

NWFSC MGF 1130 TEXT



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MGF 1130: Mathematical Thinking

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Glossary

Detailed Licensing

Detailed Licensing

Licensing

A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.

1: NUMBER THEORY

1.1: Prime and Composite Numbers

1.2: Visualize Fractions

1.3: Decimals

1.4: Understand Percent

1.5: Solve General Applications of Percent

1.6: Solve Sales Tax, Commission, and Discount Applications

1.1: Prime and Composite Numbers



Figure 1.1.1: Computers are protected using encryption based on prime numbers. (credit: "Data Security" by Blogtrepreneur/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Apply divisibility rules.
2. Define and identify numbers that are prime or composite.
3. Find the prime factorization of composite numbers.
4. Find the greatest common divisor.
5. Use the greatest common divisor to solve application problems.
6. Find the least common multiple.
7. Use the least common multiple to solve application problems.

Encryption, which is needed for the secure exchange of information (i.e., online banking or shopping) is based on prime numbers. Encryption uses a composite number that is the product of two very large prime numbers. To break the encryption, the two primes that were used to form the composite number need to be determined. If the two prime numbers used are sufficiently large, even the fastest computer cannot determine those two prime numbers in a reasonable amount of time. It would take a computer 300 trillion years to crack the current encryption standard.

Applying Divisibility Rules

Before we begin our investigation of divisibility, we need to know some facts about important sets of numbers:

- The counting numbers are referred to as the **natural numbers**. This set of numbers, $\{1, 2, 3, 4, \dots\}$, is denoted with the symbol $\mathbb{N}$.
- The **whole numbers** are very similar to the natural numbers, but we will add the number 0 to the set. So the set of whole numbers is $\{0, 1, 2, 3, 4, \dots\}$
- Another important set of numbers is the **integers**. The integers are the natural numbers, along with 0, and the negatives of the natural numbers. This set is often written as $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. We denote the integers with the symbol $\mathbb{Z}$.

All the ideas of this section apply to the natural numbers, while only some apply to all the integers.

Divisibility is when the integer n is divisible by m , if n can be written as m times another integer. Equivalently, there is no remainder when n is divided by m . There are many occasions when separating items into equal groups comes into play to ensure an equal distribution of whole items. For example, Francis, a preschool art teacher, has 15 students in one class. Francis has 225 sheets of construction paper and wants to provide each student with an equal number of pieces. To know if he will use all the construction paper, Francis is really asking if 225 can be evenly divided into 15 groups.

Example 1.1.1

Determining if a Number Divides Another Number

Determine if 36 is divisible by 4.

Answer available at the end of this section.

Your Turn 1.1.1

Determine if 54 is divisible by 9.

You can quickly check if a number is divisible by 2, 3, 4, 5, 6, 9, 10, and 12. Each has an easy-to-identify feature, or rule, that indicates the divisibility by those numbers, as shown in the following table.

Divisor	Rule
2	Last digit is even
3	Add the digits of the number together. If that sum is divisible by 3, then so is the original number
4	Look at only the last two digits. If this number is divisible by 4, so is the original number
5	Look at only the last digit. If it is a 5 or a 0, then the original number is divisible by 5
6	If the number passes the rule for divisibility by 2 and for 3, then the number is divisible by 6
9	Add the digits of the number together. If that number is divisible by 9, then so is the original number
10	Look at only the last digit. If it is a 0, then the original number is divisible by 10
12	If the number passes the rule for 3 and 4, the number is divisible by 12

Example 1.1.2

Using Divisibility Rules

Using divisibility rules, determine if 245 is divisible by 5.

Answer available at the end of this section.

Your Turn 1.1.2

Using divisibility rules, determine if 45,730 is divisible by 5.

Example 1.1.3

Using Divisibility Rules

Using divisibility rules, determine if 25,983 is divisible by 9.

Answer available at the end of this section.

Your Turn 1.1.3

Using divisibility rules, determine if 342,887 is divisible by 9.

Example 1.1.4

Using Divisibility Rules

Can 298 coins be stacked into 6 stacks with an equal number of coins in each stack?

Answer available at the end of this section.

Your Turn 1.1.4

Can 43,568 pieces of mail be separated into 6 bins with the same number of pieces of mail per bin?

Example 1.1.5

Using Divisibility Rules

Using divisibility rules, determine if 4,259 is divisible by 10.

Answer available at the end of this section.

Your Turn 1.1.5

Using divisibility rules, determine if 87,762 is divisible by 10.

Example 1.1.6

Using Divisibility Rules

Using divisibility rules, determine if 936,276 is divisible by 4.

Answer available at the end of this section.

Your Turn 1.1.6

Using divisibility rules, determine if 43,568 is divisible by 4.

Prime and Composite Numbers

Sometimes, a natural number has only two unique divisors, 1 and itself. For instance, 7 and 19 are prime. In other words, there is no way to divide a prime number into groups with an equal number of things, unless there is only one group, or those groups have one item per group. Other natural numbers have more than two unique divisors, such as 4, or 26. These numbers are called composite. The number 1 is special; it is neither prime nor composite.

To determine if a number is prime or composite, you have to determine if the number has any divisors other than 1 and itself. The divisibility rules are useful here, and can quickly show you if a number has a divisor on that list.

However, if none of those divide the number, you still have to check all other possible prime divisors. What are the prime numbers

that are possibly divisors of the number you are checking? You need only check the prime numbers up to the square root of the number in question. For instance, if you want to know if 2,117 is prime, you need to determine if any primes up to the square root of 2,117 (which is 46.0 when rounded to one decimal place) divide 2,117. If any of those primes are divisors of the number in question, then the number is composite. If none of those primes work, then the number is itself prime.

We can check divisibility with whatever tool we wish. Divisibility rules are quick for some prime divisors (2 and 5 come to mind) but aren't quick for other values (like 11). In place of divisibility rules, we could just use a calculator. If the prime number divides the number evenly (that is, there is no decimal or fractional part), then the number is divisible by that prime. Table 1.1.1 is a quick list of the prime numbers up to 50. There are 15 prime numbers less than 50.

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47

Table 1.1.1 Prime Numbers Less than 50

Example 1.1.7

Determining If a Number Is Prime or Composite

Determine if 2,117 is prime or composite.

Answer available at the end of this section.

Your Turn 1.1.7

Determine if 1,429 is prime or composite.

Example 1.1.8

Determining if a Number Is Prime or Composite

Determine if 423 is prime or composite.

Answer available at the end of this section.

Your Turn 1.1.8

Determine if 859 is prime or composite.

Example 1.1.9

Determining if a Number Is Prime or Composite

Determine if 1,034 is prime or composite

Answer available at the end of this section.

Your Turn 1.1.9

Determine if 5,067,322 is prime or composite.

Example 1.1.10

Determining if a Number Is Prime or Composite

Determine if 2,917 is prime or composite.

Answer available at the end of this section.

Your Turn 1.1.10

Determine if 1,477 is prime.

Who Knew?

ILLEGAL PRIMES

Large primes are a hot commodity. Using two very large primes (some have more than 22 million digits!) is necessary for secure encryption. Anyone who has a new prime that is large enough can use that prime to create a new encryption. Of course, whoever discovers a large prime could sell it to a security company. These primes are so useful for encryption, it is necessary to protect that intellectual property. In fact, at least one prime number was declared illegal.

Video

[Illegal Prime Number](#)

People in Mathematics

Sophie Germain



SOPHIE GERMAIN (portrait del., à l'âge de 11 ans.)

Figure 1.1.2 Sophie Germain (credit: "Sophie Germain at 14 years," Illustration from *histoire du socialisme*, approx. 1880, Wikimedia Commons, public domain)

Born into a wealthy French family in 1776, Sophie Germain discovered and fell in love with mathematics by browsing her father's books. Clandestine study, hard and tenacious work, and a mathematical mindset did not lead to college, however, as she was not allowed to attend. She did manage, through friends, to obtain problem sets and submit brilliant solutions under the name Monsieur LaBlanc. One of her great interests was number theory, which is the study of properties of integers. One

of her theorems, titled “Sophie Germain’s Theorem,” partially solved one of the great mathematical mysteries, Fermat’s Last Theorem. From this she discovered what are now known as Sophie Germain Primes. A Sophie Germain Prime is a prime number that can be written in the form $2p + 1$, where p is a prime number. For instance, 23 is prime: $2(23) + 1 = 47$, which is prime, so 47 is a Sophie Germain Prime. It should be noted that $2p + 1$, where p is a prime number, may or may not be prime (check for $p = 7$).

Finding the Prime Factorization of Composite Numbers

Before we can start with prime factorization, let’s remind ourselves what it means to factor a number. We factor a number by identifying two (or more) numbers that, when multiplied, result in the original number. For instance, 3 and 24, when multiplied, give 72. So, 72 can be factored into 3×24 . Notice that we could have factored the 72 differently, say as $72 = 6 \times 12$, or $72 = 2 \times 36$, or even as $72 = 3 \times 4 \times 6$.

Finding the prime factorization of a composite number means writing the number as the product of all of its prime factors. For example, $80 = 2 \times 2 \times 2 \times 2 \times 5$. Notice that all the numbers being multiplied on the right-hand side are prime numbers. Sometimes prime numbers repeat themselves in the factorization. When prime factors do repeat, we may write the prime factorization using exponents, as in $80 = 2^4 \times 5$. In that equation, the 2 is raised to the 4th power. The 4 is the **exponent**, and the 2 is the **base**. More generally, in the exponent notation a^b , the number represented by **Math input error** is the exponent.

One has to wonder if finding the prime factorization could result in different factorizations. The **Fundamental Theorem of Arithmetic** tells us that there is only one prime factorization for a given natural number.

Fundamental Theorem of Arithmetic

Every natural number, other than 1, can be expressed in exactly one way, apart from the arrangement, as a product of primes.

The process of finding the prime factorization of a number is iterative, which means we do a step, then repeat it until we cannot do the step any longer. The step we use is to identify one prime factor of the number, then write the number as the prime factor times another factor. We repeat this step on the other, newly found, factor. This step is repeated until no more primes can be factored from the remaining factor. This is easier to see and explain with an example.

Example 1.1.11

Finding the Prime Factorization

Find the prime factorization of 140.

Answer available at the end of this section.

Your Turn 1.1.11

Find the prime factorization of 90.

Factor Trees

A useful tool for helping with prime factorization is a **factor tree**. To create a factor tree for the natural number n (where n is not 1), perform the following steps:

Step 1: If n is prime, you’re done. If n is composite, continue to the next step.

Step 2: Identify two divisors of n , call them a and b .

Step 3: Write the number n down, and draw two branches extending down (or to the right) of the number n .

Step 4: Label the end of one branch a and the end of the other branch b .

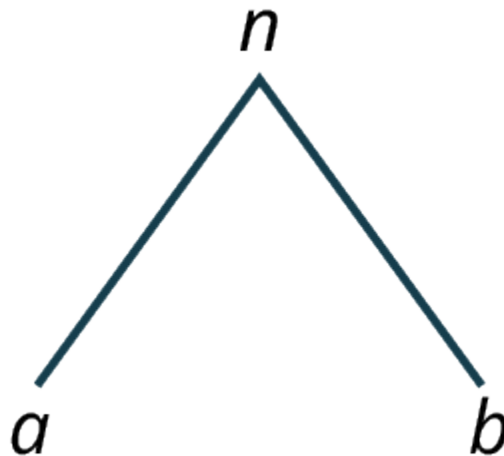


Figure 1.1.3: Example of a factor tree

Step 5: If a and b are prime, the tree is complete. When a number at this step is a prime number, we refer to it as a leaf of the tree diagram.

Step 6: If either a or b are composite, repeat Steps 2 through 4 for a and b .

Step 7: The process stops when the leaves are all prime.

Step 8: The prime factorization is then the product of all the leaves.

This is best seen in an example.

Example 1.1.12

Finding the Prime Factorization

Find the prime factorization of 66.

Answer available at the end of this section.

Your Turn 1.1.12

Find the prime factorization of 85.

Video

[Using a Factor Tree to Find the Prime Factorization](#)

Example 1.1.13

Finding the Prime Factorization

Find the prime factorization of 135.

Answer available at the end of this section.

Your Turn 1.1.13

Find the prime factorization of 280.

Example 1.1.14

Identifying Prime Factors

How many different prime factors does 10,241 have?

Answer available at the end of this section.

Your Turn 1.1.14

Find the number of different prime factors of 180.

Video

[Finding the Prime Factorization of 168](#)

Finding the Greatest Common Divisor

Two numbers often have more than one divisor in common (all pairs of natural numbers have the common divisor 1). When listing the common divisors, it's often the case that the largest is of interest. This divisor is called the greatest common divisor and is denoted **GCD**. It is also sometimes referred to as the greatest common factor (GCF).

For instance, 6 is the greatest common divisor of 12 and 18. We can see this by listing all the divisors of each number and, by inspection, select the largest value that shows up in each list.

The divisors of 12	1, 2, 3, 4, 6, 12
The divisors of 18	1, 2, 3, 6, 9, 18

It is easy to see that 6 is the largest value that appears in both lists.

Example 1.1.15

Finding the Greatest Common Divisor Using Lists

Find the greatest common divisor of 1,400 and 250 by listing all divisors of each number.

Answer available at the end of this section.

Your Turn 1.1.15

Find the greatest common divisor of 270 and 99 by listing all divisors of each number.

Listing all the divisors of the numbers in the set will always work, but for some relatively small numbers, the set of all divisors can become pretty big, and finding them all can be a chore. Another approach to finding the greatest common divisor is to use the prime factorization of the numbers. To do so, use the following steps:

Step 1: Find the prime factorization of the numbers.

Step 2: Identify the prime factors that appear in every number's prime factorization. These are called the common prime factors.

Step 3: Identify the smallest exponent of each prime factor identified in Step 2 in the prime factorizations.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3. The result is the greatest common divisor.

Example 1.1.16

Finding the Greatest Common Divisor Using Prime Factorization

Find the greatest common divisor of 1,400 and 250 by using their prime factorizations.

Answer available at the end of this section.

Your Turn 1.1.16

Using prime factorization, determine the greatest common divisor of 36 and 128.

Example 1.1.17

Finding the Greatest Common Divisor Using Prime Factorization

Find the greatest common divisor of 600 and 784 by using their prime factorizations.

Answer available at the end of this section.

Your Turn 1.1.17

Using prime factorization, determine the greatest common divisor of 120 and 200.

People in Mathematics

Srinivasa Ramanujan



Figure 1.1.9 Srinivasa Ramanujan (credit: Srinivasa Ramanujan, photo by Konrad Jacobs/Oberwolfach Photo Collection/public domain)

Ramanujan was born in southern India in 1887, during British rule. He was a self-taught mathematician, who, while in high school, began working through a two-volume text of mathematical theorems and results. His work included examination of the distribution of primes. He eventually came to the attention of British mathematician, G.H. Hardy. During one visit, Hardy mentioned to Ramanujan that his taxicab number was 1,729, remarking that 1,729 appeared to be a rather dull number. To which Ramanujan responded, "It is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."

Applications of the Greatest Common Divisor

The greatest common divisor has uses that are related to other mathematics (reducing fractions) but also in everyday applications. We'll look at two such applications, which have very similar underlying structures. In each case, something must divide the groups or measurements equally.

Example 1.1.18

Calculating Floor Tile Size

Suppose you have a rectangular room that is 570 cm wide and 450 cm long. You wish to cover the floor of the room with square tiles. What's the largest size square tile that can be used to cover this floor?

Answer available at the end of this section.

Your Turn 1.1.18

You are designing a brick patio made of square bricks 5 cm thick, but you need to determine the width and length of those bricks. The patio will be 400 cm by 540 cm. What are the largest size square bricks that can be used so that you do not need to cut any bricks?

Example 1.1.19

Organizing Books Per Shelf

Suppose you want to organize books onto shelves, and you want the shelves to hold the same number of books. Each shelf will only contain one genre of book. You have 24 sci-fi, 42 fantasy, and 30 horror books. How many books can go on each shelf?

Answer available at the end of this section.

Your Turn 1.1.19

There are three gym classes. The number of students in the classes is 21, 35, and 28. What is the largest team size that can be formed if teams from every class must have the same number of students?

Finding the Least Common Multiple

The flip side to a divisor of a number is a multiple of a number. For example, 5 is a divisor of 45 and so 45 is a multiple of 5. More generally, if the number a divides the number b , then b is a multiple of a .

This drives the idea of the least common multiple of a set of numbers. A common multiple of a set of numbers is a multiple of each of those numbers. For instance, 45 is a common multiple of 9 and 5, because 45 is a multiple of 9 (9 divides 45) and 45 is also a multiple of 5 (5 divides 45). The least common multiple (LCM) of a set of number is the smallest positive common multiple of that set of numbers.

There are (at least) three ways to find the LCM of a set of numbers, and we will explore two of them. One way is to create a list of multiples of each number in the set, and then identify the smallest multiple that appears in those lists.

Example 1.1.20

Finding the Least Common Multiple Using Lists

Find the LCM of 24 and 90 by listing multiples and choosing the smallest common multiple.

Answer available at the end of this section.

Your Turn 1.1.20

Use lists to find the LCM of 12 and 15.

The second method we can use is to find the prime factorizations of the number in the set to build the LCM of the numbers based on the prime divisors of the numbers. The LCM can be built from the prime factorization of the numbers in the set in a similar way as when finding the greatest common divisor. Here are the steps for using the prime factorization process for finding the LCM.

Step 1: Find the prime factorization of each number.

Step 2: Identify each prime that is present in any of the prime factorizations.

Step 3: Identify the largest exponent of each prime identified in Step 2 in the prime factorizations.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

Example 1.1.21

Finding the Least Common Multiple Using Prime Factorization

Use the prime factorizations of 24 and 90 to identify their LCM.

Answer available at the end of this section.

Your Turn 1.1.21

Use prime factorization to find the LCM of 20 and 28.

Example 1.1.22

Finding the Least Common Multiple Using Prime Factorization

Use the prime factorizations of 36, 66, and 250 to identify the LCM.

Answer available at the end of this section.

Your Turn 1.1.22

Use the prime factorization method to find the LCM of 150, 240, and 462.

Using lists for three or more numbers, particularly larger numbers, could take quite a bit of time. Frequently, as in this example, the prime factorization process is much quicker. In practice, you can use either listing or prime factorization to find the LCM.

Example 1.1.23**Finding the Least Common Multiple Using Both Methods**

Find the LCM of 20, 36, and 45 using lists and prime factorization.

Answer available at the end of this section.

Your Turn 1.1.23

Find the LCM of 18, 24, and 40 using lists and prime factorization.

Applications of the Least Common Multiple

Some applications of LCM involve events that repeat at fixed intervals, such as visits to a location. Other applications involve getting things to be of equal magnitude when using parts that are different sizes (see Example 1.1.25, for instance). In each case, we may be looking to determine when two processes “line up.”

Example 1.1.24**Determining Scheduling Overlap Using the Least Common Multiple**

Two students, João, and Amelia, meet one day at an assisted living facility where they volunteer. João volunteers every 6 days. Amelia volunteers every 10 days. How many days will it be until they are both volunteering on the same day again?

Answer available at the end of this section.

Your Turn 1.1.24

The sun, Venus, and Jupiter all line up on a given day. Venus orbits the sun once every 255 days. Jupiter orbits the sun every 4,330 days (we'll ignore the decimal values of days for each orbit). How many days will it be until they line up again?

Answer available at the end of this section.

Example 1.1.25**Determining the Minimum Height Using the LCM**

A team-building exercise has teams build a house of cards as high as possible. However, the cards for different teams are of different sizes. Team 1 uses 10 cm $\times$ 10 cm cards, while Team 2 uses 8 cm $\times$ 8 cm cards. What is the minimum height when the two teams will be tied? Ignore the width of the cards.

Answer available at the end of this section.

Your Turn 1.1.25

In an Internet giveaway, every 130th person who submits a survey receives \$250, and every 900th person receives a free cell phone. How many submissions must be received for the first person to receive both prizes?

Answers

We could divide 36 by 4 and see if there is a remainder, or we could see if we can write 36 as 4 times another integer. If we divide 36 by 4, we see $36 \div 4 = 9$ with no remainder. We see that 36 is divisible by 4. We can write 36 as 4 times another integer, $36 = 4 \times 9$. By the definition of divisibility, 36 is divisible by 4.

Since the last digit is a 5, the number 245 is divisible by 5 because the rule states if the last digit of the number is a 5 or a 0, then the original number is divisible by 5.

The divisibility rule for 9 is when the digits of the number are added, the sum is divisible by 9. So, we calculate the sum of the digits.

$2 + 5 + 9 + 8 + 3 = 27$. Since 27 is divisible by 9, so is the original number 25,983.

In order for the coins to be in equal-sized stacks, 298 would need to be divisible by 6. The divisibility rule for 6 is that the number passes the divisibility rules for both 2 and 3. Since the last digit is even, 298 is divisible by 2. To determine if 298 is divisible by 3, we first add the digits of the number: $2 + 9 + 8 = 19$. Since 19 is not divisible by 3, neither is 298. Because 298 is not divisible by 3, it is also not divisible by 6, which means they cannot be put into 6 equal stacks of coins.

The divisibility rule for 10 is that the last digit of the number is 0. Since the last digit of 4,259 is not 0, then 4,259 is not divisible by 10.

The divisibility rule for 4 is to check the last two digits of the number. If the number formed by the last two digits of the original number is divisible by 4, then so is the original number. The last two digits make the number 76 and 76 is divisible by 4, since $76 = 4 \times 19$. Since 76 is divisible by 4, so is 936,276.

The square root of 2,117 is 46.0 (rounded to one decimal place). So, we need to check if 2,117 is divisible by any prime up to 46.

Step 1: First we'll use the rules of divisibility we learned earlier:

- We can tell 2,117 is not divisible by 2, as the last digit isn't even.
- 2,117 is not divisible by 5 (the last digit isn't 0 or 5).
- Add the digits of 2,117 to get 11, which is not divisible by 3. So, 2,117 is also not divisible by 3.

Step 2: Now we repeat the process for all the primes up to 46.

Using a calculator, we find that 2,117 divided by the prime numbers 7, 11, 13, 17, 19, and 23 results in a remainder, a decimal part. So, we know that 2,117 is not divisible by these prime numbers. (You should check these results yourself.)

Moving on, we check the next prime: 29. Using the calculator to divide 2,117 by 29 results in 73. Since there is no decimal part, 2,117 is divisible by 29.

This means that 2,117 is not a prime number, but rather, a composite number. Writing 2,117 as the product of 29 and another natural number, $2,117 = 29 \times 73$.

The square root of 423 is 20.57 (rounded to two decimal places). So, we need to check if 423 is divisible by any prime up to 20.

Step 1: Check 2. We can tell 423 is not divisible by 2, as the last digit isn't even.

Step 2: Check 5. It is not divisible by 5 (the last digit isn't 0 or 5).

Step 3: Check 3. To check if 423 is divisible by 3, we use the divisibility rule for 3. When we take the sum of the digits of 423, the result is 9. Since 9 is divisible by 3, so is 423.

Since 423 is divisible by 3, then 423 is a composite number. Writing 423 as the product of 3 and another natural number, $423 = 3 \times 141$.

A quick inspection of 1,034 shows it is divisible by 2 since the last digit is even, and so 1,034 is a composite number.

The square root of 2,917 is 50.01 (rounded to two decimal places). So, we need to check if 2,917 is divisible by any prime up to 50.

Step 1: Check 2. We can tell 2,917 is not divisible by 2, as the last digit isn't even.

Step 2: Check 5. It is not divisible by 5 (the last digit isn't 0 or 5).

Step 3: Check 3. Using the divisibility rule for 3, we take the sum of the digits of 2,917, which is 19. Since 19 is not divisible by 3, neither is 2,917.

Step 4: Check the rest of the primes up to 50 using a calculator. When 2,917 is divided by every prime number up to 50, the result has a decimal part.

Since no prime up to 50 divides 2,917, it is a prime number.

Step 1: Identify a prime number that divides 140. Since 140 is even (the last digit is even), 2 divides 140. We then factor the 2 out of the 140, giving us $140 = 2 \times 70$.

Step 2: With the other factor, 70, find a prime factor of 70. Since 70 is also even, 2 divides 70. We factor the 2 out of the 70 and the factorization is now $140 = 2 \times 2 \times 35 = 2^2 \times 35$.

Notice that we expressed the two factors of 2 as 2^2 .

Step 3: Look to the remaining factor, 35. The last digit of 35 is 5, so 5 is a factor of 35. We factor the 5 out of the 35. The factorization is now $140 = 2^2 \times 5 \times 7$.

Step 4: Look to the remaining factor, 7. Since 7 is prime, the process is complete.

The prime factorization of 140 is $2^2 \times 5 \times 7$.

Since 66 is even, 2 is a factor.

Step 1: Factor out the 2. The factorization is $66 = 2 \times 33$.

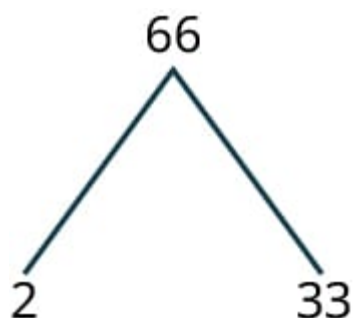


Figure 1.1.4: Factor tree showing $66 = 2 \times 33$

Since 2 is a prime factor, that branch is done, and 2 is a leaf.

Step 2: The 33, though, is divisible by 3, and is the product of 3 and 11. We attach that to the factor tree (Figure 1.1.5).

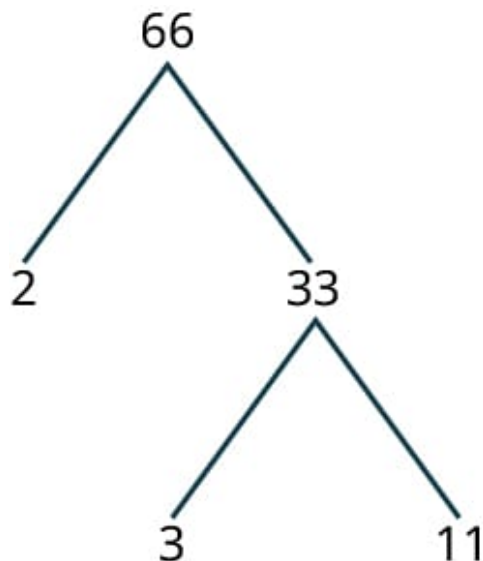


Figure 1.1.5: Factor tree showing $66 = 2 \times 3 \times 11$

Since the 2, 3, and 11 are all prime, the factor tree is done.

The prime factorization of 66 is the product of the leaves, so $66 = 2 \times 3 \times 11$. The factorization is complete.

The number 135 is divisible by 3, and so 3 is a factor of 135.

Step 1: Factor out the 3. The factorization is $135 = 3 \times 45$, shown in Figure 1.1.6.

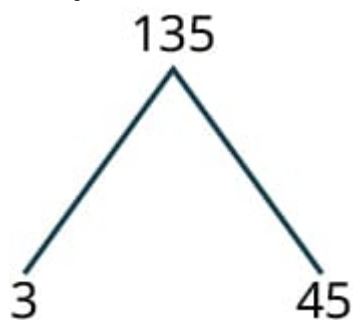


Figure 1.1.6: Factor tree showing $135 = 3 \times 45$

45 is also divisible by 3.

Step 2: Factor out a 3 from 45. The other factor is 15. The factor tree is shown in Figure 1.1.7.

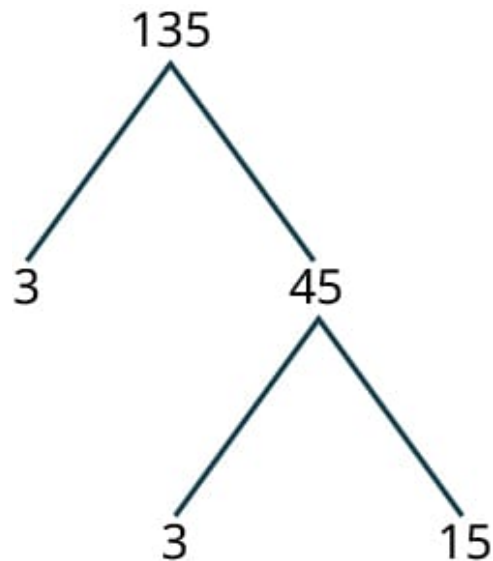


Figure 1.1.7: Factor tree showing $135 = 3 \times 3 \times 15$
15 is also divisible by 3.

Step 3: The factors of 15 are 3 and 5. The factor tree is shown in Figure 1.1.8.

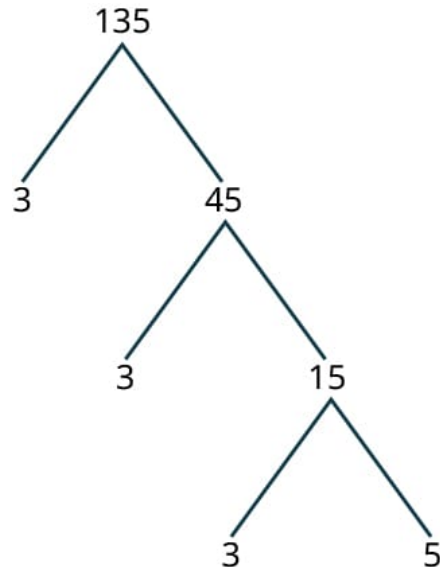


Figure 1.1.8: Final factor tree for 135, showing $135 = 3 \times 3 \times 3 \times 5$

All the leaves are prime, so the process is complete. The prime factorization of 135 is $3^3 \times 5$.

To know how many different prime factors 10,241 has, we need the prime factorization of 10,241.

Step 1: Use divisibility rules, to see that the number 10,241 is not divisible by 2 or by 3 (the sum of the digits is 8), or by 5. However, it is divisible by 7.

Step 2: After factoring the 7, the factorization is $10,241 = 7 \times 1463$; 1,463 is also divisible by 7.

Step 3: After factoring the 7, the factorization is $10,241 = 7 \times 7 \times 209 = 7^2 \times 209$. The number 209 is not divisible by 7.

Step 4: Check the next prime number: 11; 11 does divide 1,463.

Step 5: After factoring the 11, the factorization is $10,241 = 7^2 \times 11 \times 19$.

Since 19 is prime, the prime factorization of 10,241 is complete. We see that 10,241 has three different prime factors: 7, 11, and 19.

We create a list of all the divisors of 1,400 and of 250, and choose the largest one.

The divisors of 1,400 are

1, 2, 4, 5, 7, 8, 10, 14, 20, 25, 28, 35, 40, 50, 56, 70, 100, 140, 175, 200, 280, 350, 700, 1,400.

The divisors of 250 are

1, 2, 5, 10, 25, 50, 125, 250.

The largest value that appears on both lists is 50, so the greatest common divisor of 1,400 and 250 is 50.

Step 1: Find the prime factorizations of the numbers.

The prime factorization of 1,400 is $2^3 \times 5^2 \times 7$.

The prime factorization of 250 is 2×5^3 .

Step 2: Identify the prime factors that appear in every number's prime factorization.

The common prime factors are 2 and 5.

Step 3: Identify the smallest exponent of each prime identified in Step 2 in the prime factorizations.

The exponent of common prime factor 2 in the prime factorization of 1,400 is 3, and in the prime factorization of 250 is 1. The smallest of those exponents is 1. The exponent of the common prime factor 5 in the prime factorization of 1,400 is 2 and in the prime factorization of 250 is 3. The smallest of these exponents is 2.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^1 \times 5^2 = 50$. The greatest common divisor of 1,400 and 250 is $2 \times 5^2 = 50$.

Notice that the answer matches the one we found in Example 1.1.15.

Step 1: Find the prime factorizations of the numbers.

The prime factorization of 600 is $2^3 \times 3 \times 5^2$.

The prime factorization of 784 is $2^4 \times 7^2$.

Step 2: Identify the prime factors that appear in every number's prime factorization.

There is only one common prime factor, 2.

Step 3: Identify the smallest exponent of each prime from identified in Step 2 in the prime factorizations.

The exponent of 2 in the prime factorization of 600 is 3. The exponent of 2 in the prime factorization of 784 is 4. So, the smallest exponent of 2 is 3.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^3 = 8$. The greatest common divisor of 600 and 784 is 8.

Using squares means that the length and width of the tiles are equal. To ensure we are using full tiles, the side length of the square tiles must divide the length of the room and the width of the room. Since we want the largest square tiles, we need the GCD of the width and length of the room or the GCD of 570 and 450.

Step 1: Find the prime factorizations of the numbers.

The prime factorization of 570 is $2 \times 3 \times 5 \times 19$.

The prime factorization of 450 is $2 \times 3^2 \times 5^2$.

Step 2: Identify the prime factors that appear in every number's prime factorization.

The common prime factors are 2, 3, and 5.

Step 3: Identify the smallest exponent of each prime identified in Step 2 in the prime factorizations.

The exponents of 2 in the prime factorizations of 570 and 450 are 1 and 1. So the smallest exponent for 2 is 1.

The exponents of 3 in the prime factorizations of 570 and 450 are 1 and 2, so the smallest exponent for 3 is 1.

The exponents of 5 in the prime factorizations of 570 and 450 are 1 and 2, so the smallest exponent for 5 is 1.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^1 \times 3^1 \times 5^1 = 30$. The GCD of 450 and 570 is 30, so the largest size square tile that can be used to cover the floor is 30 cm by 30 cm.

Since we want shelves that hold an equal number of books, and a shelf can only hold one genre of book, we need a number that will equally divide 24, 42, and 30. So, we need the GCD of the number of books of each genre or the GCD of 24, 42, and 30.

Step 1: Find the prime factorizations of the numbers.

The prime factorization of 24 is $2^3 \times 3$.

The prime factorization of 42 is $2 \times 3 \times 7$.

The prime factorization of 30 is $2 \times 3 \times 5$.

Step 2: Identify the prime factors that appear in every number's prime factorization.

The common prime factors are 2 and 3.

Step 3: Identify the smallest exponent of each prime identified in Step 2 in the prime factorizations.

The smallest exponent of 2 and 3 in the factorizations is 1.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^1 \times 3^1 = 6$. The GCD of 24, 42, and 30 is 6, so the largest number of books that can go on a shelf is 6.

Create a list of the multiples of each number.

Step 1: The first 15 multiples for 24:

24, 48, 72, 96, 120, 144, 168, 192, 216, 240, 264, 288, 312, 336, 360.

Step 2: The first 15 multiples for 90:

90, 180, 270, 360, 450, 540, 630, 720, 810, 900, 990, 1,080, 1,170, 1,260, 1,350.

There is one multiple common to these lists, which is 360. So, 360 is the LCM of 24 and 90.

Step 1: Find the prime factorization of each number.

$24 = 2^3 \times 3$

$90 = 2 \times 3^2 \times 5$

Step 2: Identify each prime that is present in any of the prime factorizations.

The prime numbers present in the prime factorizations are 2, 3, and 5.

Step 3: Identify the largest exponent of each prime identified in Step 2 in the prime factorizations.

Prime	2	3	5
Largest exponent	3	2	1

Step 4: Compute the LCM by multiplying the prime factors identified in Step 2 raised to the powers identified in Step 3.

The LCM for 24 and 90 is $2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5^1 = 360$.

Step 1: Find the prime factorization of each number.

$$36 = 2^2 \times 3^2$$

$$66 = 2 \times 3 \times 11$$

$$250 = 2 \times 5^3$$

Step 2: Identify each prime that is present in any of the prime factorizations.

The prime numbers present in the prime factorizations are 2, 3, 5, and 11.

Step 3: Identify the largest exponent of each prime identified in Step 2 in the prime factorizations.

Prime	2	3	5	11
Largest exponent	2	2	3	1

Step 4: Compute the LCM by multiplying the prime factors identified in Step 2 raised to the powers identified in Step 3.

The LCM for 36, 66, and 250 is $2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 5 \times 11 = 2^2 \times 3^2 \times 5^3 \times 11^1 = 49,500$.

Step 1: Use listing. List the multiples:

20	20, 40, 60, 80, 100, 120, 140, 160, 180, 200, 220
36	36, 72, 108, 144, 180, 216, 252, 288, 324, 360
45	45, 90, 135, 180, 225, 270, 315, 360, 405, 450, 495, 540

The smallest value that appears on all the lists is 180, so 180 is the LCM of 20, 36, and 45.

Step 2: Find the prime factorization of each number.

$$20 = 2^2 \times 5$$

$$36 = 2^2 \times 3^2$$

$$45 = 3^2 \times 5$$

Step 3: Identify each prime that is present in any of the prime factorizations.

The prime numbers present in the prime factorizations are 2, 3, 5.

Step 4: Identify the largest exponent of each prime identified in Step 3 in the prime factorizations.

Prime	2	3	5
Largest exponent	2	2	1

Step 5: Compute the LCM by multiplying the prime factors identified in Step 3 raised to the powers identified in Step 4.

The LCM for 20, 36, and 45 is $2^2 \times 3^2 \times 5^1 = 180$.

Both listing and prime factorization produced the same result: the LCM is 180.

If we list the days that each student will volunteer, it becomes clear that we could solve this problem using the LCM of 6 and 10.

João will be at the assisted living facility 6, 12, 18, 24, 30, 36, 42, and 48 days later.

Amelia will be at the assisted living facility 10, 20, 30, 40, 50, and 60 days later.

The smallest number appearing on both lists is 30. João and Amelia will once more be volunteering together 30 days later.

This is an example where we want to put together objects with different sizes. We want to know the minimum height when they are tied, or when the houses of cards line up the first time. The heights of the towers built using the 10 cm $\times$ 10 cm cards will be 10, 20, 30, 40, 50, and 60 cm tall. When the 8 cm $\times$ 8 cm cards are used, the tower will be 8, 16, 24, 32, 40, 48, and 56 cm tall. The smallest number appearing on both lists is 40. The first time they are tied is when the two towers are 40 cm tall.

1.2: Visualize Fractions

Learning Objectives

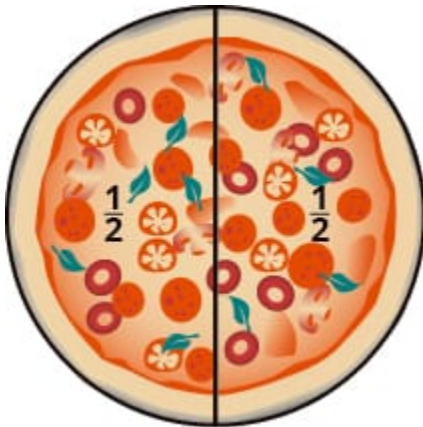
After completing this section, you should be able to:

1. Understand the meaning of fractions
2. Model improper fractions and mixed numbers
3. Convert between improper fractions and mixed numbers
4. Model equivalent fractions
5. Find equivalent fractions

Understand the Meaning of Fractions

Andy and Bobby love pizza. On Monday night, they share a pizza equally. How much of the pizza does each one get? Are you thinking that each boy gets half of the pizza? That's right. There is one whole pizza, evenly divided into two parts, so each boy gets one of the two equal parts.

In math, we write $\frac{1}{2}$ to mean one out of two parts.



On Tuesday, Andy and Bobby share a pizza with their parents, Fred and Christy, with each person getting an equal amount of the whole pizza. How much of the pizza does each person get? There is one whole pizza, divided evenly into four equal parts. Each person has one of the four equal parts, so each has $\frac{1}{4}$ of the pizza.



On Wednesday, the family invites some friends over for a pizza dinner. There are a total of 12 people. If they share the pizza equally, each person would get $\frac{1}{12}$ of the pizza.



Fractions

A fraction is written $\frac{a}{b}$, where a and b are integers and $b \neq 0$. In a fraction, a is called the numerator and b is called the denominator.

A fraction is a way to represent parts of a whole. The denominator, b , represents the number of equal parts the whole has been divided into, and the numerator, a , represents how many parts are included. The denominator, b , cannot equal zero because division by zero is undefined.

In Figure 1.2.1, the circle has been divided into three parts of equal size. Each part represents $\frac{1}{3}$ of the circle. This type of model is called a fraction circle. Other shapes, such as rectangles, can also be used to model fractions.

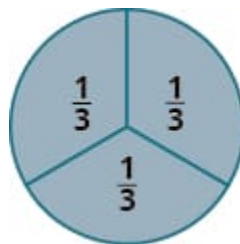


Figure 1.2.1

What does the fraction $\frac{2}{3}$ represent? The fraction $\frac{2}{3}$ means two of three equal parts.

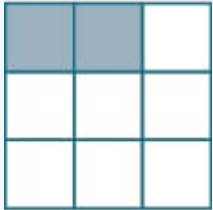


Example 1.2.1

Name the fraction of the shape that is shaded in each of the figures.



(a)



(b)

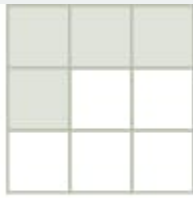
Answer available at the end of this section.

Your Turn 1.2.1

Name the fraction of the shape that is shaded in each figure:



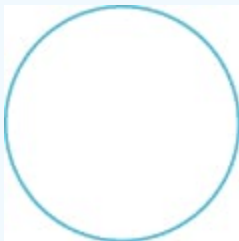
(a)



(b)

Example 1.2.2

Shade $\frac{3}{4}$ of the circle.



Answer available at the end of this section.

Your Turn 1.2.2

Shade $\frac{6}{8}$ of the circle.



In Example 1.2.1 and Example 1.2.2, we used circles and rectangles to model fractions. Fractions can also be modeled as manipulatives called fraction tiles, as shown in Figure 1.2.2. Here, the whole is modeled as one long, undivided rectangular tile. Beneath it are tiles of equal length divided into different numbers of equally sized parts.



Figure 1.2.2

We'll be using fraction tiles to discover some basic facts about fractions. Refer to Figure 1.2.2 to answer the following questions:

How many $\frac{1}{2}$ tiles does it take to make one whole tile?	It takes two halves to make a whole, so two out of two is $\frac{2}{2} = 1$.
How many $\frac{1}{3}$ tiles does it take to make one whole tile?	It takes three thirds, so three out of three is $\frac{3}{3} = 1$.
How many $\frac{1}{4}$ tiles does it take to make one whole tile?	It takes four fourths, so four out of four is $\frac{4}{4} = 1$.
How many $\frac{1}{6}$ tiles does it take to make one whole tile?	It takes six sixths, so six out of six is $\frac{6}{6} = 1$.
What if the whole were divided into 24 equal parts? (We have not shown fraction tiles to represent this, but try to visualize it in your mind.) How many $\frac{1}{24}$ tiles does it take to make one whole tile?	It takes 24 twenty-fourths, so $\frac{24}{24} = 1$.

This leads us to the *Property of One*.

Property of One

Any number, except zero, divided by itself is one.

$$\frac{a}{a} = 1 \quad (a \neq 0)$$

What if we have more fraction pieces than we need for 1 whole? We'll look at this in the next example.

Example 1.2.3

Use fraction circles to make wholes using the following pieces:

1. Ⓐ 3 halves
2. Ⓑ 8 fifths
3. Ⓒ 7 thirds

Answer available at the end of this section.

Your Turn 1.2.3

Use fraction circles to make wholes with the following pieces: 5 thirds.

Model Improper Fractions and Mixed Numbers

In Example 1.2.3 (b), you had eight equal fifth pieces. You used five of them to make one whole, and you had three fifths left over. Let us use fraction notation to show what happened. You had eight pieces, each of them one fifth, $\frac{1}{5}$, so altogether you had eight fifths, which we can write as $\frac{8}{5}$. The fraction $\frac{8}{5}$ is one whole, 1, plus three fifths, $\frac{3}{5}$, or $1\frac{3}{5}$, which is read as *one and three-fifths*.

The number $1\frac{3}{5}$ is called a mixed number. A mixed number consists of a whole number and a fraction.

Mixed Numbers

A mixed number consists of a whole number a and a fraction $\frac{b}{c}$ where $c \neq 0$. It is written as follows.

$a\frac{b}{c}$

Fractions such as $\frac{5}{4}$, $\frac{3}{2}$, $\frac{5}{5}$, and $\frac{7}{3}$ are called improper fractions. In an improper fraction, the numerator is greater than or equal to the denominator, so its value is greater than or equal to one. When a fraction has a numerator that is smaller than the denominator, it is called a proper fraction, and its value is less than one. Fractions such as $\frac{1}{2}$, $\frac{3}{7}$, and $\frac{11}{18}$ are proper fractions.

Proper and Improper Fractions

The fraction $\frac{a}{b}$ is a proper fraction if $a < b$ and an improper fraction if $a \geq b$.

Example 1.2.4

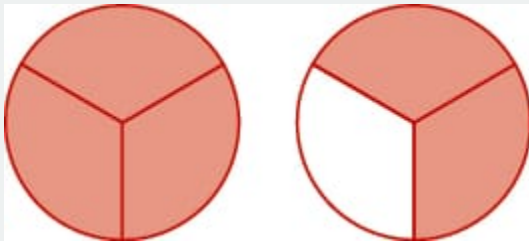
Name the improper fraction modeled. Then write the improper fraction as a mixed number.



Answer available at the end of this section.

Your Turn 1.2.4

Name the improper fraction. Then write it as a mixed number.



Example 1.2.5

Draw a figure to model $\frac{11}{8}$.

Answer available at the end of this section.

Your Turn 1.2.5

Draw a figure to model $\frac{7}{6}$.

Example 1.2.6

Use a model to rewrite the improper fraction $\frac{11}{6}$ as a mixed number.

Answer available at the end of this section.

Your Turn 1.2.6

Use a model to rewrite the improper fraction as a mixed number: $\frac{9}{7}$

Example 1.2.7

Use a model to rewrite the mixed number $1\frac{4}{5}$ as an improper fraction.

Answer available at the end of this section.

Your Turn 1.2.7

Use a model to rewrite the mixed number as an improper fraction: $1\frac{3}{8}$.

Convert between Improper Fractions and Mixed Numbers

In Example 1.2.6, we converted the improper fraction $\frac{11}{6}$ to the mixed number $1\frac{5}{6}$ using fraction circles. We did this by grouping six sixths together to make a whole; then we looked to see how many of the 11 pieces were left. We saw that $\frac{11}{6}$ made one whole group of six sixths plus five more sixths, showing that $\frac{11}{6} = 1\frac{5}{6}$.

The division expression $\frac{11}{6}$ (which can also be written as $6\overline{)11}$) tells us to find how many groups of 6 are in 11. To convert an improper fraction to a mixed number without fraction circles, we divide.

Example 1.2.8

Convert $\frac{11}{6}$ to a mixed number.

Answer available at the end of this section.

Your Turn 1.2.8

Convert the improper fraction to a mixed number: $\frac{13}{7}$

How To**Convert an improper fraction to a mixed number.**

1. Divide the denominator into the numerator.
2. Identify the quotient, remainder, and divisor.
3. Write the mixed number as quotient remainderdivisorremainderdivisor.

Example 1.2.9

Convert the improper fraction $\frac{33}{8}$ to a mixed number.

Answer available at the end of this section.

Your Turn 1.2.9

Convert the improper fraction to a mixed number: $\frac{23}{7}$.

In Example 1.2.7, we changed $1\frac{4}{5}$ to an improper fraction by first seeing that the whole is a set of five fifths. So we had five fifths and four more fifths.

$$\frac{5}{5} + \frac{4}{5} = \frac{9}{5}$$

Where did the nine come from? There are nine fifths—one whole (five fifths) plus four fifths. Let us use this idea to see how to convert a mixed number to an improper fraction.

Example 1.2.10

Convert the mixed number $4\frac{2}{3}$ to an improper fraction.

Answer available at the end of this section.

Your Turn 1.2.10

Convert the mixed number to an improper fraction: $3\frac{5}{7}$.

How To

Convert a mixed number to an improper fraction.

1. Multiply the whole number by the denominator.
2. Add the numerator to the product found in Step 1.
3. Write the final sum over the original denominator.

Example 1.2.11

Convert the mixed number $10\frac{2}{7}$ to an improper fraction.

Answer available at the end of this section.

Your Turn 1.2.11

Convert the mixed number to an improper fraction: $4\frac{6}{11}$.

Model Equivalent Fractions

Let's think about Andy and Bobby and their favorite food again. If Andy eats $\frac{1}{2}$ of a pizza and Bobby eats $\frac{2}{4}$ of the pizza, have they eaten the same amount of pizza? In other words, does $\frac{1}{2} = \frac{2}{4}$? We can use fraction tiles to find out whether Andy and Bobby have

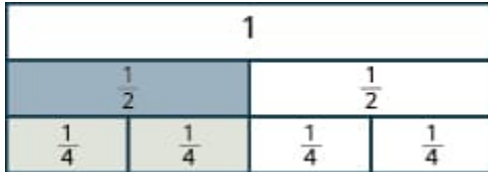
eaten *equivalent* parts of the pizza.

Equivalent Fractions

Equivalent fractions are fractions that have the same value.

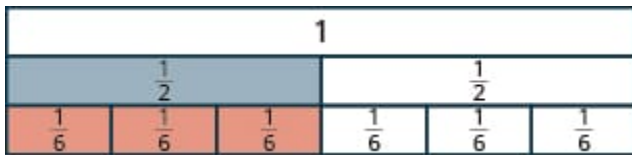
Fraction tiles serve as a useful model of equivalent fractions. You may want to use fraction tiles to do the following activity. Or you might make a copy of Figure 1.2.2 and extend it to include eighths, tenths, and twelfths.

Start with a $\frac{1}{2}$ tile. How many fourths equal one-half? How many of the $\frac{1}{4}$ tiles exactly cover the $\frac{1}{2}$ tile?



Since two $\frac{1}{4}$ tiles cover the $\frac{1}{2}$ tile, we see that $\frac{2}{4}$ is the same as $\frac{1}{2}$, or $\frac{2}{4} = \frac{1}{2}$.

How many of the $\frac{1}{6}$ tiles cover the $\frac{1}{2}$ tile?



Since three $\frac{1}{6}$ tiles cover the $\frac{1}{2}$ tile, we see that $\frac{3}{6}$ is the same as $\frac{1}{2}$.

So, $\frac{3}{6} = \frac{1}{2}$. The fractions are equivalent fractions.

Example 1.2.12

Use fraction tiles to find equivalent fractions. Show your result with a figure.

1. Ⓐ How many eighths equal one-half?
2. Ⓑ How many tenths equal one-half?
3. Ⓒ How many twelfths equal one-half?

Answer available at the end of this section.

Your Turn 1.2.12

Use fraction tiles to find equivalent fractions: How many eighths equal one-fourth?

Find Equivalent Fractions

We used fraction tiles to show that there are many fractions equivalent to $\frac{1}{2}$.

We can show this with pizzas, too. Figure 1.2.3(a) shows a single pizza, cut into two equal pieces with $\frac{1}{2}$ shaded. Figure 1.2.3(b) shows a second pizza of the same size, cut into eight pieces with $\frac{4}{8}$ shaded.

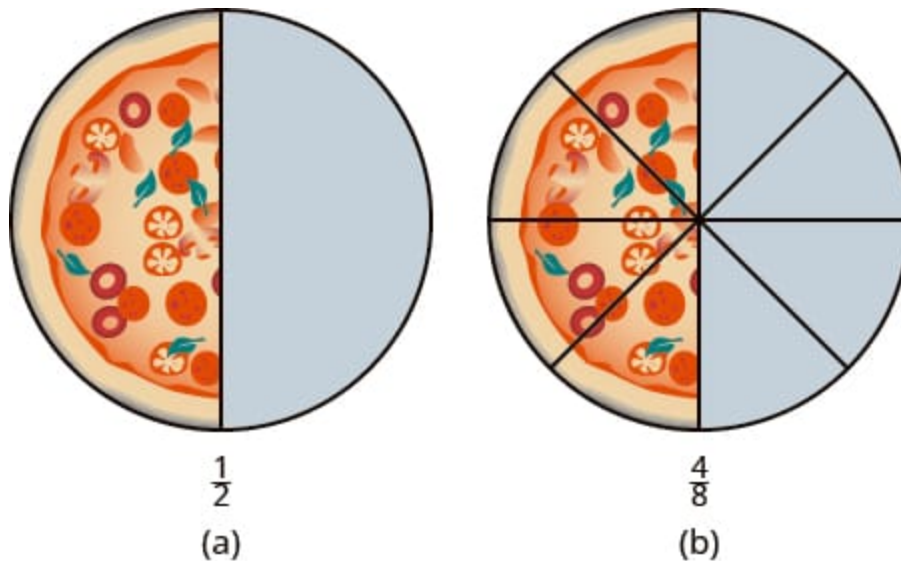


Figure 1.2.3

This is another way to show that $\frac{1}{2}$ is equivalent to $\frac{4}{8}$.

How can we use mathematics to change $\frac{1}{2}$ into $\frac{4}{8}$? How could you take a pizza that is cut into two pieces and cut it into eight pieces? You could cut each of the two larger pieces into four smaller pieces! The whole pizza would then be cut into eight pieces instead of just two. Mathematically, what we've described could be written as:

$$\frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$$

These models lead to the Equivalent Fractions Property, which states that if we multiply the numerator and denominator of a fraction by the same number, the value of the fraction does not change.

Equivalent Fractions Property

If a , b , and c are numbers where $b \neq 0$ and $c \neq 0$, then

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$$

When working with fractions, it is often necessary to express the same fraction in different forms. To find equivalent forms of a fraction, we can use the Equivalent Fractions Property. For example, consider the fraction one-half.

$$\frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6} \quad \text{so} \quad \frac{1}{2} = \frac{3}{6}$$

$$\frac{1 \cdot 2}{2 \cdot 2} = \frac{2}{4} \quad \text{so} \quad \frac{1}{2} = \frac{2}{4}$$

$$\frac{1 \cdot 10}{2 \cdot 10} = \frac{10}{20} \quad \text{so} \quad \frac{1}{2} = \frac{10}{20}$$

So, we say that $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, and $\frac{10}{20}$ are equivalent fractions.

Example 1.2.13

Find three fractions equivalent to $\frac{2}{5}$.

Answer available at the end of this section.

Your Turn 1.2.13

Find three fractions equivalent to $\frac{3}{5}$.

Example 1.2.14

Find a fraction with a denominator of 21 that is equivalent to $\frac{2}{7}$.

Answer available at the end of this section.

Your Turn 1.2.14

Find a fraction with a denominator of 21 that is equivalent to $\frac{6}{7}$.

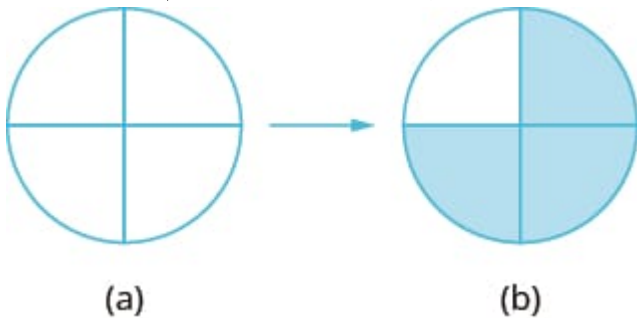
Answers

We need to ask two questions. First, how many equal parts are there? This will be the denominator. Second, of these equal parts, how many are shaded? This will be the numerator.

Ⓐ How many equal parts are there? There are eight equal parts.
How many are shaded? Five parts are shaded.
Five out of eight parts are shaded. Therefore, the fraction of the circle that is shaded is $\frac{5}{8}$.

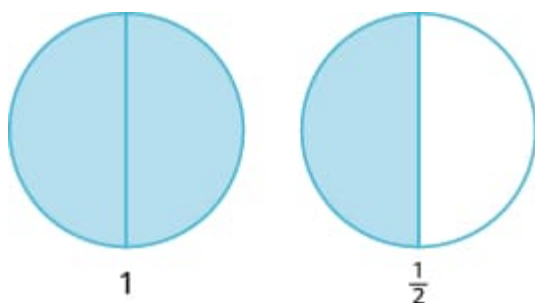
Ⓑ How many equal parts are there? There are nine equal parts.
How many are shaded? Two parts are shaded.
Two out of nine parts are shaded. Therefore, the fraction of the square that is shaded is $\frac{2}{9}$.

The denominator is 4, so we divide the circle into four equal parts Ⓐ.
The numerator is 3, so we shade three of the four parts Ⓑ.

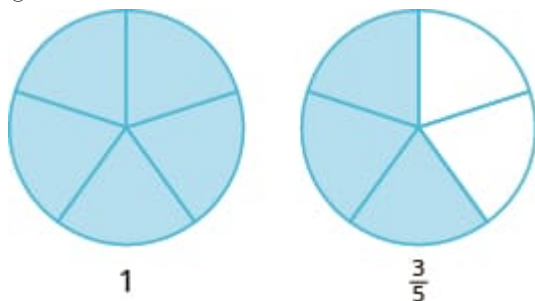


$\frac{3}{4}$ of the circle is shaded.

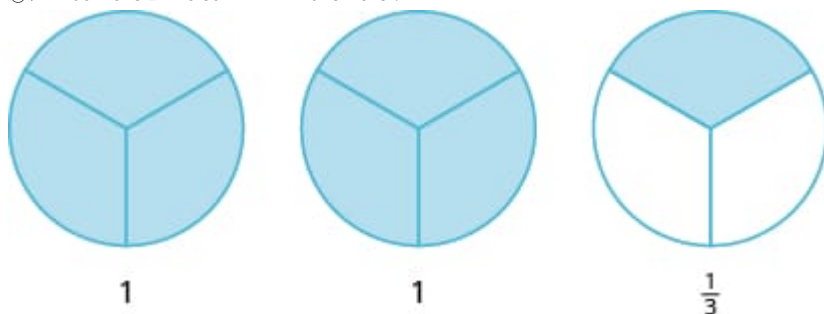
Ⓐ 3 halves make 1 whole with 1 half left over.



Ⓓ 8 fifths make 1 whole with 3 fifths left over.

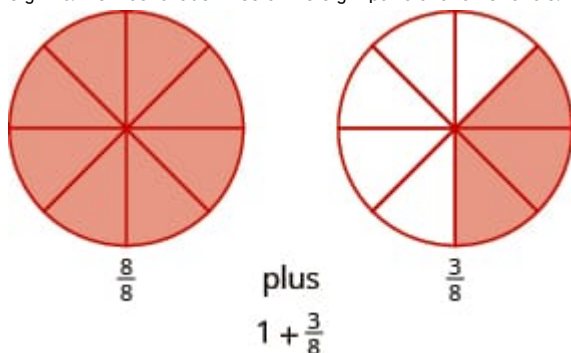


Ⓒ 7 thirds make 2 wholes with 1 third left over.



Each circle is divided into three pieces, so each piece is $\frac{1}{3}$ of the circle. There are four pieces shaded, so there are four thirds or $\frac{4}{3}$. The figure shows that we also have one whole circle and one third, which is $1\frac{1}{3}$. So, $\frac{4}{3} = 1\frac{1}{3}$.

The denominator of the improper fraction is 8. Draw a circle divided into eight pieces and shade all of them. This takes care of eight eighths, but we have 11 eighths. We must shade three of the eight parts of another circle.



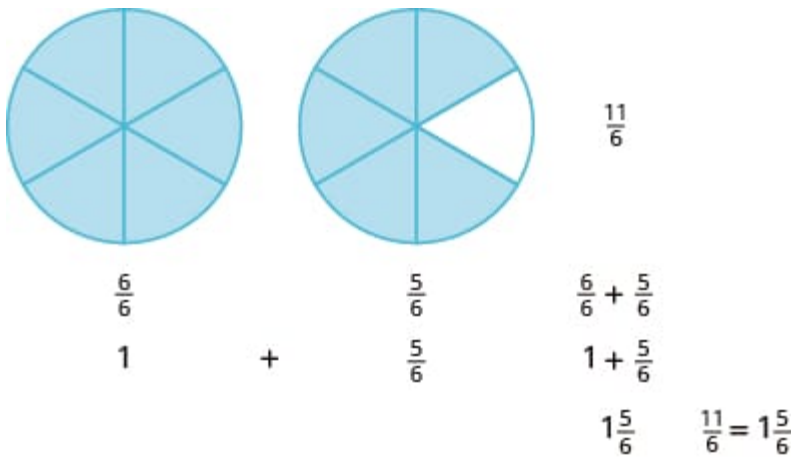
So, $\frac{11}{8} = 1\frac{3}{8}$.

We start with 11 sixths ($\frac{11}{6}$). We know that six sixths makes one whole.

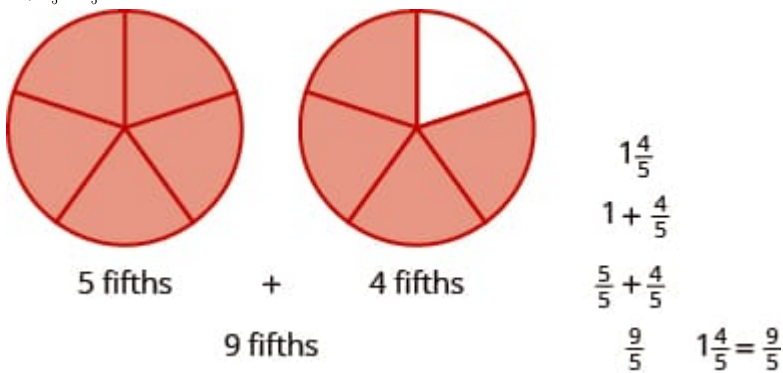
$$\frac{6}{6} = 1$$

That leaves us with five more sixths, which is $\frac{5}{6}$ (11 sixths minus 6 sixths is 5 sixths.)

So, $\frac{11}{6} = 1\frac{5}{6}$.



The mixed number $1\frac{4}{5}$ means one whole plus four fifths. The denominator is 5, so the whole is $\frac{5}{5}$. Together five fifths and four fifths equals nine fifths. So, $1\frac{4}{5} = \frac{9}{5}$.



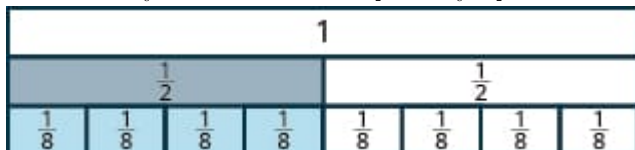
	$\frac{11}{6}$
Divide the denominator into the numerator.	Remember $\frac{11}{6}$ means $11 \div 6$.
	divisor $\rightarrow$ $\begin{array}{r} 1 \\ 6 \overline{)11} \\ \underline{6} \\ 5 \end{array}$ $\leftarrow$ quotient $\leftarrow$ remainder
Identify the quotient, remainder and divisor.	
Write the mixed number as quotient remainderdivisorremainderdivisor.	$1\frac{5}{6}$
So, $\frac{11}{6} = 1\frac{5}{6}$	

	$\frac{33}{8}$
Divide the denominator into the numerator.	Remember, $\frac{33}{8}$ means $8 \overline{)33}$.
Identify the quotient, remainder, and divisor.	divisor $\rightarrow$ $\begin{array}{r} 4 \\ 8 \overline{)33} \\ \underline{32} \\ 1 \end{array}$ $\leftarrow$ quotient $\leftarrow$ remainder
Write the mixed number as quotient remainderdivisorremainderdivisor.	$4\frac{1}{8}$

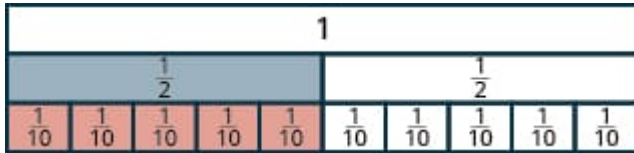
So, $\frac{33}{8} = 4\frac{1}{8}$	
	$4\frac{2}{3}$
Multiply the whole number by the denominator.	
The whole number is 4 and the denominator is 3.	$\frac{4 \cdot 3 + \square}{\square}$
Simplify.	$\frac{12 + \square}{\square}$
Add the numerator to the product.	
The numerator of the mixed number is 2.	$\frac{12 + 2}{\square}$
Simplify.	$\frac{14}{\square}$
Write the final sum over the original denominator.	
The denominator is 3.	$\frac{14}{3}$

	$10\frac{2}{7}$
Multiply the whole number by the denominator.	
The whole number is 10 and the denominator is 7.	$\frac{10 \cdot 7 + \square}{\square}$
Simplify.	$\frac{70 + \square}{\square}$
Add the numerator to the product.	
The numerator of the mixed number is 2.	$\frac{70 + 2}{\square}$
Simplify.	$\frac{72}{\square}$
Write the final sum over the original denominator.	
The denominator is 7.	$\frac{72}{7}$

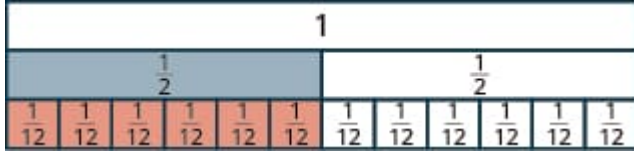
Ⓐ It takes four $\frac{1}{8}$ tiles to exactly cover the $\frac{1}{2}$ tile, so $\frac{4}{8} = \frac{1}{2}$.



⑤ It takes five $\frac{1}{10}$ tiles to exactly cover the $\frac{1}{2}$ tile, so $\frac{5}{10} = \frac{1}{2}$.



⑥ It takes six $\frac{1}{12}$ tiles to exactly cover the $\frac{1}{2}$ tile, so $\frac{6}{12} = \frac{1}{2}$.



Suppose you had tiles marked $\frac{1}{20}$. How many of them would it take to equal $\frac{1}{2}$? Are you thinking ten tiles? If you are, you're right, because $\frac{10}{20} = \frac{1}{2}$.

We have shown that $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{6}{12}$, and $\frac{10}{20}$ are all equivalent fractions.

To find a fraction equivalent to $\frac{2}{5}$, we multiply the numerator and denominator by the same number (but not zero). Let us multiply them by 2, 3, and 5.

$$\frac{2 \cdot 2}{5 \cdot 2} = \frac{4}{10} \quad \frac{2 \cdot 3}{5 \cdot 3} = \frac{6}{15} \quad \frac{2 \cdot 5}{5 \cdot 5} = \frac{10}{25}$$

So, $\frac{4}{10}$, $\frac{6}{15}$, and $\frac{10}{25}$ are equivalent to $\frac{2}{5}$.

To find equivalent fractions, we multiply the numerator and denominator by the same number. In this case, we need to multiply the denominator by a number that will result in 21.

Since we can multiply 7 by 3 to get 21, we can find the equivalent fraction by multiplying both the numerator and denominator by 3.

$$\frac{2}{7} = \frac{2 \cdot 3}{7 \cdot 3} = \frac{6}{21}$$

1.3: Decimals

Learning Objectives

After completing this section, you should be able to:

1. Name decimals
2. Write decimals
3. Convert decimals to fractions or mixed numbers
4. Round decimals

Name Decimals

You probably already know quite a bit about decimals based on your experience with money. Suppose you buy a sandwich and a bottle of water for lunch. If the sandwich costs \$3.45, the bottle of water costs \$1.25, and the total sales tax is \$0.33, what is the total cost of your lunch?

\$3.45	Sandwich
\$1.25	Water
+ \$0.33	Tax
\$5.03	Total

The total is \$5.03. Suppose you pay with a \$5 bill and 3 pennies. Should you wait for change? No, \$5 and 3 pennies is the same as \$5.03.

Because 100 pennies = \$1, each penny is worth $\frac{1}{100}$ of a dollar. We write the value of one penny as \$0.01, since $0.01 = \frac{1}{100}$.

Writing a number with a decimal is known as decimal notation. It is a way of showing parts of a whole when the whole is a power of ten. In other words, decimals are another way of writing fractions whose denominators are powers of ten. Just as the counting numbers are based on powers of ten, decimals are based on powers of ten. Table 1.3.1 shows the counting numbers.

Counting number	Name
1	One
$10 = 10$	Ten
$10 \cdot 10 = 100$	One hundred
$10 \cdot 10 \cdot 10 = 1000$	One thousand
$10 \cdot 10 \cdot 10 \cdot 10 = 10,000$	Ten thousand

Table 1.3.1

How are decimals related to fractions? Table 1.3.2 shows the relation.

Decimal	Fraction	Name
0.1	$\frac{1}{10}$	One tenth
0.01	$\frac{1}{100}$	One hundredth
0.001	$\frac{1}{1,000}$	One thousandth

Decimal	Fraction	Name
0.0001	$\frac{1}{10,000}$	One ten-thousandth

Table 1.3.2

When we name a whole number, the name corresponds to the place value based on the powers of ten. We know to read 10,000 as *ten thousand*. Likewise, the names of the decimal places correspond to their fraction values. Notice how the place value names in Figure 1.3.1 relate to the names of the fractions from Table 1.3.2.

Place Value											
Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones	.	Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths

Figure 1.3.1 This chart illustrates place values to the left and right of the decimal point.

Notice two important facts shown in Figure 1.3.1.

- The “th” at the end of the name means the number is a fraction. “One thousand” is a number larger than one, but “one thousandth” is a number smaller than one.
- The tenths place is the first place to the right of the decimal, but the tens place is two places to the left of the decimal.

Remember that \$5.03 lunch? We read \$5.03 as *five dollars and three cents*. Naming decimals (those that don’t represent money) is done in a similar way. We read the number 5.03 as *five and three hundredths*.

We sometimes need to translate a number written in decimal notation into words. As shown in Figure 1.3.2, we write the amount on a check in both words and numbers.



Figure 1.3.2: When we write a check, we write the amount as a decimal number as well as in words. The bank looks at the check to make sure both numbers match. This helps prevent errors.

Let's try naming a decimal, such as 15.68.	
We start by naming the number to the left of the decimal.	fifteen_____
We use the word "and" to indicate the decimal point.	fifteen and_____
Then we name the number to the right of the decimal point as if it were a whole number.	fifteen and sixty-eight_____
Last, name the decimal place of the last digit.	fifteen and sixty-eight hundredths

How To

Name a decimal number.

- Name the number to the left of the decimal point.
- Write "and" for the decimal point.
- Name the "number" part to the right of the decimal point as if it were a whole number.
- Name the decimal place of the last digit.

Example 1.3.1

Name each decimal: Ⓐ 4.3 Ⓑ 2.45 Ⓒ 0.009 Ⓓ -15.571.

Answer available at the end of this section.

Your Turn 1.3.1

Name each decimal:

Ⓐ 6.7 Ⓑ 19.58 Ⓒ 0.018 Ⓓ -2.053

Write Decimals

Now we will translate the name of a decimal number into decimal notation. We will reverse the procedure we just used.

Let's start by writing the number six and seventeen hundredths:

	six and seventeen hundredths
The word <i>and</i> tells us to place a decimal point.	____.____
The word before <i>and</i> is the whole number; write it to the left of the decimal point.	6.____
The decimal part is seventeen hundredths. Mark two places to the right of the decimal point for hundredths.	6._ _
Write the numerals for seventeen in the places marked.	6.17

Example 1.3.2

Write fourteen and thirty-seven hundredths as a decimal.

Answer available at the end of this section.

Your Turn 1.3.2

Write as a decimal: thirteen and sixty-eight hundredths.

How To

Write a decimal number from its name.

1. Look for the word “and”—it locates the decimal point.
2. Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.
 - Place a decimal point under the word “and.” Translate the words before “and” into the whole number and place it to the left of the decimal point.
 - If there is no “and,” write a “0” with a decimal point to its right.
3. Translate the words after “and” into the number to the right of the decimal point. Write the number in the spaces—putting the final digit in the last place.
4. Fill in zeros for place holders as needed.

The second bullet in Step 2 is needed for decimals that have no whole number part, like ‘nine thousandths’. We recognize them by the words that indicate the place value after the decimal – such as ‘tenths’ or ‘hundredths.’ Since there is no whole number, there is no ‘and.’ We start by placing a zero to the left of the decimal and continue by filling in the numbers to the right, as we did above.

Example 1.3.3

Write twenty-four thousandths as a decimal.

Answer available at the end of this section.

Your Turn 1.3.3

Write as a decimal: fifty-eight thousandths.

Before we move on to our next objective, think about money again. We know that \$1 is the same as \$1.00. The way we write \$1 (or \$1.00) depends on the context. In the same way, integers can be written as decimals with as many zeros as needed to the right of the decimal.

$$\begin{array}{rcl} 5 & = & 5.0 \\ 5 & = & 5.00 \\ 5 & = & 5.000 \end{array} \qquad \begin{array}{rcl} -2 & = & -2.0 \\ -2 & = & -2.00 \\ -2 & = & -2.000 \end{array}$$

and so on...

Convert Decimals to Fractions or Mixed Numbers

We often need to rewrite decimals as fractions or mixed numbers. Let’s go back to our lunch order to see how we can convert decimal numbers to fractions. We know that \$5.03 means 5 dollars and 3 cents. Since there are 100 cents in one dollar, 3 cents means $\frac{3}{100}$ of a dollar, so $0.03 = \frac{3}{100}$.

We convert decimals to fractions by identifying the place value of the farthest right digit. In the decimal 0.03, the 3 is in the hundredths place, so 100 is the denominator of the fraction equivalent to 0.03.

$$0.03 = \frac{3}{100}$$

For our \$5.03 lunch, we can write the decimal 5.03 as a mixed number.

$$5.03 = 5\frac{3}{100}$$

Notice that when the number to the left of the decimal is zero, we get a proper fraction. When the number to the left of the decimal is not zero, we get a mixed number.

How To

Convert a decimal number to a fraction or mixed number.

1. Look at the number to the left of the decimal.
 - If it is zero, the decimal converts to a proper fraction.
 - If it is not zero, the decimal converts to a mixed number.
 - Write the whole number.
2. Determine the place value of the final digit.
3. Write the fraction.
 - numerator—the 'numbers' to the right of the decimal point
 - denominator—the place value corresponding to the final digit
4. Simplify the fraction, if possible.

Example 1.3.4

Write each of the following decimal numbers as a fraction or a mixed number:

Ⓐ 4.09 Ⓑ 3.7 Ⓒ -0.286

Answer available at the end of this section.

Your Turn 1.3.4

Write as a fraction or mixed number. Simplify the answer if possible.

Ⓐ 5.3 Ⓑ 6.07 Ⓒ -0.234

Equivalent Decimals

Two decimals are equivalent decimals if they convert to equivalent fractions.

Remember, writing zeros at the end of a decimal does not change its value. 0.31 and 0.310 are equivalent decimals. They have the same value.

Round Decimals

In the United States, gasoline prices are usually written with the decimal part as thousandths of a dollar. For example, a gas station might post the price of unleaded gas at \$3.279.

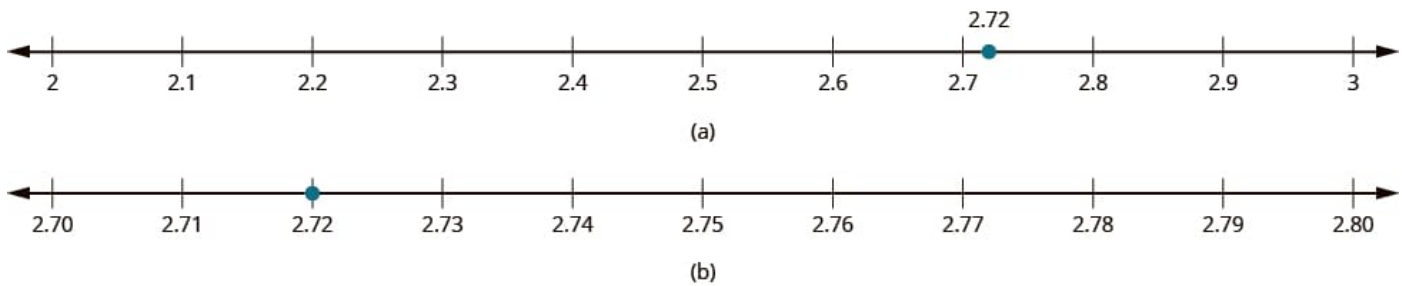


Figure 1.3.3 (a) We see that 2.72 is closer to 3 than to 2. So, 2.72 rounded to the nearest whole number is 3.

(b) We see that 2.72 is closer to 2.70 than 2.80. So we say that 2.72 rounded to the nearest tenth is 2.7.

Can we round decimals without number lines? Yes! We use a method based on the one we used to round whole numbers.

How To

Round a decimal.

1. Locate the given place value and mark it with an arrow.
2. Underline the digit to the right of the given place value.
3. Is this digit greater than or equal to 5?
 - Yes - add 1 to the digit in the given place value.
 - No - do not change the digit in the given place value
4. Rewrite the number, removing all digits to the right of the given place value.

Example 1.3.5

Round 18.379 to the nearest hundredth.

Answer available at the end of this section.

Your Turn 1.3.5

Round to the nearest hundredth: 1.047.

Example 1.3.6

Round 18.379 to the nearest (a) tenth (b) whole number.

Answer available at the end of this section.

Your Turn 1.3.6

Round 6.582 to the nearest (a) hundredth (b) tenth (c) whole number.

Media

ACCESS ADDITIONAL ONLINE RESOURCES

- [Introduction to Decimal Notation](#)
- [Write a Number in Decimal Notation from Words](#)
- [Rounding Decimals](#)
- [Writing a Decimal as a Simplified Fraction](#)

Answers

Ⓐ	
	4.3
Name the number to the left of the decimal point.	four_____
Write "and" for the decimal point.	four and_____
Name the number to the right of the decimal point as if it were a whole number.	four and three_____
Name the decimal place of the last digit.	four and three tenths

Ⓑ	
	2.45
Name the number to the left of the decimal point.	two_____
Write "and" for the decimal point.	two and_____
Name the number to the right of the decimal point as if it were a whole number.	two and forty-five_____
Name the decimal place of the last digit.	two and forty-five hundredths

Ⓒ	
	0.009
Name the number to the left of the decimal point.	Zero is the number to the left of the decimal; it is not included in the name.
Name the number to the right of the decimal point as if it were a whole number.	nine_____
Name the decimal place of the last digit.	nine thousandths

Ⓓ	
	-15.571
Name the number to the left of the decimal point.	negative fifteen
Write "and" for the decimal point.	negative fifteen and_____
Name the number to the right of the decimal point as if it were a whole number.	negative fifteen and five hundred seventy-one_____
Name the decimal place of the last digit.	negative fifteen and five hundred seventy-one thousandths

	fourteen and thirty-seven hundredths
Place a decimal point under the word 'and'.	_____' _____
Translate the words before 'and' into the whole number and place it to the left of the decimal point.	14. _____
Mark two places to the right of the decimal point for "hundredths".	14.____

Translate the words after "and" and write the number to the right of the decimal point.	14.37
	Fourteen and thirty-seven hundredths is written 14.37.

	twenty-four thousandths
Look for the word "and".	There is no "and" so start with 0.
To the right of the decimal point, put three decimal places for thousandths.	0. — — — tenth hundredth thousandth
Write the number 24 with the 4 in the thousandths place.	0. — <u>2</u> <u>4</u> tenth hundredth thousandth
Put zeros as placeholders in the remaining decimal places.	0.024
	So, twenty-four thousandths is written 0.024

Ⓐ	
	4.09
There is a 4 to the left of the decimal point. Write "4" as the whole number part of the mixed number.	$4\frac{\square}{\square}$
Determine the place value of the final digit.	4. 0 9 tenth hundredth
Write the fraction. Write 9 in the numerator as it is the number to the right of the decimal point.	$4\frac{9}{\square}$
Write 100 in the denominator as the place value of the final digit, 9, is hundredth.	$4\frac{9}{100}$
The fraction is in simplest form.	So, $4.09 = 4\frac{9}{100}$

Did you notice that the number of zeros in the denominator is the same as the number of decimal places?

Ⓑ	
	3.7
There is a 3 to the left of the decimal point. Write "3" as the whole number part of the mixed number.	$3\frac{\square}{\square}$
Determine the place value of the final digit.	3. 7 tenth

Write the fraction. Write 7 in the numerator as it is the number to the right of the decimal point.	$3\frac{7}{\square}$
Write 10 in the denominator as the place value of the final digit, 7, is tenths.	$3\frac{7}{10}$
The fraction is in simplest form.	So, $3.7 = 3\frac{7}{10}$

©	
	-0.286
There is a 0 to the left of the decimal point. Write a negative sign before the fraction.	$-\frac{\square}{\square}$
Determine the place value of the final digit and write it in the denominator.	-0. 2 8 6 tenths hundredths thousandths
Write the fraction. Write 286 in the numerator as it is the number to the right of the decimal point. Write 1,000 in the denominator as the place value of the final digit, 6, is thousandths.	$-\frac{286}{1000}$
We remove a common factor of 2 to simplify the fraction.	$-\frac{143}{500}$

	18.379
Locate the hundredths place and mark it with an arrow.	hundredths place ↓ 18.379
Underline the digit to the right of the 7.	hundredths place ↓ 18.37 <u>9</u>
Because 9 is greater than or equal to 5, add 1 to the 7.	18.379 add 1 delete
Rewrite the number, deleting all digits to the right of the hundredths place.	18.38
	18.38 is 18.379 rounded to the nearest hundredth.

Ⓐ Round 18.379 to the nearest tenth.	
--------------------------------------	--

	18.379
Locate the tenths place and mark it with an arrow.	tenths place  18.379
Underline the digit to the right of the tenths digit.	tenths place  18.<u>3</u>79
Because 7 is greater than or equal to 5, add 1 to the 3.	 18.379
Rewrite the number, deleting all digits to the right of the tenths place.	18.4
	So, 18.379 rounded to the nearest tenth is 18.4.

⑥ Round 18.379 to the nearest whole number.	
	18.379
Locate the ones place and mark it with an arrow.	ones place  18.379
Underline the digit to the right of the ones place.	ones place  18.<u>3</u>79
Since 3 is not greater than or equal to 5, do not add 1 to the 8.	 18.379
Rewrite the number, deleting all digits to the right of the ones place.	18
	So 18.379 rounded to the nearest whole number is 18.

1.4: Understand Percent

Learning Objectives

After completing this section, you should be able to:

1. Use the definition of percent
2. Convert percents to fractions and decimals
3. Convert decimals and fractions to percents

Use the Definition of Percent

How many cents are in one dollar? There are 100 cents in a dollar. How many years are in a century? There are 100 years in a century. Does this give you a clue about what the word “percent” means? It is really two words, “per cent,” and means per one hundred. A percent is a ratio whose denominator is 100. We use the percent symbol %, to show percent.

Percent

A percent is a ratio whose denominator is 100.

According to data from the American Association of Community Colleges (2015), out of 100 community college students, 57 are female. This is demonstrated visually in Figure 1.4.1. Out of the 100 squares on the grid, 57 are shaded, which we write as the ratio $\frac{57}{100}$.

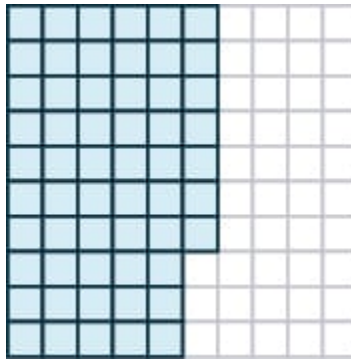


Figure 1.4.1 Among every 100 community college students, 57 are female.

Similarly, 25% means a ratio of $\frac{25}{100}$, 3% means a ratio of $\frac{3}{100}$ and 100% means a ratio of $\frac{100}{100}$. In words, “one hundred percent” means the total 100% is $\frac{100}{100}$, and since $\frac{100}{100} = 1$, we see that 100% means 1 whole.

Example 1.4.1

According to the Public Policy Institute of California (2010), 44% of parents of public school children would like their youngest child to earn a graduate degree. Write this percent as a ratio.

Answer available at the end of this section.

Your Turn 1.4.1

Write the percent as a ratio.

According to a survey, 89% of college students have a smartphone.

Example 1.4.2

In 2007, according to a U.S. Department of Education report, 21 out of every 100 first-time freshmen college students at 4-year public institutions took at least one remedial course. Write this as a ratio and then as a percent.

Answer available at the end of this section.

Your Turn 1.4.2

Write as a ratio and then as a percent: The American Association of Community Colleges reported that 62 out of 100 full-time community college students balance their studies with full-time or part-time employment.

Convert Percents to Fractions and Decimals

Since percents are ratios, they can easily be expressed as fractions. Remember that percent means "per 100", so the denominator of the fraction is 100.

How To

Convert a percent to a fraction.

1. Write the percent as a ratio with the denominator 100.
2. Simplify the fraction, if possible.

Example 1.4.3

Convert each percent to a fraction:

1. Ⓐ 36%
2. Ⓑ 125%

Answer available at the end of this section.

Your Turn 1.4.3

Convert each percent to a fraction:

1. Ⓐ 48%
2. Ⓑ 110%

The previous example shows that a percent can be greater than 1. We saw that 125% means $\frac{125}{100}$, or $\frac{5}{4}$. This is an improper fraction,

since its value is greater than one.

Example 1.4.4

Convert each percent to a fraction:

1. Ⓐ 24.5%
2. Ⓑ $33\frac{1}{3}\%$

Answer available at the end of this section.

Your Turn 1.4.4

Convert each percent to a fraction:

1. Ⓐ 64.4%
2. Ⓑ $66\frac{2}{3}\%$

As a review, to convert a fraction to a decimal, you can divide in your calculator. For example, to convert $\frac{1}{5}$, you type $1 \div 5$ into your calculator. The answer is 0.2. So $\frac{1}{5} = 0.2$.

To convert a percent to a decimal, we first convert it to a fraction and then change the fraction to a decimal.

How To

Convert a percent to a decimal.

1. Write the percent as a ratio with the denominator 100.
2. Convert the fraction to a decimal by dividing the numerator by the denominator.

Example 1.4.5

Convert each percent to a decimal:

1. Ⓐ 6%
2. Ⓑ 78%

Answer available at the end of this section.

Your Turn 1.4.5

Convert each percent to a decimal:

1. Ⓐ 9%
2. Ⓑ 87%

Example 1.4.6

Convert each percent to a decimal:

1. Ⓐ 135%
2. Ⓑ 12.5%

Answer available at the end of this section.

Your Turn 1.4.6

Convert each percent to a decimal:

1. Ⓐ 115%
2. Ⓑ 23.5%

Let's summarize the results from the previous examples in Table 1.4.1, and look for a pattern we could use to quickly convert a percent number to a decimal number.

Percent	Decimal
6%	0.06
78%	0.78
135%	1.35
12.5%	0.125

Table 1.4.1

Do you see the pattern?

To convert a percent number to a decimal number, we move the decimal point two places to the left and remove the % sign. (Sometimes the decimal point does not appear in the percent number, but just like we can think of the integer 6 as 6.0, we can think of 6% as 6.0%.) Notice that we may need to add zeros in front of the number when moving the decimal to the left.

Figure 1.4.2 uses the percents in Table 1.4.1 and shows visually how to convert them to decimals by moving the decimal point two places to the left.

Percent	Decimal
006.%	0.06
078.%	0.78
135.%	1.35
012.5%	0.125

Figure 1.4.2

Example 1.4.7

Among a group of business leaders, 77% believe that poor math and science education in the U.S. will lead to higher unemployment rates.

Convert the percent to: Ⓐ a fraction Ⓑ a decimal

Answer available at the end of this section.

Your Turn 1.4.7

Convert the percent to: Ⓐ a fraction and Ⓑ a decimal

Twitter's share of web traffic jumped 24% when one celebrity tweeted live on air.

Example 1.4.8

There are four suits of cards in a deck of cards—hearts, diamonds, clubs, and spades. The probability of randomly choosing a heart from a shuffled deck of cards is 25%. Convert the percent to:

1. Ⓐ a fraction
2. Ⓑ a decimal



Figure 1.4.3 (credit: Riles32807, Wikimedia Commons)

Answer available at the end of this section.

Your Turn 1.4.8

Convert the percent to: Ⓐ a fraction, and Ⓑ a decimal

The probability that it will rain Monday is 30%.

Convert Decimals and Fractions to Percents

To convert a decimal to a percent, remember that percent means per hundred. If we change the decimal to a fraction whose denominator is 100, it is easy to change that fraction to a percent.

How To

Convert a decimal to a percent.

1. Write the decimal as a fraction.
2. If the denominator of the fraction is not 100, rewrite it as an equivalent fraction with denominator 100.
3. Write this ratio as a percent.

Example 1.4.9

Convert each decimal to a percent: Ⓐ 0.05 Ⓑ 0.83

Answer available at the end of this section.

Your Turn 1.4.9

Convert each decimal to a percent: Ⓐ 0.01 Ⓑ 0.17.

To convert a mixed number to a percent, we first write it as an improper fraction.

Example 1.4.10

Convert each decimal to a percent: Ⓐ 1.05 Ⓑ 0.075

Answer available at the end of this section.

Your Turn 1.4.10

Convert each decimal to a percent: Ⓐ 1.75 Ⓑ 0.0825

Let's summarize the results from the previous examples in Table 1.4.2 so we can look for a pattern.

Decimal	Percent
0.05	5%
0.83	83%
1.05	105%
0.075	7.5%

Table 1.4.2

Do you see the pattern? To convert a decimal to a percent, we move the decimal point two places to the right and then add the percent sign.

We know how to convert fractions to decimals. Now, we also know how to change decimals to percents. So to convert a fraction to a percent, we first change it to a decimal and then convert that decimal to a percent.

How To

Convert a fraction to a percent.

1. Convert the fraction to a decimal.
2. Convert the decimal to a percent.

Example 1.4.11

Convert each fraction or mixed number to a percent: Ⓐ $\frac{3}{4}$ Ⓑ $\frac{11}{8}$ Ⓒ $2\frac{1}{5}$

Answer available at the end of this section.

Your Turn 1.4.11

Convert each fraction or mixed number to a percent: Ⓐ $\frac{5}{8}$ Ⓑ $\frac{11}{4}$ Ⓒ $3\frac{2}{5}$

Sometimes when changing a fraction to a decimal, the division continues for many decimal places and we will round off the quotient. The number of decimal places we round to will depend on the situation. If the decimal involves money, we round to the hundredths place.

Example 1.4.12

Convert $\frac{5}{7}$ to a percent.

Answer available at the end of this section.

Your Turn 1.4.12

Convert the fraction to a percent: $\frac{3}{7}$

When we first looked at fractions and decimals, we saw that some fractions converted to a repeating decimal. For example, when we converted the fraction $\frac{4}{3}$ to a decimal, we wrote the answer as $1.\bar{3}$. We will use this same notation, as well as fraction notation, when we convert fractions to percents in the next example.

Example 1.4.13

An article in a medical journal claimed that approximately $\frac{1}{3}$ of American adults are obese. Convert the fraction $\frac{1}{3}$ to a percent.

Answer available at the end of this section.

Your Turn 1.4.13

Convert the fraction to a percent:

According to the U.S. Census Bureau, about $\frac{1}{9}$ of United States housing units have just 1 bedroom.

Answers

The amount we want to convert is 44%.	44%
Write the percent as a ratio. Remember that <i>percent</i> means per 100.	$\frac{44}{100}$

The amount we want to convert is 21 out of 100.	21 out of 100
Write as a ratio.	$\frac{21}{100}$
Convert the 21 per 100 to percent.	21%

Ⓐ	
	36%
Write as a ratio with denominator 100.	$\frac{36}{100}$
Simplify by dividing out the common factor of 4.	$\frac{9}{25}$

Ⓑ	
	125%
Write as a ratio with denominator 100.	$\frac{125}{100}$
Simplify by dividing out the common factor of 25.	$\frac{5}{4}$

Ⓐ	
	24.5%
Write as a ratio with denominator 100.	$\frac{24.5}{100}$
Clear the decimal by multiplying numerator and denominator by 10.	$\frac{24.5(10)}{100(10)}$
Multiply.	$\frac{245}{1000}$
Rewrite showing common factors.	$\frac{5 \cdot 49}{5 \cdot 200}$
Simplify.	$\frac{49}{200}$

Ⓑ	
	$33\frac{1}{3}\%$
Write as a ratio with denominator 100.	$\frac{33\frac{1}{3}}{100}$
Write the numerator as an improper fraction.	$\frac{\frac{100}{3}}{100}$
Rewrite as fraction division, replacing 100 with $\frac{100}{1}$.	$\frac{100}{3} \div \frac{100}{1}$
Multiply by the reciprocal.	$\frac{100}{3} \cdot \frac{1}{100}$
Simplify.	$\frac{1}{3}$

Because we want to change to a decimal, we will leave the fractions with denominator 100 instead of removing common factors.

Ⓐ	
	6%
Write as a ratio with denominator 100.	$\frac{6}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.06

Ⓑ	
	78%
Write as a ratio with denominator 100.	$\frac{78}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.78

Ⓐ	
	135%
Write as a ratio with denominator 100.	$\frac{135}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	1.35

Ⓑ	
	12.5%
Write as a ratio with denominator 100.	$\frac{12.5}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.125

Ⓐ	
	77%
Write as a ratio with denominator 100.	$\frac{77}{100}$

Ⓑ	
	$\frac{77}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.77

Ⓐ	
	25%
Write as a ratio with denominator 100.	$\frac{25}{100}$
Simplify.	$\frac{1}{4}$

Ⓑ	$\frac{1}{4}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.25

Ⓐ	
	0.05
Write as a fraction. The denominator is 100.	$\frac{5}{100}$
Write this ratio as a percent.	5%

⑥	
	0.83
The denominator is 100.	$\frac{83}{100}$
Write this ratio as a percent.	83%

⑦	
	1.05
Write as a fraction.	$1\frac{5}{100}$
Write as an improper fraction. The denominator is 100.	$\frac{105}{100}$
Write this ratio as a percent.	105%

Notice that since $1.05 > 1$, the result is more than 100%.

⑧	
	0.075
Write as a fraction. The denominator is 1,000.	$\frac{75}{1,000}$
Divide the numerator and denominator by 10, so that the denominator is 100.	$\frac{7.5}{100}$
Write this ratio as a percent.	7.5%

To convert a fraction to a decimal, divide the numerator by the denominator.

⑨	
Change to a decimal.	$\frac{3}{4} = 0.75$
Write as a percent by moving the decimal two places.	0.75 = 75%

⑩	
Change to a decimal.	$\frac{11}{8} = 1.375$
Write as a percent by moving the decimal two places.	1.375 = 137.5%

⑪	
Write as an improper fraction.	$2\frac{1}{5} = \frac{11}{5}$
Change to a decimal.	$\frac{11}{5} = 2.2$
Write as a percent by moving the decimal two places.	2.20 = 220%

Notice that we needed to add a zero at the end of the number when moving the decimal two places to the right in example c.

To change a fraction to a decimal, we divide the numerator by the denominator.

	$\frac{5}{7}$
Change to a decimal—rounding to the nearest thousandth.	0.714
Write as a percent.	71.4%

	$\frac{1}{3}$
--	---------------

Change to a decimal.	$\begin{array}{r} 0.33... \\ 3 \overline{)1.00} \\ \underline{9} \\ 10 \\ \underline{9} \\ 1 \end{array}$
Write as a repeating decimal.	0.333 ...
Write as a percent.	$33\frac{1}{3}\%$

We could also write the percent as $33.\overline{3}\%$.

1.5: Solve General Applications of Percent

Learning Objectives

After completing this section, you should be able to:

1. Translate and solve basic percent equations
2. Solve applications of percent
3. Find percent increase and percent decrease

Translate and Solve Basic Percent Equations

In the past, you may have solved percent problems by setting them up as proportions. This method may continue to be used, or we will learn another method in this section that involves solving an equation. We will use the basic equation **amount = percent (as a decimal) x base**.

We'll look at a common application of percent—tips to a server at a restaurant—to see how to set up a basic percent application.

When Aolani and her friends ate dinner at a restaurant, the bill came to \$80. They wanted to leave a 20% tip. What amount would the tip be?

To solve this, we want to find what *amount* is 20% of \$80. We can solve this by plugging into the formula $\text{amount} = \text{percent (as a decimal)} \times \text{base}$. Here, we have a base of \$80, and they want to tip 20% (0.20 as a decimal). So we can plug into the formula to get $\text{amount} = 0.20 \times 80 = 16$. So Aolani and her friends should tip \$16.

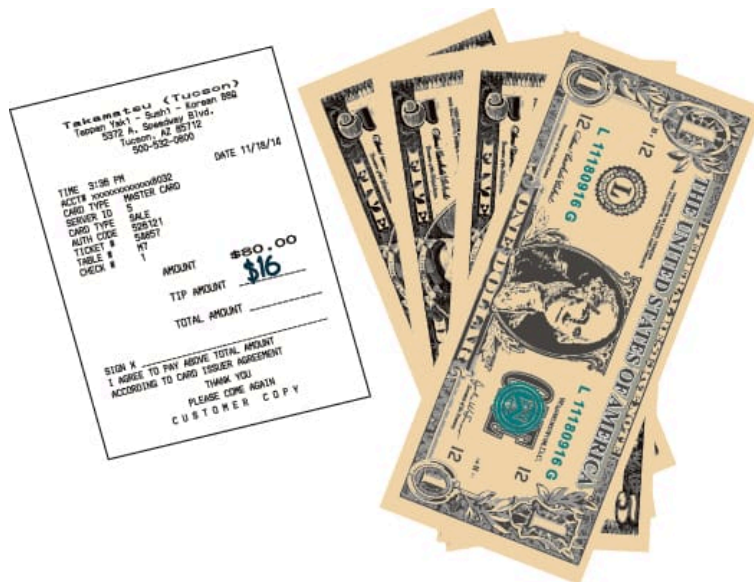


Figure 1.5.1: A 20% tip for an \$80 restaurant bill comes out to \$16.

In the next examples, we will find the amount. We must be sure to change the given percent to a decimal when we translate the words into an equation.

Example 1.5.1

What number is 35% of 90?

Answer available at the end of this section.

Your Turn 1.5.1

What number is 45% of 80?

Example 1.5.2

125% of 28 is what number?

Answer available at the end of this section.

Your Turn 1.5.2

150% of 78 is what number?

In the next examples, we are asked to find the base.

Example 1.5.3

Translate and solve: 36 is 75% of what number?

Answer available at the end of this section.

Your Turn 1.5.3

17 is 25% of what number?

Example 1.5.4

6.5% of what number is \$1.17?

Answer available at the end of this section.

Your Turn 1.5.4

7.5% of what number is \$1.95?

In the next examples, we will solve for the percent.

Example 1.5.5

What percent of 36 is 9?

Answer available at the end of this section.

Your Turn 1.5.5

What percent of 76 is 57?

Example 1.5.6

144 is what percent of 96?

Answer available at the end of this section.

Your Turn 1.5.6

110 is what percent of 88?

Solve Applications of Percent

Many applications of percent occur in our daily lives, such as tips, sales tax, discount, and interest. To solve these applications, we'll translate to a basic percent equation, just like those we solved in the previous examples in this section. Once you translate the sentence into a percent equation, you know how to solve it.

We will use the generic problem solving strategy for solving application problems listed below.

How To**Solve an application**

1. Identify what you are asked to find and choose a variable to represent it.
2. Write a sentence that gives the information to find it.
3. Translate the sentence into an equation.
4. Solve the equation using good algebra techniques.
5. Check the answer in the problem and make sure it makes sense.
6. Write a complete sentence that answers the question.

Now that we have the strategy to refer to, and have practiced solving basic percent equations, we are ready to solve percent applications. Be sure to ask yourself if your final answer makes sense—since many of the applications we'll solve involve everyday situations, you can rely on your own experience.

Example 1.5.7

Dezohn and his girlfriend enjoyed a dinner at a restaurant, and the bill was \$68.50. They want to leave an 18% tip. If the tip will be 18% of the total bill, how much should the tip be?

Answer available at the end of this section.

Your Turn 1.5.7

Cierra and her sister enjoyed a special dinner in a restaurant, and the bill was \$81.50. If she wants to leave 18% of the total bill as her tip, how much should she leave?

Example 1.5.8

The label on Masao's breakfast cereal said that one serving of cereal provides 85 milligrams (mg) of potassium, which is 2% of the recommended daily amount. What is the total recommended daily amount of potassium?

Nutrition Facts		
Serving Size: 1 cup (47g)		
Servings Per Container: About 7		
Amount Per Serving	Cereal	With Milk
Calories	180	230
Calories from Fat	10	20
% Daily Value*		
Total Fat 1g	2%	2%
Saturated Fat 0g	0%	0%
Trans Fat 0g		
Polyunsaturated Fat 0.5g		
Monounsaturated Fat 0.5g		
Cholesterol 0mg	0%	0%
Sodium 190mg	8%	11%
Potassium 85mg	2%	8%
Total Carbohydrate 40g	13%	15%
Dietary Fiber 1g	4%	4%
Sugars 8g		
Protein 3g		

Answer available at the end of this section.

Your Turn 1.5.8

One serving of wheat square cereal has 7 grams of fiber, which is 29% of the recommended daily amount. What is the total recommended daily amount of fiber?

Example 1.5.9

Mitzi received some gourmet brownies as a gift. The wrapper said each brownie was 480 calories, and had 240 calories of fat. What percent of the total calories in each brownie comes from fat?

Answer available at the end of this section.

Your Turn 1.5.9

Veronica is planning to make muffins from a mix. The package says each muffin will be 230 calories and 60 calories will be from fat. What percent of the total calories is from fat? (Round to the nearest whole percent.)

Find Percent Increase and Percent Decrease

People in the media often talk about how much an amount has increased or decreased over a certain period of time. They usually express this increase or decrease as a percent.

To find the percent increase, first we find the amount of increase, which is the difference between the new amount and the original amount. Then we find what percent the amount of increase is of the original amount.

How To

Find Percent Increase

1. Find the amount of increase.
 - $\text{increase} = \text{new amount} - \text{original}$
2. Find the percent increase as a percent of the original amount.

Example 1.5.10

In 2011, the California governor proposed raising community college fees from \$26 per unit to \$36 per unit. Find the percent increase. (Round to the nearest tenth of a percent.)

Answer available at the end of this section.

Your Turn 1.5.10

In 2011, the IRS increased the deductible mileage cost to 55.5 cents from 51 cents. Find the percent increase. (Round to the nearest tenth of a percent.)

Finding the percent decrease is very similar to finding the percent increase, but now the amount of decrease is the difference between the original amount and the final amount. Then we find what percent the amount of decrease is of the original amount.

How To

Find percent decrease

1. Find the amount of decrease.
 - $\text{decrease} = \text{original amount} - \text{new}$
2. Find the percent decrease as a percent of the original amount.

Example 1.5.11

The average price of a gallon of gas in one city in June 2014 was \$3.71. The average price in that city in July was \$3.64. Find the percent decrease.

Answer available at the end of this section.

Your Turn 1.5.11

The population of one city was about 672,000 in 2010. The population of the city is projected to be about 630,000 in 2020. Find the percent decrease. (Round to the nearest tenth of a percent.)

Media

ACCESS ADDITIONAL ONLINE RESOURCES

- [Percent Increase and Percent Decrease Visualization](#)

Answers

Translate into algebra. Let n = the number.	$\underbrace{\text{What number}}_n \text{ is } \underbrace{35\%}_{= 0.35} \text{ of } \underbrace{90}_{90}?$
Multiply.	$n = 31.5$
	31.5 is 35% of 90

Translate into algebra. Let a = the number.	$\underbrace{125\%}_{1.25} \text{ of } \underbrace{28}_{28} \text{ is } \underbrace{\text{what number?}}_a$
Multiply.	$35 = a$
	125% of 28 is 35.

Remember that a percent over 100 is a number greater than 1. We found that 125% of 28 is 35, which is greater than 28.

Translate. Let b = the number.	$\underbrace{36}_{36} \text{ is } \underbrace{75\%}_{= 0.75} \text{ of } \underbrace{\text{what number?}}_b$
Divide both sides by 0.75.	$\frac{36}{0.75} = \frac{0.75b}{0.75}$

Simplify.	$48 = b$ 36 is 75% of 48.
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Translate. Let b = the number.	$\begin{array}{ccccccc} \underbrace{6.5\%}_{0.065} & \underbrace{\text{of}} & \underbrace{\text{what number}} & \underbrace{\text{is}} & \underbrace{\$1.17?}_{1.17} \\ & & b & = & \end{array}$
Divide both sides by 0.065.	$\frac{0.065n}{0.065} = \frac{1.17}{0.065}$
Simplify.	$n = 18$ 6.5% of \$18 is \$1.17.

Translate into algebra. Let p = the percent.	$\begin{array}{ccccccc} \underbrace{\text{What percent}} & \underbrace{\text{of}} & \underbrace{36} & \underbrace{\text{is}} & \underbrace{9?} \\ & & p & . & 36 & = & 9 \end{array}$
Divide by 36.	$\frac{36p}{36} = \frac{9}{36}$
Simplify.	$p = \frac{1}{4}$
Convert to decimal form.	$p = 0.25$
Convert to percent.	$p = 25\%$ 25% of 36 is 9.

Translate into algebra. Let p = the percent.	$\begin{array}{ccccccc} \underbrace{144} & \underbrace{\text{is}} & \underbrace{\text{what percent}} & \underbrace{\text{of}} & \underbrace{96?} \\ 144 & = & p & . & 96 \end{array}$
Divide by 96.	$\frac{144}{96} = \frac{96p}{96}$
Simplify.	$1.5 = p$
Convert to percent.	$150\% = p$ 144 is 150% of 96.

What are you asked to find?	the amount of the tip
Choose a variable to represent it.	Let t = amount of tip.
Write a sentence that gives the information to find it.	The tip is 18% of the total bill.

Translate the sentence into an equation.	$\underbrace{\text{The tip}}_t \underbrace{\text{is}}_= \underbrace{18\%}_{0.18} \underbrace{\text{of}}_{\cdot} \underbrace{\$68.50}_{\$68.50}$
Solve by multiplying.	$t = 12.33$
Check. Is this answer reasonable?	If we approximate the bill to \$70 and the percent to 20%, we would have a tip of \$14. So a tip of \$12.33 seems reasonable.
Write a complete sentence that answers the question.	The couple should leave a tip of \$12.33.

What are you asked to find?	the total amount of potassium recommended
Choose a variable to represent it.	Let a = total amount of potassium.
Write a sentence that gives the information to find it.	85 mg is 2% of the total amount.
Translate the sentence into an equation.	$\underbrace{85 \text{ mg}}_{85} \underbrace{\text{is}}_= \underbrace{2\%}_{0.02} \underbrace{\text{of}}_{\cdot} \underbrace{a}_{a}?$
Solve by dividing both sides by 0.02.	$\frac{85}{0.02} = \frac{0.02a}{0.02}$
Simplify.	$4,250 = a$
Check: Is this answer reasonable?	Yes. 2% is a small percent and 85 is a small part of 4,250.
Write a complete sentence that answers the question.	The amount of potassium that is recommended is 4250 mg.

What are you asked to find?	the percent of the total calories from fat
Choose a variable to represent it.	Let p = percent from fat.
Write a sentence that gives the information to find it.	What percent of 480 is 240?
Translate the sentence into an equation.	$\underbrace{\text{What percent}}_p \underbrace{\text{of}}_{\cdot} \underbrace{480}_{480} \underbrace{\text{is}}_= \underbrace{240}_{240}?$
Solve by dividing both sides by 480.	$\frac{p \cdot 480}{480} = \frac{240}{480}$
Simplify.	$p = 0.5$
Convert to percent form.	$p = 50\%$
Check. Is this answer reasonable?	Yes. 240 is half of 480, so 50% makes sense.
Write a complete sentence that answers the question.	Of the total calories in each brownie, 50% is fat.

What are you asked to find?	the percent increase
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Choose a variable to represent it.	Let p = percent.
Find the amount of increase.	$\underbrace{36}_{\text{new amount}} - \underbrace{26}_{\text{original amount}} = \underbrace{10}_{\text{increase}}$
Find the percent increase.	The increase is what percent of the original amount?
Translate to an equation.	$\underbrace{10}_{10} \underbrace{\text{is}}{=} \underbrace{\text{what percent}}_p \underbrace{\text{of}}_{.} \underbrace{26}_{26}?$
Divide both sides by 26.	$\frac{10}{26} = \frac{26p}{26}$
Round to the nearest thousandth.	$0.385 = p$
Convert to percent form.	$38.5\% = p$
Write a complete sentence.	The new fees represent a 38.5% increase over the old fees.

What are you asked to find?	the percent decrease
Choose a variable to represent it.	Let p = percent.
Find the amount of decrease.	$\underbrace{3.71}_{\text{original amount}} - \underbrace{3.64}_{\text{new amount}} = \underbrace{0.07}_{\text{decrease}}$
Find the percent of decrease.	The decrease is what percent of the original amount?
Translate to an equation.	$\underbrace{0.07}_{0.07} \underbrace{\text{is}}{=} \underbrace{\text{what percent}}_p \underbrace{\text{of}}_{.} \underbrace{3.71}_{3.71}?$
Divide both sides by 3.71.	$\frac{0.07}{3.71} = \frac{3.71p}{3.71}$
Round to the nearest thousandth.	$0.019 = p$
Convert to percent form.	$1.9\% = p$
Write a complete sentence.	The price of gas decreased 1.9%.

1.6: Solve Sales Tax, Commission, and Discount Applications

Learning Objectives

After completing this section, you should be able to:

1. Solve sales tax applications
2. Solve commission applications
3. Solve discount applications
4. Solve mark-up applications

Sales tax and commissions are applications of percent in our everyday lives.

We will continue working with the percent equation that we learned in Section 1.5, while learning about new applications. Many of these applications should be familiar to you from your everyday life, so remember to always check that the answer makes sense.

Remember that whatever the application, once we write the sentence with the given information (Step 2 in our How To Solve an Application Problem), we can translate it to a percent equation and then solve it.

Solve Sales Tax Applications

Do you pay a tax when you shop in your city or state? In many parts of the United States, sales tax is added to the purchase price of an item. See Figure 1.6.1. The sales tax is determined by computing a percent of the purchase price.

To find the sales tax multiply the purchase price by the sales tax rate. Remember to convert the sales tax rate from a percent to a decimal number. Once the sales tax is calculated, it is added to the purchase price. The result is the total cost—this is what the customer pays.

BOULEVARD		
ONE MISSION STREET SAN FRANCISCO, CA 94105 (415) 543-8084 DINING ROOM		
5213 KEN		
Tbl 32/1	Chk 5265	Gst 2
Apr'14'08 08:05PM		
1 D SOUP		13.75
1 D LOBSTER LINGUI		18.75
1 D LAMB		32.00
1 1/2 GL SAUV BLANC		4.25
1 D BANANAS FOSTER		9.50
SUBTOTAL		78.25
Tax (8%)		6.26
Total		84.51
BOULEVARD COOKBOOKS ARE NOW AVAILABLE PLEASE ASK YOUR SERVER THANK YOU FOR DINING WITH US		

Figure 1.6.1: The sales tax is calculated as a percent of the purchase price.

Sales Tax

The sales tax is a percent of the purchase price.

$$\text{Sales Tax} = \text{Tax Rate (as a decimal number)} \times \text{Purchase Price}$$

$$\text{Total Cost} = \text{Purchase Price} + \text{Sales Tax}$$

Example 1.6.1

Cathy bought a bicycle in Washington, where the sales tax rate was 6.5% of the purchase price. What was

- Ⓐ the sales tax and
- Ⓑ the total cost of a bicycle if the purchase price of the bicycle was \$392?

Answer available at the end of this section.

Your Turn 1.6.1

Find Ⓐ the sales tax and Ⓑ the total cost: Alexandra bought a television set for \$724 in Boston, where the sales tax rate was 6.25% of the purchase price.

Example 1.6.2

Evelyn bought a new smartphone for \$499 plus tax. She was surprised when she got the receipt and saw that the tax was \$42.42. What was the sales tax rate for this purchase?

Answer available at the end of this section.

Your Turn 1.6.2

Diego bought a new car for \$26,525. He was surprised that the dealer then added \$2,387.25. What was the sales tax rate for this purchase?

Solve Commission Applications

Sales people often receive a commission, or percent of total sales, for their sales. Their income may be just the commission they earn, or it may be their commission added to their hourly wages or salary. The commission they earn is calculated as a certain percent of the price of each item they sell. That percent is called the **rate of commission**.

Commission

A commission is a percentage of total sales as determined by the rate of commission.

$$\text{commission} = \text{rate of commission (as a decimal number)} \times \text{total sales}$$

To find the commission on a sale, multiply the rate of commission by the total sales. Just as we did for computing sales tax, remember to first convert the rate of commission from a percent to a decimal.

Example 1.6.3

Helene is a realtor. She receives 3% commission when she sells a house. How much commission will she receive for selling a house that costs \$260,000?

Answer available at the end of this section.

Your Turn 1.6.3

Bob is a travel agent. He receives 7% commission when he books a cruise for a customer. How much commission will he receive for booking a \$3,900 cruise?

Example 1.6.4

Rikki earned \$87 commission when she sold a \$1,450 stove. What rate of commission did she get?

Answer available at the end of this section.

Your Turn 1.6.4

Homer received \$1,140 commission when he sold a car for \$28,500. What rate of commission did he get?

Solve Discount Applications

Applications of discount are very common in retail settings. When you buy an item on sale, the **original price** of the item has been reduced by some dollar amount. The **discount rate**, usually given as a percent, is used to determine the amount of the discount. To determine the **amount of discount**, we multiply the discount rate by the original price. We summarize the discount model in the box below.



Figure 1.6.2: Applications of discounts are common in everyday life. (credit: Charleston's TheDigitel, Flickr)

Discount

An amount of discount is a percent off the original price.

$$\text{Amount of Discount} = \text{Discount Rate (as a decimal number)} \times \text{Original Price}$$

$$\text{Sale Price} = \text{Original Price} - \text{Amount of Discount}$$

The sale price should always be less than the original price. In some cases, the amount of discount is a fixed dollar amount. Then we just find the sale price by subtracting the amount of discount from the original price.

Example 1.6.5

Jason bought a pair of sunglasses that were on sale for \$10 off. The original price of the sunglasses was \$39. What was the sale price of the sunglasses?

Answer available at the end of this section.

Your Turn 1.6.5

Marta bought a dishwasher that was on sale for \$75 off. The original price of the dishwasher was \$525. What was the sale price of the dishwasher?

In Example 1.6.5, the amount of discount was a set amount. Let's look at an example now where the amount of discount is a percent of the original price.

Example 1.6.6

Elise bought a dress that was discounted 35% off of the original price of \$140. What was Ⓐ the amount of discount and Ⓑ the sale price of the dress?

Answer available at the end of this section.

Your Turn 1.6.6

Find Ⓐ the amount of discount and Ⓑ the sale price: Sergio bought a belt that was discounted 40% from an original price of \$29.

There may be times when you buy something on sale and want to know the discount rate. The next example will show this case.

Example 1.6.7

Jeannette bought a swimsuit at a sale price of \$13.95. The original price of the swimsuit was \$31. Find the Ⓐ amount of discount and Ⓑ discount rate.

Answer available at the end of this section.

Your Turn 1.6.7

Find Ⓐ the amount of discount and Ⓑ the discount rate: Lena bought a kitchen table at the sale price of \$375.20. The original price of the table was \$560.

Solve Mark-up Applications

Applications of mark-up are very common in retail settings. The price a retailer pays for an item is called the **wholesale price**. The retailer then adds a **mark-up** to the wholesale price to get the **list price**, the price he sells the item for. The mark-up is usually calculated as a percent of the wholesale price. The percent is called the **mark-up rate**. To determine the amount of mark-up, multiply the mark-up rate by the wholesale price. We summarize the mark-up model in the box below.

Mark-up

The mark-up is the amount added to the wholesale price.

$$\text{Amount of Mark-up} = \text{Mark-up Rate (as a decimal number)} \times \text{Wholesale Price}$$

$$\text{List Price} = \text{Wholesale Price} + \text{Amount of Mark-Up}$$

The list price should always be more than the wholesale price.

Example 1.6.8

Adam's art gallery bought a photograph at the wholesale price of \$250. Adam marked the price up 40%. Find the Ⓐ amount of mark-up and Ⓑ the list price of the photograph.

Answer available at the end of this section.

Your Turn 1.6.8

Jim's music store bought a guitar at wholesale price \$1,200. Jim marked the price up 50%. Find the Ⓐ amount of mark-up and Ⓑ the list price.

Answers

Ⓐ	
Identify what you are asked to find.	What is the sales tax?
Choose a variable to represent it.	Let t = sales tax.
Write a sentence that gives the information to find it.	The sales tax is 6.5% of the purchase price.
Translate into an equation. (Remember to change the percent to a decimal).	$\underbrace{\text{The sales tax}}_t \text{ is } \underbrace{6.5\%}_{0.065} \text{ of } \underbrace{\text{the \$392 purchase price.}}_{392}$

Solve the equation by simplifying.	$t = 25.48$
Check: Is this answer reasonable?	Yes, because the sales tax is less than 10% of the purchase price.
Write a complete sentence that answers the question.	The sales tax is \$25.48.

⑥	
Identify what you are asked to find.	What is the total cost of the bicycle?
Choose a variable to represent it.	Let c = total cost of bicycle.
Write a sentence that gives the information to find it.	The total cost is the purchase price plus the sales tax.
Translate into an equation.	$\underbrace{\text{The total cost}}_c \quad \text{is} \quad \underbrace{\$392}_{392} \quad \text{plus} \quad \underbrace{\$25.48}_{25.48}$ $c = 392 + 25.48$
Solve by simplifying.	$c = 417.48$
Check: Is this answer reasonable?	Yes, because the total cost is a little more than the purchase price.
Write a complete sentence that answers the question.	The total cost of the bicycle is \$417.48.

Identify what you are asked to find.	What is the sales tax rate?
Choose a variable to represent it.	Let r = sales tax.
Write a sentence that gives the information to find it.	What percent of the price is the sales tax?
Translate into an equation.	$\underbrace{\text{What percent}}_r \quad \text{of} \quad \underbrace{\text{the \$499 price}}_{499} \quad \text{is} \quad \underbrace{\text{the \$42.42 tax?}}_{42.42}$ $r \cdot 499 = 42.42$
Solve by dividing.	$\frac{499r}{499} = \frac{42.42}{499}$
Simplify.	$r = 0.085$
Check. Is this answer reasonable?	Yes, because 8.5% is close to 10%. 10% of \$500 is \$50, which is close to \$42.42.
Write a complete sentence that answers the question.	The sales tax rate is 8.5%.

Identify what you are asked to find.	What is the commission?
Choose a variable to represent it.	Let c = the commission.
Write a sentence that gives the information to find it.	The commission is 3% of the price.

Translate into an equation.	$\underbrace{\text{The commission}}_c \text{ is } \underbrace{3\%}_{= 0.03} \text{ of } \underbrace{\text{the \$260,000 price.}}_{260,000}$
Solve by simplifying.	$c = 7800$
Check. Is this answer reasonable?	Yes. 1% of \$260,000 is \$2,600. And \$7,800 is three times \$2,600.
Write a complete sentence that answers the question.	Helene will receive a commission of \$7,800.

Identify what you are asked to find.	What is the rate of commission?
Choose a variable to represent it.	Let r = the rate of commission.
Write a sentence that gives the information to find it.	The commission is what percent of the sale?
Translate into an equation.	$\underbrace{\text{The \$87 commission}}_{87} \text{ is } \underbrace{\text{what percent}}_r \text{ of } \underbrace{\text{the \$1450 sale}}_{1450} ?$
Solve by dividing.	$\frac{87}{1450} = \frac{1450r}{1450}$
Simplify.	$0.06 = r$
Change to percent form.	$r = 6\%$
Check if this answer is reasonable.	Yes. A 10% commission would have been \$145. The 6% commission, \$87, is a little more than half of that.
Write a complete sentence that answers the question.	The commission was 6% of the price of the stove.

Identify what you are asked to find.	What is the sale price?
Choose a variable to represent it.	Let s = the sale price.
Write a sentence that gives the information to find it.	The sale price is the original price minus the discount.
Translate into an equation.	$\underbrace{\text{The sale price}}_s \text{ is } \underbrace{\text{the original \$39 price}}_{39} \text{ minus } \underbrace{\text{the \$10 discount.}}_{10}$
Simplify.	$s = 29$
Check if this answer is reasonable.	Yes. The sale price, \$29, is less than the original price, \$39.
Write a complete sentence that answers the question.	The sale price of the sunglasses was \$29.

Ⓐ Before beginning, you may find it helpful to organize the information in a list.
Original price = \$140

Discount rate = 35%
Amount of discount = ?

Identify what you are asked to find.	What is the amount of discount?
Choose a variable to represent it.	Let d = the amount of discount.
Write a sentence that gives the information to find it.	The discount is 35% of the original price.
Translate into an equation.	$\underbrace{\text{The discount}}_s \text{ is } \underbrace{35\%}_{0.35} \text{ of } \underbrace{\text{the \$140 original price.}}_{140}$
Solve by simplifying.	$d = 49$
Check if this answer is reasonable.	Yes. A \$49 discount is reasonable for a \$140 dress.
Write a complete sentence that answers the question.	The amount of discount was \$49.

⑤
Original price = \$140
Amount of discount = \$49
Sale price = ?

Identify what you are asked to find.	What is the sale price of the dress?
Choose a variable to represent it.	Let s = the sale price.
Write a sentence that gives the information to find it.	The sale price is the original price minus the discount.
Translate into an equation.	$\underbrace{\text{The sale price}}_s \text{ is } \underbrace{\text{the \$140}}_{140} \text{ minus } \underbrace{\text{the \$49 discount.}}_{49}$
Solve by simplifying.	$s = 91$
Check if this answer is reasonable.	Yes. The sale price, \$91, is less than the original price, \$140.
Write a complete sentence that answers the question.	The sale price of the dress was \$91.

⑥ Before beginning, you may find it helpful to organize the information in a list.
Original price = \$31
Amount of discount = ?
Sale price = \$13.95

Identify what you are asked to find.	What is the amount of discount?
Choose a variable to represent it.	Let d = the amount of discount.
Write a sentence that gives the information to find it.	The discount is the original price minus the sale price.
Translate into an equation.	$\underbrace{\text{The discount}}_d \text{ is } \underbrace{\text{the \$31 original price}}_{31} \text{ minus } \underbrace{\text{the \$13.95 sale price.}}_{13.95}$

Simplify.	$d = 17.05$
Check if this answer is reasonable.	Yes. The \$17.05 discount is less than the original price of \$31.
Write a complete sentence that answers the question.	The amount of discount was \$17.05.

⑥ Before beginning, you may find it helpful to organize the information in a list.

Original price = \$31

Amount of discount = \$17.05

Discount rate = ?

Identify what you are asked to find.	What is the discount rate?
Choose a variable to represent it.	Let r = the discount rate.
Write a sentence that gives the information to find it.	The discount is what percent of the original price?
Translate into an equation.	The discount of \$17.05 is what percent of the \$31 original price. $\underbrace{\text{The discount of } \$17.05}_d = \underbrace{\text{is}}_{=} \underbrace{\text{what percent}}_r \underbrace{\text{of}}_{\cdot} \underbrace{\text{the } \$31 \text{ original price.}}_{31}$
Solve by dividing.	$\frac{17.05}{31} = \frac{r(31)}{31}$
Simplify.	$0.55 = r$
Check if this answer is reasonable.	Yes. The rate of discount was a little more than 50% and the amount of discount is a little more than half of \$31.
Write a complete sentence that answers the question.	The rate of discount was 55%.

⑦	
Identify what you are asked to find.	What is the amount of mark-up?
Choose a variable to represent it.	Let m = the amount of each mark-up.
Write a sentence that gives the information to find it.	The mark-up is 40% of the wholesale price.
Translate into an equation.	The mark-up is 40% of the \$250 wholesale price. $\underbrace{\text{The mark-up}}_m = \underbrace{\text{is}}_{=} \underbrace{40\%}_{0.40} \underbrace{\text{of}}_{\cdot} \underbrace{\text{the } \$250 \text{ wholesale price.}}_{250}$
Solve by simplifying.	$m = 100$
Check if this answer is reasonable.	Yes. The mark-up rate is less than 50% and \$100 is less than half of \$250.
Write a complete sentence that answers the question.	The mark-up on the photograph was \$100.

⑧	
Identify what you are asked to find.	What is the list price?

Choose a variable to represent it.	Let p = the list price.
Write a sentence that gives the information to find it.	The list price is the wholesale price plus the mark-up.
Translate into an equation.	$\underbrace{\text{The list price}}_p \text{ is } \underbrace{\text{the \$250 wholesale price}}_{250} \text{ plus } \underbrace{\text{the \$100 mark-up.}}_{100}$
Solve by simplifying.	$p = 350$
Check if this answer is reasonable.	Yes. The list price, \$350, is more than the wholesale price, \$250.
Write a complete sentence that answers the question.	The list price of the photograph was \$350.

2: PROBLEM SOLVING

2.1: Problem Solving and Estimating

2.2: Linear Equations in One Variable with Applications

2.3: Quadratic Equations with Applications

2.4: Linear (Algebraic) Growth

2.5: When good models go bad

2.6: Exponential (Geometric) Growth

2.1: Problem Solving and Estimating

Learning Objectives

After completing this section, you should be able to:

1. Solve real-world problems using the problem solving process.

In many problems, it is tempting to take the given information, plug it into whatever formulas you have handy, and hope that the result is what you were supposed to find. Chances are, this approach has served you well in other math classes.

This approach does not work well with real life problems. Instead, problem solving is best approached by first starting at the end: identifying exactly what you are looking for. From there, you then work backwards, asking “what information and procedures will I need to find this?” Very few interesting questions can be answered in one mathematical step; often times you will need to chain together a solution pathway, or a series of steps, that will allow you to answer the question.

Problem Solving Process

1. Identify the question you’re trying to answer.
2. Work backwards, identifying the information you will need and the relationships you will use to answer that question.
3. Continue working backwards, creating a solution pathway.
4. If you are missing necessary information, look it up or estimate it. If you have unnecessary information, ignore it.
5. Solve the problem, following your solution pathway.

In most problems we work, we will be approximating a solution, because we will not have perfect information. We will begin with a few examples where we will be able to approximate the solution using basic knowledge from our lives.

Example 2.1.1

How many times does your heart beat in a year?

Solution

This question is asking for the rate of heart beats per year. Since a year is a long time to measure heart beats for, if we knew the rate of heart beats per minute, we could scale that quantity up to a year. So the information we need to answer this question is heart beats per minute. This is something you can easily measure by counting your pulse while watching a clock for a minute.

Suppose you count 80 beats in a minute. To convert this beats per year:

$$\frac{80 \text{ beats}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{24 \text{ hours}}{1 \text{ day}} \cdot \frac{365 \text{ days}}{1 \text{ year}} = 42,048,000 \text{ beats per year}$$

Example 2.1.2

How thick is a single sheet of paper? How much does it weigh?

Solution

While you might have a sheet of paper handy, trying to measure it would be tricky. Instead we might imagine a stack of paper, and then scale the thickness and weight to a single sheet.

If you've ever bought paper for a printer or copier, you probably bought a ream, which contains 500 sheets. We could estimate that a ream of paper is about 2 inches thick and weighs about 5 pounds. Scaling these down,

$$\frac{2 \text{ inches}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}} = 0.004 \text{ inches per sheet}$$

$$\frac{5 \text{ pounds}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}} = 0.01 \text{ pounds per sheet, or } 0.16 \text{ ounces per sheet.}$$

Example 2.1.3

A recipe for zucchini muffins states that it yields 12 muffins, with 250 calories per muffin. You instead decide to make mini-muffins, and the recipe yields 20 muffins. If you eat 4, how many calories will you consume?

Solution

There are several possible solution pathways to answer this question. We will explore one.

To answer the question of how many calories 4 mini-muffins will contain, we would want to know the number of calories in each mini-muffin. To find the calories in each mini-muffin, we could first find the total calories for the entire recipe, then divide it by the number of mini-muffins produced. To find the total calories for the recipe, we could multiply the calories per standard muffin by the number per muffin. Notice that this produces a multi-step solution pathway. It is often easier to solve a problem in small steps, rather than trying to find a way to jump directly from the given information to the solution.

We can now execute our plan:

$$12 \text{ muffins} \cdot \frac{250 \text{ calories}}{\text{muffin}} = 3000 \text{ calories for the whole recipe}$$

$$\frac{3000 \text{ calories}}{20 \text{ mini-muffins}} \text{ gives } 150 \text{ calories per mini-muffin}$$

$$4 \text{ mini muffins} \cdot \frac{150 \text{ calories}}{\text{mini-muffin}} \text{ totals } 600 \text{ calories consumed.}$$

Example 2.1.4

You need to replace the boards on your deck. About how much will the materials cost?

Solution

There are two approaches we could take to this problem: 1) estimate the number of boards we will need and find the cost per board, or 2) estimate the area of the deck and find the approximate cost per square foot for deck boards. We will take the latter approach.

For this solution pathway, we will be able to answer the question if we know the cost per square foot for decking boards and the square footage of the deck. To find the cost per square foot for decking boards, we could compute the area of a single board, and divide it into the cost for that board. We can compute the square footage of the deck using geometric formulas. So first we need information: the dimensions of the deck, and the cost and dimensions of a single deck board.

Suppose that measuring the deck, it is rectangular, measuring 16 ft by 24 ft, for a total area of 384ft^2 .

From a visit to the local home improvement store, you find that an 8 foot by 4 inch cedar deck board costs about \$7.50. The area of this board, doing the necessary conversion from inches to feet, is:

$$8 \text{ feet} \cdot 4 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} = 2.667\text{ft}^2. \text{ The cost per square foot is then}$$

$$\frac{\$7.50}{2.667\text{ft}^2} = \$2.8125 \text{ per ft}^2.$$

This will allow us to estimate the material cost for the whole 384ft^2 deck

$$\$384\text{ft}^2 \cdot \frac{\$2.8125}{\text{ft}^2} = \$1080 \text{ total cost.}$$

Of course, this cost estimate assumes that there is no waste, which is rarely the case. It is common to add at least 10% to the cost estimate to account for waste.

Example 2.1.5

Is it worth buying a Hyundai Sonata hybrid instead the regular Hyundai Sonata?

Solution

To make this decision, we must first decide what our basis for comparison will be. For the purposes of this example, we'll focus on fuel and purchase costs, but environmental impacts and maintenance costs are other factors a buyer might consider.

It might be interesting to compare the cost of gas to run both cars for a year. To determine this, we will need to know the miles per gallon both cars get, as well as the number of miles we expect to drive in a year. From that information, we can find the number of gallons required from a year. Using the price of gas per gallon, we can find the running cost.

From Hyundai's website, the 2013 Sonata will get 24 miles per gallon (mpg) in the city, and 35 mpg on the highway. The hybrid will get 35 mpg in the city, and 40 mpg on the highway.

An average driver drives about 12,000 miles a year. Suppose that you expect to drive about 75% of that in the city, so 9,000 city miles a year, and 3,000 highway miles a year.

We can then find the number of gallons each car would require for the year.

Sonata:

$$9000 \text{ city miles} \cdot \frac{1 \text{ gallon}}{24 \text{ city miles}} + 3000 \text{ highway miles} \cdot \frac{1 \text{ gallon}}{35 \text{ highway miles}} = 460.7 \text{ gallons}$$

Hybrid:

$$9000 \text{ city miles} \cdot \frac{1 \text{ gallon}}{35 \text{ city miles}} + 3000 \text{ highway miles} \cdot \frac{1 \text{ gallon}}{40 \text{ highway miles}} = 332.1 \text{ gallons}$$

If gas in your area averages about \$3.50 per gallon, we can use that to find the running cost:

$$\text{Sonata: } 460.7 \text{ gallons} \cdot \frac{\$3.50}{\text{gallon}} = \$1612.45$$

$$\text{Hybrid: } 332.1 \text{ gallons} \cdot \frac{\$3.50}{\text{gallon}} = \$1162.35$$

The hybrid will save \$450.10 a year. The gas costs for the hybrid are about $\frac{\$450.10}{\$1612.45} = 0.279 = 27.9\%$ lower than the costs for the standard Sonata.

While both the absolute and relative comparisons are useful here, they still make it hard to answer the original question, since "is it worth it" implies there is some tradeoff for the gas savings. Indeed, the hybrid Sonata costs about \$25,850, compared to the base model for the regular Sonata, at \$20,895.

To better answer the "is it worth it" question, we might explore how long it will take the gas savings to make up for the additional initial cost. The hybrid costs \$4965 more. With gas savings of \$451.10 a year, it will take about 11 years for the gas savings to make up for the higher initial costs.

We can conclude that if you expect to own the car 11 years, the hybrid is indeed worth it. If you plan to own the car for less than 11 years, it may still be worth it, since the resale value of the hybrid may be higher, or for other non-monetary reasons. This is a case where math can help guide your decision, but it can't make it for you.

Example 2.1.6

If traveling from Seattle, WA to Spokane WA for a three-day conference, does it make more sense to drive or fly?

Solution available at the end of this section.

Answers

Example 2.1.6

There is not enough information provided to answer the question, so we will have to make some assumptions, and look up some values.

Assumptions:

- a) We own a car. Suppose it gets 24 miles to the gallon. We will only consider gas cost.
- b) We will not need to rent a car in Spokane, but will need to get a taxi from the airport to the conference hotel downtown and back.
- c) We can get someone to drop us off at the airport, so we don't need to consider airport parking.
- d) We will not consider whether we will lose money by having to take time off work to drive.

Values looked up (your values may be different)

- a) Flight cost: \$184
- b) Taxi cost: \$25 each way (estimate, according to hotel website)
- c) Driving distance: 280 miles each way
- d) Gas cost: \$3.79 a gallon

Cost for flying: \$184 flight cost + \$50 in taxi fares = \$234.

Cost for driving: 560 miles round trip will require 23.3 gallons of gas, costing \$88.31.

Based on these assumptions, driving is cheaper. However, our assumption that we only include gas cost may not be a good one. Tax law allows you deduct \$0.55 (in 2012) for each mile driven, a value that accounts for gas as well as a portion of the car cost, insurance, maintenance, etc. Based on this number, the cost of driving would be \$319.

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2.2: Linear Equations in One Variable with Applications

Learning Objectives

After completing this section, you should be able to:

1. Solve linear equations in one variable using properties of equations.
2. Construct a linear equation to solve applications.
3. Solve a formula for a given variable.



Figure 2.2.1: Most gyms have a monthly membership fee. (credit: modification of work "Morning PT after the Holidays 2021" by Fort Drum & 10th Mountain Division (LI)/Flickr, Public Domain Mark 1.0)

In this section, we will study linear equations in one variable. There are several real-world scenarios that can be represented by linear equations: taxi rentals with a flat fee and a rate per mile; cell phone bills that charge a monthly fee plus a separate rate per text; gym memberships with a monthly fee plus a rate per class taken; etc. For example, if you join your local gym at \$10 per month and pay \$5 per class, how many classes can you take if your gym budget is \$75 per month?

Linear Equations and Applications

Solving any equation is like discovering the answer to a puzzle. The purpose of solving an equation is to find the value or values of the variable that makes the equation a true statement. Any value of the variable that makes the equation true is called a solution to the equation. It is the answer to the puzzle! There are many types of equations that we will learn to solve. In this section, we will focus on a linear equation, which is an equation in one variable that can be written as

$$ax + b = 0$$

where a and b are real numbers and $a \neq 0$, such that a is the coefficient of x and b is the constant.

To solve a linear equation, it is a good idea to have an overall strategy that can be used to solve any linear equation. We will follow the steps below in examples 2.2.1 and 2.2.2.

How To

Solve a Linear Equation

1. Simplify each side of the equation as much as possible.
2. Collect all variable terms on one side of the equation.
3. Collect constant terms on the other side of the equation.
4. Make the coefficient of the variable term equal to 1.
5. Check the solution.

Example 2.2.1

Solving a Linear Equation Using a General Strategy

Solve $7(n - 3) - 8 = -15$

Answer available at the end of this section.

Your Turn 2.2.1

Solve $2(x + 1) - 3 = 5$

In Example 2.2.1, we used both the addition and division property of equations. All the properties of equations are summarized in table below. Basically, what you do to one side of the equation, you must do to the other side of the equation to preserve equality.

Operation	Property	Example
Addition	If $a = b$ Then $a + c = b + c$	$2 = 2$ $2 + 3 = 2 + 3$ $5 = 5$
Subtraction	If $a = b$ Then $a - c = b - c$	$5 = 5$ $5 - 2 = 5 - 2$ $3 = 3$
Multiplication	If $a = b$ Then $a \bullet c = b \bullet c$	$3 = 3$ $3 \bullet 4 = 3 \bullet 4$ $12 = 12$
Division	If $a = b$ Then $a \div c = b \div c$ for $c \neq 0$	$8 = 8$ $8 \div 2 = 8 \div 2$ $4 = 4$

Example 2.2.2

Solving a Linear Equation Using Properties of Equations

Solve $9(y - 2) - y = 16 + 7y$.

Answer available at the end of this section.

Your Turn 2.2.2

Solve $6(y - 2) - 5y = 4(y + 3) - 4(y - 1)$.

Who Knew?

Who Invented the Symbol for Equals ?

Before the creation of a symbol for equality, it was usually expressed with a word that meant equals, such as *aequales* (Latin), *esgale* (French), or *gleich* (German). Welsh mathematician and physician Robert Recorde is given credit for inventing the modern sign. It first appears in writing in *The Whetstone of Witte*, a book Recorde wrote about algebra, which was published in 1557. In this book, Recorde states, "I will set as I do often in work use, a pair of parallels, or Gemowe (twin) lines of one length, thus: $==$, because no two things can be more equal." Although his version of the sign was a bit longer than the one we use today, his idea stuck and " $=$ " is used throughout the world to indicate equality in mathematics.

Working with percents in Sections 1.5 and 1.6, you translated an English sentence into an equation. In this section, we take that one step further and translate an English paragraph into an equation, and then we solve the equation. We can go back to the opening question in this section: *If you join your local gym at \$10 per month and pay \$5 per class, how many classes can you take if your gym budget is \$75 per month?* We can create an equation for this scenario and then solve the equation (see Example 2.2.4).

Example 2.2.3

Constructing a Linear Equation to Solve an Application

The Beaudrie family has two cats, Basil and Max. Together, they weigh 23 pounds. Basil weighs 16 pounds. How much does Max weigh?

Answer available at the end of this section.

Your Turn 2.2.3

Sam and Henry are roommates. Together, they have 68 books. Sam has 26 books. How many books does Henry have?

Example 2.2.4

Constructing a Linear Equation to Solve Another Application

If you join your local gym at \$10 per month and pay \$5 per class, how many classes can you take if your gym budget is \$75 per month?

Answer available at the end of this section.

Your Turn 2.2.4

On June 7, 2021, the national average price for regular gasoline was \$3.053 per gallon. If Aiko fills up his car with 16 gallons, how much is the total cost? Round to the nearest cent.

Example 2.2.5

Constructing an Application from a Linear Equation

Write an application that can be solved using the equation $50x + 35 = 185$. Then solve your application.

Answer available at the end of this section.

Your Turn 2.2.5

Write an application that can be solved using the equation $25x + 75 = 200$. Then solve your application.

Solving a Formula for a Given Variable

You are probably familiar with some geometry formulas. A formula is a mathematical description of the relationship between variables. Formulas are also used in the sciences, such as chemistry, physics, and biology. In medicine, they are used for calculations for dispensing medicine or determining body mass index. Spreadsheet programs rely on formulas to make calculations. It is important to be able to manipulate formulas and solve for specific variables.

To solve a formula for a specific variable means to isolate that variable on one side of the equal sign with a coefficient of 1. All other variables and constants are on the other side of the equal sign. To see how to solve a formula for a specific variable, we will start with the distance, rate, and time formula.

Example 2.2.6

Solving for a Given Variable with Distance, Rate, and Time

Solve the formula $d = rt$ for t . This is the distance formula where d stands for distance, r stands for rate, and t stands for time.

Answer available at the end of this section.

Your Turn 2.2.6

Solve the formula $I = Prt$. This formula is used to calculate simple interest I , for a principal P , invested at a rate r , for t years.

Video

[Solving for a Variable in an Equation](#)

Example 2.2.7

Solving for a Given Variable in the Area Formula for a Triangle

Solve the formula $A = \frac{1}{2}bh$ for h . This is the area formula of a triangle where A = area, b = base, and h = height.

Answer available at the end of this section.

Your Turn 2.2.7

Solve the formula $V = \frac{1}{3}\pi r^2 h$ for h . This formula is used to calculate the volume V of a right circular cone with radius r and height h .

Answers

Step 1: Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is now simplified as much as possible.	$7(n - 3) - 8 = -15$ $7n - 21 - 8 = -15$ $7n - 29 = -15$
Step 2: Collect all variable terms on one side of the equation.	Nothing to do; all n -terms are on the left side.	$7n - 29 = -15$
Step 3: Collect constant terms on the other side of the equation.	To get constants only on the right, add 29 to each side. Simplify.	$7n - 29 + 29 = -15$ $+29$ $7n = 14$
Step 4: Make the coefficient of the variable term equal to 1.	Divide each side by 7. Simplify.	$\frac{7n}{7} = \frac{14}{7}$ $n = 2$
Step 5: Check the solution.	Let $n = 2$ Subtract.	Check: $7(n - 3) - 8 = -15$ $7(2 - 3) - 8 \stackrel{?}{=} -15$ $7(-1) - 8 \stackrel{?}{=} -15$ $-7 - 8 \stackrel{?}{=} -15$ $-15 = -15$

Step 1: Simplify each side.

$$\begin{aligned}
 9(y - 2) - y &= 16 + 7y \\
 9y - 18 - y &= 16 + 7y \\
 8y - 18 &= 16 + 7y
 \end{aligned}$$

Step 2: Collect all variables on one side by subtracting $7y$ from both sides.

$$\begin{aligned}
 8y - 18 - 7y &= 16 + 7y - 7y \\
 y - 18 &= 16
 \end{aligned}$$

Step 3: Collect constant terms on one side by adding 18 to both sides.

$$\begin{aligned}
 y - 18 + 18 &= 16 + 18 \\
 y &= 34
 \end{aligned}$$

Step 4: Make the coefficient of the variable 1. Already done!

Step 5: Check.

$$\begin{aligned}
 9(34) - 18 - (34) &\stackrel{?}{=} 16 + 7(34) \\
 306 - 18 - 34 &\stackrel{?}{=} 16 + 238 \\
 288 - 34 &\stackrel{?}{=} 254 \\
 254 &= 254 \checkmark
 \end{aligned}$$

Steps 1 and 2:

Let b = Basil's weight and m = Max's weight.

$$b + m = 23$$

We also know that Basil weighs 16 pounds so replace b with 16 in the equation.

$$16 + m = 23$$

Step 3: Since both sides are simplified and the variable is on one side of the equation, we start solving the equation by collecting the constants on one side.

$$\begin{aligned}
 16 + m - 16 &= 23 - 16 \\
 m &= 7
 \end{aligned}$$

Step 4: is already done, so we go to Step 5.

Step 5:

$$\begin{aligned}
 16 + 7 &\stackrel{?}{=} 23 \\
 23 &= 23 \checkmark
 \end{aligned}$$

Basil weighs 16 pounds and Max weighs 7 pounds.

If we let x = number of classes, the expression $5x+10$ would represent what you pay per month if each class is \$5 and there's a \$10 monthly fee per class. \$10 is your constant. If you want to know how many classes you can take if you have a \$75 monthly gym budget, set the equation equal to 75. Then solve the equation $5x + 10 = 75$ for x .

Steps 1 and 2:

$$5x + 10 = 75$$

Step 3:

$$5x + 10 - 10 = 75 - 10$$

$$5x = 65$$

Step 4:

$$\frac{5x}{5} = \frac{65}{5}$$

$$x = 13$$

Step 5:

$$5(13) + 10 \stackrel{?}{=} 75$$

$$65 + 10 \stackrel{?}{=} 75$$

$$75 = 75 \checkmark$$

The solution is 13 classes. You can take 13 classes on a \$75 monthly gym budget.

Answers will vary. Let's say you want to rent a snowblower for a huge winter storm coming up. If x = the number of days you rent a snowblower, then the expression $50x+35$ represents what you pay if, for each day, it costs \$50 to rent the snowblower and there is a \$35 flat rental fee. \$35 is the constant. To find out how many days you can rent a snowblower for \$185, set the expression equal to 185. Then solve the equation $50x+35=185$ for x .

Steps 1 and 2:

$$50x + 35 = 185$$

Step 3:

$$50x + 35 - 35 = 185 - 35$$

$$50x = 150$$

Step 4:

$$\frac{50x}{50} = \frac{150}{50}$$

$$x = 3$$

Step 5:

$$50(3) + 35 \stackrel{?}{=} 185$$

$$150 + 35 \stackrel{?}{=} 185$$

$$185 = 185 \checkmark$$

The equation is $50x+35=185$ and the solution is 3 days. You can rent a snowblower for 3 days on a \$185 budget.

Divide both sides by r : $d/r = rt/r$

Simplify.

$$d/r = t$$

Step 1: Multiply both sides by 2.

$$2A = 2(\frac{1}{2}bh)$$

$$2A = bh$$

Step 2: Divide both sides by b .

$$\frac{2A}{b} = \frac{bh}{b}$$

$$\frac{2A}{b} = h$$

2.3: Quadratic Equations with Applications

Learning Objectives

After completing this section, you should be able to:

1. Multiply polynomials.
2. Factor trinomials.
3. Solve quadratic equations by factoring.
4. Solve quadratic equations using the quadratic formula.
5. Solve real-world applications modeled by quadratic equations.

In this section, we will discuss quadratic equations. There are several real-world scenarios that can be represented by the graph of a quadratic equation. Think of the Gateway Arch in St. Louis, Missouri. Both ends of the arch are 630 feet apart and the arch is 630 feet tall. You can plot these points on a coordinate system and create a parabola to graph the quadratic equation.



Figure 2.3.1: The Gateway Arch in St. Louis, Missouri (credit: modification of work "Gateway Arch - St. Louis - Missouri" by Sam valadi/Flickr, CC BY 2.0)

Before we begin working with solving quadratic equations, let's do some quick review. A **monomial** is a term of the form ax^m where a is any real number and m is a whole number. Some examples of monomials are $3x$, $5x^7$, and 6 . We can add (or subtract) several monomials together to form a **polynomial**. Some examples of polynomials are $3x - 2$, -123 , $y^2 + 4y - \frac{1}{3}$, and $2x^7 + 5x$. Polynomials can be named more specifically using the number of terms they have. If a polynomial has two terms, it is called a **binomial** (examples: $3x - 2$ or $2x^7 + 5x$). If a polynomial has three terms, it is called a **trinomial** (example: $y^2 + 4y - \frac{1}{3}$).

Multiply Polynomials

Distribution is the key to multiplying polynomials. You are used to distributing a number when solving equations. For example $2(x+3)$ can be rewritten as $2x + 6$. Let's look at what happens if we add a variable to the monomial outside the parentheses.

Example 2.3.1

Multiply $3x(2x + 1)$.

Answer

	$3x(2x + 1)$
Distribute the $3x$.	$3x \cdot 2x + 3x \cdot 1$
Simplify.	$6x^2 + 3x$

Your Turn 2.3.1

Multiply $4x(5x - 7)$.

Now let's try multiplying two binomials. You may have learned the Box Method or FOIL in a previous course to help you remember the process for multiplying two binomials. Both of these are really just distributing each term from the first binomial to each term of the second binomial.

Example 2.3.2

Multiply $(2x - 5)(x + 6)$

Solution

	$(2x - 5)(x + 6)$
Distribute the $2x$ to each term of the second binomial. Distribute the -5 to each term of the second binomial.	$2x \cdot x + 2x \cdot 6 - 5 \cdot x - 5 \cdot 6$
Multiply the terms.	$2x^2 + 12x - 5x - 30$
Simplify.	$2x^2 + 7x - 30$

Your Turn 2.3.2

Multiply $(x - 8)(4x + 3)$.

You may remember factoring trinomials of the form $x^2 + bx + c$ in previous courses. You should look for two numbers that add to b and multiply to c . Let's look at an example as a review.

Factor $x^2 - 16x + 63$

Answer

Here we are looking for two numbers that add to -16 and multiply to 63 .

Let's start by thinking through the factors of 63 . We have 1 and 63 , 3 and 21 , or 7 and 9 .

Which pair of factors adds to -16 ? 7 and 9 add to 16 , but we are looking for -16 .

If we make both the 7 and 9 negative, we have $-7 + -9 = -16$ and $-7 \cdot -9 = 63$.

So the trinomial factors into $(x-7)(x-9)$.

Your Turn 2.3.3

Factor $x^2 - 5x - 14$

Video

[Factoring with the Box Method \(Area Model\)](#)

Now that we've spent some time reviewing, let's look at quadratic equations.

Quadratic Equation

A quadratic equation has a polynomial where the highest exponent is 2, and is often written with the polynomial equal to zero. It must be able to be written in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers.

Since we are working with equations now, we will want to solve them. This means we are trying to find the value(s) of x that make the equation true.

Solving Quadratic Equations by Factoring

One way to solve a quadratic equation is by factoring. We will use the Zero Product Property that says that if the product of two quantities is zero, it must be that at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

Example 2.3.4

Solving a Quadratic Equation by Factoring

Solve $x^2 + 2x - 15 = 0$.

Answer

Factor the trinomial on the left hand side. What two numbers add to 2 and multiply to -15?	Factors of -15 are -1 and 15, 1 and -15, -3 and 5, or 3 and -5. The pair that adds to 2 is -3 and 5. So the trinomial factors into $(x - 3)(x + 5)$.
Rewrite the equation with the factored form.	$(x - 3)(x + 5) = 0$
Apply the Zero Product Property.	$x - 3 = 0$ or $x + 5 = 0$
Solve each equation.	$x = 3$ or $x = -5$

Your Turn 2.3.4

Solve $x^2 + 11x + 30 = 0$

Video

[Solving Quadratics with the Zero Property](#)

Solving Quadratic Equations Using the Quadratic Formula

The other method we will look at for solving quadratic equations is the quadratic formula. This method works for all quadratic equations, even quadratic equations we could not factor!

Quadratic Formula

The solutions to an equation of the form $ax^2 + bx + c = 0$ can be found using the formula $x =$ [REDACTED]

Example 2.3.5

Solve using the quadratic formula: $3x^2 - 5x + 2 = 0$.

Answer

Write the equation in standard form.	Already done!
Identify the values of a, b, and c.	$a = 3$, $b = -5$, and $c = 2$
Plug these values into the quadratic formula.	$x = \frac{5 \pm \sqrt{(-5)^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 3}$
Simplify.	$x = \frac{5 \pm \sqrt{1}}{6} = \frac{5 \pm 1}{6}$
Solve for each value.	$x = \frac{5+1}{6} = 1$ or $x = \frac{5-1}{6} = \frac{4}{6} = \frac{2}{3}$

Your Turn 2.3.5

Solve using the quadratic formula: $4x^2 + 8x + 7 = 4$.

Video

[Solving Quadratics with the Quadratic Formula](#)

Solving Real-World Applications Modeled by Quadratic Equations

There are problem solving strategies that will work well for applications that translate to quadratic equations. Here's a problem-solving strategy to solve word problems:

Step 1: Read the problem. Make sure all the words and ideas are understood.

Step 2: Identify what we are looking for.

Step 3: Name what we are looking for. Choose a variable to represent that quantity.

Step 4: Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.

Step 5: Solve the equation using good algebra techniques.

Step 6: Check the answer in the problem and make sure it makes sense.

Step 7: Answer the question with a complete sentence.

Example 2.3.6

The product of two consecutive integers is 132. Find the integers.

Answer available at the end of this section.

Your Turn 2.3.6

Two consecutive integers have a product of 72. What are the integers?

Quadratic equations can be used to model projectile motion. When you launch an object into the air, it will go up for awhile, but eventually gravity will take over, and the object will start falling. The path of the object is a parabola, similar to the St. Louis arch we looked at during the start of this section.

Example 2.3.7

A baseball player hits a ball into the air. The height (in feet) t seconds after the ball is hit can be given by the equation $h = -16t^2 + 47t + 3$. Assuming no one catches the ball, how long will it be in the air?

Solution

If we want to know how long the ball is in the air, we want to know at what time the ball will hit the ground. When the ball is on the ground, the height is 0 feet. So let's set the height equal to zero and solve the equation.

$$0 = -16t^2 + 47t + 3.$$

We need to solve this equation. We could try factoring it, or we could use the quadratic equation. Let's try the quadratic equation this time.

$$t = \frac{-47 \pm \sqrt{47^2 - 4(-16)(3)}}{2(-16)}$$

$$t = \frac{-47 \pm \sqrt{2401}}{-32} = \frac{-47 \pm 49}{-32}$$

$$\text{So } t = \frac{-47+49}{-32} = \frac{2}{-32} = -0.0625 \text{ or } t = \frac{-47-49}{-32} = \frac{-96}{-32} = 3$$

So, we get that the ball is on the ground at -0.0625 seconds and at 3 seconds. The -0.0625 does not make sense in the problem, since time cannot be negative. So, we will say that the ball lands on the ground after 3 seconds.

Your Turn 2.3.7

A child throws a rock into a river. The height of the rock t seconds after being thrown can be given using $h = -16t^2 - 12t + 10$. How long will it take before the rock lands in the river?

Answers

Example 2.3.6

Step 1: Read the problem.

Step 2: Identify what we are looking for.

We are looking for two consecutive integers.

Step 3: Name what we are looking for. Choose a variable to represent that quantity.

We are looking for two integers that are consecutive. If we call the first one x , then the next consecutive integer would be $x + 1$.

Step 4: Translate into an equation.

The product of the consecutive integers is 132. So $x(x + 1) = 132$.

Step 5: Solve the equation using good algebra techniques.

Write the equation in standard form.	$x^2 + x = 132$ $x^2 + x - 132 = 0$
Solve the equation by factoring. (You could also use the quadratic formula here.)	$(x + 12)(x - 11) = 0$ $x + 12 = 0 \text{ or } x - 11 = 0$ $x = -12 \text{ or } x = 11$

Step 6: Check the answer in the problem and make sure it makes sense.

If $x = -12$, then the next consecutive integer would be -11 . $-12 \cdot -11 = 132$, so that solution works!

If $x = 11$, then the next consecutive integer would be 12 . $11 \cdot 12 = 132$, so that solution also works!

Step 7: Answer the question with a complete sentence.

-12 and -11 are consecutive integers whose product is 132. 11 and 12 are another set of consecutive integers whose product is 132.

2.4: Linear (Algebraic) Growth

Learning Objectives

After completing this section, you should be able to:

1. Write recursive and explicit formulas to model linear growth
2. Use explicit formulas to find future population values of linear growth
3. Find the time needed for a population to reach a certain value

Marco is a collector of antique soda bottles. His collection currently contains 437 bottles. Every year, he budgets enough money to buy 32 new bottles. Can we determine how many bottles he will have in 5 years, and how long it will take for his collection to reach 1000 bottles?

While both of these questions you could probably solve without an equation or formal mathematics, we are going to formalize our approach to this problem to provide a means to answer more complicated questions.

Suppose that P_n represents the number, or population, of bottles Marco has after n years. So P_0 would represent the number of bottles now, P_1 would represent the number of bottles after 1 year, P_2 would represent the number of bottles after 2 years, and so on. We could describe how Marco's bottle collection is changing using:

$$P_0 = 437$$

$$P_n = P_{n-1} + 32$$

This is called a **recursive relationship**. A recursive relationship is a formula which relates the next value in a sequence to the previous values. Here, the number of bottles in year n can be found by adding 32 to the number of bottles in the previous year, P_{n-1} . Using this relationship, we could calculate:

$$P_1 = P_0 + 32 = 437 + 32 = 469$$

$$P_2 = P_1 + 32 = 469 + 32 = 501$$

$$P_3 = P_2 + 32 = 501 + 32 = 533$$

$$P_4 = P_3 + 32 = 533 + 32 = 565$$

$$P_5 = P_4 + 32 = 565 + 32 = 597$$

We have answered the question of how many bottles Marco will have in 5 years. However, solving how long it will take for his collection to reach 1000 bottles would require a lot more calculations.

While recursive relationships are excellent for describing simply and cleanly *how* a quantity is changing, they are not convenient for making predictions or solving problems that stretch far into the future. For that, a closed or explicit form for the relationship is preferred. An **explicit equation** allows us to calculate P_n directly, without needing to know P_{n-1} . While you may already be able to guess the explicit equation, let us derive it from the recursive formula. We can do so by selectively not simplifying as we go:

$$\begin{aligned} P_1 &= 437 + 32 &&= 437 + 1(32) \\ P_2 &= P_1 + 32 = 437 + 32 + 32 &&= 437 + 2(32) \\ P_3 &= P_2 + 32 = (437 + 2(32)) + 32 &&= 437 + 3(32) \\ P_4 &= P_3 + 32 = (437 + 3(32)) + 32 &&= 437 + 4(32) \end{aligned}$$

You can probably see the pattern now, and generalize that

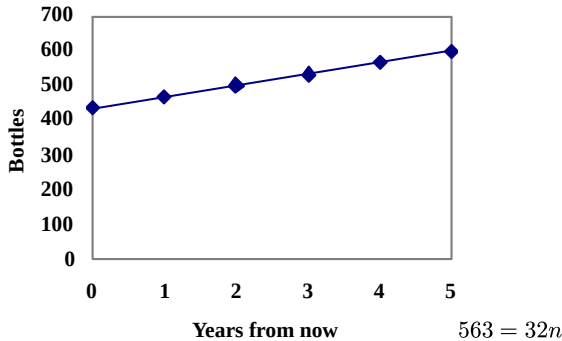
$$P_n = 437 + n(32) = 437 + 32n$$

Using this equation, we can calculate how many bottles he'll have after 5 years:

$$P_5 = 437 + 32(5) = 437 + 160 = 597$$

We can now also solve for when the collection will reach 1000 bottles by substituting in 1000 for P_n and solving for n .

$$1000 = 437 + 32n$$



$$n = 563/32 = 17.59$$

So Marco will reach 1000 bottles in 18 years.

In the previous example, Marco's collection grew by the *same number* of bottles every year. This constant change is the defining characteristic of linear growth. Plotting the values we calculated for Marco's collection, we can see the values form a straight line, the shape of linear growth.

Linear Growth

If a quantity starts at size P_0 and grows by d every time period, then the quantity after n time periods can be determined using either of these relations:

Recursive form:

$$P_n = P_{n-1} + d$$

Explicit form:

$$P_n = P_0 + dn$$

In this equation, d represents the **common difference** – the amount that the population changes each time n increases by 1

Connection to prior learning – slope and intercept

You may recognize the common difference, d , in our linear equation as slope. In fact, the entire explicit equation should look familiar – it is the same linear equation you learned in algebra, probably stated as $y = mx + b$.

In the standard algebraic equation $y = mx + b$, b was the y -intercept, or the y value when x was zero. In the form of the equation we're using, we are using P_0 to represent that initial amount.

In the $y = mx + b$ equation, recall that m was the slope. You might remember this as “rise over run”, or the change in y divided by the change in x . Either way, it represents the same thing as the common difference, d , we are using – the amount the output P_n changes when the input n increases by 1.

The equations $y = mx + b$ and $P_n = P_0 + dn$ mean the same thing and can be used the same ways, we're just writing it somewhat differently.

Example 2.4.1

The population of elk in a national forest was measured to be 12,000 in 2003, and was measured again to be 15,000 in 2007. If the population continues to grow linearly at this rate, what will the elk population be in 2014?

Solution

To begin, we need to define how we're going to measure n . Remember that P_0 is the population when $n = 0$, so we probably don't want to literally use the year 0. Since we already know the population in 2003, let us define $n = 0$ to be the year 2003. Then $P_0 = 12000$

Next we need to find d . Remember d is the growth per time period, in this case growth per year. Between the two measurements, the population grew by $15,000 - 12,000 = 3,000$, but it took $2007 - 2003 = 4$ years to grow that much. To find the growth per year, we can divide: $3000 \text{ elk} / 4 \text{ years} = 750 \text{ elk in 1 year}$.

Alternatively, you can use the slope formula from algebra to determine the common difference, noting that the population is the output of the formula, and time is the input.

$$d = \text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{15,000 - 12,000}{2007 - 2003} = \frac{3000}{4} = 750$$

We can now write our equation in whichever form is preferred.

Recursive form:

$$P_0 = 12,000$$

$$P_n = P_{n-1} + 750$$

Explicit form:

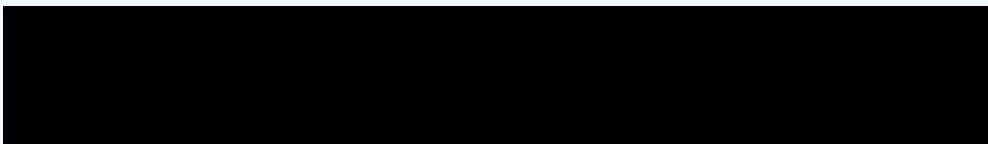
$$P_n = 12,000 + 750n$$

To answer the question, we need to first note that the year 2014 will be $n = 11$, since 2014 is 11 years after 2003. The explicit form will be easier to use for this calculation:

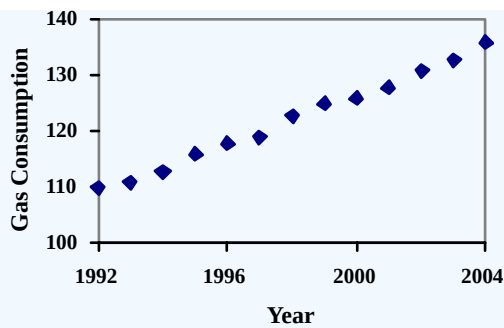
$$P_{11} = 12,000 + 750(11) = 20,250 \text{ elk}$$

Example 2.4.2

Gasoline consumption in the US has been increasing steadily. Consumption data from 1992 to 2004 is shown below[1]. Find a model for this data, and use it to predict consumption in 2016. If the trend continues, when will consumption reach 200 billion gallons?



Solution



Plotting this data, it appears to have an approximately linear relationship:

While there are more advanced statistical techniques that can be used to find an equation to model the data, to get an idea of what is happening, we can find an equation by using two pieces of the data – perhaps the data from 1993 and 2003.

Letting $n = 0$ correspond with 1993 would give $P_0 = 111$ billion gallons.

To find d , we need to know how much the gas consumption increased each year, on average. From 1993 to 2003 the gas consumption increased from 111 billion gallons to 133 billion gallons, a total change of $133 - 111 = 22$ billion gallons, over 10 years. This gives us an average change of $22 \text{ billion gallons} / 10 \text{ year} = 2.2 \text{ billion gallons per year}$.

Equivalently,

$$d = \text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{133-111}{10-0} = \frac{22}{10} = 2.2 \text{ billion gallons per year}$$

We can now write our equation in whichever form is preferred.

Recursive form:

$$P_0 = 111$$

$$P_n = P_{n-1} + 2.2$$

Explicit form:

$$P_n = 111 + 2.2n$$

Calculating values using the explicit form and plotting them with the original data shows how well our model fits the data.

We can now use our model to make predictions about the future, assuming that the previous trend continues unchanged. To predict the gasoline consumption in 2016:

$$n = 23(2016 - 1993 = 23 \text{ years later})$$

$$P_{23} = 111 + 2.2(23) = 161.6$$

Our model predicts that the US will consume 161.6 billion gallons of gasoline in 2016 if the current trend continues.

To find when the consumption will reach 200 billion gallons, we would set $P_n = 200$, and solve for n :

$$\begin{array}{ll} P_n = 200 & \text{Replace } P_n \text{ with our model} \\ 111 + 2.2n = 200 & \text{Subtract 111 from both sides} \\ 2.2n = 89 & \text{Divide both sides by 2.2} \\ n = 40.4545 & \end{array}$$

This tells us that consumption will reach 200 billion about 40 years after 1993, which would be in the year 2033.

Example 2.4.3

The cost, in dollars, of a gym membership for n months can be described by the explicit equation $P_n = 70 + 30n$. What does this equation tell us?

Solution

The value for P_0 in this equation is 70, so the initial starting cost is \$70. This tells us that there must be an initiation or start-up fee of \$70 to join the gym.

The value for d in the equation is 30, so the cost increases by \$30 each month. This tells us that the monthly membership fee for the gym is \$30 a month.

Your Turn 2.4.1

The number of stay-at-home fathers in Canada has been growing steadily[2]. While the trend is not perfectly linear, it is fairly linear. Use the data from 1976 and 2010 to find an explicit formula for the number of stay-at-home fathers, then use it to predict the number if 2020.



[1] www.bts.gov/publications/national/tables/tables_04_10.html

[2] www.fira.ca/article.php?id=140

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2.5: When good models go bad

When using mathematical models to predict future behavior, it is important to keep in mind that very few trends will continue indefinitely.

Example 2.5.1

Suppose a four year old boy is currently 39 inches tall, and you are told to expect him to grow 2.5 inches a year.

Solution

We can set up a growth model, with $n = 0$ corresponding to 4 years old.

Recursive form:

$$P_0 = 39$$

$$P_n = P_{n-1} + 2.5$$

Explicit form:

$$P_n = 39 + 2.5n$$

So at 6 years old, we would expect him to be

$$P_2 = 39 + 2.5(2) = 44 \text{ inches tall}$$

Any mathematical model will break down eventually. Certainly, we shouldn't expect this boy to continue to grow at the same rate all his life. If he did, at age 50 he would be

$$P_{46} = 39 + 2.5(46) = 154 \text{ inches tall} = 12.8 \text{ feet tall!}$$

When using any mathematical model, we have to consider which inputs are reasonable to use. Whenever we extrapolate, or make predictions into the future, we are assuming the model will continue to be valid.

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2.6: Exponential (Geometric) Growth

Learning Objectives

After completing this section, you should be able to:

1. Write recursive and explicit formulas to model exponential growth
2. Use explicit formulas to find future population values of exponential growth

Suppose that every year, only 10% of the fish in a lake have surviving offspring. If there were 100 fish in the lake last year, there would now be 110 fish. If there were 1000 fish in the lake last year, there would now be 1100 fish. Absent any inhibiting factors, populations of people and animals tend to grow by a percent of the existing population each year.

Suppose our lake began with 1000 fish, and 10% of the fish have surviving offspring each year. Since we start with 1000 fish, $P_0 = 1000$. How do we calculate P_1 ? The new population will be the old population, plus an additional 10%. Symbolically:

$$P_1 = P_0 + 0.10P_0$$

Notice this could be condensed to a shorter form by factoring:

$$P_1 = P_0 + 0.10P_0 = 1P_0 + 0.10P_0 = (1 + 0.10)P_0 = 1.10P_0$$

While 10% is the **growth rate**, 1.10 is the **growth multiplier**. Notice that 1.10 can be thought of as “the original 100% plus an additional 10%”

For our fish population,

$$P_1 = 1.10(1000) = 1100$$

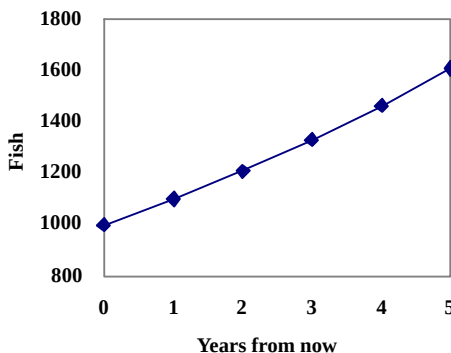
We could then calculate the population in later years:

$$P_2 = 1.10P_1 = 1.10(1100) = 1210$$

$$P_3 = 1.10P_2 = 1.10(1210) = 1331$$

Notice that in the first year, the population grew by 100 fish, in the second year, the population grew by 110 fish, and in the third year the population grew by 121 fish.

While there is a constant percentage growth, the actual increase in number of fish is increasing each year.



Graphing these values we see that this growth doesn't quite appear linear.

To get a better picture of how this percentage-based growth affects things, we need an explicit form, so we can quickly calculate values further out in the future.

Like we did for the linear model, we will start building from the recursive equation:

$$P_1 = 1.10P_0 = 1.10(1000)$$

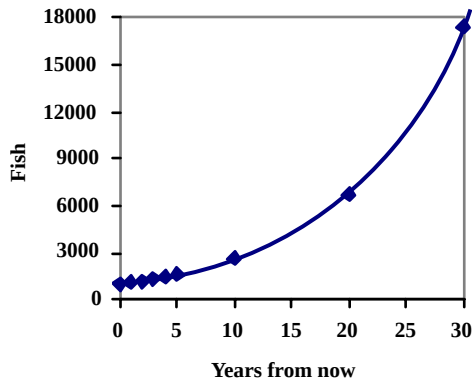
$$P_2 = 1.10P_1 = 1.10(1.10(1000)) = 1.10^2(1000)$$

$$P_3 = 1.10P_2 = 1.10(1.10^2(1000)) = 1.10^3(1000)$$

$$P_4 = 1.10P_3 = 1.10(1.10^3(1000)) = 1.10^4(1000)$$

Observing a pattern, we can generalize the explicit form to be:

$$P_n = 1.10^n(1000), \text{ or equivalently, } P_n = 1000(1.10^n)$$



From this, we can quickly calculate the number of fish in 10, 20, or 30 years:

$$P_{10} = 1.10^{10}(1000) = 2594$$

$$P_{20} = 1.10^{20}(1000) = 6727$$

$$P_{30} = 1.10^{30}(1000) = 17449$$

Adding these values to our graph reveals a shape that is definitely not linear. If our fish population had been growing linearly, by 100 fish each year, the population would have only reached 4000 in 30 years compared to almost 18000 with this percent-based growth, called exponential growth.

In exponential growth, the population grows proportional to the size of the population, so as the population gets larger, the same percent growth will yield a larger numeric growth.

Exponential Growth

If a quantity starts at size P_0 and grows by $R\%$ (written as a decimal, r) every time period, then the quantity after n time periods can be determined using either of these relations:

Recursive form:

$$P_n = (1 + r)P_{n-1}$$

Explicit form:

$$P_n = (1 + r)^n P_0 \text{ or equivalently, } P_n = P_0(1 + r)^n$$

We call r the **growth rate**.

The term $(1 + r)$ is called the **growth multiplier**, or common ratio.

Example 2.6.1

Between 2007 and 2008, Olympia, WA grew almost 3% to a population of 245 thousand people. If this growth rate was to continue, what would the population of Olympia be in 2014?

Solution

As we did before, we first need to define what year will correspond to $n = 0$. Since we know the population in 2008, it would make sense to have 2008 correspond to $n = 0$, so $P_0 = 245,000$. The year 2014 would then be $n = 6$.

We know the growth rate is 3%, giving $r = 0.03$.

Using the explicit form:

$$P_6 = (1 + 0.03)^6(245,000) = 1.19405(245,000) = 292,542.25$$

The model predicts that in 2014, Olympia would have a population of about 293 thousand people.

Evaluating exponents on the calculator

To evaluate expressions like $(1.03)^6$, it will be easier to use a calculator than multiply 1.03 by itself six times. Most scientific calculators have a button for exponents. It is typically either labeled like:

$[^]$, $[y^x]$, or $[x^y]$.

To evaluate 1.03^6 we'd type 1.03 $[^]$ 6, or 1.03 $[y^x]$ 6. Try it out - you should get an answer around 1.1940523.

Example 2.6.2

India is the second most populous country in the world, with a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year. If this trend continues, what will India's population grow to by 2020?

Answer available at the end of this section.

Example 2.6.3

A friend is using the equation $P_n = 4600(1.072)^n$ to predict the annual tuition at a local college. She says the formula is based on years after 2010. What does this equation tell us?

Solution

In the equation, $P_0 = 4600$, which is the starting value of the tuition when $n = 0$. This tells us that the tuition in 2010 was \$4,600.

The growth multiplier is 1.072, so the growth rate is 0.072, or 7.2%. This tells us that the tuition is expected to grow by 7.2% each year.

Putting this together, we could say that the tuition in 2010 was \$4,600, and is expected to grow by 7.2% each year.

Example 2.6.4

In 1990, the residential energy use in the US was responsible for 962 million metric tons of carbon dioxide emissions. By the year 2000, that number had risen to 1182 million metric tons[1]. If the emissions grow exponentially and continue at the same rate, what will the emissions grow to by 2050?

Solution

Similar to before, we will correspond $n = 0$ with 1990, as that is the year for the first piece of data we have. That will make $P_0 = 962$ (million metric tons of CO_2). In this problem, we are not given the growth rate, but instead are given that $P_{10} = 1182$.

When $n = 10$, the explicit equation looks like:

$$P_{10} = (1 + r)^{10} P_0$$

We know the value for P_0 , so we can put that into the equation:

$$P_{10} = (1 + r)^{10} 962$$

We also know that $P_{10} = 1182$, so substituting that in, we get

$$1182 = (1 + r)^{10} 962$$

We can now solve this equation for the growth rate, r . Start by dividing by 962.

$$\frac{1182}{962} = (1 + r)^{10} \quad \text{Next, take the 10th root of both sides}$$

$$\sqrt[10]{\frac{1182}{962}} = 1 + r \quad \text{Next, subtract 1 from both sides}$$

$$r = \sqrt[10]{\frac{1182}{962}} - 1 = 0.0208 = 2.08\%$$

So if the emissions are growing exponentially, they are growing by about 2.08% per year. We can now predict the emissions in 2050 by finding P_{60}

$$P_{60} = (1 + 0.0208)^{60} 962 = 3308.4 \text{ million metric tons of } \text{CO}_2 \text{ in } 2050$$

Rounding

As a note on rounding, notice that if we had rounded the growth rate to 2.1%, our calculation for the emissions in 2050 would have been 3347. Rounding to 2% would have changed our result to 3156. A very small difference in the growth rates gets magnified greatly in exponential growth. For this reason, it is recommended to round the growth rate as little as possible.

If you need to round, **keep at least three significant digits** - numbers after any leading zeros. So 0.4162 could be reasonably rounded to 0.416. A growth rate of 0.001027 could be reasonably rounded to 0.00103.

Evaluating roots on the calculator

In the previous example, we had to calculate the 10th root of a number. This is different than taking the basic square root, $\sqrt{\quad}$. Many scientific calculators have a button for general roots. It is typically labeled like:

$$[\sqrt[n]{\quad}], [\sqrt[n]{\quad}], \text{ or } [\sqrt[n]{\quad}]$$

To evaluate the 3rd root of 8, for example, we'd either type 3 $[\sqrt[n]{\quad}]$ 8, or 8 $[\sqrt[n]{\quad}]$ 3, depending on the calculator. Try it on yours to see which to use - you should get an answer of 2.

If your calculator does not have a general root button, all is not lost. You can instead use the property of exponents which states that $\sqrt[n]{a} = a^{1/n}$. So, to compute the 3rd root of 8, you could use your calculator's exponent key. To do this, type:

8 [y^x] (1 [÷] 3)

The parentheses tell the calculator to divide 1/3 before doing the exponent.

Example 2.6.5

The number of users on a social networking site was 45 thousand in February when they officially went public, and grew to 60 thousand by October. If the site is growing exponentially, and growth continues at the same rate, how many users should they expect two years after they went public?

Answer available at the end of this section.

Example 2.6.6

Looking back at Example 2.6.4, for the sake of comparison, what would the carbon emissions be in 2050 if emissions grow linearly at the same rate?

Solution

Again we will get $n = 0$ correspond with 1990, giving $P_0 = 962$. To find d , we could take the same approach as earlier, noting that the emissions increased by 220 million metric tons in 10 years, giving a common difference of 22 million metric tons each year.

Alternatively, we could use an approach similar to that which we used to find the exponential equation. When $n = 10$, the explicit linear equation looks like:

$$P_{10} = P_0 + 10d$$

We know the value for P_0 , so we can put that into the equation:

$$P_{10} = 962 + 10d$$

Since we know that $P_{10} = 1182$, substituting that in we get

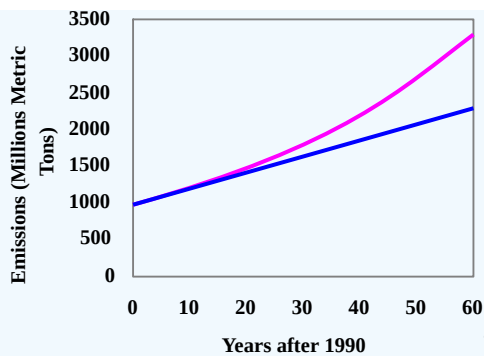
$$1182 = 962 + 10d$$

We can now solve this equation for the common difference, d .

$$1182 - 962 = 10d$$

$$220 = 10d$$

$$d = 22$$



This tells us that if the emissions are changing linearly, they are growing by 22 million metric tons each year. Predicting the emissions in 2050,

$$P_{60} = 962 + 22(60) = 2282 \text{ million metric tons.}$$

You will notice that this number is substantially smaller than the prediction from the exponential growth model. Calculating and plotting more values helps illustrate the differences.

So how do we know which growth model to use when working with data? There are two approaches which should be used together whenever possible:

- 1) Find more than two pieces of data. Plot the values, and look for a trend. Does the data appear to be changing like a line, or do the values appear to be curving upwards?
- 2) Consider the factors contributing to the data. Are they things you would expect to change linearly or exponentially? For example, in the case of carbon emissions, we could expect that, absent other factors, they would be tied closely to population values, which tend to change exponentially.

[1] www.eia.doe.gov/oiaf/1605/ggrpt/carbon.html

Answers

Example 2.6.2

Using $n = 0$ corresponding with 2008,

$$P_{12} = (1 + 0.0134)^{12}(1.14) = \text{about 1.337 billion people in 2020}$$

Example 2.6.5

Here we will measure n in months rather than years, with $n = 0$ corresponding to the February when they went public. This gives $P_0 = 45$ thousand. October is 8 months later, so $P_8 = 60$.

$$P_8 = (1 + r)^8 P_0$$

$$60 = (1 + r)^8 45$$

$$\frac{60}{45} = (1 + r)^8$$

$$\sqrt[8]{\frac{60}{45}} = 1 + r$$

$$r = \sqrt[8]{\frac{60}{45}} - 1 = 0.0366, \text{ or } 3.66\%$$

The general explicit equation is $P_n = (1.0366)^n 45$

Predicting 24 months (2 years) after they went public:

$$P_{24} = (1.0366)^{24} 45 = 106.63 \text{ thousand users.}$$

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3: GEOMETRY

3.1: Systems of Measurement

3.2: Use Properties of Angles, Triangles, and the Pythagorean Theorem

3.3: Use Properties of Rectangles, Triangles, and Trapezoids

3.4: Solve Geometry Applications- Circles and Irregular Figures

3.5: Solve Geometry Applications- Volume and Surface Area

3.1: Systems of Measurement

Learning Objectives

After completing this section, you should be able to:

1. Make unit conversions in the U.S. system
2. Use mixed units of measurement in the U.S. system
3. Make unit conversions in the metric system
4. Use mixed units of measurement in the metric system
5. Convert between the U.S. and the metric systems of measurement
6. Convert between Fahrenheit and Celsius temperatures

In this section we will see how to convert among different types of units, such as feet to miles or kilograms to pounds. The basic idea in all of the unit conversions will be to use a form of 1, the multiplicative identity, to change the units but not the value of a quantity.

Make Unit Conversions in the U.S. System

There are two systems of measurement commonly used around the world. Most countries use the metric system. The United States uses a different system of measurement, usually called the U.S. system. We will look at the U.S. system first.

The U.S. system of measurement uses units of inch, foot, yard, and mile to measure length and pound and ton to measure weight. For capacity, the units used are cup, pint, quart and gallons. Both the U.S. system and the metric system measure time in seconds, minutes, or hours.

The equivalencies among the basic units of the U.S. system of measurement are listed in Table 3.1.1. The table also shows, in parentheses, the common abbreviations for each measurement.

U.S. System Units	
Length	Volume
1 foot (ft) = 12 inches (in) 1 yard (yd) = 3 feet (ft) 1 mile (mi) = 5280 feet (ft)	3 teaspoons (t) = 1 tablespoon (T) 16 Tablespoons (T) = 1 cup (C) 1 cup (C) = 8 fluid ounces (fl oz) 1 pint (pt) = 2 cups (C) 1 quart (qt) = 2 pints (pt) 1 gallon (gal) = 4 quarts (qt)
Weight	Time
1 pound (lb) = 16 ounces (oz) 1 ton = 2000 pounds (lb)	1 minute (min) = 60 seconds (s) 1 hour (h) = 60 minutes (min) 1 day = 24 hours (h) 1 week (wk) = 7 days 1 year (yr) = 365 days

Table 3.1.1

In many real-life applications, we need to convert between units of measurement. We will use the identity property of multiplication to do these conversions. We'll restate the Identity Property of Multiplication here for easy reference.

For any real number a , $a \cdot 1 = a$

To use the identity property of multiplication, we write 1 in a form that will help us convert the units. For example, suppose we

want to convert inches to feet. We know that 1 foot is equal to 12 inches, so we can write 1 as the fraction $\frac{1\text{ft}}{12\text{in}}$. When we multiply by this fraction, we do not change the value but just change the units.

But $\frac{12\text{in}}{1\text{ft}}$ also equals 1. How do we decide whether to multiply by $\frac{1\text{ft}}{12\text{in}}$ or $\frac{12\text{in}}{1\text{ft}}$? We choose the fraction that will make the units we want to convert *from* divide out. For example, suppose we wanted to convert 60 inches to feet. If we choose the fraction that has inches in the denominator, we can eliminate the inches.

$$60\text{in} \cdot \frac{1\text{ft}}{12\text{in}} = 5\text{ft}$$

On the other hand, if we wanted to convert 5 feet to inches, we would choose the fraction that has feet in the denominator.

$$5\text{ft} \cdot \frac{12\text{in}}{1\text{ft}} = 60\text{in}$$

We treat the unit words like factors and ‘divide out’ common units like we do common factors.

How To

Make unit conversions

1. Multiply the measurement to be converted by 1; write 1 as a fraction relating the units given and the units needed.
2. Multiply.
3. Simplify the fraction, performing the indicated operations and removing the common units.

Example 3.1.1

Mary Anne is 66 inches tall. What is her height in feet?

Answer available at the end of this section.

Your Turn 3.1.1

Lexie is 30 inches tall. Convert her height to feet.

When we use the Identity Property of Multiplication to convert units, we need to make sure the units we want to change from will divide out. Usually this means we want the conversion fraction to have those units in the denominator.

Example 3.1.2

Ndula, an elephant at the San Diego Safari Park, weighs almost 3.2 tons. Convert her weight to pounds.



Figure 3.1.1: (credit: Guldo Da Rozze, Flickr)

Answer

	3.2 tons
Multiply the measurement to be converted by 1.	$3.2 \text{ tons} \cdot 1$
Write 1 as a fraction relating tons and pounds.	$3.2 \text{ tons} \cdot \frac{2000 \text{ lbs}}{1 \text{ ton}}$
Simplify.	$\frac{3.2 \cancel{\text{ tons}} \cdot 2000 \text{ lbs}}{1 \cancel{\text{ ton}}}$
Multiply.	6400
	Ndula weighs almost 6,400 pounds.

Your Turn 3.1.2

Arnold's SUV weighs about 4.3 tons. Convert the weight to pounds.

Sometimes to convert from one unit to another, we may need to use several other units in between, so we will need to multiply several fractions.

Example 3.1.3

Juliet is going with her family to their summer home. She will be away for 9 weeks. Convert the time to minutes.

Answer available at the end of this section.

Your Turn 3.1.3

The distance between Earth and the moon is about 250,000 miles. Convert this length to yards.

Example 3.1.4

How many fluid ounces are in 1 gallon of milk?



Figure 3.1.2: (credit: www.bluewaikiki.com, Flickr)

Answer

Use conversion factors to get the right units: convert gallons to quarts, quarts to pints, pints to cups, and cups to fluid ounces.

	1 gallon
Multiply the measurement to be converted by 1.	$\frac{1\text{ gal}}{1} \cdot \frac{4\text{ qt}}{1\text{ gal}} \cdot \frac{2\text{ pt}}{1\text{ qt}} \cdot \frac{2\text{ C}}{1\text{ pt}} \cdot \frac{8\text{ floz}}{1\text{ C}}$
Cancel the units, and multiply.	128
Simplify.	128 fluid ounces
	There are 128 fluid ounces in a gallon.

Your Turn 3.1.4

How many cups are in 1 gallon?

Use Mixed Units of Measurement in the U.S. System

Performing arithmetic operations on measurements with mixed units of measures requires care. Be sure to add or subtract like units.

Example 3.1.5

Charlie bought three steaks for a barbecue. Their weights were 14 ounces, 1 pound 2 ounces, and 1 pound 6 ounces. How many total pounds of steak did he buy?

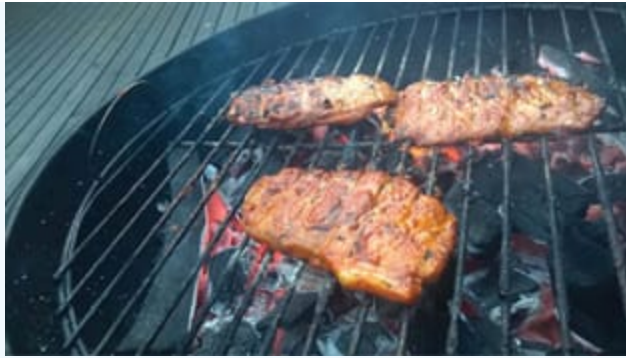


Figure 3.1.3: (credit: Helen Penjam, Flickr)

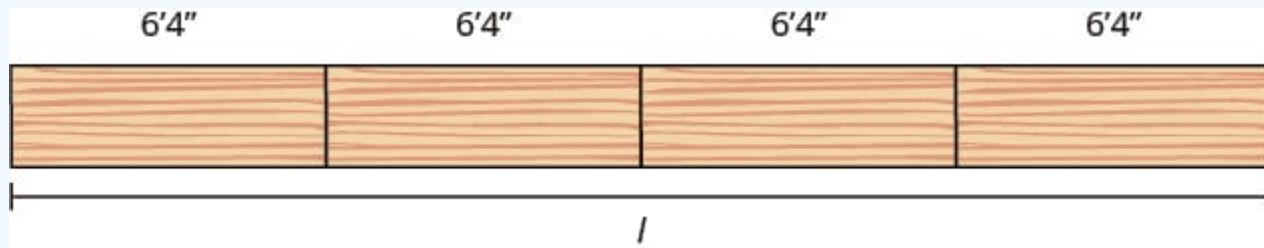
Answer available at the end of this section.

Your Turn 3.1.5

Laura gave birth to triplets weighing 3 pounds 12 ounces, 3 pounds 3 ounces, and 2 pounds 9 ounces. What was the total birth weight of the three babies?

Example 3.1.6

Anthony bought four planks of wood that were each 6 feet 4 inches long. If the four planks are placed end-to-end, what is the total length of the wood?



Answer available at the end of this section.

Your Turn 3.1.6

Henri wants to triple his spaghetti sauce recipe, which calls for 1 pound 8 ounces of ground turkey. How many pounds of ground turkey will he need?

Make Unit Conversions in the Metric System

In the metric system, units are related by powers of 10. The root words of their names reflect this relation. For example, the basic unit for measuring length is a meter. One kilometer is 1000 meters; the prefix *kilo-* means thousand. One centimeter is $\frac{1}{100}$ of a meter, because the prefix *centi-* means one one-hundredth (just like one cent is $\frac{1}{100}$ of one dollar).

The equivalencies of measurements in the metric system are shown in Table 3.1.2. The common abbreviations for each measurement are given in parentheses.

Metric Measurements		
Length	Mass	Volume/Capacity
1 kilometer (km) = 1000 m 1 hectometer (hm) = 100 m 1 dekameter (dam) = 10 m 1 meter (m) = 1 m 1 decimeter (dm) = 0.1 m 1 centimeter (cm) = 0.01 m 1 millimeter (mm) = 0.001 m	1 kilogram (kg) = 1000 g 1 hectogram (hg) = 100 g 1 dekagram (dag) = 10 g 1 gram (g) = 1 g 1 decigram (dg) = 0.1 g 1 centigram (cg) = 0.01 g 1 milligram (mg) = 0.001 g	1 kiloliter (kL) = 1000 L 1 hectoliter (hL) = 100 L 1 dekaliter (daL) = 10 L 1 liter (L) = 1 L 1 deciliter (dL) = 0.1 L 1 centiliter (cL) = 0.01 L 1 milliliter (mL) = 0.001 L
1 meter = 100 centimeters 1 meter = 1000 millimeters	1 gram = 100 centigrams 1 gram = 1000 milligrams	1 liter = 100 centiliters 1 liter = 1000 milliliters

Table 3.1.2

To make conversions in the metric system, we will use the same technique we did in the U.S. system. Using the identity property of multiplication, we will multiply by a conversion factor of one to get to the correct units.

Have you ever run a 5K or 10K race? The lengths of those races are measured in kilometers. The metric system is commonly used in the United States when talking about the length of a race.

Example 3.1.7

Nick ran a 10-kilometer race. How many meters did he run?



Figure 3.1.4: (credit: William Warby, Flickr)

Answer

We will convert kilometers to meters using the Identity Property of Multiplication and the equivalencies in Table 3.1.2.

	10 kilometers
Multiply the measurement to be converted by 1.	$10 \text{ km} \cdot 1$
Write 1 as a fraction relating kilometers and meters.	$10 \text{ km} \cdot \frac{1000 \text{ m}}{1 \text{ km}}$
Simplify.	$\frac{10 \cancel{\text{km}} \cdot 1000 \text{ m}}{1 \cancel{\text{km}}}$
Multiply.	10,000 m
	Nick ran 10,000 meters.

Your Turn 3.1.7

Sandy completed her first 5 km race. How many meters did she run?

Example 3.1.8

Eleanor's newborn baby weighed 3200 grams. How many kilograms did the baby weigh?

Answer available at the end of this section.

Your Turn 3.1.8

Kari's newborn baby weighed 2800 grams. How many kilograms did the baby weigh?

Since the metric system is based on multiples of ten, conversions involve multiplying by multiples of ten. In Decimal Operations, we learned how to simplify these calculations by just moving the decimal.

To multiply by 10, 100, or 1000, we move the decimal to the right 1, 2, places, respectively. To multiply by 0.1, 0.01, or 0.001 we move the decimal to the left 1, 2, places respectively.

We can apply this pattern when we make measurement conversions in the metric system.

In Example 3.1.8, we changed 3200 grams to kilograms by multiplying by $\frac{1}{1000}$ (or 0.001). This is the same as moving the decimal 3 places to the left.

$$3200 \cdot \frac{1}{1000} = 3.2$$


Example 3.1.9

Convert:

1. Ⓐ 350 liters to kiloliters
2. Ⓑ 4.1 liters to milliliters.

Answer available at the end of this section.

Your Turn 3.1.9

Convert: Ⓐ 7.25 L to kL Ⓑ 6.3 L to mL.

Use Mixed Units of Measurement in the Metric System

Performing arithmetic operations on measurements with mixed units of measures in the metric system requires the same care we used in the U.S. system. But it may be easier because of the relation of the units to the powers of 10. We still must make sure to add or subtract like units.

Example 3.1.10

Ryland is 1.6 meters tall. His younger brother is 85 centimeters tall. How much taller is Ryland than his younger brother?

Answer available at the end of this section.

Your Turn 3.1.10

Mariella is 1.58 meters tall. Her daughter is 75 centimeters tall. How much taller is Mariella than her daughter? Write the answer in centimeters.

Example 3.1.11

Dena's recipe for lentil soup calls for 150 milliliters of olive oil. Dena wants to triple the recipe. How many liters of olive oil will she need?

Answer available at the end of this section.

Your Turn 3.1.11

A recipe for Alfredo sauce calls for 250 milliliters of milk. Renata is making pasta with Alfredo sauce for a big party and needs to multiply the recipe amounts by 8. How many liters of milk will she need?

Convert Between U.S. and Metric Systems of Measurement

Many measurements in the United States are made in metric units. A drink may come in 2 liter bottles, calcium may come in 500 mg capsules, and we may run a 5-K race. To work easily in both systems, we need to be able to convert between the two systems.

Table 3.1.3 shows some of the most common conversions.

Conversion Factors Between U.S. and Metric Systems		
Length	Weight	Volume
1 in = 2.54 cm 1 ft = 0.305 m 1 yd = 0.914 m 1 mi = 1.61 km	1 lb = 0.45 kg 1 oz = 28 g	1 qt = 0.95 L 1 fl oz = 30 mL
1 m = 3.28 ft	1 kg = 2.2 lb	1 L = 1.06 qt

Table 3.1.3

We make conversions between the systems just as we do within the systems—by multiplying by unit conversion factors.

Example 3.1.12

Lee's water bottle holds 500 mL of water. How many fluid ounces are in the bottle? Round to the nearest tenth of an ounce.

Answer available at the end of this section.

Your Turn 3.1.12

How many quarts of soda are in a 2 liter bottle?

The conversion factors in Table 3.1.3 are not exact, but the approximations they give are close enough for everyday purposes. In Example 3.1.12, we rounded the number of fluid ounces to the nearest tenth.

Example 3.1.13

Soleil lives in Minnesota but often travels in Canada for work. While driving on a Canadian highway, she passes a sign that says the next rest stop is in 100 kilometers. How many miles until the next rest stop? Round your answer to the nearest mile.

Answer available at the end of this section.

Your Turn 3.1.13

The height of Mount Kilimanjaro is 5,895 meters. Convert the height to feet. Round to the nearest foot.

Convert Between Fahrenheit and Celsius Temperatures

Have you ever been in a foreign country and heard the weather forecast? If the forecast is for 22°C. What does that mean?

The U.S. and metric systems use different scales to measure temperature. The U.S. system uses degrees Fahrenheit, written °F. Figure 3.1.5 shows the relationship between the two systems.

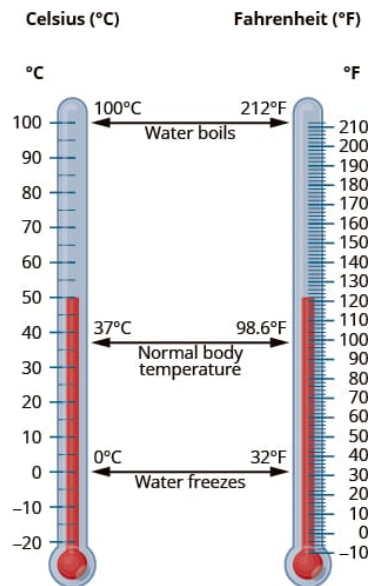


Figure 3.1.5: A temperature of 37°C is equivalent to 98.6°F.

If we know the temperature in one system, we can use a formula to convert it to the other system.

Temperature Conversion

To convert from Fahrenheit temperature, F , to Celsius temperature, C , use the formula

$$C = \frac{5}{9}(F - 32)$$

To convert from Celsius temperature, C , to Fahrenheit temperature, F , use the formula

$$F = \frac{9}{5}C + 32$$

Example 3.1.14

Convert 50° into degrees Celsius.

Answer available at the end of this section.

Your Turn 3.1.14

Convert the Fahrenheit temperatures to degrees Celsius: 59°F.

Example 3.1.15

The weather forecast for Paris predicts a high of 20°C. Convert the temperature into degrees Fahrenheit.

Answer available at the end of this section.

Your Turn 3.1.15

Convert the Celsius temperatures to degrees Fahrenheit:

The temperature in Helsinki, Finland was 15°C.

Media

ACCESS ADDITIONAL ONLINE RESOURCES

- [American Unit Conversion](#)
- [Time Conversions](#)
- [Metric Unit Conversions](#)
- [American and Metric Conversions](#)
- [Convert from Celsius to Fahrenheit](#)
- [Convert from Fahrenheit to Celsius](#)

Answers

Convert 66 inches into feet.	
Multiply the measurement to be converted by 1.	66 inches · 1
Write 1 as a fraction relating the units given and the units needed.	$66\text{in} \cdot \frac{1\text{ft}}{12\text{in}}$
Multiply and simplify the fraction.	$\frac{66\text{ft}}{12} = 5.5\text{ft}$

Notice that the when we simplified the fraction, we first divided out the inches.

Mary Anne is 5.5 feet tall.

To convert weeks into minutes, we will convert weeks to days, days to hours, and then hours to minutes. To do this, we will multiply by conversion factors of 1.

	9 weeks
Write 1 as a fraction.	$\frac{9 \text{ wk}}{1} \cdot \frac{7 \text{ days}}{1 \text{ wk}} \cdot \frac{24 \text{ hr}}{1 \text{ day}} \cdot \frac{60 \text{ min}}{1 \text{ hr}}$
Cancel common units.	$\frac{9 \cancel{\text{wk}}}{1} \cdot \frac{7 \cancel{\text{days}}}{1 \cancel{\text{wk}}} \cdot \frac{24 \cancel{\text{hr}}}{1 \cancel{\text{day}}} \cdot \frac{60 \text{ min}}{1 \cancel{\text{hr}}}$
Multiply.	$\frac{9 \cdot 7 \cdot 24 \cdot 60 \text{ min}}{1 \cdot 1 \cdot 1 \cdot 1} = 90720 \text{ min}$

	Juliet will be away for 90,720 minutes.
--	---

We will add the weights of the steaks to find the total weight of the steaks.

Add the ounces. Then add the pounds.	$ \begin{array}{r} 14 \text{ ounces} \\ 1 \text{ pound} \quad 2 \text{ ounces} \\ + 1 \text{ pound} \quad 6 \text{ ounces} \\ \hline 2 \text{ pounds} \quad 22 \text{ ounces} \end{array} $
Convert 22 ounces to pounds and ounces.	We know one pound is 16 ounces, so 22 ounces is 1 pound 6 ounces.
Add the pounds.	2 pounds + 1 pound, 6 ounces 3 pounds, 6 ounces
	Charlie bought 3 pounds 6 ounces of steak.

We will multiply the length of one plank by 4 to find the total length.

Multiply the inches and then the feet.	$ \begin{array}{r} 6 \text{ feet} \quad 4 \text{ inches} \\ \times \qquad \qquad \qquad 4 \\ \hline 24 \text{ feet} \quad 16 \text{ inches} \end{array} $
Convert 16 inches to feet.	24 feet + 1 foot 4 inches
Add the feet.	25 feet 4 inches
	Anthony bought 25 feet 4 inches of wood.

We will convert grams to kilograms.

	3200 grams
Multiply the measurement to be converted by 1.	$3200 \text{ g} \cdot 1$
Write 1 as a fraction relating kilograms and grams.	$3200 \text{ g} \cdot \frac{1 \text{ kg}}{1000 \text{ g}}$
Simplify.	$3200 \cancel{\text{g}} \cdot \frac{1 \text{ kg}}{1000 \cancel{\text{g}}}$
Multiply.	$ \frac{3200 \text{ kilograms}}{1000} $
Divide.	3.2 kilograms
	The baby weighed 3.2 kilograms.

Ⓐ We will convert liters to kiloliters. In Table 3.1.2, we see that 1 kiloliter = 1000 liters.

	350 L
--	-------

Multiply by 1, writing 1 as a fraction relating liters to kiloliters.	$350 \text{ L} \cdot \frac{1 \text{ kL}}{1000 \text{ L}}$
Simplify.	$350 \cancel{\text{L}} \cdot \frac{1 \text{ kL}}{1000 \cancel{\text{L}}}$
Move the decimal 3 units to the left.	$0.350 \cancel{\text{L}} \cdot \frac{1 \text{ kL}}{1000 \cancel{\text{L}}}$
	0.35 kL

⑤ We will convert liters to milliliters. In Table 7.3, we see that 1 liter = 1000 milliliters.

	4.1 L
Multiply by 1, writing 1 as a fraction relating milliliters to liters.	$4.1 \text{ L} \cdot \frac{1000 \text{ mL}}{1 \text{ L}}$
Simplify.	$4.1 \cancel{\text{L}} \cdot \frac{1000 \text{ mL}}{1 \cancel{\text{L}}}$
Move the decimal 3 units to the left.	4.100 mL
	4100 mL

We will subtract the lengths in meters. Convert 85 centimeters to meters by moving the decimal 2 places to the left; 85 cm is the same as 0.85 m. Now that both measurements are in meters, subtract to find out how much taller Ryland is than his brother.

$$\begin{array}{r} 1.60 \text{ m} \\ -0.85 \text{ m} \\ \hline 0.75 \text{ m} \end{array}$$

Ryland is 0.75 meters taller than his brother.

We will find the amount of olive oil in milliliters then convert to liters.

	Triple 150 mL
Translate to algebra.	$3 \cdot 150 \text{ mL}$
Multiply.	450 mL
Convert to liters.	$450 \text{ mL} \cdot \frac{0.001 \text{ L}}{1 \text{ mL}}$
Simplify.	0.45 L
	Dena needs 0.45 liter of olive oil.

	500 mL
Multiply by a unit conversion factor relating mL and ounces.	$500 \text{ mL} \cdot \frac{1 \text{ fl oz}}{30 \text{ mL}}$
Simplify.	$\frac{500 \text{ fl oz}}{30}$
Divide.	16.7 fl. oz.
	The water bottle holds 16.7 fluid ounces.

	100 kilometers
Multiply by a unit conversion factor relating kilometers and miles.	100 kilometers $\cdot \frac{1 \text{ mile}}{1.61 \text{ kilometers}}$
Simplify.	$\frac{100 \text{ mi}}{1.61}$
Divide.	62 mi
	It is about 62 miles to the next rest stop.

We will substitute 50°F into the formula to find C.

Use the formula for converting °F to °C	$C = \frac{5}{9}(F - 32)$
Substitute 50 for F.	$C = \frac{5}{9}(50 - 32)$
Simplify in parentheses.	$C = \frac{5}{9}(18)$
Multiply.	$C = 10$
	A temperature of 50°F is equivalent to 10°C.

We will substitute 20° into the formula to find F.

Use the formula for converting °F to °C	$F = \frac{9}{5}C + 32$
Substitute 20 for C.	$F = \frac{9}{5}(20) + 32$
Multiply.	$F = 36 + 32$
Add.	$F = 68$
	So 20°C is equivalent to 68°F.

3.2: Use Properties of Angles, Triangles, and the Pythagorean Theorem

Learning Objectives

After completing this section, you should be able to:

1. Use the properties of angles
2. Use the properties of triangles
3. Use the Pythagorean Theorem

In the next few sections, we will apply our problem-solving strategies to some common geometry problems.

Use the Properties of Angles

Are you familiar with the phrase ‘do a 180’?

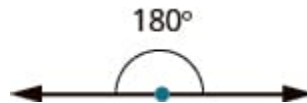


Figure 3.2.1

An angle is formed by two rays that share a common endpoint. Each ray is called a side of the angle and the common endpoint is called the vertex. An angle is named by its vertex. In Figure 3.2.2, $\angle A$ is the angle with vertex at point A . The measure of $\angle A$ is written $m\angle A$.

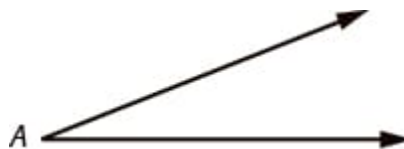


Figure 3.2.2: $\angle A$ is the angle with vertex at point A .

We measure angles in degrees, and use the symbol $^\circ$ to represent degrees. We use the abbreviation m for the *measure* of an angle. So if $\angle A$ is 27° , we would write $m\angle A = 27$.

If the sum of the measures of two angles is 180° , as in Figure 3.2.3, each pair of angles is supplementary because their measures add to 180° . Each angle is the *supplement* of the other.

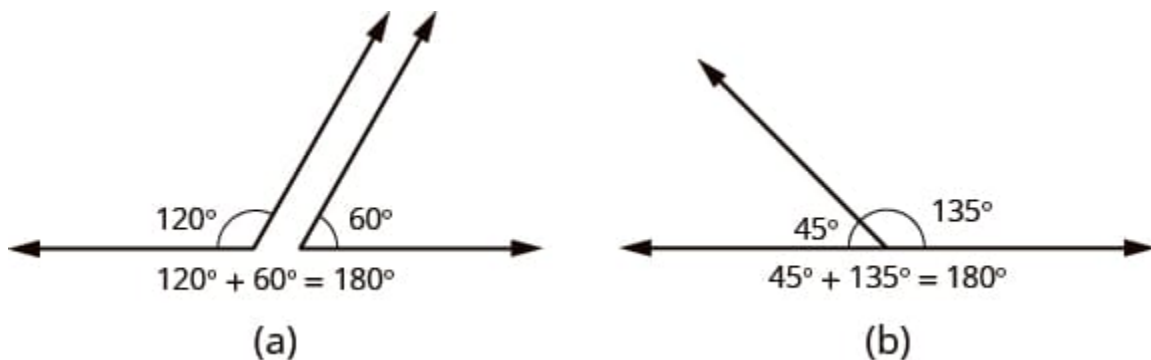


Figure 3.2.3: The sum of the measures of supplementary angles is 180° .

If the sum of the measures of two angles is 90° , as in Figure 3.2.4, each pair of angles is complementary, because their measures add to 90° . Each angle is the *complement* of the other.

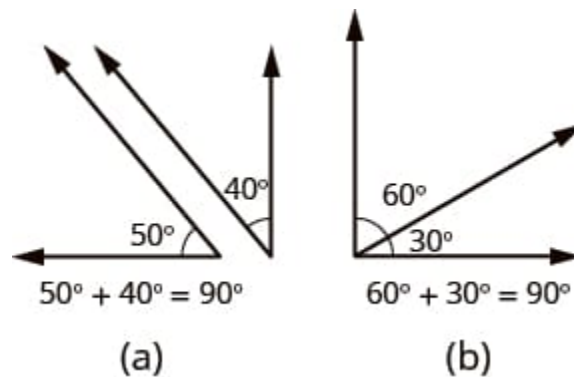


Figure 3.2.4: The sum of the measures of complementary angles is 90° .

Supplementary and Complementary Angles

If the sum of the measures of two angles is 180° , then the angles are supplementary.

If $\angle A$ and $\angle B$ are supplementary, then $m\angle A + m\angle B = 180^\circ$.

If the sum of the measures of two angles is 90° , then the angles are complementary.

If $\angle A$ and $\angle B$ are complementary, then $m\angle A + m\angle B = 90^\circ$.

In the next few sections, you will be introduced to some common geometry formulas. We will use the Problem Solving Strategy for Geometry Applications below. The geometry formula will name the variables and give us the equation to solve. In addition, since these applications will all involve geometric shapes, it will be helpful to draw a figure and then label it with the information from the problem. We will include this step in the Problem Solving Strategy for Geometry Applications.

How To

Use a Problem Solving Strategy for Geometry Applications.

1. **Read** the problem and make sure you understand all the words and ideas. Draw a figure and label it with the given information.
2. **Identify** what you are looking for.
3. **Name** what you are looking for and choose a variable to represent it.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

The next example will show how you can use the Problem Solving Strategy for Geometry Applications to answer questions about supplementary and complementary angles.

Example 3.2.1

An angle measures 40° . Find **a** its supplement, and **b** its complement.

Answer available at the end of this section.

Your Turn 3.2.1

An angle measures 25° . Find its: ④ supplement ⑤ complement.

Did you notice that the words complementary and supplementary are in alphabetical order just like 90 and 180 are in numerical order?

Example 3.2.2

Two angles are supplementary. The larger angle is 30° more than the smaller angle. Find the measure of both angles.

Answer available at the end of this section.

Your Turn 3.2.2

Two angles are supplementary. The larger angle is 100° more than the smaller angle. Find the measures of both angles.

Use the Properties of Triangles

What do you already know about triangles? Triangles have three sides and three angles. Triangles are named by their vertices. The triangle in Figure 3.2.5 is called $\triangle ABC$, read 'triangle ABC'. We label each side with a lower case letter to match the upper case letter of the opposite vertex.

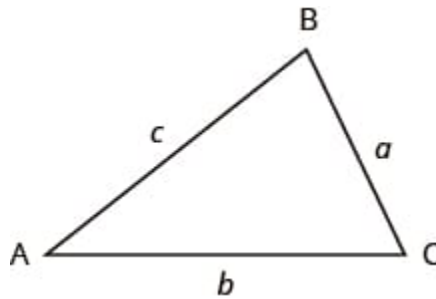


Figure 3.2.5: $\triangle ABC$ has vertices A , B , and C and sides a , b , and c .

The three angles of a triangle are related in a special way. The sum of their measures is 180° .

$$m\angle A + m\angle B + m\angle C = 180^\circ$$

Sum of the Measures of the Angles of a Triangle

For any $\triangle ABC$, the sum of the measures of the angles is 180° .

$$m\angle A + m\angle B + m\angle C = 180^\circ$$

Example 3.2.3

The measures of two angles of a triangle are 55° and 82° . Find the measure of the third angle.

Answer available at the end of this section.

Your Turn 3.2.3

The measures of two angles of a triangle are 31° and 128° . Find the measure of the third angle.

Right Triangles

Some triangles have special names. We will look first at the right triangle. A right triangle has one 90° .

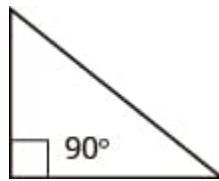


Figure 3.2.6: a right triangle

If we know that a triangle is a right triangle, we know that one angle measures 90° so we only need the measure of one of the other angles in order to determine the measure of the third angle.

Example 3.2.4

One angle of a right triangle measures 28° . What is the measure of the third angle?

Answer available at the end of this section.

Your Turn 3.2.4

One angle of a right triangle measures 56° . What is the measure of the other angle?

In the examples so far, we could draw a figure and label it directly after reading the problem. In the next example, we will have to define one angle in terms of another. So we will wait to draw the figure until we write expressions for all the angles we are looking for.

Example 3.2.5

The measure of one angle of a right triangle is 20° more than the measure of the smallest angle. Find the measures of all three angles.

Answer available at the end of this section.

Your Turn 3.2.5

The measure of one angle of a right triangle is 50° more than the measure of the smallest angle. Find the measures of all three angles.

Similar Triangles

When we use a map to plan a trip, a sketch to build a bookcase, or a pattern to sew a dress, we are working with similar figures. In geometry, if two figures have exactly the same shape but different sizes, we say they are similar figures. One is a scale model of the other. The corresponding sides of the two figures have the same ratio, and all their corresponding angles have the same measures.

The two triangles in Figure 3.2.7 are similar. Each side of $\triangle ABC$ is four times the length of the corresponding side of $\triangle XYZ$ and their corresponding angles have equal measures.

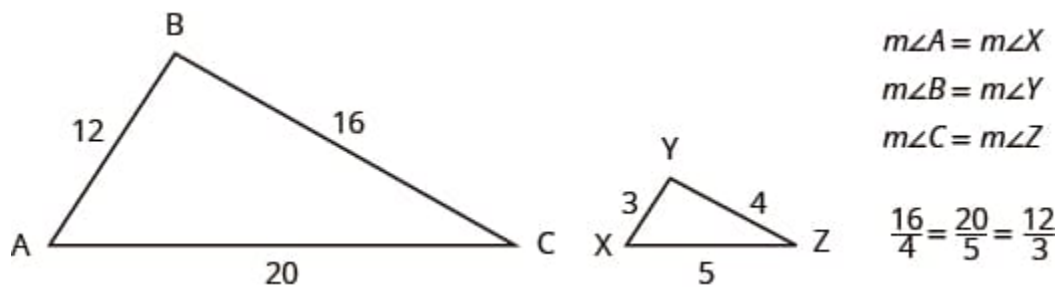
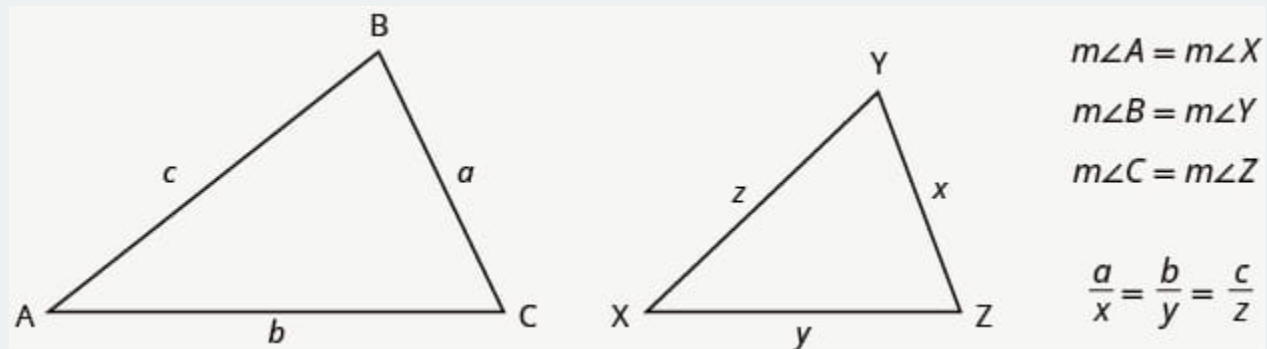


Figure 3.2.7: $\triangle ABC$ and $\triangle XYZ$ are similar triangles. Their corresponding sides have the same ratio and the corresponding angles have the same measure.

Properties of Similar Triangles

If two triangles are similar, then their corresponding angle measures are equal and their corresponding side lengths are in the same ratio.



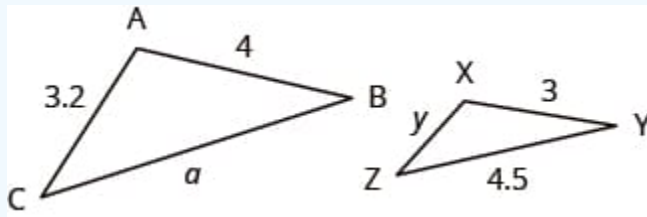
The length of a side of a triangle may be referred to by its endpoints, two vertices of the triangle. For example, in $\triangle ABC$

- The length a can also be written BC
- The length b can also be written AC
- The length c can also be written AB

We will often use this notation when we solve similar triangles because it will help us match up the corresponding side lengths.

Example 3.2.6

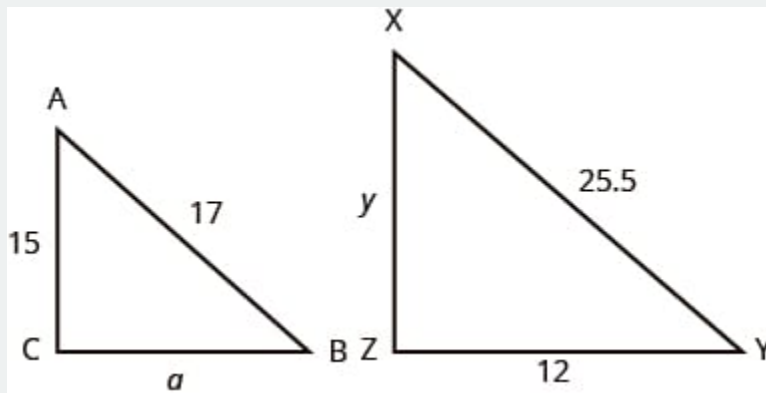
$\triangle ABC$ and $\triangle XYZ$ are similar triangles. The lengths of two sides of each triangle are shown. Find the lengths of the third side of each triangle.



Answer available at the end of this section.

Your Turn 3.2.6

$\triangle ABC$ is similar to $\triangle XYZ$. Find a .



Use the Pythagorean Theorem

The Pythagorean Theorem is a special property of right triangles that has been used since ancient times. It is named after the Greek philosopher and mathematician Pythagoras who lived around 500 BCE. Remember that a right triangle has a 90° angle.

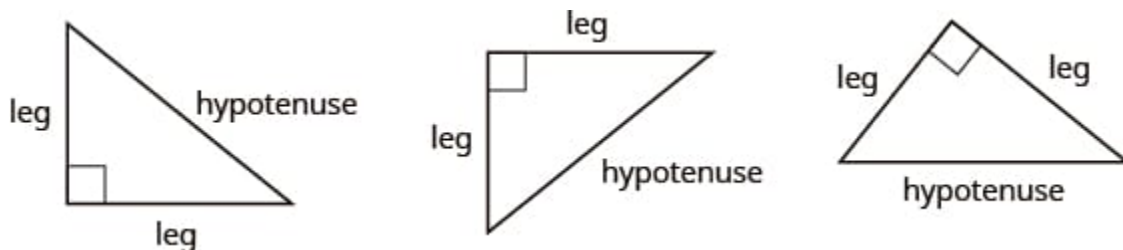


Figure 3.2.8: In a right triangle, the side opposite the 90° angle is called the hypotenuse and each of the other sides is called a leg.

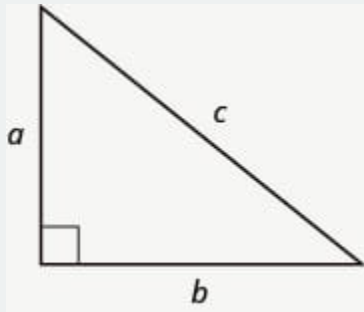
The Pythagorean Theorem tells how the lengths of the three sides of a right triangle relate to each other. It states that in any right triangle, the sum of the squares of the two legs equals the square of the hypotenuse.

The Pythagorean Theorem

In any right triangle $\triangle ABC$,

$$a^2 + b^2 = c^2$$

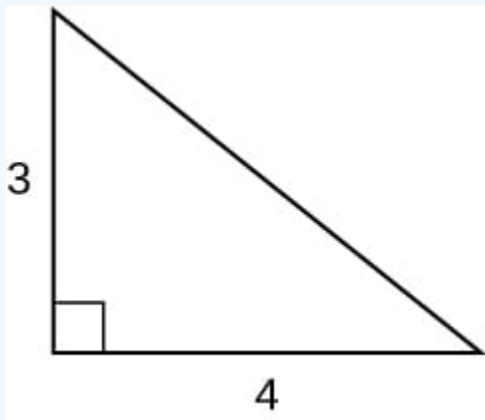
where c is the length of the hypotenuse a and b are the lengths of the legs.



To solve problems that use the Pythagorean Theorem, we will need to find square roots. As a review, $\sqrt{25}$ is 5 because $5^2 = 25$.

Example 3.2.7

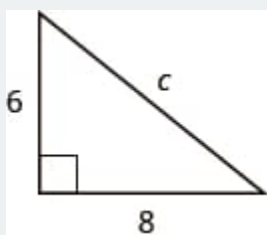
Use the Pythagorean Theorem to find the length of the hypotenuse.



Answer available at the end of this section.

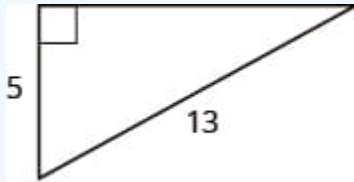
Your Turn 3.2.7

Use the Pythagorean Theorem to find the length of the hypotenuse.



Example 3.2.8

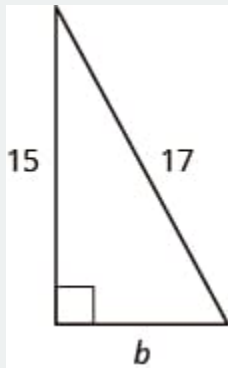
Use the Pythagorean Theorem to find the length of the longer leg.



Answer available at the end of this section.

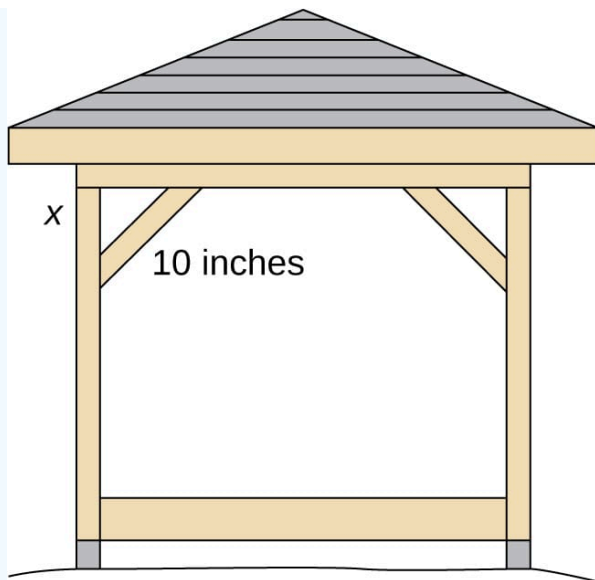
Your Turn 3.2.8

Use the Pythagorean Theorem to find the length of the leg.



Example 3.2.9

Kelvin is building a gazebo and wants to brace each corner by placing a 10 inch wooden bracket diagonally as shown. How far below the corner should he fasten the bracket if he wants the distances from the corner to each end of the bracket to be equal? Approximate to the nearest tenth of an inch.



Answer available at the end of this section.

Your Turn 3.2.9

John puts the base of a 13 foot ladder 5 feet from the wall of his house. How far up the wall does the ladder reach?



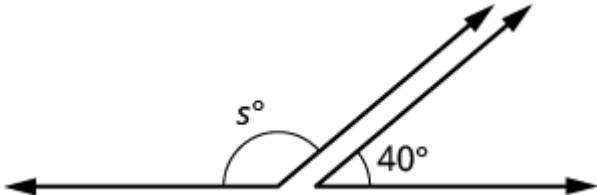
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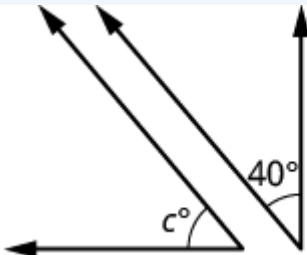
ACCESS ADDITIONAL ONLINE RESOURCES

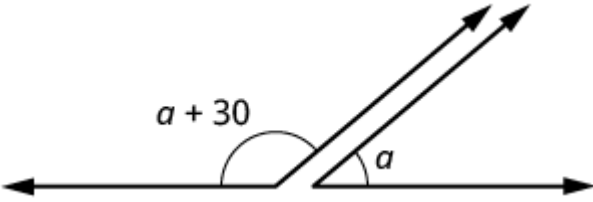
- [Animation: The Sum of the Interior Angles of a Triangle](#)
- [Similar Polygons](#)
- [Example: Determine the Length of the Hypotenuse of a Right Triangle](#)

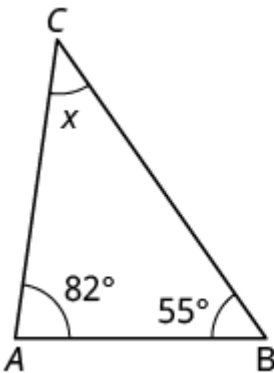
Answers

a	
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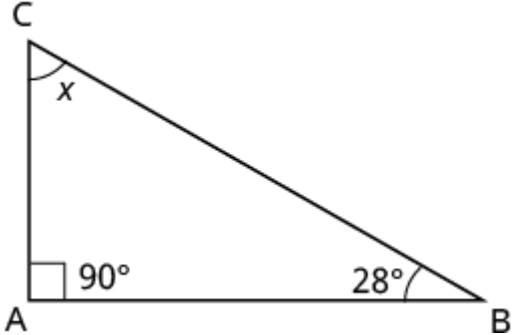
Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the supplement of a 40° angle.
Step 3. Name. Choose a variable to represent it.	let s = the measure of the supplement
Step 4. Translate. Write the appropriate formula for the situation and substitute in the given information.	$m\angle A + m\angle B = 180$ $s + 40 = 180$
Step 5. Solve the equation.	$s = 140$
Step 6. Check.	$140 + 40 \stackrel{?}{=} 180$ $180 = 180 \checkmark$
Step 7. Answer the question.	The supplement of the 40° angle is 140°.

⑥	
Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the complement of a 40° angle.
Step 3. Name. Choose a variable to represent it.	let c = the measure of the complement
Step 4. Translate. Write the appropriate formula for the situation and substitute in the given information.	$m\angle A + m\angle B = 90$
Step 5. Solve the equation.	$c + 40 = 90$ $c = 50$
Step 6. Check:	$50 + 40 \stackrel{?}{=} 90$ $90 = 90 \checkmark$
Step 7. Answer the question.	The complement of the 40° angle is 50°.

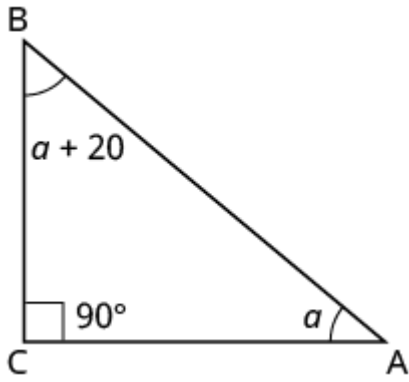
Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the measures of both angles
Step 3. Name. Choose a variable to represent it. The larger angle is 30° more than the smaller angle.	let a = measure of smaller angle $a + 30$ = measure of larger angle
Step 4. Translate. Write the appropriate formula and substitute.	$m\angle A + m\angle B = 180$
Step 5. Solve the equation.	$(a + 30) + a = 180$ $2a + 30 = 180$ $2a = 150$ $a = 75$ measure of smaller angle $a + 30$ measure of larger angle $75 + 30$ 105
Step 6. Check:	$m\angle A + m\angle B = 180$ $75 + 105 \stackrel{?}{=} 180$ $180 = 180 \checkmark$
Step 7. Answer the question.	The measures of the angles are 75° and 105°.

Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the measure of the third angle in a triangle
Step 3. Name. Choose a variable to represent it.	let x = the measure of the angle

Step 4. Translate. Write the appropriate formula and substitute.	$m\angle A + m\angle B + m\angle C = 180$
Step 5. Solve the equation.	$55 + 82 + x = 180$ $137 + x = 180$ $x = 43$
Step 6. Check:	$55 + 82 + 43 \stackrel{?}{=} 180$ $180 = 180 \checkmark$
Step 7. Answer the question.	The measure of the third angle is 43 degrees.

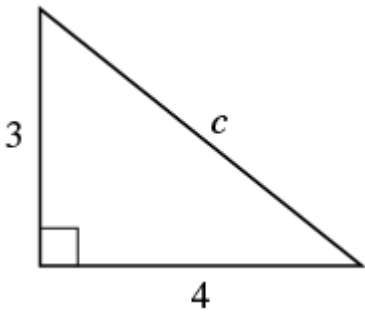
Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the measure of an angle
Step 3. Name. Choose a variable to represent it.	let x = the measure of the angle
Step 4. Translate. Write the appropriate formula and substitute.	$m\angle A + m\angle B + m\angle C = 180$
Step 5. Solve the equation.	$x + 90 + 28 = 180$ $x + 118 = 180$ $x = 62$
Step 6. Check:	$180 \stackrel{?}{=} 90 + 28 + 62$ $180 = 180 \checkmark$
Step 7. Answer the question.	The measure of the third angle is 62° .

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the measures of all three angles

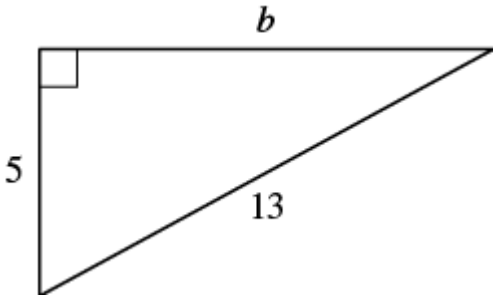
Step 3. Name. Choose a variable to represent it. Now draw the figure and label it with the given information.	Let $a = 1^{\text{st}}$ angle $a + 20 = 2^{\text{nd}}$ angle $90 = 3^{\text{rd}}$ angle (the right angle) 
Step 4. Translate. Write the appropriate formula and substitute into the formula.	$m\angle A + m\angle B + m\angle C = 180$ $a + (a + 20) + 90 = 180$
Step 5. Solve the equation.	$2a + 110 = 180$ $2a = 70$ $a = 35 \text{ first angle}$ $a + 20$ second angle $35 + 20$ 55 90 third angle
Step 6. Check:	$35 + 55 + 90 \stackrel{?}{=} 180$ $180 = 180 \checkmark$
Step 7. Answer the question.	The three angles measure 35°, 55°, and 90°.

Step 1. Read the problem. Draw the figure and label it with the given information.	The figure is provided.
Step 2. Identify what you are looking for.	The length of the sides of similar triangles
Step 3. Name. Choose a variable to represent it.	Let $a =$ length of the third side of $\triangle ABC$ $y =$ length of the third side $\triangle XYZ$

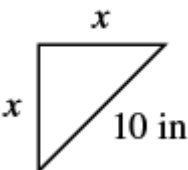
Step 4. Translate.	The triangles are similar, so the corresponding sides are in the same ratio. So $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$ Since the side $AB = 4$ corresponds to the side $XY = 3$, we will use the ratio $\frac{AB}{XY} = \frac{4}{3}$ to find the other sides. Be careful to match up corresponding sides correctly. To find a: sides of large triangle $\longrightarrow \frac{AB}{XY} = \frac{BC}{YZ}$ sides of small triangle $\longrightarrow \frac{4}{3} = \frac{a}{4.5}$ To find y: $\frac{AB}{XY} = \frac{AC}{XZ}$ $\frac{4}{3} = \frac{3.2}{y}$	
Step 5. Solve the equation.	$3a = 4(4.5)$ $3a = 18$ $a = 6$	$4y = 3(3.2)$ $4y = 9.6$ $y = 2.4$
Step 6. Check:	$\frac{4}{3} \stackrel{?}{=} \frac{6}{4.5}$ $4(4.5) \stackrel{?}{=} 6(3)$ $18 = 18 \checkmark$	$\frac{4}{3} \stackrel{?}{=} \frac{3.2}{2.4}$ $4(2.4) \stackrel{?}{=} 3.2(3)$ $9.6 = 9.6 \checkmark$
Step 7. Answer the question.	The third side of $\triangle ABC$ is 6 and the third side of $\triangle XYZ$ is 2.4.	

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length of the hypotenuse of the triangle
Step 3. Name. Choose a variable to represent it.	Let c = the length of the hypotenuse. 
Step 4. Translate. Write the appropriate formula. Substitute.	$a^2 + b^2 = c^2$ $3^2 + 4^2 = c^2$

Step 5. Solve the equation.	$9 + 16 = c^2$ $25 = c^2$ $\sqrt{25} = c^2$ $5 = c$
Step 6. Check:	$3^2 + 4^2 = 5^2$ $9 + 16 \stackrel{?}{=} 25$ $25 = 25 \checkmark$
Step 7. Answer the question.	The length of the hypotenuse is 5.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	The length of the leg of the triangle
Step 3. Name. Choose a variable to represent it.	Let b = the leg of the triangle Label side b 
Step 4. Translate. Write the appropriate formula. Substitute.	$a^2 + b^2 = c^2$ $5^2 + b^2 = 13^2$
Step 5. Solve the equation. Isolate the variable term. Use the definition of the square root. Simplify.	$25 + b^2 = 169$ $b^2 = 144$ $b = \sqrt{144}$ $b = 12$
Step 6. Check:	$5^2 + 12^2 \stackrel{?}{=} 13^2$ $25 + 144 \stackrel{?}{=} 169$ $169 = 169 \checkmark$
Step 7. Answer the question.	The length of the leg is 12.

Step 1. Read the problem.	
----------------------------------	--

Step 2. Identify what you are looking for.	the distance from the corner that the bracket should be attached
Step 3. Name. Choose a variable to represent it.	Let x = the distance from the corner 
Step 4. Translate. Write the appropriate formula. Substitute.	$a^2 + b^2 = c^2$ $x^2 + x^2 = 10^2$
Step 5. Solve the equation. Isolate the variable. Use the definition of the square root. Simplify. Approximate to the nearest tenth.	$2x^2 = 100$ $x^2 = 50$ $\sqrt{x^2} = x = \sqrt{50}$ $b \approx 7.1$
Step 6. Check:	$a^2 + b^2 = c^2$ $(7.1)^2 + (7.1)^2 \stackrel{?}{\approx} 10^2$
Step 7. Answer the question.	Kelvin should fasten each piece of wood approximately 7.1" from the corner.

3.3: Use Properties of Rectangles, Triangles, and Trapezoids

Learning Objectives

After completing this section, you should be able to:

- Understand linear, square, and cubic measure
- Use properties of rectangles
- Use properties of triangles
- Use properties of trapezoids

In this section, we'll continue working with geometry applications. We will add some more properties of triangles, and we'll learn about the properties of rectangles and trapezoids.

Understand Linear, Square, and Cubic Measure

When you measure your height or the length of a garden hose, you use a ruler or tape measure (Figure 3.3.1). A tape measure might remind you of a line—you use it for linear measure, which measures length. Inch, foot, yard, mile, centimeter and meter are units of linear measure.



Figure 3.3.1: This tape measure measures inches along the top and centimeters along the bottom.

When you want to know how much tile is needed to cover a floor, or the size of a wall to be painted, you need to know the area, a measure of the region needed to cover a surface. Area is measured in square units. We often use square inches, square feet, square centimeters, or square miles to measure area. A square centimeter is a square that is one centimeter (cm) on each side. A square inch is a square that is one inch on each side (Figure 3.3.2).

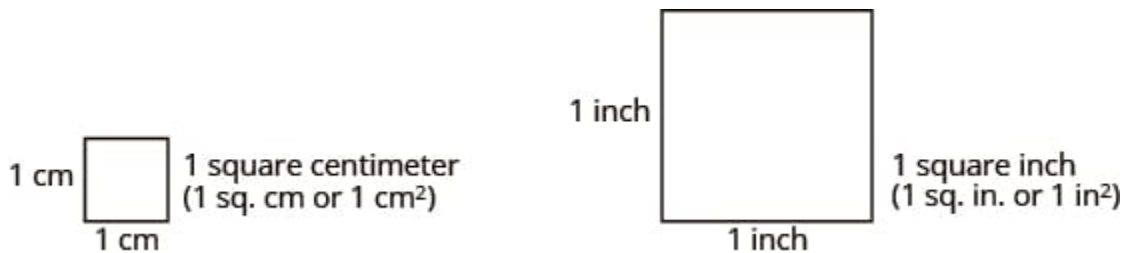


Figure 3.3.2: Square measures have sides that are each 1 unit in length.

Figure 3.3.3 shows a rectangular rug that is 2 feet long by 3 feet wide. Each square is 1 foot wide by 1 foot long, or 1 square foot. The rug is made of 6 squares. The area of the rug is 6 square feet.

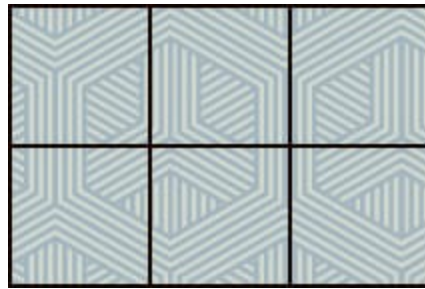


Figure 3.3.3: The rug contains six squares of 1 square foot each, so the total area of the rug is 6 square feet.

When you measure how much it takes to fill a container, such as the amount of gasoline that can fit in a tank, or the amount of medicine in a syringe, you are measuring volume. Volume is measured in cubic units such as cubic inches or cubic centimeters. When measuring the volume of a rectangular solid, you measure how many cubes fill the container. We often use cubic centimeters, cubic inches, and cubic feet. A cubic centimeter is a cube that measures one centimeter on each side, while a cubic inch is a cube that measures one inch on each side (Figure 3.3.4).

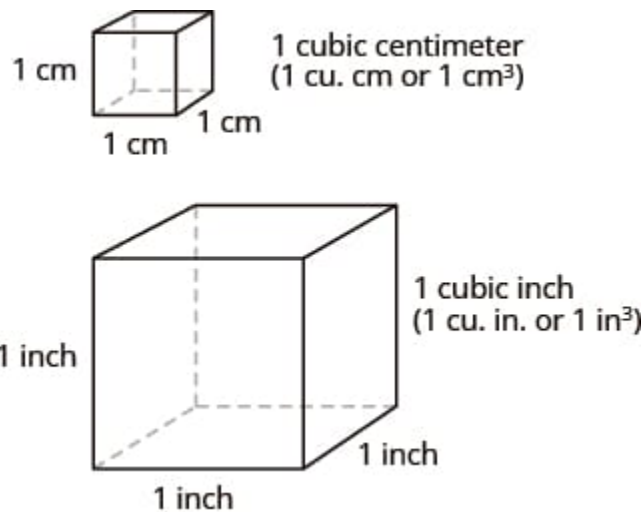


Figure 3.3.4: Cubic measures have sides that are 1 unit in length.

Suppose the cube in Figure 3.3.5 measures 3 inches on each side and is cut on the lines shown. How many little cubes does it contain? If we were to take the big cube apart, we would find 27 little cubes, with each one measuring one inch on all sides. So each little cube has a volume of 1 cubic inch, and the volume of the big cube is 27 cubic inches.

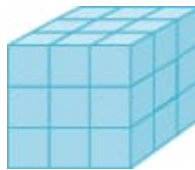


Figure 3.3.5: A cube that measures 3 inches on each side is made up of 27 one-inch cubes, or 27 cubic inches.

Example 3.3.1

For each item, state whether you would use linear, square, or cubic measure:

1. Ⓐ amount of carpeting needed in a room
2. Ⓑ extension cord length
3. Ⓒ amount of sand in a sandbox
4. Ⓓ length of a curtain rod
5. Ⓔ amount of flour in a canister

6. ① size of the roof of a doghouse.

Answer available at the end of this section.

Your Turn 3.3.1

Determine whether you would use linear, square, or cubic measure for each item.

- Ⓐ amount of paint in a can Ⓑ height of a tree Ⓒ floor of your bedroom Ⓓ diameter of bike wheel Ⓔ size of a piece of sod
 ① amount of water in a swimming pool

Many geometry applications will involve finding the perimeter or the area of a figure. There are also many applications of perimeter and area in everyday life, so it is important to make sure you understand what they each mean.

Picture a room that needs new floor tiles. The tiles come in squares that are a foot on each side—one square foot. How many of those squares are needed to cover the floor? This is the area of the floor.

Next, think about putting new baseboard around the room, once the tiles have been laid. To figure out how many strips are needed, you must know the distance around the room. You would use a tape measure to measure the number of feet around the room. This distance is the perimeter.

Perimeter and Area

The perimeter is a measure of the distance around a figure.

The area is a measure of the surface covered by a figure.

Figure 3.3.6 shows a square tile that is 1 inch on each side. If an ant walked around the edge of the tile, it would walk 4 inches. This distance is the perimeter of the tile.

Since the tile is a square that is 1 inch on each side, its area is one square inch. The area of a shape is measured by determining how many square units cover the shape.

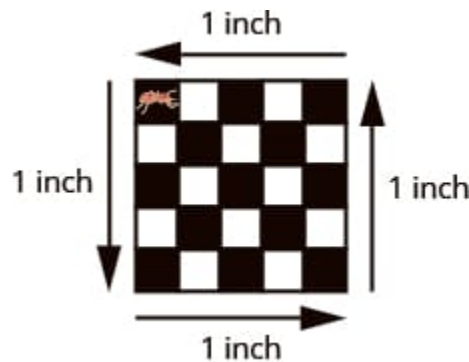
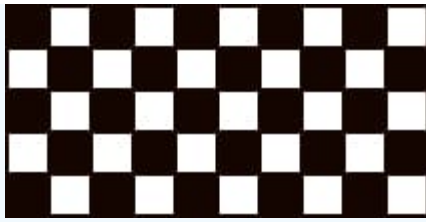


Figure 3.3.6: When the ant walks completely around the tile on its edge, it is tracing the perimeter of the tile. The perimeter is 4 inches. The area of the tile is one square inch.

Example 3.3.2

Each of two square tiles is 1 square inch. Two tiles are shown together.

1. Ⓐ What is the perimeter of the figure?



2. ⑥ What is the area?

Answer available at the end of this section.

Your Turn 3.3.2

Each box in the figure below is 1 square inch. Find the ③ perimeter and ⑥ area of the figure:



Use the Properties of Rectangles

A rectangle has four sides and four right angles. The opposite sides of a rectangle are the same length. We refer to one side of the rectangle as the length, L , and the other side as the width, W .

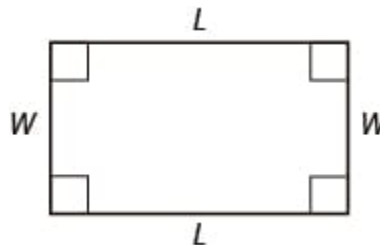


Figure 3.3.7: A rectangle has four sides and four right angles. The sides are labeled L for length and W for width.

The perimeter, P , of the rectangle is the distance around the rectangle. If you started at one corner and walked around the rectangle, you would walk $L + W + L + W$ units, or two lengths and two widths. The perimeter then is

$$P = L + W + L + W$$

$$P = 2L + 2W$$

What about the area of a rectangle? Remember the rectangular rug from the beginning of this section in Figure 3.3.3. It was 6 square feet. Since $A = 2 \cdot 3$ we see that the area equals the length times the width.

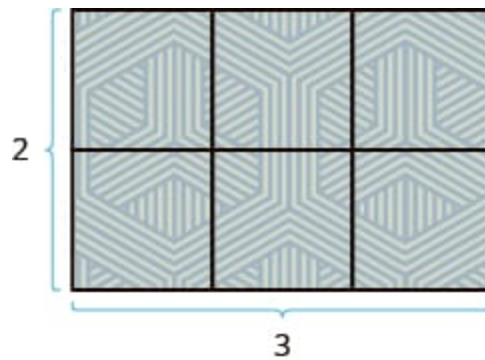


Figure 3.3.3 (again): The area of this rectangular rug is 6 square feet, its length times its width.

Properties of Rectangles

- Rectangles have four sides and four right (90°) angles.
- The lengths of opposite sides are equal.
- The perimeter, P , of a rectangle is found by adding all of the sides.

$$P = 2L + 2W$$

- The area, A , of a rectangle is the length times the width.

$$A = L \cdot W$$

For easy reference as we work the examples in this section, we will restate the Problem Solving Strategy for Geometry Applications here.

How To

Use a Problem Solving Strategy for Geometry Applications

1. **Read** the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.
2. **Identify** what you are looking for.
3. **Name** what you are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Example 3.3.3

The length of a rectangle is 32 meters and the width is 20 meters. Find ① the perimeter, and ② the area.

Answer available at the end of this section.

Your Turn 3.3.3

The length of a rectangle is 120 yards and the width is 50 yards. Find ① the perimeter and ② the area.

Example 3.3.4

Find the length of a rectangle with perimeter 50 inches and width 10 inches.

Answer available at the end of this section.

Your Turn 3.3.4

Find the length of a rectangle with a perimeter of 80 inches and width of 25 inches.

In the next example, the width is defined in terms of the length. We'll wait to draw the figure until we write an expression for the width so that we can label one side with that expression.

Example 3.3.5

The width of a rectangle is two inches less than the length. The perimeter is 52 inches. Find the length and width.

Answer available at the end of this section.

Your Turn 3.3.5

The width of a rectangle is seven meters less than the length. The perimeter is 58 meters. Find the length and width.

Example 3.3.6

The length of a rectangle is four centimeters more than twice the width. The perimeter is 32 centimeters. Find the length and width.

Answer available at the end of this section.

Your Turn 3.3.6

The length of a rectangle is eight more than twice the width. The perimeter is 64 feet. Find the length and width.

Example 3.3.7

The area of a rectangular room is 168 square feet. The length is 14 feet. What is the width?

Answer available at the end of this section.

Your Turn 3.3.7

The area of a rectangle is 598 square feet. The length is 23 feet. What is the width?

Use the Properties of Triangles

We now know how to find the area of a rectangle. We can use this fact to help us visualize the formula for the area of a triangle. In the rectangle in Figure 3.3.8, we've labeled the length b and the width h , so its area is bh .

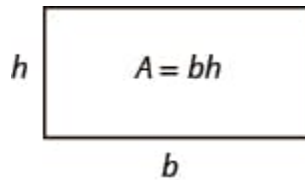


Figure 3.3.8: The area of a rectangle is the base, b , times the height, h .

We can divide this rectangle into two congruent triangles (Figure 3.3.9). Triangles that are congruent have identical side lengths and angles, and so their areas are equal. The area of each triangle is one-half the area of the rectangle, or $\frac{1}{2}bh$. This example helps us see why the formula for the area of a triangle is $A = \frac{1}{2}bh$.

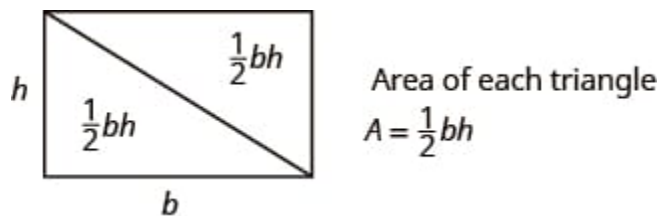


Figure 3.3.9: A rectangle can be divided into two triangles of equal area. The area of each triangle is one-half the area of the rectangle.

The formula for the area of a triangle is $A = \frac{1}{2}bh$, where b is the base and h is the height.

To find the area of the triangle, you need to know its base and height. The base is the length of one side of the triangle, usually the side at the bottom. The height is the length of the line that connects the base to the opposite vertex, and makes a 90° angle. Figure 3.3.10 shows three triangles with the base and height of each marked.

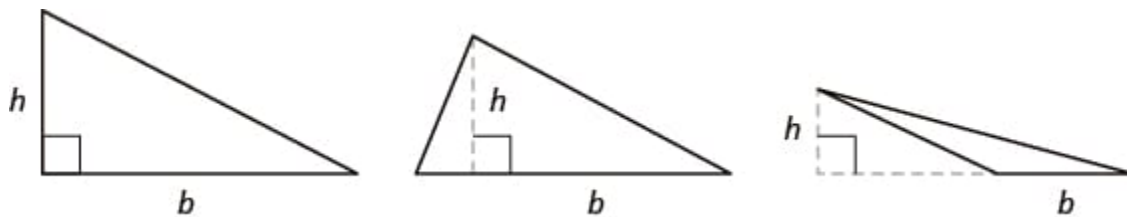


Figure 3.3.10: The height h of a triangle is the length of a line segment that connects the the base to the opposite vertex and makes a 90° angle with the base.

Triangle Properties

For any triangle $\triangle ABC$, the sum of the measures of the angles is 180° .

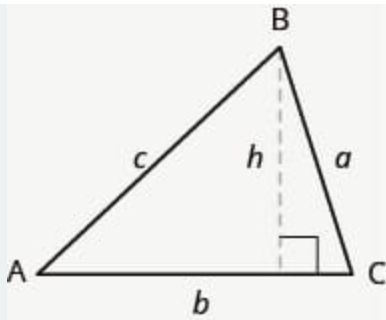
$$m\angle A + m\angle B + m\angle C = 180^\circ$$

The perimeter of a triangle is the sum of the lengths of the sides.

$$P = a + b + c$$

The area of a triangle is one-half the base, b times the height, h

$$A = \frac{1}{2}bh$$



Example 3.3.8

Find the area of a triangle whose base is 11 inches and whose height is 8 inches.

Answer available at the end of this section.

Your Turn 3.3.8

Find the area of a triangle with base 13 inches and height 2 inches.

Example 3.3.9

The perimeter of a triangular garden is 24 feet. The lengths of two sides are 4 feet and 9 feet. How long is the third side?

Answer available at the end of this section.

Your Turn 3.3.9

The perimeter of a triangular garden is 48 feet. The lengths of two sides are 18 feet and 22 feet. How long is the third side?

Example 3.3.10

The area of a triangular church window is 90 square meters. The base of the window is 15 meters. What is the window's height?

Answer available at the end of this section.

Your Turn 3.3.10

The area of a triangular painting is 126 square inches. The base is 18 inches. What is the height?

Isosceles and Equilateral Triangles

Besides the right triangle, some other triangles have special names. A triangle with two sides of equal length is called an isosceles triangle. A triangle that has three sides of equal length is called an equilateral triangle. Figure 3.3.11 shows both types of triangles.

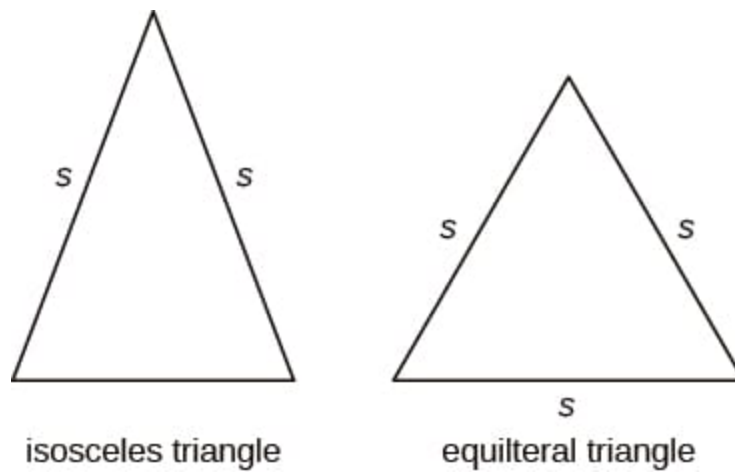


Figure 3.3.11: In an isosceles triangle, two sides have the same length, and the third side is the base. In an equilateral triangle, all three sides have the same length.

Isosceles and Equilateral Triangles

An **isosceles** triangle has two sides the same length.

An **equilateral** triangle has three sides of equal length.

Example 3.3.11

The perimeter of an equilateral triangle is 93 inches. Find the length of each side.

Answer available at the end of this section.

Your Turn 3.3.11

Find the length of each side of an equilateral triangle with perimeter 39 inches.

Example 3.3.12

Arianna has 156 inches of beading to use as trim around a scarf. The scarf will be an isosceles triangle with a base of 60 inches. How long can she make the two equal sides?

Answer available at the end of this section.

Your Turn 3.3.12

A backyard deck is in the shape of an isosceles triangle with a base of 20 feet. The perimeter of the deck is 48 feet. How long is each of the equal sides of the deck?

Use the Properties of Trapezoids

A trapezoid is a four-sided figure, a *quadrilateral*, with two sides that are parallel and two sides that are not. The parallel sides are called the bases. We call the length of the smaller base b , and the length of the bigger base B . The height, h , of a trapezoid is the distance between the two bases as shown in Figure 3.3.12.

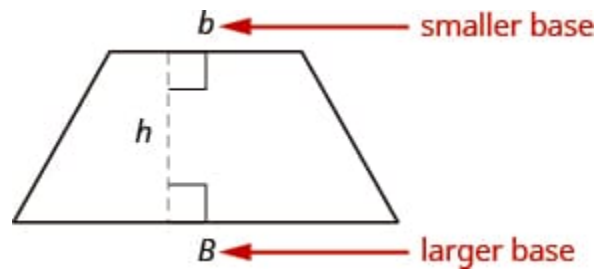


Figure 3.3.12: A trapezoid has a larger base, B , and a smaller base, b . The height h is the distance between the bases.

The formula for the area of a trapezoid is:

$$\text{Area of trapezoid} = \frac{1}{2}h(b + B)$$

Splitting the trapezoid into two triangles may help us understand the formula. The area of the trapezoid is the sum of the areas of the two triangles. See Figure 3.3.13.

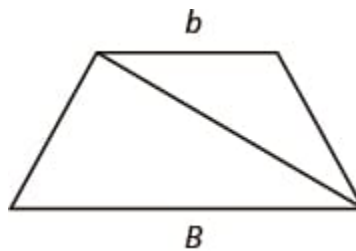


Figure 3.3.13: Splitting a trapezoid into two triangles may help you understand the formula for its area.

The formula for the area of a trapezoid is

$$\text{Area}_{\text{trapezoid}} = \frac{1}{2}h(b + B)$$

If we distribute, we get,

$$\text{Area}_{\text{trapezoid}} = \frac{1}{2}bh + \frac{1}{2}Bh$$

$$\text{Area}_{\text{trapezoid}} = A_{\text{blue}\Delta} + A_{\text{red}\Delta}$$

Properties of Trapezoids

- A trapezoid has four sides.
- Two of its sides are parallel and two sides are not.
- The area, A , of a trapezoid is $A = \frac{1}{2}h(b + B)$.

Example 3.3.13

Find the area of a trapezoid whose height is 6 inches and whose bases are 14 and 11 inches.

Answer available at the end of this section.

Your Turn 3.3.13

The height of a trapezoid is 14 yards and the bases are 7 and 16 yards. What is the area?

Example 3.3.14

Find the area of a trapezoid whose height is 5 feet and whose bases are 10.3 and 13.7 feet.

Answer available at the end of this section.

Your Turn 3.3.14

The height of a trapezoid is 7 centimeters and the bases are 4.6 and 7.4 centimeters. What is the area?

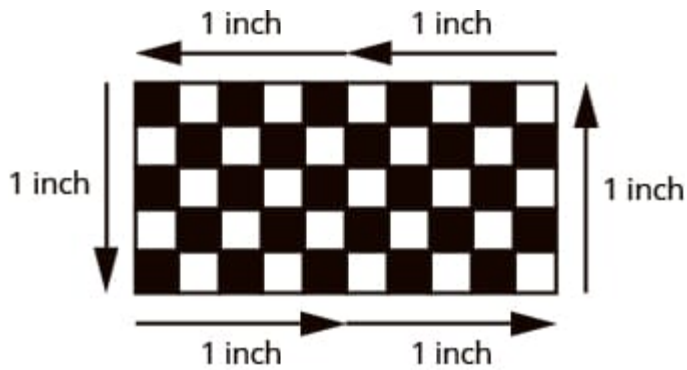
Media**ACCESS ADDITIONAL ONLINE RESOURCES**

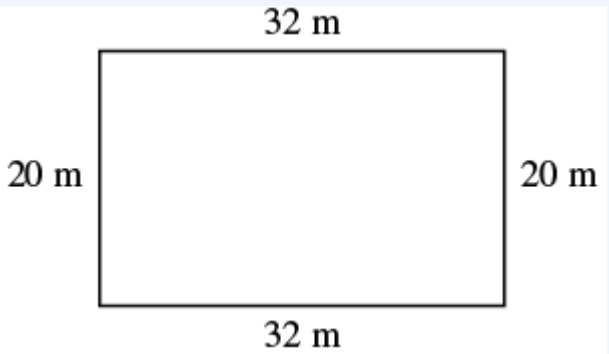
- [Perimeter of a Rectangle](#)
- [Area of a Rectangle](#)
- [Perimeter and Area Formulas](#)
- [Area of a Triangle](#)
- [Area of a Triangle with Fractions](#)
- [Area of a Trapezoid](#)

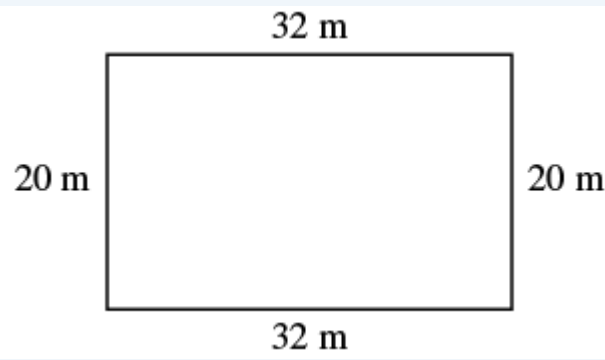
Answers

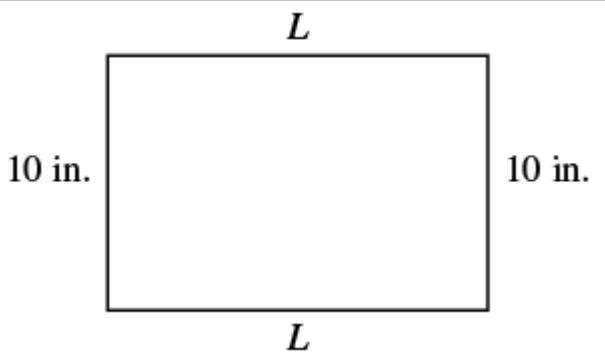
Ⓐ You are measuring how much surface the carpet covers, which is the area.	square measure
Ⓑ You are measuring how long the extension cord is, which is the length.	linear measure
Ⓒ You are measuring the volume of the sand.	cubic measure
Ⓓ You are measuring the length of the curtain rod.	linear measure
Ⓔ You are measuring the volume of the flour.	cubic measure
Ⓕ You are measuring the area of the roof.	square measure

- Ⓐ The perimeter is the distance around the figure. The perimeter is 6 inches.
Ⓑ The area is the surface covered by the figure. There are 2 square inch tiles so the area is 2 square inches.



Ⓐ	
Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the perimeter of a rectangle
Step 3. Name. Choose a variable to represent it.	Let P = the perimeter
Step 4. Translate. Write the appropriate formula. Substitute.	$\begin{array}{ccccccc} P & = & 2L & + & 2W \\ \underbrace{P} & = & \underbrace{2L} & + & \underbrace{2W} \\ P & = & 2(32) & + & 2(20) \end{array}$
Step 5. Solve the equation.	$\begin{array}{l} P = 64 + 40 \\ P = 104 \end{array}$
Step 6. Check:	$\begin{array}{l} P \stackrel{?}{=} 104 \\ 20 + 32 + 20 + 32 \stackrel{?}{=} 104 \\ 104 = 104 \checkmark \end{array}$
Step 7. Answer the question.	The perimeter of the rectangle is 104 meters.
Ⓑ	

Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the area of a rectangle
Step 3. Name. Choose a variable to represent it.	Let A = the area
Step 4. Translate. Write the appropriate formula. Substitute.	$\underbrace{A}_{A} = \underbrace{L}_{32 \text{ m}} \cdot \underbrace{W}_{20 \text{ m}}$
Step 5. Solve the equation.	$A = 640$
Step 6. Check:	$A \stackrel{?}{=} 640$ $32 \cdot 20 \stackrel{?}{=} 640$ $640 = 640 \checkmark$
Step 7. Answer the question.	The area of the rectangle is 640 square meters.

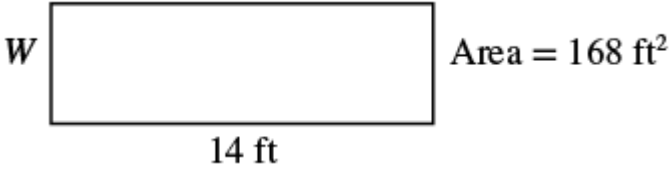
Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the length of the rectangle
Step 3. Name. Choose a variable to represent it.	Let L = the length
Step 4. Translate. Write the appropriate formula. Substitute.	$\underbrace{P}_{50} = \underbrace{2L}_{2L} + \underbrace{2W}_{2(10)}$

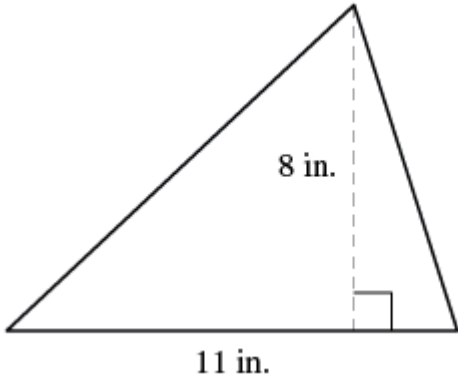
Step 5. Solve the equation.	$50 - 20 = 2L + 20 - 20$ $30 = 2L$ $\frac{30}{2} = \frac{2L}{2}$ $15 = L$
Step 6. Check:	$A \stackrel{?}{=} 640$ $32 \cdot 20 \stackrel{?}{=} 640$ $640 = 640 \checkmark$
Step 7. Answer the question.	The length is 15 inches.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length and width of the rectangle
Step 3. Name. Choose a variable to represent it. Now we can draw a figure using these expressions for the length and width.	Since the width is defined in terms of the length, we let L = length. The width is two feet less than the length, so we let $L - 2$ = width L  L The diagram shows a rectangle. The top side is labeled L, the bottom side is labeled L, the left side is labeled $L - 2$, and the right side is labeled $L - 2$.
Step 4. Translate. Write the appropriate formula. The formula for the perimeter of a rectangle relates all the information. Substitute in the given information.	$\underbrace{P}_{52} = \underbrace{2L}_{2L} + \underbrace{2W}_{2(L-2)}$
Step 5. Solve the equation.	$52 = 2L + 2L - 4$
Combine like terms.	$52 = 4L - 4$
Add 4 to each side.	$56 = 4L$
Divide by 4.	$\frac{56}{4} = \frac{4L}{4}$
	$14 = L$
	The length is 14 inches.
Now we need to find the width.	

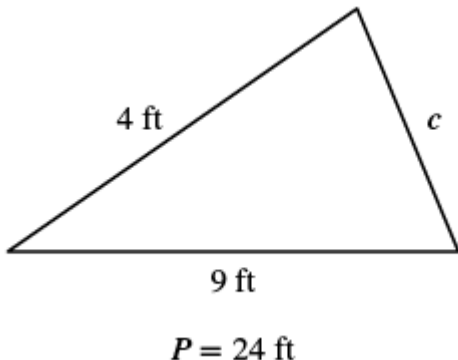
The width is $L - 2$.	$L - 2$ $14 - 2$ 12 The width is 12 inches.
Step 6. Check:	Since $14 + 12 + 14 + 12 = 52$, this works!
Step 7. Answer the question.	The length is 14 feet and the width is 12 feet.

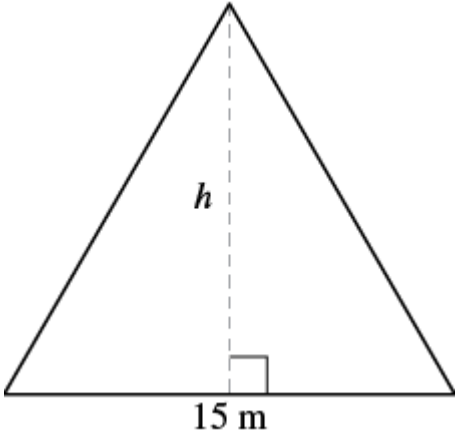
Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length and width
Step 3. Name. Choose a variable to represent it.	let W = width The length is four more than twice the width. $2w + 4$ = length $2w + 4$  $2w + 4$
Step 4. Translate. Write the appropriate formula and substitute in the given information.	$\underbrace{P}_{32} = \underbrace{\quad}_{= 2(2w + 4)} + \underbrace{\quad}_{+ 2w}$
Step 5. Solve the equation.	$32 = 4w + 8 + 2w$ $32 = 6w + 8$ $24 = 6w$ $4 = w$ width $2w + 4$ length $2(4) + 4$ 12 The length is 12 cm.
Step 6. Check:	$p = 2L + 2W$ $32 \stackrel{?}{=} 2 \cdot 12 + 2 \cdot 4$ $32 = 32$
Step 7. Answer the question.	The length is 12 cm and the width is 4 cm.

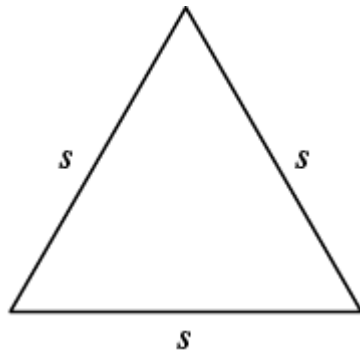
Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the width of a rectangular room
Step 3. Name. Choose a variable to represent it.	Let W = width
Step 4. Translate. Write the appropriate formula and substitute in the given information.	$A = LW$ $168 = 14W$
Step 5. Solve the equation.	$\frac{168}{14} = \frac{14W}{14}$ $12 = W$
Step 6. Check:	$A = LW$ $168 \stackrel{?}{=} 14 \cdot 12$ $168 = 168 \checkmark$
Step 7. Answer the question.	The width of the room is 12 feet.

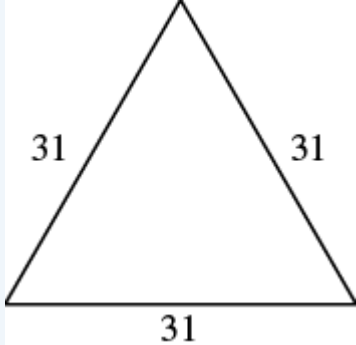
Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the area of the triangle
Step 3. Name. Choose a variable to represent it.	let A = area of the triangle
Step 4. Translate. Write the appropriate formula. Substitute.	$A = \frac{1}{2} \cdot b \cdot h$ $A = \frac{1}{2} \cdot 11 \cdot 8$
Step 5. Solve the equation.	$A = 44$ square inches

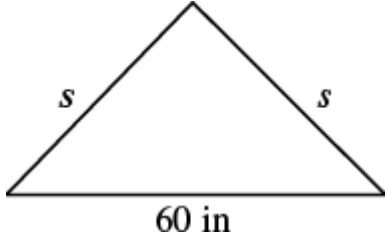
Step 6. Check:	$A = \frac{1}{2}bh$ $44 \stackrel{?}{=} \frac{1}{2}(11)8$ $44 = 44 \checkmark$
Step 7. Answer the question.	The area is 44 square inches.

Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	length of the third side of a triangle
Step 3. Name. Choose a variable to represent it.	Let c = the third side
Step 4. Translate. Write the appropriate formula. Substitute in the given information.	$\underbrace{P}_{24} = \underbrace{a}_4 + \underbrace{b}_9 + \underbrace{c}_c$
Step 5. Solve the equation.	$24 = 13 + c$ $11 = c$
Step 6. Check:	$P = a + b + c$ $24 \stackrel{?}{=} 4 + 9 + 11$ $24 = 24 \checkmark$
Step 7. Answer the question.	The third side is 11 feet long.

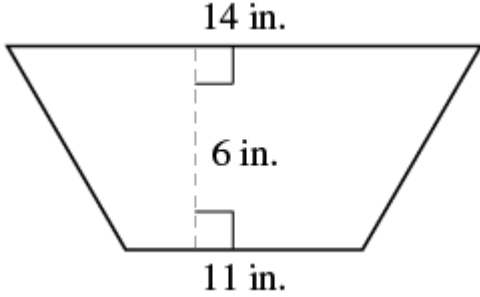
Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	height of a triangle
Step 3. Name. Choose a variable to represent it.	Let h = the height
Step 4. Translate. Write the appropriate formula. Substitute in the given information.	$\underbrace{A}_{90} = \underbrace{\frac{1}{2}} \cdot \underbrace{b}_{15} \cdot \underbrace{h}_h$
Step 5. Solve the equation.	$90 = \frac{15}{2}h$ $12 = h$
Step 6. Check:	$A = \frac{1}{2}bh$ $90 \stackrel{?}{=} \frac{1}{2} \cdot 15 \cdot 12$ $90 = 90 \checkmark$
Step 7. Answer the question.	The height of the triangle is 12 meters.

Step 1. Read the problem. Draw the figure and label it with the given information.	 Perimeter = 93 in.
Step 2. Identify what you are looking for.	length of the sides of an equilateral triangle

Step 3. Name. Choose a variable to represent it.	Let s = length of each side
Step 4. Translate. Write the appropriate formula. Substitute.	$\underbrace{P}_{93} = \underbrace{a}_s + \underbrace{b}_s + \underbrace{c}_s$
Step 5. Solve the equation.	$93 = 3s$ $31 = s$
Step 6. Check:	 $93 \stackrel{?}{=} 31 + 31 + 31$ $93 = 93 \checkmark$
Step 7. Answer the question.	Each side is 31 inches.

Step 1. Read the problem. Draw the figure and label it with the given information.	 $P = 156$ in.
Step 2. Identify what you are looking for.	the lengths of the two equal sides
Step 3. Name. Choose a variable to represent it.	Let s = the length of each side
Step 4. Translate. Write the appropriate formula. Substitute in the given information.	$\underbrace{P}_{156} = \underbrace{a}_s + \underbrace{b}_{60} + \underbrace{c}_s$
Step 5. Solve the equation.	$156 = 2s + 60$ $96 = 2s$ $48 = s$

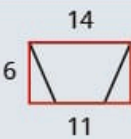
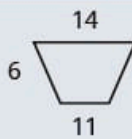
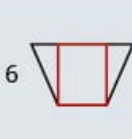
Step 6. Check:	$p = a + b + c$ $156 \stackrel{?}{=} 48 + 60 + 48$ $156 = 156 \checkmark$
Step 7. Answer the question.	Arianna can make each of the two equal sides 48 inches long.

Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the area of the trapezoid
Step 3. Name. Choose a variable to represent it.	Let A = the area of the trapezoid.
Step 4. Translate. Write the appropriate formula. Substitute.	$A = \frac{1}{2} \cdot h \cdot (b + B)$ $A = \frac{1}{2} \cdot 6 \cdot (11 + 14)$
Step 5. Solve the equation.	$A = \frac{1}{2} \cdot 6(25)$ $A = 3(25)$ $A = 75 \text{ square inches}$

Step 6.

Check: Is this answer reasonable?

If we draw a rectangle around the trapezoid that has the same big base B and a height h , its area should be greater than that of the trapezoid. If we draw a rectangle inside the trapezoid that has the same little base b and a height h , its area should be smaller than that of the trapezoid.

		
$A_{\text{rectangle}} = bh$	$A_{\text{trapezoid}} = \frac{1}{2}h(b + B)$	$A_{\text{rectangle}} = bh$
$A_{\text{rectangle}} = 14 \cdot 6$	$A_{\text{trapezoid}} = \frac{1}{2} \cdot 6(11 + 14)$	$A_{\text{rectangle}} = 11 \cdot 6$
$A_{\text{rectangle}} = 84 \text{ sq. in.}$	$A_{\text{trapezoid}} = 75 \text{ sq. in.}$	$A_{\text{rectangle}} = 66 \text{ sq. in.}$

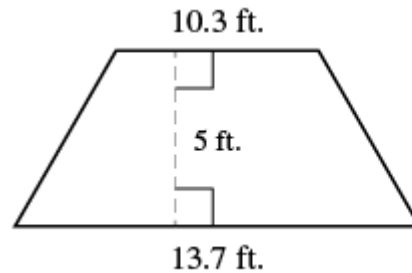
The area of the larger rectangle is 84 square inches and the area of the smaller rectangle is 66 square inches. So it makes sense that the area of the trapezoid is between 84 and 66 square inches.

Step 7.

Answer the question.

The area of the trapezoid is 75 square inches.

Step 1. **Read** the problem. Draw the figure and label it with the given information.



Step 2. **Identify** what you are looking for.

the area of the trapezoid

Step 3. **Name.** Choose a variable to represent it.

Let A = the area

Step 4. **Translate.** Write the appropriate formula. Substitute.

$$\underbrace{A}_{\text{Area}} = \underbrace{\frac{1}{2}}_{\text{Fraction}} \cdot \underbrace{h}_{\text{Height}} \cdot \underbrace{(b + B)}_{\text{Sum of bases}}$$

$$A = \frac{1}{2} \cdot 5 \cdot (10.3 + 13.7)$$

Step 5. **Solve** the equation.

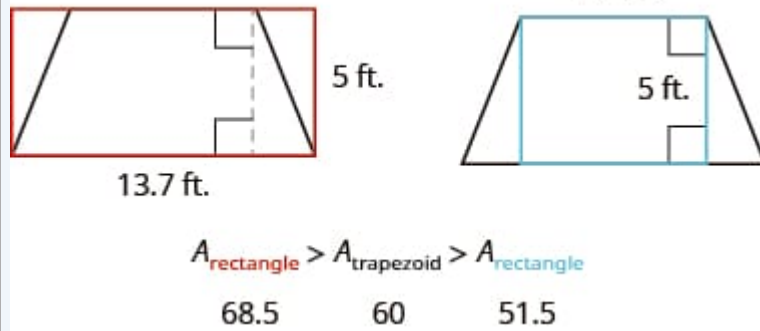
$$A = \frac{1}{2} \cdot 5(24)$$

$$A = 12 \cdot 5$$

$$A = 60 \text{ square feet}$$

Step 6. **Check:** Is this answer reasonable?

The area of the trapezoid should be less than the area of a rectangle with base 13.7 and height 5, but more than the area of a rectangle with base 10.3 and height 5.



Step 7. **Answer** the question.

The area of the trapezoid is 60 square feet.

3.4: Solve Geometry Applications- Circles and Irregular Figures

Learning Objectives

After completing this section, you should be able to:

1. Use the properties of circles
2. Find the area of irregular figures

In this section, we'll continue working with geometry applications. We will add several new formulas to our collection of formulas. To help you as you do the examples and exercises in this section, we will show the Problem Solving Strategy for Geometry Applications here.

How To

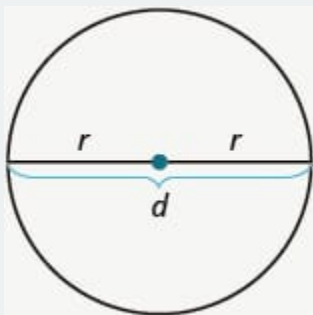
Problem Solving Strategy for Geometry Applications

1. **Read** the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.
2. **Identify** what you are looking for.
3. **Name** what you are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Use the Properties of Circles

Circles are defined as the set of points equidistant from a center point.

Properties of Circles



- r is the length of the radius
- d is the length of the diameter
- $d = 2r$
- Circumference is the perimeter of a circle. The formula for circumference is

$$C = 2\pi r$$

- The formula for area of a circle is

$$A = \pi r^2$$

Remember, that we will use the π button in our calculators to get an accurate answer to each problem. Make sure to read and follow rounding directions carefully.

Example 3.4.1

A circular sandbox has a radius of 2.5 feet. Find the ③ circumference and ④ area of the sandbox. Round both answers to the nearest tenth.

Answer available at the end of this section.

Your Turn 3.4.1

A circular mirror has radius of 5 inches. Find the ③ circumference and ④ area of the mirror.

We usually see the formula for circumference in terms of the radius r of the circle:

$$C = 2\pi r$$

But since the diameter of a circle is two times the radius, we could write the formula for the circumference in terms of d .

	$C = 2\pi r$
Using the commutative property, we can rewrite the formula.	$C = \pi 2r$
Substituting d for $2r$	$C = \pi d$

We will use this form of the circumference when we're given the length of the diameter instead of the radius.

Example 3.4.2

A circular table has a diameter of four feet. What is the circumference of the table? Round your answer to the nearest hundredth.

Answer available at the end of this section.

Your Turn 3.4.2

Find the circumference of a circular fire pit whose diameter is 5.5 feet.

Example 3.4.3

Find the diameter of a circle with a circumference of 47.1 centimeters. Round to the nearest whole number.

Answer available at the end of this section.

Your Turn 3.4.3

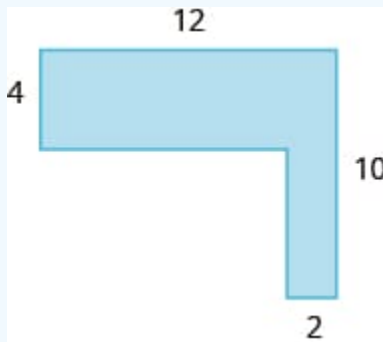
Find the diameter of a circle with circumference of 94.2 centimeters.

Find the Area of Irregular Figures

So far, we have found area for rectangles, triangles, trapezoids, and circles. An irregular figure is a figure that is not a standard geometric shape. Its area cannot be calculated using any of the standard area formulas. But some irregular figures are made up of two or more standard geometric shapes. To find the area of one of these irregular figures, we can split it into figures whose formulas we know and then add the areas of the figures.

Example 3.4.4

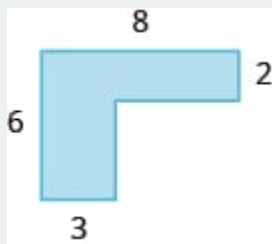
Find the area of the shaded region.



Answer available at the end of this section.

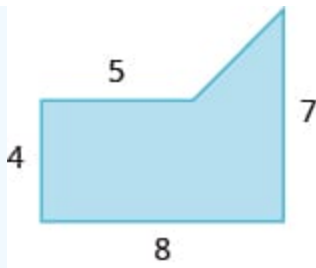
Your Turn 3.4.4

Find the area of each shaded region:



Example 3.4.5

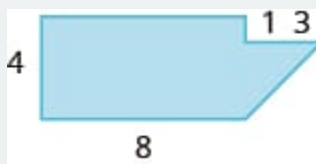
Find the area of the shaded region.



Answer available at the end of this section.

Your Turn 3.4.5

Find the area of each shaded region.



Example 3.4.6

A high school track is shaped like a rectangle with a semi-circle (half a circle) on each end. The rectangle has length 105 meters and width 68 meters. Find the area enclosed by the track. Round your answer to the nearest hundredth.



Answer available at the end of this section.

Your Turn 3.4.6

Find the area:

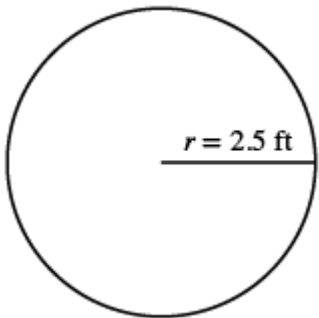
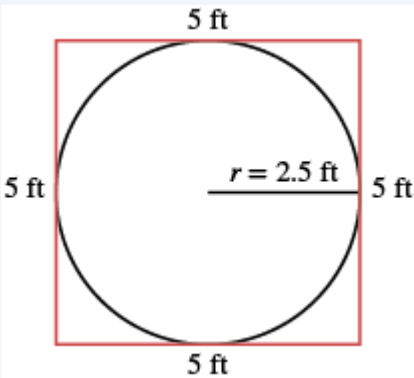


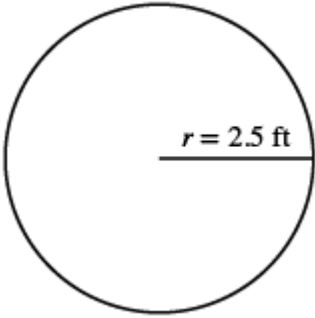
Media

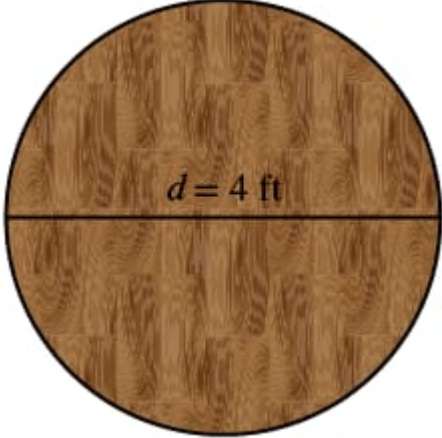
ACCESS ADDITIONAL ONLINE RESOURCES

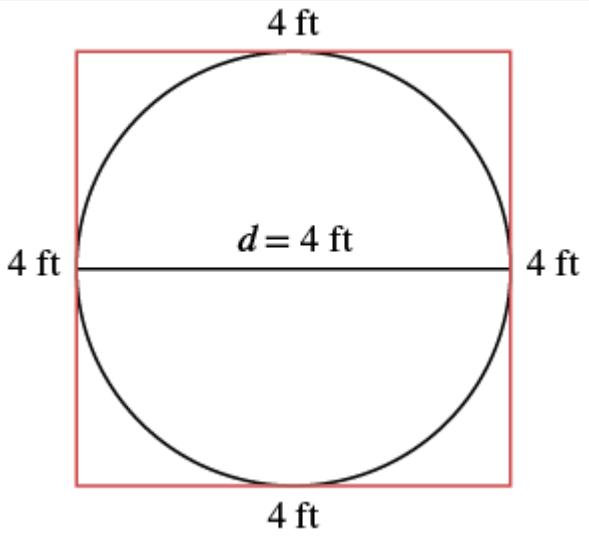
- Circumference of a Circle
- Area of a Circle
- Area of an L-shaped polygon
- Area of an L-shaped polygon with Decimals
- Perimeter Involving a Rectangle and Circle
- Area Involving a Rectangle and Circle

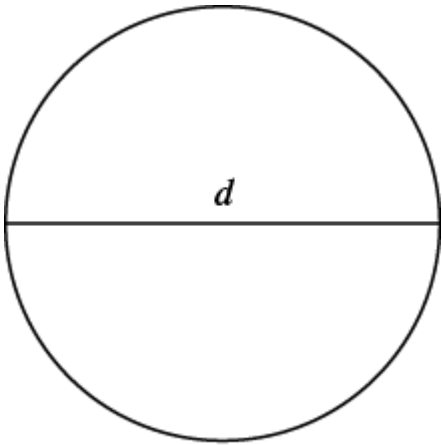
Answers

Ⓐ Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the circumference of the circle
Step 3. Name. Choose a variable to represent it.	Let C = circumference of the circle
Step 4. Translate. Write the appropriate formula Substitute	$C = 2\pi r$ $C = 2\pi (2.5)$
Step 5. Solve the equation.	$C \approx 15.7$ ft
Step 6. Check. Does this answer make sense?	Yes. If we draw a square around the circle, its sides would be 5 ft (twice the radius), so its perimeter would be 20 ft. This is slightly more than the circle's circumference, 15.7 ft. 
Step 7. Answer the question.	The circumference of the sandbox is 15.7 feet.

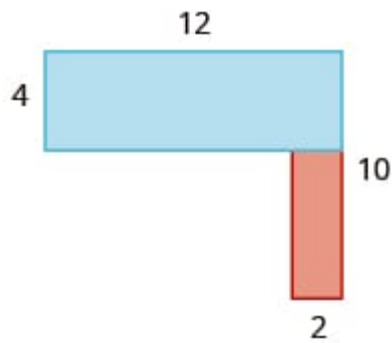
⑥ Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the area of the circle
Step 3. Name. Choose a variable to represent it.	Let A = the area of the circle
Step 4. Translate. Write the appropriate formula Substitute	$A =$ $A = \pi(2.5)^2$
Step 5. Solve the equation.	$A \approx 19.6$ sq. ft.
Step 6. Check.	Yes. If we draw a square around the circle, its sides would be 5 ft, as shown in part ⑤. So the area of the square would be 25 sq. ft. This is slightly more than the circle's area, 19.6 sq. ft.
Step 7. Answer the question.	The area of the circle is 19.6 square feet.

Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the circumference of the table
Step 3. Name. Choose a variable to represent it.	Let C = the circumference of the table
Step 4. Translate. Write the appropriate formula for the situation. Substitute.	$C = \pi d$ $C = \pi(4)$
Step 5. Solve the equation.	$C \approx 12.57$ feet

Step 6. Check:	If we put a square around the circle, its side would be 4. The perimeter would be 16. It makes sense that the circumference of the circle, 12.57, is a little less than 16. 
Step 7. Answer the question.	The diameter of the table is 12.57 square feet.

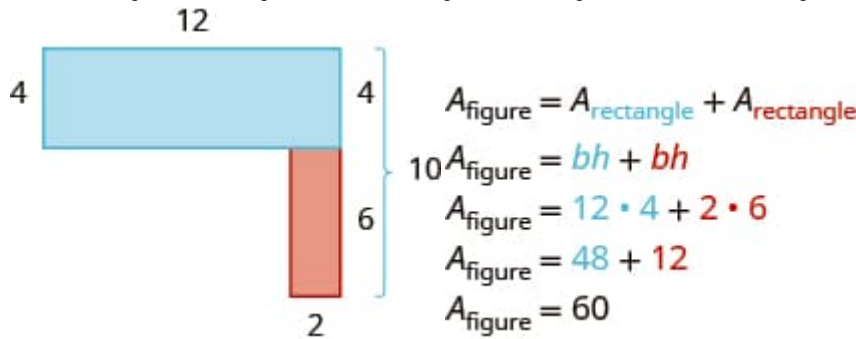
Step 1. Read the problem. Draw the figure and label it with the given information.	 $C = 47.1 \text{ cm}$
Step 2. Identify what you are looking for.	the diameter of the circle
Step 3. Name. Choose a variable to represent it.	Let d = the diameter of the circle
Step 4. Translate. Write the formula. Substitute.	$C = \pi d$ $47.1 = \pi d$
Step 5. Solve by dividing both sides by pi.	$\frac{47.1}{\pi} = \frac{\pi d}{\pi}$ $d \approx 15$
Step 6. Check:	$C = \pi d$ $C = \pi 15$ $C \approx 47.1$
Step 7. Answer the question.	The diameter of the circle is approximately 15 centimeters.

The given figure is irregular, but we can break it into two rectangles. The area of the shaded region will be the sum of the areas of both rectangles.



$$A_{\text{figure}} = A_{\text{blue rectangle}} + A_{\text{red rectangle}}$$

The blue rectangle has a width of 12 and a length of 4. The red rectangle has a width of 2, but its length is not labeled. The right side of the figure is the length of the red rectangle plus the length of the blue rectangle. Since the right side of the blue rectangle is 4 units long, the length of the red rectangle must be 6 units.



The area of the figure is 60 square units.

Is there another way to split this figure into two rectangles? Try it, and make sure you get the same area.

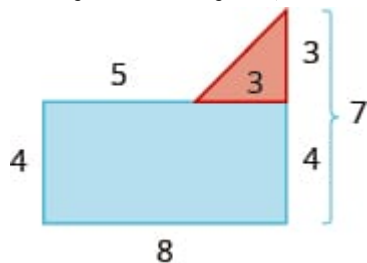
We can break this irregular figure into a triangle and rectangle. The area of the figure will be the sum of the areas of triangle and rectangle.

The rectangle has a length of 8 units and a width of 4 units.

We need to find the base and height of the triangle.

Since both sides of the rectangle are 4, the vertical side of the triangle is 3, which is $7 - 4$.

The length of the rectangle is 8, so the base of the triangle will be 3, which is $8 - 4$.



Now we can add the areas to find the area of the irregular figure.

$$A_{\text{figure}} = A_{\text{rectangle}} + A_{\text{triangle}}$$

$$A_{\text{figure}} = lw + \frac{1}{2}bh$$

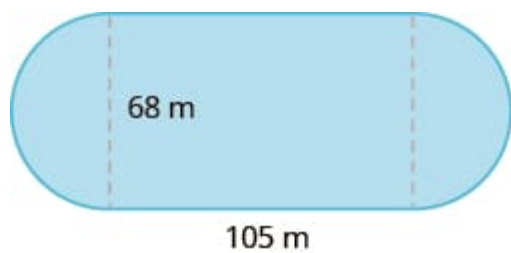
$$A_{\text{figure}} = 8 \cdot 4 + \frac{1}{2} \cdot 3 \cdot 3$$

$$A_{\text{figure}} = 32 + 4.5$$

$$A_{\text{figure}} = 36.5 \text{ sq. units}$$

The area of the figure is 36.5 square units.

We will break the figure into a rectangle and two semi-circles. The area of the figure will be the sum of the areas of the rectangle and the semicircles.



The rectangle has a length of 105 m and a width of 68 m. The semi-circles have a diameter of 68 m, so each has a radius of 34 m.

Area of figure = Area of rectangle + Area of two semicircles

$$\text{Area of figure} = bh + 2 \left(\frac{1}{2} \pi r^2 \right)$$

$$\text{Area of figure} = 105 \bullet 68 + 2 \left(\frac{1}{2} \pi 34^2 \right)$$

$$\text{Area of figure} \approx 7140 + 3631.68 = 10771.68 \text{ square meters}$$

3.5: Solve Geometry Applications- Volume and Surface Area

Learning Objectives

After completing this section, you should be able to:

1. Find volume and surface area of rectangular solids
2. Find volume and surface area of spheres
3. Find volume and surface area of cylinders
4. Find volume of cones

In this section, we will finish our study of geometry applications. We find the volume and surface area of some three-dimensional figures. Since we will be solving applications, we will once again show our Problem-Solving Strategy for Geometry Applications.

How To

Problem Solving Strategy for Geometry Applications

1. **Read** the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.
2. **Identify** what you are looking for.
3. **Name** what you are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Find Volume and Surface Area of Rectangular Solids

A cheerleading coach is having the squad paint wooden crates with the school colors to stand on at the games. (See Figure 3.5.1). The amount of paint needed to cover the outside of each box is the surface area, a square measure of the total area of all the sides. The amount of space inside the crate is the volume, a cubic measure.

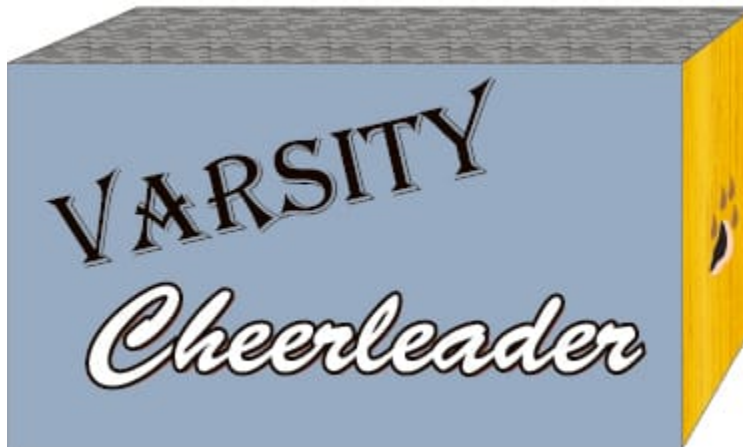


Figure 3.5.1: This wooden crate is in the shape of a rectangular solid.

Each crate is in the shape of a rectangular solid. Its dimensions are the length, width, and height. The rectangular solid shown in Figure 3.5.2 has length 4 units, width 2 units, and height 3 units. Can you tell how many cubic units there are altogether? Let's

look layer by layer.

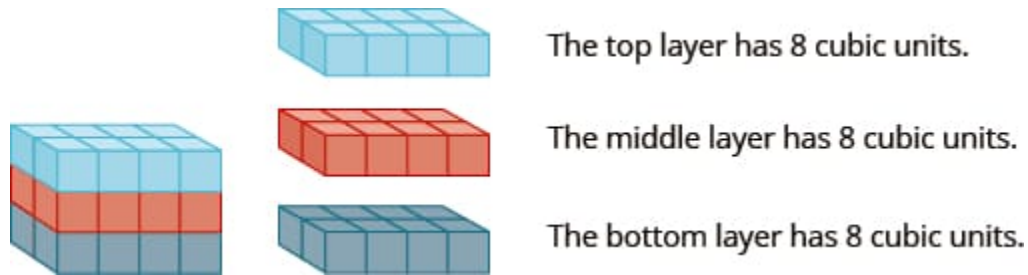


Figure 3.5.2: Breaking a rectangular solid into layers makes it easier to visualize the number of cubic units it contains. This 4 by 2 by 3 rectangular solid has 24 cubic units.

Altogether there are 24 cubic units. Notice that 24 is the length $\times$ width $\times$ height.

$$\underbrace{V}_{24} = \underbrace{L}_{4} \cdot \underbrace{W}_{2} \cdot \underbrace{H}_{3}$$

The volume, V , of any rectangular solid is the product of the length, width, and height.

$$V = LWH$$

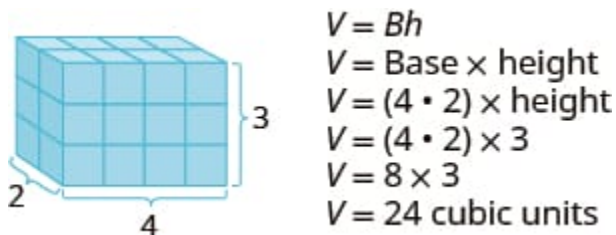
We could also write the formula for volume of a rectangular solid in terms of the area of the base. The area of the base, B , is equal to length $\times$ width.

$$B = L \cdot W$$

We can substitute B for $L \cdot W$ in the volume formula to get another form of the volume formula.

$$\begin{aligned} V &= L \cdot W \cdot H \\ V &= (L \cdot W) \cdot H \\ V &= Bh \end{aligned}$$

We now have another version of the volume formula for rectangular solids. Let's see how this works with the $4 \times 2 \times 3$ shape from Figure 3.5.2.



To find the *surface area* of a rectangular solid, think about finding the area of each of its faces. How many faces does the rectangular solid above have? You can see three of them.

$$A_{\text{front}} = L \cdot W = 4 \cdot 3 = 12$$

$$A_{\text{side}} = L \cdot W = 2 \cdot 3 = 6$$

$$A_{\text{top}} = L \cdot W = 4 \cdot 2 = 8$$

Notice for each of the three faces you see, there is an identical opposite face that does not show.

$$S = (\text{front} + \text{back}) + (\text{left side} + \text{right side}) + (\text{top} + \text{bottom})$$

$$S = (2 \cdot \text{front}) + (2 \cdot \text{left side}) + (2 \cdot \text{top})$$

$$S = 2 \cdot 12 + 2 \cdot 6 + 2 \cdot 8$$

$$S = 24 + 12 + 16$$

$$S = 52 \text{ square units}$$

The surface area of Figure 3.5.2 is 52 square units.

In general, to find the surface area of a rectangular solid, remember that each face is a rectangle, so its area is the product of its two dimensions, either length and width, length and height, or width and height (see Figure 3.5.3). Find the area of each face that you see and then multiply each area by two to account for the face on the opposite side.

$$S = 2LH + 2LW + 2WH$$

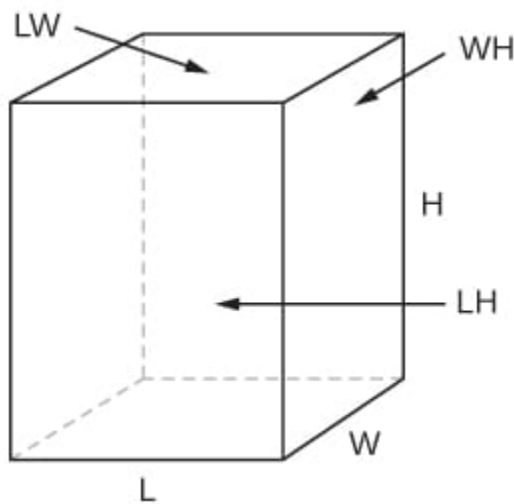
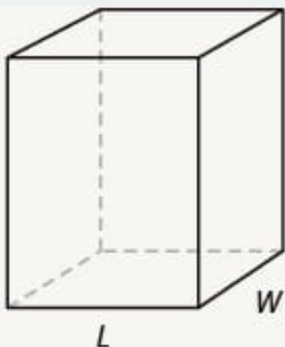


Figure 3.5.3: For each face of the rectangular solid facing you, there is another face on the opposite side. There are 6 faces in all.

Volume and Surface Area of a Rectangular Solid

For a rectangular solid with length L , width W , and height H :



Volume: $V = LWH$

Surface Area: $S = 2LH + 2LW + 2WH$

Example 3.5.1

For a rectangular solid with length 14 cm, height 17 cm, and width 9 cm, find the ① volume and ② surface area.

Answer available at the end of this section.

Your Turn 3.5.1

Find the ④ volume and ⑤ surface area of rectangular solid with the: length 8 feet, width 9 feet, and height 11 feet.

Example 3.5.2

A rectangular crate has a length of 30 inches, width of 25 inches, and height of 20 inches. Find its ④ volume and ⑤ surface area.

Answer available at the end of this section.

Your Turn 3.5.2

A rectangular box has length 9 feet, width 4 feet, and height 6 feet. Find its ④ volume and ⑤ surface area.

Volume and Surface Area of a Cube

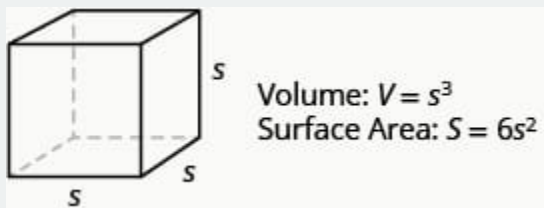
A cube is a rectangular solid whose length, width, and height are equal. See Volume and Surface Area of a Cube, below. Substituting, s for the length, width and height into the formulas for volume and surface area of a rectangular solid, we get:

$$V = LWH = s \cdot s \cdot s = s^3$$

$$S = 2LH + 2LW + 2WH = 2 \cdot s \cdot s + 2 \cdot s \cdot s + 2 \cdot s \cdot s = 2s^2 + 2s^2 + 2s^2 = 6s^2$$

Volume and Surface Area of a Cube

For any cube with sides of length s ,



Example 3.5.3

A cube is 2.5 inches on each side. Find its ④ volume and ⑤ surface area.

Answer available at the end of this section.

Your Turn 3.5.3

For a cube with side 4.5 meters, find the ① volume and ② surface area of the cube.

Example 3.5.4

A notepad cube measures 2 inches on each side. Find its ① volume and ② surface area.

Answer available at the end of this section.

Your Turn 3.5.4

A packing box is a cube measuring 4 feet on each side. Find its ① volume and ② surface area.

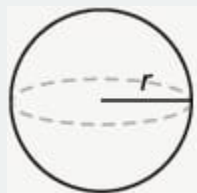
Find the Volume and Surface Area of Spheres

A sphere is the shape of a basketball, like a three-dimensional circle. Just like a circle, the size of a sphere is determined by its radius, which is the distance from the center of the sphere to any point on its surface. The formulas for the volume and surface area of a sphere are given below.

Showing where these formulas come from, like we did for a rectangular solid, is beyond the scope of this course. We will again use the π button on our calculator to get accurate answers.

Volume and Surface Area of a Sphere

For a sphere with radius r :



$$\text{Volume: } V = \frac{4}{3} \pi r^3$$

$$\text{Surface Area: } S = 4\pi r^2$$

Example 3.5.5

A sphere has a radius of 6 inches. Find its ① volume and ② surface area. Round your answers to the nearest hundredth.

Answer available at the end of this section.

Your Turn 3.5.5

Find the ① volume and ② surface area of a sphere with radius 3 centimeters. Round your answers to the nearest tenth.

Example 3.5.6

A globe of Earth is in the shape of a sphere with radius 14 inches. Find its ④ volume and ⑤ surface area. Round the answers to the nearest hundredth.

Answer available at the end of this section.

Your Turn 3.5.6

A beach ball is in the shape of a sphere with radius of 9 inches. Find its ④ volume and ⑤ surface area. Round your answers to the nearest hundredth.

Find the Volume and Surface Area of a Cylinder

If you have ever seen a can of soda, you know what a cylinder looks like. A cylinder is a solid figure with two parallel circles of the same size at the top and bottom. The top and bottom of a cylinder are called the bases. The height h of a cylinder is the distance between the two bases. For all the cylinders we will work with here, the sides and the height, h , will be perpendicular to the bases.

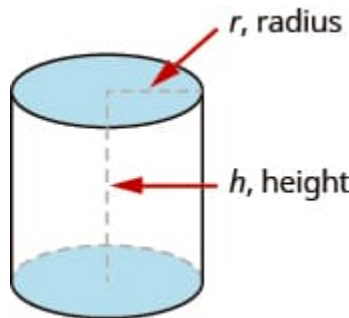


Figure 3.5.4: A cylinder has two circular bases of equal size. The height is the distance between the bases.

Rectangular solids and cylinders are somewhat similar because they both have two bases and a height. The formula for the volume of a rectangular solid, $V = Bh$, can also be used to find the volume of a cylinder.

Figure 3.5.5 compares how the formula $V = Bh$ is used for rectangular solids and cylinders. In a rectangular solid, the base is a rectangle. In a cylinder, the base is a circle.

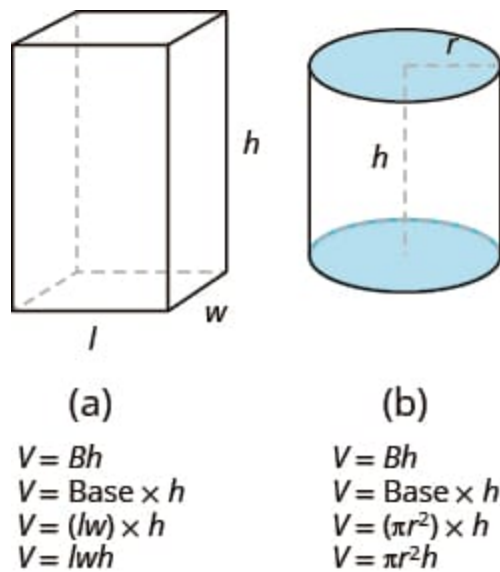


Figure 3.5.5: Seeing how a cylinder is similar to a rectangular solid may make it easier to understand the formula for the volume of a cylinder.

To understand the formula for the surface area of a cylinder, think of a can of vegetables. It has three surfaces: the top, the bottom, and the piece that forms the sides of the can. If you carefully cut the label off the side of the can and unroll it, you will see that it is a rectangle. See Figure 3.5.6.

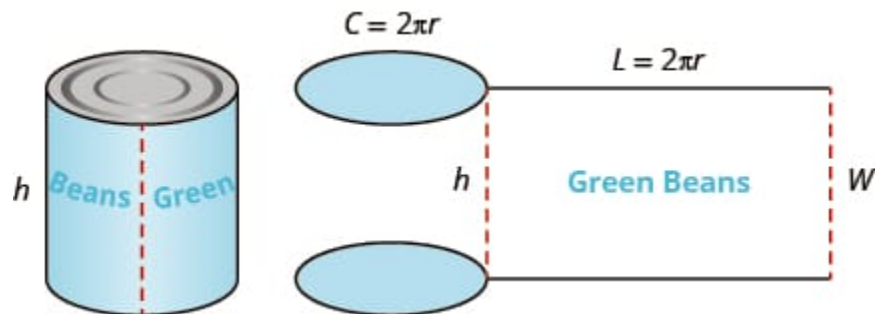


Figure 3.5.6: By cutting and unrolling the label of a can of vegetables, we can see that the surface of a cylinder is a rectangle. The length of the rectangle is the circumference of the cylinder's base, and the width is the height of the cylinder.

The distance around the edge of the can is the circumference of the cylinder's base. It is also the length L of the rectangular label. The height of the cylinder is the width W of the rectangular label. So the area of the label can be represented as

$$A = L \cdot W$$

$$A = 2\pi r \cdot h$$

To find the total surface area of the cylinder, we add the areas of the two circles to the area of the rectangle.



$$S = A_{\text{top circle}} + A_{\text{bottom circle}} + A_{\text{rectangle}}$$

$$S = \underbrace{\pi r^2 + \pi r^2}_{2\pi r^2} + 2\pi r \cdot h$$

$$S = 2\pi r^2 + 2\pi rh$$

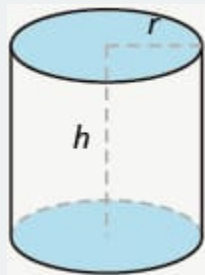
$$S = 2\pi r^2 + 2\pi rh$$

The surface area of a cylinder with radius r and height h , is

$$S = 2\pi r^2 + 2\pi rh$$

Volume and Surface Area of a Cylinder

For a cylinder with radius r and height h :



Volume: $V = \pi r^2 h$ or $V = Bh$
 Surface Area: $S = 2\pi r^2 + 2\pi rh$

Example 3.5.7

A cylinder has height 5 inches and radius 3 inches. Find the ① volume and ② surface area. Round your answers to the nearest tenth.

Answer available at the end of this section.

Your Turn 3.5.7

Find the ① volume and ② surface area of the cylinder with radius 4 cm and height 7 cm. Round your answers to the nearest hundredth.

Example 3.5.8

Find the ① volume and ② surface area of a can of soda. The radius of the base is 4 centimeters and the height is 13 centimeters. Assume the can is shaped exactly like a cylinder. Round your answers to the nearest hundredth.

Your Turn 3.5.8

Find the **a** volume and **b** surface area of a can of paint with radius 8 centimeters and height 19 centimeters. Assume the can is shaped exactly like a cylinder.

Find the Volume of Cones

The first image that many of us have when we hear the word ‘cone’ is an ice cream cone. There are many other applications of cones (but most are not as tasty as ice cream cones). In this section, we will see how to find the volume of a cone.

In geometry, a cone is a solid figure with one circular base and a vertex. The height of a cone is the distance between its base and the vertex. The cones that we will look at in this section will always have the height perpendicular to the base. See Figure 3.5.7.

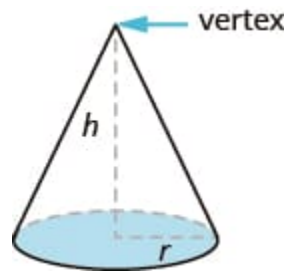


Figure 3.5.7: The height of a cone is the distance between its base and the vertex.

Earlier in this section, we saw that the volume of a cylinder is $V = \pi r^2 h$. Figure 3.5.8 shows a cone placed inside a cylinder with the same height and same base. If we compare the volume of the cone and the cylinder, we can see that the volume of the cone is less than that of the cylinder.

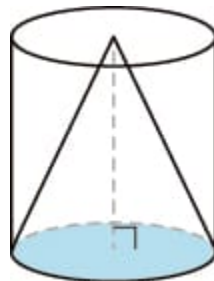


Figure 3.5.8: The volume of a cone is less than the volume of a cylinder with the same base and height.

In fact, the volume of a cone is exactly one-third of the volume of a cylinder with the same base and height. The volume of a cone is

$$V = \frac{1}{3} Bh$$

Since the base of a cone is a circle, we can substitute the formula of area of a circle, πr^2 , for B to get the formula for volume of a cone.

$$V = \frac{1}{3} \pi r^2 h$$

In this book, we will only find the volume of a cone, and not its surface area.

Volume of a Cone

For a cone with radius r and height h ,



Example 3.5.9

Find the volume of a cone with height 6 inches and radius of its base 2 inches. Round your answer to the nearest thousandth.

Answer available at the end of this section.

Your Turn 3.5.9

Find the volume of a cone with height 7 inches and radius 3 inches. Round your answer to the nearest tenth.

Example 3.5.10

Marty's favorite gastro pub serves french fries in a paper wrap shaped like a cone. What is the volume of a conic wrap that is 8 inches tall and 5 inches in radius? Round the answer to the nearest hundredth.

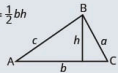
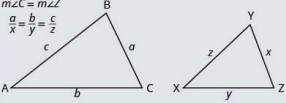
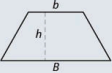
Answer available at the end of this section.

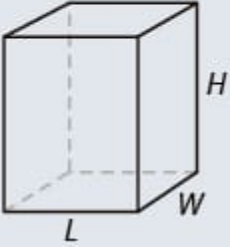
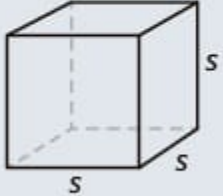
Your Turn 3.5.10

How many cubic inches of candy will fit in a cone-shaped piñata that is 18 inches long and 12 inches across its base? Round the answer to the nearest hundredth.

Summary of Geometry Formulas

The following charts summarize all of the formulas covered in this chapter.

Supplementary and Complementary Angles $m\angle A + m\angle B = 180^\circ$ for supplementary angles A and B $m\angle C + m\angle D = 90^\circ$ for complementary angles C and D	Triangle For $\triangle ABC$, angle measures: $m\angle A + m\angle B + m\angle C = 180^\circ$ Perimeter: $P = a + b + c$ $A = \frac{1}{2}bh$ 
Similar Triangles $m\angle A = m\angle X$ $m\angle B = m\angle Y$ $m\angle C = m\angle Z$ $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$ 	Trapezoid Area: $A = \frac{1}{2}h(b + B)$ 
Sphere Volume: $V = \frac{4}{3}\pi r^3$ Surface Area: $S = 4\pi r^2$ 	Cylinder Volume: $V = \pi r^2 h$ or $V = Bh$ Surface Area: $S = 2\pi r^2 + 2\pi rh$ 
Circle Circumference: $C = 2\pi r$ or $C = \pi d$ Area: $A = \pi r^2$ 	Cone vertex  $V = \frac{1}{3}\pi r^2 h$

Rectangular Solid Volume: $V = LWH$ Surface Area: $S = 2LH + 2LW + 2WH$ 	Rectangle Perimeter: $P = 2L + 2W$ Area: $A = LW$ 	Cube Volume: $V = s^3$ Surface Area: $S = 6s^2$ 
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Media

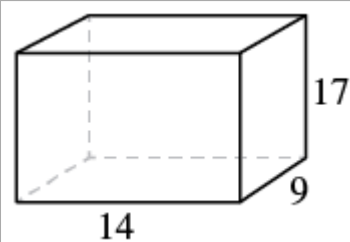
ACCESS ADDITIONAL ONLINE RESOURCES

- [Volume of a Cone](#)

Answers

Step 1 is the same for both Ⓐ and Ⓑ, so we will show it just once.

Step 1. **Read** the problem. Draw the figure and label it with the given information.



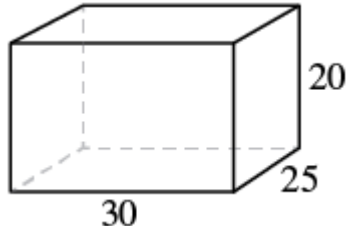
Ⓐ

Step 2. **Identify** what you are looking for.

the volume of the rectangular solid

Step 3. Name. Choose a variable to represent it.	Let V = volume
Step 4. Translate. Write the appropriate formula. Substitute.	$V = LWH$ $V = 14 \cdot 9 \cdot 17$
Step 5. Solve the equation.	$V = 2,142$
Step 6. Check	We leave it to you to check your calculations.
Step 7. Answer the question.	The volume is 2,142 cubic centimeters.
ⓑ	
Step 2. Identify what you are looking for.	the surface area of the solid
Step 3. Name. Choose a variable to represent it.	Let S = surface area
Step 4. Translate. Write the appropriate formula. Substitute.	$S = 2LH + 2LW + 2WH$ $S = 2(14 \cdot 17) + 2(14 \cdot 9) + 2(9 \cdot 17)$
Step 5. Solve the equation.	$S = 1,034$
Step 6. Check:	Double-check with a calculator.
Step 7. Answer the question.	The surface area is 1,034 square centimeters.

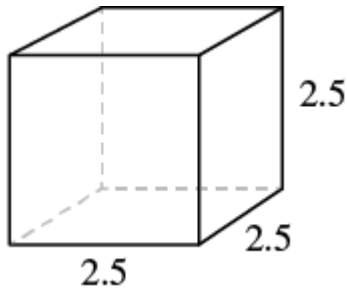
Step 1 is the same for both ⓐ and ⓑ, so we will show it just once.

Step 1. Read the problem. Draw the figure and label it with the given information.	
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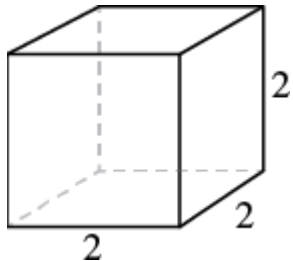
ⓐ	
Step 2. Identify what you are looking for.	the volume of the crate
Step 3. Name. Choose a variable to represent it.	let V = volume
Step 4. Translate. Write the appropriate formula. Substitute.	$V = LWH$ $V = 30 \cdot 25 \cdot 20$
Step 5. Solve the equation.	$V = 15,000$
Step 6. Check.	Double check your math.
Step 7. Answer the question.	The volume is 15,000 cubic inches.
ⓑ	
Step 2. Identify what you are looking for.	the surface area of the crate
Step 3. Name. Choose a variable to represent it.	let S = surface area
Step 4. Translate. Write the appropriate formula. Substitute.	$S = 2LH + 2LW + 2WH$ $S = 2(30 \cdot 20) + 2(30 \cdot 25) + 2(25 \cdot 20)$
Step 5. Solve the equation.	$S = 3,700$
Step 6. Check.	Double check your math.

Step 7. Answer the question.	The surface area is 3,700 square inches.
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Step 1 is the same for both (a) and (b), so we will show it just once.

Step 1. Read the problem. Draw the figure and label it with the given information.	
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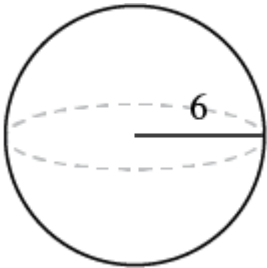
(a)	
Step 2. Identify what you are looking for.	the volume of the cube
Step 3. Name. Choose a variable to represent it.	let V = volume
Step 4. Translate. Write the appropriate formula.	$V = s^3$
Step 5. Solve. Substitute and solve.	$V = (2.5)^3$ $V = 15.625$
Step 6. Check.	Check your work.
Step 7. Answer the question.	The volume is 15.625 cubic inches.
(b)	
Step 2. Identify what you are looking for.	the surface area of the cube
Step 3. Name. Choose a variable to represent it.	let S = surface area
Step 4. Translate.	$S = 6s^2$
Step 5. Solve. Substitute and solve.	$S = 6 \cdot (2.5)^2$ $S = 37.5$
Step 6. Check.	The check is left to you.
Step 7. Answer the question.	The surface area is 37.5 square inches.

Step 1. Read the problem. Draw the figure and label it with the given information.	
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(a)	
Step 2. Identify what you are looking for.	the volume of the cube
Step 3. Name. Choose a variable to represent it.	let V = volume
Step 4. Translate. Write the appropriate formula.	$V = s^3$

Step 5. Solve the equation.	$V = 2^3$ $V = 8$
Step 6. Check.	Check that you did the calculations correctly.
Step 7. Answer the question.	The volume is 8 cubic inches.
ⓑ	
Step 2. Identify what you are looking for.	the surface area of the cube
Step 3. Name. Choose a variable to represent it.	let S = surface area
Step 4. Translate. Write the appropriate formula.	$S = 6s^2$
Step 5. Solve the equation.	$S = 6 \cdot 2^2$ $S = 24$
Step 6. Check.	The check is left to you.
Step 7. Answer the question.	The surface area is 24 square inches.

Step 1 is the same for both ⓐ and ⓑ, so we will show it just once.

Step 1. Read the problem. Draw the figure and label it with the given information.	
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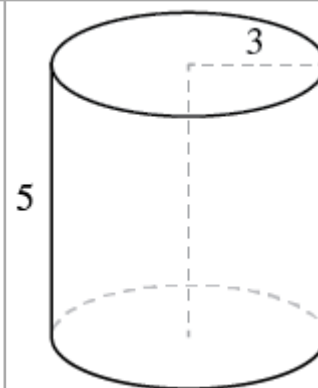
ⓐ	
Step 2. Identify what you are looking for.	the volume of the sphere
Step 3. Name. Choose a variable to represent it.	let V = volume
Step 4. Translate. Write the appropriate formula.	$V = \frac{4}{3}\pi r^3$
Step 5. Solve.	$V = \frac{4}{3}\pi 6^3$ $V \approx 904.78$
Step 6. Check.	Double-check your math on a calculator.
Step 7. Answer the question.	The volume is approximately 904.78 cubic inches.
ⓑ	
Step 2. Identify what you are looking for.	the surface area of the sphere
Step 3. Name. Choose a variable to represent it.	let S = surface area
Step 4. Translate. Write the appropriate formula.	$S = 4\pi r^2$
Step 5. Solve.	$S = 4\pi 6^2$ $S \approx 452.39$
Step 6. Check.	We leave it to you to check your calculations.
Step 7. Answer the question.	The surface area is approximately 452.39 square inches.

Step 1. **Read** the problem. Draw a figure with the given information and label it.



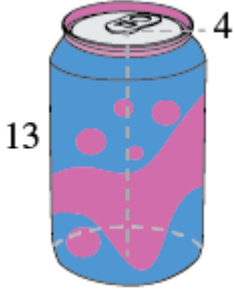
Ⓐ	
Step 2. Identify what you are looking for.	the volume of the sphere
Step 3. Name. Choose a variable to represent it.	let V = volume
Step 4. Translate. Write the appropriate formula. Substitute.	$V = \frac{4}{3}\pi r^3$ $V = \frac{4}{3}\pi 14^3$
Step 5. Solve.	$V \approx 11,494.04$
Step 6. Check:	We leave it to you to check your calculations.
Step 7. Answer the question.	The volume is approximately 11,494.03 cubic inches.
Ⓑ	
Step 2. Identify what you are looking for.	the surface area of the sphere
Step 3. Name. Choose a variable to represent it.	let S = surface area
Step 4. Translate. Write the appropriate formula. Substitute.	$S = 4\pi r^2$ $S = 4\pi 14^2$
Step 5. Solve.	$S \approx 2463.01$
Step 6. Check:	We leave it to you to check your calculations.
Step 7. Answer the question.	The surface area is approximately 2463.01 square inches.

Step 1. **Read** the problem. Draw the figure and label it with the given information.

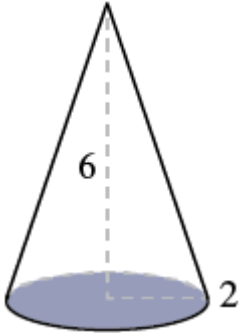


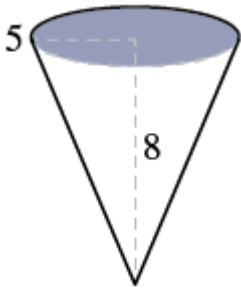
Ⓐ	
Step 2. Identify what you are looking for.	the volume of the cylinder
Step 3. Name. Choose a variable to represent it.	let V = volume

Step 4. Translate. Write the appropriate formula. Substitute.	$V = \pi r^2 h$ $V = \pi 3^2 \cdot 5$
Step 5. Solve.	$V \approx 141.4$
Step 6. Check.	We leave it to you to check the calculations.
Step 7. Answer the question.	The volume is approximately 141.4 cubic inches.
⑥	
Step 2. Identify what you are looking for.	the surface area of the cylinder
Step 3. Name. Choose a variable to represent it.	let S = surface area
Step 4. Translate. Write the appropriate formula. Substitute.	$S = 2\pi r^2 + 2\pi r h$ $S = 2\pi 3^2 + 2\pi 3 \cdot 5$
Step 5. Solve.	$S \approx 150.8$
Step 6. Check.	We leave it to you to check your calculations.
Step 7. Answer the question.	The surface area is approximately 150.8 square inches.

Step 1. Read the problem. Draw the figure and label it with the given information.	
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⑦	
Step 2. Identify what you are looking for.	the volume of the cylinder
Step 3. Name. Choose a variable to represent it.	let V = volume
Step 4. Translate. Write the appropriate formula. Substitute.	$V = \pi r^2 h$ $V = \pi 4^2 \cdot 13$
Step 5. Solve.	$V \approx 653.45$
Step 6. Check:	We leave it to you to check.
Step 7. Answer the question.	The volume is approximately 653.45 cubic centimeters.
⑧	
Step 2. Identify what you are looking for.	the surface area of the cylinder
Step 3. Name. Choose a variable to represent it.	let S = surface area
Step 4. Translate. Write the appropriate formula. Substitute.	$S = 2\pi r^2 + 2\pi r h$ $S = 2\pi 4^2 + 2\pi 4 \cdot 13$
Step 5. Solve.	$S \approx 427.26$
Step 6. Check.	We leave it to you to check your calculations.
Step 7. Answer the question.	The surface area is approximately 427.26 square centimeters.

Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the volume of the cone
Step 3. Name. Choose a variable to represent it.	let V = volume
Step 4. Translate. Write the appropriate formula. Substitute.	$V = \frac{1}{3}\pi r^2 h$ $V = \frac{1}{3}\pi 2^2 \cdot 6$
Step 5. Solve.	$V \approx 25.133$
Step 6. Check.	We leave it to you to check your calculations.
Step 7. Answer the question.	The volume is approximately 25.133 cubic inches.

Step 1. Read the problem. Draw the figure and label it with the given information. Notice here that the base is the circle at the top of the cone.	
Step 2. Identify what you are looking for.	the volume of the cone
Step 3. Name. Choose a variable to represent it.	let V = volume
Step 4. Translate. Write the appropriate formula. Substitute.	$V = \frac{1}{3}\pi r^2 h$ $V = \frac{1}{3}\pi 5^2 \cdot 8$
Step 5. Solve.	$V \approx 209.44$
Step 6. Check.	We leave it to you to check your calculations.
Step 7. Answer the question.	The volume of the wrap is approximately 209.44 cubic inches.

4: SET THEORY

4.1: Introduction

4.2: Basic Set Concepts

4.3: Subsets

4.4: Understanding Venn Diagrams

4.5: Set Operations with Two Sets

4.6: Set Operations with Three Sets

4.1: Introduction



Figure 4.1.1: A flatware drawer is like a set in that it contains distinct objects. (credit: modification of work “silverware” by Jo Naylor/Flickr, CC BY 2.0)

Think of a drawer in your kitchen used to store flatware. This drawer likely holds forks, spoons, and knives, and possibly other items such as a meat thermometer and a can opener. The drawer in this case represents a tool used to group a collection of objects. The members of the group are the individual items in the drawer, such as a fork or a spoon.

The members of a set can be anything, such as people, numbers, or letters of the alphabet. In statistical studies, a set is a well-defined collection of objects used to identify an entire population of interest. For example, in a research study examining the effects of a new medication, there can be two sets of people: one set that is given the medication and a different set that is given a placebo (control group). In this module, we will discuss sets and Venn diagrams, which are graphical ways to show relationships between different groups.

4.2: Basic Set Concepts



Figure 4.2.1: A spoon, fork, and knife are elements of the set of flatware. (credit: modification of work “Cupofjoy” Wikimedia CC0 1.0 Public Domain Dedication)

Learning Objectives

After completing this section, you should be able to:

1. Represent sets in a variety of ways.
2. Represent well-defined sets and the empty set with proper set notation.
3. Compute the cardinal value of a set.
4. Differentiate between finite and infinite sets.
5. Differentiate between equal and equivalent sets.

Sets and Ways to Represent Them

Think back to your kitchen organization. If the drawer is the set, then the forks and knives are elements in the set. Sets can be described in a number of different ways: by roster notation, by set-builder notation, by interval notation, by graphing on a number line, and by Venn diagrams. Sets are typically designated with capital letters. The simplest way to represent a set with only a few members is the **roster method**, in which the elements in a set are listed, enclosed by curly braces and separated by commas. For example, if F represents our set of flatware, we can represent F by using the following set notation with the roster method:

$$F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$$

Example 4.2.1

Writing a Set Using the Roster Method

Write a set consisting of your three favorite sports and label it with a capital S .

Answer available at the end of this section.

Your Turn 4.2.1

Write a set consisting of four small hand tools that might be in a toolbox and label it with a capital T .

All the sets we have considered so far have been well-defined sets. A well-defined set clearly communicates whether an element is a member of the set or not. The members of a well-defined set are fixed and do not change over time. Consider the following question. What are your top 10 songs of 2021? You could create a list of your top 10 favorite songs from 2021, but the list your friend creates will not necessarily contain the same 10 songs. So, the set of your top 10 songs of 2021 is not a well-defined set. On the other hand, the set of the letters in your name is a well-defined set because it does not vary (unless of course you change your name). The NFL wide receiver, Chad Johnson, famously changed his name to Chad Ochocinco to match his jersey number of 85.

Example 4.2.2

Identifying Well-Defined Sets

For each of the following collections, determine if it represents a well-defined set.

1. The group of all past vice presidents of the United States.
2. A group of old cats.

Answer available at the end of this section.

Your Turn 4.2.2

For each of the following collections, determine if it represents a well-defined set.

1. A collection of medium-sized potatoes.
2. The original members of the Black Eyed Peas musical group.

On January 20, 2021, Kamala Harris was sworn in as the first woman vice president of the United States of America. If we were to consider the set of all women vice presidents of the United States of America prior to January 20, 2021, this set would be known as an empty set. The number of people in this set is 0, since there were no women vice presidents before Harris. The empty set, also called the null set, is written symbolically using a pair of braces, $\{ \}$, or a zero with a slash through it, $\emptyset$.

Checkpoint

The set containing the number 0, $\{0\}$, is a set with one element in it. It is not the same as the empty set, $\{ \}$, which does not have any elements in it. Symbolically: $\{0\} \neq \{ \}$.

Example 4.2.3

Representing the Empty Set Symbolically

Represent each of the following sets symbolically.

1. The set of prime numbers less than 2.
2. The set of birds that are also mammals.

Answer available at the end of this section.

Your Turn 4.2.3

Represent the set of all numbers divisible by 0 symbolically.

Who Knew?

The Number Zero

We use the number zero to represent the concept of nothing every day. The machine language of computers is binary, consisting only of zeros and ones, and even way before that, the number zero was a powerful invention that allowed our understanding of mathematics and science to develop. The historical record shows the Babylonians first used zeros around 300 B.C., while the Mayans developed and began using zero separately around 350 A.D. What is considered the first formal use of zero in arithmetic operations was developed by the Indian mathematician Brahmagupta around 650 A.D.

Brahmagupta, Mathematician and Astronomer

Another interesting feature of the number zero is that although it is an even number, it is the only number that is neither negative nor positive.

For larger sets that have a natural ordering, sometimes an ellipsis is used to indicate that the pattern continues. It is common practice to list the first three elements of a set to establish a pattern, write the ellipsis, and then provide the last element. Consider the set of all lowercase letters of the English alphabet, A . This set can be written symbolically as $A = \{a, b, c, \dots, z\}$.

The sets we have been discussing so far are finite sets. They all have a limited or fixed number of elements. We also use an ellipsis for infinite sets, which have an unlimited number of elements, to indicate that the pattern continues. For example, in set theory, the set of natural numbers, which is the set of all positive counting numbers, is represented as $\mathbb{N} = \{1, 2, 3, \dots\}$.

Notice that for this set, there is no element following the ellipsis. This is because there is no largest natural number; you can always add one more to get to the next natural number. Because the set of natural numbers grows without bound, it is an infinite set.

Example 4.2.4

Writing a Finite Set Using the Roster Method and an Ellipsis

Write the set of even natural numbers including and between 2 and 100, and label it with a capital E . Include an ellipsis.

Answer available at the end of this section.

Your Turn 4.2.4

Use an ellipsis to write the set of single digit numbers greater than or equal to zero and label it with a capital D .

Our number system is made up of several different infinite sets of numbers. The set of integers, $\mathbb{Z}$, is another infinite set of numbers. It includes all the positive and negative counting numbers and the number zero. There is no largest or smallest integer.

Example 4.2.5

Writing an Infinite Set Using the Roster Method and Ellipses

Write the set of integers using the roster method, and label it with a $\mathbb{Z}$.

Answer available at the end of this section.

Your Turn 4.2.5

Write the set of odd numbers greater than 0 and label it with a capital M .

A shorthand way to write sets is with the use of set builder notation, which is a verbal description or formula for the set. For example, the set of all lowercase letters of the English alphabet, A , written in set builder notation is:

$$A = \{x | x \text{ is a lowercase letter of the English alphabet.}\}$$

This is read as, "Set A is the set of all elements x such that x is a lowercase letter of the English alphabet."

Example 4.2.6

Writing a Set Using Set Builder Notation

Using set builder notation, write the set B of all types of balls. Explain what the notation means.

Answer available at the end of this section.

Your Turn 4.2.6

Using set builder notation, write the set C of all types of cars.

Example 4.2.7

Writing Sets Using Various Methods

Consider the set of letters in the word "happy." Determine the best way to represent this set, and then write the set using either the roster method or set builder notation, whichever is more appropriate.

Answer available at the end of this section.

Your Turn 4.2.7

Use the roster method or set builder notation to represent the collection of all musical instruments.

Computing the Cardinal Value of a Set

Almost all the sets most people work with outside of pure mathematics are finite sets. For these sets, the **cardinal value** or **cardinality** of the set is the number of elements in the set. For finite set A , the cardinality is denoted symbolically as $n(A)$. For example, a set that contains four elements has a cardinality of 4.

How do we measure the cardinality of infinite sets? The 'smallest' infinite set is the set of natural numbers, or counting numbers, $\mathbb{N} = \{1, 2, 3, \dots\}$. This set has a cardinality of $\aleph_0$ (pronounced "aleph-null"). All sets that have the same cardinality as the set of natural numbers are countably infinite. This concept, as well as notation using aleph, was introduced by mathematician Georg Cantor who once said, "A set is a Many that allows itself to be thought of as a One."

Example 4.2.8

Computing the Cardinal Value of a Set

Write the cardinal value of each of the following sets in symbolic form.

1. $F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$
2. The empty set.

Answer available at the end of this section.

Your Turn 4.2.8

Write the cardinal value of each of the following sets in symbolic form.

1. Set P is the set of prime numbers less than 2.
2. Set A is the set of lowercase letters of the English alphabet, $A = \{a, b, c, \dots, z\}$.

Now that we have learned to represent finite and infinite sets using both the roster method and set builder notation, we should also be able to determine if a set is finite or infinite based on its verbal or symbolic description. One way to determine if a set is finite or not is to determine the cardinality of the set. If the cardinality of a set is a natural number, then the set is finite.

Example 4.2.9

Differentiating Between Finite and Infinite Sets

Classify each of the following sets as infinite or finite.

1. $E = \{2, 4, 6, 8, 10\}$
2. A is the set of lowercase letters of the English Alphabet, $A = \{a, b, c, \dots, z\}$.
3. $\mathbb{Q} = \left\{ \frac{p}{q} \mid p \text{ and } q \text{ are integers and } q \neq 0 \right\}$

Answer available at the end of this section.

Your Turn 4.2.9

Classify each of the following sets as infinite or finite.

1. $B = \{b, a, k, e\}$
2. $\mathbb{R} = \{x \mid x \text{ is a real number}\}$

Equal versus Equivalent Sets

When speaking or writing we tend to use equal and equivalent interchangeably, but there is an important distinction between their meanings. Consider a new Ford Escape Hybrid and a new Toyota Rav4 Hybrid. Both cars are hybrid electric sport utility vehicles; in that sense, they are equivalent. They will both get you from place to place in a relatively fuel-efficient way. In this example we are comparing the single member set {Toyota Rav4 Hybrid} to the single member set {Ford Escape Hybrid}. Since these two sets have the same number of elements, they are also equivalent mathematically, meaning they have the same cardinality. But they are not equal, because the two cars have different looks and features, and probably even handle differently. Each manufacturer will emphasize the features unique to their vehicle to persuade you to buy it. If the SUVs were truly equal, there would be no reason to choose one over the other.

Now consider two Honda CR-Vs that are made with exactly the same parts, on the same assembly line within a few minutes of each other—these SUVs are equal. They are identical to each other, containing the same elements without regard to order, and the only differentiator when making a purchasing decision would be varied pricing at different dealerships. The set {Honda CR-V} is equal to the set {Honda CR-V}. Symbolically, we represent equal sets as $A = B$ and equivalent sets as $A \sim B$.

Now, let us consider a Toyota dealership that has 10 RAV4s on the lot, 8 Prii, 7 Highlanders, and 12 Camrys. There is a one-to-one relationship between the set of vehicles on the lot and the set consisting of the number of each type of vehicle on the lot. Therefore, these two sets are equivalent, but not equal. The set {RAV4, Prius, Highlander, Camry} is equivalent to the set {10, 8, 7, 12} because they have the same number of elements.

Video

Equal and Equivalent Sets

Checkpoint

If two sets are equal, they are also equivalent, because equal sets also have the same cardinality.

Example 4.2.10

Differentiating Between Equivalent and Equal Sets

Determine if the following pairs of sets are equal, equivalent, or neither.

1. $E = \{2, 4, 6, 8, 10\}$ and $F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$
2. The empty set and the set of prime numbers less than 2.
3. The set of vowels in the word "happiness" and the set of consonants in the word happiness.

Answer available at the end of this section.

Your Turn 4.2.10

Determine if the following pairs of sets are equal, equivalent, or neither.

1. $B = \{b, a, k, e\}$ and $A = \{a, b, e, k\}$
2. $B = \{b, a, k, e\}$ and $F = \{f, a, k, e\}$
3. $B = \{b, a, k, e\}$ and $C = \{c, a, k, e\}$

People in Mathematics

Georg Cantor

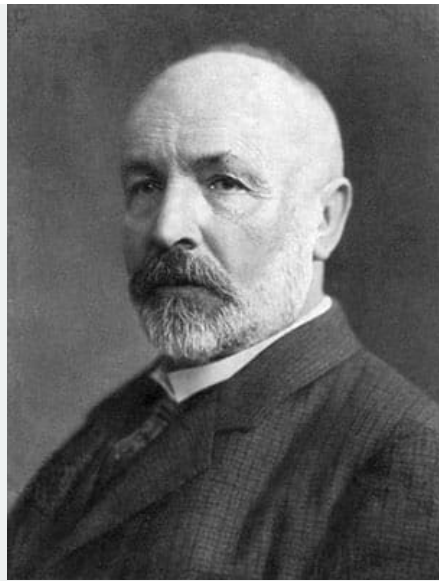


Figure 4.2.2: Georg Cantor (credit: Wikimedia, public domain)

Georg Cantor, the father of modern set theory, was born during the year 1845 in Saint Petersburg, Russia and later moved to Germany as a youth. Besides being an accomplished mathematician, he also played the violin. Cantor received his doctoral degree in Mathematics at the age of 22.

In 1870, at the age of 25, he established the uniqueness theorem for trigonometric series. His most significant work happened between 1874 and 1884, when he established the existence of transcendental numbers (also called irrational numbers). He proved that the set of real numbers are uncountably infinite—despite the objections of his former professor Leopold Kronecker.

Cantor published his final treatise on set theory in 1897 at the age of 52, and was awarded the Sylvester Medal from the Royal Society of London in 1904 for his contributions to the field. At the heart of Cantor's work was his goal to solve the continuum problem, which later influenced the works of David Hilbert and Ernst Zermelo.

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Wikipedia contributors. "Cantor." *Wikipedia, The Free Encyclopedia*, 23 Mar. 2021. Web. 20 Jul. 2021.

Akihiro Kanamori, "Set Theory from Cantor to Cohen," Editor(s): Dov M. Gabbay, Akihiro Kanamori, John Woods, *Handbook of the History of Logic, North-Holland, Volume 6*, 2012.

Answers

There are multiple possible answers depending on what your three favorite sports are, but any answer must list three different sports separated by commas, such as the following:

$$S = \{\text{hockey, basketball, soccer}\}$$

1. The group of all past vice presidents of the United States is a well-defined set, because you can clearly identify if any individual was or was not a member of that group. For example, Britney Spears is not a member of this set, but Joe Biden is a member of this set.
2. A group of old cats is not a well-defined set because the word "old" is ambiguous. Some people might consider a seven-year-old cat to be old, while others might think a cat is not old until it is 13 years old. Because people can disagree on what is and what is not a member of this group, the set is not well-defined.
1. A prime number is a natural number greater than 1 that is only divisible by one and itself. Since there are no prime numbers less than 2, this set is empty, and we can represent it symbolically as follows: $\{ \}$ or $\emptyset$. These two different symbols for the empty set can be used interchangeably.
2. The set of birds and the set of mammals do not intersect, so the set of birds that are also mammals is empty, and we can represent it symbolically as $\emptyset$ or $\{ \}$.

Write the label, E , followed by an equal sign and then a bracket. Write the first three even numbers separated by commas, beginning with the number two to establish a pattern. Next, write the ellipsis followed by a comma and the last number in the list, 100. Finally, write the closing bracket to complete the set.

Write the label, E , followed by an equal sign and then a bracket.

$$E = \{$$

Write the first three even numbers separated by commas, beginning with the number 2 to establish a pattern.

$$E = \{2, 4, 6,$$

Next, write the ellipsis followed by a comma and the last number in the list, 100.

$$E = \{2, 4, 6, \dots, 100$$

Finally, write the closing bracket to complete the set.

$$E = \{2, 4, 6, \dots, 100\}$$

Step 1: As always, we write the label and then the opening bracket. Because the negative counting numbers are infinite, to represent that the pattern continues without bound to the left, we must use an ellipsis as the first element in our list.

Step 2: We place a comma and follow it with at least three consecutive integers separated by commas to establish a pattern.

Step 3: Add an ellipsis to the end of the list to show that the set of integers continues without bound to the right.

Complete the list with a closing bracket. The set of integers may be represented as follows: $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.

The verbal description of the set is, "Set B is the set of all elements b such that b is a ball." This set can be written in set builder notation as follows:
 $B = \{b | b \text{ is a ball.}\}$

Because the letters in the word "happy" consist of a small finite set, the best way to represent this set is with the roster method. Choose a label to represent the set, such as H .

$$H = \{h, a, p, y\}.$$

Notice that the letter "p" is only represented one time. This occurs because when representing members of a set, each unique element is only listed once no matter how many times it occurs. Duplicate elements are never repeated when representing members of a set.

1. There are 5 distinct elements in set F : a fork, a spoon, a knife, a meat thermometer, and a can opener. Therefore, the cardinal value of set F is 5 and written symbolically as $n(F) = 5$.
2. Because the empty set does not have any elements in it, the cardinality of the empty set is zero. Symbolically we write this as: $n(\emptyset) = 0$.
1. $n(E) = 5$. Since 5 is a natural number, the set is finite.
2. $n(A) = 26$. Since 26 is a natural number, the set is finite.
3. Set $\mathbb{Q}$ is the set of rational numbers or fractions. Because the set of integers is a subset of the set of rational numbers, and the set of integers is infinite, the set of rational numbers is also infinite. There is no smallest or largest rational number.
1. Sets E and F both have a cardinal value of 5, but the elements in these sets are different. So, the two sets are equivalent, but they are not equal. $E \sim F$.
2. The set of prime numbers consists of the set of counting numbers greater than one that can only be divided evenly by one and itself. The set of prime numbers less than 2 is an empty set, since there are no prime numbers less than 2. Therefore, these two sets are equal (and equivalent).
3. The set of vowels in the word happiness is $\{a, i, e\}$ and the set of consonants in the word happiness is $\{h, p, n, s\}$. The cardinal value of these sets two sets is $n(\{a, i, e\}) = 3$ and $n(\{h, p, n, s\}) = 4$. Because the cardinality of the two sets differs, they are not equivalent. Further, their elements are not identical, so they are also not equal.

4.3: Subsets



Figure 4.3.1: The players on a soccer team who are actively participating in a game are a subset of the greater set of team members. (Credit: “PAFC-Mezokovesd-108” by Puskás Akadémia/Flickr, Public Domain Mark 1.0)

Learning Objectives

After completing this section, you should be able to:

1. Represent subsets and proper subsets symbolically.
2. Compute the number of subsets of a set.
3. Apply concepts of subsets and equivalent sets to finite and infinite sets.

The rules of Major League Soccer (MLS) allow each team to have up to 30 players on their team. However, only 18 of these players can be listed on the game day roster. And of the 18 listed, 11 players must be selected to start the game. How the coaches and general managers form the team and choose the starters for each game will determine the success of the team in any given year.

The entire group of 30 players is each team's set. The group of game day players is a subset of the team set, and the group of 11 starters is a subset of both the team set and the set of players on the game day roster.

Subset

Set A is a subset of set B if every member of set A is also a member of set B . Symbolically, this relationship is written as $A \subseteq B$.

Sets can be related to each other in several different ways: they may not share any members in common, they may share some members in common, or they may share all members in common. In this section, we will explore the way we can select a group of members from the whole set.

Checkpoint

Every set is also a subset of itself, $B \subseteq B$

Recall the set of flatware in our kitchen drawer from Section 4.1, $F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$. Suppose you are preparing to eat dinner, so you pull a fork and a knife from the drawer to set the table. The set $D = \{\text{knife, fork}\}$ is a subset of set F , because every member or element of set D is also a member of set F . More specifically, set D is a proper subset of set F , because there are other members of set F not in set D . This is written as $D \subset F$.

Proper Subset

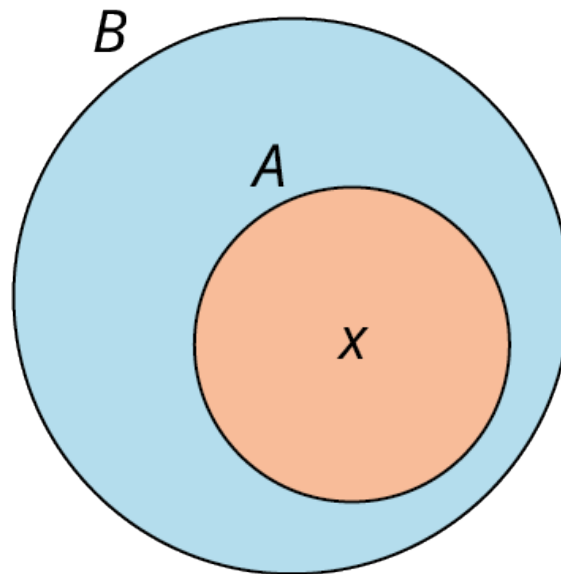
Set A is a proper subset of set B if every member of set A is also a member of set B , but B also contains at least one element that is not in A . Symbolically, this relationship is written as $A \subset B$.

The only subset of a set that is not a proper subset of the set would be the set itself.

Checkpoint

The empty set or null set, $\emptyset$, is a proper subset of every set, except itself.

Graphically, sets are often represented as circles. In the following graphic, set A is represented as a circle completely enclosed inside the circle representing set B , showing that set A is a proper subset of set B . The element x represents an element that is in both set A and set B .



Checkpoint

While we can list all the subsets of a finite set, it is not possible to list all the possible subsets of an infinite set, as it would take an infinitely long time.

Example 4.3.1

Listing All the Proper Subsets of a Finite Set

Set L is a set of reading materials available in a shop at the airport, $L = \{\text{newspaper, magazine, book}\}$. List all the subsets of set L .

Answer available at the end of this section.

Your Turn 4.3.1

Consider the set of possible outcomes when you flip a coin, $S = \{\text{heads, tails}\}$. List all the possible subsets of set S .

Example 4.3.2

Determining Whether a Set Is a Proper Subset

Consider the set of common political parties in the United States, $P = \{\text{Democratic, Green, Libertarian, Republican}\}$. Determine if the following sets are proper subsets of P .

1. $M = \{\text{Democratic, Republican}\}$
2. $G = \{\text{Green}\}$
3. $V = \{\text{Republican, Libertarian, Green, Democratic}\}$

Answer available at the end of this section.

Your Turn 4.3.2

Consider the set of generation I legendary Pokémon, $L = \{\text{Articuno, Zapdos, Moltres, Mewtwo}\}$. Give an example of a proper subset containing:

1. one member
2. three members
3. no members

Example 4.3.3

Expressing the Relationship between Sets Symbolically

Consider the subsets of a standard deck of cards: $S = \{\text{spades, hearts, diamonds, clubs}\}$; $R = \{\text{hearts, diamonds}\}$; $B = \{\text{spades, clubs}\}$; and $C = \{\text{clubs}\}$.

Express the relationship between the following sets symbolically.

1. Set S and set B .
2. Set C and set B .
3. Set R and R .

Answer available at the end of this section.

Your Turn 4.3.3

Express the relationship between the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \dots\}$ and the set of even numbers, $E = \{2, 4, 6, \dots\}$.

Number of Subsets

So far, we have figured out how many subsets exist in a finite set by listing them. Recall that in Example 4.3.1, when we listed all the subsets of the three-element set $L = \{\text{newspaper, magazine, book}\}$, we discovered that there were eight subsets. In Your Turn 4.3.1, we discovered that there are four subsets of the two-element subset, $S = \{\text{heads, tails}\}$. A one-element set has two subsets, the empty set and itself. The only subset of the empty set is the empty set itself. But how can we easily figure out the number of

subsets in a very large finite set? It turns out that the number of subsets can be found by raising 2 to the number of elements in the set.

FORMULA

The number of subsets of a finite set A is equal to 2 raised to the power of $n(A)$, where $n(A)$ is the number of elements in set A : Number of Subsets of Set $A = 2^{n(A)}$.

Checkpoint

Note that $2^0 = 1$, so this formula works for the empty set, also.

Example 4.3.4

Computing the Number of Subsets of a Set

Find the number of subsets of each of the following sets.

1. The set of top five scorers of all time in the NBA:
 $S = \{\text{LeBron James, Kareem Abdul-Jabbar, Karl Malone, Kobe Bryant, Michael Jordan}\}$.
2. The set of the top four bestselling albums of all time:
 $A = \{\text{Thriller, Hotel California, The Beatles White Album, Led Zepplin IV}\}$.
3. $R = \{\text{Snap, Crackle, Pop}\}$.

Answer available at the end of this section.

Your Turn 4.3.4

Compute the total number of subsets in the set of the top six tennis grand slam singles winners, $T = \{\text{Margaret Court, Serena Williams, Steffi Graff, Serena Williams, Rafael Nadal, Novak Djokovic}\}$.

Who Knew?

Employment Opportunities

You can make a career out of working with sets. Applications of equivalent sets include relational database design and analysis.

Relational databases that store data are tables of related information. Each row of a table has the same number of columns as every other row in the table; in this way, relational databases are examples of set equivalences for finite sets. In a relational database, a primary key is set up to identify all related information. There is a one-to-one relationship between the primary key and any other information associated with it.

Database design and analysis is a high demand career with a median entry-level salary of about \$85,000 per year, according to salary.com.

Answers

Step 1: It is best to begin with the set itself, as every set is a subset of itself. In our example, the cardinality of set L is $n(L) = 3$. There is only one subset of set L that has the same number of elements of set L : $\{\text{newspaper, magazine, book}\}$.

Step 2: Next, list all the proper subsets of the set containing $n(L) - 1$ elements. In this case, $3 - 1 = 2$. There are three subsets that each contain two elements: $\{\text{newspaper, magazine}\}$, $\{\text{newspaper, book}\}$, and $\{\text{magazine, book}\}$.

Step 3: Continue this process by listing all the proper subsets of the set containing $n(L) - 2$ elements. In this case, $3 - 2 = 1$. There are three subsets that contain

one element: {newspaper}, {magazine}, and {book}.

Step 4: Finally, list the subset containing 0 elements, or the empty set: { }.

1. M is a proper subset of P , written symbolically as $M \subset P$ because every member of M is a member of set P , but P also contains at least one element that is not in M .
2. G is a single member proper subset of P , written symbolically as $G \subset P$, because Green is a member of set P , but P also contains other members (such as Democratic) that are not in G .
3. V is subset of P because every member of V is also a member of P , but it is not a proper subset of P because there are no members of V that are not also in set P . We can represent the relationship symbolically as $V \subseteq P$, or more precisely, set V is equal to set P , $V = P$.
1. $B \subset S$. B is a proper subset of set S .
2. $C \subset B$. C is a proper subset of set B .
3. $R \subseteq R$ or $R = R$. R is subset of itself, but not a proper subset of itself because R is equal to itself.
1. $n(S) = 5$. So, the total number of subsets of S is $2^5 = 32$.
2. $n(A) = 4$. Therefore, the total number of subsets of A is $2^4 = 16$.
3. $n(R) = 3$. So, the total number of subsets of R is $2^3 = 8$.

4.4: Understanding Venn Diagrams

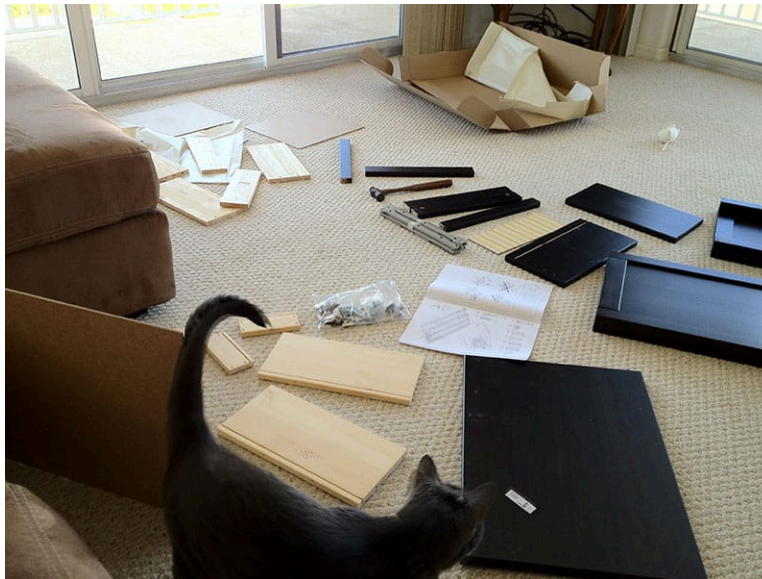


Figure 4.4.1: When assembling furniture, instructions with images are easier to follow, just like how set relationships are easier to understand when depicted graphically. (credit: "Time to assemble more Ikea furniture!" by Rod Herrea/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Utilize a universal set with two sets to interpret a Venn diagram.
2. Utilize a universal set with two sets to create a Venn diagram.
3. Determine the complement of a set.

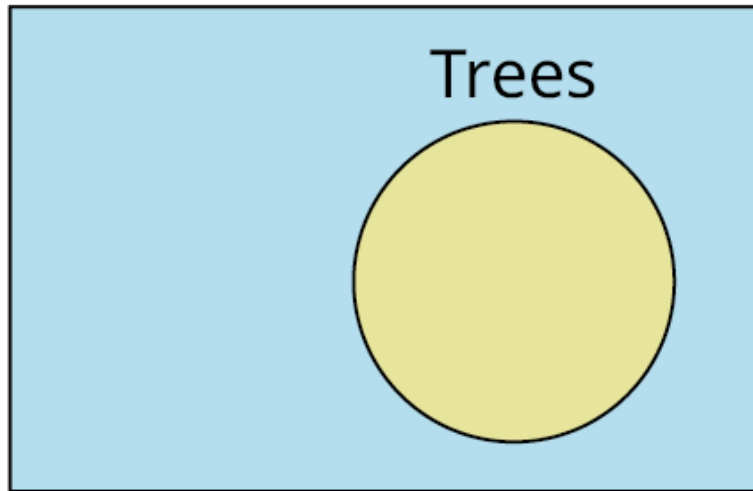
Have you ever ordered a new dresser or bookcase that required assembly? When your package arrives you excitedly open it and spread out the pieces. Then you check the assembly guide and verify that you have all the parts required to assemble your new dresser. Now, the work begins. Luckily for you, the assembly guide includes step-by-step instructions with images that show you how to put together your product. If you are really lucky, the manufacturer may even provide a URL or QR code connecting you to an online video that demonstrates the complete assembly process. We can likely all agree that assembly instructions are much easier to follow when they include images or videos, rather than just written directions. The same goes for the relationships between sets.

Interpreting Venn Diagrams

Venn diagrams are the graphical tools or pictures that we use to visualize and understand relationships between sets. Venn diagrams are named after the mathematician John Venn, who first popularized their use in the 1880s. When we use a Venn diagram to visualize the relationships between sets, the entire set of data under consideration is drawn as a rectangle, and subsets of this set are drawn as circles completely contained within the rectangle. The entire set of data under consideration is known as the universal set.

Consider the statement: All trees are plants. This statement expresses the relationship between the set of all plants and the set of all trees. Because every tree is a plant, the set of trees is a subset of the set of plants. To represent this relationship using a Venn diagram, the set of plants will be our universal set and the set of trees will be the subset. Recall that this relationship is expressed symbolically as: $\text{Trees} \subset \text{Plants}$. To create a Venn diagram, first we draw a rectangle and label the universal set " $U = \text{Plants}$." Then we draw a circle within the universal set and label it with the word "Trees."

$U = \text{Plants}$



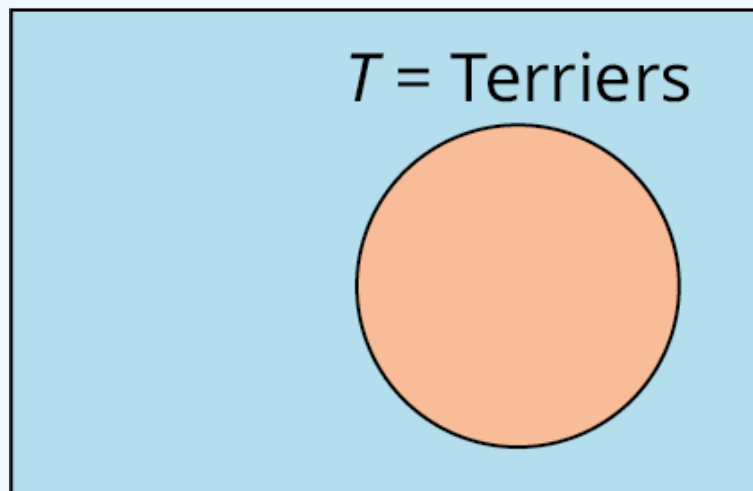
This section will introduce how to interpret and construct Venn diagrams. In future sections, as we expand our knowledge of relationships between sets, we will also develop our knowledge and use of Venn diagrams to explore how multiple sets can be combined to form new sets.

Example 4.4.1

Interpreting the Relationship between Sets in a Venn Diagram

Write the relationship between the sets in the following Venn diagram, in words and symbolically.

$U = \text{Dogs}$

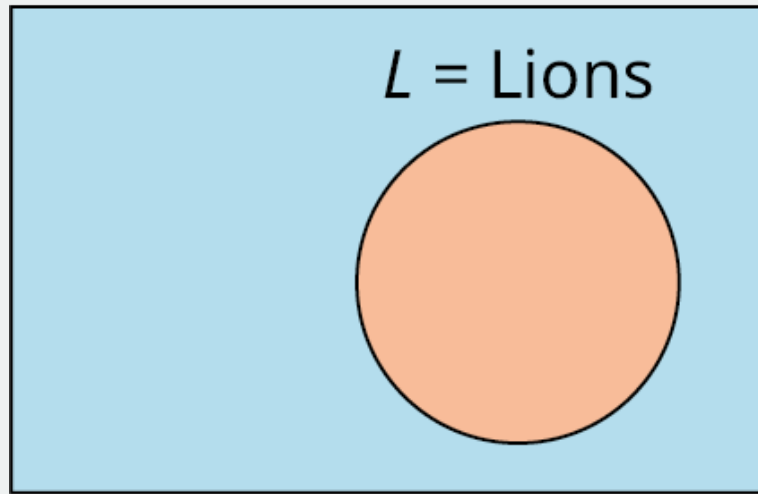


Answer available at the end of this section.

Your Turn 4.4.1

Write the relationship between the sets in the following Venn diagram, in words and symbolically.

$U = \text{Cats}$



So far, the only relationship we have been considering between two sets is the subset relationship, but sets can be related in other ways. Lions and tigers are both different types of cats, but no lions are tigers, and no tigers are lions. Because the set of all lions and the set of all tigers do not have any members in common, we call these two sets disjoint sets, or non-overlapping sets.

Disjoint Sets

Two sets A and B are disjoint sets if they do not share any elements in common. That is, if a is a member of set A , then a is not a member of set B . If b is a member of set B , then b is not a member of set A .

To represent the relationship between the set of all cats and the sets of lions and tigers using a Venn diagram, we draw the universal set of cats as a rectangle and then draw a circle for the set of lions and a separate circle for the set of tigers within the rectangle, ensuring that the two circles representing the set of lions and the set of tigers do not touch or overlap in any way.

$U = \text{Cats}$

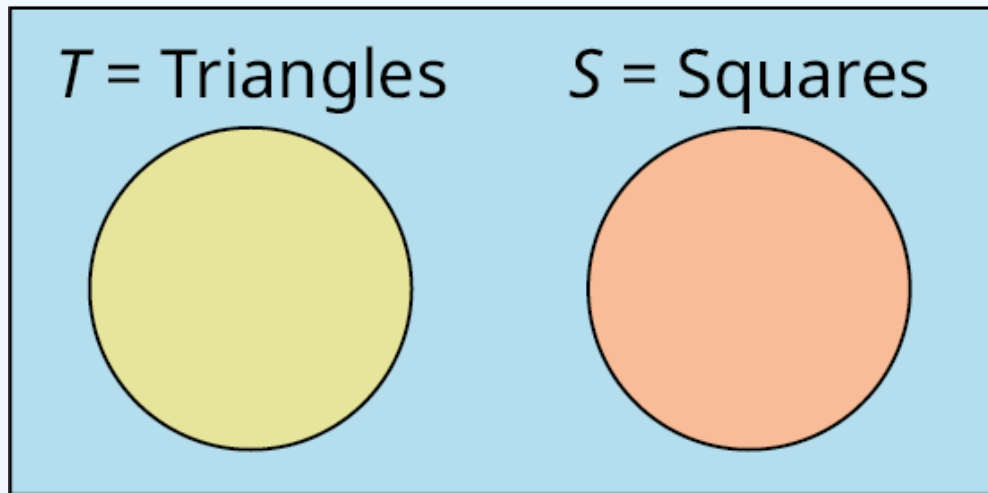


Example 4.4.2

Describing the Relationship between Sets

Describe the relationship between the sets in the following Venn diagram.

$U = 2D \text{ Figures}$

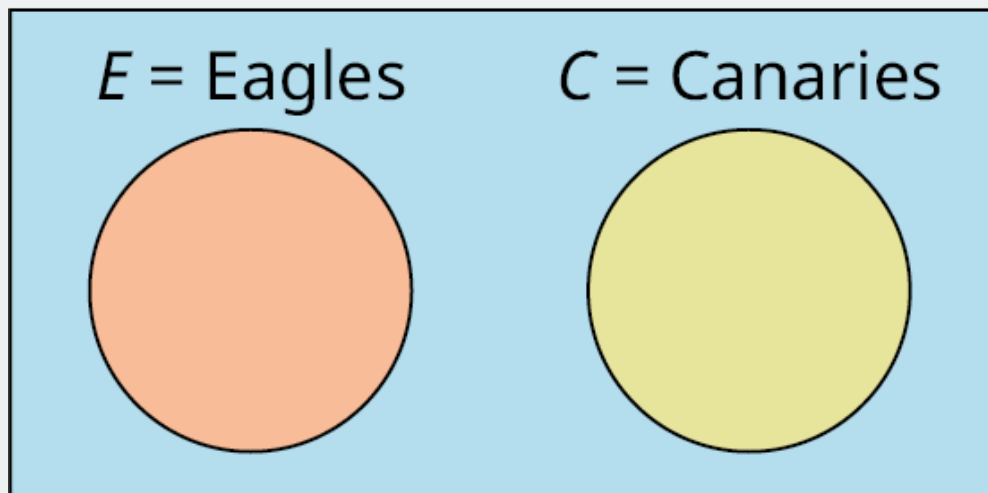


Answer available at the end of this section.

Your Turn 4.4.2

Describe the relationship between the sets in the following Venn diagram.

$U = \text{Birds}$



Creating Venn Diagrams

The main purpose of a Venn diagram is to help you visualize the relationship between sets. As such, it is necessary to be able to draw Venn diagrams from a written or symbolic description of the relationship between sets.

How To

Create a Venn diagram

1. Draw a rectangle to represent the universal set, and label it U .
2. Draw a circle within the rectangle to represent a subset of the universal set and label it with the set name.

Checkpoint

If there are multiple disjoint subsets of the universal set, their separate circles should not touch or overlap.

Example 4.4.3

Drawing a Venn Diagram to Represent the Relationship Between Two Sets

Draw a Venn diagram to represent the relationship between each of the sets.

1. All rectangles are parallelograms.
2. All women are people.

Answer available at the end of this section.

Your Turn 4.4.3

Draw a Venn diagram to represent the relationship between each of the sets.

1. All natural numbers are integers.
2. $A \subseteq U$

Example 4.4.4

Drawing a Venn Diagram to Represent the Relationship Between Three Sets

All bicycles and all cars have wheels, but no bicycle is a car. Draw a Venn diagram to represent this relationship.

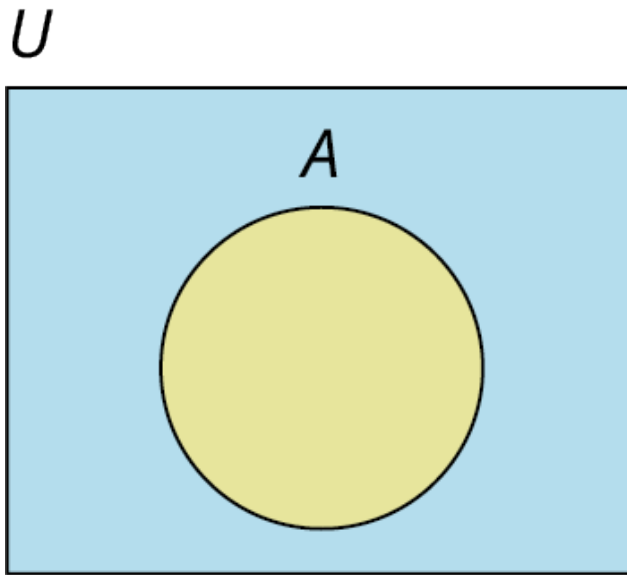
Answer available at the end of this section.

Your Turn 4.4.4

Airplanes and birds can fly, but no birds are airplanes. Draw a Venn diagram to represent this relationship.

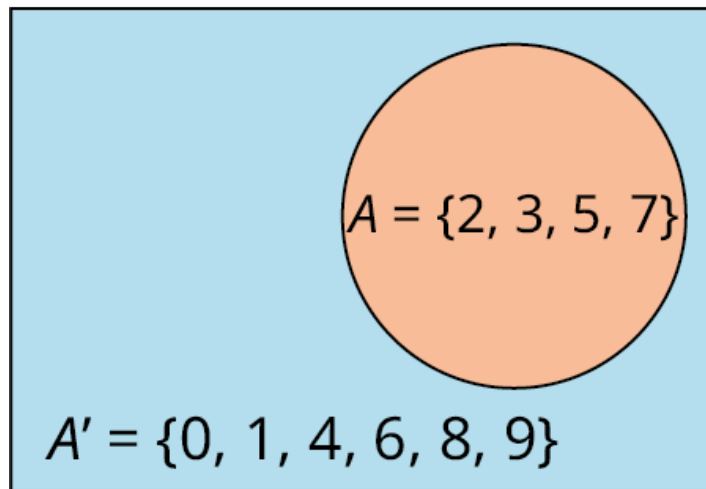
The Complement of a Set

Recall that if set A is a proper subset of set U , the universal set, (written symbolically as $A \subset U$), then there is at least one element in set U that is not in set A . The set of all the elements in the universal set U that are not in the subset A is called the complement of set A , A' . In set builder notation this is written symbolically as: $A' = \{x \in U | x \notin A\}$. The symbol $\in$ is used to represent the phrase, "is a member of," and the symbol $\notin$ is used to represent the phrase, "is not a member of." In the Venn diagram below, the complement of set A is the region that lies outside the circle and inside the rectangle. The universal set U includes all of the elements in set A and all of the elements in the complement of set A , and nothing else.



Consider the set of digit numbers. Let this be our universal set, $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Now, let set A be the subset of U consisting of all the prime numbers in set U , $A = \{2, 3, 5, 7\}$. The complement of set A is $A' = \{0, 1, 4, 6, 8, 9\}$. The following Venn diagram represents this relationship graphically.

$$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$



Example 4.4.5

Finding the Complement of a Set

For both of the questions below, A is a proper subset of U .

1. Given the universal set $U = \{\text{Billie Eilish, Donald Glover, Bruno Mars, Adele, Ed Sheeran}\}$ and set $A = \{\text{Donald Glover, Bruno Mars, Ed Sheeran}\}$, find A' .
2. Given the universal set $U = \{d | d \text{ is a dog}\}$ and $B = \{b \in U | b \text{ is a beagle}\}$, find B' .

Answer available at the end of this section.

Your Turn 4.4.5

For both of the questions below, A is a proper subset of U .

1. Given the universal set $U = \{\text{red, orange, yellow, green, blue, indigo, violet}\}$ and set $A = \{\text{yellow, red, blue}\}$, find A' .
2. Given the universal set $U = \{c \mid c \text{ is a cat}\}$ and set $A = \{c \text{ in } U \mid c \text{ is not a lion}\}$, find A' .

Answers

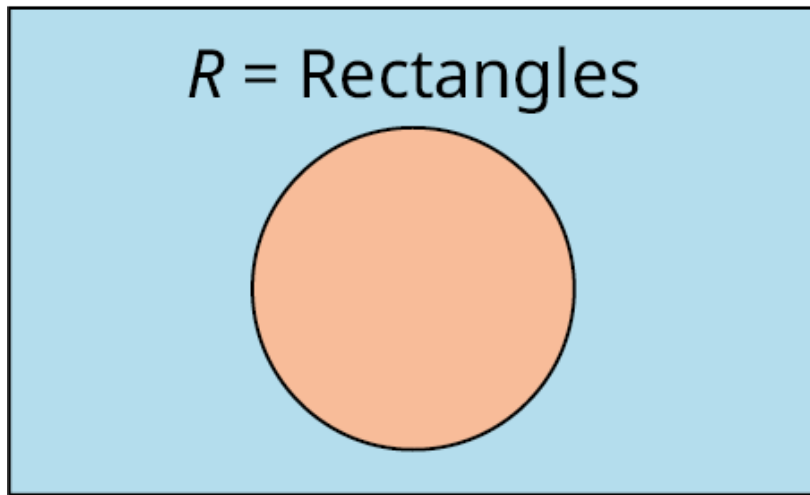
The set of terriers is a subset of the universal set of dogs. In other words, the Venn diagram depicts the relationship that all terriers are dogs. This is expressed symbolically as $T \subset U$.

The set of triangles and the set of squares are two disjoint subsets of the universal set of two-dimensional figures. The set of triangles does not share any elements in common with the set of squares. No triangles are squares and no squares are triangles, but both squares and triangles are 2D figures.

1. The set of rectangles is a subset of the set of parallelograms.

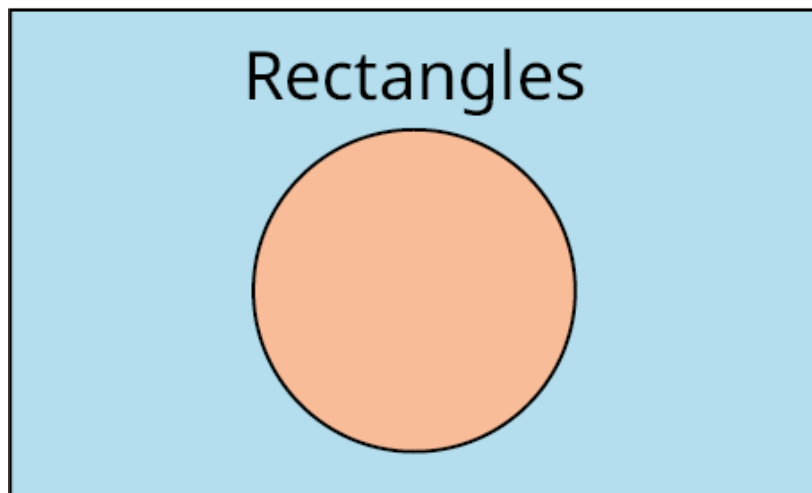
First, draw a rectangle to represent the universal set and label it with $U = \text{parallelograms}$, then draw a circle completely within the rectangle, and label it with the name of the set it represents, $R = \text{Rectangles}$.

$U = \text{Parallelograms}$

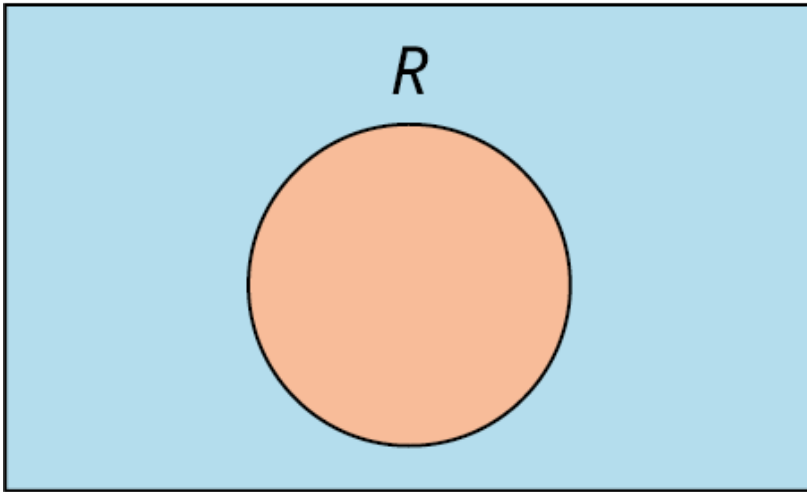


In this example, both letters and names are used to represent the sets involved, but this is not necessary. You may use either letters or names alone, as long as the relationship is clearly depicted in the diagram, as shown below.

Parallelograms

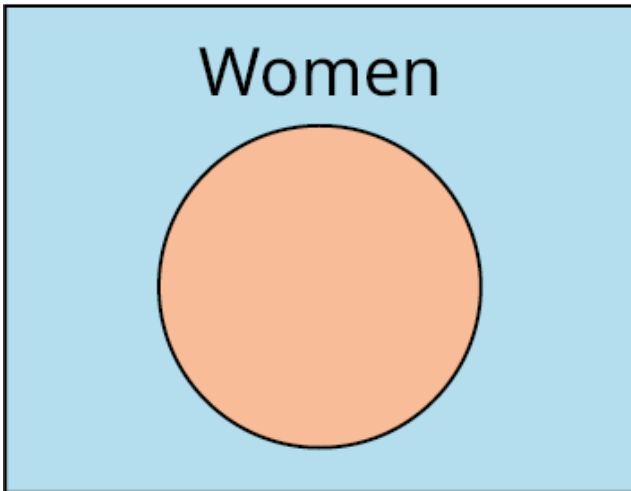


U



2. The universal set is the set of people, and the set of all women is a subset of the set of people.

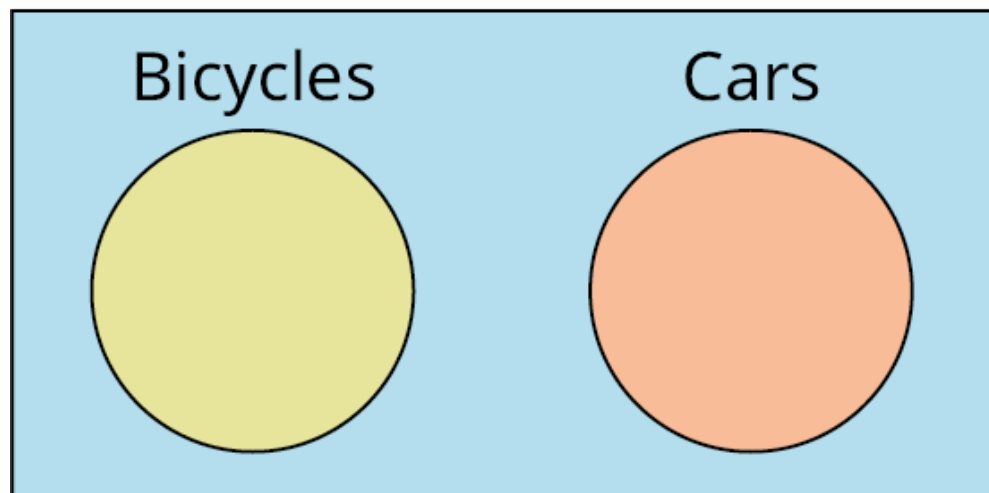
People



Step 1: The set of bicycles and the set of cars are both subsets of the set of things with wheels. The universal set is the set of things with wheels, so we first draw a rectangle and label it with $U = \text{Things with Wheels}$.

Step 2: Because the set of bicycles and the set of cars do not share any elements in common, these two sets are disjoint and must be drawn as two circles that do not touch or overlap with the universal set.

U = Things with Wheels



1. The complement of set A is the set of all elements in the universal set U that are not in set A . $A' = \{\text{Billie Eilish, Adele}\}$.
2. The complement of set B is the set of all dogs that are not beagles. All members of set B' are in the universal set because they are dogs, but they are not in set B , because they are not beagles. This relationship can be expressed in set build notation as follows: $B' = \{\text{All dogs that are not beagles.}\}$, $B' = \{d \in U \mid d \text{ is not a beagle.}\}$, or $B' = \{d \in U \mid d \notin B\}$.

4.5: Set Operations with Two Sets



Figure 4.5.1: A large, multigenerational family contains an intersection and a union of sets. (credit: "Family Photo Shoot Bani Syakur" by Mainur Risyada/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Determine the intersection of two sets.
2. Determine the union of two sets.
3. Determine the cardinality of the union of two sets.
4. Apply the concepts of AND and OR to set operations.
5. Draw conclusions from Venn diagrams with two sets.

The movie *Yours, Mine, and Ours* was originally released in 1968 and starred Lucille Ball and Henry Fonda. This movie, which is loosely based on a true story, is about the marriage of Helen, a widow with eight children, and Frank, a widower with ten children, who then have an additional child together. The movie is a comedy that plays on the interpersonal and organizational struggles of feeding, bathing, and clothing twenty people in one household.

If we consider the set of Helen's children and the set of Frank's children, then the child they had together is the intersection of these two sets, and the collection of all their children combined is the union of these two sets. In this section, we will explore the operations of union and intersection as it relates to two sets.

The Intersection of Two Sets

Intersection

The members that the two sets share in common are included in the intersection of two sets. To be in the intersection of two sets, an element must be in both the first set and the second set. In this way, the intersection of two sets is a logical AND statement. Symbolically, A intersection B is written as: $A \cap B$. A intersection B is written in set builder notation as: $A \cap B = \{x | x \in A \text{ and } x \in B\}$.

Let us look at Helen's and Frank's children from the movie *Yours, Mine, and Ours*. Helen's children consist of the set

$H = \{\text{Colleen, Nick, Janette, Tommy, Jean, Phillip, Gerald, Theresa, Joseph}\}$ and Frank's children are included in the set $F = \{\text{Mike, Rusty, Greg, Rosemary, Loise, Susan, Veronica, Mary, Germaine, Joan, Joseph}\}$. H intersection F is the set of children they had together. $H \cap F = \{\text{Joseph}\}$, because Joseph is in both set H and set F .

Example 4.5.1

Finding the Intersection of Set A and Set B

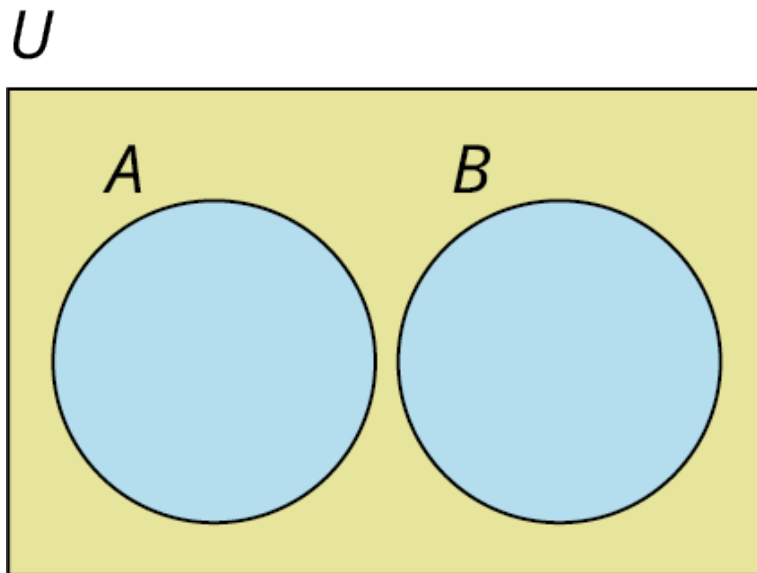
Set $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 5, 7\}$. Find $A \cap B$

Answer available at the end of this section.

Your Turn 4.5.1

Set $A = \{h, a, p, y\}$ and $B = \{s, a, d\}$. Find $A \cap B$.

Notice that if sets A and B are disjoint sets, then they do not share any elements in common, and $A \cap B$ is the empty set, as shown in the Venn diagram below.



Example 4.5.2

Determining the Intersection of Disjoint Sets

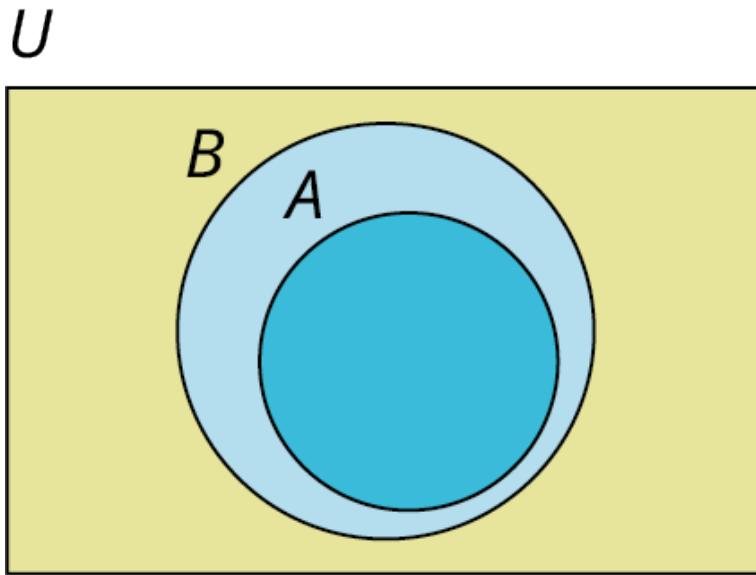
Set $A = \{0, 2, 4, 6, 8\}$ and set $B = \{1, 3, 5, 7, 9\}$. Find $A \cap B$.

Answer available at the end of this section.

Your Turn 4.5.2

Set $A = \{\text{red, yellow, blue}\}$ and Set $B = \{\text{orange, green, purple}\}$. Find $A \cap B$

Notice that if set A is a subset of set B , then A intersection B is equal to set A , as shown in the Venn diagram below.



Example 4.5.3

Finding the Intersection of a Set and a Subset

Set $A = \{1, 3, 5, \dots\}$ and set $B = \mathbb{N} = \{1, 2, 3, \dots\}$ Find $A \cap B$.

Answer available at the end of this section.

Your Turn 4.5.3

Set $A = \{a, b, c, \dots, z\}$ and set $B = \{a, e, i, o, u\}$. Find $A \cap B$

The Union of Two Sets

Like the union of two families in marriage, the union of two sets includes all the members of the first set and all the members of the second set.

Union

To be in the union of two sets, an element must be in the first set, the second set, or both. In this way, the union of two sets is a logical inclusive OR statement. Symbolically, A union B is written as: $A \cup B$. A union B is written in set builder notation as: $A \cup B = \{x | x \in A \text{ or } x \in B\}$.

Let us consider the sets of Helen's and Frank's children from the movie *Yours, Mine, and Ours* again. Helen's children is set $H = \{\text{Colleen, Nick, Janette, Tommy, Jean, Phillip, Gerald, Theresa, Joseph}\}$ and Frank's children is set $F = \{\text{Mike, Rusty, Greg, Rosemary, Loise, Susan, Veronica, Mary, Germaine, Joan, Joseph}\}$. The union of these two sets is the collection of all nineteen of their children, $H \cup F = \{\text{Colleen, Nick, Janette, Tommy, Jean, Phillip, Gerald, Theresa, Joseph, Mike, Rusty, Greg, Rosemary, Loise, Susan, Veronica, Mary, Germaine, Joan}\}$. Notice, Joseph is in both set H and set F , but he is only one child, so, he is only listed once in the union.

Example 4.5.4

Finding the Union of Sets A and B When A and B Overlap

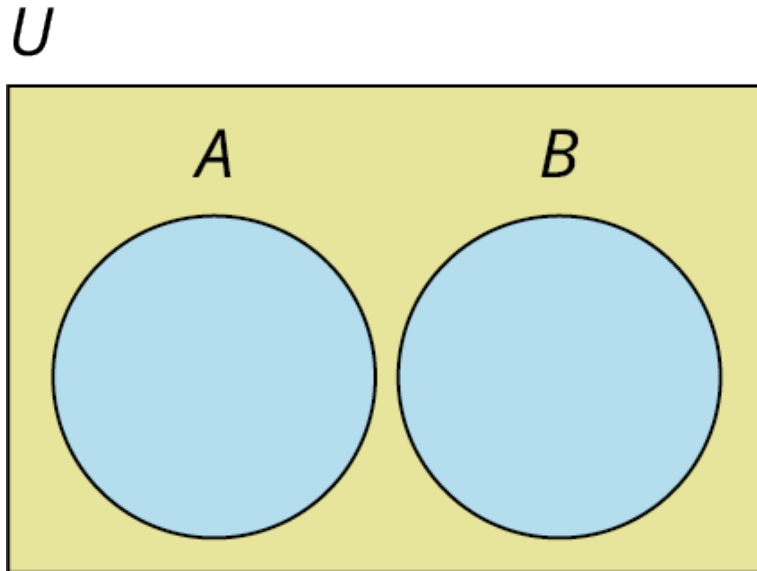
Set $A = \{1, 3, 5, 7, 9\}$ and set $B = \{2, 3, 5, 7\}$. Find $A \cup B$.

Answer available at the end of this section.

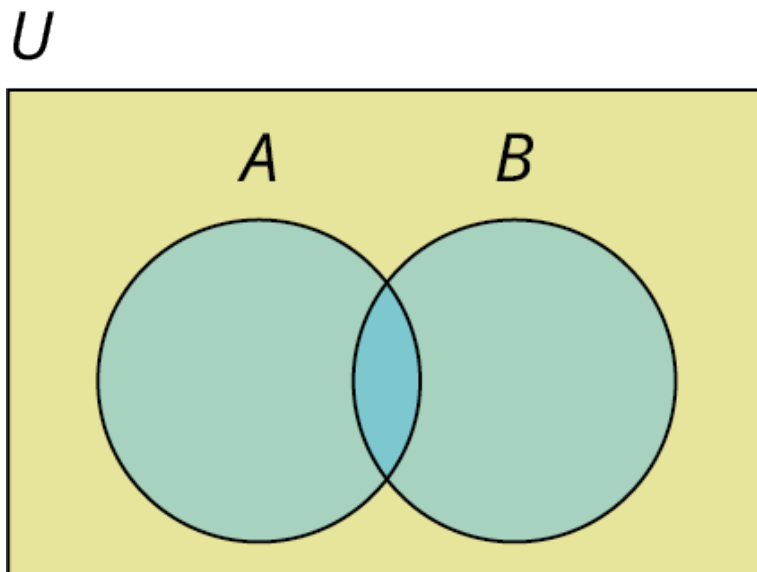
Your Turn 4.5.4

Set $A = \{h, a, p, y\}$ and set $B = \{s, a, d\}$. Find $A \cup B$.

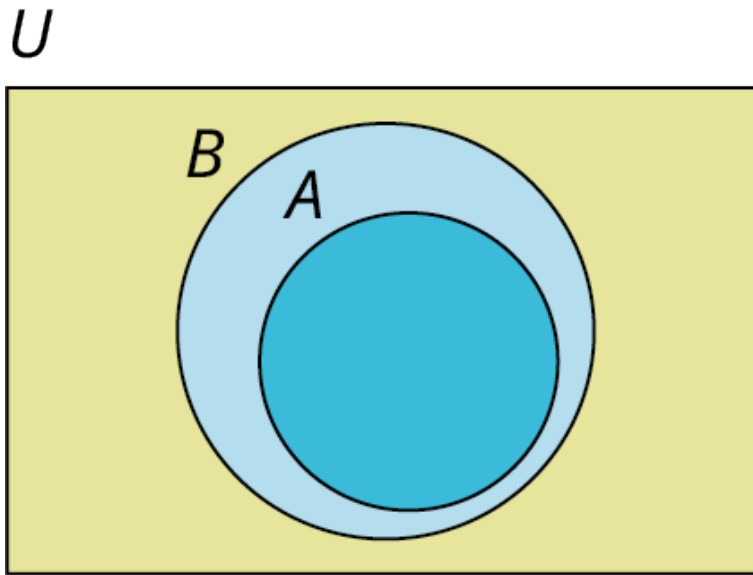
When observing the union of sets A and B , notice that both set A and set B are subsets of $A \cup B$. Graphically, $A \cup B$ can be represented in several different ways depending on the members that they have in common. If A and B are disjoint sets, then $A \cup B$ would be represented with two disjoint circles within the universal set, as shown in the Venn diagram below.



If sets A and B share some, but not all, members in common, then the Venn diagram is drawn as two separate circles that overlap.



If every member of set A is also a member of set B , then A is a subset of set B , and $A \cup B$ would be equal to set B . To draw the Venn diagram, the circle representing set A should be completely enclosed in the circle containing set B .



Example 4.5.5

Finding the Union of Sets A and B When A and B Are Disjoint

Set $A = \{0, 2, 4, 6, 8\}$ and set $B = \{1, 3, 5, 7, 9\}$. Find $A \cup B$.

Answer available at the end of this section.

Your Turn 4.5.5

Set $A = \{\text{red, yellow, blue}\}$ and Set $B = \{\text{orange, green, purple}\}$. Find $A \cup B$.

Example 4.5.6

Finding the Union of Sets A and B When One Set is a Subset of the Other

Set $A = \{1, 3, 5, \dots\}$ and set $B = \mathbb{N} = \{1, 2, 3, \dots\}$. Find $A \cup B$.

Answer available at the end of this section.

Your Turn 4.5.6

Set $A = \{a, b, c, \dots, z\}$ and set $B = \{a, e, i, o, u\}$. Find $A \cup B$.

Video

[The Basics of Intersection of Sets, Union of Sets and Venn Diagrams](#)

Let's think back to our example of the children from the movie *Yours, Mine, and Ours*. We had $n(H) = 9$, since Helen had 9 children. We had $n(F) = 11$, since Frank had 11 children. What was $n(H \cup F)$? You may be tempted to guess that it was $9 + 11 = 20$. But $n(H \cup F)$ was actually 19. Remember that the child Helen and Frank had together (Joseph) is included in set H and set F , but he should only be counted once in $H \cup F$. We generalize this result in the formula below.

Cardinality of the Union of Two Sets

For any sets A and B , $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Example 4.5.7

Determining the Cardinality of the Union of Two Sets

The number of elements in set A is 10, the number of elements in set B is 20, and the number of elements in A intersection B is 4. Find the number of elements in $A \cup B$.

Answer available at the end of this section.

Your Turn 4.5.7

If $n(A) = 23$, $n(B) = 17$, and $n(A \cap B) = 7$. Find $n(A \cup B)$.

Example 4.5.8

Determining the Cardinality of the Union of Two Disjoint Sets

If A and B are disjoint sets and the cardinality of set A is 37 and the cardinality of set B is 43, find the cardinality of $A \cup B$.

Answer available at the end of this section.

Your Turn 4.5.8

If $A \cap B = \emptyset$, $n(A) = 35$, and $n(B) = 78$, then find $n(A \cup B)$.

Applying Concepts of “AND” and “OR” to Set Operations

To become a licensed driver, you must pass some form of written test and a road test, along with several other requirements depending on your age. To keep this example simple, let us focus on the road test and the written test. If you pass the written test but fail the road test, you will not receive your license. If you fail the written test, you will not be allowed to take the road test and you will not receive a license to drive. To receive a driver's license, you must pass the written test AND the road test. For an “AND” statement to be true, both conditions that make up the statement must be true. Similarly, the intersection of two sets A and B is the set of elements that are in both set A and set B . To be a member of $A \cap B$, an element must be in set A and also must be in set B . The intersection of two sets corresponds to a logical “AND” statement.

The union of two sets is a logical inclusive “OR” statement. Say you are at a birthday party and the host offers Leah, Lenny, Maya, and you some cake or ice cream for dessert. Leah asks for cake, Lenny accepts both cake and ice cream, Maya turns down both, and you choose only ice cream. Leah, Lenny, and you are all having dessert. The “OR” statement is true if at least one of the components is true. Maya is the only one who did not have cake or ice cream; therefore, she did not have dessert and the “OR” statement is false. To be in the union of two sets A and B , an element must be in set A or set B or both set A and set B .

Example 4.5.9

Applying the “AND” or “OR” Operation

$A = \{0, 3, 6, 9, 12\}$, $B = \{0, 4, 8, 12, 16\}$, and $C = \{1, 2, 3, 5, 8, 13\}$.

Find the set consisting of elements in:

1. A and B .
2. A or B .
3. A or C .
4. $(B \text{ and } C) \text{ or } A$.

Answer available at the end of this section.

Your Turn 4.5.9

Set $A = \{h, a, p, y\}$, set $B = \{a, w, e, s, o, m\}$, and set $C = \{m, a, t, h\}$.
Find the set consisting of elements in:

1. A or B
2. A and C
3. B or C
4. $(A \text{ and } C) \text{ and } B$

Example 4.5.10

Determine and Apply the Appropriate Set Operations to Solve the Problem

Don Woods is serving cake and ice cream at his Juneteenth celebration. The party has a total of 54 guests in attendance. Suppose 30 guests requested cake, 20 guests asked for ice cream, and 12 guests did not have either cake or ice cream.

1. How many guests had cake or ice cream?
2. How many guests had cake and ice cream?

Answer available at the end of this section.

Your Turn 4.5.10

Ravi and Priya are serving soup and salad along with the main course at their wedding reception. The reception will have a total of 150 guests in attendance. A total of 92 soups and 85 salads were ordered, while 23 guests did not order any soup or salad.

1. How many guests had soup or salad or both?
2. How many guests had both soup and a salad?

Who Knew?

The Real Inventor of the Venn Diagram

John Venn, in his writings, references works by both John Boole and Augustus De Morgan, who referred to the circle diagrams commonly used to present logical relationships as Euler's circles. Leonhard Euler's works were published over 100 years prior to Venn's, and Euler may have been influenced by the works of Gottfried Leibniz.

So, why does John Venn get all the credit for these graphical depictions? Venn was the first to formalize the use of these diagrams in his book *Symbolic Logic*, published in 1881. Further, he made significant improvements in their design, including shading to highlight the region of interest. The mathematician C.L. Dodgson, also known as Lewis Carroll, built upon Venn's work by adding an enclosing universal set.

Invention is not necessarily coming up with an initial idea. It is about seeing the potential of an idea and applying it to a new situation.

References:

Margaret E. Baron. "A Note on the Historical Development of Logic Diagrams: Leibniz, Euler and Venn." *The Mathematical Gazette*, vol. 53, no. 384, 1969, pp. 113-125. *JSTOR*, www.jstor.org/stable/3614533. Accessed 15 July 2021.

Deborah Bennett. "Drawing Logical Conclusions." *Math Horizons*, vol. 22, no. 3, 2015, pp. 12-15. *JSTOR*, www.jstor.org/stable/10.4169/mathhorizons.22.3.12. Accessed 15 July 2021.

Drawing Conclusions from a Venn Diagram with Two Sets

All Venn diagrams will display the relationships between the sets, such as subset, intersecting, and/or disjoint. In addition to displaying the relationship between the two sets, there are two main additional details that Venn diagrams can include: the individual members of the sets or the cardinality of each disjoint subset of the universal set.

A Venn diagram with two subsets will partition the universal set into 3 or 4 sections depending on whether they are disjoint or intersecting sets. Recall that the complement of set A , written A' , is the set of all elements in the universal set that are not in set A .

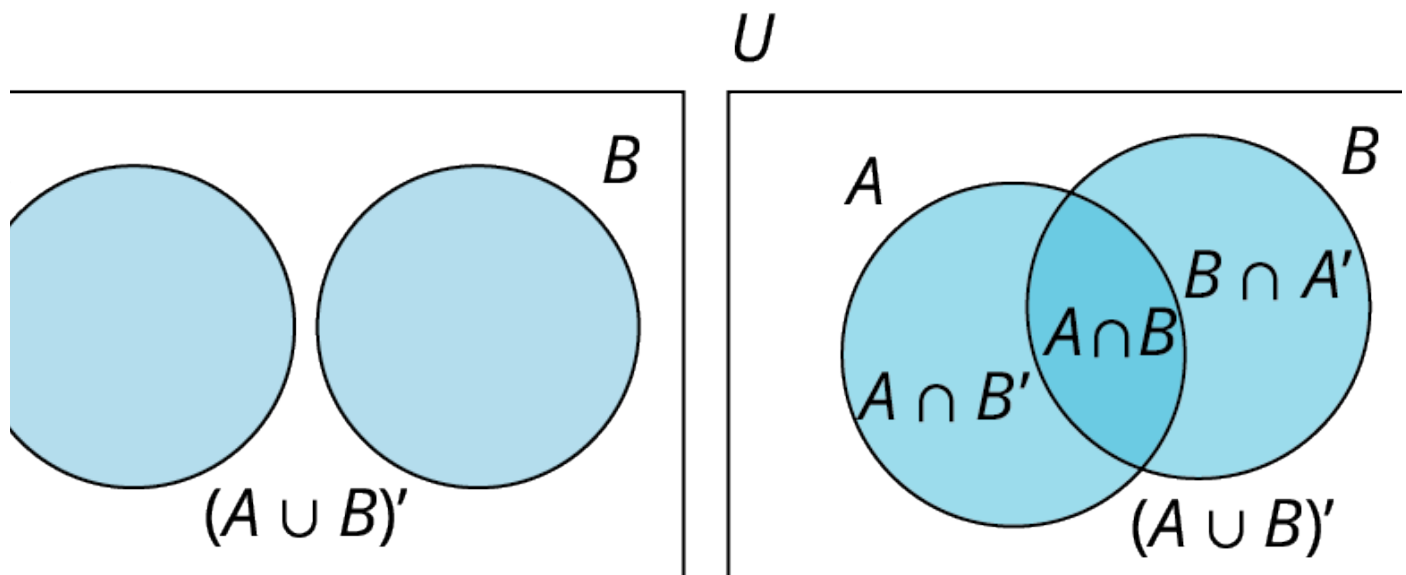


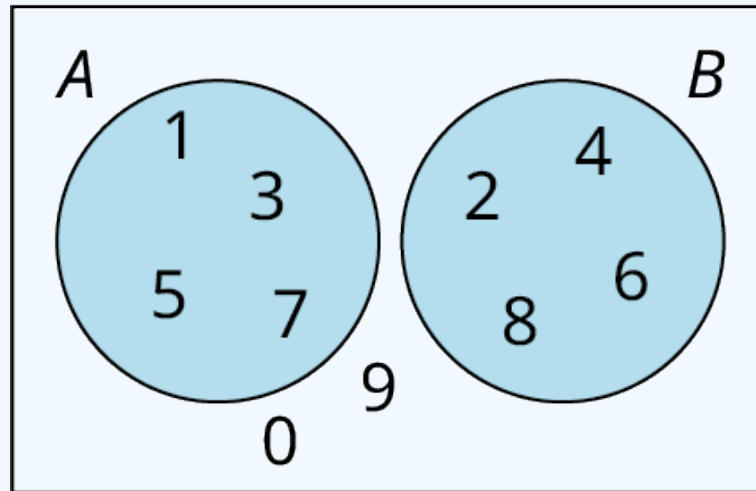
Figure 4.5.2:

Side-by-side Venn diagrams with disjoint and intersecting sets, respectively.

Example 4.5.11

Using a Venn Diagram to Draw Conclusions about Set Membership

$$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

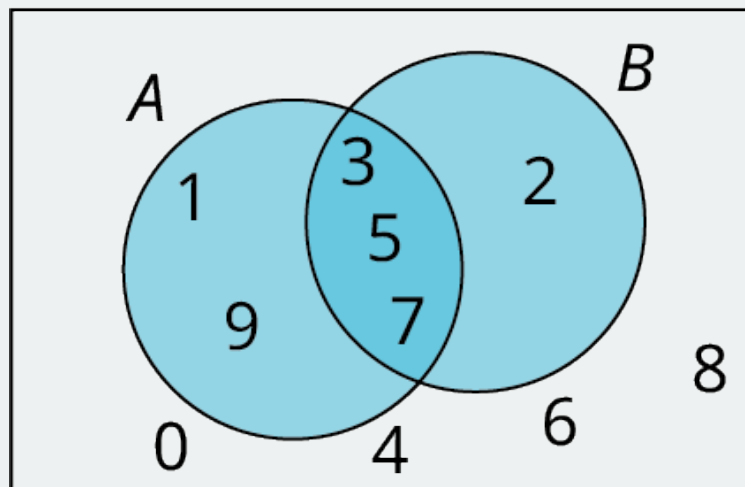


1. Find $A \cup B$.
2. Find $A \cap B$.
3. Find B' .
4. Find $n(B')$.

Answer available at the end of this section.

Your Turn 4.5.11

$$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

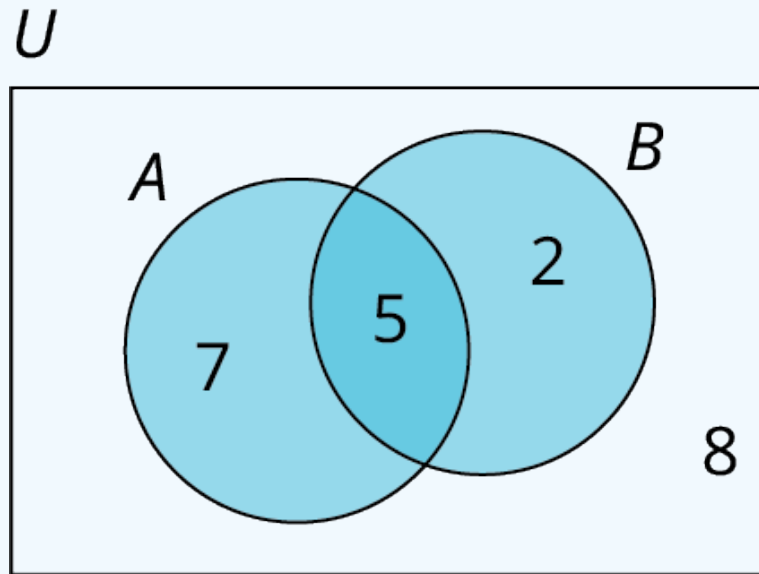


1. Find $A \cap B$
2. Find $A \cup B$
3. Find $A \cap B'$
4. Find $n(A \cap B')$

Example 4.5.12

Using a Venn Diagram to Draw Conclusions about Set Cardinality

In the Venn diagram below, the cardinality is listed in each section.

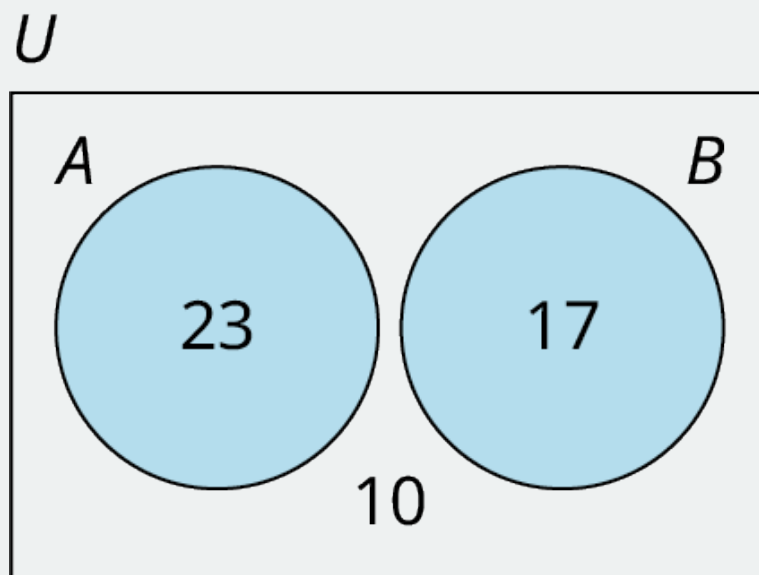


1. Find $n(A \text{ or } B)$.
2. Find $n(A \text{ and } B)$.
3. Find $n(A)$.

Answer available at the end of this section.

Your Turn 4.5.12

The Venn diagram below shows the cardinality of each set.



1. Find $n(A \text{ or } B)$.
2. Find $n(A \text{ and } B)$.
3. Find $n(A')$.

Answers

The intersection of sets A and B include the elements that set A and B have in common: 3, 5, and 7. $A \cap B = \{3, 5, 7\}$.

Because sets A and B are disjoint, they do not share any elements in common. So, the intersection of set A and set B is the empty set. $A \cap B = \emptyset$.

Because set A is a subset of set B , A intersection B is equal to set A . $A \cap B = A = \{1, 3, 5, \dots\}$, the set of odd natural numbers.

A union B is the set formed by including all the unique elements in set A , set B , or both sets A and B : $A \cup B = \{1, 3, 5, 7, 9, 2\}$. The first five elements of the union are the five unique elements in set A . Even though 3, 5, and 7 are also members of set B , these elements are only listed one time. Lastly, set B includes the unique element 2, so 2 is also included as part of the union of sets A and B .

Because sets A and B are disjoint, the union is simply the set containing all the elements in both set A and set B . $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Because set A is a subset of set B , A union B is equal to set B . $A \cup B = \mathbb{N} = \{1, 2, 3, \dots\} = B$.

Using the formula for determining the cardinality of the union of two sets, we can say $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 10 + 20 - 4 = 26$.

To find the cardinality of A union B , apply the formula, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. Because sets A and B are disjoint, $A \cap B$ is the empty set, therefore $n(A \cap B) = n(\emptyset) = 0$ and $n(A \cup B) = 37 + 43 - 0 = 80$.

1. A and $B = A \cap B = \{0, 12\}$, because only the elements 0 and 12 are members of both set A and set B .
2. A or $B = A \cup B = \{0, 3, 4, 6, 8, 9, 12, 16\}$, because the set A or B is the collection of all elements in set A or set B , or both.
3. A or $C = A \cup C = \{0, 1, 2, 3, 5, 6, 8, 9, 12, 13\}$, because the set A or C is the collection of all elements in set A or set C , or both.
4. $(B \text{ and } C) \text{ or } A = (B \cap C) \cup A$. Parentheses are evaluated first: $(B \text{ and } C) = B \cap C = \{8\}$, because the only member that both set B and set C share in common is 8. So, now we need to find $\{8\}$ or $\{0, 3, 6, 9, 12\}$. Because the word translates to the union operation, the problem becomes $\{8\} \cup \{0, 3, 6, 9, 12\}$, which is equal to $\{0, 3, 6, 8, 9, 12\}$.
1. The total number of people at the party is 54, and 12 people did not have cake or ice cream. Recall that the total number of elements in the universal set is always equal to the number of elements in a subset plus the number of elements in the complement of the set, $n(U) = n(A) + n(A')$. That means $54 = n(\text{cake or ice cream}) + n(\text{not (cake or ice cream)})$, or equivalently, $n(\text{cake} \cup \text{ice cream}) = 54 - n((\text{cake} \cup \text{ice cream})') = 54 - 12 = 42$. A total of 42 people at the party had cake or ice cream.
2. To determine the number of people who had both cake and ice cream, we need to find the intersection of the set of people who had cake and the set of people who had ice cream. From Question 1, the number of people who had cake or ice cream is 42. This is the union of the two sets. The formula for the union of two sets is $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. Use the information given in the problem and substitute the known values into the formula to solve for the number of people in the intersection: $42 = 30 + 20 - n(A \cap B)$. Adding 30 and 20, the equation simplifies to $42 = 50 - n(\text{cake and ice cream})$. Which means $n(\text{cake and ice cream}) = 50 - 42 = 8$.
1. $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$, because $A \cup B$ is the collection of all elements in set A or set B or both.
2. Because A and B are disjoint sets, there are no elements that are in both A and B . Therefore, $A \cap B$ is the empty set, $A \cap B = \emptyset$.
3. The complement of set B is the set of all elements in the universal set that are not in set B : $B' = \{0, 1, 3, 5, 7, 9\}$.
4. The cardinality, or number of elements in set B' , is $n(B') = 6$.
1. The number of elements in A or B is the number of elements in $A \cup B$: $n(A \cup B) = 2 + 5 + 7 = 14$.
2. The number of elements in A and B is the number of elements in $A \cap B$: $n(A \cap B) = 5$.
3. The number of elements in set A is the sum of all the numbers enclosed in the circle representing set A : $n(A) = 5 + 7 = 12$.

4.6: Set Operations with Three Sets



Figure 4.6.1: Companies like Google collect data on how you use their services, but the data requires analysis to really mean something. (credit: “Man holding smartphone and searches through Google” by Nenad Stojkovic/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

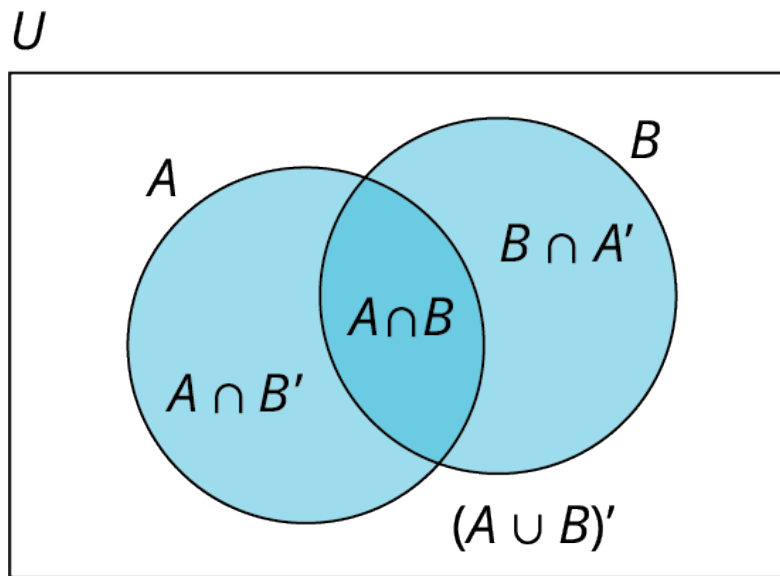
1. Interpret Venn diagrams with three sets.
2. Create Venn diagrams with three sets.
3. Apply set operations to three sets.
4. Prove equality of sets using Venn diagrams.

Have you ever searched for something on the Internet and then soon after started seeing multiple advertisements for that item while browsing other web pages? Large corporations have built their business on data collection and analysis. As we start working with larger data sets, the analysis becomes more complex. In this section, we will extend our knowledge of set relationships by including a third set.

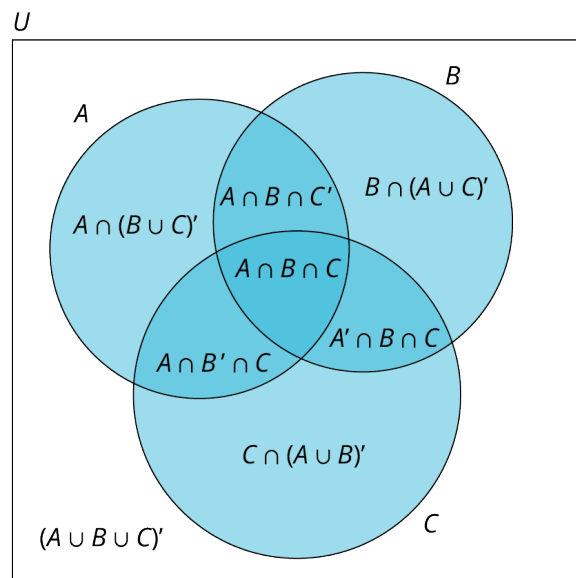
A Venn diagram with two intersecting sets breaks up the universal set into four regions; simply adding one additional set will increase the number of regions to eight, doubling the complexity of the problem.

Venn Diagrams with Three Sets

Below is a Venn diagram with two intersecting sets, which breaks the universal set up into four distinct regions.



Next, we see a Venn diagram with three intersecting sets, which breaks up the universal set into eight distinct regions.



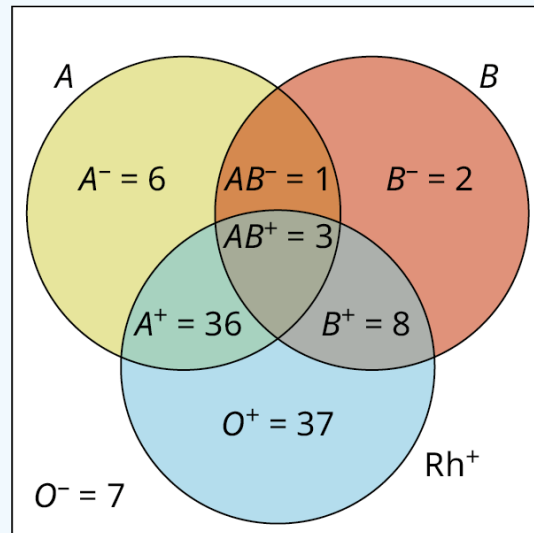
In the next example, we will explore the three main blood factors, A, B and Rh. The following background information about blood types will help explain the relationships between the sets of blood factors. If an individual has blood factor A or B, those will be included in their blood type. The Rh factor is indicated with a + or a -. For example, if a person has all three blood factors, then their blood type would be AB^+ . In the Venn diagram, they would be in the intersection of all three sets, $A \cap B \cap Rh^+$. If a person did not have any of these three blood factors, then their blood type would be O^- , and they would be in the set $(A \cup B \cup Rh^+)'$ which is the region outside all three circles.

Example 4.6.1

Interpreting a Venn Diagram with Three Sets

Use the Venn diagram below, which shows the blood types of 100 people who donated blood at a local clinic, to answer the following questions.

U = Blood types of 100 People



1. How many people with a type A blood factor donated blood?
2. Julio has blood type B^+ . If he needs to have surgery that requires a blood transfusion, he can accept blood from anyone who does not have a type A blood factor. How many people donated blood that Julio can accept?
3. How many people who donated blood do not have the Rh^+ blood factor?
4. How many people had type A and type B blood?

Answer available at the end of this section.

Your Turn 4.6.1

Use the same Venn diagram in Example 4.6.1 to answer the following questions.

1. How many people donated blood with a type B blood factor?
2. How many people who donated blood did not have a type B blood factor?
3. How many people who donated blood had a type B blood factor or were Rh^+ ?

Who Knew?

Blood Types

Most people know their main blood type of A, B, AB, or O and whether they are Rh^+ or Rh^- , but did you know that the International Society of Blood Transfusion recognizes twenty-eight additional blood types that have important implications for organ transplants and successful pregnancy?

Creating Venn Diagrams with Three Sets

In general, when creating Venn diagrams from data involving three subsets of a universal set, the strategy is to work from the inside out. Start with the intersection of the three sets, then address the regions that involve the intersection of two sets. Next, complete the regions that involve a single set, and finally address the region in the universal set that does not intersect with any of the three sets. This method can be extended to any number of sets. The key is to start with the region involving the most overlap, working your way from the center out.

Example 4.6.2

Creating a Venn Diagram with Three Sets

A teacher surveyed her class of 43 students to find out how they prepared for their last test. She found that 24 students made flash cards, 14 studied their notes, and 27 completed the review assignment. Of the entire class of 43 students, 12 completed the review and made flash cards, nine completed the review and studied their notes, and seven made flash cards and studied their notes, while only five students completed all three of these tasks. The remaining students did not do any of these tasks. Create a Venn diagram with subsets labeled: “Notes,” “Flash Cards,” and “Review” to represent how the students prepared for the test.

Answer available at the end of this section.

Your Turn 4.6.2

A group of 50 people attending a conference who preordered their lunch were able to select their choice of soup, salad, or sandwich. A total of 17 people selected soup, 29 people selected salad and 35 people selected a sandwich. Of these orders, 11 attendees selected soup and salad, 10 attendees selected soup and a sandwich, and 18 selected a salad and a sandwich, while eight people selected a soup, a salad, and a sandwich. Create a Venn diagram with subsets labeled “Soup,” “Salad,” and “Sandwich,” and label the cardinality of each section of the Venn diagram as indicated by the data.

Applying Set Operations to Three Sets

Set operations are applied between two sets at a time. Parentheses indicate which operation should be performed first. As with numbers, the inner most parentheses are applied first. Next, find the complement of any sets, then perform any union or intersections that remain.

Example 4.6.3

Applying Set Operations to Three Sets

Perform the set operations as indicated on the following sets: $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A = \{0, 1, 2, 3, 4, 5, 6\}$, $B = \{0, 2, 4, 6, 8, 10, 12\}$, and $C = \{0, 3, 6, 9, 12\}$.

1. Find $(A \cap B) \cap C$.
2. Find $A \cap (B \cup C)$.
3. Find $(A \cap B) \cup C'$.

Answer available at the end of this section.

Your Turn 4.6.3

Using the same sets from Example 4.6.3, perform the set operations indicated.

1. Find $A \cap (B \cap C)$.
2. Find $(A \cap B) \cup (A \cap C)$.
3. Find $(A \cup C') \cap (B \cup C')$.

Notice that the answers to the Your Turn are the same as those in the Example. This is not a coincidence. The following equivalences hold true for sets:

- $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$. These are the associative property for set intersection and set union.
- $A \cap B = B \cap A$ and $A \cup B = B \cup A$. These are the commutative property for set intersection and set union.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. These are the distributive property for sets over union and intersection, respectively.

Proving Equality of Sets Using Venn Diagrams

To prove set equality using Venn diagrams, the strategy is to draw a Venn diagram to represent each side of the equality, then look at the resulting diagrams to see if the regions under consideration are identical.

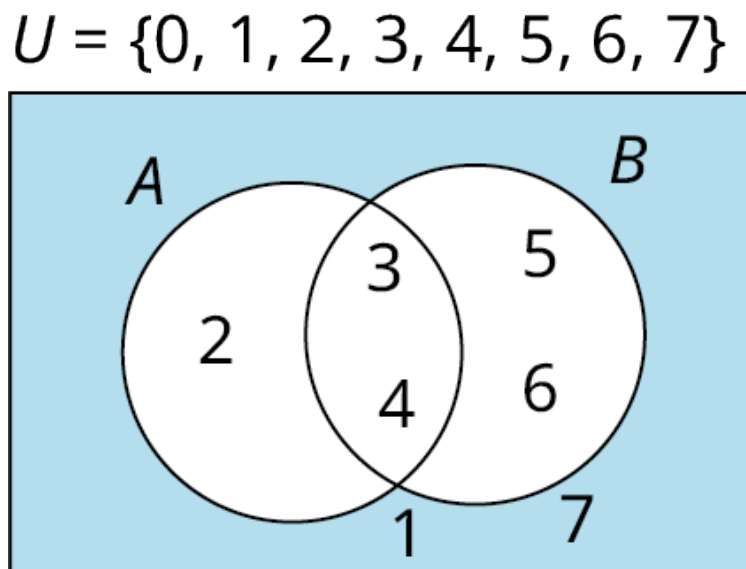
Augustus De Morgan was an English mathematician known for his contributions to set theory and logic. De Morgan's law for set complement over union states that $(A \cup B)' = A' \cap B'$. In the next example, we will use Venn diagrams to prove De Morgan's law for set complement over union is true. But before we begin, let us confirm De Morgan's law works for a specific example. While showing something is true for one specific example is not a proof, it will provide us with some reason to believe that it may be true for all cases.

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{2, 3, 4\}$, and $B = \{3, 4, 5, 6\}$. We will use these sets in the equation $(A \cup B)' = A' \cap B'$. To begin, find the value of the set defined by each side of the equation.

Step 1: $A \cup B$ is the collection of all unique elements in set A or set B or both. $A \cup B = \{2, 3, 4, 5, 6\}$. The complement of A union B , $(A \cup B)'$, is the set of all elements in the universal set that are not in $A \cup B$. So, the left side the equation $(A \cup B)'$ is equal to the set $\{1, 7\}$.

Step 2: The right side of the equation is $A' \cap B'$. A' is the set of all members of the universal set U that are not in set A . $A' = \{1, 5, 6, 7\}$. Similarly, $B' = \{1, 2, 7\}$.

Step 3: Finally, $A' \cap B'$ is the set of all elements that are in both A' and B' . The numbers 1 and 7 are common to both sets, therefore, $A' \cap B' = \{1, 7\}$. Because, $\{1, 7\} = \{1, 7\}$ we have demonstrated that De Morgan's law for set complement over union works for this particular example. The Venn diagram below depicts this relationship.



Example 4.6.4

Proving De Morgan's Law for Set Complement over Union Using a Venn Diagram

De Morgan's Law for the complement of the union of two sets A and B states that: $(A \cup B)' = A' \cap B'$. Use a Venn diagram to prove that De Morgan's Law is true.

Answer available at the end of this section.

Your Turn 4.6.4

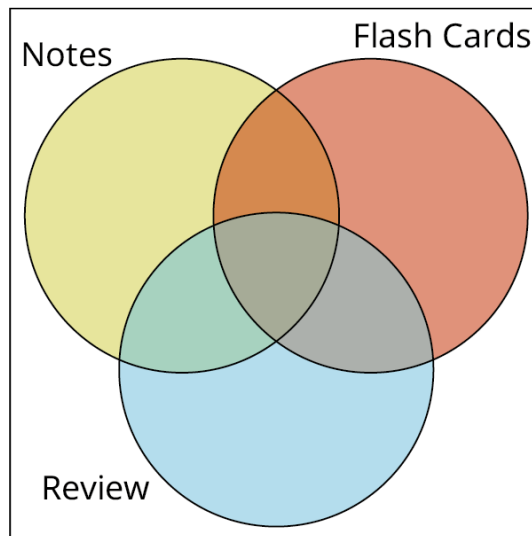
De Morgan's Law for the complement of the intersection of two sets A and B states that $(A \cap B)' = A' \cup B'$. Use a Venn diagram to prove that De Morgan's Law is true.

Answers

1. The number of people who donated blood with a type A blood factor will include the sum of all the values included in the A circle. It will be the union of sets A^- , A^+ , AB^- and AB^+ . $n(A) = n(A^-) + n(A^+) + n(AB^-) + n(AB^+) = 6 + 36 + 1 + 3 = 46$.
2. In part 1, it was determined that the number of donors with a type A blood factor is 46. To determine the number of people who did not have a type A blood factor, use the following property, A' union A is equal to U , which means $n(A) + n(A') = n(U)$, and $n(A') = n(U) - n(A) = 100 - 46 = 54$. Thus, 54 people donated blood that Julio can accept.
3. This would be everyone outside the Rh^+ circle, or everyone with a negative Rh factor, $n(Rh^-) = n(O^-) + n(A^-) + n(AB^-) + n(B^-) = 7 + 6 + 1 + 2 = 16$.
4. To have both blood type A and blood type B, a person would need to be in the intersection of sets A and B . The two circles overlap in the regions labeled AB^- and AB^+ . Add up the number of people in these two regions to get the total: $1 + 3 = 4$. This can be written symbolically as $n(A \text{ and } B) = n(A \cap B) = n(AB^-) + n(AB^+) = 1 + 3 = 4$.

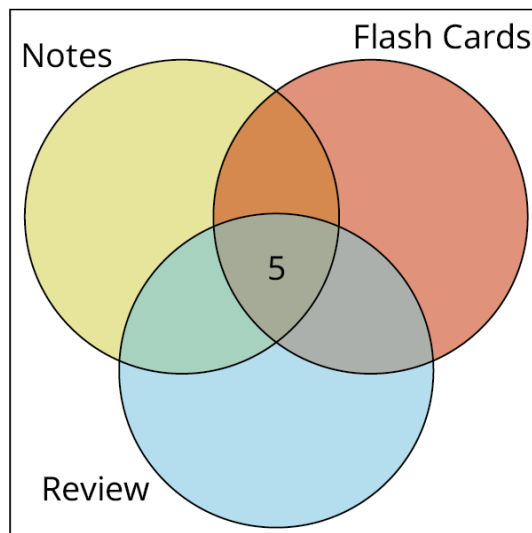
Step 1: First, draw a Venn diagram with three intersecting circles to represent the three intersecting sets: Notes, Flash Cards, and Review. Label the universal set with the cardinality of the class.

$U = \text{Class} = 43$



Step 2: Next, in the region where all three sets intersect, enter the number of students who completed all three tasks.

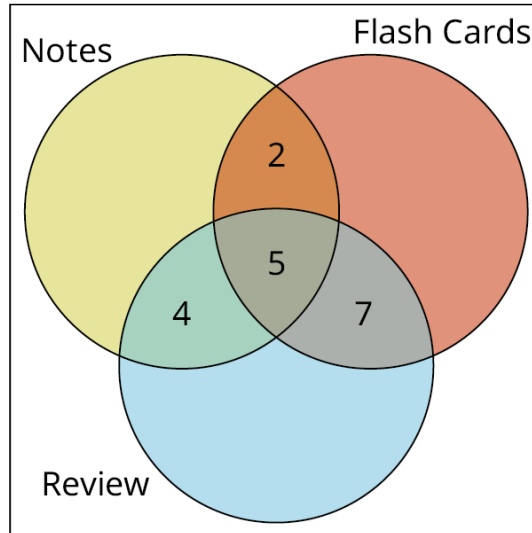
$U = \text{Class} = 43$



Step 3: Next, calculate the value and label the three sections where just two sets overlap.

- Review and flash card overlap.** A total of 12 students completed the review and made flash cards, but five of these twelve students did all three tasks, so we need to subtract: $12 - 5 = 7$. This is the value for the region where the flash card set intersects with the review set.
- Review and notes overlap.** A total of 9 students completed the review and studied their notes, but again, five of these nine students completed all three tasks. So, we subtract: $9 - 5 = 4$. This is the value for the region where the review set intersects with the notes set.
- Flash card and notes overlap.** A total of 7 students made flash cards and studied their notes; subtracting the five students that did all three tasks from this number leaves 2 students who only studied their notes and made flash cards. Add these values to the Venn diagram.

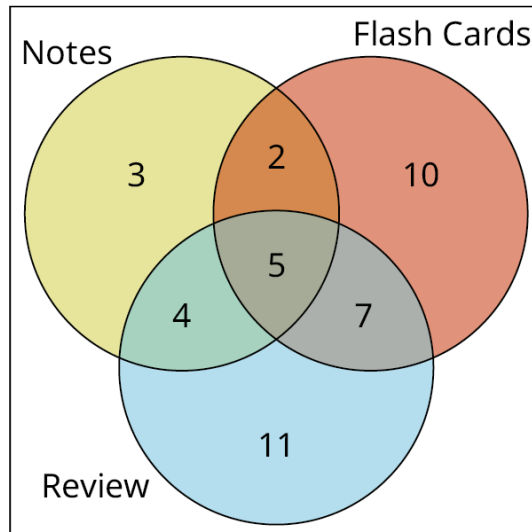
$$U = \text{Class} = 43$$



Step 4: Now, repeat this process to find the number of students who only completed one of these three tasks.

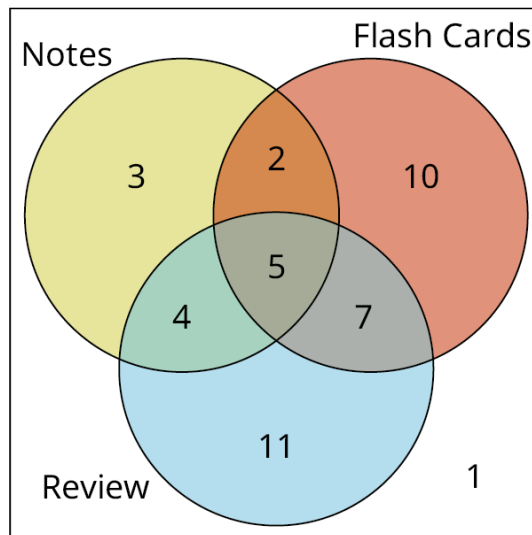
- A total of 24 students completed flash cards, but we have already accounted for $2 + 5 + 7 = 14$ of these. Thus, $24 - 14 = 10$ students who just made flash cards.
- A total of 14 students studied their notes, but we have already accounted for $4 + 5 + 2 = 11$ of these. Thus, $14 - 11 = 3$ students only studied their notes.
- A total of 27 students completed the review assignment, but we have already accounted for $4 + 5 + 7 = 16$ of these, which means $27 - 16 = 11$ students only completed the review assignment.
- Add these values to the Venn diagram.

$$U = \text{Class} = 43$$



Step 5: Finally, compute how many students did not do any of these three tasks. To do this, we add together each value that we have already calculated for the separate and intersecting sections of our three sets: $3 + 2 + 4 + 5 + 10 + 7 + 11 = 42$. Because there are 43 students in the class, and $43 - 42 = 1$, this means only one student did not complete any of these tasks to prepare for the test. Record this value somewhere in the rectangle, but outside of all the circles, to complete the Venn diagram.

$U = \text{Class} = 43$



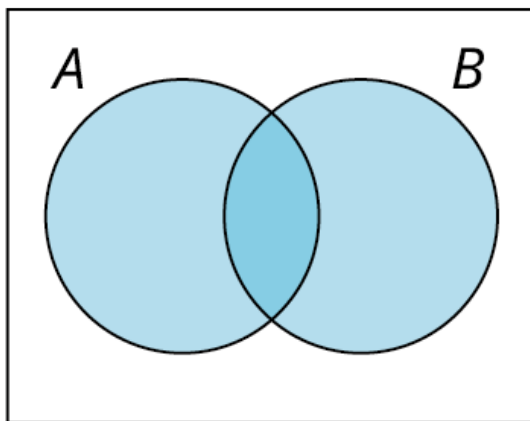
1. Parentheses first, A intersection B equals $A \cap B = \{0, 2, 4, 6\}$, the elements common to both A and B . $(A \cap B) \cap C = \{0, 2, 4, 6\} \cap \{0, 3, 6, 9, 12\} = \{0, 6\}$, because the only elements that are in both sets are 0 and 6.
2. Parentheses first, B union C equals $B \cup C = \{0, 2, 3, 4, 6, 8, 9, 10, 12\}$, the collection of all elements in set B or set C or both. $A \cap (B \cup C) = \{0, 1, 2, 3, 4, 5, 6\} \cap \{0, 2, 3, 4, 6, 8, 9, 10, 12\} = \{0, 2, 3, 4, 6\}$, because the intersection of these two sets is the set of elements that are common to both sets.
3. Parentheses first, A intersection B equals $A \cap B = \{0, 2, 4, 6\}$. Next, find C' . The complement of set C is the set of elements in the universal set U that are not in set C . $C' = \{1, 2, 4, 5, 7, 8, 10, 11\}$. Finally, find $(A \cap B) \cup C' = \{0, 2, 4, 6\} \cup \{1, 2, 4, 5, 7, 8, 10, 11\} = \{0, 1, 2, 4, 5, 6, 7, 8, 10, 11\}$.

Step 1: First, draw a Venn diagram representing the left side of the equality. The regions of interest are shaded to highlight the sets of interest. $A \cup B$ is shaded on the left, and $(A \cup B)'$ is shaded on the right.

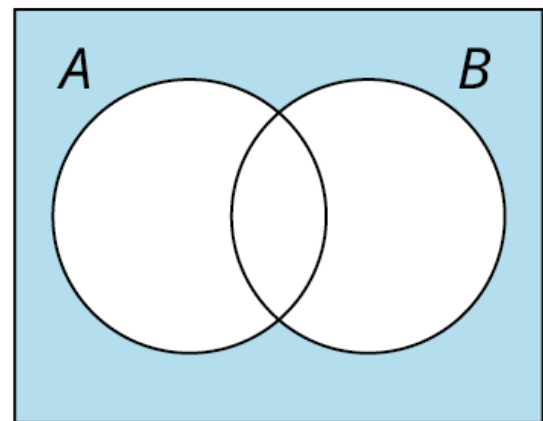
$A \cup B$

$(A \cup B)'$

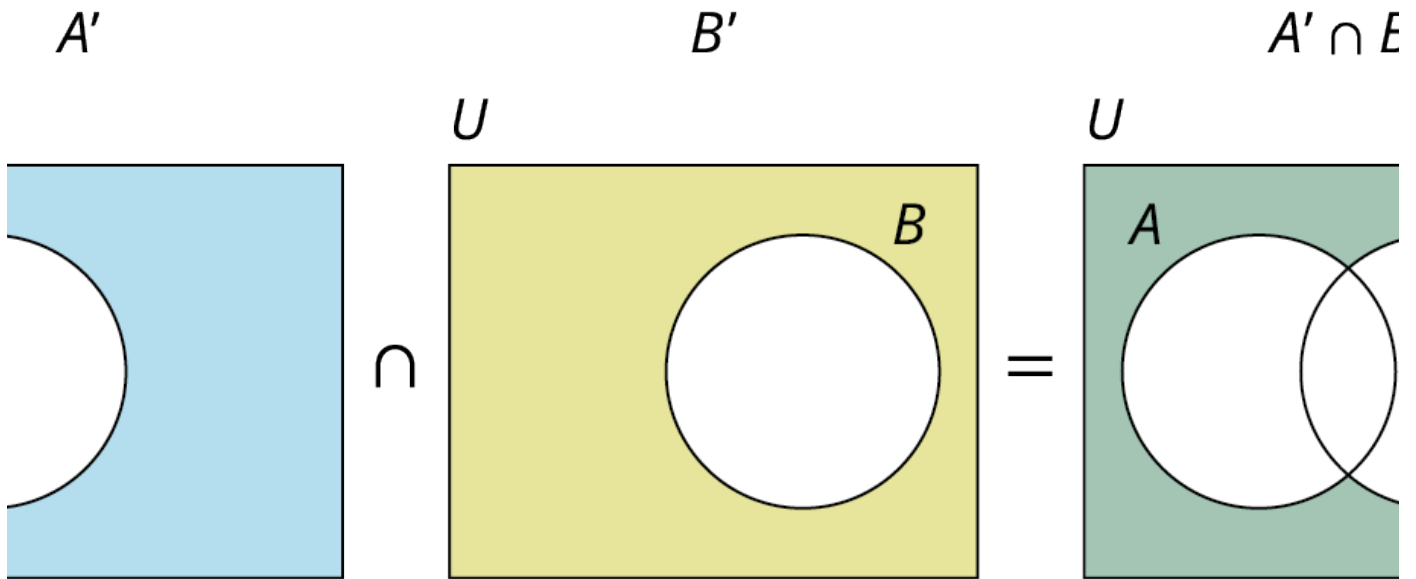
U



U



Step 2: Next, draw a Venn diagram to represent the right side of the equation. A' is shaded and B' is shaded. Because A' and B' mix to form $A' \cap B'$ is also shaded.



Step 3: Verify the conclusion. Because the shaded region in the Venn diagram for $(A \cup B)'$ matches the shaded region in the Venn diagram for $A' \cap B'$, the two sides of the equation are equal, and the statement is true. This completes the proof that De Morgan's law is valid.

5: LOGIC

5.1: Conditional Statements

5.2: Truth Tables- Conjunction (and), Disjunction (or), Negation (not)

5.3: Truth Tables- Conditional, Biconditional

5.4: De Morgan's Laws

5.5: Arguments

5.6: Evaluating Deductive Arguments with Truth Tables

5.1: Conditional Statements

Learning Objectives

After completing this section, you should be able to:

1. Write a conditional statement for a given situation

Beyond searching, Boolean logic is commonly used in spreadsheet applications like Excel to do conditional calculations. A **statement** is something that is either true or false. A statement like $3 < 5$ is true; a statement like “a rat is a fish” is false. A statement like $x < 5$ is true for some values of x and false for others. When an action is taken or not depending on the value of a statement, it forms a **conditional**.

Statements and Conditionals

A **statement** is either true or false.

A **conditional** is a compound statement of the form

“if p then q ” or “if p then q , else s ”

Example 5.1.1

In common language, an example of a conditional statement would be “If it is raining, then we’ll go to the mall. Otherwise we’ll go for a hike.”

The statement “If it is raining” is the condition – this may be true or false for any given day. If the condition is true, then we will follow the first course of action, and go to the mall. If the condition is false, then we will use the alternative, and go for a hike.

Example 5.1.2

As mentioned earlier, conditional statements are commonly used in spreadsheet applications like Excel or Google Sheets. In Excel, you can enter an expression like

$= \text{IF} (A1 < 2000, A1 + 1, A1 \times 2)$

Notice that after the IF, there are three parts. The first part is the condition, and the second two are calculations. Excel will look at the value in cell A1 and compare it to 2000. If that condition is true, then the first calculation is used, and 1 is added to the value of A1 and the result is stored. If the condition is false, then the second calculation is used, and A1 is multiplied by 2 and the result is stored.

In other words, this statement is equivalent to saying “If the value of A1 is less than 2000, then add 1 to the value in A1. Otherwise, multiply A1 by 2”.

Example 5.1.3

The expression $= \text{IF} (A1 > 5, 2 \times A1, 3 \times A1)$ is used. Find the result if A1 is 3, and the result if A1 is 8.

Solution

If $A1 > 5$, then calculate $2 \times A1$. Otherwise, calculate $3 \times A1$.

If $A1$ is 3, then the condition is false, since $3 > 5$ is not true, so we do the alternate action, and multiply by 3, giving $3 \times 3 = 9$

If $A1$ is 8, then the condition is true, since $8 > 5$, so we multiply the value by 2, giving $2 \times 8 = 16$

Example 5.1.4

An accountant needs to withhold 15% of income for taxes if the income is below \$30,000, and 20% of income if the income is \$30,000 or more. Write an expression that would calculate the amount to withhold.

Our conditional needs to compare the value to 30,000. If the income is less than 30,000, we need to calculate 15% of the income: $0.15 \times \text{income}$. If the income is more than 30,000, we need to calculate 20% of the income: $0.20 \times \text{income}$.

In words we could write "If income < 30,000, then multiply by 0.15, otherwise multiply by 0.20". In Excel, we would write:

= IF (A1 < 30000, 0.15 × A1, 0.20 × A1)

As we did earlier, we can create more complex conditions by using the operators *and*, *or*, and *not* to join simpler conditions together.

Example 5.1.5

A parent might say to their child "if you clean your room and take out the garbage, then you can have ice cream."

Here, there are two simpler conditions:

- 1) The child cleaning her room
- 2) The child taking out the garbage

Since these conditions were joined with *and*, the combined conditional will be true only if both simpler conditions are true; if either chore is not completed, then the parent's condition is not met.

Notice that if the parent had said "if you clean your room *or* take out the garbage, then you can have ice cream", then the child would need to complete only one chore to meet the condition.

Suppose you wanted to have something happen when a certain value is between 100 and 300. To create the condition " $A1 < 300$ and $A1 > 100$ " in Excel, you would need to enter " $\text{AND}(A1 < 300, A1 > 100)$ ". Likewise, for the condition " $A1=4$ or $A1=6$ " you would enter " $\text{OR}(A1=4, A1=6)$ "

Example 5.1.6

In a spreadsheet, cell A1 contains annual income, and A2 contains number of dependents.

A certain tax credit applies if someone with no dependents earns less than \$10,000, or if someone with dependents earns less than \$20,000. Write a rule that describes this.

There are two ways the rule is met:

income is less than 10,000 *and* dependents is 0, *or*

income is less than 20,000 *and* dependents is not 0.

Informally, we could write these as

$(A1 < 10000 \text{ and } A2 = 0) \text{ or } (A1 < 20000 \text{ and } A2 > 0)$

In Excel's format, we'd write

IF (OR(AND(A1 < 10000, A2 = 0), AND(A1 < 20000, A2 > 0)), "you qualify", "you don't qualify")

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5.2: Truth Tables- Conjunction (and), Disjunction (or), Negation (not)

Learning Objectives

After completing this section, you should be able to:

1. Convert sentences to symbols and vice versa.
2. Create truth tables for compound statements involving and, or, and not

Before we focus on truth tables, we're going to introduce some symbols that are commonly used for *and*, *or*, and *not*.

Symbols

The symbol $\wedge$ is used for and: A and B is notated $A \wedge B$

The symbol $\vee$ is used for or: A or B is notated $A \vee B$

The symbol $\sim$ is used for not: not A is notated $\sim A$

You can remember the first two symbols by relating them to the shapes for the union and intersection. $A \wedge B$ would be the elements that exist in both sets, in $A \cap B$. Likewise, $A \vee B$ would be the elements that exist in either set, in $A \cup B$. When we are working with sets, we use the rounded version of the symbols; when we are working with statements, we use the pointy version.

Example 5.2.1

Translate each statement into symbolic notation. Let P represent "I like Pepsi" and let C represent "I like Coke".

- a. I like Pepsi or I like Coke.
- b. I like Pepsi and I like Coke.
- c. I do not like Pepsi.
- d. It is not the case that I like Pepsi or Coke.
- e. I like Pepsi and I do not like Coke.

Solution

- a. $P \vee C$
- b. $P \wedge C$
- c. $\sim P$
- d. $\sim (P \vee C)$
- e. $P \wedge \sim C$

As you can see, we can use parentheses to organize more complicated statements.

Your Turn 5.2.1

Translate "We have carrots or we will not make soup" into symbols. Let C represent "we have carrots" and let S represent "we will make soup".

Answer available at the end of this section.

Because complex Boolean statements can get tricky to think about, we can create a **truth table** to keep track of what truth values for the simple statements make the complex statement true and false.

Truth Table

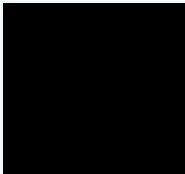
A table showing what the resulting truth value of a complex statement is for all the possible truth values for the simple statements.

Example 5.2.2

Suppose you're picking out a new couch, and your significant other says "get a sectional *or* something with a chaise".

This is a complex statement made of two simpler conditions: "is a sectional", and "has a chaise". For simplicity, let's use *S* to designate "is a sectional", and *C* to designate "has a chaise".

A truth table for this situation would look like this:

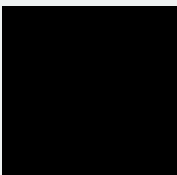


In the table, T is used for true, and F for false. In the first row, if *S* is true and *C* is also true, then the complex statement "*S or C*" is true. This would be a sectional that also has a chaise, which meets our desire. (Remember that *or* in logic is not exclusive; if the couch has both features, it meets the condition.) The second and third rows of the table also show that the complex statement "*S or C*" is true if just one of the statements is true. In the last row, we see that if the couch does not have a sectional and does not have a chaise, then it does not meet the requirements.

In the previous example about the couch, the truth table was really just summarizing what we already know about how the *or* statement works. The truth tables for the basic *and*, *or*, and *not* statements are shown below.

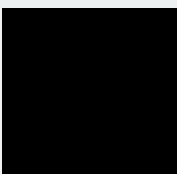
Basic truth tables

Conjunction



Notice that we ended the truth table with one true and three falses. We can say a conjunction is true only when **both** of the statements are true.

Disjunction



Notice that we ended the truth table with three trues and one false. We can say a disjunction is false only when **both** of the statements are false.

Negation



Notice that when we negated the statement, the truth value changed. A true statement became false, and a false statement became true.

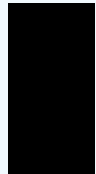
Truth tables really become useful when we analyze more complex Boolean statements.

Example 5.2.3

Create a truth table for the statement $A \vee \sim B$

Solution

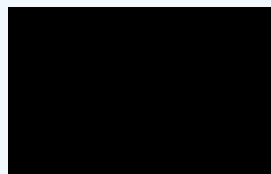
When we create the truth table, we need to list all the possible truth value combinations for A and B . Notice how the first column contains 2 Ts followed by 2 Fs, and the second column alternates T, F, T, F. This pattern ensures that all 4 combinations are considered.



After creating columns with those initial values, we create a third column for the expression $\sim B$. Now we will temporarily ignore the column for A and write the truth values for $\sim B$



Next we can find the truth values of $A \vee \sim B$, using the first and third columns.



The truth table shows that $A \vee \sim B$ is true in three cases and false in one case. If you're wondering what the point of this is, suppose it is the last day of the baseball season and two teams, who are not playing each other, are competing for the final playoff spot. Anaheim will make the playoffs if it wins its game or if Boston does not win its game. (Anaheim owns the tie-breaker; if both teams win, or if both teams lose, then Anaheim gets the playoff spot.) If A = Anaheim wins its game and B = Boston wins its game, then $A \vee \sim B$ represents the situation "Anaheim wins its game or Boston does not win its game". The truth table shows us the different scenarios related to Anaheim making the playoffs. In the first row, Anaheim wins its game and Boston wins its game, so it is true that Anaheim makes the playoffs. In the second row, Anaheim wins and Boston

does not win, so it is true that Anaheim makes the playoffs. In the third row, Anaheim does not win its game and Boston wins its game, so it is false that Anaheim makes the playoffs. In the fourth row, Anaheim does not win and Boston does not win, so it is true that Anaheim makes the playoffs.

Your Turn 5.2.2

Create a truth table for this statement: $\sim A \wedge B$

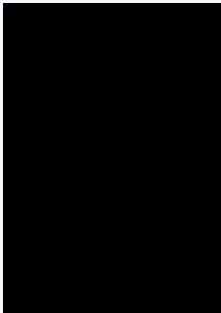
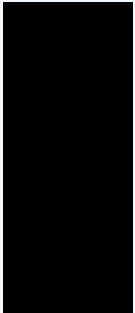
Answer available at the end of this section.

Example 5.2.4

Create a truth table for the statement $A \wedge \sim (B \vee C)$

Solution

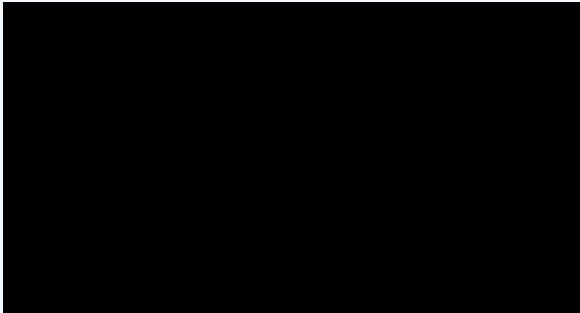
It helps to work from the inside out when creating a truth table, and to create columns in the table for intermediate operations. We start by listing all the possible truth value combinations for A , B , and C . Notice how the first column contains 4 Ts followed by 4Fs, the second column contains 2Ts, 2Fs, then repeats, and the last column alternates T, F, T, F... This pattern ensures that all 8 combinations are considered. After creating columns with those initial values, we create a fourth column for the innermost expression, $B \vee C$. Now we will temporarily ignore the column for A and focus on B and C , writing the truth values for $B \vee C$



Next we can find the negation of $B \vee C$, working off the $B \vee C$ column we just created. (Ignore the first three columns and simply negate the values in the $B \vee C$ column.)



Finally, we find the values of A and $\sim (B \vee C)$. (Ignore the second, third, and fourth columns.)



It turns out that this complex expression is true in only one case: when A is true, B is false, and C is false. To illustrate this situation, suppose that Anaheim will make the playoffs if: (1) Anaheim wins, and (2) neither Boston nor Cleveland wins. T,F,F is the only scenario in which Anaheim will make the playoffs.

Your Turn 5.2.3

Create a truth table for this statement: $(\sim A \wedge B) \vee \sim B$

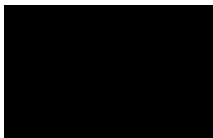
Answer available at the end of this section.

Answers

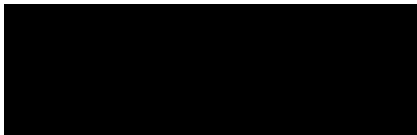
Your Turn 5.2.1

$$C \vee \sim S$$

Your Turn 5.2.2



Your Turn 5.2.3



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5.3: Truth Tables- Conditional, Biconditional

Learning Objectives

After completing this section, you should be able to:

1. Create a truth table for statements involving a conditional or biconditional.
2. Write the inverse, converse, and contrapositive of a conditional statement.
3. Negate a conditional statement.

We discussed conditional statements earlier, in which we take an action based on the value of the condition. We are now going to look at another version of a conditional, sometimes called an implication, which states that the second part must logically follow from the first.

Conditional

A conditional is a logical compound statement in which a statement p , called the antecedent, implies a statement q , called the consequent.

A conditional is written as $p \rightarrow q$ and is translated as "if p , then q ".

Example 5.3.1

The English statement "If it is raining, then there are clouds in the sky" is a conditional statement. It makes sense because if the antecedent "it is raining" is true, then the consequent "there are clouds in the sky" must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining; there might be clouds in the sky, or there might not. If the antecedent is false, then the consequent becomes irrelevant.

Example 5.3.2

Suppose you order a team jersey online on Tuesday and want to receive it by Friday so you can wear it to Saturday's game. The website says that if you pay for expedited shipping, you will receive the jersey by Friday. In what situation is the website telling a lie?

There are four possible outcomes:

- 1) You pay for expedited shipping and receive the jersey by Friday
- 2) You pay for expedited shipping and don't receive the jersey by Friday
- 3) You don't pay for expedited shipping and receive the jersey by Friday
- 4) You don't pay for expedited shipping and don't receive the jersey by Friday

Only one of these outcomes proves that the website was lying: the second outcome in which you pay for expedited shipping but don't receive the jersey by Friday. The first outcome is exactly what was promised, so there's no problem with that. The third outcome is not a lie because the website never said what would happen if you didn't pay for expedited shipping; maybe the jersey would arrive by Friday whether you paid for expedited shipping or not. The fourth outcome is not a lie because, again, the website didn't make any promises about when the jersey would arrive if you didn't pay for expedited shipping.

It may seem strange that the third outcome in the previous example, in which the first part is false but the second part is true, is not a lie. Remember, though, that if the antecedent is false, we cannot make any judgment about the consequent. The website never said that paying for expedited shipping was the *only* way to receive the jersey by Friday.

Example 5.3.3

A friend tells you “If you upload that picture to Facebook, you’ll lose your job.” Under what conditions can you say that your friend was wrong?

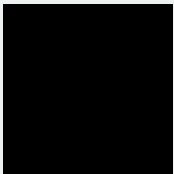
There are four possible outcomes:

- 1) You upload the picture and lose your job
- 2) You upload the picture and don’t lose your job
- 3) You don’t upload the picture and lose your job
- 4) You don’t upload the picture and don’t lose your job

There is only one possible case in which you can say your friend was wrong: the second outcome in which you upload the picture but still keep your job. In the last two cases, your friend didn’t say anything about what would happen if you didn’t upload the picture, so you can’t say that their statement was wrong. Even if you didn’t upload the picture and lost your job anyway, your friend never said that you were guaranteed to keep your job if you didn’t upload the picture; you might lose your job for missing a shift or punching your boss instead.

In traditional logic, a conditional is considered true as long as there are no cases in which the antecedent is true and the consequent is false.

Truth table for the conditional



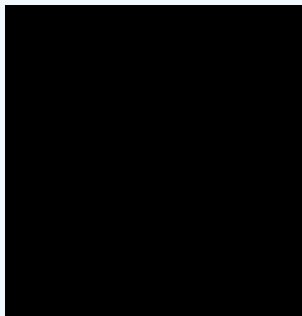
Again, if the antecedent p is false, we cannot prove that the statement is a lie, so the result of the third and fourth rows is true.

Example 5.3.4

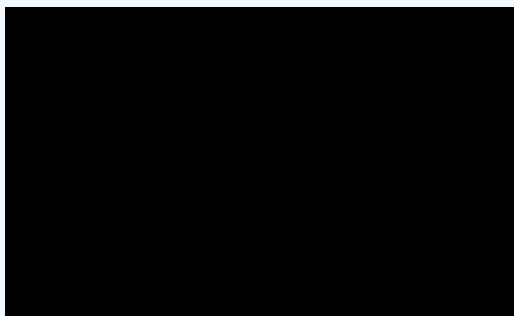
Construct a truth table for the statement $(m \wedge \sim p) \rightarrow r$

Solution

We start by constructing a truth table with 8 rows to cover all possible scenarios. Next, we can focus on the antecedent, $m \wedge \sim p$.



Now we can create a column for the conditional. Because it can be confusing to keep track of all the Ts and Fs, why don't we copy the column for r to the right of the column for $m \wedge \sim p$? This makes it a lot easier to read the conditional from left to right.



When m is true, p is false, and r is false- the fourth row of the table-then the antecedent $m \wedge \sim p$ will be true but the consequent false, resulting in an invalid conditional; every other case gives a valid conditional.

If you want a real-life situation that could be modeled by $(m \wedge \sim p) \rightarrow r$, consider this: let m = we order meatballs, p = we order pasta, and r = Rob is happy. The statement $(m \wedge \sim p) \rightarrow r$ is "if we order meatballs and don't order pasta, then Rob is happy". If m is true (we order meatballs), p is false (we don't order pasta), and r is false (Rob is not happy), then the statement is false, because we satisfied the antecedent but Rob did not satisfy the consequent.

For any conditional, there are three related statements: the converse, the inverse, and the contrapositive.

Related Statements

The original conditional is "if p , then q " $p \rightarrow q$

The converse is "if q , then p " $q \rightarrow p$

The inverse is "if not p , then not q " $\sim p \rightarrow \sim q$

The contrapositive is "if not q , then not p " $\sim q \rightarrow \sim p$

Example 5.3.5

Consider again the conditional "If it is raining, then there are clouds in the sky." It seems reasonable to assume that this is true.

The converse would be "If there are clouds in the sky, then it is raining." This is not always true.

The inverse would be "If it is not raining, then there are not clouds in the sky." Likewise, this is not always true.

The contrapositive would be "If there are not clouds in the sky, then it is not raining." This statement is true, and is equivalent to the original conditional.

Looking at truth tables, we can see that the original conditional and the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.

		Conditional	Converse	Inverse	Contrapositive
p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Equivalent

Equivalence

A conditional statement and its contrapositive are logically equivalent.

The converse and inverse of a conditional statement are logically equivalent.

In other words, the original statement and the contrapositive must agree with each other; they must both be true, or they must both be false. Similarly, the converse and the inverse must agree with each other; they must both be true, or they must both be false.

Be aware that symbolic logic cannot represent the English language perfectly. For example, we may need to change the verb tense to show that one thing occurred before another.

Example 5.3.6

Suppose this statement is true: “If I eat this giant cookie, then I will feel sick.” Which of the following statements must also be true?

- a. If I feel sick, then I ate that giant cookie.
- b. If I don’t eat this giant cookie, then I won’t feel sick.
- c. If I don’t feel sick, then I didn’t eat that giant cookie.

Solution

- a. This is the converse, which is not necessarily true. I could feel sick for some other reason, such as drinking sour milk.
- b. This is the inverse, which is not necessarily true. Again, I could feel sick for some other reason; avoiding the cookie doesn’t guarantee that I won’t feel sick.
- c. This is the contrapositive, which is true, but we have to think somewhat backwards to explain it. If I ate the cookie, I would feel sick, but since I don’t feel sick, I must not have eaten the cookie.

Notice again that the original statement and the contrapositive have the same truth value (both are true), and the converse and the inverse have the same truth value (both are false).

Your Turn 5.3.1

“If you microwave salmon in the staff kitchen, then I will be mad at you.” If this statement is true, which of the following statements must also be true?

- a. If you don’t microwave salmon in the staff kitchen, then I won’t be mad at you.
- b. If I am not mad at you, then you didn’t microwave salmon in the staff kitchen.
- c. If I am mad at you, then you microwaved salmon in the staff kitchen.

Answer available at the end of this section.

Your Turn 5.3.2

Consider the statement “If you park here, then you will get a ticket.” What set of conditions would prove this statement false?

- a. You don’t park here and you get a ticket.
- b. You don’t park here and you don’t get a ticket.
- c. You park here and you don’t get a ticket.

Answer available at the end of this section.

The Negation of a Conditional

The negation of a conditional statement is logically equivalent to a conjunction of the antecedent and the negation of the consequent.

$$\sim (p \rightarrow q) \text{ is equivalent to } p \wedge \sim q$$

Example 5.3.7

Which of the following statements is equivalent to the negation of “If you don’t grease the pan, then the food will stick to it”?

- a. I didn't grease the pan and the food didn't stick to it.
- b. I didn't grease the pan and the food stuck to it.
- c. I greased the pan and the food didn't stick to it.

Solution

- a. This is correct; it is the conjunction of the antecedent and the negation of the consequent. To disprove that not greasing the pan will cause the food to stick, I have to not grease the pan and have the food not stick.
- b. This is essentially the original statement with no negation; the "if...then" has been replaced by "and".
- c. This essentially agrees with the original statement and cannot disprove it.

Your Turn 5.3.3

"If you go swimming less than an hour after eating lunch, then you will get cramps." Which of the following statements is equivalent to the negation of this statement?

- a. I went swimming more than an hour after eating lunch and I got cramps.
- b. I went swimming less than an hour after eating lunch and I didn't get cramps.
- c. I went swimming more than an hour after eating lunch and I didn't get cramps.

Answer available at the end of this section.

In everyday life, we often have a stronger meaning in mind when we use a conditional statement. Consider "If you submit your hours today, then you will be paid next Friday." What the payroll rep really means is "If you submit your hours today, then you will be paid next Friday, and if you don't submit your hours today, then you won't be paid next Friday." The conditional statement if t , then p also includes the inverse of the statement: if not t , then not p . A more compact way to express this statement is "You will be paid next Friday *if and only if* you submit your timesheet today." A statement of this form is called a **biconditional**.

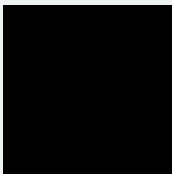
Biconditional

A biconditional is a logical conditional statement in which the antecedent and consequent are interchangeable.

A biconditional is written as $p \leftrightarrow q$ and is translated as " p if and only if q ".

Because a biconditional statement $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$, we may think of it as a conditional statement combined with its converse: if p , then q and if q , then p . The double-headed arrow shows that the conditional statement goes from left to right and from right to left. A biconditional is considered true as long as the antecedent and the consequent have the same truth value; that is, they are either both true or both false.

Truth table for the biconditional



Notice that we ended the truth table with two trues and two falses. The biconditional is true when the two statements have the same truth value (both are true or both are false).

Notice that the fourth row, where both components are false, is true; if you don't submit your timesheet and you don't get paid, the person from payroll told you the truth.

Example 5.3.8

Suppose this statement is true: "The garbage truck comes down my street if and only if it is Thursday morning." Which of the following statements could be true?

- It is noon on Thursday and the garbage truck did not come down my street this morning.
- It is Monday and the garbage truck is coming down my street.
- It is Wednesday at 11:59PM and the garbage truck did not come down my street today.

Solution

- This cannot be true. This is like the second row of the truth table; it is true that I just experienced Thursday morning, but it is false that the garbage truck came.
- This cannot be true. This is like the third row of the truth table; it is false that it is Thursday, but it is true that the garbage truck came.
- This could be true. This is like the fourth row of the truth table; it is false that it is Thursday, but it is also false that the garbage truck came, so everything worked out like it should.

Your Turn 5.3.4

Suppose this statement is true: "I wear my running shoes if and only if I am exercising." Determine whether each of the following statements must be true or false.

- I am exercising and I am not wearing my running shoes.
- I am wearing my running shoes and I am not exercising.
- I am not exercising and I am not wearing my running shoes.

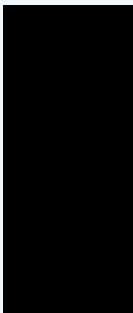
Answer available at the end of this section.

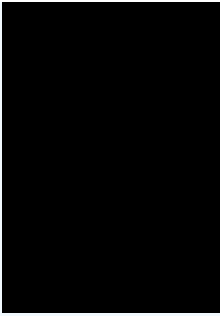
Example 5.3.9

Create a truth table for the statement $(A \vee B) \leftrightarrow \sim C$

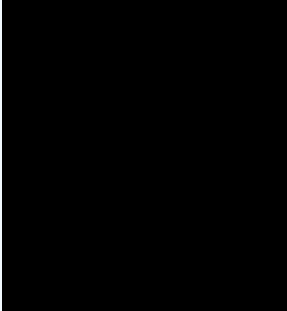
Solution

Whenever we have three component statements, we start by listing all the possible truth value combinations for A , B , and C . After creating those three columns, we can create a fourth column for the antecedent, $A \vee B$. Now we will temporarily ignore the column for C and focus on A and B , writing the truth values for $A \vee B$.

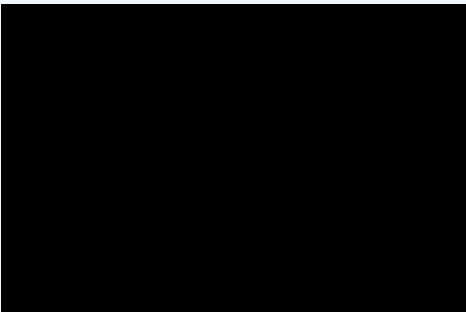




Next we can create a column for the negation of C . (Ignore the $A \vee B$ column and simply negate the values in the C column.)



Finally, we find the truth values of $(A \vee B) \leftrightarrow \sim C$. Remember, a biconditional is true when the truth value of the two parts match, but it is false when the truth values do not match.



To illustrate this situation, suppose your boss needs you to do either project A or project B (or both, if you have the time). If you do one of the projects, you will not get a crummy review (C is for crummy). So $(A \vee B) \leftrightarrow \sim C$ means "You will not get a crummy review if and only if you do project A or project B ." Looking at a few of the rows of the truth table, we can see how this works out. In the first row, A , B , and C are all true: you did both projects and got a crummy review, which is not what your boss told you would happen! That is why the final result of the first row is false. In the fourth row, A is true, B is false, and C is false: you did project A and did not get a crummy review. This is what your boss said would happen, so the final result of this row is true. And in the eighth row, A , B , and C are all false: you didn't do either project and did not get a crummy review. This is not what your boss said would happen, so the final result of this row is false. (Even though you may be happy that your boss didn't follow through on the threat, the truth table shows that your boss lied about what would happen.)

Answers

Your Turn 5.3.1

Choice b is correct because it is the contrapositive of the original statement.

Your Turn 5.3.2

The first two statements are irrelevant because we don't know what will happen if you park somewhere else. The third statement, however contradicts the conditional statement "If you park here, then you will get a ticket" because you parked here but didn't get a ticket. This example demonstrates a general rule; the negation of a conditional can be written as a conjunction: "It is not the case that if you park here, then you will get a ticket" is equivalent to "You park here *and* you do not get a ticket."

Your Turn 5.3.3

Choice b is equivalent to the negation; it keeps the first part the same and negates the second part.

Your Turn 5.3.4

Choices a & b are false; c is true.

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5.4: De Morgan's Laws

Learning Objectives

After completing this section, you should be able to:

1. Negate a conjunction and a disjunction using DeMorgan's Laws.

A contemporary of Boole's, Augustus De Morgan, formalized two rules of logic that had previously been known informally. They allow us to rewrite the negation of a conjunction as a disjunction, and vice-versa.

For example, suppose you want to schedule a meeting with two colleagues at 4 : 30PM on Friday, and you need both of them to be available at that time. What situation would make it impossible to have the meeting? It is NOT the case that colleague a is available AND colleague b is available: $\sim (a \wedge b)$. This situation is equivalent to either colleague a NOT being available OR colleague b NOT being available: $\sim a \vee \sim b$

De Morgan's Laws

The negation of a conjunction is equivalent to the disjunction of the negation of the statements making up the conjunction. To negate an "and" statement, negate each part and change the "and" to "or".

$$\sim (p \wedge q) \text{ is equivalent to } \sim p \vee \sim q$$

The negation of a disjunction is equivalent to the conjunction of the negation of the statements making up the disjunction. To negate an "or" statement, negate each part and change the "or" to "and".

$$\sim (p \vee q) \text{ is equivalent to } \sim p \wedge \sim q$$

Example 5.4.1

For Valentine's Day, you did not get your sweetie flowers or candy: Which of the following statements is logically equivalent?

- a. You did not get them flowers or did not get them candy.
- b. You did not get them flowers and did not get them candy.
- c. You got them flowers or got them candy.

Solution

- a. This statement does not go far enough; it leaves open the possibility that you got them one of the two things.
- b. This statement is equivalent to the original; $\sim (f \vee c)$ is equivalent to $\sim f \wedge \sim c$
- c. This statement says that you got them something, but we know that you did not.

Your Turn 5.4.1

Negate the statement "It is raining and I cannot go to the beach."

Solution

The original statement is a conjunction. So the negation will be a disjunction. We also need to negate each simple statement.

It is not raining or I can go to the beach.

Example 5.4.2

To serve as the President of the US, a person must have been born in the US, must be at least 35 years old, and must have lived in the US for at least 14 years. What minimum set of conditions would disqualify someone from serving as President?

Answer available at the end of this section.

Answers

Example 5.4.2

Failing to meet just one of the three conditions is all it takes to be disqualified. A person is disqualified if they were not born in the US, or are not at least 35 years old, or have not lived in the US for at least 14 years. The key word in the negation is “or” instead of “and”.

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5.5: Arguments

Learning Objectives

After completing this section, you should be able to:

1. Determine if an argument is inductive or deductive.

A logical argument is a claim that a set of premises support a conclusion. There are two general types of arguments: inductive and deductive arguments.

Argument types

An **inductive** argument uses a collection of specific examples as its premises and uses them to propose a general conclusion.

A **deductive** argument uses a collection of general statements as its premises and uses them to propose a specific situation as the conclusion.

Example 5.5.1

The argument “When I went to the store last week I forgot my purse, and when I went today I forgot my purse. I always forget my purse when I go the store” is an inductive argument.

The premises are:

- I forgot my purse last week
- I forgot my purse today

The conclusion is:

I always forget my purse

Notice that the premises are specific situations, while the conclusion is a general statement. In this case, this is a fairly weak argument, since it is based on only two instances.

Example 5.5.2

The argument “Every day for the past year, a plane flies over my house at 2:00 P.M. A plane will fly over my house every day at 2:00 P.M.” is a stronger inductive argument, since it is based on a larger set of evidence. While it is not necessarily true—the airline may have cancelled its afternoon flight—it is probably a safe bet.

Evaluating inductive arguments

An inductive argument is never able to prove the conclusion true, but it can provide either weak or strong evidence to suggest that it may be true.

Many scientific theories, such as the Big Bang Theory, can never be proven. Instead, they are inductive arguments supported by a wide variety of evidence. Usually in science, an idea is considered a hypothesis until it has been well tested, at which point it

graduates to being considered a theory. Common scientific theories, like Newton's Theory of Gravity, have all stood up to years of testing and evidence, though sometimes they need to be adjusted based on new evidence, such as when Einstein proposed the Theory of General Relativity.

A deductive argument is more clearly valid or not, which makes it easier to evaluate.

Evaluating deductive arguments

A deductive argument is considered valid if, assuming that all the premises are true, the conclusion follows logically from those premises. In other words, when the premises are all true, the conclusion *must* be true.

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5.6: Evaluating Deductive Arguments with Truth Tables

Learning Objectives

After completing this section, you should be able to:

1. Determine if a deductive argument is valid using a truth table.

Arguments can also be analyzed using truth tables, although this can be a lot of work.

Analyzing arguments using truth tables

To analyze an argument with a truth table:

1. Represent each of the premises symbolically
2. Create a conditional statement, joining all the premises to form the antecedent, and using the conclusion as the consequent.
3. Create a truth table for the statement. If it is always true, then the argument is valid.

Example 5.6.1

Consider the argument

Premise: If you bought bread, then you went to the store.
Premise: You bought bread.
Conclusion: You went to the store.

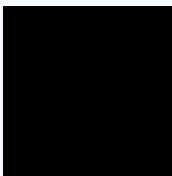
Solution

While this example is fairly obviously a valid argument, we can analyze it using a truth table by representing each of the premises symbolically. We can then form a conditional statement showing that the premises together imply the conclusion. If the truth table is a tautology (always true), then the argument is valid.

We'll let b represent "you bought bread" and s represent "you went to the store". Then the argument becomes:

Premise: $b \rightarrow s$
Premise: b
Conclusion: s

To test the validity, we look at whether the combination of both premises implies the conclusion; is it true that $[(b \rightarrow s) \wedge b] \rightarrow s$?





Since the truth table for $[(b \rightarrow s) \wedge b] \rightarrow s$ is always true, this is a valid argument.

Your Turn 5.6.1

Determine whether the argument is valid:

Premise: If I have a shovel, I can dig a hole.

Premise: I dug a hole.

Conclusion: Therefore, I had a shovel.

Answer available at the end of this section.

Example 5.6.2

Premise: If I go to the mall, then I'll buy new jeans.

Premise: If I buy new jeans, I'll buy a shirt to go with it.

Conclusion: If I go to the mall, I'll buy a shirt.

Solution

Let m = I go to the mall, j = I buy jeans, and s = I buy a shirt.

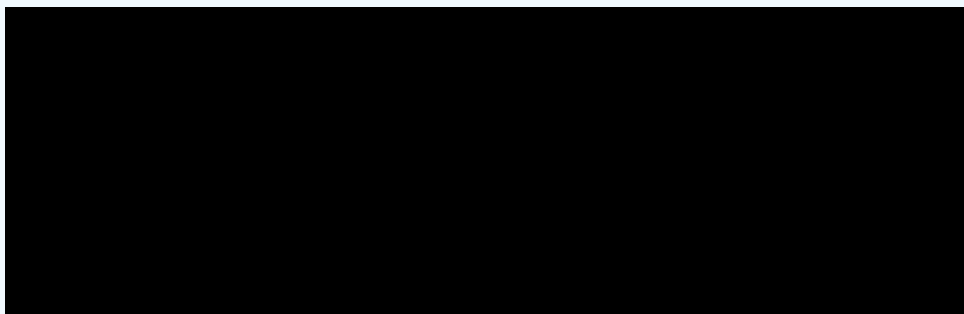
The premises and conclusion can be stated as:

Premise: $m \rightarrow j$

Premise: $j \rightarrow s$

Conclusion: $m \rightarrow s$

We can construct a truth table for $[(m \rightarrow j) \wedge (j \rightarrow s)] \rightarrow (m \rightarrow s)$. Try to recreate each step and see how the truth table was constructed.

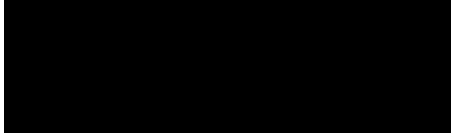


From the final column of the truth table, we can see this is a valid argument.

Answers

Your Turn 5.6.1

Let S = have a shovel, D = dig a hole. The first premise is equivalent to $S \rightarrow D$. The second premise is D . The conclusion is S . We are testing $[(S \rightarrow D) \wedge D] \rightarrow S$



This is not a tautology, so this is an invalid argument.

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