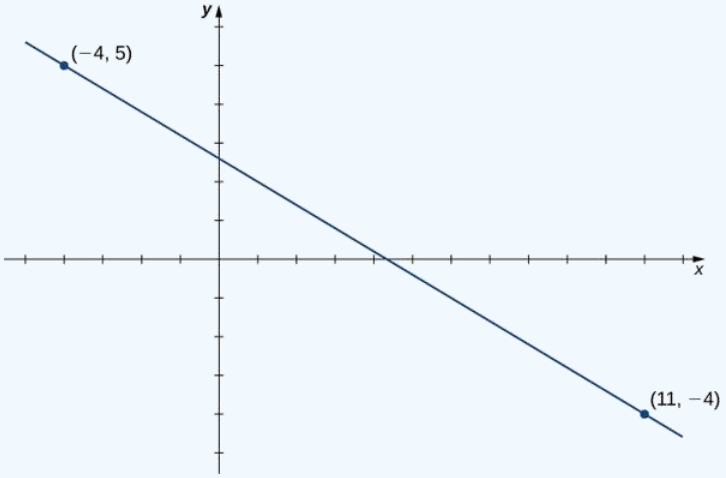


Figure 2.2.3: Finding the equation of a linear function with a graph that is a line between two given points.

1. Find the slope of the line.
2. Find an equation for this linear function in point-slope form.
3. Find an equation for this linear function in slope-intercept form.

Solution

1. The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-4)}{-4 - 11} = -\frac{9}{15} = -\frac{3}{5}.$$

2. To find an equation for the linear function in point-slope form, use the slope $m = -3/5$ and choose any point on the line. If we choose the point $(11, -4)$, we get the equation

$$f(x) + 4 = -\frac{3}{5}(x - 11).$$

3. To find an equation for the linear function in slope-intercept form, solve the equation in part b. for $f(x)$. When we do this, we get the equation

$$f(x) = -\frac{3}{5}x + \frac{13}{5}.$$

Exercise 2.2.1

Consider the line passing through points $(-3, 2)$ and $(1, 4)$.

a. Find the slope of the line.
b. Find an equation of that line in point-slope form.
c. Find an equation of that line in slope-intercept form.

Hint

The slope $m = \Delta y/\Delta x$.

Answer a

$m = 1/2$.

Answer b

The point-slope form is $y - 4 = \frac{1}{2}(x - 1)$.

Answer c

The slope-intercept form is $y = \frac{1}{2}x + \frac{7}{2}$.

Example 2.2.2:

Jessica leaves her house at 5:50 a.m. and goes for a 9-mile run. She returns to her house at 7:08 a.m. Answer the following questions, assuming Jessica runs at a constant pace.

a. Describe the distance D (in miles) Jessica runs as a linear function of her run time t (in minutes).
b. Sketch a graph of D.
c. Interpret the meaning of the slope.

Solution

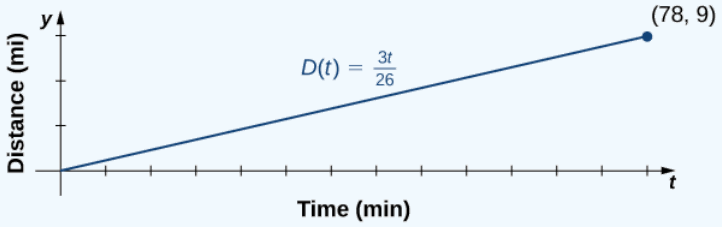
a. At time $t = 0$, Jessica is at her house, so $D(0) = 0$. At time $t = 78$ minutes, Jessica has finished running 9 mi, so $D(78) = 9$. The slope of the linear function is

$$m = \frac{9 - 0}{78 - 0} = \frac{3}{26}.$$

The y-intercept is $(0, 0)$, so the equation for this linear function is

$$D(t) = \frac{3}{26}t.$$

b. To graph D, use the fact that the graph passes through the origin and has slope $m = 3/26$.



c. The slope $m = 3/26 \approx 0.115$ describes the distance (in miles) Jessica runs per minute, or her average velocity.

2.2.2 Polynomials

A linear function is a special type of a more general class of functions: polynomials. A polynomial function is any function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

for some integer $n \geq 0$ and constants $a_n, a_{n-1}, \ldots, a_0$, where $a_n \neq 0$. In the case when $n = 0$, we allow for $a_0 = 0$; if $a_0 = 0$, the function $f(x) = 0$ is called the **zero function**. The value n is called the **degree** of the polynomial; the constant a_n is called the **leading coefficient**. A linear function of the form $f(x) = mx + b$ is a polynomial of degree 1 if $m \neq 0$ and degree 0 if $m = 0$. A polynomial of degree 0 is also called a **constant function**. A polynomial function of degree 2 is called a **quadratic function**. In particular, a quadratic function has the form

$$f(x) = ax^2 + bx + c,$$

where $a \neq 0$. A polynomial function of degree 3 is called a **cubic function**.

2.2.3 Power Functions

Some polynomial functions are power functions. A **power function** is any function of the form $f(x) = ax^b$, where a and b are any real numbers. The exponent in a power function can be any real number, but here we consider the case when the exponent is a positive integer. (We consider other cases later.) If the exponent is a positive integer, then $f(x) = ax^n$ is a polynomial. If n is even, then $f(x) = ax^n$ is an even function because $f(-x) = a(-x)^n = ax^n$ if n is even. If n is odd, then $f(x) = ax^n$ is an odd function because $f(-x) = a(-x)^n = -ax^n$ if n is odd (Figure 2.2.4).

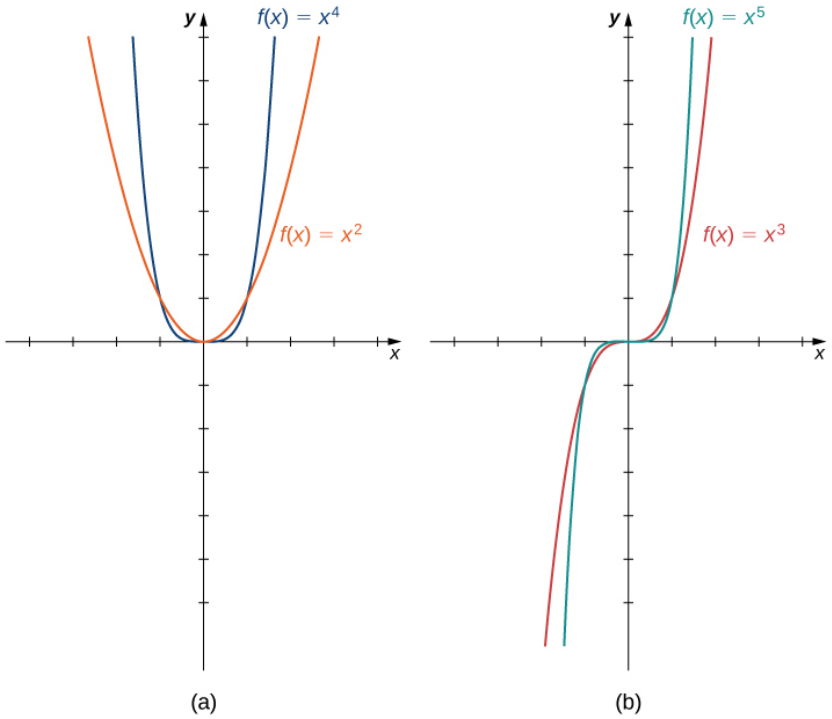


Figure 2.2.4: (a) For any even integer $n, f(x) = ax^n$ is an even function. (b) For any odd integer $n, f(x) = ax^n$ is an odd function.

2.2.4 Behavior at Infinity

To determine the behavior of a function f as the inputs approach infinity, we look at the values $f(x)$ as the inputs, x, become larger. For some functions, the values of $f(x)$ approach a finite number. For example, for the function $f(x) = 2 + 1/x$, the values $1/x$ become closer and closer to zero for all values of x as they get larger and larger. For this function, we say "$f(x)$ approaches two as x goes to infinity," and we write $f(x) \to 2$ as $x \to \infty$. The line $y = 2$ is a horizontal asymptote for the function $f(x) = 2 + 1/x$ because the graph of the function gets closer to the line as x gets larger.

For other functions, the values $f(x)$ may not approach a finite number but instead may become larger for all values of x as they get larger. In that case, we say "$f(x)$ approaches infinity as x approaches infinity," and we write $f(x) \to \infty$ as $x \to \infty$. For example, for the function $f(x) = 3x^2$, the outputs $f(x)$ become larger as the inputs x get larger. We can conclude that the function $f(x) = 3x^2$ approaches infinity as x approaches infinity, and we write $3x^2 \to \infty$ as $x \to \infty$. The behavior as $x \to -\infty$ and the meaning of $f(x) \to -\infty$ as $x \to \infty$ or $x \to -\infty$ can be defined similarly. We can describe what happens to the values of $f(x)$ as $x \to \infty$ and as $x \to -\infty$ as the **end behavior** of the function.

To understand the end behavior for polynomial functions, we can focus on quadratic and cubic functions. The behavior for higher-degree polynomials can be analyzed similarly. Consider a quadratic function $f(x) = ax^2 + bx + c$. If $a > 0$, the values $f(x) \to \infty$ as $x \to \pm\infty$. If $a < 0$, the values $f(x) \to -\infty$ as $x \to \pm\infty$. Since the graph of a quadratic function is a parabola, the parabola opens upward if $a > 0$.; the parabola opens downward if $a < 0$ (Figure 2.2.5*a*).

Now consider a cubic function $f(x) = ax^3 + bx^2 + cx + d$. If $a > 0$, then $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. If $a < 0$, then $f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$. As we can see from both of these graphs, the leading term of the polynomial determines the end *behavior* (Figure 2.2.5*b*).

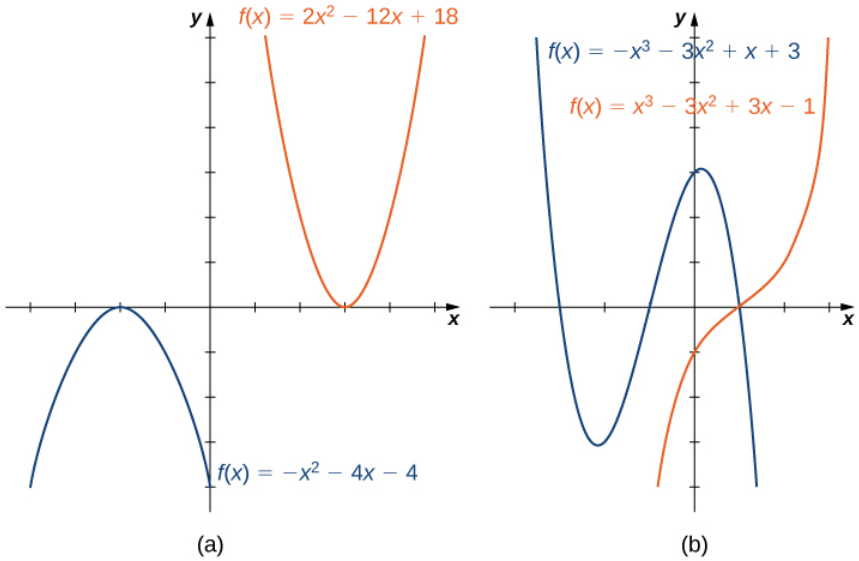


Figure 2.2.5: (a) For a quadratic function, if the leading coefficient $a > 0$,the parabola opens upward. If $a < 0$, the parabola opens downward. (b) For a cubic function f, if the leading coefficient $a > 0$, the values $f(x) \to \infty$ as $x \to \infty$ and the values $f(x) \to -\infty$ as $x \to -\infty$. If the leading coefficient $a < 0$, the opposite is true.

2.2.5 Zeros of Polynomial Functions

Another characteristic of the graph of a polynomial function is where it intersects the x-axis. To determine where a function f intersects the x-axis, we need to solve the equation $f(x) = 0$ for x. In the case of the linear function $f(x) = mx + b$, the x-intercept is given by solving the equation $mx + b = 0$. In this case, we see that the x-intercept is given by $(-b/m, 0)$. In the case of a quadratic function, finding the x-intercept(s) requires finding the zeros of a quadratic equation: $ax^2 + bx + c = 0$. In some cases, it is easy to factor the polynomial $ax^2 + bx + c$ to find the zeros. If not, we make use of the quadratic formula.

The Quadratic Formula

Consider the quadratic equation

$$ax^2 + bx + c = 0,$$

where $a \neq 0$. The solutions of this equation are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.2.1}$$

If the discriminant $b^2 - 4ac > 0$, Equation 2.2.1 tells us there are two real numbers that satisfy the quadratic equation. If $b^2 - 4ac = 0$, this formula tells us there is only one solution, and it is a real number. If $b^2 - 4ac < 0$, no real numbers satisfy the quadratic equation.

In the case of higher-degree polynomials, it may be more complicated to determine where the graph intersects the x-axis. In some instances, it is possible to find the x-intercepts by factoring the polynomial to find its zeros. In other cases, it is impossible to calculate the exact values of the x-intercepts. However, as we see later in the text, in cases such as this, we can use analytical tools to approximate (to a very high degree) where the x-intercepts are located. Here we focus on the graphs of polynomials for which we can calculate their zeros explicitly.

Example 2.2.3: Graphing Polynomial Functions

For the following functions,

a. $f(x) = -2x^2 + 4x - 1$
b. $f(x) = x^3 - 3x^2 - 4x$

i. describe the behavior of $f(x)$ as $x \to \pm\infty$,

ii. find all zeros of f, and
iii. sketch a graph of f.

Solution

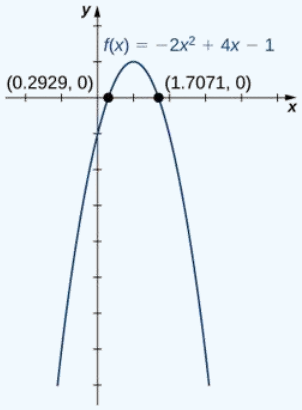
1. The function $f(x) = -2x^2 + 4x - 1$ is a quadratic function.

    1. Because $a = -2 < 0$, as $x \to \pm\infty$, $f(x) \to -\infty$.

    2. To find the zeros of f, use the quadratic formula. The zeros are

$$x = \frac{-4 \pm \sqrt{4^2 - 4(-2)(-1)}}{2(-2)} = \frac{-4 \pm \sqrt{8}}{-4} = \frac{-4 \pm 2\sqrt{2}}{-4} = \frac{2 \pm \sqrt{2}}{2}.$$

    3. To sketch the graph of f,use the information from your previous answers and combine it with the fact that the graph is a parabola opening downward.



2. The function $f(x) = x^3 - 3x^2 - 4x$ is a cubic function.

    1. Because $a = 1 > 0$, as $x \to \infty$, $f(x) \to \infty$. As $x \to -\infty$, $f(x) \to -\infty$.

    2. To find the zeros of f, we need to factor the polynomial. First, when we factor x out of all the terms, we find

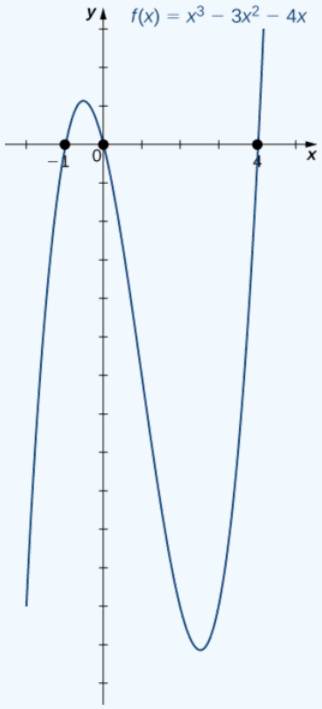
$$f(x) = x(x^2 - 3x - 4).$$

    Then, when we factor the quadratic function $x^2 - 3x - 4$, we find

$$f(x) = x(x-4)(x+1).$$

    Therefore, the zeros of f are $x = 0, 4, -1$.

3. Combining the results from parts i. and ii., draw a rough sketch of f.



? Exercise 2.2.2

Consider the quadratic function $f(x) = 3x^2 - 6x + 2$. Find the zeros of f. Does the parabola open upward or downward?

Hint

Use the quadratic formula.

Answer

The zeros are $x = 1 \pm \sqrt{3}/3$. The parabola opens upward.

2.2.6 Mathematical Models

A large variety of real-world situations can be described using **mathematical models**. A mathematical model is a method of simulating real-life situations with mathematical equations. Physicists, engineers, economists, and other researchers develop models by combining observation with quantitative data to develop equations, functions, graphs, and other mathematical tools to describe the behavior of various systems accurately. Models are useful because they help predict future outcomes. Examples of mathematical models include the study of population dynamics, investigations of weather patterns, and predictions of product sales.

As an example, let's consider a mathematical model that a company could use to describe its revenue for the sale of a particular item. The amount of revenue R a company receives for the sale of n items sold at a price of p dollars per item is described by the equation $R = p \cdot n$. The company is interested in how the sales change as the price of the item changes. Suppose the data in Table 2.2.1 show the number of units a company sells as a function of the price per item.

Table 2.2.1: Number of Units Sold n (in Thousands) as a Function of Price per Unit p (in Dollars)

p	6	8	10	12	14
n	19.4	18.5	16.2	13.8	12.2

In Figure 2.2.6, we see the graph the number of units sold (in thousands) as a function of price (in dollars). We note from the shape of the graph that the number of units sold is likely a linear function of price per item, and the data can be closely approximated by the linear function $n = -1.04p + 26$ for $0 \le p \le 25$, where n predicts the number of units sold in thousands. Using this linear function, the revenue (in thousands of dollars) can be estimated by the quadratic function

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p \text{ for } 0 \le p \le 25.$$

In Example 2.2.4, we use this quadratic function to predict the amount of revenue the company receives depending on the price the company charges per item. Note that we cannot conclude definitively the actual number of units sold for values of p, for which no data are collected. However, given the other data values and the graph shown, it seems reasonable that the number of units sold (in thousands) if the price charged is p dollars may be close to the values predicted by the linear function $n = -1.04p + 26$.

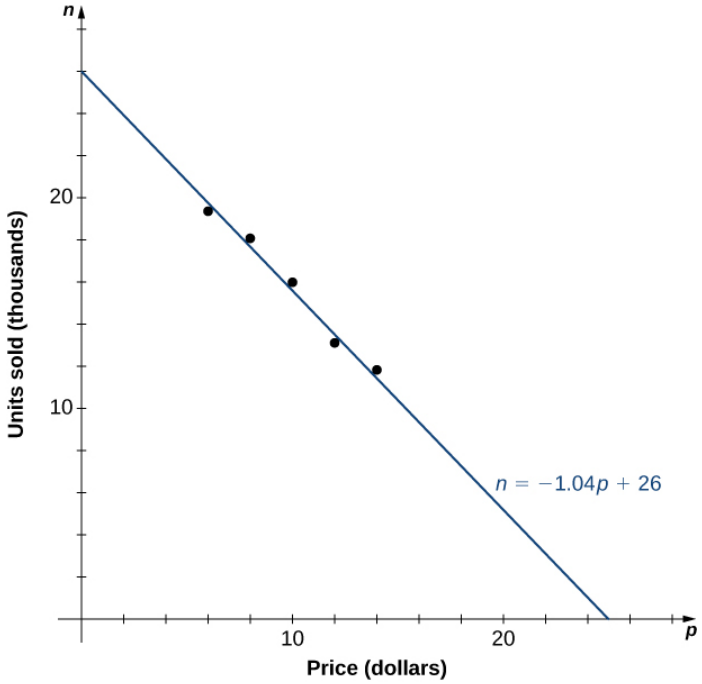


Figure 2.2.6: The data collected for the number of items sold as a function of price is roughly linear. We use the linear function $n = -1.04p + 26$ to estimate this function.

Example 2.2.4: Maximizing Revenue

A company is interested in predicting the amount of revenue it will receive depending on the price it charges for a particular item. Using the data from Table 2.2.1, the company arrives at the following quadratic function to model revenue R as a function of price per item p :

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p$$

for $0 \le p \le 25$.

a. Predict the revenue if the company sells the item at a price of $p = \$5$ and $p = \$17$.
b. Find the zeros of this function and interpret the meaning of the zeros.
c. Sketch a graph of R.
d. Use the graph to determine the value of p that maximizes revenue. Find the maximum revenue.

Solution

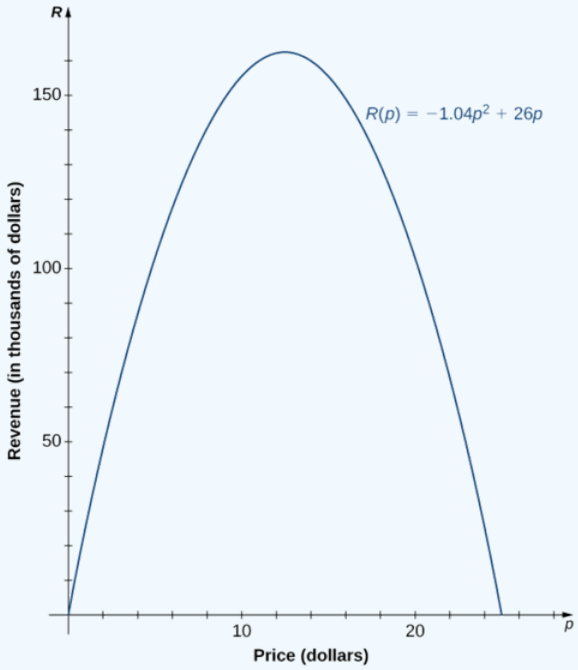
a. Evaluating the revenue function at $p = 5$ and $p = 17$, we can conclude that

$R(5) = -1.04(5)^2 + 26(5) = 104$, so revenue $= \$104,000$;

$R(17) = -1.04(17)^2 + 26(17) = 141.44$, so revenue $= \$141,440$.

b. The zeros of this function can be found by solving the equation $-1.04p^2 + 26p = 0$. When we factor the quadratic expression, we get $p(-1.04p + 26) = 0$. The solutions to this equation are given by $p = 0, 25$. For these values of p, the revenue is zero. When $p = \$0$, the revenue is zero because the company is giving away its merchandise for free. When $p = \$25$,the revenue is zero because the price is too high, and no one will buy any items.

c. Knowing the fact that the function is quadratic, we also know the graph is a parabola. Since the leading coefficient is negative, the parabola opens downward. One property of parabolas is that they are symmetric about the axis of symmetry, so since the zeros are at $p = 0$ and $p = 25$, the parabola must be symmetric about the line halfway between them, or $p = 12.5$.



d. The function is a parabola with zeros at $p = 0$ and $p = 25$, and it is symmetric about the line $p = 12.5$, so the maximum revenue occurs at a price of $p = \$12.50$ per item. At that price, the revenue is $R(p) = -1.04(12.5)^2 + 26(12.5) = \$162,500$.

2.2.7 Algebraic Functions

By allowing for quotients and fractional powers in polynomial functions, we create a larger class of functions. An **algebraic function** is one that involves addition, subtraction, multiplication, division, rational powers, and roots. Two types of algebraic functions are rational functions and root functions.

Just as rational numbers are quotients of integers, rational functions are quotients of polynomials. In particular, a **rational function** is any function of the form $f(x) = p(x)/q(x)$,where $p(x)$ and $q(x)$ are polynomials. For example,

$$f(x) = \frac{3x-1}{5x+2} \text{ and } g(x) = \frac{4}{x^2+1}$$

are rational functions. A **root function** is a power function of the form $f(x) = x^{1/n}$, where n is a positive integer greater than one. For example, $f(x) = x^{1/2} = \sqrt{x}$ is the square-root function and $g(x) = x^{1/3} = \sqrt[3]{x}$ is the cube-root function. By allowing for compositions of root functions and rational functions, we can create other algebraic functions. For example, $f(x) = \sqrt{4-x^2}$ is an algebraic function.

Example 2.2.5: Finding Domain and Range for Algebraic Functions

For each of the following functions, find the domain and range.

a. $f(x) = \dfrac{3x-1}{5x+2}$
b. $f(x) = \sqrt{4-x^2}$

Solution

1. It is not possible to divide by zero, so the domain is the set of real numbers x such that $x \neq -2/5$. To find the range, we need to find the values y for which there exists a real number x such that

$$y = \frac{3x-1}{5x+2}$$

When we multiply both sides of this equation by $5x+2$, we see that x must satisfy the equation

$$5xy + 2y = 3x - 1.$$

From this equation, we can see that x must satisfy

$$2y + 1 = x(3 - 5y).$$

If y=$3/5$, this equation has no solution. On the other hand, as long as $y \neq 3/5$,

$$x = \frac{2y+1}{3-5y}$$

satisfies this equation. We can conclude that the range of f is $\{y \mid y \neq 3/5\}$.

2. To find the domain of f, we need $4 - x^2 \geq 0$. When we factor, we write $4 - x^2 = (2-x)(2+x) \geq 0$. This inequality holds if and only if both terms are positive or both terms are negative. For both terms to be positive, we need to find x such that

$$2 - x \geq 0 \text{ and } 2 + x \geq 0.$$

These two inequalities reduce to $2 \geq x$ and $x \geq -2$. Therefore, the set $\{x \mid -2 \leq x \leq 2\}$ must be part of the domain. For both terms to be negative, we need

$$2 - x \leq 0 \text{ and } 2 + x \leq 0.$$

These two inequalities also reduce to $2 \leq x$ and $x \leq -2$. There are no values of x that satisfy both of these inequalities. Thus, we can conclude the domain of this function is $\{x \mid -2 \leq x \leq 2\}$.

If $-2 \leq x \leq 2$, then $0 \leq 4 - x^2 \leq 4$. Therefore, $0 \leq \sqrt{4 - x2} \leq 2$, and the range of f is $\{y \mid 0 \leq y \leq 2\}$.

? Exercise 2.2.3

Find the domain and range for the function $f(x) = (5x+2)/(2x-1)$.

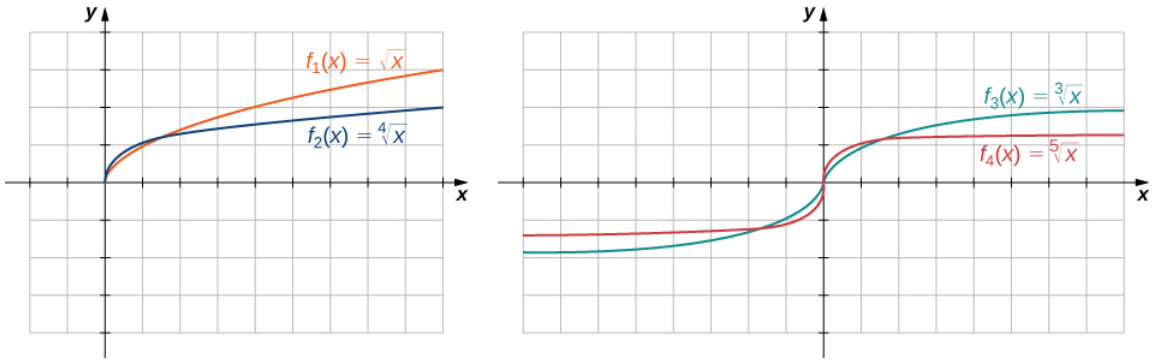
Hint

The denominator cannot be zero. Solve the equation $y = (5x+2)/(2x-1)$ for x to find the range.

Answer

The domain is the set of real numbers x such that $x \neq 1/2$. The range is the set $\{y \mid y \neq 5/2\}$

The root functions $f(x) = x^{1/n}$ have defining characteristics depending on whether n is odd or even. For all even integers $n \geq 2$, the domain of $f(x) = x^{1/n}$ is the interval $[0, \infty)$. For all odd integers $n \geq 1$, the domain of $f(x) = x^{1/n}$ is the set of all real numbers. Since $x^{1/n} = (-x)^{1/n}$ for odd integers $n, f(x) = x^{1/n}$ is an odd function if$n$ is odd. See the graphs of root functions for different values of n in Figure 2.2.7.



(a) (b)

Figure 2.2.7: (a) If n is even, the domain of $f(x) = \sqrt[n]{x}$ is $[0, \infty)$. (b) If n is odd, the domain of $f(x) = \sqrt[n]{x}$ is $(-\infty, \infty)$ and the function $f(x) = \sqrt[n]{x}$ is an odd function.

Example 2.2.6: Finding Domains for Algebraic Functions

For each of the following functions, determine the domain of the function.

a. $f(x) = \dfrac{3}{x^2 - 1}$
b. $f(x) = \dfrac{2x + 5}{3x^2 + 4}$
c. $f(x) = \sqrt{4 - 3x}$
d. $f(x) = \sqrt[3]{2x - 1}$

Solution

a. You cannot divide by zero, so the domain is the set of values x such that $x^2 - 1 \neq 0$. Therefore, the domain is $\{x \mid x \neq \pm 1\}$.
b. You need to determine the values of x for which the denominator is zero. Since $3x^2 + 4 \geq 4$ for all real numbers x, the denominator is never zero. Therefore, the domain is $(-\infty, \infty)$.
c. Since the square root of a negative number is not a real number, the domain is the set of values x for which $4 - 3x \geq 0$. Therefore, the domain is $\{x \mid x \leq 4/3\}$.
d. The cube root is defined for all real numbers, so the domain is the interval $(-\infty, \infty)$.

Exercise 2.2.4

Find the domain for each of the following functions: $f(x) = (5 - 2x)/(x^2 + 2)$ and $g(x) = \sqrt{5x - 1}$.

Hint

Determine the values of x when the expression in the denominator of f is nonzero, and find the values of x when the expression inside the radical of g is nonnegative.

Answer

The domain of f is $(-\infty, \infty)$. The domain of g is $\{x \mid x \geq 1/5\}$.

2.2.8 Piecewise-Defined Functions

Sometimes a function is defined by different formulas on different parts of its domain. A function with this property is known as a piecewise-defined function. The absolute value function is an example of a piecewise-defined function because the formula changes with the sign of x:

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}.$$

Other piecewise-defined functions may be represented by completely different formulas, depending on the part of the domain in which a point falls. To graph a piecewise-defined function, we graph each part of the function in its respective domain, on the same coordinate system. If the formula for a function is different for $x < a$ and $x > a$, we need to pay special attention to what happens at $x = a$ when we graph the function. Sometimes the graph needs to include an open or closed circle to indicate the value of the function at $x = a$. We examine this in the next example.

Example 2.2.8: Graphing a Piecewise-Defined Function

Sketch a graph of the following piecewise-defined function:

$$f(x) = \begin{cases} x+3, & \text{if } x < 1 \\ (x-2)^2, & \text{if } x \geq 1 \end{cases}$$

Solution

Graph the linear function $y = x + 3$ on the interval $(-\infty, 1)$ and graph the quadratic function $y = (x-2)^2$ on the interval $[1, \infty)$. Since the value of the function at $x = 1$ is given by the formula $f(x) = (x-2)^2$, we see that $f(1) = 1$. To indicate this on the graph, we draw a closed circle at the point $(1, 1)$. The value of the function is given by $f(x) = x + 3$ for all $x < 1$, but not at $x = 1$. To indicate this on the graph, we draw an open circle at $(1, 4)$.

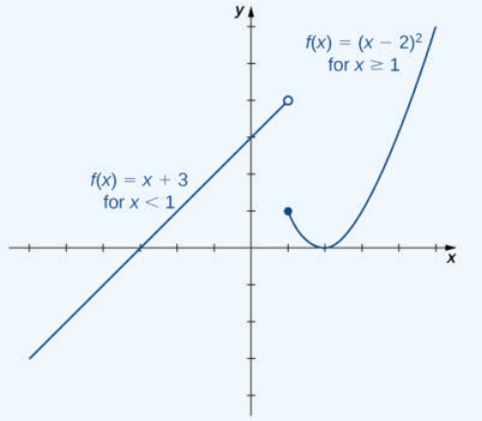
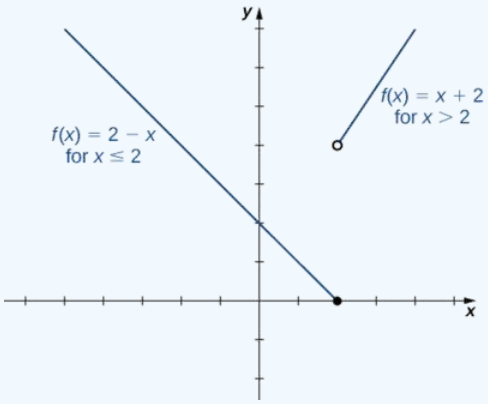


Figure 2.2.8: This piecewise-defined function is linear for $x < 1$ and quadratic for $x \geq 1$.

2) Sketch a graph of the function

$$f(x) = \begin{cases} 2-x, & \text{if } x \leq 2 \\ x+2, & \text{if } x > 2 \end{cases}.$$

Solution:



https://math.libretexts.org/@go/page/149859

Example 2.2.9: Parking Fees Described by a Piecewise-Defined Function

In a big city, drivers are charged variable rates for parking in a parking garage. They are charged $10 for the first hour or any part of the first hour and an additional $2 for each hour or part thereof up to a maximum of $30 for the day. The parking garage is open from 6 a.m. to 12 midnight.

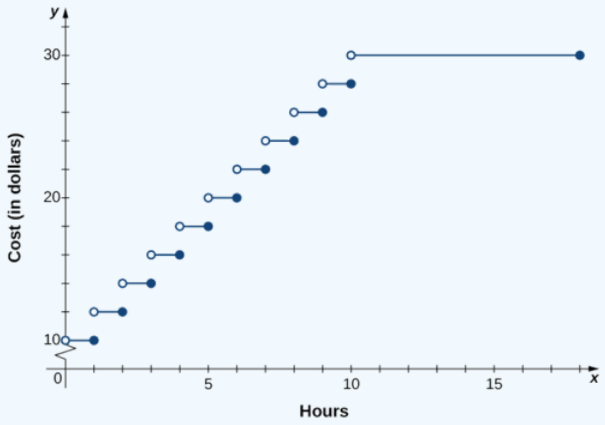
a. Write a piecewise-defined function that describes the cost C to park in the parking garage as a function of hours parked x.
b. Sketch a graph of this function $C(x)$.

Solution

1.Since the parking garage is open 18 hours each day, the domain for this function is $\{x \mid 0 < x \leq 18\}$. The cost to park a car at this parking garage can be described piecewise by the function

$$C(x) = \begin{cases} 10, & \text{for } 0 < x \leq 1 \\ 12, & \text{for } 1 < x \leq 2 \\ 14, & \text{for } 2 < x \leq 3 \\ 16, & \text{for } 3 < x \leq 4 \\ \vdots & \\ 30, & \text{for } 10 < x \leq 18 \end{cases}.$$

2.The graph of the function consists of several horizontal line segments.



Exercise 2.2.6

The cost of mailing a letter is a function of the weight of the letter. Suppose the cost of mailing a letter is $49¢$ for the first ounce and $21¢$ for each additional ounce. Write a piecewise-defined function describing the cost C as a function of the weight x for $0 < x \leq 3$, where C is measured in cents and x is measured in ounces.

Hint

The piecewise-defined function is constant on the intervals $(0, 1]$, $(1, 2]$, ...

Answer

$$C(x) = \begin{cases} 49, 0 < x \leq 1 \\ 70, 1 < x \leq 2 \\ 91, 2 < x \leq 3 \end{cases}$$

2.2.9 Transformations of Functions

We have seen several cases in which we have added, subtracted, or multiplied constants to form variations of simple functions. In the previous example, for instance, we subtracted 2 from the argument of the function $y = x^2$ to get the function $f(x) = (x-2)^2$. This subtraction represents a shift of the function $y = x^2$ two units to the right. A shift, horizontally or vertically, is a type of **transformation of a function**. Other transformations include horizontal and vertical scalings, and reflections about the axes.

A vertical shift of a function occurs if we add or subtract the same constant to each output y. For $c > 0$, the graph of $f(x) + c$ is a shift of the graph of $f(x)$ up c units, whereas the graph of $f(x) - c$ is a shift of the graph of $f(x)$ down c units. For example, the graph of the function $f(x) = x^3 + 4$ is the graph of $y = x^3$ shifted up 4 units; the graph of the function $f(x) = x^3 - 4$ is the graph of $y = x^3$ shifted down 4 units (Figure 2.2.9).

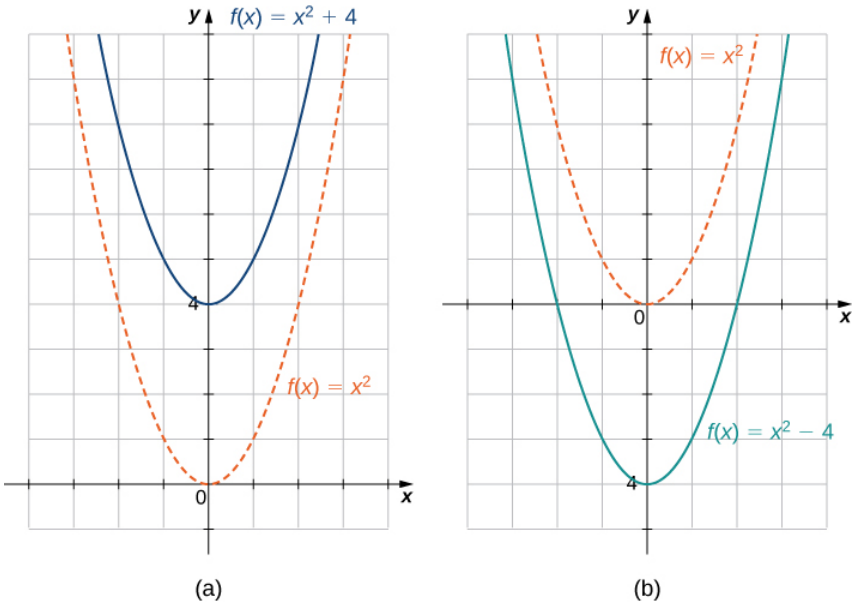


Figure 2.2.9: (a) For $c > 0$, the graph of $y = f(x) + c$ is a vertical shift up c units of the graph of $y = f(x)$. (b) For $c > 0$, the graph of $y = f(x) - c$ is a vertical shift down c units of the graph of $y = f(x)$.

A horizontal shift of a function occurs if we add or subtract the same constant to each input x. For $c > 0$, the graph of $f(x + c)$ is a shift of the graph of $f(x)$ to the left c units; the graph of $f(x - c)$ is a shift of the graph of $f(x)$ to the right c units. Why does the graph shift left when adding a constant and shift right when subtracting a constant? To answer this question, let's look at an example.

Consider the function $f(x) = |x + 3|$ and evaluate this function at $x - 3$. Since $f(x - 3) = |x|$ and $x - 3 < x$, the graph of $f(x) = |x + 3|$ is the graph of $y = |x|$ shifted left 3 units. Similarly, the graph of $f(x) = |x - 3|$ is the graph of $y = |x|$ shifted right 3 units (Figure 2.2.10).

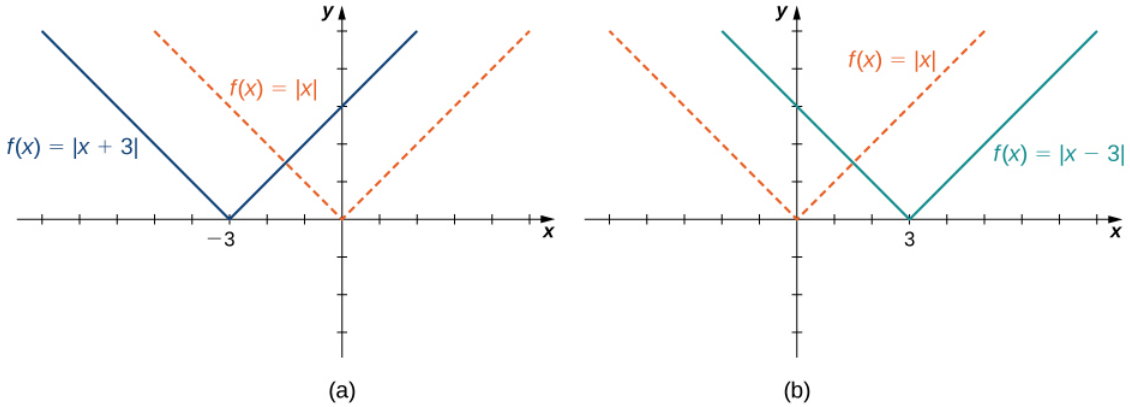


Figure 2.2.10: (a) For $c > 0$, the graph of $y = f(x + c)$ is a horizontal shift left c units of the graph of $y = f(x)$. (b) For $c > 0$, the graph of $y = f(x - c)$ is a horizontal shift right c units of the graph of $y = f(x)$.

A vertical scaling of a graph occurs if we multiply all outputs y of a function by the same positive constant. For $c > 0$, the graph of the function $cf(x)$ is the graph of $f(x)$ scaled vertically by a factor of c. If $c > 1$, the values of the outputs for the function $cf(x)$ are larger than the values of the outputs for the function $f(x)$; therefore, the graph has been stretched vertically. If $0 < c < 1$, then the outputs of the function $cf(x)$ are smaller, so the graph has been compressed. For example, the graph of the function $f(x) = 3x^2$ is the graph of $y = x^2$ stretched vertically by a factor of 3, whereas the graph of $f(x) = x^2/3$ is the graph of $y = x^2$ compressed vertically by a factor of 3 (Figure 2.2.11b).

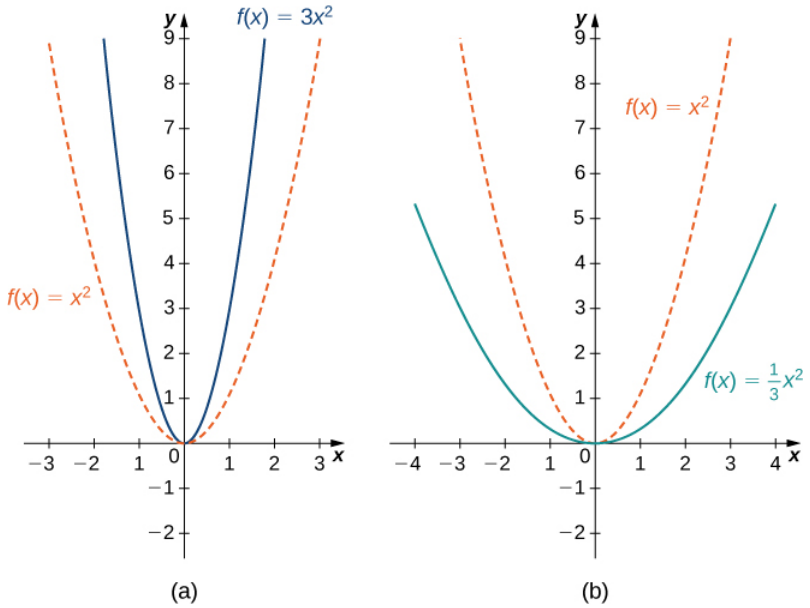


Figure 2.2.11: (a) If $c > 1$, the graph of $y = cf(x)$ is a vertical stretch of the graph of $y = f(x)$. (b) If $0 < c < 1$, the graph of $y = cf(x)$ is a vertical compression of the graph of $y = f(x)$.

The horizontal scaling of a function occurs if we multiply the inputs x by the same positive constant. For $c > 0$, the graph of the function $f(cx)$ is the graph of $f(x)$ scaled horizontally by a factor of c. If $c > 1$, the graph of $f(cx)$ is the graph of $f(x)$ compressed horizontally. If $0 < c < 1$, the graph of $f(cx)$ is the graph of $f(x)$ stretched horizontally. For example, consider the function $f(x) = \sqrt{2x}$ and evaluate f at $x/2$. Since $f(x/2) = \sqrt{x}$, the graph of $f(x) = \sqrt{2x}$ is the graph of $y = \sqrt{x}$ compressed horizontally. The graph of $y = \sqrt{x/2}$ is a horizontal stretch of the graph of $y = \sqrt{x}$ (Figure 2.2.12).

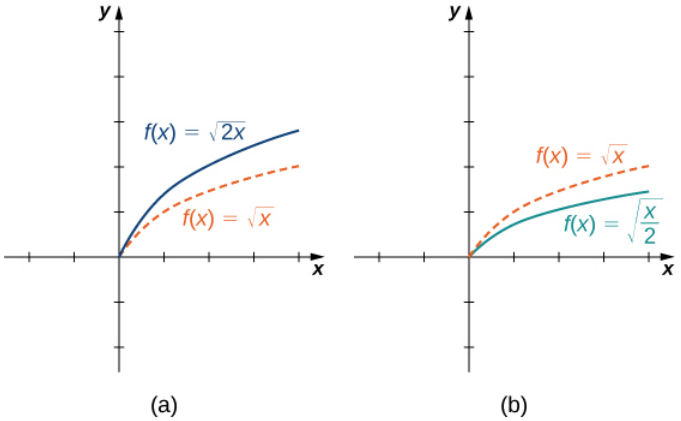


Figure 2.2.12: (a) If $c > 1$, the graph of $y = f(cx)$ is a horizontal compression of the graph of $y = f(x)$. (b) If $0 < c < 1$, the graph of $y = f(cx)$ is a horizontal stretch of the graph of $y = f(x)$.

We have explored what happens to the graph of a function f when we multiply f by a constant $c > 0$ to get a new function $cf(x)$. We have also discussed what happens to the graph of a function $f$when we multiply the independent variable x by $c > 0$ to get a new function $f(cx)$. However, we have not addressed what happens to the graph of the function if the constant c is negative. If we have a constant $c < 0$, we can write c as a positive number multiplied by -1; but, what kind of transformation do we get when we multiply the function or its argument by -1? When we multiply all the outputs by -1, we get a reflection about the x-axis. When we multiply all inputs by -1, we get a reflection about the y-axis. For example, the graph of $f(x) = -(x^3 + 1)$ is the graph of $y = (x^3 + 1)$ reflected about the x-axis. The graph of $f(x) = (-x)^3 + 1$ is the graph of $y = x^3 + 1$ reflected about the y-axis (Figure 2.2.13).

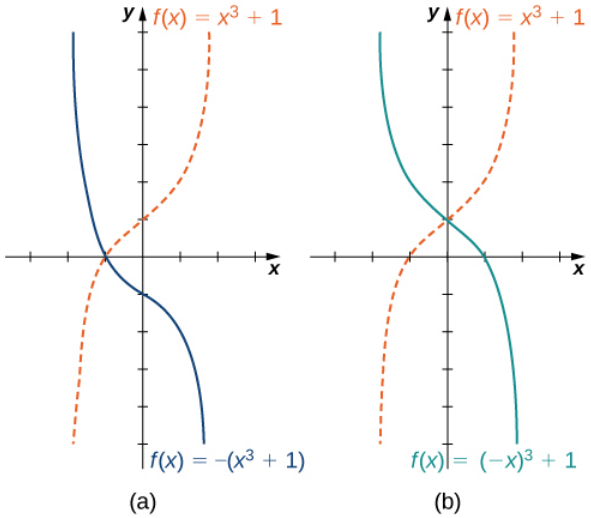


Figure 2.2.13: (a) The graph of $y=-f(x)$ is the graph of $y=f(x)$ reflected about the x-axis. (b) The graph of $y=f(-x)$ is the graph of $y=f(x)$ reflected about the y-axis.

If the graph of a function consists of more than one transformation of another graph, it is important to transform the graph in the correct order. Given a function $f(x)$, the graph of the related function $y=cf(a(x+b))+d$ can be obtained from the graph of $y=f(x)$by performing the transformations in the following order.

- Horizontal shift of the graph of $y=f(x)$. If $b>0$, shift left. If $b<0$ shift right.
- Horizontal scaling of the graph of $y=f(x+b)$ by a factor of $|a|$. If $a<0$, reflect the graph about the y-axis.
- Vertical scaling of the graph of $y=f(a(x+b))$ by a factor of $|c|$. If $c<0$, reflect the graph about the x -axis.
- Vertical shift of the graph of $y=cf(a(x+b))$. If $d>0$, shift up. If $d<0$, shift down.

We can summarize the different transformations and their related effects on the graph of a function in the following table.

Transformation of $f(c>0)$	Effect of the graph of f
$f(x)+c$	Vertical shift up c units
$f(x)-c$	Vertical shift down c units
$f(x+c)$	Shift left by c units
$f(x-c)$	Shift right by c units
$cf(x)$	Vertical stretch if $c>1$; vertical compression if $0<c<1$
$f(cx)$	Horizontal stretch if $0<c<1$; horizontal compression if $c>1$
$-f(x)$	Reflection about the x-axis
$f(-x)$	Reflection about the y-axis

Example 2.2.10: Transforming a Function

For each of the following functions, a. and b., sketch a graph by using a sequence of transformations of a well-known function.

a. $f(x)=-|x+2|-3$
b. $f(x)=3\sqrt{-x}+1$

Solution

1.Starting with the graph of $y=|x|$, shift 2 units to the left, reflect about the x-axis, and then shift down 3 units.

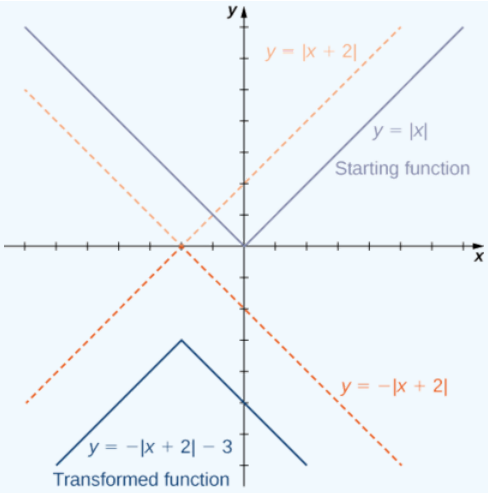


Figure 2.2.14: The function $f(x) = -|x+2| - 3$ can be viewed as a sequence of three transformations of the function $y = |x|$.

2. Starting with the graph of $y = \sqrt{x}$, reflect about the y-axis, stretch the graph vertically by a factor of 3, and move up 1 unit.

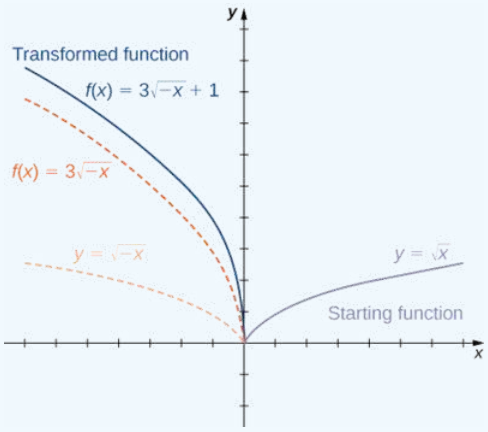


Figure 2.2.15: The function $f(x) = 3\sqrt{-x} + 1$ can be viewed as a sequence of three transformations of the function $y = \sqrt{x}$.

? Exercise 2.2.7

Describe how the function $f(x) = -(x+1)^2 - 4$ can be graphed using the graph of $y = x^2$ and a sequence of transformations

Answer

Shift the graph $y = x^2$ to the left 1 unit, reflect about the x-axis, then shift down 4 units.

2.2.10 Key Concepts

- The power function $f(x) = x^n$ is an even function if n is even and $n \neq 0$, and it is an odd function if n is odd.
- The root function $f(x) = x^{1/n}$ has the domain $[0, \infty)$ if n is even and the domain $(-\infty, \infty)$ if n is odd. If n is odd, then $f(x) = x^{1/n}$ is an odd function.
- The domain of the rational function $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomial functions, is the set of x such that $q(x) \neq 0$.
- Functions that involve the basic operations of addition, subtraction, multiplication, division, and powers are algebraic functions.
- A polynomial function f with degree $n \geq 1$ satisfies $f(x) \to \pm\infty$ as $x \to \pm\infty$. The sign of the output as $x \to \infty$ depends on the sign of the leading coefficient only and on whether n is even or odd.
- Vertical and horizontal shifts, vertical and horizontal scalings, and reflections about the x- and y-axes are examples of transformations of functions.

2.2.11 Key Equations

- **Point-slope equation of a line**

$$y - y_1 = m(x - x_1)$$

- **Slope-intercept form of a line**

$$y = mx + b$$

- **Standard form of a line**

$$ax + by = c$$

- **Polynomial function**

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

2.2.12 Glossary

algebraic function
a function involving any combination of only the basic operations of addition, subtraction, multiplication, division, powers, and roots applied to an input variable x

cubic function
a polynomial of degree 3; that is, a function of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$

degree
for a polynomial function, the value of the largest exponent of any term

linear function
a function that can be written in the form $f(x) = mx + b$

mathematical model
A method of simulating real-life situations with mathematical equations

piecewise-defined function
a function that is defined differently on different parts of its domain

point-slope equation
equation of a linear function indicating its slope and a point on the graph of the function

polynomial function
a function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

power function
a function of the form $f(x) = x^n$ for any positive integer $n \geq 1$

quadratic function
a polynomial of degree 2; that is, a function of the form $f(x) = ax^2 + bx + c$ where $a \neq 0$

rational function
a function of the form $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials

root function
a function of the form $f(x) = x^{1/n}$ for any integer $n \geq 2$

slope
the change in y for each unit change in x

slope-intercept form

equation of a linear function indicating its slope and y-intercept

transformation of a function

a shift, scaling, or reflection of a function

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2.2E: Exercises for Section 2.2

In exercises 1 - 8, for each pair of points,

a. find the slope of the line passing through the points and

b. indicate whether the line is increasing, decreasing, horizontal, or vertical.

1) $(-2, 4)$ and $(1, 1)$

Answer

a. $m = -1$
b. Decreasing

2) $(-1, 4)$ and $(3, -1)$

3) $(3, 5)$ and $(-1, 2)$

Answer

a. $m = 3/4$
b. Increasing

4) $(6, 4)$ and $(4, -3)$

5) $(2, 3)$ and $(5, 7)$

Answer

a. $m = 4/3$
b. Increasing

6) $(1, 9)$ and $(-8, 5)$

7) $(2, 4)$ and $(1, 4)$

Answer

a. $m = 0$
b. Horizontal

8) $(1, 4)$ and $(1, 0)$

In exercises 9 - 16, write the equation of the line satisfying the given conditions in slope-intercept form.

9) Slope $=-6$, passes through $(1, 3)$

Answer

$y = -6x + 9$

10) Slope $=3$, passes through $(-3, 2)$

11) Slope $=\frac{1}{3}$, passes through $(0, 4)$

Answer

$y = \frac{1}{3}x + 4$

12) Slope $=\frac{2}{5}$, x-intercept $=8$

13) Passing through $(2, 1)$ and $(-2, -1)$

Answer

$y = \frac{1}{2}x$

14) Passing through $(-3, 7)$ and $(1, 2)$

15) x-intercept =5 and y-intercept =-3

Answer

$y = \frac{3}{5}x - 3$

16) x-Intercept =-6 and y-intercept =9

In exercises 17 - 24, for each linear equation,

a. give the slope m and y-intercept b, if any, and

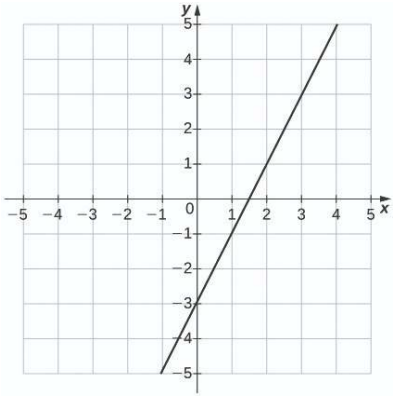
b. graph the line.

17) $y = 2x - 3$

Answer

a. $m = 2,\ b = -3$

b.



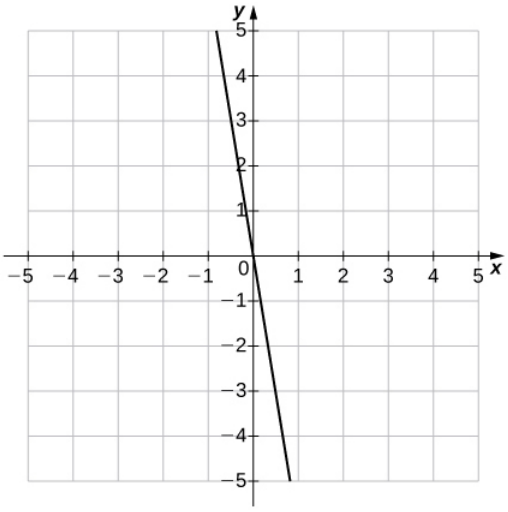
18) $y = -\frac{1}{7}x + 1$

19) $f(x) = -6x$

Answer

a. $m = -6,\ b = 0$

b.



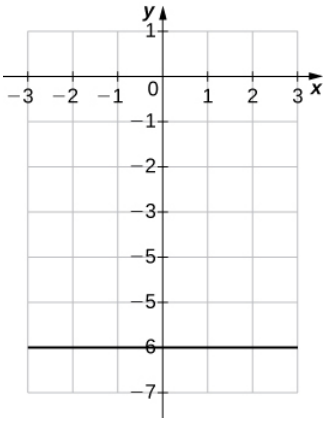
20) $f(x) = -5x + 4$

21) $4y + 24 = 0$

Answer

a. $m = 0,\ b = -6$

b.



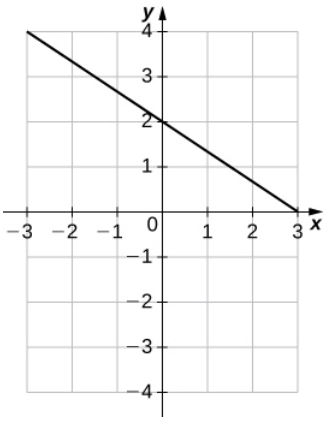
22) $8x - 4 = 0$

23) $2x + 3y = 6$

Answer

a. $m = -\frac{2}{3},\ b = 2$

b.



24) $6x - 5y + 15 = 0$

In exercises 25 - 29, for each polynomial,

a. find the degree;

b. find the zeros, if any;

c. find the y-intercept(s), if any;

d. use the leading coefficient to determine the graph's end behavior; and

e. determine algebraically whether the polynomial is even, odd, or neither.

25) $f(x) = 2x^2 - 3x - 5$

Answer

a. 2
b. $\frac{5}{2},\ -1$;
c. -5
d. Both ends rise
e. Neither

26) $f(x) = -3x^2 + 6x$

27) $f(x) = \frac{1}{2}x^2 - 1$

Answer

a. 2
b. $\pm\sqrt{2}$
c. -1
d. Both ends rise
e. Even

28) $f(x) = x^3 + 3x^2 - x - 3$

29) $f(x) = 3x - x^3$

Answer

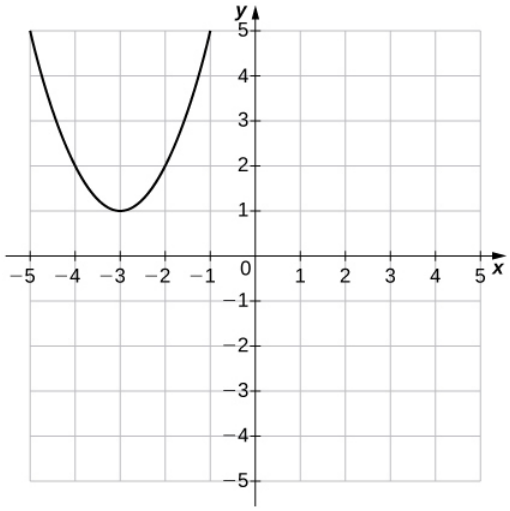
a. 3
b. $0, \pm\sqrt{3}$
c. 0
d. Left end rises, right end falls
e. Odd

For exercises 30 - 31, use the graph of $f(x) = x^2$ to graph each transformed function g.

30) $g(x) = x^2 - 1$

31) $g(x) = (x+3)^2 + 1$

Answer

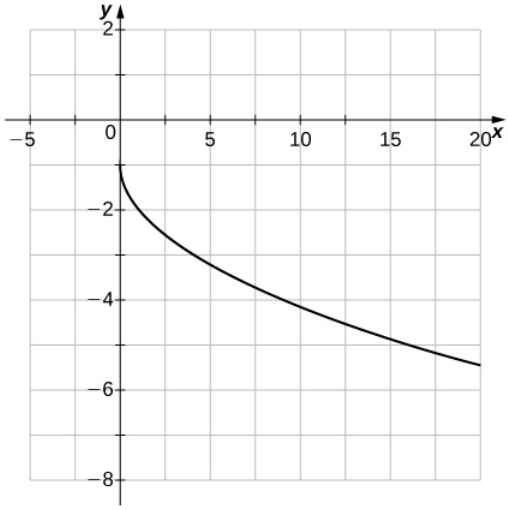


For exercises 32 - 33, use the graph of $f(x) = \sqrt{x}$ to graph each transformed function g.

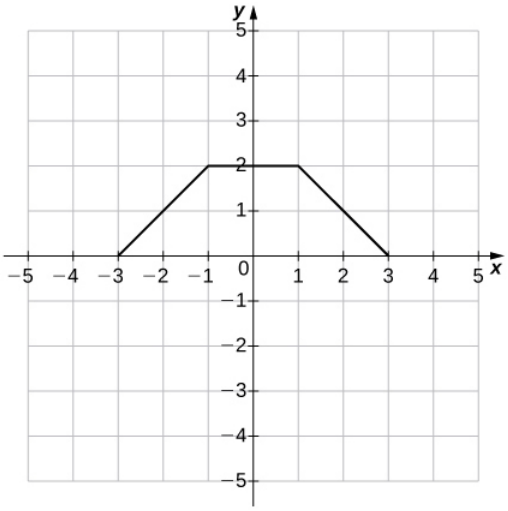
32) $g(x) = \sqrt{x+2}$

33) $g(x) = -\sqrt{x} - 1$

Answer



For exercises 34 - 35, use the graph of $y = f(x)$ to graph each transformed function g.

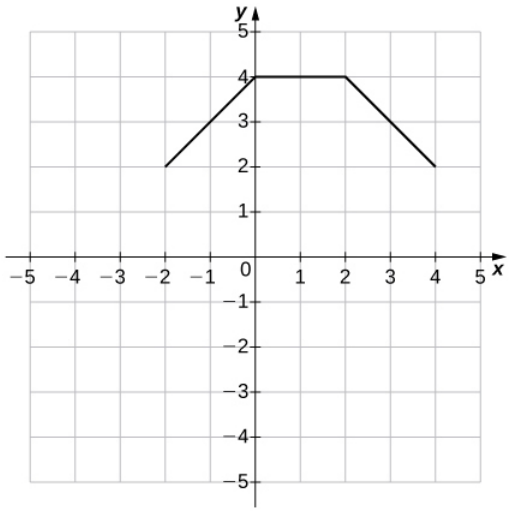


34) $g(x) = f(x) + 1$

35) $g(x) = f(x - 1) + 2$

Answer

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In exercises 36 - 39, for each of the piecewise-defined functions,

a. evaluate at the given values of the independent variable, and

b. sketch the graph.

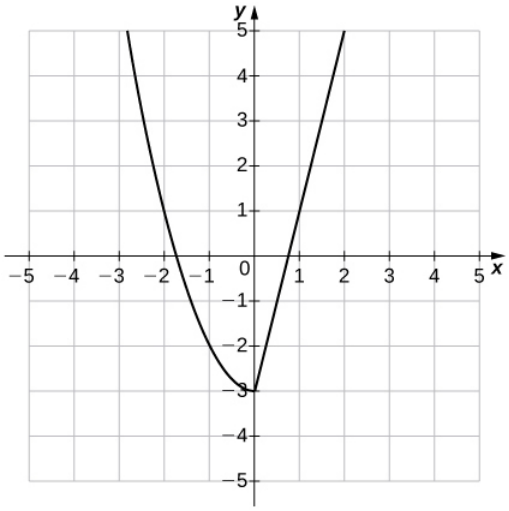
36) $f(x)=\begin{cases}4x+3, & \text{if } x\le 0\\ -x+1, & \text{if } x>0\end{cases}$; $f(-3);\ f(0);\ f(2)$

37) $f(x)=\begin{cases}x^2-3, & \text{if } x\le 0\\ 4x-3, & \text{if } x>0\end{cases}$; $f(-4);\ f(0);\ f(2)$

Answer

a. $f(-4)=13,\quad f(0)=-3,\quad f(2)=5$

b.



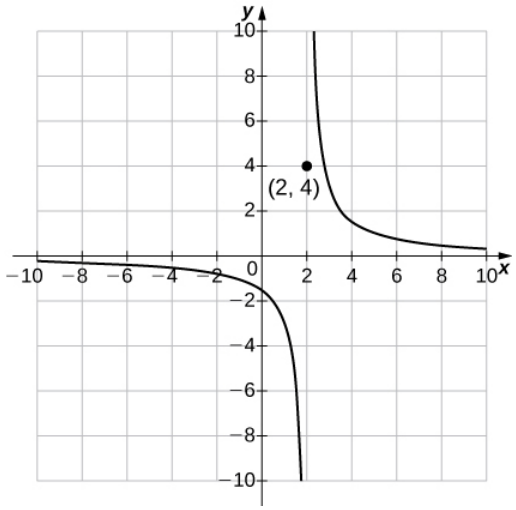
38) $h(x)=\begin{cases}x+1, & \text{if } x\le 5\\ 4, & \text{if } x>5\end{cases}$; $h(0);\ h(\pi);\ h(5)$

39) $g(x)=\begin{cases}\dfrac{3}{x-2}, & \text{if } x\ne 2\\ 4, & \text{if } x=2\end{cases}$; $g(0);\ g(-4);\ g(2)$

Answer

a. $g(0)=-\frac{3}{2},\ g(-4)=-\frac{1}{2},\ g(2)=4$

b.



In exercises 40 - 44, determine whether the statement is *true or false*. Explain why.

40) $f(x)=\dfrac{4x+1}{7x-2}$ is an algebraic function.

41) $g(x)=\sqrt[3]{x}$ is an odd root function.

Answer

True; $n=3$

42) A logarithmic function is an algebraic function.

43) A function of the form $f(x)=x^b$, where b is a real valued constant, is an exponential function.

Answer

False; $f(x)=x^b$, where b is a real-valued constant, is a power function

44) The domain of an even root function is all real numbers.

45) [T] A company purchases some computer equipment for $20,500. At the end of a 3-year period, the value of the equipment has decreased linearly to $12,301.

a. Find a function $y=V(t)$ that determines the value V of the equipment at the end of t years.

b. Find and interpret the meaning of the x- and y-intercepts for this situation.

c. What is the value of the equipment at the end of 5 years?

d. When will the value of the equipment be $3000?

Answer

a. $V(t)=-2733t+20500$

b. $(0,20,500)$means that the initial purchase price of the equipment is $20,500; $(7.5,0)$ means that in 7.5 years the computer equipment has no value.

c. $6835

d. In approximately 6.4 years

46) [T] Total online shopping during the Christmas holidays has increased dramatically during the past 5 years. In 2012 $(t=0)$,total online holiday sales were $42.3 billion, whereas in 2013 they were $48.1 billion.

a. Find a linear function S that estimates the total online holiday sales in the year t.

b. Interpret the slope of the graph of S.

c. Use part a. to predict the year when online shopping during Christmas will reach $60 billion.

47) [T] A family bakery makes cupcakes and sells them at local outdoor festivals. For a music festival, there is a fixed cost of $125 to set up a cupcake stand. The owner estimates that it costs $0.75 to make each cupcake. The owner is interested in determining the total cost C as a function of number of cupcakes made.

a. Find a linear function that relates cost C to x, the number of cupcakes made.

b. Find the cost to bake 160 cupcakes.

c. If the owner sells the cupcakes for $1.50 apiece, how many cupcakes does she need to sell to start making profit? (Hint: Use the INTERSECTION function on a calculator to find this number.)

Answer

a. $C = 0.75x + 125$
b. $245
c. 167 cupcakes

48) [T] A house purchased for $250,000 is expected to be worth twice its purchase price in 18 years.

a. Find a linear function that models the price P of the house versus the number of years t since the original purchase.

b. Interpret the slope of the graph of P.

c. Find the price of the house 15 years from when it was originally purchased.

49) [T] A car was purchased for $26,000. The value of the car depreciates by $1500 per year.

a. Find a linear function that models the value V of the car after t years.

b. Find and interpret $V(4)$.

Answer

a. $V(t) = -1500t + 26,000$
b. In 4 years, the value of the car is $20,000.

50) [T] A condominium in an upscale part of the city was purchased for $432,000. In 35 years it is worth $60,500. Find the rate of depreciation.

51) [T] The total cost C (in thousands of dollars) to produce a certain item is modeled by the function $C(x) = 10.50x + 28,500$, where x is the number of items produced. Determine the cost to produce 175 items.

Answer

$30,337,500

52) [T] A professor asks her class to report the amount of time t they spent writing two assignments. Most students report that it takes them about 45 minutes to type a four-page assignment and about 1.5 hours to type a nine-page assignment.

a. Find the linear function $y = N(t)$ that models this situation, where N is the number of pages typed and t is the time in minutes.

b. Use part a. to determine how many pages can be typed in 2 hours.

c. Use part a. to determine how long it takes to type a 20-page assignment.

53) [T] The output (as a percent of total capacity) of nuclear power plants in the United States can be modeled by the function $P(t) = 1.8576t + 68.052$, where t is time in years and $t = 0$ corresponds to the beginning of 2000. Use the model to predict the percentage output in 2015.

Answer

96% of the total capacity

54) [T] The admissions office at a public university estimates that 65% of the students offered admission to the class of 2019 will actually enroll.

a. Find the linear function $y = N(x)$, where N is the number of students that actually enroll and x is the number of all students offered admission to the class of 2019.

b. If the university wants the 2019 freshman class size to be 1350, determine how many students should be admitted.

2.2E.1 Contributors

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2.3: A Preview of Differential Calculus

Learning Objectives

- Describe the tangent problem and how it led to the idea of a derivative.
- Explain how the idea of a limit is involved in solving the tangent problem.
- Recognize a tangent to a curve at a point as the limit of secant lines.
- Identify instantaneous velocity as the limit of average velocity over a small time interval.

As we embark on our study of calculus, we shall see how its development arose from common solutions to practical problems in areas such as engineering physics—like the space travel problem posed in the chapter opener. Two key problems led to the initial formulation of calculus: (1) the tangent problem, or how to determine the slope of a line tangent to a curve at a point; and (2) the area problem, or how to determine the area under a curve.

2.3.1 The Tangent Problem and Differential Calculus

Rate of change is one of the most critical concepts in calculus. We begin our investigation of rates of change by looking at the graphs of the three lines $f(x) = -2x - 3$, $g(x) = \frac{1}{2}x + 1$, and $h(x) = 2$, shown in Figure 2.3.1.

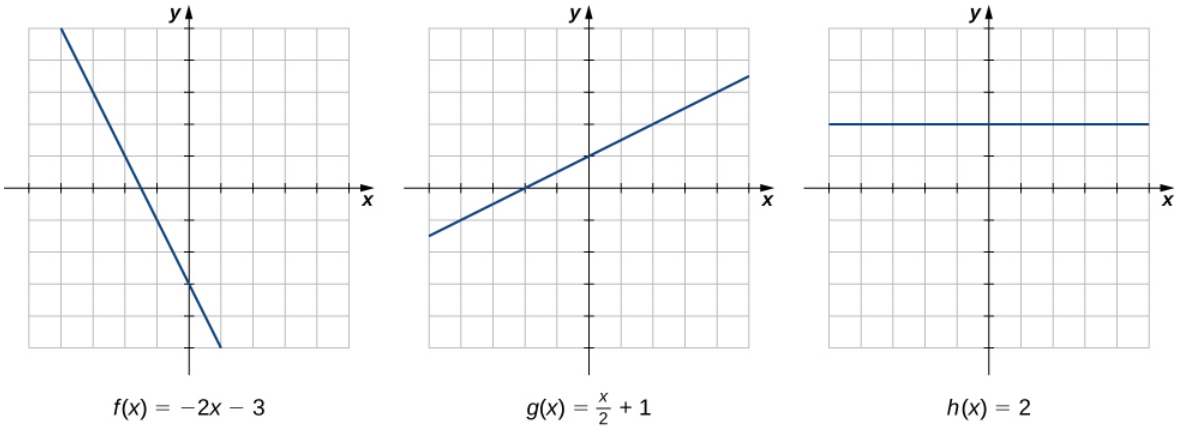


Figure 2.3.1: The rate of change of a linear function is constant in each of these three graphs, with the constant determined by the slope.

As we move from left to right along the graph of $f(x) = -2x - 3$, we see that the graph decreases at a constant rate. For every 1 unit we move to the right along the x-axis, the y-coordinate decreases by 2 units. This **rate of change** is determined by the slope (-2) of the line. Similarly, the slope of $1/2$ in the function $g(x)$ tells us that for every change in x of 1 unit there is a corresponding change in y of $1/2$ unit. The function $h(x) = 2$ has a slope of zero, indicating that the values of the function remain constant. We see that the slope of each linear function indicates the rate of change of the function.

Compare the graphs of these three functions with the graph of $k(x) = x^2$ (Figure 2.3.2). The graph of $k(x) = x^2$ starts from the left by decreasing rapidly, then begins to decrease more slowly and level off, and then finally begins to increase—slowly at first, followed by an increasing rate of increase as it moves toward the right. Unlike a linear function, no single number represents the rate of change for this function. We quite naturally ask: How do we measure the rate of change of a nonlinear function?

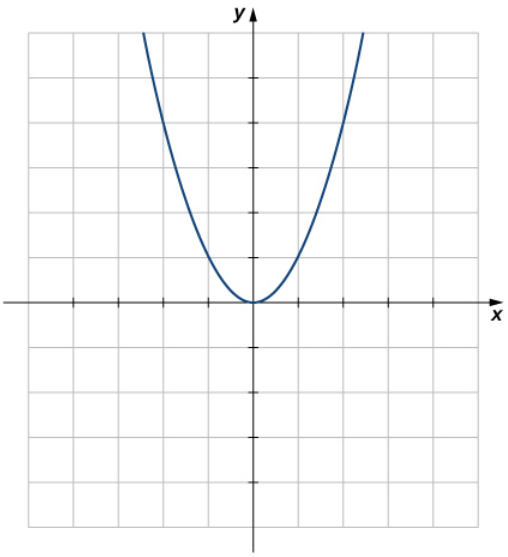


Figure 2.3.2: The function $k(x) = x^2$ does not have a constant rate of change.

We can approximate the rate of change of a function $f(x)$ at a point $(a, f(a))$ on its graph by taking another point $(x, f(x))$ on the graph of $f(x)$, drawing a line through the two points, and calculating the slope of the resulting line. Such a line is called a secant line. Figure 2.3.3 shows a *secant line* to a function $f(x)$ at a point $(a, f(a))$.

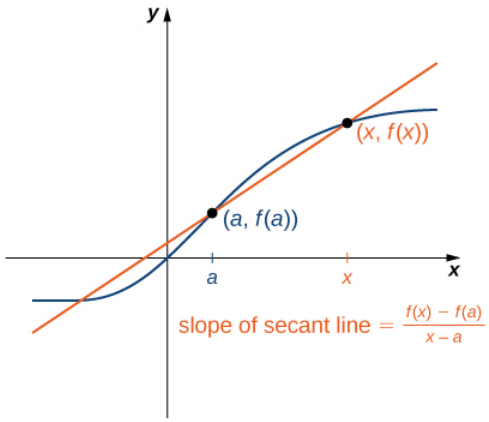


Figure 2.3.3: The slope of a secant line through a point $(a, f(a))$ estimates the rate of change of the function at the point $(a, f(a))$.

We formally define a secant line as follows:

Definition: Secant Line

The *secant* to the function $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is given by

$$m_{sec} = \frac{f(x) - f(a)}{x - a}. \tag{2.3.1}$$

The accuracy of approximating the rate of change of the function with a secant line depends on how close x is to a. As we see in Figure 2.3.4, if x is closer to a, the slope of the secant line is a better measure of the rate of change of $f(x)$ at a.

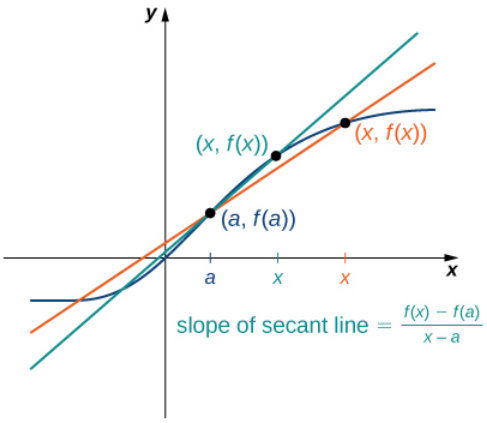


Figure 2.3.4: As x gets closer to a, the slope of the secant line becomes a better approximation to the rate of change of the function $f(x)$ at a.

The secant lines themselves approach a line that is called the **tangent** to the function $f(x)$ at a (Figure 2.3.5). The slope of the tangent line to the graph at a measures the rate of change of the function at a. This value also represents the derivative of the function $f(x)$ at a, or the rate of change of the function at a. This derivative is denoted by $f'(a)$. **Differential calculus** is the field of calculus concerned with the study of derivatives and their applications.

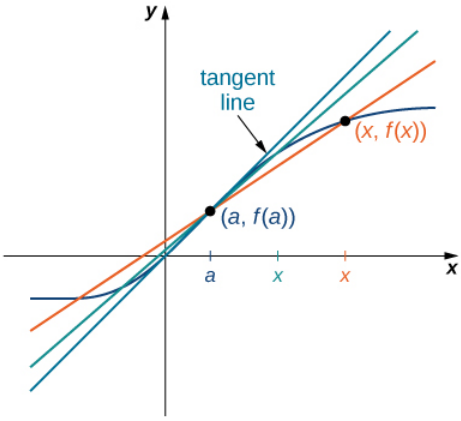


Figure 2.3.5: Solving the Tangent Problem: As x approaches a, the secant lines approach the tangent line.

Example 2.3.1 illustrates how to find slopes of secant lines. These slopes estimate the slope of the tangent line or, equivalently, the rate of change of the function at the point at which the slopes are calculated.

Example 2.3.1: Finding Slopes of Secant Lines

Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and each of the following points on the graph of $f(x) = x^2$.

a. $(2, 4)$
b. $\left(\frac{3}{2}, \frac{9}{4}\right)$

Solution:

Use the formula for the slope of a secant line (Equation 2.3.1).

a. $m_{sec} = \frac{4-1}{2-1} = 3$

b. $m_{sec} = \frac{\frac{9}{4}-1}{\frac{3}{2}-1} = \frac{5}{2} = 2.5$

The point in part b. is closer to the point $(1,1)$, so the slope of 2.5 is closer to the slope of the tangent line. A good estimate for the slope of the tangent would be in the range of 2 to 2.5 (Figure $2.3.6$).

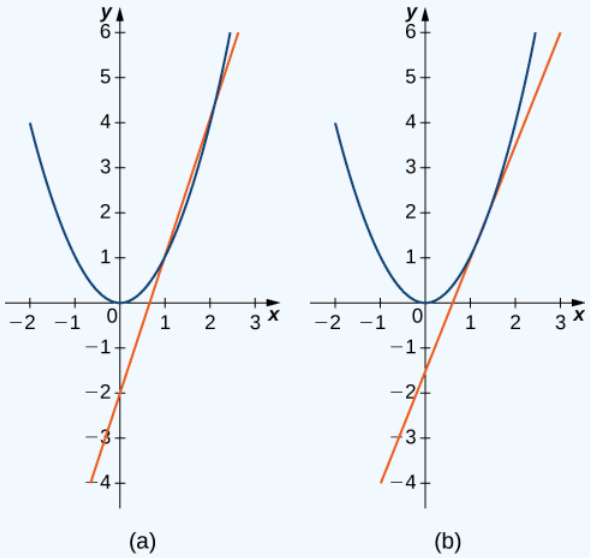


Figure $2.3.6$: The secant lines to $f(x)=x^2$ at $(1,1)$ through (a) $(2,4)$ and (b) $(\frac{3}{2},\frac{9}{4})$ provide successively closer approximations to the tangent line to $f(x)=x^2$ at $(1,1)$.

? Exercise 2.3.1

Estimate the slope of the tangent line (rate of change) to $f(x)=x^2$ at $x=1$ by finding slopes of secant lines through $(1,1)$ and the point $(\frac{5}{4},\frac{25}{16})$ on the graph of $f(x)=x^2$.

Answer

2.25

We continue our investigation by exploring a related question. Keeping in mind that velocity may be thought of as the rate of change of position, suppose that we have a function, $s(t)$, that gives the position of an object along a coordinate axis at any given time t. Can we use these same ideas to create a reasonable definition of the instantaneous velocity at a given time $t=a$? We start by approximating the instantaneous velocity with an average velocity. First, recall that the speed of an object traveling at a constant rate is the ratio of the distance traveled to the length of time it has traveled. We define the *average velocity* of an object over a time period to be the change in its position divided by the length of the time period.

Definition: Average Velocity

Let $s(t)$ be the position of an object moving along a coordinate axis at time t. The *average velocity* of the object over a time interval $[a,t]$ where $a<t$ (or $[t,a]$ if $t<a$) is

$$v_{ave}=\frac{s(t)-s(a)}{t-a}. \tag{2.3.2}$$

As t is chosen closer to a, the average velocity becomes closer to the instantaneous velocity. Note that finding the average velocity of a position function over a time interval is essentially the same as finding the slope of a secant line to a function. Furthermore, to find the slope of a tangent line at a point a, we let the x-values approach a in the slope of the secant line. Similarly, to find the instantaneous velocity at time a, we let the t-values approach a in the average velocity. This process of letting x or t approach a in an expression is called taking a **limit**. Thus, we may define the *instantaneous velocity* as follows.

Definition: Instantaneous Velocity

For a position function $s(t)$, the *instantaneous velocity* at a time $t = a$ is the value that the average velocities approach on intervals of the form $[a, t]$ and $[t, a]$ as the values of t become closer to a, provided such a value exists.

Example $2.3.2$ illustrates this concept of limits and average velocity.

Example 2.3.2: Finding Average Velocity

A rock is dropped from a height of 64 ft. It is determined that its height (in feet) above ground t seconds later (for $0 \leq t \leq 2$) is given by $s(t) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time $t = 0.5$.

a. $[0.49, 0.5]$
b. $[0.5, 0.51]$

Solution

Substitute the data into Equation 2.3.2 for the definition of average velocity.

a.
$$v_{ave} = \frac{s(0.49) - s(0.5)}{0.49 - 0.5} = -15.84$$

b.
$$v_{ave} = \frac{s(0.51) - s(0.5)}{0.51 - 0.5} = -16.016$$

The instantaneous velocity is somewhere between −15.84 and −16.16 ft/sec. A good guess might be −16 ft/sec.

Exercise 2.3.2

An object moves along a coordinate axis so that its position at time t is given by $s(t) = t^3$. Estimate its instantaneous velocity at time $t = 2$ by computing its average velocity over the time interval $[2, 2.001]$.

Hint

Use Equation 2.3.2 with $v_{ave} = \frac{s(2.001) - s(2)}{2.001 - 2}$.

Answer

12.006001

2.3.2 Key Concept

- Differential calculus arose from trying to solve the problem of determining the slope of a line tangent to a curve at a point. The slope of the tangent line indicates the rate of change of the function, also called the *derivative*. Calculating a derivative requires finding a limit.

2.3.3 Key Equations

- **Slope of a Secant Line**

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

- **Average Velocity over Interval [a,t]**

$$v_{ave} = \frac{s(t) - s(a)}{t - a}$$

2.3.4 Glossary

average velocity

the change in an object's position divided by the length of a time period; the average velocity of an object over a time interval [t, a] (if $t < a$ or $[a, t]$ if $t > a$), with a position given by $s(t)$, that is $v_{ave} = \dfrac{s(t) - s(a)}{t - a}$

differential calculus

the field of calculus concerned with the study of derivatives and their applications

instantaneous velocity

The instantaneous velocity of an object with a position function that is given by $s(t)$ is the value that the average velocities on intervals of the form $[t, a]$ and $[a, t]$ approach as the values of t move closer to a, provided such a value exists

limit

the process of letting x or t approach a in an expression; the limit of a function $f(x)$ as x approaches a is the value that $f(x)$ approaches as x approaches a

secant

A secant line to a function $f(x)$ at a is a line through the point $(a, f(a))$ and another point on the function; the slope of the secant line is given by $m_{sec} = \dfrac{f(x) - f(a)}{x - a}$

tangent

A tangent line to the graph of a function at a point $(a, f(a))$ is the line that secant lines through $(a, f(a))$ approach as they are taken through points on the function with x-values that approach a; the slope of the tangent line to a graph at a measures the rate of change of the function at a

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2.3E: Exercises for Section 2.3

For exercises 1 - 3 , points $P(1,2)$ and $Q(x,y)$ are on the graph of the function $f(x)=x^2+1$.

1) [T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x,y)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.

Answer

a. 2.2100000
b. 2.0201000
c. 2.0020010
d. 2.0002000
e. (1.1000000, 2.2100000)
f. (1.0100000, 2.0201000)
g. (1.0010000, 2.0020010)
h. (1.0001000, 2.0002000)
i. 2.1000000
j. 2.0100000
k. 2.0010000
l. 2.0001000

2) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to f at $x=1$.

3) Use the value in the preceding exercise to find the equation of the tangent line at point P. Graph $f(x)$ and the tangent line.

Answer

$y=2x$

For the exercises 4-6, points $P(1,1)$ and $Q(x,y)$ are on the graph of the function $f(x)=x^3$.

4) [T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x,y)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.2

5) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $x=1$.

Answer

3

6) Use the value in the preceding exercise to find the equation of the tangent line at point P. Graph $f(x)$ and the tangent line.

For the exercises 7 - 9, points $P(4,2)$ and $Q(x,y)$ are on the graph of the function $f(x)=\sqrt{x}$.

7) [T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x,y)$	m_{sec}
4.1	a.	e.	i.
4.01	b.	f.	j.
4.001	c.	g.	k.
4.0001	d.	h.	l.

Answer

a. 2.0248457
b. 2.0024984
c. 2.0002500
d. 2.0000250
e. (4.1000000,2.0248457)
f. (4.0100000,2.0024984)
g. (4.0010000,2.0002500)
h. (4.00010000,2.0000250)
i. 0.24845673
j. 0.24984395
k. 0.24998438
l. 0.24999844

8) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $x=4$.

9) Use the value in the preceding exercise to find the equation of the tangent line at point P.

Answer

$y=\frac{x}{4}+1$

For exercises 10 - 12, points $P(1.5,0)$ and $Q(\phi,y)$ are on the graph of the function $f(\phi)=\cos(\pi\phi)$.

10) [T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(\phi,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(\phi,y)$	m_{sec}
1.4	a.	e.	i.
1.49	b.	f.	j.
1.499	c.	g.	k.
1.4999	d.	h.	l.

11) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $\phi=1.5$.

Answer

π

12) Use the value in the preceding exercise to find the equation of the tangent line at point P.

For exercises 13 - 15, points $P(-1,-1)$ and $Q(x,y)$ are on the graph of the function $f(x)=\frac{1}{x}$.

13) [T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x,y)$	m_{sec}
-1.05	a.	e.	i.
-1.01	b.	f.	j.
-1.005	c.	g.	k.
-1.001	d.	h.	l.

Answer

a. −0.95238095
b. −0.99009901
c. −0.99502488
d. −0.99900100
e. (−1;.0500000,−0;.95238095)
f. (−1;.0100000,−0;.9909901)
g. (−1;.0050000,−0;.99502488)
h. (1.0010000,−0;.99900100)
i. −0.95238095
j. −0.99009901
k. −0.99502488
l. −0.99900100

14) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to f at $x=-1$.

15) Use the value in the preceding exercise to find the equation of the tangent line at point P.

Answer

$y=-x-2$

For exercises 16 - 17, the position function of a ball dropped from the top of a 200-meter tall building is given by $s(t)=200-4.9t^2$, where position s is measured in meters and time t is measured in seconds. Round your answer to eight significant digits.

16) [T] Compute the average velocity of the ball over the given time intervals.

a. [4.99,5]

b. [5,5.01]

c. [4.999,5]

d. [5,5.001]

17) Use the preceding exercise to guess the instantaneous velocity of the ball at $t=5$ sec.

Answer

−49 m/sec (velocity of the ball is 49 m/sec downward)

For exercises 18 - 19, consider a stone tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters at time t seconds is $h(t)=15t-4.9t^2$.

18) [T] Compute the average velocity of the stone over the given time intervals.

a. [1,1.05]

b. [1,1.01]

c. [1,1.005]

d. [1,1.001]

19) Use the preceding exercise to guess the instantaneous velocity of the stone at $t = 1$ sec.

Answer

5.2 m/sec

For exercises 20 - 21, consider a rocket shot into the air that then returns to Earth. The height of the rocket in meters is given by $h(t) = 600 + 78.4t - 4.9t^2$, where t is measured in seconds.

20) [T] Compute the average velocity of the rocket over the given time intervals.

a. [9,9.01]

b. [8.99,9]

c. [9,9.001]

d. [8.999,9]

21) Use the preceding exercise to guess the instantaneous velocity of the rocket at $t = 9$ sec.

Answer

-9.8 m/sec

For exercises, consider an athlete running a 40-m dash. The position of the athlete is given by $d(t) = \frac{t^3}{6} + 4t$, where d is the position in meters and t is the time elapsed, measured in seconds.

22) [T] Compute the average velocity of the runner over the given time intervals.

a. [1.95,2.05]

b. [1.995,2.005]

c. [1.9995,2.0005]

d. [2,2.00001]

23) Use the preceding exercise to guess the instantaneous velocity of the runner at $t = 2$ sec.

Answer

6 m/sec

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2.4: The Limit of a Function

Learning Objectives

- Using correct notation, describe the limit of a function.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
- Use a graph to estimate the limit of a function or to identify when the limit does not exist.
- Define one-sided limits and provide examples.
- Explain the relationship between one-sided and two-sided limits.
- Using correct notation, describe an infinite limit.
- Define a vertical asymptote.

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach. At the end of this chapter, armed with a conceptual understanding of limits, we examine the formal definition of a limit.

We begin our exploration of limits by taking a look at the graphs of the functions

- $f(x) = \dfrac{x^2 - 4}{x - 2}$,
- $g(x) = \dfrac{|x - 2|}{x - 2}$, and
- $h(x) = \dfrac{1}{(x - 2)^2}$,

which are shown in Figure 2.4.1. In particular, let's focus our attention on the behavior of each graph at and around $x = 2$.

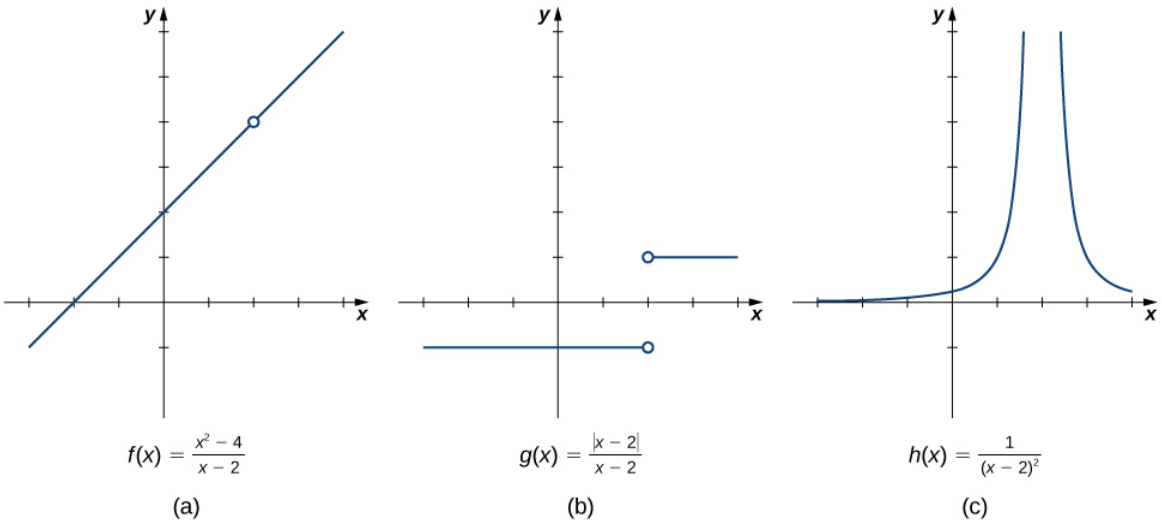


Figure 2.4.1: These graphs show the behavior of three different functions around $x = 2$.

Each of the three functions is undefined at $x = 2$, but if we make this statement and no other, we give a very incomplete picture of how each function behaves in the vicinity of $x = 2$. To express the behavior of each graph in the vicinity of 2 more completely, we need to introduce the concept of a limit.

2.4.1 Intuitive Definition of a Limit

Let's first take a closer look at how the function $f(x) = (x^2 - 4)/(x - 2)$ behaves around $x = 2$ in Figure 2.4.1. As the values of x approach 2 from either side of 2, the values of $y = f(x)$ approach 4. Mathematically, we say that the limit of $f(x)$ as x approaches 2 is 4. Symbolically, we express this limit as

$$\lim_{x \to 2} f(x) = 4$$

From this very brief informal look at one limit, let's start to develop an *intuitive definition of the limit*. We can think of the limit of a function at a number a as being the one real number L that the functional values approach as the x-values approach a, provided such a real number L exists. Stated more carefully, we have the following definition:

Definition (Intuitive): Limit

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If all values of the function $f(x)$ approach the real number L as the values of $x (\neq a)$ approach the number a, then we say that the limit of $f(x)$ as x approaches a is L. (More succinct, as x gets closer to a, $f(x)$ gets closer and stays close to L.) Symbolically, we express this idea as

$$\lim_{x \to a} f(x) = L. \tag{2.4.1}$$

We can estimate limits by constructing tables of functional values and by looking at their graphs. This process is described in the following Problem-Solving Strategy.

Problem-Solving Strategy: Evaluating a Limit Using a Table of Functional Values

1. To evaluate $\lim_{x \to a} f(x)$, we begin by completing a table of functional values. We should choose two sets of x-values—one set of values approaching a and less than a, and another set of values approaching a and greater than a. Table $2.4.1$ demonstrates what your tables might look like.

Table 2.4.1

x	$f(x)$	x	$f(x)$
$a - 0.1$	$f(a - 0.1)$	$a + 0.1$	$f(a + 0.1)$
$a - 0.01$	$f(a - 0.01)$	$a + 0.01$	$f(a + 0.01)$
$a - 0.001$	$f(a - 0.001)$	$a + 0.001$	$f(a + 0.001)$
$a - 0.0001$	$f(a - 0.0001)$	$a + 0.0001$	$f(a + 0.0001)$
Use additional values as necessary.		Use additional values as necessary.	

2. Next, let's look at the values in each of the $f(x)$ columns and determine whether the values seem to be approaching a single value as we move down each column. In our columns, we look at the sequence $f(a-0.1)$, $f(a-0.01)$, $f(a-0.001)$, $f(a-0.0001)$, and so on, and $f(a+0.1)$, $f(a+0.01)$, $f(a+0.001)$, $f(a+0.0001)$, and so on. (Note: Although we have chosen the x-values $a \pm 0.1$, $a \pm 0.01$, $a \pm 0.001$, $a \pm 0.0001$, and so forth, and these values will probably work nearly every time, on very rare occasions we may need to modify our choices.)

3. If both columns approach a common y-value L, we state $\lim_{x \to a} f(x) = L$. We can use the following strategy to confirm the result obtained from the table or as an alternative method for estimating a limit.

4. Using a graphing calculator or computer software that allows us graph functions, we can plot the function $f(x)$, making sure the functional values of $f(x)$ for x-values near a are in our window. We can use the trace feature to move along the graph of the function and watch the y-value readout as the x-values approach a. If the y-values approach L as our x-values approach a from both directions, then $\lim_{x \to a} f(x) = L$. We may need to zoom in on our graph and repeat this process several times.

We apply this Problem-Solving Strategy to compute a limit in Examples $2.4.1A$ and $2.4.1B$.

Example 2.4.1A: Evaluating a Limit Using a Table of Functional Values

Evaluate $\lim_{x\to 0}\dfrac{\sin x}{x}$ using a table of functional values.

Solution

We have calculated the values of $f(x)=\dfrac{\sin x}{x}$ for the values of x listed in Table 2.4.2.

Table 2.4.2

x	$\frac{\sin x}{x}$	x	$\frac{\sin x}{x}$
-0.1	0.998334166468	0.1	0.998334166468
-0.01	0.999983333417	0.01	0.999983333417
-0.001	0.999999833333	0.001	0.999999833333
-0.0001	0.999999998333	0.0001	0.999999998333

Note: The values in this table were obtained using a calculator and using all the places given in the calculator output.

As we read down each $\dfrac{\sin x}{x}$ column, we see that the values in each column appear to be approaching one. Thus, it is fairly reasonable to conclude that $\lim_{x\to 0}\dfrac{\sin x}{x}=1$. A calculator-or computer-generated graph of $f(x)=\dfrac{\sin x}{x}$ would be similar to that shown in Figure 2.4.2, and it confirms our estimate.

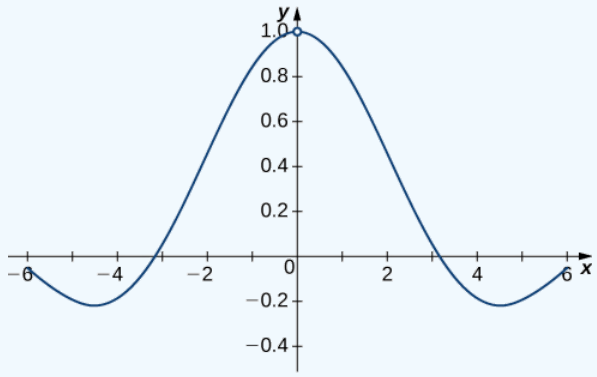


Figure 2.4.2: The graph of $f(x)=(\sin x)/x$ confirms the estimate from Table 2.4.2.

Example 2.4.1B: Evaluating a Limit Using a Table of Functional Values

Evaluate $\lim_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}$ using a table of functional values.

Solution

As before, we use a table—in this case, Table 2.4.3—to list the values of the function for the given values of x.

Table 2.4.3

x	$\frac{\sqrt{x}-2}{x-4}$	x	$\frac{\sqrt{x}-2}{x-4}$
3.9	0.251582341869	4.1	0.248456731317
3.99	0.25015644562	4.01	0.24984394501
3.999	0.250015627	4.001	0.249984377
3.9999	0.250001563	4.0001	0.249998438
3.99999	0.25000016	4.00001	0.24999984

After inspecting this table, we see that the functional values less than 4 appear to be decreasing toward 0.25 whereas the functional values greater than 4 appear to be increasing toward 0.25. We conclude that $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}=0.25$. We confirm this estimate using the graph of $f(x)=\dfrac{\sqrt{x}-2}{x-4}$ shown in Figure 2.4.3.

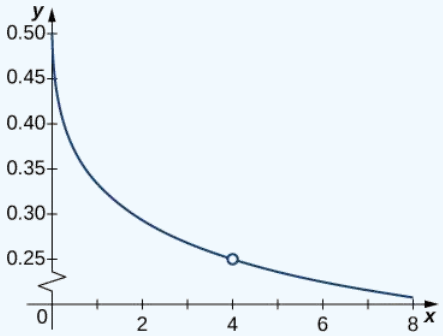


Figure 2.4.3: The graph of $\frac{\sqrt{x}-2}{x-4}$ confirms the estimate from Table 2.4.3.

? Exercise 2.4.1

Estimate $\lim\limits_{x\to 1}\dfrac{\frac{1}{x}-1}{x-1}$ using a table of functional values. Use a graph to confirm your estimate.

Hint

Use 0.9, 0.99, 0.999, 0.9999, 0.99999 and 1.1, 1.01, 1.001, 1.0001, 1.00001 as your table values.

Answer

$$\lim_{x\to 1}\frac{\frac{1}{x}-1}{x-1}=-1$$

At this point, we see from Examples 2.4.1A and 2.4.1b that it may be just as easy, if not easier, to estimate a limit of a function by inspecting its graph as it is to estimate the limit by using a table of functional values. In Example 2.4.2, we evaluate a limit exclusively by looking at a graph rather than by using a table of functional values.

✓ Example 2.4.2: Evaluating a Limit Using a Graph

For $g(x)$ shown in Figure 2.4.4, evaluate $\lim\limits_{x\to -1} g(x)$.

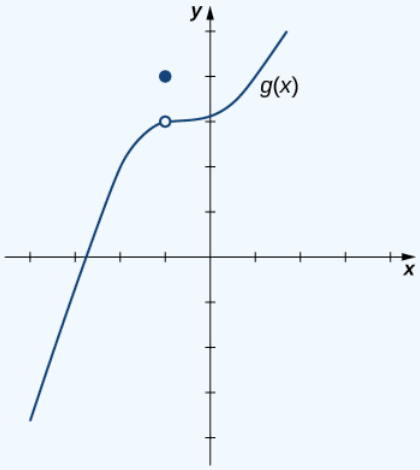


Figure 2.4.4: The graph of $g(x)$ includes one value not on a smooth curve.

Solution:

Despite the fact that $g(-1) = 4$, as the x-values approach -1 from either side, the $g(x)$ values approach 3. Therefore, $\lim_{x \to -1} g(x) = 3$. Note that we can determine this limit without even knowing the algebraic expression of the function.

Based on Example 2.4.2, we make the following observation: It is possible for the limit of a function to exist at a point, and for the function to be defined at this point, but the limit of the function and the value of the function at the point may be different.

Exercise 2.4.2

Use the graph of $h(x)$ in Figure 2.4.5 to evaluate $\lim_{x \to 2} h(x)$, if possible.

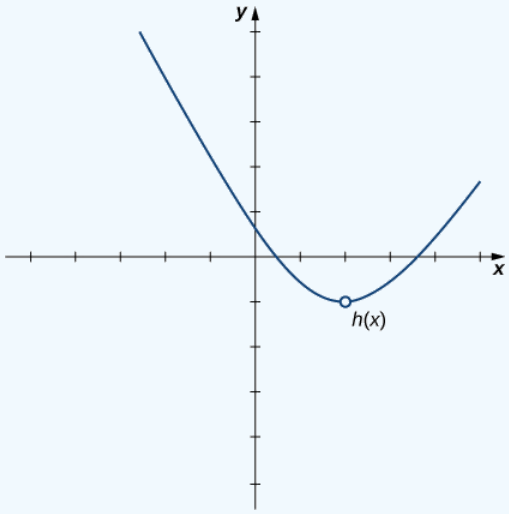


Figure 2.4.5:

Hint

What y-value does the function approach as the x-values approach 2?

Solution

$\lim_{x \to 2} h(x) = -1$.

Looking at a table of functional values or looking at the graph of a function provides us with useful insight into the value of the limit of a function at a given point. However, these techniques rely too much on guesswork. We eventually need to develop alternative methods of evaluating limits. These new methods are more algebraic in nature and we explore them in the next section; however, at this point we introduce two special limits that are foundational to the techniques to come.

Two Important Limits

Let a be a real number and c be a constant.

i. $\lim_{x \to a} x = a$
ii. $\lim_{x \to a} c = c$

We can make the following observations about these two limits.

i. For the first limit, observe that as x approaches a, so does $f(x)$, because $f(x) = x$. Consequently, $\lim_{x \to a} x = a$.

ii. For the second limit, consider Table 2.4.4.

Table 2.4.4

x	$f(x) = c$	x	$f(x) = c$
$a - 0.1$	c	$a + 0.1$	c

x	$f(x)=c$	x	$f(x)=c$
$a-0.01$	c	$a+0.01$	c
$a-0.001$	c	$a+0.001$	c
$a-0.0001$	c	$a+0.0001$	c

Observe that for all values of x (regardless of whether they are approaching a), the values $f(x)$ remain constant at c. We have no choice but to conclude $\lim\limits_{x\to a} c = c$.

2.4.2 The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach a single real-number value at that point. If the functional values do not approach a single value, then the limit does not exist.

Example 2.4.3: Evaluating a Limit That Fails to Exist

Evaluate $\lim\limits_{x\to 0} \sin(1/x)$ using a table of values.

Solution

Table 2.4.5 lists values for the function $\sin(1/x)$ for the given values of x.

Table 2.4.5

x	$\sin(1/x)$	x	$\sin(1/x)$
-0.1	0.544021110889	0.1	−0.544021110889
-0.01	0.50636564111	0.01	−0.50636564111
-0.001	−0.8268795405312	0.001	0.8268795405312
-0.0001	0.305614388888	0.0001	−0.305614388888
-0.00001	−0.035748797987	0.00001	0.035748797987
-0.000001	0.349993504187	0.000001	−0.349993504187

After examining the table of functional values, we can see that the y-values do not seem to approach any one single value. It appears the limit does not exist. Before drawing this conclusion, let's take a more systematic approach. Take the following sequence of x-values approaching 0:

$$\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \frac{2}{9\pi}, \frac{2}{11\pi}, \ldots.$$

The corresponding y-values are

$$1, -1, 1, -1, 1, -1, \ldots.$$

At this point we can indeed conclude that $\lim\limits_{x\to 0} \sin(1/x)$ does not exist. (Mathematicians frequently abbreviate "does not exist" as DNE. Thus, we would write $\lim\limits_{x\to 0} \sin(1/x)$ DNE.) The graph of $f(x) = \sin(1/x)$ is shown in Figure 2.4.6 and it gives a clearer picture of the behavior of $\sin(1/x)$ as x approaches 0. You can see that $\sin(1/x)$ oscillates ever more wildly between -1 and 1 as x approaches 0.

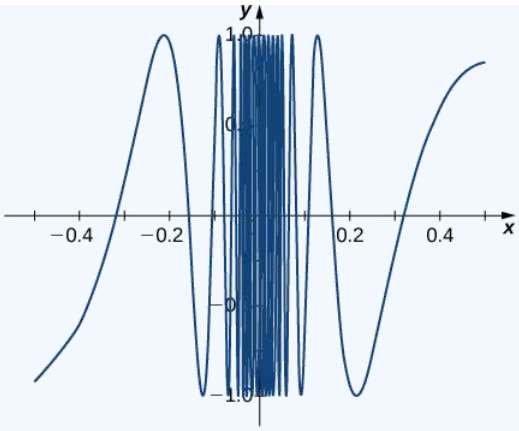


Figure 2.4.6: The graph of $f(x) = \sin(1/x)$ oscillates rapidly between -1 and 1 as x approaches 0.

? Exercise 2.4.3

Use a table of functional values to evaluate $\lim\limits_{x \to 2} \dfrac{|x^2 - 4|}{x - 2}$, if possible.

Hint

Use x-values 1.9, 1.99, 1.999, 1.9999, 1.99999 and 2.1, 2.01, 2.001, 2.0001, 2.00001 in your table.

Answer

$\lim\limits_{x \to 2} \dfrac{|x^2 - 4|}{x - 2}$ does not exist.

2.4.3 One-Sided Limits

Sometimes indicating that the limit of a function fails to exist at a point does not provide us with enough information about the behavior of the function at that particular point. To see this, we now revisit the function $g(x) = |x - 2|/(x - 2)$ introduced at the beginning of the section (see Figure 2.4.1(b)). As we pick values of x close to 2, $g(x)$ does not approach a single value, so the limit as x approaches 2 does not exist—that is, $\lim\limits_{x \to 2} g(x)$ DNE. However, this statement alone does not give us a complete picture of the behavior of the function around the x-value 2. To provide a more accurate description, we introduce the idea of a **one-sided limit**. For all values to the left of 2 (or the negative side of 2), $g(x) = -1$. Thus, as x approaches 2 from the left, $g(x)$ approaches -1. Mathematically, we say that the limit as x approaches 2 from the left is -1. Symbolically, we express this idea as

$$\lim_{x \to 2^-} g(x) = -1.$$

Similarly, as x approaches 2 from the right (or from the positive side), $g(x)$ approaches 1. Symbolically, we express this idea as

$$\lim_{x \to 2^+} g(x) = 1.$$

We can now present an informal definition of one-sided limits.

Definition: One-sided Limits

We define two types of one-sided limits.

Limit from the left:

Let $f(x)$ be a function defined at all values in an open interval of the form (z, a), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x < a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the left. Symbolically, we express this idea as

$$\lim_{x \to a^-} f(x) = L.$$

Limit from the right:

Let $f(x)$ be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x > a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the right. Symbolically, we express this idea as

$$\lim_{x \to a^+} f(x) = L.$$

Example 2.4.4: Evaluating One-Sided Limits

For the function $f(x) = \begin{cases} x+1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits.

a. $\lim_{x \to 2^-} f(x)$
b. $\lim_{x \to 2^+} f(x)$

Solution

We can use tables of functional values again. Observe in Table 2.4.6 that for values of x less than 2, we use $f(x) = x + 1$ and for values of x greater than 2, we use $f(x) = x^2 - 4$.

Table 2.4.6

x	$f(x) = x + 1$	x	$f(x) = x^2 - 4$
1.9	2.9	2.1	0.41
1.99	2.99	2.01	0.0401
1.999	2.999	2.001	0.004001
1.9999	2.9999	2.0001	0.00040001
1.99999	2.99999	2.00001	0.0000400001

Based on this table, we can conclude that a. $\lim_{x \to 2^-} f(x) = 3$ and b. $\lim_{x \to 2^+} f(x) = 0$. Therefore, the (two-sided) limit of $f(x)$ does not exist at $x = 2$. Figure 2.4.7 shows a graph of $f(x)$ and reinforces our conclusion about these limits.

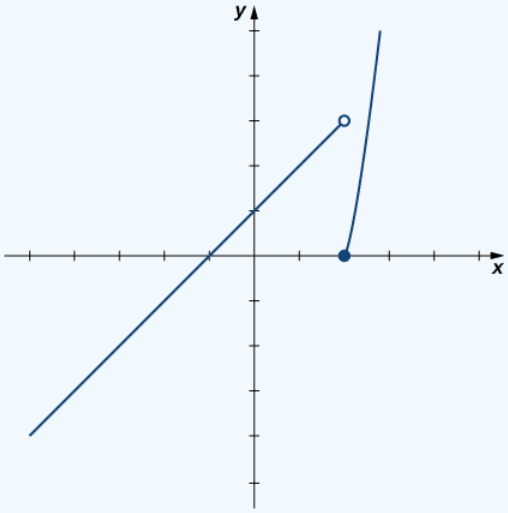


Figure 2.4.7: The graph of $f(x) = \begin{cases} x+1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$ has a break at $x = 2$.

Exercise 2.4.4

Use a table of functional values to estimate the following limits, if possible.

a. $\lim\limits_{x\to 2^-} \dfrac{|x^2-4|}{x-2}$

b. $\lim\limits_{x\to 2^+} \dfrac{|x^2-4|}{x-2}$

Hint

Use x-values 1.9, 1.99, 1.999, 1.9999, 1.99999 to estimate $\lim\limits_{x\to 2^-} \dfrac{|x^2-4|}{x-2}$.

Use x-values 2.1, 2.01, 2.001, 2.0001, 2.00001 to estimate $\lim\limits_{x\to 2^+} \dfrac{|x^2-4|}{x-2}$.

(These tables are available from a previous Checkpoint problem.)

Solution a

a. $\lim\limits_{x\to 2^-} \dfrac{|x^2-4|}{x-2} = -4$

Solution b

$\lim\limits_{x\to 2^+} \dfrac{|x^2-4|}{x-2} = 4$

Let us now consider the relationship between the limit of a function at a point and the limits from the right and left at that point. It seems clear that if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. Similarly, if the limit from the left and the limit from the right take on different values, the limit of the function does not exist. These conclusions are summarized in Note.

Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. Then,

$$\lim_{x\to a} f(x) = L$$

if and only if $\lim\limits_{x\to a^-} f(x) = L$ and $\lim\limits_{x\to a^+} f(x) = L$.

2.4.4 Infinite Limits

Evaluating the limit of a function at a point or evaluating the limit of a function from the right and left at a point helps us to characterize the behavior of a function around a given value. As we shall see, we can also describe the behavior of functions that do not have finite limits.

We now turn our attention to $h(x) = 1/(x-2)^2$, the third and final function introduced at the beginning of this section (see Figure 2.4.1(c)). From its graph we see that as the values of x approach 2, the values of $h(x) = 1/(x-2)^2$ become larger and larger and, in fact, become infinite. Mathematically, we say that the limit of $h(x)$ as x approaches 2 is positive infinity. Symbolically, we express this idea as

$$\lim_{x\to 2} h(x) = +\infty.$$

More generally, we define *infinite limits* as follows:

Definitions: Infinite Limits

We define three types of *infinite limits*.

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

i. If the values of $f(x)$ increase without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^-} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is negative infinity and we write

$$\lim_{x \to a^-} f(x) = -\infty.$$

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

i. If the values of $f(x)$ increase without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is positive infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is negative infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty.$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a

i. If the values of $f(x)$ increase without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is positive infinity and we write

$$\lim_{x \to a} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is negative infinity and we write

$$\lim_{x \to a} f(x) = -\infty.$$

It is important to understand that when we write statements such as $\lim_{x \to a} f(x) = +\infty$ or $\lim_{x \to a} f(x) = -\infty$ we are describing the behavior of the function, as we have just defined it. We are not asserting that a limit exists. For the limit of a function $f(x)$ to exist at a, it must approach a real number L as x approaches a. That said, if, for example, $\lim_{x \to a} f(x) = +\infty$, we always write $\lim_{x \to a} f(x) = +\infty$ rather than $\lim_{x \to a} f(x)$ DNE.

Example 2.4.5: Recognizing an Infinite Limit

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x$ to confirm your conclusion.

a. $\lim_{x \to 0^-} \frac{1}{x}$
b. $\lim_{x \to 0^+} \frac{1}{x}$
c. $\lim_{x \to 0} \frac{1}{x}$

Solution

Begin by constructing a table of functional values.

Table 2.4.7

x	$\frac{1}{x}$	x	$\frac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10,000	0.0001	10,000
-0.00001	-100,000	0.00001	100,000
-0.000001	-1,000,000	0.000001	1,000,000

a. The values of $1/x$ decrease without bound as x approaches 0 from the left. We conclude that

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

b. The values of $1/x$ increase without bound as x approaches 0 from the right. We conclude that

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty.$$

c. Since $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ have different values, we conclude that

$$\lim_{x\to 0} \frac{1}{x} \quad \text{DNE.}$$

The graph of $f(x) = 1/x$ in Figure 2.4.8 confirms these conclusions.

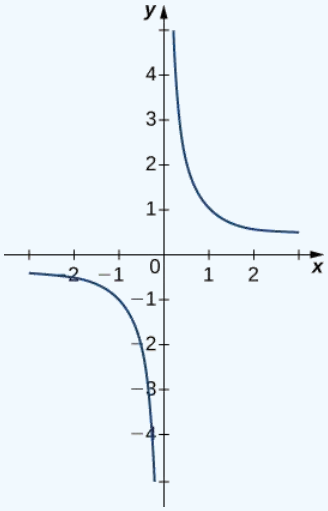


Figure 2.4.8: The graph of $f(x) = 1/x$ confirms that the limit as x approaches 0 does not exist.

Exercise 2.4.5

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x^2$ to confirm your conclusion.

a. $\lim_{x\to 0^-} \frac{1}{x^2}$

b. $\lim_{x\to 0^+} \frac{1}{x^2}$

c. $\lim_{x\to 0} \frac{1}{x^2}$

Hint

Follow the procedures from Example 2.4.5

Answer

a. $\lim\limits_{x\to 0^-} \dfrac{1}{x^2} = +\infty$;

b. $\lim\limits_{x\to 0^+} \dfrac{1}{x^2} = +\infty$;

c. $\lim\limits_{x\to 0} \dfrac{1}{x^2} = +\infty$

It is useful to point out that functions of the form $f(x) = 1/(x-a)^n$, where n is a positive integer, have infinite limits as x approaches a from either the left or right (Figure 2.4.9). These limits are summarized in the above definitions.

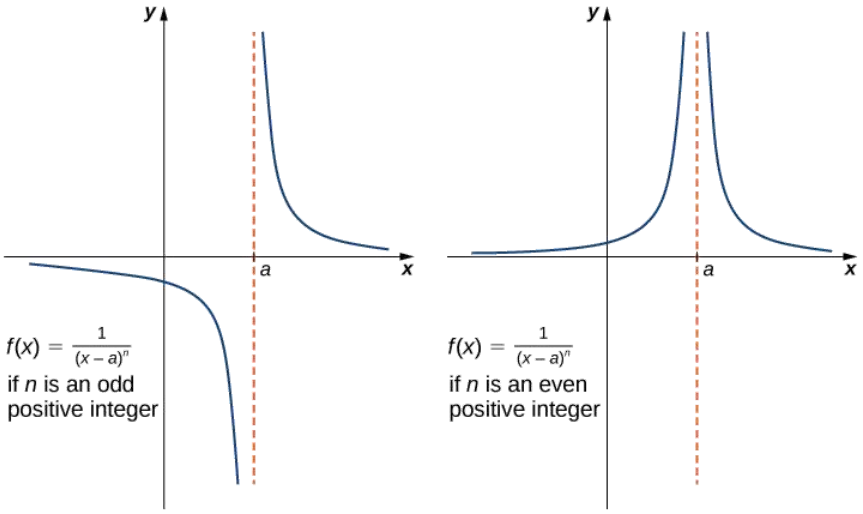


Figure 2.4.9: The function $f(x) = 1/(x-a)^n$ has infinite limits at a.

2.4.5 Infinite Limits from Positive Integers

If n is a positive even integer, then

$$\lim_{x\to a} \frac{1}{(x-a)^n} = +\infty. \tag{2.4.2}$$

If n is a positive odd integer, then

$$\lim_{x\to a^+} \frac{1}{(x-a)^n} = +\infty \tag{2.4.3}$$

and

$$\lim_{x\to a^-} \frac{1}{(x-a)^n} = -\infty. \tag{2.4.4}$$

We should also point out that in the graphs of $f(x) = 1/(x-a)^n$, points on the graph having x-coordinates very near to a are very close to the vertical line $x = a$. That is, as x approaches a, the points on the graph of $f(x)$ are closer to the line $x = a$. The line $x = a$ is called a vertical asymptote of the graph. We formally define a *vertical asymptote* as follows:

Definition: Vertical Asymptotes

Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a vertical asymptote of $f(x)$.

$$\lim_{x\to a^-} f(x) = +\infty$$

$$\lim_{x \to a^-} f(x) = -\infty$$

$$\lim_{x \to a^+} f(x) = +\infty$$

$$\lim_{x \to a^+} f(x) = -\infty$$

$$\lim_{x \to a} f(x) = +\infty$$

$$\lim_{x \to a} f(x) = -\infty$$

Example 2.4.6: Finding a Vertical Asymptote

Evaluate each of the following limits using Equations 2.4.2, 2.4.3, and 2.4.4 above. Identify any vertical asymptotes of the function $f(x) = 1/(x+3)^4$.

a. $\lim_{x \to -3^-} \dfrac{1}{(x+3)^4}$
b. $\lim_{x \to -3^+} \dfrac{1}{(x+3)^4}$
c. $\lim_{x \to -3} \dfrac{1}{(x+3)^4}$

Solution

We can use the above equations directly.

a. $\lim_{x \to -3^-} \dfrac{1}{(x+3)^4} = +\infty$
b. $\lim_{x \to -3^+} \dfrac{1}{(x+3)^4} = +\infty$
c. $\lim_{x \to -3} \dfrac{1}{(x+3)^4} = +\infty$

The function $f(x) = 1/(x+3)^4$ has a vertical asymptote of $x = -3$.

Exercise 2.4.6

Evaluate each of the following limits. Identify any vertical asymptotes of the function $f(x) = \dfrac{1}{(x-2)^3}$.

a. $\lim_{x \to 2^-} \dfrac{1}{(x-2)^3}$
b. $\lim_{x \to 2^+} \dfrac{1}{(x-2)^3}$
c. $\lim_{x \to 2} \dfrac{1}{(x-2)^3}$

Answer a

$$\lim_{x \to 2^-} \frac{1}{(x-2)^3} = -\infty$$

Answer b

$$\lim_{x \to 2^+} \frac{1}{(x-2)^3} = +\infty$$

Answer c

$\lim_{x \to 2} \dfrac{1}{(x-2)^3}$ DNE. The line $x = 2$ is the vertical asymptote of $f(x) = 1/(x-2)^3$.

In the next example we put our knowledge of various types of limits to use to analyze the behavior of a function at several different points.

Example 2.4.7: Behavior of a Function at Different Points

Use the graph of $f(x)$ in Figure 2.4.10 to determine each of the following values:

a. $\lim\limits_{x\to -4^-} f(x)$; $\lim\limits_{x\to -4^+} f(x)$; $\lim\limits_{x\to -4} f(x)$; $f(-4)$
b. $\lim\limits_{x\to -2^-} f(x)$; $\lim\limits_{x\to -2^+} f(x)$; $\lim\limits_{x\to -2} f(x)$; $f(-2)$
c. $\lim\limits_{x\to 1^-} f(x)$; $\lim\limits_{x\to 1^+} f(x)$; $\lim\limits_{x\to 1} f(x)$; $f(1)$
d. $\lim\limits_{x\to 3^-} f(x)$; $\lim\limits_{x\to 3^+} f(x)$; $\lim\limits_{x\to 3} f(x)$; $f(3)$

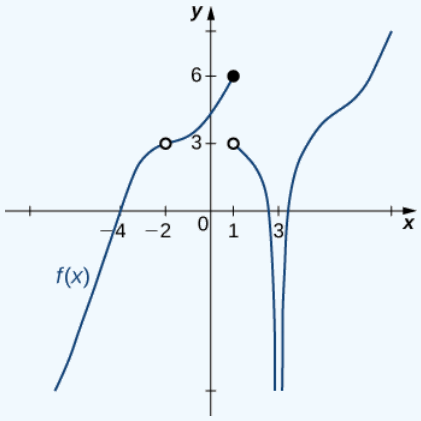


Figure 2.4.10: The graph shows $f(x)$.

Solution

Using the definitions above and the graph for reference, we arrive at the following values:

a. $\lim\limits_{x\to -4^-} f(x) = 0$; $\lim\limits_{x\to -4^+} f(x) = 0$; $\lim\limits_{x\to -4} f(x) = 0$; $f(-4) = 0$
b. $\lim\limits_{x\to -2^-} f(x) = 3$; $\lim\limits_{x\to -2^+} f(x) = 3$; $\lim\limits_{x\to -2} f(x) = 3$; $f(-2)$ is undefined
c. $\lim\limits_{x\to 1^-} f(x) = 6$; $\lim\limits_{x\to 1^+} f(x) = 3$; $\lim\limits_{x\to 1} f(x)$ DNE; $f(1) = 6$
d. $\lim\limits_{x\to 3^-} f(x) = -\infty$; $\lim\limits_{x\to 3^+} f(x) = -\infty$; $\lim\limits_{x\to 3} f(x) = -\infty$; $f(3)$ is undefined

Exercise 2.4.7

Evaluate $\lim\limits_{x\to 1} f(x)$ for $f(x)$ shown here:

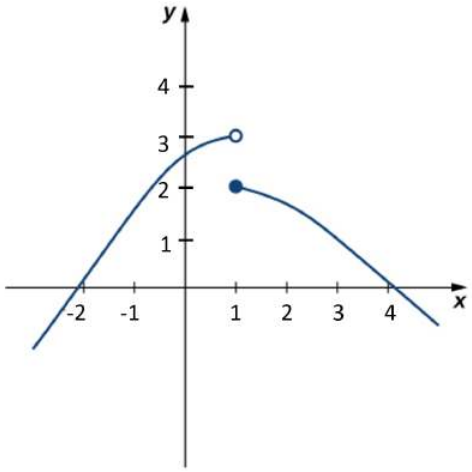


Figure 2.4.11. The graph of a piecewise function f.

Hint

Compare the limit from the right with the limit from the left.

Answer

$\lim_{x \to 1} f(x)$ does not exist

Example 2.4.8: Einstein's Equation

In the Chapter opener we mentioned briefly how Albert Einstein showed that a limit exists to how fast any object can travel. Given Einstein's equation for the mass of a moving object

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

what is the value of this bound?

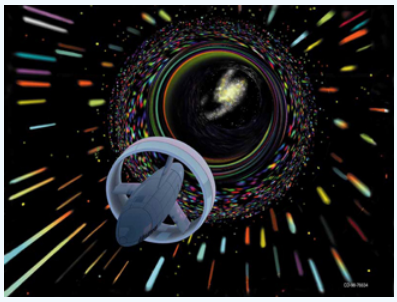


Figure 2.4.12. (Credit:NASA)

Solution

Our starting point is Einstein's equation for the mass of a moving object,

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. To see how the mass changes at high speeds, we can graph the ratio of masses m/m_0 as a function of the ratio of speeds, v/c (Figure 2.4.13).

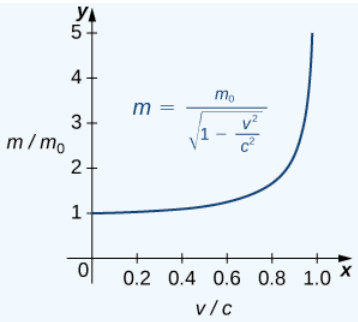


Figure 2.4.13: This graph shows the ratio of masses as a function of the ratio of speeds in Einstein's equation for the mass of a moving object.

We can see that as the ratio of speeds approaches 1—that is, as the speed of the object approaches the speed of light—the ratio of masses increases without bound. In other words, the function has a vertical asymptote at $v/c = 1$. We can try a few values of this ratio to test this idea.

Table 2.4.8

v/c	$\sqrt{1-\frac{v^2}{c^2}}$	m/m_o
0.99	0.1411	7.089
0.999	0.0447	22.37
0.9999	0.0141	70.7

Thus, according to Table 2.4.8:, if an object with mass 100 kg is traveling at 0.9999c, its mass becomes 7071 kg. Since no object can have an infinite mass, we conclude that no object can travel at or more than the speed of light.

2.4.6 Key Concepts

- A table of values or graph may be used to estimate a limit.
- If the limit of a function at a point does not exist, it is still possible that the limits from the left and right at that point may exist.
- If the limits of a function from the left and right exist and are equal, then the limit of the function is that common value.
- We may use limits to describe infinite behavior of a function at a point.

2.4.7 Key Equations

- **Intuitive Definition of the Limit**

$$\lim_{x \to a} f(x) = L$$

- **Two Important Limits**

$$\lim_{x \to a} x = a \qquad \lim_{x \to a} c = c$$

- **One-Sided Limits**

$$\lim_{x \to a^-} f(x) = L \qquad \lim_{x \to a^+} f(x) = L$$

- **Infinite Limits from the Left**

$$\lim_{x \to a^-} f(x) = +\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

- **Infinite Limits from the Right**

$$\lim_{x \to a^+} f(x) = +\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

- **Two-Sided Infinite Limits**

$$\lim_{x \to a} f(x) = +\infty \text{: } \lim_{x \to a^-} f(x) = +\infty \text{ and } \lim_{x \to a^+} f(x) = +\infty$$

$\lim_{x \to a} f(x) = -\infty$: $\lim_{x \to a^-} f(x) = -\infty$ and $\lim_{x \to a^+} f(x) = -\infty$

2.4.8 Glossary

infinite limit

A function has an infinite limit at a point a if it either increases or decreases without bound as it approaches a

intuitive definition of the limit

If all values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach a, $f(x)$ approaches L

one-sided limit

A one-sided limit of a function is a limit taken from either the left or the right

vertical asymptote

A function has a vertical asymptote at $x = a$ if the limit as x approaches a from the right or left is infinite

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2.4E: Exercises for Section 2.4

2.4E.1 Intuitive Definition of Limits

For exercises 1 - 2, consider the function $f(x) = \dfrac{x^2 - 1}{|x - 1|}$.

1) [T] Complete the following table for the function. Round your solutions to four decimal places.

x	$f(x)$	x	$f(x)$
0.9	a.	1.1	e.
0.99	b.	1.01	f.
0.999	c.	1.001	g.
0.9999	d.	1.0001	h.

2) What do your results in the preceding exercise indicate about the two-sided limit $\lim\limits_{x \to 1} f(x)$? Explain your response.

Answer

$\lim\limits_{x \to 1} f(x)$ does not exist because $\lim\limits_{x \to 1^-} f(x) = -2 \neq \lim\limits_{x \to 1^+} f(x) = 2$.

For exercises 3 - 5, consider the function $f(x) = (1 + x)^{1/x}$.

3) [T] Make a table showing the values of f for $x = -0.01,\ -0.001,\ -0.0001,\ -0.00001$ and for $x = 0.01,\ 0.001,\ 0.0001,\ 0.00001$. Round your solutions to five decimal places.

x	$f(x)$	x	$f(x)$
-0.01	a.	0.01	e.
-0.001	b.	0.001	f.
-0.0001	c.	0.0001	g.
-0.00001	d.	0.00001	h.

4) What does the table of values in the preceding exercise indicate about the function $f(x) = (1 + x)^{1/x}$?

Answer

$\lim\limits_{x \to 0} (1 + x)^{1/x} \approx 2.7183.$

5) To which mathematical constant do the values in the preceding exercise appear to be approaching? This is the actual limit here.

In exercises 6 - 8, use the given values to set up a table to evaluate the limits. Round your solutions to eight decimal places.

6) [T] $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$; $\pm 0.1,\ \pm 0.01,\ \pm 0.001,\ \pm .0001$

x	$\frac{\sin 2x}{x}$	x	$\frac{\sin 2x}{x}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

Answer

a. 1.98669331; b. 1.99986667; c. 1.99999867; d. 1.99999999; e. 1.98669331; f. 1.99986667; g. 1.99999867; h. 1.99999999;

$\lim_{x\to 0}\dfrac{\sin 2x}{x}=2$

7) [T] $\lim_{x\to 0}\dfrac{\sin 3x}{x}$ $\pm 0.1,\ \pm 0.01,\ \pm 0.001,\ \pm 0.0001$

x	$\frac{\sin 3x}{x}$	x	$\frac{\sin 3x}{x}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

8) Use the preceding two exercises to conjecture (guess) the value of the following limit: $\lim_{x\to 0}\dfrac{\sin ax}{x}$ for a, a positive real value.

Answer

$\lim_{x\to 0}\dfrac{\sin ax}{x}=a$

[T] In exercises 9 - 14, set up a table of values to find the indicated limit. Round to eight significant digits.

9) $\lim_{x\to 2}\dfrac{x^2-4}{x^2+x-6}$

x	$\frac{x^2-4}{x^2+x-6}$	x	$\frac{x^2-4}{x^2+x-6}$
1.9	a.	2.1	e.
1.99	b.	2.01	f.
1.999	c.	2.001	g.
1.9999	d.	2.0001	h.

10) $\lim_{x\to 1}(1-2x)$

x	$1-2x$	x	$1-2x$
0.9	a.	1.1	e.
0.99	b.	1.01	f.
0.999	c.	1.001	g.
0.9999	d.	1.0001	h.

Answer

a. −0.80000000; b. −0.98000000; c. −0.99800000; d. −0.99980000; e. −1.2000000; f. −1.0200000; g. −1.0020000; h. −1.0002000;

$\lim_{x\to 1}(1-2x)=-1$

11) $\lim_{x\to 0}\dfrac{5}{1-e^{1/x}}$

x	$\frac{5}{1-e^{1/x}}$	x	$\frac{5}{1-e^{1/x}}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

12) $\lim_{z\to 0} \frac{z-1}{z^2(z+3)}$

z	$\frac{z-1}{z^2(z+3)}$	z	$\frac{z-1}{z^2(z+3)}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

Answer

a. −37.931034; b. −3377.9264; c. −333,777.93; d. −33,337,778; e. −29.032258; f. −3289.0365; g. −332,889.04; h. −33,328,889

$\lim_{x\to 0} \frac{z-1}{z^2(z+3)} = -\infty$

13) $\lim_{t\to 0^+} \frac{\cos t}{t}$

t	$\frac{\cos t}{t}$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

14) $\lim_{x\to 2} \frac{1-\frac{2}{x}}{x^2-4}$

x	$\frac{1-\frac{2}{x}}{x^2-4}$	x	$\frac{1-\frac{2}{x}}{x^2-4}$
1.9	a.	2.1	e.
1.99	b.	2.01	f.
1.999	c.	2.001	g.
1.9999	d.	2.0001	h.

Answer

a. 0.13495277; b. 0.12594300; c. 0.12509381; d. 0.12500938; e. 0.11614402; f. 0.12406794; g. 0.12490631; h. 0.12499063;

$\therefore \lim_{x\to 2} \frac{1-\frac{2}{x}}{x^2-4} = 0.1250 = \frac{1}{8}$

[T] In exercises 15 - 16, set up a table of values and round to eight significant digits. Based on the table of values, make a guess about what the limit is. Then, use a calculator to graph the function and determine the limit. Was the conjecture correct? If not, why does the method of tables fail?

15) $\lim_{\theta \to 0} \sin\left(\frac{\pi}{\theta}\right)$

θ	$\sin\left(\frac{\pi}{\theta}\right)$	θ	$\sin\left(\frac{\pi}{\theta}\right)$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

16) $\lim_{\alpha \to 0^+} \frac{1}{\alpha} \cos\left(\frac{\pi}{\alpha}\right)$

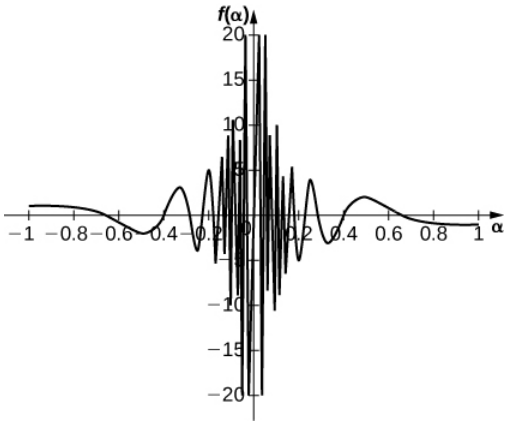
a	$\frac{1}{\alpha} \cos\left(\frac{\pi}{\alpha}\right)$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

Answer

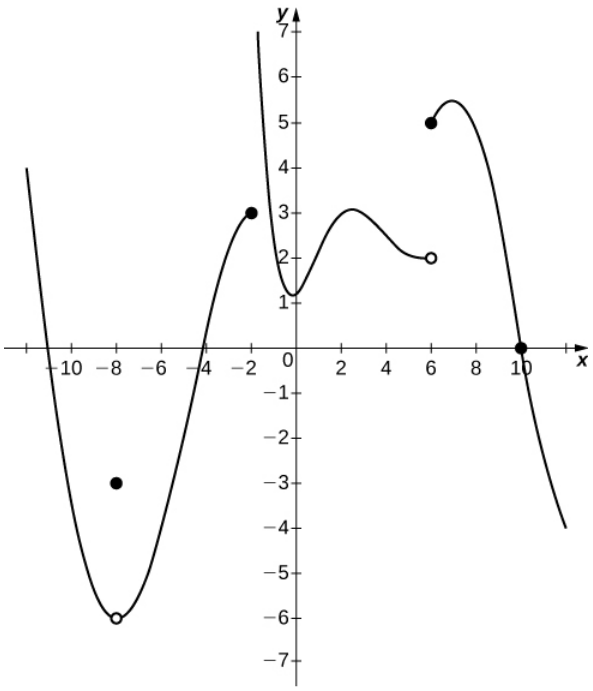
a. 10.00000; b. 100.00000; c. 1000.0000; d. 10,000.000;

Guess: $\lim_{\alpha \to 0^+} \frac{1}{\alpha} \cos\left(\frac{\pi}{\alpha}\right) = \infty$;

Actual: DNE , since the graph shows the function oscillates wildly between values approaching positive infinity and values approaching negative infinity, as the value of α approaches 0 from the positive side.



In exercises 17 - 20, consider the graph of the function $y = f(x)$ shown here. Which of the statements about $y = f(x)$ are true and which are false? Explain why a statement is false.



17) $\lim_{x\to 10} f(x) = 0$

18) $\lim_{x\to -2^+} f(x) = 3$

Answer

False; $\lim_{x\to -2^+} f(x) = +\infty$

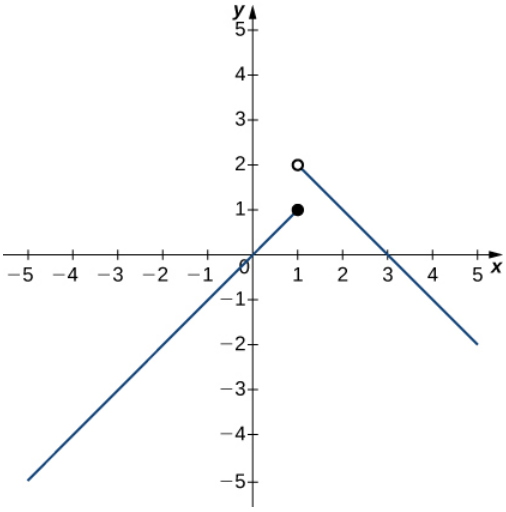
19) $\lim_{x\to -8} f(x) = f(-8)$

20) $\lim_{x\to 6} f(x) = 5$

Answer

False; $\lim_{x\to 6} f(x)$ DNE since $\lim_{x\to 6^-} f(x) = 2$ and $\lim_{x\to 6^+} f(x) = 5$.

In exercises 21 - 25, use the following graph of the function $y = f(x)$ to find the values, if possible. Estimate when necessary.



21) $\lim_{x \to 1^-} f(x)$

22) $\lim_{x \to 1^+} f(x)$

Answer

2

23) $\lim_{x \to 1} f(x)$

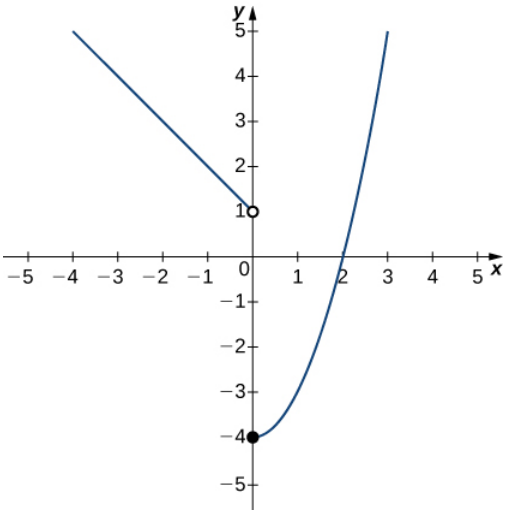
24) $\lim_{x \to 2} f(x)$

Answer

1

25) $f(1)$

In exercises 26 - 29, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



26) $\lim_{x \to 0^-} f(x)$

Answer

1

27) $\lim_{x \to 0^+} f(x)$

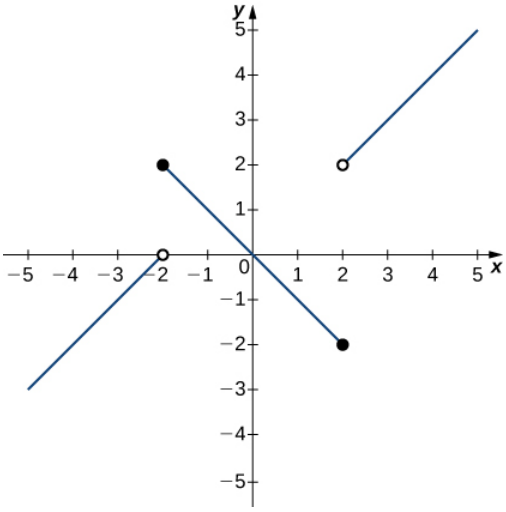
28) $\lim_{x \to 0} f(x)$

Answer

DNE

29) $\lim_{x \to 2} f(x)$

In exercises 30 - 35, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



30) $\lim_{x \to -2^-} f(x)$

Answer

0

31) $\lim_{x \to -2^+} f(x)$

32) $\lim_{x \to -2} f(x)$

Answer

DNE

33) $\lim_{x \to 2^-} f(x)$

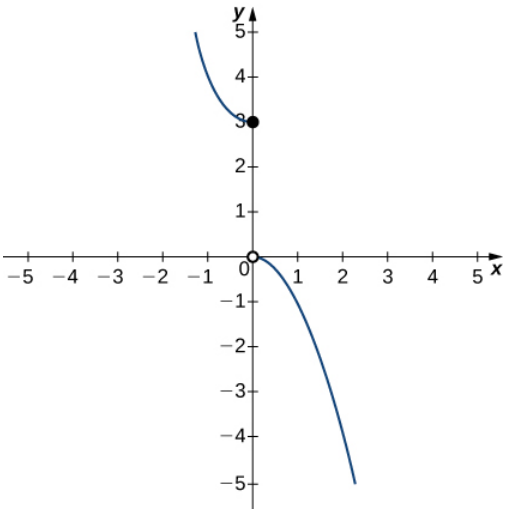
34) $\lim_{x \to 2^+} f(x)$

Answer

2

35) $\lim_{x \to 2} f(x)$

In exercises 36 - 38, use the graph of the function $y = g(x)$ shown here to find the values, if possible. Estimate when necessary.



36) $\lim_{x \to 0^-} g(x)$

Answer

3

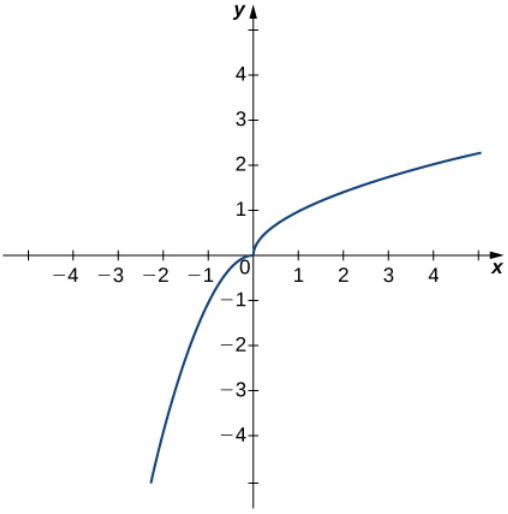
37) $\lim_{x \to 0^+} g(x)$

38) $\lim_{x \to 0} g(x)$

Answer

DNE

In exercises 39 - 41, use the graph of the function $y = h(x)$ shown here to find the values, if possible. Estimate when necessary.



39) $\lim_{x \to 0^-} h(x)$

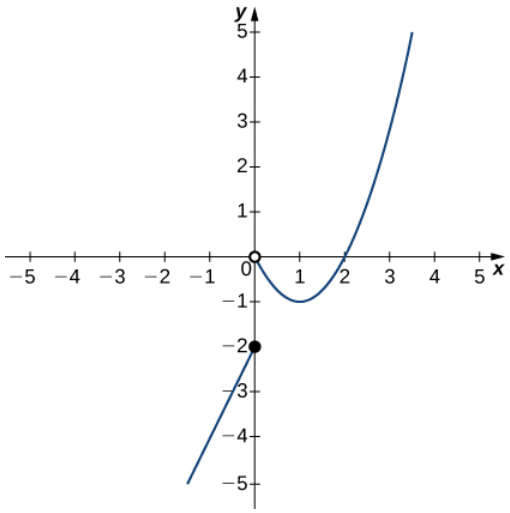
40) $\lim_{x \to 0^+} h(x)$

Answer

0

41) $\lim_{x \to 0} h(x)$

In exercises 42 - 46, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



42) $\lim_{x \to 0^-} f(x)$

Answer

-2

43) $\lim_{x \to 0^+} f(x)$

44) $\lim_{x \to 0} f(x)$

Answer

DNE

45) $\lim_{x \to 1} f(x)$

46) $\lim_{x \to 2} f(x)$

Answer

0

2.4E.2 Infinite Limits

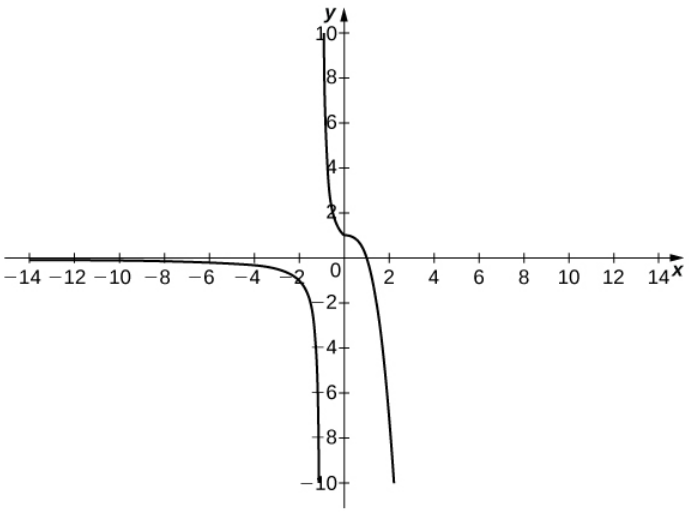
In exercises 47 - 51, sketch the graph of a function with the given properties.

47) $\lim_{x \to 2} f(x) = 1, \quad \lim_{x \to 4^-} f(x) = 3, \quad \lim_{x \to 4^+} f(x) = 6, \quad x = 4$ is not defined.

48) $\lim_{x \to -\infty} f(x) = 0, \quad \lim_{x \to -1^-} f(x) = -\infty, \quad \lim_{x \to -1^+} f(x) = \infty, \quad \lim_{x \to 0} f(x) = f(0), \quad f(0) = 1, \quad \lim_{x \to \infty} f(x) = -\infty$

Answer

Answers may vary

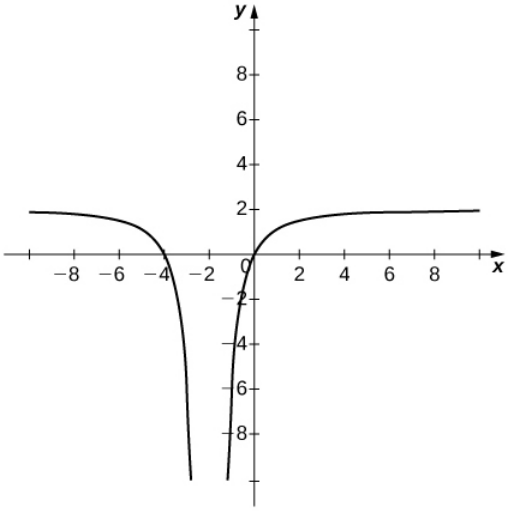


49) $\lim\limits_{x\to-\infty} f(x)=2,\quad \lim\limits_{x\to3^-} f(x)=-\infty,\quad \lim\limits_{x\to3^+} f(x)=\infty,\quad \lim\limits_{x\to\infty} f(x)=2,\quad f(0)=-\dfrac{1}{3}$

50) $\lim\limits_{x\to-\infty} f(x)=2,\quad \lim\limits_{x\to-2} f(x)=-\infty,\quad \lim\limits_{x\to\infty} f(x)=2,\quad f(0)=0$

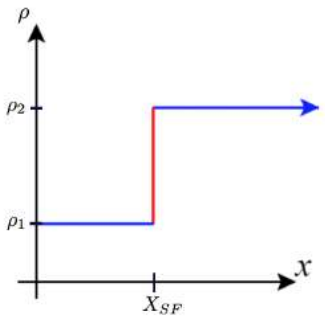
Answer

Answer may vary



51) $\lim\limits_{x\to-\infty} f(x)=0,\quad \lim\limits_{x\to-1^-} f(x)=\infty,\quad \lim\limits_{x\to-1^+} f(x)=-\infty,\quad f(0)=-1,\quad \lim\limits_{x\to1^-} f(x)=-\infty,\quad \lim\limits_{x\to1^+} f(x)=\infty,$
$\lim\limits_{x\to\infty} f(x)=0$

52) Shock waves arise in many physical applications, ranging from supernovas to detonation waves. A graph of the density of a shock wave with respect to distance, x, is shown here. We are mainly interested in the location of the front of the shock, labeled X_{SF} in the diagram.



a. Evaluate $\lim_{x \to X_{SF}^{+}} \rho(x)$.

b. Evaluate $\lim_{x \to X_{SF}^{-}} \rho(x)$.

c. Evaluate $\lim_{x \to X_{SF}} \rho(x)$. Explain the physical meanings behind your answers.

Answer

a. ρ_2 b. ρ_1 c. DNE unless $\rho_1 = \rho_2$. As you approach X_{SF} from the right, you are in the high-density area of the shock. When you approach from the left, you have not experienced the "shock" yet and are at a lower density.

53) A track coach uses a camera with a fast shutter to estimate the position of a runner with respect to time. A table of the values of position of the athlete versus time is given here, where x is the position in meters of the runner and t is time in seconds. What is $\lim_{t \to 2} x(t)$? What does it mean physically?

$t(sec)$	$x(m)$
1.75	4.5
1.95	6.1
1.99	6.42
2.01	6.58
2.05	6.9
2.25	8.5

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2.5: The Limit Laws

Learning Objectives

- Recognize the basic limit laws.
- Use the limit laws to evaluate the limit of a function.
- Evaluate the limit of a function by factoring.
- Use the limit laws to evaluate the limit of a polynomial or rational function.
- Evaluate the limit of a function by factoring or by using conjugates.

In the previous section, we evaluated limits by looking at graphs or by constructing a table of values. In this section, we establish laws for calculating limits and learn how to apply these laws. In the Student Project at the end of this section, you have the opportunity to apply these limit laws to derive the formula for the area of a circle by adapting a method devised by the Greek mathematician Archimedes. We begin by restating two useful limit results from the previous section. These two results, together with the limit laws, serve as a foundation for calculating many limits.

2.5.1 Evaluating Limits with the Limit Laws

The first two limit laws were stated previously and we repeat them here. These basic results, together with the other limit laws, allow us to evaluate limits of many algebraic functions.

Basic Limit Results

For any real number a and any constant c,

I. $\lim\limits_{x \to a} x = a$
II. $\lim\limits_{x \to a} c = c$

Example 2.5.1: Evaluating a Basic Limit

Evaluate each of the following limits using "Basic Limit Results."

a. $\lim\limits_{x \to 2} x$
b. $\lim\limits_{x \to 2} 5$

Solution

a. The limit of x as x approaches a is a: $\lim\limits_{x \to 2} x = 2$.
b. The limit of a constant is that constant: $\lim\limits_{x \to 2} 5 = 5$.

We now take a look at the **limit laws**, the individual properties of limits. The proofs that these laws hold are omitted here.

Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

- **Sum law for limits**:

$$\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$$

- **Difference law for limits**:

$$\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

- **Constant multiple law for limits**:

$$\lim_{x\to a} cf(x) = c\cdot \lim_{x\to a} f(x) = cL$$

- **Product law for limits**:

$$\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x) = L\cdot M$$

- **Quotient law for limits**:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \frac{L}{M}$$

for $M \neq 0$.

- **Power law for limits**:

$$\lim_{x\to a}\big(f(x)\big)^n = \big(\lim_{x\to a} f(x)\big)^n = L^n$$

for every positive integer n.

- **Root law for limits**:

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$$

for all L if n is odd and for $L \geq 0$ if n is even.

We now practice applying these limit laws to evaluate a limit.

Example 2.5.2A: Evaluating a Limit Using Limit Laws

Use the limit laws to evaluate

$$\lim_{x\to -3}(4x+2).$$

Solution

Let's apply the limit laws one step at a time to be sure we understand how they work. We need to keep in mind the requirement that, at each application of a limit law, the new limits must exist for the limit law to be applied.

$$\begin{aligned}
\lim_{x\to -3}(4x+2) &= \lim_{x\to -3} 4x + \lim_{x\to -3} 2 && \text{Apply the sum law.}\\
&= 4\cdot \lim_{x\to -3} x + \lim_{x\to -3} 2 && \text{Apply the constant multiple law.}\\
&= 4\cdot(-3)+2 = -10. && \text{Apply the basic limit results and simplify.}
\end{aligned}$$

Example 2.5.2B: Using Limit Laws Repeatedly

Use the limit laws to evaluate

$$\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4}.$$

Solution

To find this limit, we need to apply the limit laws several times. Again, we need to keep in mind that as we rewrite the limit in terms of other limits, each new limit must exist for the limit law to be applied.

$$\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4}=\frac{\lim_{x\to 2}(2x^2-3x+1)}{\lim_{x\to 2}(x^3+4)}$$ Apply the quotient law, make sure that $(2)^3+4\neq 0$.

$$=\frac{2\cdot\lim_{x\to 2}x^2-3\cdot\lim_{x\to 2}x+\lim_{x\to 2}1}{\lim_{x\to 2}x^3+\lim_{x\to 2}4}$$ Apply the sum law and constant multiple law.

$$=\frac{2\cdot\left(\lim_{x\to 2}x\right)^2-3\cdot\lim_{x\to 2}x+\lim_{x\to 2}1}{\left(\lim_{x\to 2}x\right)^3+\lim_{x\to 2}4}$$ Apply the power law.

$$=\frac{2(4)-3(2)+1}{(2)^3+4}=\frac{1}{4}.$$ Apply the basic limit laws and simplify.

Exercise 2.5.2

Use the limit laws to evaluate $\lim_{x\to 6}(2x-1)\sqrt{x+4}$. In each step, indicate the limit law applied.

Hint

Begin by applying the product law.

Answer

$11\sqrt{10}$

2.5.2 Limits of Polynomial and Rational Functions

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim_{x\to a}f(x)=f(a)$. This is not always true, but it does hold for all polynomials for any choice of a and for all rational functions at all values of a for which the rational function is defined.

Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then,

$$\lim_{x\to a}p(x)=p(a)$$

$$\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{p(a)}{q(a)}$$

when $q(a)\neq 0$.

To see that this theorem holds, consider the polynomial

$$p(x)=c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0.$$

By applying the sum, constant multiple, and power laws, we end up with

$$\begin{aligned}\lim_{x\to a}p(x)&=\lim_{x\to a}(c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0)\\&=c_n\left(\lim_{x\to a}x\right)^n+c_{n-1}\left(\lim_{x\to a}x\right)^{n-1}+\cdots+c_1\left(\lim_{x\to a}x\right)+\lim_{x\to a}c_0\\&=c_na^n+c_{n-1}a^{n-1}+\cdots+c_1a+c_0\\&=p(a)\end{aligned}$$

It now follows from the quotient law that if $p(x)$ and $q(x)$ are polynomials for which $q(a)\neq 0$,

then

$$\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{p(a)}{q(a)}.$$

Example 2.5.3: Evaluating a Limit of a Rational Function

Evaluate the $\lim_{x\to 3}\dfrac{2x^2-3x+1}{5x+4}$.

Solution

Since 3 is in the domain of the rational function $f(x)=\dfrac{2x^2-3x+1}{5x+4}$, we can calculate the limit by substituting 3 for x into the function. Thus,

$$\lim_{x\to 3}\frac{2x^2-3x+1}{5x+4}=\frac{10}{19}.$$

Exercise 2.5.3

Evaluate $\lim_{x\to -2}(3x^3-2x+7)$.

Hint

Use LIMITS OF POLYNOMIAL AND RATIONAL FUNCTIONS as reference

Answer

-13

2.5.3 Additional Limit Evaluation Techniques

As we have seen, we may evaluate easily the limits of polynomials and limits of some (but not all) rational functions by direct substitution. However, as we saw in the introductory section on limits, it is certainly possible for $\lim_{x\to a}f(x)$ to exist when $f(a)$ is undefined. The following observation allows us to evaluate many limits of this type:

If for all $x\neq a,\ f(x)=g(x)$ over some open interval containing a, then

$$\lim_{x\to a}f(x)=\lim_{x\to a}g(x).$$

To understand this idea better, consider the limit $\lim_{x\to 1}\dfrac{x^2-1}{x-1}$.

The function

$$f(x)=\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}$$

and the function $g(x)=x+1$ are identical for all values of $x\neq 1$. The graphs of these two functions are shown in Figure 2.5.1.

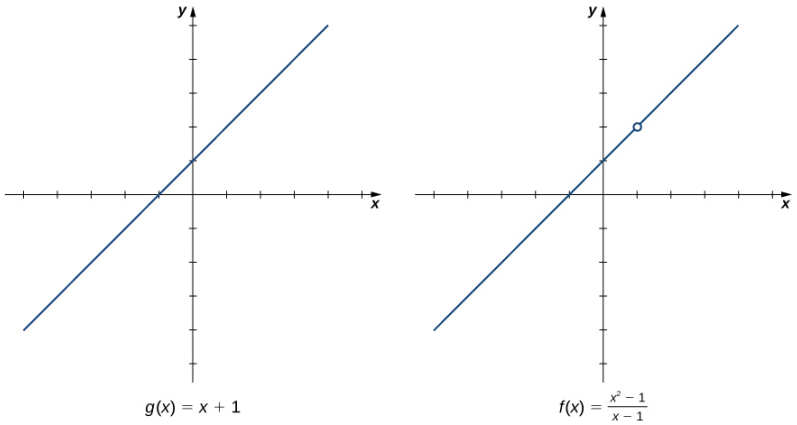


Figure 2.5.1: The graphs of $f(x)$ and $g(x)$ are identical for all $x \neq 1$. Their limits at 1 are equal.

We see that

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}=\lim_{x\to 1}(x+1)=2.$$

The limit has the form $\lim_{x\to a} f(x)/g(x)$, where $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$. (In this case, we say that $f(x)/g(x)$ has the indeterminate form $0/0$.) The following Problem-Solving Strategy provides a general outline for evaluating limits of this type.

Problem-Solving Strategy: Calculating a Limit When $f(x)/g(x)$ has the Indeterminate Form $0/0$

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
2. We then need to find a function that is equal to $h(x)=f(x)/g(x)$ for all $x \neq a$ over some interval containing a. To do this, we may need to try one or more of the following steps:
   a. If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
   b. If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
   c. If $f(x)/g(x)$ is a complex fraction, we begin by simplifying it.
3. Last, we apply the limit laws.

The next examples demonstrate the use of this Problem-Solving Strategy. Example 2.5.4 illustrates the factor-and-cancel technique; Example 2.5.5 shows multiplying by a conjugate. In Example 2.5.6, we look at simplifying a complex fraction.

Example 2.5.4: Evaluating a Limit by Factoring and Canceling

Evaluate $\lim_{x\to 3}\dfrac{x^2-3x}{2x^2-5x-3}$.

Solution

Step 1. The function $f(x)=\dfrac{x^2-3x}{2x^2-5x-3}$ is undefined for $x=3$. In fact, if we substitute 3 into the function we get $0/0$, which is undefined. Factoring and canceling is a good strategy:

$$\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3}=\lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)}$$

Step 2. For all $x \neq 3$, $\dfrac{x^2-3x}{2x^2-5x-3}=\dfrac{x}{2x+1}$. Therefore,

$$\lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)}=\lim_{x\to 3}\frac{x}{2x+1}.$$

Step 3. Evaluate using the limit laws:

$$\lim_{x\to 3}\frac{x}{2x+1}=\frac{3}{7}.$$

Exercise 2.5.4

Evaluate $\lim_{x\to -3}\dfrac{x^2+4x+3}{x^2-9}$.

Hint

Follow the steps in the Problem-Solving Strategy

Answer

$\dfrac{1}{3}$

Example 2.5.5: Evaluating a Limit by Multiplying by a Conjugate

Evaluate $\lim_{x\to -1}\dfrac{\sqrt{x+2}-1}{x+1}$.

Solution

Step 1. $\dfrac{\sqrt{x+2}-1}{x+1}$ has the form $0/0$ at -1. Let's begin by multiplying by $\sqrt{x+2}+1$, the conjugate of $\sqrt{x+2}-1$, on the numerator and denominator:

$$\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}=\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}\cdot\frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}.$$

Step 2. We then multiply out the numerator. We don't multiply out the denominator because we are hoping that the $(x+1)$ in the denominator cancels out in the end:

$$=\lim_{x\to -1}\frac{x+1}{(x+1)(\sqrt{x+2}+1)}.$$

Step 3. Then we cancel:

$$=\lim_{x\to -1}\frac{1}{\sqrt{x+2}+1}.$$

Step 4. Last, we apply the limit laws:

$$\lim_{x\to -1}\frac{1}{\sqrt{x+2}+1}=\frac{1}{2}.$$

Exercise 2.5.5

Evaluate $\lim_{x\to 5}\dfrac{\sqrt{x-1}-2}{x-5}$.

Hint

Follow the steps in the Problem-Solving Strategy

Answer

$\dfrac{1}{4}$

Example 2.5.6: Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim\limits_{x\to 1}\dfrac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}$.

Solution

Step 1. $\dfrac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}$ has the form $0/0$ at 1. We simplify the algebraic fraction by multiplying by $2(x+1)/2(x+1)$:

$$\lim_{x\to 1}\frac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}=\lim_{x\to 1}\frac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}\cdot\frac{2(x+1)}{2(x+1)}.$$

Step 2. Next, we multiply through the numerators. Do not multiply the denominators because we want to be able to cancel the factor $(x-1)$:

$$=\lim_{x\to 1}\frac{2-(x+1)}{2(x-1)(x+1)}.$$

Step 3. Then, we simplify the numerator:

$$=\lim_{x\to 1}\frac{-x+1}{2(x-1)(x+1)}.$$

Step 4. Now we factor out -1 from the numerator:

$$=\lim_{x\to 1}\frac{-(x-1)}{2(x-1)(x+1)}.$$

Step 5. Then, we cancel the common factors of $(x-1)$:

$$=\lim_{x\to 1}\frac{-1}{2(x+1)}.$$

Step 6. Last, we evaluate using the limit laws:

$$\lim_{x\to 1}\frac{-1}{2(x+1)}=-\frac{1}{4}.$$

Exercise 2.5.6

Evaluate $\lim\limits_{x\to -3}\dfrac{\dfrac{1}{x+2}+1}{x+3}$.

Hint

Follow the steps in the Problem-Solving Strategy

Answer

-1

Example 2.5.7 does not fall neatly into any of the patterns established in the previous examples. However, with a little creativity, we can still use these same techniques.

Example 2.5.7: Evaluating a Limit When the Limit Laws Do Not Apply

Evaluate $\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)$.

Solution:

Both $1/x$ and $5/x(x-5)$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that

$$\frac{1}{x}+\frac{5}{x(x-5)}=\frac{x-5+5}{x(x-5)}=\frac{x}{x(x-5)}.$$

Thus,

$$\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)=\lim_{x\to 0}\frac{x}{x(x-5)}=\lim_{x\to 0}\frac{1}{x-5}=-\frac{1}{5}.$$

Exercise 2.5.7

Evaluate $\lim_{x\to 3}\left(\frac{1}{x-3}-\frac{4}{x^2-2x-3}\right)$.

Hint

Use the same technique as Example 2.5.7 Don't forget to factor x^2-2x-3 before getting a common denominator.

Answer

$\frac{1}{4}$

Let's now revisit one-sided limits. Simple modifications in the limit laws allow us to apply them to one-sided limits. For example, to apply the limit laws to a limit of the form $\lim_{x\to a^-}h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (b,a); for a limit of the form $\lim_{x\to a^+}h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (a,c). Example 2.5.8A illustrates this point.

Example 2.5.8A: Evaluating a One-Sided Limit Using the Limit Laws

Evaluate each of the following limits, if possible.

a. $\lim_{x\to 3^-}\sqrt{x-3}$

b. $\lim_{x\to 3^+}\sqrt{x-3}$

Solution

Figure 2.5.2 illustrates the function $f(x)=\sqrt{x-3}$ and aids in our understanding of these limits.

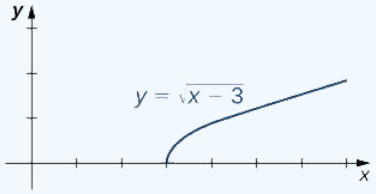


Figure 2.5.2: The graph shows the function $f(x)=\sqrt{x-3}$.

a. The function $f(x)=\sqrt{x-3}$ is defined over the interval $[3,+\infty)$. Since this function is not defined to the left of 3, we cannot apply the limit laws to compute $\lim_{x\to 3^-}\sqrt{x-3}$. In fact, since $f(x)=\sqrt{x-3}$ is undefined to the left of 3, $\lim_{x\to 3^-}\sqrt{x-3}$

does not exist.

b. Since $f(x) = \sqrt{x-3}$ is defined to the right of 3, the limit laws do apply to $\lim\limits_{x \to 3^+} \sqrt{x-3}$. By applying these limit laws we obtain $\lim\limits_{x \to 3^+} \sqrt{x-3} = 0$.

In Example $2.5.8B$ we look at one-sided limits of a piecewise-defined function and use these limits to draw a conclusion about a two-sided limit of the same function.

Example $2.5.8B$: Evaluating a Two-Sided Limit Using the Limit Laws

For $f(x) = \begin{cases} 4x-3, & \text{if } x < 2 \\ (x-3)^2, & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits:

a. $\lim\limits_{x \to 2^-} f(x)$
b. $\lim\limits_{x \to 2^+} f(x)$
c. $\lim\limits_{x \to 2} f(x)$

Solution

Figure $2.5.3$ illustrates the function $f(x)$ and aids in our understanding of these limits.

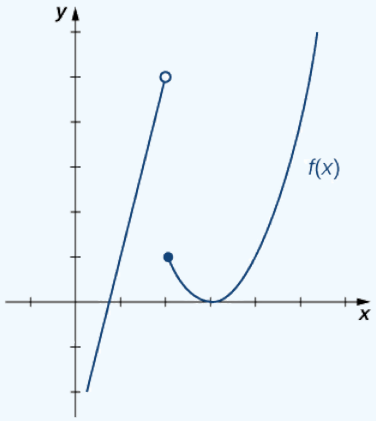


Figure 2.5.3: This graph shows a function $f(x)$.

a. Since $f(x) = 4x-3$ for all x in $(-\infty, 2)$, replace $f(x)$ in the limit with $4x-3$ and apply the limit laws:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4x-3) = 5$$

b. Since $f(x) = (x-3)^2$ for all x in $(2, +\infty)$, replace $f(x)$ in the limit with $(x-3)^2$ and apply the limit laws:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} (x-3)^2 = 1.$$

c. Since $\lim\limits_{x \to 2^-} f(x) = 5$ and $\lim\limits_{x \to 2^+} f(x) = 1$, we conclude that $\lim\limits_{x \to 2} f(x)$ does not exist.

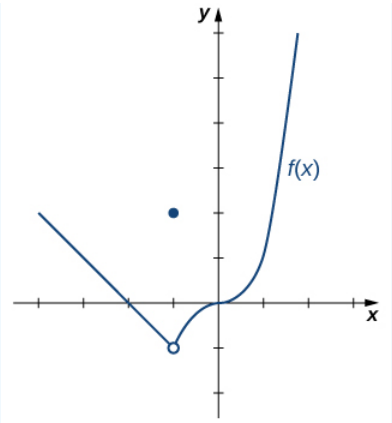
Exercise $2.5.8$

Graph $f(x) = \begin{cases} -x-2, & \text{if } x < -1 \\ 2, & \text{if } x = -1 \\ x^3, & \text{if } x > -1 \end{cases}$ and evaluate $\lim\limits_{x \to -1^-} f(x)$.

Hint

Use the method in Example $2.5.8B$ to evaluate the limit.

Answer



$$\lim_{x \to -1^-} f(x) = -1$$

We now turn our attention to evaluating a limit of the form $\lim_{x \to a} \dfrac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = K$, where $K \neq 0$ and $\lim_{x \to a} g(x) = 0$. That is, $f(x)/g(x)$ has the form $K/0, K \neq 0$ at a.

Example 2.5.9: Evaluating a Limit of the Form $K/0$, $K \neq 0$ Using the Limit Laws

Evaluate $\lim_{x \to 2^-} \dfrac{x-3}{x^2-2x}$.

Solution

Step 1. After substituting in $x = 2$, we see that this limit has the form $-1/0$. That is, as x approaches 2 from the left, the numerator approaches -1; and the denominator approaches 0. Consequently, the magnitude of $\dfrac{x-3}{x(x-2)}$ becomes infinite. To get a better idea of what the limit is, we need to factor the denominator:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = \lim_{x \to 2^-} \frac{x-3}{x(x-2)}$$

Step 2. Since $x-2$ is the only part of the denominator that is zero when 2 is substituted, we then separate $1/(x-2)$ from the rest of the function:

$$= \lim_{x \to 2^-} \frac{x-3}{x} \cdot \frac{1}{x-2}$$

Step 3. Using the Limit Laws, we can write:

$$= \left(\lim_{x \to 2^-} \frac{x-3}{x} \right) \cdot \left(\lim_{x \to 2^-} \frac{1}{x-2} \right).$$

Step 4. $\lim_{x \to 2^-} \dfrac{x-3}{x} = -\dfrac{1}{2}$ and $\lim_{x \to 2^-} \dfrac{1}{x-2} = -\infty$. Therefore, the product of $(x-3)/x$ and $1/(x-2)$ has a limit of $+\infty$:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = +\infty.$$

Exercise 2.5.9

Evaluate $\lim_{x \to 1} \dfrac{x+2}{(x-1)^2}$.

Solution

https://math.libretexts.org/@go/page/149865

Use the methods from Example 2.5.9

Answer

$+\infty$

2.5.4 Key Concepts

- The limit laws allow us to evaluate limits of functions without having to go through step-by-step processes each time.
- For polynomials and rational functions,

$$\lim_{x \to a} f(x) = f(a).$$

- You can evaluate the limit of a function by factoring and canceling, by multiplying by a conjugate, or by simplifying a complex fraction.

2.5.5 Key Equations

- **Basic Limit Results**

$$\lim_{x \to a} x = a \qquad \lim_{x \to a} c = c$$

2.5.6 Glossary

constant multiple law for limits

the limit law

$$\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$$

difference law for limits

the limit law

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

limit laws

the individual properties of limits; for each of the individual laws, let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a; assume that L and M are real numbers so that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$; let c be a constant

power law for limits

the limit law

$$\lim_{x \to a} (f(x))^n = (\lim_{x \to a} f(x))^n = L^n$$

for every positive integer n

product law for limits

the limit law

$$\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$$

quotient law for limits

the limit law $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L}{M}$ for M≠0

root law for limits

the limit law $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even

sum law for limits

The limit law $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

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2.5.1: EXERCISES FOR SECTION 2.5

In exercises 1 - 4, use the limit laws to evaluate each limit. Justify each step by indicating the appropriate limit law(s).

1) $\lim\limits_{x\to 0}(4x^2-2x+3)$

Answer

Use constant multiple law and difference law:

$\lim\limits_{x\to 0}(4x^2-2x+3)=4\lim\limits_{x\to 0}x^2-2\lim\limits_{x\to 0}x+\lim\limits_{x\to 0}3=0+0+3=3$

2) $\lim\limits_{x\to 1}\dfrac{x^3+3x^2+5}{4-7x}$

3) $\lim\limits_{x\to -2}\sqrt{x^2-6x+3}$

Answer

Use root law: $\lim\limits_{x\to -2}\sqrt{x^2-6x+3}=\sqrt{\lim\limits_{x\to -2}(x^2-6x+3)}=\sqrt{19}$

4) $\lim\limits_{x\to -1}(9x+1)^2$

In exercises 5 - 10, use direct substitution to evaluate the limit of each continuous function.

5) $\lim\limits_{x\to 7}x^2$

Answer

$\lim\limits_{x\to 7}x^2=49$

6) $\lim\limits_{x\to -2}(4x^2-1)$

7) $\lim\limits_{x\to 0}\dfrac{1}{1+\sin x}$

Answer

$\lim\limits_{x\to 0}\dfrac{1}{1+\sin x}=1$

8) $\lim\limits_{x\to 2}e^{2x-x^2}$

9) $\lim\limits_{x\to 1}\dfrac{2-7x}{x+6}$

Answer

$\lim\limits_{x\to 1}\dfrac{2-7x}{x+6}=-\dfrac{5}{7}$

10) $\lim\limits_{x\to 3}\ln e^{3x}$

In exercises 11 - 20, use direct substitution to show that each limit leads to the indeterminate form $0/0$. Then, evaluate the limit analytically.

11) $\lim\limits_{x\to 4}\dfrac{x^2-16}{x-4}$

Answer

When $x=4$, $\dfrac{x^2-16}{x-4}=\dfrac{16-16}{4-4}=\dfrac{0}{0}$;

then, $\lim\limits_{x\to 4}\dfrac{x^2-16}{x-4}=\lim\limits_{x\to 4}\dfrac{(x+4)(x-4)}{x-4}=\lim\limits_{x\to 4}(x+4)=4+4=8$

12) $\lim_{x\to 2}\dfrac{x-2}{x^2-2x}$

13) $\lim_{x\to 6}\dfrac{3x-18}{2x-12}$

Answer

When $x=6$, $\dfrac{3x-18}{2x-12}=\dfrac{18-18}{12-12}=\dfrac{0}{0}$;

then, $\lim_{x\to 6}\dfrac{3x-18}{2x-12}=\lim_{x\to 6}\dfrac{3(x-6)}{2(x-6)}=\lim_{x\to 6}\dfrac{3}{2}=\dfrac{3}{2}$

14) $\lim_{h\to 0}\dfrac{(1+h)^2-1}{h}$

15) $\lim_{t\to 9}\dfrac{t-9}{\sqrt{t}-3}$

Answer

When $t=9$, $\dfrac{t-9}{\sqrt{t}-3}=\dfrac{9-9}{3-3}=\dfrac{0}{0}$;

then, $\lim_{t\to 9}\dfrac{t-9}{\sqrt{t}-3}=\lim_{t\to 9}\dfrac{t-9}{\sqrt{t}-3}\dfrac{\sqrt{t}+3}{\sqrt{t}+3}=\lim_{t\to 9}\dfrac{(t-9)(\sqrt{t}+3)}{t-9}=\lim_{t\to 9}(\sqrt{t}+3)=\sqrt{9}+3=6$

16) $\lim_{h\to 0}\dfrac{\dfrac{1}{a+h}-\dfrac{1}{a}}{h}$, where a is a real-valued constant

17) $\lim_{\theta\to\pi}\dfrac{\sin\theta}{\tan\theta}$

Answer

When $\theta=\pi$, $\dfrac{\sin\theta}{\tan\theta}=\dfrac{\sin\pi}{\tan\pi}=\dfrac{0}{0}$;

then, $\lim_{\theta\to\pi}\dfrac{\sin\theta}{\tan\theta}=\lim_{\theta\to\pi}\dfrac{\sin\theta}{\frac{\sin\theta}{\cos\theta}}=\lim_{\theta\to\pi}\cos\theta=\cos\pi=-1$

18) $\lim_{x\to 1}\dfrac{x^3-1}{x^2-1}$

19) $\lim_{x\to 1/2}\dfrac{2x^2+3x-2}{2x-1}$

Answer

When $x=1/2$, $\dfrac{2x^2+3x-2}{2x-1}=\dfrac{\frac{1}{2}+\frac{3}{2}-2}{1-1}=\dfrac{0}{0}$;

then, $\lim_{x\to 1/2}\dfrac{2x^2+3x-2}{2x-1}=\lim_{x\to 1/2}\dfrac{(2x-1)(x+2)}{2x-1}=\lim_{x\to 1/2}(x+2)=\dfrac{1}{2}+2=\dfrac{5}{2}$

20) $\lim_{x\to -3}\dfrac{\sqrt{x+4}-1}{x+3}$

In exercises 21 - 24, use direct substitution to obtain an undefined expression. Then, use the method used in Example 9 of this section to simplify the function and determine the limit.

21) $\lim_{x\to -2^-}\dfrac{2x^2+7x-4}{x^2+x-2}$

Answer

$-\infty$

22) $\lim_{x\to -2^+} \dfrac{2x^2+7x-4}{x^2+x-2}$

23) $\lim_{x\to 1^-} \dfrac{2x^2+7x-4}{x^2+x-2}$

Answer

$-\infty$

24) $\lim_{x\to 1^+} \dfrac{2x^2+7x-4}{x^2+x-2}$

In exercises 25 - 32, assume that $\lim_{x\to 6} f(x)=4$, $\lim_{x\to 6} g(x)=9$, and $\lim_{x\to 6} h(x)=6$. Use these three facts and the limit laws to evaluate each limit.

25) $\lim_{x\to 6} 2f(x)g(x)$

Answer

$\lim_{x\to 6} 2f(x)g(x) = 2\left(\lim_{x\to 6} f(x)\right)\left(\lim_{x\to 6} g(x)\right) = 2(4)(9) = 72$

26) $\lim_{x\to 6} \dfrac{g(x)-1}{f(x)}$

27) $\lim_{x\to 6} \left(f(x)+\dfrac{1}{3}g(x)\right)$

Answer

$\lim_{x\to 6} \left(f(x)+\dfrac{1}{3}g(x)\right) = \lim_{x\to 6} f(x) + \dfrac{1}{3}\lim_{x\to 6} g(x) = 4+\dfrac{1}{3}(9) = 7$

28) $\lim_{x\to 6} \dfrac{\left(h(x)\right)^3}{2}$

29) $\lim_{x\to 6} \sqrt{g(x)-f(x)}$

Answer

$\lim_{x\to 6} \sqrt{g(x)-f(x)} = \sqrt{\lim_{x\to 6} g(x) - \lim_{x\to 6} f(x)} = \sqrt{9-4} = \sqrt{5}$

30) $\lim_{x\to 6} x\cdot h(x)$

31) $\lim_{x\to 6}[(x+1)\cdot f(x)]$

Answer

$\lim_{x\to 6}[(x+1)f(x)] = \left(\lim_{x\to 6}(x+1)\right)\left(\lim_{x\to 6} f(x)\right) = 7(4) = 28$

32) $\lim_{x\to 6}(f(x)\cdot g(x)-h(x))$

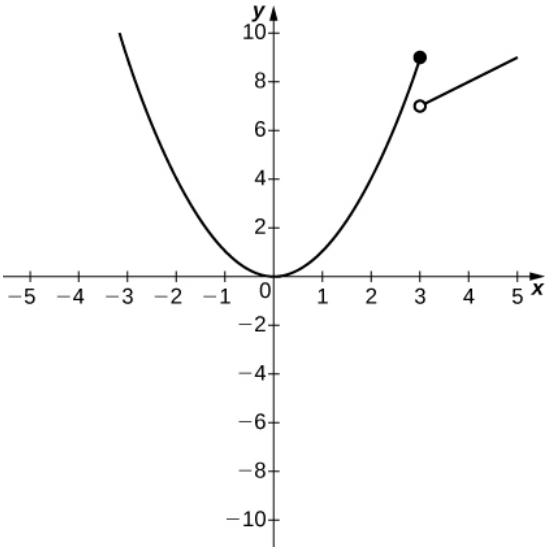
[T] In exercises 33 - 35, use a calculator to draw the graph of each piecewise-defined function and study the graph to evaluate the given limits.

33) $f(x)=\begin{cases} x^2, & x\le 3 \\ x+4, & x>3 \end{cases}$

a. $\lim_{x\to 3^-} f(x)$

b. $\lim_{x\to 3^+} f(x)$

Answer



a. 9; b.7

34) $g(x) = \begin{cases} x^3 - 1, & x \leq 0 \\ 1, & x > 0 \end{cases}$

a. $\lim\limits_{x \to 0^-} g(x)$

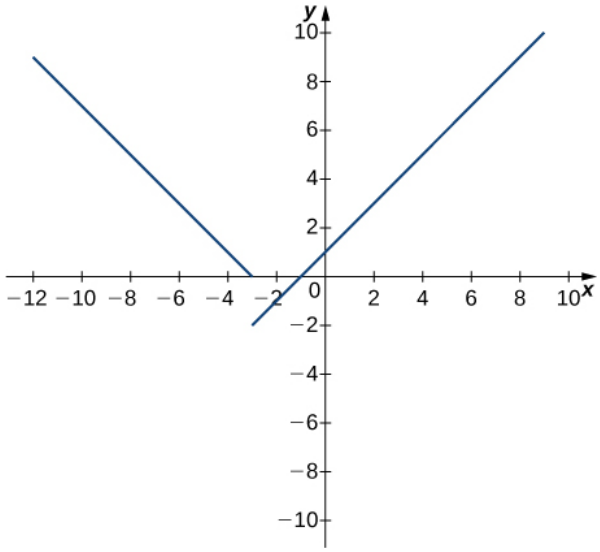
b. $\lim\limits_{x \to 0^+} g(x)$

35) $h(x) = \begin{cases} x^2 - 2x + 1, & x < 2 \\ 3 - x, & x \geq 2 \end{cases}$

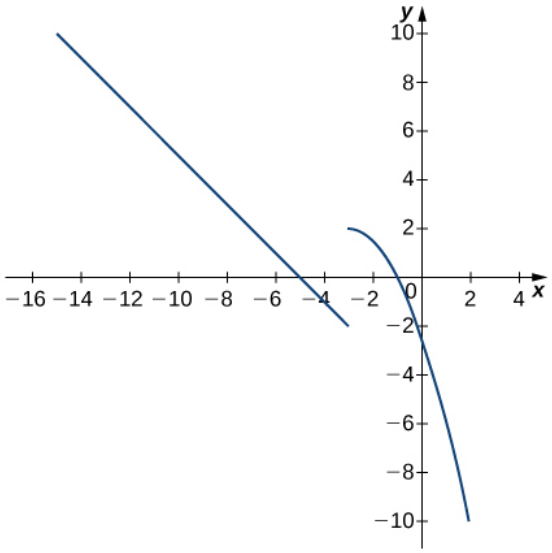
a. $\lim\limits_{x \to 2^-} h(x)$

b. $\lim\limits_{x \to 2^+} h(x)$

In exercises 36 - 43, use the following graphs and the limit laws to evaluate each limit.



$y = f(x)$



$y = g(x)$

36) $\lim_{x \to -3^+} (f(x) + g(x))$

37) $\lim_{x \to -3^-} (f(x) - 3g(x))$

Answer

$$\lim_{x \to -3^-} (f(x) - 3g(x)) = \lim_{x \to -3^-} f(x) - 3 \lim_{x \to -3^-} g(x) = 0 + 6 = 6$$

38) $\lim_{x \to 0} \dfrac{f(x)g(x)}{3}$

39) $\lim_{x \to -5} \dfrac{2 + g(x)}{f(x)}$

Answer

$$\lim_{x \to -5} \frac{2 + g(x)}{f(x)} = \frac{2 + \left(\lim_{x \to -5} g(x) \right)}{\lim_{x \to -5} f(x)} = \frac{2 + 0}{2} = 1$$

40) $\lim_{x\to 1}(f(x))^2$

41) $\lim_{x\to 1}\sqrt[3]{f(x)-g(x)}$

Answer

$$\lim_{x\to 1}\sqrt[3]{f(x)-g(x)} = \sqrt[3]{\lim_{x\to 1} f(x) - \lim_{x\to 1} g(x)} = \sqrt[3]{2+5} = \sqrt[3]{7}$$

42) $\lim_{x\to -7}(x\cdot g(x))$

43) $\lim_{x\to -9}[x\cdot f(x)+2\cdot g(x)]$

Answer

$$\lim_{x\to -9}(xf(x)+2g(x)) = \left(\lim_{x\to -9} x\right)\left(\lim_{x\to -9} f(x)\right) + 2\lim_{x\to -9} g(x) = (-9)(6)+2(4) = -46$$

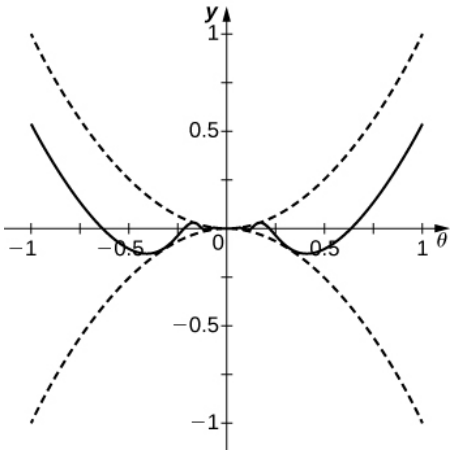
For exercises 44 - 46, evaluate the limit using the squeeze theorem. Use a calculator to graph the functions $f(x)$, $g(x)$, and $h(x)$ when possible.

44) [T] True or False? If $2x-1 \le g(x) \le x^2-2x+3$, then $\lim_{x\to 2} g(x) = 0$.

45) [T] $\lim_{\theta\to 0}\theta^2\cos\left(\frac{1}{\theta}\right)$

Answer

The limit is zero.



46) $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} 0, & x \text{ rational} \\ x^2, & x \text{ irrrational} \end{cases}$

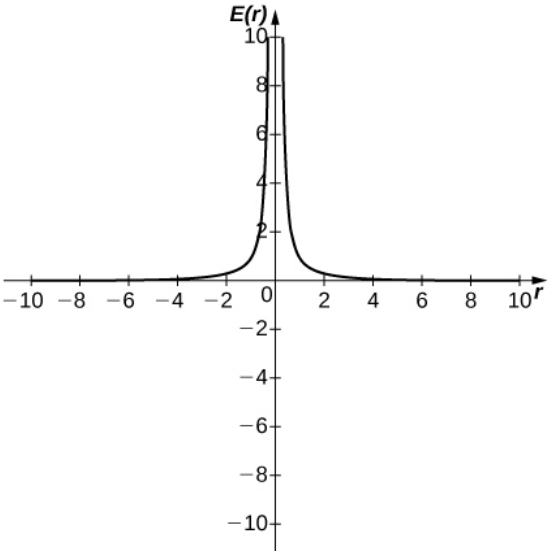
47) [T] In physics, the magnitude of an electric field generated by a point charge at a distance r in vacuum is governed by Coulomb's law: $E(r) = \frac{q}{4\pi\varepsilon_0 r^2}$, where E represents the magnitude of the electric field, q is the charge of the particle, r is the distance between the particle and where the strength of the field is measured, and $\frac{1}{4\pi\varepsilon_0}$ is Coulomb's constant: $8.988\times 109 N\cdot m^2/C^2$.

a. Use a graphing calculator to graph $E(r)$ given that the charge of the particle is $q = 10^{-10}$.

b. Evaluate $\lim_{r\to 0^+} E(r)$. What is the physical meaning of this quantity? Is it physically relevant? Why are you evaluating from the right?

Answer

a.



b. ∞. The magnitude of the electric field as you approach the particle q becomes infinite. It does not make physical sense to evaluate negative distance.

48) [T] The density of an object is given by its mass divided by its volume: $\rho = m/V$.

a. Use a calculator to plot the volume as a function of density $(V = m/\rho)$, assuming you are examining something of mass 8 kg ($m = 8$).

b. Evaluate $\lim_{x \to 0^+} V(\rho)$ and explain the physical meaning.

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2.6: Continuity

Learning Objectives

- Explain the three conditions for continuity at a point.
- Describe three kinds of discontinuities.
- Define continuity on an interval.
- Provide an example of the intermediate value theorem.

Many functions have the property that their graphs can be traced with a pencil without lifting the pencil from the page. Such functions are called *continuous*. Other functions have points at which a break in the graph occurs, but satisfy this property over intervals contained in their domains. They are continuous on these intervals and are said to have a *discontinuity* at a point where a break occurs.

We begin our investigation of continuity by exploring what it means for a function to have *continuity at a point*. Intuitively, a function is continuous at a particular point if there is no break in its graph at that point.

2.6.1 Continuity at a Point

Before we look at a formal definition of what it means for a function to be continuous at a point, let's consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point. We then create a list of conditions that prevent such failures.

Our first function of interest is shown in Figure 2.6.1. We see that the graph of $f(x)$ has a hole at a. In fact, $f(a)$ is undefined. At the very least, for $f(x)$ to be continuous at a, we need the following condition:

2.6.1.1 i. $f(a)$ is defined

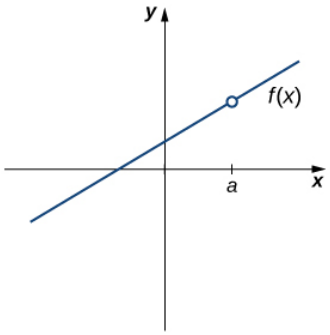


Figure 2.6.1: The function $f(x)$ is not continuous at a because $f(a)$ is undefined.

However, as we see in Figure 2.6.2, this condition alone is insufficient to guarantee continuity at the point a. Although $f(a)$ is defined, the function has a gap at a. In this example, the gap exists because $\lim\limits_{x \to a} f(x)$ does not exist. We must add another condition for continuity at a—namely,

2.6.1.2 ii. $\lim\limits_{x \to a} f(x)$ exists

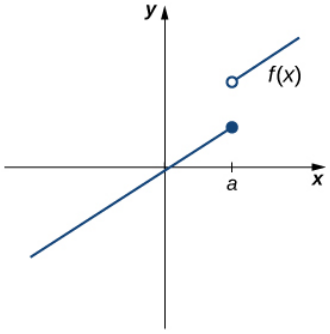


Figure 2.6.2: The function $f(x)$ is not continuous at a because $\lim\limits_{x \to a} f(x)$ does not exist.

However, as we see in Figure 2.6.3, these two conditions by themselves do not guarantee continuity at a point. The function in this figure satisfies both of our first two conditions, but is still not continuous at a. We must add a third condition to our list:

2.6.1.3 iii. $\lim_{x\to a} f(x) = f(a)$

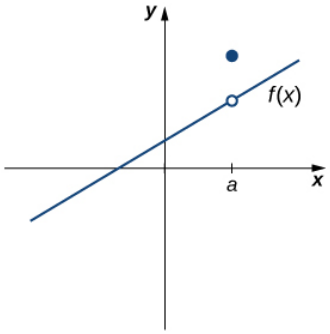


Figure 2.6.3: The function $f(x)$ is not continuous at a because $\lim_{x\to a} f(x) \neq f(a)$.

Now we put our list of conditions together and form a definition of continuity at a point.

Definition: Continuous at a Point

A function $f(x)$ is **continuous at a point a** if and only if the following three conditions are satisfied:

i. $f(a)$ is defined
ii. $\lim_{x\to a} f(x)$ exists
iii. $\lim_{x\to a} f(x) = f(a)$

A function is **discontinuous at a point a** if it fails to be continuous at a.

The following procedure can be used to analyze the continuity of a function at a point using this definition.

Problem-Solving Strategy: Determining Continuity at a Point

1. Check to see if $f(a)$ is defined. If $f(a)$ is undefined, we need go no further. The function is not continuous at a. If $f(a)$ is defined, continue to step 2.
2. Compute $\lim_{x\to a} f(x)$. In some cases, we may need to do this by first computing $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$. If $\lim_{x\to a} f(x)$ does not exist (that is, it is not a real number), then the function is not continuous at a and the problem is solved. If $\lim_{x\to a} f(x)$ exists, then continue to step 3.
3. Compare $f(a)$ and $\lim_{x\to a} f(x)$. If $\lim_{x\to a} f(x) \neq f(a)$, then the function is not continuous at a. If $\lim_{x\to a} f(x) = f(a)$, then the function is continuous at a.

The next three examples demonstrate how to apply this definition to determine whether a function is continuous at a given point. These examples illustrate situations in which each of the conditions for continuity in the definition succeed or fail.

Example 2.6.1A: Determining Continuity at a Point, Condition 1

Using the definition, determine whether the function $f(x) = \dfrac{x^2-4}{x-2}$ is continuous at $x = 2$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(2)$. We can see that $f(2) = 0/0$, which is undefined. Therefore, $f(x) = \dfrac{x^2-4}{x-2}$ is discontinuous at 2 because $f(2)$ is undefined. The graph of $f(x)$ is shown in Figure 2.6.4.

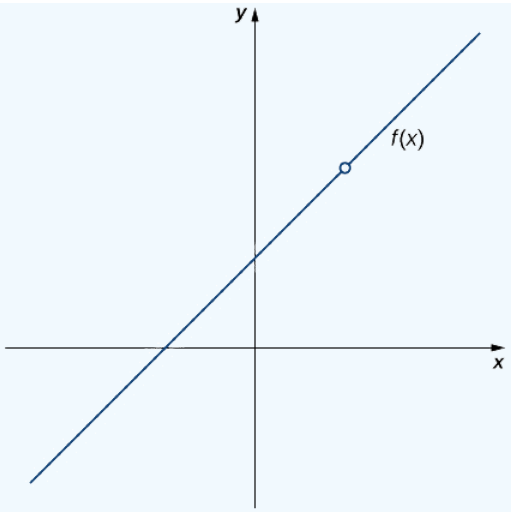


Figure 2.6.4: The function $f(x)$ is discontinuous at 2 because $f(2)$ is undefined.

Example 2.6.1B: Determining Continuity at a Point, Condition 2

Using the definition, determine whether the function $f(x)=\begin{cases} -x^2+4, & \text{if } x\leq 3 \\ 4x-8, & \text{if } x>3 \end{cases}$ is continuous at $x=3$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(3)$.

$$f(3)=-(3^2)+4=-5\ .$$

Thus, $f(3)$ is defined. Next, we calculate $\lim\limits_{x\to 3} f(x)$. To do this, we must compute $\lim\limits_{x\to 3^-} f(x)$ and $\lim\limits_{x\to 3^+} f(x)$:

$$\lim_{x\to 3^-} f(x)=-(3^2)+4=-5$$

and

$$\lim_{x\to 3^+} f(x)=4(3)-8=4\ .$$

Therefore, $\lim\limits_{x\to 3} f(x)$ does not exist. Thus, $f(x)$ is not continuous at 3. The graph of $f(x)$ is shown in Figure 2.6.5.

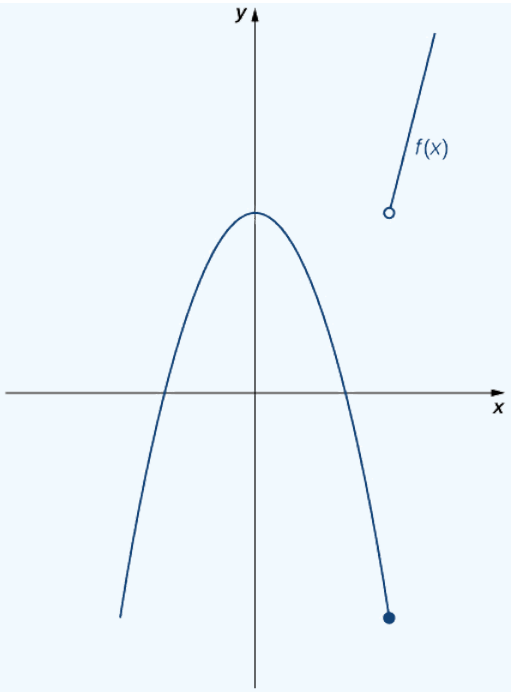


Figure 2.6.5: The function $f(x)$ is not continuous at 3 because $\lim_{x\to 3} f(x)$ does not exist.

Example 2.6.1C: Determining Continuity at a Point, Condition 3

Using the definition, determine whether the function $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

Solution

First, observe that

$$f(0) = 1$$

Next,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1\,.$$

Last, compare $f(0)$ and $\lim_{x\to 0} f(x)$. We see that

$$f(0) = 1 = \lim_{x\to 0} f(x).$$

Since all three of the conditions in the definition of continuity are satisfied, $f(x)$ is continuous at $x = 0$.

Exercise 2.6.1

Using the definition, determine whether the function $f(x) = \begin{cases} 2x+1, & \text{if } x < 1 \\ 2, & \text{if } x = 1 \\ -x+4, & \text{if } x > 1 \end{cases}$ is continuous at $x = 1$. If the function is not continuous at 1, indicate the condition for continuity at a point that fails to hold.

Hint

Check each condition of the definition.

Answer

f is not continuous at 1 because $f(1) = 2 \neq 3 = \lim_{x\to 1} f(x)$.

By applying the definition of continuity and previously established theorems concerning the evaluation of limits, we can state the following theorem.

Theorem 2.6.1: Continuity of Polynomials and Rational Functions

Polynomials and rational functions are continuous at every point in their domains.

Proof

Previously, we showed that if $p(x)$ and $q(x)$ are polynomials, $\lim_{x\to a} p(x) = p(a)$ for every polynomial $p(x)$ and $\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ as long as $q(a) \neq 0$. Therefore, polynomials and rational functions are continuous on their domains.

□

We now apply Theorem 2.6.1 to determine the points at which a given rational function is continuous.

Example 2.6.2:Continuity of a Rational Function

For what values of x is $f(x) = \frac{x+1}{x-5}$ continuous?

Solution

The rational function $f(x) = \frac{x+1}{x-5}$ is continuous for every value of x except $x = 5$.

Exercise 2.6.2

For what values of x is $f(x) = 3x^4 - 4x^2$ continuous?

Hint

Use the Continuity of Polynomials and Rational Functions stated above.

Answer

$f(x)$ is continuous at every real number.

2.6.2 Types of Discontinuities

As we have seen in Example 2.6.1A and Example 2.6.1B, discontinuities take on several different appearances. We classify the types of discontinuities we have seen thus far as removable discontinuities, infinite discontinuities, or jump discontinuities. Intuitively, a **removable discontinuity** is a discontinuity for which there is a hole in the graph, a **jump discontinuity** is a noninfinite discontinuity for which the sections of the function do not meet up, and an **infinite discontinuity** is a discontinuity located at a vertical asymptote. Figure 2.6.6 illustrates the differences in these types of discontinuities. Although these terms provide a handy way of describing three common types of discontinuities, keep in mind that not all discontinuities fit neatly into these categories.

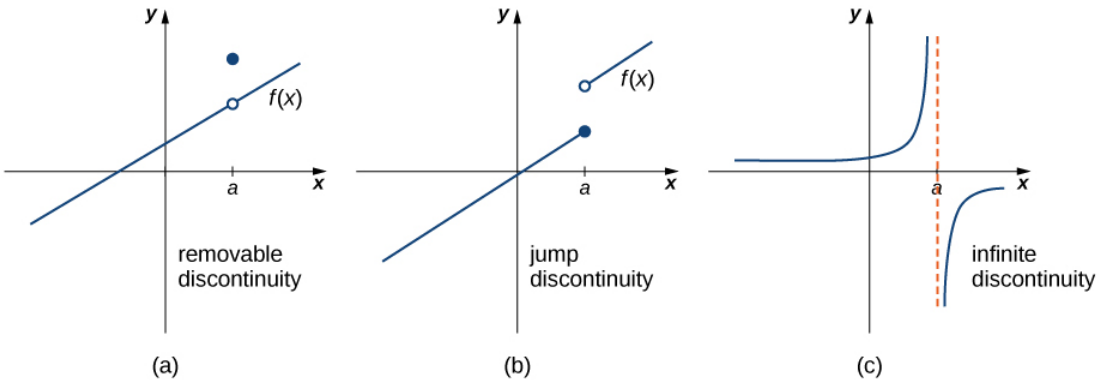


Figure 2.6.6: Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.

These three discontinuities are formally defined as follows:

Definition

If $f(x)$ is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim\limits_{x\to a} f(x)$ exists. (Note: When we state that $\lim\limits_{x\to a} f(x)$ exists, we mean that $\lim\limits_{x\to a} f(x) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist, but $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$. (Note: When we state that $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an **infinite discontinuity** at a if $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$.

Example 2.6.3: Classifying a Discontinuity

In Example 2.6.1A, we showed that $f(x) = \dfrac{x^2-4}{x-2}$ is discontinuous at $x = 2$. Classify this discontinuity as removable, jump, or infinite.

Solution

To classify the discontinuity at 2 we must evaluate $\lim\limits_{x\to 2} f(x)$:

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2} \frac{x^2-4}{x-2} \\ &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2} (x+2) \\ &= 4.\end{aligned}$$

Since f is discontinuous at 2 and $\lim\limits_{x\to 2} f(x)$ exists, f has a removable discontinuity at $x = 2$.

Example 2.6.4: Classifying a Discontinuity

In Example 2.6.1B, we showed that $f(x) = \begin{cases} -x^2+4, & \text{if } x \leq 3 \\ 4x-8, & \text{if } x > 3 \end{cases}$ is discontinuous at $x = 3$. Classify this discontinuity as removable, jump, or infinite.

Solution

Earlier, we showed that f is discontinuous at 3 because $\lim\limits_{x\to 3} f(x)$ does not exist. However, since $\lim\limits_{x\to 3^-} f(x) = -5$ and $\lim\limits_{x\to 3^+} f(x) = 4$ both exist, we conclude that the function has a jump discontinuity at 3.

Example 2.6.5: Classifying a Discontinuity

Determine whether $f(x) = \dfrac{x+2}{x+1}$ is continuous at -1. If the function is discontinuous at -1, classify the discontinuity as removable, jump, or infinite.

Solution

The function value $f(-1)$ is undefined. Therefore, the function is not continuous at -1. To determine the type of discontinuity, we must determine the limit at -1. We see that $\lim\limits_{x\to -1^-} \dfrac{x+2}{x+1} = -\infty$ and $\lim\limits_{x\to -1^+} \dfrac{x+2}{x+1} = +\infty$. Therefore, the function has an infinite discontinuity at -1.

Exercise 2.6.3

For $f(x) = \begin{cases} x^2, & \text{if } x \neq 1 \\ 3, & \text{if } x = 1 \end{cases}$, decide whether f is continuous at 1. If f is not continuous at 1, classify the discontinuity as removable, jump, or infinite.

Hint

Consider the definitions of the various kinds of discontinuity stated above. If the function is discontinuous at 1, look at $\lim\limits_{x\to 1} f(x)$

Answer

Discontinuous at 1; removable

2.6.3 Continuity over an Interval

Now that we have explored the concept of continuity at a point, we extend that idea to **continuity over an interval**. As we develop this idea for different types of intervals, it may be useful to keep in mind the intuitive idea that a function is continuous over an interval if we can use a pencil to trace the function between any two points in the interval without lifting the pencil from the paper. In preparation for defining continuity on an interval, we begin by looking at the definition of what it means for a function to be continuous from the right at a point and continuous from the left at a point.

Definition: Continuity from the Right and from the Left

A function $f(x)$ is said to be **continuous from the right** at a if $\lim\limits_{x\to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be **continuous from the left** at a if $\lim\limits_{x\to a^-} f(x) = f(a)$

A function is continuous over an open interval if it is continuous at every point in the interval. A function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous from the right at a and is continuous from the left at b. Analogously, a function $f(x)$ is continuous over an interval of the form $(a, b]$ if it is continuous over (a, b) and is continuous from the left at b. Continuity over other types of intervals are defined in a similar fashion.

Requiring that $\lim\limits_{x\to a^+} f(x) = f(a)$ and $\lim\limits_{x\to b^-} f(x) = f(b)$ ensures that we can trace the graph of the function from the point $(a, f(a))$ to the point $(b, f(b))$ without lifting the pencil. If, for example, $\lim\limits_{x\to a^+} f(x) \neq f(a)$, we would need to lift our pencil to jump from $f(a)$ to the graph of the rest of the function over $(a, b]$.

Example 2.6.6: Continuity on an Interval

State the interval(s) over which the function $f(x) = \dfrac{x-1}{x^2+2x}$ is continuous.

Solution

Since $f(x) = \dfrac{x-1}{x^2+2x}$ is a rational function, it is continuous at every point in its domain. The domain of $f(x)$ is the set $(-\infty, -2) \cup (-2, 0) \cup (0, +\infty)$. Thus, $f(x)$ is continuous over each of the intervals $(-\infty, -2)$, $(-2, 0)$, and $(0, +\infty)$.

Example 2.6.7: Continuity over an Interval

State the interval(s) over which the function $f(x) = \sqrt{4-x^2}$ is continuous.

Solution

From the limit laws, we know that $\lim\limits_{x \to a} \sqrt{4-x^2} = \sqrt{4-a^2}$ for all values of a in $(-2, 2)$. We also know that $\lim\limits_{x \to -2^+} \sqrt{4-x^2} = 0$ exists and $\lim\limits_{x \to 2^-} \sqrt{4-x^2} = 0$ exists. Therefore, $f(x)$ is continuous over the interval $[-2, 2]$.

Exercise 2.6.4

State the interval(s) over which the function $f(x) = \sqrt{x+3}$ is continuous.

Hint

Use Example 2.6.7as a guide.

Answer

$[-3, +\infty)$

2.6.4 The Intermediate Value Theorem

Functions that are continuous over intervals of the form $[a, b]$, where a and b are real numbers, exhibit many useful properties. Throughout our study of calculus, we will encounter many powerful theorems concerning such functions. The first of these theorems is the **Intermediate Value Theorem.**

The Intermediate Value Theorem

Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$ in Figure 2.6.7.

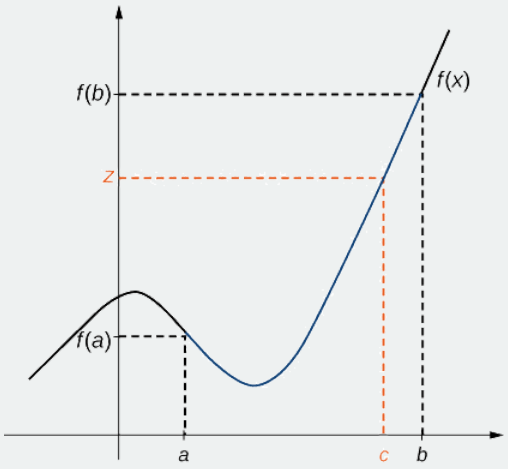


Figure 2.6.7: There is a number $c \in [a, b]$ that satisfies $f(c) = z$.

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Example 2.6.9: Application of the Intermediate Value Theorem

Show that $f(x) = x^5 + x^2 + 5$ has at least one zero.

Solution

Since $f(x) = x^5 + x^2 + 5$ is continuous over $(-\infty, +\infty)$, it is continuous over any closed interval of the form $[a, b]$. If you can find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs, you can use the Intermediate Value Theorem to conclude there must be a real number c in (a, b) that satisfies $f(c) = 0$. Note that

$$f(-2) = -23 < 0$$

and

$$f(-1) = 5 > 0\,.$$

Using the Intermediate Value Theorem, we can see that there must be a real number c in $[-2, -1]$ that satisfies $f(c) = 0$. Therefore, $f(x) = x^5 + x^2 + 5$ has at least one zero.

Example 2.6.10: When Can You Apply the Intermediate Value Theorem?

If $f(x)$ is continuous over $[0, 2]$, $f(0) > 0$ and $f(2) > 0$, can we use the Intermediate Value Theorem to conclude that $f(x)$ has no zeros in the interval $[0, 2]$? Explain.

Solution

No. The Intermediate Value Theorem only allows us to conclude that we can find a value between $f(0)$ and $f(2)$; it doesn't allow us to conclude that we can't find other values. To see this more clearly, consider the function $f(x) = (x-1)^2$. It satisfies $f(0) = 1 > 0$, $f(2) = 1 > 0$, and $f(1) = 0$.

Example 2.6.11: When Can You Apply the Intermediate Value Theorem?

For $f(x) = 1/x$, $f(-1) = -1 < 0$ and $f(1) = 1 > 0$. Can we conclude that $f(x)$ has a zero in the interval $[-1, 1]$?

Solution

No. The function is not continuous over $[-1, 1]$. The Intermediate Value Theorem does not apply here.

Exercise 2.6.5

Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero over the interval $[0, 1]$.

Hint

Find $f(0)$ and $f(1)$. Apply the Intermediate Value Theorem.

Answer

$f(0) = 1 > 0,\ f(1) = -2 < 0;\ f(x)$ is continuous over $[0, 1]$. It must have a zero on this interval.

2.6.5 Key Concepts

- For a function to be continuous at a point, it must be defined at that point, its limit must exist at the point, and the value of the function at that point must equal the value of the limit at that point.
- Discontinuities may be classified as removable, jump, or infinite.
- A function is continuous over an open interval if it is continuous at every point in the interval. It is continuous over a closed interval if it is continuous at every point in its interior and is continuous at its endpoints.
- The Intermediate Value Theorem guarantees that if a function is continuous over a closed interval, then the function takes on every value between the values at its endpoints.

2.6.6 Glossary

continuity at a point

A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied: (1) $f(a)$ is defined, (2) $\lim\limits_{x \to a} f(x)$ exists, and (3) $\lim x \to a f(x) = f(a)$

continuity from the left

A function is continuous from the left at b if $\lim\limits_{x \to b^-} f(x) = f(b)$

continuity from the right

A function is continuous from the right at a if $\lim\limits_{x \to a^+} f(x) = f(a)$

continuity over an interval

a function that can be traced with a pencil without lifting the pencil; a function is continuous over an open interval if it is continuous at every point in the interval; a function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b), and it is continuous from the right at a and from the left at b

discontinuity at a point

A function is discontinuous at a point or has a discontinuity at a point if it is not continuous at the point

infinite discontinuity

An infinite discontinuity occurs at a point a if $\lim\limits_{x \to a^-} f(x) = \pm\infty$ or $\lim\limits_{x \to a^+} f(x) = \pm\infty$

Intermediate Value Theorem

Let f be continuous over a closed bounded interval $[a, b]$ if z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$

jump discontinuity

A jump discontinuity occurs at a point a if $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ both exist, but $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$

removable discontinuity

A removable discontinuity occurs at a point a if $f(x)$ is discontinuous at a, but $\lim\limits_{x \to a} f(x)$ exists

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2.6E: EXERCISES FOR SECTION 2.6

For exercises 1 - 7, determine the point(s), if any, at which each function is discontinuous. Classify any discontinuity as *jump, removable, infinite,* or *other*.

1) $f(x) = \dfrac{1}{\sqrt{x}}$

Answer

The function is defined for all x in the interval $(0, \infty)$.

2) $f(x) = \dfrac{2}{x^2 + 1}$

3) $f(x) = \dfrac{x}{x^2 - x}$

Answer

Removable discontinuity at $x = 0$; infinite discontinuity at $x = 1$.

4) $g(t) = t^{-1} + 1$

5) $f(x) = \dfrac{5}{e^x - 2}$

Answer

Infinite discontinuity at $x = \ln 2$

6) $f(x) = \dfrac{|x - 2|}{x - 2}$

7) $f(t) = \dfrac{t + 3}{t^2 + 5t + 6}$

For exercises 8 - 13, decide if the function continuous at the given point. If it is discontinuous, what type of discontinuity is it?

8) $\dfrac{2x^2 - 5x + 3}{x - 1}$ at $x = 1$

Answer

No. It is a removable discontinuity.

9) $h(t) = \dfrac{2t + 1}{t - 4}$ at $t = 4$

10) $g(u) = \begin{cases} \dfrac{6u^2 + u - 2}{2u - 1}, & \text{if } u \neq \frac{1}{2} \\ \frac{7}{2}, & \text{if } u = \frac{1}{2} \end{cases}$, at $u = \frac{1}{2}$

Answer

Yes. It is continuous.

11) $f(y) = \dfrac{y^2 - y}{5y - 5}$, at $y = 1$

12) $f(x) = \begin{cases} x^2 - e^x, & \text{if } x < 0 \\ x - 1, & \text{if } x \geq 0 \end{cases}$, at $x = 0$

Answer

Yes. It is continuous.

13) $f(x) = \dfrac{x}{x^2 + 1}$, at $x = 0$

In exercises 14- 17, find the value(s) of k that makes each function continuous over the given interval.

14) $f(x) = \begin{cases} 3x + 2, & \text{if } x < k \\ 2x - 3, & \text{if } k \leq x \leq 8 \end{cases}$

Answer

$k = -5$

15) $f(x) = \begin{cases} \dfrac{x^2+3x+2}{x+2}, & \text{if } x \neq -2 \\ k, & \text{if } x = -2 \end{cases}$

Answer

$k = -1$

16) $f(x) = \begin{cases} e^{kx}, & \text{if } 0 \leq x < 4 \\ x+3, & \text{if } 4 \leq x \leq 8 \end{cases}$

17) $f(x) = \begin{cases} \sqrt{kx}, & \text{if } 0 \leq x \leq 3 \\ x+1, & \text{if } 3 < x \leq 10 \end{cases}$

Answer

$k = \frac{16}{3}$

In exercises 18-19, use the Intermediate Value Theorem (IVT).

18) Let $h(x) = \begin{cases} 3x^2-4, & \text{if } x \leq 2 \\ 5+4x, & \text{if } x > 2 \end{cases}$ Over the interval $[0,4]$, there is no value of x such that $h(x) = 10$, although $h(0) < 10$ and $h(4) > 10$. Explain why this does not contradict the IVT.

19) A particle moving along a line for time t has a position function $s(t)$, which is continuous. Assume $s(2) = 5$ and $s(5) = 2$. Another particle moves such that its position is given by $h(t) = s(t) - t$. Explain why there must be a value c for $2 < c < 5$ such that $h(c) = 0$.

Answer

Since both s and $y = t$ are continuous everywhere, then $h(t) = s(t) - t$ is continuous everywhere and, in particular, it is continuous over the closed interval $[2,5]$. Also, $h(2) = 3 > 0$ and $h(5) = -3 < 0$. Therefore, by the IVT, there is a value $x = c$ such that $h(c) = 0$.

20) [T] Use the statement “The cosine of t is equal to t cubed."

a. Write a mathematical equation of the statement.

b. Prove that the equation in part a. has at least one real solution.

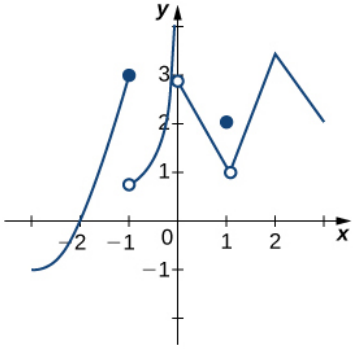
c. Use a calculator to find an interval of length 0.01 that contains a solution.

21) Apply the IVT to determine whether $2^x = x^3$ has a solution in one of the intervals $[1.25, 1.375]$ or $[1.375, 1.5]$. Briefly explain your response for each interval.

Answer

The function $f(x) = 2^x - x^3$ is continuous over the interval $[1.25, 1.375]$ and has opposite signs at the endpoints.

22) Consider the graph of the function $y = f(x)$ shown in the following graph.



a. Find all values for which the function is discontinuous.

b. For each value in part a., state why the formal definition of continuity does not apply.

c. Classify each discontinuity as either jump, removable, or infinite.

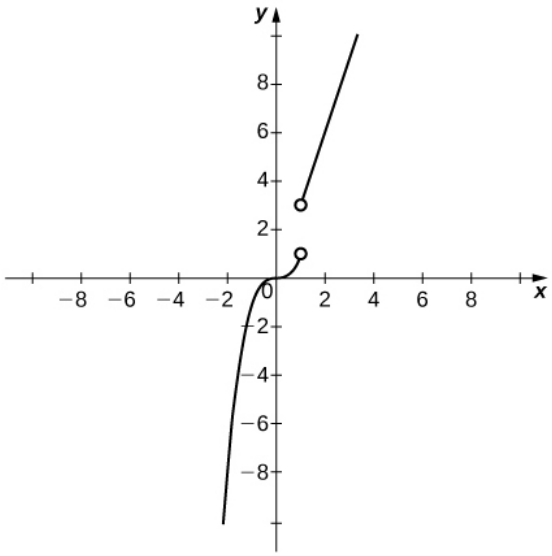
23) Let $f(x) = \begin{cases} 3x, & \text{if } x > 1 \\ x^3, & \text{if } x < 1 \end{cases}$.

a. Sketch the graph of f.

b. Is it possible to find a value k such that $f(1) = k$, which makes $f(x)$ continuous for all real numbers? Briefly explain.

Answer

a.



b. It is not possible to redefine $f(1)$ since the discontinuity is a jump discontinuity.

24) Let $f(x) = \dfrac{x^4 - 1}{x^2 - 1}$ for $x \neq -1, 1$.

a. Sketch the graph of f.

b. Is it possible to find values k_1 and k_2 such that $f(-1) = k$ and $f(1) = k_2$, and that makes $f(x)$ continuous for all real numbers? Briefly explain.

25) Sketch the graph of the function $y = f(x)$ with properties i. through vii.

i. The domain of f is $(-\infty, +\infty)$.

ii. f has an infinite discontinuity at $x = -6$.

iii. $f(-6) = 3$

iv. $\lim\limits_{x \to -3^-} f(x) = \lim\limits_{x \to -3^+} f(x) = 2$

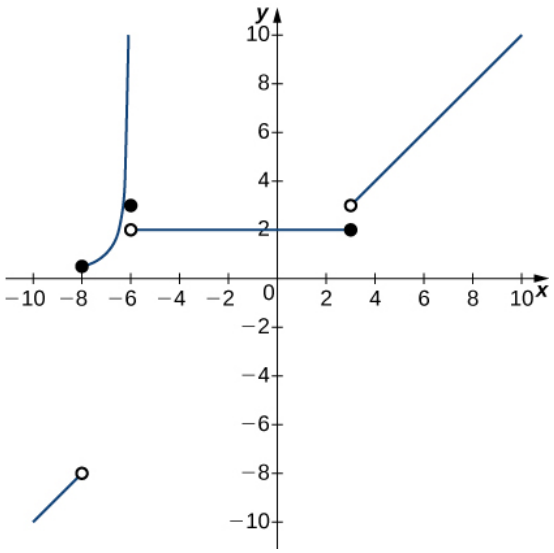
v. $f(-3) = 3$

vi. f is left continuous but not right continuous at $x = 3$.

vii. $\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to +\infty} f(x) = +\infty$

Answer

Answers may vary; see the following example:



26) Sketch the graph of the function $y = f(x)$ with properties i. through iv.

i. The domain of f is $[0, 5]$.

ii. $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$ exist and are equal.

iii. $f(x)$ is left continuous but not continuous at $x = 2$, and right continuous but not continuous at $x = 3$.

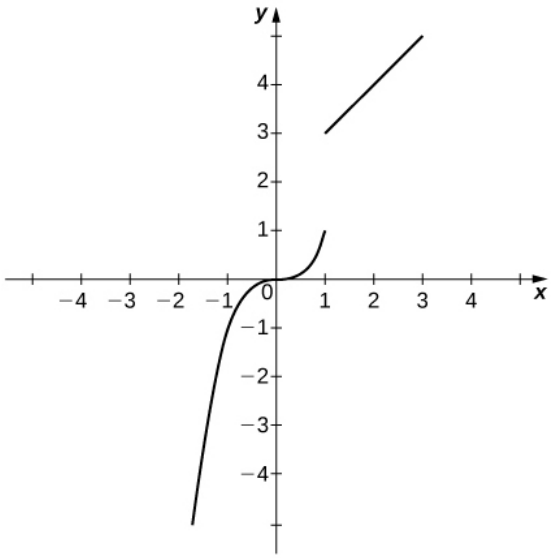
iv. $f(x)$ has a removable discontinuity at $x = 1$, a jump discontinuity at $x = 2$, and the following limits hold: $\lim_{x \to 3^-} f(x) = -\infty$ and $\lim_{x \to 3^+} f(x) = 2$.

In exercises 27-28, suppose $y = f(x)$ is defined for all x. For each description, sketch a graph with the indicated property.

27) Discontinuous at $x = 1$ with $\lim_{x \to -1} f(x) = -1$ and $\lim_{x \to 2} f(x) = 4$

Answer

Answers may vary; see the following example:



28) Discontinuous at $x = 2$ but continuous elsewhere with $\lim_{x \to 0} f(x) = \frac{1}{2}$

Determine whether each of the given statements is true. Justify your response with an explanation or counterexample.

29) $f(t) = \dfrac{2}{e^t - e^{-t}}$ is continuous everywhere.

Answer

False. It is continuous over $(-\infty, 0) \cup (0, \infty)$.

30) If the left- and right-hand limits of $f(x)$ as $x \to a$ exist and are equal, then f cannot be discontinuous at $x = a$.

31) If a function is not continuous at a point, then it is not defined at that point.

Answer

False. Consider $f(x) = \begin{cases} x, & \text{if } x \neq 0 \\ 4, & \text{if } x = 0 \end{cases}$.

32) If $f(x)$ is continuous such that $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has exactly one solution in $[a, b]$.

Answer

False. Consider $f(x) = \cos(x)$ on $[-\pi, 2\pi]$.

33) The function $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 1}$ is continuous over the interval $[0, 3]$.

34) If $f(x)$ is continuous everywhere and $f(a), f(b) > 0$, then there is no root of $f(x)$ in the interval $[a, b]$.

Answer

False. The IVT does not work in reverse! Consider $(x-1)^2$ over the interval $[-2, 2]$.

[T] The following problems consider the scalar form of Coulomb's law, which describes the electrostatic force between two point charges, such as electrons. It is given by the equation $F(r) = k_e \dfrac{|q_1 q_2|}{r^2}$, where k_e is Coulomb's constant, q_i are the magnitudes of the charges of the two particles, and r is the distance between the two particles.

35) To simplify the calculation of a model with many interacting particles, after some threshold value $r = R$, we approximate F as zero.

a. Explain the physical reasoning behind this assumption.

b. What is the force equation?

c. Evaluate the force F using both Coulomb's law and our approximation, assuming two protons with a charge magnitude of 1.6022×10^{-19} **coulombs (C)**, and the Coulomb constant $k_e = 8.988 \times 10^9 Nm^2/C^2$ are 1 m apart. Also, assume $R < 1$ m. How much inaccuracy does our approximation generate? Is our approximation reasonable?

d. Is there any finite value of R for which this system remains continuous at R?

36) Instead of making the force 0 at R, we let the force be $10 - 20$ for $r \geq R$. Assume two protons, which have a magnitude of charge 1.6022×10^{-19} C, and the Coulomb constant $k_e = 8.988 \times 10^9$ Nm^2/C^2. Is there a value R that can make this system continuous? If so, find it.

Answer

$R = 0.0001519$ m

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2.7: Defining the Derivative

Learning Objectives

- Recognize the meaning of the tangent to a curve at a point.
- Calculate the slope of a tangent line.
- Identify the derivative as the limit of a difference quotient.
- Calculate the derivative of a given function at a point.
- Describe the velocity as a rate of change.
- Explain the difference between average velocity and instantaneous velocity.
- Estimate the derivative from a table of values.

Now that we have both a conceptual understanding of a limit and the practical ability to compute limits, we have established the foundation for our study of calculus, the branch of mathematics in which we compute derivatives and integrals. Most mathematicians and historians agree that calculus was developed independently by the Englishman Isaac **Newton (1643–1727)** and the German Gottfried **Leibniz (1646–1716)**, whose images appear in Figure 2.7.1. When we credit Newton and Leibniz with developing calculus, we are really referring to the fact that Newton and Leibniz were the first to understand the relationship between the derivative and the integral. Both mathematicians benefited from the work of predecessors, such as Barrow, Fermat, and Cavalieri. The initial relationship between the two mathematicians appears to have been amicable; however, in later years a bitter controversy erupted over whose work took precedence. Although it seems likely that Newton did, indeed, arrive at the ideas behind calculus first, we are indebted to Leibniz for the notation that we commonly use today.



Figure 2.7.1: Newton and Leibniz are credited with developing calculus independently.

2.7.1 Tangent Lines

We begin our study of calculus by revisiting the notion of secant lines and tangent lines. Recall that we used the slope of a secant line to a function at a point $(a, f(a))$ to estimate the rate of change, or the rate at which one variable changes in relation to another variable. We can obtain the slope of the secant by choosing a value of x near a and drawing a line through the points $(a, f(a))$ and $(x, f(x))$, as shown in Figure 2.7.2. The slope of this line is given by an equation in the form of a difference quotient:

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

We can also calculate the slope of a secant line to a function at a value a by using this equation and replacing x with $a + h$, where h is a value close to a. We can then calculate the slope of the line through the points $(a, f(a))$ and $(a + h, f(a + h))$. In this case, we find the secant line has a slope given by the following difference quotient with increment h:

$$m_{sec} = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$

Definition: Difference Quotient

Let f be a function defined on an interval I containing a. If $x \neq a$ is in I, then

$$Q = \frac{f(x) - f(a)}{x - a}$$

is a *difference quotient.*

Also, if $h \neq 0$ is chosen so that $a + h$ is in I, then

$$Q = \frac{f(a+h) - f(a)}{h}$$

is a difference quotient with increment h.

These two expressions for calculating the slope of a secant line are illustrated in Figure 2.7.2. We will see that each of these two methods for finding the slope of a secant line is of value. Depending on the setting, we can choose one or the other. The primary consideration in our choice usually depends on ease of calculation.

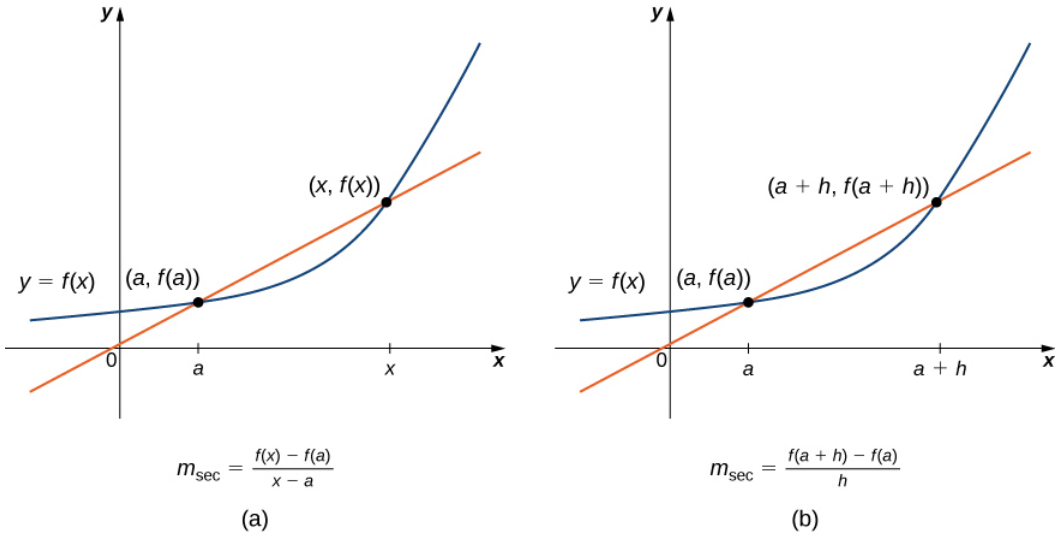


Figure 2.7.2: We can calculate the slope of a secant line in either of two ways.

In Figure 2.7.3*a* we see that, as the values of x approach a, the slopes of the secant lines provide better estimates of the rate of change of the function at a. Furthermore, the secant lines themselves approach the tangent line to the function at a, which represents the limit of the secant lines. Similarly, Figure 2.7.3*b* shows that as the values of h get closer to 0, the secant lines also approach the tangent line. The slope of the tangent line at a is the rate of change of the function at a, as shown in Figure 2.7.3*c*.

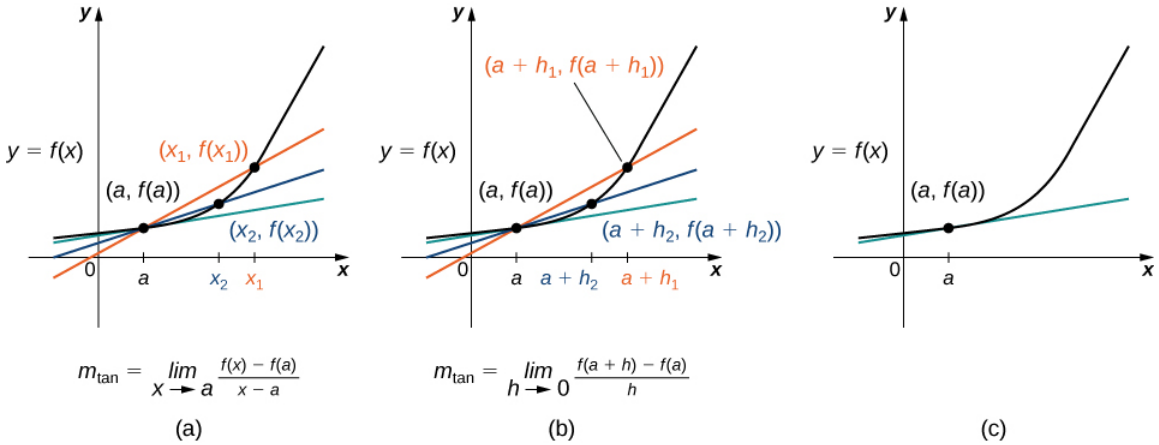


Figure 2.7.3: The secant lines approach the tangent line (shown in green) as the second point approaches the first.

In Figure 2.7.4 we show the graph of $f(x) = \sqrt{x}$ and its tangent line at $(1, 1)$ in a series of tighter intervals about $x = 1$. As the intervals become narrower, the graph of the function and its tangent line appear to coincide, making the values on the tangent line a good approximation to the values of the function for choices of x close to 1. In fact, the graph of $f(x)$ itself appears to be **locally linear** in the immediate vicinity of $x = 1$.

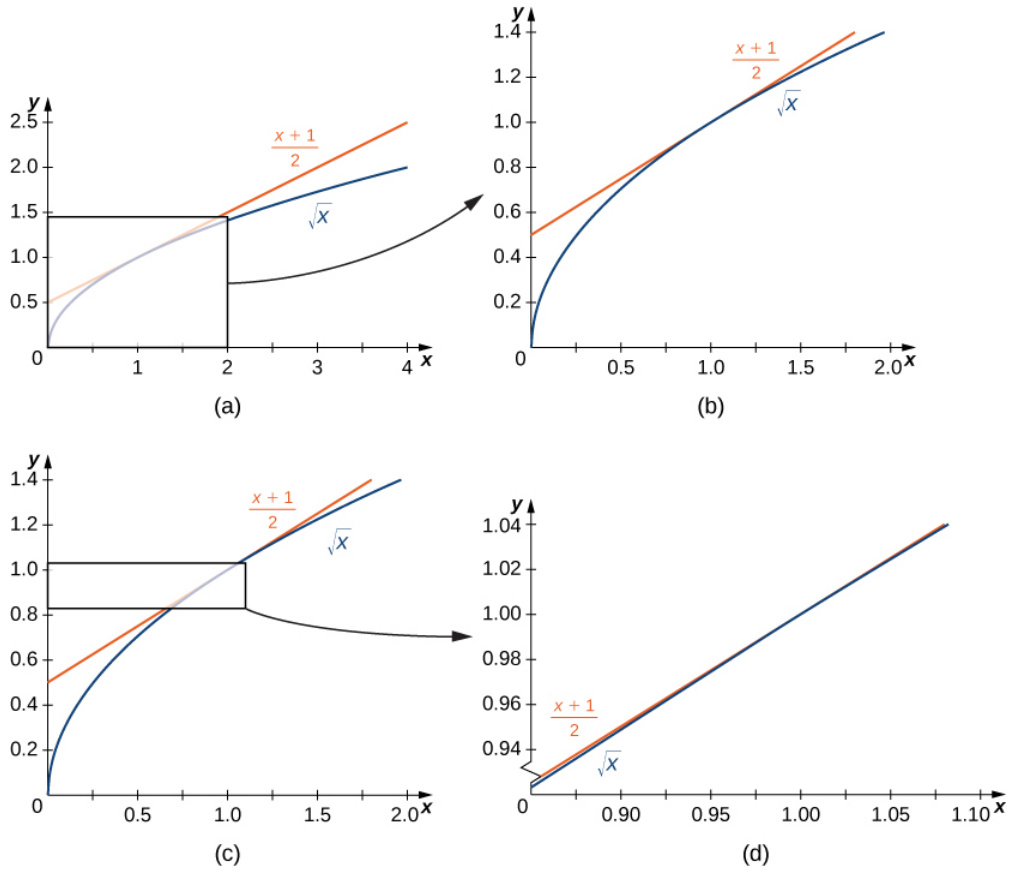


Figure 2.7.4: For values of x close to 1, the graph of $f(x) = \sqrt{x}$ and its tangent line appear to coincide.

Formally we may define the tangent line to the graph of a function as follows.

Definition: Tangent Line

Let $f(x)$ be a function defined in an open interval containing a. The tangent line to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{2.7.1}$$

provided this limit exists.

Equivalently, we may define the *tangent line* to $f(x)$ at a to be the line passing through the point $(a, f(a))$ having slope

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{2.7.2}$$

provided this limit exists.

Just as we have used two different expressions to define the slope of a secant line, we use two different forms to define the slope of the tangent line. In this text we use both forms of the definition. As before, the choice of definition will depend on the setting. Now that we have formally defined a tangent line to a function at a point, we can use this definition to find equations of tangent lines.

Example 2.7.1: Finding a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution

First find the slope of the tangent line. In this example, use Equation 2.7.1.

$$m_{tan} = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} \qquad \text{Apply the definition.}$$

$$= \lim_{x \to 3} \frac{x^2 - 9}{x - 3} \qquad \text{Substitute } f(x) = x^2 \text{ and } f(3) = 9$$

$$= \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \to 3}(x+3) = 6 \qquad \text{Factor the numerator to evaluate the limit.}$$

Next, find a point on the tangent line. Since the line is tangent to the graph of $f(x)$ at $x = 3$, it passes through the point $(3, f(3))$. We have $f(3) = 9$, so the tangent line passes through the point $(3, 9)$.

Using the point-slope equation of the line with the slope $m = 6$ and the point $(3, 9)$, we obtain the line $y - 9 = 6(x - 3)$. Simplifying, we have $y = 6x - 9$. The graph of $f(x) = x^2$ and its tangent line at 3 are shown in Figure 2.7.5.

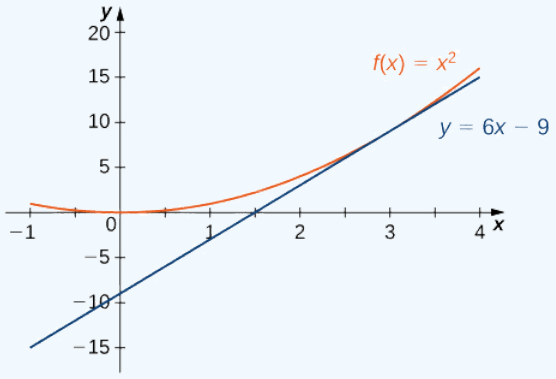


Figure 2.7.5: The tangent line to $f(x)$ at $x = 3$.

Example 2.7.2: The Slope of a Tangent Line Revisited

Use Equation 2.7.2 to find the slope of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution

The steps are very similar to Example 2.7.1. See Equation 2.7.2 for the definition.

$$m_{tan} = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \qquad \text{Apply the definition.}$$

$$= \lim_{h \to 0} \frac{(3+h)^2 - 9}{h} \qquad \text{Substitute } f(3+h) = (3+h)^2 \text{ and } f(3) = 9$$

$$= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} \qquad \text{Expand and simplify to evaluate the limit.}$$

$$= \lim_{h \to 0} \frac{h(6+h)}{h} = \lim_{h \to 0}(6+h) = 6$$

We obtained the same value for the slope of the tangent line by using the other definition, demonstrating that the formulas can be interchanged.

Example 2.7.3: Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = 1/x$ at $x = 2$.

Solution

We can use Equation 2.7.1, but as we have seen, the results are the same if we use Equation 2.7.2.

$$
\begin{aligned}
m_{tan} &= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} && \text{Apply the definition.}\\
&= \lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2} && \text{Substitute } f(x)=\frac{1}{x} \text{ and } f(2)=\frac{1}{2}\\
&= \lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2}\cdot\frac{2x}{2x} && \text{Multiply numerator and denominator by } 2x \text{ to simplify fractions.}\\
&= \lim_{x\to 2} \frac{(2-x)}{(x-2)(2x)} && \text{Simplify.}\\
&= \lim_{x\to 2} \frac{-1}{2x} && \text{Simplify using } \frac{2-x}{x-2}=-1, \text{ for } x\neq 2.\\
&= -\frac{1}{4} && \text{Evaluate the limit.}
\end{aligned}
$$

We now know that the slope of the tangent line is $-\frac{1}{4}$. To find the equation of the tangent line, we also need a point on the line. We know that $f(2) = \frac{1}{2}$. Since the tangent line passes through the point $(2, \frac{1}{2})$ we can use the point-slope equation of a line to find the equation of the tangent line. Thus the tangent line has the equation $y = -\frac{1}{4}x + 1$. The graphs of $f(x) = \frac{1}{x}$ and $y = -\frac{1}{4}x + 1$ are shown in Figure 2.7.6.

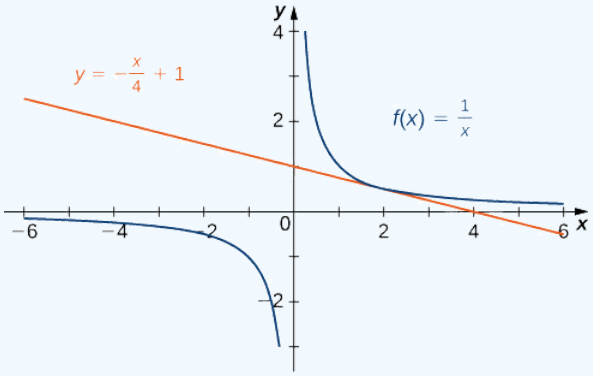


Figure 2.7.6:The line is tangent to $f(x)$ at $x = 2$.

Exercise 2.7.1

Find the slope of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$.

Hint

Use either Equation 2.7.1or Equation 2.7.2 Multiply the numerator and the denominator by a conjugate.

Answer

$\frac{1}{4}$

2.7.2 The Derivative of a Function at a Point

The type of limit we compute in order to find the slope of the line tangent to a function at a point occurs in many applications across many disciplines. These applications include velocity and acceleration in physics, marginal profit functions in business, and

growth rates in biology. This limit occurs so frequently that we give this value a special name: the **derivative.** The process of finding a derivative is called **differentiation**.

Definition: Derivative

Let $f(x)$ be a function defined in an open interval containing a. The **derivative** of the function $f(x)$ at a, denoted by $f\,'(a)$, is defined by

$$f\,'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{2.7.3}$$

provided this limit exists.

Alternatively, we may also define the derivative of $f(x)$ at a as

$$f\,'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{2.7.4}$$

Example 2.7.4: Estimating a Derivative

For $f(x) = x^2$, use a table to estimate $f\,'(3)$ using Equation 2.7.3.

Solution

Create a table using values of x just below 3 and just above 3.

x	$\frac{x^2 - 9}{x - 3}$
2.9	5.9
2.99	5.99
2.999	5.999
3.001	6.001
3.01	6.01
3.1	6.1

After examining the table, we see that a good estimate is $f\,'(3) = 6$.

Exercise 2.7.2

For $f(x) = x^2$, use a table to estimate $f\,'(3)$ using Equation 2.7.4.

Hint

Evaluate $\frac{(x+h)^2 - x^2}{h}$ at $h = -0.1,\ -0.01,\ -0.001,\ 0.001,\ 0.01,\ 0.1$

Answer

6

Example 2.7.6: Finding a Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f\,'(2)$ by using Equation 2.7.3.

Solution

Substitute the given function and value directly into the equation.

$$
\begin{aligned}
f'(x) &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} && \text{Apply the definition.}\\
&= \lim_{x\to 2}\frac{(3x^2-4x+1)-5}{x-2} && \text{Substitute } f(x)=3x^2-4x+1 \text{ and } f(2)=5.\\
&= \lim_{x\to 2}\frac{(x-2)(3x+2)}{x-2} && \text{Simplify and factor the numerator.}\\
&= \lim_{x\to 2}(3x+2) && \text{Cancel the common factor.}\\
&= 8 && \text{Evaluate the limit.}
\end{aligned}
$$

Example 2.7.7: Revisiting the Derivative

For $f(x)=3x^2-4x+1$, find $f'(2)$ by using Equation 2.7.4.

Solution

Using this equation, we can substitute two values of the function into the equation, and we should get the same value as in Example 2.7.6.

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} && \text{Apply the definition.}\\
&= \lim_{h\to 0}\frac{(3(2+h)^2-4(2+h)+1)-5}{h} && \text{Substitute } f(2)=5 \text{ and } f(2+h)=3(2+h)^2-4(2+h)+1.\\
&= \lim_{h\to 0}\frac{3(4+4h+h^2)-8-4h+1-5}{h} && \text{Expand the numerator.}\\
&= \lim_{h\to 0}\frac{12+12h+3h^2-12-4h}{h} && \text{Distribute and begin simplifying the numerator.}\\
&= \lim_{h\to 0}\frac{3h^2+8h}{h} && \text{Finish simplifying the numerator.}\\
&= \lim_{h\to 0}\frac{h(3h+8)}{h} && \text{Factor the numerator.}\\
&= \lim_{h\to 0}(3h+8) && \text{Cancel the common factor.}\\
&= 8 && \text{Evaluate the limit.}
\end{aligned}
$$

The results are the same whether we use Equation 2.7.3 or Equation 2.7.4.

Exercise 2.7.4

For $f(x)=x^2+3x+2$, find $f'(1)$.

Hint

Use either Equation 2.7.3, Equation 2.7.4, or try both.

Answer

$f'(1)=5$

2.7.3 Velocities and Rates of Change

Now that we can evaluate a derivative, we can use it in velocity applications. Recall that if $s(t)$ is the position of an object moving along a coordinate axis, the **average velocity** of the object over a time interval $[a,t]$ if $t>a$ or $[t,a]$ if $t<a$ is given by the difference quotient

$$v_{ave} = \frac{s(t) - s(a)}{t - a}. \tag{2.7.5}$$

As the values of t approach a, the values of v_{ave} approach the value we call the **instantaneous velocity** at a. That is, instantaneous velocity at a, denoted $v(a)$, is given by

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}. \tag{2.7.6}$$

To better understand the relationship between average velocity and instantaneous velocity, see Figure 2.7.7. In this figure, the slope of the tangent line (shown in red) is the instantaneous velocity of the object at time $t = a$ whose position at time t is given by the function $s(t)$. The slope of the secant line (shown in green) is the average velocity of the object over the time interval $[a, t]$.

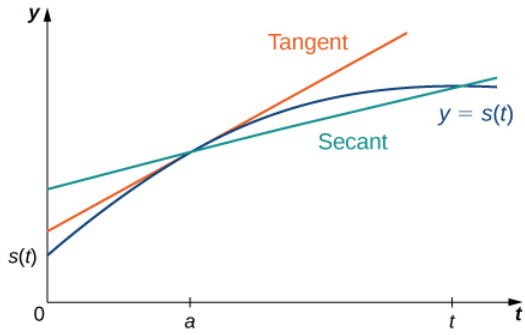


Figure 2.7.7: The slope of the secant line is the average velocity over the interval $[a, t]$. The slope of the tangent line is the instantaneous velocity.

We can use Equation 2.7.6 to calculate the instantaneous velocity, or we can estimate the velocity of a moving object by using a table of values. We can then confirm the estimate by using Equation 2.7.5.

? Exercise 2.7.5

A rock is dropped from a height of 64 feet. Its height above ground at time t seconds later is given by $s(t) = -16t^2 + 64,\ 0 \le t \le 2$. Find its instantaneous velocity 1 second after it is dropped, using Equation 2.7.6.

Hint

$v(t) = s'(t)$. Follow the earlier examples of the derivative using Equation 2.7.6

Answer

−32 ft/s

As we have seen throughout this section, the slope of a tangent line to a function and instantaneous velocity are related concepts. Each is calculated by computing a derivative and each measures the instantaneous rate of change of a function, or the rate of change of a function at any point along the function.

Definition: Instantaneous Rate of Change

The **instantaneous rate of change** of a function $f(x)$ at a value a is its derivative $f'(a)$.

✓ Example 2.7.9: Chapter Opener: Estimating Rate of Change of Velocity

Reaching a top speed of 270.49 mph, the Hennessey Venom GT is one of the fastest cars in the world. In tests it went from 0 to 60 mph in 3.05 seconds, from 0 to 100 mph in 5.88 seconds, from 0 to 200 mph in 14.51 seconds, and from 0 to 229.9 mph in 19.96 seconds. Use this data to draw a conclusion about the rate of change of velocity (that is, its **acceleration**) as it approaches 229.9 mph. Does the rate at which the car is accelerating appear to be increasing, decreasing, or constant?



Figure 2.7.9: (credit: modification of work by Codex41, Flickr)

Solution: First observe that 60 mph = 88 ft/s, 100 mph ≈146.67 ft/s, 200 mph ≈293.33 ft/s, and 229.9 mph ≈337.19 ft/s. We can summarize the information in a table.

Table 2.7.3: $v(t)$ at different values of t

t	$v(t)$
0	0
3.05	88
5.88	147.67
14.51	293.33
19.96	337.19

Now compute the average acceleration of the car in feet per second on intervals of the form $[t, 19.96]$ as t approaches 19.96, as shown in the following table.

Average acceleration

t	$\frac{v(t)-v(19.96)}{t-19.96}=\frac{v(t)-337.19}{t-19.96}$
0.0	16.89
3.05	14.74
5.88	13.46
14.51	8.05

The rate at which the car is accelerating is decreasing as its velocity approaches 229.9 mph (337.19 ft/s).

Example 2.7.10: Rate of Change of Temperature

A homeowner sets the thermostat so that the temperature in the house begins to drop from $70^\circ F$ at 9 p.m., reaches a low of 60° during the night, and rises back to 70° by 7 a.m. the next morning. Suppose that the temperature in the house is given by $T(t) = 0.4t^2 - 4t + 70$ for $0 \le t \le 10$, where t is the number of hours past 9 p.m. Find the instantaneous rate of change of the temperature at midnight.

Solution

Since midnight is 3 hours past 9 p.m., we want to compute $T'(3)$. Refer to Equation 2.7.3.

$$
\begin{aligned}
T'(3) &= \lim_{t\to 3}\frac{T(t)-T(3)}{t-3} && \text{Apply the definition.}\\
&= \lim_{t\to 3}\frac{0.4t^2-4t+70-61.6}{t-3} && \text{Substitute } T(t)=0.4t^2-4t+70 \text{ and } T(3)=61.6.\\
&= \lim_{t\to 3}\frac{0.4t^2-4t+8.4}{t-3} && \text{Simplify.}\\
&= \lim_{t\to 3}\frac{0.4(t-3)(t-7)}{t-3}\\
&= \lim_{t\to 3}0.4(t-7) && \text{Cancel.}\\
&= -1.6 && \text{Evaluate the limit.}
\end{aligned}
$$

The instantaneous rate of change of the temperature at midnight is $-1.6^\circ F$ per hour.

Example 2.7.11: Rate of Change of Profit

A toy company can sell x electronic gaming systems at a price of $p=-0.01x+400$ dollars per gaming system. The cost of manufacturing x systems is given by $C(x)=100x+10,000$ dollars. Find the rate of change of profit when $10,000$ games are produced. Should the toy company increase or decrease production?

Solution

The profit $P(x)$ earned by producing x gaming systems is $R(x)-C(x)$, where $R(x)$ is the revenue obtained from the sale of x games. Since the company can sell x games at $p=-0.01x+400$ per game,

$$R(x)=xp=x(-0.01x+400)=-0.01x^2+400x.$$

Consequently,

$$P(x)=-0.01x^2+300x-10,000.$$

Therefore, evaluating the rate of change of profit gives

$$
\begin{aligned}
P'(10000) &= \lim_{x\to 10000}\frac{P(x)-P(10000)}{x-10000}\\
&= \lim_{x\to 10000}\frac{-0.01x^2+300x-10000-1990000}{x-10000}\\
&= \lim_{x\to 10000}\frac{-0.01x^2+300x-2000000}{x-10000}\\
&= 100
\end{aligned}
$$

Since the rate of change of profit $P'(10,000)>0$ and $P(10,000)>0$, the company should increase production.

Exercise 2.7.6

A coffee shop determines that the daily profit on scones obtained by charging s dollars per scone is $P(s)=-20s^2+150s-10$. The coffee shop currently charges $\$3.25$ per scone. Find $P'(3.25)$, the rate of change of profit when the price is $\$3.25$ and decide whether or not the coffee shop should consider raising or lowering its prices on scones.

Hint

Use Example 2.7.11for a guide.

Answer

$P'(3.25)=20>0$; raise prices

2.7.4 Key Concepts

- The slope of the tangent line to a curve measures the instantaneous rate of change of a curve. We can calculate it by finding the limit of the difference quotient or the difference quotient with increment h.
- The derivative of a function $f(x)$ at a value a is found using either of the definitions for the slope of the tangent line.
- Velocity is the rate of change of position. As such, the velocity $v(t)$ at time t is the derivative of the position $s(t)$ at time t. Average velocity is given by

$$v_{ave} = \frac{s(t) - s(a)}{t - a}.$$

  Instantaneous velocity is given by

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}.$$

- We may estimate a derivative by using a table of values.

2.7.5 Key Equations

- **Difference quotient**

$$Q = \frac{f(x) - f(a)}{x - a}$$

- **Difference quotient with increment h**

$$Q = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$

- **Slope of tangent line**

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- **Derivative of f(x) at a**

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- **Average velocity**

$$v_{ave} = \frac{s(t) - s(a)}{t - a}$$

- **Instantaneous velocity**

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

2.7.6 Glossary

derivative

the slope of the tangent line to a function at a point, calculated by taking the limit of the difference quotient, is the derivative

difference quotient

of a function $f(x)$ at a is given by

$$\frac{f(a+h) - f(a)}{h} \text{ or } \frac{f(x) - f(a)}{x - a}$$

differentiation

the process of taking a derivative

instantaneous rate of change

the rate of change of a function at any point along the function a, also called $f'(a)$, or the derivative of the function at a

---

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2.7E: EXERCISES FOR SECTION 2.7

For exercises 1 - 10, use the equation $m_{\text{sec}} = \dfrac{f(x)-f(a)}{x-a}$ to find the slope of the secant line between the values x_1 and x_2 for each function $y = f(x)$.

1) $f(x) = 4x + 7; \quad x_1 = 2, \quad x_2 = 5$

Answer

$m_{\text{sec}} = 4$

2) $f(x) = 8x - 3; \quad x_1 = -1, \quad x_2 = 3$

3) $f(x) = x^2 + 2x + 1; \quad x_1 = 3, \quad x_2 = 3.5$

Answer

$m_{\text{sec}} = 8.5$

4) $f(x) = -x^2 + x + 2; \quad x_1 = 0.5, \quad x_2 = 1.5$

5) $f(x) = \dfrac{4}{3x-1}; \quad x_1 = 1, \quad x_2 = 3$

Answer

$m_{\text{sec}} = -\frac{3}{4}$

6) $f(x) = \dfrac{x-7}{2x+1}; \quad x_1 = -2, \quad x_2 = 0$

7) $f(x) = \sqrt{x}; \quad x_1 = 1, \quad x_2 = 16$

Answer

$m_{\text{sec}} = 0.2$

8) $f(x) = \sqrt{x-9}; \quad x_1 = 10, \quad x_2 = 13$

9) $f(x) = x^{1/3} + 1; \quad x_1 = 0, \quad x_2 = 8$

Answer

$m_{\text{sec}} = 0.25$

10) $f(x) = 6x^{2/3} + 2x^{1/3}; \quad x_1 = 1, \quad x_2 = 27$

For the functions in exercises 11 - 20,

a. use the equation $m_{\tan} = \lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h}$ to find the slope of the tangent line $m_{\tan} = f'(a)$, and

b. find the equation of the tangent line to f at $x = a$.

11) $f(x) = 3 - 4x, \quad a = 2$

Answer

a. $m_{\tan} = -4$

b. $y = -4x + 3$

12) $f(x) = \dfrac{x}{5} + 6, \quad a = -1$

13) $f(x) = x^2 + x, \quad a = 1$

Answer

a. $m_{\tan} = 3$

b. $y = 3x - 1$

14) $f(x) = 1 - x - x^2, \quad a = 0$

15) $f(x) = \dfrac{7}{x}, \quad a = 3$

Answer

a. $m_{\tan} = \frac{-7}{9}$
b. $y = \frac{-7}{9}x + \frac{14}{3}$

16) $f(x) = \sqrt{x+8}, \quad a = 1$

17) $f(x) = 2 - 3x^2, \quad a = -2$

Answer

a. $m_{\tan} = 12$
b. $y = 12x + 14$

18) $f(x) = \dfrac{-3}{x-1}, \quad a = 4$

19) $f(x) = \dfrac{2}{x+3}, \quad a = -4$

Answer

a. $m_{\tan} = -2$
b. $y = -2x - 10$

20) $f(x) = \dfrac{3}{x^2}, \quad a = 3$

For the functions $y = f(x)$ in exercises 21 - 30, find $f'(a)$ using the equation $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

21) $f(x) = 5x + 4, \quad a = -1$

Answer

$f'(-1) = 5$

22) $f(x) = -7x + 1, \quad a = 3$

23) $f(x) = x^2 + 9x, \quad a = 2$

Answer

$f'(2) = 13$

24) $f(x) = 3x^2 - x + 2, \quad a = 1$

25) $f(x) = \sqrt{x}, \quad a = 4$

Answer

$f'(4) = \frac{1}{4}$

26) $f(x) = \sqrt{x-2}, \quad a = 6$

27) $f(x) = \dfrac{1}{x}, \quad a = 2$

Answer

$f'(2) = -\frac{1}{4}$

28) $f(x) = \dfrac{1}{x-3}, \quad a = -1$

29) $f(x) = \dfrac{1}{x^3}, \quad a = 1$

Answer

$f'(1) = -3$

30) $f(x) = \dfrac{1}{\sqrt{x}}, \quad a = 4$

For the following exercises, given the function $y = f(x)$,

a. find the slope of the secant line PQ for each point $Q(x, f(x))$ with x value given in the table.

b. Use the answers from a. to estimate the value of the slope of the tangent line at P.

c. Use the answer from b. to find the equation of the tangent line to f at point P.

31) [T] $f(x) = x^2 + 3x + 4, \quad P(1, 8)$ (Round to 6 decimal places.)

x	Slope m_{PQ}	x	Slope m_{PQ}
1.1	(i)	0.9	(vii)
1.01	(ii)	0.99	(viii)
1.001	(iii)	0.999	(ix)
1.0001	(iv)	0.9999	(x)
1.00001	(v)	0.99999	(xi)
1.000001	(vi)	0.999999	(xii)

Answer

$a. (i) 5.100000, (ii) 5.010000, (iii) 5.001000, (iv) 5.000100, (v) 5.000010, (vi) 5.000001, (vii) 4.900000, (viii) 4.990000, (ix) 4.999000, (x) 4.999900, (xi) 4.999990, (x) 4.999999$

b. $m_{\tan} = 5$

c. $y = 5x + 3$

32) [T] $f(x) = \dfrac{x+1}{x^2-1}, \quad P(0, -1)$

x	Slope m_{PQ}	x	Slope m_{PQ}
0.1	(i)	−0.1	(vii)
0.01	(ii)	−0.01	(viii)
0.001	(iii)	−0.001	(ix)
0.0001	(iv)	−0.0001	(x)
0.00001	(v)	−0.00001	(xi)
0.000001	(vi)	−0.000001	(xii)

33) [T] $f(x) = 10e^{0.5x}, \quad P(0, 10)$ (Round to 4 decimal places.)

x	Slope m_{PQ}
−0.1	(i)
−0.01	(ii)
−0.001	(iii)
−0.0001	(iv)
−0.00001	(v)
−0.000001	(vi)

Answer

a. $(i) 4.8771, (ii) 4.9875, (iii) 4.9988, (iv) 4.9999, (v) 4.9999, (vi) 4.9999$

b. $m_{\tan} = 5$

c. $y = 5x + 10$

[T] For the following position functions $y = s(t)$, an object is moving along a straight line, where t is in seconds and s is in meters. Find

a. the simplified expression for the average velocity from $t = 2$ to $t = 2 + h$;

b. the average velocity between $t = 2$ and $t = 2 + h$, where $(i)\ h = 0.1, (ii)\ h = 0.01, (iii)\ h = 0.001$, and $(iv)\ h = 0.0001$; and

c. use the answer from a. to estimate the instantaneous velocity at $t = 2$ second.

35) $s(t) = \frac{1}{3}t + 5$

Answer

a. $\frac{1}{3}$;

b. $(i)\ \frac{1}{3}$ m/s, $(ii)\ \frac{1}{3}$ m/s, $(iii)\ \frac{1}{3}$ m/s, $(iv)\ \frac{1}{3}$ m/s;

c. $\frac{1}{3}$ m/s

36) $s(t) = t^2 - 2t$

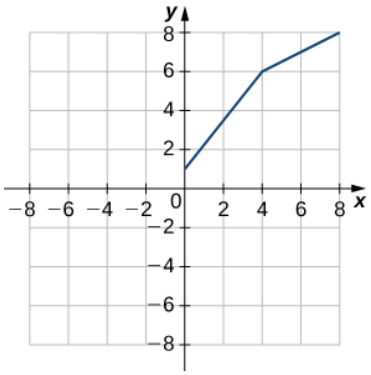
37) $s(t) = 2t^3 + 3$

Answer

a. $2(h^2+6h+12)$;
b. (i) 25.22 m/s, (ii) 24.12 m/s, (iii) 24.01 m/s, (iv) 24 m/s;
c. 24 m/s

38) $s(t)=\dfrac{16}{t^2}-\dfrac{4}{t}$

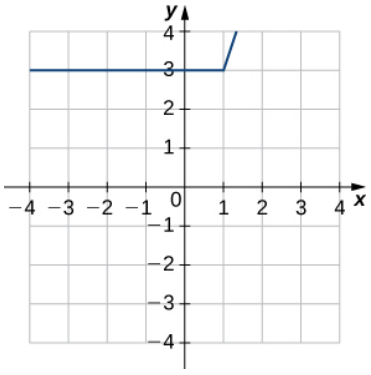
39) Use the following graph to evaluate a. $f'(1)$ and b. $f'(6)$.



Answer

a. 1.25; b. 0.5

40) Use the following graph to evaluate a. $f'(-3)$ and b. $f'(1.5)$.



For the following exercises, use the limit definition of derivative to show that the derivative does not exist at $x=a$ for each of the given functions.

41) $f(x)=x^{1/3}, \quad x=0$

Answer

$$\lim_{x\to 0^-}\frac{x^{1/3}-0}{x-0}=\lim_{x\to 0^-}\frac{1}{x^{2/3}}=\infty$$

42) $f(x)=x^{2/3}, \quad x=0$

43) $f(x)=\begin{cases}1, & \text{if } x<1\\ x, & \text{if } x\geq 1\end{cases}, \quad x=1$

Answer

$$\lim_{x\to 1^-}\frac{1-1}{x-1}=0\neq 1=\lim_{x\to 1^+}\frac{x-1}{x-1}$$

44) $f(x)=\dfrac{|x|}{x}, \quad x=0$

45) [T] The position in feet of a race car along a straight track after t seconds is modeled by the function $s(t)=8t^2-\frac{1}{16}t^3$.

a. Find the average velocity of the vehicle over the following time intervals to four decimal places:

i. $[4, 4.1]$

ii. $[4, 4.01]$

iii. $[4, 4.001]$

iv. $[4, 4.0001]$

b. Use a. to draw a conclusion about the instantaneous velocity of the vehicle at $t = 4$ seconds.

Answer

a. $(i) 61.7244 ft/s,\ (ii) 61.0725 ft/s,\ (iii) 61.0072 ft/s,\ (iv) 61.0007 ft/s$

b. At 4 seconds the race car is traveling at a rate/velocity of 61 ft/s.

46) [T] The distance in feet that a ball rolls down an incline is modeled by the function $s(t) = 14t^2$,

where t is seconds after the ball begins rolling.

a. Find the average velocity of the ball over the following time intervals:

i. [5, 5.1]

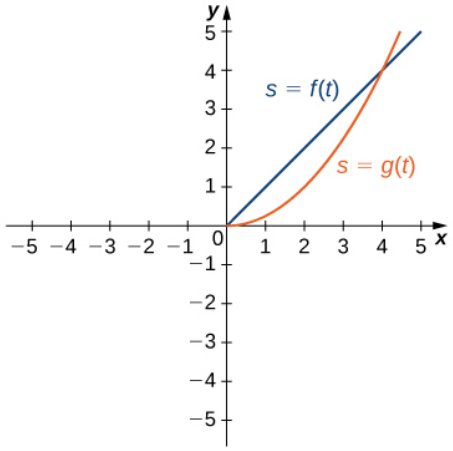
ii. [5, 5.01]

iii. [5, 5.001]

iv. [5, 5.0001]

b. Use the answers from a. to draw a conclusion about the instantaneous velocity of the ball at $t = 5$ seconds.

47) Two vehicles start out traveling side by side along a straight road. Their position functions, shown in the following graph, are given by $s = f(t)$ and $s = g(t)$, where s is measured in feet and t is measured in seconds.



a. Which vehicle has traveled farther at $t = 2$ seconds?

b. What is the approximate velocity of each vehicle at $t = 3$ seconds?

c. Which vehicle is traveling faster at $t = 4$ seconds?

d. What is true about the positions of the vehicles at $t = 4$ seconds?

Answer

a. The vehicle represented by $f(t)$, because it has traveled 2 feet, whereas $g(t)$ has traveled 1 foot.

b. The velocity of $f(t)$ is constant at 1 ft/s, while the velocity of $g(t)$ is approximately 2 ft/s.

c. The vehicle represented by $g(t)$, with a velocity of approximately 4 ft/s.

d. Both have traveled 4 feet in 4 seconds.

48) [T] The total cost $C(x)$, in hundreds of dollars, to produce x jars of mayonnaise is given by $C(x) = 0.000003x^3 + 4x + 300$.

a. Calculate the average cost per jar over the following intervals:

i. [100, 100.1]

ii. [100, 100.01]

iii. [100, 100.001]

iv. [100, 100.0001]

b. Use the answers from a. to estimate the average cost to produce 100 jars of mayonnaise.

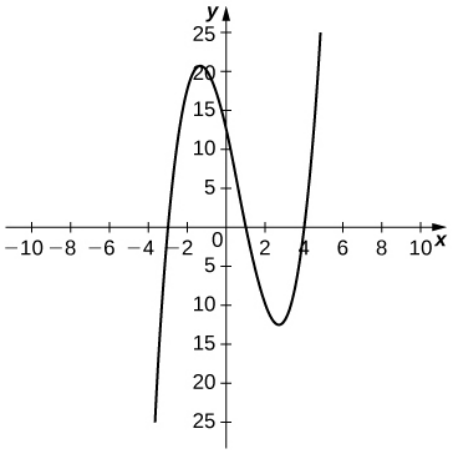
49) [T] For the function $f(x) = x^3 - 2x^2 - 11x + 12$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the ZOOM feature on the calculator to approximate the two values of $x = a$ for which $m_{tan} = f'(a) = 0$.

Answer

a.



b. $a \approx -1.361,\ 2.694$

50) [T] For the function $f(x) = \dfrac{x}{1+x^2}$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the ZOOM feature on the calculator to approximate the values of $x = a$ for which $m_{\tan} = f'(a) = 0$.

51) Suppose that $N(x)$ computes the number of gallons of gas used by a vehicle traveling x miles. Suppose the vehicle gets 30 mpg.

a. Find a mathematical expression for $N(x)$.

b. What is $N(100)$? Explain the physical meaning.

c. What is $N'(100)$? Explain the physical meaning.

Answer

a. $N(x) = \dfrac{x}{30}$

b. ~ 3.3 gallons. When the vehicle travels 100 miles, it has used 3.3 gallons of gas.

c. $\frac{1}{30}$. The rate of gas consumption in gallons per mile that the vehicle is achieving after having traveled 100 miles.

52) [T] For the function $f(x) = x^4 - 5x^2 + 4$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the $nDeriv$ function, which numerically finds the derivative, on a graphing calculator to estimate $f'(-2)$, $f'(-0.5)$, $f'(1.7)$, and $f'(2.718)$.

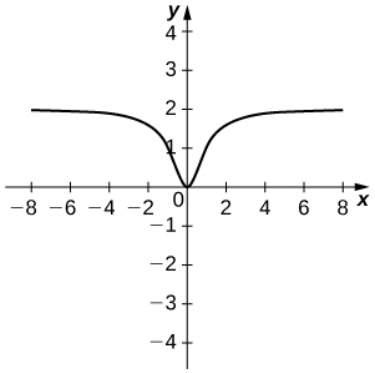
53) [T] For the function $f(x) = \dfrac{x^2}{x^2+1}$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the $nDeriv$ function on a graphing calculator to find $f'(-4)$, $f'(-2)$, $f'(2)$, and $f'(4)$.

Answer

a.



b. $-0.028, -0.16, 0.16, 0.028$

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2.8: The Derivative as a Function

Learning Objectives

- Define the derivative function of a given function.
- Graph a derivative function from the graph of a given function.
- State the connection between derivatives and continuity.
- Describe three conditions for when a function does not have a derivative.
- Explain the meaning of a higher-order derivative.

As we have seen, the derivative of a function at a given point gives us the rate of change or slope of the tangent line to the function at that point. If we differentiate a position function at a given time, we obtain the velocity at that time. It seems reasonable to conclude that knowing the derivative of the function at every point would produce valuable information about the behavior of the function. However, the process of finding the derivative at even a handful of values using the techniques of the preceding section would quickly become quite tedious. In this section we define the derivative function and learn a process for finding it.

2.8.1 Derivative Functions

The derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined. We can formally define a derivative function as follows.

Definition: Derivative Function

Let f be a function. The **derivative function**, denoted by f', is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{2.8.1}$$

A function $f(x)$ is said to be **differentiable at** a if $f'(a)$ exists. More generally, a function is said to be **differentiable on** S if it is differentiable at every point in an open set S, and a differentiable function is one in which $f'(x)$ exists on its domain.

In the next few examples we use Equation 2.8.1 to find the derivative of a function.

Example 2.8.1: Finding the Derivative of a Square-Root Function

Find the derivative of $f(x) = \sqrt{x}$.

Solution

Start directly with the definition of the derivative function.

Substitute $f(x+h) = \sqrt{x+h}$ and $f(x) = \sqrt{x}$ into $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$	
$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$	Multiply numerator and denominator by $\sqrt{x+h} + \sqrt{x}$ without distributing in the denominator.
$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$	Multiply the numerators and simplify.
$= \lim_{h \to 0} \frac{1}{\left(\sqrt{x+h} + \sqrt{x}\right)}$	Cancel the h.
$= \frac{1}{2\sqrt{x}}$	Evaluate the limit

Example 2.8.2: Finding the Derivative of a Quadratic Function

Find the derivative of the function $f(x) = x^2 - 2x$.

Solution

Follow the same procedure here, but without having to multiply by the conjugate.

Substitute $f(x+h) = (x+h)^2 - 2(x+h)$ and $f(x) = x^2 - 2x$ into $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

$f'(x) = \lim_{h \to 0} \dfrac{((x+h)^2 - 2(x+h)) - (x^2 - 2x)}{h}$	
$= \lim_{h \to 0} \dfrac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h}$	Expand $(x+h)^2 - 2(x+h)$.
$= \lim_{h \to 0} \dfrac{2xh - 2h + h^2}{h}$	Simplify
$= \lim_{h \to 0} \dfrac{h(2x - 2 + h)}{h}$	Factor out h from the numerator
$= \lim_{h \to 0} (2x - 2 + h)$	Cancel the common factor of h
$= 2x - 2$	Evaluate the limit

Exercise 2.8.1

Find the derivative of $f(x) = x^2$.

Hint

Use Equation 2.8.1 and follow the example.

Answer

$f'(x) = 2x$

We use a variety of different notations to express the derivative of a function. In Example 2.8.2 we showed that if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$. If we had expressed this function in the form $y = x^2 - 2x$, we could have expressed the derivative as $y' = 2x - 2$ or $\dfrac{dy}{dx} = 2x - 2$. We could have conveyed the same information by writing $\dfrac{d}{dx}\left(x^2 - 2x\right) = 2x - 2$. Thus, for the function $y = f(x)$, each of the following notations represents the derivative of $f(x)$:

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}\left(f(x)\right).$$

In place of $f'(a)$ we may also use $\left.\dfrac{dy}{dx}\right|_{x=a}$. Use of the $\dfrac{dy}{dx}$ notation (called Leibniz notation) is quite common in engineering and physics. To understand this notation better, recall that the derivative of a function at a point is the limit of the slopes of secant lines as the secant lines approach the tangent line. The slopes of these secant lines are often expressed in the form $\dfrac{\Delta y}{\Delta x}$ where Δy is the difference in the y values corresponding to the difference in the x values, which are expressed as Δx (Figure 2.8.1). Thus the derivative, which can be thought of as the instantaneous rate of change of y with respect to x, is expressed as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

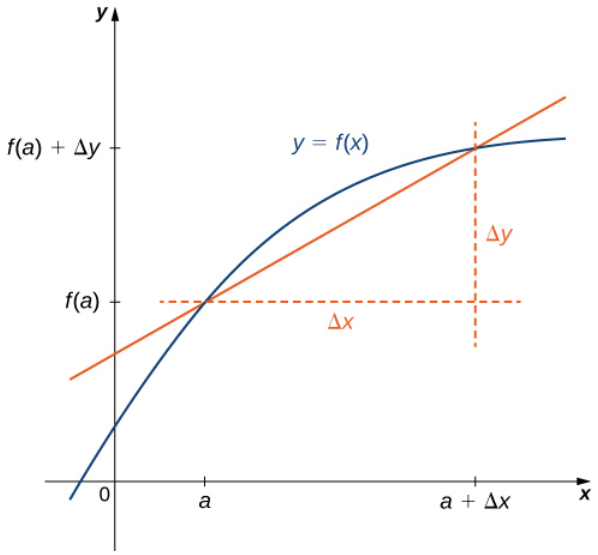


Figure 2.8.1: The derivative is expressed as $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

2.8.2 Graphing a Derivative

We have already discussed how to graph a function, so given the equation of a function or the equation of a derivative function, we could graph it. Given both, we would expect to see a correspondence between the graphs of these two functions, since $f'(x)$ gives the rate of change of a function $f(x)$ (or slope of the tangent line to $f(x)$).

In Example 2.8.1, we found that for $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$. If we graph these functions on the same axes, as in Figure 2.8.2, we can use the graphs to understand the relationship between these two functions. First, we notice that $f(x)$ is increasing over its entire domain, which means that the slopes of its tangent lines at all points are positive. Consequently, we expect $f'(x) > 0$ for all values of x in its domain. Furthermore, as x increases, the slopes of the tangent lines to $f(x)$ are decreasing and we expect to see a corresponding decrease in $f'(x)$. We also observe that $f(0)$ is undefined and that $\lim_{x \to 0^+} f'(x) = +\infty$, corresponding to a vertical tangent to $f(x)$ at 0.

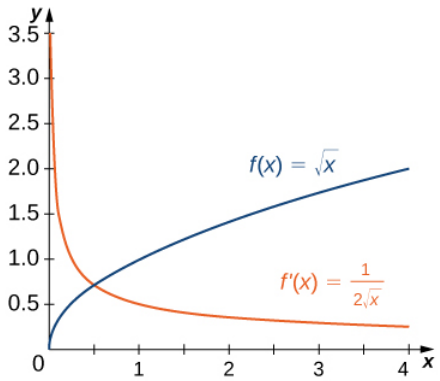


Figure 2.8.2: The derivative $f'(x)$ is positive everywhere because the function $f(x)$ is increasing.

In Example 2.8.2, we found that for $f(x) = x^2 - 2x$, $f'(x) = 2x - 2$. The graphs of these functions are shown in Figure 2.8.3. Observe that $f(x)$ is decreasing for $x < 1$. For these same values of x, $f'(x) < 0$. For values of $x > 1$, $f(x)$ is increasing and $f'(x) > 0$. Also, $f(x)$ has a horizontal tangent at $x = 1$ and $f'(1) = 0$.

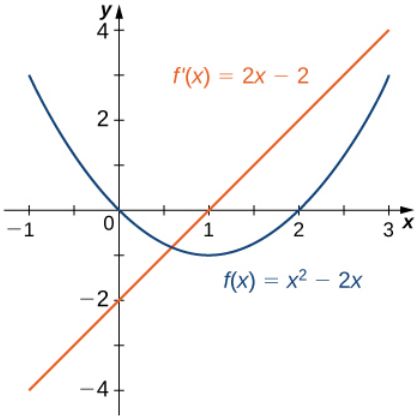
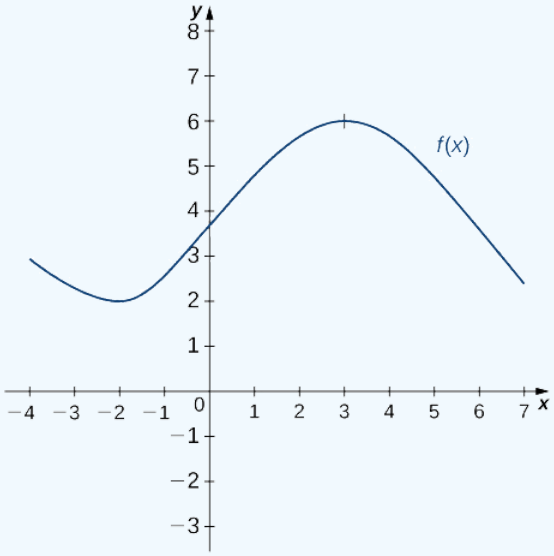


Figure 2.8.3: The derivative $f'(x) < 0$ where the function $f(x)$ is decreasing and $f'(x) > 0$ where $f(x)$ is increasing. The derivative is zero where the function has a horizontal tangent

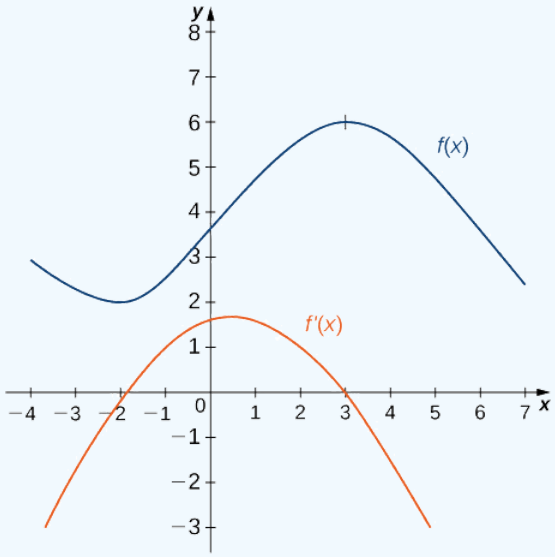
Example 2.8.3: Sketching a Derivative Using a Function

Use the following graph of $f(x)$ to sketch a graph of $f'(x)$.



Solution

The solution is shown in the following graph. Observe that $f(x)$ is increasing and $f'(x) > 0$ on $(-2, 3)$. Also, $f(x)$ is decreasing and $f'(x) < 0$ on $(-\infty, -2)$ and on $(3, +\infty)$. Also note that $f(x)$ has horizontal tangents at -2 and 3, and $f'(-2) = 0$ and $f'(3) = 0$.



Exercise 2.8.2

Sketch the graph of $f(x) = x^2 - 4$. On what interval is the graph of $f'(x)$ above the x-axis?

Hint

The graph of $f'(x)$ is positive where $f(x)$ is increasing.

Answer

$(0, +\infty)$

2.8.3 Derivatives and Continuity

Now that we can graph a derivative, let's examine the behavior of the graphs. First, we consider the relationship between differentiability and continuity. We will see that if a function is differentiable at a point, it must be continuous there; however, a function that is continuous at a point need not be differentiable at that point. In fact, a function may be continuous at a point and fail to be differentiable at the point for one of several reasons.

Differentiability Implies Continuity

Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at a, then f is continuous at a.

Proof

If $f(x)$ is differentiable at a, then $f'(a)$ exists and, if we let $h = x - a$, we have $x = a + h$, and as $h = x - a \to 0$, we can see that $x \to a$.

Then

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

can be rewritten as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We want to show that $f(x)$ is continuous at a by showing that $\lim_{x \to a} f(x) = f(a)$. Thus,

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\ \big(f(x)-f(a)+f(a)\big)\\ &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)+f(a)\right) && \text{Multiply and divide } (f(x)-f(a)) \text{ by } x-a.\\ &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\cdot\Big(\lim_{x\to a}(x-a)\Big)+\lim_{x\to a} f(a)\\ &= f'(a)\cdot 0+f(a)\\ &= f(a).\end{aligned}$$

Therefore, since $f(a)$ is defined and $\lim_{x\to a} f(x)=f(a)$, we conclude that f is continuous at a.

□

We have just proven that differentiability implies continuity, but now we consider whether continuity implies differentiability. To determine an answer to this question, we examine the function $f(x)=|x|$. This function is continuous everywhere; however, $f'(0)$ is undefined. This observation leads us to believe that continuity does not imply differentiability. Let's explore further. For $f(x)=|x|$,

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{|x|-|0|}{x-0}=\lim_{x\to 0}\frac{|x|}{x}\ .$$

This limit does not exist because

$$\lim_{x\to 0^-}\frac{|x|}{x}=-1 \text{ and } \lim_{x\to 0^+}\frac{|x|}{x}=1.$$

See Figure 2.8.4.

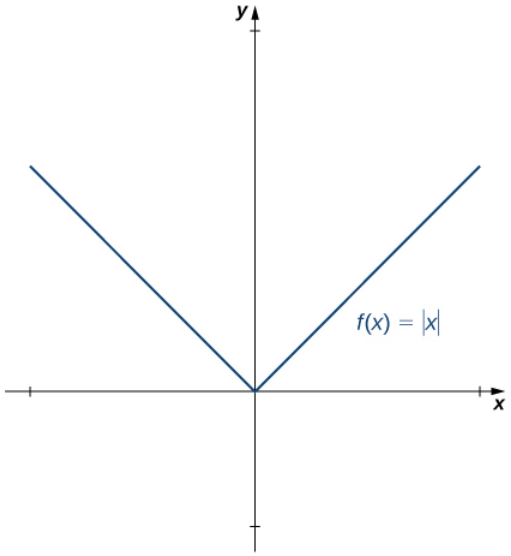


Figure 2.8.4: The function $f(x)=|x|$ is continuous at 0 but is not differentiable at 0.

Let's consider some additional situations in which a continuous function fails to be differentiable. Consider the function $f(x)=\sqrt[3]{x}$:

$$f'(0)=\lim_{x\to 0}\frac{\sqrt[3]{x}-0}{x-0}=\lim_{x\to 0}\frac{1}{\sqrt[3]{x^2}}=+\infty\ .$$

Thus $f'(0)$ does not exist. A quick look at the graph of $f(x)=\sqrt[3]{x}$ clarifies the situation. The function has a vertical tangent line at 0 (Figure 2.8.5).

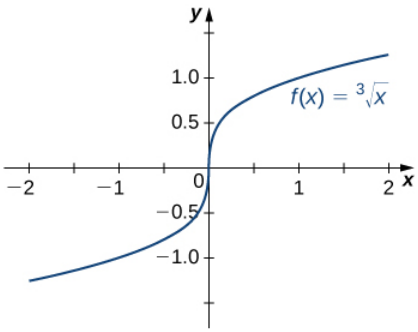


Figure 2.8.5: The function $f(x) = \sqrt[3]{x}$ has a vertical tangent at $x = 0$. It is continuous at 0 but is not differentiable at 0.

In summary:

1. We observe that if a function is not continuous, it cannot be differentiable, since every differentiable function must be continuous. However, if a function is continuous, it may still fail to be differentiable.
2. We saw that $f(x) = |x|$ failed to be differentiable at 0 because the limit of the slopes of the tangent lines on the left and right were not the same. Visually, this resulted in a sharp corner on the graph of the function at 0. From this we conclude that in order to be differentiable at a point, a function must be "smooth" at that point.
3. As we saw in the example of $f(x) = \sqrt[3]{x}$, a function fails to be differentiable at a point where there is a vertical tangent line.

Example 2.8.4: A Piecewise Function that is Continuous and Differentiable

A toy company wants to design a track for a toy car that starts out along a parabolic curve and then converts to a straight line (Figure 2.8.7). The function that describes the track is to have the form $f(x) = \begin{cases} \frac{1}{10}x^2 + bx + c, & \text{if } x < -10 \\ -\frac{1}{4}x + \frac{5}{2}, & \text{if } x \geq -10 \end{cases}$ where x and $f(x)$ are in inches. For the car to move smoothly along the track, the function $f(x)$ must be both continuous and differentiable at -10. Find values of b and c that make $f(x)$ both continuous and differentiable.

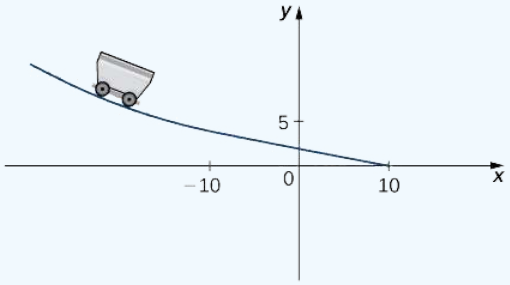


Figure 2.8.7: For the car to move smoothly along the track, the function must be both continuous and differentiable.

Solution

For the function to be continuous at $x = -10$, $\lim\limits_{x \to 10^-} f(x) = f(-10)$. Thus, since

$$\lim_{x \to -10^-} f(x) = \frac{1}{10}(-10)^2 - 10b + c = 10 - 10b + c$$

and $f(-10) = 5$, we must have $10 - 10b + c = 5$. Equivalently, we have $c = 10b - 5$.

For the function to be differentiable at -10,

$$f'(10) = \lim_{x \to -10} \frac{f(x) - f(-10)}{x + 10}$$

must exist. Since $f(x)$ is defined using different rules on the right and the left, we must evaluate this limit from the right and the left and then set them equal to each other:

$$\begin{aligned}\lim_{x\to -10^-}\frac{f(x)-f(-10)}{x+10} &= \lim_{x\to -10^-}\frac{\frac{1}{10}x^2+bx+c-5}{x+10}\\ &= \lim_{x\to -10^-}\frac{\frac{1}{10}x^2+bx+(10b-5)-5}{x+10} && \text{Substitute } c=10b-5.\\ &= \lim_{x\to -10^-}\frac{x^2-100+10bx+100b}{10(x+10)}\\ &= \lim_{x\to -10^-}\frac{(x+10)(x-10+10b)}{10(x+10)} && \text{Factor by grouping}\\ &= b-2\end{aligned}$$

We also have

$$\begin{aligned}\lim_{x\to -10^+}\frac{f(x)-f(-10)}{x+10} &= \lim_{x\to -10^+}\frac{-\frac{1}{4}x+\frac{5}{2}-5}{x+10}\\ &= \lim_{x\to -10^+}\frac{-(x+10)}{4(x+10)}\\ &= -\frac{1}{4}\end{aligned}$$

This gives us $b-2=-\frac{1}{4}$. Thus $b=\frac{7}{4}$ and $c=10(\frac{7}{4})-5=\frac{25}{2}$.

? Exercise 2.8.3

Find values of a and b that make $f(x)=\begin{cases} ax+b, & \text{if } x<3\\ x^2, & \text{if } x\geq 3\end{cases}$ both continuous and differentiable at 3.

Hint

Use Example 2.8.4as a guide.

Answer

$a=6$ and $b=-9$

2.8.4 Higher-Order Derivatives

The derivative of a function is itself a function, so we can find the derivative of a derivative. For example, the derivative of a position function is the rate of change of position, or velocity. The derivative of velocity is the rate of change of velocity, which is acceleration. The new function obtained by differentiating the derivative is called the second derivative. Furthermore, we can continue to take derivatives to obtain the third derivative, fourth derivative, and so on. Collectively, these are referred to as **higher-order derivatives**. The notation for the higher-order derivatives of $y=f(x)$ can be expressed in any of the following forms:

$$\begin{gathered}f''(x),\ f'''(x),\ f^{(4)}(x),\ \ldots,\ f^{(n)}(x)\\ y''(x),\ y'''(x),\ y^{(4)}(x),\ \ldots,\ y^{(n)}(x)\\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dy^3},\ \frac{d^4y}{dy^4},\ \cdots,\ \frac{d^ny}{dy^n}.\end{gathered}$$

It is interesting to note that the notation for $\frac{d^2y}{dx^2}$ may be viewed as an attempt to express $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ more compactly.

Analogously, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3}$.

Example 2.8.5: Finding a Second Derivative

For $f(x) = 2x^2 - 3x + 1$, find $f''(x)$.

Solution

First find $f'(x)$.

Substitute $f(x) = 2x^2 - 3x + 1$ and $f(x+h) = 2(x+h)^2 - 3(x+h) + 1$ into $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$f'(x) = \lim_{h \to 0} \frac{(2(x+h)^2 - 3(x+h) + 1) - (2x^2 - 3x + 1)}{h}$	
$= \lim_{h \to 0} \frac{4xh + 2h^2 - 3h}{h}$	Simplify the numerator.
$= \lim_{h \to 0} (4x + 2h - 3)$	Factor out the h in the numerator and cancel with the h in the denominator.
$= 4x - 3$	Take the limit.

Next, find $f''(x)$ by taking the derivative of $f'(x) = 4x - 3$.

$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$	Use $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ with $f'(x)$ in place of $f(x)$.
$= \lim_{h \to 0} \frac{(4(x+h) - 3) - (4x - 3)}{h}$	Substitute $f'(x+h) = 4(x+h) - 3$ and $f'(x) = 4x - 3$.
$= \lim_{h \to 0} 4$	Simplify.
$= 4$	Take the limit.

Exercise 2.8.4

Find $f''(x)$ for $f(x) = x^2$.

Hint

We found $f'(x) = 2x$ in a previous checkpoint. Use Equation 2.8.1 to find the derivative of $f'(x)$

Answer

$f''(x) = 2$

Example 2.8.6: Finding Acceleration

The position of a particle along a coordinate axis at time t (in seconds) is given by $s(t) = 3t^2 - 4t + 1$ (in meters). Find the function that describes its acceleration at time t.

Solution

Since $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$, we begin by finding the derivative of $s(t)$:

$$\begin{aligned} s'(t) &= \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} \\ &= \lim_{h \to 0} \frac{3(t+h)^2 - 4(t+h) + 1 - (3t^2 - 4t + 1)}{h} \\ &= 6t - 4. \end{aligned}$$

Next,

$$\begin{aligned} s''(t) &= \lim_{h \to 0} \frac{s'(t+h) - s'(t)}{h} \\ &= \lim_{h \to 0} \frac{6(t+h) - 4 - (6t-4)}{h} \\ &= 6. \end{aligned}$$

Thus, $a = 6 \text{ m/s}^2$.

? Exercise 2.8.5

For $s(t) = t^3$, find $a(t)$.

Hint

Use Example 2.8.6as a guide.

Answer

$a(t) = 6t$

2.8.5 Key Concepts

- The derivative of a function $f(x)$ is the function whose value at x is $f'(x)$.
- The graph of a derivative of a function $f(x)$ is related to the graph of $f(x)$. Where $f(x)$ has a tangent line with positive slope, $f'(x) > 0$. Where $f(x)$ has a tangent line with negative slope, $f'(x) < 0$. Where $f(x)$ has a horizontal tangent line, $f'(x) = 0$.
- If a function is differentiable at a point, then it is continuous at that point. A function is not differentiable at a point if it is not continuous at the point, if it has a vertical tangent line at the point, or if the graph has a sharp corner or cusp.
- Higher-order derivatives are derivatives of derivatives, from the second derivative to the nth derivative.

2.8.6 Key Equations

- **The derivative function**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2.8.7 Glossary

derivative function
gives the derivative of a function at each point in the domain of the original function for which the derivative is defined

differentiable at a
a function for which $f'(a)$ exists is differentiable at a

differentiable on S
a function for which $f'(x)$ exists for each x in the open set S is differentiable on S

differentiable function
a function for which $f'(x)$ exists is a differentiable function

higher-order derivative
a derivative of a derivative, from the second derivative to the nth derivative, is called a higher-order derivative

2.8.8 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

- Paul Seeburger (Monroe Community College) added explanation of the alternative definition of the derivative used in the proof of that differentiability implies continuity.

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2.8E: EXERCISES FOR SECTION 2.8

For exercises 1 - 10, use the definition of a derivative to find $f'(x)$.

1) $f(x) = 6$

2) $f(x) = 2 - 3x$

Answer

$f'(x) = -3$

3) $f(x) = \dfrac{2x}{7} + 1$

4) $f(x) = 4x^2$

Answer

$f'(x) = 8x$

5) $f(x) = 5x - x^2$

6) $f(x) = \sqrt{2x}$

Answer

$f'(x) = \dfrac{1}{\sqrt{2x}}$

7) $f(x) = \sqrt{x - 6}$

8) $f(x) = \dfrac{9}{x}$

Answer

$f'(x) = \dfrac{-9}{x^2}$

9) $f(x) = x + \dfrac{1}{x}$

10) $f(x) = \dfrac{1}{\sqrt{x}}$

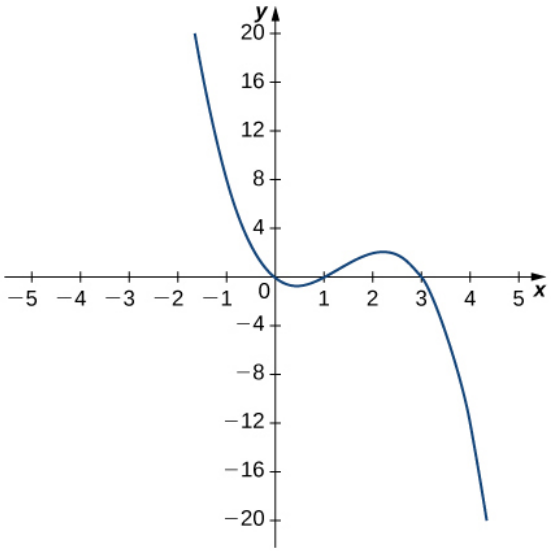
Answer

$f'(x) = \dfrac{-1}{2x^{3/2}}$

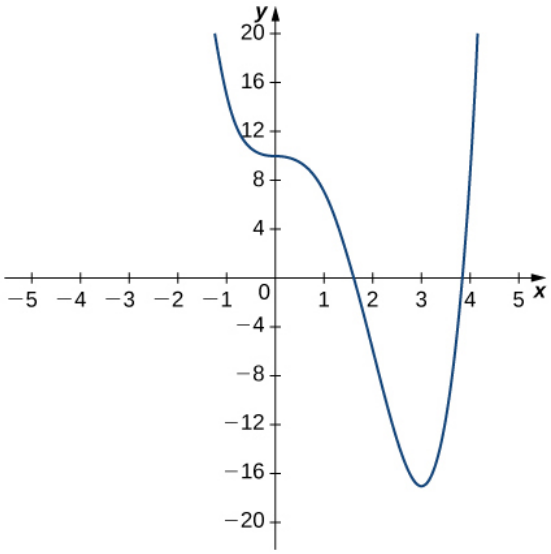
For the exercises 11 - 14, use the graph of $y = f(x)$ to sketch the graph of its derivative $f'(x)$.

11)

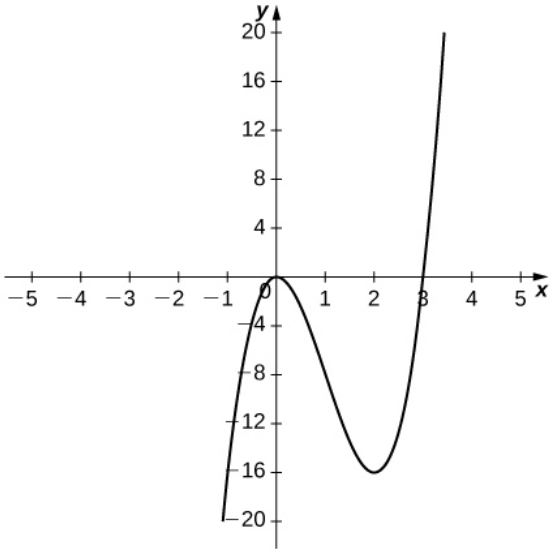
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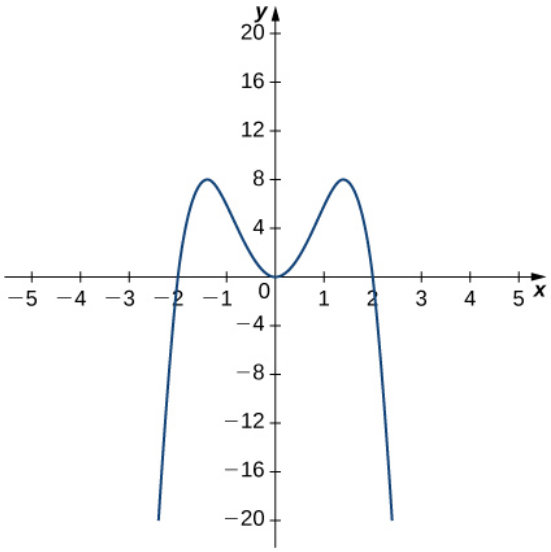
12)


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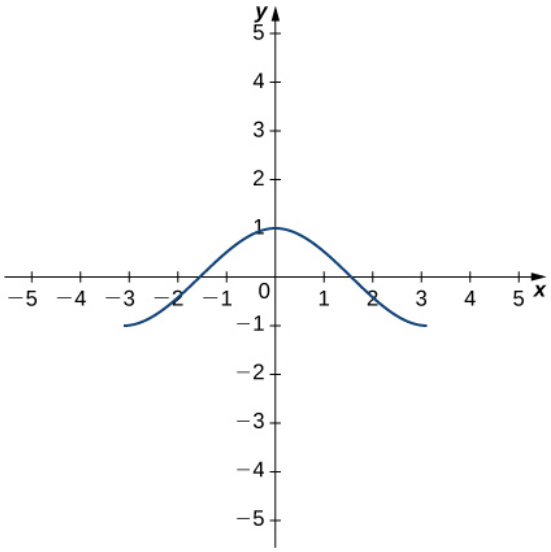
Answer


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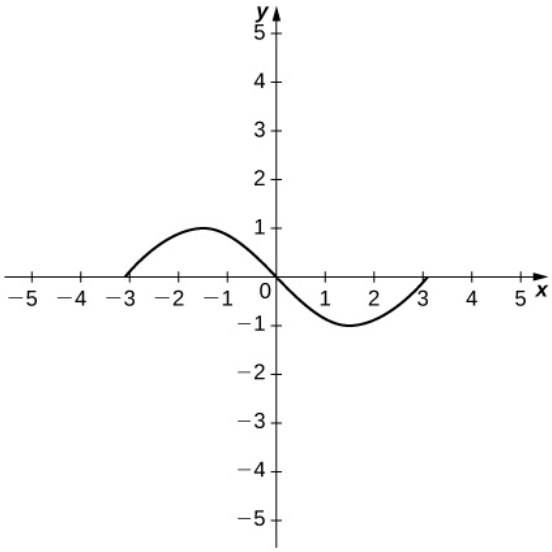
13)


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14)


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−5 −4 −3 −2 −1 0 1 2 3 4 5
−1
−2
−3
−4
−5

Answer



For exercises 15 - 20, the given limit represents the derivative of a function $y = f(x)$ at $x = a$. Find $f(x)$ and a.

15) $\lim\limits_{h \to 0} \dfrac{(1+h)^{2/3} - 1}{h}$

16) $\lim\limits_{h \to 0} \dfrac{[3(2+h)^2 + 2] - 14}{h}$

Answer

$f(x) = 3x^2 + 2, \quad a = 2$

18) $\lim\limits_{h \to 0} \dfrac{(2+h)^4 - 16}{h}$

Answer

$f(x) = x^4, \quad a = 2$

19) $\lim\limits_{h \to 0} \dfrac{[2(3+h)^2 - (3+h)] - 15}{h}$

For the functions in exercises 21 - 24,

a. sketch the graph and

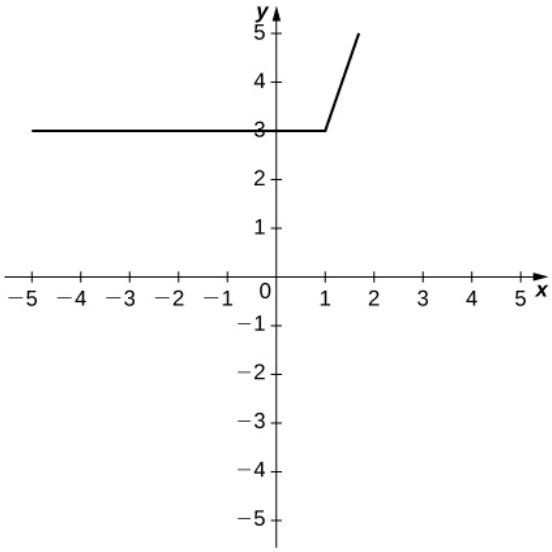
b. use the definition of a derivative to show that the function is not differentiable at $x = 1$.

21) $f(x) = \begin{cases} 2\sqrt{x}, & \text{if } 0 \le x \le 1 \\ 3x - 1, & \text{if } x > 1 \end{cases}$

22) $f(x) = \begin{cases} 3, & \text{if } x < 1 \\ 3x, & \text{if } x \ge 1 \end{cases}$

Answer

a.



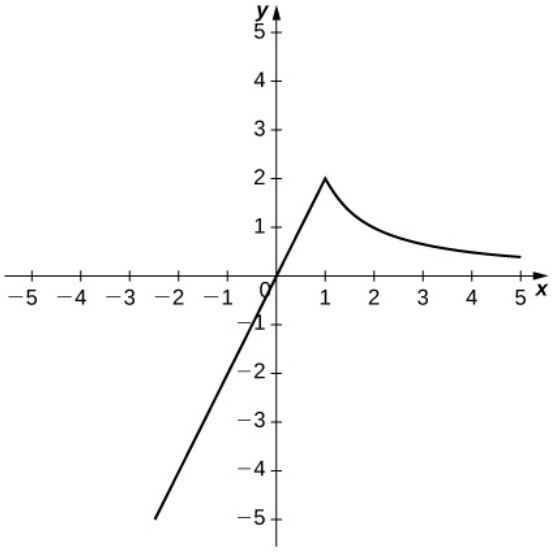
b. $\lim_{h\to 1^-} \dfrac{3-3}{h} \neq \lim_{h\to 1^+} \dfrac{3h}{h}$

23) $f(x) = \begin{cases} -x^2+2, & \text{if } x \leq 1 \\ x, & \text{if } x > 1 \end{cases}$

24) $f(x) = \begin{cases} 2x, & \text{if } x \leq 1 \\ \frac{2}{x}, & \text{if } x > 1 \end{cases}$

Answer

a.



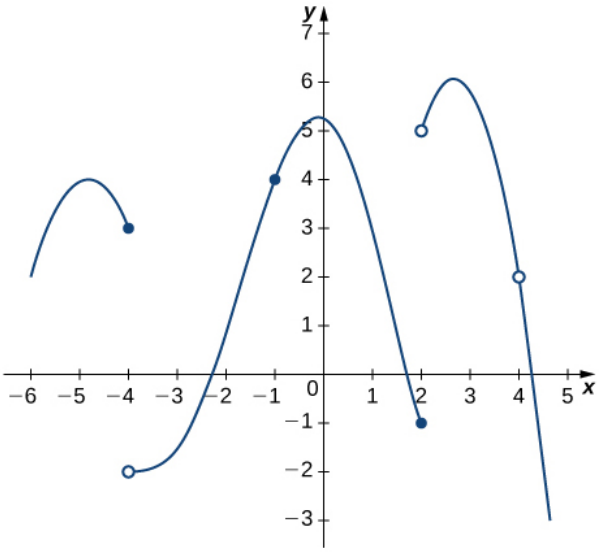
b. $\lim_{h\to 1^-} \dfrac{2h}{h} \neq \lim_{h\to 1^+} \dfrac{\frac{2}{x+h} - 2x}{h}$.

For the graphs in exercises 25 - 26,

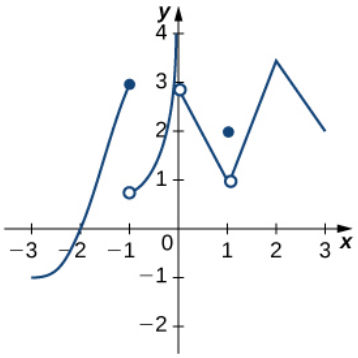
a. determine for which values of $x = a$ the $\lim_{x\to a} f(x)$ exists but f is not continuous at $x = a$, and

b. determine for which values of $x = a$ the function is continuous but not differentiable at $x = a$.

25)



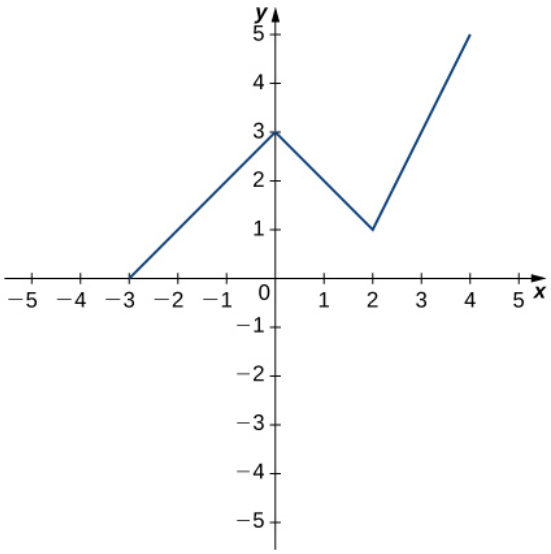
26)



Answer

a. $x = 1$,
b. $x = 2$

27) Use the graph to evaluate a. $f'(-0.5)$, b. $f'(0)$, c. $f'(1)$, d. $f'(2)$, and e. $f'(3)$, if it exists.



For the functions in exercises 28 - 30, use $f''(x) = \lim_{h \to 0} \dfrac{f'(x+h) - f'(x)}{h}$ to find $f''(x)$.

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28) $f(x) = 2 - 3x$

Answer

$f''(x) = 0$

29) $f(x) = 4x^2$

30) $f(x) = x + \dfrac{1}{x}$

Answer

$f''(x) = \dfrac{2}{x^3}$

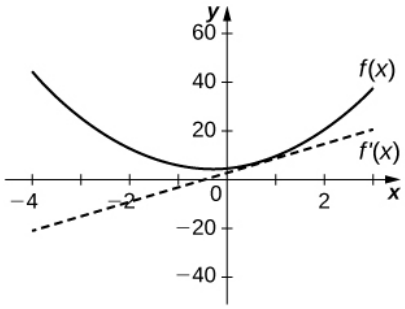
For exercises 31 - 36, use a calculator to graph $f(x)$. Determine the function $f'(x)$, then use a calculator to graph $f'(x)$.

31) [T] $f(x) = -\dfrac{5}{x}$

32) [T] $f(x) = 3x^2 + 2x + 4$.

Answer

$f'(x) = 6x + 2$

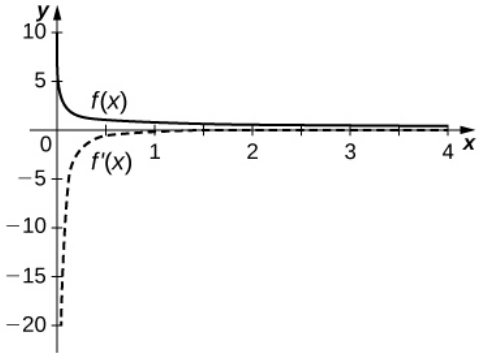


33) [T] $f(x) = \sqrt{x} + 3x$

34) [T] $f(x) = \dfrac{1}{\sqrt{2x}}$

Answer

$f'(x) = -\dfrac{1}{(2x)^{3/2}}$

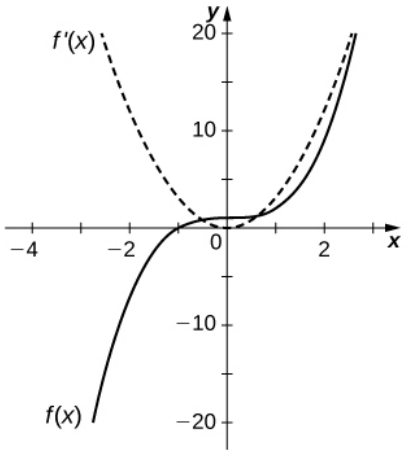


35) [T] $f(x) = 1 + x + \dfrac{1}{x}$

36) [T] $f(x) = x^3 + 1$

Answer

$f'(x) = 3x^2$



For exercises 37 - 42, describe what the two expressions represent in terms of each of the given situations. Be sure to include units.

a. $\dfrac{f(x+h)-f(x)}{h}$

b. $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

37) $P(x)$ denotes the population of a city at time x in years.

38) $C(x)$ denotes the total amount of money (in thousands of dollars) spent on concessions by x customers at an amusement park.

Answer

a. Average rate at which customers spent on concessions in thousands per customer.

b. Rate (in thousands per customer) at which x customers spent money on concessions in thousands per customer.

39) $R(x)$ denotes the total cost (in thousands of dollars) of manufacturing x clock radios

40) $g(x)$ denotes the grade (in percentage points) received on a test, given x hours of studying.

Answer

a. Average grade received on the test with an average study time between two values.

b. Rate (in percentage points per hour) at which the grade on the test increased or decreased for a given average study time of x hours.

41) $B(x)$denotes the cost (in dollars) of a sociology textbook at university bookstores in the United States in x years since 1990.

42) $p(x)$ denotes atmospheric pressure at an altitude of x feet.

Answer

a. Average change of atmospheric pressure between two different altitudes.

b. Rate (torr per foot) at which atmospheric pressure is increasing or decreasing at x feet.

43) Sketch the graph of a function $y=f(x)$ with all of the following properties:

a. $f'(x)>0$ for $-2\le x<1$

b. $f'(2)=0$

c. $f'(x)>0$ for $x>2$

d. $f(2)=2$ and $f(0)=1$

e. $\lim\limits_{x\to-\infty}f(x)=0$ and $\lim\limits_{x\to\infty}f(x)=\infty$

f. $f'(1)$ does not exist.

44) Suppose temperature T in degrees Fahrenheit at a height x in feet above the ground is given by $y=T(x)$.

a. Give a physical interpretation, with units, of $T'(x)$.

b. If we know that $T'(1000)=-0.1,$ explain the physical meaning.

Answer

a. The rate (in degrees per foot) at which temperature is increasing or decreasing for a given height x.

b. The rate of change of temperature as altitude changes at 1000 feet is -0.1 degrees per foot.

45) Suppose the total profit of a company is $y = P(x)$ thousand dollars when x units of an item are sold.

a. What does $\dfrac{P(b) - P(a)}{b - a}$ for $0 < a < b$ measure, and what are the units?

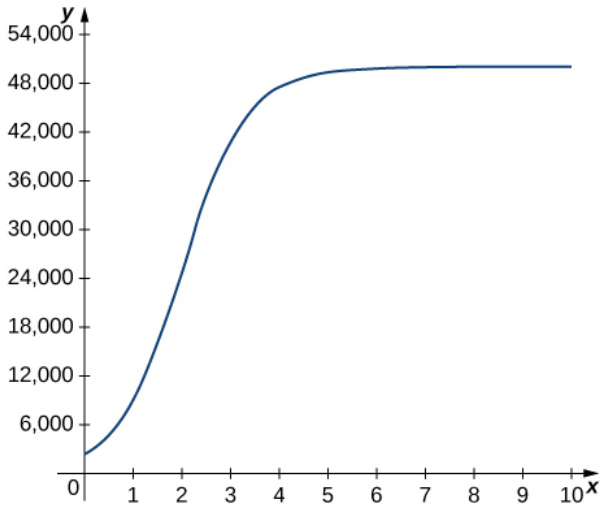
b. What does $P'(x)$ measure, and what are the units?

c. Suppose that $P'(30) = 5$, what is the approximate change in profit if the number of items sold increases from 30 to 31?

46) The graph in the following figure models the number of people $N(t)$ who have come down with the flu t weeks after its initial outbreak in a town with a population of 50,000 citizens.

a. Describe what $N'(t)$ represents and how it behaves as t increases.

b. What does the derivative tell us about how this town is affected by the flu outbreak?



Answer

a. The rate at which the number of people who have come down with the flu is changing t weeks after the initial outbreak.

b. The rate is increasing sharply up to the third week, at which point it slows down and then becomes constant.

For exercises 47 - 52, use the following table, which shows the height h of the Saturn V rocket for the Apollo 11 mission t seconds after launch.

Time (seconds)	Height (meters)
0	0
1	2
2	4
3	13
4	25
5	32

47) What is the physical meaning of $h'(t)$? What are the units?

48) [T] Construct a table of values for $h'(t)$ and graph both $h(t)$ and $h'(t)$ on the same graph. (Hint: for interior points, estimate both the left limit and right limit and average them.)

Answer

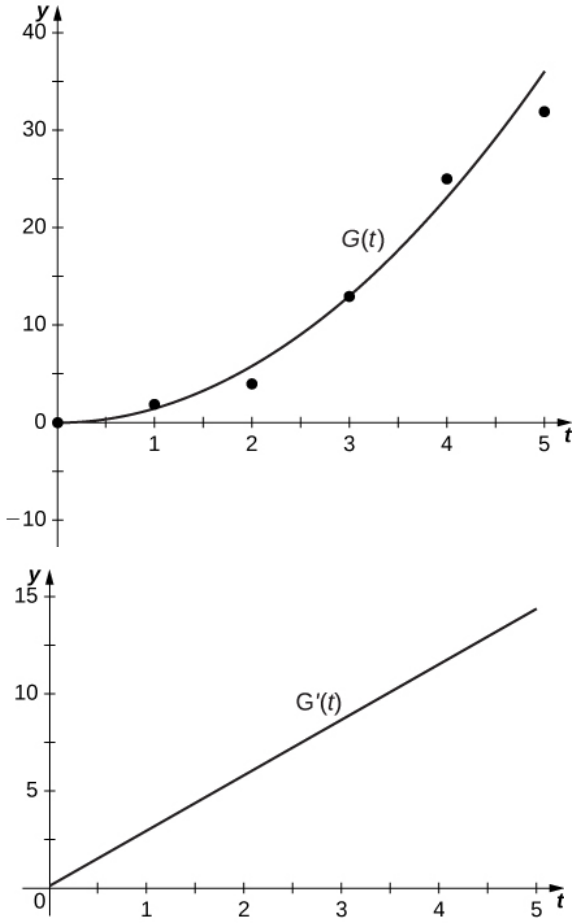
Time (seconds)	$h'(t)$ (m/s)
0	2
1	2
2	5.5
3	10.5
4	9.5
5	7

49) [T] The best linear fit to the data is given by $H(t) = 7.229t - 4.905$, where H is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $H'(t)$. Graph $H(t$ with the given data and, on a separate coordinate plane, graph $H'(t)$.

50) [T] The best quadratic fit to the data is given by $G(t) = 1.429t^2 + 0.0857t - 0.1429,$ where G is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $G'(t)$. Graph $G(t)$ with the given data and, on a separate coordinate plane, graph $G'(t)$.

Answer

$G'(t) = 2.858t + 0.0857$



51) [T] The best cubic fit to the data is given by $F(t) = 0.2037t^3 + 2.956t^2 - 2.705t + 0.4683$, where F is the height of the rocket (in m) and t is the time elapsed since take off. From this equation, determine $F'(t)$. Graph $F(t)$ with the given data and, on a separate coordinate plane, graph $F'(t)$. Does the linear, quadratic, or cubic function fit the data best?

52) Using the best linear, quadratic, and cubic fits to the data, determine what $H''(t)$, $G''(t)$ and $F''(t)$ are. What are the physical meanings of $H''(t)$, $G''(t)$ and $F''(t)$, and what are their units?

Answer

$H''(t) = 0$, $G''(t) = 2.858$ and $f''(t) = 1.222t + 5.912$ represent the acceleration of the rocket, with units of meters per second squared $(\mathrm{m/s^2})$.

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CHAPTER OVERVIEW

Unit 3: Derivatives

Calculating velocity and changes in velocity are important uses of calculus, but it is far more widespread than that. Calculus is important in all branches of mathematics, science, and engineering, and it is critical to analysis in business and health as well. In this chapter, we explore one of the main tools of calculus, the derivative, and show convenient ways to calculate derivatives. We apply these rules to a variety of functions in this chapter so that we can then explore applications of these techniques.

3.1: Differentiation Rules

3.1E: Exercises for Section 3.1

3.2: Derivatives as Rates of Change

3.2E: Exercises for Section 3.2

3.3: The Chain Rule

3.3E: Exercises for Section 3.3

3.4: Implicit Differentiation

3.4E: Exercises for Section 3.4

3.5: Exponential and Logarithmic Functions

3.5E: Exercises for Section 3.5

3.6: Derivatives of Exponential and Logarithmic Functions

3.6E: Exercises for Section 3.6

3.7: Maxima and Minima

3.7E: Exercises for Section 3.7

3.8: Derivatives and the Shape of a Graph

3.8E: Exercises for Section 3.8

3.9: Limits at Infinity, Asymptotes, and Rational Functions

3.9E: Exercises for Section 3.9

3.10: Applied Optimization Problems

3.10E: Exercises for Section 3.10

3.R: Chapter 3 Review Exercises

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3.1: Differentiation Rules

Learning Objectives

- State the constant, constant multiple, and power rules.
- Apply the sum and difference rules to combine derivatives.
- Use the product rule for finding the derivative of a product of functions.
- Use the quotient rule for finding the derivative of a quotient of functions.
- Extend the power rule to functions with negative exponents.
- Combine the differentiation rules to find the derivative of a polynomial or rational function.

Finding derivatives of functions by using the definition of the derivative can be a lengthy and, for certain functions, a rather challenging process. For example, previously we found that

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

by using a process that involved multiplying an expression by a conjugate prior to evaluating a limit.

The process that we could use to evaluate $\frac{d}{dx}(\sqrt[3]{x})$ using the definition, while similar, is more complicated.

In this section, we develop rules for finding derivatives that allow us to bypass this process. We begin with the basics.

3.1.1 The Basic Rules

The functions $f(x) = c$ and $g(x) = x^n$ where n is a positive integer are the building blocks from which all polynomials and rational functions are constructed. To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

3.1.2 The Constant Rule

We first apply the limit definition of the derivative to find the derivative of the constant function, $f(x) = c$. For this function, both $f(x) = c$ and $f(x+h) = c$, so we obtain the following result:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{c - c}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 = 0. \end{aligned}$$

The rule for differentiating constant functions is called the **constant rule**. It states that the derivative of a constant function is zero; that is, since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is 0. We restate this rule in the following theorem.

The Constant Rule

Let c be a constant. If $f(x) = c$, then $f'(x) = 0$.

Alternatively, we may express this rule as

$$\frac{d}{dx}(c) = 0.$$

Example 3.1.1: Applying the Constant Rule

Find the derivative of $f(x) = 8$.

Solution

This is just a one-step application of the rule: $f'(8) = 0$.

Exercise 3.1.1

Find the derivative of $g(x) = -3$.

Hint

Use the preceding example as a guide

Answer

0

3.1.3 The Power Rule

We have shown that

$$\frac{d}{dx}\left(x^2\right)=2x \quad \text{and} \quad \frac{d}{dx}\left(x^{1/2}\right)=\frac{1}{2}x^{-1/2}.$$

At this point, you might see a pattern beginning to develop for derivatives of the form $\frac{d}{dx}(x^n)$. We continue our examination of derivative formulas by differentiating power functions of the form $f(x)=x^n$ where n is a positive integer. We develop formulas for derivatives of this type of function in stages, beginning with positive integer powers. Before stating and proving the general rule for derivatives of functions of this form, we take a look at a specific case, $\frac{d}{dx}(x^3)$. As we go through this derivation, pay special attention to the portion of the expression in boldface, as the technique used in this case is essentially the same as the technique used to prove the general case.

Example 3.1.2: Differentiating x^3

Find $\frac{d}{dx}\left(x^3\right)$.

Solution:

$\frac{d}{dx}\left(x^3\right)=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$	
$=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$	Notice that the first term in the expansion of $(x+h)^3$ is x^3 and the second term is $3x^2h$. All other terms contain powers of h that are two or greater
$=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$	In this step the x^3 terms have been cancelled, leaving only terms containing h.
$=\lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}$	Factor out the common factor of h.
$=\lim_{h\to 0}(3x^2+3xh+h^2)$	After cancelling the common factor of h,the only term not containing h is $3x^2$.
$=3x^2$	Let h go to 0.

Exercise 3.1.2

Find $\frac{d}{dx}\left(x^4\right)$.

Hint

Use $(x+h)^4=x^4+4x^3h+6x^2h^2+4xh^3+h^4$ and follow the procedure outlined in the preceding example.

Answer

$\frac{d}{dx}\left(x^4\right)=4x^3$

As we shall see, the procedure for finding the derivative of the general form $f(x)=x^n$ is very similar. Although it is often unwise to draw general conclusions from specific examples, we note that when we differentiate $f(x)=x^3$, the power on x becomes the coefficient of x^2 in the derivative and the power on x in the derivative decreases by 1. The following theorem states that the **power rule** holds for all positive integer powers of x. We will eventually extend this result to negative integer powers. Later, we will see that this rule may also be extended first to rational powers of x and then to arbitrary powers of x. Be aware, however, that this rule does not apply to functions in which a constant is raised to a variable power, such as $f(x)=3^x$.

The Power Rule

Let n be a positive integer. If $f(x)=x^n$,then

$$f'(x)=nx^{n-1}.$$

Alternatively, we may express this rule as

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof

For $f(x) = x^n$ where n is a positive integer, we have

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

Since

$$(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n,$$

we see that

$$(x+h)^n - x^n = nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n.$$

Next, divide both sides by h:

$$\frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n}{h}.$$

Thus,

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-2} + h^{n-1}.$$

Finally,

$$f'(x) = \lim_{h \to 0}(nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-2} + h^{n-1})$$
$$= nx^{n-1}.$$

□

Example 3.1.3: Applying the Power Rule

Find the derivative of the function $f(x) = x^{10}$ by applying the power rule.

Solution

Using the power rule with $n = 10$, we obtain

$$f'(x) = 10x^{10-1} = 10x^9.$$

Exercise 3.1.3

Find the derivative of $f(x) = x^7$.

Hint

Use the power rule with $n = 7$.

Answer

$f'(x) = 7x^6$

3.1.4 The Sum, Difference, and Constant Multiple Rules

We find our next differentiation rules by looking at derivatives of sums, differences, and constant multiples of functions. Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract, or multiply by a constant. These rules are summarized in the following theorem.

Sum, Difference, and Constant Multiple Rules

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equations holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.

$$\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}\big(f(x)\big) + \frac{d}{dx}\big(g(x)\big);$$

that is,

$$\text{for } s(x) = f(x) + g(x), \quad s'(x) = f'(x) + g'(x).$$

Difference Rule. The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g :

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x));$$

that is,

$$\text{for } d(x) = f(x) - g(x), \quad d'(x) = f'(x) - g'(x).$$

Constant Multiple Rule. The derivative of a constant k multiplied by a function f is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}\big(kf(x)\big) = k\frac{d}{dx}\big(f(x)\big);$$

that is,

$$\text{for } m(x) = kf(x), \quad m'(x) = kf'(x).$$

Proof

We provide only the proof of the sum rule here. The rest follow in a similar manner.

For differentiable functions $f(x)$ and $g(x)$, we set $s(x) = f(x) + g(x)$. Using the limit definition of the derivative we have

$$s'(x) = \lim_{h \to 0} \frac{s(x+h) - s(x)}{h}.$$

By substituting $s(x+h) = f(x+h) + g(x+h)$ and $s(x) = f(x) + g(x)$, we obtain

$$s'(x) = \lim_{h \to 0} \frac{\big(f(x+h) + g(x+h)\big) - \big(f(x) + g(x)\big)}{h}.$$

Rearranging and regrouping the terms, we have

$$s'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right).$$

We now apply the sum law for limits and the definition of the derivative to obtain

$$s'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x).$$

□

Example 3.1.4: Applying the Constant Multiple Rule

Find the derivative of $g(x) = 3x^2$ and compare it to the derivative of $f(x) = x^2$.

Solution

We use the power rule directly:

$$g'(x) = \frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2) = 3(2x) = 6x.$$

Since $f(x) = x^2$ has derivative $f'(x) = 2x$, we see that the derivative of $g(x)$ is 3 times the derivative of $f(x)$. This relationship is illustrated in Figure 3.1.1.

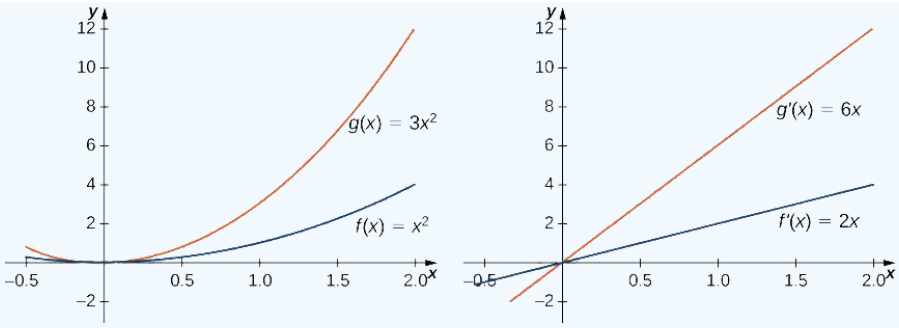


Figure 3.1.1: The derivative of $g(x)$ is 3 times the derivative of $f(x)$.

Example 3.1.5: Applying Basic Derivative Rules

Find the derivative of $f(x) = 2x^5 + 7$.

Solution

We begin by applying the rule for differentiating the sum of two functions, followed by the rules for differentiating constant multiples of functions and the rule for differentiating powers. To better understand the sequence in which the differentiation rules are applied, we use Leibniz notation throughout the solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x^5 + 7\right) \\ &= \frac{d}{dx}(2x^5) + \frac{d}{dx}(7) && \text{Apply the sum rule.} \\ &= 2\frac{d}{dx}(x^5) + \frac{d}{dx}(7) && \text{Apply the constant multiple rule.} \\ &= 2(5x^4) + 0 && \text{Apply the power rule and the constant rule.} \\ &= 10x^4 && \text{Simplify.} \end{aligned}$$

Exercise 3.1.4

Find the derivative of $f(x) = 2x^3 - 6x^2 + 3$.

Hint

Use the preceding example as a guide.

Answer

$f'(x) = 6x^2 - 12x$.

Example 3.1.6: Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2 - 4x + 6$ at $x = 1$

Solution

To find the equation of the tangent line, we need a point and a slope. To find the point, compute

$$f(1) = 1^2 - 4(1) + 6 = 3.$$

This gives us the point $(1, 3)$. Since the slope of the tangent line at 1 is $f'(1)$, we must first find $f'(x)$. Using the definition of a derivative, we have

$$f'(x) = 2x - 4$$

so the slope of the tangent line is $f'(1) = -2$. Using the point-slope formula, we see that the equation of the tangent line is

$$y - 3 = -2(x - 1).$$

Putting the equation of the line in slope-intercept form, we obtain

$$y = -2x + 5.$$

Exercise 3.1.5

Find the equation of the line tangent to the graph of $f(x) = 3x^2 - 11$ at $x = 2$. Use the point-slope form.

Hint

Use the preceding example as a guide.

Answer

$y = 12x - 23$

3.1.5 The Product Rule

Now that we have examined the basic rules, we can begin looking at some of the more advanced rules. The first one examines the derivative of the product of two functions. Although it might be tempting to assume that the derivative of the product is the product of the derivatives, similar to the sum and difference rules, the **product rule** does not follow this pattern. To see why we cannot use this pattern, consider the function $f(x) = x^2$, whose derivative is $f\,'(x) = 2x$ and not $\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$.

Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x).$$

That is,

$$\text{if } p(x) = f(x)g(x), \quad \text{then } p\,'(x) = f\,'(x)g(x) + g\,'(x)f(x).$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

Proof

We begin by assuming that $f(x)$ and $g(x)$ are differentiable functions. At a key point in this proof we need to use the fact that, since $g(x)$ is differentiable, it is also continuous. In particular, we use the fact that since $g(x)$ is continuous, $\lim_{h \to 0} g(x+h) = g(x)$.

By applying the limit definition of the derivative to $p(x) = f(x)g(x)$, we obtain

$$p\,'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

By adding and subtracting $f(x)g(x+h)$ in the numerator, we have

$$p\,'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$

After breaking apart this quotient and applying the sum law for limits, the derivative becomes

$$p\,'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h \to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}.$$

Rearranging, we obtain

$$\begin{aligned} p\,'(x) &= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h) \right) + \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \cdot f(x) \right) \\ &= \left(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right) \cdot \left(\lim_{h \to 0} g(x+h) \right) + \left(\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \right) \cdot f(x) \end{aligned}$$

By using the continuity of $g(x)$, the definition of the derivatives of $f(x)$ and $g(x)$, and applying the limit laws, we arrive at the product rule,

$$p\,'(x) = f\,'(x)g(x) + g\,'(x)f(x).$$

□

Example 3.1.7: Applying the Product Rule to Constant Functions

For $p(x) = f(x)g(x)$, use the product rule to find $p'(2)$ if $f(2) = 3,\ f'(2) = -4,\ g(2) = 1$, and $g'(2) = 6$.

Solution

Since $p(x) = f(x)g(x)$, $p'(x) = f'(x)g(x) + g'(x)f(x)$, and hence

$p'(2) = f'(2)g(2) + g'(2)f(2) = (-4)(1) + (6)(3) = 14.$

Example 3.1.8: Applying the Product Rule to Binomials

For $p(x) = (x^2 + 2)(3x^3 - 5x),$ find $p'(x)$ by applying the product rule. Check the result by first finding the product and then differentiating.

Solution

If we set $f(x) = x^2 + 2$ and $g(x) = 3x^3 - 5x$, then $f'(x) = 2x$ and $g'(x) = 9x^2 - 5$. Thus,

$p'(x) = f'(x)g(x) + g'(x)f(x) = (2x)(3x^3 - 5x) + (9x^2 - 5)(x^2 + 2).$

Simplifying, we have

$$p'(x) = 15x^4 + 3x^2 - 10.$$

To check, we see that $p(x) = 3x^5 + x^3 - 10x$ and, consequently, $p'(x) = 15x^4 + 3x^2 - 10.$

Exercise 3.1.6

Use the product rule to obtain the derivative of $p(x) = 2x^5(4x^2 + x)$.

Hint

Set $f(x) = 2x^5$ and $g(x) = 4x^2 + x$ and use the preceding example as a guide.

Answer

$p'(x) = 10x^4(4x^2 + x) + (8x + 1)(2x^5) = 56x^6 + 12x^5.$

3.1.6 The Quotient Rule

Having developed and practiced the product rule, we now consider differentiating quotients of functions. As we see in the following theorem, the derivative of the quotient is not the quotient of the derivatives; rather, it is the derivative of the function in the numerator times the function in the denominator minus the derivative of the function in the denominator times the function in the numerator, all divided by the square of the function in the denominator. In order to better grasp why we cannot simply take the quotient of the derivatives, keep in mind that

$$\frac{d}{dx}(x^2) = 2x, \text{ not } \frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(x)} = \frac{3x^2}{1} = 3x^2.$$

The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{\left(g(x)\right)^2}.$$

That is, if

$$q(x) = \frac{f(x)}{g(x)}$$

then

$$q'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{\left(g(x)\right)^2}.$$

The proof of the quotient rule is very similar to the proof of the product rule, so it is omitted here. Instead, we apply this new rule for finding derivatives in the next example.

Example 3.1.9: Applying the Quotient Rule

Use the quotient rule to find the derivative of $q(x) = \dfrac{5x^2}{4x+3}$.

Solution

Let $f(x) = 5x^2$ and $g(x) = 4x+3$. Thus, $f'(x) = 10x$ and $g'(x) = 4$.

Substituting into the quotient rule, we have

$$q'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} = \frac{10x(4x+3) - 4(5x^2)}{(4x+3)^2}.$$

Simplifying, we obtain

$$q'(x) = \frac{20x^2 + 30x}{(4x+3)^2}$$

Exercise 3.1.7

Find the derivative of $h(x) = \dfrac{3x+1}{4x-3}$.

Hint

Apply the quotient rule with $f(x) = 3x+1$ and $g(x) = 4x-3$.

Answer

$h'(x) = -\dfrac{13}{(4x-3)^2}$.

It is now possible to use the quotient rule to extend the power rule to find derivatives of functions of the form x^k where k is a negative integer.

Extended Power Rule

If k is a negative integer, then

$$\frac{d}{dx}(x^k) = kx^{k-1}.$$

Proof

If k is a negative integer, we may set $n = -k$, so that n is a positive integer with $k = -n$. Since for each positive integer n, $x^{-n} = \dfrac{1}{x^n}$, we may now apply the quotient rule by setting $f(x) = 1$ and $g(x) = x^n$. In this case, $f'(x) = 0$ and $g'(x) = nx^{n-1}$. Thus,

$$\frac{d}{dx}(x^{-n}) = \frac{0(x^n) - 1(nx^{n-1})}{(x^n)^2}.$$

Simplifying, we see that

$$\begin{aligned}\frac{d}{dx}(x^{-n}) &= \frac{-nx^{n-1}}{x^{2n}} \\ &= -nx^{(n-1)-2n} \\ &= -nx^{-n-1}.\end{aligned}$$

Finally, observe that since $k = -n$, by substituting we have

$$\frac{d}{dx}(x^k) = kx^{k-1}.$$

□

Example 3.1.10: Using the Extended Power Rule

Find $\dfrac{d}{dx}(x^{-4})$.

Solution

By applying the extended power rule with $k = -4$, we obtain

$$\frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5}.$$

Example 3.1.11: Using the Extended Power Rule and the Constant Multiple Rule

Use the extended power rule and the constant multiple rule to find $f(x) = \frac{6}{x^2}$.

Solution

It may seem tempting to use the quotient rule to find this derivative, and it would certainly not be incorrect to do so. However, it is far easier to differentiate this function by first rewriting it as $f(x) = 6x^{-2}$.

$$f'(x) = \frac{d}{dx}\left(\frac{6}{x^2}\right) = \frac{d}{dx}(6x^{-2}) \qquad \text{Rewrite } \frac{6}{x^2} \text{ as } 6x^{-2}.$$

$$= 6\frac{d}{dx}(x^{-2}) \qquad \text{Apply the constant multiple rule.}$$

$$= 6(-2x^{-3}) \qquad \text{Use the extended power rule to differentiate } x^{-2}.$$

$$= -12x^{-3} \qquad \text{Simplify.}$$

Exercise 3.1.8

Find the derivative of $g(x) = \frac{1}{x^7}$ using the extended power rule.

Hint

Rewrite $g(x) = \frac{1}{x^7} = x^{-7}$. Use the extended power rule with $k = -7$.

Answer

$g'(x) = -7x^{-8}$.

3.1.7 Combining Differentiation Rules

As we have seen throughout the examples in this section, it seldom happens that we are called on to apply just one differentiation rule to find the derivative of a given function. At this point, by combining the differentiation rules, we may find the derivatives of any polynomial or rational function. Later on we will encounter more complex combinations of differentiation rules. A good rule of thumb to use when applying several rules is to apply the rules in reverse of the order in which we would evaluate the function.

Example 3.1.12: Combining Differentiation Rules

For $k(x) = 3h(x) + x^2 g(x)$, find $k'(x)$.

Solution: Finding this derivative requires the sum rule, the constant multiple rule, and the product rule.

$k'(x) = \frac{d}{dx}(3h(x) + x^2 g(x)) = \frac{d}{dx}(3h(x)) + \frac{d}{dx}(x^2 g(x))$	Apply the sum rule.
$= 3\frac{d}{dx}(h(x)) + \left(\frac{d}{dx}(x^2)g(x) + \frac{d}{dx}(g(x))x^2\right)$	Apply the constant multiple rule to differentiate $3h(x)$ and the product rule to differentiate $x^2 g(x)$.
$= 3h'(x) + 2xg(x) + g'(x)x^2$	

Example 3.1.13: Extending the Product Rule

For $k(x) = f(x)g(x)h(x)$, express $k'(x)$ in terms of $f(x), g(x), h(x)$, and their derivatives.

Solution

We can think of the function $k(x)$ as the product of the function $f(x)g(x)$ and the function $h(x)$. That is, $k(x) = (f(x)g(x)) \cdot h(x)$. Thus,

$$k'(x) = \frac{d}{dx}(f(x)g(x)) \cdot h(x) + \frac{d}{dx}(h(x)) \cdot (f(x)g(x)). \qquad \text{Apply the product rule to the product of } f(x)g(x) \text{ and } h(x).$$

$$= (f'(x)g(x) + g'(x)f(x))h(x) + h'(x)f(x)g(x) \qquad \text{Apply the product rule to } f(x)g(x)$$

$$= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x). \qquad \text{Simplify.}$$

Example 3.1.14: Combining the Quotient Rule and the Product Rule

For $h(x)=\dfrac{2x^3k(x)}{3x+2}$, find $h'(x)$.

Solution

This procedure is typical for finding the derivative of a rational function.

$$h'(x)=\frac{\frac{d}{dx}(2x^3k(x))\cdot(3x+2)-\frac{d}{dx}(3x+2)\cdot(2x^3k(x))}{(3x+2)^2}$$ Apply the quotient rule.

$$=\frac{(6x^2k(x)+k'(x)\cdot 2x^3)(3x+2)-3(2x^3k(x))}{(3x+2)^2}$$ Apply the product rule to find $\frac{d}{dx}(2x^3k(x))$. Use $\frac{d}{dx}(3x+2)=3$.

$$=\frac{-6x^3k(x)+18x^3k(x)+12x^2k(x)+6x^4k'(x)+4x^3k'(x)}{(3x+2)^2}$$ Simplify

Exercise 3.1.9

Find $\dfrac{d}{dx}(3f(x)-2g(x))$.

Hint

Apply the difference rule and the constant multiple rule.

Answer

$3f'(x)-2g'(x)$.

Example 3.1.15: Determining Where a Function Has a Horizontal Tangent

Determine the values of x for which $f(x)=x^3-7x^2+8x+1$ has a horizontal tangent line.

Solution

To find the values of x for which $f(x)$ has a horizontal tangent line, we must solve $f'(x)=0$.

Since $f'(x)=3x^2-14x+8=(3x-2)(x-4)$,

we must solve $(3x-2)(x-4)=0$. Thus we see that the function has horizontal tangent lines at $x=\dfrac{2}{3}$ and $x=4$ as shown in the following graph.

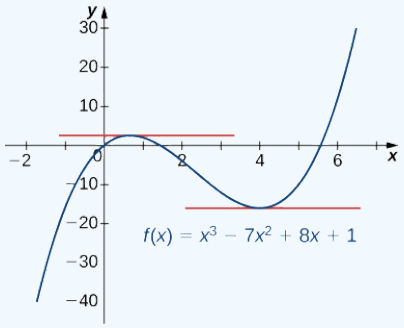


Figure 3.1.2: This function has horizontal tangent lines at $x=2/3$ and $x=4$.

Example 3.1.16: Finding a Velocity

The position of an object on a coordinate axis at time t is given by $s(t)=\dfrac{t}{t^2+1}$. What is the initial velocity of the object?

Solution

Since the initial velocity is $v(0)=s'(0)$, begin by finding $s'(t)$ by applying the quotient rule:

$$s'(t)=\frac{1(t^2+1)-2t(t)}{(t^2+1)^2}=\frac{1-t^2}{(t^2+1)^2}\ .$$

After evaluating, we see that $v(0)=1$.

Exercise 3.1.10

Find the values of x for which the line tangent to the graph of $f(x) = 4x^2 - 3x + 2$ has a tangent line parallel to the line $y = 2x + 3$.

Hint

Solve $f'(x) = 2$.

Answer

$\frac{5}{8}$

Formula One Grandstands

Formula One car races can be very exciting to watch and attract a lot of spectators. Formula One track designers have to ensure sufficient grandstand space is available around the track to accommodate these viewers. However, car racing can be dangerous, and safety considerations are paramount. The grandstands must be placed where spectators will not be in danger should a driver lose control of a car (Figure 3.1.3).



Figure 3.1.3: The grandstand next to a straightaway of the Circuit de Barcelona-Catalunya race track, located where the spectators are not in danger.

Safety is especially a concern on turns. If a driver does not slow down enough before entering the turn, the car may slide off the racetrack. Normally, this just results in a wider turn, which slows the driver down. But if the driver loses control completely, the car may fly off the track entirely, on a path tangent to the curve of the racetrack.

Suppose you are designing a new Formula One track. One section of the track can be modeled by the function $f(x) = x^3 + 3x^2 + x$ (Figure 3.1.4). The current plan calls for grandstands to be built along the first straightaway and around a portion of the first curve. The plans call for the front corner of the grandstand to be located at the point $(-1.9, 2.8)$. We want to determine whether this location puts the spectators in danger if a driver loses control of the car.

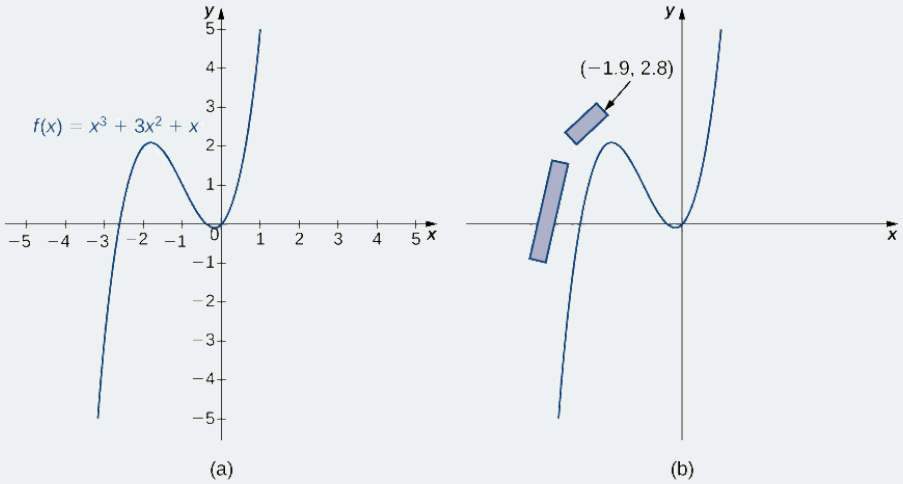


Figure 3.1.4: (a) One section of the racetrack can be modeled by the function $f(x) = x^3 + 3x^2 + x$. (b) The front corner of the grandstand is located at $(-1.9, 2.8)$.

1. Physicists have determined that drivers are most likely to lose control of their cars as they are coming into a turn, at the point where the slope of the tangent line is 1. Find the (x, y) coordinates of this point near the turn.
2. Find the equation of the tangent line to the curve at this point.
3. To determine whether the spectators are in danger in this scenario, find the x-coordinate of the point where the tangent line crosses the line $y = 2.8$. Is this point safely to the right of the grandstand? Or are the spectators in danger?

4. What if a driver loses control earlier than the physicists project? Suppose a driver loses control at the point $(-2.5, 0.625)$. What is the slope of the tangent line at this point?
5. If a driver loses control as described in part 4, are the spectators safe?
6. Should you proceed with the current design for the grandstand, or should the grandstands be moved?

3.1.8 Key Concepts

- The derivative of a constant function is zero.
- The derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1.
- The derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative.
- The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.
- The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g.
- The derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.
- The derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function.
- We used the limit definition of the derivative to develop formulas that allow us to find derivatives without resorting to the definition of the derivative. These formulas can be used singly or in combination with each other.

3.1.9 Glossary

constant multiple rule

the derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative: $\frac{d}{dx}\big(cf(x)\big) = cf\,'(x)$

constant rule

the derivative of a constant function is zero: $\frac{d}{dx}(c) = 0$, where c is a constant

difference rule

the derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g: $\frac{d}{dx}\big(f(x) - g(x)\big) = f\,'(x) - g'(x)$

power rule

the derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1: If n is an integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$

product rule

the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function: $\frac{d}{dx}\big(f(x)g(x)\big) = f\,'(x)g(x) + g'(x)f(x)$

quotient rule

the derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f\,'(x)g(x) - g'(x)f(x)}{\big(g(x)\big)^2}$

sum rule

the derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g: $\frac{d}{dx}\big(f(x) + g(x)\big) = f\,'(x) + g'(x)$

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3.1E: EXERCISES FOR SECTION 3.1

In exercises 1 - 12, find $f'(x)$ for each function.

1) $f(x) = x^7 + 10$

2) $f(x) = 5x^3 - x + 1$

Answer

$f'(x) = 15x^2 - 1$

3) $f(x) = 4x^2 - 7x$

4) $f(x) = 8x^4 + 9x^2 - 1$

Answer

$f'(x) = 32x^3 + 18x$

5) $f(x) = x^4 + 2x$

6) $f(x) = 3x\left(18x^4 + \dfrac{13}{x+1}\right)$

Answer

$f'(x) = 270x^4 + \dfrac{39}{(x+1)^2}$

7) $f(x) = (x+2)(2x^2 - 3)$

8) $f(x) = x^2\left(\dfrac{2}{x^2} + \dfrac{5}{x^3}\right)$

Answer

$f'(x) = \dfrac{-5}{x^2}$

9) $f(x) = \dfrac{x^3 + 2x^2 - 4}{3}$

10) $f(x) = \dfrac{4x^3 - 2x + 1}{x^2}$

Answer

$f'(x) = \dfrac{4x^4 + 2x^2 - 2x}{x^4}$

11) $f(x) = \dfrac{x^2 + 4}{x^2 - 4}$

12) $f(x) = \dfrac{x + 9}{x^2 - 7x + 1}$

Answer

$f'(x) = \dfrac{-x^2 - 18x + 64}{(x^2 - 7x + 1)^2}$

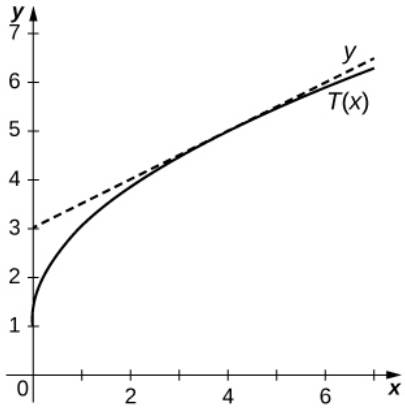
In exercises 13 - 16, find the equation of the tangent line $T(x)$ to the graph of the given function at the indicated point. Use a graphing calculator to graph the function and the tangent line.

13) [T] $y = 3x^2 + 4x + 1$ at $(0, 1)$

14) [T] $y = 2\sqrt{x} + 1$ at $(4, 5)$

Answer

$T(x) = \frac{1}{2}x + 3$

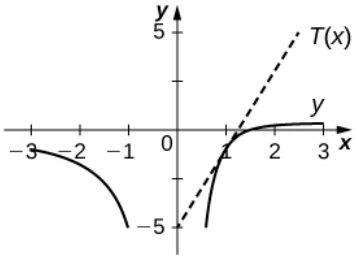


15) [T] $y = \dfrac{2x}{x-1}$ at $(-1, 1)$

16) [T] $y = \dfrac{2}{x} - \dfrac{3}{x^2}$ at $(1, -1)$

Answer

$T(x) = 4x - 5$



In exercises 17 - 20, assume that $f(x)$ and $g(x)$ are both differentiable functions for all x. Find the derivative of each of the functions $h(x)$.

17) $h(x) = 4f(x) + \dfrac{g(x)}{7}$

18) $h(x) = x^3 f(x)$

Answer

$h'(x) = 3x^2 f(x) + x^3 f'(x)$

19) $h(x) = \dfrac{f(x)g(x)}{2}$

20) $h(x) = \dfrac{3f(x)}{g(x) + 2}$

Answer

$h'(x) = \dfrac{3f'(x)(g(x) + 2) - 3f(x)g'(x)}{(g(x) + 2)^2}$

For exercises 21 - 24, assume that $f(x)$ and $g(x)$ are both differentiable functions with values as given in the following table. Use the following table to calculate the following derivatives.

x	1	2	3	4
$f(x)$	3	5	−2	0
$g(x)$	2	3	−4	6
$f'(x)$	−1	7	8	−3
$g'(x)$	4	1	2	9

21) Find $h'(1)$ if $h(x) = xf(x) + 4g(x)$.

22) Find $h'(2)$ if $h(x) = \dfrac{f(x)}{g(x)}$.

Answer

$h'(2) = \frac{16}{9}$

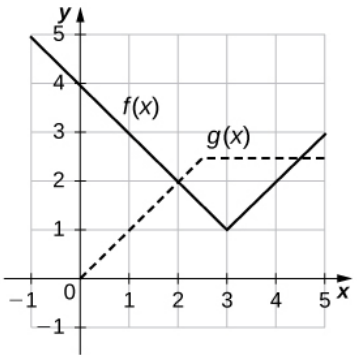
23) Find $h'(3)$ if $h(x) = 2x + f(x)g(x)$.

24) Find $h'(4)$ if $h(x) = \dfrac{1}{x} + \dfrac{g(x)}{f(x)}$.

Answer

$h'(4)$ is undefined.

In exercises 25 - 27, use the following figure to find the indicated derivatives, if they exist.



25) Let $h(x) = f(x) + g(x)$. Find

a) $h'(1)$,

b) $h'(3)$, and

c) $h'(4)$.

26) Let $h(x) = f(x)g(x)$. Find

a) $h'(1)$,

b) $h'(3)$, and

c) $h'(4)$.

Answer

a. $h'(1) = 2$,
b. $h'(3)$ does not exist,
c. $h'(4) = 2.5$

27) Let $h(x) = \dfrac{f(x)}{g(x)}$. Find

a) $h'(1)$,

b) $h'(3)$, and

c) $h'(4)$.

In exercises 28 - 31,

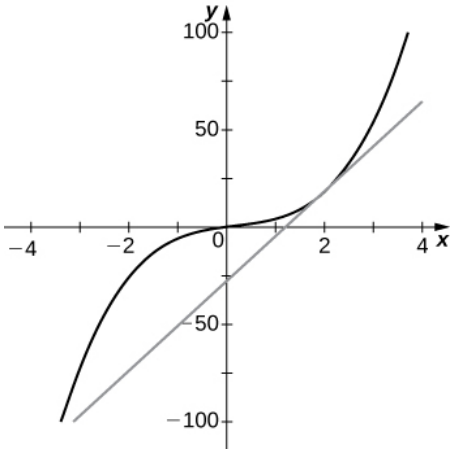
a) evaluate $f'(a)$, and

b) graph the function $f(x)$ and the tangent line at $x = a$.

28) [T] $f(x) = 2x^3 + 3x - x^2, \quad a = 2$

Answer

a. 23
b. $y = 23x - 28$

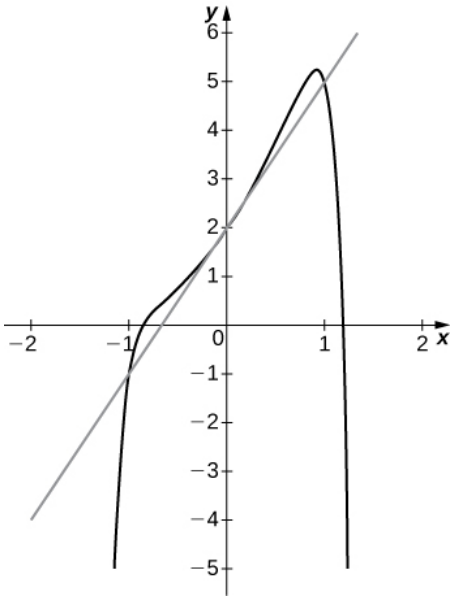


29) [T] $f(x) = \dfrac{1}{x} - x^2, \quad a = 1$

30) [T] $f(x) = x^2 - x^{12} + 3x + 2, \quad a = 0$

Answer

a. 3
b. $y = 3x + 2$



31) [T] $f(x) = \dfrac{1}{x} - x^{2/3}, \quad a = -1$

32) Find the equation of the tangent line to the graph of $f(x) = 2x^3 + 4x^2 - 5x - 3$ at $x = -1$.

Answer

$y = -7x - 3$

33) Find the equation of the tangent line to the graph of $f(x) = x^2 + \dfrac{4}{x} - 10$ at $x = 8$.

34) Find the equation of the tangent line to the graph of $f(x) = (3x - x^2)(3 - x - x^2)$ at $x = 1$.

Answer

$y = -5x + 7$

35) Find the point on the graph of $f(x) = x^3$ such that the tangent line at that point has an x-intercept of $(6, 0)$.

36) Find the equation of the line passing through the point $P(3, 3)$ and tangent to the graph of $f(x) = \dfrac{6}{x-1}$.

Answer

$y = -\frac{3}{2}x + \frac{15}{2}$

37) Determine all points on the graph of $f(x) = x^3 + x^2 - x - 1$ for which the slope of the tangent line is

a. horizontal

b. −1.

38) Find a quadratic polynomial such that $f(1) = 5,\ f'(1) = 3$ and $f''(1) = -6$.

Answer

$y = -3x^2 + 9x - 1$

39) A car driving along a freeway with traffic has traveled $s(t) = t^3 - 6t^2 + 9t$ meters in t seconds.

a. Determine the time in seconds when the velocity of the car is 0.

b. Determine the acceleration of the car when the velocity is 0.

40) [T] A herring swimming along a straight line has traveled $s(t) = \dfrac{t^2}{t^2+2}$ feet in t seconds. Determine the velocity of the herring when it has traveled 3 seconds.

Answer

$\frac{12}{121}$ or 0.0992 ft/s

41) The population in millions of arctic flounder in the Atlantic Ocean is modeled by the function $P(t) = \dfrac{8t+3}{0.2t^2+1}$, where t is measured in years.

a. Determine the initial flounder population.

b. Determine $P'(10)$ and briefly interpret the result.

42) [T] The concentration of antibiotic in the bloodstream t hours after being injected is given by the function $C(t) = \dfrac{2t^2+t}{t^3+50}$, where C is measured in milligrams per liter of blood.

a. Find the rate of change of $C(t)$.

b. Determine the rate of change for $t = 8, 12, 24$,and 36.

c. Briefly describe what seems to be occurring as the number of hours increases.

Answer

a. $\dfrac{-2t^4 - 2t^3 + 200t + 50}{(t^3+50)^2}$

b. -0.02395 mg/L-hr, -0.01344 mg/L-hr, -0.003566 mg/L-hr, -0.001579 mg/L-hr

c. The rate at which the concentration of drug in the bloodstream decreases is slowing to 0 as time increases.

43) A book publisher has a cost function given by $C(x) = \dfrac{x^3+2x+3}{x^2}$, where x is the number of copies of a book in thousands and C is the cost, per book, measured in dollars. Evaluate $C'(2)$and explain its meaning.

44) [T] According to Newton's law of universal gravitation, the force F between two bodies of constant mass m_1 and m_2 is given by the formula $F = \dfrac{Gm_1m_2}{d^2}$, where G is the gravitational constant and d is the distance between the bodies.

a. Suppose that G, m_1, and m_2 are constants. Find the rate of change of force F with respect to distance d.

b. Find the rate of change of force F with gravitational constant $G = 6.67 \times 10^{-11}\text{Nm}^2/\text{kg}^2$, on two bodies 10 meters apart, each with a mass of 1000 kilograms.

Answer

a. $F'(d) = \dfrac{-2Gm_1m_2}{d_3}$

b. -1.33×10^{-7} N/m

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3.2: Derivatives as Rates of Change

Learning Objectives

- Determine a new value of a quantity from the old value and the amount of change.
- Calculate the average rate of change and explain how it differs from the instantaneous rate of change.
- Apply rates of change to displacement, velocity, and acceleration of an object moving along a straight line.
- Predict the future population from the present value and the population growth rate.
- Use derivatives to calculate marginal cost and revenue in a business situation.

In this section we look at some applications of the derivative by focusing on the interpretation of the derivative as the rate of change of a function. These applications include **acceleration** and velocity in physics, **population growth rates** in biology, and marginal functions in economics.

3.2.1 Amount of Change Formula

One application for derivatives is to estimate an unknown value of a function at a point by using a known value of a function at some given point together with its rate of change at the given point. If $f(x)$ is a function defined on an interval $[a, a+h]$, then the **amount of change** of $f(x)$ over the interval is the change in the y values of the function over that interval and is given by

$$f(a+h)-f(a).$$

The **average rate of change** of the function f over that same interval is the ratio of the amount of change over that interval to the corresponding change in the x values. It is given by

$$\frac{f(a+h)-f(a)}{h}.$$

As we already know, the instantaneous rate of change of $f(x)$ at a is its derivative

$$f\,'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

For small enough values of h, $f\,'(a)\approx\frac{f(a+h)-f(a)}{h}$. We can then solve for $f(a+h)$ to get the amount of change formula:

$$f(a+h)\approx f(a)+f\,'(a)h. \tag{3.2.1}$$

We can use this formula if we know only $f(a)$and $f\,'(a)$ and wish to estimate the value of $f(a+h)$. For example, we may use the current population of a city and the rate at which it is growing to estimate its population in the near future. As we can see in Figure 3.2.1, we are approximating $f(a+h)$ by the y coordinate at a+h on the line tangent to $f(x)$ at $x=a$. Observe that the accuracy of this estimate depends on the value of h as well as the value of $f\,'(a)$.

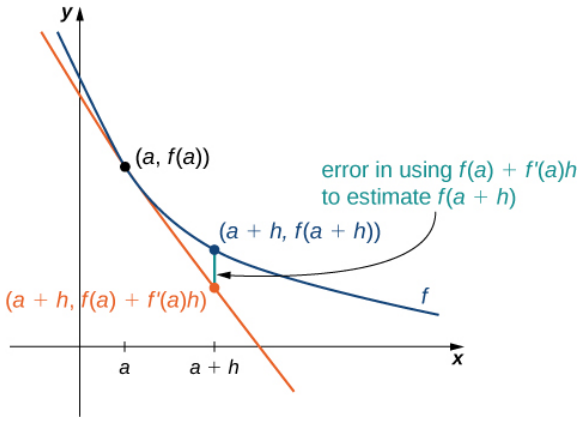


Figure 3.2.1: The new value of a changed quantity equals the original value plus the rate of change times the interval of change: $f(a+h)\approx f(a)+f\,'(a)h$.

Example 3.2.1: Estimating the Value of a Function

If $f(3) = 2$ and $f'(3) = 5$, estimate $f(3.2)$.

Solution

Begin by finding h. We have $h = 3.2 - 3 = 0.2$. Thus,

$$f(3.2) = f(3 + 0.2) \approx f(3) + (0.2)f'(3) = 2 + 0.2(5) = 3.$$

Exercise 3.2.1

Given $f(10) = -5$ and $f'(10) = 6$, estimate $f(10.1)$.

Hint

Use the same process as in the preceding example.

Answer

-4.4

3.2.2 Motion along a Line

Another use for the derivative is to analyze motion along a line. We have described velocity as the rate of change of position. If we take the derivative of the velocity, we can find the acceleration, or the rate of change of velocity. It is also important to introduce the idea of **speed**, which is the magnitude of velocity. Thus, we can state the following mathematical definitions.

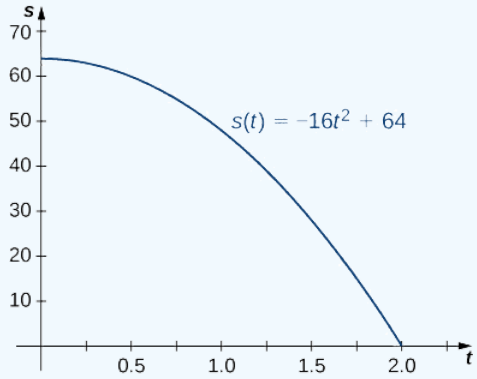
Definition

Let $s(t)$ be a function giving the position of an object at time t.

- The velocity of the object at time t is given by $v(t) = s'(t)$.
- The speed of the object at time t is given by $|v(t)|$.
- The acceleration of the object at t is given by $a(t) = v'(t) = s''(t)$.

Example 3.2.2: Comparing Instantaneous Velocity and Average Velocity

A ball is dropped from a height of 64 feet. Its height above ground (in feet) t seconds later is given by $s(t) = -16t^2 + 64$.



a. What is the instantaneous velocity of the ball when it hits the ground?
b. What is the average velocity during its fall?

Solution

The first thing to do is determine how long it takes the ball to reach the ground. To do this, set $s(t) = 0$. Solving $-16t^2 + 64 = 0$, we get $t = 2$, so it takes 2 seconds for the ball to reach the ground.

a. The instantaneous velocity of the ball as it strikes the ground is $v(2)$. Since $v(t) = s'(t) = -32t$, we obtain $v(t) = -64$ ft/s.

b. The average velocity of the ball during its fall is

$$v_{ave} = \frac{s(2)-s(0)}{2-0} = \frac{0-64}{2} = -32 \text{ ft/s.}$$

Example 3.2.3: Interpreting the Relationship between $v(t)$ and $a(t)$

A particle moves along a coordinate axis in the positive direction to the right. Its position at time t is given by $s(t) = t^3 - 4t + 2$. Find $v(1)$ and $a(1)$ and use these values to answer the following questions.

a. Is the particle moving from left to right or from right to left at time $t = 1$?

b. Is the particle speeding up or slowing down at time $t = 1$?

Solution

Begin by finding $v(t)$ and $a(t)$.

$v(t) = s'(t) = 3t^2 - 4$ and $a(t) = v'(t) = s''(t) = 6t$.

Evaluating these functions at $t = 1$, we obtain $v(1) = -1$ and $a(1) = 6$.

a. Because $v(1) < 0$, the particle is moving from right to left.

b. Because $v(1) < 0$ and $a(1) > 0$, velocity and acceleration are acting in opposite directions. In other words, the particle is being accelerated in the direction opposite the direction in which it is traveling, causing $|v(t)|$ to decrease. The particle is slowing down.

Example 3.2.4: Position and Velocity

The position of a particle moving along a coordinate axis is given by $s(t) = t^3 - 9t^2 + 24t + 4,\ t \geq 0$.

a. Find $v(t)$.

b. At what time(s) is the particle at rest?

c. On what time intervals is the particle moving from left to right? From right to left?

d. Use the information obtained to sketch the path of the particle along a coordinate axis.

Solution

a. The velocity is the derivative of the position function:

$$v(t) = s'(t) = 3t^2 - 18t + 24.$$

b. The particle is at rest when $v(t) = 0$, so set $3t^2 - 18t + 24 = 0$. Factoring the left-hand side of the equation produces $3(t-2)(t-4) = 0$. Solving, we find that the particle is at rest at $t = 2$ and $t = 4$.

c. The particle is moving from left to right when $v(t) > 0$ and from right to left when $v(t) < 0$. Figure 3.2.2 gives the analysis of the sign of $v(t)$ for $t \geq 0$, but it does not represent the axis along which the particle is moving.



Figure 3.2.2: The sign of $v(t)$ determines the direction of the particle.

- Since $3t^2 - 18t + 24 > 0$ on $[0, 2) \cup (4, +\infty)$, the particle is moving from left to right on these intervals.
- Since $3t^2 - 18t + 24 < 0$ on $(2, 4)$, the particle is moving from right to left on this interval.

d. Before we can sketch the graph of the particle, we need to know its position at the time it starts moving $(t = 0)$ and at the times that it changes direction $(t = 2, 4)$. We have $s(0) = 4$, $s(2) = 24$, and $s(4) = 20$. This means that the particle begins on the coordinate axis at 4 and changes direction at 24 and 20 on the coordinate axis. The path of the particle is shown on a coordinate axis in Figure 3.2.3.

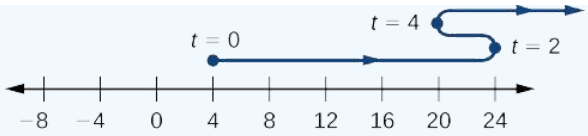


Figure 3.2.3: The path of the particle can be determined by analyzing $v(t)$.

Exercise 3.2.2

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = t^2 - 5t + 1$. Is the particle moving from right to left or from left to right at time $t = 3$?

Hint

Find $v(3)$ and look at the sign.

Answer

left to right

3.2.3 Population Change

In addition to analyzing velocity, speed, acceleration, and position, we can use derivatives to analyze various types of populations, including those as diverse as bacteria colonies and cities. We can use a current population, together with a growth rate, to estimate the size of a population in the future. The population growth rate is the rate of change of a population and consequently can be represented by the derivative of the size of the population.

Definition

If $P(t)$ is the number of entities present in a population, then the population growth rate of $P(t)$ is defined to be $P'(t)$.

Example 3.2.5: Estimating a Population

The population of a city is tripling every 5 years. If its current population is 10,000, what will be its approximate population 2 years from now?

Solution

Let $P(t)$ be the population (in thousands) t years from now. Thus, we know that $P(0) = 10$ and based on the information, we anticipate $P(5) = 30$. Now estimate $P'(0)$, the current growth rate, using

$$P'(0) \approx \frac{P(5) - P(0)}{5 - 0} = \frac{30 - 10}{5} = 4 \, .$$

By applying Equation 3.2.1 to $P(t)$, we can estimate the population 2 years from now by writing

$$P(2) \approx P(0) + (2)P'(0) \approx 10 + 2(4) = 18 \, ;$$

thus, in 2 years the population will be 18,000.

Exercise 3.2.3

The current population of a mosquito colony is known to be 3,000; that is, $P(0) = 3,000$. If $P'(0) = 100$, estimate the size of the population in 3 days, where t is measured in days.

Hint

Use $P(3) \approx P(0) + 3P'(0)$

Answer

3,300

3.2.4 Changes in Cost and Revenue

In addition to analyzing motion along a line and population growth, derivatives are useful in analyzing changes in cost, revenue, and profit. The concept of a marginal function is common in the fields of business and economics and implies the use of derivatives. The marginal cost is the derivative of the cost function. The marginal revenue is the derivative of the revenue function. The marginal profit is the derivative of the profit function, which is based on the cost function and the revenue function.

Definition

- If $C(x)$ is the cost of producing x items, then the **marginal cost** $MC(x)$ is $MC(x) = C'(x)$.
- If $R(x)$ is the revenue obtained from selling x items, then the marginal revenue $MR(x)$ is $MR(x) = R'(x)$.
- If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items, then the **marginal profit** $MP(x)$ is defined to be $MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$.

We can roughly approximate

$$MC(x) = C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h}$$

by choosing an appropriate value for h. Since x represents objects, a reasonable and small value for h is 1. Thus, by substituting $h = 1$, we get the approximation $MC(x) = C'(x) \approx C(x+1) - C(x)$. Consequently, $C'(x)$ for a given value of x can be thought of as the change in cost associated with producing one additional item. In a similar way, $MR(x) = R'(x)$ approximates the revenue obtained by selling one additional item, and $MP(x) = P'(x)$ approximates the profit obtained by producing and selling one additional item.

Example 3.2.6: Applying Marginal Revenue

Assume that the number of barbeque dinners that can be sold, x, can be related to the price charged, p, by the equation $p(x) = 9 - 0.03x, 0 \leq x \leq 300$.

In this case, the revenue in dollars obtained by selling x barbeque dinners is given by

$$R(x) = xp(x) = x(9 - 0.03x) = -0.03x^2 + 9x \quad \text{for } 0 \leq x \leq 300\,.$$

Use the marginal revenue function to estimate the revenue obtained from selling the 101st barbeque dinner. Compare this to the actual revenue obtained from the sale of this dinner.

Solution

First, find the marginal revenue function: $MR(x) = R'(x) = -0.06x + 9$.

Next, use $R'(100)$ to approximate $R(101) - R(100)$, the revenue obtained from the sale of the 101st dinner. Since $R'(100) = 3$, the revenue obtained from the sale of the 101st dinner is approximately \$3.

The actual revenue obtained from the sale of the 101st dinner is

$$R(101) - R(100) = 602.97 - 600 = 2.97, \text{ or } \$2.97.$$

The marginal revenue is a fairly good estimate in this case and has the advantage of being easy to compute.

Exercise 3.2.4

Suppose that the profit obtained from the sale of x fish-fry dinners is given by $P(x) = -0.03x^2 + 8x - 50$. Use the marginal profit function to estimate the profit from the sale of the 101st fish-fry dinner.

Hint

Use $P'(100)$ to approximate $P(101) - P(100)$.

Answer

\$2

3.2.5 Key Concepts

- Using $f(a+h) \approx f(a) + f'(a)h$, it is possible to estimate $f(a+h)$ given $f'(a)$ and $f(a)$.
- The rate of change of position is velocity, and the rate of change of velocity is acceleration. Speed is the absolute value, or magnitude, of velocity.
- The population growth rate and the present population can be used to predict the size of a future population.
- Marginal cost, marginal revenue, and marginal profit functions can be used to predict, respectively, the cost of producing one more item, the revenue obtained by selling one more item, and the profit obtained by producing and selling one more item.

3.2.6 Glossary

acceleration

is the rate of change of the velocity, that is, the derivative of velocity

amount of change

the amount of a function $f(x)$ over an interval $[x, x+h] is f(x+h) - f(x)$

average rate of change

is a function $f(x)$ over an interval $[x, x+h]$ is $\frac{f(x+h)-f(a)}{b-a}$

marginal cost

is the derivative of the cost function, or the approximate cost of producing one more item

marginal revenue

is the derivative of the revenue function, or the approximate revenue obtained by selling one more item

marginal profit

is the derivative of the profit function, or the approximate profit obtained by producing and selling one more item

population growth rate

is the derivative of the population with respect to time

speed

is the absolute value of velocity, that is, $|v(t)|$ is the speed of an object at time t whose velocity is given by $v(t)$

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3.2E: EXERCISES FOR SECTION 3.2

In exercises 1 - 3, the given functions represent the position of a particle traveling along a horizontal line.

a. Find the velocity and acceleration functions.

b. Determine the time intervals when the object is slowing down or speeding up.

1) $s(t) = 2t^3 - 3t^2 - 12t + 8$

2) $s(t) = 2t^3 - 15t^2 + 36t - 10$

Answer

a. $v(t) = 6t^2 - 30t + 36, \quad a(t) = 12t - 30;$

b. speeds up for $(2, 2.5) \cup (3, \infty)$, slows down for $(0, 2) \cup (2.5, 3)$

3) $s(t) = \dfrac{t}{1 + t^2}$

4) A rocket is fired vertically upward from the ground. The distance s in feet that the rocket travels from the ground after t seconds is given by $s(t) = -16t^2 + 560t$.

a. Find the velocity of the rocket 3 seconds after being fired.

b. Find the acceleration of the rocket 3 seconds after being fired.

Answer

a. $464 \text{ ft}/\text{s}^2$

b. $-32 \text{ ft}/\text{s}^2$

5) A ball is thrown downward with a speed of 8 ft/s from the top of a 64-foot-tall building. After t seconds, its height above the ground is given by $s(t) = -16t^2 - 8t + 64$.

a. Determine how long it takes for the ball to hit the ground.

b. Determine the velocity of the ball when it hits the ground.

6) The position function $s(t) = t^2 - 3t - 4$ represents the position of the back of a car backing out of a driveway and then driving in a straight line, where s is in feet and t is in seconds. In this case, $s(t) = 0$ represents the time at which the back of the car is at the garage door, so $s(0) = -4$ is the starting position of the car, 4 feet inside the garage.

a. Determine the velocity of the car when $s(t) = 0$.

b. Determine the velocity of the car when $s(t) = 14$.

Answer

a. 5 ft/s

b. 9 ft/s

7) The position of a hummingbird flying along a straight line in t seconds is given by $s(t) = 3t^3 - 7t$ meters.

a. Determine the velocity of the bird at $t = 1$ sec.

b. Determine the acceleration of the bird at $t = 1$ sec.

c. Determine the acceleration of the bird when the velocity equals 0.

8) A potato is launched vertically upward with an initial velocity of 100 ft/s from a potato gun at the top of an 85-foot-tall building. The distance in feet that the potato travels from the ground after t seconds is given by $s(t) = -16t^2 + 100t + 85$.

a. Find the velocity of the potato after 0.5 s and 5.75 s.

b. Find the speed of the potato at 0.5 s and 5.75 s.

c. Determine when the potato reaches its maximum height.

d. Find the acceleration of the potato at 0.5 s and 1.5 s.

e. Determine how long the potato is in the air.

f. Determine the velocity of the potato upon hitting the ground.

Answer

a. 84 ft/s, −84 ft/s
b. 84 ft/s
c. $\frac{25}{8}$ s
d. -32 ft/s^2 in both cases
e. $\frac{1}{8}(25+\sqrt{965})$ s
f. $-4\sqrt{965}$ ft/s

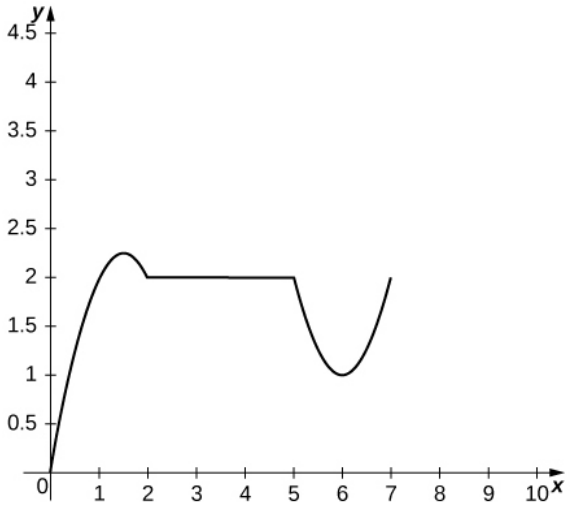
9) The position function $s(t)=t^3-8t$ gives the position in miles of a freight train where east is the positive direction and t is measured in hours.

a. Determine the direction the train is traveling when $s(t)=0$.

b. Determine the direction the train is traveling when $a(t)=0$.

c. Determine the time intervals when the train is slowing down or speeding up.

10) The following graph shows the position $y=s(t)$ of an object moving along a straight line.



a. Use the graph of the position function to determine the time intervals when the velocity is positive, negative, or zero.

b. Sketch the graph of the velocity function.

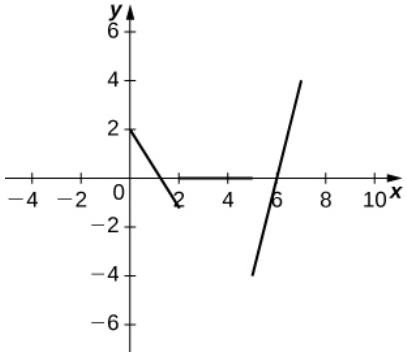
c. Use the graph of the velocity function to determine the time intervals when the acceleration is positive, negative, or zero.

d. Determine the time intervals when the object is speeding up or slowing down.

Answer

a. Velocity is positive on $(0,1.5)\cup(6,7)$, negative on $(1.5,2)\cup(5,6)$, and zero on $(2,5)$.

b.



c. Acceleration is positive on $(5,7)$, negative on $(0,2)$, and zero on $(2,5)$.

d. The object is speeding up on $(6,7)\cup(1.5,2)$ and slowing down on $(0,1.5)\cup(5,6)$.

https://math.libretexts.org/@go/page/149876

11) The cost function, in dollars, of a company that manufactures food processors is given by $C(x) = 200 + \frac{7}{x} + \frac{x}{27}$, where x is the number of food processors manufactured.

a. Find the marginal cost function.

b. Find the marginal cost of manufacturing 12 food processors.

c. Find the actual cost of manufacturing the thirteenth food processor.

12) The price p (in dollars) and the demand x for a certain digital clock radio is given by the price–demand function $p = 10 - 0.001x$.

a. Find the revenue function $R(x)$

b. Find the marginal revenue function.

c. Find the marginal revenue at $x = 2000$ and 5000.

Answer

a. $R(x) = 10x - 0.001x^2$
b.$R'(x) = 10 - 0.002x$
c. \$6 per item, \$0 per item

13) [T] A profit is earned when revenue exceeds cost. Suppose the profit function for a skateboard manufacturer is given by $P(x) = 30x - 0.3x^2 - 250$, where x is the number of skateboards sold.

a. Find the exact profit from the sale of the thirtieth skateboard.

b. Find the marginal profit function and use it to estimate the profit from the sale of the thirtieth skateboard.

14) [T] In general, the profit function is the difference between the revenue and cost functions: $P(x) = R(x) - C(x)$.

Suppose the price-demand and cost functions for the production of cordless drills is given respectively by $p = 143 - 0.03x$ and $C(x) = 75,000 + 65x$, where x is the number of cordless drills that are sold at a price of p dollars per drill and $C(x)$ is the cost of producing x cordless drills.

a. Find the marginal cost function.

b. Find the revenue and marginal revenue functions.

c. Find $R'(1000)$ and $R'(4000)$. Interpret the results.

d. Find the profit and marginal profit functions.

e. Find $P'(1000)$ and $P'(4000)$. Interpret the results.

Answer

a. $C'(x) = 65$
b. $R(x) = 143x - 0.03x^2$,$R'(x) = 143 - 0.06x$
c. $R'(1000) = 83, \quad R'(4000) = -97$. At a production level of 1000 cordless drills, revenue is increasing at a rate of \$83 per drill; at a production level of 4000 cordless drills, revenue is decreasing at a rate of \$97 per drill.
d. $P(x) = -0.03x^2 + 78x - 75000, \quad P'(x) = -0.06x + 78$
e. $P'(1000) = 18, \quad P'(4000) = -162$. At a production level of 1000 cordless drills, profit is increasing at a rate of \$18 per drill; at a production level of 4000 cordless drills, profit is decreasing at a rate of \$162 per drill.

15) A small town in Ohio commissioned an actuarial firm to conduct a study that modeled the rate of change of the town's population. The study found that the town's population (measured in thousands of people) can be modeled by the function $P(t) = -\frac{1}{3}t^3 + 64t + 3000$, where t is measured in years.

a. Find the rate of change function $P'(t)$ of the population function.

b. Find $P'(1), \ P'(2), \ P'(3)$, and $P'(4)$. Interpret what the results mean for the town.

c. Find $P''(1), \ P''(2), \ P''(3)$, and $P''(4)$. Interpret what the results mean for the town's population.

16) [T] A culture of bacteria grows in number according to the function $N(t) = 3000(1 + \frac{4t}{t^2 + 100})$, where t is measured in hours.

a. Find the rate of change of the number of bacteria.

b. Find $N'(0), \ N'(10), \ N'(20)$, and $N'(30)$.

c. Interpret the results in (b).

d. Find $N''(0), \ N''(10), \ N''(20)$, and $N''(30)$. Interpret what the answers imply about the bacteria population growth.

Answer

a. $N'(t) = 3000\left(\dfrac{-4t^2+400}{(t^2+100)^2}\right)$

b. $120, 0, -14.4, -9.6$

c. The bacteria population increases from time 0 to 10 hours; afterwards, the bacteria population decreases.

d. $0, -6, 0.384, 0.432$. The rate at which the bacteria is increasing is decreasing during the first 10 hours. Afterwards, the bacteria population is decreasing at a decreasing rate.

17) The centripetal force of an object of mass m is given by $F(r) = \dfrac{mv^2}{r}$, where v is the speed of rotation and r is the distance from the center of rotation.

a. Find the rate of change of centripetal force with respect to the distance from the center of rotation.

b. Find the rate of change of centripetal force of an object with mass 1000 kilograms, velocity of 13.89 m/s, and a distance from the center of rotation of 200 meters.

The following questions concern the population (in millions) of London by decade in the 19th century, which is listed in the following table.

Year Since 1800	Population (millions)
1	0.8975
11	1.040
21	1.264
31	1.516
41	1.661
51	2.000
61	2.634
71	3.272
81	3.911
91	4.422

Population of LondonSource: http://en.Wikipedia.org/wiki/Demographics_of_London

18) [T]

a. Using a calculator or a computer program, find the best-fit linear function to measure the population.

b. Find the derivative of the equation in a. and explain its physical meaning.

c. Find the second derivative of the equation and explain its physical meaning.

Answer

a. $P(t) = 0.03983 + 0.4280$

b. $P'(t) = 0.03983$. The population is increasing.

c. $P''(t) = 0$. The rate at which the population is increasing is constant.

19) [T]

a. Using a calculator or a computer program, find the best-fit quadratic curve through the data.

b. Find the derivative of the equation and explain its physical meaning.

c. Find the second derivative of the equation and explain its physical meaning.

For the following exercises, consider an astronaut on a large planet in another galaxy. To learn more about the composition of this planet, the astronaut drops an electronic sensor into a deep trench. The sensor transmits its vertical position every second in relation to the astronaut's position. The summary of the falling sensor data is displayed in the following table.

Time after dropping (s)	Position (m)
0	0
1	−1
2	−2
3	−5
4	−7
5	−14

20) [T]

a. Using a calculator or computer program, find the best-fit quadratic curve to the data.

b. Find the derivative of the position function and explain its physical meaning.

c. Find the second derivative of the position function and explain its physical meaning.

Answer

a. $p(t) = -0.6071x^2 + 0.4357x - 0.3571$
b. $p'(t) = -1.214x + 0.4357$. This is the velocity of the sensor.
c. $p''(t) = -1.214$. This is the acceleration of the sensor; it is a constant acceleration downward.

21) [T]

a. Using a calculator or computer program, find the best-fit cubic curve to the data.

b. Find the derivative of the position function and explain its physical meaning.

c. Find the second derivative of the position function and explain its physical meaning.

d. Using the result from c. explain why a cubic function is not a good choice for this problem.

The following problems deal with the Holling type I, II, and III equations. These equations describe the ecological event of growth of a predator population given the amount of prey available for consumption.

22) [T] The **Holling type I equation** is described by $f(x) = ax$, where x is the amount of prey available and $a > 0$ is the rate at which the predator meets the prey for consumption.

a. Graph the Holling type I equation, given $a = 0.5$.

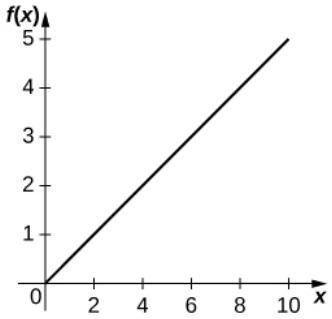
b. Determine the first derivative of the Holling type I equation and explain physically what the derivative implies.

c. Determine the second derivative of the Holling type I equation and explain physically what the derivative implies.

d. Using the interpretations from b. and c. explain why the Holling type I equation may not be realistic.

Answer

a.



b. $f'(x) = a$. The more increase in prey, the more growth for predators.
c. $f''(x) = 0$. As the amount of prey increases, the rate at which the predator population growth increases is constant.
d. This equation assumes that if there is more prey, the predator is able to increase consumption linearly. This assumption is unphysical because we would expect there to be some saturation point at which there is too much prey for the predator to consume adequately.

23) [T] The Holling type II equation is described by $f(x) = \dfrac{ax}{n+x}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

a. Graph the Holling type II equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type I and II equations?

b. Take the first derivative of the Holling type II equation and interpret the physical meaning of the derivative.

c. Show that $f(n) = \frac{1}{2}a$ and interpret the meaning of the parameter n.

d. Find and interpret the meaning of the second derivative. What makes the Holling type II function more realistic than the Holling type I function?

24) [T] The Holling type III equation is described by $f(x) = \dfrac{ax^2}{n^2 + x^2}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

a. Graph the Holling type III equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type II and III equations?

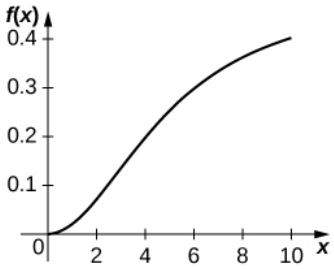
b. Take the first derivative of the Holling type III equation and interpret the physical meaning of the derivative.

c. Find and interpret the meaning of the second derivative (it may help to graph the second derivative).

d. What additional ecological phenomena does the Holling type III function describe compared with the Holling type II function?

Answer

a.



b. $f'(x) = \dfrac{2axn^2}{(n^2+x^2)^2}$. When the amount of prey increases, the predator growth increases.

c. $f''(x) = \dfrac{2an^2(n^2-3x^2)}{(n^2+x^2)^3}$. When the amount of prey is extremely small, the rate at which predator growth is increasing is increasing, but when the amount of prey reaches above a certain threshold, the rate at which predator growth is increasing begins to decrease.

d. At lower levels of prey, the prey is more easily able to avoid detection by the predator, so fewer prey individuals are consumed, resulting in less predator growth.

25) [T] The populations of the snowshoe hare (in thousands) and the lynx (in hundreds) collected over 7 years from 1937 to 1943 are shown in the following table. The snowshoe hare is the primary prey of the lynx.

Population of snowshoe hare (thousands)	Population of lynx (hundreds)
20	10
5	15
65	55
95	60

Snowshoe Hare and Lynx PopulationsSource: http://www.biotopics.co.uk/newgcse/predatorprey.html.

a. Graph the data points and determine which Holling-type function fits the data best.

b. Using the meanings of the parameters a and n, determine values for those parameters by examining a graph of the data. Recall that n measures what prey value results in the half-maximum of the predator value.

c. Plot the resulting Holling-type I, II, and III functions on top of the data. Was the result from part a. correct?

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3.3: The Chain Rule

Learning Objectives

- State the chain rule for the composition of two functions.
- Apply the chain rule together with the power rule.
- Apply the chain rule and the product/quotient rules correctly in combination when both are necessary.

We have seen the techniques for differentiating basic functions (for example, polynomials) as well as sums, differences, products, quotients, and constant multiples of these functions. However, these techniques do not allow us to differentiate compositions of functions, such as $k(x)=\sqrt{3x^2+1}$. In this section, we study the rule for finding the derivative of the composition of two or more functions.

3.3.1 Deriving the Chain Rule

When we have a function that is a composition of two or more functions, we could use all of the techniques we have already learned to differentiate it. However, using all of those techniques to break down a function into simpler parts that we are able to differentiate can get cumbersome. Instead, we use the *chain rule*, which states that the derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

To put this rule into context, let's take a look at an example: $h(x)=(x^2+3)^2$. We can think of the derivative of this function with respect to x as the rate of change of $(x^2+3)^2$ relative to the change in x. Consequently, we want to know how $(x^2+3)^2$ changes as x changes. We can think of this event as a chain reaction: As x changes, x^2+3 changes, which leads to a change in $(x^2+3)^2$. This chain reaction gives us hints as to what is involved in computing the derivative of $(x^2+3)^2$. First of all, a change in x forcing a change in x^2+3 suggests that somehow the derivative of x^2+3 is involved. In addition, the change in x^2+3 forcing a change in $(x^2+3)^2$ suggests that the derivative of u^2 with respect to u, where $u=x^2+3$, is also part of the final derivative.

We can take a more formal look at the derivative of $h(x)=(x^2+3)^2$ by setting up the limit that would give us the derivative at a specific value a in the domain of $h(x)=(x^2+3)^2$.

$$h'(a)=\lim_{x\to a}\frac{(x^2+3)^2-(a^2+3)^2}{x-a}$$

This expression does not seem particularly helpful; however, we can modify it by expanding the numerator to obtain

$$h'(a)=\lim_{x\to a}\frac{x^4+6x^2+9-a^4-6a^2-9}{x-a} \tag{3.3.1}$$

Simplifying further, and rearranging terms, we have

$$h'(a)=\lim_{x\to a}\frac{x^4-a^4+6x^2-6a^2}{x-a} \tag{3.3.2}$$

Factoring the numerator, we have

$$h'(a)=\lim_{x\to a}\frac{(x^2+a^2)(x^2-a^2)+6(x^2-a^2)}{x-a} \tag{3.3.3}$$

Factoring further,

$$h'(a)=\lim_{x\to a}\frac{(x^2-a^2)[x^2+a^2+6]}{x-a} \tag{3.3.4}$$

Factoring further, and reducing the common factor of $x-a$, we are at long last left with

$$h'(a)=\lim_{x\to a}(x+a)(x^2+a^2+6) \tag{3.3.5}$$

Thus, using direct substitution, $h'(a)=2a(2a^2+6)=2a\cdot 2(a^2+3)$

In other words, if $h(x)=(x^2+3)^2$, then $h'(x)=2(x^2+3)\cdot 2x$. Thus, if we think of $h(x)=(x^2+3)^2$ as the composition $(f\circ g)(x)=f\big(g(x)\big)$ where $f(x)=x^2$ and $g(x)=x^2+3$, then the derivative of $h(x)=(x^2+3)^2$ is the product of the

derivative of $g(x) = x^2 + 3$ and the derivative of the function $f(x) = x^2$ evaluated at the function $g(x) = x^2 + 3$.

Now that we have derived a special case of the chain rule, we state the general case and then apply it in a general form to other composite functions. An informal proof is provided at the end of the section.

Rule: The Chain Rule

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function

$$h(x) = (f \circ g)(x) = f\big(g(x)\big)$$

is given by

$$h'(x) = f'\big(g(x)\big) \cdot g'(x).$$

Alternatively, if y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Problem-Solving Strategy: Applying the Chain Rule

1. To differentiate $h(x) = f\big(g(x)\big)$, begin by identifying $f(x)$ and $g(x)$.
2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'\big(g(x)\big)$.
3. Find $g'(x)$.
4. Write $h'(x) = f'\big(g(x)\big) \cdot g'(x)$.

Note: When applying the chain rule to the composition of two or more functions, keep in mind that we work our way from the outside function in. It is also useful to remember that the derivative of the composition of two functions can be thought of as having two parts; the derivative of the composition of three functions has three parts; and so on. Also, remember that we never evaluate a derivative at a derivative.

3.3.2 The Chain and Power Rules Combined

We can now apply the chain rule to composite functions, but note that we often need to use it with other rules. For example, to find derivatives of functions of the form $h(x) = \big(g(x)\big)^n$, we need to use the chain rule combined with the power rule. To do so, we can think of $h(x) = \big(g(x)\big)^n$ as $f\big(g(x)\big)$ where $f(x) = x^n$. Then $f'(x) = nx^{n-1}$. Thus, $f'\big(g(x)\big) = n\big(g(x)\big)^{n-1}$. This leads us to the derivative of a power function using the chain rule,

$$h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x)$$

Rule: Power Rule for Composition of Functions (General Power Rule)

For all values of x for which the derivative is defined, if

$$h(x) = \big(g(x)\big)^n,$$

Then

$$h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x). \tag{3.3.6}$$

Example 3.3.1: Using the Chain and Power Rules

Find the derivative of $h(x) = \dfrac{1}{(3x^2+1)^2}$.

Solution

First, rewrite $h(x) = \dfrac{1}{(3x^2+1)^2} = (3x^2+1)^{-2}$.

Applying the power rule with $g(x) = 3x^2 + 1$, we have

$$h'(x) = -2(3x^2 + 1)^{-3} \cdot 6x \, .$$

Rewriting back to the original form gives us

$$h'(x) = \frac{-12x}{(3x^2 + 1)^3}$$

? Exercise 3.3.1

Find the derivative of $h(x) = (2x^3 + 2x - 1)^4$.

Hint

Use the General Power Rule (Equation 3.3.6) with $g(x) = 2x^3 + 2x - 1$.

Answer

$h'(x) = 4(2x^3 + 2x - 1)^3(6x^2 + 2) = 8(3x^2 + 1)(2x^3 + 2x - 1)^3$

✓ Example 3.3.2: Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $h(x) = \dfrac{1}{(3x - 5)^2}$ at $x = 2$.

Solution

Because we are finding an equation of a line, we need a point. The x-coordinate of the point is 2. To find the y-coordinate, substitute 2 into $h(x)$. Since $h(2) = \dfrac{1}{(3(2) - 5)^2} = 1$, the point is $(2, 1)$.

For the slope, we need $h'(2)$. To find $h'(x)$, first we rewrite $h(x) = (3x - 5)^{-2}$ and apply the power rule to obtain

$$h'(x) = -2(3x - 5)^{-3}(3) = -6(3x - 5)^{-3} \, .$$

By substituting, we have $h'(2) = -6(3(2) - 5)^{-3} = -6$.

Therefore, the line has equation $y - 1 = -6(x - 2)$. Rewriting, the equation of the line is $y = -6x + 13$.

? Exercise 3.3.2

Find the equation of the line tangent to the graph of $f(x) = (x^2 - 2)^3$ at $x = -2$.

Hint

Use the preceding example as a guide.

Answer

$y = -48x - 88$

3.3.3 Combining the Chain Rule with Other Rules

Now that we can combine the chain rule and the power rule, we examine how to combine the chain rule with the other rules we have learned. In particular, we can use it with the product rule or quotient rule.

✓ Example 3.3.3: Combining the Chain Rule with the Product Rule

Find the derivative of $h(x) = (2x + 1)^5(3x - 2)^7$.

Solution

First apply the product rule, then apply the chain rule to each term of the product.

$$
\begin{aligned}
h'(x) &= \frac{d}{dx}\left((2x+1)^5\right)\cdot(3x-2)^7 + \frac{d}{dx}\left((3x-2)^7\right)\cdot(2x+1)^5 && \text{Apply the product rule.}\\
&= 5(2x+1)^4\cdot 2\cdot(3x-2)^7 + 7(3x-2)^6\cdot 3\cdot(2x+1)^5 && \text{Apply the chain rule.}\\
&= 10(2x+1)^4(3x-2)^7 + 21(3x-2)^6(2x+1)^5 && \text{Simplify.}\\
&= (2x+1)^4(3x-2)^6(10(3x-2)+21(2x+1)) && \text{Factor out } (2x+1)^4(3x-2)^6\\
&= (2x+1)^4(3x-2)^6(72x+1) && \text{Simplify.}
\end{aligned}
$$

Exercise 3.3.3

Find the derivative of $h(x) = \dfrac{x}{(2x+3)^3}$.

Hint

Start out by applying the quotient rule. Remember to use the chain rule to differentiate the denominator.

Answer

$$h'(x) = \frac{3-4x}{(2x+3)^4}$$

Example 3.3.4: Using the Chain Rule in a Velocity Problem

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = (t^2+3t-4)^3$ feet, with time measured in seconds. What is the velocity of the particle at time $t = 2$?

Solution

To find $v(t)$, the velocity of the particle at time t, we must differentiate $s(t)$. Thus,

$$v(t) = s'(t) = 3(t^2+3t-4)^2\cdot(2t+3)ft/sec \tag{3.3.7}$$

$$v(2) = 3(2^2+3\cdot 2-4)^2\cdot(2\cdot 2+3) = 756ft/sec \tag{3.3.8}$$

Proof of Chain Rule

At this point, we present a very informal proof of the chain rule. For simplicity's sake we ignore certain issues: For example, we assume that $g(x) \neq g(a)$ for $x \neq a$ in some open interval containing a. We begin by applying the limit definition of the derivative to the function $h(x)$ to obtain $h'(a)$:

$$h'(a) = \lim_{x\to a}\frac{f\big(g(x)\big) - f\big(g(a)\big)}{x-a}.$$

Rewriting, we obtain

$$h'(a) = \lim_{x\to a}\frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x)-g(a)}\cdot\frac{g(x)-g(a)}{x-a}.$$

Although it is clear that

$$\lim_{x\to a}\frac{g(x)-g(a)}{x-a} = g'(a),$$

it is not obvious that

$$\lim_{x\to a}\frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x)-g(a)} = f'\big(g(a)\big).$$

To see that this is true, first recall that since g is differentiable at a, g is also continuous at a. Thus,

$$\lim_{x \to a} g(x) = g(a).$$

Next, make the substitution $y = g(x)$ and $b = g(a)$ and use change of variables in the limit to obtain

$$\lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} = \lim_{y \to b} \frac{f(y) - f(b)}{y - b} = f'(b) = f'\big(g(a)\big).$$

Finally,

$$h'(a) = \lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} = f'\big(g(a)\big) \cdot g'(a).$$

□

Example 3.3.5: Using the Chain Rule with Functional Values

Let $h(x) = f\big(g(x)\big)$. If $g(1) = 4, g'(1) = 3$, and $f'(4) = 7$, find $h'(1)$.

Solution

Use the chain rule, then substitute.

$$\begin{aligned} h'(1) &= f'\big(g(1)\big) \cdot g'(1) && \text{Apply the chain rule.} \\ &= f'(4) \cdot 3 && \text{Substitute } g(1) = 4 \text{ and } g'(1) = 3. \\ &= 7 \cdot 3 && \text{Substitute } f'(4) = 7. \\ &= 21 && \text{Simplify.} \end{aligned}$$

Exercise 3.3.5

Given $h(x) = f(g(x))$. If $g(2) = -3, g'(2) = 4,$ and $f'(-3) = 7$, find $h'(2)$.

Hint

Follow Example 3.3.5

Answer

28

3.3.4 The Chain Rule Using Leibniz's Notation

As with other derivatives that we have seen, we can express the chain rule using Leibniz's notation. This notation for the chain rule is used heavily in physics applications.

For $h(x) = f(g(x))$, let $u = g(x)$ and $y = h(x) = f(u)$. Thus,

$$h'(x) = \frac{dy}{dx}$$

$$f'(g(x)) = f'(u) = \frac{dy}{du}$$

and

$$g'(x) = \frac{du}{dx}.$$

Consequently,

$$\frac{dy}{dx} = h'(x) = f'\big(g(x)\big) \cdot g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Rule: Chain Rule Using Leibniz's Notation

If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 3.3.6: Taking a Derivative Using Leibniz's Notation I

Find the derivative of $y = \left(\frac{x}{3x+2}\right)^5$.

Solution

First, let $u = \frac{x}{3x+2}$. Thus, $y = u^5$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$. Using the quotient rule,

$$\frac{du}{dx} = \frac{2}{(3x+2)^2}$$

and

$$\frac{dy}{du} = 5u^4.$$

Finally, we put it all together.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the chain rule.} \\ &= 5u^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } \frac{dy}{du} = 5u^4 \text{ and } \frac{du}{dx} = \frac{2}{(3x+2)^2}. \\ &= 5\left(\frac{x}{3x+2}\right)^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } u = \frac{x}{3x+2}. \\ &= \frac{10x^4}{(3x+2)^6} && \text{Simplify.} \end{aligned}$$

It is important to remember that, when using the Leibniz form of the chain rule, the final answer must be expressed entirely in terms of the original variable given in the problem.

Key Concepts

- The chain rule allows us to differentiate compositions of two or more functions. It states that for $h(x) = f\big(g(x)\big)$,

$$h'(x) = f'\big(g(x)\big) \cdot g'(x).$$

  In Leibniz's notation this rule takes the form

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

- We can use the chain rule with other rules that we have learned, and we can derive formulas for some of them.
- The chain rule combines with the power rule to form a new rule:

$$\text{If } h(x) = \big(g(x)\big)^n, \text{ then } h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x).$$

- When applied to the composition of three functions, the chain rule can be expressed as follows: If $h(x) = f\Big(g\big(k(x)\big)\Big)$, then $h'(x) = f'\Big(g\big(k(x)\big)\Big) \cdot g'\big(k(x)\big) \cdot k'(x)$.

3.3.5 Key Equations

- **The chain rule**

$$h'(x) = f'\big(g(x)\big) \cdot g'(x)$$

- **The power rule for functions**

$$h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x)$$

3.3.6 Glossary

chain rule

the chain rule defines the derivative of a composite function as the derivative of the outer function evaluated at the inner function times the derivative of the inner function

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3.3E: EXERCISES FOR SECTION 3.3

In exercises 1 - 3, given $y = f(u)$ and $u = g(x)$, find $\frac{dy}{dx}$ by using Leibniz's notation for the chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

1) $y = 3u - 6, \quad u = 2x^2$

2) $y = 6u^3, \quad u = 7x - 4$

Answer

$$\frac{dy}{dx} = 18u^2 \cdot 7 = 18(7x-4)^2 \cdot 7 = 126(7x-4)^2$$

3) $y = \sqrt{4u+3}, \quad u = x^2 - 6x$

Answer

$$\frac{dy}{dx} = \frac{8x-24}{2\sqrt{4u+3}} = \frac{4x-12}{\sqrt{4x^2-24x+3}}$$

For each of the following exercises,

a. decompose each function in the form $y = f(u)$ and $u = g(x)$, and

b. find $\frac{dy}{dx}$ as a function of x.

4) $y = (3x-2)^6$

5) $y = (3x^2+1)^3$

Answer

a. $f(u) = u^3, \quad u = 3x^2 + 1$;

b. $\frac{dy}{dx} = 18x(3x^2+1)^2$

6) $y = \left(\frac{x}{7} + \frac{7}{x}\right)^7$

Answer

a. $f(u) = u^7, \quad u = \frac{x}{7} + \frac{7}{x}$;

b. $\frac{dy}{dx} = 7\left(\frac{x}{7} + \frac{7}{x}\right)^6 \cdot \left(\frac{1}{7} - \frac{7}{x^2}\right)$

In exercises 7-11, find $\frac{dy}{dx}$ for each function.

7) $y = (3x^2 + 3x - 1)^4$

8) $y = (5-2x)^{-2}$

Answer

$$\frac{dy}{dx} = \frac{4}{(5-2x)^3}$$

9) $y = (2x^3 - x^2 + 6x + 1)^3$

Answer

$$\frac{dy}{dx} = 6(2x^3 - x^2 + 6x + 1)^2 \cdot (3x^2 - x + 3)$$

10) $y = \frac{2}{\sqrt{x^2+9}}$

11) $y = \dfrac{1}{3(5-7x)^4}$

Answer

$$\frac{dy}{dx} = -12\cot^2(4x+1)\cdot\csc^2(4x+1)$$

12) Let $y = \big[f(x)\big]^3$ and suppose that $f'(1) = 4$ and $\frac{dy}{dx} = 10$ for $x = 1$. Find $f(1)$.

13) Let $y = \big(f(x) + 5x^2\big)^4$ and suppose that $f(-1) = -4$ and $\frac{dy}{dx} = 3$ when $x = -1$. Find $f'(-1)$

Answer

$f'(-1) = 10\frac{3}{4}$

14) Let $y = (f(u) + 3x)^2$ and $u = x^3 - 2x$. If $f(4) = 6$ and $\frac{dy}{dx} = 18$ when $x = 2$, find $f'(4)$.

15) [T] Find the equation of the tangent line to $y = -\sqrt{9-x}$ at the origin. Use a calculator to graph the function and the tangent line together.

Answer

$y = \frac{1}{6}x - 3$

16) [T] Find the equation of the tangent line to $y = \big(3x + \frac{1}{x}\big)^2$ at the point $(1, 16)$. Use a calculator to graph the function and the tangent line together.

17) Find the x -coordinates at which the tangent line to $y = \big(x - \frac{6}{x}\big)^8$ is horizontal.

Answer

$x = \pm\sqrt{6}$

For exercises 18-24, use the information in the following table to find $h'(a)$ at the given value for a.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	5	0	2
1	1	−2	3	0
2	4	4	1	−1
3	3	−3	2	3

18) $h(x) = f\big(g(x)\big); \quad a = 0$

Answer

$h'(0) = 10$

19) $h(x) = g\big(f(x)\big); \quad a = 0$

20) $h(x) = \big(x^4 + g(x)\big)^{-2}; \quad a = 1$

Answer

$h'(1) = -\frac{1}{8}$

21) $h(x) = \left(\dfrac{f(x)}{g(x)}\right)^2; \quad a = 3$

22) $h(x) = f\big(x + f(x)\big); \quad a = 1$

Answer

$h'(1) = -4$

23) $h(x) = \big(1 + g(x)\big)^3; \quad a = 2$

24) $h(x) = g\big(2 + f(x^2)\big); \quad a = 1$

Answer

$h'(1) = -12$

25) [T] The position function of a freight train is given by $s(t) = 100(t+1)^{-2}$, with s in meters and t in seconds. At time $t = 6$ s, find the train's

a. velocity and

b. acceleration.

c. Considering your results in parts a. and b., is the train speeding up or slowing down?

Answer

a. $v(6) = -\frac{200}{343}$ m/s,

b. $a(6) = \frac{600}{2401}$ m/s^2,

c. The train is slowing down since velocity and acceleration have opposite signs.

26) [T] The total cost to produce x boxes of Thin Mint Girl Scout cookies is C dollars, where $C = 0.0001x^3 - 0.02x^2 + 3x + 300$. In t weeks production is estimated to be $x = 1600 + 100t$ boxes.

a. Find the marginal cost $C'(x)$.

b. Use Leibniz's notation for the chain rule, $\frac{dC}{dt} = \frac{dC}{dx} \cdot \frac{dx}{dt}$, to find the rate with respect to time t that the cost is changing.

c. Use your result in part b. to determine how fast costs are increasing when $t = 2$ weeks. Include units with the answer.

Answer

a. $C'(x) = 0.0003x^2 - 0.04x + 3$

b. $\frac{dC}{dt} = 100 \cdot (0.0003x^2 - 0.04x + 3) = 100 \cdot (0.0003(1600 + 100t)^2 - 0.04(1600 + 100t) + 3) = 300t^2 + 9200t + 70700$

c. Approximately $90,300 per week

27) [T] The formula for the area of a circle is $A = \pi r^2$, where r is the radius of the circle. Suppose a circle is expanding, meaning that both the area A and the radius r (in inches) are expanding.

a. Suppose $r = 2 - \frac{100}{(t+7)^2}$ where t is time in seconds. Use the chain rule $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the area is expanding.

b. Use your result in part a. to find the rate at which the area is expanding at $t = 4$ s.

28) [T] The formula for the volume of a sphere is $S = \frac{4}{3}\pi r^3$, where r (in feet) is the radius of the sphere. Suppose a spherical snowball is melting in the sun.

a. Suppose $r = \frac{1}{(t+1)^2} - \frac{1}{12}$ where t is time in minutes. Use the chain rule $\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the snowball is melting.

b. Use your result in part a. to find the rate at which the volume is changing at $t = 1$ min.

Answer

a. $\frac{dS}{dt} = -\frac{8\pi r^2}{(t+1)^3} = -\frac{8\pi\left(\frac{1}{(t+1)^2} - \frac{1}{12}\right)^2}{(t+1)^3}$

b. The volume is decreasing at a rate of $-\frac{\pi}{36}$ ft^3/min

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3.4: Implicit Differentiation

Learning Objectives

- Find the derivative of a complicated function by using implicit differentiation.
- Use implicit differentiation to determine the equation of a tangent line.

We have already studied how to find equations of tangent lines to functions and the rate of change of a function at a specific point. In all these cases we had the explicit equation for the function and differentiated these functions explicitly. Suppose instead that we want to determine the equation of a tangent line to an arbitrary curve or the rate of change of an arbitrary curve at a point. In this section, we solve these problems by finding the derivatives of functions that define y implicitly in terms of x.

3.4.1 Implicit Differentiation

In most discussions of math, if the dependent variable y is a function of the independent variable x, we express y in terms of x. If this is the case, we say that y is an explicit function of x. For example, when we write the equation $y = x^2 + 1$, we are defining y explicitly in terms of x. On the other hand, if the relationship between the function y and the variable x is expressed by an equation where y is not expressed entirely in terms of x, we say that the equation defines y implicitly in terms of x. For example, the equation $y - x^2 = 1$ defines the function $y = x^2 + 1$ implicitly.

Implicit differentiation allows us to find slopes of tangents to curves that are clearly not functions (they fail the vertical line test). We are using the idea that portions of y are functions that satisfy the given equation, but that y is not actually a function of x.

In general, an equation defines a function implicitly if the function satisfies that equation. An equation may define many different functions implicitly. For example, the functions

$$y = \sqrt{25 - x^2}$$

and

$$y = \begin{cases} \sqrt{25 - x^2}, & \text{if } -5 \le x < 0 \\ -\sqrt{25 - x^2}, & \text{if } 0 \le x \le 5 \end{cases}$$

which are illustrated in Figure 3.4.1, are just two of the many functions defined implicitly by the equation $x^2 + y^2 = 25$.

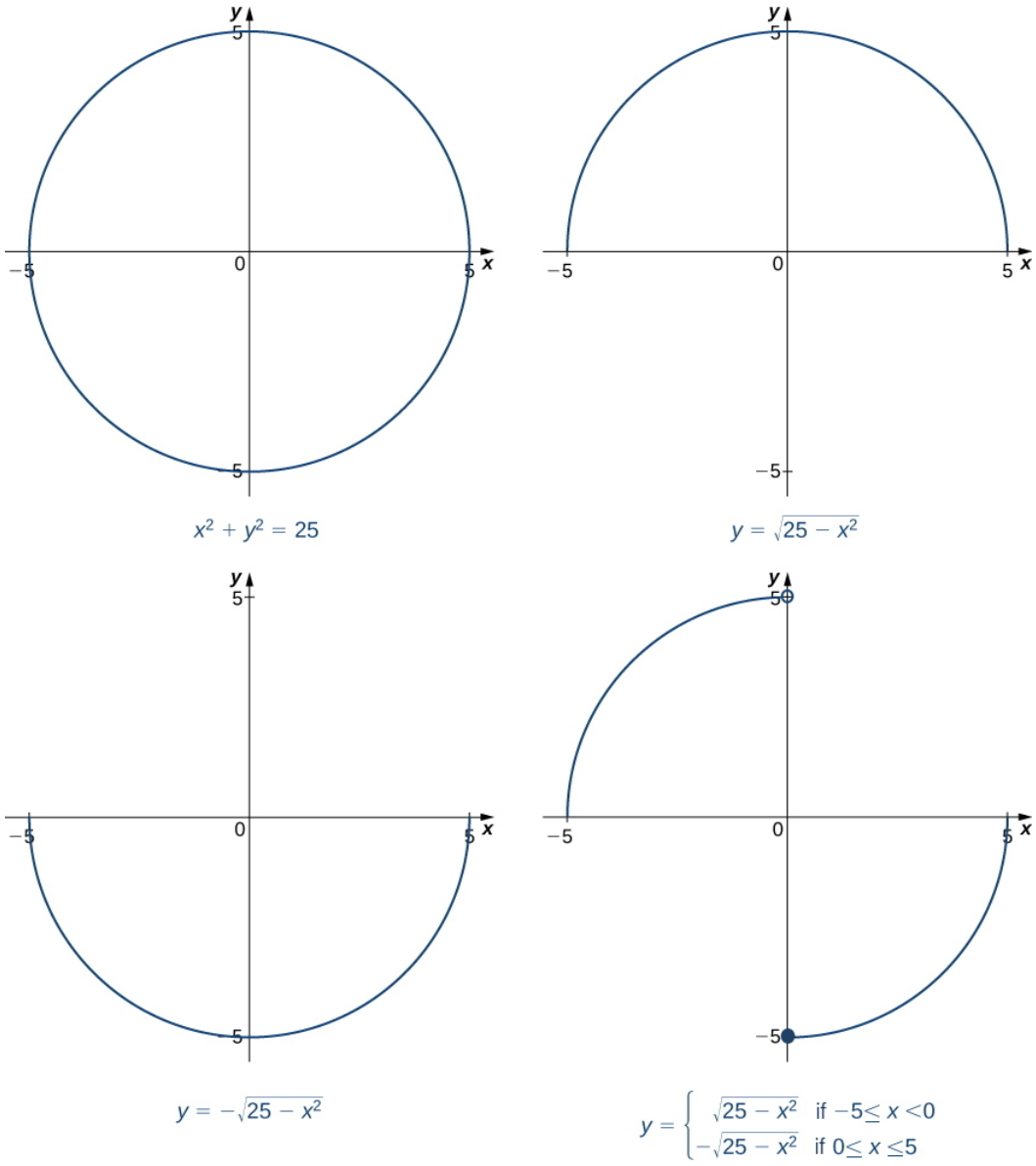


Figure 3.4.1:The equation $x^2 + y^2 = 25$ defines many functions implicitly.

If we want to find the slope of the line tangent to the graph of $x^2 + y^2 = 25$ at the point $(3, 4)$, we could evaluate the derivative of the function $y = \sqrt{25 - x^2}$ at $x = 3$. On the other hand, if we want the slope of the tangent line at the point $(3, -4)$, we could use the derivative of $y = -\sqrt{25 - x^2}$. However, it is not always easy to solve for a function defined implicitly by an equation. Fortunately, the technique of **implicit differentiation** allows us to find the derivative of an implicitly defined function without ever solving for the function explicitly. The process of finding $\dfrac{dy}{dx}$ using implicit differentiation is described in the following problem-solving strategy.

Problem-Solving Strategy: Implicit Differentiation

To perform *implicit differentiation* on an equation that defines a function y implicitly in terms of a variable x, use the following steps:

1. Take the derivative of both sides of the equation. Keep in mind that y is a function of x. Consequently,

$$\frac{d}{dx}(x^3) = 3x^2 \tag{3.4.1}$$

and

$$\frac{d}{dx}(y^3) = 3y^2 \cdot \frac{dy}{dx} \tag{3.4.2}$$

because we must use the chain rule to differentiate y^3 with respect to x.

2. Rewrite the equation so that all terms containing dy/dx are on the left and all terms that do not contain dy/dx are on the right.
3. Factor out dy/dx on the left.
4. Solve for dy/dx by dividing both sides of the equation by an appropriate algebraic expression.

Example 3.4.1: Using Implicit Differentiation

Assuming that y is defined implicitly by the equation $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

Solution

Follow the steps in the problem-solving strategy.

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$	Step 1. Differentiate both sides of the equation.
$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$	Step 1.1. Use the sum rule on the left. On the right $\frac{d}{dx}(25) = 0$.
$2x + 2y\frac{dy}{dx} = 0$	Step 1.2. Take the derivatives, so $\frac{d}{dx}(x^2) = 2x$ and $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$.
$2y\frac{dy}{dx} = -2x$	Step 2. Keep the terms with $\frac{dy}{dx}$ on the left. Move the remaining terms to the right.
$\frac{dy}{dx} = -\frac{x}{y}$	Step 4. Divide both sides of the equation by $2y$. (Step 3 does not apply in this case.)

Analysis

Note that the resulting expression for $\frac{dy}{dx}$ is in terms of both the independent variable x and the dependent variable y. Although in some cases it may be possible to express $\frac{dy}{dx}$ in terms of x only, it is generally not possible to do so.

Example 3.4.2: Using Implicit Differentiation and the Product Rule

Assuming that y is defined implicitly by the equation $x^3y + y = 4x + 3$, find $\frac{dy}{dx}$.

Solution

$\frac{d}{dx}(x^3y + y) = \frac{d}{dx}(4x + 3)$	Step 1: Differentiate both sides of the equation.
$\frac{d}{dx}(x^3y) + \frac{d}{dx}(y) = 4$	Step 1.1: Apply the sum rule on the left. On the right, $\frac{d}{dx}(4x + 3) = 4$.
$\left(\frac{d}{dx}(x^3) \cdot y + \frac{d}{dx}(y) \cdot x^3\right) + \frac{dy}{dx} = 4$	Step 1.2: Use the product rule to find $\frac{d}{dx}(x^3y)$. Observe that $\frac{d}{dx}(y) = \frac{dy}{dx}$.

$3x^2y+(\frac{dy}{dx})\cdot x^3+\frac{dy}{dx}=4$	Step 1.3: We know $\frac{d}{dx}(x^3)=3x^2$. We also know that $\frac{d}{dx}(y)=\frac{dy}{dx}$.
$x^3\frac{dy}{dx}+\frac{dy}{dx}=4-3x^2y$	Step 2: Keep all terms containing $\frac{dy}{dx}$ on the left. Move all other terms to the right.
$\frac{dy}{dx}(x^3+1)=4-3x^2y$	Step 3: Factor out $\frac{dy}{dx}$ on the left.
$\frac{dy}{dx}=\frac{4-3x^2y}{x^3+1}$	Step 4: Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by x^3+1.

Example 3.4.3: Using Implicit Differentiation to Find a Second Derivative

Find $\frac{d^2y}{dx^2}$ if $x^2+y^2=25$.

Solution

In Example 3.4.1, we showed that $\frac{dy}{dx}=-\frac{x}{y}$. We can take the derivative of both sides of this equation to find $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2}=\frac{d}{dy}\left(-\frac{x}{y}\right) \qquad \text{Differentiate both sides of } \frac{dy}{dx}=-\frac{x}{y}.$$

$$=-\frac{\left(1\cdot y-x\frac{dy}{dx}\right)}{y^2} \qquad \text{Use the quotient rule to find } \frac{d}{dy}\left(-\frac{x}{y}\right).$$

$$=\frac{-y+x\frac{dy}{dx}}{y^2} \qquad \text{Simplify.}$$

$$=\frac{-y+x\left(-\frac{x}{y}\right)}{y^2} \qquad \text{Substitute } \frac{dy}{dx}=-\frac{x}{y}.$$

$$=\frac{-y^2-x^2}{y^3} \qquad \text{Simplify.}$$

At this point we have found an expression for $\frac{d^2y}{dx^2}$. If we choose, we can simplify the expression further by recalling that $x^2+y^2=25$ and making this substitution in the numerator to obtain $\frac{d^2y}{dx^2}=-\frac{25}{y^3}$.

Exercise 3.4.1

Find $\frac{dy}{dx}$ for y defined implicitly by the equation $4x^5+y^3=y^2+5x$.

Hint

Follow the problem solving strategy, remembering to apply the chain rule to differentiate y^3 and y^2.

Answer

$$\frac{dy}{dx}=\frac{5-20x^4}{3y^2-2y}$$

3.4.2 Finding Tangent Lines Implicitly

Now that we have seen the technique of implicit differentiation, we can apply it to the problem of finding equations of tangent lines to curves described by equations.

Example 3.4.4: Finding a Tangent Line to a Circle

Find the equation of the line tangent to the curve $x^2+y^2=25$ at the point $(3,-4)$.

Solution

Although we could find this equation without using implicit differentiation, using that method makes it much easier. In Example 3.4.1, we found $\frac{dy}{dx}=-\frac{x}{y}$.

The slope of the tangent line is found by substituting $(3,-4)$ into this expression. Consequently, the slope of the tangent line is $\frac{dy}{dx}\Big|_{(3,-4)}=-\frac{3}{-4}=\frac{3}{4}$.

Using the point $(3,-4)$ and the slope $\frac{3}{4}$ in the point-slope equation of the line, we obtain the equation $y=\frac{3}{4}x-\frac{25}{4}$ (Figure 3.4.2).

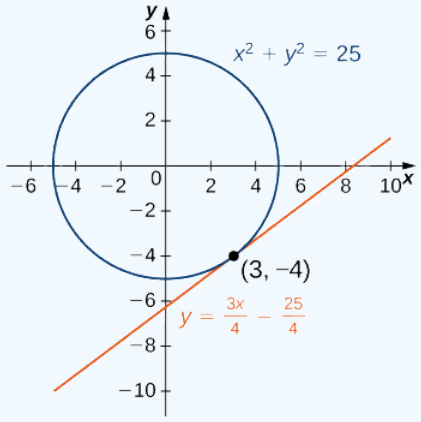


Figure 3.4.2: The line $y=\frac{3}{4}x-\frac{25}{4}$ is tangent to $x^2+y^2=25$ at the point (3, −4).

Example 3.4.5: Finding the Equation of the Tangent Line to a Curve

Find the equation of the line tangent to the graph of $y^3+x^3-3xy=0$ at the point $\left(\frac{3}{2},\frac{3}{2}\right)$ (Figure 3.4.3). This curve is known as the folium (or leaf) of Descartes.

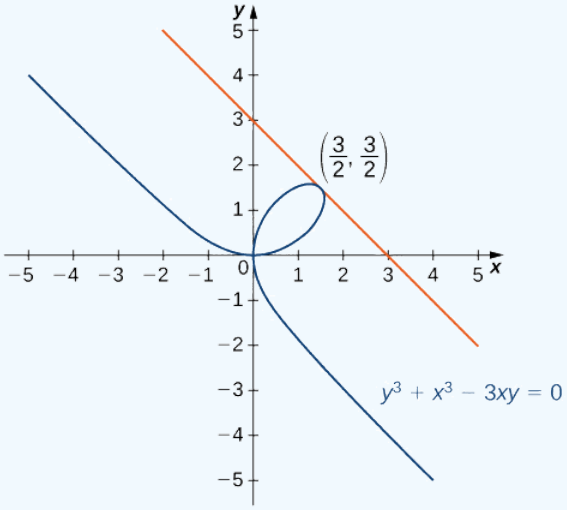


Figure 3.4.3: Finding the tangent line to the folium of Descartes at $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Solution

Begin by finding $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(y^3 + x^3 - 3xy\right) = \frac{d}{dx}(0)$$
$$3y^2\frac{dy}{dx} + 3x^2 - \left(3y + 3x\frac{dy}{dx}\right) = 0$$
$$3y^2\frac{dy}{dx} + 3x^2 - 3y - 3x\frac{dy}{dx} = 0$$
$$\left(3y^2 - 3x\right)\frac{dy}{dx} = 3y - 3x^2$$
$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}.$$

Next, substitute $\left(\frac{3}{2}, \frac{3}{2}\right)$ into $\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}$ to find the slope of the tangent line:

$$\left.\frac{dy}{dx}\right|_{\left(\frac{3}{2}, \frac{3}{2}\right)} = -1.$$

Finally, substitute into the point-slope equation of the line to obtain

$$y = -x + 3.$$

Example 3.4.6: Applying Implicit Differentiation

In a simple video game, a rocket travels in an elliptical orbit whose path is described by the equation $4x^2 + 25y^2 = 100$. The rocket can fire missiles along lines tangent to its path. The object of the game is to destroy an incoming asteroid traveling along the positive x-axis toward $(0, 0)$. If the rocket fires a missile when it is located at $\left(3, \frac{8}{5}\right)$, where will it intersect the x-axis?

Solution

To solve this problem, we must determine where the line tangent to the graph of

$4x^2 + 25y^2 = 100$ at $\left(3, \frac{8}{5}\right)$ intersects the x-axis. Begin by finding $\frac{dy}{dx}$ implicitly.

Differentiating, we have

$$8x+50y\frac{dy}{dx}=0.$$

Solving for $\frac{dy}{dx}$,

we have

$$\frac{dy}{dx}=-\frac{4x}{25y}.$$

The slope of the tangent line is $\left.\frac{dy}{dx}\right|_{\left(3,\frac{8}{5}\right)}=-\frac{3}{10}$. The equation of the tangent line is $y=-\frac{3}{10}x+\frac{5}{2}$. To determine where the line intersects the x-axis, solve $0=-\frac{3}{10}x+\frac{5}{2}$. The solution is $x=\frac{25}{3}$. The missile intersects the x-axis at the point $\left(\frac{25}{3},0\right)$.

? Exercise 3.4.2

Find the equation of the line tangent to the hyperbola $x^2-y^2=16$ at the point $(5,3)$.

Hint

$\frac{dy}{dx}=\frac{x}{y}$

Answer

$y=\frac{5}{3}x-\frac{16}{3}$

3.4.3 Key Concepts

- We use implicit differentiation to find derivatives of implicitly defined functions (functions defined by equations).
- By using implicit differentiation, we can find the equation of a tangent line to the graph of a curve.

3.4.4 Glossary

implicit differentiation

is a technique for computing $\frac{dy}{dx}$ for a function defined by an equation, accomplished by differentiating both sides of the equation (remembering to treat the variable y as a function) and solving for $\frac{dy}{dx}$

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3.4E: EXERCISES FOR SECTION 3.4

In exercises 1 - 10, use implicit differentiation to find $\frac{dy}{dx}$.

1) $x^2 - y^2 = 4$

2) $6x^2 + 3y^2 = 12$

Answer

$$\frac{dy}{dx} = \frac{-2x}{y}$$

3) $x^2y = y - 7$

4) $3x^3 + 9xy^2 = 5x^3$

Answer

$$\frac{dy}{dx} = \frac{x}{3y} - \frac{y}{2x}$$

5) $xy - 1/y = 1$

6) $y\sqrt{x+4} = xy + 8$

Answer

$$\frac{dy}{dx} = \frac{y - \frac{y}{2\sqrt{x+4}}}{\sqrt{x+4} - x}$$

7) $-xy - 2 = \frac{x}{7}$

8) $(xy)^2 + 3x = y^2$

9) $x^3y + xy^3 = -8$

Answer

$$\frac{dy}{dx} = \frac{-3x^2y - y^3}{x^3 + 3xy^2}$$

For exercises 10-12, find the equation of the tangent line to the graph of the given equation at the indicated point.

10) [T] $x^4y - xy^3 = -2, \quad (-1, -1)$

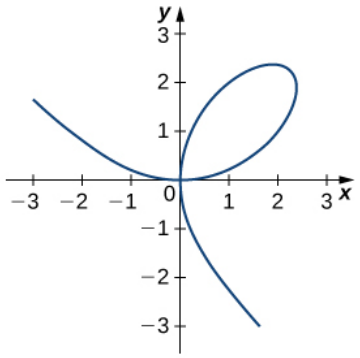
11) [T] $x^2y^2 + 5xy = 14, \quad (2, 1)$

Answer

$y = -\frac{1}{2}x + 2$

12) [T] $\frac{x}{y} + 5x - 7 = -\frac{3}{4}y, \quad (1, 2)$

13) [T] The graph of a folium of Descartes with equation $2x^3 + 2y^3 - 9xy = 0$ is given in the following graph.



a. Find the equation of the tangent line at the point $(2,1)$. Graph the tangent line along with the folium.

b. Find the equation of the normal line to the tangent line in a. at the point $(2,1)$.

14) For the equation $x^2+2xy-3y^2=0$,

a. Find the equation of the normal to the tangent line at the point $(1,1)$.

b. At what other point does the normal line in a. intersect the graph of the equation?

Answer

a. $y=-x+2$
b. $(3,-1)$

15) Find all points on the graph of $y^3-27y=x^2-90$ at which the tangent line is vertical.

16) For the equation $x^2+xy+y^2=7$,

a. Find the x-intercept(s).

b.Find the slope of the tangent line(s) at the x-intercept(s).

c. What does the value(s) in part b. indicate about the tangent line(s)?

Answer

a. $(\pm\sqrt{7},0)$
b. -2
c. They are parallel since the slope is the same at both intercepts.

17) Find the equation of the tangent line to the graph of the equation $\sin^{-1}x+\sin^{-1}y=\frac{\pi}{6}$ at the point $\left(0,\frac{1}{2}\right)$.

18) Find the equation of the tangent line to the graph of the equation $\tan^{-1}(x+y)=x^2+\frac{\pi}{4}$ at the point $(0,1)$.

Answer

$y=-x+1$

19) Find y' and y'' for $x^2+6xy-2y^2=3$.

20) [T] The number of cell phones produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $60x^{3/4}y^{1/4}=3240$.

a. Find $\frac{dy}{dx}$ and evaluate at the point $(81,16)$.

b. Interpret the result of a.

Answer

a. $\frac{dy}{dx}=-0.5926$
b. When \$81 is spent on labor and \$16 is spent on capital, the amount spent on capital is decreasing by \$0.5926 per \$1 spent on labor.

21) [T] The number of cars produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $30x^{1/3}y^{2/3}=360$.

(Both $x$and y are measured in thousands of dollars.)

a. Find $\frac{dy}{dx}$ and evaluate at the point $(27,8)$.

b. Interpret the result of part a.

22) The volume of a right circular cone of radius x and height y is given by $V=\frac{1}{3}\pi x^2y$. Suppose that the volume of the cone is $85\pi\,\text{cm}^3$. Find $\dfrac{dy}{dx}$ when $x=4$ and $y=16$.

Answer

$\dfrac{dy}{dx}=-8$

For exercises 23 - 24, consider a closed rectangular box with a square base with side x and height y.

23) Find an equation for the surface area of the rectangular box, $S(x,y)$.

24) If the surface area of the rectangular box is 78 square feet, find $\frac{dy}{dx}$ when $x = 3$ feet and $y = 5$ feet.

Answer

$\frac{dy}{dx} = -2.67$

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3.5: Exponential and Logarithmic Functions

Learning Objectives

- Identify the form of an exponential function.
- Explain the difference between the graphs of x^b and b^x.
- Recognize the significance of the number e.
- Identify the form of a logarithmic function.
- Explain the relationship between exponential and logarithmic functions.
- Describe how to calculate a logarithm to a different base.

In this section we examine exponential and logarithmic functions. We use the properties of these functions to solve equations involving exponential or logarithmic terms, and we study the meaning and importance of the number e.

3.5.1 Exponential Functions

Exponential functions arise in many applications. One common example is **population growth.** For example, if a population starts with P_0 individuals and then grows at an annual rate of 2%, its population after 1 year is

$$P(1) = P_0 + 0.02P_0 = P_0(1 + 0.02) = P_0(1.02).$$

Its population after 2 years is

$$P(2) = P(1) + 0.02P(1) = P(1)(1.02) = P_0(1.02)^2.$$

In general, its population after t years is

$$P(t) = P_0(1.02)^t,$$

which is an exponential function. More generally, any function of the form $f(x) = b^x$, where $b > 0$, $b \neq 1$, is an exponential function with **base** b and **exponent** x. Exponential functions have constant bases and variable exponents. Note that a function of the form $f(x) = x^b$ for some constant b is not an exponential function but a power function.

To see the difference between an exponential function and a power function, we compare the functions $y = x^2$ and $y = 2^x$. In Table 3.5.1, we see that both 2^x and x^2 approach infinity as $x \to \infty$. Eventually, however, 2^x becomes larger than x^2 and grows more rapidly as $x \to \infty$. In the opposite direction, as $x \to -\infty$, $x^2 \to \infty$, whereas $2^x \to 0$. The line $y = 0$ is a horizontal asymptote for $y = 2^x$.

Table 3.5.1

x	-3	-2	-1	0	1	2	3	4	5	6
x^2	9	4	1	0	1	4	9	16	25	36
2^x	1/8	1/4	1/2	1	2	4	8	16	32	64

In Figure 3.5.1, we graph both $y = x^2$ and $y = 2^x$ to show how the graphs differ.

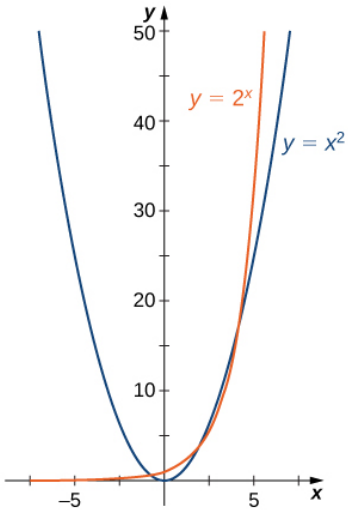


Figure 3.5.1: Both 2^x and x^2 approach infinity as $x \to \infty$, but 2^x grows more rapidly than x^2. As $x \to -\infty$, $x^2 \to \infty$, whereas $2^x \to 0$.

3.5.2 Evaluating Exponential Functions

Recall the properties of exponents: If x is a positive integer, then we define $b^x = b \cdot b \cdots b$ (with x factors of b). If x is a negative integer, then $x = -y$ for some positive integer y, and we define $b^x = b^{-y} = 1/b^y$. Also, b^0 is defined to be 1. If x is a rational number, then $x = p/q$, where p and q are integers and $b^x = b^{p/q} = \sqrt[q]{b^p}$. For example, $9^{3/2} = \sqrt{9^3} = \left(\sqrt{9}\right)^3 = 27$. However, how is b^x defined if x is an irrational number? For example, what do we mean by $2^{\sqrt{2}}$? This is too complex a question for us to answer fully right now; however, we can make an approximation.

Table 3.5.2: Values of 2^x for a List of Rational Numbers Approximating $\sqrt{2}$

x	1.4	1.41	1.414	1.4142	1.41421	1.414213
2^x	2.639	2.65737	2.66475	2.665119	2.665138	2.665143

In Table 3.5.2, we list some rational numbers approaching $\sqrt{2}$, and the values of 2^x for each rational number x are presented as well. We claim that if we choose rational numbers x getting closer and closer to $\sqrt{2}$, the values of 2^x get closer and closer to some number L. We define that number L to be $2^{\sqrt{2}}$.

Example 3.5.1: Bacterial Growth

Suppose a particular population of bacteria is known to double in size every 4 hours. If a culture starts with 1000 bacteria, the number of bacteria after 4 hours is $n(4) = 1000 \cdot 2$. The number of bacteria after 8 hours is $n(8) = n(4) \cdot 2 = 1000 \cdot 2^2$. In general, the number of bacteria after $4m$ hours is $n(4m) = 1000 \cdot 2^m$. Letting $t = 4m$, we see that the number of bacteria after t hours is $n(t) = 1000 \cdot 2^{t/4}$. Find the number of bacteria after 6 hours, 10 hours, and 24 hours.

Solution

The number of bacteria after 6 hours is given by

$$n(6) = 1000 \cdot 2^{6/4} \approx 2828 \text{ bacteria.}$$

The number of bacteria after 10 hours is given by

$$n(10) = 1000 \cdot 2^{10/4} \approx 5657 \text{ bacteria.}$$

The number of bacteria after 24 hours is given by $n(24) = 1000 \cdot 2^6 = 64,000$ bacteria.

Exercise 3.5.1

Given the exponential function $f(x) = 100 \cdot 3^{x/2}$, evaluate $f(4)$ and $f(10)$.

Answer

$f(4) = 900$

$f(10) = 24,300$

3.5.3 Graphing Exponential Functions

For any base $b > 0$, $b \neq 1$, the exponential function $f(x) = b^x$ is defined for all real numbers x and $b^x > 0$. Therefore, the domain of $f(x) = b^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. To graph b^x, we note that for $b > 1$, b^x is increasing on $(-\infty, \infty)$ and $b^x \to \infty$ as $x \to \infty$, whereas $b^x \to 0$ as $x \to -\infty$. On the other hand, if $0 < b < 1$, $f(x) = b^x$ is decreasing on $(-\infty, \infty)$ and $b^x \to 0$ as $x \to \infty$ whereas $b^x \to \infty$ as $x \to -\infty$ (Figure 3.5.2).

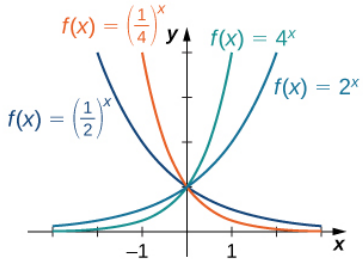


Figure 3.5.2: If $b > 1$, then b^x is increasing on $(-\infty, \infty)$. If $0 < b < 1$, then b^x is decreasing on $(-\infty, \infty)$.

Note that exponential functions satisfy the general laws of exponents. To remind you of these laws, we state them as rules.

Laws of Exponents

For any constants $a > 0$, $b > 0$, and for all x and y,

1. $b^x \cdot b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x b^x$
5. $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

Example 3.5.2: Using the Laws of Exponents

Use the laws of exponents to simplify each of the following expressions.

a. $\dfrac{(2x^{2/3})^3}{(4x^{-1/3})^2}$

b. $\dfrac{(x^3 y^{-1})^2}{(xy^2)^{-2}}$

Soution

a. We can simplify as follows:

$$\frac{(2x^{2/3})^3}{(4x^{-1/3})^2} = \frac{2^3(x^{2/3})^3}{4^2(x^{-1/3})^2} = \frac{8x^2}{16x^{-2/3}} = \frac{x^2 x^{2/3}}{2} = \frac{x^{8/3}}{2}.$$

b. We can simplify as follows:

$$\frac{(x^3y^{-1})^2}{(xy^2)^{-2}} = \frac{(x^3)^2(y^{-1})^2}{x^{-2}(y^2)^{-2}} = \frac{x^6y^{-2}}{x^{-2}y^{-4}} = x^6x^2y^{-2}y^4 = x^8y^2.$$

? Exercise 3.5.2

Use the laws of exponents to simplify $\dfrac{6x^{-3}y^2}{12x^{-4}y^5}$.

Hint

$x^a/x^b = x^{a-b}$

Answer

$x/(2y^3)$

3.5.4 The Number e

A special type of exponential function appears frequently in real-world applications. To describe it, consider the following example of exponential growth, which arises from **compounding interest** in a savings account. Suppose a person invests P dollars in a savings account with an annual interest rate r, compounded annually. The amount of money after 1 year is

$$A(1) = P + rP = P(1+r) \ .$$

The amount of money after 2 years is

$$A(2) = A(1) + rA(1) = P(1+r) + rP(1+r) = P(1+r)^2 \quad .$$

More generally, the amount after t years is

$$A(t) = P(1+r)^t \ .$$

If the money is compounded 2 times per year, the amount of money after half a year is

$$A\left(\frac{1}{2}\right) = P + \left(\frac{r}{2}\right)P = P\left(1 + \left(\frac{r}{2}\right)\right) \ .$$

The amount of money after 1 year is

$$A(1) = A\left(\frac{1}{2}\right) + \left(\frac{r}{2}\right)A\left(\frac{1}{2}\right) = P\left(1+\frac{r}{2}\right) + \frac{r}{2}\left(\left(P(1+\frac{r}{2}\right)\right) = P\left(1+\frac{r}{2}\right)^2.$$

After t years, the amount of money in the account is

$$A(t) = P\left(1+\frac{r}{2}\right)^{2t}.$$

More generally, if the money is compounded n times per year, the amount of money in the account after t years is given by the function

$$A(t) = P\left(1+\frac{r}{n}\right)^{nt}.$$

What happens as $n \to \infty$? To answer this question, we let $m = n/r$ and write

$$\left(1+\frac{r}{n}\right)^{nt} = \left(1+\frac{1}{m}\right)^{mrt},$$

and examine the behavior of $(1+1/m)^m$ as $m \to \infty$, using a table of values (Table 3.5.3).

Table 3.5.3: Values of $\left(1+\dfrac{1}{m}\right)^m$ as $m \to \infty$

m	10	100	1000	10,000	100,000	1,000,000

$\left(1+\frac{1}{m}\right)^m$	2.5937	2.7048	2.71692	2.71815	2.718268	2.718280

Looking at this table, it appears that $(1+1/m)^m$ is approaching a number between 2.7 and 2.8 as $m \to \infty$. In fact, $(1+1/m)^m$ does approach some number as $m \to \infty$. We call this number e. To six decimal places of accuracy,

$$e \approx 2.718282.$$

Leonhard Euler

The letter e was first used to represent this number by the Swiss mathematician Leonhard Euler during the 1720s. Although Euler did not discover the number, he showed many important connections between e and logarithmic functions. We still use the notation e today to honor Euler's work because it appears in many areas of mathematics and because we can use it in many practical applications.

Returning to our savings account example, we can conclude that if a person puts P dollars in an account at an annual interest rate r, compounded continuously, then $A(t) = Pe^{rt}$. This function may be familiar. Since functions involving base e arise often in applications, we call the function $f(x) = e^x$ the **natural exponential function**. Not only is this function interesting because of the definition of the number e, but also, as discussed next, its graph has an important property.

Since $e > 1$, we know $f(x) = e^x$ is increasing on $(-\infty, \infty)$. In Figure 3.5.3, we show a graph of $f(x) = e^x$ along with a tangent line to the graph of f at $x = 0$. We give a precise definition of tangent line in the next chapter; but, informally, we say a tangent line to a graph of f at $x = a$ is a line that passes through the point $(a, f(a))$ and has the same "slope" as f at that point . The function $f(x) = e^x$ is the only exponential function b^x with tangent line at $x = 0$ that has a slope of 1. As we see later in the text, having this property makes the natural exponential function the most simple exponential function to use in many instances.

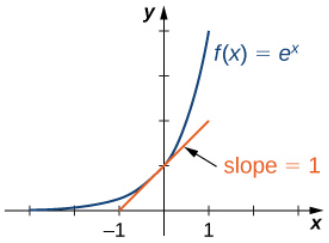


Figure 3.5.3: The graph of $f(x) = e^x$ has a tangent line with slope 1 at $x = 0$.

Example 3.5.3: Compounding Interest

Suppose $\$500$ is invested in an account at an annual interest rate of $r = 5.5\%$, compounded continuously.

a. Let t denote the number of years after the initial investment and $A(t)$ denote the amount of money in the account at time t. Find a formula for $A(t)$.
b. Find the amount of money in the account after 10 years and after 20 years.

Solution

a. If P dollars are invested in an account at an annual interest rate r, compounded continuously, then $A(t) = Pe^{rt}$. Here $P = \$500$ and $r = 0.055$. Therefore, $A(t) = 500e^{0.055t}$.

b. After 10 years, the amount of money in the account is

$$A(10) = 500e^{0.055 \cdot 10} = 500e^{0.55} \approx \$866.63.$$

After 20 years, the amount of money in the account is

$$A(20) = 500e^{0.055 \cdot 20} = 500e^{1.1} \approx \$1,502.08.$$

Exercise 3.5.3

If $\$750$ is invested in an account at an annual interest rate of 4%, compounded continuously, find a formula for the amount of money in the account after t years. Find the amount of money after 30 years.

Hint

$A(t) = Pe^{rt}$

Answer

$A(t) = 750e^{0.04t}$. After 30 years, there will be approximately $\$2,490.09$

3.5.5 Logarithmic Functions

Using our understanding of exponential functions, we can discuss their inverses, which are the logarithmic functions. These come in handy when we need to consider any phenomenon that varies over a wide range of values, such as the pH scale in chemistry or decibels in sound levels.

The exponential function $f(x) = b^x$ is *one-to-one*, with domain $(-\infty, \infty)$ and range $(0, \infty)$. Therefore, it has an inverse function, called the logarithmic function with base b. For any $b > 0$, $b \neq 1$, the logarithmic function with base b, denoted $\log_b$, has domain $(0, \infty)$ and range $(-\infty, \infty)$,and satisfies

$$\log_b(x) = y$$

if and only if $b^y = x$.

For example,

$$\log_2(8) = 3$$

since $2^3 = 8$,

$$\log_{10}\left(\frac{1}{100}\right) = -2$$

since $10^{-2} = \dfrac{1}{10^2} = \dfrac{1}{100}$,

$$\log_b(1) = 0$$

since $b^0 = 1$ for any base $b > 0$.

Furthermore, since $y = \log_b(x)$ and $y = b^x$ are inverse functions,

$$\log_b(b^x) = x$$

and

$$b^{\log_b(x)} = x.$$

The most commonly used logarithmic function is the function $\log_e$. Since this function uses natural e as its base, it is called the **natural logarithm**. Here we use the notation $\ln(x)$ or $\ln x$ to mean $\log_e(x)$. For example,

$$\begin{aligned} \ln(e) &= \log_e(e) = 1 \\ \ln(e^3) &= \log_e(e^3) = 3 \\ \ln(1) &= \log_e(1) = 0. \end{aligned}$$

Since the functions $f(x) = e^x$ and $g(x) = \ln(x)$ are inverses of each other,

$$\ln(e^x) = x \text{ and } e^{\ln x} = x,$$

and their graphs are symmetric about the line $y = x$ (Figure $3.5.4$).

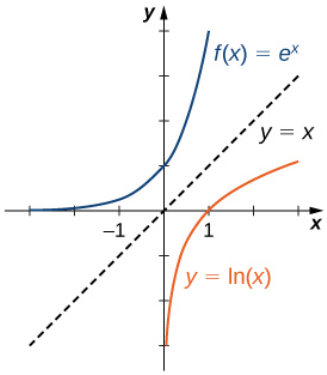


Figure 3.5.4: The functions $y = e^x$ and $y = \ln(x)$ are inverses of each other, so their graphs are symmetric about the line $y = x$.

In general, for any base $b > 0$, $b \neq 1$, the function $g(x) = \log_b(x)$ is symmetric about the line $y = x$ with the function $f(x) = b^x$. Using this fact and the graphs of the exponential functions, we graph functions $\log_b$ for several values of $b > 1$ (Figure 3.5.5).

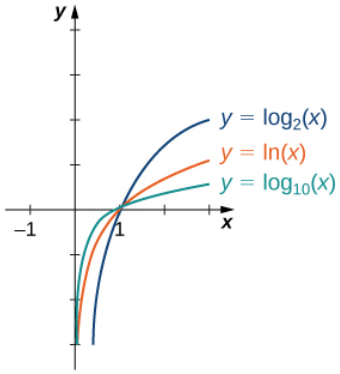


Figure 3.5.5: Graphs of $y = \log_b(x)$ are depicted for $b = 2,\ e,\ 10$.

Before solving some equations involving exponential and logarithmic functions, let's review the basic properties of logarithms.

Properties of Logarithms

If $a,\ b,\ c > 0,\ b \neq 1$, and r is any real number, then

- **Product property**

$$\log_b(ac) = \log_b(a) + \log_b(c) \tag{3.5.1}$$

- **Quotient property**

$$\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c) \tag{3.5.2}$$

- **Power property**

$$\log_b(a^r) = r\log_b(a) \tag{3.5.3}$$

Example 3.5.4: Solving Equations Involving Exponential Functions

Solve each of the following equations for x.

a. $5^x = 2$
b. $e^x + 6e^{-x} = 5$

Solution

a. Applying the natural logarithm function to both sides of the equation, we have

$$\ln 5^x = \ln 2\,.$$

Using the power property of logarithms,

$$x \ln 5 = \ln 2.$$

Therefore,

$$x = \frac{\ln 2}{\ln 5}.$$

b. Multiplying both sides of the equation by e^x,we arrive at the equation

$$e^{2x} + 6 = 5e^x \, .$$

Rewriting this equation as

$$e^{2x} - 5e^x + 6 = 0 \, ,$$

we can then rewrite it as a quadratic equation in e^x:

$$(e^x)^2 - 5(e^x) + 6 = 0.$$

Now we can solve the quadratic equation. Factoring this equation, we obtain

$$(e^x - 3)(e^x - 2) = 0.$$

Therefore, the solutions satisfy $e^x = 3$ and $e^x = 2$. Taking the natural logarithm of both sides gives us the solutions $x = \ln 3, \ln 2$.

Exercise 3.5.4

Solve

$$e^{2x}/(3 + e^{2x}) = 1/2.$$

Hint

First solve the equation for e^{2x}

Answer

$$x = \frac{\ln 3}{2} \, .$$

Example 3.5.5: Solving Equations Involving Logarithmic Functions

Solve each of the following equations for x.

a. $\ln\left(\frac{1}{x}\right) = 4$
b. $\log_{10} \sqrt{x} + \log_{10} x = 2$
c. $\ln(2x) - 3\ln(x^2) = 0$

Solution

a. By the definition of the natural logarithm function,

$$\ln\left(\frac{1}{x}\right) = 4$$

- if and only if $e^4 = \frac{1}{x}$.

Therefore, the solution is $x = 1/e^4$.

b. Using the product (Equation 3.5.1) and power (Equation 3.5.3) properties of logarithmic functions, rewrite the left-hand side of the equation as

$$\begin{aligned}\log_{10}\sqrt{x}+\log_{10}x &= \log_{10}x\sqrt{x}\\ &= \log_{10}x^{3/2}\\ &= \frac{3}{2}\log_{10}x.\end{aligned}$$

Therefore, the equation can be rewritten as

$$\frac{3}{2}\log_{10}x = 2$$

or

$$\log_{10}x = \frac{4}{3}.$$

The solution is $x = 10^{4/3} = 10\sqrt[3]{10}$.

c. Using the power property (Equation 3.5.3) of logarithmic functions, we can rewrite the equation as $\ln(2x) - \ln(x^6) = 0$.

Using the quotient property (Equation 3.5.2), this becomes

$$\ln\left(\frac{2}{x^5}\right) = 0$$

Therefore, $2/x^5 = 1$, which implies $x = \sqrt[5]{2}$. We should then check for any extraneous solutions.

Exercise 3.5.5

Solve $\ln(x^3) - 4\ln(x) = 1$.

Hint

First use the power property, then use the product property of logarithms.

Answer

$$x = \frac{1}{e}$$

When evaluating a logarithmic function with a calculator, you may have noticed that the only options are $\log_{10}$ or log, called the **common logarithm**, or ln, which is the natural logarithm. However, exponential functions and logarithm functions can be expressed in terms of any desired base b. If you need to use a calculator to evaluate an expression with a different base, you can apply the change-of-base formulas first. Using this change of base, we typically write a given exponential or logarithmic function in terms of the natural exponential and natural logarithmic functions.

Rule: Change-of-Base Formulas

Let $a > 0$, $b > 0$, and $a \neq 1$, $b \neq 1$.

1. $a^x = b^{x\log_b a}$ for any real number x.

   If $b = e$, this equation reduces to $a^x = e^{x\log_e a} = e^{x\ln a}$.

2. $\log_a x = \dfrac{\log_b x}{\log_b a}$ for any real number $x > 0$.

   If $b = e$, this equation reduces to $\log_a x = \dfrac{\ln x}{\ln a}$.

Proof

For the first change-of-base formula, we begin by making use of the power property of logarithmic functions. We know that for any base $b > 0,\ b \neq 1$, $\log_b(a^x) = x \log_b a$. Therefore,

$$b^{\log_b(a^x)} = b^{x \log_b a}.$$

In addition, we know that b^x and $\log_b(x)$ are inverse functions. Therefore,

$$b^{\log_b(a^x)} = a^x.$$

Combining these last two equalities, we conclude that $a^x = b^{x \log_b a}$.

To prove the second property, we show that

$$(\log_b a) \cdot (\log_a x) = \log_b x.$$

Let $u = \log_b a, v = \log_a x$, and $w = \log_b x$. We will show that $u \cdot v = w$. By the definition of logarithmic functions, we know that $b^u = a,\ a^v = x$, and $b^w = x$. From the previous equations, we see that

$$b^{uv} = (b^u)^v = a^v = x = b^w.$$

Therefore, $b^{uv} = b^w$. Since exponential functions are one-to-one, we can conclude that $u \cdot v = w$.

□

Example 3.5.6: Changing Bases

Use a calculating utility to evaluate $\log_3 7$ with the change-of-base formula presented earlier.

Solution

Use the second equation with $a = 3$ and $b = e$: $\log_3 7 = \dfrac{\ln 7}{\ln 3} \approx 1.77124.$

Exercise 3.5.6

Use the change-of-base formula and a calculating utility to evaluate $\log_4 6$.

Hint

Use the change of base to rewrite this expression in terms of expressions involving the natural logarithm function.

Answer

$\log_4 6 = \dfrac{\ln 6}{\ln 4} \approx 1.29248$

Example 3.5.7: The Richter Scale for Earthquakes

In 1935, Charles Richter developed a scale (now known as the Richter scale) to measure the magnitude of an **earthquake**. The scale is a base-10 logarithmic scale, and it can be described as follows: Consider one earthquake with magnitude R_1 on the Richter scale and a second earthquake with magnitude R_2 on the Richter scale. Suppose $R_1 > R_2$, which means the earthquake of magnitude R_1 is stronger, but how much stronger is it than the other earthquake?



Figure 3.5.6: (credit: modification of work by Robb Hannawacker, NPS)

A way of measuring the intensity of an earthquake is by using a seismograph to measure the amplitude of the earthquake waves. If A_1 is the amplitude measured for the first earthquake and A_2 is the amplitude measured for the second earthquake, then the amplitudes and magnitudes of the two earthquakes satisfy the following equation:

$$R_1 - R_2 = \log_{10}\left(\frac{A1}{A2}\right).$$

Consider an earthquake that measures 8 on the Richter scale and an earthquake that measures 7 on the Richter scale. Then,

$$8 - 7 = \log_{10}\left(\frac{A1}{A2}\right).$$

Therefore,

$$\log_{10}\left(\frac{A1}{A2}\right) = 1,$$

which implies $A_1/A_2 = 10$ or $A_1 = 10A_2$. Since A_1 is 10 times the size of A_2, we say that the first earthquake is 10 times as intense as the second earthquake. On the other hand, if one earthquake measures 8 on the Richter scale and another measures 6, then the relative intensity of the two earthquakes satisfies the equation

$$\log_{10}\left(\frac{A1}{A2}\right) = 8 - 6 = 2\,.$$

Therefore, $A_1 = 100A_2$.That is, the first earthquake is 100 times more intense than the second earthquake.

How can we use logarithmic functions to compare the relative severity of the magnitude 9 earthquake in Japan in 2011 with the magnitude 7.3 earthquake in Haiti in 2010?

Solution

To compare the Japan and Haiti earthquakes, we can use an equation presented earlier:

$$9 - 7.3 = \log_{10}\left(\frac{A1}{A2}\right).$$

Therefore, $A_1/A_2 = 10^{1.7}$, and we conclude that the earthquake in Japan was approximately 50 times more intense than the earthquake in Haiti.

? Exercise 3.5.7

Compare the relative severity of a magnitude 8.4 earthquake with a magnitude 7.4 earthquake.

Hint

$R_1 - R_2 = \log_{10}(A1/A2)$.

Answer

The magnitude 8.4 earthquake is roughly 10 times as severe as the magnitude 7.4 earthquake.

3.5.6 Key Concepts

- The exponential function $y = b^x$ is increasing if $b > 1$ and decreasing if $0 < b < 1$. Its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$.
- The logarithmic function $y = \log_b(x)$ is the inverse of $y = b^x$. Its domain is $(0, \infty)$ and its range is $(-\infty, \infty)$.
- The natural exponential function is $y = e^x$ and the natural logarithmic function is $y = \ln x = \log_e x$.
- Given an exponential function or logarithmic function in base a, we can make a change of base to convert this function to any base $b > 0$, $b \neq 1$. We typically convert to base e.

3.5.7 Glossary

base

the number b in the exponential function $f(x) = b^x$ and the logarithmic function $f(x) = \log_b x$

exponent

the value x in the expression b^x

natural exponential function

the function $f(x) = e^x$

natural logarithm

the function $\ln x = \log_e x$

number e

as m gets larger, the quantity $(1 + (1/m)^m$ gets closer to some real number; we define that real number to be e; the value of e is approximately 2.718282

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3.5E: Exercises for Section 3.5

In exercises 1 - 4, evaluate the given exponential functions as indicated, accurate to two significant digits after the decimal.

1) $f(x) = 5^x$

a. $x = 3$

b. $x = \frac{1}{2}$

c. $x = \sqrt{2}$

Answer

a. 125
b. 2.24
c. 9.74

2) $f(x) = (0.3)^x$

a. $x = -1$

b. $x = 4$

c. $x = -1.5$

3) $f(x) = 10^x$

a. $x = -2$

b. $x = 4$

c. $x = \frac{5}{3}$

Answer

a. 0.01
b. $10,000$
c. 46.42

4) $f(x) = e^x$

a. $x = 2$

b. $x = -3.2$

c. $x = \pi$

In exercises 5 - 10, match the exponential equation to the correct graph.

a. $y = 4^{-x}$

b. $y = 3^{x-1}$

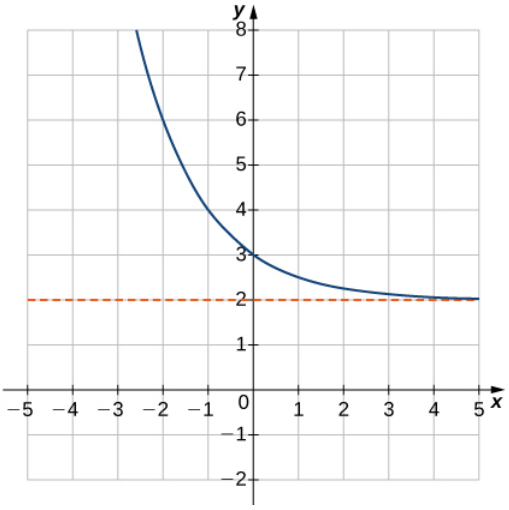
c. $y = 2^{x+1}$

d. $y = \left(\frac{1}{2}\right)^x + 2$

e. $y = -3^{-x}$

f. $y = 1 - 5^x$

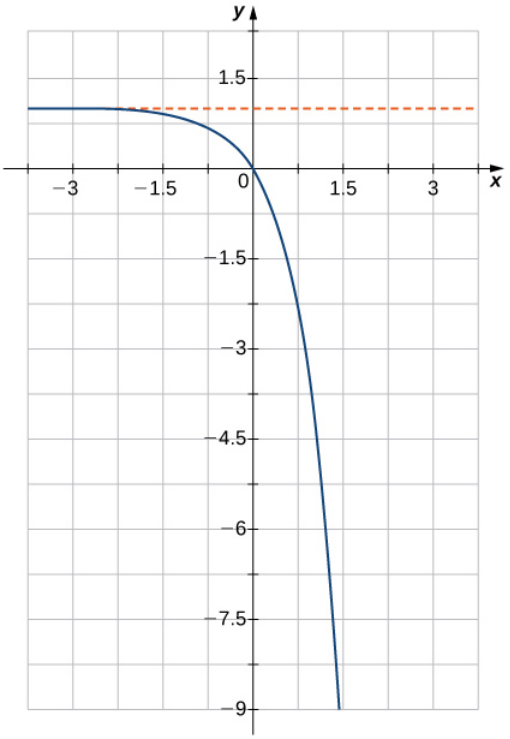
5)



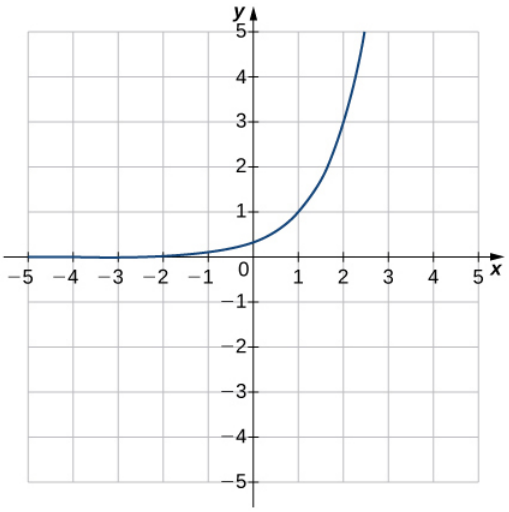
Answer

d

6)



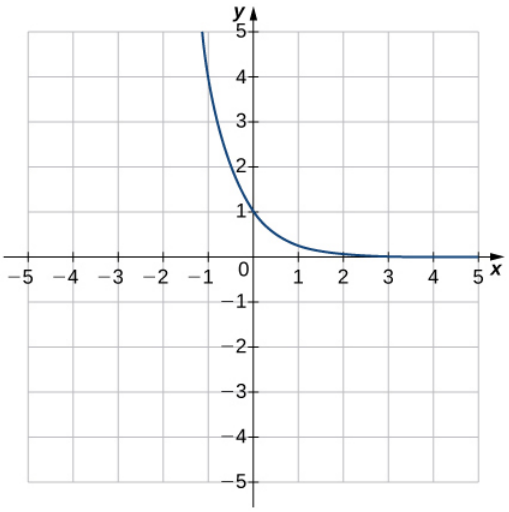
7)


y
x
5
4
3
2
1
0
−1
−2
−3
−4
−5
−5 −4 −3 −2 −1 1 2 3 4 5

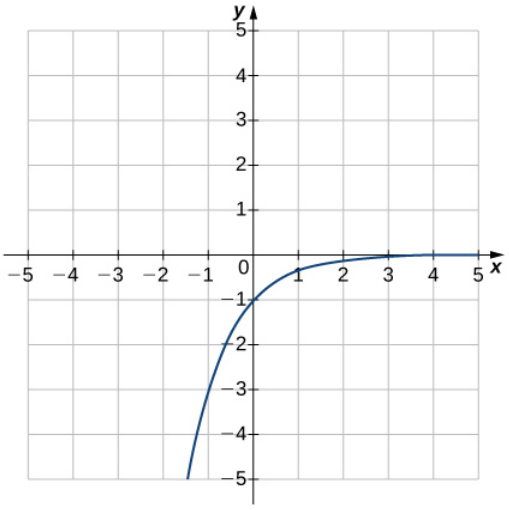
Answer

b

8)


y
x
5
4
3
2
1
0
−1
−2
−3
−4
−5
−5 −4 −3 −2 −1 1 2 3 4 5

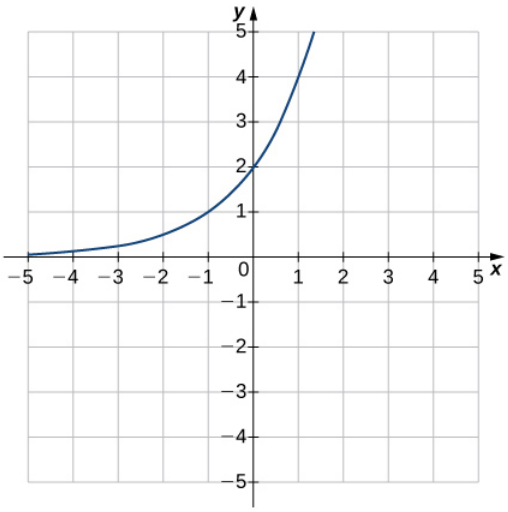
9)


y
x
5
4
3
2
1
0
−1
−2
−3
−4
−5
−5 −4 −3 −2 −1 1 2 3 4 5

Answer

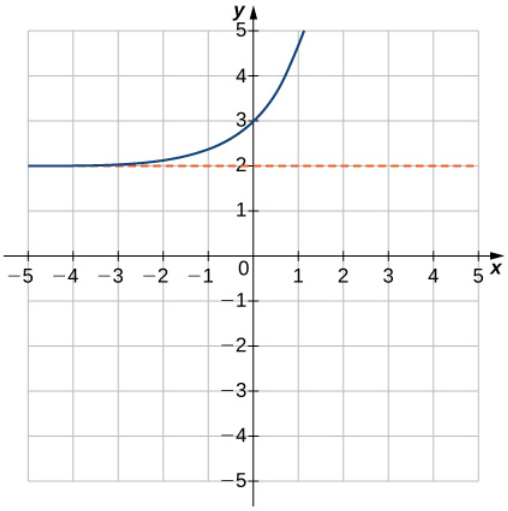
e

10)



In exercises 11 - 17, sketch the graph of the exponential function. Determine the domain, range, and horizontal asymptote.

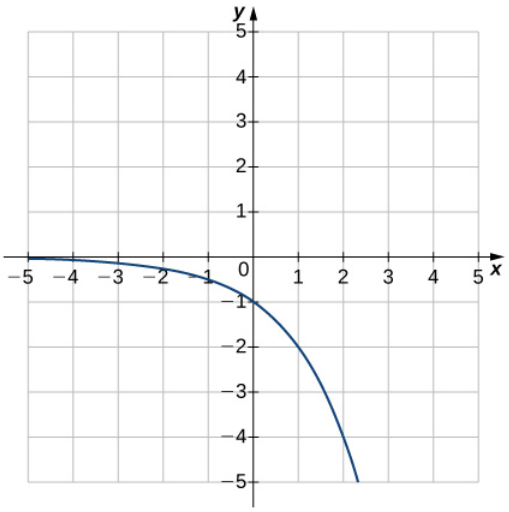
11) $f(x) = e^x + 2$



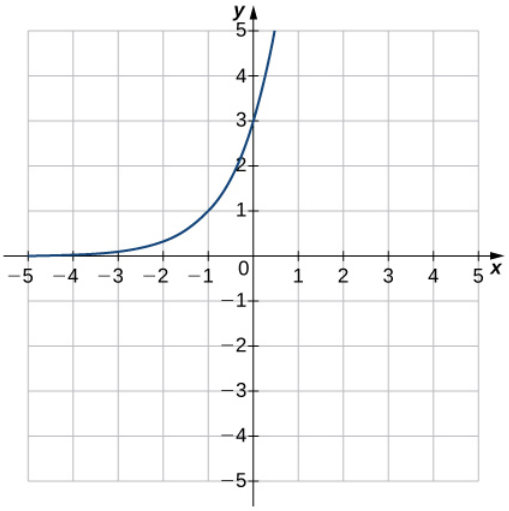
Answer

Domain: all real numbers, Range: $(2, \infty)$, $y = 2$

12) $f(x) = -2^x$



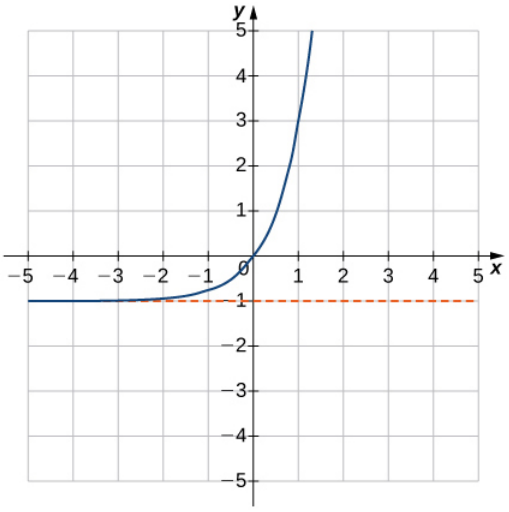
13) $f(x) = 3^{x+1}$



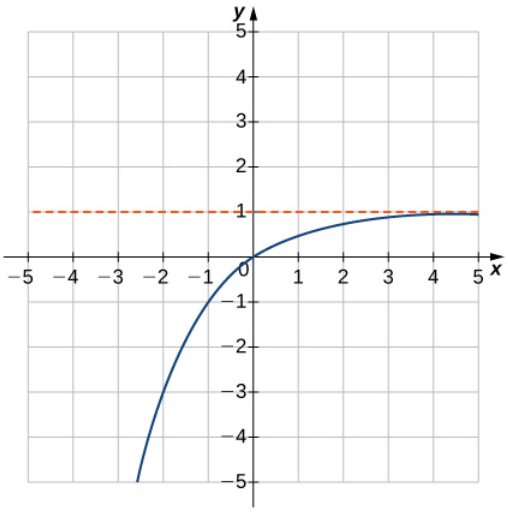
Answer

Domain: all real numbers, Range: $(0, \infty),\ y = 0$

14) $f(x) = 4^x - 1$



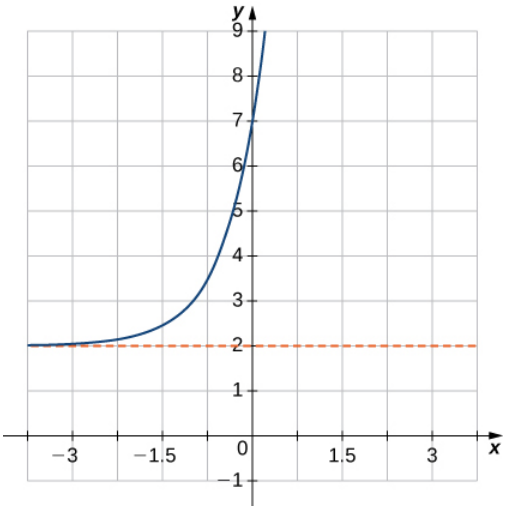
15) $f(x) = 1 - 2^{-x}$



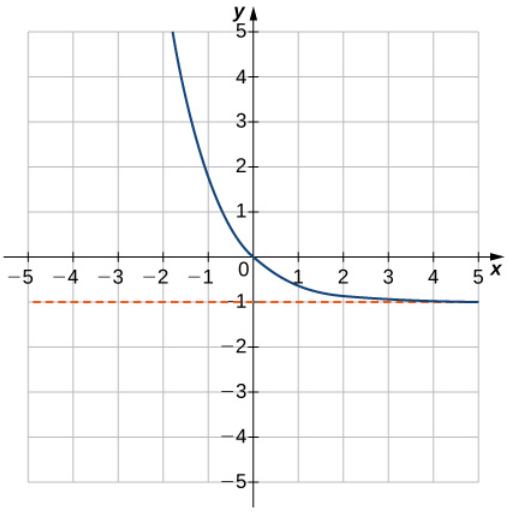
Answer

Domain: all real numbers, Range: $(-\infty, 1),\ y = 1$

16) $f(x) = 5^{x+1} + 2$



17) $f(x) = e^{-x} - 1$



Answer

Domain: all real numbers, Range: $(-1, \infty),\ y = -1$

In exercises 18 - 25, write the equation in equivalent exponential form.

18) $\log_3 81 = 4$

19) $\log_8 2 = \frac{1}{3}$

Answer

$8^{1/3} = 2$

20) $\log_5 1 = 0$

21) $\log_5 25 = 2$

Answer

$5^2 = 25$

22) $\log 0.1 = -1$

23) $\ln\left(\frac{1}{e^3}\right) = -3$

Answer

$e^{-3} = \dfrac{1}{e^3}$

24) $\log_9 3 = 0.5$

25) $\ln 1 = 0$

Answer

$e^0 = 1$

In exercises 26 - 35, write the equation in equivalent logarithmic form.

26) $2^3 = 8$

27) $4^{-2} = \frac{1}{16}$

Answer

$\log_4\left(\frac{1}{16}\right) = -2$

28) $10^2 = 100$

29) $9^0 = 1$

Answer

$\log_9 1 = 0$

30) $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$

31) $\sqrt[3]{64} = 4$

Answer

$\log_{64} 4 = \frac{1}{3}$

32) $e^x = y$

33) $9^y = 150$

Answer

$\log_9 150 = y$

34) $b^3 = 45$

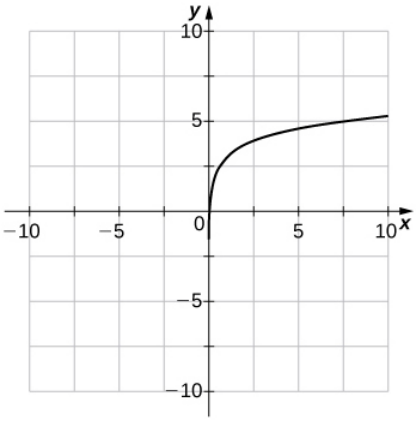
35) $4^{-3/2} = 0.125$

Answer

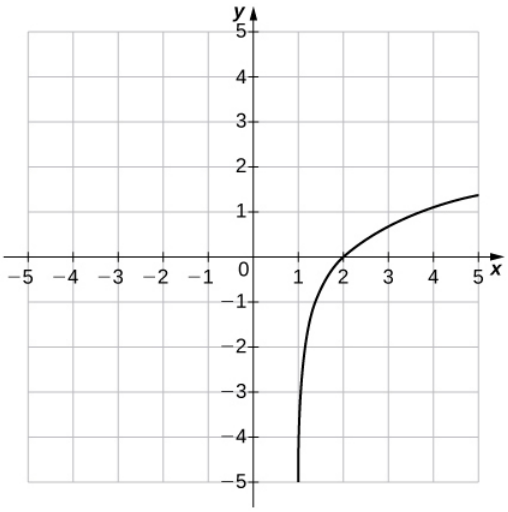
$\log_4 0.125 = -\frac{3}{2}$

In exercises 36 - 41, sketch the graph of the logarithmic function. Determine the domain, range, and vertical asymptote.

36) $f(x) = 3 + \ln x$



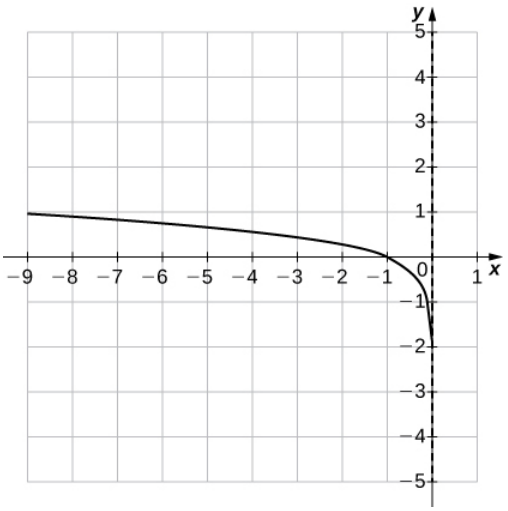
37) $f(x) = \ln(x - 1)$



Answer

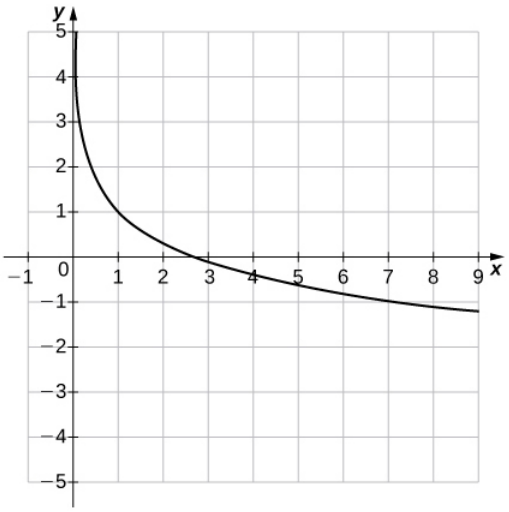
Domain: $(1, \infty)$, Range: $(-\infty, \infty)$, $x = 1$

38) $f(x) = \ln(-x)$



39) $f(x) = 1 - \ln x$

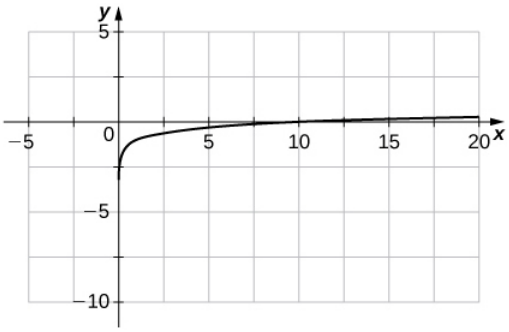
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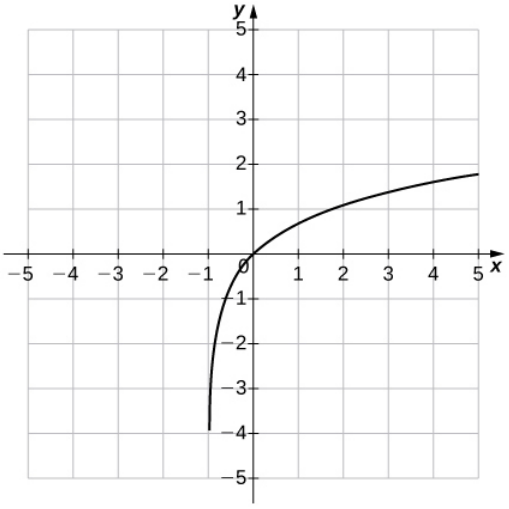
Answer

Domain: $(0, \infty)$, Range: $(-\infty, \infty)$, $x = 0$

40) $f(x) = \log x - 1$



41) $f(x) = \ln(x + 1)$



Answer

Domain: $(-1, \infty)$, Range: $(-\infty, \infty)$, $x = -1$

In exercises 42 - 47, use properties of logarithms to write the expressions as a sum, difference, and/or product of logarithms.

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42) $\log x^4 y$

43) $\log_3 \frac{9a^3}{b}$

Answer

$2 + 3\log_3 a - \log_3 b$

44) $\ln a\sqrt[3]{b}$

45) $\log_5 \sqrt{125xy^3}$

Answer

$\frac{3}{2} + \frac{1}{2}\log_5 x + \frac{3}{2}\log_5 y$

46) $\log_4 \frac{\sqrt[3]{xy}}{64}$

47) $\ln\left(\frac{6}{\sqrt{e^3}}\right)$

Answer

$-\frac{3}{2} + \ln 6$

In exercises 48 - 55, solve the exponential equation exactly.

48) $5^x = 125$

49) $e^{3x} - 15 = 0$

Answer

$\frac{\ln 15}{3}$

50) $8^x = 4$

51) $4^{x+1} - 32 = 0$

Answer

$\frac{3}{2}$

52) $3^{x/14} = \frac{1}{10}$

53) $10^x = 7.21$

Answer

$\log 7.21$

54) $4 \cdot 2^{3x} - 20 = 0$

55) $7^{3x-2} = 11$

Answer

$\frac{2}{3} + \frac{\log 11}{3\log 7}$

In exercises 56 - 63, solve the logarithmic equation exactly, if possible.

56) $\log_3 x = 0$

57) $\log_5 x = -2$

Answer

$x = \frac{1}{25}$

58) $\log_4(x+5)=0$

59) $\log(2x-7)=0$

Answer

$x=4$

60) $\ln\sqrt{x+3}=2$

61) $\log_6(x+9)+\log_6 x=2$

Answer

$x=3$

62) $\log_4(x+2)-\log_4(x-1)=0$

63) $\ln x+\ln(x-2)=\ln 4$

Answer

$1+\sqrt{5}$

In exercises 64 - 69, use the change-of-base formula and either base 10 or base e to evaluate the given expressions. Answer in exact form and in approximate form, rounding to four decimal places.

64) $\log_5 47$

65) $\log_7 82$

Answer

$\dfrac{\log 82}{\log 7}\approx 2.2646$

66) $\log_6 103$

67) $\log_{0.5} 211$

Answer

$\dfrac{\log 211}{\log 0.5}\approx -7.7211$

68) $\log_2 \pi$

69) $\log_{0.2} 0.452$

Answer

$\dfrac{\log 0.452}{\log 0.2}\approx 0.4934$

70) [T] The number of bacteria N in a culture after t days can be modeled by the function $N(t)=1300\cdot(2)^{t/4}$. Find the number of bacteria present after 15 days.

Answer

$\sim 17,491$

71) [T] The demand D (in millions of barrels) for oil in an oil-rich country is given by the function $D(p)=150\cdot(2.7)^{-0.25p}$, where p is the price (in dollars) of a barrel of oil. Find the amount of oil demanded (to the nearest million barrels) when the price is between $15 and $20.

72) [T] The amount A of a $100,000 investment paying continuously and compounded for t years is given by $A(t)=100,000\cdot e^{0.055t}$. Find the amount A accumulated in 5 years.

Answer

Approximately $131,653 is accumulated in 5 years.

73) [T] An investment is compounded monthly, quarterly, or yearly and is given by the function $A = P\left(1 + \frac{j}{n}\right)^{nt}$, where A is the value of the investment at time t, P is the initial principle that was invested, j is the annual interest rate, and n is the number of time the interest is compounded per year. Given a yearly interest rate of 3.5% and an initial principle of $100,000, find the amount A accumulated in 5 years for interest that is compounded a. daily, b., monthly, c. quarterly, and d. yearly.

74) [T] The concentration of hydrogen ions in a substance is denoted by $[H+]$, measured in moles per liter. The pH of a substance is defined by the logarithmic function $pH = -\log[H+]$. This function is used to measure the acidity of a substance. The pH of water is 7. A substance with a pH less than 7 is an acid, whereas one that has a pH of more than 7 is a base.

a. Find the pH of the following substances. Round answers to one digit.

b. Determine whether the substance is an acid or a base.

i. Eggs: $[H+] = 1.6 \times 10^{-8}$ mol/L

ii. Beer: $[H+] = 3.16 \times 10^{-3}$ mol/L

iii. Tomato Juice: $[H+] = 7.94 \times 10^{-5}$ mol/L

Answer

i. a. pH = 8 b. Base
ii. a. pH = 3 b. Acid
iii. a. pH = 4 b. Acid

75) [T] Iodine-131 is a radioactive substance that decays according to the function $Q(t) = Q_0 \cdot e^{-0.08664t}$, where Q_0 is the initial quantity of a sample of the substance and t is in days. Determine how long it takes (to the nearest day) for 95% of a quantity to decay.

76) [T] According to the World Bank, at the end of 2013 $(t = 0)$ the U.S. population was 316 million and was increasing according to the following model:

$$P(t) = 316e^{0.0074t},$$

where P is measured in millions of people and t is measured in years after 2013.

a. Based on this model, what will be the population of the United States in 2020?

b. Determine when the U.S. population will be twice what it is in 2013.

Answer

a. ~ 333 million
b. 94 years from 2013, or in 2107

77) [T] The amount A accumulated after 1000 dollars is invested for t years at an interest rate of 4% is modeled by the function $A(t) = 1000(1.04)^t$.

a. Find the amount accumulated after 5 years and 10 years.

b. Determine how long it takes for the original investment to triple.

78) [T] A bacterial colony grown in a lab is known to double in number in 12 hours. Suppose, initially, there are 1000 bacteria present.

a. Use the exponential function $Q = Q_0 e^{kt}$ to determine the value k, which is the growth rate of the bacteria. Round to four decimal places.

b. Determine approximately how long it takes for $200,000$ bacteria to grow.

Answer

a. $k \approx 0.0578$
b. ≈ 92 hours

79) [T] The rabbit population on a game reserve doubles every 6 months. Suppose there were 120 rabbits initially.

a. Use the exponential function $P = P_0 a^t$ to determine the growth rate constant a. Round to four decimal places.

b. Use the function in part a. to determine approximately how long it takes for the rabbit population to reach 3500.

80) [T] The 1906 earthquake in San Francisco had a magnitude of 8.3 on the Richter scale. At the same time, in Japan, an earthquake with magnitude 4.9 caused only minor damage. Approximately how much more energy was released by the San Francisco earthquake than by the Japanese earthquake?

Answer

The San Francisco earthquake had $10^{3.4}$ or ~ 2512 times more energy than the Japan earthquake.

3.5E.1 Contributors

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3.6: Derivatives of Exponential and Logarithmic Functions

Learning Objectives

- Find the derivative of exponential functions.
- Find the derivative of logarithmic functions.
- Use logarithmic differentiation to determine the derivative of a function.

So far, we have learned how to differentiate a variety of functions, but all primarily *algebraic* functions (polynomial, root, rational, etc.). In this section, we explore derivatives of exponential and logarithmic functions. As we discussed in our last section, exponential functions play an important role in modeling population growth and the decay of radioactive materials. Logarithmic functions can help rescale large quantities and are particularly helpful for rewriting complicated expressions.

3.6.1 Derivative of the Exponential Function

Just as when we found the derivatives of other functions, we can find the derivatives of exponential and logarithmic functions using formulas. As we develop these formulas, we need to make certain basic assumptions. The proofs that these assumptions hold are beyond the scope of this course.

First of all, we begin with the assumption that the function $B(x) = b^x,\ b > 0,$ is defined for every real number and is continuous. In previous courses, the values of exponential functions for all rational numbers were defined—beginning with the definition of b^n, where n is a positive integer—as the product of b multiplied by itself n times. Later, we defined $b^0 = 1, b^{-n} = \dfrac{1}{b^n}$, for a positive integer n, and $b^{s/t} = (\sqrt[t]{b})^s$ for positive integers s and t. These definitions leave open the question of the value of b^r where r is an arbitrary real number. By assuming the continuity of $B(x) = b^x, b > 0$, we may interpret b^r as $\lim\limits_{x \to r} b^x$ where the values of x as we take the limit are rational. For example, we may view 4^π as the number satisfying

$$4^3 < 4^\pi < 4^4, \quad 4^{3.1} < 4^\pi < 4^{3.2}, \quad 4^{3.14} < 4^\pi < 4^{3.15},$$
$$4^{3.141} < 4^\pi < 4^{3.142}, \quad 4^{3.1415} < 4^\pi < 4^{3.1416}, \quad \ldots.$$

As we see in the following table, $4^\pi \approx 77.88$.

x	4^x	x	4^x
4^3	64	$4^{3.141593}$	77.8802710486
$4^{3.1}$	73.5166947198	$4^{3.1416}$	77.8810268071
$4^{3.14}$	77.7084726013	$4^{3.142}$	77.9242251944
$4^{3.141}$	77.8162741237	$4^{3.15}$	78.7932424541
$4^{3.1415}$	77.8702309526	$4^{3.2}$	84.4485062895
$4^{3.14159}$	77.8799471543	4^4	256

Approximating a Value of 4^π

We also assume that for $B(x) = b^x,\ b > 0$, the value $B'(0)$ of the derivative exists. In this section, we show that by making this one additional assumption, it is possible to prove that the function $B(x)$ is differentiable everywhere.

We make one final assumption: that there is a unique value of $b > 0$ for which $B'(0) = 1$. We define e to be this unique value, as we did in Introduction to Functions and Graphs. Figure 3.6.1 provides graphs of the functions $y = 2^x$, $y = 3^x$, $y = 2.7^x$, and $y = 2.8^x$. A visual estimate of the slopes of the tangent lines to these functions at 0 provides evidence that the value of e lies somewhere between 2.7 and 2.8. The function $E(x) = e^x$ is called the **natural exponential function**. Its inverse, $L(x) = \log_e x = \ln x$ is called the **natural logarithmic function.**

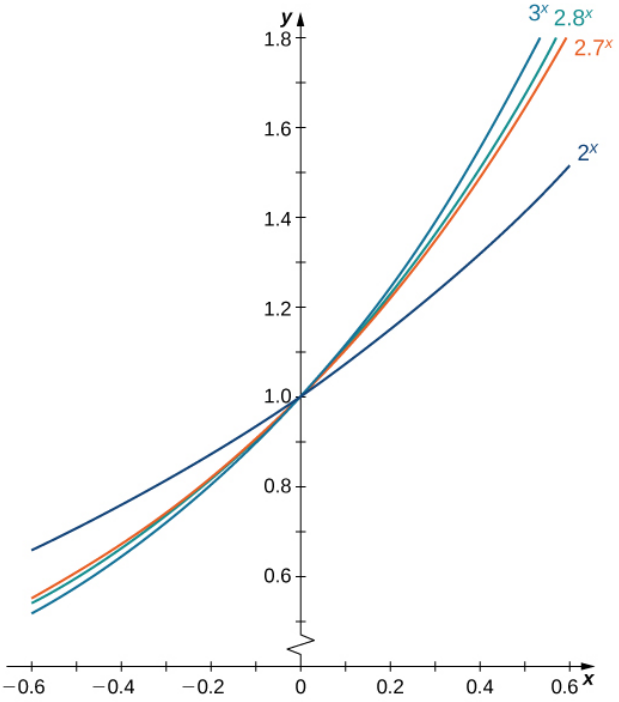


Figure 3.6.1: The graph of $E(x) = e^x$ is between $y = 2^x$ and $y = 3^x$.

For a better estimate of e, we may construct a table of estimates of $B'(0)$ for functions of the form $B(x) = b^x$. Before doing this, recall that

$$B'(0) = \lim_{x \to 0} \frac{b^x - b^0}{x - 0} = \lim_{x \to 0} \frac{b^x - 1}{x} \approx \frac{b^x - 1}{x}$$

for values of x very close to zero. For our estimates, we choose $x = 0.00001$ and $x = -0.00001$

to obtain the estimate

$$\frac{b^{-0.00001} - 1}{-0.00001} < B'(0) < \frac{b^{0.00001} - 1}{0.00001}.$$

See the following table.

Table : Estimating the value of e

b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$.	b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$.
2	$0.693145 < B'(0) < 0.69315$	2.7183	$1.000002 < B'(0) < 1.000012$
2.7	$0.993247 < B'(0) < 0.993257$	2.719	$1.000259 < B'(0) < 1.000269$
2.71	$0.996944 < B'(0) < 0.996954$	2.72	$1.000627 < B'(0) < 1.000637$
2.718	$0.999891 < B'(0) < 0.999901$	2.8	$1.029614 < B'(0) < 1.029625$
2.7182	$0.999965 < B'(0) < 0.999975$	3	$1.098606 < B'(0) < 1.098618$

The evidence from the table suggests that $2.7182 < e < 2.7183$.

The graph of $E(x) = e^x$ together with the line $y = x + 1$ are shown in Figure 3.6.2. This line is tangent to the graph of $E(x) = e^x$ at $x = 0$.

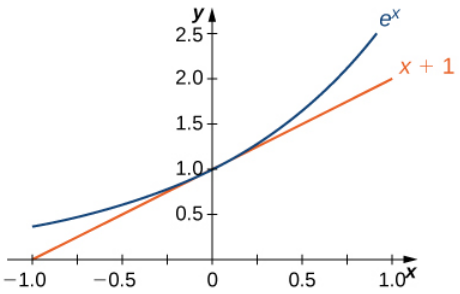


Figure 3.6.2: The tangent line to $E(x) = e^x$ at $x = 0$ has slope 1.

Now that we have laid out our basic assumptions, we begin our investigation by exploring the derivative of $B(x) = b^x$, $b > 0$. Recall that we have assumed that $B'(0)$ exists. By applying the limit definition to the derivative we conclude that

$$B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}$$

Turning to $B'(x)$, we obtain the following.

$$\begin{aligned}
B'(x) &= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} && \text{Apply the limit definition of the derivative.} \\
&= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} && \text{Note that } b^{x+h} = b^x b^h. \\
&= \lim_{h \to 0} \frac{b^x (b^h - 1)}{h} && \text{Factor out } b^x. \\
&= b^x \lim_{h \to 0} \frac{b^h - 1}{h} && \text{Apply a property of limits.} \\
&= b^x B'(0) && \text{Use } B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}.
\end{aligned}$$

We see that on the basis of the assumption that $B(x) = b^x$ is differentiable at 0, $B(x)$ is not only differentiable everywhere, but its derivative is

$$B'(x) = b^x B'(0).$$

For $E(x) = e^x$, $E'(0) = 1$. Thus, we have $E'(x) = e^x$. (The value of $B'(0)$ for an arbitrary function of the form $B(x) = b^x$, $b > 0$, will be derived later.)

Derivative of the Natural Exponential Function

Let $E(x) = e^x$ be the natural exponential function. Then

$$E'(x) = e^x.$$

In general,

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x)$$

Example 3.6.1: Derivative of an Exponential Function

Find the derivative of $f(x) = e^{5x^2}$.

Solution:

Using the derivative formula and the chain rule,

$$f'(x) = e^{5x^2} \frac{d}{dx}\left(5x^2\right) = e^{5x^2} \cdot 10x = 10x e^{5x^2} \tag{3.6.1}$$

Example 3.6.2: Combining Differentiation Rules

Find the derivative of $y = \dfrac{e^{x^2}}{x}$.

Solution

Use the derivative of the natural exponential function, the quotient rule, and the chain rule.

$$y' = \frac{(e^{x^2} \cdot 2)x \cdot x - 1 \cdot e^{x^2}}{x^2} \qquad \text{Apply the quotient rule.}$$

$$= \frac{e^{x^2}(2x^2 - 1)}{x^2} \qquad \text{Simplify.}$$

Exercise 3.6.1

Find the derivative of $h(x) = xe^{2x}$.

Hint

Don't forget to use the product rule.

Answer

$h'(x) = e^{2x} + 2xe^{2x}$

Example 3.6.3: Applying the Natural Exponential Function

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'(t)$, to the population, $A(t)$ is constant.

Solution

First find $A'(t)$. By using the chain rule, we have $A'(t) = 300e^{0.3t}$. Thus, the ratio of the rate of change of the population to the population is given by

$$\frac{A'(t)}{A(t)} = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3.$$

The ratio of the rate of change of the population to the population is the constant 0.3.

Exercise 3.6.2

If $A(t) = 1000e^{0.3t}$ describes the mosquito population after t days, as in the preceding example, what is the rate of change of $A(t)$ after 4 days?

Hint

Find $A'(4)$.

Answer

996

3.6.2 Derivative of the Logarithmic Function

Now that we have the derivative of the natural exponential function, we can use implicit differentiation to find the derivative of its inverse, the natural logarithmic function.

Definition: The Derivative of the Natural Logarithmic Function

If $x > 0$ and $y = \ln x$, then

$$\frac{dy}{dx} = \frac{1}{x}.$$

More generally, let $g(x)$ be a differentiable function. For all values of x for which $g'(x) > 0$, the derivative of $h(x) = \ln(g(x))$ is given by

$$h'(x) = \frac{1}{g(x)} g'(x).$$

Proof

If $x > 0$ and $y = \ln x$, then $e^y = x$. Differentiating both sides of this equation results in the equation

$$e^y \frac{dy}{dx} = 1.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

Finally, we substitute $x = e^y$ to obtain

$$\frac{dy}{dx} = \frac{1}{x}.$$

The graph of $y = \ln x$ and its derivative $\frac{dy}{dx} = \frac{1}{x}$ are shown in Figure 3.6.3.

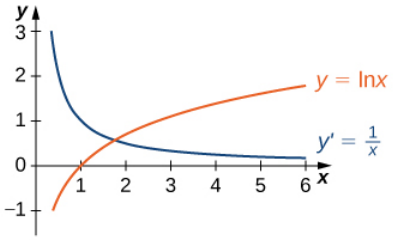


Figure 3.6.3: The function $y = \ln x$ is increasing on $(0, +\infty)$. Its derivative $y' = \frac{1}{x}$ is greater than zero on $(0, +\infty)$

Example 3.6.4: Taking a Derivative of a Natural Logarithm

Find the derivative of $f(x) = \ln(x^3 + 3x - 4)$.

Solution

$$f'(x) = \frac{1}{x^3 + 3x - 4} \cdot (3x^2 + 3) \qquad \text{Use } g(x) = x^3 + 3x - 4 \text{ in } h'(x) = \frac{1}{g(x)} g'(x).$$

$$= \frac{3x^2 + 3}{x^3 + 3x - 4} \qquad \text{Rewrite.}$$

Example 3.6.5: Using Properties of Logarithms in a Derivative

Find the derivative of $f(x) = \ln\left(\frac{x^2 + 3x}{2x + 1}\right)$.

Solution

At first glance, taking this derivative appears rather complicated. However, by using the properties of logarithms prior to finding the derivative, we can make the problem much simpler.

$f(x) = \ln\left(\dfrac{x^2+3x}{2x+1}\right) = \ln(x^2+3x) - \ln(2x+1)$ Apply properties of logarithms.

$f'(x) = \dfrac{2x+3}{x^2+3x} - \dfrac{2}{2x+1}$ Apply sum rule and $h'(x) = \dfrac{1}{g(x)}g'(x)$.

Exercise 3.6.3

Differentiate: $f(x) = \ln(3x+2)^5$.

Hint

Use a property of logarithms to simplify before taking the derivative.

Answer

$f'(x) = \dfrac{15}{3x+2}$

Now that we can differentiate the natural logarithmic function, we can use this result to find the derivatives of $y = \log_b x$ and $y = b^x$ for $b > 0$, $b \neq 1$.

Derivatives of General Exponential and Logarithmic Functions

Let $b > 0, b \neq 1,$ and let $g(x)$ be a differentiable function.

i. If $y = \log_b x$, then

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

More generally, if $h(x) = \log_b(g(x))$, then for all values of x for which $g(x) > 0$,

$$h'(x) = \frac{g'(x)}{g(x)\ln b}. \tag{3.6.2}$$

ii. If $y = b^x$, then

$$\frac{dy}{dx} = b^x \ln b.$$

More generally, if $h(x) = b^{g(x)}$, then

$$h'(x) = b^{g(x)} g'(x) \ln b \tag{3.6.3}$$

Proof

If $y = \log_b x$, then $b^y = x$. It follows that $\ln(b^y) = \ln x$. Thus $y \ln b = \ln x$. Solving for y, we have $y = \dfrac{\ln x}{\ln b}$. Differentiating and keeping in mind that $\ln b$ is a constant, we see that

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

The derivative in Equation 3.6.2 now follows from the chain rule.

If $y = b^x$. then $\ln y = x \ln b$. Using implicit differentiation, again keeping in mind that $\ln b$ is constant, it follows that $\dfrac{1}{y}\dfrac{dy}{dx} = \ln b$. Solving for $\dfrac{dy}{dx}$ and substituting $y = b^x$, we see that

$$\frac{dy}{dx} = y \ln b = b^x \ln b.$$

The more general derivative (Equation 3.6.3) follows from the chain rule.

□

Example 3.6.6: Applying Derivative Formulas

Find the derivative of $h(x) = \dfrac{3^x}{3^x + 2}$.

Solution

Use the quotient rule and Note.

$$h'(x) = \frac{3^x \ln 3(3^x + 2) - 3^x \ln 3(3^x)}{(3^x + 2)^2} \qquad \text{Apply the quotient rule.}$$

$$= \frac{2 \cdot 3^x \ln 3}{(3x + 2)^2} \qquad \text{Simplify.}$$

Example 3.6.7: Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

Solution

To find the slope, we must evaluate $\dfrac{dy}{dx}$ at $x = 1$. Using Equation 3.6.2, we see that

$$\frac{dy}{dx} = \frac{3}{(3x + 1)\ln 2}.$$

By evaluating the derivative at $x = 1$, we see that the tangent line has slope

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{3}{4 \ln 2} = \frac{3}{\ln 16}.$$

Exercise 3.6.4

Find the slope for the line tangent to $y = 3^x$ at $x = 2$.

Hint

Evaluate the derivative at $x = 2$.

Answer

$9 \ln(3)$

3.6.3 Key Concepts

- On the basis of the assumption that the exponential function $y = b^x$, $b > 0$ is continuous everywhere and differentiable at 0, this function is differentiable everywhere and there is a formula for its derivative.
- We can use a formula to find the derivative of $y = \ln x$, and the relationship $\log_b x = \dfrac{\ln x}{\ln b}$ allows us to extend our differentiation formulas to include logarithms with arbitrary bases.

3.6.4 Key Equations

- **Derivative of the natural exponential function**

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}\,g'(x)$$

- **Derivative of the natural logarithmic function**

$$\frac{d}{dx}\left(\ln g(x)\right) = \frac{1}{g(x)}\,g'(x)$$

- **Derivative of the general exponential function**

$$\frac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)}\,g'(x)\ln b$$

- **Derivative of the general logarithmic function**

$$\frac{d}{dx}\left(\log_b g(x)\right) = \frac{g'(x)}{g(x)\ln b}$$

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3.6E: EXERCISES FOR SECTION 3.6

In exercises 1 - 15, find $f'(x)$ for each function.

1) $f(x) = x^2 e^x$

Answer

$$f'(x) = 2xe^x + x^2 e^x$$

2) $f(x) = \dfrac{e^{-x}}{x}$

3) $f(x) = e^{x^3 \ln x}$

Answer

$$f'(x) = e^{x^3 \ln x}\left(3x^2 \ln x + x^2\right)$$

4) $f(x) = \sqrt{e^{2x} + 2x}$

5) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Answer

$$f'(x) = \frac{4}{(e^x + e^{-x})^2}$$

6) $f(x) = \dfrac{10^x}{\ln 10}$

7) $f(x) = 2^{4x} + 4x^2$

Answer

$$f'(x) = 2^{4x+2} \cdot \ln 2 + 8x$$

8) $f(x) = 3^{4x+1}$

9) $f(x) = x^\pi \cdot \pi^x$

Answer

$$f'(x) = \pi x^{\pi-1} \cdot \pi^x + x^\pi \cdot \pi^x \ln \pi$$

10) $f(x) = \ln(4x^3 + x)$

11) $f(x) = \ln \sqrt{5x - 7}$

Answer

$$f'(x) = \frac{5}{2(5x - 7)}$$

12) $f(x) = x^2 \ln 9x$

13) $f(x) = \log(x^3 + 2x + 1)$

Answer

$$f'(x) = \frac{3x^2 + 2}{(\ln 10)(x^3 + 2x + 1)}$$

14) $f(x) = \log_7 (6x^4 + 3)^5$

15) $f(x) = 2^x \cdot \log_3 7^{x^2 - 4}$

Answer

$$f'(x) = 2^x \cdot \ln 2 \cdot \log_3 7^{x^2-4} + 2^x \cdot \frac{2x \ln 7}{\ln 3}$$

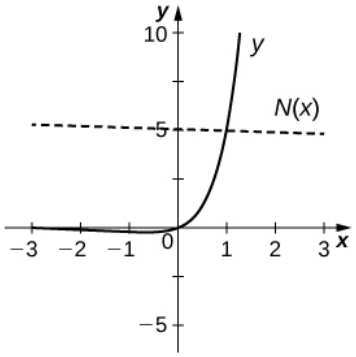
16) [T] Find an equation of the tangent line to the graph of $f(x) = 4xe^{(x^2-1)}$ at the point where

$x = -1$. Graph both the function and the tangent line.

17) [T] Find the equation of the line that is normal to the graph of $f(x) = x \cdot 5^x$ at the point where $x = 1$. Graph both the function and the normal line.

Answer

$y = \frac{-1}{5+5\ln 5}x + \left(5 + \frac{1}{5+5\ln 5}\right)$



18) [T] Find the equation of the tangent line to the graph of $x^3 - x\ln y + y^3 = 2x + 5$ at the point where $x = 2$. (Hint: Use implicit differentiation to find $\frac{dy}{dx}$.) Graph both the curve and the tangent line.

19) [T] The population of Toledo, Ohio, in 2000 was approximately 500,000. Assume the population is increasing at a rate of 5% per year.

a. Write the exponential function that relates the total population as a function of t.

b. Use part a. to determine the rate at which the population is increasing in t years.

c. Use part b. to determine the rate at which the population is increasing in 10 years

Answer

a. $P = 500,000(1.05)^t$ individuals
b. $P'(t) = 24395 \cdot (1.05)^t$ individuals per year
c. $39,737$ individuals per year

20)[T] An isotope of the element erbium has a half-life of approximately 12 hours. Initially there are 9 grams of the isotope present.

a. Write the exponential function that relates the amount of substance remaining as a function of t, measured in hours.

b. Use a. to determine the rate at which the substance is decaying in t hours.

c. Use b. to determine the rate of decay at $t = 4$ hours.

212) [T] The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1964 is modeled by the function $N(t) = 5.3e^{0.093t^2 - 0.87t}$, $(0 \le t \le 4)$, where $N(t)$ gives the number of cases (in thousands) and t is measured in years, with $t = 0$ corresponding to the beginning of 1960.

a. Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.

b. Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in the United States.

Answer

a. At the beginning of 1960 there were 5.3 thousand cases of the disease in New York City. At the beginning of 1963 there were approximately 723 cases of the disease in the United States.
b. At the beginning of 1960 the number of cases of the disease was decreasing at rate of -4.611 thousand per year; at the beginning of 1963, the number of cases of the disease was decreasing at a rate of -0.2808 thousand per year.

22) [T] The relative rate of change of a differentiable function $y = f(x)$ is given by $\frac{100 \cdot f'(x)}{f(x)}$ One model for population growth is a Gompertz growth function, given by $P(x) = ae^{-b \cdot e^{-cx}}$ where a, b, and c are constants.

a. Find the relative rate of change formula for the generic Gompertz function.

b. Use part a. to find the relative rate of change of a population in $x = 20$ months when $a = 204,\ b = 0.0198,$ and $c = 0.15$.

c. Briefly interpret what the result of part b. means.

For exercises 23-26 use the population of New York City from 1790 to 1860, given in the following table.

Year since 1790	Population
0	33,131
10	60,515
20	96,373
30	123,706
40	202,300
50	312,710
60	515,547
70	813,669

New York City Population Over TimeSource: http://en.Wikipedia.org/wiki/Largest..._United_States_by_population_by_decade

23) [T] Using a computer program or a calculator, fit a growth curve to the data of the form $p = ab^t$.

Answer

$p = 35741(1.045)^t$

24) [T] Using the exponential best fit for the data, write a table containing the derivatives evaluated at each year.

25) [T] Using the exponential best fit for the data, write a table containing the second derivatives evaluated at each year.

Answer

Year since 1790	P”
0	69.25
10	107.5
20	167.0
30	259.4
40	402.8
50	625.5
60	971.4
70	1508.5

26) [T] Using the tables of first and second derivatives and the best fit, answer the following questions:

a. Will the model be accurate in predicting the future population of New York City? Why or why not?

b. Estimate the population in 2010. Was the prediction correct from part a.?

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3.7: Maxima and Minima

Learning Objectives

- Define absolute extrema.
- Define local extrema.
- Explain how to find the critical points of a function over a closed interval.
- Describe how to use critical points to locate absolute extrema over a closed interval.

Given a particular function, we are often interested in determining the largest and smallest values of the function. This information is important in creating accurate graphs. Finding the maximum and minimum values of a function also has practical significance, because we can use this method to solve optimization problems, such as maximizing profit, minimizing the amount of material used in manufacturing an aluminum can, or finding the maximum height a rocket can reach. In this section, we look at how to use derivatives to find the largest and smallest values for a function.

3.7.1 Absolute Extrema

Consider the function $f(x) = x^2 + 1$ over the interval $(-\infty, \infty)$. As $x \to \pm\infty$, $f(x) \to \infty$. Therefore, the function does not have a largest value. However, since $x^2 + 1 \geq 1$ for all real numbers x and $x^2 + 1 = 1$ when $x = 0$, the function has a smallest value, 1, when $x = 0$. We say that 1 is the absolute minimum of $f(x) = x^2 + 1$ and it occurs at $x = 0$. We say that $f(x) = x^2 + 1$ does not have an absolute maximum (Figure 3.7.1).

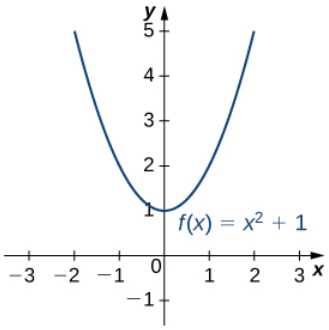


Figure 3.7.1: The given function has an absolute minimum of 1 at $x = 0$. The function does not have an absolute maximum.

Definition: Absolute Extrema

Let f be a function defined over an interval I and let $c \in I$. We say f has an *absolute maximum* on I at c if $f(c) \geq f(x)$ for all $x \in I$. We say f has an *absolute minimum* on I at c if $f(c) \leq f(x)$ for all $x \in I$. If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has *an absolute extremum* on I at c.

Before proceeding, let's note two important issues regarding this definition. First, the term absolute here does not refer to absolute value. An absolute extremum may be positive, negative, or zero. Second, if a function f has an absolute extremum over an interval I at c, the absolute extremum is $f(c)$. The real number c is a point in the domain at which the absolute extremum occurs. For example, consider the function $f(x) = 1/(x^2 + 1)$ over the interval $(-\infty, \infty)$. Since

$$f(0) = 1 \geq \frac{1}{x^2 + 1} = f(x)$$

for all real numbers x, we say f has an absolute maximum over $(-\infty, \infty)$ at $x = 0$. The absolute maximum is $f(0) = 1$. It occurs at $x = 0$, as shown in Figure 3.7.2(b).

A function may have both an absolute maximum and an absolute minimum, just one extremum, or neither. Figure 3.7.2 shows several functions and some of the different possibilities regarding absolute extrema. However, the following theorem, called the **Extreme Value Theorem**, guarantees that a continuous function f over a closed, bounded interval $[a, b]$ has both an absolute maximum and an absolute minimum.

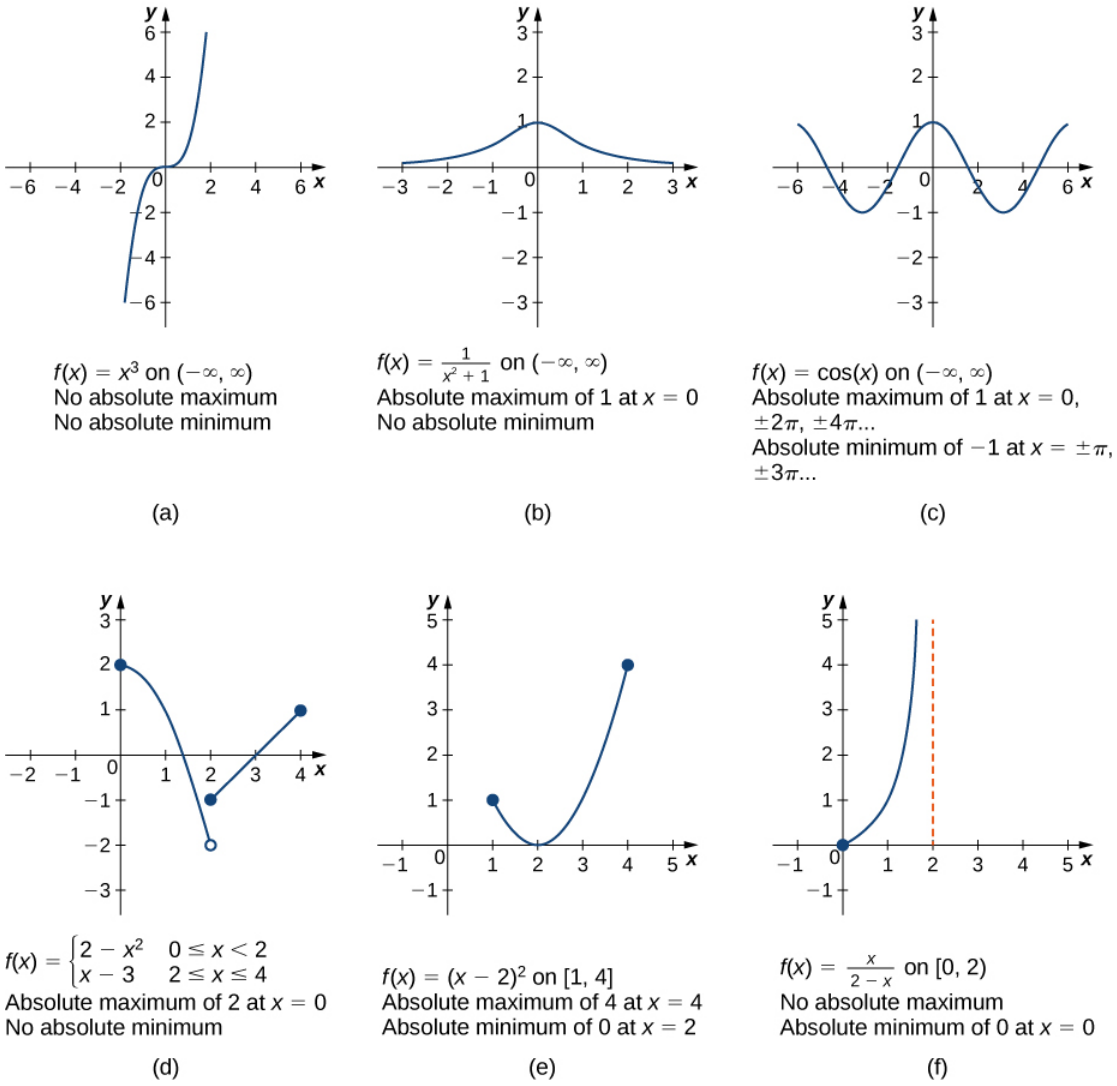


Figure 3.7.2: Graphs (a), (b), and (c) show several possibilities for absolute extrema for functions with a domain of $(-\infty, \infty)$. Graphs (d), (e), and (f) show several possibilities for absolute extrema for functions with a domain that is a bounded interval.

Theorem 3.7.1: Extreme Value Theorem

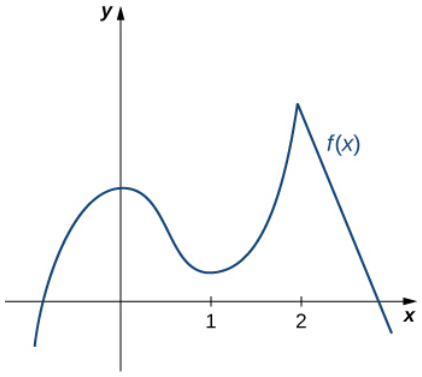
If f is a continuous function over the closed, bounded interval $[a, b]$, then there is a point in $[a, b]$ at which f has an absolute maximum over $[a, b]$ and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b]$.

The proof of the extreme value theorem is beyond the scope of this text. Typically, it is proved in a course on real analysis. There are a couple of key points to note about the statement of this theorem. For the extreme value theorem to apply, the function must be continuous over a closed, bounded interval. If the interval I is open or the function has even one point of discontinuity, the function may not have an absolute maximum or absolute minimum over I. For example, consider the functions shown in Figure 3.7.2(d), (e), and (f). All three of these functions are defined over bounded intervals. However, the function in graph (e) is the only one that has both an absolute maximum and an absolute minimum over its domain. The extreme value theorem cannot be applied to the functions in graphs (d) and (f) because neither of these functions is continuous over a closed, bounded interval. Although the function in graph (d) is defined over the closed interval $[0, 4]$, the function is discontinuous at $x = 2$. The function has an absolute maximum over $[0, 4]$ but does not have an absolute minimum. The function in graph (f) is continuous over the half-open interval $[0, 2)$, but is not defined at $x = 2$, and therefore is not continuous over a closed, bounded interval. The function has an absolute minimum over $[0, 2)$, but does not have an absolute maximum over $[0, 2)$. These two graphs illustrate why a function over a bounded interval may fail to have an absolute maximum and/or absolute minimum.

Before looking at how to find absolute extrema, let's examine the related concept of local extrema. This idea is useful in determining where absolute extrema occur.

3.7.2 Local Extrema and Critical Points

Consider the function f shown in Figure 3.7.3. The graph can be described as two mountains with a valley in the middle. The absolute maximum value of the function occurs at the higher peak, at $x = 2$. However, $x = 0$ is also a point of interest. Although $f(0)$ is not the largest value of f, the value $f(0)$ is larger than $f(x)$ for all x near 0. We say f has a local maximum at $x = 0$. Similarly, the function f does not have an absolute minimum, but it does have a local minimum at $x = 1$ because $f(1)$ is less than $f(x)$ for x near 1.



$f(x)$ defined on $(-\infty, \infty)$
Local maxima at $x = 0$ and $x = 2$
Local minimum at $x = 1$

Figure 3.7.3: This function f has two local maxima and one local minimum. The local maximum at $x = 2$ is also the absolute maximum.

Definition: Local Extrema

A function f has a *local maximum* at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$. A function f has a *local minimum* at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$. A function f has a *local extremum* at c if f has a local maximum at c or f has a local minimum at c.

Note that if f has an absolute extremum at c and f is defined over an interval containing c, then $f(c)$ is also considered a local extremum. If an absolute extremum for a function f occurs at an endpoint, we do not consider that to be a local extremum, but instead refer to that as an endpoint extremum.

Given the graph of a function f, it is sometimes easy to see where a local maximum or local minimum occurs. However, it is not always easy to see, since the interesting features on the graph of a function may not be visible because they occur at a very small scale. Also, we may not have a graph of the function. In these cases, how can we use a formula for a function to determine where these extrema occur?

To answer this question, let's look at Figure 3.7.3 again. The local extrema occur at $x = 0, x = 1$, and $x = 2$. Notice that at $x = 0$ and $x = 1$, the derivative $f'(x) = 0$. At $x = 2$, the derivative $f'(x)$ does not exist, since the function f has a corner there. In fact, if f has a local extremum at a point $x = c$, the derivative $f'(c)$ must satisfy one of the following conditions: either $f'(c) = 0$ or $f'(c)$ is undefined. Such a value c is known as a critical point and it is important in finding extreme values for functions.

Definition: Critical Points

Let c be an interior point in the domain of f. We say that c is a *critical point* of f if $f'(c) = 0$ or $f'(c)$ is undefined.

As mentioned earlier, if f has a local extremum at a point $x = c$, then c must be a critical point of f. This fact is known as **Fermat's theorem.**

Theorem 3.7.2: Fermat's Theorem

If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.

Proof

Suppose f has a local extremum at c and f is differentiable at c. We need to show that $f'(c) = 0$. To do this, we will show that $f'(c) \geq 0$ and $f'(c) \leq 0$, and therefore $f'(c) = 0$. Since f has a local extremum at c, f has a local maximum or local minimum at c. Suppose f has a local maximum at c. The case in which f has a local minimum at c can be handled similarly. There then exists an open interval I such that $f(c) \geq f(x)$ for all $x \in I$. Since f is differentiable at c, from the definition of the derivative, we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Since this limit exists, both one-sided limits also exist and equal $f'(c)$. Therefore,

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c,} \tag{3.7.1}$$

and

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}.$$

Since $f(c)$ is a local maximum, we see that $f(x) - f(c) \leq 0$ for x near c. Therefore, for x near c, but $x > c$, we have $\frac{f(x)-f(c)}{x-c} \leq 0$. From Equation 3.7.1 we conclude that $f'(c) \leq 0$. Similarly, it can be shown that $f'(c) \geq 0$. Therefore, $f'(c) = 0$.

□

From Fermat's theorem, we conclude that if f has a local extremum at c, then either $f'(c) = 0$ or $f'(c)$ is undefined. In other words, local extrema can only occur at critical points.

Note this theorem does not claim that a function f must have a local extremum at a critical point. Rather, it states that critical points are candidates for local extrema. For example, consider the function $f(x) = x^3$. We have $f'(x) = 3x^2 = 0$ when $x = 0$. Therefore, $x = 0$ is a critical point. However, $f(x) = x^3$ is increasing over $(-\infty, \infty)$, and thus f does not have a local extremum at $x = 0$. In Figure 3.7.4, we see several different possibilities for critical points. In some of these cases, the functions have local extrema at critical points, whereas in other cases the functions do not. Note that these graphs do not show all possibilities for the behavior of a function at a critical point.

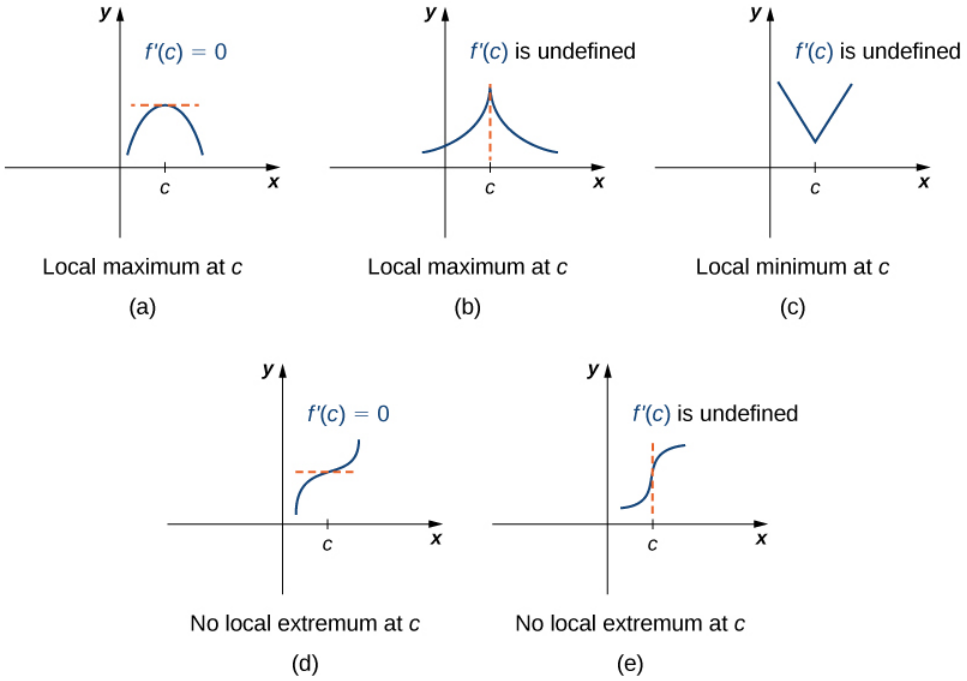


Figure 3.7.4: (a–e) A function f has a critical point at c if $f'(c) = 0$ or $f'(c)$ is undefined. A function may or may not have a local extremum at a critical point.

Later in this chapter we look at analytical methods for determining whether a function actually has a local extremum at a critical point. For now, let's turn our attention to finding critical points. We will use graphical observations to determine whether a critical point is associated with a local extremum.

Example 3.7.1: Locating Critical Points

For each of the following functions, find all critical points. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.

a. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$
b. $f(x) = (x^2 - 1)^3$
c. $f(x) = \frac{4x}{1+x^2}$

Solution

a. The derivative $f'(x) = x^2 - 5x + 4$ is defined for all real numbers x. Therefore, we only need to find the values for x where $f'(x) = 0$. Since $f'(x) = x^2 - 5x + 4 = (x-4)(x-1)$, the critical points are $x = 1$ and $x = 4$. From the graph of f in Figure 3.7.5, we see that f has a local maximum at $x = 1$ and a local minimum at $x = 4$.

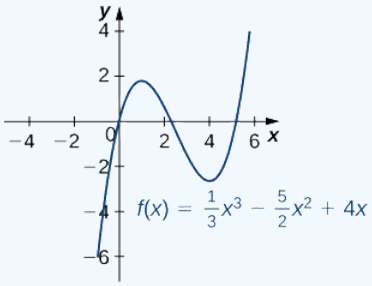


Figure 3.7.5: This function has a local maximum and a local minimum.

b. Using the chain rule, we see the derivative is

$$f'(x) = 3(x^2 - 1)^2(2x) = 6x(x^2 - 1)^2.$$

Therefore, f has critical points when $x = 0$ and when $x^2 - 1 = 0$. We conclude that the critical points are $x = 0, \pm 1$. From the graph of f in Figure 3.7.6, we see that f has a local (and absolute) minimum at $x = 0$, but does not have a local extremum

at $x = 1$ or $x = -1$.

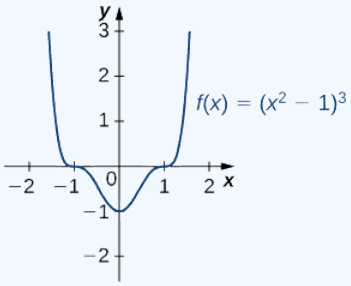


Figure 3.7.6: This function has three critical points: $x = 0$, $x = 1$, and $x = -1$. The function has a local (and absolute) minimum at $x = 0$, but does not have extrema at the other two critical points.

c. By the quotient rule, we see that the derivative is

$$f'(x) = \frac{4(1+x^2)-4x(2x)}{(1+x^2)^2} = \frac{4-4x^2}{(1+x^2)^2}\ .$$

The derivative is defined everywhere. Therefore, we only need to find values for x where $f'(x) = 0$. Solving $f'(x) = 0$, we see that $4 - 4x^2 = 0,$ which implies $x = \pm 1$. Therefore, the critical points are $x = \pm 1$. From the graph of f in Figure 3.7.7, we see that f has an absolute maximum at $x = 1$ and an absolute minimum at $x = -1$. Hence, f has a local maximum at $x = 1$ and a local minimum at $x = -1$. (Note that if f has an absolute extremum over an interval I at a point c that is not an endpoint of I, then f has a local extremum at c.)

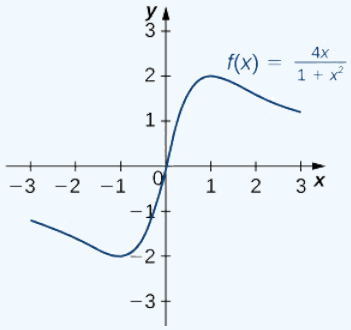


Figure 3.7.7: This function has an absolute maximum and an absolute minimum.

? Exercise 3.7.1

Find all critical points for $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$.

Hint

Calculate $f'(x)$.

Answer

$x = \frac{-2}{3}, x = 1$

3.7.3 Locating Absolute Extrema

The extreme value theorem states that a continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. As shown in Figure 3.7.2, one or both of these absolute extrema could occur at an endpoint. If an absolute extremum does not occur at an endpoint, however, it must occur at an interior point, in which case the absolute extremum is a local extremum. Therefore, by Fermat's Theorem, the point c at which the local extremum occurs must be a critical point. We summarize this result in the following theorem.

Theorem 3.7.3: Location of Absolute Extrema

Let f be a continuous function over a closed, bounded interval I. The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in I.

With this idea in mind, let's examine a procedure for locating absolute extrema.

Problem-Solving Strategy: Locating Absolute Extrema over a Closed Interval

Consider a continuous function f defined over the closed interval $[a, b]$.

1. Evaluate f at the endpoints $x = a$ and $x = b$.
2. Find all critical points of f that lie over the interval (a, b) and evaluate f at those critical points.
3. Compare all values found in (1) and (2). From "Location of Absolute Extrema," the absolute extrema must occur at endpoints or critical points. Therefore, the largest of these values is the absolute maximum of f. The smallest of these values is the absolute minimum of f.

Now let's look at how to use this strategy to find the absolute maximum and absolute minimum values for continuous functions.

Example 3.7.2: Locating Absolute Extrema

For each of the following functions, find the absolute maximum and absolute minimum over the specified interval and state where those values occur.

a. $f(x) = -x^2 + 3x - 2$ over $[1, 3]$.
b. $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$.

Solution

a. Step 1. Evaluate f at the endpoints $x = 1$ and $x = 3$.

$$f(1) = 0 \text{ and } f(3) = -2$$

Step 2. Since $f'(x) = -2x + 3$, f' is defined for all real numbers x. Therefore, there are no critical points where the derivative is undefined. It remains to check where $f'(x) = 0$. Since $f'(x) = -2x + 3 = 0$ at $x = \frac{3}{2}$ and $\frac{3}{2}$ is in the interval $[1, 3]$, $f(\frac{3}{2})$ is a candidate for an absolute extremum of f over $[1, 3]$. We evaluate $f(\frac{3}{2})$ and find

$$f\left(\tfrac{3}{2}\right) = \tfrac{1}{4}.$$

Step 3. We set up the following table to compare the values found in steps 1 and 2.

x	$f(x)$	Conclusion
1	0	
$\frac{3}{2}$	$\frac{1}{4}$	Absolute maximum
3	-2	Absolute minimum

From the table, we find that the absolute maximum of f over the interval [1, 3] is $\frac{1}{4}$, and it occurs at $x = \frac{3}{2}$. The absolute minimum of f over the interval $[1, 3]$ is -2, and it occurs at $x = 3$ as shown in Figure 3.7.8.

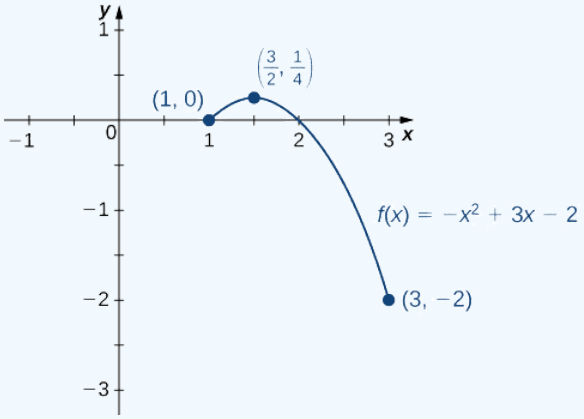


Figure 3.7.8: This function has both an absolute maximum and an absolute minimum.

b. Step 1. Evaluate f at the endpoints $x = 0$ and $x = 2$.

$$f(0) = 0 \text{ and } f(2) = 4 - 3(2)^{2/3} \approx -0.762$$

Step 2. The derivative of f is given by

$$f'(x) = 2x - \frac{2}{x^{1/3}} = \frac{2x^{4/3} - 2}{x^{1/3}}$$

for $x \neq 0$. The derivative is zero when $2x^{4/3} - 2 = 0$, which implies $x = \pm 1$. The derivative is undefined at $x = 0$. Therefore, the critical points of f are $x = 0, 1, -1$. The point $x = 0$ is an endpoint, so we already evaluated $f(0)$ in step 1. The point $x = -1$ is not in the interval of interest, so we need only evaluate $f(1)$. We find that

$$f(1) = -2.$$

Step 3. We compare the values found in steps 1 and 2, in the following table.

x	$f(x)$	Conclusion
0	0	Absolute maximum
1	-2	Absolute minimum
2	-0.762	

We conclude that the absolute maximum of f over the interval $[0, 2]$ is zero, and it occurs at $x = 0$. The absolute minimum is -2, and it occurs at $x = 1$ as shown in Figure 3.7.9.

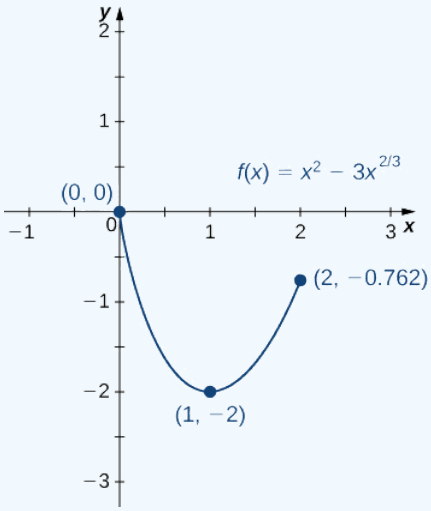


Figure 3.7.9: This function has an absolute maximum at an endpoint of the interval.

? Exercise 3.7.2

Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over the interval $[1, 4]$.

Hint

Look for critical points. Evaluate f at all critical points and at the endpoints.

Answer

The absolute maximum is 3 and it occurs at $x = 4$. The absolute minimum is -1 and it occurs at $x = 2$.

At this point, we know how to locate absolute extrema for continuous functions over closed intervals. We have also defined local extrema and determined that if a function f has a local extremum at a point c, then c must be a critical point of f. However, c being a critical point is not a sufficient condition for f to have a local extremum at c. Later in this chapter, we show how to determine whether a function actually has a local extremum at a critical point. First, however, we need to introduce the Mean Value Theorem, which will help as we analyze the behavior of the graph of a function.

3.7.4 Key Concepts

- A function may have both an absolute maximum and an absolute minimum, have just one absolute extremum, or have no absolute maximum or absolute minimum.
- If a function has a local extremum, the point at which it occurs must be a critical point. However, a function need not have a local extremum at a critical point.
- A continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. Each extremum occurs at a critical point or an endpoint.

3.7.5 Glossary

absolute extremum

if f has an absolute maximum or absolute minimum at c, we say f has an absolute extremum at c

absolute maximum

if $f(c) \geq f(x)$ for all x in the domain of f, we say f has an absolute maximum at c

absolute minimum

if $f(c) \leq f(x)$ for all x in the domain of f, we say f has an absolute minimum at c

critical point

if $f'(c) = 0$ or $f'(c)$ is undefined, we say that c is a critical point of f

extreme value theorem

if f is a continuous function over a finite, closed interval, then f has an absolute maximum and an absolute minimum

Fermat's theorem

if f has a local extremum at c, then c is a critical point of f

local extremum

if f has a local maximum or local minimum at c, we say f has a local extremum at c

local maximum

if there exists an interval I such that $f(c) \geq f(x)$ for all $x \in I$, we say f has a local maximum at c

local minimum

if there exists an interval I such that $f(c) \leq f(x)$ for all $x \in I$, we say f has a local minimum at c

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3.7E: EXERCISES FOR SECTION 3.7

1) In precalculus, you learned a formula for the position of the maximum or minimum of a quadratic equation $y = ax^2 + bx + c$, which was $m = -\frac{b}{2a}$. Prove this formula using calculus.

2) If you are finding an absolute minimum over an interval $[a, b]$, why do you need to check the endpoints? Draw a graph that supports your hypothesis.

Answer

On a closed interval, the endpoints often lie above or below any local (relative) extrema. Answers may vary for the graph.

3) If you are examining a function over an interval (a, b), for a and b finite, is it possible not to have an absolute maximum or absolute minimum?

4) When you are checking for critical points to locate the extrema of a function f, explain why you also need to determine points where $f'(x)$ is undefined. Draw a graph to support your explanation.

Answer

Points on the graph of f where there is a corner, a cusp, or a jump discontinuity or removable discontinuity can easily be absolute (or local) extrema of the function. Answers may vary for the graph.

5) Can you have a finite absolute maximum for $y = ax^2 + bx + c$ over $(-\infty, \infty)$? Explain why or why not using graphical arguments.

6) Can you have a finite absolute maximum for $y = ax^3 + bx^2 + cx + d$ over $(-\infty, \infty)$ assuming a is non-zero? Explain why or why not using graphical arguments.

Answer

No; answers will vary

7) Let m be the number of local minima and M be the number of local maxima. Can you create a function where $M > m + 2$? Draw a graph to support your explanation.

8) Is it possible to have more than one absolute maximum? Use a graphical argument to prove your hypothesis.

Answer

Since the absolute maximum is the function (output) value rather than the x value, the answer is no; answers will vary

9) Is it possible to have no absolute minimum or maximum for a function? If so, construct such a function. If not, explain why this is not possible.

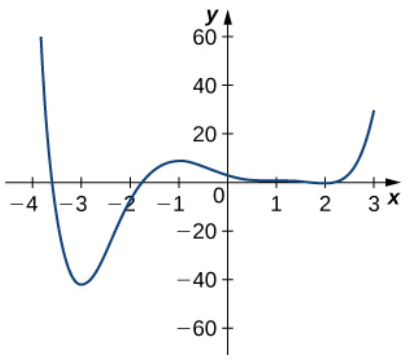
10) [T] Graph the function $y = e^{ax}$. For which values of a, on any infinite domain, will you have an absolute minimum and absolute maximum?

Answer

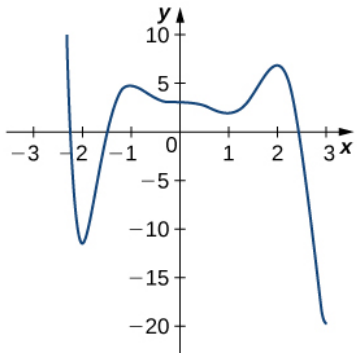
When $a = 0$

In exercises 11 - 14, determine where the local and absolute maxima and minima occur on the graph given. Assume domains are closed intervals unless otherwise specified.

11)



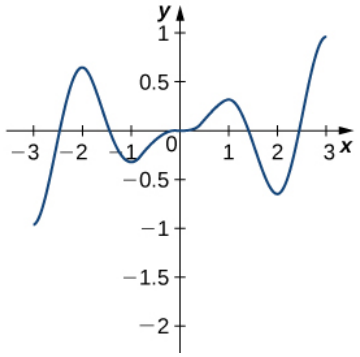
12)



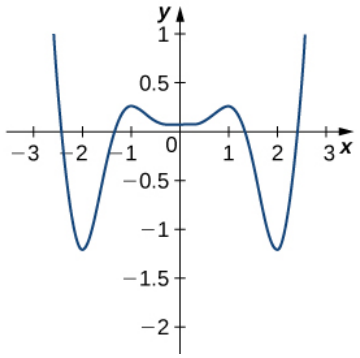
Answer

Absolute minimum at 3; Absolute maximum at −2.2; local minima at −2, 1; local maxima at −1, 2

13)



14)



Answer

Absolute minima at −2, 2; absolute maxima at −2.5, 2.5; local minimum at 0; local maxima at −1, 1

For exercises 15 - 18, draw graphs of $f(x)$, which is continuous, over the interval $[-4, 4]$ with the following properties:

15) Absolute maximum at $x = 2$ and absolute minima at $x = \pm 3$

16) Absolute minimum at $x = 1$ and absolute maximum at $x = 2$

Answer

Answers may vary.

17) Absolute maximum at $x = 4$, absolute minimum at $x = -1$, local maximum at $x = -2$, and a critical point that is not a maximum or minimum at $x = 2$

18) Absolute maxima at $x = 2$ and $x = -3$, local minimum at $x = 1$, and absolute minimum at $x = 4$

Answer

Answers may vary.

In exercises 19 - 28, find the critical points in the domains of the given functions.

19) $y = 4x^3 - 3x$

20) $y = 4\sqrt{x} - x^2$

Answer

$x = 1$

21) $y = \dfrac{1}{x-1}$

22) $y = \ln(x-2)$

Answer

None

23) $y = 6x^5 + 33x^4 - 30x^3 + 100$

24) $y = \sqrt{4-x^2}$

Answer

$x = 0$

25) $y = x^{3/2} - 3x^{5/2}$

26) $y = \dfrac{x^2-1}{x^2+2x-3}$

Answer

None

27) $y = 10xe^{3-x^2}$

28) $y = x + \dfrac{1}{x}$

Answer

$x = -1$ and $x = 1$

In exercises 29 - 39, find the local and/or absolute maxima for the functions over the specified domain.

29) $y = x^2 + 4$ over $[-1, 2]$

Answer

Absolute maximum: $x = 2, y = 16$; absolute minimum: $x = 0, y = 4$

30) $f(x) = x^2 + 3$ over $[-1, 4]$

31) $y = x^2 + \dfrac{2}{x}$ over $[1, 4]$

Answer

Absolute maximum: $x = 4, y = \frac{33}{2}$; absolute minimum: $x = 1, y = 3$

32) $y = (x - x^2)^2$ over $[-1, 1]$

33) $y = \dfrac{1}{x - x^2}$ over $[0, 1]$

Answer

Absolute minimum: $x = \frac{1}{2}, y = 4$

34) $y = \sqrt{9-x}$ over $[1, 9]$

35) $y = \ln x + \dfrac{1}{x}$ over $[0.5, 2]$

Answer

Absolute maximum: $x = 0.5, y = \ln 0.5 + 2$; absolute minimum: $x = 1, y = 1$

36) $y = \dfrac{x}{1+x}$ over $[0, 100]$

37) $y = |x+1| + |x-1|$ over $[-3, 2]$

Answer

Absolute maximum: $x = -3, y = 6$; absolute minimum: $-1 \le x \le 1, y = 2$

38) $y = \sqrt{x} - \sqrt{x^3}$ over $[0, 4]$

39) $y = xe^{-x}$ over $[-1, 3]$

Answer

Absolute maximum: $x = 1, y = e^{-1}$; absolute minimum: $x = -1, y = -e$

In exercises 40 - 45, find the local and absolute minima and maxima for the functions over $(-\infty, \infty)$.

40) $y = x^2 + 4x + 5$

Answer

Absolute minimum: $x = -2, y = 1$

41) $y = x^3 - 12x$

42) $y = 3x^4 + 8x^3 - 18x^2$

Answer

Absolute minimum: $x = -3, y = -135$; local maximum: $x = 0, y = 0$; local minimum: $x = 1, y = -7$

43) $y = x^3(1-x)^6$

44) $y = \dfrac{x^2 + x + 6}{x - 1}$

Answer

Local maximum: $x = 1 - 2\sqrt{2}, y = 3 - 4\sqrt{2}$; local minimum: $x = 1 + 2\sqrt{2}, y = 3 + 4\sqrt{2}$

45) $y = \dfrac{x^2 - 1}{x - 1}$

In exercises 46 - 50, use a calculator to graph the function and to estimate the absolute and local maxima and minima. Then, solve for them explicitly.

46) [T] $y = 3x\sqrt{1 - x^2}$

Answer

Absolute maximum: $x = \frac{\sqrt{2}}{2}, y = \frac{3}{2}$; absolute minimum: $x = -\frac{\sqrt{2}}{2}, y = -\frac{3}{2}$

48) [T] $y = 12x^5 + 45x^4 + 20x^3 - 90x^2 - 120x + 3$

Answer

Local maximum: $x = -2, y = 59$; local minimum: $x = 1, y = -130$

49) [T] $y = \dfrac{x^3 + 6x^2 - x - 30}{x - 2}$

50) [T] $y = \dfrac{\sqrt{4 - x^2}}{\sqrt{4 + x^2}}$

Answer

Absolute maximum: $x = 0, y = 1$; absolute minimum: $x = -2, 2, y = 0$

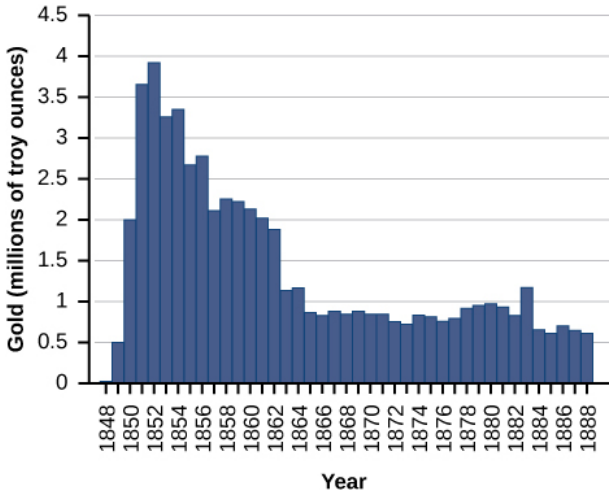
51) A company that produces cell phones has a cost function of $C = x^2 - 1200x + 36,400$, where C is cost in dollars and x is number of cell phones produced (in thousands). How many units of cell phone (in thousands) minimizes this cost function?

52) A ball is thrown into the air and its position is given by $h(t) = -4.9t^2 + 60t + 5$ meters. Find the height at which the ball stops ascending. How long after it is thrown does this happen?

Answer

$h = \frac{9245}{49}$ m, $t = \frac{300}{49}$ s

For exercises 53-54, consider the production of gold during the California gold rush (1848–1888). The production of gold can be modeled by $G(t) = \frac{(25t)}{(t^2 + 16)}$, where t is the number of years since the rush began $(0 \le t \le 40)$ and G is ounces of gold produced (in millions). A summary of the data is shown in the following figure.



53) Find when the maximum (local and global) gold production occurred, and the amount of gold produced during that maximum.

54) Find when the minimum (local and global) gold production occurred. What was the amount of gold produced during this minimum?

Answer

The global minimum was in 1848, when no gold was produced.

In exercises 55 & 56, find the critical points, maxima, and minima for the given piecewise functions.

55) $y = \begin{cases} x^2 - 4x, & \text{if } 0 \le x \le 1 \\ x^2 - 4, & \text{if } 1 < x \le 2 \end{cases}$

56) $y = \begin{cases} x^2 + 1, & \text{if } x \le 1 \\ x^2 - 4x + 5, & \text{if } x > 1 \end{cases}$

Answer

Absolute minima: $x = 0, x = 2, y = 1$; local maximum at $x = 1, y = 2$

In exercises 57 - 58, find the critical points of the following generic functions. Are they maxima, minima, or neither? State the necessary conditions.

57) $y = ax^2 + bx + c$, given that $a > 0$

58) $y = (x - 1)^a$, given that $a > 1$

Answer

No maxima/minima if a is odd, minimum at $x = 1$ if a is even

3.7E.1 CONTRIBUTORS AND ATTRIBUTIONS

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

- Paul Seeburger (Monroe Community College) added answers for exercises 2 and 4.

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3.8: Derivatives and the Shape of a Graph

Learning Objectives

- Explain how the sign of the first derivative affects the shape of a function's graph.
- State the first derivative test for critical points.
- Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
- Explain the concavity test for a function over an open interval.
- Explain the relationship between a function and its first and second derivatives.
- State the second derivative test for local extrema.

Earlier in this chapter we stated that if a function f has a local extremum at a point c, then c must be a critical point of f. However, a function is not guaranteed to have a local extremum at a critical point. For example, $f(x) = x^3$ has a critical point at $x = 0$ since $f'(x) = 3x^2$ is zero at $x = 0$, but f does not have a local extremum at $x = 0$. Using the results from the previous section, we are now able to determine whether a critical point of a function actually corresponds to a local extreme value. In this section, we also see how the second derivative provides information about the shape of a graph by describing whether the graph of a function curves upward or curves downward.

3.8.1 The First Derivative Test

Corollary 3 of the Mean Value Theorem showed that if the derivative of a function is positive over an interval I then the function is increasing over I. On the other hand, if the derivative of the function is negative over an interval I, then the function is decreasing over I as shown in the following figure.

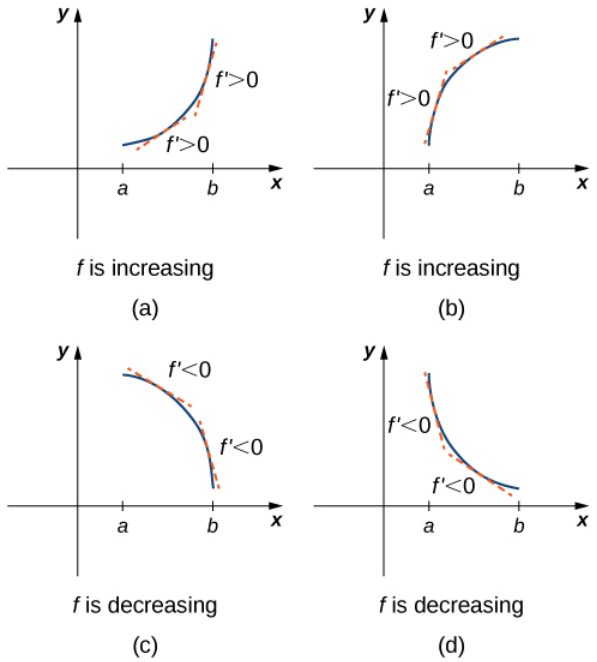


Figure 3.8.1: Both functions are increasing over the interval (a, b). At each point x, the derivative $f'(x) > 0$. Both functions are decreasing over the interval (a, b). At each point x, the derivative $f'(x) < 0$.

A continuous function f has a local maximum at point c if and only if f switches from increasing to decreasing at point c. Similarly, f has a local minimum at c if and only if f switches from decreasing to increasing at c. If f is a continuous function over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) at point c is if f' changes sign as x increases through c. If f is differentiable at c, the only way that f' can change sign as x increases through c is if $f'(c) = 0$. Therefore, for a function f that is continuous over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) is if $f'(c) = 0$ or $f'(c)$ is undefined. Consequently, to locate local extrema for a function f, we look for points c in the domain of f such that $f'(c) = 0$ or $f'(c)$ is undefined. Recall that such points are called critical points of f.

Note that f need not have a local extrema at a critical point. The critical points are candidates for local extrema only. In Figure 3.8.2, we show that if a continuous function f has a local extremum, it must occur at a critical point, but a function may not have a local extremum at a critical point. We show that if f has a local extremum at a critical point, then the sign of f' switches as x increases through that point.

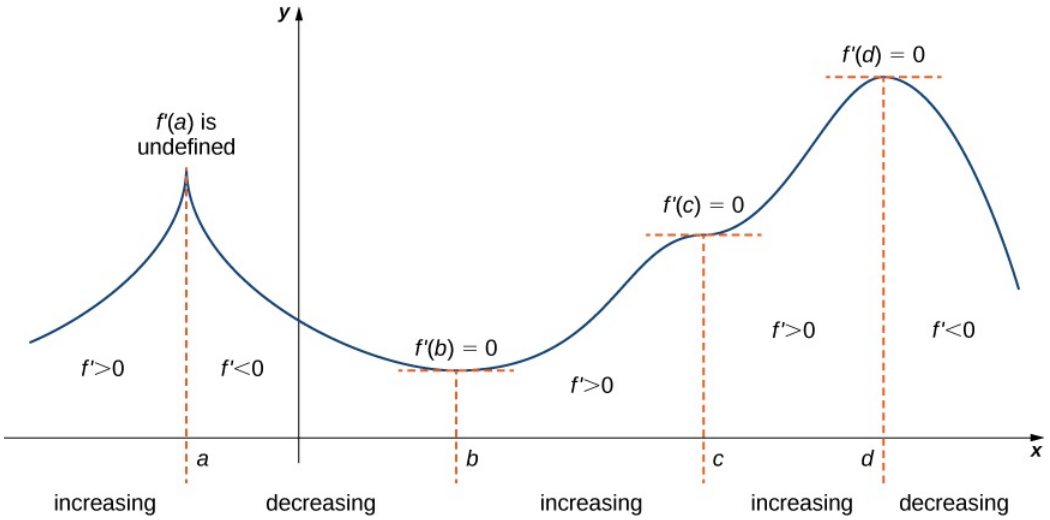


Figure 3.8.2: The function f has four critical points: a, b, c ,and d. The function f has local maxima at a and d, and a local minimum at b. The function f does not have a local extremum at c. The sign of f' changes at all local extrema.

Using Figure 3.8.2, we summarize the main results regarding local extrema.

- If a continuous function f has a local extremum, it must occur at a critical point c.
- The function has a local extremum at the critical point c if and only if the derivative f' switches sign as x increases through c.
- Therefore, to test whether a function has a local extremum at a critical point c, we must determine the sign of $f'(x)$ to the left and right of c.

This result is known as the **first derivative test.**

First Derivative Test

Suppose that f is a continuous function over an interval I containing a critical point c. If f is differentiable over I, except possibly at point c, then $f(c)$ satisfies one of the following descriptions:

i. If f' changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a local maximum of f.
ii. If f' changes sign from negative when $x < c$ to positive when $x > c$, then $f(c)$ is a local minimum of f.
iii. If f' has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of f

Now let's look at how to use this strategy to locate all local extrema for particular functions.

Example 3.8.1: Using the First Derivative Test to Find Local Extrema

Use the first derivative test to find the location of all local extrema for $f(x) = x^3 - 3x^2 - 9x - 1$. Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is $f'(x) = 3x^2 - 6x - 9$. To find the critical points, we need to find where $f'(x) = 0$. Factoring the polynomial, we conclude that the critical points must satisfy

$$3(x^2 - 2x - 3) = 3(x - 3)(x + 1) = 0.$$

Therefore, the critical points are $x = 3, -1$. Now divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1), (-1, 3)$ and $(3, \infty)$.

Step 2. Since f' is a continuous function, to determine the sign of $f'(x)$ over each subinterval, it suffices to choose a point over each of the intervals $(-\infty, -1), (-1, 3)$ and $(3, \infty)$ and determine the sign of f' at each of these points. For example, let's choose $x = -2$, $x = 0$, and $x = 4$ as test points.

Table: 3.8.1: First Derivative Test for $f(x) = x^3 - 3x^2 - 9x - 1$.

Interval	Test Point	Sign of $f'(x) = 3(x-3)(x+1)$ at Test Point	Conclusion
$(-\infty, -1)$	$x = -2$	(+)(−)(−)=+	f is increasing.
$(-1, 3)$	$x = 0$	(+)(−)(+)=-	f is decreasing.
$(3, \infty)$	$x = 4$	(+)(+)(+)=+	f is increasing.

Step 3. Since f' switches sign from positive to negative as x increases through -1, f has a local maximum at $x = -1$. Since f' switches sign from negative to positive as x increases through 3, f has a local minimum at $x = 3$. These analytical results agree with the following graph.

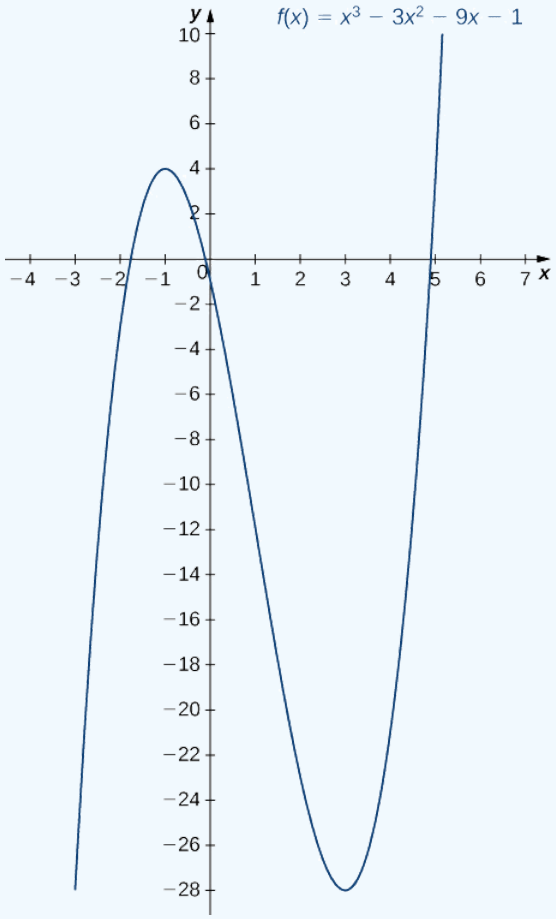


Figure 3.8.3: The function f has a maximum at $x = -1$ and a minimum at $x = 3$

? Exercise 3.8.1

Use the first derivative test to locate all local extrema for $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$.

Hint

Find all critical points of f and determine the signs of $f'(x)$ over particular intervals determined by the critical points.

Answer

f has a local minimum at -2 and a local maximum at 3.

Example 3.8.2: Using the First Derivative Test

Use the first derivative test to find the location of all local extrema for $f(x) = 5x^{1/3} - x^{5/3}$. Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is

$$f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \frac{5}{3x^{2/3}} - \frac{5x^{2/3}}{3} = \frac{5 - 5x^{4/3}}{3x^{2/3}} = \frac{5(1 - x^{4/3})}{3x^{2/3}}.$$

The derivative $f'(x) = 0$ when $1 - x^{4/3} = 0$. Therefore, $f'(x) = 0$ at $x = \pm 1$. The derivative $f'(x)$ is undefined at $x = 0$. Therefore, we have three critical points: $x = 0$, $x = 1$, and $x = -1$. Consequently, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$.

Step 2: Since f' is continuous over each subinterval, it suffices to choose a test point x in each of the intervals from step 1 and determine the sign of f' at each of these points. The points $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 2$ are test points for these intervals.

Table: 3.8.2: First Derivative Test for $f(x) = 5x^{1/3} - x^{5/3}$.

Interval	Test Point	Sign of $f'(x) = \frac{5(1-x^{4/3})}{3x^{2/3}}$ at Test Point	Conclusion
$(-\infty, -1)$	$x = -2$	$\frac{(+)(-)}{+} = -$	f is decreasing.
$(-1, 0)$	$x = -\frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(0, 1)$	$x = \frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(1, \infty)$	$x = 2$	$\frac{(+)(-)}{+} = -$	f is decreasing.

Step 3: Since f is decreasing over the interval $(-\infty, -1)$ and increasing over the interval $(-1, 0)$, f has a local minimum at $x = -1$. Since f is increasing over the interval $(-1, 0)$ and the interval $(0, 1)$, f does not have a local extremum at $x = 0$. Since f is increasing over the interval $(0, 1)$ and decreasing over the interval $(1, \infty)$, f has a local maximum at $x = 1$. The analytical results agree with the following graph.

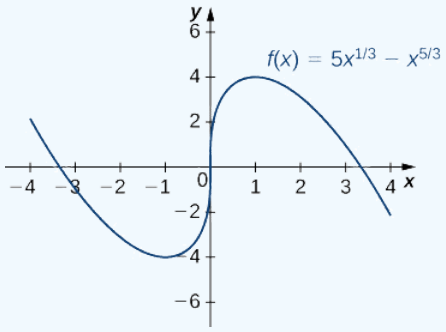


Figure 3.8.4: The function f has a local minimum at $x = -1$ and a local maximum at $x = 1$

Exercise 3.8.2

Use the first derivative test to find all local extrema for $f(x) = \dfrac{3}{x - 1}$.

https://math.libretexts.org/@go/page/149887

Hint

The only critical point of f is $x = 1$.

Answer

f has no local extrema because f' does not change sign at $x = 1$.

3.8.2 Concavity and Points of Inflection

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

Figure $3.8.5a$ shows a function f with a graph that curves upward. As x increases, the slope of the tangent line increases. Thus, since the derivative increases as x increases, f' is an increasing function. We say this function f is concave up. Figure $3.8.5b$ shows a function f that curves downward. As x increases, the slope of the tangent line decreases. Since the derivative decreases as x increases, f' is a decreasing function. We say this function f is concave down.

Definition: concavity test

Let f be a function that is differentiable over an open interval I. If f' is increasing over I, we say f is **concave up** over I. If f' is decreasing over I, we say f is **concave down** over I.

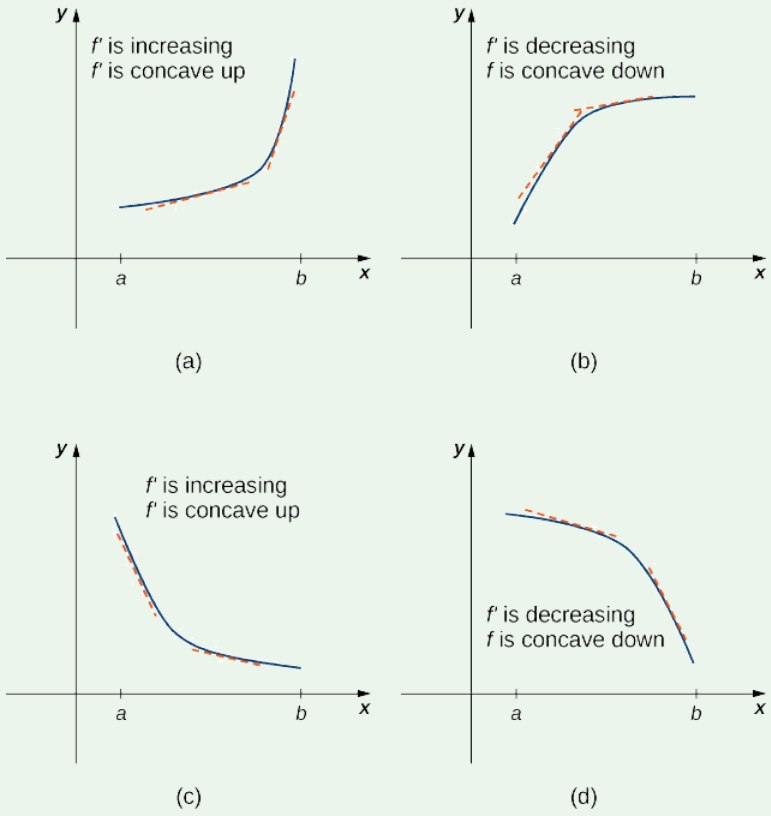


Figure 3.8.5: (a), (c) Since f' is increasing over the interval (a, b), we say f is concave up over (a, b). $(b), (d)$ Since f' is decreasing over the interval (a, b), we say f is concave down over (a, b).

In general, without having the graph of a function f, how can we determine its concavity? By definition, a function f is concave up if f' is increasing. From Corollary 3, we know that if f' is a differentiable function, then f' is increasing if its derivative $f''(x) > 0$. Therefore, a function f that is twice differentiable is concave up when $f''(x) > 0$. Similarly, a function f is concave down if f' is decreasing. We know that a differentiable function f' is decreasing if its derivative $f''(x) < 0$. Therefore, a twice-differentiable function f is concave down when $f''(x) < 0$. Applying this logic is known as the *concavity test*.

Test for Concavity

Let f be a function that is twice differentiable over an interval I.

i. If $f''(x) > 0$ for all $x \in I$, then f is concave up over I
ii. If $f''(x) < 0$ for all $x \in I$, then f is concave down over I.

We conclude that we can determine the concavity of a function f by looking at the second derivative of f. In addition, we observe that a function f can switch concavity (Figure 3.8.6). However, a continuous function can switch concavity only at a point x if $f''(x) = 0$ or $f''(x)$ is undefined. Consequently, to determine the intervals where a function f is concave up and concave down, we look for those values of x where $f''(x) = 0$ or $f''(x)$ is undefined. When we have determined these points, we divide the domain of f into smaller intervals and determine the sign of f'' over each of these smaller intervals. If f'' changes sign as we pass through a point x, then f changes concavity. It is important to remember that a function f may not change concavity at a point x even if $f''(x) = 0$ or $f''(x)$ is undefined. If, however, f does change concavity at a point a and f is continuous at a, we say the point $(a, f(a))$ is an **inflection point** of f.

Definition: inflection point

If f is continuous at a and f changes concavity at a, the point $(a,\ f(a))$ is an *inflection point* of f.

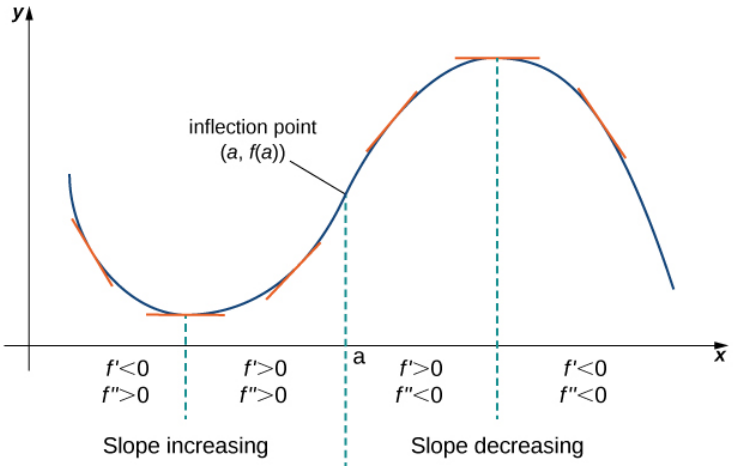


Figure 3.8.6: Since $f''(x) > 0$ for $x < a$, the function f is concave up over the interval $(-\infty, a)$. Since $f''(x) < 0$ for $x > a$, the function f is concave down over the interval (a, ∞). The point $(a, f(a))$ is an inflection point of f.

Example 3.8.3: Testing for Concavity

For the function $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f. Use a graphing utility to confirm your results.

Solution

To determine concavity, we need to find the second derivative $f''(x)$. The first derivative is $f'(x) = 3x^2 - 12x + 9$, so the second derivative is $f''(x) = 6x - 12$. If the function changes concavity, it occurs either when $f''(x) = 0$ or $f''(x)$ is undefined. Since f'' is defined for all real numbers x, we need only find where $f''(x) = 0$. Solving the equation $6x - 12 = 0$, we see that $x = 2$ is the only place where f could change concavity. We now test points over the intervals $(-\infty, 2)$ and $(2, \infty)$ to determine the concavity of f. The points $x = 0$ and $x = 3$ are test points for these intervals.

Table: 3.8.3: Test for Concavity for $f(x) = x^3 - 6x^2 + 9x + 30$.

Interval	Test Point	Sign of $f''(x) = 6x - 12$ at Test Point	Conclusion
$(-\infty, 2)$	$x = 0$	$-$	f is concave down
$(2, \infty)$	$x = 3$	$+$	f is concave up

We conclude that f is concave down over the interval $(-\infty, 2)$ and concave up over the interval $(2, \infty)$. Since f changes concavity at $x = 2$, the point $(2, f(2)) = (2, 32)$ is an inflection point. Figure 3.8.7 confirms the analytical results.

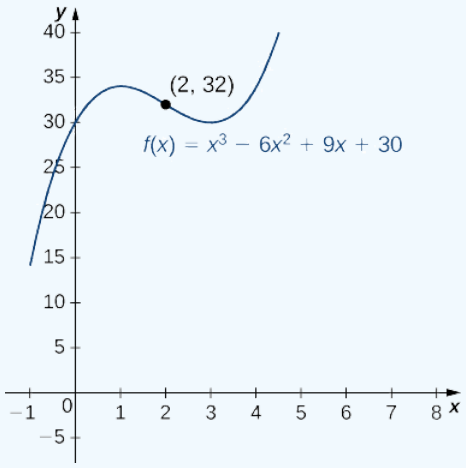


Figure 3.8.7: The given function has a point of inflection at $(2, 32)$ where the graph changes concavity.

Exercise 3.8.3

For $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$, find all intervals where f is concave up and all intervals where f is concave down.

Hint

Find where $f''(x) = 0$

Answer

f is concave up over the interval $(-\infty, \frac{1}{2})$ and concave down over the interval $(\frac{1}{2}, \infty)$

We now summarize, in Table 3.8.4, the information that the first and second derivatives of a function f provide about the graph of f, and illustrate this information in Figure 3.8.8.

Table: 3.8.4: What Derivatives Tell Us about Graphs

Sign of f'	Sign of f''	Is f increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

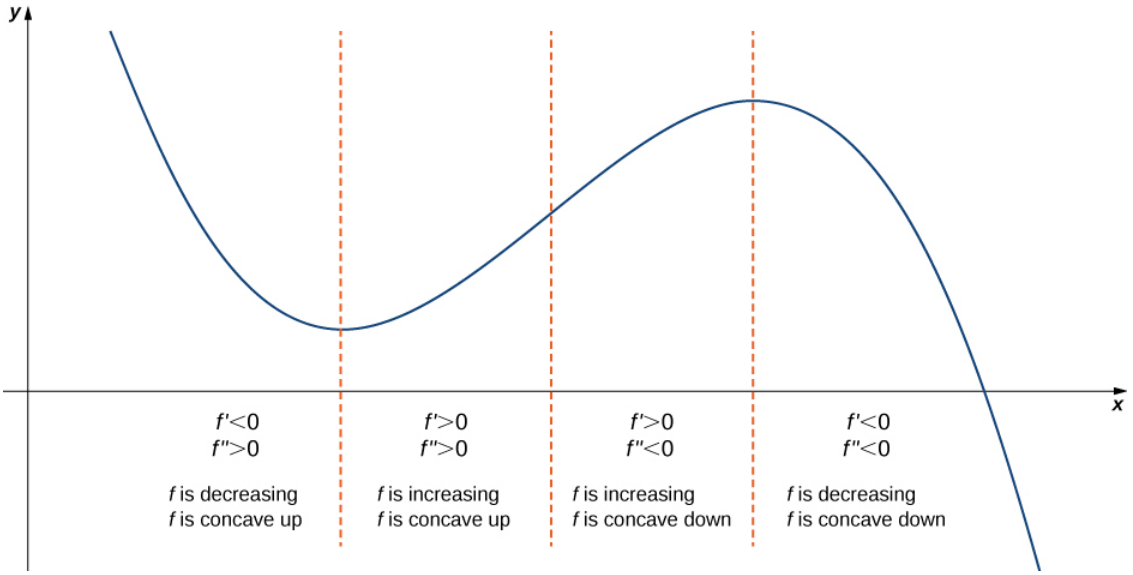


Figure 3.8.8:Consider a twice-differentiable function f over an open interval I. If $f'(x) > 0$ for all $x \in I$, the function is increasing over I. If $f'(x) < 0$ for all $x \in I$, the function is decreasing over I. If $f''(x) > 0$ for all $x \in I$, the function is concave up. If $f''(x) < 0$ for all $x \in I$, the function is concave down on I.

3.8.3 The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extremum, it must occur at a critical point. However, a function need not have a local extremum at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let f be a twice-differentiable function such that $f'(a) = 0$ and f'' is continuous over an open interval I containing a. Suppose $f''(a) < 0$. Since f'' is continuous over I, $f''(x) < 0$ for all $x \in I$ (Figure 3.8.9). Then, by Corollary 3, f' is a decreasing function over I. Since $f'(a) = 0$, we conclude that for all $x \in I$, $f'(x) > 0$ if $x < a$ and $f'(x) < 0$ if $x > a$. Therefore, by the first derivative test, f has a local maximum at $x = a$.

On the other hand, suppose there exists a point b such that $f'(b) = 0$ but $f''(b) > 0$. Since f'' is continuous over an open interval I containing b, then $f''(x) > 0$ for all $x \in I$ (Figure 3.8.9). Then, by Corollary 3, f' is an increasing function over I. Since $f'(b) = 0$, we conclude that for all $x \in I$, $f'(x) < 0$ if $x < b$ and $f'(x) > 0$ if $x > b$. Therefore, by the first derivative test, f has a local minimum at $x = b$.

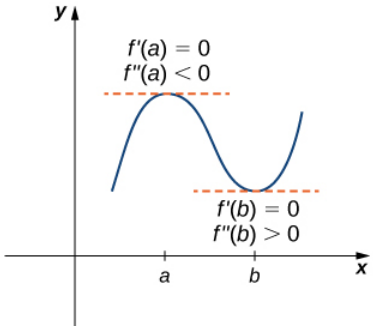


Figure 3.8.9: Consider a twice-differentiable function f such that f'' is continuous. Since $f'(a) = 0$ and $f''(a) < 0$, there is an interval I containing a such that for all x in I, f is increasing if $x < a$ and f is decreasing if $x > a$. As a result, f has a local maximum at $x = a$. Since $f'(b) = 0$ and $f''(b) > 0$, there is an interval I containing b such that for all x in I, f is decreasing if $x < b$ and f is increasing if $x > b$. As a result, f has a local minimum at $x = b$.

Second Derivative Test

Suppose $f'(c) = 0$ and f'' is continuous over an interval containing c.

i. If $f''(c) > 0$, then f has a local minimum at c.
ii. If $f''(c) < 0$, then f has a local maximum at c.
iii. If $f''(c) = 0$, then the test is inconclusive.

Note that for case iii. when $f''(c) = 0$, then f may have a local maximum, local minimum, or neither at c. For example, the functions $f(x) = x^3$, $f(x) = x^4$, and $f(x) = -x^4$ all have critical points at $x = 0$. In each case, the second derivative is zero at $x = 0$. However, the function $f(x) = x^4$ has a local minimum at $x = 0$ whereas the function $f(x) = -x^4$ has a local maximum at $x = 0$, and the function $f(x) = x^3$ does not have a local extremum at $x = 0$.

Let's now look at how to use the second derivative test to determine whether f has a local maximum or local minimum at a critical point c where $f'(c) = 0$.

Example 3.8.4: Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$.

Solution

To apply the second derivative test, we first need to find critical points c where $f'(c) = 0$. The derivative is $f'(x) = 5x^4 - 15x^2$. Therefore, $f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3) = 0$ when $x = 0, \pm\sqrt{3}$.

To determine whether f has a local extremum at any of these points, we need to evaluate the sign of f'' at these points. The second derivative is

$$f''(x) = 20x^3 - 30x = 10x(2x^2 - 3).$$

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether f has a local maximum or local minimum at any of these points.

Table: 3.8.5: Second Derivative Test for $f(x) = x^5 - 5x^3$.

x	$f''(x)$	Conclusion
$-\sqrt{3}$	$-30\sqrt{3}$	Local maximum
0	0	Second derivative test is inconclusive
$\sqrt{3}$	$30\sqrt{3}$	Local minimum

By the second derivative test, we conclude that f has a local maximum at $x = -\sqrt{3}$ and f has a local minimum at $x = \sqrt{3}$. The second derivative test is inconclusive at $x = 0$. To determine whether f has a local extrema at $x = 0$, we apply the first derivative test. To evaluate the sign of $f'(x) = 5x^2(x^2 - 3)$ for $x \in (-\sqrt{3}, 0)$ and $x \in (0, \sqrt{3})$, let $x = -1$ and $x = 1$ be the two test points. Since $f'(-1) < 0$ and $f'(1) < 0$, we conclude that f is decreasing on both intervals and, therefore, f does not have a local extrema at $x = 0$ as shown in the following graph.

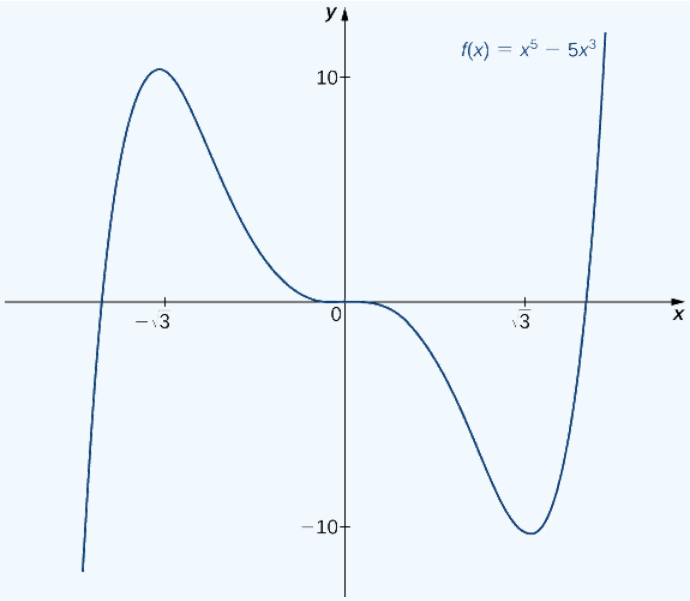


Figure 3.8.10:The function f has a local maximum at $x=-\sqrt{3}$ and a local minimum at $x=\sqrt{3}$

? Exercise 3.8.4

Consider the function $f(x)=x^3-(\frac{3}{2})x^2-18x$. The points $c=3,\ -2$ satisfy $f'(c)=0$. Use the second derivative test to determine whether f has a local maximum or local minimum at those points.

Hint

$f''(x)=6x-3$

Answer

f has a local maximum at -2 and a local minimum at 3.

We have now developed the tools we need to determine where a function is increasing and decreasing, as well as acquired an understanding of the basic shape of the graph. In the next section we discuss what happens to a function as $x\to\pm\infty$. At that point, we have enough tools to provide accurate graphs of a large variety of functions.

3.8.4 Key Concepts

- If c is a critical point of f and $f'(x)>0$ for $x<c$ and $f'(x)<0$ for $x>c$, then f has a local maximum at c.
- If c is a critical point of f and $f'(x)<0$ for $x<c$ and $f'(x)>0$ for $x>c$, then f has a local minimum at c.
- If $f''(x)>0$ over an interval I, then f is concave up over I.
- If $f''(x)<0$ over an interval I, then f is concave down over I.
- If $f'(c)=0$ and $f''(c)>0$, then f has a local minimum at c.
- If $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at c.
- If $f'(c)=0$ and $f''(c)=0$, then evaluate $f'(x)$ at a test point x to the left of c and a test point x to the right of c, to determine whether f has a local extremum at c.

3.8.5 Glossary

concave down

if f is differentiable over an interval I and f' is decreasing over I, then f is concave down over I

concave up

if f is differentiable over an interval I and f' is increasing over I, then f is concave up over I

concavity

the upward or downward curve of the graph of a function

concavity test

suppose f is twice differentiable over an interval I; if $f'' > 0$ over I, then f is concave up over I; if $f'' <$ over I, then f is concave down over I

first derivative test

let f be a continuous function over an interval I containing a critical point c such that f is differentiable over I except possibly at c; if f' changes sign from positive to negative as x increases through c, then f has a local maximum at c; if f' changes sign from negative to positive as x increases through c, then f has a local minimum at c; if f' does not change sign as x increases through c, then f does not have a local extremum at c

inflection point

if f is continuous at c and f changes concavity at c, the point $(c, f(c))$ is an inflection point of f

second derivative test

suppose $f'(c) = 0$ and f'' is continuous over an interval containing c; if $f''(c) > 0$, then f has a local minimum at c; if $f''(c) < 0$, then f has a local maximum at c; if $f''(c) = 0$, then the test is inconclusive

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3.8E: EXERCISES FOR SECTION 3.8

1) If c is a critical point of $f(x)$, when is there no local maximum or minimum at c? Explain.

2) For the function $y = x^3$, is $x = 0$ both an inflection point and a local maximum/minimum?

Answer

It is not a local maximum/minimum because f' does not change sign

3) For the function $y = x^3$, is $x = 0$ an inflection point?

4) Is it possible for a point c to be both an inflection point and a local extremum of a twice differentiable function?

Answer

No

5) Why do you need continuity for the first derivative test? Come up with an example.

6) Explain whether a concave-down function has to cross $y = 0$ for some value of x.

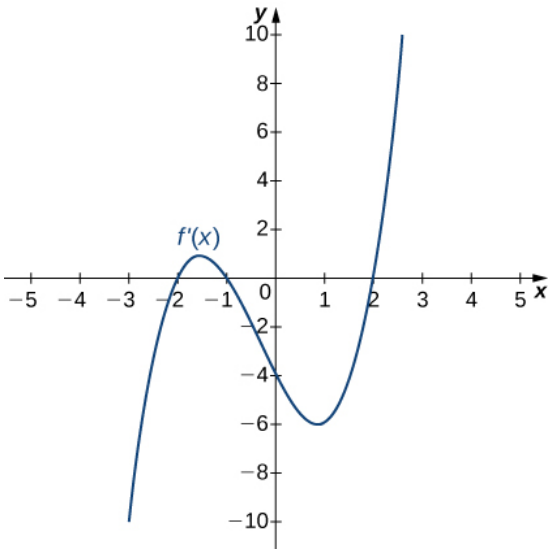
Answer

False; for example, $y = \sqrt{x}$.

7) Explain whether a polynomial of degree 2 can have an inflection point.

In exercises 8 - 12, analyze the graphs of f', then list all intervals where f is increasing or decreasing.

8)

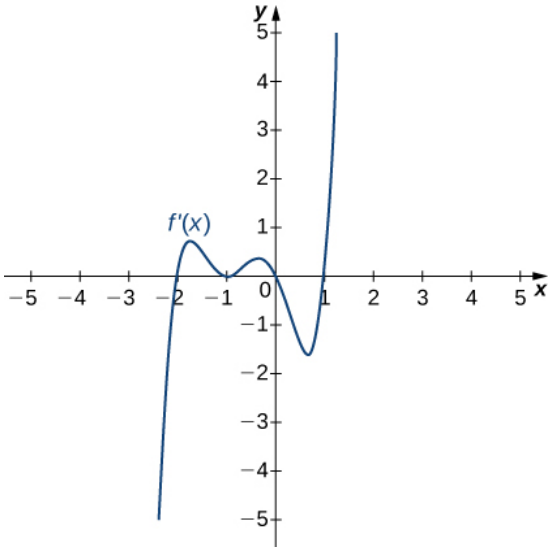


Answer

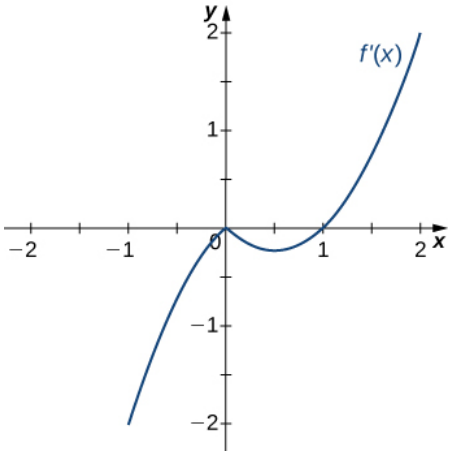
Increasing for $-2 < x < -1$ and $x > 2$;
Decreasing for $x < -2$ and $-1 < x < 2$

9)

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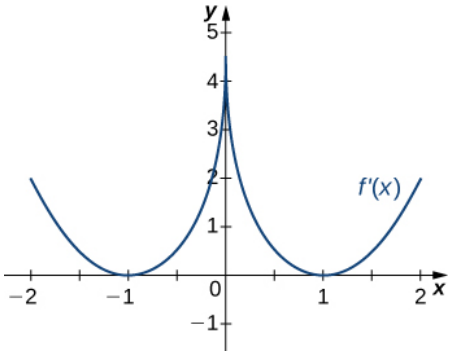
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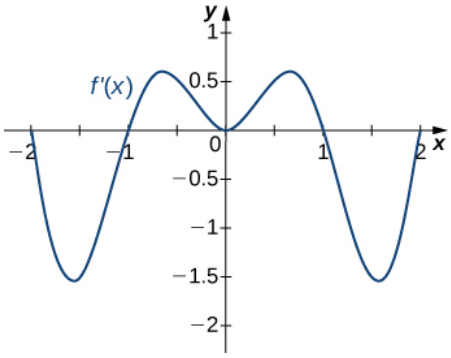
Answer

Decreasing for $x < 1$,
Increasing for $x > 1$

11)



12)



Answer

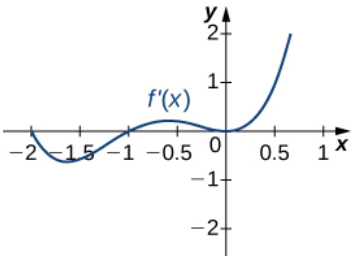
Decreasing for $-2 < x < -1$ and $1 < x < 2$;
Increasing for $-1 < x < 1$ and $x < -2$ and $x > 2$

In exercises 13 - 17, analyze the graphs of f', then list all intervals where

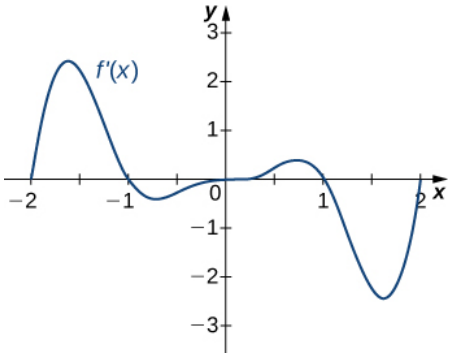
a. f is increasing and decreasing and

b. the minima and maxima are located.

13)



14)

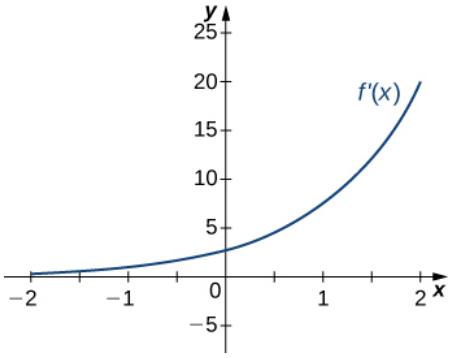


Answer

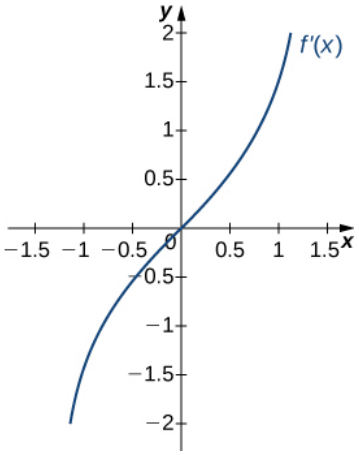
a. Increasing over $-2 < x < -1,\ 0 < x < 1, x > 2$, Decreasing over $x < -2,\ -1 < x < 0,\ 1 < x < 2$;
b. Maxima at $x = -1$ and $x = 1$, Minima at $x = -2$ and $x = 0$ and $x = 2$

15)

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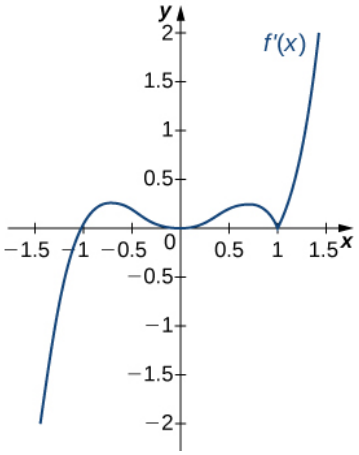
16)



Answer

a. Increasing over $x > 0$, Decreasing over $x < 0$;
b. Minimum at $x = 0$

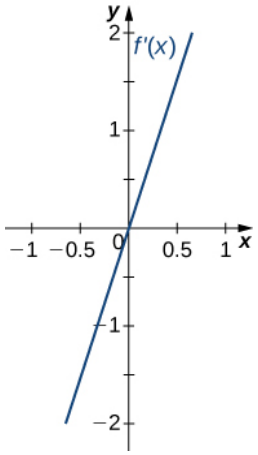
17)



In exercises 18 - 22, analyze the graphs of f', then list all inflection points and intervals f that are concave up and concave down.

18)

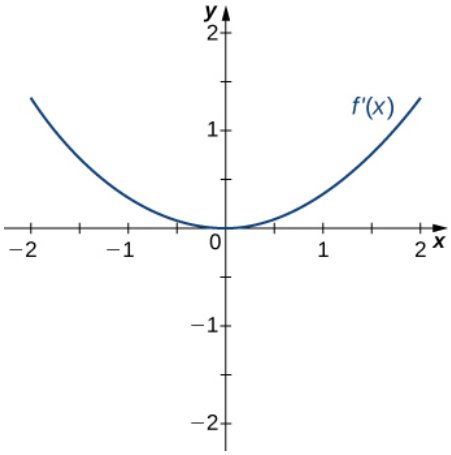
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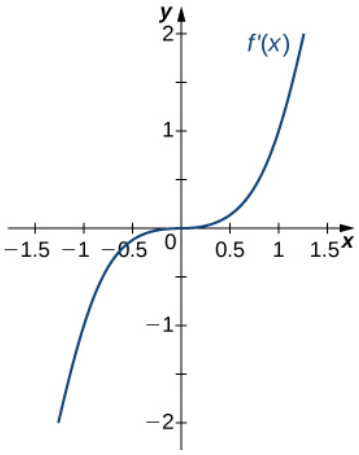
Answer

Concave up for all x,
No inflection points

19)



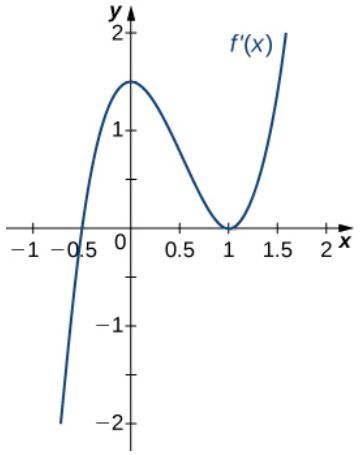
20)



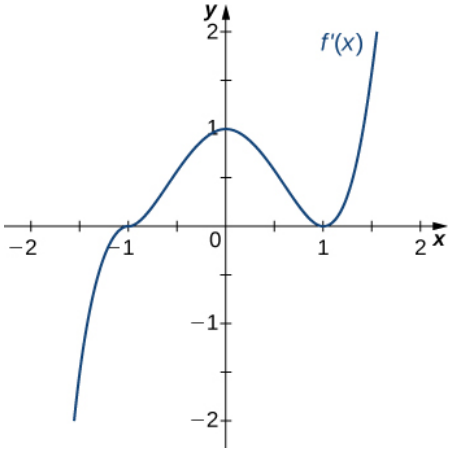
Answer

Concave up for all x,
No inflection points

21)



22)



Answer

Concave up for $x < 0$ and $x > 1$,
Concave down for $0 < x < 1$,
Inflection points at $x = 0$ and $x = 1$

For exercises 23 - 27, draw a graph that satisfies the given specifications for the domain $x = [-3, 3]$. The function does not have to be continuous or differentiable.

23) $f(x) > 0,\ f'(x) > 0$ over $x > 1,\ -3 < x < 0,\ f'(x) = 0$ over $0 < x < 1$

24) $f'(x) > 0$ over $x > 2,\ -3 < x < -1,\ f'(x) < 0$ over $-1 < x < 2,\ f''(x) < 0$ for all x

Answer

Answers will vary

25) $f''(x) < 0$ over $-1 < x < 1,\ f''(x) > 0,\ -3 < x < -1,\ 1 < x < 3,$ local maximum at $x = 0$, local minima at $x = \pm 2$

26) There is a local maximum at $x = 2$, local minimum at $x = 1$, and the graph is neither concave up nor concave down.

Answer

Answers will vary

27) There are local maxima at $x = \pm 1$, the function is concave up for all x, and the function remains positive for all x.

For the following exercises, determine

a. intervals where f is increasing or decreasing and

b. local minima and maxima of f.

28) $f(x) = x^2 + 10x - 2$

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Answer

a. Increasing over $-5 < x < \infty$, decreasing over $-\infty < x < -5$

b. Local minimum at $x = -5$; No local maximum

29) $f(x) = x^3 - 6x^2 + 9x + 4$

30) $f(x) = 3x^4 - 24x^2 + 5$

Answer

a. Increasing over $-2 < x < 0, 2 < x < \infty$, decreasing over $-\infty < x < -2, 0 < x < 2$

b. Local minimums at $x = -2, x = 2$; Local maximum at $x = 0$

For exercise 31, determine

a. intervals where f is concave up or concave down, and

b. the inflection points of f.

31) $f(x) = x^3 - 4x^2 + x + 2$

Answer

a. Concave up for $x > \frac{4}{3}$, concave down for $x < \frac{4}{3}$

b. Inflection point at $x = \frac{4}{3}$

For exercises 32 - 38, determine

a. intervals where f is increasing or decreasing,

b. local minima and maxima of f,

c. intervals where f is concave up and concave down, and

d. the inflection points of f.

32) $f(x) = x^2 - 6x$

33) $f(x) = x^3 - 6x^2$

Answer

a. Increasing over $x < 0$ and $x > 4$, decreasing over $0 < x < 4$

b. Maximum at $x = 0$, minimum at $x = 4$

c. Concave up for $x > 2$, concave down for $x < 2$

d. Inflection point at $x = 2$

34) $f(x) = x^4 - 6x^3$

35) $f(x) = x^{11} - 6x^{10}$

Answer

a. Increasing over $x < 0$ and $x > \frac{60}{11}$, decreasing over $0 < x < \frac{60}{11}$

b. Maximum at $x = 0$, minimum at $x = \frac{60}{11}$

c. Concave down for $x < \frac{54}{11}$, concave up for $x > \frac{54}{11}$

d. Inflection point at $x = \frac{54}{11}$

36) $f(x) = x + x^2 - x^3$

37) $f(x) = x^2 + x + 1$

Answer

a. Increasing over $x > -\frac{1}{2}$, decreasing over $x < -\frac{1}{2}$

b. Minimum at $x = -\frac{1}{2}$

c. Concave up for all x

d. No inflection points

38) $f(x) = x^3 + x^4$

For exercises 39-41, determine

a. intervals where f is increasing or decreasing,

b. local minima and maxima of f,

c. intervals where f is concave up and concave down, and

d. the inflection points of f. Sketch the curve, then use a calculator to compare your answer. If you cannot determine the exact answer analytically, use a calculator.

39) $f(x) = \ln x\sqrt{x}, \quad x > 0$

40) $f(x) = \frac{1}{4}\sqrt{x} + \frac{1}{x}, \quad x > 0$

Answer

a. Increasing over $x > 4$, decreasing over $0 < x < 4$
b. Minimum at $x = 4$
c. Concave up for $0 < x < 8\sqrt[3]{2}$, concave down for $x > 8\sqrt[3]{2}$
d. Inflection point at $x = 8\sqrt[3]{2}$

41) $f(x) = \dfrac{e^x}{x}, \quad x \neq 0$

In exercises 42-46, interpret the sentences in terms of f, f', and f''.

42) The population is growing more slowly. Here f is the population.

Answer

$f > 0,\ f' > 0,\ f'' < 0$

43) A bike accelerates faster, but a car goes faster. Here $f =$ Bike's position minus Car's position.

44) The airplane lands smoothly. Here f is the plane's altitude.

Answer

$f > 0,\ f' < 0,\ f'' > 0$

45) Stock prices are at their peak. Here $f$is the stock price.

46) The economy is picking up speed. Here f is a measure of the economy, such as GDP.

Answer

$f > 0,\ f' > 0,\ f'' > 0$

For exercises 47-51, consider a third-degree polynomial $f(x)$, which has the properties $f'(1) = 0$ and $f'(3) = 0$.

Determine whether the following statements are true or false. Justify your answer.

47) $f(x) = 0$ for some $1 \leq x \leq 3$.

48) $f''(x) = 0$ for some $1 \leq x \leq 3$.

Answer

True, by the Mean Value Theorem

49) There is no absolute maximum at $x = 3$.

50) If $f(x)$ has three roots, then it has 1 inflection point.

Answer

True, examine derivative

51) If $f(x)$ has one inflection point, then it has three real roots.

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3.9: Limits at Infinity, Asymptotes, and Rational Functions

Learning Objectives

- Calculate the limit of a function as x increases or decreases without bound.
- Recognize a horizontal asymptote on the graph of a function.
- Estimate the end behavior of a function as x increases or decreases without bound.
- Recognize an oblique asymptote on the graph of a function.
- Analyze a function and its derivatives to draw its graph.

We have shown how to use the first and second derivatives of a function to describe the shape of a graph. To graph a function f defined on an unbounded domain, we also need to know the behavior of f as $x \to \pm\infty$. In this section, we define limits at infinity and show how these limits affect the graph of a function. At the end of this section, we outline a strategy for graphing an arbitrary function f.

We begin by examining what it means for a function to have a finite limit at infinity. Then we study the idea of a function with an infinite limit at infinity. Back in Introduction to Functions and Graphs, we looked at vertical asymptotes; in this section we deal with horizontal and oblique asymptotes.

3.9.1 Limits at Infinity and Horizontal Asymptotes

Recall that $\lim\limits_{x\to a} f(x) = L$ means $f(x)$ becomes arbitrarily close to L as long as x is sufficiently close to a. We can extend this idea to limits at infinity. For example, consider the function $f(x) = 2 + \frac{1}{x}$. As can be seen graphically in Figure 3.9.1 and numerically in Table 3.9.1, as the values of x get larger, the values of $f(x)$ approach 2. We say the limit as x approaches ∞ of $f(x)$ is 2 and write $\lim\limits_{x\to\infty} f(x) = 2$. Similarly, for $x < 0$, as the values $|x|$ get larger, the values of $f(x)$ approach 2. We say the limit as x approaches $-\infty$ of $f(x)$ is 2 and write $\lim\limits_{x\to-\infty} f(x) = 2$.

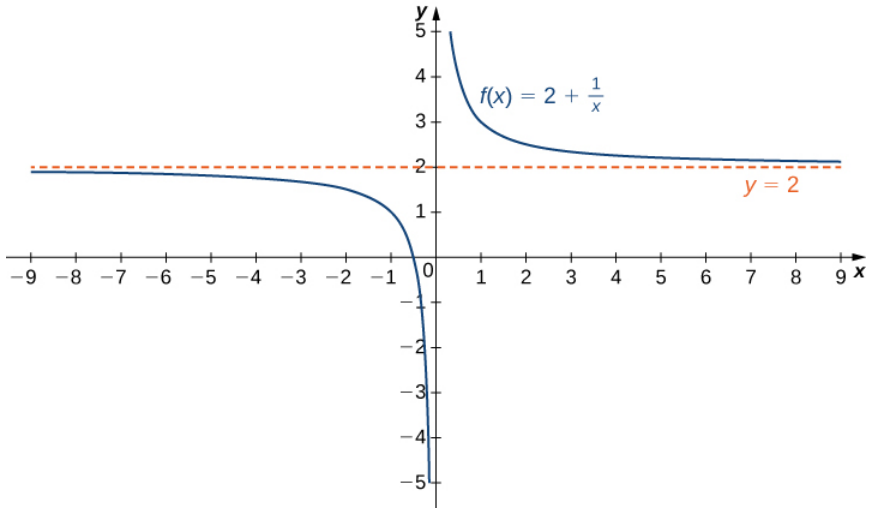


Figure 3.9.1: The function approaches the asymptote $y = 2$ as x approaches $\pm\infty$.

Table 3.9.1: Values of a function f as $x \to \pm\infty$

x	10	100	1,000	10,000
$2+\frac{1}{x}$	2.1	2.01	2.001	2.0001
x	−10	−100	−1000	−10,000
$2+\frac{1}{x}$	1.9	1.99	1.999	1.9999

More generally, for any function f, we say the limit as $x \to \infty$ of $f(x)$ is L if $f(x)$ becomes arbitrarily close to L as long as x is sufficiently large. In that case, we write $\lim\limits_{x\to\infty} f(x) = L$. Similarly, we say the limit as $x \to -\infty$ of $f(x)$ is L if $f(x)$ becomes

arbitrarily close to L as long as $x < 0$ and $|x|$ is sufficiently large. In that case, we write $\lim_{x \to -\infty} f(x) = L$. We now look at the definition of a function having a limit at infinity.

Definition: Limit at Infinity (Informal)

If the values of $f(x)$ become arbitrarily close to L as x becomes sufficiently large, we say the function f has a **limit at infinity** and write

$$\lim_{x \to \infty} f(x) = L.$$

If the values of $f(x)$ becomes arbitrarily close to L for $x < 0$ as $|x|$ becomes sufficiently large, we say that the function f has a limit at negative infinity and write

$$\lim_{x \to -\infty} f(x) = L.$$

If the values $f(x)$ are getting arbitrarily close to some finite value L as $x \to \infty$ or $x \to -\infty$, the graph of f approaches the line $y = L$. In that case, the line $y = L$ is a horizontal asymptote of f (Figure 3.9.2). For example, for the function $f(x) = \dfrac{1}{x}$, since $\lim_{x \to \infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of $f(x) = \dfrac{1}{x}$.

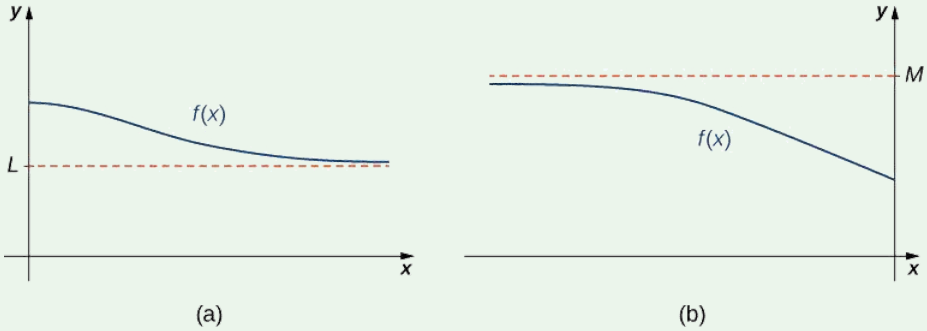


Figure 3.9.2: (a) As $x \to \infty$, the values of f are getting arbitrarily close to L. The line $y = L$ is a horizontal asymptote of f. (b) As $x \to -\infty$, the values of f are getting arbitrarily close to M. The line $y = M$ is a horizontal asymptote of f.

Definition: Horizontal Asymptote

If $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, we say the line $y = L$ is a *horizontal asymptote* of f.

A function cannot cross a vertical asymptote because the graph must approach infinity (or $-\infty$) from at least one direction as x approaches the vertical asymptote. However, a function may cross a horizontal asymptote. For example, the function $f(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$ crosses its horizontal asymptote $y = 3$ at $x = 6$.

The algebraic limit laws we introduced in Introduction to Limits also apply to limits at infinity. We illustrate how to use these laws to compute several limits at infinity.

Example 3.9.1: Computing Limits at Infinity

Given $f(x) = 5 - \dfrac{2}{x^2}$, evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$. Determine the horizontal asymptote(s) for f.

Solution

Using the algebraic limit laws, we have

$$\lim_{x \to \infty} \left(5 - \frac{2}{x^2}\right) = \lim_{x \to \infty} 5 - 2 \left(\lim_{x \to \infty} \frac{1}{x}\right) \cdot \left(\lim_{x \to \infty} \frac{1}{x}\right) = 5 - 2 \cdot 0 = 5.$$

Similarly, $\lim_{x\to-\infty} f(x) = 5$. Therefore, $f(x) = 5 - \dfrac{2}{x^2}$ has a horizontal asymptote of $y = 5$ and f approaches this horizontal asymptote as $x \to \pm\infty$ as shown in the following graph.

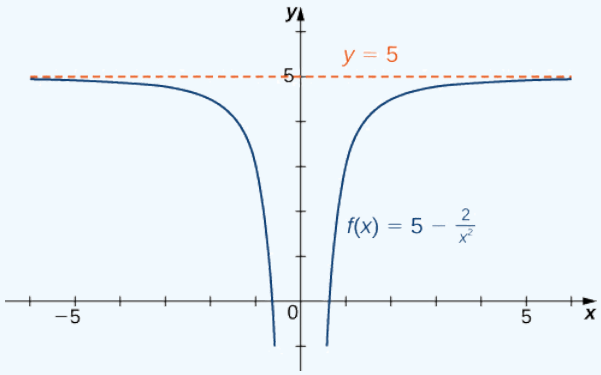


Figure 3.9.4: This function approaches a horizontal asymptote as $x \to \pm\infty$.

? Exercise 3.9.1

Evaluate $\lim_{x\to-\infty}\left(3+\dfrac{4}{x}\right)$ and $\lim_{x\to\infty}\left(3+\dfrac{4}{x}\right)$. Determine the horizontal asymptotes of $f(x) = 3 + \frac{4}{x}$, if any.

Hint

$$\lim_{x\to\pm\infty}\frac{1}{x} = 0$$

Answer

Both limits are 3. The line $y = 3$ is a horizontal asymptote.

3.9.2 Infinite Limits at Infinity

Sometimes the values of a function f become arbitrarily large as $x \to \infty$(or as $x \to -\infty$). In this case, we write $\lim_{x\to\infty} f(x) = \infty$ (or $\lim_{x\to-\infty} f(x) = \infty$). On the other hand, if the values of f are negative but become arbitrarily large in magnitude as $x \to \infty$ (or as $x \to -\infty$), we write $\lim_{x\to\infty} f(x) = -\infty$ (or $\lim_{x\to-\infty} f(x) = -\infty$).

For example, consider the function $f(x) = x^3$. As seen in Table 3.9.2 and Figure 3.9.8, as $x \to \infty$ the values $f(x)$ become arbitrarily large. Therefore, $\lim_{x\to\infty} x^3 = \infty$. On the other hand, as $x \to -\infty$, the values of $f(x) = x^3$ are negative but become arbitrarily large in magnitude. Consequently, $\lim_{x\to-\infty} x^3 = -\infty$.

Table 3.9.2: Values of a power function as $x \to \pm\infty$

x	10	20	50	100	1000
x^3	1000	8000	125,000	1,000,000	1,000,000,000
x	−10	−20	−50	−100	−1000
x^3	−1000	−8000	−125,000	−1,000,000	−1,000,000,000

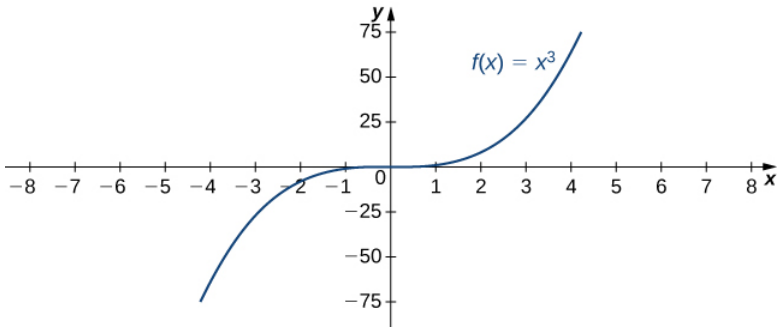


Figure 3.9.8: For this function, the functional values approach ±infinity as $x \to \pm\infty$.

Definition: Infinite Limit at Infinity (Informal)

We say a function f has an infinite limit at infinity and write

$$\lim_{x\to\infty} f(x) = \infty.$$

if $f(x)$ becomes arbitrarily large for x sufficiently large. We say a function has a negative infinite limit at infinity and write

$$\lim_{x\to\infty} f(x) = -\infty.$$

if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large for x sufficiently large. Similarly, we can define infinite limits as $x \to -\infty$.

3.9.3 End Behavior

The behavior of a function as $x \to \pm\infty$ is called the function's **end behavior**. At each of the function's ends, the function could exhibit one of the following types of behavior:

1. The function $f(x)$ approaches a horizontal asymptote $y = L$.
2. The function $f(x) \to \infty$ or $f(x) \to -\infty$.
3. The function does not approach a finite limit, nor does it approach ∞ or $-\infty$. In this case, the function may have some oscillatory behavior.

Let's consider several classes of functions here and look at the different types of end behaviors for these functions.

3.9.4 End Behavior for Polynomial Functions

Consider the power function $f(x) = x^n$ where n is a positive integer. From Figure 3.9.11 and Figure 3.9.12, we see that

$$\lim_{x\to\infty} x^n = \infty;\ n = 1, 2, 3, \ldots$$

and

$$\lim_{x\to-\infty} x^n = \begin{cases} \infty, & n = 2, 4, 6, \ldots \\ -\infty, & n = 1, 3, 5, \ldots . \end{cases}$$

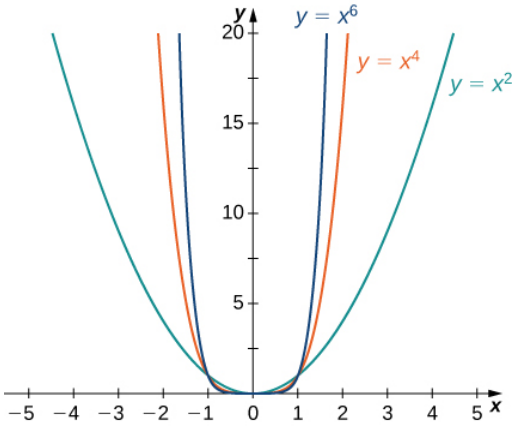


Figure 3.9.11: For power functions with an even power n, $\lim\limits_{x\to\infty} x^n = \infty = \lim\limits_{x\to-\infty} x^n$.

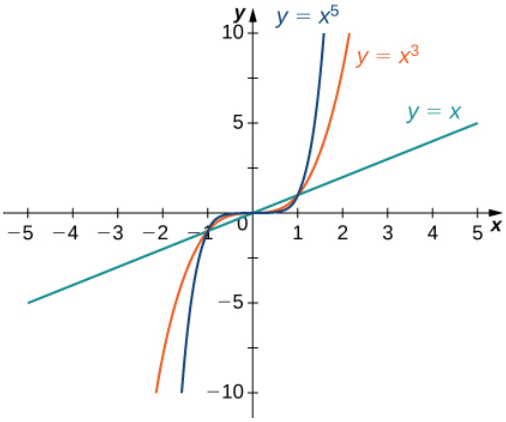


Figure 3.9.12: For power functions with an odd power n, $\lim\limits_{x\to\infty} x^n = \infty$ and $\lim\limits_{x\to-\infty} x^n = -\infty$.

Using these facts, it is not difficult to evaluate $\lim\limits_{x\to\infty} cx^n$ and $\lim\limits_{x\to-\infty} cx^n$, where c is any constant and n is a positive integer. If $c > 0$, the graph of $y = cx^n$ is a vertical stretch or compression of $y = x^n$, and therefore

$$\lim_{x\to\infty} cx^n = \lim_{x\to\infty} x^n \text{ and } \lim_{x\to-\infty} cx^n = \lim_{x\to-\infty} x^n \text{ if } c > 0.$$

If $c < 0$, the graph of $y = cx^n$ is a vertical stretch or compression combined with a reflection about the x-axis, and therefore

$$\lim_{x\to\infty} cx^n = -\lim_{x\to\infty} x^n \text{ and } \lim_{x\to-\infty} cx^n = -\lim_{x\to-\infty} x^n \text{ if } c < 0.$$

If $c = 0, y = cx^n = 0,$ in which case $\lim\limits_{x\to\infty} cx^n = 0 = \lim\limits_{x\to-\infty} cx^n$.

Example 3.9.4: Limits at Infinity for Power Functions

For each function f, evaluate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

a. $f(x) = -5x^3$
b. $f(x) = 2x^4$

Solution

a. Since the coefficient of x^3 is -5, the graph of $f(x) = -5x^3$ involves a vertical stretch and reflection of the graph of $y = x^3$ about the x-axis. Therefore, $\lim\limits_{x\to\infty}(-5x^3) = -\infty$ and $\lim\limits_{x\to-\infty}(-5x^3) = \infty$.

b. Since the coefficient of x^4 is 2, the graph of $f(x) = 2x^4$ is a vertical stretch of the graph of $y = x^4$. Therefore, $\lim\limits_{x\to\infty} 2x^4 = \infty$ and $\lim\limits_{x\to-\infty} 2x^4 = \infty$.

Exercise 3.9.4

Let $f(x) = -3x^4$. Find $\lim_{x\to\infty} f(x)$.

Hint

The coefficient -3 is negative.

Answer

$-\infty$

We now look at how the limits at infinity for power functions can be used to determine $\lim_{x\to\pm\infty} f(x)$ for any polynomial function f. Consider a polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a^1 x + a^0$$

of degree $n \geq 1$ so that $a_n \neq 0$.

Factoring, we see that

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n}\frac{1}{x} + \ldots + \frac{a_1}{a_n}\frac{1}{x^{n-1}} + \frac{a_0}{a_n}\frac{1}{x^n}\right).$$

As $x \to \pm\infty$, all the terms inside the parentheses approach zero except the first term. We conclude that

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} a_n x^n.$$

For example, the function $f(x) = 5x^3 - 3x^2 + 4$ behaves like $g(x) = 5x^3$ as $x \to \pm\infty$ as shown in Figure 3.9.13 and Table 3.9.3.

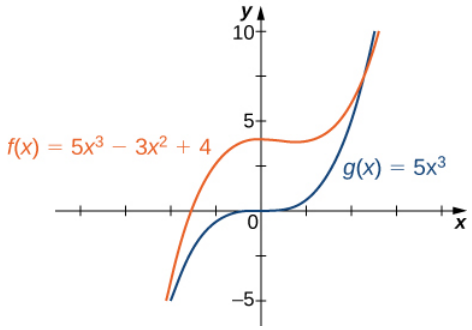


Figure 3.9.13: The end behavior of a polynomial is determined by the behavior of the term with the largest exponent.

Table 3.9.3: A polynomial's end behavior is determined by the term with the largest exponent

x	10	100	1000
$f(x) = 5x^3 - 3x^2 + 4$	4704	4,970,004	4,997,000,004
$g(x) = 5x^3$	5000	5,000,000	5,000,000,000
x	−10	−100	−000
$f(x) = 5x^3 - 3x^2 + 4$	−5296	−5,029,996	−5,002,999,996
$g(x) = 5x^3$	−5000	−5,000,000	−5,000,000,000

3.9.5 End Behavior for Algebraic Functions

The end behavior for rational functions and functions involving radicals is a little more complicated than for polynomials. In Example 3.9.5, we show that the limits at infinity of a rational function $f(x) = \frac{p(x)}{q(x)}$ depend on the relationship between the

degree of the numerator and the degree of the denominator. To evaluate the limits at infinity for a rational function, we divide the numerator and denominator by the highest power of x appearing in the denominator. This determines which term in the overall expression dominates the behavior of the function at large values of x.

Example 3.9.5: Determining End Behavior for Rational Functions

For each of the following functions, determine the limits as $x \to \infty$ and $x \to -\infty$. Then, use this information to describe the end behavior of the function.

a. $f(x) = \dfrac{3x-1}{2x+5}$ (Note: The degree of the numerator and the denominator are the same.)

b. $f(x) = \dfrac{3x^2+2x}{4x^3-5x+7}$ (Note: The degree of numerator is less than the degree of the denominator.)

c. $f(x) = \dfrac{3x^2+4x}{x+2}$ (Note: The degree of the numerator is greater than the degree of the denominator.)

Solution

a. The highest power of x in the denominator is x. Therefore, dividing the numerator and denominator by x and applying the algebraic limit laws, we see that

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{3x-1}{2x+5} &= \lim_{x\to\pm\infty} \frac{3-1/x}{2+5/x} \\ &= \frac{\lim_{x\to\pm\infty}(3-1/x)}{\lim_{x\to\pm\infty}(2+5/x)} \\ &= \frac{\lim_{x\to\pm\infty} 3 - \lim_{x\to\pm\infty} 1/x}{\lim_{x\to\pm\infty} 2 + \lim_{x\to\pm\infty} 5/x} \\ &= \frac{3-0}{2+0} = \frac{3}{2}.\end{aligned}$$

Since $\lim\limits_{x\to\pm\infty} f(x) = \dfrac{3}{2}$, we know that $y = \frac{3}{2}$ is a horizontal asymptote for this function as shown in the following graph.

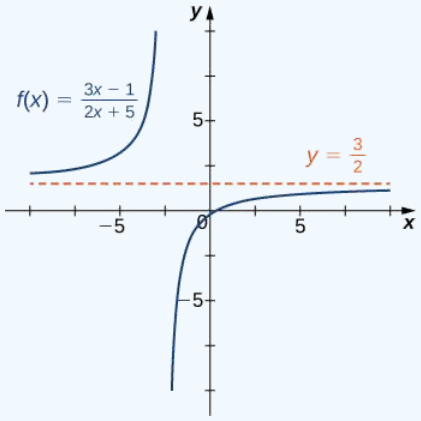


Figure 3.9.14: The graph of this rational function approaches a horizontal asymptote as $x \to \pm\infty$.

b. Since the largest power of x appearing in the denominator is x^3, divide the numerator and denominator by x^3. After doing so and applying algebraic limit laws, we obtain

$$\lim_{x\to\pm\infty} \frac{3x^2+2x}{4x^3-5x+7} = \lim_{x\to\pm\infty} \frac{3/x+2/x^2}{4-5/x^2+7/x^3} = \frac{3\cdot 0+2\cdot 0}{4-5\cdot 0+7\cdot 0} = \frac{0}{4} = 0.$$

Therefore f has a horizontal asymptote of $y = 0$ as shown in the following graph.

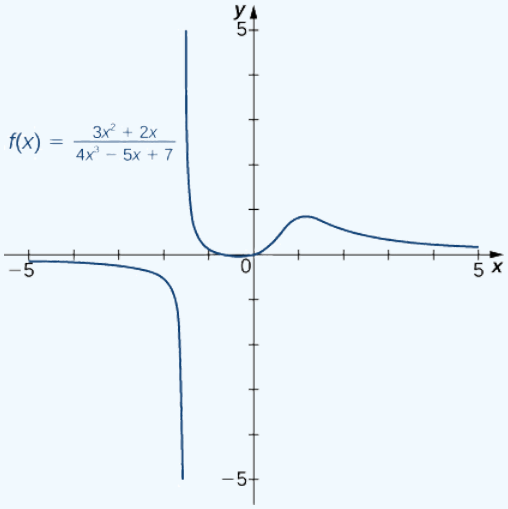


Figure 3.9.15: The graph of this rational function approaches the horizontal asymptote $y = 0$ as $x \to \pm\infty$.

c. Dividing the numerator and denominator by x, we have

$$\lim_{x\to\pm\infty} \frac{3x^2+4x}{x+2} = \lim_{x\to\pm\infty} \frac{3x+4}{1+2/x}.$$

As $x \to \pm\infty$, the denominator approaches 1. As $x \to \infty$, the numerator approaches $+\infty$. As $x \to -\infty$, the numerator approaches $-\infty$. Therefore $\lim_{x\to\infty} f(x) = \infty$, whereas $\lim_{x\to-\infty} f(x) = -\infty$ as shown in the following figure.

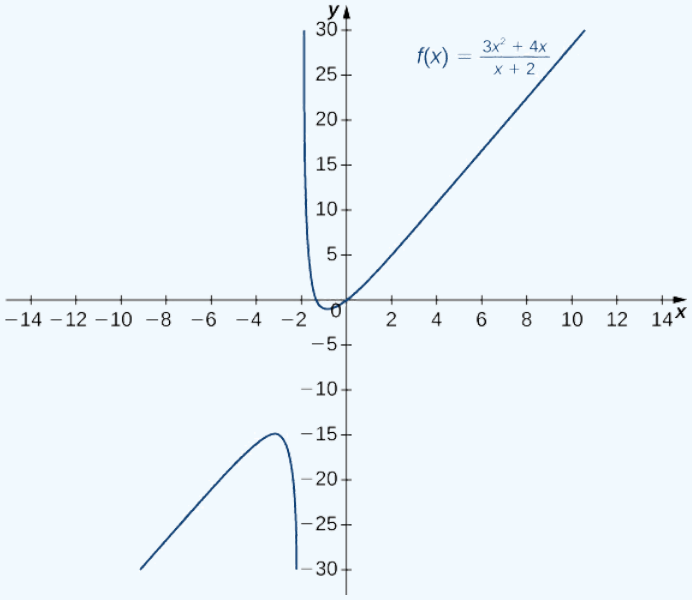


Figure 3.9.16: As $x \to \infty$, the values $f(x) \to \infty$. As $x \to -\infty$, the values $f(x) \to -\infty$.

Exercise 3.9.5

Evaluate $\lim_{x\to\pm\infty} \frac{3x^2+2x-1}{5x^2-4x+7}$ and use these limits to determine the end behavior of $f(x) = \frac{3x^2+2x-1}{5x^2-4x+7}$.

Hint

Divide the numerator and denominator by x^2.

Answer

$\frac{3}{5}$

Before proceeding, consider the graph of $f(x)=\dfrac{3x^2+4x}{x+2}$ shown in Figure 3.9.16. As $x\to\infty$ and $x\to-\infty$, the graph of f appears almost linear. Although f is certainly not a linear function, we now investigate why the graph of f seems to be approaching a linear function. First, using long division of polynomials, we can write

$$f(x)=\frac{3x^2+4x}{x+2}=3x-2+\frac{4}{x+2}.$$

Since $\dfrac{4}{x+2}\to 0$ as $x\to\pm\infty$, we conclude that

$$\lim_{x\to\pm\infty}\left(f(x)-(3x-2)\right)=\lim_{x\to\pm\infty}\frac{4}{x+2}=0.$$

Therefore, the graph of f approaches the line $y=3x-2$ as $x\to\pm\infty$. This line is known as an **oblique asymptote** for f (Figure 3.9.17).

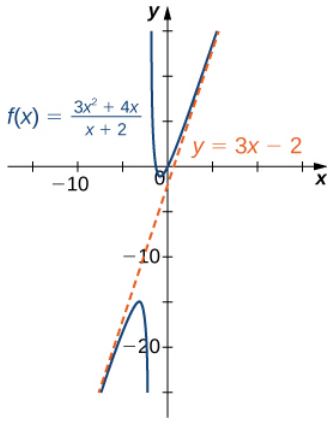


Figure 3.9.17: The graph of the rational function $f(x)=(3x^2+4x)/(x+2)$ approaches the oblique asymptote $y=3x-2$ as $x\to\pm\infty$.

We can summarize the results of Example 3.9.5 to make the following conclusion regarding end behavior for rational functions. Consider a rational function

$$f(x)=\frac{p(x)}{q(x)}=\frac{a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+\ldots+b_1x+b_0},$$

where $a_n\neq 0$ and $b_m\neq 0$.

1. If the degree of the numerator is the same as the degree of the denominator $(n=m)$, then f has a horizontal asymptote of $y=a_n/b_m$ as $x\to\pm\infty$.
2. If the degree of the numerator is less than the degree of the denominator $(n<m)$, then f has a horizontal asymptote of $y=0$ as $x\to\pm\infty$.
3. If the degree of the numerator is greater than the degree of the denominator $(n>m)$, then f does not have a horizontal asymptote. The limits at infinity are either positive or negative infinity, depending on the signs of the leading terms. In addition, using long division, the function can be rewritten as

   $$f(x)=\frac{p(x)}{q(x)}=g(x)+\frac{r(x)}{q(x)},$$

   where the degree of $r(x)$ is less than the degree of $q(x)$. As a result, $\lim\limits_{x\to\pm\infty}r(x)/q(x)=0$. Therefore, the values of $[f(x)-g(x)]$ approach zero as $x\to\pm\infty$. If the degree of $p(x)$ is exactly one more than the degree of $q(x)$ (i.e., $n=m+1$), the function $g(x)$ is a linear function. In this case, we call $g(x)$ an oblique asymptote.

Now let's consider the end behavior for functions involving a radical.

Example 3.9.6: Determining End Behavior for a Function Involving a Radical

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \dfrac{3x-2}{\sqrt{4x^2+5}}$ and describe the end behavior of f.

Solution

Let's use the same strategy as we did for rational functions: divide the numerator and denominator by a power of x. To determine the appropriate power of x, consider the expression $\sqrt{4x^2+5}$ in the denominator. Since

$$\sqrt{4x^2+5} \approx \sqrt{4x^2} = 2|x|$$

for large values of x in effect x appears just to the first power in the denominator. Therefore, we divide the numerator and denominator by $|x|$. Then, using the fact that $|x| = x$ for $x > 0$, $|x| = -x$ for $x < 0$, and $|x| = \sqrt{x^2}$ for all x, we calculate the limits as follows:

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x-2}{\sqrt{4x^2+5}} &= \lim_{x\to\infty} \frac{(1/|x|)(3x-2)}{(1/|x|)\sqrt{4x^2+5}} \\
&= \lim_{x\to\infty} \frac{(1/x)(3x-2)}{\sqrt{(1/x^2)(4x^2+5)}} \\
&= \lim_{x\to\infty} \frac{3-2/x}{\sqrt{4+5/x^2}} = \frac{3}{\sqrt{4}} = \frac{3}{2} \\
\lim_{x\to-\infty} \frac{3x-2}{\sqrt{4x^2+5}} &= \lim_{x\to-\infty} \frac{(1/|x|)(3x-2)}{(1/|x|)\sqrt{4x^2+5}} \\
&= \lim_{x\to-\infty} \frac{(-1/x)(3x-2)}{\sqrt{(1/x^2)(4x^2+5)}} \\
&= \lim_{x\to-\infty} \frac{-3+2/x}{\sqrt{4+5/x^2}} = \frac{-3}{\sqrt{4}} = \frac{-3}{2}.
\end{aligned}$$

Therefore, $f(x)$ approaches the horizontal asymptote $y = \frac{3}{2}$ as $x \to \infty$ and the horizontal asymptote $y = -\frac{3}{2}$ as $x \to -\infty$ as shown in the following graph.

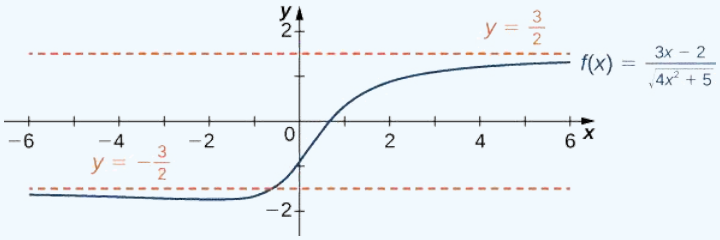


Figure 3.9.18: This function has two horizontal asymptotes and it crosses one of the asymptotes.

Exercise 3.9.6

Evaluate $\lim_{x\to\infty} \dfrac{\sqrt{3x^2+4}}{x+6}$.

Hint

Divide the numerator and denominator by x.

Answer

$\sqrt{3}$

3.9.6 Determining End Behavior for Exponential and Logarithmic Functions

Recall that for any base $b > 0,\ b \neq 1$, the function $y = b^x$ is an exponential function with domain $(-\infty, \infty)$ and range $(0, \infty)$. If $b > 1,\ y = b^x$ is increasing over $(-\infty, \infty)$. If $0 < b < 1,\ y = b^x$ is decreasing over $(-\infty, \infty)$. For the natural exponential function $f(x) = e^x,\ e \approx 2.718 > 1$. Therefore, $f(x) = e^x$ is increasing on $(-\infty, \infty)$ and the range is $(0, \infty)$. The exponential function $f(x) = e^x$ approaches ∞ as $x \to \infty$ and approaches 0 as $x \to -\infty$ as shown in Table 3.9.4 and Figure 3.9.21.

Table 3.9.4: End behavior of the natural exponential function

x	−5	−2	0	2	5
e^x	0.00674	0.135	1	7.389	148.413

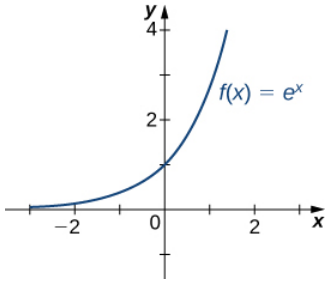


Figure 3.9.21: The exponential function approaches zero as $x \to -\infty$ and approaches ∞ as $x \to \infty$.

Recall that the natural logarithm function $f(x) = \ln(x)$ is the inverse of the natural exponential function $y = e^x$. Therefore, the domain of $f(x) = \ln(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$. The graph of $f(x) = \ln(x)$ is the reflection of the graph of $y = e^x$ about the line $y = x$. Therefore, $\ln(x) \to -\infty$ as $x \to 0^+$ and $\ln(x) \to \infty$ as $x \to \infty$ as shown in Figure 3.9.22 and Table 3.9.5.

Table 3.9.5: End behavior of the natural logarithm function

x	0.01	0.1	1	10	100
$\ln(x)$	−4.605	−2.303	0	2.303	4.605

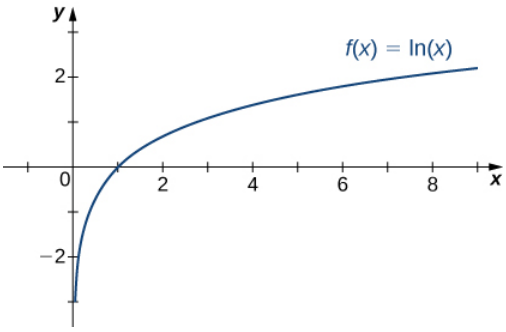


Figure 3.9.22: The natural logarithm function approaches ∞ as $x \to \infty$.

Example 3.9.7: Determining End Behavior for a Transcendental Function

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \dfrac{2 + 3e^x}{7 - 5e^x}$ and describe the end behavior of f.

Solution

To find the limit as $x \to \infty$, divide the numerator and denominator by e^x:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{2 + 3e^x}{7 - 5e^x}$$

$$= \lim_{x \to \infty} \frac{(2/e^x) + 3}{(7/e^x) - 5.}$$

As shown in Figure 3.9.21, $e^x \to \infty$ as $x \to \infty$. Therefore,

$$\lim_{x\to\infty}\frac{2}{e^x}=0=\lim_{x\to\infty}\frac{7}{e^x}.$$

We conclude that $\lim_{x\to\infty} f(x)=-\frac{3}{5}$, and the graph of f approaches the horizontal asymptote $y=-\frac{3}{5}$ as $x\to\infty$. To find the limit as $x\to-\infty$, use the fact that $e^x\to 0$ as $x\to-\infty$ to conclude that $\lim_{x\to-\infty} f(x)=\frac{2}{7}$, and therefore the graph of $f(x)$ approaches the horizontal asymptote $y=\frac{2}{7}$ as $x\to-\infty$.

? Exercise 3.9.7

Find the limits as $x\to\infty$ and $x\to-\infty$ for $f(x)=\dfrac{3e^x-4}{5e^x+2}$.

Hint

$\lim_{x\to\infty} e^x=\infty$ and $\lim_{x\to-\infty} e^x=0$.

Answer

$\lim_{x\to\infty} f(x)=\frac{3}{5}$, $\quad\lim_{x\to-\infty} f(x)=-2$

3.9.7 Guidelines for Drawing the Graph of a Function

We now have enough analytical tools to draw graphs of a wide variety of algebraic and transcendental functions. Before showing how to graph specific functions, let's look at a general strategy to use when graphing any function.

Problem-Solving Strategy: Drawing the Graph of a Function

Given a function f, use the following steps to sketch a graph of f:

1. Determine the domain of the function.
2. Locate the x- and y-intercepts.
3. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ to determine the end behavior. If either of these limits is a finite number L, then $y=L$ is a horizontal asymptote. If either of these limits is ∞ or $-\infty$, determine whether f has an oblique asymptote. If f is a rational function such that $f(x)=\dfrac{p(x)}{q(x)}$, where the degree of the numerator is greater than the degree of the denominator, then f can be written as
$$f(x)=\frac{p(x)}{q(x)}=g(x)+\frac{r(x)}{q(x),}$$
where the degree of $r(x)$ is less than the degree of $q(x)$. The values of $f(x)$ approach the values of $g(x)$ as $x\to\pm\infty$. If $g(x)$ is a linear function, it is known as an oblique asymptote.
4. Determine whether f has any vertical asymptotes.
5. Calculate f'. Find all critical points and determine the intervals where f is increasing and where f is decreasing. Determine whether f has any local extrema.
6. Calculate f''. Determine the intervals where f is concave up and where f is concave down. Use this information to determine whether f has any inflection points. The second derivative can also be used as an alternate means to determine or verify that f has a local extremum at a critical point.

Now let's use this strategy to graph several different functions. We start by graphing a polynomial function.

Example 3.9.8: Sketching a Graph of a Polynomial

Sketch a graph of $f(x) = (x-1)^2(x+2)$.

Solution

Step 1: Since f is a polynomial, the domain is the set of all real numbers.

Step 2: When $x = 0$, $f(x) = 2$. Therefore, the y-intercept is $(0, 2)$. To find the x-intercepts, we need to solve the equation $(x-1)^2(x+2) = 0$, which gives us the x-intercepts $(1, 0)$ and $(-2, 0)$

Step 3: We need to evaluate the end behavior of f. As $x \to \infty$, $(x-1)^2 \to \infty$ and $(x+2) \to \infty$. Therefore, $\lim\limits_{x\to\infty} f(x) = \infty$.

As $x \to -\infty$, $(x-1)^2 \to \infty$ and $(x+2) \to -\infty$. Therefore, $\lim\limits_{x\to-\infty} f(x) = -\infty$.

To get even more information about the end behavior of f, we can multiply the factors of f. When doing so, we see that

$$f(x) = (x-1)^2(x+2) = x^3 - 3x + 2.$$

Since the leading term of f is x^3, we conclude that f behaves like $y = x^3$ as $x \to \pm\infty$.

Step 4: Since f is a polynomial function, it does not have any vertical asymptotes.

Step 5: The first derivative of f is

$$f'(x) = 3x^2 - 3.$$

Therefore, f has two critical points: $x = 1, -1$. Divide the interval $(-\infty, \infty)$ into the three smaller intervals: $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. Then, choose test points $x = -2, x = 0$, and $x = 2$ from these intervals and evaluate the sign of $f'(x)$ at each of these test points, as shown in the following table.

Interval	Test point	Sign of Derivative $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$	Conclusion
$(-\infty, -1)$	$x = -2$	$(+)(-)(-) = +$	f is increasing
$(-1, 1)$	$x = 0$	$(+)(-)(+) = -$	f decreasing
$(1, \infty)$	$x = 2$	$(+)(+)(+) = +$	f is increasing

From the table, we see that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. Evaluating $f(x)$ at those two points, we find that the local maximum value is $f(-1) = 4$ and the local minimum value is $f(1) = 0$.

Step 6: The second derivative of f is

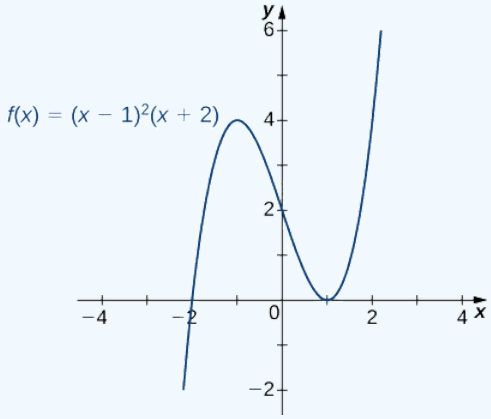
$$f''(x) = 6x.$$

The second derivative is zero at $x = 0$. Therefore, to determine the concavity of f, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0)$ and $(0, \infty)$, and choose test points $x = -1$ and $x = 1$ to determine the concavity of f on each of these smaller intervals as shown in the following table.

Interval	Test Point	Sign of $f''(x) = 6x$	Conclusion
$(-\infty, 0)$	$x = -1$	$-$	f is concave down.
$(0, \infty)$	$x = 1$	$+$	f is concave up.

We note that the information in the preceding table confirms the fact, found in step 5, that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. In addition, the information found in step 5—namely, f has a local maximum at $x = -1$ and a local minimum at $x = 1$, and $f'(x) = 0$ at those points—combined with the fact that f'' changes sign only at $x = 0$ confirms the results found in step 6 on the concavity of f.

Combining this information, we arrive at the graph of $f(x) = (x-1)^2(x+2)$ shown in the following graph.



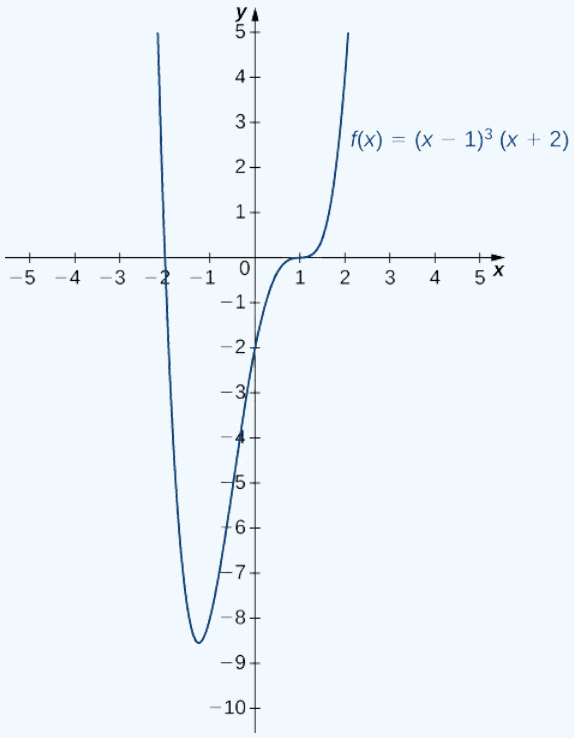
Exercise 3.9.8

Sketch a graph of $f(x) = (x-1)^3(x+2)$.

Hint

f is a fourth-degree polynomial.

Answer



Example 3.9.9: Sketching a Rational Function

Sketch the graph of $f(x) = \dfrac{x^2}{1-x^2}$.

Solution

Step 1: The function f is defined as long as the denominator is not zero. Therefore, the domain is the set of all real numbers x except $x = \pm 1$.

Step 2: Find the intercepts. If $x=0$, then $f(x)=0$, so 0 is an intercept. If $y=0$, then $\dfrac{x^2}{1-x^2}=0$, which implies $x=0$. Therefore, $(0,0)$ is the only intercept.

Step 3: Evaluate the limits at infinity. Since $f$is a rational function, divide the numerator and denominator by the highest power in the denominator: x^2.We obtain

$$\lim_{x\to\pm\infty}\frac{x^2}{1-x^2}=\lim_{x\to\pm\infty}\frac{1}{\frac{1}{x^2}-1}=-1.$$

Therefore, f has a horizontal asymptote of $y=-1$ as $x\to\infty$ and $x\to-\infty$.

Step 4: To determine whether f has any vertical asymptotes, first check to see whether the denominator has any zeroes. We find the denominator is zero when $x=\pm1$. To determine whether the lines $x=1$ or $x=-1$ are vertical asymptotes of f, evaluate $\lim\limits_{x\to1}f(x)$ and $\lim\limits_{x\to-1}f(x)$. By looking at each one-sided limit as $x\to1$, we see that

$$\lim_{x\to1^+}\frac{x^2}{1-x^2}=-\infty \text{ and } \lim_{x\to1^-}\frac{x^2}{1-x^2}=\infty.$$

In addition, by looking at each one-sided limit as $x\to-1$, we find that

$$\lim_{x\to-1^+}\frac{x^2}{1-x^2}=\infty \text{ and } \lim_{x\to-1^-}\frac{x^2}{1-x^2}=-\infty.$$

Step 5: Calculate the first derivative:

$$f\,'(x)=\frac{(1-x^2)(2x)-x^2(-2x)}{\left(1-x^2\right)^2}=\frac{2x}{\left(1-x^2\right)^2}.$$

Critical points occur at points x where $f\,'(x)=0$ or $f\,'(x)$ is undefined. We see that $f\,'(x)=0$ when $x=0$. The derivative $f\,'$ is not undefined at any point in the domain of f. However, $x=\pm1$ are not in the domain of f. Therefore, to determine where f is increasing and where f is decreasing, divide the interval $(-\infty,\infty)$ into four smaller intervals: $(-\infty,-1)$, $(-1,0)$, $(0,1)$, and $(1,\infty)$, and choose a test point in each interval to determine the sign of $f\,'(x)$ in each of these intervals. The values $x=-2$, $x=-\frac{1}{2}$, $x=\frac{1}{2}$, and $x=2$ are good choices for test points as shown in the following table.

Interval	Test point	Sign of $f\,'(x)=\frac{2x}{(1-x^2)^2}$	Conclusion
$(-\infty,-1)$	$x=-2$	$-/+=-$	f is decreasing.
$(-1,0)$	$x=-1/2$	$-/+=-$	f is decreasing.
$(0,1)$	$x=1/2$	$+/+=+$	f is increasing.
$(1,\infty)$	$x=2$	$+/+=+$	f is increasing.

From this analysis, we conclude that f has a local minimum at $x=0$ but no local maximum.

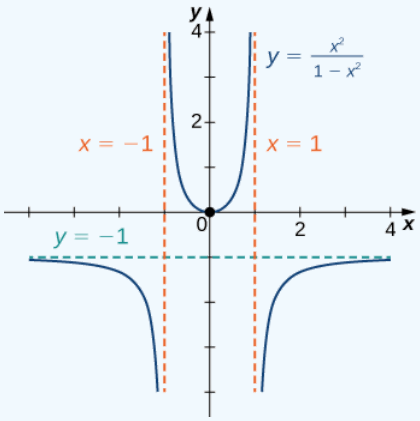
Step 6: Calculate the second derivative:

$$f''(x) = \frac{(1-x^2)^2(2) - 2x(2(1-x^2)(-2x))}{(1-x^2)^4}$$
$$= \frac{(1-x^2)[2(1-x^2)+8x^2]}{\left(1-x^2\right)^4}$$
$$= \frac{2(1-x^2)+8x^2}{\left(1-x^2\right)^3}$$
$$= \frac{6x^2+2}{\left(1-x^2\right)^3}.$$

To determine the intervals where f is concave up and where f is concave down, we first need to find all points x where $f''(x) = 0$ or $f''(x)$ is undefined. Since the numerator $6x^2 + 2 \neq 0$ for any x, $f''(x)$ is never zero. Furthermore, f'' is not undefined for any x in the domain of f. However, as discussed earlier, $x = \pm 1$ are not in the domain of f. Therefore, to determine the concavity of f, we divide the interval $(-\infty, \infty)$ into the three smaller intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$, and choose a test point in each of these intervals to evaluate the sign of $f''(x)$. The values $x = -2$, $x = 0$, and $x = 2$ are possible test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) = \frac{6x^2+2}{(1-x^2)^3}$	Conclusion
$(-\infty, -1)$	$x = -2$	$+/- = -$	f is concave down.
$(-1, 1)$	$x = 0$	$+/+ = +$	f is concave up
$(1, \infty)$	$x = 2$	$+/- = -$	f is concave down.

Combining all this information, we arrive at the graph of f shown below. Note that, although f changes concavity at $x = -1$ and $x = 1$, there are no inflection points at either of these places because f is not continuous at $x = -1$ or $x = 1$.



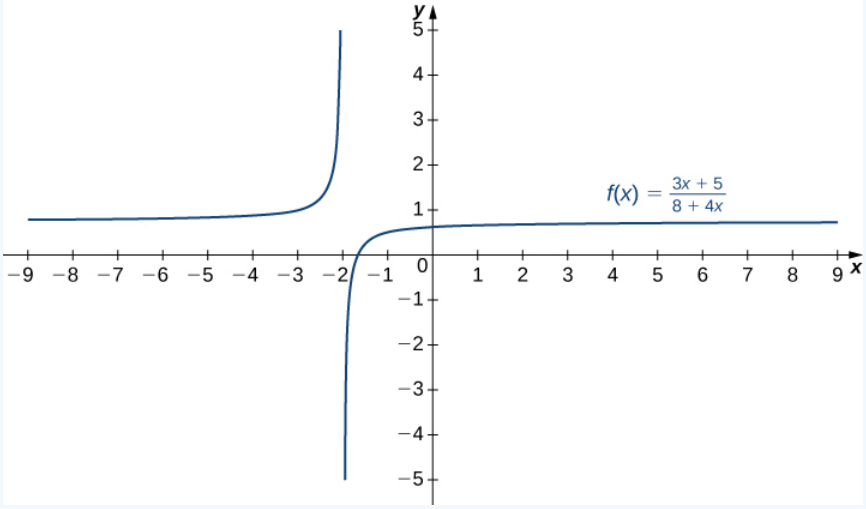
Exercise 3.9.9

Sketch a graph of $f(x) = \dfrac{3x+5}{8+4x}$.

Hint

A line $y = L$ is a horizontal asymptote of f if the limit as $x \to \infty$ or the limit as $x \to -\infty$ of $f(x)$ is L. A line $x = a$ is a vertical asymptote if at least one of the one-sided limits of f as $x \to a$ is ∞ or $-\infty$.

Answer



Example 3.9.10: Sketching a Rational Function with an Oblique Asymptote

Sketch the graph of $f(x)=\dfrac{x^2}{x-1}$

Solution

Step 1: The domain of f is the set of all real numbers x except $x=1$.

Step 2: Find the intercepts. We can see that when $x=0,\ f(x)=0$, so $(0,0)$ is the only intercept.

Step 3: Evaluate the limits at infinity. Since the degree of the numerator is one more than the degree of the denominator, f must have an oblique asymptote. To find the oblique asymptote, use long division of polynomials to write

$$f(x)=\frac{x^2}{x-1}=x+1+\frac{1}{x-1}\ .$$

Since $\dfrac{1}{x-1}\to 0$ as $x\to\pm\infty$, $f(x)$ approaches the line $y=x+1$ as $x\to\pm\infty$. The line $y=x+1$ is an oblique asymptote for f.

Step 4: To check for vertical asymptotes, look at where the denominator is zero. Here the denominator is zero at $x=1$. Looking at both one-sided limits as $x\to 1$, we find

$$\lim_{x\to 1^+}\frac{x^2}{x-1}=\infty \text{ and } \lim_{x\to 1^-}\frac{x^2}{x-1}=-\infty.$$

Therefore, $x=1$ is a vertical asymptote, and we have determined the behavior of f as x approaches 1 from the right and the left.

Step 5: Calculate the first derivative:

$$f'(x)=\frac{(x-1)(2x)-x^2(1)}{(x-1)^2}=\frac{x^2-2x}{(x-1)^2}.$$

We have $f'(x)=0$ when $x^2-2x=x(x-2)=0$. Therefore, $x=0$ and $x=2$ are critical points. Since f is undefined at $x=1$, we need to divide the interval $(-\infty,\infty)$ into the smaller intervals $(-\infty,0),(0,1),(1,2)$,and $(2,\infty)$, and choose a test point from each interval to evaluate the sign of $f'(x)$ in each of these smaller intervals. For example, let $x=-1, x=\frac{1}{2}, x=\frac{3}{2}$, and $x=3$ be the test points as shown in the following table.

Interval	Test Point	Sign of $f'(x)=\dfrac{x^2-2x}{(x-1)^2}$	Conclusion
$(-\infty,0)$	$x=-1$	(−)(−)/+=+	f is increasing.

Interval	Test Point	Sign of $f'(x) = \dfrac{x^2-2x}{(x-1)^2}$	Conclusion
$(0,1)$	$x=1/2$	$(+)(-)/+=-$	f is decreasing.
$(1,2)$	$x=3/2$	$(+)(-)/+=-$	f is decreasing.
$(2,\infty)$	$x=3$	$(+)(+)/+=+$	f is increasing.

From this table, we see that f has a local maximum at $x=0$ and a local minimum at $x=2$. The value of f at the local maximum is $f(0)=0$ and the value of f at the local minimum is $f(2)=4$. Therefore, $(0,0)$ and $(2,4)$ are important points on the graph.

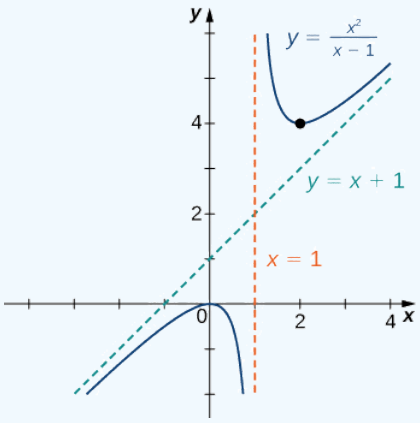
Step 6. Calculate the second derivative:

$$\begin{aligned} f''(x) &= \frac{(x-1)^2(2x-2)-2(x-1)(x^2-2x)}{(x-1)^4} \\ &= \frac{2(x-1)[(x-1)^2-(x^2-2x)]}{(x-1)^4} \\ &= \frac{2[x^2-2x+1-x^2+2x]}{(x-1)^3} \\ &= \frac{2}{(x-1)^3}. \end{aligned}$$

We see that $f''(x)$ is never zero or undefined for x in the domain of f. Since f is undefined at $x=1$, to check concavity we just divide the interval $(-\infty,\infty)$ into the two smaller intervals $(-\infty,1)$ and $(1,\infty)$, and choose a test point from each interval to evaluate the sign of $f''(x)$ in each of these intervals. The values $x=0$ and $x=2$ are possible test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) = \dfrac{2}{(x-1)^3}$	Conclusion
$(-\infty,1)$	$x=0$	$+/-=-$	f is concave down.
$(1,\infty)$	$x=2$	$+/+=+$	f is concave up

From the information gathered, we arrive at the following graph for f.



Exercise 3.9.10

Find the oblique asymptote for $f(x) = \dfrac{3x^3 - 2x + 1}{2x^2 - 4}$.

Hint

Use long division of polynomials.

Answer

$y = \frac{3}{2}x$

Example 3.9.11: Sketching the Graph of a Function with a Cusp

Sketch a graph of $f(x) = (x-1)^{2/3}$

Solution

Step 1: Since the cube-root function is defined for all real numbers x and $(x-1)^{2/3} = (\sqrt[3]{x-1})^2$, the domain of f is all real numbers.

Step 2: To find the y-intercept, evaluate $f(0)$. Since $f(0) = 1$, the y-intercept is $(0, 1)$. To find the x-intercept, solve $(x-1)^{2/3} = 0$. The solution of this equation is $x = 1$, so the x-intercept is $(1, 0)$.

Step 3: Since $\lim\limits_{x \to \pm\infty} (x-1)^{2/3} = \infty$, the function continues to grow without bound as $x \to \infty$ and $x \to -\infty$.

Step 4: The function has no vertical asymptotes.

Step 5: To determine where f is increasing or decreasing, calculate f'. We find

$$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$$

This function is not zero anywhere, but it is undefined when $x = 1$. Therefore, the only critical point is $x = 1$. Divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose test points in each of these intervals to determine the sign of $f'(x)$ in each of these smaller intervals. Let $x = 0$ and $x = 2$ be the test points as shown in the following table.

Interval	Test Point	Sign of $f'(x) = \frac{2}{3(x-1)^{1/3}}$	Conclusion
$(-\infty, 1)$	$x = 0$	$+/- = -$	f is decreasing
$(1, \infty)$	$x = 2$	$+/+ = +$	f is increasing

We conclude that f has a local minimum at $x = 1$. Evaluating f at $x = 1$, we find that the value of f at the local minimum is zero. Note that $f'(1)$ is undefined, so to determine the behavior of the function at this critical point, we need to examine $\lim\limits_{x \to 1} f'(x)$. Looking at the one-sided limits, we have

$$\lim_{x \to 1^+} \frac{2}{3(x-1)^{1/3}} = \infty \text{ and } \lim_{x \to 1^-} \frac{2}{3(x-1)^{1/3}} = -\infty.$$

Therefore, f has a cusp at $x = 1$.

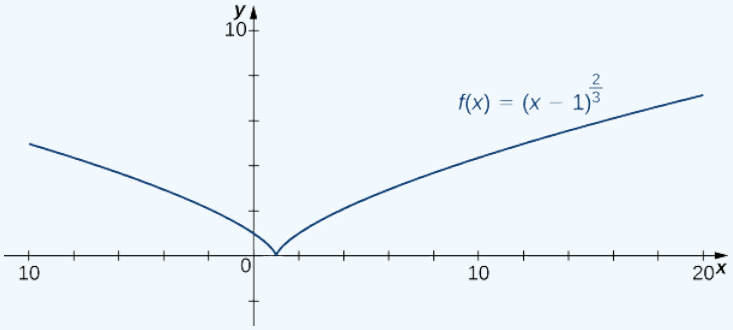
Step 6: To determine concavity, we calculate the second derivative of f:

$$f''(x) = -\frac{2}{9}(x-1)^{-4/3} = \frac{-2}{9(x-1)^{4/3}}.$$

We find that $f''(x)$ is defined for all x, but is undefined when $x = 1$. Therefore, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose test points to evaluate the sign of $f''(x)$ in each of these intervals. As we did earlier, let $x = 0$ and $x = 2$ be test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) = \frac{-2}{9(x-1)^{4/3}}$	Conclusion
$(-\infty, 1)$	$x = 0$	$-/+ = -$	f is concave down
$(1, \infty)$	$x = 2$	$-/+ = -$	f is concave down

From this table, we conclude that f is concave down everywhere. Combining all of this information, we arrive at the following graph for f.



Exercise 3.9.11

Consider the function $f(x) = 5 - x^{2/3}$. Determine the point on the graph where a cusp is located. Determine the end behavior of f.

Hint

A function f has a cusp at a point a if $f(a)$ exists, $f'(a)$ is undefined, one of the one-sided limits as $x \to a$ of $f'(x)$ is $+\infty$, and the other one-sided limit is $-\infty$.

Answer

The function f has a cusp at $(0, 5)$, since $\lim_{x \to 0^-} f'(x) = \infty$ and $\lim_{x \to 0^+} f'(x) = -\infty$. For end behavior, $\lim_{x \to \pm\infty} f(x) = -\infty$.

3.9.8 Key Concepts

- The limit of $f(x)$ is L as $x \to \infty$ (or as $x \to -\infty$) if the values $f(x)$ become arbitrarily close to L as x becomes sufficiently large.
- The limit of $f(x)$ is ∞ as $x \to \infty$ if $f(x)$ becomes arbitrarily large as x becomes sufficiently large. The limit of $f(x)$ is $-\infty$ as $x \to \infty$ if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large as x becomes sufficiently large. We can define the limit of $f(x)$ as x approaches $-\infty$ similarly.
- For a polynomial function $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_n \neq 0$, the end behavior is determined by the leading term $a_n x^n$. If $n \neq 0, p(x)$ approaches ∞ or $-\infty$at each end.
- For a rational function $f(x) = \frac{p(x)}{q(x),}$ the end behavior is determined by the relationship between the degree of p and the degree of q. If the degree of p is less than the degree of q, the line $y = 0$ is a horizontal asymptote for f. If the degree of p is equal to the degree of q, then the line $y = \frac{a_n}{b_n}$ is a horizontal asymptote, where a_n and b_n are the leading coefficients of p and q, respectively. If the degree of p is greater than the degree of q, then f approaches ∞ or $-\infty$ at each end.

3.9.9 Glossary

end behavior

the behavior of a function as $x \to \infty$ and $x \to -\infty$

horizontal asymptote

if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then $y = L$ is a horizontal asymptote of f

infinite limit at infinity

a function that becomes arbitrarily large as x becomes large

limit at infinity

a function that approaches a limit value L as x becomes large

oblique asymptote

the line $y = mx + b$ if $f(x)$ approaches it as $x \to \infty$ or$x \to -\infty$

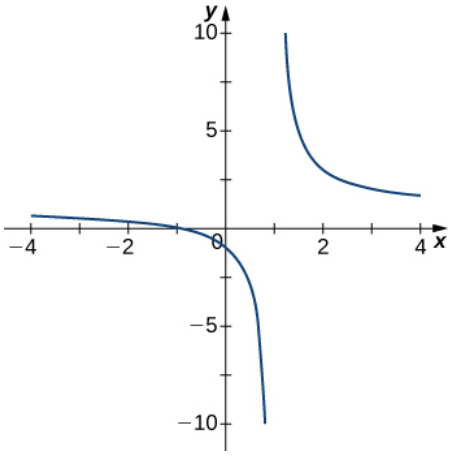
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3.9E: EXERCISES FOR SECTION 3.9

For exercises 1 - 5, examine the graphs. Identify where the vertical asymptotes are located.

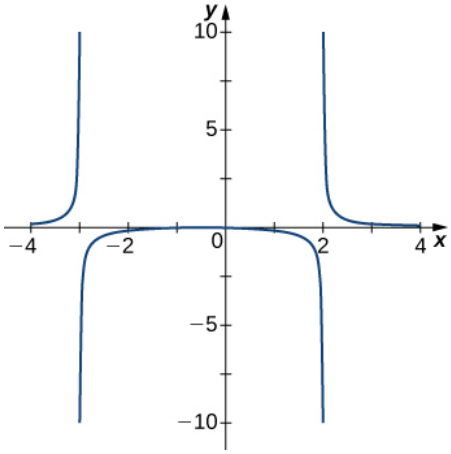
1)



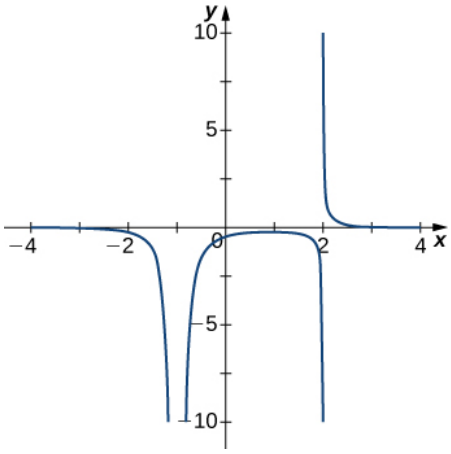
Answer

$x = 1$

2)



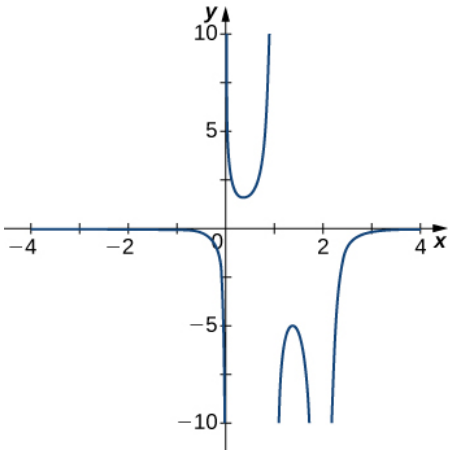
3)



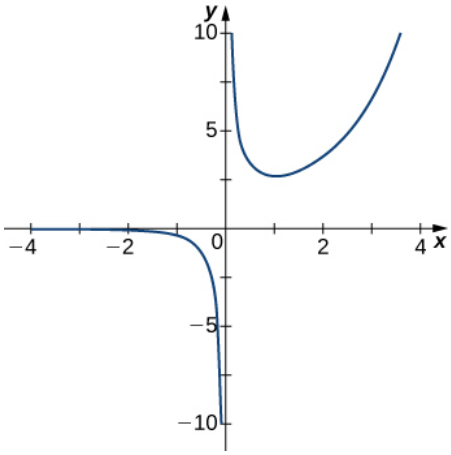
Answer

$x = -1,\ x = 2$

4)



5)



Answer

$x = 0$

For the functions $f(x)$ in exercises 6 - 10, determine whether there is an asymptote at $x = a$. Justify your answer without graphing on a calculator.

6) $f(x) = \dfrac{x+1}{x^2+5x+4}, \quad a = -1$

7) $f(x) = \dfrac{x}{x-2}, \quad a = 2$

Answer

Yes, there is a vertical asymptote at $x = 2$.

8) $f(x) = (x+2)^{3/2}, \quad a = -2$

9) $f(x) = (x-1)^{-1/3}, \quad a = 1$

Answer

Yes, there is vertical asymptote at $x = 1$.

10) $f(x) = 1 + x^{-2/5}, \quad a = 1$

In exercises 11 - 20, evaluate the limit.

11) $\lim\limits_{x\to\infty} \dfrac{1}{3x+6}$

Answer

$$\lim_{x\to\infty}\frac{1}{3x+6}=0$$

12) $\lim_{x\to\infty}\dfrac{2x-5}{4x}$

13) $\lim_{x\to\infty}\dfrac{x^2-2x+5}{x+2}$

Answer

$$\lim_{x\to\infty}\frac{x^2-2x+5}{x+2}=\infty$$

14) $\lim_{x\to-\infty}\dfrac{3x^3-2x}{x^2+2x+8}$

15) $\lim_{x\to-\infty}\dfrac{x^4-4x^3+1}{2-2x^2-7x^4}$

Answer

$$\lim_{x\to-\infty}\frac{x^4-4x^3+1}{2-2x^2-7x^4}=-\frac{1}{7}$$

16) $\lim_{x\to\infty}\dfrac{3x}{\sqrt{x^2+1}}$

17) $\lim_{x\to-\infty}\dfrac{\sqrt{4x^2-1}}{x+2}$

Answer

$$\lim_{x\to-\infty}\frac{\sqrt{4x^2-1}}{x+2}=-2$$

18) $\lim_{x\to\infty}\dfrac{4x}{\sqrt{x^2-1}}$

19) $\lim_{x\to-\infty}\dfrac{4x}{\sqrt{x^2-1}}$

Answer

$$\lim_{x\to-\infty}\frac{4x}{\sqrt{x^2-1}}=-4$$

20) $\lim_{x\to\infty}\dfrac{2\sqrt{x}}{x-\sqrt{x}+1}$

For exercises 21 - 29, find the horizontal and vertical asymptotes.

21) $f(x)=x-\dfrac{9}{x}$

Answer

Horizontal: none,
Vertical: $x=0$

22) $f(x)=\dfrac{1}{1-x^2}$

23) $f(x)=\dfrac{x^3}{4-x^2}$

Answer

Horizontal: none,
Vertical: $x=\pm 2$

24) $f(x) = \dfrac{x^2+3}{x^2+1}$

25) $f(x) = \dfrac{3x^2-3x}{x^2+x-12}$

Answer

Horizontal: $y = 3$
Vertical: $x = -4, x = 3$

26) $f(x) = \dfrac{1}{x^3+x^2}$

Answer

Horizontal: $y = 0$,
Vertical: $x = 0$ and $x = -1$

27) $f(x) = \dfrac{1}{x-1} - 2x$

28) $f(x) = \dfrac{x^3+1}{x^3-1}$

Answer

Horizontal: $y = 1$,
Vertical: $x = 1$

29) $f(x) = \dfrac{1}{x} - \sqrt{x}$

For exercises 30-33, construct a function $f(x)$ that has the given asymptotes.

30) $x = 1$ and $y = 2$

Answer

Answers will vary, for example: $y = \dfrac{2x}{x-1}$

31) $x = 1$ and $y = 0$

32) $y = 4,\ x = -1$

Answer

Answers will vary, for example: $y = \dfrac{4x}{x+1}$

33) $x = 0$

In exercises 34-38 graph the function on a graphing calculator on the window $x = [-5, 5]$ and estimate the horizontal asymptote or limit. Then, calculate the actual horizontal asymptote or limit.

34) [T] $f(x) = \dfrac{1}{x+10}$

Answer

$\lim\limits_{x\to\infty} \dfrac{1}{x+10} = 0$ so f has a horizontal asymptote of $y = 0$.

35) [T] $f(x) = \dfrac{x+1}{x^2+7x+6}$

36) [T] $\lim\limits_{x\to-\infty} x^2+10x+25$

Answer

$\lim\limits_{x\to-\infty} x^2+10x+25 = \infty$

37) [T] $\lim_{x\to-\infty} \dfrac{x+2}{x^2+7x+6}$

38) [T] $\lim_{x\to\infty} \dfrac{3x+2}{x+5}$

Answer

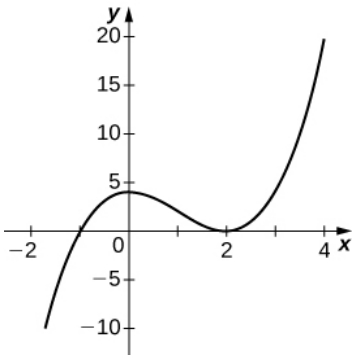
$\lim_{x\to\infty} \dfrac{3x+2}{x+5} = 3$ so this function has a horizontal asymptote of $y = 3$.

In exercises 39-47, draw a graph of the functions without using a calculator. Be sure to notice all important features of the graph: local maxima and minima, inflection points, and asymptotic behavior.

39) $y = 3x^2 + 2x + 4$

40) $y = x^3 - 3x^2 + 4$

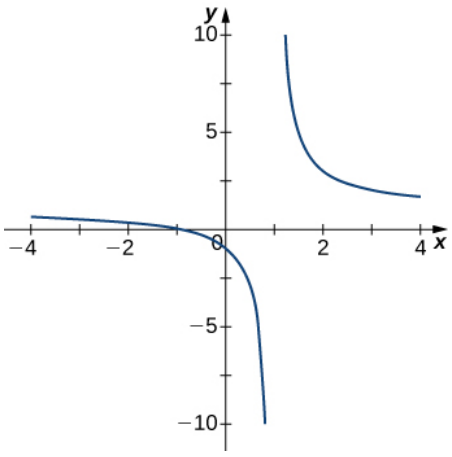
Answer



41) $y = \dfrac{2x+1}{x^2+6x+5}$

42) $y = \dfrac{2x+2}{x-1}$

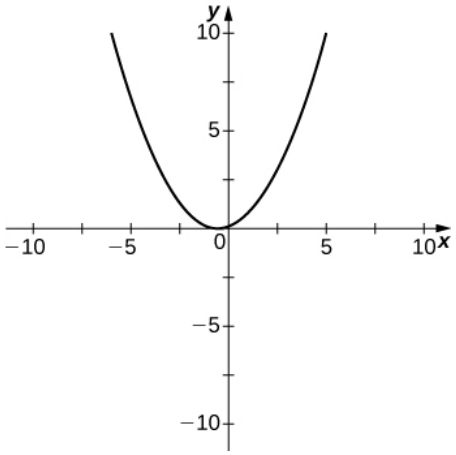
Answer



43) $y = \dfrac{x^3+4x^2+3x}{3x+9}$

Answer

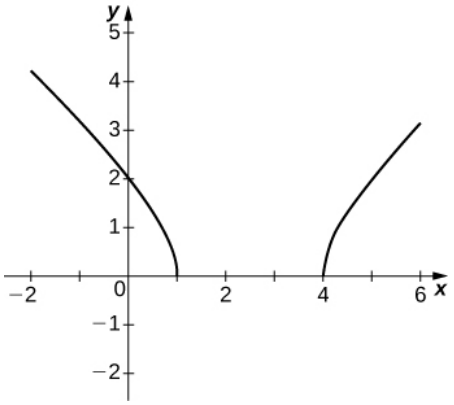
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44) $y = \dfrac{x^2 + x - 2}{x^2 - 3x - 4}$

45) $y = \sqrt{x^2 - 5x + 4}$

Answer



46) $y = e^x - x^3$

47) $y = x\ln(x), \quad x > 0$

48) For $f(x) = \dfrac{P(x)}{Q(x)}$ to have an asymptote at $y = 2$ then the polynomials $P(x)$ and $Q(x)$ must have what relation?

49) For $f(x) = \dfrac{P(x)}{Q(x)}$ to have an asymptote at $x = 0$, then the polynomials $P(x)$ and $Q(x)$. must have what relation?

Answer

$Q(x)$. must have have x^{k+1} as a factor, where $P(x)$ has x^k as a factor.

50) If $f'(x)$ has asymptotes at $y = 3$ and $x = 1$, then $f(x)$ has what asymptotes?

51) Both $f(x) = \dfrac{1}{x-1}$ and $g(x) = \dfrac{1}{(x-1)^2}$ have asymptotes at $x = 1$ and $y = 0$. What is the most obvious difference between these two functions?

Answer

$\lim\limits_{x \to 1^-} f(x) = -\infty$ and $\lim\limits_{x \to 1^-} g(x) = \infty$

52) True or false: Every ratio of polynomials has vertical asymptotes.

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3.10: Applied Optimization Problems

Learning Objectives

- Set up and solve optimization problems in several applied fields.

One common application of calculus is calculating the minimum or maximum value of a function. For example, companies often want to minimize production costs or maximize revenue. In manufacturing, it is often desirable to minimize the amount of material used to package a product with a certain volume. In this section, we show how to set up these types of minimization and maximization problems and solve them by using the tools developed in this chapter.

3.10.1 Solving Optimization Problems over a Closed, Bounded Interval

The basic idea of the optimization problems that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied. For example, in Example 3.10.1, we are interested in maximizing the area of a rectangular garden. Certainly, if we keep making the side lengths of the garden larger, the area will continue to become larger. However, what if we have some restriction on how much fencing we can use for the perimeter? In this case, we cannot make the garden as large as we like. Let's look at how we can maximize the area of a rectangle subject to some constraint on the perimeter.

Example 3.10.1: Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides (Figure 3.10.1). Given 100 ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

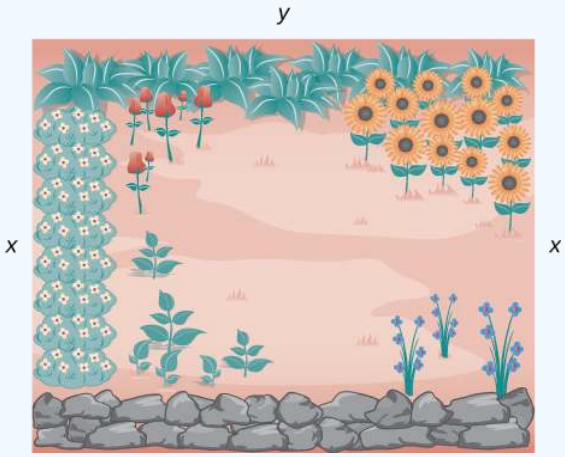


Figure 3.10.1: We want to determine the measurements x and y that will create a garden with a maximum area using 100 ft of fencing.

Solution

Let x denote the length of the side of the garden perpendicular to the rock wall and y denote the length of the side parallel to the rock wall. Then the area of the garden is

$$A = x \cdot y.$$

We want to find the maximum possible area subject to the constraint that the total fencing is 100 ft. From Figure 3.10.1, the total amount of fencing used will be $2x + y$. Therefore, the constraint equation is

$$2x + y = 100.$$

Solving this equation for y, we have $y = 100 - 2x$. Thus, we can write the area as

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2.$$

Before trying to maximize the area function $A(x) = 100x - 2x^2$, we need to determine the domain under consideration. To construct a rectangular garden, we certainly need the lengths of both sides to be positive. Therefore, we need $x > 0$ and $y > 0$. Since $y = 100 - 2x$, if $y > 0$, then $x < 50$. Therefore, we are trying to determine the maximum value of $A(x)$ for x over the open interval $(0, 50)$. We do not know that a function necessarily has a maximum value over an open interval. However, we do know that a continuous function has an absolute maximum (and absolute minimum) over a closed interval. Therefore, let's consider the function $A(x) = 100x - 2x^2$ over the closed interval $[0, 50]$. If the maximum value occurs at an interior point, then we have found the value x in the open interval $(0, 50)$ that maximizes the area of the garden.

Therefore, we consider the following problem:

Maximize $A(x) = 100x - 2x^2$ over the interval $[0, 50]$.

As mentioned earlier, since A is a continuous function on a closed, bounded interval, by the extreme value theorem, it has a maximum and a minimum. These extreme values occur either at endpoints or critical points. At the endpoints, $A(x) = 0$. Since the area is positive for all x in the open interval $(0, 50)$, the maximum must occur at a critical point. Differentiating the function $A(x)$, we obtain

$$A'(x) = 100 - 4x.$$

Therefore, the only critical point is $x = 25$ (Figure 3.10.2). We conclude that the maximum area must occur when $x = 25$.

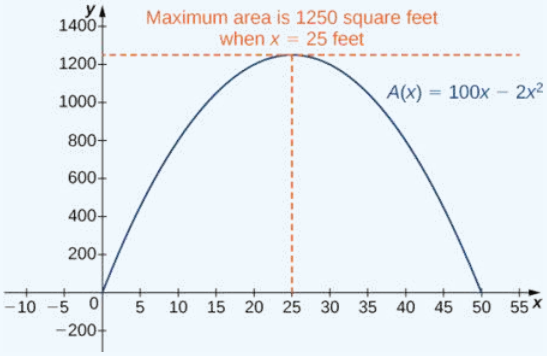


Figure 3.10.2: To maximize the area of the garden, we need to find the maximum value of the function $A(x) = 100x - 2x^2$.

Then we have $y = 100 - 2x = 100 - 2(25) = 50$. To maximize the area of the garden, let $x = 25$ ft and $y = 50$ ft. The area of this garden is 1250 ft^2.

Exercise 3.10.1

Determine the maximum area if we want to make the same rectangular garden as in Figure 3.10.2, but we have 200 ft of fencing.

Hint

We need to maximize the function $A(x) = 200x - 2x^2$ over the interval $[0, 100]$.

Answer

The maximum area is 5000 ft^2.

Now let's look at a general strategy for solving optimization problems similar to Example 3.10.1.

Problem-Solving Strategy: Solving Optimization Problems

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.

4. Write any equations relating the independent variables in the formula from step 3. Use these equations to write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain of consideration for the function in step 4 based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from step 4. This step typically involves looking for critical points and evaluating a function at endpoints.

Now let's apply this strategy to maximize the volume of an open-top box given a constraint on the amount of material to be used.

Example 3.10.2: Maximizing the Volume of a Box

An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

Solution

Step 1: Let x be the side length of the square to be removed from each corner (Figure 3.10.3). Then, the remaining four flaps can be folded up to form an open-top box. Let V be the volume of the resulting box.

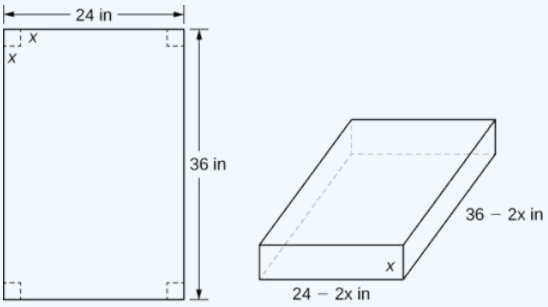


Figure 3.10.3: A square with side length x inches is removed from each corner of the piece of cardboard. The remaining flaps are folded to form an open-top box.

Step 2: We are trying to maximize the volume of a box. Therefore, the problem is to maximize V.

Step 3: As mentioned in step 2, are trying to maximize the volume of a box. The volume of a box is

$$V = L \cdot W \cdot H,$$

where L, W, and H are the length, width, and height, respectively.

Step 4: From Figure 3.10.3, we see that the height of the box is x inches, the length is $36 - 2x$ inches, and the width is $24 - 2x$ inches. Therefore, the volume of the box is

$$\begin{aligned} V(x) &= (36 - 2x)(24 - 2x)x \\ &= 4x^3 - 120x^2 + 864x \end{aligned}.$$

Step 5: To determine the domain of consideration, let's examine Figure 3.10.3. Certainly, we need $x > 0$. Furthermore, the side length of the square cannot be greater than or equal to half the length of the shorter side, 24 in.; otherwise, one of the flaps would be completely cut off. Therefore, we are trying to determine whether there is a maximum volume of the box for x over the open interval $(0, 12)$. Since V is a continuous function over the closed interval $[0, 12]$, we know V will have an absolute maximum over the closed interval. Therefore, we consider V over the closed interval $[0, 12]$ and check whether the absolute maximum occurs at an interior point.

Step 6: Since $V(x)$ is a continuous function over the closed, bounded interval $[0, 12]$, V must have an absolute maximum (and an absolute minimum). Since $V(x) = 0$ at the endpoints and $V(x) > 0$ for $0 < x < 12$, the maximum must occur at a critical point. The derivative is

$$V'(x) = 12x^2 - 240x + 864.$$

To find the critical points, we need to solve the equation

$$12x^2 - 240x + 864 = 0.$$

Dividing both sides of this equation by 12, the problem simplifies to solving the equation

$$x^2 - 20x + 72 = 0.$$

Using the quadratic formula, we find that the critical points are

$$\begin{aligned} x &= \frac{20 \pm \sqrt{(-20)^2 - 4(1)(72)}}{2} \\ &= \frac{20 \pm \sqrt{112}}{2} \\ &= \frac{20 \pm 4\sqrt{7}}{2} \\ &= 10 \pm 2\sqrt{7} \end{aligned} .$$

Since $10 + 2\sqrt{7}$ is not in the domain of consideration, the only critical point we need to consider is $10 - 2\sqrt{7}$. Therefore, the volume is maximized if we let $x = 10 - 2\sqrt{7}$ in. The maximum volume is

$$V(10 - 2\sqrt{7}) = 640 + 448\sqrt{7} \approx 1825 \text{ in}^3.$$

as shown in the following graph.

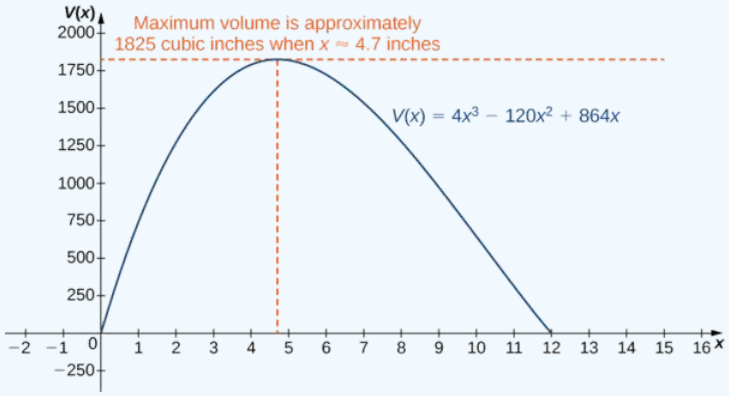


Figure 3.10.4: Maximizing the volume of the box leads to finding the maximum value of a cubic polynomial.

Exercise 3.10.2

Suppose the dimensions of the cardboard in Example 3.10.2 are 20 in. by 30 in. Let x be the side length of each square and write the volume of the open-top box as a function of x. Determine the domain of consideration for x.

Hint

The volume of the box is $L \cdot W \cdot H$.

Answer

$V(x) = x(20 - 2x)(30 - 2x)$. The domain is $[0, 10]$.

Example 3.10.3: Minimizing Travel Time

An island is 2 mi due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 mi west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 mph and swims at a rate of 3 mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?

Solution

Step 1: Let x be the distance running and let y be the distance swimming (Figure 3.10.5). Let T be the time it takes to get from the cabin to the island.

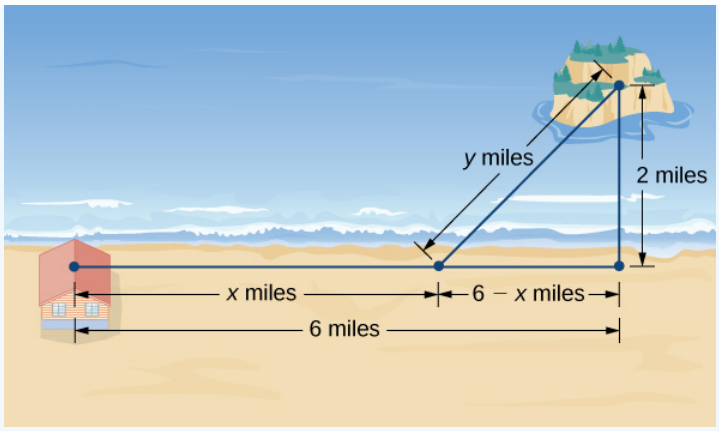


Figure 3.10.5: How can we choose x and y to minimize the travel time from the cabin to the island?

Step 2: The problem is to minimize T.

Step 3: To find the time spent traveling from the cabin to the island, add the time spent running and the time spent swimming. Since Distance = Rate × Time $(D = R \times T)$, the time spent running is

$$T_{running} = \frac{D_{running}}{R_{running}} = \frac{x}{8},$$

and the time spent swimming is

$$T_{swimming} = \frac{D_{swimming}}{R_{swimming}} = \frac{y}{3}.$$

Therefore, the total time spent traveling is

$$T = \frac{x}{8} + \frac{y}{3}.$$

Step 4: From Figure 3.10.5, the line segment of y miles forms the hypotenuse of a right triangle with legs of length 2 mi and $6 - x$ mi. Therefore, by the Pythagorean theorem, $2^2 + (6-x)^2 = y^2$, and we obtain $y = \sqrt{(6-x)^2 + 4}$. Thus, the total time spent traveling is given by the function

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6-x)^2 + 4}}{3}.$$

Step 5: From Figure 3.10.5, we see that $0 \le x \le 6$. Therefore, $[0, 6]$ is the domain of consideration.

Step 6: Since $T(x)$ is a continuous function over a closed, bounded interval, it has a maximum and a minimum. Let's begin by looking for any critical points of T over the interval $[0, 6]$. The derivative is

$$\begin{aligned} T'(x) &= \frac{1}{8} - \frac{1}{2}\frac{[(6-x)^2 + 4]^{-1/2}}{3} \cdot 2(6-x) \\ &= \frac{1}{8} - \frac{(6-x)}{3\sqrt{(6-x)^2 + 4}} \end{aligned}$$

If $T'(x) = 0$,, then

$$\frac{1}{8} = \frac{6-x}{3\sqrt{(6-x)^2 + 4}} \tag{3.10.1}$$

Therefore,

$$3\sqrt{(6-x)^2 + 4} = 8(6-x). \tag{3.10.2}$$

Squaring both sides of this equation, we see that if x satisfies this equation, then x must satisfy

$$9[(6-x)^2 + 4] = 64(6-x)^2,$$

which implies

$$55(6-x)^2 = 36.$$

We conclude that if x is a critical point, then x satisfies

$$(x-6)^2 = \frac{36}{55}.$$

[Note that since we are squaring, $(x-6)^2 = (6-x)^2$.]

Therefore, the possibilities for critical points are

$$x = 6 \pm \frac{6}{\sqrt{55}}.$$

Since $x = 6 + 6/\sqrt{55}$ is not in the domain, it is not a possibility for a critical point. On the other hand, $x = 6 - 6/\sqrt{55}$ is in the domain. Since we squared both sides of Equation 3.10.2 to arrive at the possible critical points, it remains to verify that $x = 6 - 6/\sqrt{55}$ satisfies Equation 3.10.1. Since $x = 6 - 6/\sqrt{55}$ does satisfy that equation, we conclude that $x = 6 - 6/\sqrt{55}$ is a critical point, and it is the only one. To justify that the time is minimized for this value of x, we just need to check the values of $T(x)$ at the endpoints $x = 0$ and $x = 6$, and compare them with the value of $T(x)$ at the critical point $x = 6 - 6/\sqrt{55}$. We find that $T(0) \approx 2.108$ h and $T(6) \approx 1.417$ h, whereas

$$T(6 - 6/\sqrt{55}) \approx 1.368 \text{ h}.$$

Therefore, we conclude that T has a local minimum at $x \approx 5.19$ mi.

? Exercise 3.10.3

Suppose the island is 1 mi from shore, and the distance from the cabin to the point on the shore closest to the island is 15 mi. Suppose a visitor swims at the rate of 2.5 mph and runs at a rate of 6 mph. Let x denote the distance the visitor will run before swimming, and find a function for the time it takes the visitor to get from the cabin to the island.

Hint

The time $T = T_{running} + T_{swimming}$.

Answer

$$T(x) = \frac{x}{6} + \frac{\sqrt{(15-x)^2 + 1}}{2.5}$$

In business, companies are interested in **maximizing revenue.** In the following example, we consider a scenario in which a company has collected data on how many cars it is able to lease, depending on the price it charges its customers to rent a car. Let's use these data to determine the price the company should charge to maximize the amount of money it brings in.

✓ Example 3.10.4: Maximizing Revenue

Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \le p \le 200$, the number of cars n they rent per day can be modeled by the linear function $n(p) = 1000 - 5p$. If they charge $\$50$ per day or less, they will rent all their cars. If they charge $\$200$ per day or more, they will not rent any cars. Assuming the owners plan to charge customers between $\$50$ per day and $\$200$ per day to rent a car, how much should they charge to maximize their revenue?

Solution

Step 1: Let p be the price charged per car per day and let n be the number of cars rented per day. Let R be the revenue per day.

Step 2: The problem is to maximize R.

Step 3: The revenue (per day) is equal to the number of cars rented per day times the price charged per car per day—that is, $R = n \times p$.

Step 4: Since the number of cars rented per day is modeled by the linear function $n(p) = 1000 - 5p$, the revenue R can be represented by the function

$$\begin{aligned} R(p) &= n \times p \\ &= (1000 - 5p)p \\ &= -5p^2 + 1000p. \end{aligned}$$

Step 5: Since the owners plan to charge between $\$50$ per car per day and $\$200$ per car per day, the problem is to find the maximum revenue $R(p)$ for p in the closed interval $[50, 200]$.

Step 6: Since R is a continuous function over the closed, bounded interval $[50, 200]$, it has an absolute maximum (and an absolute minimum) in that interval. To find the maximum value, look for critical points. The derivative is $R'(p) = -10p + 1000$. Therefore, the critical point is $p = 100$. When $p = 100, R(100) = \$50,000$. When $p = 50, R(p) = \$37,500$. When $p = 200, R(p) = \$0$.

Therefore, the absolute maximum occurs at $p = \$100$. The car rental company should charge $\$100$ per day per car to maximize revenue as shown in the following figure.

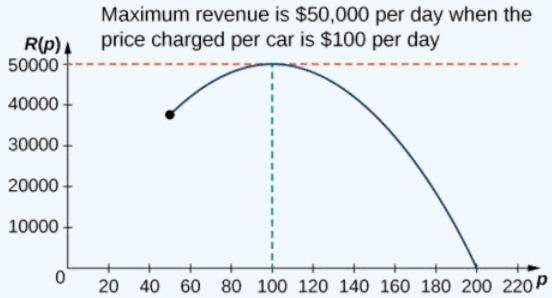


Figure 3.10.6: To maximize revenue, a car rental company has to balance the price of a rental against the number of cars people will rent at that price.

Exercise 3.10.4

A car rental company charges its customers p dollars per day, where $60 \leq p \leq 150$. It has found that the number of cars rented per day can be modeled by the linear function $n(p) = 750 - 5p$. How much should the company charge each customer to maximize revenue?

Hint

$R(p) = n \times p$, where n is the number of cars rented and p is the price charged per car.

Answer

The company should charge $\$75$ per car per day.

Example 3.10.5: Maximizing the Area of an Inscribed Rectangle

A rectangle is to be inscribed in the ellipse

$$\frac{x^2}{4} + y^2 = 1.$$

What should the dimensions of the rectangle be to maximize its area? What is the maximum area?

Solution

Step 1: For a rectangle to be inscribed in the ellipse, the sides of the rectangle must be parallel to the axes. Let L be the length of the rectangle and W be its width. Let A be the area of the rectangle.

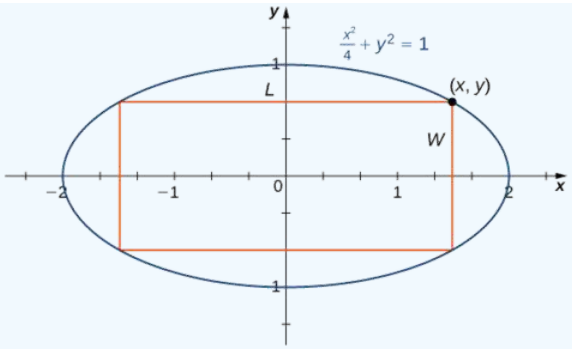


Figure 3.10.7: We want to maximize the area of a rectangle inscribed in an ellipse.

Step 2: The problem is to maximize A.

Step 3: The area of the rectangle is $A = LW$.

Step 4: Let (x, y) be the corner of the rectangle that lies in the first quadrant, as shown in Figure 3.10.7. We can write length $L = 2x$ and width $W = 2y$. Since $\frac{x^2}{4} + y^2 = 1$ and $y > 0$, we have $y = \sqrt{1 - \frac{x^2}{4}}$. Therefore, the area is

$$A = LW = (2x)(2y) = 4x\sqrt{1 - \frac{x^2}{4}} = 2x\sqrt{4 - x^2}$$

Step 5: From Figure 3.10.7, we see that to inscribe a rectangle in the ellipse, the x-coordinate of the corner in the first quadrant must satisfy $0 < x < 2$. Therefore, the problem reduces to looking for the maximum value of $A(x)$ over the open interval $(0, 2)$. Since $A(x)$ will have an absolute maximum (and absolute minimum) over the closed interval $[0, 2]$, we consider $A(x) = 2x\sqrt{4 - x^2}$ over the interval $[0, 2]$. If the absolute maximum occurs at an interior point, then we have found an absolute maximum in the open interval.

Step 6: As mentioned earlier, $A(x)$ is a continuous function over the closed, bounded interval $[0, 2]$. Therefore, it has an absolute maximum (and absolute minimum). At the endpoints $x = 0$ and $x = 2$, $A(x) = 0$. For $0 < x < 2$, $A(x) > 0$.

Therefore, the maximum must occur at a critical point. Taking the derivative of $A(x)$, we obtain

$$\begin{aligned} A'(x) &= 2\sqrt{4 - x^2} + 2x \cdot \frac{1}{2\sqrt{4 - x^2}}(-2x) \\ &= 2\sqrt{4 - x^2} - \frac{2x^2}{\sqrt{4 - x^2}} \\ &= \frac{8 - 4x^2}{\sqrt{4 - x^2}}. \end{aligned}$$

To find critical points, we need to find where $A'(x) = 0$. We can see that if x is a solution of

$$\frac{8 - 4x^2}{\sqrt{4 - x^2}} = 0, \tag{3.10.3}$$

then x must satisfy

$$8 - 4x^2 = 0.$$

Therefore, $x^2 = 2$. Thus, $x = \pm\sqrt{2}$ are the possible solutions of Equation 3.10.3. Since we are considering x over the interval $[0, 2]$, $x = \sqrt{2}$ is a possibility for a critical point, but $x = -\sqrt{2}$ is not. Therefore, we check whether $\sqrt{2}$ is a solution of Equation 3.10.3. Since $x = \sqrt{2}$ is a solution of Equation 3.10.3, we conclude that $\sqrt{2}$ is the only critical point of $A(x)$ in the interval $[0, 2]$.

Therefore, $A(x)$ must have an absolute maximum at the critical point $x = \sqrt{2}$. To determine the dimensions of the rectangle, we need to find the length L and the width W. If $x = \sqrt{2}$ then

$$y = \sqrt{1 - \frac{(\sqrt{2})^2}{4}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Therefore, the dimensions of the rectangle are $L = 2x = 2\sqrt{2}$ and $W = 2y = \dfrac{2}{\sqrt{2}} = \sqrt{2}$. The area of this rectangle is $A = LW = (2\sqrt{2})(\sqrt{2}) = 4$.

? Exercise 3.10.5

Modify the area function A if the rectangle is to be inscribed in the unit circle $x^2 + y^2 = 1$. What is the domain of consideration?

Hint

If (x, y) is the vertex of the square that lies in the first quadrant, then the area of the square is $A = (2x)(2y) = 4xy$.

Answer

$A(x) = 4x\sqrt{1 - x^2}$. The domain of consideration is $[0, 1]$.

3.10.2 Solving Optimization Problems when the Interval Is Not Closed or Is Unbounded

In the previous examples, we considered functions on closed, bounded domains. Consequently, by the extreme value theorem, we were guaranteed that the functions had absolute extrema. Let's now consider functions for which the domain is neither closed nor bounded.

Many functions still have at least one absolute extrema, even if the domain is not closed or the domain is unbounded. For example, the function $f(x) = x^2 + 4$ over $(-\infty, \infty)$ has an absolute minimum of 4 at $x = 0$. Therefore, we can still consider functions over unbounded domains or open intervals and determine whether they have any absolute extrema. In the next example, we try to minimize a function over an unbounded domain. We will see that, although the domain of consideration is $(0, \infty)$, the function has an absolute minimum.

In the following example, we look at constructing a box of least surface area with a prescribed volume. It is not difficult to show that for a closed-top box, by symmetry, among all boxes with a specified volume, a cube will have the smallest surface area. Consequently, we consider the modified problem of determining which open-topped box with a specified volume has the smallest surface area.

✓ Example 3.10.6: Minimizing Surface Area

A rectangular box with a square base, an open top, and a volume of $216\ \text{in}^3$ is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

Solution

Step 1: Draw a rectangular box and introduce the variable x to represent the length of each side of the square base; let y represent the height of the box. Let S denote the surface area of the open-top box.

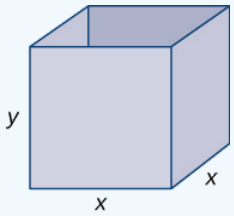


Figure 3.10.8: We want to minimize the surface area of a square-based box with a given volume.

Step 2: We need to minimize the surface area. Therefore, we need to minimize S.

Step 3: Since the box has an open top, we need only determine the area of the four vertical sides and the base. The area of each of the four vertical sides is $x \cdot y$. The area of the base is x^2. Therefore, the surface area of the box is

$$S = 4xy + x^2.$$

Step 4: Since the volume of this box is x^2y and the volume is given as $216\,\text{in}^3$, the constraint equation is

$$x^2y = 216.$$

Solving the constraint equation for y, we have $y = \dfrac{216}{x^2}$. Therefore, we can write the surface area as a function of x only:

$$S(x) = 4x\left(\frac{216}{x^2}\right) + x^2.$$

Therefore, $S(x) = \dfrac{864}{x} + x^2$.

Step 5: Since we are requiring that $x^2y = 216$, we cannot have $x = 0$. Therefore, we need $x > 0$. On the other hand, x is allowed to have any positive value. Note that as x becomes large, the height of the box y becomes correspondingly small so that $x^2y = 216$. Similarly, as x becomes small, the height of the box becomes correspondingly large. We conclude that the domain is the open, unbounded interval $(0, \infty)$. Note that, unlike the previous examples, we cannot reduce our problem to looking for an absolute maximum or absolute minimum over a closed, bounded interval. However, in the next step, we discover why this function must have an absolute minimum over the interval $(0, \infty)$.

Step 6: Note that as $x \to 0^+$, $S(x) \to \infty$. Also, as $x \to \infty$, $S(x) \to \infty$. Since S is a continuous function that approaches infinity at the ends, it must have an absolute minimum at some $x \in (0, \infty)$. This minimum must occur at a critical point of S. The derivative is

$$S'(x) = -\frac{864}{x^2} + 2x.$$

Therefore, $S'(x) = 0$ when $2x = \dfrac{864}{x^2}$. Solving this equation for x, we obtain $x^3 = 432$, so $x = \sqrt[3]{432} = 6\sqrt[3]{2}$. Since this is the only critical point of S, the absolute minimum must occur at $x = 6\sqrt[3]{2}$ (see Figure 3.10.9).

When $x = 6\sqrt[3]{2}$, $y = \dfrac{216}{(6\sqrt[3]{2})^2} = 3\sqrt[3]{2}\,\text{in}$. Therefore, the dimensions of the box should be $x = 6\sqrt[3]{2}\,\text{in}$. and $y = 3\sqrt[3]{2}\,\text{in}$. With these dimensions, the surface area is

$$S(6\sqrt[3]{2}) = \frac{864}{6\sqrt[3]{2}} + (6\sqrt[3]{2})^2 = 108\sqrt[3]{4}\,\text{in}^2$$

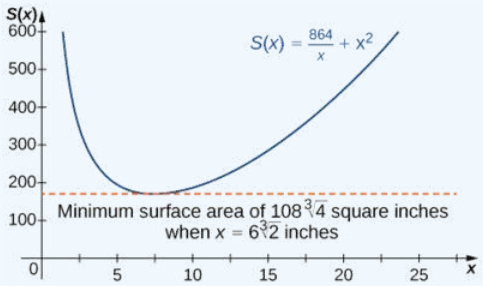


Figure 3.10.9: We can use a graph to determine the dimensions of a box of given the volume and the minimum surface area.

Exercise 3.10.6

Consider the same open-top box, which is to have volume $216\,\text{in}^3$. Suppose the cost of the material for the base is $20¢/\text{in}^2$ and the cost of the material for the sides is $30¢/\text{in}^2$ and we are trying to minimize the cost of this box. Write the cost as a function of the side lengths of the base. (Let x be the side length of the base and y be the height of the box.)

Hint

If the cost of one of the sides is $30¢/\text{in}^2$,the cost of that side is $0.30xy$dollars.

Answer

$$c(x) = \frac{259.2}{x} + 0.2x^2 \text{ dollars}$$

3.10.3 Key Concepts

- To solve an optimization problem, begin by drawing a picture and introducing variables.
- Find an equation relating the variables.
- Find a function of one variable to describe the quantity that is to be minimized or maximized.
- Look for critical points to locate local extrema.

3.10.4 Glossary

optimization problems

problems that are solved by finding the maximum or minimum value of a function

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3.10E: EXERCISES FOR SECTION 3.10

For exercises 1 - 4, answer by proof, counterexample, or explanation.

1) When you find the maximum for an optimization problem, why do you need to check the sign of the derivative around the critical points?

Answer

The critical points can be the minima, maxima, or neither.

2) Why do you need to check the endpoints for optimization problems?

3) True or False. For every continuous nonlinear function, you can find the value x that maximizes the function.

Answer

False; $y = -x^2$ has a minimum only

4) True or False. For every continuous non-constant function on a closed, finite domain, there exists at least one x that minimizes or maximizes the function.

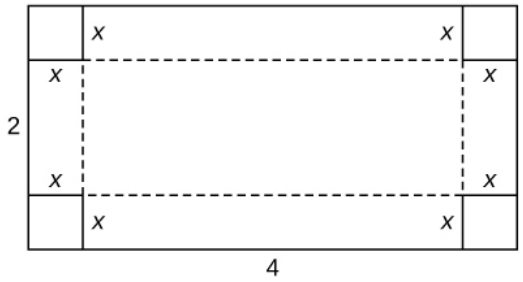
In exercises 5 - 8, set up and evaluate each optimization problem.

5) To carry a suitcase on an airplane, the $\text{length} + \text{width} + \text{height}$ of the box must be less than or equal to 62 in. Assuming the height is fixed, show that the maximum volume is $V = h\left(31 - \frac{1}{2}h\right)^2$. What height allows you to have the largest volume?

Answer

$h = \frac{62}{3}$ in.

6) You are constructing a cardboard box with the dimensions 2 m by 4 m. You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?



7) Find the positive integer that minimizes the sum of the number and its reciprocal.

Answer

1

8) Find two positive integers such that their sum is 10, and minimize and maximize the sum of their squares.

In exercises 9 - 11, consider the construction of a pen to enclose an area.

9) You have $400\,\text{ft}$ of fencing to construct a rectangular pen for cattle. What are the dimensions of the pen that maximize the area?

Answer

$100\,\text{ft}$ by $100\,\text{ft}$

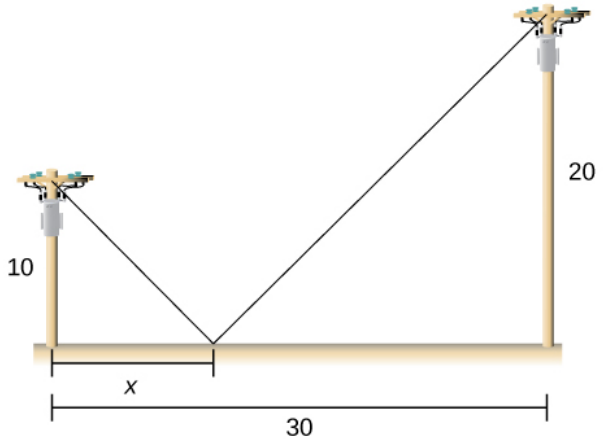
10) You have $800\,\text{ft}$ of fencing to make a pen for hogs. If you have a river on one side of your property, what is the dimension of the rectangular pen that maximizes the area?

11) You need to construct a fence around an area of $1600\,\text{ft}^2$. What are the dimensions of the rectangular pen to minimize the amount of material needed?

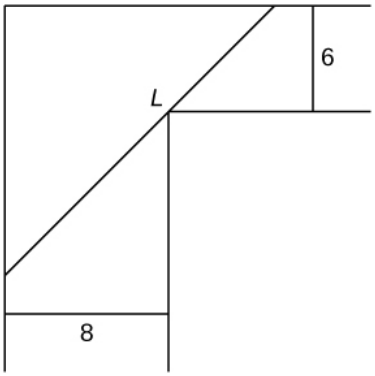
Answer

$40\,\text{ft}$ by $40\,\text{ft}$

12) Two poles are connected by a wire that is also connected to the ground. The first pole is $20\,\text{ft}$ tall and the second pole is $10\,\text{ft}$ tall. There is a distance of $30\,\text{ft}$ between the two poles. Where should the wire be anchored to the ground to minimize the amount of wire needed?



13) [T] You are moving into a new apartment and notice there is a corner where the hallway narrows from $8\,\text{ft}$ to $6\,\text{ft}$. What is the length of the longest item that can be carried horizontally around the corner?



Answer

$19.73\,\text{ft}$

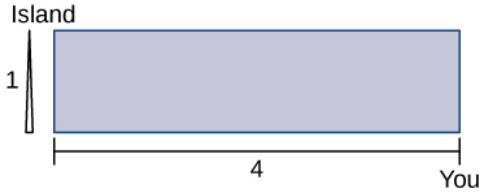
14) A patient's pulse measures $70\,\text{bpm}$, $80\,\text{bpm}$, then $120\,\text{bpm}$. To determine an accurate measurement of pulse, the doctor wants to know what value minimizes the expression $(x-70)^2+(x-80)^2+(x-120)^2$? What value minimizes it?

15) In the previous problem, assume the patient was nervous during the third measurement, so we only weight that value half as much as the others. What is the value that minimizes $(x-70)^2+(x-80)^2+\frac{1}{2}(x-120)^2$?

Answer

$84\,\text{bpm}$

16) You can run at a speed of 6 mph and swim at a speed of 3 mph and are located on the shore, 4 miles east of an island that is 1 mile north of the shoreline. How far should you run west to minimize the time needed to reach the island?



For exercises 17-18, consider a limousine that gets $m(v)=\frac{120-2v}{5}\,\text{mi/gal}$ at speed v, the chauffeur costs $\$15/\text{h}$, and gas is $\$3.50/\text{gal}$.

17) Find the cost per mile at speed v.

18) Find the cheapest driving speed.

Answer

approximately 34.02 mph

For exercises 19-21, consider a pizzeria that sell pizzas for a revenue of $R(x) = ax$ and costs $C(x) = b + cx + dx^2$, where x represents the number of pizzas.

19) Find the profit function for the number of pizzas. How many pizzas gives the largest profit per pizza?

20) Assume that $R(x) = 10x$ and $C(x) = 2x + x^2$. How many pizzas sold maximizes the profit?

Answer

Selling 4 pizzas will maximize the profit.

21) Assume that $R(x) = 15x$, and $C(x) = 60 + 3x + \frac{1}{2}x^2$. How many pizzas sold maximizes the profit?

For exercises 22-23, consider a wire 4 ft long cut into two pieces. One piece forms a circle with radius r and the other forms a square of side x.

22) Choose x to maximize the sum of their areas.

Answer

$x = 0$

23) Choose x to minimize the sum of their areas.

For exercises 24-27, consider two nonnegative numbers x and y such that $x + y = 10$. Maximize and minimize the quantities.

24) xy

Answer

Maximal: $x = 5$, $y = 5$;
Minimal: $x = 0$, $y = 10$ and $y = 0$, $x = 10$

25) x^2y^2

26) $y - \dfrac{1}{x}$

Answer

Maximal: $x = 1$, $y = 9$;
Minimal: none

27) $x^2 - y$

In exercises 28-33, draw the given optimization problem and solve.

28) Find the volume of the largest right circular cylinder that fits in a sphere of radius 1.

Answer

$V = \frac{4\pi}{3\sqrt{3}}$

29) Find the volume of the largest right cone that fits in a sphere of radius 1.

30) Find the area of the largest rectangle that fits into the triangle with sides $x = 0$, $y = 0$ and $\dfrac{x}{4} + \dfrac{y}{6} = 1$.

Answer

$A = 6$

31) Find the largest volume of a cylinder that fits into a cone that has base radius R and height h.

32) Find the dimensions of the closed cylinder volume $V = 16\pi$ that has the least amount of surface area.

Answer

$r = 2$, $h = 4$

33) Find the dimensions of a right cone with surface area $S = 4\pi$ that has the largest volume.

For exercises 34-37, consider the points on the graphs of the given equations. Use a calculator to graph the functions.

34) [T] Where is the line $y = 5 - 2x$ closest to the origin?

Answer

$(2, 1)$

35) [T] Where is the line $y = 5 - 2x$ closest to point $(1, 1)$?

36) [T] Where is the parabola $y = x^2$ closest to point $(2, 0)$?

Answer

$(0.8351, 0.6974)$

37) [T] Where is the parabola $y = x^2$ closest to point $(0, 3)$?

In exercises 38-42, set up, but do not evaluate, each optimization problem.

38) A window is composed of a semicircle placed on top of a rectangle. If you have $20\,\text{ft}$ of window-framing materials for the outer frame, what is the maximum size of the window you can create? Use r to represent the radius of the semicircle.



Answer

$A = 20r - 2r^2 - \frac{1}{2}\pi r^2$

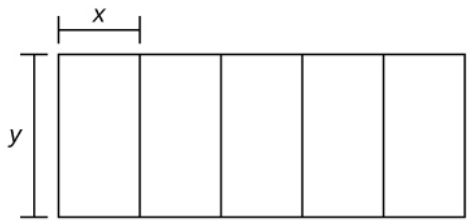
39) You have a garden row of 20 watermelon plants that produce an average of 30 watermelons apiece. For any additional watermelon plants planted, the output per watermelon plant drops by one watermelon. How many extra watermelon plants should you plant?

40) You are constructing a box for your cat to sleep in. The plush material for the square bottom of the box costs $\$5/\text{ft}^2$ and the material for the sides costs $\$2/\text{ft}^2$. You need a box with volume $4\,\text{ft}^3$. Find the dimensions of the box that minimize cost. Use x to represent the length of the side of the box.

Answer

$C(x) = 5x^2 + \dfrac{32}{x}$

41) You are building five identical pens adjacent to each other with a total area of $1000\,\text{m}^2$, as shown in the following figure. What dimensions should you use to minimize the amount of fencing?



42) You are the manager of an apartment complex with 50 units. When you set rent at $\$800/\text{month}$, all apartments are rented. As you increase rent by $\$25/\text{month}$, one fewer apartment is rented. Maintenance costs run $\$50/\text{month}$ for each occupied unit. What is the rent that maximizes the total amount of profit?

Answer

$P(x) = (50 - x)(800 + 25x - 50)$

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3.R: Chapter 3 Review Exercises

3.R.1 Chapter Review Exercises

True or False? Justify the answer with a proof or a counterexample.

1) Every function has a derivative.

Answer

False

2) A continuous function has a continuous derivative.

3) A continuous function has a derivative.

Answer

False

4) If a function is differentiable, it is continuous.

In exercises 5 and 6, use the limit definition of the derivative to exactly evaluate the derivative.

5) $f(x) = \sqrt{x+4}$

Answer

$$f'(x) = \frac{1}{2\sqrt{x+4}}$$

6) $f(x) = \frac{3}{x}$

In exercises 7 - 15, find the derivatives of the given functions.

7) $f(x) = 3x^3 - \frac{4}{x^2}$

Answer

$f'(x) = 9x^2 + \frac{8}{x^3}$

9) $f(x) = (4 - x^2)^3$

10) $f(x) = e^{-x^2+3}$

Answer

$f'(x) = -2x \cdot e^{-x^2+3}$

11) $f(x) = \ln(x+2)$

12) $f(x) = \frac{3x-7}{4x+1}$

Answer

$$f'(x) = \frac{31}{(4x+1)^2}$$

13) $f(x) = \sqrt{3x^2+2}$

14) $f(x) = \frac{8}{\sqrt{3x^2+4}}$

Answer

$$f'(x) = \frac{-24x}{(3x^2+4)^{\frac{3}{2}}}$$

15) $x^2y = (y+2) + 4x$

In exercises 16 - 18, find the indicated derivatives of various orders.

16) Second derivative of $y = \dfrac{1}{x}$

Answer

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

17) Third derivative of $y = (3x+2)^2$

18) Second derivative of $y = \dfrac{3x}{4x+1}$

Answer

$$\frac{d^2y}{dx^2} = \frac{-24}{(4x+1)^3}$$

In exercises 19 and 20, find the equation of the tangent line to the following equations at the specified point.

19) $y = x^2 - 5x + 3$ at $x = 1$

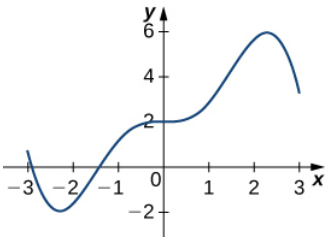
20) $y = x + e^x - \dfrac{1}{x}$ at $x = 1$

Answer

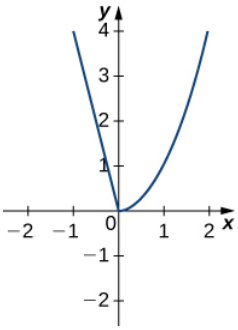
$y = (2+e)x - 2$

In exercises 21 and 22, draw the derivative of the functions with the given graphs.

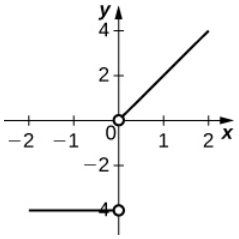
21)



22)



Answer



Questions 23 and 24 we are given the following information: The cost function for a company that manufactures widgets is given by $C(q) = 100 + \frac{2}{q} + \frac{q}{4}$, where q is the number of widgets manufactured.

23) Find the equation for the marginal cost function, $MC(q)$.

24) Find $MC(20)$. What is the meaning of this value?

Answer

$MC(20) = 0.245$. This is an approximation to the cost of the 21st widget manufactured.

Questions 25 and 26 consider the wind speeds of Hurricane Katrina, which affected New Orleans, Louisiana, in August 2005. The data are displayed in a table.

Hours after Midnight, August 26	Wind Speed (mph)
1	45
5	75
11	100
29	115
49	145
58	175
73	155
81	125
85	95
107	35

Wind Speeds of Hurricane KatrinaSource: news.nationalgeographic.com/n..._timeline.html.

25) Using the table, estimate the derivative of the wind speed at hour 39. What is the physical meaning?

26) Estimate the derivative of the wind speed at hour 83. What is the physical meaning?

Answer

-7.5. The wind speed is decreasing at a rate of 7.5 mph/hr

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CHAPTER OVERVIEW

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4.1: Antiderivatives

Learning Objectives

- Find the general antiderivative of a given function.
- Explain the terms and notation used for an indefinite integral.
- State the power rule for integrals.
- Use antidifferentiation to solve simple initial-value problems.

At this point, we have seen how to calculate derivatives of many functions and have been introduced to a variety of their applications. We now ask a question that turns this process around: Given a function f, how do we find a function with the derivative f and why would we be interested in such a function?

We answer the first part of this question by defining antiderivatives. The antiderivative of a function f is a function with a derivative f. Why are we interested in antiderivatives? The need for antiderivatives arises in many situations, and we look at various examples throughout the remainder of the text. Here we examine one specific example that involves rectilinear motion. In our examination in Derivatives of rectilinear motion, we showed that given a position function $s(t)$ of an object, then its velocity function $v(t)$ is the derivative of $s(t)$—that is, $v(t) = s'(t)$. Furthermore, the acceleration $a(t)$ is the derivative of the velocity $v(t)$—that is, $a(t) = v'(t) = s''(t)$. Now suppose we are given an acceleration function a, but not the velocity function v or the position function s. Since $a(t) = v'(t)$, determining the velocity function requires us to find an antiderivative of the acceleration function. Then, since $v(t) = s'(t)$, determining the position function requires us to find an antiderivative of the velocity function. Rectilinear motion is just one case in which the need for antiderivatives arises. We will see many more examples throughout the remainder of the text. For now, let's look at the terminology and notation for antiderivatives, and determine the antiderivatives for several types of functions. We examine various techniques for finding antiderivatives of more complicated functions later in the text (Introduction to Techniques of Integration).

4.1.0.1 The Reverse of Differentiation

At this point, we know how to find derivatives of various functions. We now ask the opposite question. Given a function f, how can we find a function with derivative f? If we can find a function F with derivative f, we call F an antiderivative of f.

Definition: Antiderivative

A function F is an **antiderivative** of the function f if

$$F'(x) = f(x)$$

for all x in the domain of f.

Consider the function $f(x) = 2x$. Knowing the power rule of differentiation, we conclude that $F(x) = x^2$ is an antiderivative of f since $F'(x) = 2x$.

Are there any other antiderivatives of f?

Yes; since the derivative of any constant C is zero, $x^2 + C$ is also an antiderivative of $2x$. Therefore, $x^2 + 5$ and $x^2 - \sqrt{2}$ are also antiderivatives.

Are there any others that are not of the form $x^2 + C$ for some constant C?

The answer is no. From Corollary 2 of the Mean Value Theorem, we know that if F and G are differentiable functions such that $F'(x) = G'(x)$, then $F(x) - G(x) = C$ for some constant C. This fact leads to the following important theorem.

Theorem 4.1.1: General Form of an Antiderivative

Let F be an antiderivative of f over an interval I. Then,

I. for each constant C, the function $F(x) + C$ is also an antiderivative of f over I;
II. if G is an antiderivative of f over I, there is a constant C for which $G(x) = F(x) + C$ over I.

In other words, the most general form of the antiderivative of f over I is $F(x) + C$.

We use this fact and our knowledge of derivatives to find all the antiderivatives for several functions.

Example 4.1.1: Finding Antiderivatives

For each of the following functions, find all antiderivatives.

a. $f(x) = 3x^2$

b. $f(x) = \dfrac{1}{x}$

c. $f(x) = e^x$

Solution:

a. Because

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

then $F(x) = x^3$ is an antiderivative of $3x^2$. Therefore, every antiderivative of $3x^2$ is of the form $x^3 + C$ for some constant C, and every function of the form $x^3 + C$ is an antiderivative of $3x^2$.

b. Let $f(x) = \ln|x|$.

For $x > 0,\ f(x) = \ln|x| = \ln(x)$ and

$$\frac{d}{dx}\Big(\ln x\Big) = \frac{1}{x}.$$

For $x < 0,\ f(x) = \ln|x| = \ln(-x)$ and

$$\frac{d}{dx}\Big(\ln(-x)\Big) = -\frac{1}{-x} = \frac{1}{x}.$$

Therefore,

$$\frac{d}{dx}\Big(\ln|x|\Big) = \frac{1}{x}.$$

Thus, $F(x) = \ln|x|$ is an antiderivative of $\dfrac{1}{x}$. Therefore, every antiderivative of $\dfrac{1}{x}$ is of the form $\ln|x| + C$ for some constant C and every function of the form $\ln|x| + C$ is an antiderivative of $\dfrac{1}{x}$.

c. Since

$$\frac{d}{dx}(e^x) = e^x,$$

then $F(x) = e^x$ is an antiderivative of e^x. Therefore, every antiderivative of e^x is of the form $e^x + C$ for some constant C and every function of the form $e^x + C$ is an antiderivative of e^x.

Exercise 4.1.1

Find all antiderivatives of $f(x) = 5$.

Hint

What function has a derivative of 5?

Answer

$F(x) = 5x + C$

4.1.1 Indefinite Integrals

We now look at the formal notation used to represent antiderivatives and examine some of their properties. These properties allow us to find antiderivatives of more complicated functions. Given a function f, we use the notation $f'(x)$ or $\frac{df}{dx}$ to denote the derivative of f. Here we introduce notation for antiderivatives. If F is an antiderivative of f, we say that $F(x)+C$ is the most general antiderivative of f and write

$$\int f(x)\,dx = F(x)+C.$$

The symbol $\int$ is called an integral sign, and $\int f(x)\,dx$ is called the indefinite integral of f.

Definition: Indefinite Integrals

Given a function f, the indefinite integral of f, denoted

$$\int f(x)\,dx,$$

is the most general antiderivative of f. If F is an antiderivative of f, then

$$\int f(x)\,dx = F(x)+C.$$

The expression $f(x)$ is called the integrand and the variable x is the variable of integration.

Given the terminology introduced in this definition, the act of finding the antiderivatives of a function f is usually referred to as integrating f.

For a function f and an antiderivative F, the functions $F(x)+C$, where C is any real number, is often referred to as the family of antiderivatives of f. For example, since x^2 is an antiderivative of $2x$ and any antiderivative of $2x$ is of the form x^2+C, we write

$$\int 2x\,dx = x^2+C.$$

The collection of all functions of the form x^2+C, where C is any real number, is known as the family of antiderivatives of $2x$. Figure 4.1.1 shows a graph of this family of antiderivatives.

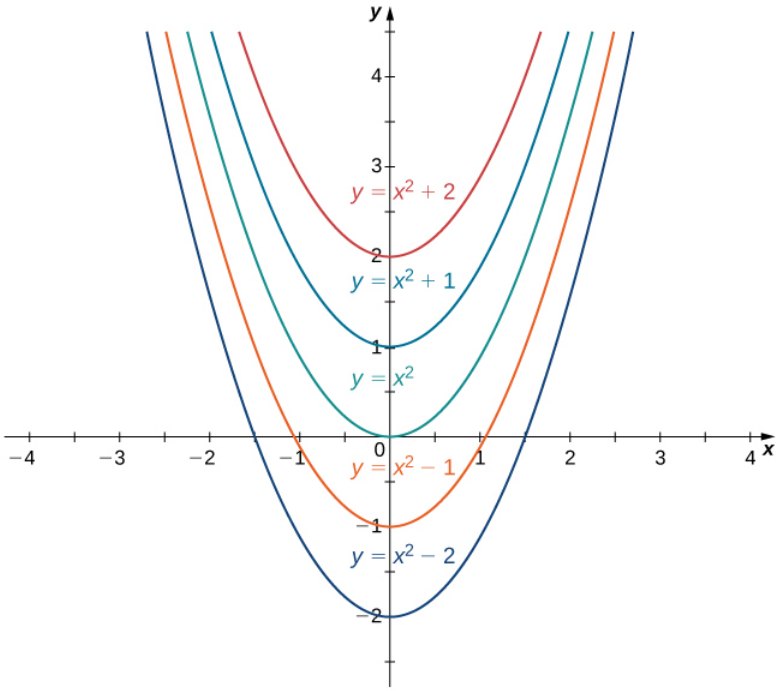


Figure 4.1.1: The family of antiderivatives of $2x$ consists of all functions of the form $x^2 + C$, where C is any real number.

For some functions, evaluating indefinite integrals follows directly from properties of derivatives. For example, for $n \neq -1$,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C,$$

which comes directly from

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = (n+1)\frac{x^n}{n+1} = x^n \ .$$

This fact is known as the power rule for integrals.

Power Rule for Integrals

For $n \neq -1$,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C.$$

Evaluating indefinite integrals for some other functions is also a straightforward calculation. The following table lists the indefinite integrals for several common functions. A more complete list appears in Appendix B.

Table 4.1.1: Integration Formulas

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}\left(k\right) = 0$	$\int k \, dx = \int kx^0 \, dx = kx + C$
$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\frac{d}{dx}\left(\ln\lvert x\rvert\right) = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}\left(e^x\right) = e^x$	$\int e^x \, dx = e^x + C$

From the definition of indefinite integral of f, we know

$$\int f(x)\,dx = F(x) + C$$

if and only if F is an antiderivative of f.

Therefore, when claiming that

$$\int f(x)\,dx = F(x) + C$$

it is important to check whether this statement is correct by verifying that $F'(x) = f(x)$.

Example 4.1.2: Verifying an Indefinite Integral

Each of the following statements is of the form $\int f(x)\,dx = F(x) + C$. Verify that each statement is correct by showing that $F'(x) = f(x)$.

a. $\int \left(x + e^x\right) dx = \dfrac{x^2}{2} + e^x + C$

b. $\int xe^x\,dx = xe^x - e^x + C$

Solution:

a. Since

$$\frac{d}{dx}\left(\frac{x^2}{2} + e^x + C\right) = x + e^x\ ,$$

the statement

$$\int \left(x + e^x\right) dx = \frac{x^2}{2} + e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a sum. Furthermore, $\dfrac{x^2}{2}$ and e^x are antiderivatives of x and e^x, respectively, and the sum of the antiderivatives is an antiderivative of the sum. We discuss this fact again later in this section.

b. Using the product rule, we see that

$$\frac{d}{dx}(xe^x - e^x + C) = e^x + xe^x - e^x = xe^x.$$

Therefore, the statement

$$\int xe^x\,dx = xe^x - e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a product. The antiderivative $xe^x - e^x$ is not a product of the antiderivatives. Furthermore, the product of antiderivatives, $x^2e^x/2$ is not an antiderivative of xe^x since

$$\frac{d}{dx}\left(\frac{x^2e^x}{2}\right) = xe^x + \frac{x^2e^x}{2} \neq xe^x\ .$$

In general, the product of antiderivatives is not an antiderivative of a product.

Exercise 4.1.2

Verify that $\int x \ln x\ dx = \dfrac{x^2}{2}\ln x - \dfrac{x^2}{4} + C$.

Hint

Calculate $\frac{d}{dx}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4} + C\right)$

Answer

$\frac{d}{dx}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4} + C\right) = \frac{x^2}{2}\cdot\frac{1}{x} + \ln x \cdot x - \frac{x}{2} = x\ln x$

In Table 4.1.1, we listed the indefinite integrals for many elementary functions. Let's now turn our attention to evaluating indefinite integrals for more complicated functions. For example, consider finding an antiderivative of a sum $f+g$. In Example 4.1.2a we showed that an antiderivative of the sum $x+e^x$ is given by the sum $\frac{x^2}{2}+e^x$ —that is, an antiderivative of a sum is given by a sum of antiderivatives. This result was not specific to this example. In general, if F and G are antiderivatives of any functions f and g, respectively, then

$$\frac{d}{dx}\big(F(x)+G(x)\big) = F'(x)+G'(x) = f(x)+g(x).$$

Therefore, $F(x)+G(x)$ is an antiderivative of $f(x)+g(x)$ and we have

$$\int \big(f(x)+g(x)\big)\,dx = F(x)+G(x)+C.$$

Similarly,

$$\int \big(f(x)-g(x)\big)\,dx = F(x)-G(x)+C.$$

In addition, consider the task of finding an antiderivative of $kf(x)$, where k is any real number. Since

$$\frac{d}{dx}\Big(kF(x)\Big) = k\frac{d}{dx}\Big(F(x)\Big) = kF'(x)$$

for any real number k, we conclude that

$$\int kf(x)\,dx = kF(x)+C.$$

These properties are summarized next.

Properties of Indefinite Integrals

Let F and G be antiderivatives of f and g, respectively, and let k be any real number.

Sums and Differences

$$\int \big(f(x)\pm g(x)\big)\,dx = F(x)\pm G(x)+C$$

Constant Multiples

$$\int kf(x)\,dx = kF(x)+C$$

From this theorem, we can evaluate any integral involving a sum, difference, or constant multiple of functions with antiderivatives that are known. Evaluating integrals involving products, quotients, or compositions is more complicated. (See Example 4.1.2b for an example involving an antiderivative of a product.) We look at and address integrals involving these more complicated functions in Introduction to Integration. In the next example, we examine how to use this theorem to calculate the indefinite integrals of several functions.

Example 4.1.3: Evaluating Indefinite Integrals

Evaluate each of the following indefinite integrals:

a. $\int \left(5x^3 - 7x^2 + 3x + 4\right) dx$

b. $\int \dfrac{x^2 + 4\sqrt[3]{x}}{x}\, dx$

Solution:

a. Using Properties of Indefinite Integrals, we can integrate each of the four terms in the integrand separately. We obtain

$$\int \left(5x^3 - 7x^2 + 3x + 4\right) dx = \int 5x^3\, dx - \int 7x^2\, dx + \int 3x\, dx + \int 4\, dx.$$

From the second part of Properties of Indefinite Integrals, each coefficient can be written in front of the integral sign, which gives

$$\int 5x^3\, dx - \int 7x^2\, dx + \int 3x\, dx + \int 4\, dx = 5\int x^3\, dx - 7\int x^2\, dx + 3\int x\, dx + 4\int 1\, dx.$$

Using the power rule for integrals, we conclude that

$$\int \left(5x^3 - 7x^2 + 3x + 4\right) dx = \frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C.$$

b. Rewrite the integrand as

$$\frac{x^2 + 4\sqrt[3]{x}}{x} = \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x}.$$

Then, to evaluate the integral, integrate each of these terms separately. Using the power rule, we have

$$\begin{aligned}\int \left(x + \frac{4}{x^{2/3}}\right) dx &= \int x\, dx + 4\int x^{-2/3}\, dx \\ &= \frac{1}{2}x^2 + 4\frac{1}{\left(\frac{-2}{3}\right)+1}x^{(-2/3)+1} + C \\ &= \frac{1}{2}x^2 + 12x^{1/3} + C.\end{aligned}$$

Exercise 4.1.3

Evaluate $\int \left(4x^3 - 5x^2 + x - 7\right) dx$.

Hint

Integrate each term in the integrand separately, making use of the power rule.

Answer

$$\int \left(4x^3 - 5x^2 + x - 7\right) dx = \quad x^4 - \frac{5}{3}x^3 + \frac{1}{2}x^2 - 7x + C$$

4.1.2 Initial-Value Problems

We look at techniques for integrating a large variety of functions involving products, quotients, and compositions later in the text. Here we turn to one common use for antiderivatives that arises often in many applications: solving differential equations.

A differential equation is an equation that relates an unknown function and one or more of its derivatives. The equation

$$\frac{dy}{dx} = f(x) \tag{4.1.1}$$

is a simple example of a differential equation. Solving this equation means finding a function y with a derivative f. Therefore, the solutions of Equation 4.1.1 are the antiderivatives of f. If F is one antiderivative of f, every function of the form $y = F(x) + C$ is a solution of that differential equation. For example, the solutions of

$$\frac{dy}{dx} = 6x^2$$

are given by

$$y = \int 6x^2\, dx = 2x^3 + C.$$

Sometimes we are interested in determining whether a particular solution curve passes through a certain point (x_0, y_0) —that is, $y(x_0) = y_0$. The problem of finding a function y that satisfies a differential equation

$$\frac{dy}{dx} = f(x)$$

with the additional condition

$$y(x_0) = y_0$$

is an example of an initial-value problem. The condition $y(x_0) = y_0$ is known as an initial condition. For example, looking for a function y that satisfies the differential equation

$$\frac{dy}{dx} = 6x^2$$

and the initial condition

$$y(1) = 5$$

is an example of an initial-value problem. Since the solutions of the differential equation are $y = 2x^3 + C,$ to find a function y that also satisfies the initial condition, we need to find C such that $y(1) = 2(1)^3 + C = 5$. From this equation, we see that $C = 3$, and we conclude that $y = 2x^3 + 3$ is the solution of this initial-value problem as shown in the following graph.

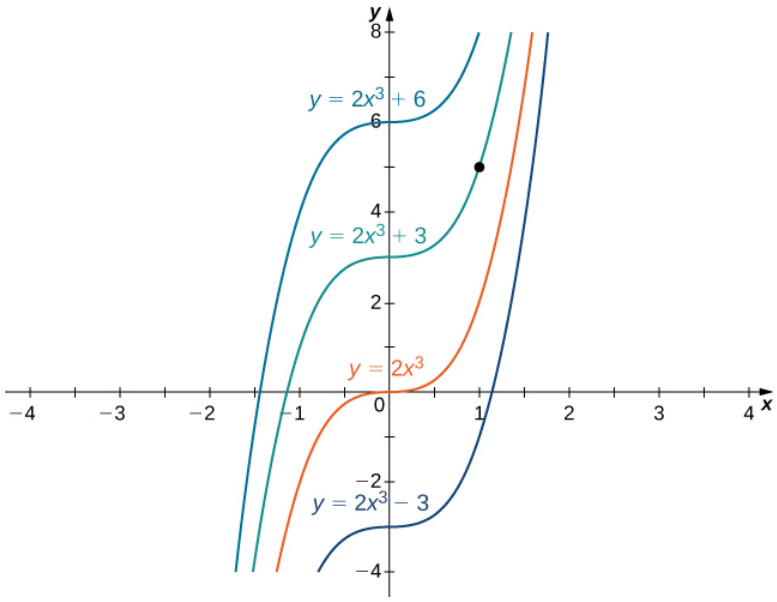


Figure 4.1.2: Some of the solution curves of the differential equation $\frac{dy}{dx} = 6x^2$ are displayed. The function $y = 2x^3 + 3$ satisfies the differential equation and the initial condition $y(1) = 5$.

Example 4.1.4: Solving an Initial-Value Problem

Solve the initial-value problem

$$\frac{dy}{dx} = \sqrt{x}, \quad y(4) = 5.$$

Solution

First we need to solve the differential equation. If $\frac{dy}{dx} = \sqrt{x}$, then

$$y = \int \sqrt{x}\, dx = \int x^{\frac{1}{2}}\, dx = \frac{2}{3}x^{\frac{3}{2}} + C.$$

Next we need to look for a solution y that satisfies the initial condition. The initial condition $y(4) = 5$ means we need a constant C such that $\frac{2}{3}(4)^{\frac{3}{2}} + C = 5$. Therefore,

$$C = 5 - \frac{2}{3}(4)^{\frac{3}{2}} = -\frac{1}{3}.$$

The solution of the initial-value problem is $y = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}$.

Exercise 4.1.4

Solve the initial value problem $\frac{dy}{dx} = 3x^{-2}, \quad y(1) = 2$.

Hint

Find all antiderivatives of $f(x) = 3x^{-2}$.

Answer

$$y = -\frac{3}{x} + 5$$

Initial-value problems arise in many applications. Next we consider a problem in which a driver applies the brakes in a car. We are interested in how long it takes for the car to stop. Recall that the velocity function $v(t)$ is the derivative of a position function $s(t)$, and the acceleration $a(t)$ is the derivative of the velocity function. In earlier examples in the text, we could calculate the velocity from the position and then compute the acceleration from the velocity. In the next example we work the other way around. Given an acceleration function, we calculate the velocity function. We then use the velocity function to determine the position function.

Example 4.1.5:

A car is traveling at the rate of 88 ft/sec (60 mph) when the brakes are applied. The car begins decelerating at a constant rate of 15 ft/sec^2.

a. How many seconds elapse before the car stops?
b. How far does the car travel during that time?

Solution

a. First we introduce variables for this problem. Let t be the time (in seconds) after the brakes are first applied. Let $a(t)$ be the acceleration of the car (in feet per seconds squared) at time t. Let $v(t)$ be the velocity of the car (in feet per second) at time t. Let $s(t)$ be the car's position (in feet) beyond the point where the brakes are applied at time t.

The car is traveling at a rate of 88 ft/sec. Therefore, the initial velocity is $v(0) = 88$ ft/sec. Since the car is decelerating, the acceleration is

$$a(t) = -15 \text{ ft/sec}^2.$$

The acceleration is the derivative of the velocity,

$$v'(t) = -15.$$

Therefore, we have an initial-value problem to solve:

$$v'(t) = -15, \quad v(0) = 88.$$

Integrating, we find that

$$v(t) = -15t + C.$$

Since $v(0) = 88, C = 88$. Thus, the velocity function is

$$v(t) = -15t + 88.$$

To find how long it takes for the car to stop, we need to find the time t such that the velocity is zero. Solving $-15t + 88 = 0$, we obtain $t = \dfrac{88}{15}$ sec.

b. To find how far the car travels during this time, we need to find the position of the car after $\dfrac{88}{15}$ sec. We know the velocity $v(t)$ is the derivative of the position $s(t)$. Consider the initial position to be $s(0) = 0$. Therefore, we need to solve the initial-value problem

$$s'(t) = -15t + 88, \quad s(0) = 0.$$

Integrating, we have

$$s(t) = -\frac{15}{2}t^2 + 88t + C.$$

Since $s(0) = 0$, the constant is $C = 0$. Therefore, the position function is

$$s(t) = -\frac{15}{2}t^2 + 88t.$$

After $t = \frac{88}{15}$ sec, the position is $s\left(\frac{88}{15}\right) \approx 258.133$ ft.

? Exercise 4.1.5

Suppose the car is traveling at the rate of 44 ft/sec. How long does it take for the car to stop? How far will the car travel?

Hint

$v(t) = -15t + 44.$

Answer

$2.93\text{sec}, 64.5\text{ft}$

4.1.3 Key Concepts

- If F is an antiderivative of f, then every antiderivative of f is of the form $F(x) + C$ for some constant C.
- Solving the initial-value problem

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0$$

requires us first to find the set of antiderivatives of f and then to look for the particular antiderivative that also satisfies the initial condition.

4.1.4 Glossary

antiderivative

a function F such that $F'(x) = f(x)$ for all x in the domain of f is an antiderivative of f

indefinite integral

the most general antiderivative of $f(x)$ is the indefinite integral of f; we use the notation $\int f(x)\,dx$ to denote the indefinite integral of f

initial value problem

a problem that requires finding a function y that satisfies the differential equation $\frac{dy}{dx} = f(x)$ together with the initial condition $y(x_0) = y_0$

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4.1E: EXERCISES FOR SECTION 4.1

In exercises 1 - 14, find the antiderivative $F(x)$ of each function $f(x)$.

1) $f(x) = \dfrac{1}{x^2} + x$

2) $f(x) = e^x - 3x^2 + \dfrac{1}{x}$

Answer

$F(x) = e^x - x^3 + \ln|x| + C$

3) $f(x) = e^x + 3x - x^2$

4) $f(x) = x - 1 + \dfrac{4}{\sqrt{x}}$

Answer

$F(x) = \dfrac{x^2}{2} - x + 8\sqrt{x} + C$

5) $f(x) = 5x^4 + 4x^5$

6) $f(x) = x + 12x^2$

Answer

$F(x) = \frac{1}{2}x^2 + 4x^3 + C$

7) $f(x) = \dfrac{1}{\sqrt{x}}$

8) $f(x) = (\sqrt{x})^3$

Answer

$F(x) = \frac{2}{5}(\sqrt{x})^5 + C$

9) $f(x) = x^{1/3} + (2x)^{1/3}$

10) $f(x) = \dfrac{x^{1/3}}{x^{2/3}}$

Answer

$F(x) = \frac{3}{2}x^{2/3} + C$

11) $f(x) = \frac{x^2+4x+1}{x}$

12) $f(x) = (2x+1)(x-4)$ Hint: Multiply out first!

Answer

$F(x) = \frac{2}{3}x^3 - \frac{7}{2}x^2 - 4x + C$

13) $f(x) = 5$

14) $f(x) = 0$

Answer

$F(x) = C$

For exercises 15-23, evaluate the integral.

15) $\displaystyle\int (-1)\,dx$

16) $\displaystyle\int (x+2)\,dx$

Answer

$$\int (x+2)\,dx = \frac{1}{2}x^2 + 2x + C$$

17) $\int \left(4x + \sqrt{x}\right)\,dx$

18) $\int \dfrac{3x^2+2}{x^2}\,dx$

Answer

$$\int \frac{3x^2+2}{x^2}\,dx = 3x - \frac{2}{x} + C$$

19) $\int (x+2)(3x+2)\,dx$

20) $\int \left(4\sqrt{x} + \sqrt[4]{x}\right)\,dx$

Answer

$$\int \left(4\sqrt{x} + \sqrt[4]{x}\right)\,dx = \frac{8}{3}x^{3/2} + \frac{4}{5}x^{5/4} + C$$

21) $\int \left(x^{-1/3} - x^{2/3}\right)\,dx$

22) $\int \dfrac{14x^3+2x+1}{x^3}\,dx$

Answer

$$\int \frac{14x^3+2x+1}{x^3}\,dx = 14x - \frac{2}{x} - \frac{1}{2x^2} + C$$

23) $\int \left(e^x + \dfrac{2}{x}\right)\,dx$

In exercises 24-28, solve the initial value problem.

24) $f'(x) = x^{-3}, \quad f(1) = 1$

Answer

$$f(x) = -\frac{1}{2x^2} + \frac{3}{2}$$

25) $f'(x) = \sqrt{x} + x^2, \quad f(0) = 2$

26) $f'(x) = 6x^2 + 2x - 1, \quad f(3) = 2$

Answer

$$f(x) = 2x^3 + x^2 - x - 58$$

27) $f'(x) = x^3 - 8x^2 + 16x + 1, \quad f(0) = 0$

28) $f'(x) = \dfrac{2}{x^2} - \dfrac{x^2}{2}, \quad f(1) = 0$

Answer

$$f(x) = -\tfrac{1}{6}x^3 - \frac{2}{x} + \frac{13}{6}$$

In exercises 29-31, find two possible functions f given the second- or third-order derivatives

29) $f''(x) = x^2 + 2$

30) $f''(x) = e^{-x}$

Answer

Answers may vary; one possible answer is $f(x) = e^{-x}$

31) $f''(x) = 1 + x$

32) A car is being driven at a rate of 40 mph when the brakes are applied. The car decelerates at a constant rate of $10\,\text{ft}/\text{sec}^2$. How long before the car stops?

Answer

5.867 sec

33) In the preceding problem, calculate how far the car travels in the time it takes to stop.

34) You are merging onto the freeway, accelerating at a constant rate of $12\,\text{ft}/\text{sec}^2$. How long does it take you to reach merging speed at 60 mph?

Answer

7.333 sec

35) Based on the previous problem, how far does the car travel to reach merging speed?

36) A car company wants to ensure its newest model can stop in 8 sec when traveling at 75 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

Answer

$13.75\,\text{ft}/\text{sec}^2$

37) A car company wants to ensure its newest model can stop in less than 450 ft when traveling at 60 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

In exercises 38-40, find the antiderivative of the function, assuming $F(0) = 0$.

38) [T] $f(x) = x^2 + 2$

Answer

$F(x) = \frac{1}{3}x^3 + 2x$

39) [T] $f(x) = 4x - \sqrt{x}$

40) [T] $f(x) = e^x$

Answer

$F(x) = e^x - 1$

In exercises 41-44, determine whether the statement is true or false. Either prove it is true or find a counterexample if it is false.

41) If $f(x)$ is the antiderivative of $v(x)$, then $2f(x)$ is the antiderivative of $2v(x)$.

Answer

True

42) If $f(x)$ is the antiderivative of $v(x)$, then $f(2x)$ is the antiderivative of $v(2x)$.

43) If $f(x)$ is the antiderivative of $v(x)$, then $f(x) + 1$ is the antiderivative of $v(x) + 1$.

Answer

False

44) If $f(x)$ is the antiderivative of $v(x)$, then $(f(x))^2$ is the antiderivative of $(v(x))^2$.

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4.2: Approximating Areas

Learning Objectives

- Use sigma (summation) notation to calculate sums and powers of integers.
- Use the sum of rectangular areas to approximate the area under a curve.
- Use Riemann sums to approximate area.

Archimedes was fascinated with calculating the areas of various shapes—in other words, the amount of space enclosed by the shape. He used a process that has come to be known as the **method of exhaustion**, which used smaller and smaller shapes, the areas of which could be calculated exactly, to fill an irregular region and thereby obtain closer and closer approximations to the total area. In this process, an area bounded by curves is filled with rectangles, triangles, and shapes with exact area formulas. These areas are then summed to approximate the area of the curved region.

In this section, we develop techniques to approximate the area between a curve, defined by a function $f(x)$, and the x-axis on a closed interval $[a, b]$. Like Archimedes, we first approximate the area under the curve using shapes of known area (namely, rectangles). By using smaller and smaller rectangles, we get closer and closer approximations to the area. Taking a limit allows us to calculate the exact area under the curve.

Let's start by introducing some notation to make the calculations easier. We then consider the case when $f(x)$ is continuous and nonnegative. Later in the chapter, we relax some of these restrictions and develop techniques that apply in more general cases.

4.2.1 Sigma (Summation) Notation

As mentioned, we will use shapes of known area to approximate the area of an irregular region bounded by curves. This process often requires adding up long strings of numbers. To make it easier to write down these lengthy sums, we look at some new notation here, called **sigma notation** (also known as **summation notation**). The Greek capital letter Σ, sigma, is used to express long sums of values in a compact form. For example, if we want to add all the integers from 1 to 20 without sigma notation, we have to write

$$1+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+17+18+19+20.$$

We could probably skip writing a couple of terms and write

$$1+2+3+4+\cdots+19+20,$$

which is better, but still cumbersome. With sigma notation, we write this sum as

$$\sum_{i=1}^{20} i$$

which is much more compact. Typically, sigma notation is presented in the form

$$\sum_{i=1}^{n} a_i$$

where a_i describes the terms to be added, and the i is called the $index$. Each term is evaluated, then we sum all the values, beginning with the value when $i = 1$ and ending with the value when $i = n$. For example, an expression like $\sum_{i=2}^{7} s_i$ is interpreted as $s_2 + s_3 + s_4 + s_5 + s_6 + s_7$. Note that the index is used only to keep track of the terms to be added; it does not factor into the calculation of the sum itself. The index is therefore called a **dummy variable**. We can use any letter we like for the index. Typically, mathematicians use i, j, k, m, and n for indices.

Let's try a couple of examples of using sigma notation.

Example 4.2.1: Using Sigma Notation

a. Write in sigma notation and evaluate the sum of terms 3^i for $i = 1, 2, 3, 4, 5$.
b. Write the sum in sigma notation:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}.$$

Solution

a. Write

$$\sum_{i=1}^{5} 3^i = 3 + 3^2 + 3^3 + 3^4 + 3^5 = 363.$$

b. The denominator of each term is a perfect square. Using sigma notation, this sum can be written as $\sum_{i=1}^{5} \frac{1}{i^2}$.

Exercise 4.2.1

Write in sigma notation and evaluate the sum of terms 2^i for $i = 3, 4, 5, 6$.

Hint

Use the solving steps in Example 4.2.1as a guide.

Answer

$$\sum_{i=3}^{6} 2^i = 2^3 + 2^4 + 2^5 + 2^6 = 120$$

The properties associated with the summation process are given in the following rule.

Rule: Properties of Sigma Notation

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m, with $1 \leq m \leq n$.

i. $\sum_{i=1}^{n} c = nc$
ii. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
iii. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
iv. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$
v. $\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$

Proof

We prove properties (ii.) and (iii.) here, and leave proof of the other properties to the Exercises.

(ii.) We have

$$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + ca_3 + \cdots + ca_n = c(a_1 + a_2 + a_3 + \cdots + a_n) = c\sum_{i=1}^{n} a_i.$$

(iii.) We have

$$\sum_{i=1}^{n}(a_i+b_i) = (a_1+b_1)+(a_2+b_2)+(a_3+b_3)+\cdots+(a_n+b_n) \tag{4.2.1}$$

$$= (a_1+a_2+a_3+\cdots+a_n)+(b_1+b_2+b_3+\cdots+b_n) \tag{4.2.2}$$

$$= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i. \tag{4.2.3}$$

□

A few more formulas for frequently found functions simplify the summation process further. These are shown in the next rule, for **sums and powers of integers**, and we use them in the next set of examples.

Rule: Sums and Powers of Integers

1. The sum of n integers is given by

$$\sum_{i=1}^{n} i = 1+2+\cdots+n = \frac{n(n+1)}{2}. \tag{4.2.4}$$

2. The sum of consecutive integers squared is given by

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{4.2.5}$$

3. The sum of consecutive integers cubed is given by

$$\sum_{i=1}^{n} i^3 = 1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}. \tag{4.2.6}$$

Example 4.2.2: Evaluation Using Sigma Notation

Write using sigma notation and evaluate:

a. The sum of the terms $(i-3)^2$ for $i = 1, 2, \ldots, 200$.
b. The sum of the terms (i^3-i^2) for $i = 1, 2, 3, 4, 5, 6$

Solution

a. Multiplying out $(i-3)^2$, we can break the expression into three terms.

$$\begin{aligned}
\sum_{i=1}^{200}(i-3)^2 &= \sum_{i=1}^{200}(i^2-6i+9) \\
&= \sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} 6i + \sum_{i=1}^{200} 9 \\
&= \sum_{i=1}^{200} i^2 - 6\sum_{i=1}^{200} i + \sum_{i=1}^{200} 9 \\
&= \frac{200(200+1)(400+1)}{6} - 6\left[\frac{200(200+1)}{2}\right] + 9(200) \\
&= 2,686,700 - 120,600 + 1800 \\
&= 2,567,900
\end{aligned}$$

b. Use sigma notation property iv. and the rules for the sum of squared terms and the sum of cubed terms.

$$\begin{aligned}\sum_{i=1}^{6}(i^3-i^2) &= \sum_{i=1}^{6} i^3 - \sum_{i=1}^{6} i^2 \\ &= \frac{6^2(6+1)^2}{4} - \frac{6(6+1)(2(6)+1)}{6} \\ &= \frac{1764}{4} - \frac{546}{6} \\ &= 350\end{aligned}$$

Exercise 4.2.2

Find the sum of the values of $4+3i$ for $i = 1, 2, \ldots, 100$.

Hint

Use the properties of sigma notation to solve the problem.

Answer

$15,550$

Example 4.2.3: Finding the Sum of the Function Values

Find the sum of the values of $f(x) = x^3$ over the integers $1, 2, 3, \ldots, 10$.

Solution

Using Equation 4.2.6, we have

$$\sum_{i=0}^{10} i^3 = \frac{(10)^2(10+1)^2}{4} = \frac{100(121)}{4} = 3025$$

Exercise 4.2.3

Evaluate the sum indicated by the notation $\sum_{k=1}^{20}(2k+1)$.

Hint

Use the rule on sum and powers of integers (Equations 4.2.4-4.2.6).

Answer

440

4.2.2 Approximating Area

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area A bounded by $f(x)$ above, the x-axis below, the line $x = a$ on the left, and the line $x = b$ on the right (Figure 4.2.1).

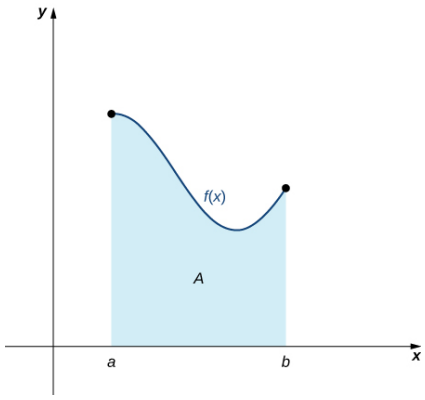


Figure 4.2.1: An area (shaded region) bounded by the curve $f(x)$ at top, the x-axis at bottom, the line $x = a$ to the left, and the line $x = b$ at right.

How do we approximate the area under this curve? The approach is a geometric one. By dividing a region into many small shapes that have known area formulas, we can sum these areas and obtain a reasonable estimate of the true area. We begin by dividing the interval $[a, b]$ into n subintervals of equal width, $\dfrac{b-a}{n}$. We do this by selecting equally spaced points $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = a, x_n = b,$ and

$$x_i - x_{i-1} = \frac{b-a}{n}$$

for $i = 1, 2, 3, \ldots, n$.

We denote the width of each subinterval with the notation Δx, so $\Delta x = \frac{b-a}{n}$ and

$$x_i = x_0 + i\Delta x$$

for $i = 1, 2, 3, \ldots, n$. This notion of dividing an interval $[a, b]$ into subintervals by selecting points from within the interval is used quite often in approximating the area under a curve, so let's define some relevant terminology.

Definition: Partitions

A set of points $P = x_i$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ is called a **partition** of $[a, b]$. If the subintervals all have the same width, the set of points forms a **regular partition** (or uniform partition) of the interval $[a, b]$.

We can use this regular partition as the basis of a method for estimating the area under the curve. We next examine two methods: the left-endpoint approximation and the right-endpoint approximation.

Rule: Left-Endpoint Approximation

On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \ldots, n$), construct a rectangle with width Δx and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1})\Delta x$. Adding the areas of all these rectangles, we get an approximate value for A (Figure 4.2.2). We use the notation L_n to denote that this is a **left-endpoint approximation** of A using n subintervals.

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

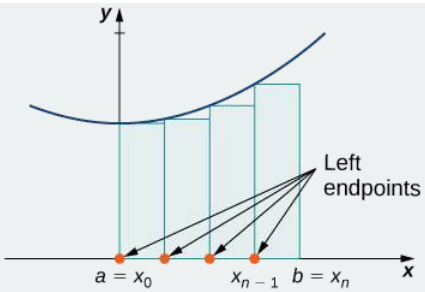


Figure 4.2.2: In the left-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the left of each subinterval.

The second method for approximating area under a curve is the right-endpoint approximation. It is almost the same as the left-endpoint approximation, but now the heights of the rectangles are determined by the function values at the right of each subinterval.

Rule: Right-Endpoint Approximation

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i)\,\Delta x$ and the approximation for A is given by

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$$

The notation R_n indicates this is a **right-endpoint approximation** for A (Figure 4.2.3).

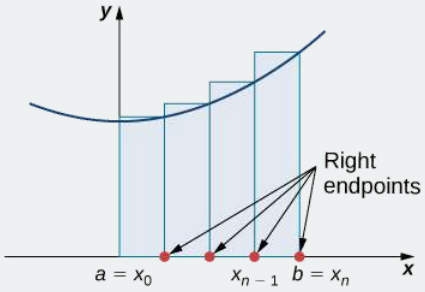


Figure 4.2.3: In the right-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the right of each subinterval. Note that the right-endpoint approximation differs from the left-endpoint approximation in Figure 4.2.2.

The graphs in Figure 4.2.4 represent the curve $f(x) = \dfrac{x^2}{2}$. In Figure 4.2.4*b* we divide the region represented by the interval $[0, 3]$ into six subintervals, each of width 0.5. Thus, $\Delta x = 0.5$. We then form six rectangles by drawing vertical lines perpendicular to x_{i-1}, the left endpoint of each subinterval. We determine the height of each rectangle by calculating $f(x_{i-1})$ for $i = 1, 2, 3, 4, 5, 6$. The intervals are $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3]$. We find the area of each rectangle by multiplying the height by the width. Then, the sum of the rectangular areas approximates the area between $f(x)$ and the x-axis. When the left endpoints are used to calculate height, we have a left-endpoint approximation. Thus,

$$\begin{aligned} A \approx L_6 &= \sum_{i=1}^{6} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\ &= f(0)0.5 + f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 \\ &= (0)0.5 + (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 \\ &= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\ &= 3.4375 \text{ units}^2 \end{aligned}$$

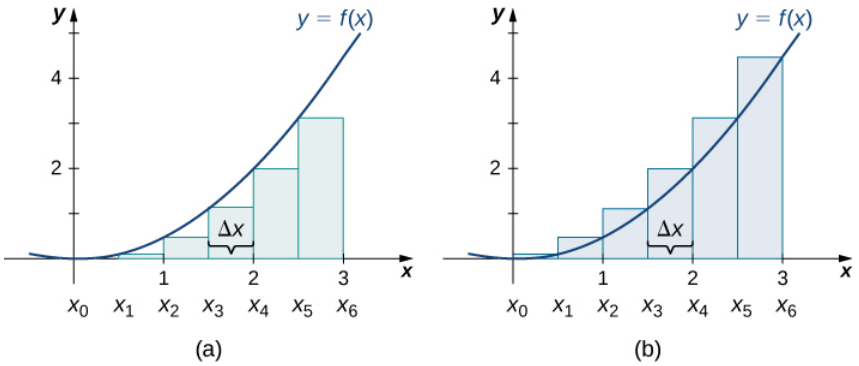


Figure 4.2.4: Methods of approximating the area under a curve by using (a) the left endpoints and (b) the right endpoints.

In Figure 4.2.4b, we draw vertical lines perpendicular to x_i such that x_i is the right endpoint of each subinterval, and calculate $f(x_i)$ for $i = 1, 2, 3, 4, 5, 6$. We multiply each $f(x_i)$ by Δx to find the rectangular areas, and then add them. This is a right-endpoint approximation of the area under $f(x)$. Thus,

$$\begin{aligned} A \approx R_6 &= \sum_{i=1}^{6} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\ &= f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 + f(3)0.5 \\ &= (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 + (4.5)0.5 \\ &= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\ &= 5.6875 \text{ units}^2. \end{aligned}$$

Example 4.2.4: Approximating the Area Under a Curve

Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.

Solution

First, divide the interval $[0, 2]$ into n equal subintervals. Using $n = 4$, $\Delta x = \dfrac{(2-0)}{4} = 0.5$. This is the width of each rectangle. The intervals $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]$ are shown in Figure 4.2.5. Using a left-endpoint approximation, the heights are $f(0) = 0$, $f(0.5) = 0.25$, $f(1) = 1$, and $f(1.5) = 2.25$. Then,

$$\begin{aligned} L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= 0(0.5) + 0.25(0.5) + 1(0.5) + 2.25(0.5) \\ &= 1.75 \text{ units}^2 \end{aligned}$$

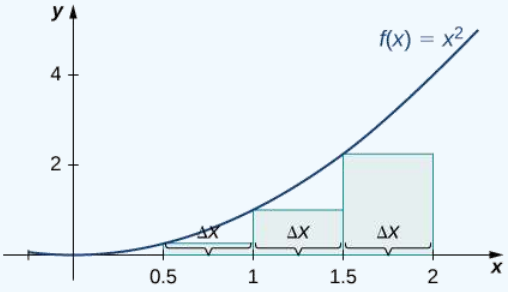


Figure 4.2.5: The graph shows the left-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The right-endpoint approximation is shown in Figure 4.2.6. The intervals are the same, $\Delta x = 0.5$, but now use the right endpoint to calculate the height of the rectangles. We have

$$\begin{aligned} R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= 0.25(0.5) + 1(0.5) + 2.25(0.5) + 4(0.5) \\ &= 3.75 \text{ units}^2 \end{aligned}$$

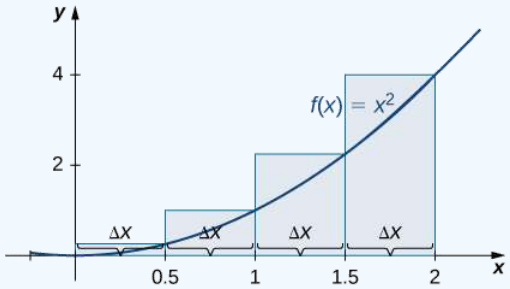


Figure 4.2.6: The graph shows the right-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The left-endpoint approximation is 1.75 units^2; the right-endpoint approximation is 3.75 units^2.

Exercise 4.2.4

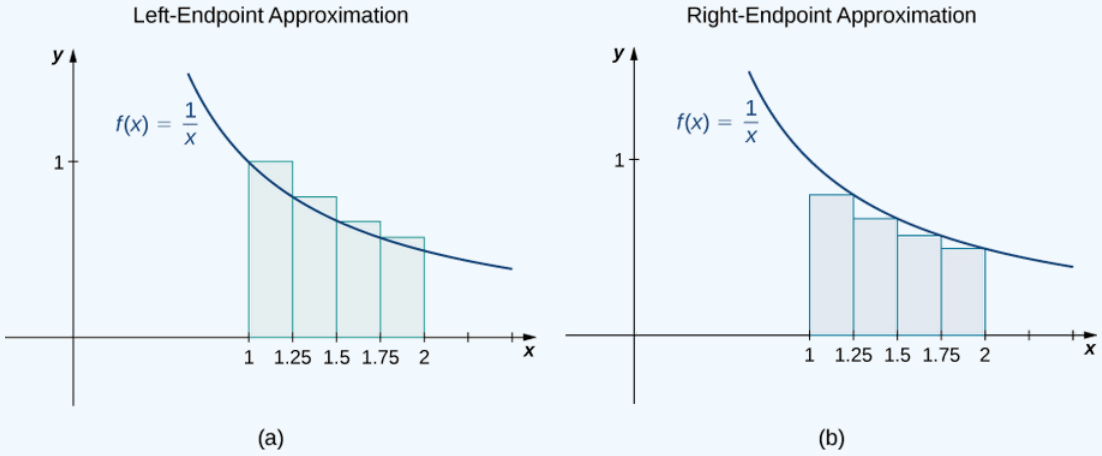
Sketch left-endpoint and right-endpoint approximations for $f(x) = \dfrac{1}{x}$ on $[1, 2]$; use $n = 4$. Approximate the area using both methods.

Hint

Follow the solving strategy in Example 4.2.4step-by-step.

Answer

The left-endpoint approximation is 0.7595 units^2. The right-endpoint approximation is 0.6345 units^2. See the below Media.



Looking at Figure 4.2.4 and the graphs in Example 4.2.4, we can see that when we use a small number of intervals, neither the left-endpoint approximation nor the right-endpoint approximation is a particularly accurate estimate of the area under the curve. However, it seems logical that if we increase the number of points in our partition, our estimate of A will improve. We will have more rectangles, but each rectangle will be thinner, so we will be able to fit the rectangles to the curve more precisely.

We can demonstrate the improved approximation obtained through smaller intervals with an example. Let's explore the idea of increasing n, first in a left-endpoint approximation with four rectangles, then eight rectangles, and finally 32 rectangles. Then, let's do the same thing in a right-endpoint approximation, using the same sets of intervals, of the same curved region. Figure 4.2.7 shows the area of the region under the curve $f(x) = (x-1)^3 + 4$ on the interval $[0, 2]$ using a left-endpoint approximation where $n = 4$. The width of each rectangle is

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}.$$

The area is approximated by the summed areas of the rectangles, or

$$L_4 = f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)0.5 = 7.5 \text{ units}^2$$

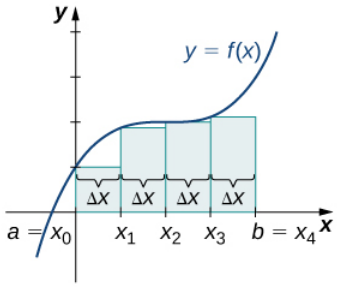


Figure 4.2.7: With a left-endpoint approximation and dividing the region from a to b into four equal intervals, the area under the curve is approximately equal to the sum of the areas of the rectangles.

Figure 4.2.8 shows the same curve divided into eight subintervals. Comparing the graph with four rectangles in Figure 4.2.7 with this graph with eight rectangles, we can see there appears to be less white space under the curve when $n = 8$. This white space is area under the curve we are unable to include using our approximation. The area of the rectangles is

$$L_8 = f(0)(0.25) + f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) = 7.75 \text{ units}^2$$

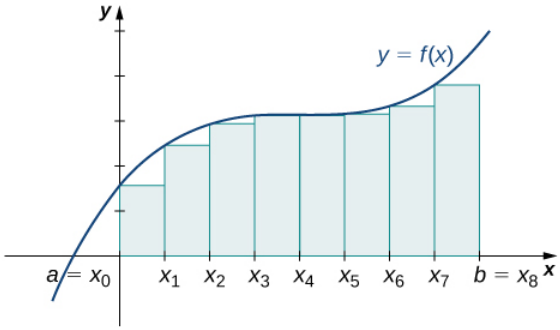


Figure 4.2.8: The region under the curve is divided into $n = 8$ rectangular areas of equal width for a left-endpoint approximation.

The graph in Figure 4.2.9 shows the same function with 32 rectangles inscribed under the curve. There appears to be little white space left. The area occupied by the rectangles is

$$L_{32} = f(0)(0.0625) + f(0.0625)(0.0625) + f(0.125)(0.0625) + \cdots + f(1.9375)(0.0625) = 7.9375 \text{ units}^2.$$

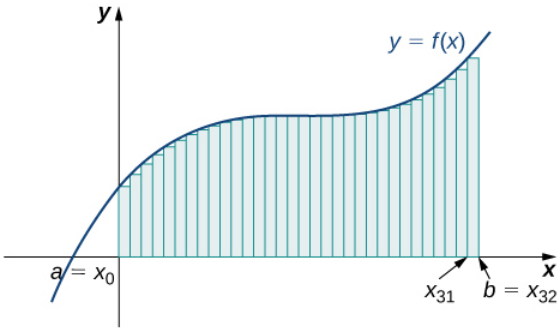


Figure 4.2.9: Here, 32 rectangles are inscribed under the curve for a left-endpoint approximation.

We can carry out a similar process for the right-endpoint approximation method. A right-endpoint approximation of the same curve, using four rectangles (Figure 4.2.10), yields an area

$$R_4 = f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) = 8.5 \text{ units}^2.$$

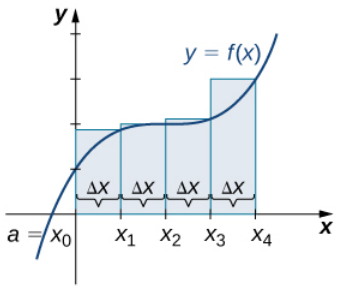


Figure 4.2.10: Now we divide the area under the curve into four equal subintervals for a right-endpoint approximation.

Dividing the region over the interval $[0, 2]$ into eight rectangles results in $\Delta x = \dfrac{2-0}{8} = 0.25$. The graph is shown in Figure 4.2.11. The area is

$$R_8 = f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) + f(2)(0.25) = 8.25 \text{ units}^2$$

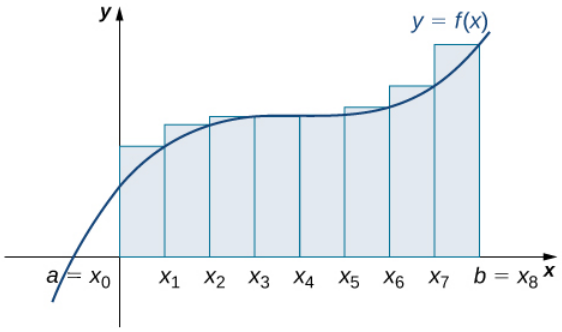


Figure 4.2.11: Here we use right-endpoint approximation for a region divided into eight equal subintervals.

Last, the right-endpoint approximation with $n = 32$ is close to the actual area (Figure 4.2.12). The area is approximately

$$R_{32} = f(0.0625)(0.0625) + f(0.125)(0.0625) + f(0.1875)(0.0625) + \cdots + f(2)(0.0625) = 8.0625 \text{ units}^2$$

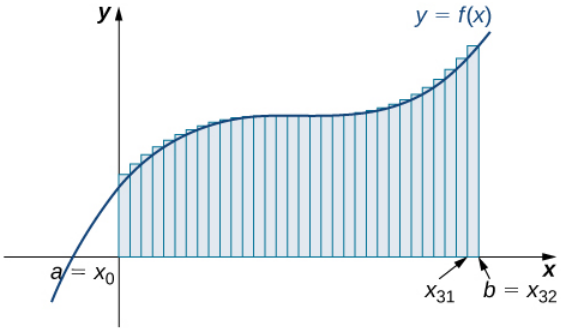


Figure 4.2.12: The region is divided into 32 equal subintervals for a right-endpoint approximation.

Based on these figures and calculations, it appears we are on the right track; the rectangles appear to approximate the area under the curve better as n gets larger. Furthermore, as n increases, both the left-endpoint and right-endpoint approximations appear to approach an area of 8 square units. Table 4.2.15 shows a numerical comparison of the left- and right-endpoint methods. The idea that the approximations of the area under the curve get better and better as n gets larger and larger is very important, and we now explore this idea in more detail.

Table 4.2.15: Converging Values of Left- and Right-Endpoint Approximations as n Increases

Value of n	Approximate Area L_n	Approximate Area R_n
$n = 4$	7.5	8.5
$n = 8$	7.75	8.25

Value of n	Approximate Area L_n	Approximate Area R_n
$n = 32$	7.94	8.06

4.2.2.1 Forming Riemann Sums

So far we have been using rectangles to approximate the area under a curve. The heights of these rectangles have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_{i-1}, x_i]$. In reality, there is no reason to restrict evaluation of the function to one of these two points only. We could evaluate the function at any point x_i^* in the subinterval $[x_{i-1}, x_i]$, and use $f(x_i^*)$ as the height of our rectangle. This gives us an estimate for the area of the form

$$A \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

A sum of this form is called a Riemann sum, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

Definition: Riemann sum

Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be any partition of $[a, b]$. Let Δx_i be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1},\ x_i]$. A Riemann sum is defined for $f(x)$ as

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

At this point, we'll choose a regular partition P, as we have in our examples above. This forces all Δx_i to be equal to $\Delta x = \dfrac{b-a}{n}$ for any natural number of intervals n.

Recall that with the left- and right-endpoint approximations, the estimates seem to get better and better as n get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of n. We are now ready to define the area under a curve in terms of Riemann sums.

Definition: Area Under the Curve

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\displaystyle\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ be a Riemann sum for $f(x)$ with a regular partition P. Then, the **area under the curve** $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

See a graphical demonstration of the construction of a Riemann sum.

Some subtleties here are worth discussing. First, note that taking the limit of a sum is a little different from taking the limit of a function $f(x)$ as x goes to infinity. Limits of sums are discussed in detail in the chapter on Sequences and Series; however, for now we can assume that the computational techniques we used to compute limits of functions can also be used to calculate limits of sums.

Second, we must consider what to do if the expression converges to different limits for different choices of x_i^*. Fortunately, this does not happen. Although the proof is beyond the scope of this text, it can be shown that if $f(x)$ is continuous on the closed interval $[a, b]$, then $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique (in other words, it does not depend on the choice of x_i^*).

We look at some examples shortly. But, before we do, let's take a moment and talk about some specific choices for x_i^*. Although any choice for x_i^* gives us an estimate of the area under the curve, we don't necessarily know whether that estimate is too high

(overestimate) or too low (underestimate). If it is important to know whether our estimate is high or low, we can select our value for x_i^* to guarantee one result or the other.

If we want an overestimate, for example, we can choose x_i^* such that for $i = 1, 2, 3, \ldots, n, f(x_i^*) \geq f(x)$ for all $x \in [x_i - 1, x_i]$. In other words, we choose x_i^* so that for $i = 1, 2, 3, \ldots, n, f(x_i^*)$ is the maximum function value on the interval $[x_{i-1}, x_i]$. If we select x_i^* in this way, then the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called an **upper sum**. Similarly, if we want an underestimate, we can choose $x * i$ so that for $i = 1, 2, 3, \ldots, n$, $f(x_i^*)$ is the minimum function value on the interval $[x_{i-1}, x_i]$. In this case, the associated Riemann sum is called a **lower sum**. Note that if $f(x)$ is either increasing or decreasing throughout the interval $[a, b]$, then the maximum and minimum values of the function occur at the endpoints of the subintervals, so the upper and lower sums are just the same as the left- and right-endpoint approximations.

Example 4.2.5: Finding Lower and Upper Sums

Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$ subintervals.

Solution

With $n = 4$ over the interval $[1, 2]$, $\Delta x = \frac{1}{4}$. We can list the intervals as $[1, 1.25]$, $[1.25, 1.5]$, $[1.5, 1.75]$, and $[1.75, 2]$. Because the function is decreasing over the interval $[1, 2]$, Figure shows that a lower sum is obtained by using the right endpoints.

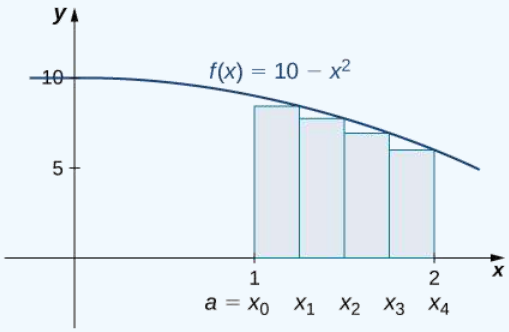


Figure 4.2.13: The graph of $f(x) = 10 - x^2$ is set up for a right-endpoint approximation of the area bounded by the curve and the x-axis on $[1, 2]$, and it shows a lower sum.

The Riemann sum is

$$\begin{aligned}\sum_{k=1}^{4}(10 - x^2)(0.25) &= 0.25[10 - (1.25)^2 + 10 - (1.5)^2 + 10 - (1.75)^2 + 10 - (2)^2] \\ &= 0.25[8.4375 + 7.75 + 6.9375 + 6] \\ &= 7.28 \text{ units}^2.\end{aligned}$$

The area of 7.28 units^2 is a lower sum and an underestimate.

Exercise 4.2.5

a. Find an upper sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$.
b. Sketch the approximation.

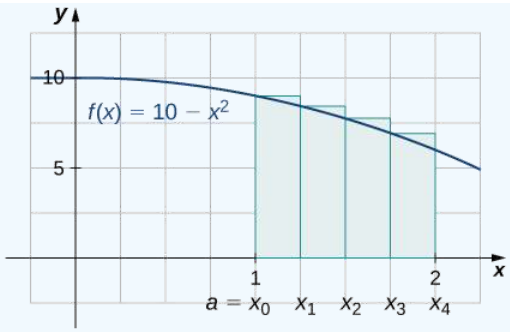
Hint

$f(x)$ is decreasing on $[1, 2]$, so the maximum function values occur at the left endpoints of the subintervals.

Answer

a. Upper sum=8.0313 units^2.

b.



4.2.3 Key Concepts

- The use of sigma (summation) notation of the form $\sum_{i=1}^{n} a_i$ is useful for expressing long sums of values in compact form.
- For a continuous function defined over an interval $[a, b]$, the process of dividing the interval into n equal parts, extending a rectangle to the graph of the function, calculating the areas of the series of rectangles, and then summing the areas yields an approximation of the area of that region.
- When using a regular partition, the width of each rectangle is $\Delta x = \dfrac{b-a}{n}$.
- Riemann sums are expressions of the form $\sum_{i=1}^{n} f(x_i^*)\Delta x$, and can be used to estimate the area under the curve $y = f(x)$. Left- and right-endpoint approximations are special kinds of Riemann sums where the values of x_i^* are chosen to be the left or right endpoints of the subintervals, respectively.
- Riemann sums allow for much flexibility in choosing the set of points x_i^* at which the function is evaluated, often with an eye to obtaining a lower sum or an upper sum.

4.2.4 Key Equations

- **Properties of Sigma Notation**

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$$

- **Sums and Powers of Integers**

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

- **Left-Endpoint Approximation**

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

- **Right-Endpoint Approximation**

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

4.2.5 Glossary

left-endpoint approximation
an approximation of the area under a curve computed by using the left endpoint of each subinterval to calculate the height of the vertical sides of each rectangle

lower sum
a sum obtained by using the minimum value of $f(x)$ on each subinterval

partition
a set of points that divides an interval into subintervals

regular partition
a partition in which the subintervals all have the same width

riemann sum
an estimate of the area under the curve of the form $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$

right-endpoint approximation
the right-endpoint approximation is an approximation of the area of the rectangles under a curve using the right endpoint of each subinterval to construct the vertical sides of each rectangle

sigma notation
(also, **summation notation**) the Greek letter sigma (Σ) indicates addition of the values; the values of the index above and below the sigma indicate where to begin the summation and where to end it

upper sum
a sum obtained by using the maximum value of $f(x)$ on each subinterval

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4.2E: EXERCISES FOR SECTION 4.2

1.) State whether the given sums are equal or unequal.

a. $\sum_{i=1}^{10} i$ and $\sum_{k=1}^{10} k$

b. $\sum_{i=1}^{10} i$ and $\sum_{i=6}^{15} (i-5)$

c. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{j=0}^{9} (j+1)j$

d. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{k=1}^{10} (k^2-k)$

Answer

a. They are equal; both represent the sum of the first 10 whole numbers.
b. They are equal; both represent the sum of the first 10 whole numbers.
c. They are equal by substituting $j = i-1$.
d. They are equal; the first sum factors the terms of the second.

In exercises 2 - 3, use the rules for sums of powers of integers to compute the sums.

2) $\sum_{i=5}^{10} i$

3) $\sum_{i=5}^{10} i^2$

Answer

$$\sum_{i=5}^{10} i^2 = 385 - 30 = 355$$

Suppose that $\sum_{i=1}^{100} a_i = 15$ and $\sum_{i=1}^{100} b_i = -12$.

In exercises 4 - 7, compute the sums.

4) $\sum_{i=1}^{100} (a_i + b_i)$

5) $\sum_{i=1}^{100} (a_i - b_i)$

Answer

$$\sum_{i=1}^{100} (a_i - b_i) = 15 - (-12) = 27$$

6) $\sum_{i=1}^{100} (3a_i - 4b_i)$

7) $\sum_{i=1}^{100} (5a_i + 4b_i)$

Answer

$$\sum_{i=1}^{100} (5a_i + 4b_i) = 5(15) + 4(-12) = 27$$

In exercises 8 - 11, use summation properties and formulas to rewrite and evaluate the sums.

8) $\sum_{k=1}^{20} 100(k^2 - 5k + 1)$

9) $\sum_{j=1}^{50} (j^2 - 2j)$

Answer

$$\sum_{j=1}^{50} j^2 - 2\sum_{j=1}^{50} j = \frac{(50)(51)(101)}{6} - \frac{2(50)(51)}{2} = 40,375$$

10) $\sum_{j=11}^{20} (j^2 - 10j)$

11) $\sum_{k=1}^{25} [(2k)^2 - 100k]$

Answer

$$4\sum_{k=1}^{25} k^2 - 100\sum_{k=1}^{25} k = \frac{4(25)(26)(51)}{9} - 50(25)(26) = -10,400$$

Let L_n denote the left-endpoint sum using n subintervals and let R_n denote the corresponding right-endpoint sum.

In exercises 12 - 19, compute the indicated left and right sums for the given functions on the indicated interval. Round to 3 decimal places as needed.

12) L_4 for $f(x) = \dfrac{1}{x-1}$ on $[2,3]$

13) R_4 for $g(x) = \ln x$ on $[1,2]$.

Answer

$R_4 = 1.881$

14) L_6 for $f(x) = \dfrac{1}{x(x-1)}$ on $[2,5]$

15) R_6 for $f(x) = \dfrac{1}{x(x-1)}$ on $[2,5]$

Answer

$R_6 = 0.372$

16) R_4 for $\dfrac{1}{x^2+1}$ on $[-2,2]$

17) L_4 for $\dfrac{1}{x^2+1}$ on $[-2,2]$

Answer

$L_4 = 2.20$

18) R_4 for $x^2 - 2x + 1$ on $[0,2]$

19) L_8 for $x^2 - 2x + 1$ on $[0,2]$

Answer

$L_8 = 0.6875$

20) Compute the left and right Riemann sums— L_4 and R_4, respectively—for $f(x) = (2 - |x|)$ on $[-2,2]$. Compute their average value and compare it with the area under the graph of f.

21) Compute the left and right Riemann sums— L_6 and R_6, respectively—for $f(x) = (3 - |3 - x|)$ on $[0,6]$. Compute their average value and compare it with the area under the graph of f.

Answer

$L_6 = 9.000 = R_6$. The graph of f is a triangle with area $9\,\text{units}^2$.

22) Compute the left and right Riemann sums— L_4 and R_4, respectively—for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ and compare their values.

23) Compute the left and right Riemann sums— L_6 and R_6, respectively—for $f(x) = \sqrt{9 - (x-3)^2}$ on $[0, 6]$ and compare their values.

Answer

$L_6 = 13.12899 = R_6$. They are equal.

For exercises 24 - 26, express the following endpoint sums in sigma notation but do not evaluate them.

24) L_{30} for $f(x) = x^2$ on $[1, 2]$

25) L_{10} for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$

Answer

$$L_{10} = \frac{4}{10} \sum_{i=1}^{10} \sqrt{4 - (-2 + 4\frac{(i-1)}{10})}$$

26) R_{100} for $\ln x$ on $[1, e]$

Answer

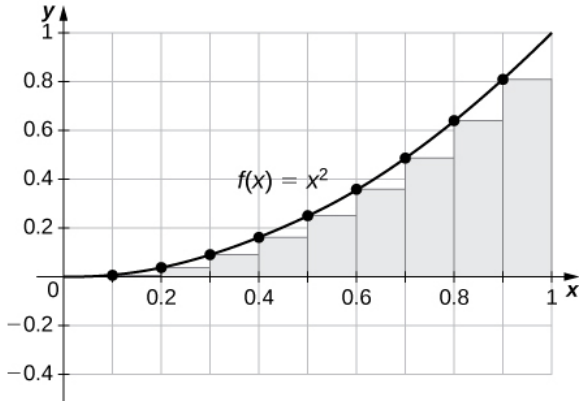
$$R_{100} = \frac{e-1}{100} \sum_{i=1}^{100} ln(1 + (e-1)\frac{i}{100})$$

In exercises 27-31, graph the function then use a calculator or a computer program to evaluate the following left and right endpoint sums. Is the area under the curve between the left and right endpoint sums?

27) [T] L_{100} and R_{100} for $y = x^2 - 3x + 1$ on the interval $[-1, 1]$

28) [T] L_{100} and R_{100} for $y = x^2$ on the interval $[0, 1]$

Answer



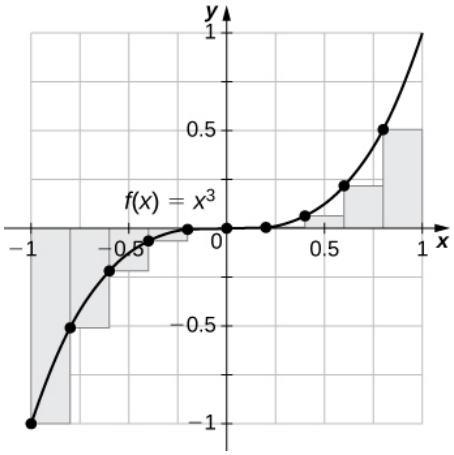
$R_{100} = 0.33835, \quad L_{100} = 0.32835.$

The plot shows that the left Riemann sum is an underestimate because the function is increasing. Similarly, the right Riemann sum is an overestimate. The area lies between the left and right Riemann sums. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

29) [T] L_{50} and R_{50} for $y = \dfrac{x+1}{x^2-1}$ on the interval $[2, 4]$

30) [T] L_{100} and R_{100} for $y = x^3$ on the interval $[-1, 1]$

Answer

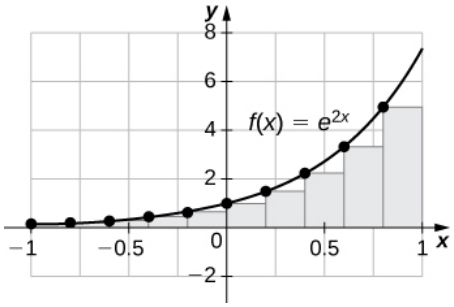


$L_{100} = -0.02, \quad R_{100} = 0.02$.

The left endpoint sum is an underestimate because the function is increasing. Similarly, a right endpoint approximation is an overestimate. The area lies between the left and right endpoint estimates.

31) [T] L_{100} and R_{100} for $y = e^{2x}$ on the interval $[-1, 1]$

Answer



$L_{100} = 3.555, \quad R_{100} = 3.670$.

The plot shows that the left Riemann sum is an underestimate because the function is increasing. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

32) Let t_j denote the time that it took Tejay van Garteren to ride the jth stage of the Tour de France in 2014. If there were a total of 21 stages, interpret $\sum_{j=1}^{21} t_j$.

33) Let r_j denote the total rainfall in Portland on the jth day of the year in 2009. Interpret $\sum_{j=1}^{31} r_j$.

Answer

The sum represents the cumulative rainfall in January 2009.

34) Let d_j denote the hours of daylight and δ_j denote the increase in the hours of daylight from day $j-1$ to day j in Fargo, North Dakota, on the jth day of the year. Interpret $d1 + \sum_{j=2}^{365} \delta_j$.

35) To help get in shape, Joe gets a new pair of running shoes. If Joe runs 1 mi each day in week 1 and adds $\frac{1}{10}$ mi to his daily routine each week, what is the total mileage on Joe's shoes after 25 weeks?

Answer

The total mileage is $7 \times \sum_{i=1}^{25} (1 + \frac{(i-1)}{10}) = 7 \times 25 + \frac{7}{10} \times 12 \times 25 = 385$ mi.

36) The following table gives approximate values of the average annual atmospheric rate of increase in carbon dioxide (CO_2) each decade since 1960, in parts per million (ppm). Estimate the total increase in atmospheric CO_2 between 1964 and 2013.

Decade	Ppm/y
1964-1973	1.07
1976-1983	1.34
1984-1993	1.40
1994-2003	1.87
2004-2013	2.07

Average Annual Atmospheric CO2 Increase, 1964–2013 Source: http://www.esrl.noaa.gov/gmd/ccgg/trends/.

37) The following table gives the approximate increase in sea level in inches over 20 years starting in the given year. Estimate the net change in mean sea level from 1870 to 2010.

Starting Year	20- Year Change
1870	0.3
1890	1.5
1910	0.2
1930	2.8
1950	0.7
1970	1.1
1990	1.5

Approximate 20-Year Sea Level Increases, 1870–1990

Source: http://link.springer.com/article/10....712-011-9119-1

Answer

Add the numbers to get 8.1-in. net increase.

38) The following table gives the approximate increase in dollars in the average price of a gallon of gas per decade since 1950. If the average price of a gallon of gas in 2010 was $2.60, what was the average price of a gallon of gas in 1950?

Starting Year	10- Year Change
1950	0.03
1960	0.05
1970	0.86
1980	−0.03
1990	0.29
2000	1.12

Approximate 10-Year Gas Price Increases, 1950–2000

Source: epb.lbl.gov/homepages/Rick_Di...011-trends.pdf.

39) The following table gives the percent growth of the U.S. population beginning in July of the year indicated. If the U.S. population was 281,421,906 in July 2000, estimate the U.S. population in July 2010.

Year	% Change/Year
2000	1.12
2001	0.99
2002	0.93
2003	0.86
2004	0.93
2005	0.93
2006	0.97
2007	0.96
2008	0.95
2009	0.88

Annual Percentage Growth of U.S. Population, 2000–2009

Source: www.census.gov/popest/data.

(Hint: To obtain the population in July 2001, multiply the population in July 2000 by 1.0112 to get 284,573,831.)

Answer

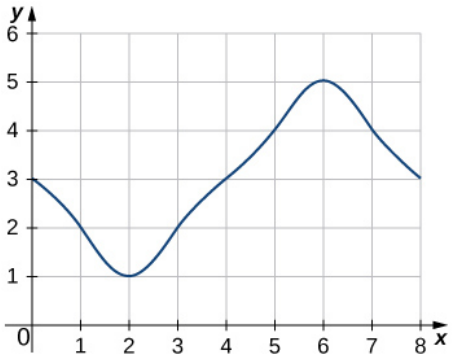
309,389,957

In exercises 40-43, estimate the areas under the curves by computing the left Riemann sums, L_8.

40)



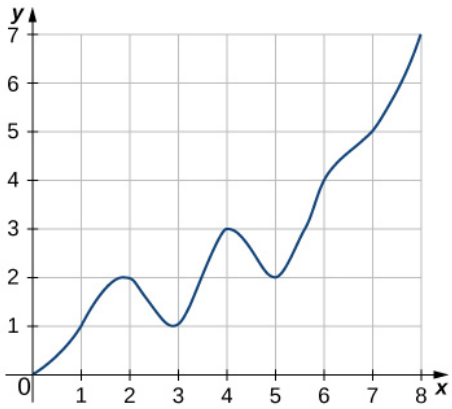
41)



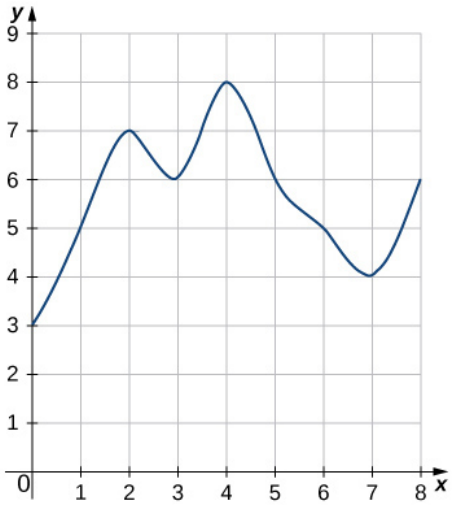
Answer

$L_8 = 3+2+1+2+3+4+5+4 = 24$

42)



43)



Answer

$L_8 = 3+5+7+6+8+6+5+4 = 44$

44) [T] Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \sqrt{1-x^2}$ on $[-1,1]$.

45) [T] Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \dfrac{1}{\sqrt{1+x^2}}$ on $[-1,1]$.

Answer

$L_{10} \approx 1.7604, \quad L_{30} \approx 1.7625, \quad L_{50} \approx 1.76265$

In exercises 46-47, use a calculator or a computer program to evaluate the endpoint sums R_N and L_N for $N = 1, 10, 100$.

46) [T] $y = x^4 - 5x^2 + 4$ on the interval $[-2,2]$, which has an exact area of $\frac{32}{15}$

Answer

$R_1 = 0, \quad L_1 = 0,$
$R_{10} = 2.4499, \quad L_{10} = 2.4499,$
$R_{100} = 2.1365, \quad L_{100} = 2.1365$

47) [T] $y = \ln x$ on the interval $[1,2]$, which has an exact area of $2\ln(2) - 1$

48) Explain why, if $f(a) \geq 0$ and f is increasing on $[a,b]$, that the left endpoint estimate is a lower bound for the area below the graph of f on $[a,b]$.

Answer

If $[c,d]$ is a subinterval of $[a,b]$ under one of the left-endpoint sum rectangles, then the area of the rectangle contributing to the left-endpoint estimate is $f(c)(d-c)$. But, $f(c) \leq f(x)$ for $c \leq x \leq d$, so the area under the graph of f between c and d is $f(c)(d-c)$ plus the area below the graph of f but above the horizontal line segment at height $f(c)$, which is positive. As this is true for each left-endpoint sum interval, it follows that the left Riemann sum is less than or equal to the area below the graph of f on $[a,b]$.

49) Explain why, if $f(b) \geq 0$ and f is decreasing on $[a,b]$, that the left endpoint estimate is an upper bound for the area below the graph of f on $[a,b]$.

50) Show that, in general, $R_N - L_N = (b-a) \times \dfrac{f(b)-f(a)}{N}$.

Answer

$L_N = \dfrac{b-a}{N}\sum_{i=1}^{N} f(a+(b-a)\dfrac{i-1}{N}) = \dfrac{b-a}{N}\sum_{i=0}^{N-1} f(a+(b-a)\dfrac{i}{N})$ and $R_N = \dfrac{b-a}{N}\sum_{i=1}^{N} f(a+(b-a)\dfrac{i}{N})$. The left sum has a term corresponding to $i = 0$ and the right sum has a term corresponding to $i = N$. In $R_N - L_N$, any term

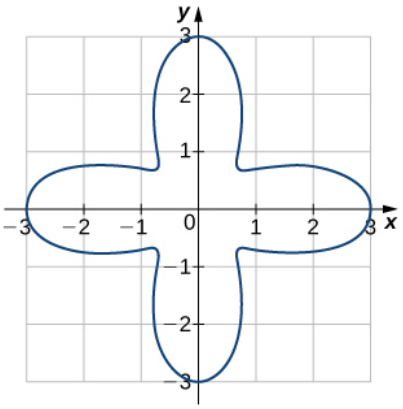
corresponding to $i = 1, 2, \ldots, N-1$ occurs once with a plus sign and once with a minus sign, so each such term cancels and one is left with $R_N - L_N = \frac{b-a}{N}(f(a+(b-a))\frac{N}{N}) - (f(a)+(b-a)\frac{0}{N}) = \frac{b-a}{N}(f(b)-f(a))$.

51) Explain why, if f is increasing on $[a,b]$, the error between either L_N or R_N and the area A below the graph of f is at most $(b-a)\frac{f(b)-f(a)}{N}$.

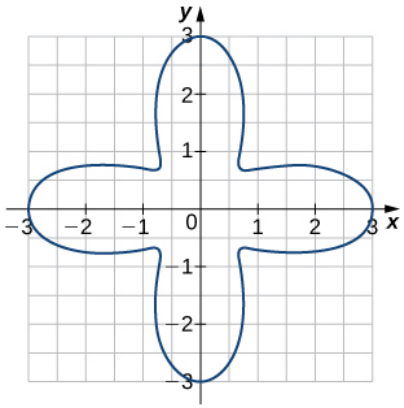
52) For each of the three graphs:

a. Obtain a lower bound $L(A)$ for the area enclosed by the curve by adding the areas of the squares enclosed completely by the curve.

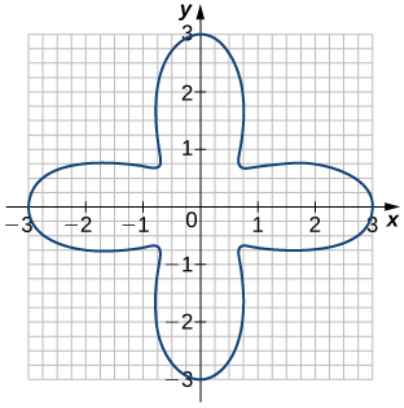
b. Obtain an upper bound $U(A)$ for the area by adding to $L(A)$ the areas $B(A)$ of the squares enclosed partially by the curve.



Graph 1



Graph 2



Graph 3

Answer

https://math.libretexts.org/@go/page/149898

Graph 1: a. $L(A) = 0, \quad B(A) = 20; \quad b. U(A) = 20.$
Graph 2: $a. L(A) = 9; \quad b. B(A) = 11, \quad U(A) = 20.$
Graph 3: a. $L(A) = 11.0; \quad b. B(A) = 4.5, \quad U(A) = 15.5.$

53) In the previous exercise, explain why $L(A)$ gets no smaller while $U(A)$ gets no larger as the squares are subdivided into four boxes of equal area.

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4.3: The Definite Integral

Learning Objectives

- State the definition of the definite integral.
- Explain the terms integrand, limits of integration, and variable of integration.
- Explain when a function is integrable.
- Describe the relationship between the definite integral and net area.
- Use geometry and the properties of definite integrals to evaluate them.
- Calculate the average value of a function.

In the preceding section we defined the area under a curve in terms of Riemann sums:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative. Unfortunately, real-world problems don't always meet these restrictions. In this section, we look at how to apply the concept of the area under the curve to a broader set of functions through the use of the definite integral.

4.3.1 Definition and Notation

The definite integral generalizes the concept of the area under a curve. We lift the requirements that $f(x)$ be continuous and nonnegative, and define the definite integral as follows.

Definition: Definite Integral

If $f(x)$ is a function defined on an interval $[a, b]$, the ***definite integral*** of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an integrable function.

The integral symbol in the previous definition should look familiar. We have seen similar notation earlier in the course, where we used the indefinite integral symbol (without the a and b above and below) to represent an antiderivative. Although the notation for indefinite integrals may look similar to the notation for a definite integral, they are not the same. A definite integral is a number. An indefinite integral is a family of functions. Later in this chapter we examine how these concepts are related. However, close attention should always be paid to notation so we know whether we're working with a definite integral or an indefinite integral.

Integral notation goes back to the late seventeenth century and is one of the contributions of Gottfried Wilhelm **Leibniz**, who is often considered to be the codiscoverer of calculus, along with Isaac Newton. The integration symbol $\int$ is an elongated S, suggesting sigma or summation. On a definite integral, above and below the summation symbol are the boundaries of the interval, $[a, b]$. The numbers a and b are x-values and are called the **limits of integration**; specifically, a is the lower limit and b is the upper limit. To clarify, we are using the word limit in two different ways in the context of the definite integral. First, we talk about the limit of a sum as $n \to \infty$. Second, the boundaries of the region are called the *limits of integration.*

We call the function $f(x)$ the **integrand**, and the dx indicates that $f(x)$ is a function with respect to x, called the **variable of integration**. Note that, like the index in a sum, the variable of integration is a **dummy variable**, and has no impact on the computation of the integral. We could use any variable we like as the variable of integration:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

Previously, we discussed the fact that if $f(x)$ is continuous on $[a, b]$, then the limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique. This leads to the following theorem, which we state without proof.

Continuous Functions Are Integrable

If $f(x)$ is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Functions that are not continuous on $[a,b]$ may still be integrable, depending on the nature of the discontinuities. For example, functions with a finite number of jump discontinuities or removable discontinuities on a closed interval are integrable.

It is also worth noting here that we have retained the use of a regular partition in the Riemann sums. This restriction is not strictly necessary. Any partition can be used to form a Riemann sum. However, if a nonregular partition is used to define the definite integral, it is not sufficient to take the limit as the number of subintervals goes to infinity. Instead, we must take the limit as the width of the largest subinterval goes to zero. This introduces a little more complex notation in our limits and makes the calculations more difficult without really gaining much additional insight, so we stick with regular partitions for the Riemann sums.

Example 4.3.1: Evaluating an Integral Using the Definition

Use the definition of the definite integral to evaluate $\int_0^2 x^2\,dx$. Use a right-endpoint approximation to generate the Riemann sum.

Solution

We first want to set up a Riemann sum. Based on the limits of integration, we have $a=0$ and $b=2$. For $i=0,1,2,\ldots,n$, let $P=x_i$ be a regular partition of $[0,2]$. Then

$$\Delta x=\frac{b-a}{n}=\frac{2}{n}.$$

Since we are using a right-endpoint approximation to generate Riemann sums, for each i, we need to calculate the function value at the right endpoint of the interval $[x_{i-1},x_i]$. The right endpoint of the interval is x_i, and since P is a regular partition,

$$x_i=x_0+i\Delta x=0+i\left[\frac{2}{n}\right]=\frac{2i}{n}.$$

Thus, the function value at the right endpoint of the interval is

$$f(x_i)=x_i^2=\left(\frac{2i}{n}\right)^2=\frac{4i^2}{n^2}.$$

Then the Riemann sum takes the form

$$\sum_{i=1}^{n}f(x_i)\Delta x=\sum_{i=1}^{n}\left(\frac{4i^2}{n^2}\right)\frac{2}{n}=\sum_{i=1}^{n}\frac{8i^2}{n^3}=\frac{8}{n^3}\sum_{i=1}^{n}i^2.$$

Using the summation formula for $\sum_{i=1}^{n}i^2$, we have

$$\begin{aligned}\sum_{i=1}^{n}f(x_i)\Delta x&=\frac{8}{n^3}\sum_{i=1}^{n}i^2\\&=\frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\\&=\frac{8}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right]\\&=\frac{16n^3+24n^2+n}{6n^3}\\&=\frac{8}{3}+\frac{4}{n}+\frac{1}{6n^2}.\end{aligned}$$

Now, to calculate the definite integral, we need to take the limit as $n\to\infty$. We get

$$\begin{aligned}\int_0^2 x^2 dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \\ &= \lim_{n\to\infty}\left(\frac{8}{3}+\frac{4}{n}+\frac{1}{6n^2}\right) \\ &= \lim_{n\to\infty}\left(\frac{8}{3}\right)+\lim_{n\to\infty}\left(\frac{4}{n}\right)+\lim_{n\to\infty}\left(\frac{1}{6n^2}\right) \\ &= \frac{8}{3}+0+0=\frac{8}{3}.\end{aligned}$$

? Exercise 4.3.1

Use the definition of the definite integral to evaluate $\int_0^3 (2x-1)\,dx$.

Use a right-endpoint approximation to generate the Riemann sum.

Hint

Use the solving strategy from Example 4.3.1

Answer

6

4.3.2 Evaluating Definite Integrals

Evaluating definite integrals this way can be quite tedious because of the complexity of the calculations. Later in this chapter we develop techniques for evaluating definite integrals *without* taking limits of Riemann sums. However, for now, we can rely on the fact that definite integrals represent the area under the curve, and we can evaluate definite integrals by using geometric formulas to calculate that area. We do this to confirm that definite integrals do, indeed, represent areas, so we can then discuss what to do in the case of a curve of a function dropping below the x-axis.

✓ Example 4.3.2: Using Geometric Formulas to Calculate Definite Integrals

Use the formula for the area of a circle to evaluate $\int_3^6 \sqrt{9-(x-3)^2}\,dx$.

Solution

The function describes a semicircle with radius 3. To find

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx$$

we want to find the area under the curve over the interval $[3,6]$. The formula for the area of a circle is $A=\pi r^2$. The area of a semicircle is just one-half the area of a circle, or $A=\frac{1}{2}\pi r^2$. The shaded area in Figure 4.3.1 covers one-half of the semicircle, or $A=\frac{1}{4}\pi r^2$. Thus,

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx=\frac{1}{4}\pi(3)^2=\frac{9}{4}\pi\approx 7.069.$$

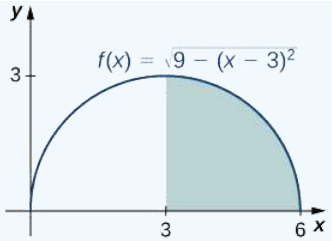


Figure 4.3.1: The value of the integral of the function $f(x)$ over the interval $[3, 6]$ is the area of the shaded region.

? Exercise 4.3.2

Use the formula for the area of a trapezoid to evaluate $\int_2^4 (2x+3)\,dx$.

Hint

Graph the function $f(x)$ and calculate the area under the function on the interval $[2, 4]$.

Answer

18 square units

4.3.3 Area and the Definite Integral

When we defined the definite integral, we lifted the requirement that $f(x)$ be nonnegative. But how do we interpret "the area under the curve" when $f(x)$ is negative?

4.3.3.1 Net Signed Area

Let us return to the Riemann sum. Consider, for example, the function $f(x) = 2 - 2x^2$ (shown in Figure 4.3.2) on the interval $[0, 2]$. Use $n = 8$ and choose $\{x_i^*\}$ as the left endpoint of each interval. Construct a rectangle on each subinterval of height $f(x_i^*)$ and width Δx. When $f(x_i^*)$ is positive, the product $f(x_i^*)\Delta x$ represents the area of the rectangle, as before. When $f(x_i^*)$ is negative, however, the product $f(x_i^*)\Delta x$ represents the negative of the area of the rectangle. The Riemann sum then becomes

$$\sum_{i=1}^{8} f(x_i^*)\Delta x = (\text{Area of rectangles above the } x\text{-axis}) - (\text{Area of rectangles below the } x\text{-axis})$$

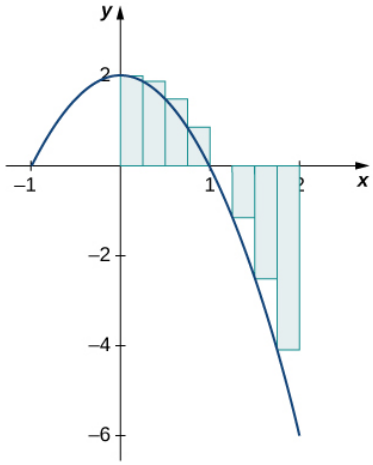


Figure 4.3.2: For a function that is partly negative, the Riemann sum is the area of the rectangles above the x-axis less the area of the rectangles below the x-axis.

Taking the limit as $n \to \infty$, the Riemann sum approaches the area between the curve above the x-axis and the x-axis, less the area between the curve below the x-axis and the x-axis, as shown in Figure 4.3.3. Then,

$$\int_0^2 f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x = A_1 - A_2.$$

The quantity $A_1 - A_2$ is called the **net signed area.**

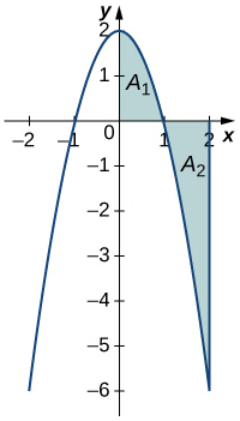


Figure 4.3.3: In the limit, the definite integral equals area A_1 less area A_2, or the net signed area.

Notice that net signed area can be positive, negative, or zero. If the area above the x-axis is larger, the net signed area is positive. If the area below the x-axis is larger, the net signed area is negative. If the areas above and below the x-axis are equal, the net signed area is zero.

Example 4.3.3: Finding the Net Signed Area

Find the net signed area between the curve of the function $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Solution

The function produces a straight line that forms two triangles: one from $x = -3$ to $x = 0$ and the other from $x = 0$ to $x = 3$ (Figure 4.3.4). Using the geometric formula for the area of a triangle, $A = \frac{1}{2}bh$, the area of triangle A_1, above the axis, is

$$A_1 = \frac{1}{2}3(6) = 9\,,$$

where 3 is the base and $2(3) = 6$ is the height. The area of triangle A_2, below the axis, is

$$A_2 = \frac{1}{2}(3)(6) = 9,$$

where 3 is the base and 6 is the height. Thus, the net area is

$$\int_{-3}^{3} 2x\,dx = A_1 - A_2 = 9 - 9 = 0.$$

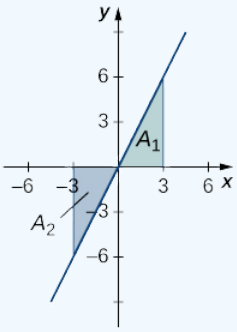


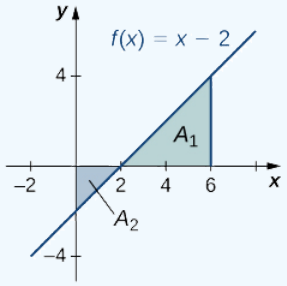
Figure 4.3.4: The area above the curve and below the x-axis equals the area below the curve and above the x-axis.

Analysis

If A_1 is the area above the x-axis and A_2 is the area below the x-axis, then the net area is $A_1 - A_2$. Since the areas of the two triangles are equal, the net area is zero.

Exercise 4.3.3

Find the net signed area of $f(x) = x - 2$ over the interval $[0, 6]$, illustrated in the following image.



Hint

Use the solving method described in Example 4.3.3

Answer

6

4.3.4 Total Area

One application of the definite integral is finding **displacement** when given a velocity function. If $v(t)$ represents the velocity of an object as a function of time, then the area under the curve tells us how far the object is from its original position. This is a very important application of the definite integral, and we examine it in more detail later in the chapter. For now, we're just going to look at some basics to get a feel for how this works by studying constant velocities.

When velocity is a constant, the area under the curve is just velocity times time. This idea is already very familiar. If a car travels away from its starting position in a straight line at a speed of 70 mph for 2 hours, then it is 140 miles away from its original position (Figure 4.3.5). Using integral notation, we have

$$\int_0^2 70\,dt = 140 \text{ miles.}$$

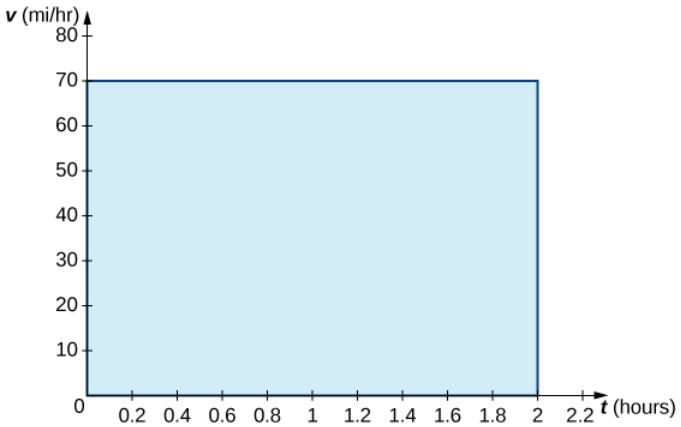


Figure 4.3.5: The area under the curve $v(t) = 70$ tells us how far the car is from its starting point at a given time.

In the context of displacement, net signed area allows us to take direction into account. If a car travels straight north at a speed of 60 mph for 2 hours, it is 120 miles north of its starting position. If the car then turns around and travels south at a speed of 40 mph for 3 hours, it will be back at it starting position (Figure 4.3.6). Again, using integral notation, we have

$$\int_0^2 60\,dt + \int_2^5 -40\,dt = 120 - 120 = 0.$$

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In this case the displacement is zero.

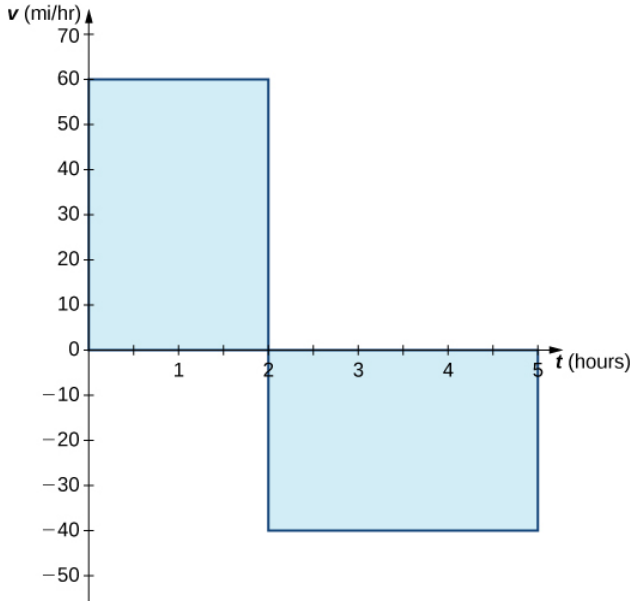


Figure 4.3.6: The area above the axis and the area below the axis are equal, so the net signed area is zero.

Suppose we want to know how far the car travels overall, regardless of direction. In this case, we want to know the area between the curve and the t-axis, regardless of whether that area is above or below the axis. This is called the **total area.**

Graphically, it is easiest to think of calculating total area by adding the areas above the axis and the areas below the axis (rather than subtracting the areas below the axis, as we did with net signed area). To accomplish this mathematically, we use the absolute value function. Thus, the total distance traveled by the car is

$$\int_0^2 |60|\,dt + \int_2^5 |-40|\,dt = \int_0^2 60\,dt + \int_2^5 40\,dt = 120 + 120 = 240.$$

Bringing these ideas together formally, we state the following definitions.

Definition: Net Signed Area

Let $f(x)$ be an integrable function defined on an interval $[a, b]$. Let A_1 represent the area between $f(x)$ and the x-axis that lies above the axis and let A_2 represent the area between $f(x)$ and the x-axis that lies below the axis. Then, the *net signed area* between $f(x)$ and the x-axis is given by

$$\int_a^b f(x)\,dx = A_1 - A_2.$$

The total area between $f(x)$ and the x-axis is given by

$$\int_a^b |f(x)|\,dx = A_1 + A_2.$$

Example 4.3.4: Finding the Total Area

Find the total area between $f(x) = x - 2$ and the x-axis over the interval $[0, 6]$.

Solution

Calculate the x-intercept as $(2, 0)$ (set $y = 0$, solve for x). To find the total area, take the area below the x-axis over the subinterval $[0, 2]$ and add it to the area above the x-axis on the subinterval $[2, 6]$ (Figure 4.3.7).

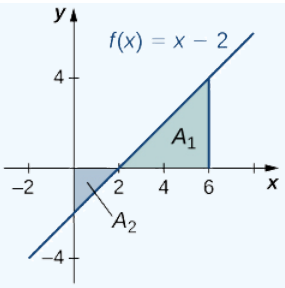


Figure 4.3.7: The total area between the line and the x-axis over $[0,6]$ is A_2 plus A_1.

We have

$$\int_0^6 |(x-2)|\,dx = A_2 + A_1.$$

Then, using the formula for the area of a triangle, we obtain

$$A_2 = \frac{1}{2}bh = \frac{1}{2}\cdot 2\cdot 2 = 2$$

$$A_1 = \frac{1}{2}bh = \frac{1}{2}\cdot 4\cdot 4 = 8.$$

The total area, then, is

$$A_1 + A_2 = 8 + 2 = 10 \text{ units}^2.$$

? Exercise 4.3.4

Find the total area between the function $f(x) = 2x$ and the x-axis over the interval $[-3,3]$.

Hint

Review the solving strategy in Example 4.3.4

Answer

18 units^2

4.3.5 Properties of the Definite Integral

The properties of indefinite integrals apply to definite integrals as well. Definite integrals also have properties that relate to the limits of integration. These properties, along with the rules of integration that we examine later in this chapter, help us manipulate expressions to evaluate definite integrals.

Rule: Properties of the Definite Integral

1.

$$\int_a^a f(x)\,dx = 0$$

If the limits of integration are the same, the integral is just a line and contains no area.

2.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

If the limits are reversed, then place a negative sign in front of the integral.

3.

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

The integral of a sum is the sum of the integrals.

4.

$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

The integral of a difference is the difference of the integrals.

5.

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

for constant c. The integral of the product of a constant and a function is equal to the constant multiplied by the integral of the function.

6.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Although this formula normally applies when c is between a and b, the formula holds for all values of a, b, and c, provided $f(x)$ is integrable on the largest interval.

Example 4.3.5: Using the Properties of the Definite Integral

Use the properties of the definite integral to express the definite integral of $f(x) = -3x^3 + 2x + 2$ over the interval $[-2, 1]$ as the sum of three definite integrals.

Solution

Using integral notation, we have $\int_{-2}^{1} (-3x^3 + 2x + 2)\,dx$. We apply properties 3. and 5. to get

$$\begin{aligned}\int_{-2}^{1} (-3x^3 + 2x + 2)\,dx &= \int_{-2}^{1} -3x^3\,dx + \int_{-2}^{1} 2x\,dx + \int_{-2}^{1} 2\,dx \\ &= -3\int_{-2}^{1} x^3\,dx + 2\int_{-2}^{1} x\,dx + \int_{-2}^{1} 2\,dx.\end{aligned}$$

Exercise 4.3.5

Use the properties of the definite integral to express the definite integral of $f(x) = 6x^3 - 4x^2 + 2x - 3$ over the interval $[1, 3]$ as the sum of four definite integrals.

Hint

Use the solving strategy from Example 4.3.5and the properties of definite integrals.

Answer

$$6\int_1^3 x^3\,dx - 4\int_1^3 x^2\,dx + 2\int_1^3 x\,dx - \int_1^3 3\,dx$$

Example 4.3.6: Using the Properties of the Definite Integral

If it is known that $\int_0^8 f(x)\,dx = 10$ and $\int_0^5 f(x)\,dx = 5$, find the value of $\int_5^8 f(x)\,dx$.

Solution

By property 6,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Thus,

$$\begin{aligned}\int_0^8 f(x)\,dx &= \int_0^5 f(x)\,dx + \int_5^8 f(x)\,dx \\ 10 &= 5 + \int_5^8 f(x)\,dx \\ 5 &= \int_5^8 f(x)\,dx.\end{aligned}$$

Exercise 4.3.6

If it is known that $\int_1^5 f(x)\,dx = -3$ and $\int_2^5 f(x)\,dx = 4$, find the value of $\int_1^2 f(x)\,dx$.

Hint

Use the solving strategy from Example 4.3.6and the rule on properties of definite integrals.

Answer

-7

4.3.6 Comparison Properties of Integrals

A picture can sometimes tell us more about a function than the results of computations. Comparing functions by their graphs as well as by their algebraic expressions can often give new insight into the process of integration. Intuitively, we might say that if a function $f(x)$ is above another function $g(x)$, then the area between $f(x)$ and the x-axis is greater than the area between $g(x)$ and the x-axis. This is true depending on the interval over which the comparison is made. The properties of definite integrals are valid whether $a < b$, $a = b$, or $a > b$. The following properties, however, concern only the case $a \le b$, and are used when we want to compare the sizes of integrals.

Comparison Theorem

i. If $f(x) \ge 0$ for $a \le x \le b$, then

$$\int_a^b f(x)\,dx \ge 0.$$

ii. If $f(x) \ge g(x)$ for $a \le x \le b$, then

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx.$$

iii. If m and M are constants such that $m \le f(x) \le M$ for $a \le x \le b$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Example 4.3.7: Comparing Two Functions over a Given Interval

Compare $f(x) = \sqrt{1+x^2}$ and $g(x) = \sqrt{1+x}$ over the interval $[0, 1]$.

Solution

Graphing these functions is necessary to understand how they compare over the interval $[0, 1]$. Initially, when graphed on a graphing calculator, $f(x)$ appears to be above $g(x)$ everywhere. However, on the interval $[0, 1]$, the graphs appear to be on top of each other. We need to zoom in to see that, on the interval $[0, 1]$, $g(x)$is above $f(x)$. The two functions intersect at $x = 0$ and $x = 1$ (Figure 4.3.8).

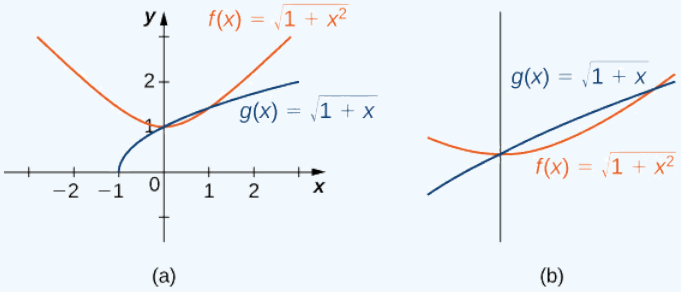


Figure 4.3.8: (a) The function $f(x)$ appears above the function $g(x)$ except over the interval $[0, 1]$ (b) Viewing the same graph with a greater zoom shows this more clearly.

We can see from the graph that over the interval $[0, 1]$, $g(x) \geq f(x)$. Comparing the integrals over the specified interval $[0, 1]$, we also see that $\int_0^1 g(x)\,dx \geq \int_0^1 f(x)\,dx$ (Figure 4.3.9). The thin, red-shaded area shows just how much difference there is between these two integrals over the interval $[0, 1]$.

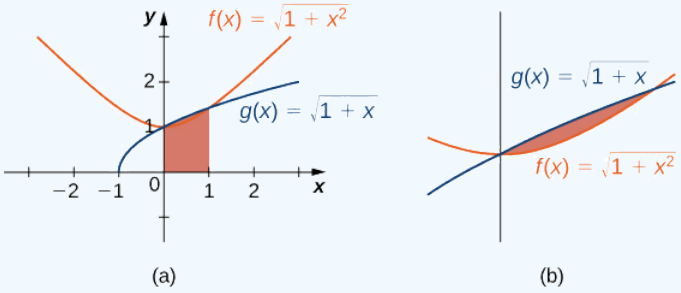


Figure 4.3.9: (a) The graph shows that over the interval $[0, 1], g(x) \geq f(x)$, where equality holds only at the endpoints of the interval. (b) Viewing the same graph with a greater zoom shows this more clearly.

4.3.7 Average Value of a Function

We often need to find the average of a set of numbers, such as an average test grade. Suppose you received the following test scores in your algebra class: 89, 90, 56, 78, 100, and 69. Your semester grade is your average of test scores and you want to know what grade to expect. We can find the average by adding all the scores and dividing by the number of scores. In this case, there are six test scores. Thus,

$$\frac{89+90+56+78+100+69}{6} = \frac{482}{6} \approx 80.33.$$

Therefore, your average test grade is approximately 80.33, which translates to a B− at most schools.

Suppose, however, that we have a function $v(t)$ that gives us the speed of an object at any time t, and we want to find the object's average speed. The function $v(t)$ takes on an infinite number of values, so we can't use the process just described. Fortunately, we can use a definite integral to find the average value of a function such as this.

Let $f(x)$ be continuous over the interval $[a, b]$ and let $[a, b]$ be divided into n subintervals of width $\Delta x = (b-a)/n$. Choose a representative x_i^* in each subinterval and calculate $f(x_i^*)$ for $i = 1, 2, \ldots, n$. In other words, consider each $f(x_i^*)$ as a sampling of the function over each subinterval. The average value of the function may then be approximated as

$$f_{ave} \approx \frac{f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)}{n},$$

which is basically the same expression used to calculate the average of discrete values.

But we know $\Delta x = \frac{b-a}{n}$, so $n = \frac{b-a}{\Delta x}$, and we get

$$f_{ave} \approx \frac{f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)}{n} = \frac{f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)}{\left(\frac{b-a}{\Delta x}\right)}.$$

Following through with the algebra, the numerator is a sum that is represented as $\sum_{i=1}^{n} f(x*i)$, and we are dividing by a fraction. To divide by a fraction, invert the denominator and multiply. Thus, an approximate value for the average value of the function is given by

$$\frac{\sum_{i=1}^{n} f(x_i^*)}{\left(\frac{b-a}{\Delta x}\right)} = \left(\frac{\Delta x}{b-a}\right)\sum_{i=1}^{n} f(x_i^*) = \left(\frac{1}{b-a}\right)\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

This is a Riemann sum. Then, to get the exact average value, take the limit as n goes to infinity. Thus, the average value of a function is given by

$$\frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{b-a}\int_a^b f(x)dx.$$

Definition: Average Value of a Function

Let $f(x)$ be continuous over the interval $[a,b]$. Then, the *average value of the function* $f(x)$ (or f_{ave}) on $[a,b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{4.3.1}$$

Example 4.3.8: Finding the Average Value of a Linear Function

Find the average value of $f(x) = x+1$ over the interval $[0,5]$.

Solution

First, graph the function on the stated interval, as shown in Figure 4.3.10

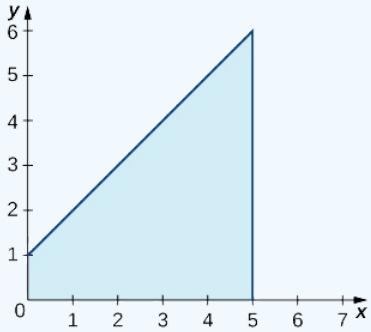


Figure 4.3.10:The graph shows the area under the function $(x) = x+1$ over $[0,5]$.

The region is a trapezoid lying on its side, so we can use the area formula for a trapezoid $A = \frac{1}{2}h(a+b)$, where h represents height, and a and b represent the two parallel sides. Then,

$$\int_0^5 x+1\,dx = \frac{1}{2}h(a+b) = \frac{1}{2}\cdot 5\cdot(1+6) = \frac{35}{2} \quad .$$

Thus the average value of the function is

$$\frac{1}{5-0}\int_0^5 x+1\,dx=\frac{1}{5}\cdot\frac{35}{2}=\frac{7}{2}\ .$$

? Exercise 4.3.7

Find the average value of $f(x)=6-2x$ over the interval $[0,3]$.

Hint

Use the average value formula (Equation 4.3.1), and use geometry to evaluate the integral.

Answer

3

4.3.8 Key Concepts

- The definite integral can be used to calculate net signed area, which is the area above the x-axis less the area below the x-axis. Net signed area can be positive, negative, or zero.
- The component parts of the definite integral are the integrand, the variable of integration, and the limits of integration.
- Continuous functions on a closed interval are integrable. Functions that are not continuous may still be integrable, depending on the nature of the discontinuities.
- The properties of definite integrals can be used to evaluate integrals.
- The area under the curve of many functions can be calculated using geometric formulas.
- The average value of a function can be calculated using definite integrals.

4.3.9 Key Equations

- **Definite Integral**

$$\int_a^b f(x)\,dx=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i^*)\Delta x$$

- **Properties of the Definite Integral**

$$\int_a^a f(x)\,dx=0$$

$$\int_b^a f(x)\,dx=-\int_a^b f(x)\,dx$$

$$\int_a^b [f(x)+g(x)]\,dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx$$

$$\int_a^b [f(x)-g(x)]\,dx=\int_a^b f(x)\,dx-\int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx=c\int_a^b f(x)\,dx,\text{ for constant } c$$

$$\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$$

4.3.10 Glossary

average value of a function

(or f_{ave}) the average value of a function on an interval can be found by calculating the definite integral of the function and dividing that value by the length of the interval

definite integral

a primary operation of calculus; the area between the curve and the x-axis over a given interval is a definite integral

integrable function

a function is integrable if the limit defining the integral exists; in other words, if the limit of the Riemann sums as n goes to infinity exists

integrand

the function to the right of the integration symbol; the integrand includes the function being integrated

limits of integration

these values appear near the top and bottom of the integral sign and define the interval over which the function should be integrated

net signed area

the area between a function and the x-axis such that the area below the x-axis is subtracted from the area above the x-axis; the result is the same as the definite integral of the function

total area

total area between a function and the x-axis is calculated by adding the area above the x-axis and the area below the x-axis; the result is the same as the definite integral of the absolute value of the function

variable of integration

indicates which variable you are integrating with respect to; if it is x, then the function in the integrand is followed by dx

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4.3E: EXERCISES FOR SECTION 4.3

In exercises 1 - 2, express the limits as integrals.

1) $\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^*)\Delta x$ over $[1,3]$

2) $\lim_{n\to\infty}\sum_{i=1}^{n}(5(x_i^*)^2-3(x_i^*)^3)\Delta x$ over $[0,2]$

Answer

$$\int_0^2 (5x^2-3x^3)\,dx$$

In exercises 3-7, given L_n or R_n as indicated, express their limits as $n\to\infty$ as definite integrals, identifying the correct intervals.

3) $L_n=\frac{1}{n}\sum_{i=1}^{n}\frac{i-1}{n}$

4) $R_n=\frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}$

Answer

$$\int_0^1 x\,dx$$

5) $Ln=\frac{2}{n}\sum_{i=1}^{n}(1+2\frac{i-1}{n})$

6) $R_n=\frac{3}{n}\sum_{i=1}^{n}(3+3\frac{i}{n})$

Answer

$$\int_3^6 x\,dx$$

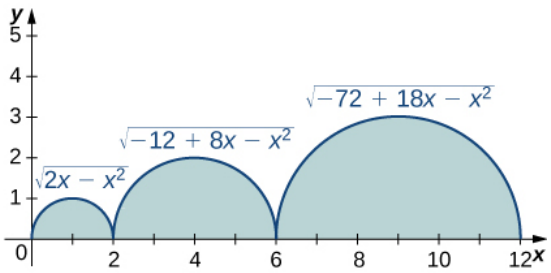
7) $R_n=\frac{1}{n}\sum_{i=1}^{n}(1+\frac{i}{n})\log((1+\frac{i}{n})^2)$

Answer

$$\int_1^2 x\log(x^2)\,dx$$

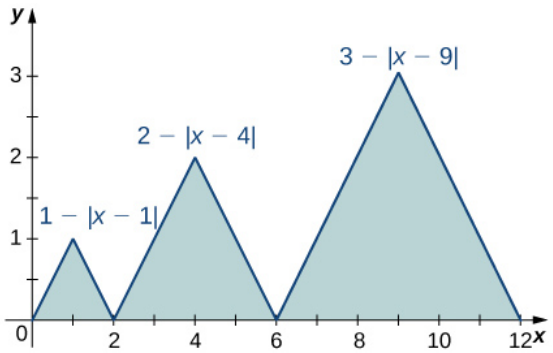
In exercises 8-13, evaluate the integrals of the functions graphed using the formulas for areas of triangles and circles, and subtracting the areas below the x-axis.

8)



9)

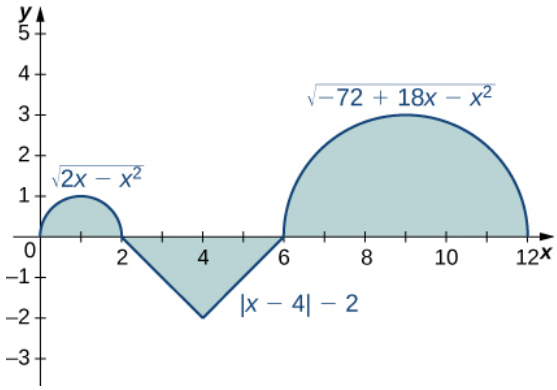
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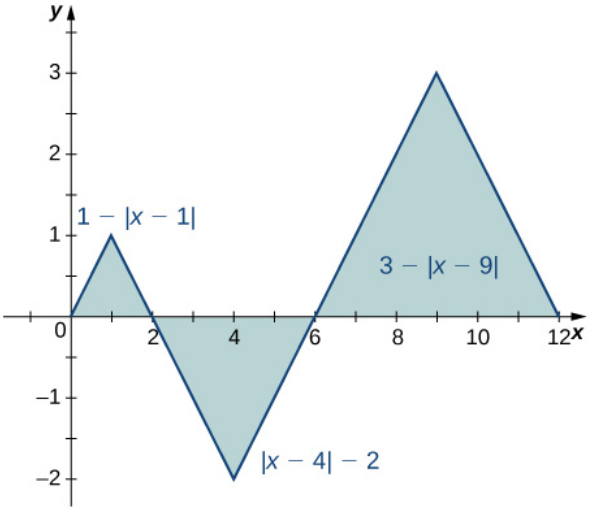
Answer

$1+2\cdot 2+3\cdot 3=14$

10)



11)

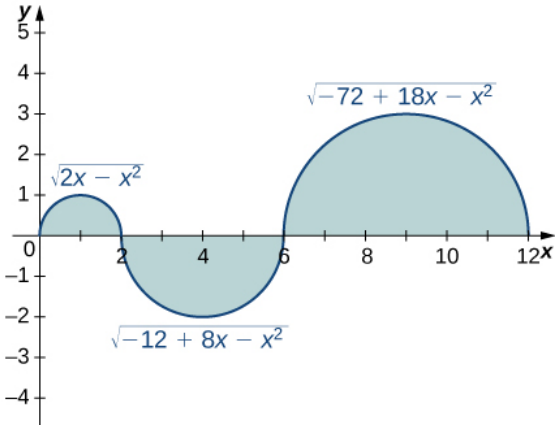


Answer

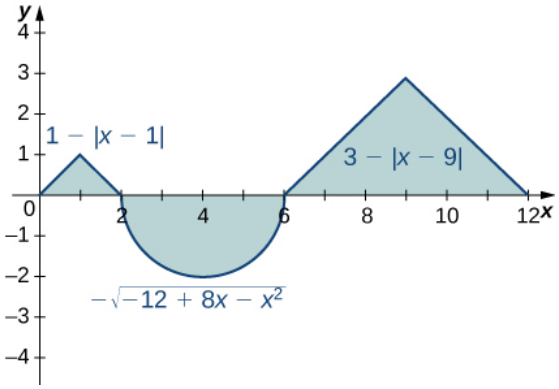
$1-4+9=6$

12)

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13)



Answer

$1-2\pi+9=10-2\pi$

In exercises 14-21, evaluate the integral using area formulas.

14) $\int_0^3 (3-x)\,dx$

15) $\int_2^3 (3-x)\,dx$

Answer

The integral is the area of the triangle, $\frac{1}{2}$.

16) $\int_{-3}^3 (3-|x|)\,dx$

17) $\int_0^6 (3-|x-3|)\,dx$

Answer

The integral is the area of the triangle, 9.

18) $\int_{-2}^2 \sqrt{4-x^2}\,dx$

19) $\int_1^5 \sqrt{4-(x-3)^2}\,dx$

Answer

The integral is the area $\frac{1}{2}\pi r^2=2\pi$.

20) $\int_0^{12} \sqrt{36-(x-6)^2}\, dx$

21) $\int_{-2}^{3} (3-|x|)\, dx$

Answer

The integral is the area of the "big" triangle less the "missing" triangle, $9-\frac{1}{2}$.

In exercises 22-25, use averages of values at the left (L) and right (R) endpoints to compute the integrals of the piecewise linear functions with graphs that pass through the given list of points over the indicated intervals.

22) $(0,0),(2,1),(4,3),(5,0),(6,0),(8,3)$ over $[0,8]$

23) $(0,2),(1,0),(3,5),(5,5),(6,2),(8,0)$ over $[0,8]$

Answer

$$L=2+0+10+5+4=21,\ R=0+10+10+2+0=22,\ \frac{L+R}{2}=21.5$$

24) $(-4,-4),(-2,0),(0,-2),(3,3),(4,3)$ over $[-4,4]$

25) $(-4,0),(-2,2),(0,0),(1,2),(3,2),(4,0)$ over $[-4,4]$

Answer

$$L=0+4+0+4+2=10,\ R=4+0+2+4+0=10,\ \frac{L+R}{2}=10$$

Suppose that $\int_0^4 f(x)\, dx=5$ and $\int_0^2 f(x)\, dx=-3$, and $\int_0^4 g(x)\, dx=-1$ and $\int_0^2 g(x)\, dx=2$. In exercises 26-31, compute the integrals.

26) $\int_0^4 (f(x)+g(x))\, dx$

27) $\int_2^4 (f(x)+g(x))\, dx$

Answer

$$\int_2^4 f(x)\, dx+\int_2^4 g(x)\, dx=8-3=5$$

28) $\int_0^2 (f(x)-g(x))\, dx$

29) $\int_2^4 (f(x)-g(x))\, dx$

Answer

$$\int_2^4 f(x)\, dx-\int_2^4 g(x)\, dx=8+3=11$$

30) $\int_0^2 (3f(x)-4g(x))\, dx$

31) $\int_2^4 (4f(x)-3g(x))\, dx$

Answer

$$4\int_2^4 f(x)\, dx-3\int_2^4 g(x)\, dx=32+9=41$$

In exercises 32-35, use the identity $\int_{-A}^{A} f(x)\, dx=\int_{-A}^{0} f(x)\, dx+\int_0^A f(x)\, dx$ to compute the integrals.

32) $\int_{-\pi}^{\pi} \frac{\sin t}{1+t^2} dt$ (Hint: $\sin(-t) = -\sin(t)$)

33) $\int_{\sqrt{-\pi}}^{\sqrt{\pi}} \frac{t}{1+\cos t} dt$

Answer

The integrand is odd; the integral is zero.

34) $\int_1^3 (2-x)\, dx$ (Hint: Look at the graph of f.)

35) $\int_2^4 (x-3)^3\, dx$ (Hint: Look at the graph of f.)

Answer

The integrand is antisymmetric with respect to $x = 3$. The integral is zero.

In exercises 36-41, given that $\int_0^1 x\, dx = \frac{1}{2}$, $\int_0^1 x^2\, dx = \frac{1}{3}$, and $\int_0^1 x^3\, dx = \frac{1}{4}$, compute the integrals.

36) $\int_0^1 (1+x+x^2+x^3)\, dx$

37) $\int_0^1 (1-x+x^2-x^3)\, dx$

Answer

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$

38) $\int_0^1 (1-x)^2\, dx$

39) $\int_0^1 (1-2x)^3\, dx$

Answer

$\int_0^1 (1-6x+12x^2-8x^3)\, dx = 1 - 6\left(\frac{1}{2}\right) + 12\left(\frac{1}{3}\right) - 8\left(\frac{1}{4}\right) = 1-3+4-2 = 0$

40) $\int_0^1 \left(6x - \frac{4}{3}x^2\right)\, dx$

41) $\int_0^1 (7-5x^3)\, dx$

Answer

$7 - \frac{5}{4} = \frac{23}{4}$

In exercises 42-45, use the comparison theorem.

42) Show that $\int_0^3 (x^2-6x+9)\, dx \geq 0$.

43) Show that $\int_{-2}^3 (x-3)(x+2)\, dx \leq 0$.

Answer

The integrand is negative over $[-2, 3]$.

44) Show that $\int_0^1 \sqrt{1+x^3}\, dx \leq \int_0^1 \sqrt{1+x^2}\, dx$.

45) Show that $\int_1^2 \sqrt{1+x}\,dx \le \int_1^2 \sqrt{1+x^2}\,dx$.

Answer

$x \le x^2$ over $[1,2]$, so $\sqrt{1+x} \le \sqrt{1+x^2}$ over $[1,2]$.

In exercises 46-49, find the average value f_{ave} of f between a and b, and find a point c, where $f(c) = f_{ave}$

46) $f(x) = x^2,\ a = -1,\ b = 1$

47) $f(x) = x^5,\ a = -1,\ b = 1$

Answer

$f_{ave} = 0;\ c = 0$

48) $f(x) = \sqrt{4-x^2},\ a = 0,\ b = 2$

49) $f(x) = 3 - |x|,\ a = -3,\ b = 3$

Answer

$\frac{3}{2}$ when $c = \pm\frac{3}{2}$

In exercises 50-52, approximate the average value using Riemann sums L_{100} and R_{100}. How does your answer compare with the exact given answer?

50) [T] $y = \ln(x)$ over the interval $[1,4]$; the exact solution is $\dfrac{\ln(256)}{3} - 1$.

51) [T] $y = e^{x/2}$ over the interval $[0,1]$; the exact solution is $2(\sqrt{e}-1)$.

Answer

$L_{100} = 1.294,\ R_{100} = 1.301;$ the exact average is between these values.

52) [T] $y = \dfrac{x+1}{\sqrt{4-x^2}}$ over the interval $[-1,1]$; the exact solution is $\dfrac{\pi}{6}$.

Answer

$L_{100} \times (\frac{1}{2}) = 0.5178,\ R_{100} \times (\frac{1}{2}) = 0.5294$

In exercises 53-55, compute the average value using the left Riemann sums L_N for $N = 1, 10, 100$. How does the accuracy compare with the given exact value?

53) [T] $y = x^2 - 4$ over the interval $[0,2]$; the exact solution is $-\frac{8}{3}$.

54) [T] $y = xe^{x^2}$ over the interval $[0,2]$; the exact solution is $\frac{1}{4}(e^4 - 1)$.

Answer

$L_1 = 0,\ L_{10} \times (\frac{1}{2}) = 8.743493,\ L_{100} \times (\frac{1}{2}) = 12.861728.$ The exact answer ≈ 26.799, so L_{100} is not accurate.

55) [T] $y = \left(\dfrac{1}{2}\right)^x$ over the interval $[0,4]$; the exact solution is $\dfrac{15}{64\ln(2)}$.

56) Explain why the graphs of a quadratic function (parabola) $p(x)$ and a linear function $\ell(x)$ can intersect in at most two points. Suppose that $p(a) = \ell(a)$ and $p(b) = \ell(b)$, and that $\int_a^b p(t)\,dt > \int_a^b \ell(t)dt$. Explain why $\int_c^d p(t) > \int_c^d \ell(t)\,dt$ whenever $a \le c < d \le b$.

57) Suppose that parabola $p(x) = ax^2 + bx + c$ opens downward $(a < 0)$ and has a vertex of $y = \dfrac{-b}{2a} > 0$. For which interval $[A,B]$ is $\int_A^B (ax^2 + bx + c)\,dx$ as large as possible?

Answer

The integral is maximized when one uses the largest interval on which p is nonnegative. Thus, $A = \frac{-b-\sqrt{b^2-4ac}}{2a}$ and $B = \frac{-b+\sqrt{b^2-4ac}}{2a}$.

58) Suppose $[a,b]$ can be subdivided into subintervals $a = a_0 < a_1 < a_2 < \cdots < a_N = b$ such that either $f \geq 0$ over $[a_{i-1}, a_i]$ or $f \leq 0$ over $[a_{i-1}, a_i]$. Set $A_i = \int_{a_{i-1}}^{a_i} f(t)\, dt$.

a. Explain why $\int_a^b f(t)\, dt = A_1 + A_2 + \cdots + A_N$.

b. Then, explain why $\int_a^b f(t)\, dt \leq \int_a^b |f(t)|\, dt$.

59) Suppose f and g are continuous functions such that $\int_c^d f(t)\, dt \leq \int_c^d g(t)\, dt$ for every subinterval $[c,d]$ of $[a,b]$. Explain why $f(x) \leq g(x)$ for all values of x.

Answer

If $f(t_0) > g(t_0)$ for some $t_0 \in [a,b]$, then since $f - g$ is continuous, there is an interval containing t_0 such that $f(t) > g(t)$ over the interval $[c,d]$, and then $\int_d^d f(t)\, dt > \int_c^d g(t)\, dt$over this interval.

60) Suppose the average value of f over $[a,b]$ is 1 and the average value of f over $[b,c]$ is 1 where $a < c < b$. Show that the average value of f over $[a,c]$ is also 1.

61) Suppose that $[a,b]$ can be partitioned. taking $a = a_0 < a_1 < \cdots < a_N = b$ such that the average value of f over each subinterval $[a_{i-1}, a_i] = 1$ is equal to 1 for each $i = 1, \ldots, N$. Explain why the average value of f over $[a,b]$ is also equal to 1.

Answer

The integral of f over an interval is the same as the integral of the average of f over that interval. Thus,

$\int_a^b f(t)\, dt = \int_{a_0}^{a_1} f(t)\, dt + \int_a^{a_2} 1_f(t)\, dt + \cdots + \int_{a_{N+1}}^{a_N} f(t)\, dt = \int_{a_0}^{a_1} 1\, dt + \int_{a_1}^{a_2} 1\, dt + \cdots + \int_{a_{N+1}}^{a_N} 1\, dt$

$= (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_N - a_{N-1}) = a_N - a_0 = b - a$.

Dividing through by $b - a$ gives the desired identity.

62) Suppose that for each i such that $1 \leq i \leq N$ one has $\int_{i-1}^{i} f(t)\, dt = i$. Show that $\int_0^N f(t)\, dt = \frac{N(N+1)}{2}$.

63) Suppose that for each i such that $1 \leq i \leq N$ one has $\int_{i-1}^{i} f(t)\, dt = i^2$. Show that $\int_0^N f(t)\, dt = \frac{N(N+1)(2N+1)}{6}$.

Answer

$$\int_0^N f(t)\, dt = \sum_{i=1}^{N} \int_{i-1}^{i} f(t)\, dt = \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$$

64) [T] Compute the left and right Riemann sums L_{10} and R_{10} and their average $\frac{L_{10}+R_{10}}{2}$ for $f(t) = t^2$over $[0,1]$. Given that $\int_0^1 t^2\, dt = 1/3$, to how many decimal places is $\frac{L_{10}+R_{10}}{2}$ accurate?

65) [T] Compute the left and right Riemann sums, L_10 and R_{10}, and their average $\frac{L_{10}+R_{10}}{2}$ for $f(t) = (4 - t^2)$ over $[1,2]$. Given that $\int_1^2 (4 - t^2)\, dt = 1.66$, to how many decimal places is $\frac{L_{10}+R_{10}}{2}$ accurate?

Answer

$L_{10} = 1.815,\ R_{10} = 1.515,\ \frac{L_{10}+R_{10}}{2} = 1.665,$ so the estimate is accurate to two decimal places.

66) If $\int_1^5 \sqrt{1+t^4}\, dt = 41.7133...,$ what is $\int_1^5 \sqrt{1+u^4}\, du$?

67) Estimate $\int_0^1 t\,dt$ using the left and right endpoint sums, each with a single rectangle. How does the average of these left and right endpoint sums compare with the actual value $\int_0^1 t\,dt$?

Answer

The average is $1/2$, which is equal to the integral in this case.

68) Estimate $\int_0^1 t\,dt$ by comparison with the area of a single rectangle with height equal to the value of t at the midpoint $t = \frac{1}{2}$. How does this midpoint estimate compare with the actual value $\int_0^1 t\,dt$?

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4.4: The Fundamental Theorem of Calculus

Learning Objectives

- Describe the meaning of the Mean Value Theorem for Integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 1.
- Use the Fundamental Theorem of Calculus, Part 1, to evaluate derivatives of integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 2.
- Use the Fundamental Theorem of Calculus, Part 2, to evaluate definite integrals.
- Explain the relationship between differentiation and integration.

In the previous two sections, we looked at the definite integral and its relationship to the area under the curve of a function. Unfortunately, so far, the only tools we have available to calculate the value of a definite integral are geometric area formulas and limits of Riemann sums, and both approaches are extremely cumbersome. In this section we look at some more powerful and useful techniques for evaluating definite integrals.

These new techniques rely on the relationship between differentiation and integration. This relationship was discovered and explored by both Sir Isaac Newton and Gottfried Wilhelm Leibniz (among others) during the late 1600s and early 1700s, and it is codified in what we now call the **Fundamental Theorem of Calculus**, which has two parts that we examine in this section. Its very name indicates how central this theorem is to the entire development of calculus.

Isaac Newton's contributions to mathematics and physics changed the way we look at the world. The relationships he discovered, codified as Newton's laws and the law of universal gravitation, are still taught as foundational material in physics today, and his calculus has spawned entire fields of mathematics.

Before we get to this crucial theorem, however, let's examine another important theorem, the Mean Value Theorem for Integrals, which is needed to prove the Fundamental Theorem of Calculus.

4.4.1 The Mean Value Theorem for Integrals

The Mean Value Theorem for Integrals states that a continuous function on a closed interval takes on its average value at the same point in that interval. The theorem guarantees that if $f(x)$ is continuous, a point c exists in an interval $[a, b]$ such that the value of the function at c is equal to the average value of $f(x)$ over $[a, b]$. We state this theorem mathematically with the help of the formula for the average value of a function that we presented at the end of the preceding section.

Theorem 4.4.1: The Mean Value Theorem for Integrals

If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

This formula can also be stated as

$$\int_a^b f(x)\, dx = f(c)(b-a). \tag{4.4.1}$$

Since $f(x)$ is continuous on $[a, b]$, by the extreme value theorem (see section on Maxima and Minima), it assumes minimum and maximum values—m and M, respectively—on $[a, b]$. Then, for all x in $[a, b]$, we have $m \le f(x) \le M$. Therefore, by the comparison theorem (see Section on The Definite Integral), we have

$$m(b-a) \le \int_a^b f(x)\, dx \le M(b-a).$$

Proof

Since $f(x)$ is continuous on $[a,b]$, by the extreme value theorem (see section on Maxima and Minima), it assumes minimum and maximum values—m and M, respectively—on $[a,b]$. Then, for all x in $[a,b]$, we have $m \le f(x) \le M$. Therefore, by the comparison theorem (see Section on The Definite Integral), we have

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Dividing by $b-a$ gives us

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

Since $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$ is a number between m and M, and since $f(x)$ is continuous and assumes the values m and M over $[a,b]$, by the Intermediate Value Theorem, there is a number c over $[a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

and the proof is complete.

□

Example 4.4.1: Finding the Average Value of a Function

Find the average value of the function $f(x) = 8 - 2x$ over the interval $[0,4]$ and find c such that $f(c)$ equals the average value of the function over $[0,4]$.

Solution

The formula states the mean value of $f(x)$ is given by

$$\frac{1}{4-0}\int_0^4 (8-2x)\,dx.$$

We can see in Figure 4.4.1 that the function represents a straight line and forms a right triangle bounded by the x- and y-axes. The area of the triangle is $A = \frac{1}{2}(base)(height)$. We have

$$A = \frac{1}{2}(4)(8) = 16.$$

The average value is found by multiplying the area by $1/(4-0)$. Thus, the average value of the function is

$$\frac{1}{4}(16) = 4$$

Set the average value equal to $f(c)$ and solve for c.

$$8 - 2c = 4$$
$$c = 2$$

At $c = 2$, $f(2) = 4$.

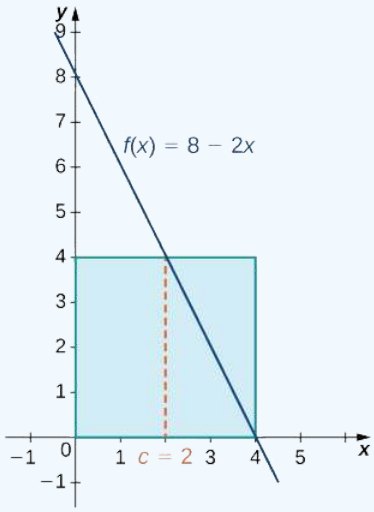


Figure 4.4.1: By the Mean Value Theorem, the continuous function $f(x)$ takes on its average value at c at least once over a closed interval.

Exercise 4.4.1

Find the average value of the function $f(x) = \dfrac{x}{2}$ over the interval $[0, 6]$ and find c such that $f(c)$ equals the average value of the function over $[0, 6]$.

Hint

Use the procedures from Example 4.4.1to solve the problem

Answer

The average value is 1.5 and $c = 3$.

Example 4.4.2: Finding the Point Where a Function Takes on Its Average Value

Given $\int_0^3 x^2\,dx = 9$, find c such that $f(c)$ equals the average value of $f(x) = x^2$ over $[0, 3]$.

Solution

We are looking for the value of c such that

$$f(c) = \frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}(9) = 3.$$

Replacing $f(c)$ with c^2, we have

$$c^2 = 3$$
$$c = \pm\sqrt{3}.$$

Since $-\sqrt{3}$ is outside the interval, take only the positive value. Thus, $c = \sqrt{3}$ (Figure 4.4.2).

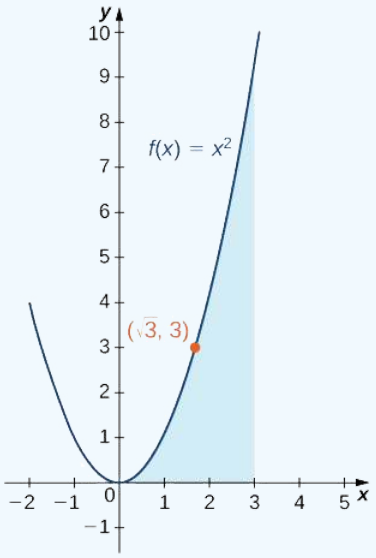


Figure 4.4.2: Over the interval $[0,3]$, the function $f(x)=x^2$ takes on its average value at $c=\sqrt{3}$.

? Exercise 4.4.2

Given $\int_0^3 (2x^2-1)\,dx = 15$, find c such that $f(c)$ equals the average value of $f(x)=2x^2-1$ over $[0,3]$.

Hint

Use the procedures from Example 4.4.2to solve the problem.

Answer

$c=\sqrt{3}$

4.4.2 Fundamental Theorem of Calculus Part 1: Integrals and Antiderivatives

As mentioned earlier, the Fundamental Theorem of Calculus is an extremely powerful theorem that establishes the relationship between differentiation and integration, and gives us a way to evaluate definite integrals without using Riemann sums or calculating areas. The theorem is comprised of two parts, the first of which, the **Fundamental Theorem of Calculus, Part 1,** is stated here. Part 1 establishes the relationship between differentiation and integration.

Theorem 4.4.2: The Fundamental Theorem of Calculus, Part 1

If $f(x)$ is continuous over an interval $[a,b]$, and the function $F(x)$ is defined by

$$F(x)=\int_a^x f(t)\,dt,$$

then $F'(x)=f(x)$ over $[a,b]$.

Before we delve into the proof, a couple of subtleties are worth mentioning here. First, a comment on the notation. Note that we have defined a function, $F(x)$, as the definite integral of another function, $f(t)$, from the point a to the point x. At first glance, this is confusing, because we have said several times that a definite integral is a number, and here it looks like it's a function. The key here is to notice that for any particular value of x, the definite integral is a number. So the function $F(x)$ returns a number (the value of the definite integral) for each value of x.

Second, it is worth commenting on some of the key implications of this theorem. There is a reason it is called the *Fundamental Theorem of Calculus*. Not only does it establish a relationship between integration and differentiation, but also it guarantees that any integrable function has an antiderivative. Specifically, it guarantees that any continuous function has an antiderivative.

Proof: Fundamental Theorem of Calculus, Part 1

Applying the definition of the derivative, we have

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt. \end{aligned}$$

Looking carefully at this last expression, we see $\frac{1}{h}\int_x^{x+h} f(t)\,dt$ is just the average value of the function $f(x)$ over the interval $[x, x+h]$. Therefore, by Equation 4.4.1, there is some number c in $[x, x+h]$ such that

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c).$$

In addition, since c is between x and h, c approaches x as h approaches zero. Also, since $f(x)$ is continuous, we have

$$\lim_{h\to 0} f(c) = \lim_{c\to x} f(c) = f(x)$$

Putting all these pieces together, we have

$$F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0} f(c) = f(x),$$

and the proof is complete.

□

Example 4.4.3: Finding a Derivative with the Fundamental Theorem of Calculus

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of

$$g(x) = \int_1^x \frac{1}{t^3+1}\,dt.$$

Solution

According to the Fundamental Theorem of Calculus, the derivative is given by

$$g'(x) = \frac{1}{x^3+1}.$$

Exercise 4.4.3

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of $g(r) = \int_0^r \sqrt{x^2+4}\,dx$.

Hint

Follow the procedures from Example 4.4.3to solve the problem.

Answer

$g'(r) = \sqrt{r^2+4}$

4.4.3 Fundamental Theorem of Calculus, Part 2: The Evaluation Theorem

The Fundamental Theorem of Calculus, Part 2, is perhaps the most important theorem in calculus. After tireless efforts by mathematicians for approximately 500 years, new techniques emerged that provided scientists with the necessary tools to explain many phenomena. Using calculus, astronomers could finally determine distances in space and map planetary orbits. Everyday financial problems such as calculating marginal costs or predicting total profit could now be handled with simplicity and accuracy. Engineers could calculate the bending strength of materials or the three-dimensional motion of objects. Our view of the world was forever changed with calculus.

After finding approximate areas by adding the areas of n rectangles, the application of this theorem is straightforward by comparison. It almost seems too simple that the area of an entire curved region can be calculated by just evaluating an antiderivative at the first and last endpoints of an interval.

Theorem 4.4.3: The Fundamental Theorem of Calculus, Part 2

If $f(x)$ is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{4.4.2}$$

We often see the notation $F(x)|_a^b$ to denote the expression $F(b) - F(a)$. We use this vertical bar and associated limits a and b to indicate that we should evaluate the function $F(x)$ at the upper limit (in this case, b), and subtract the value of the function $F(x)$ evaluated at the lower limit (in this case, a).

The **Fundamental Theorem of Calculus, Part 2** (also known as the **evaluation theorem**) states that if we can find an antiderivative for the integrand, then we can evaluate the definite integral by evaluating the antiderivative at the endpoints of the interval and subtracting.

Proof

Let $P = x_i, i = 0, 1, \ldots, n$ be a regular partition of $[a, b]$. Then, we can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \ldots + [F(x_1) - F(x_0)] \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]. \end{aligned}$$

Now, we know F is an antiderivative of f over $[a, b]$, so by the Mean Value Theorem for derivatives (see The Mean Value Theorem) for $i = 0, 1, \ldots, n$ we can find c_i in $[x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\,\Delta x.$$

Then, substituting into the previous equation, we have

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\,\Delta x.$$

Taking the limit of both sides as $n \to \infty$, we obtain

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\,dx.$$

□

Example 4.4.6: Evaluating an Integral with the Fundamental Theorem of Calculus

Use Equation 4.4.2 to evaluate

$$\int_{-2}^{2}(t^2-4)\,dt.$$

Solution

Recall the power rule for Antiderivatives:

If $y=x^n$,

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C.$$

Use this rule to find the antiderivative of the function and then apply the theorem. We have

$$\begin{aligned}\int_{-2}^{2}(t^2-4)dt &= \left(\frac{t^3}{3}-4t\right)\Big|_{-2}^{2}\\ &= \left[\frac{(2)^3}{3}-4(2)\right]-\left[\frac{(-2)^3}{3}-4(-2)\right]\\ &= \left[\frac{8}{3}-8\right]-\left[-\frac{8}{3}+8\right]\\ &= \frac{8}{3}-8+\frac{8}{3}-8\\ &= \frac{16}{3}-16=-\frac{32}{3}.\end{aligned}$$

Analysis

Notice that we did not include the "$+C$" term when we wrote the antiderivative. The reason is that, according to the Fundamental Theorem of Calculus, Part 2 (Equation 4.4.2), any antiderivative works. So, for convenience, we chose the antiderivative with $C=0$. If we had chosen another antiderivative, the constant term would have canceled out. This always happens when evaluating a definite integral.

The region of the area we just calculated is depicted in Figure 4.4.3. Note that the region between the curve and the x-axis is all below the x-axis. Area is always positive, but a definite integral can still produce a negative number (a net signed area). For example, if this were a profit function, a negative number indicates the company is operating at a loss over the given interval.

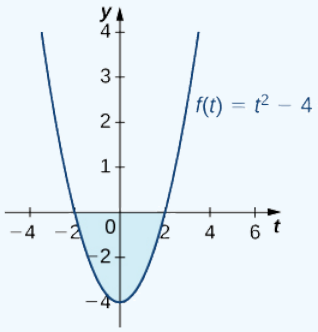


Figure 4.4.3: The evaluation of a definite integral can produce a negative value, even though area is always positive.

Example 4.4.7: Evaluating a Definite Integral Using the Fundamental Theorem of Calculus, Part 2

Evaluate the following integral using the Fundamental Theorem of Calculus, Part 2 (Equation 4.4.2):

$$\int_{1}^{9}\frac{x-1}{\sqrt{x}}dx.$$

Solution

First, eliminate the radical by rewriting the integral using rational exponents. Then, separate the numerator terms by writing each one over the denominator:

$$\int_1^9 \frac{x-1}{x^{1/2}}\,dx = \int_1^9 \left(\frac{x}{x^{1/2}} - \frac{1}{x^{1/2}}\right) dx.$$

Use the properties of exponents to simplify:

$$\int_1^9 \left(\frac{x}{x^{1/2}} - \frac{1}{x^{1/2}}\right) dx = \int_1^9 (x^{1/2} - x^{-1/2})\,dx.$$

Now, integrate using the power rule:

$$\begin{aligned}\int_1^9 (x^{1/2} - x^{-1/2})\,dx &= \left(\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{1/2}}{\frac{1}{2}}\right)\Bigg|_1^9 \\ &= \left[\frac{(9)^{3/2}}{\frac{3}{2}} - \frac{(9)^{1/2}}{\frac{1}{2}}\right] - \left[\frac{(1)^{3/2}}{\frac{3}{2}} - \frac{(1)^{1/2}}{\frac{1}{2}}\right] \\ &= \left[\frac{2}{3}(27) - 2(3)\right] - \left[\frac{2}{3}(1) - 2(1)\right] \\ &= 18 - 6 - \frac{2}{3} + 2 = \frac{40}{3}.\end{aligned}$$

See Figure 4.4.4.

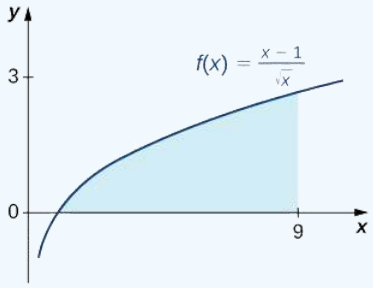


Figure 4.4.4: The area under the curve from $x = 1$ to $x = 9$ can be calculated by evaluating a definite integral.

? Exercise 4.4.6

Use Note to evaluate $\int_1^2 x^{-4}\,dx$.

Hint

Use the power rule.

Answer

$\frac{7}{24}$

✔ Example 4.4.8: A Roller-Skating Race

James and Kathy are racing on roller skates. They race along a long, straight track, and whoever has gone the farthest after 6 sec wins a prize. If James can skate at a velocity of $f(t) = 5 + 2t$ ft/sec and Kathy can skate at a velocity of $g(t) = 12 - 0.06t^2$ ft/sec, who is going to win the race?

Solution

We need to integrate both functions over the interval $[0, 6]$ and see which value is bigger. For James, we want to calculate

$$\int_0^6 (5+2t)\,dt.$$

Using the power rule, we have

$$\begin{aligned}\int_0^6 (5+2t)\,dt &= \left(5t+t^2\right)\Big|_0^6 \\ &= (30+36) \\ &= 66.\end{aligned}$$

Thus, James has skated 66 ft after 6 sec. Turning now to Kathy, we want to calculate

$$\int_0^6 (12-0.06t^2)\,dt.$$

Using the power rule, we have

$$\begin{aligned}\int_0^6 (12-0.06t^2)\,dt &= \left(12t-0.02t^3\right)\Big|_0^6 \\ &= (72-4.32) \\ &= 67.68.\end{aligned}$$

Kathy has skated 67.68 ft after 6 sec, while James has skated 66 ft. Kathy wins, but not by much!

? Exercise 4.4.7

Suppose James and Kathy have a rematch, but this time the official stops the contest after only 3 sec. Does this change the outcome?

Hint

Change the limits of integration from those in Example 4.4.7.

Answer

Kathy still wins, but by a much larger margin: James skates 24 ft in 3 sec, but Kathy skates 35.46 ft in 3 sec.

A Parachutist in Free Fall

Julie is an avid skydiver with more than 300 jumps under her belt and has mastered the art of making adjustments to her body position in the air to control how fast she falls. If she arches her back and points her belly toward the ground, she reaches a terminal velocity of approximately 120 mph (176 ft/sec). If, instead, she orients her body with her head straight down, she falls faster, reaching a terminal velocity of 150 mph (220 ft/sec).



Figure 4.4.5: Skydivers can adjust the velocity of their dive by changing the position of their body during the free fall. (credit: Jeremy T. Lock)

Since Julie will be moving (falling) in a downward direction, we assume the downward direction is positive to simplify our calculations. Julie executes her jumps from an altitude of 12,500 ft. After she exits the aircraft, she immediately starts falling at a velocity given by $v(t) = 32t$.

She continues to accelerate according to this velocity function until she reaches terminal velocity. After she reaches terminal velocity, her speed remains constant until she pulls her ripcord and slows down to land.

On her first jump of the day, Julie orients herself in the slower "belly down" position (terminal velocity is 176 ft/sec). Using this information, answer the following questions.

1. How long after she exits the aircraft does Julie reach terminal velocity?
2. Based on your answer to question 1, set up an expression involving one or more integrals that represents the distance Julie falls after 30 sec.
3. If Julie pulls her ripcord at an altitude of 3000 ft, how long does she spend in a free fall?
4. Julie pulls her ripcord at 3000 ft. It takes 5 sec for her parachute to open completely and for her to slow down, during which time she falls another 400 ft. After her canopy is fully open, her speed is reduced to 16 ft/sec. Find the total time Julie spends in the air, from the time she leaves the airplane until the time her feet touch the ground. On Julie's second jump of the day, she decides she wants to fall a little faster and orients herself in the "head down" position. Her terminal velocity in this position is 220 ft/sec. Answer these questions based on this velocity:
5. How long does it take Julie to reach terminal velocity in this case?
6. Before pulling her ripcord, Julie reorients her body in the "belly down" position so she is not moving quite as fast when her parachute opens. If she begins this maneuver at an altitude of 4000 ft, how long does she spend in a free fall before beginning the reorientation?

Some jumpers wear "**wingsuits**" (Figure 4.4.6). These suits have fabric panels between the arms and legs and allow the wearer to glide around in a free fall, much like a flying squirrel. (Indeed, the suits are sometimes called "flying squirrel suits.") When wearing these suits, terminal velocity can be reduced to about 30 mph (44 ft/sec), allowing the wearers a much longer time in the air. Wingsuit flyers still use parachutes to land; although the vertical velocities are within the margin of safety, horizontal velocities can exceed 70 mph, much too fast to land safely.



Figure 4.4.6: The fabric panels on the arms and legs of a wingsuit work to reduce the vertical velocity of a skydiver's fall. (credit: Richard Schneider)

Answer the following question based on the velocity in a wingsuit.

7. If Julie dons a wingsuit before her third jump of the day, and she pulls her ripcord at an altitude of 3000 ft, how long does she get to spend gliding around in the air

4.4.4 Key Concepts

- The Mean Value Theorem for Integrals states that for a continuous function over a closed interval, there is a value c such that $f(c)$ equals the average value of the function.
- The Fundamental Theorem of Calculus, Part 1 shows the relationship between the derivative and the integral.
- The Fundamental Theorem of Calculus, Part 2 is a formula for evaluating a definite integral in terms of an antiderivative of its integrand. The total area under a curve can be found using this formula.

4.4.5 Key Equations

- **Mean Value Theorem for Integrals**

  If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

- **Fundamental Theorem of Calculus, Part 1**

  If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t)\, dt,$$

  then

$$F'(x) = f(x).$$

- **Fundamental Theorem of Calculus, Part 2**

  If f is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

4.4.6 Glossary

fundamental theorem of calculus

the theorem, central to the entire development of calculus, that establishes the relationship between differentiation and integration

fundamental theorem of calculus, part 1

uses a definite integral to define an antiderivative of a function

fundamental theorem of calculus, part 2

(also, **evaluation theorem**) we can evaluate a definite integral by evaluating the antiderivative of the integrand at the endpoints of the interval and subtracting

mean value theorem for integrals

guarantees that a point c exists such that $f(c)$ is equal to the average value of the function

---

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4.4E: EXERCISES FOR SECTION 4.4

1) Consider two athletes running at variable speeds $v_1(t)$ and $v_2(t)$. The runners start and finish a race at exactly the same time. Explain why the two runners must be going the same speed at some point.

2) Two mountain climbers start their climb at base camp, taking two different routes, one steeper than the other, and arrive at the peak at exactly the same time. Is it necessarily true that, at some point, both climbers increased in altitude at the same rate?

Answer

Yes. It is implied by the Mean Value Theorem for Integrals.

3) To get on a certain toll road a driver has to take a card that lists the mile entrance point. The card also has a timestamp. When going to pay the toll at the exit, the driver is surprised to receive a speeding ticket along with the toll. Explain how this can happen.

4) Set $F(x) = \int_1^x (1-t)\,dt$. Find $F'(2)$ and the average value of F' over $[1,2]$.

Answer

$F'(2) = -1$; average value of F' over $[1,2]$ is $-1/2$.

In exercises 5 - 16, use the Fundamental Theorem of Calculus, Part 1, to find each derivative.

5) $\dfrac{d}{dx}\left[\int_1^x e^{-t^2}\,dt\right]$

6) $\dfrac{d}{dx}\left[\int_1^x e^{cost}\,dt\right]$

Answer

$$\frac{d}{dx}\left[\int_1^x e^{cost}\,dt\right] = e^{\cos t}$$

7) $\dfrac{d}{dx}\left[\int_3^x \sqrt{9-y^2}\,dy\right]$

8) $\dfrac{d}{dx}\left[\int_3^x \dfrac{ds}{\sqrt{16-s^2}}\right]$

Answer

$$\frac{d}{dx}\left[\int_3^x \frac{ds}{\sqrt{16-s^2}}\right] = \frac{1}{\sqrt{16-x^2}}$$

9) $\dfrac{d}{dx}\left[\int^{2x} x_t\,dt\right]$

10) $\dfrac{d}{dx}\left[\int_0^{\sqrt{x}} t\,dt\right]$

Answer

$$\frac{d}{dx}\left[\int_0^{\sqrt{x}} t\,dt\right] = \sqrt{x}\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2}$$

11) $\dfrac{d}{dx}\left[\int_0^{\sin x} \sqrt{1-t^2}\,dt\right]$

12) $\dfrac{d}{dx}\left[\int_{\cos x}^1 \sqrt{1-t^2}\,dt\right]$

Answer

$$\frac{d}{dx}\left[\int_{\cos x}^1 \sqrt{1-t^2}\,dt\right] = -\sqrt{1-\cos^2 x}\,\frac{d}{dx}\left(\cos x\right) = |\sin x|\sin x$$

13) $\dfrac{d}{dx}\left[\displaystyle\int_1^{\sqrt{x}} \frac{t^2}{1+t^4}\,dt\right]$

14) $\dfrac{d}{dx}\left[\displaystyle\int_1^{x^2} \frac{\sqrt{t}}{1+t}\,dt\right]$

Answer

$$\frac{d}{dx}\left[\int_1^{x^2} \frac{\sqrt{t}}{1+t}\,dt\right] = 2x\frac{|x|}{1+x^2}$$

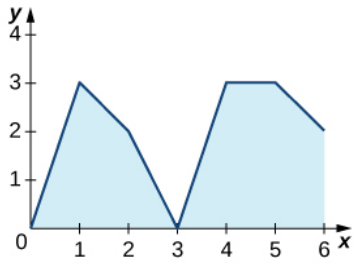
15) $\dfrac{d}{dx}\left[\displaystyle\int_0^{\ln x} e^t\,dt\right]$

16) $\dfrac{d}{dx}\left[\displaystyle\int_1^{e^x} \ln u^2\,du\right]$

Answer

$$\frac{d}{dx}\left[\int_1^{e^x} \ln u^2\,du\right] = \ln(e^{2x})\frac{d}{dx}\left(e^x\right) = 2xe^x$$

17) The graph of $y = \displaystyle\int_0^x f(t)\,dt$, where f is a piecewise constant function, is shown here.

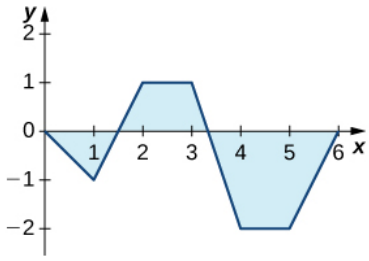


a. Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?

b. What are the maximum and minimum values of f?

c. What is the average value of f?

18) The graph of $y = \displaystyle\int_0^x f(t)\,dt$, where f is a piecewise constant function, is shown here.



a. Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?

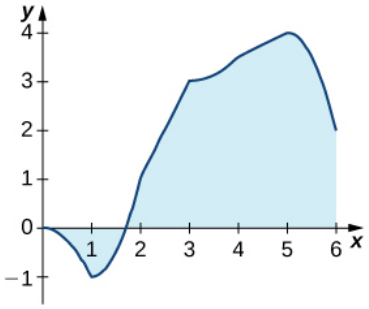
b. What are the maximum and minimum values of f?

c. What is the average value of f?

Answer

a. f is positive over $[1,2]$ and $[5,6]$, negative over $[0,1]$ and $[3,4]$, and zero over $[2,3]$ and $[4,5]$.
b. The maximum value is 2 and the minimum is -3.
c. The average value is 0.

19) The graph of $y = \displaystyle\int_0^x \ell(t)\,dt$, where ℓ is a piecewise linear function, is shown here.

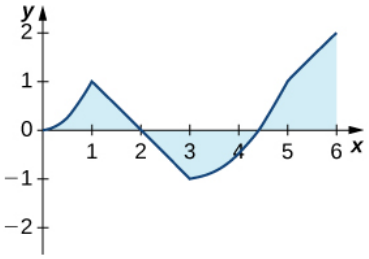


a. Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?

b. Over which intervals is ℓ increasing? Over which is it decreasing? Over which, if any, is it constant?

c. What is the average value of ℓ?

20) The graph of $y = \int_0^x \ell(t)\,dt$, where ℓ is a piecewise linear function, is shown here.



a. Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?

b. Over which intervals is ℓ increasing? Over which is it decreasing? Over which intervals, if any, is it constant?

c. What is the average value of ℓ?

Answer

a. ℓ is positive over $[0,1]$ and $[3,6]$, and negative over $[1,3]$.
b. It is increasing over $[0,1]$ and $[3,5]$, and it is constant over $[1,3]$ and $[5,6]$.
c. Its average value is $\frac{1}{3}$.

In exercises 21 - 26, use a calculator to estimate the area under the curve by computing T_{10}, the average of the left- and right-endpoint Riemann sums using $N = 10$ rectangles. Then, using the Fundamental Theorem of Calculus, Part 2, determine the exact area.

21) [T] $y = x^2$ over $[0,4]$

22) [T] $y = x^3 + 6x^2 + x - 5$ over $[-4,2]$

Answer

$$T_{10} = 49.08, \quad \int_{-2}^{3} \left(x^3 + 6x^2 + x - 5\right)\,dx = 48$$

23) [T] $y = \sqrt{x^3}$ over $[0,6]$

24) [T] $y = \sqrt{x} + x^2$ over $[1,9]$

Answer

$$T_{10} = 260.836, \quad \int_{1}^{9} \left(\sqrt{x} + x^2\right)\,dx = 260$$

25) [T] $\int (e^x)\,dx$ over $[0,2]$

26) [T] $\int \frac{4}{x^2}\,dx$ over $[1,4]$

Answer

$T_{10} = 3.058, \quad \int_1^4 \frac{4}{x^2}\,dx = 3$

In exercises 27 - 40, evaluate each definite integral using the Fundamental Theorem of Calculus, Part 2.

27) $\int_{-1}^{2} (x^2 - 3x)\,dx$

28) $\int_{-2}^{3} (x^2 + 3x - 5)\,dx$

Answer

$F(x) = \frac{x^3}{3} + \frac{3x^2}{2} - 5x, \quad F(3) - F(-2) = -\frac{35}{6}$

29) $\int_{-2}^{3} (t+2)(t-3)dt$

30) $\int_{2}^{3} (t^2 - 9)(4 - t^2)dt$

Answer

$F(x) = -\frac{t^5}{5} + \frac{13t^3}{3} - 36t, \quad F(3) - F(2) = \frac{62}{15}$

31) $\int_1^2 x^9\,dx$

32) $\int_0^1 x^{99}\,dx$

Answer

$F(x) = \frac{x^{100}}{100}, \quad F(1) - F(0) = \frac{1}{100}$

33) $\int_4^8 (4t^{5/2} - 3t^{3/2})dt$

34) $\int_{1/4}^{4} \left(x^2 - \frac{1}{x^2}\right)\,dx$

Answer

$F(x) = \frac{x^3}{3} + \frac{1}{x}, \quad F(4) - F\left(\frac{1}{4}\right) = \frac{1125}{64}$

35) $\int_1^2 \frac{2}{x^3}\,dx$

36) $\int_1^4 \frac{1}{2\sqrt{x}}\,dx$

Answer

$F(x) = \sqrt{x}, \quad F(4) - F(1) = 1$

37) $\int_1^4 \frac{2 - \sqrt{t}}{t^2}\,dt$

38) $\int_1^{16} \frac{dt}{t^{1/4}}$

Answer

$F(t) = \frac{4}{3}t^{3/4}, \quad F(16) - F(1) = \frac{28}{3}$

39) $\int_1^2 \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt$

40) $\int_{-2}^{-1} \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt$

Answer

$F(t) = -\frac{1}{t} + \frac{1}{2t^2}, \quad F(-1) - F(-2) = \frac{7}{8}$

In exercises 41-42, use the evaluation theorem to express the integral as a function $F(x)$.

41) $\int_a^x t^2\, dt$

42) $\int_1^x e^t\, dt$

Answer

$F(x) = e^x - e$

In exercises 43-44, identify the roots of the integrand to remove absolute values, then evaluate using the Fundamental Theorem of Calculus, Part 2.

43) $\int_{-2}^3 |x|\, dx$

44) $\int_{-2}^4 |\, t^2 - 2t - 3\,|\, dt$

Answer

$\int_{-2}^{-1} (t^2 - 2t - 3)\, dt - \int_{-1}^3 (t^2 - 2t - 3)\, dt + \int_3^4 (t^2 - 2t - 3)\, dt = \frac{46}{3}$

45) Suppose that for a particular company, the spending on construction projects between 2010 ($t = 1$) to 2019 ($t = 20$) is modeled by the function $S(t) = 25\sqrt{t}$, where S is measured in thousands of dollars per year and t is measured in years. What is the average spending per year on construction projects between 2010 and 2019?

46) Suppose the rate of gasoline consumption for a family over a one year period ($t = 0$ to $t = 12$) can be modeled by the function $-t^2 + 12t + 25$ gal/mo.What is the average monthly consumption, and for which value(s) of t is the rate at time t equal to the average rate?

Answer

The average value is 49 gal/mo, and the rate is equal to the average value at approximately time $t = 2.54$ and $t = 9.46$.

47) Explain why, if f is continuous over $[a, b]$, there is at least one point $c \in [a, b]$ such that $f(c) = \frac{1}{b-a} \int_a^b f(t)\, dt$.

48) Explain why, if f is continuous over $[a, b]$ and is not equal to a constant, there is at least one point $M \in [a, b]$ such that $f(M) = \frac{1}{b-a} \int_a^b f(t)\, dt$ and at least one point $m \in [a, b]$ such that $f(m) < \frac{1}{b-a} \int_a^b f(t)\, dt$.

Answer

If f is not constant, then its average is strictly smaller than the maximum and larger than the minimum, which are attained over $[a, b]$ by the extreme value theorem.

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4.5: Integration Formulas and the Net Change Theorem

Learning Objectives

- Apply the basic integration formulas.
- Explain the significance of the net change theorem.
- Use the net change theorem to solve applied problems.

In this section, we use some basic integration formulas studied previously to solve some key applied problems. It is important to note that these formulas are presented in terms of indefinite integrals. Although definite and indefinite integrals are closely related, there are some key differences to keep in mind. A definite integral is either a number (when the limits of integration are constants) or a single function (when one or both of the limits of integration are variables). An indefinite integral represents a family of functions, all of which differ by a constant. As you become more familiar with integration, you will get a feel for when to use definite integrals and when to use indefinite integrals. You will naturally select the correct approach for a given problem without thinking too much about it. However, until these concepts are cemented in your mind, think carefully about whether you need a definite integral or an indefinite integral and make sure you are using the proper notation based on your choice.

4.5.1 Basic Integration Formulas

Recall the integration formulas given in the section on Antiderivatives and the properties of definite integrals. Let's look at a few examples of how to apply these formulas and properties.

Example 4.5.1: Integrating a Function Using the Power Rule

Use the power rule to integrate the function $\int_1^4 \sqrt{t}(1+t)\,dt$.

Solution

The first step is to rewrite the function and simplify it so we can apply the power rule:

$$\begin{aligned}\int_1^4 \sqrt{t}(1+t)\,dt &= \int_1^4 t^{1/2}(1+t)\,dt \\ &= \int_1^4 (t^{1/2}+t^{3/2})\,dt.\end{aligned}$$

Now apply the power rule:

$$\begin{aligned}\int_1^4 (t^{1/2}+t^{3/2})\,dt &= \left(\frac{2}{3}t^{3/2}+\frac{2}{5}t^{5/2}\right)\Big|_1^4 \\ &= \left[\frac{2}{3}(4)^{3/2}+\frac{2}{5}(4)^{5/2}\right]-\left[\frac{2}{3}(1)^{3/2}+\frac{2}{5}(1)^{5/2}\right] \\ &= \frac{256}{15}.\end{aligned}$$

Exercise 4.5.1

Find the definite integral of $f(x)=x^2-3x$ over the interval $[1,3]$.

Hint

Follow the process from Example 4.5.1to solve the problem.

Answer

$$\int_1^3 (x^2-3x)\,dx = -\frac{10}{3}$$

4.5.2 The Net Change Theorem

The **net change theorem** considers the integral of a **rate of change**. It says that when a quantity changes, the new value equals the initial value plus the integral of the rate of change of that quantity. The formula can be expressed in two ways. The second is more familiar; it is simply the definite integral.

Net Change Theorem

The new value of a changing quantity equals the initial value plus the integral of the rate of change:

$$F(b) = F(a) + \int_a^b F'(x)dx \tag{4.5.1}$$

or

$$\int_a^b F'(x)dx = F(b) - F(a). \tag{4.5.2}$$

Subtracting $F(a)$ from both sides of the Equation 4.5.1 yields Equation 4.5.2. Since they are equivalent formulas, which one we use depends on the application.

The significance of the net change theorem lies in the results. Net change can be applied to area, distance, and volume, to name only a few applications. Net change accounts for negative quantities automatically without having to write more than one integral. To illustrate, let's apply the net change theorem to a **velocity** function in which the result is **displacement.**

We looked at a simple example of this in The Definite Integral section. Suppose a car is moving due north (the positive direction) at 40 mph between 2 p.m. and 4 p.m., then the car moves south at 30 mph between 4 p.m. and 5 p.m. We can graph this motion as shown in Figure 4.5.1.

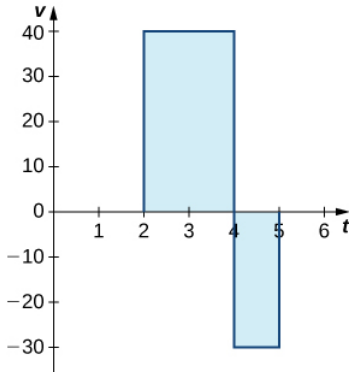


Figure 4.5.1: The graph shows speed versus time for the given motion of a car.

Just as we did before, we can use definite integrals to calculate the net displacement as well as the total distance traveled. The **net displacement** is given by

$$\int_2^5 v(t)\,dt = \int_2^4 40\,dt + \int_4^5 -30\,dt = 80 - 30 = 50.$$

Thus, at 5 p.m. the car is 50 mi north of its starting position. The **total distance** traveled is given by

$$\int_2^5 |v(t)|\,dt = \int_2^4 40\,dt + \int_4^5 30\,dt = 80 + 30 = 110.$$

Therefore, between 2 p.m. and 5 p.m., the car traveled a total of 110 mi.

To summarize, net displacement may include both positive and negative values. In other words, the velocity function accounts for both forward distance and backward distance. To find net displacement, integrate the velocity function over the interval. Total

distance traveled, on the other hand, is always positive. To find the total distance traveled by an object, regardless of direction, we need to integrate the absolute value of the velocity function.

Example 4.5.2: Finding Net Displacement

Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$, find the net displacement of the particle.

Solution

Applying the net change theorem, we have

$$\int_0^3 (3t-5)\,dt = \left(\frac{3t^2}{2} - 5t\right)\Big|_0^3 = \left[\frac{3(3)^2}{2} - 5(3)\right] - 0 = \frac{27}{2} - 15 = \frac{27}{2} - \frac{30}{2} = -\frac{3}{2}.$$

The net displacement is $-\frac{3}{2}$ m (Figure 4.5.2).

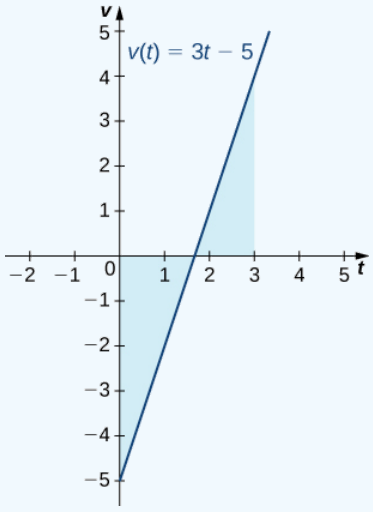


Figure 4.5.2: The graph shows velocity versus time for a particle moving with a linear velocity function.

Example 4.5.3: Finding the Total Distance Traveled

Use Example 4.5.2 to find the total distance traveled by a particle according to the velocity function $v(t) = 3t - 5$ m/sec over a time interval $[0, 3]$.

Solution

The total distance traveled includes both the positive and the negative values. Therefore, we must integrate the absolute value of the velocity function to find the total distance traveled.

To continue with the example, use two integrals to find the total distance. First, find the t-intercept of the function, since that is where the division of the interval occurs. Set the equation equal to zero and solve for t. Thus,

$$\begin{aligned} 3t - 5 &= 0 \\ 3t &= 5 \\ t &= \frac{5}{3}. \end{aligned}$$

The two subintervals are $\left[0, \frac{5}{3}\right]$ and $\left[\frac{5}{3}, 3\right]$. To find the total distance traveled, integrate the absolute value of the function. Since the function is negative over the interval $\left[0, \frac{5}{3}\right]$, we have $|v(t)| = -v(t)$ over that interval. Over $\left[\frac{5}{3}, 3\right]$, the function is positive, so $|v(t)| = v(t)$. Thus, we have

$$\begin{aligned}\int_0^3 |v(t)|\,dt &= \int_0^{5/3} -v(t)\,dt + \int_{5/3}^3 v(t)\,dt \\ &= \int_0^{5/3} 5-3t\,dt + \int_{5/3}^3 3t-5\,dt \\ &= \left(5t-\frac{3t^2}{2}\right)\Big|_0^{5/3} + \left(\frac{3t^2}{2}-5t\right)\Big|_{5/3}^3 \\ &= \left[5(\frac{5}{3})-\frac{3(5/3)^2}{2}\right]-0+\left[\frac{27}{2}-15\right]-\left[\frac{3(5/3)^2}{2}-\frac{25}{3}\right] \\ &= \frac{25}{3}-\frac{25}{6}+\frac{27}{2}-15-\frac{25}{6}+\frac{25}{3} \\ &= \frac{41}{6}\end{aligned}$$

So, the total distance traveled is $\frac{14}{6}$ m.

? Exercise 4.5.2

Find the net displacement and total distance traveled in meters given the velocity function $f(t)=\frac{1}{2}e^t-2$ over the interval $[0,2]$.

Hint

Follow the procedures from Examples 4.5.2and 4.5.3 Note that $f(t)\le 0$ for $t\le \ln 4$ and $f(t)\ge 0$ for $t\ge \ln 4$.

Answer

Net displacement: $\frac{e^2-9}{2}\approx -0.8055$ m; total distance traveled: $4\ln 4-7.5+\frac{e^2}{2}\approx 1.740$ m.

4.5.3 Applying the Net Change Theorem

The net change theorem can be applied to the flow and consumption of fluids, as shown in Example 4.5.4.

✓ Example 4.5.4: How Many Gallons of Gasoline Are Consumed?

If the motor on a motorboat is started at $t=0$ and the boat consumes gasoline at the rate of $5-t^3$ gal/hr, how much gasoline is used in the first 2 hours?

Solution

Express the problem as a definite integral, integrate, and evaluate using the Fundamental Theorem of Calculus. The limits of integration are the endpoints of the interval [0,2]. We have

$$\begin{aligned}\int_0^2 \left(5-t^3\right)\,dt &= \left(5t-\frac{t^4}{4}\right)\Big|_0^2 \\ &= \left[5(2)-\frac{(2)^4}{4}\right]-0 \\ &= 10-\frac{16}{4} \\ &= 6.\end{aligned}$$

Thus, the motorboat uses 6 gal of gas in 2 hours.

Example 4.5.5: Chapter Opener: Iceboats

As we saw at the beginning of the chapter, top **iceboat** racers can attain speeds of up to five times the wind speed. Andrew is an intermediate iceboater, though, so he attains speeds equal to only twice the wind speed.



Figure 4.5.3: (credit: modification of work by Carter Brown, Flickr)

Suppose Andrew takes his iceboat out one morning when a light 5-mph breeze has been blowing all morning. As Andrew gets his iceboat set up, though, the wind begins to pick up. During his first half hour of iceboating, the wind speed increases according to the function $v(t) = 20t + 5$. For the second half hour of Andrew's outing, the wind remains steady at 15 mph. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t+5, & \text{for } 0 \le t \le \frac{1}{2} \\ 15, & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

Recalling that Andrew's iceboat travels at twice the wind speed, and assuming he moves in a straight line away from his starting point, how far is Andrew from his starting point after 1 hour?

Solution

To figure out how far Andrew has traveled, we need to integrate his velocity, which is twice the wind speed. Then

$$\text{Distance} = \int_0^1 2v(t)\,dt.$$

Substituting the expressions we were given for $v(t)$, we get

$$\begin{aligned} \int_0^1 2v(t)\,dt &= \int_0^{1/2} 2v(t)\,dt + \int_{1/2}^1 2v(t)\,dt \\ &= \int_0^{1/2} 2(20t+5)\,dt + \int_{1/3}^1 2(15)\,dt \\ &= \int_0^{1/2} (40t+10)\,dt + \int_{1/2}^1 30\,dt \\ &= \left[20t^2+10t\right]\Big|_0^{1/2} + \left[30t\right]\Big|_{1/2}^1 \\ &= \left(\frac{20}{4}+5\right) - 0 + (30-15) \\ &= 25. \end{aligned}$$

Andrew is 25 mi from his starting point after 1 hour.

Exercise 4.5.3

Suppose that, instead of remaining steady during the second half hour of Andrew's outing, the wind starts to die down according to the function $v(t) = -10t + 15$. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t + 5, & \text{for } 0 \le t \le \frac{1}{2} \\ -10t + 15, & \text{for } \frac{1}{2} \le t \le 1 \end{cases}.$$

Under these conditions, how far from his starting point is Andrew after 1 hour?

Hint

Don't forget that Andrew's iceboat moves twice as fast as the wind.

Answer

17.5mi

4.5.4 Key Concept

- The net change theorem states that when a quantity changes, the final value equals the initial value plus the integral of the rate of change. Net change can be a positive number, a negative number, or zero.

4.5.5 Key Equation

- Net Change Theorem

$$F(b) = F(a) + \int_a^b F'(x)\,dx$$

or

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

4.5.6 Glossary

net change theorem

if we know the rate of change of a quantity, the net change theorem says the future quantity is equal to the initial quantity plus the integral of the rate of change of the quantity

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4.5E: EXERCISES FOR SECTION 4.5

In exercises 1 - 6, use basic integration formulas to evaluate the integrals.

1) $\int \left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right) dx$

Answer

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx = \int x^{1/2}\, dx - \int x^{-1/2}\, dx = \frac{2}{3}x^{3/2} + C_1 - 2x^{1/2+} C_2 = \frac{2}{3}x^{3/2} - 2x^{1/2} + C$$

2) $\int \left(e^x - x^e\right) dx$

3) $\int \dfrac{dx}{2x}$

Answer

$$\int \frac{dx}{2x} = \frac{1}{2}\ln|x| + C$$

4) $\int \dfrac{x-1}{x^2}\, dx$

5) $\int_0^1 (e^x - 2x)$

Answer

$$\int_0^1 (e^x - 2x)\, dx = e^x\big|_0^1 - (x^2)\big|_0^1 = (e-1) - (1-0) = e - 2$$

6) $\int_1^4 (\dfrac{1}{\sqrt{x}})\, dx$

7) Write an integral that expresses the increase in the perimeter $P(s)$ of a square when its side length s increases from 2 units to 4 units and evaluate the integral.

Answer

$P(s) = 4s$, so $\dfrac{dP}{ds} = 4$ and $\displaystyle\int_2^4 4\, ds = 8.$

8) Write an integral that quantifies the change in the area $A(s) = s^2$ of a square when the side length doubles from S units to $2S$ units and evaluate the integral.

9) A regular N-gon (an N-sided polygon with sides that have equal length s, such as a pentagon or hexagon) has perimeter Ns. Write an integral that expresses the increase in perimeter of a regular N-gon when the length of each side increases from 1 unit to 2 units and evaluate the integral.

Answer

$$\int_1^2 N\, ds = N$$

10) The area of a regular pentagon with side length $a > 0$ is pa^2 with $p = \frac{1}{4}\sqrt{5 + \sqrt{5 + 2\sqrt{5}}}$. The Pentagon in Washington, DC, has inner sides of length 360 ft and outer sides of length 920 ft. Write an integral to express the area of the roof of the Pentagon according to these dimensions and evaluate this area.

11) A dodecahedron is a Platonic solid with a surface that consists of 12 pentagons, each of equal area. By how much does the surface area of a dodecahedron increase as the side length of each pentagon doubles from 1 unit to 2 units?

Answer

With p as in the previous exercise, each of the 12 pentagons increases in area from $2p$ to $4p$ units so the net increase in the area of the dodecahedron is $36p$ units.

12) An icosahedron is a Platonic solid with a surface that consists of 20 equilateral triangles. By how much does the surface area of an icosahedron increase as the side length of each triangle doubles from a unit to $2a$ units?

13) Write an integral that quantifies the change in the area of the surface of a cube when its side length doubles from s unit to $2s$ units and evaluate the integral.

Answer

$18s^2 = 6\int_s^{2s} 2x\,dx$

14) Write an integral that quantifies the increase in the volume of a cube when the side length doubles from s unit to $2s$ units and evaluate the integral.

15) Write an integral that quantifies the increase in the surface area of a sphere as its radius doubles from R unit to $2R$ units and evaluate the integral.

Answer

$12\pi R^2 = 8\pi\int_R^{2R} r\,dr$

16) Write an integral that quantifies the increase in the volume of a sphere as its radius doubles from R unit to $2R$ units and evaluate the integral.

17) Suppose that a particle moves along a straight line with velocity $v(t) = 4 - 2t$, where $0 \le t \le 2$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 2$.

Answer

$d(t) = \int_0^t v(s)\,ds = 4t - t^2$. The total distance is $d(2) = 4$ m.

18) Suppose that a particle moves along a straight line with velocity defined by $v(t) = t^2 - 3t - 18$, where $0 \le t \le 6$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 6$.

19) Suppose that a particle moves along a straight line with velocity defined by $v(t) = |2t - 6|$, where $0 \le t \le 6$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 6$.

Answer

$d(t) = \int_0^t v(s)\,ds$. For $t < 3$, $d(t) = \int_0^t (6 - 2t)\,dt = 6t - t^2$. For $t > 3$, $d(t) = d(3) + \int_3^t (2t - 6)\,dt = 9 + (t^2 - 6t)$.
The total distance is $d(6) = 9$ m.

20) Suppose that a particle moves along a straight line with acceleration defined by $a(t) = t - 3$, where $0 \le t \le 6$ (in meters per second). Find the velocity and displacement at time t and the total distance traveled up to $t = 6$ if $v(0) = 3$ and $d(0) = 0$.

21) A ball is thrown upward from a height of 1.5 m at an initial speed of 40 m/sec. Acceleration resulting from gravity is $-9.8\,\text{m/sec}^2$. Neglecting air resistance, solve for the velocity $v(t)$ and the height $h(t)$ of the ball t seconds after it is thrown and before it returns to the ground.

Answer

$v(t) = 40 - 9.8t;\ h(t) = 1.5 + 40t - 4.9t^2$ m/s

22) A ball is thrown upward from a height of 3 m at an initial speed of 60 m/sec. Acceleration resulting from gravity is $-9.8\,\text{m/sec}^2$. Neglecting air resistance, solve for the velocity $v(t)$ and the height $h(t)$ of the ball t seconds after it is thrown and before it returns to the ground.

23) The area $A(t)$ of a circular shape is growing at a constant rate. If the area increases from 4π units to 9π units between times $t = 2$ and $t = 3$, find the net change in the radius during that time.

Answer

The net increase is 1 unit.

24) A spherical balloon is being inflated at a constant rate. If the volume of the balloon changes from 36π in^3 to 288π in^3 between time $t = 30$ and $t = 60$ seconds, find the net change in the radius of the balloon during that time.

25) Water flows into a conical tank with cross-sectional area πx^2 at height x and volume $\frac{\pi x^3}{3}$ up to height x. If water flows into the tank at a rate of 1 m^3/min, find the height of water in the tank after 5 min. Find the change in height between 5 min and 10 min.

Answer

At $t = 5$, the height of water is $x = \left(\frac{15}{\pi}\right)^{1/3}$ m. The net change in height from $t = 5$ to $t = 10$ is $\left(\left(\frac{30}{\pi}\right)^{1/3} - \left(\frac{15}{\pi}\right)^{1/3}\right)$ m.

26) A horizontal cylindrical tank has cross-sectional area $A(x) = 4(6x - x^2)\ \text{m}^2$ at height x meters above the bottom when $x \leq 3$.

a. The volume V between heights a and b is $\int_a^b A(x)\,dx$. Find the volume at heights between 2 m and 3 m.

b. Suppose that oil is being pumped into the tank at a rate of 50 L/min. Using the chain rule, $\frac{dx}{dt} = \frac{dx}{dV}\frac{dV}{dt}$, at how many meters per minute is the height of oil in the tank changing, expressed in terms of x, when the height is at x meters?

c. How long does it take to fill the tank to 3 m starting from a fill level of 2 m?

27) The following table lists the electrical power in gigawatts—the rate at which energy is consumed—used in a certain city for different hours of the day, in a typical 24-hour period, with hour 1 corresponding to midnight to 1 a.m.

Hour	Power	Hour	Power
1	28	13	48
2	25	14	49
3	24	15	49
4	23	16	50
5	24	17	50
6	27	18	50
7	29	19	46
8	32	20	43
9	34	21	42
10	39	22	40
11	42	23	37
12	46	24	34

Find the total amount of power in gigawatt-hours (gW-h) consumed by the city in a typical 24-hour period.

Answer

The total daily power consumption is estimated as the sum of the hourly power rates, or 911 gW-h.

28) The average residential electrical power use (in hundreds of watts) per hour is given in the following table.

Hour	Power	Hour	Power
1	8	13	12
2	6	14	13
3	5	15	14
4	4	16	15
5	5	17	17
6	6	18	19
7	7	19	18
8	8	20	17
9	9	21	16
10	10	22	16
11	10	23	13
12	11	24	11

a. Compute the average total energy used in a day in kilowatt-hours (kWh).

b. If a ton of coal generates 1842 kWh, how long does it take for an average residence to burn a ton of coal?

c. Explain why the data might fit a plot of the form $p(t) = 11.5 - 7.5\sin\left(\frac{\pi t}{12}\right)$.

29) The data in the following table are used to estimate the average power output produced by Peter Sagan for each of the last 18 sec of Stage 1 of the 2012 **Tour de France.**

Second	Watts	Second	Watts
1	600	10	1200
2	500	11	1170
3	575	12	1125
4	1050	13	1100
5	925	14	1075
6	950	15	1000
7	1050	16	950
8	950	17	900
9	1100	18	780

Average Power OutputSource: sportsexercisengineering.com

Estimate the net energy used in kilojoules (kJ), noting that 1W = 1 J/s, and the average power output by Sagan during this time interval.

Answer

17 kJ

30) The data in the following table are used to estimate the average power output produced by Peter Sagan for each 15-min interval of Stage 1 of the 2012 Tour de France.

Minutes	Watts	Minutes	Watts
15	200	165	170
30	180	180	220
45	190	195	140
60	230	210	225
75	240	225	170
90	210	240	10
105	210	255	200
1120	220	270	220
135	210	285	250
150	150	300	400

Average Power Output*Source*: sportsexercisengineering.com

Estimate the net energy used in kilojoules, noting that 1W = 1 J/s.

31) The distribution of incomes as of 2012 in the United States in $5000 increments is given in the following table. The kth row denotes the percentage of households with incomes between $\$5000xk$ and $5000xk + 4999$. The row $k = 40$ contains all households with income between $200,000 and $250,000 and $k = 41$ accounts for all households with income exceeding $250,000.

0	3.5	21	1.5
1	4.1	22	1.4
2	5.9	23	1.3
3	5.7	24	1.3
4	5.9	25	1.1
5	5.4	26	1.0
6	5.5	27	0.75
7	5.1	28	0.8
8	4.8	29	1.0
9	4.1	30	0.6
10	4.3	31	0.6
11	3.5	32	.5
12	3.7	33	0.5
13	3.2	34	0.4
14	3.0	35	0.3
15	2.8	36	0.3
16	2.5	37	0.3
17	2.2	38	0.2
18	2.2	39	1.8
19	1.8	40	2.3
20	2.1	41	

Income DistributionsSource: www.census.gov/prod/2013pubs/p60-245.pdf

a. Estimate the percentage of U.S. households in 2012 with incomes less than $55,000.

b. What percentage of households had incomes exceeding $85,000?

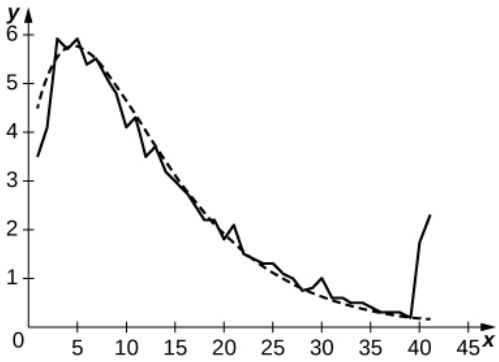
c. Plot the data and try to fit its shape to that of a graph of the form $a(x+c)e^{-b(x+e)}$ for suitable a, b, c.

Answer

a. 54.3%;

b. 27.00%;

c. The curve in the following plot is $2.35(t+3)e^{-0.15(t+3)}$.



32) Newton's law of gravity states that the gravitational force exerted by an object of mass M and one of mass m with centers that are separated by a distance r is $F = G\frac{mM}{r^2}$, with G an empirical constant $G = 6.67x10^{-11}\ \text{m}^3/(\text{kg}{\cdot}\text{s}^2)$. The work done by a variable force over an interval $[a,b]$ is defined as $W = \int_a^b F(x)\,dx$. If Earth has mass 5.97219×10^{24} and radius 6371 km, compute the amount of work to elevate a polar weather satellite of mass 1400 kg to its orbiting altitude of 850 km above Earth.

33) For a given motor vehicle, the maximum achievable **deceleration** from braking is approximately $7\ \text{m/sec}^2$ on dry concrete. On wet asphalt, it is approximately $2.5\ \text{m/sec}^2$. Given that 1 mph corresponds to 0.447 m/sec, find the total distance that a car travels in meters on dry concrete after the brakes are applied until it comes to a complete stop if the initial velocity is 67 mph (30 m/sec) or if the initial braking velocity is 56 mph (25 m/sec). Find the corresponding distances if the surface is slippery wet asphalt.

Answer

In dry conditions, with initial velocity $v_0 = 30$ m/s, $D = 64.3$ and, if $v_0 = 25,\ D = 44.64$. In wet conditions, if $v_0 = 30$, and $D = 180$ and if $v_0 = 25,\ D = 125$.

34) John is a 25-year old man who weighs 160 lb. He burns $500 - 50t$ calories/hr while riding his bike for t hours. If an oatmeal cookie has 55 cal and John eats $4t$ cookies during the tth hour, how many net calories has he lost after 3 hours riding his bike?

35) Sandra is a 25-year old woman who weighs 120 lb. She burns $300 - 50t$ cal/hr while walking on her treadmill for t hours. Her caloric intake from drinking Gatorade is $100t$ calories during the tth hour. What is her net decrease in calories after walking for 3 hours?

Answer

225 cal

36) A motor vehicle has a maximum efficiency of 33 mpg at a cruising speed of 40 mph. The efficiency drops at a rate of 0.1 mpg/mph between 40 mph and 50 mph, and at a rate of 0.4 mpg/mph between 50 mph and 80 mph. What is the efficiency in miles per gallon if the car is cruising at 50 mph? What is the efficiency in miles per gallon if the car is cruising at 80 mph? If gasoline costs $3.50/gal, what is the cost of fuel to drive 50 mi at 40 mph, at 50 mph, and at 80 mph?

37) Although some engines are more efficient at given a horsepower than others, on average, fuel efficiency decreases with horsepower at a rate of $1/25$ mpg/horsepower. If a typical 50-horsepower engine has an average fuel efficiency of 32 mpg, what is the average fuel efficiency of an engine with the following horsepower: 150, 300, 450?

Answer

$E(150) = 28,\ E(300) = 22,\ E(450) = 16$

38) [T] The following table lists the 2013 schedule of **federal income tax** versus taxable income.

Taxable Income Range	The Tax Is ...	... Of the Amount Over
$0–$8925	10%	$0
$8925–$36,250	$892.50 + 15%	$8925
$36,250–$87,850	$4,991.25 + 25%	$36,250
$87,850–$183,250	$17,891.25 + 28%	$87,850
$183,250–$398,350	$44,603.25 + 33%	$183,250
$398,350–$400,000	$115,586.25 + 35%	$398,350
> $400,000	$116,163.75 + 39.6%	$400,000

Federal Income Tax Versus Taxable IncomeSource: http://www.irs.gov/pub/irs-prior/i1040tt--2013.pdf.

Suppose that Steve just received a $10,000 raise. How much of this raise is left after federal taxes if Steve's salary before receiving the raise was $40,000? If it was $90,000? If it was $385,000?

39) [T] The following table provides hypothetical data regarding the level of service for a certain highway.

Highway Speed Range (mph)	Vehicles per Hour per Lane	Density Range (vehicles/mi)
>60	<600	<10
60-57	300-1000	10-20
57-54	1000-1500	20-30
57-54	1500-1900	30-45
46-30	1900-2100	48-70
<30	Unstable	70-200

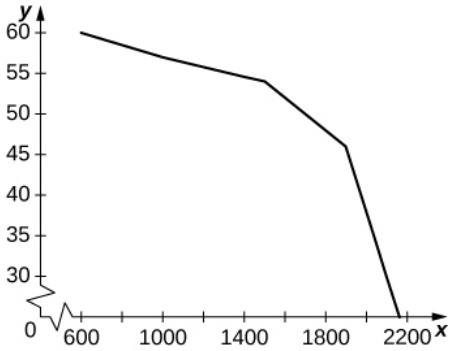
a. Plot vehicles per hour per lane on the x-axis and highway speed on the y-axis.

b. Compute the average decrease in speed (in miles per hour) per unit increase in congestion (vehicles per hour per lane) as the latter increases from 600 to 1000, from 1000 to 1500, and from 1500 to 2100. Does the decrease in miles per hour depend linearly on the increase in vehicles per hour per lane?

c. Plot minutes per mile (60 times the reciprocal of miles per hour) as a function of vehicles per hour per lane. Is this function linear?

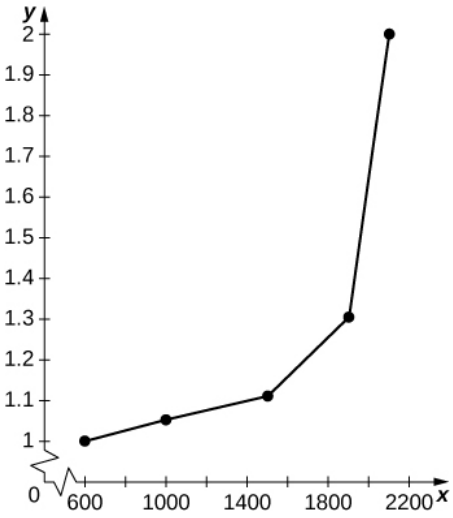
Answer

a.



b. Between 600 and 1000 the average decrease in vehicles per hour per lane is −0.0075. Between 1000 and 1500 it is −0.006 per vehicles per hour per lane, and between 1500 and 2100 it is −0.04 vehicles per hour per lane.

c.



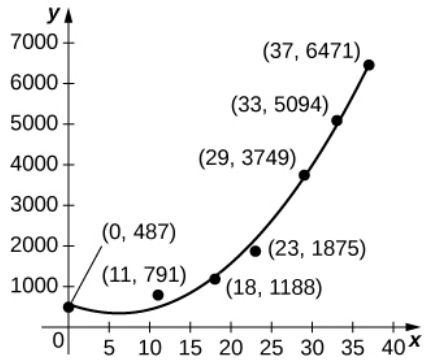
The graph is nonlinear, with minutes per mile increasing dramatically as vehicles per hour per lane reach 2000.

For the next two exercises use the data in the following table, which displays bald eagle populations from 1963 to 2000 in the continental United States.

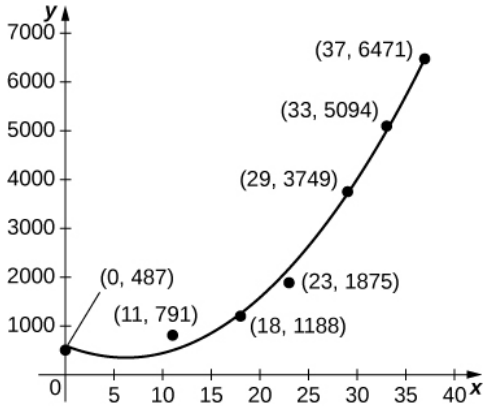
Year	Population of Breeding Pairs of Bald Eagles
1963	487
1974	791
1981	1188
1986	1875
1992	3749
1996	5094
2000	6471

Population of Breeding Bald Eagle Pairs*Source*: www.fws.gov/Midwest/eagle/pop.../chtofprs.html.

40) [T] The graph below plots the quadratic $p(t) = 6.48t^2 - 80.31t + 585.69$ against the data in preceding table, normalized so that $t = 0$ corresponds to 1963. Estimate the average number of bald eagles per year present for the 37 years by computing the average value of p over $[0, 37]$.



41) [T] The graph below plots the cubic $p(t) = 0.07t^3 + 2.42t^2 - 25.63t + 521.23$ against the data in the preceding table, normalized so that $t = 0$ corresponds to 1963. Estimate the average number of bald eagles per year present for the 37 years by computing the average value of p over $[0, 37]$.



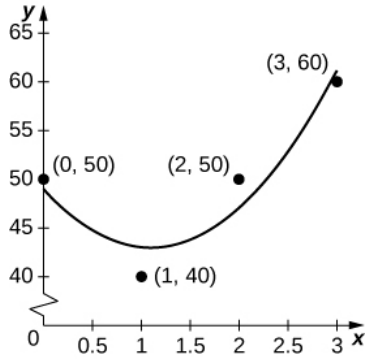
Answer

$$\frac{1}{37}\int_0^{37} p(t)\,dt = \frac{0.07(37)^3}{4} + \frac{2.42(37)^2}{3} - \frac{25.63(37)}{2} + 521.23 \approx 2037$$

Solution:

42) [T] Suppose you go on a road trip and record your speed at every half hour, as compiled in the following table. The best quadratic fit to the data is $q(t) = 5x^2 - 11x + 49$, shown in the accompanying graph. Integrate q to estimate the total distance driven over the 3 hours.

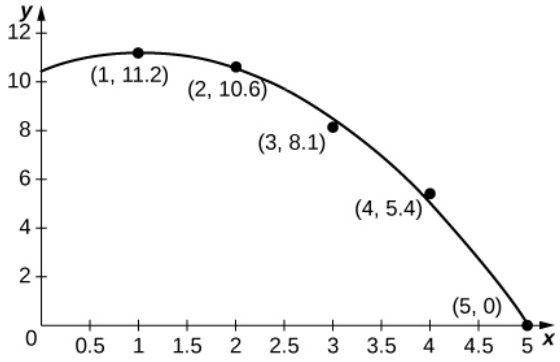
Time (hr)	Speed (m[h)
0 (start)	50
1	40
2	50
3	60



As a car accelerates, it does not accelerate at a constant rate; rather, the acceleration is variable. For the following exercises, use the following table, which contains the acceleration measured at every second as a driver merges onto a freeway.

Time (sec)	Acceleration (mph/sec)
1	11.2
2	10.6
3	8.1
4	5.4
5	0

43) [T] The accompanying graph plots the best quadratic fit, $a(t) = -0.70t^2 + 1.44t + 10.44$, to the data from the preceding table. Compute the average value of $a(t)$ to estimate the average acceleration between $t = 0$ and $t = 5$.



Answer

Average acceleration is $A = \frac{1}{5}\int_0^5 a(t)\,dt = -\frac{0.7(5^2)}{3} + \frac{1.44(5)}{2} + 10.44 \approx 8.2$ mph/s

44) [T] Using your acceleration equation from the previous exercise, find the corresponding velocity equation. Assuming the final velocity is 0 mph, find the velocity at time $t = 0$.

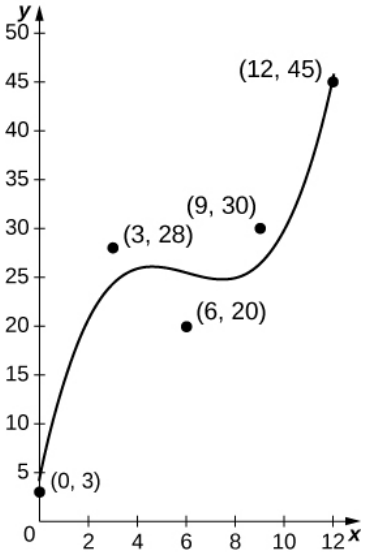
45) [T] Using your velocity equation from the previous exercise, find the corresponding distance equation, assuming your initial distance is 0 mi. How far did you travel while you accelerated your car? (Hint: You will need to convert time units.)

Answer

$d(t) = \int_0^1 |v(t)|\,dt = \int_0^t \left(\frac{7}{30}t^3 - 0.72t^2 - 10.44t + 41.033\right)\,dt = \frac{7}{120}t^4 - 0.24t^3 - 5.22t^3 + 41.033t.$ Then, $d(5) \approx 81.12\,\text{mph} \times \text{sec} \approx 119$ feet.

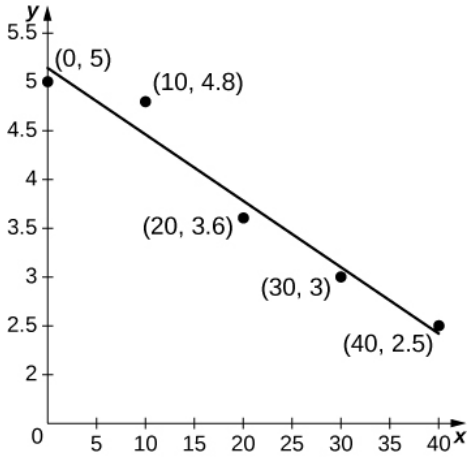
46) [T] The number of hamburgers sold at a restaurant throughout the day is given in the following table, with the accompanying graph plotting the best cubic fit to the data, $b(t) = 0.12t^3 - 2.13t^3 + 12.13t + 3.91$, with $t = 0$ corresponding to 9 a.m. and $t = 12$ corresponding to 9 p.m. Compute the average value of $b(t)$ to estimate the average number of hamburgers sold per hour.

Hours Past Midnight	No. of Burgers Sold
9	3
12	28
15	20
18	30
21	45



47) [T] An athlete runs by a motion detector, which records her speed, as displayed in the following table. The best linear fit to this data, $\ell(t) = -0.068t + 5.14$, is shown in the accompanying graph. Use the average value of $\ell(t)$ between $t = 0$ and $t = 40$ to estimate the runner's average speed.

Minutes	Speed (m/sec)
0	5
10	4.8
20	3.6
30	3.0
40	2.5



Answer

$$\frac{1}{40}\int_0^{40}(-0.068t + 5.14)\,dt = -\frac{0.068(40)}{2} + 5.14 = 3.78 \text{ m/sec}$$

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4.6: Substitution

Learning Objectives

- Use substitution to evaluate indefinite integrals.
- Use substitution to evaluate definite integrals.

The Fundamental Theorem of Calculus gave us a method to evaluate integrals without using Riemann sums. The drawback of this method, though, is that we must be able to find an antiderivative, and this is not always easy. In this section we examine a technique, called *integration by substitution*, to help us find antiderivatives. Specifically, this method helps us find antiderivatives when the integrand is the result of a chain-rule derivative.

At first, the approach to the substitution procedure may not appear very obvious. However, it is primarily a visual task—that is, the integrand shows you what to do; it is a matter of recognizing the form of the function. So, what are we supposed to see? We are looking for an integrand of the form $f\big[g(x)\big]\,g'(x)\,dx$. For example, in the integral

$$\int (x^2-3)^3\,2x\,dx. \tag{4.6.1}$$

we have

$$f(x)=x^3$$

and

$$g(x)=x^2-3.$$

Then

$$g'(x)=2x.$$

and

$$f[g(x)]g'(x)=(x^2-3)^3(2x),$$

and we see that our integrand is in the correct form. The method is called substitution because we substitute part of the integrand with the variable u and part of the integrand with du. It is also referred to as *change of variables* because we are changing variables to obtain an expression that is easier to work with for applying the integration rules.

Substitution with Indefinite Integrals

Let $u=g(x)$,, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\begin{aligned}\int f[g(x)]g'(x)\,dx &= \int f(u)\,du \\ &= F(u)+C \\ &= F(g(x))+C\end{aligned}$$

Proof

Let f, g, u, and F be as specified in the theorem. Then

$$\frac{d}{dx}\big[F(g(x))\big]=F'(g(x))g'(x)=f[g(x)]g'(x).$$

Integrating both sides with respect to x, we see that

$$\int f[g(x)]g'(x)\,dx=F(g(x))+C.$$

If we now substitute $u = g(x)$, and $du = g'(x)\,dx$, we get

$$\int f[g(x)]g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C.$$

□

Returning to the problem we looked at originally, we let $u = x^2 - 3$ and then $du = 2x\,dx$.

Rewrite the integral (Equation 4.6.1) in terms of u:

$$\int (x^2 - 3)^3 (2x\,dx) = \int u^3\,du.$$

Using the power rule for integrals, we have

$$\int u^3\,du = \frac{u^4}{4} + C.$$

Substitute the original expression for x back into the solution:

$$\frac{u^4}{4} + C = \frac{(x^2 - 3)^4}{4} + C.$$

We can generalize the procedure in the following Problem-Solving Strategy.

Problem-Solving Strategy: Integration by Substitution

1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Let's select $g(x)$. such that $g'(x)$ is also part of the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx$. into the integral.
3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.
4. Evaluate the integral in terms of u.
5. Write the result in terms of x and the expression $g(x)$.

Example 4.6.1: Using Substitution to Find an Antiderivative

Use substitution to find the antiderivative of $\int 6x(3x^2 + 4)^4\,dx$.

Solution

The first step is to choose an expression for u. We choose $u = 3x^2 + 4$ because then $du = 6x\,dx$ and we already have du in the integrand. Write the integral in terms of u:

$$\int 6x(3x^2 + 4)^4\,dx = \int u^4\,du.$$

Remember that du is the derivative of the expression chosen for u, regardless of what is inside the integrand. Now we can evaluate the integral with respect to u:

$$\int u^4\,du = \frac{u^5}{5} + C = \frac{(3x^2 + 4)^5}{5} + C.$$

Analysis

We can check our answer by taking the derivative of the result of integration. We should obtain the integrand. Picking a value for C of 1, we let $y = \frac{1}{5}(3x^2 + 4)^5 + 1$. We have

$$y = \frac{1}{5}(3x^2 + 4)^5 + 1,$$

so

$$y' = \left(\frac{1}{5}\right) 5(3x^2+4)^4 6x$$
$$= 6x(3x^2+4)^4.$$

This is exactly the expression we started with inside the integrand.

? Exercise 4.6.1

Use substitution to find the antiderivative of $\int 3x^2(x^3-3)^2\,dx$.

Hint

Let $u = x^3 - 3$.

Answer

$$\int 3x^2(x^3-3)^2\,dx = \frac{1}{3}(x^3-3)^3 + C$$

Sometimes we need to adjust the constants in our integral if they don't match up exactly with the expressions we are substituting.

✓ Example 4.6.2: Using Substitution with Alteration

Use substitution to find the antiderivative of

$$\int z\sqrt{z^2-5}\,dz.$$

Solution

Rewrite the integral as $\int z(z^2-5)^{1/2}\,dz$. Let $u = z^2-5$ and $du = 2z\,dz$. Now we have a problem because $du = 2z\,dz$ and the original expression has only $z\,dz$. We have to alter our expression for du or the integral in u will be twice as large as it should be. If we multiply both sides of the du equation by $\frac{1}{2}$. we can solve this problem. Thus,

$$u = z^2 - 5$$
$$du = 2z\,dz$$
$$\frac{1}{2}du = \frac{1}{2}(2z)\,dz = z\,dz.$$

Write the integral in terms of u, but pull the $\frac{1}{2}$ outside the integration symbol:

$$\int z(z^2-5)^{1/2}\,dz = \frac{1}{2}\int u^{1/2}\,du.$$

Integrate the expression in u:

$$\begin{aligned}\frac{1}{2}\int u^{1/2}\,du &= \left(\frac{1}{2}\right)\frac{u^{3/2}}{\frac{3}{2}}+C\\ &= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{3/2}+C\\ &= \frac{1}{3}u^{3/2}+C\\ &= \frac{1}{3}(z^2-5)^{3/2}+C\end{aligned}$$

Exercise 4.6.2

Use substitution to find the antiderivative of $\int x^2(x^3+5)^9\,dx$.

Hint

Multiply the du equation by $\frac{1}{3}$.

Answer

$$\int x^2(x^3+5)^9\,dx = \frac{(x^3+5)^{10}}{30}+C$$

4.6.1 Substitution for Definite Integrals

Substitution can be used with definite integrals, too. However, using substitution to evaluate a definite integral requires a change to the limits of integration. If we change variables in the integrand, the limits of integration change as well.

Substitution with Definite Integrals

Let $u=g(x)$ and let g' be continuous over an interval $[a,b]$, and let f be continuous over the range of $u=g(x)$. Then,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Although we will not formally prove this theorem, we justify it with some calculations here. From the substitution rule for indefinite integrals, if $F(x)$ is an antiderivative of $f(x)$, we have

$$\int f(g(x))g'(x)\,dx = F(g(x))+C.$$

Then

$$\begin{aligned}\int_a^b f[g(x)]g'(x)\,dx &= F(g(x))\Big|_{x=a}^{x=b}\\ &= F(g(b))-F(g(a))\\ &= F(u)\Big|_{u=g(a)}^{u=g(b)}\\ &= \int_{g(a)}^{g(b)} f(u)\,du\end{aligned}$$

and we have the desired result.

Example 4.6.3: Using Substitution to Evaluate a Definite Integral

Use substitution to evaluate

$$\int_0^1 x^2(1+2x^3)^5\,dx.$$

Solution

Let $u=1+2x^3$, so $du=6x^2\,dx$. Since the original function includes one factor of x^2 and $du=6x^2\,dx$, multiply both sides of the du equation by $1/6$. Then,

$$du=6x^2\,dx$$

$$\text{becomes}\quad \frac{1}{6}du=x^2\,dx.$$

To adjust the limits of integration, note that when $x=0, u=1+2(0)=1,$ and when $x=1,\ u=1+2(1)=3.$

Then

$$\int_0^1 x^2(1+2x^3)^5dx=\frac{1}{6}\int_1^3 u^5\,du.$$

Evaluating this expression, we get

$$\begin{aligned}\frac{1}{6}\int_1^3 u^5\,du &= \left(\frac{1}{6}\right)\left(\frac{u^6}{6}\right)\Big|_1^3\\ &=\frac{1}{36}\left[(3)^6-(1)^6\right]\\ &=\frac{182}{9}.\end{aligned}$$

Exercise 4.6.3

Use substitution to evaluate the definite integral $\int_{-1}^0 y(2y^2-3)^5\,dy.$

Hint

Use the steps from Example 4.6.3to solve the problem.

Answer

$$\int_{-1}^0 y(2y^2-3)^5\,dy=\frac{91}{3}$$

Exercise 4.6.4

Use substitution to evaluate $\int_0^4 \sqrt{2x+1}\,dx.$

Hint

Use the process from Example 4.6.3to solve the problem.

Answer

$$\int_0^4 \sqrt{2x+1}\,dx=\frac{26}{3}$$

Example 4.6.5: Using Substitution with an Exponential Function

Use substitution to evaluate

$$\int_0^1 xe^{4x^2+3}\,dx.$$

Solution

Let $u = 4x^3 + 3$. Then, $du = 8x\,dx$. To adjust the limits of integration, we note that when $x = 0$, $u = 3$, and when $x = 1$, $u = 7$. So our substitution gives

$$\begin{aligned}\int_0^1 xe^{4x^2+3}\,dx &= \frac{1}{8}\int_3^7 e^u\,du \\ &= \frac{1}{8}e^u\Big|_3^7 \\ &= \frac{e^7 - e^3}{8} \\ &\approx 134.568\end{aligned}$$

Just a quick note: It is possible to evaluate a definite integral using substitution without actually changing the bounds to values for "u." To do this, begin by evaluating the indefinite integral. After you have evaluated the indefinite integral using u-substitution, and expressed the result in terms of the original variable, evaluate that final result with the original bounds. This process will yield the same result. In the video below, a definite integral is evaluated using both methods, so that you can compare.



4.6.2 Key Concepts

- Substitution is a technique that simplifies the integration of functions that are the result of a chain-rule derivative. The term ‘substitution’ refers to changing variables or substituting the variable u and du for appropriate expressions in the integrand.
- When using substitution for a definite integral, we also have to change the limits of integration.

4.6.3 Key Equations

- **Substitution with Indefinite Integrals**

$$\int f[g(x)]g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C$$

- **Substitution with Definite Integrals**

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

4.6.4 Glossary

change of variables

the substitution of a variable, such as u, for an expression in the integrand

integration by substitution

a technique for integration that allows integration of functions that are the result of a chain-rule derivative

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4.6E: EXERCISES FOR SECTION 4.6

1) Why is u-substitution referred to as a *change of variable*?

2) If $f = g \circ h$, when reversing the chain rule, $\frac{d}{dx}(g \circ h)(x) = g'(h(x))h'(x)$, should you take $u = g(x)$ or $u = h(x)$?

Answer

$u = h(x)$

In exercises 3 - 7, verify each identity using differentiation. Then, using the indicated u-substitution, identify f such that the integral takes the form $\int f(u)\, du$.

3) $\int \frac{1}{\sqrt{2x+1}}\, dx = \sqrt{2x+1} + C; \quad u = 2x+1$

4) $\int xe^{x^2-3}\, dx = \frac{1}{2}e^{x^2-3} + C, \quad u = x^2 - 3$

Answer

$du = 2x\, dx; \quad f(u) = \frac{1}{2}e^u$

5) $\int x\sqrt{4x^2+9}\, dx = \frac{1}{12}(4x^2+9)^{3/2} + C; \quad u = 4x^2+9$

6) $\int \frac{x}{\sqrt{4x^2+9}}\, dx = \frac{1}{4}\sqrt{4x^2+9} + C; \quad u = 4x^2+9$

Answer

$du = 8x\, dx; \quad f(u) = \frac{1}{8\sqrt{u}}$

7) $\int \frac{x}{(4x^2+9)^2}\, dx = -\frac{1}{8(4x^2+9)} + C; \quad u = 4x^2+9$

In exercises 8 - 15, find the antiderivative using the indicated substitution.

8) $\int 3(3x+4)^4\, dx; \quad u = 3x+4$

Answer

$\int 3(3x+4)^4\, dx = \frac{1}{5}(3x+4)^5 + C$

9) $\int e^{7x+1}\, dx; \quad u = 7x+1$

10) $\int (2x-3)^{-7}\, dx; \quad u = 2x-3$

Answer

$\int (2x-3)^{-7}\, dx = -\frac{1}{12(2x-3)^6} + C$

11) $\int (3x-2)^{-1}\, dx; \quad u = 3x-2$

12) $\int \frac{x}{\sqrt{x^2+1}}\, dx; \quad u = x^2+1$

Answer

$\int \frac{x}{\sqrt{x^2+1}}\, dx = \sqrt{x^2+1} + C$

13) $\int x\sqrt{1-x^2}\, dx; \quad u = 1-x^2$

14) $\int (x-1)(x^2-2x)^3\,dx; \quad u=x^2-2x$

Answer

$$\int (x-1)(x^2-2x)^3\,dx=\frac{1}{8}(x^2-2x)^4+C$$

15) $\int (x^2-2x)(x^3-3x^2)^2\,dx; \quad u=x^3=3x^2$

In exercises 16-20, use a suitable change of variables to determine the indefinite integral.

16) $\int \frac{1}{1-2x}\,dx$

Answer

$$\int \frac{1}{1-2x}\,dx=-\frac{1}{2}\ln|1-2x|+C$$

17) $\int t(1-t^2)^{10}\,dt$

18) $\int (11x-7)^{-3}\,dx$

Answer

$$\int (11x-7)^{-3}\,dx=-\frac{1}{22(11x-7)^2}+C$$

19) $\int (7x-11)^4\,dx$

20) $\int \frac{x^2}{(x^3-3)^2}\,dx$

Answer

$$\int \frac{x^2}{(x^3-3)^2}\,dx=-\frac{1}{3(x^3-3)}+C$$

In exercises 21-26, use a change of variables to evaluate the definite integral.

21) $\int_0^1 x\sqrt{1-x^2}\,dx$

22) $\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx$

Answer

$$u=1+x^2, \quad du=2x\,dx, \quad \int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx=\frac{1}{2}\int_1^2 u^{-1/2}\,du=\sqrt{2}-1$$

23) $\int_0^2 \frac{t}{\sqrt{5+t^2}}\,dt$

24) $\int_0^1 \frac{t^2}{\sqrt{1+t^3}}\,dt$

Answer

$$u=1+t^3, \quad du=3t^2, \quad \int_0^1 \frac{t^2}{\sqrt{1+t^3}}\,dt=\frac{1}{3}\int_1^2 u^{-1/2}\,du=\frac{2}{3}(\sqrt{2}-1)$$

25) $\int_{-1}^0 e^{-2x}\,dx$

26) $\int_0^2 \dfrac{2}{4x+1}\,dx$

Answer

$u = 4x+1, \quad du = 4dx, \quad \int_0^2 \dfrac{2}{4x+1}\,dx = \dfrac{1}{2}\int_1^9 \dfrac{1}{u}du = \dfrac{1}{2}\ln 9$

27) If $h(a) = h(b)$ in $\int_a^b g'(h(x))h(x)\,dx$, what can you say about the value of the integral?

28) Is the substitution $u = 1 - x^2$ in the definite integral $\int_0^2 \dfrac{x}{1-x^2}\,dx$ okay? If not, why not?

Answer

No, because the integrand is discontinuous at $x = 1$.

In exercises 29-30, use a change of variables to show that each definite integral is equal to zero.

29) $\int_0^1 (1-2t)\,dt$

30) $\int_0^1 \dfrac{1-2t}{1+(t-\frac{1}{2})^2}\,dt$

Answer

$u = 1+(t-\frac{1}{2})^2$; the integral becomes $-\int_{5/4}^{5/4} \dfrac{1}{u}\,du$.

31) Show that the average value of $f(x)$ over an interval $[a,b]$ is the same as the average value of $f(cx)$ over the interval $\left[\frac{a}{c}, \frac{b}{c}\right]$ for $c > 0$.

Answer

Setting $u = cx$ and $du = c\,dx$ gets you $\dfrac{1}{\frac{b}{c}-\frac{a}{c}}\int_{a/c}^{b/c} f(cx)\,dx = \dfrac{c}{b-a}\int_{u=a}^{u=b} f(u)\dfrac{du}{c} = \dfrac{1}{b-a}\int_a^b f(u)\,du.$

32) Find the area under the graph of $f(t) = \dfrac{t}{(1+t^2)^a}$ between $t = 0$ and $t = x$ where $a > 0$ and $a \neq 1$ is fixed, and evaluate the limit as $x \to \infty$.

33) Find the area under the graph of $g(t) = \dfrac{t}{(1-t^2)^a}$ between $t = 0$ and $t = x$, where $0 < x < 1$ and $a > 0$ is fixed. Evaluate the limit as $x \to 1$.

Answer

$\int_0^x g(t)\,dt = \dfrac{1}{2}\int_{u=1-x^2}^1 \dfrac{du}{u^a} = \dfrac{1}{2(1-a)}u^{1-a} \mid 1u = \dfrac{1}{2(1-a)}(1-(1-x^2)^{1-a})$ As $x \to 1$ the limit is $\dfrac{1}{2(1-a)}$ if $a < 1$, and the limit diverges to $+\infty$ if $a > 1$.

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4.7: Areas between Curves

Learning Objectives

- Determine the area of a region between two curves by integrating with respect to the independent variable.
- Find the area of a compound region.
- Determine the area of a region between two curves by integrating with respect to the dependent variable.

In Introduction to Integration, we developed the concept of the definite integral to calculate the area below a curve on a given interval. In this section, we expand that idea to calculate the area of more complex regions. We start by finding the area between two curves that are functions of x, beginning with the simple case in which one function value is always greater than the other. We then look at cases when the graphs of the functions cross. Last, we consider how to calculate the area between two curves that are functions of y.

4.7.0.1 Area of a Region between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that $f(x) \geq g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions, as shown in Figure 4.7.1.

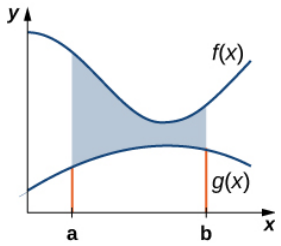


Figure 4.7.1: The area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $[a, b]$

As we did before, we are going to partition the interval on the **x-axis** and approximate the area between the graphs of the functions with rectangles. So, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g(x_i^*)$ to $f(x_i^*)$. Figure 4.7.2a shows the rectangles when x_i^* is selected to be the left endpoint of the interval and $n = 10$. Figure 4.7.2b shows a representative rectangle in detail.

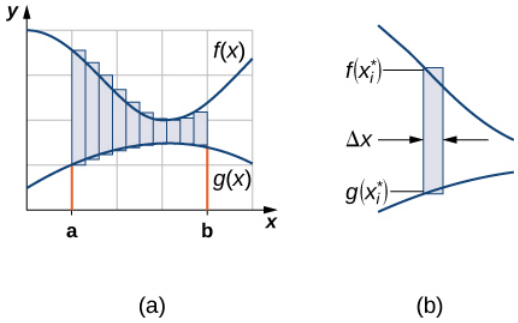


Figure 4.7.2: (a)We can approximate the area between the graphs of two functions, $f(x)$ and $g(x)$, with rectangles. (b) The area of a typical rectangle goes from one curve to the other.

The height of each individual rectangle is $f(x_i^*) - g(x_i^*)$ and the width of each rectangle is Δx. Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x.$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x = \int_a^b [f(x) - g(x)] dx.$$

These findings are summarized in the following theorem.

Finding the Area between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_a^b [f(x) - g(x)]dx.$$

We apply this theorem in the following example.

Example 4.7.1: Finding the Area of a Region between Two Curves I

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1, 4]$, find the area of region $\mathbf{R}$.

Solution

The region is depicted in the following figure.

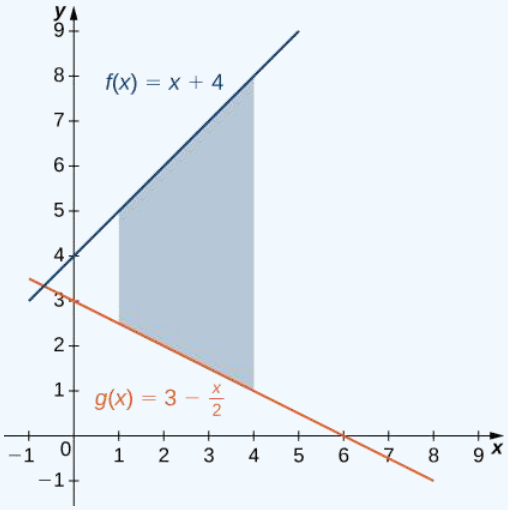


Figure 4.7.3: A region between two curves is shown where one curve is always greater than the other.

We have

$$A = \int_a^b [f(x) - g(x)]\, dx$$

$$= \int_1^4 [(x+4) - (3 - \frac{x}{2})]\, dx = \int_1^4 \left[\frac{3x}{2} + 1\right]\, dx$$

$$= [\frac{3x^2}{4} + x]\Big|_1^4 = (16 - \frac{7}{4}) = \frac{57}{4}.$$

The area of the region is $\frac{57}{4}$ units^2.

Exercise 4.7.1

If $\mathbf{R}$ is the region bounded by the graphs of the functions $f(x) = \frac{x}{2} + 5$ and $g(x) = x + \frac{1}{2}$ over the interval $[1, 5]$, find the area of region $\mathbf{R}$.

Hint

Graph the functions to determine which function's graph forms the upper bound and which forms the lower bound, then follow the process used in Example.

Answer

12 units^2

In Example $4.7.1$, we defined the interval of interest as part of the problem statement. Quite often, though, we want to define our interval of interest based on where the graphs of the two functions intersect. This is illustrated in the following example.

Example 4.7.2: Finding the Area of a Region between Two Curves II

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = 9 - (x/2)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region $\mathbf{R}$.

Solution

The region is depicted in the following figure.

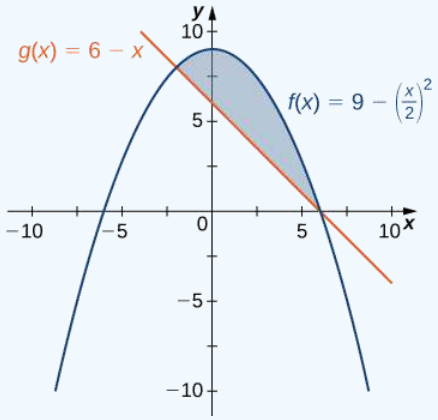


Figure 4.7.4: This graph shows the region below the graph of $f(x)$ and above the graph of $g(x)$.

We first need to compute where the graphs of the functions intersect. Setting $f(x) = g(x)$, we get

$$f(x) = g(x)$$
$$9 - (\frac{x}{2})^2 = 6 - x$$
$$9 - \frac{x^2}{4} = 6 - x$$
$$36 - x^2 = 24 - 4x$$
$$x^2 - 4x - 12 = 0$$
$$(x - 6)(x + 2) = 0.$$

The graphs of the functions intersect when $x = 6$ or $x = -2$, so we want to integrate from -2 to 6. Since $f(x) \geq g(x)$ for $-2 \leq x \leq 6$, we obtain

$$A = \int_a^b [f(x) - g(x)]\, dx$$
$$= \int_{-2}^{6} \left[9 - (\frac{x}{2})^2 - (6 - x)\right]\, dx$$
$$= \int_{-2}^{6} \left[3 - \frac{x^2}{4} + x\right]\, dx$$
$$= \left[3x - \frac{x^3}{12} + \frac{x^2}{2}\right]\Big|_{-2}^{6} = \frac{64}{3}.$$

The area of the region is $64/3$ units^2.

? Exercise 4.7.2

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = x$ and below by the graph of the function $g(x) = x^4$, find the area of region $\mathbf{R}$.

Hint

Use the process from Example 4.7.2

Answer

$\frac{3}{10}$ unit^2

4.7.1 Areas of Compound Regions

So far, we have required $f(x) \geq g(x)$ over the entire interval of interest, but what if we want to look at regions bounded by the graphs of functions that cross one another? In that case, we may need to break up the interval and evaluate two or more integrals, depending on which function is greater over a given part of the interval.

✓ Example 4.7.3: Finding the Area of a Complex Region

Consider the region depicted in Figure 4.7.6. Find the area of $\mathbf{R}$.

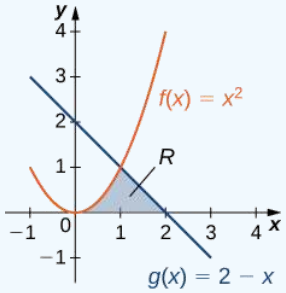


Figure 4.7.6: Two integrals are required to calculate the area of this region.

Solution

To find the area of this region, we need to divide the interval into two pieces. The graphs of the functions intersect at $x = 1$ (set $f(x) = g(x)$ and solve for x), so we evaluate two separate integrals: one over the interval $[0, 1]$ and one over the interval $[1, 2]$.

Over the interval $[0, 1]$, the region is bounded above by $f(x) = x^2$ and below by the x-axis, so we have

$$A_1 = \int_0^1 x^2 dx = \frac{x^3}{3}\,|_0^1 = \frac{1}{3}.$$

Over the interval $[1, 2]$, the region is bounded above by $g(x) = 2 - x$ and below by the **x-axis**, so we have

$$A_2 = \int_1^2 (2 - x) dx = [2x - \frac{x^2}{2}]\,|_1^2 = \frac{1}{2}.$$

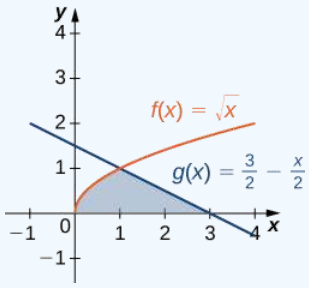
Adding these areas together, we obtain

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

The area of the region is $5/6$ units^2.

? Exercise 4.7.3

Consider the region depicted in the following figure. Find the area of $\mathbf{R}$.



Hint

The two curves intersect at **x=1**

Answer

$\frac{5}{3}$ units^2

4.7.1.1 Regions Defined with Respect to y

In Example 4.7.3, we had to evaluate two separate integrals to calculate the area of the region. However, there is another approach that requires only one integral. What if we treat the curves as functions of y, instead of as functions of x? Review Figure. Note that the left graph, shown in red, is represented by the function $y = f(x) = x^2$. We could just as easily solve this for x and represent the curve by the function $x = v(y) = \sqrt{y}$. (Note that $x = -\sqrt{y}$ is also a valid representation of the function $y = f(x) = x^2$ as a function of y. However, based on the graph, it is clear we are interested in the positive square root.) Similarly, the right graph is represented by the function $y = g(x) = 2 - x$, but could just as easily be represented by the function $x = u(y) = 2 - y$. When the graphs are represented as functions of y, we see the region is bounded on the left by the graph of one function and on the right by the graph of the other function. Therefore, if we integrate with respect to y, we need to evaluate one integral only. Let's develop a formula for this type of integration.

Let $u(y)$ and $v(y)$ be continuous functions over an interval $[c, d]$ such that $u(y) \geq v(y)$ for all $y \in [c, d]$. We want to find the area between the graphs of the functions, as shown in Figure 4.7.7.

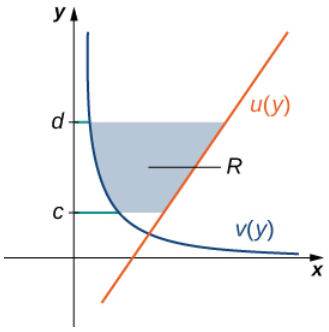


Figure 4.7.7: We can find the area between the graphs of two functions, $u(y)$ and $v(y)$.

This time, we are going to partition the interval on the **y-axis** and use horizontal rectangles to approximate the area between the functions. So, for $i = 0, 1, 2, \ldots, n$, let $Q = y_i$ be a regular partition of $[c, d]$. Then, for $i = 1, 2, \ldots, n$, choose a point $y_i^* \in [y_{i-1}, y_i]$, then over each interval $[y_{i-1}, y_i]$ construct a rectangle that extends horizontally from $v(y^0 *_i)$ to $u(y_i^*)$. Figure

4.7.8*a* shows the rectangles when y_i^* is selected to be the lower endpoint of the interval and $n = 10$. Figure 4.7.8*b* shows a representative rectangle in detail.

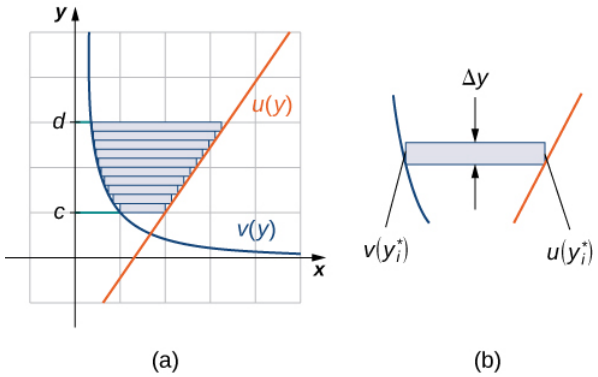


Figure 4.7.8: (a) Approximating the area between the graphs of two functions, $u(y)$ and $v(y)$, with rectangles. (b) The area of a typical rectangle.

The height of each individual rectangle is Δy and the width of each rectangle is $u(y_i^*) - v(y_i^*)$. Therefore, the area between the curves is approximately

$$A \approx \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)] \Delta y.$$

This is a Riemann sum, so we take the limit as $n \to \infty$, obtaining

$$\begin{aligned} A &= \lim_{n \to \infty} \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)] \Delta y \\ &= \int_c^d [u(y) - v(y)] dy. \end{aligned}$$

These findings are summarized in the following theorem.

Finding the Area between Two Curves, Integrating along the y-axis

Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$ for all $y \in [c, d]$. Let $\mathbf{R}$ denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, and above and below by the lines $y = d$ and $y = c$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_c^d [u(y) - v(y)] dy.$$

Example 4.7.4: Integrating with Respect to y

Let's revisit Example 4.7.3, only this time let's integrate with respect to y. Let $\mathbf{R}$ be the region depicted in Figure 4.7.9. Find the area of $\mathbf{R}$ by integrating with respect to y.

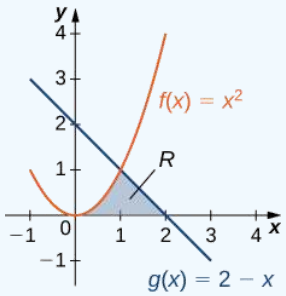


Figure 4.7.9: The area of region $\mathbf{R}$ can be calculated using one integral only when the curves are treated as functions of y.

Solution

We must first express the graphs as functions of y. As we saw at the beginning of this section, the curve on the left can be represented by the function $x = v(y) = \sqrt{y}$, and the curve on the right can be represented by the function $x = u(y) = 2 - y$.

Now we have to determine the limits of integration. The region is bounded below by the x-axis, so the lower limit of integration is $y = 0$. The upper limit of integration is determined by the point where the two graphs intersect, which is the point $(1, 1)$, so the upper limit of integration is $y = 1$. Thus, we have $[c, d] = [0, 1]$.

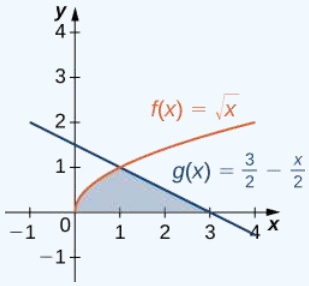
Calculating the area of the region, we get

$$\begin{aligned} A &= \int_c^d [u(y) - v(y)]dy \\ &= \int_0^1 [(2 - y) - \sqrt{y}]dy \\ &= [2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}]\Big|_0^1 \\ &= \frac{5}{6}. \end{aligned}$$

The area of the region is $5/6$ units^2.

Exercise 4.7.4

Let's revisit the checkpoint associated with Example 4.7.3, only this time, let's integrate with respect to y. Let $\mathbf{R}$ be the region depicted in the following figure. Find the area of $\mathbf{R}$ by integrating with respect to y.



Hint

Follow the process from the previous example.

Answer

$\frac{5}{3}$ units^2

4.7.2 Key Concepts

- Just as definite integrals can be used to find the area under a curve, they can also be used to find the area between two curves.
- To find the area between two curves defined by functions, integrate the difference of the functions.
- If the graphs of the functions cross, or if the region is complex, it may be necessary to evaluate two or more integrals and add the results to find the area of the region.
- Sometimes it can be easier to integrate with respect to y to find the area. The principles are the same regardless of which variable is used as the variable of integration.

4.7.3 Key Equations

- **Area between two curves, integrating on the x-axis**

$$A = \int_a^b [f(x) - g(x)]dx$$

- **Area between two curves, integrating on the y-axis**

$$A = \int_c^d [u(y) - v(y)]dy$$

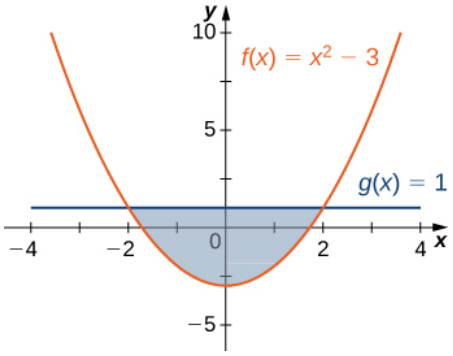
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4.7E: EXERCISES FOR SECTION 4.7

For exercises 1 - 2, determine the area of the region between the two curves in the given figure by integrating over the x-axis.

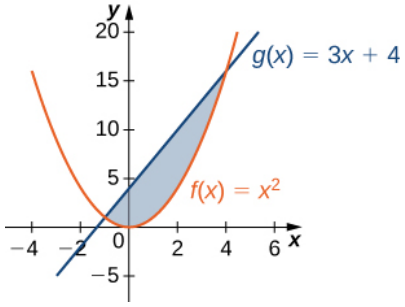
1) $y = x^2 - 3$ and $y = 1$



Answer

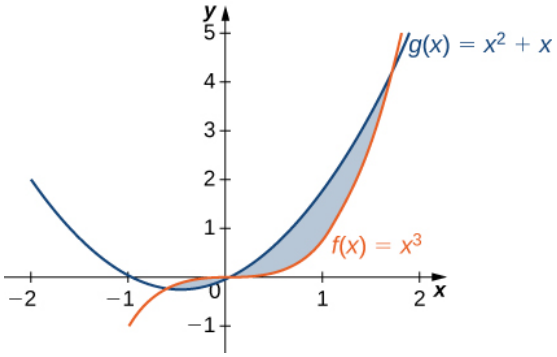
$\frac{32}{3}$ units^2

2) $y = x^2$ and $y = 3x + 4$



For exercise 3, split the region between the two curves into two smaller regions, then determine the area by integrating over the x-axis. Note that you will have two integrals to solve.

3) $y = x^3$ and $y = x^2 + x$

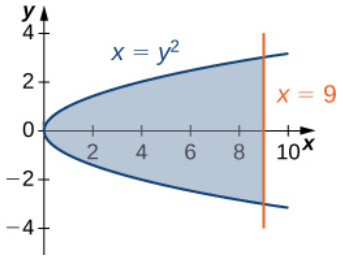


Answer

$\frac{13}{12}$ units^2

For exercises 4-5, determine the area of the region between the two curves by integrating over the y-axis.

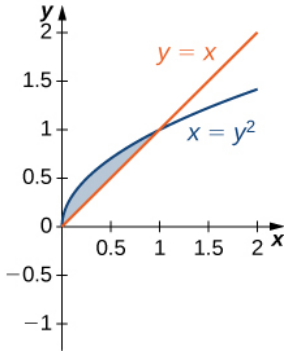
4) $x = y^2$ and $x = 9$



Answer

36 units^2

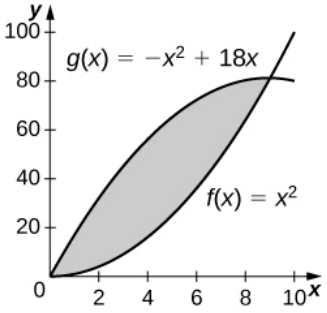
5) $y = x$ and $x = y^2$



For exercises 6-12, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis.

6) $y = x^2$ and $y = -x^2 + 18x$

Answer

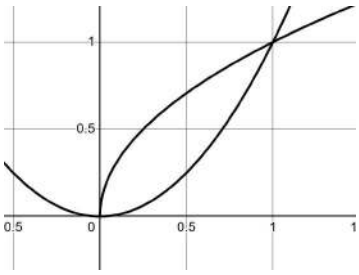


243 square units

7) $y = \dfrac{1}{x}$, $y = \dfrac{1}{x^2}$, and $x = 3$

8) $y = \sqrt{x}$ and $y = x^2$

Answer



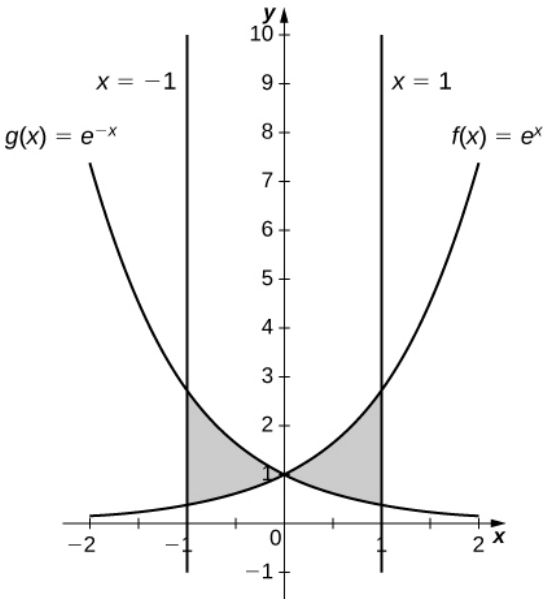
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1/3 square units

9) $y = e^x, \quad y = e^{2x-1}$, and $x = 0$

10) $y = e^x, \quad y = e^{-x}, \quad x = -1$ and $x = 1$

Answer

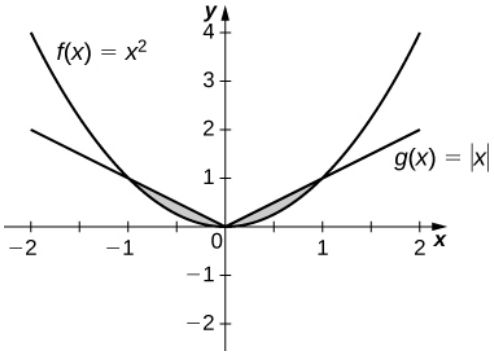


$\frac{2(e-1)^2}{e}$ units^2

11) $y = e, \quad y = e^x$, and $y = e^{-x}$

12) $y = |x|$ and $y = x^2$

Answer



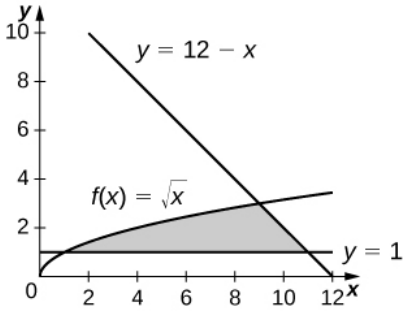
$\frac{1}{3}$ units^2

For exercises 13-18, graph the equations and shade the area of the region between the curves. If necessary, break the region into sub-regions to determine its entire area.

13) $y = 8 - 2x, \quad y = x,$ and $y = 0$

14) $y = 12 - x, \quad y = \sqrt{x},$ and $y = 1$

Answer

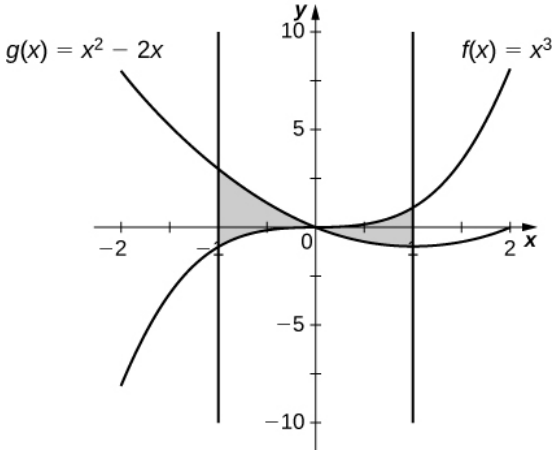


$\frac{34}{3}$ units^2

15) $y = x^2$ and $y = \frac{8}{x}$ and $y = 1$

16) $y = x^3$ and $y = x^2 - 2x$ over $x \in [-1, 1]$

Answer

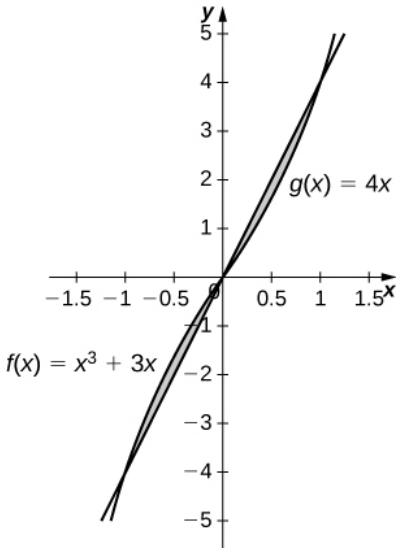


$\frac{5}{2}$ units^2

17) $y = x^2 + 9$ and $y = 10 + 2x$ over $x \in [-1, 3]$

18) $y = x^3 + 3x$ and $y = 4x$

Answer



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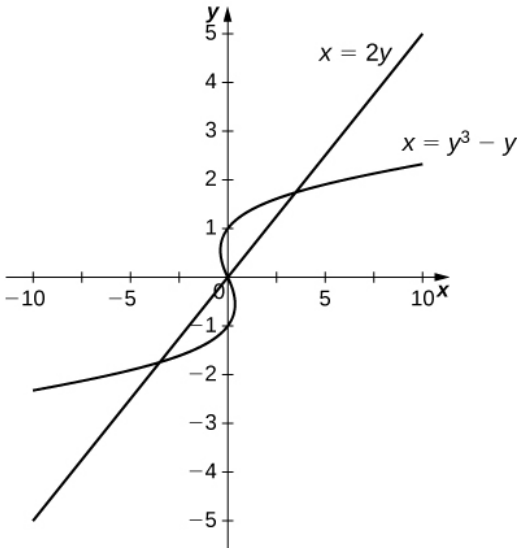
$\frac{1}{2}$ units^2

For exercises 19-22, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the y-axis.

19) $x = y^3$ and $x = 3y - 2$

20) $x = y$ and $x = y^3 - y$

Answer

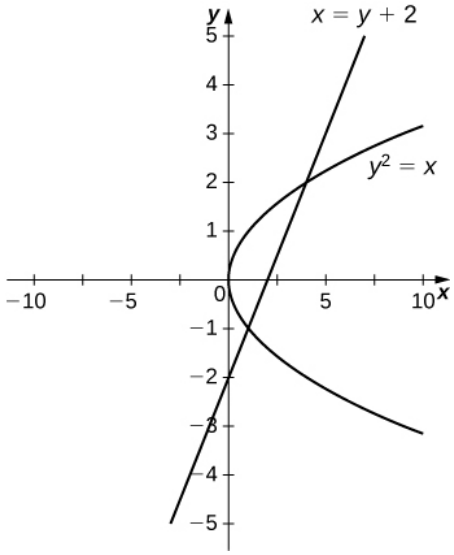


$\frac{9}{2}$ units^2

21) $x = -3 + y^2$ and $x = y - y^2$

22) $y^2 = x$ and $x = y + 2$

Answer



$\frac{9}{2}$ units^2

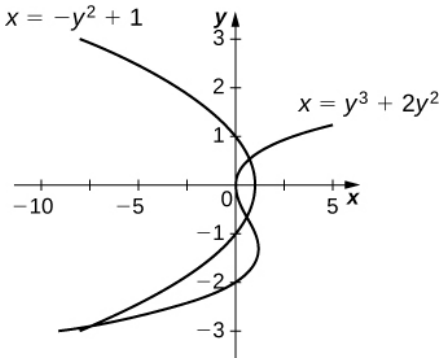
For exercises 23-29, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis or y-axis, whichever seems more convenient.

23) $x = y^4$ and $x = y^5$

24) $y = x^6$ and $y = x^4$

25) $x = y^3 + 2y^2 + 1$ and $x = -y^2 + 1$

Answer

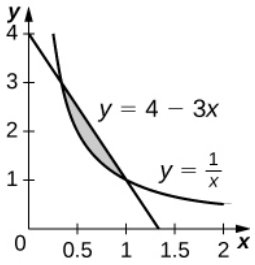


$\frac{27}{4}$ units^2

26) $y = |x|$ and $y = x^2 - 1$

27) $y = 4 - 3x$ and $y = \frac{1}{x}$

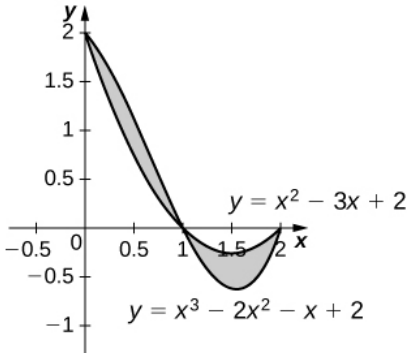
Answer



$\left(\frac{4}{3} - \ln(3)\right)$ units^2

28) $y = x^2 - 3x + 2$ and $y = x^3 - 2x^2 - x + 2$

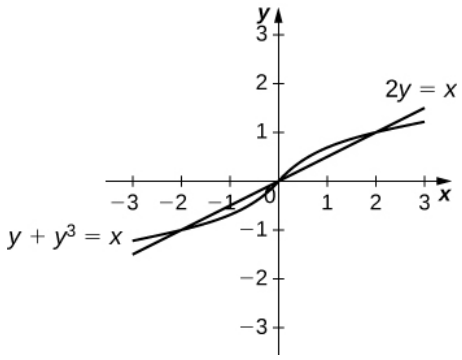
Answer



$\frac{1}{2}$ square units

29) $y + y^3 = x$ and $2y = x$

Answer



$\frac{1}{2}$ square units

30) A factory selling cell phones has a marginal cost function $C(x) = 0.01x^2 - 3x + 229$, where x represents the number of cell phones, and a marginal revenue function given by $R(x) = 429 - 2x$. Find the area between the graphs of these curves and $x = 0$. What does this area represent?

Answer

$33,333.33 total profit for **200** cell phones sold

31) An amusement park has a marginal cost function $C(x) = 1000e - x + 5$, where x represents the number of tickets sold, and a marginal revenue function given by $R(x) = 60 - 0.1x$. Find the total profit generated when selling 550 tickets. Use a calculator to determine intersection points, if necessary, to two decimal places.

4.7E.1 CONTRIBUTORS

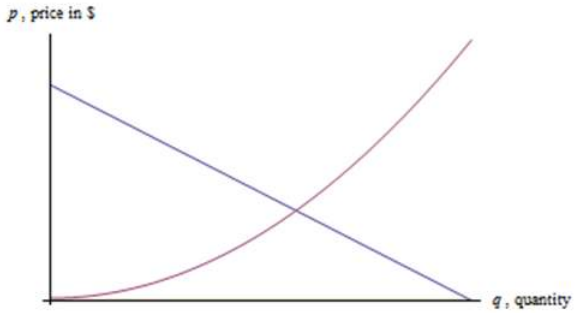
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4.8: Applications to Business

4.8.1 Consumer and Producer Surplus

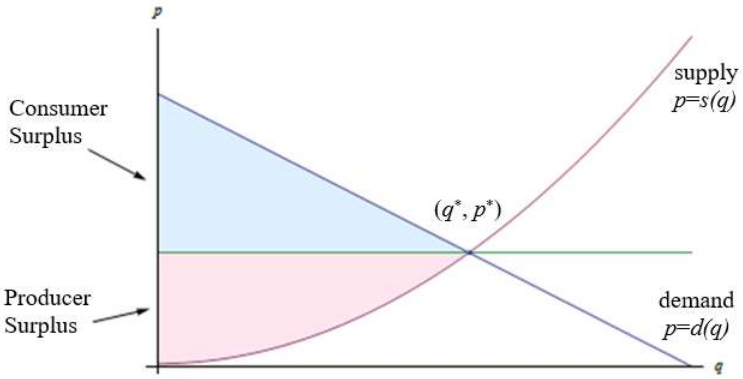
Here are a demand and a supply curve for a product. Which is which?



The demand curve is decreasing – lower prices are associated with higher quantities demanded, higher prices are associated with lower quantities demanded. Demand curves are often shown as if they were linear, but there's no reason they have to be.

The supply curve is increasing – lower prices are associated with lower supply, and higher prices are associated with higher quantities supplied.

The point where the demand and supply curve cross is called the equilibrium point (q^*, p^*).



Suppose that the price is set at the equilibrium price, so that the quantity demanded equals the quantity supplied. Now think about the folks who are represented on the left of the equilibrium point. The consumers on the left would have been willing to pay a higher price than they ended up having to pay, so the equilibrium price saved them money. On the other hand, the producers represented on the left would have been willing to supply these goods for a lower price – they made more money than they expected to. Both of these groups ended up with extra cash in their pockets!

Graphically, the amount of extra money that ended up in consumers' pockets is the area between the demand curve and the horizontal line at p^*. This is the difference in price, summed up over all the consumers who spent less than they expected to – a definite integral. Notice that since the area under the horizontal line is a rectangle, we can simplify the area integral:

$$\int_0^{q^*} (d(q) - p^*)\, dq = \int_0^{q^*} d(q)\, dq - \int_0^{q^*} p^*\, dq = \int_0^{q^*} d(q)\, dq - p^* q^*.$$

The amount of extra money that ended up in producers' pockets is the area between the supply curve and the horizontal line at p^*. This is the difference in price, summed up over all the producers who received more than they expected to. Similar to consumer surplus, this integral can be simplified:

$$\int_0^{q^*} (p^* - s(q))\, dq = \int_0^{q^*} p^*\, dq - \int_0^{q^*} s(q)\, dq = p^* q^* - \int_0^{q^*} s(q)\, dq.$$

Consumer and Producer Surplus

Given a demand function $p = d(q)$ and a supply function $p = s(q)$, and the equilibrium point (q^*, p^*)

- The **consumer surplus** is

$$\int_0^{q^*} d(q)\,dq - p^*q^*.$$

- The **producer surplus** is

$$p^*q^* - \int_0^{q^*} s(q)\,dq.$$

- The sum of the consumer surplus and producer surplus is the **total gains from trade**.

What are the units of consumer and producer surplus? The units are (price per item)(quantity of items) = money!

Example 4.8.1

Suppose the demand for a product is given by $p = d(q) = -0.8q + 150$ and the supply for the same product is given by $p = s(q) = 5.2q$. For both functions, q is the quantity and p is the price, in dollars.

1. Find the equilibrium point.
2. Find the consumer surplus at the equilibrium price.
3. Find the producer surplus at the equilibrium price.

Solution

1. The equilibrium point is where the supply and demand functions are equal.

$$\begin{aligned} -0.8q + 150 &= 5.2q \\ 150 &= 6.0q \\ q &= \frac{150}{6} \\ q &= 25 \end{aligned}$$

   The equilibrium quantity is 25 items. To find the corresponding equilibrium price we can use either the demand or supply equation.

   $s(25) = 5.2(25) = 130$. The equilibrium price is \$130.

2. The consumer surplus is

$$\int_0^{25}(-0.8q+150)\,dq-(130)(25) = \left[-0.8\frac{q^2}{2}+150q\right]_0^{25}-3250$$
$$= \left(-0.4\left(25^2\right)+150(25)\right)-\left(-0.4\left(0^2\right)+150(0)\right)-3250$$
$$= 3500-0-3250$$
$$= \$250$$

3. The producer surplus is

$$(130)(25)-\int_0^{25}5.2q\,dq = 3250-\left[5.2\frac{q^2}{2}\right]_0^{25}$$
$$= 3250-\left(\left(2.6\left(25^2\right)\right)-\left(2.6\left(0^2\right)\right)\right)$$
$$= 3250-(1625-0)$$
$$= \$1625$$

Example 4.8.2

Suppose the demand for a product is given by $p=d(q)=300-5q$ and the supply for the same product is given by $p=s(q)=q^2$. For both functions, q is the quantity and p is the price, in dollars.

1. Find the equilibrium point.
2. Find the consumer surplus at the equilibrium price.
3. Find the producer surplus at the equilibrium price.

Solution

1. This is quadratic, so we can rearrange the equation so one side is zero: $q^2+5q-300=0$.

   We could solve this using the quadratic equation: $q=\frac{-5\pm\sqrt{5^2-4(1)(-300)}}{2(1)}$

   or we could factor: $(q+20)(q-15)=0$, so $q=-20$ or $q=15$.

   We want the positive solution. The equilibrium quantity is 15 items. The equilibrium price is $s(15)=15^2=\$225$.

2. The consumer surplus is

$$\int_0^{15}(300-5q)\,dq-(225)(15) = \left[300q-5\frac{q^2}{2}\right]_0^{15}-3375$$
$$= \left(300(15)-2.5\left(15^2\right)\right)-\left(300(0)-2.5\left(0^2\right)\right)-3375$$
$$= 3937.5-0-3375$$
$$= \$562.50$$

3. The producer surplus is

$$(225)(15)-\int_0^{15}q^2\,dq = 3375-\left[\frac{q^3}{3}\right]_0^{15}s$$
$$= 3375-\left(\left(\frac{15^3}{3}\right)-\left(\frac{0^3}{3}\right)\right)$$
$$= 3375-(1125-0)$$
$$= \$2250$$

Example 4.8.3

The tables below show information about the demand and supply functions for a product. For both functions, q is the quantity and p is the price, in dollars.

q	0	100	200	300	400	500	600	700

p	70	61	53	46	40	35	31	28

q	0	100	200	300	400	500	600	700
p	14	21	28	33	40	47	54	61

1. Which is which? That is, which table represents demand and which represents supply?
2. What is the equilibrium price and quantity?
3. Find the consumer and producer surplus at the equilibrium price.

Solution

1. The first table shows decreasing price associated with increasing quantity; that is the demand function.
2. For both functions, $q = 400$ is associated with $p = 40$; the equilibrium price is \$40 and the equilibrium quantity is 400 units. Notice that we were lucky here, because the equilibrium point is actually one of the points shown. In many cases with a table, we would have to estimate.
3. The consumer surplus uses the demand function, which comes from the first table. We'll have to approximate the value of the integral using rectangles. There are 4 rectangles, and let's choose to use left endpoints.

   The consumer surplus is

$$\int_0^{400} (\text{demand})\, dq - (40)(400) \approx$$
$$(100)(70+61+53+46) - (40)(400) \;= \$7000$$

   So the consumer surplus is about \$7000.

   The producer surplus uses the supply function, which comes from the second table. Let's choose to use left endpoints for this integral also.

   The producer surplus is

$$(40)(400) - \int_0^{400} (\text{supply})\, dq \approx$$
$$(40)(400) - (100)(14+21+28+33) \;= \$6400$$

   So the producer surplus is about \$6400.

4.8.2 Continuous Income Stream

In precalculus, you learned about compound interest in that really simple situation where you made a single deposit into an interest-bearing account and let it sit undisturbed, earning interest, for some period of time. Recall:

Compound Interest Formulas

Let P be the principal (initial investment), r be the annual interest rate expressed as a decimal, and $A(t)$ be the amount in the account at the end of t years.

4.8.2.0.1 Compounding n times per year

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

4.8.2.0.1 Compounded continuously

$$A(t) = Pe^{rt}$$

If you're using this formula to find what an account will be worth in the future, $t > 0$ and $A(t)$ is called the **future value**.

If you're using the formula to find what you need to deposit today to have a certain value P sometime in the future, $t < 0$ and $A(t)$ is called the **present value**.

You may also have learned somewhat more complicated annuity formulas to deal with slightly more complicated situations – where you make equal deposits equally spaced in time.

But real life is not usually so neat.

Calculus allows us to handle situations where "deposits" are flowing continuously into an account that earns interest. As long as we can model the flow of income with a function, we can use a definite integral to calculate the present and future value of a continuous income stream. The idea here is that each little bit of income in the future needs to be multiplied by the exponential function to bring it back to the present, and then we'll add them all up (a definite integral).

Continuous Income Stream

Suppose money can earn interest at an annual interest rate of r, compounded continuously. Let $F(t)$ be a continuous income function (in dollars per year) that applies between year 0 and year T.

Then the present value of that income stream is given by

$$PV = \int_0^T F(t)e^{-rt}\,dt.$$

The future value can be computed by the ordinary compound interest formula

$$FV = PVe^{rt}.$$

This is a useful way to compare two investments – find the present value of each to see which is worth more today.

Example 4.8.4

You have an opportunity to buy a business that will earn \$75,000 per year continuously over the next eight years. Money can earn 2.8% per year, compounded continuously. Is this business worth its purchase price of \$630,000?

Solution

First, please note that we still have to make some simplifying assumptions. We have to assume that the interest rates are going to remain constant for that entire eight years. We also have to assume that the \$75,000 per year is coming in continuously, like a faucet dripping dollars into the business. Neither of these assumptions might be accurate.

But moving on: The present value of the \$630,000 is, well, \$630,000. This is one investment, where we put our \$630,000 in the bank and let it sit there.

To find the present value of the business, we think of it as an income stream. The function $F(t)$ in this case is a constant \$75,000 dollars per year, so $F(t) = 75{,}000$. The interest rate is 2.8% and the term we're interested in is 8 years, so $r = .028$, and $T = 8$:

$$PV = \int_0^8 75000e^{-0.028t}\,dt \approx 537{,}548.75$$

The present value of the business is about $537,500, which is less than the $630,000 asking price, so this is not a good deal.

While this integral could have been done using substitution, for many of the integrals in this section we don't have the techniques to use antiderivatives or, in some cases, no antiderivative exists. Technology will work quickly, and it will give you an answer that is good enough.

Example 4.8.5

A company is considering purchasing a new machine for its production floor. The machine costs $65,000. The company estimates that the additional income from the machine will be a constant $7000 for the first year, then will increase by $800 each year after that. In order to buy the machine, the company needs to be convinced that it will pay for itself by the end of 8 years with this additional income. Money can earn 1.7% per year, compounded continuously. Should the company buy the machine?

Solution

We'll assume that the income will come in continuously over the 8 years. We'll also assume that interest rates will remain constant over that 8-year time period.

We're interested in the present value of the machine, which we will compare to its $65,000 price tag. Let t be the time, in years, since the purchase of the machine. The income from the machine is different depending on the time.

From $t = 0$ to $t = 1$ (the first year), the income is constant $7000 per year. From $t = 1$ to $t = 8$, the income is increasing by $800 each year; the income flow function $F(t)$ will be $F(t) = 7000 + 800(t-1) = 6200 + 800t$. To find the present value, we'll have to divide the integral into the two pieces, one for each of the functions:

$$PV = \int_0^1 7000e^{-0.017t}\,dt + \int_1^8 (6200 + 800t)e^{-0.017t}\,dt \approx 70166.$$

The present value is greater than the cost of the machine, so the company should buy the machine.

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4.R: CHAPTER 4 REVIEW EXERCISES

In exercises 1 - 4, answer *True or False.* Justify your answer with a proof or a counterexample. Assume all functions f and g are continuous over their domains.

1) If $f(x) > 0,\ f'(x) > 0$ for all x, then the right-hand rule underestimates the integral $\int_a^b f(x)\,dx$. Use a graph to justify your answer.

Answer

False

2) $\int_a^b f(x)^2\,dx = \int_a^b f(x)\,dx$

3) If $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

Answer

True

4) All continuous functions have an antiderivative.

In exercises 5 - 8, evaluate the Riemann sums L_4 and R_4 for the given functions over the specified interval. Compare your answer with the exact answer, when possible, or use a calculator to determine the answer.

5) $y = 3x^2 - 2x + 1$ over $[-1,1]$

Answer

$L_4 = 5.25,\ R_4 = 3.25,$ exact answer: 4

6) $y = \ln(x^2+1)$ over $[0,e]$

7) $y = x^2 \sin x$ over $[0,\pi]$

Answer

$L_4 = 5.364,\ R_4 = 5.364,$ exact answer: 5.870

8) $y = \sqrt{x} + \frac{1}{x}$ over $[1,4]$

In exercises 9 - 12, evaluate the integrals.

9) $\int_{-1}^{1} (x^3 - 2x^2 + 4x)\,dx$

Answer

$-\frac{4}{3}$

10) $\int_0^4 \frac{3t}{\sqrt{1+6t^2}}\,dt$

11) $\int_0^1 \frac{dx}{1+2x}$

Answer

$-\frac{ln(3)}{2}$

12) $\int_0^2 xe^{-x^2}\,dx$

In exercises 13 - 16, find the antiderivative.

13) $\int \frac{dx}{(3x+4)^3}$

Answer

$$-\frac{1}{6(3x+4)^2}+C$$

14) $\int x\ln(x^2)\,dx$

15) $\int \frac{4x^2}{\sqrt{1-x^3}}\,dx$

Answer

$$-\frac{8}{3}\sqrt{1-x^3}+C$$

16) $\int \frac{e^{2x}}{1+e^{2x}}\,dx$

In exercises 17 - 20, find the derivative.

17) $\frac{d}{dt}\int_0^t \frac{ln(x)}{\sqrt{1+x^2}}\,dx$

Answer

$$\frac{ln(t)}{\sqrt{1+t^2}}$$

18) $\frac{d}{dx}\int_1^{x^3} \sqrt{4-t^2}\,dt$

19) $\frac{d}{dx}\int_1^{\ln(x)} (4t+e^t)\,dt$

Answer

$$4\frac{\ln x}{x}+1$$

20) $\frac{d}{dx}\int_0^{\sqrt{x}} e^{t^2}\,dt$

In exercises 21 - 23, consider the historic average cost per gigabyte of RAM on a computer.

Year	5-Year Change ($)
1980	0
1985	$-5,468,750$
1990	$-755,495$
1995	$-73,005$
2000	$-29,768$
2005	-918
2010	-177

21) If the average cost per gigabyte of RAM in 2010 is $12, find the average cost per gigabyte of RAM in 1980.

Answer

$6,328,113

Solution: $6,328,113

22) The average cost per gigabyte of RAM can be approximated by the function $C(t)=8,500,000(0.65)^t$, where t is measured in years since 1980, and C is cost in US dollars. Find the average cost per gigabyte of RAM for the period from 1980 to 2010.

23) Find the average cost of 1 GB RAM from 2005 to 2010.

Answer

$73.36

24) The velocity of a bullet from a rifle can be approximated by $v(t) = 6400t^2 - 6505t + 2686$, where t is seconds after the shot and v is the velocity measured in feet per second. This equation only models the velocity for the first half-second after the shot: $0 \leq t \leq 0.5$. What is the total distance the bullet travels in 0.5 sec?

25) What is the average velocity of the bullet for the first half-second?

Answer

$\frac{19117}{12}$ ft/sec, or about 1593 ft/sec

26) The demand function for a certain commodity is given by $D(q) = 36 - 6q$, and the supply function is given by $S(q) = 3q$.

a) Find the equilibrium point.

b) Find the Consumer Surplus.

c) Find the Producer Surplus.

27) Find the accumulated future value of a 9-year $300,000 continuous income stream that has been compounded continuously at 5.3%. Round to the nearest dollar.

Answer

$3,459,812

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