MAT 1320 FINITE MATHEMATICS, 2E



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MAT 1320 Finite Mathematics, v2

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Detailed Licensing

Detailed Licensing

Licensing

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CHAPTER OVERVIEW

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1.1: Solving Linear Equations in One Variable

Learning Objectives

- Verify linear solutions.
- Use the properties of equality to solve basic linear equations.
- Clear fractions from equations.
- Apply linear equations.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Simplify $2-6(4-7)^2$ without using a calculator.

Click here to check your answer

-52

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

2. Simplify $7x-1-4x+5$.

Click here to check your answer

$3x+4$

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

3. Evaluate $6x-4$ when $x=-2$.

Click here to check your answer

-16

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

4. Evaluate $-5x^2-x+9$ when $x=-3$.

Click here to check your answer

-33

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

Linear Equations and Their Solutions

An **equation**[129] is a statement indicating that two algebraic expressions are equal. A **linear equation with one variable**[130], x, is an equation that can be written in the standard form $ax+b=0$ where a and b are real numbers and $a\neq 0$. For example

$$3x-12=0$$

A **solution**[131] to a linear equation is any value that can replace the variable to produce a true statement. The variable in the linear equation $3x-12=0$ is x and the solution is $x=4$. To verify this, substitute the value 4 in for x and check that you obtain a true statement.

$$\begin{aligned} 3x-12 &= 0 \\ 3(4)-12 &= 0 \\ 12-12 &= 0 \\ 0 &= 0 \quad \checkmark \end{aligned}$$

Alternatively, when an equation is equal to a constant, we may verify a solution by substituting the value in for the variable and showing that the result is equal to that constant. In this sense, we say that solutions "satisfy the equation."

Example 1.1.1:

Is $a=2$ a solution to $-10a+5=-25$?

Solution

Recall that when evaluating expressions, it is a good practice to first replace all variables with parentheses, and then substitute the appropriate values. By making use of parentheses, we avoid some common errors when working the order of operations.

$$\begin{aligned} -10a+5 &= -25 \\ -10(2)+5 &= -25 \\ -20+5 &= -25 \\ -15 &\neq 25 \quad \text{✗} \end{aligned}$$

Answer:

No, $a=2$ does not satisfy the equation and is therefore not a solution.

Finding Solutions to Linear Equations

Developing techniques for solving various algebraic equations is one of our main goals in algebra. This section reviews the basic techniques used for solving linear equations with one variable. We begin by defining **equivalent equations**[132] as equations with the same solution set.

$$\left.\begin{aligned} 3x-5 &= 16 \\ 3x &= 21 \\ x &= 7 \end{aligned}\right\} \quad \textit{Equivalent equations}$$

Here we can see that the three linear equations are equivalent because they share the same solution set, namely, $\{7\}$. To obtain equivalent equations, use the following **properties of equality**[133]. Given algebraic expressions A and B, and real number c:

Addition property of equality:	If $A=B$, then $A+c=B+c$
Subtraction property of equality:	If $A=B$, then $A-c=B-c$
Multiplication property of equality:	If $A=B$, then $cA=cB$
Division property of equality:	If $A=B$, then $\frac{A}{c}=\frac{B}{c}$, $c\neq 0$

Table 1.1.1

Note

Multiplying or dividing both sides of an equation by 0 is carefully avoided. Dividing by 0 is undefined and multiplying both sides by 0 results in the equation $0=0$.

Typically linear equations are not given in standard form, and so solving them requires additional steps. When solving linear equations, the goal is to determine what value, if any, will produce a true statement when substituted in the original equation. Do this by isolating the variable using the following steps:

- Step 1: Simplify both sides of the equation using the order of operations and combine all like terms on the same side of the equal sign.
- Step 2: Use the appropriate properties of equality to combine like terms on opposite sides of the equal sign. The goal is to obtain the variable term on one side of the equation and the constant term on the other.
  - Step 2a: Add or subtract as needed to isolate the variable.
  - Step 2b: Divide or multiply as needed to isolate the variable.
- Step 3: Check to see if the answer solves the original equation.

Examples 2-4 below already have Step 1 completed, and are known as just two-step equations.

We solve algebraic equations by isolating the variable with a coefficient of 1. If given a linear equation of the form $ax+b=c$, then we can solve it in two steps. First, use the appropriate equality property of addition or subtraction to isolate the variable term. Next, isolate the variable using the equality property of multiplication or division. Checking the solution in the following examples is left to the reader.

Example 1.1.2:

Solve: $7x-2=19$.

Solution

$$\begin{aligned} 7x-2 &= 19 \\ 7x-2\color{#1a9cd8}{+2} &= 19\color{#1a9cd8}{+2} && \textit{Add 2 to both sides.} \\ 7x &= 21 \\ \frac{7x}{\color{#1a9cd8}{7}} &= \frac{21}{\color{#1a9cd8}{7}} && \textit{Divide both sides by 7.} \\ x &= 3 \end{aligned}$$

Answer:

The solution is 3.

Example 1.1.3:

Solve: $56=8+12y$.

Solution

When no sign precedes the term, it is understood to be positive. In other words, think of this as $56=+8+12y$. Therefore, we begin by subtracting 8 on both sides of the equal sign.

$$\begin{aligned} 56\color{#1a9cd8}{-8} &= 8+12y\color{#1a9cd8}{-8} \\ 48 &= 12y \\ \frac{48}{\color{#1a9cd8}{12}} &= \frac{12y}{\color{#1a9cd8}{12}} \\ 4 &= y \end{aligned}$$

It does not matter on which side we choose to isolate the variable because the **symmetric property**[134] states that $4=y$ is equivalent to $y=4$.

Answer:

The solution is 4.

Example 1.1.4:

Solve: $\frac{5}{3}x+2=-8$.

Solution

Isolate the variable term using the addition property of equality, and then multiply both sides of the equation by the reciprocal of the coefficient $\frac{5}{3}$.

$$\begin{aligned}
\frac{5}{3}x+2 &= -8 \\
\frac{5}{3}x+2-2 &= -8-2 \quad \textit{Subtract 2 on both sides.} \\
\frac{5}{3}x &= -10 \\
\frac{3}{5}\cdot\frac{5}{3}x &= \frac{3}{5}\cdot(\overset{-2}{\cancel{-10}}) \quad \textit{Multiply both sides by } \frac{3}{5}. \\
1x &= 3\cdot(-2) \\
x &= -6
\end{aligned}$$

Answer:

The solution is -6.

Exercise 1.1.1

Solve: $4+3x=-14$

Answer

$x=-6$

We will often encounter linear equations where the expressions on each side of the equal sign can be simplified. If this is the case, then it is best to simplify each side first before solving. Normally this involves combining same-side like terms.

Note

At this point in our study of algebra the use of the properties of equality should seem routine. Therefore, displaying these steps in this text, usually in blue, becomes optional.

Example 1.1.5:

Solve: $-4a+2-a=1$.

Solution

First combine the like terms on the left side of the equal sign.

$$\begin{aligned}
-4a+2-a &= 1 \quad \textit{Combine same-side like terms.} \\
-5a+2 &= 1 \quad \textit{Subtract 2 on both sides.} \\
-5a &= -1 \quad \textit{Divide both sides by } -5. \\
a &= \frac{-1}{-5} = \frac{1}{5}
\end{aligned}$$

Always use the original equation to check to see if the solution is correct.

$$\begin{aligned}
-4a+2-a &= -4\left(\frac{1}{5}\right)+2-\frac{1}{5} \\
&= -\frac{4}{5}+\frac{2}{1}\cdot\frac{5}{5}-\frac{1}{5} \\
&= \frac{-4+10+1}{5} \\
&= \frac{5}{5}=1 \ \checkmark
\end{aligned}$$

Answer:

The solution is $\frac{1}{5}$.

Given a linear equation in the form $ax+b=cx+d$, we begin the solving process by combining like terms on opposite sides of the equal sign. To do this, use the addition or subtraction property of equality to place like terms on the same side so that they can be combined. In the examples that remain, the check is left to the reader.

Example 1.1.6:

Solve: $-2y-3=5y+11$.

Solution

Subtract $5y$ on both sides so that we can combine the terms involving y on the left side.

$$\begin{aligned} -2y-3-5y &= 5y+11-5y \\ -7y-3 &= 11 \end{aligned}$$

From here, solve using the techniques developed previously.

$$\begin{aligned} -7y-3 &= 11 && \textit{Add 3 to both sides.} \\ -7y &= 14 \\ y &= \frac{14}{-7} && \textit{Divide both sides by } -7. \\ y &= -2 \end{aligned}$$

Answer:

The solution is -2.

Solving will often require the application of the distributive property in order to simplify each side of the equation.

Example 1.1.7:

Solve: $-\frac{1}{2}(10x-2)+3=7(1-2x)$.

Solution

Simplify the linear expressions on either side of the equal sign first.

$$\begin{aligned} -\frac{1}{2}(10x-2)+3 &= 7(1-2x) && \textit{Distribute} \\ -5x+1+3 &= 7-14x && \textit{Combine same}-\textit{side like terms.} \\ -5x+4 &= 7-14x && \textit{Combine opposite}-\textit{side like terms.} \\ 9x &= 3 && \textit{Solve.} \\ x &= \frac{3}{9} = \frac{1}{3} \end{aligned}$$

Answer:

The solution is $\frac{1}{3}$.

Example 1.1.8:

Solve: $5(3-a)-2(5-2a)=3$.

Solution

Begin by applying the distributive property.

$$\begin{aligned} 5(3-a)-2(5-2a) &= 3 \\ 15-5a-10+4a &= 3 \\ 5-a &= 3 \\ -a &= -2 \end{aligned}$$

Here we point out that $-a$ is equivalent to $-1a$; therefore, we choose to divide both sides of the equation by -1.

$$-a = -2$$
$$\frac{-1a}{-1} = \frac{-2}{-1}$$
$$a = 2$$

Alternatively, we can multiply both sides of $-a = -2$ by negative one and achieve the same result.

$$-a = -2$$
$$(-1)(-a) = (-1)(-2)$$
$$a = 2$$

Answer:

The solution is 2.

? Exercise 1.1.2

Solve: $6 - 3(4x - 1) = 4x - 7$.

Answer

$x = 1$

Video Solution: www.youtube.com/v/NAIAZrFjU-o

Clearing Fractions from Equations

The coefficients of linear equations may be any real number, even decimals and fractions. When this is the case it is possible to use the multiplication property of equality to clear the fractional coefficients and obtain integer coefficients in a single step. If given fractional coefficients, then multiply both sides of the equation by the least common multiple of the denominators (LCD). This will clear the equation of fractions.

✔ Example 1.1.9:

Solve: $\frac{1}{3}x + \frac{1}{5} = \frac{1}{5}x - 1$.

Solution

Clear the fractions by multiplying both sides by the least common multiple of the given denominators. In this case, it is the $LCD(3, 5) = 15$.

$$\begin{aligned} 15 \cdot \left(\frac{1}{3}x + \frac{1}{5}\right) &= 15 \cdot \left(\frac{1}{5}x - 1\right) && \textit{Multiply both sides by 15.} \\ 15 \cdot \frac{1}{3}x + 15 \cdot \frac{1}{5} &= 15 \cdot \frac{1}{5}x - 15 \cdot 1 && \textit{Simplify.} \\ 5x + 3 &= 3x - 15 && \textit{Solve.} \\ 2x &= -18 \\ x &= \frac{-18}{2} = -9 \end{aligned}$$

Answer:

The solution is -9.

It is important to know that this technique only works for equations. *Do not try to clear fractions when simplifying expressions.* As a reminder:

Expression	Equation
$\frac{1}{2}x + \frac{5}{3}$	$\frac{1}{2}x + \frac{5}{3} = 0$

Table 1.1.2

We simplify expressions and solve equations. If you multiply an expression by 6, you will change the problem. However, if you multiply both sides of an equation by 6, you obtain an equivalent equation.

Incorrect	Correct
$\frac{1}{2}x+\frac{5}{3}$ $\neq 6\cdot\left(\frac{1}{2}x+\frac{5}{3}\right)$ $=3x+10$ ✗	$\frac{1}{2}x+\frac{5}{3}=0$ $6\cdot\left(\frac{1}{2}x+\frac{5}{3}\right)=6\cdot 0$ $3x+10=0$ ✓

Table 1.1.3

? Exercise 1.1.3

Solve: $\frac{5}{6}+\frac{1}{2}x=4x-\frac{2}{3}$.

Answer

$x=\frac{3}{7}$

Note than multiplying by any factor of 2, 3, and 6 will clear the equation of fractions. While the LCD is officially 6 in this problem, multiplying by 6, 12, 18, or 36 will also work.

Applications Involving Linear Equations

Algebra simplifies the process of solving real-world problems. This is done by using letters to represent unknowns, restating problems in the form of equations, and by offering systematic techniques for solving those equations. To solve problems using algebra, first translate the wording of the problem into mathematical statements that describe the relationships between the given information and the unknowns. Usually, this translation to mathematical statements is the difficult step in the process. The key to the translation is to carefully read the problem and identify certain key words and phrases.

Key Words	Translation
Sum, increased by, more than, plus, added to, total	$+$
Difference, decreased by, subtracted from, less, minus	$-$
Product, multiplied by, of times, twice	$\cdot$
Quotient, divided by, ratio, per	$\div$
Is, total, result	$=$

Table 1.1.4

When translating sentences into mathematical statements, be sure to read the sentence several times and parse out the key words and phrases. It is important to first identify the variable, "*let x represent…*" and state in words what the unknown quantity is. This step not only makes our work more readable, but also forces us to think about what we are looking for.

✓ Example 1.1.10:

When 6 is subtracted from twice the sum of a number and 8 the result is 5. Represent this as an algebraic equation and find the number.

Solution

Let *n* represent the unknown number.

$$\underbrace{2\cdot}_{\text{"twice"}}\quad\underbrace{(n+8)}_{\text{"the sum of a number and 8"}}\quad\underbrace{-\ 6}_{\text{"6 is subtracted from"}}\quad\underbrace{=}_{\text{"result is"}}\quad 5$$

Figure 1.1.1

To understand why we included the parentheses in the set up, you must study the structure of the following two sentences and their translations:

"*twice the sum of a number and 8*"	2(n+8)
"*the sum of* ***twice a number*** *and 8*"	2n+8

Table 1.1.5

The key was to focus on the phrase "*twice the sum*," this prompted us to group the sum within parentheses and then multiply by 2. After translating the sentence into a mathematical statement we then solve.

$$\begin{aligned} 2(n+8)-6 &= 5 \\ 2n+16-6 &= 5 \\ 2n+10 &= 5 \\ 2n &= -5 \\ n &= \frac{-5}{2} \end{aligned}$$

Check.

$$\begin{aligned} 2(n+8)-6 &= 2\left(-\frac{5}{2}+8\right)-6 \\ &= 2\left(\frac{11}{2}\right)-6 \\ &= 11-6 \\ &= 5 \quad \checkmark \end{aligned}$$

Answer:

The number is $-\frac{5}{2}$.

General guidelines for setting up and solving word problems follow.

- Step 1: Read the problem several times, identify the key words and phrases, and organize the given information.
- Step 2: Identify the variables by assigning a letter or expression to the unknown quantities.
- Step 3: Translate and set up an algebraic equation that models the problem.
- Step 4: Solve the resulting algebraic equation.
- Step 5: Finally, answer the question in sentence form and make sure it makes sense (check it).

For now, set up all of your equations using only one variable. Avoid two variables by looking for a relationship between the unknowns.

Footnotes

[129] Statement indicating that two algebraic expressions are equal.

[130] An equation that can be written in the standard form $ax+b=0$, where a and b are real numbers and $a\neq 0$.

[131] Any value that can replace the variable in an equation to produce a true statement.

[132] Equations with the same solution set.

[133] Properties that allow us to obtain equivalent equations by adding, subtracting, multiplying, and dividing both sides of an equation by nonzero real numbers.

[134] Allows you to solve for the variable on either side of the equal sign, because $x = 5$ is equivalent to $5 = x$.

[135] Equations that are true for particular values.

[136] An equation that is true for all possible values.

[137] An equation that is never true and has no solution.

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1.1E: Exercises - Solving Linear Equations in One Variable

Exercise 1.1E.1

Determine whether or not the given value is a solution.

1. $-5x+4=-1; x=-1$
2. $4x-3=-7; x=-1$
3. $3y-4=5; y=\frac{9}{3}$
4. $-2y+7=12; y=-\frac{5}{2}$
5. $3a-6=18-a; a=-3$
6. $5(2t-1)=2-t; t=2$

Answer

1. No
3. Yes
5. No

Exercise 1.1E.2

Solve.

1. $5x-3=27$
2. $6x-7=47$
3. $9a+10=10$
4. $5-3a=5$
5. $-8t+5=15$
6. $-9t+12=33$
7. $7-y=22$
8. $6-y=12$
9. $\frac{2}{3}x+\frac{1}{2}=1$
10. $\frac{3}{8}x+\frac{5}{4}=\frac{3}{2}$
11. $\frac{1-3y}{5}=2$
12. $\frac{2-5y}{6}=-8$

Answer

1. 6
3. 0
5. $-\frac{5}{4}$
7. -15
9. $\frac{3}{4}$
11. -3

Exercise 1.1E.3

Solve.

1. $6x-5+2x=19$

2. $7-2x+9=24$
3. $5y-6-9y=3-2y+8$
4. $7-9y+12=3y+11-11y$
5. $\frac{1}{3}x-\frac{3}{2}+\frac{5}{2}x=\frac{5}{6}x+\frac{1}{4}$
6. $\frac{5}{8}+\frac{1}{5}x-\frac{3}{4}=\frac{3}{10}x-\frac{1}{4}$
7. $5(y+2)=3(2y-1)+10$
8. $7(y-3)=4(2y+1)-21$
9. $7-5(3t-9)=22$
10. $10-5(3t+7)=20$
11. $4(4a-1)=5(a-3)+2(a-2)$
12. $6(2b-1)+24b=8(3b-1)$
13. $\frac{2}{3}(x+18)+2=\frac{1}{3}x-13$
14. $\frac{2}{5}x-\frac{1}{2}(6x-3)=\frac{4}{3}$
15. $\frac{1}{5}(2a+3)-\frac{1}{2}=\frac{1}{3}a+\frac{1}{10}$
16. $\frac{3}{2}a=\frac{3}{4}(1+2a)-\frac{1}{5}(a+5)$

Answer

1. 3
3. $-\frac{17}{2}$
5. $\frac{7}{8}$
7. 3
9. 2
11. $-\frac{5}{3}$
13. -81
15. 0

? Exercise $1.1E.4$

Set up an algebraic equation then solve.

Number Problems

1. When 3 is subtracted from the sum of a number and 10 the result is 2. Find the number.
2. The sum of 3 times a number and 12 is equal to 3. Find the number.
3. Three times the sum of a number and 6 is equal to 5 times the number. Find the number.
4. Twice the sum of a number and 4 is equal to 3 times the sum of the number and 1. Find the number.

Answer

1. -5
3. 9

Footnotes

[138] Linear expressions related with the symbols $\leq, <, \geq$, and $>$.

[139] A real number that produces a true statement when its value is substituted for the variable.

[140] Properties used to obtain equivalent inequalities and used as a means to solve them.

[141] Inequalities that share the same solution set.

[142]Two or more inequalities in one statement joined by the word “and” or by the word “or.”

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1.2: Solving Linear Equations in Two Variables

Learning Objectives

In this section, you will learn to:

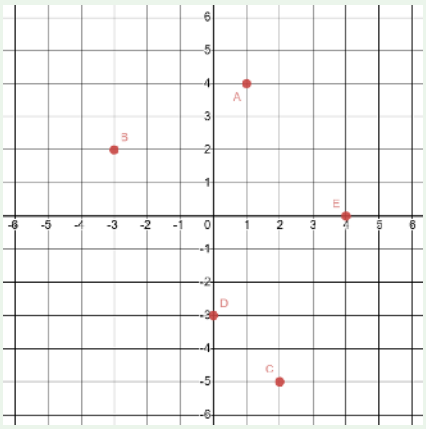
- Show all solutions to an equation by graphing a line.
- Graph a line by finding its intercepts.
- Graph and find equations of vertical and horizontal lines.
- Find the slope of a line.
- Find the slope of a line.
- Graph the line if a point and the slope are given.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

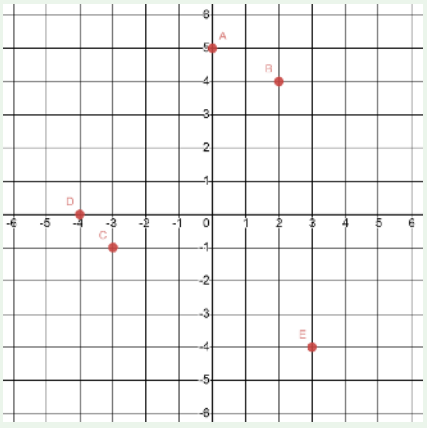
1. On a piece of graph paper, plot and label these points: A(1, 4), B(-3, 2), C(2, -5), D(0, -3), E(4, 0).

Click here to check your answer



If you missed any part of this problem, review here. (Note that this will open a different textbook in a new window.)

2. Write the coordinates of each of the points shown in this graph.



Click here to check your answer

A(0, 5)

B(2, 4)

C(-3, -1)

D(-4, 0)

E(3, -4)

If you missed any part of this problem, review here. (Note that this will open a different textbook in a new window.)

3. Solve the equation $2x - 6y = 24$ for the given values.

a. Solve for x when $y = 5$.

b. Solve for y when $x = 3$.

Click here to check your answer

a. $x = 27$

b. $y = -3$

If you missed any part of this problem, review Section 1.1. (Note that this will open a in a new window.)

4. Simplify each expression without the use of a calculator.

a. $-4 - (-7)$

b. $-4 + (-7)$

c. $-4 - 7$

d. $-4 + 7$

Click here to check your answer

a. 3

b. -11

c. -11

d. 3

If you missed any part of this problem, review here. (Note that this will open a different textbook in a new window.)

Solutions of Equations with Two Variables

Remember that a solution is the value(s) of the variable(s) that make(s) the equation true.

When an equation only has one variable, we can isolate that variable to identify the solution. In most equations with one variable, there exists only one value that can make the equation true. For example, in the equation $6 = x + 2$, the only solution to this equation is $x = 4$ since that is the only value of the variable that would make the equation true.

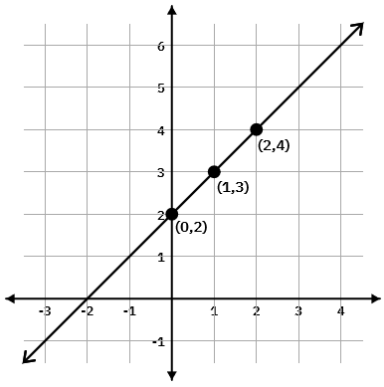
When an equation has two variables however, there exist infinite values for the variables that make the equation true. For example, consider the equation $y = x + 2$.

- If x=0, then y=2.
- If x=1, then y=3.
- If x=2, then y=4.
- If x=100, then y=102.

There are an infinite number of possibilities that can make the equation $y = x + 2$ true. Rather than try to list all solutions to the equation (which is impossible), we can **show** all solutions to the equation by graphing them on a coordinate grid. Each solution is listed as an (x, y) point, as shown here:

- If x=0, then y=2. This is shown as the point (0, 2).
- If x=1, then y=3. This is shown as the point (1, 3).
- If x=2, then y=4. This is shown as the point (2, 4).
- If x=100, then y=102. This is shown as the point (100, 102).

The graph for $y = x + 2$ would look like the graph below. Note that the points (0, 2), (1, 3), and (2, 4) are included among the points that the graph goes through. If the line were drawn out far enough, it would go through the point (100, 102) as well.



Note that this graph goes through infinite points that are not labeled here. Every point the graph touches would be a solution to the equation $y = x + 2$ though, including (3, 5), (-2, 0), and even (1.5, 3.5). You are encouraged to test each of these pairs of values into the equation to verify that they do indeed make the equation true!

Graphing a Line from its Equation

Equations whose graphs are straight lines are called **linear equations.** The following are some examples of linear equations:

$2x - 3y = 6, \quad 3x = 4y - 7, \quad y = 2x - 5, \quad 2y = 3, \quad \text{and} \quad x - 2 = 0$

A line is completely determined by two points. Therefore, to graph a linear equation we need to find the coordinates of two points. This can be accomplished by choosing an arbitrary value for x or y and then solving for the other variable.

Example 1.2.1

Graph the line: $y = 3x + 2$

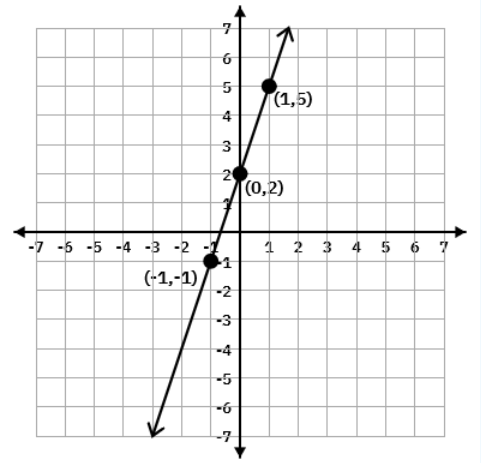
Solution

We need to find the coordinates of at least two points. We arbitrarily choose x = - 1, x = 0, and x = 1.

- If x = -1, then y = 3(-1) + 2 or -1. Therefore, (-1, -1) is a point on this line.
- If x = 0, then y = 3(0) + 2 or y = 2. Hence the point (0, 2).
- If x = 1, then y = 5, and we get the point (1, 5).

Below, the results are summarized, and the line is graphed.

x	y	(x, y)
-1	-1	(-1, -1)
0	2	(0, 2)
1	5	(1,5)



Example 1.2.2

Graph the line: $2x + y = 4$

Solution

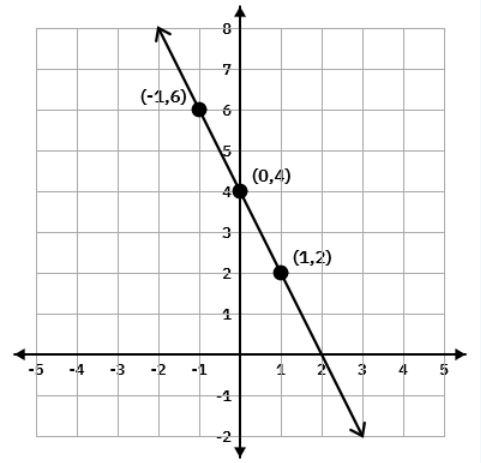
Again, we need to find coordinates of at least two points.

We arbitrarily choose x = -1, x = 0, and y = 2.

- If x = -1, then 2(-1) + y = 4 which results in y = 6. Therefore, (-1, 6) is a point on this line.
- If x = 0, then 2(0) + y = 4, which results in y = 4. Hence the point (0, 4).
- If y = 2, then 2x + 2 = 4, which yields x = 1, and gives the point (1, 2).

The table below shows the points, and the line is graphed.

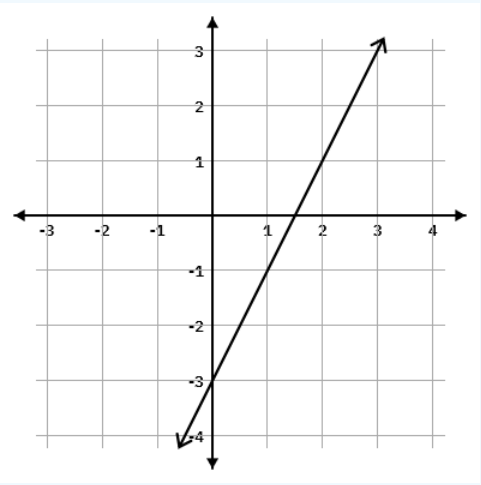
x	y	(x, y)
-1	6	(-1, 6)
0	4	(0, 4)
1	2	(1,2)



Exercise 1.2.1

Graph the line: $y = 2x - 3$.

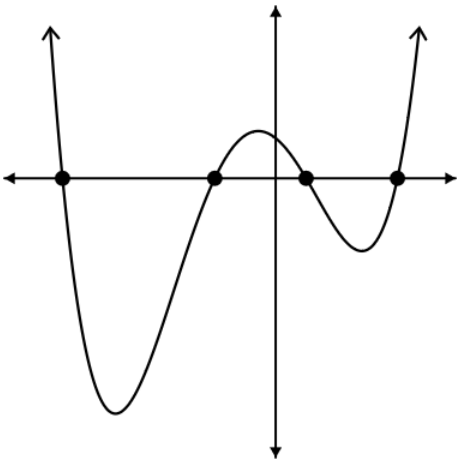
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Intercepts

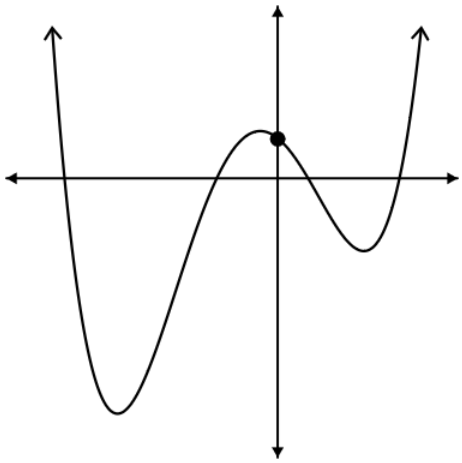
The points at which a line crosses the coordinate axes are called the **intercepts**. Since the line is touching one axis at each intercept, we know the value of one of the variables at that point.

Consider this graph below. Even though this graph isn't as "simple" as a linear graph, we can still discuss its intercepts.



The x-intercepts on this graph are identified with points. Even without numbers on this graph, we can still know **something** about the x-intercepts. What do all 4 of these points have in common? Each of them has a y-value of zero. Or to write out the coordinates of these points, they would each look like (_?_ , 0).

Likewise, we could consider the y-intercept of the same graph:



Even without numbers on this graph, we can still know **something** about the y-intercept. What is it? We know it has an x-value of zero. Or to write out the coordinate of this point, it would look like (0, _?_).

When graphing a line by plotting two points, using the intercepts is often preferred because they are often relatively easy to find.

- To find the value of the x-intercept, we let y = 0 and solve for x.
- To find the value of the y-intercept, we let x = 0 and solve for y.

Example 1.2.3

Find the intercepts of the line: $2x - 3y = 6$, and graph.

Solution

To find the x-intercept, let y = 0 in the equation and solve for x.

$$\begin{aligned} 2x - 3(0) &= 6 \\ 2x - 0 &= 6 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

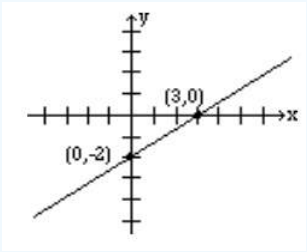
Therefore, the x-intercept is the point (3,0).

To find the y-intercept, let x = 0 in the equation and solve for y.

$$\begin{aligned} 2(0) - 3y &= 6 \\ 0 - 3y &= 6 \\ -3y &= 6 \\ y &= -2 \end{aligned}$$

Therefore, the y-intercept is the point (0, -2).

To graph the line, plot the points for the x-intercept (3,0) and the y-intercept (0, -2), and use them to draw the line.

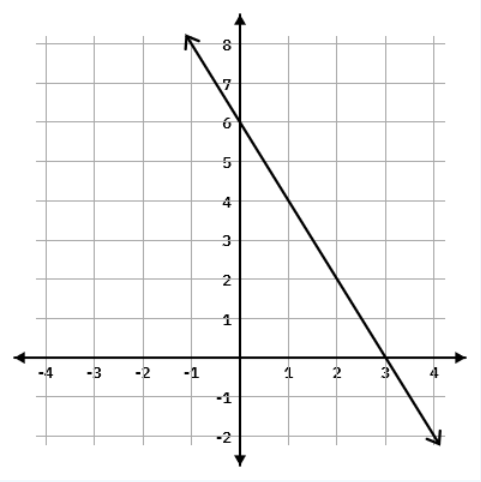


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Exercise 1.2.2

Find the intercepts of the line: $4x + 2y = 12$, and graph.

Click here to check your answer



Horizontal and Vertical Lines

When an equation of a line has only one variable, the resulting graph is a horizontal or a vertical line.

- The graph of the line $x = a$, where a is a constant, is a vertical line that passes through the point $(a, 0)$. Every point on this line has the x-coordinate equal to a, regardless of the y-coordinate.
- The graph of the line $y = b$, where b is a constant, is a horizontal line that passes through the point $(0, b)$. Every point on this line has the y-coordinate equal to b, regardless of the x-coordinate.

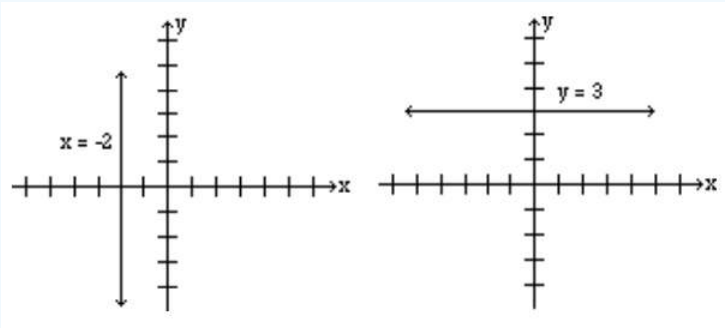
Example 1.2.4

Graph the lines: x = -2 , and y = 3.

Solution

The graph of the line x = -2 is a vertical line that has the x-coordinate -2 no matter what the y-coordinate is. The graph is a vertical line passing through point (-2, 0).

The graph of the line y = 3, is a horizontal line that has the y-coordinate 3 regardless of what the x-coordinate is. Therefore, the graph is a horizontal line that passes through point (0, 3).



Note: Most students feel that the coordinates of points must always be integers. This is not true, and in real life situations, not always possible. Do not be intimidated if your points include numbers that are fractions or decimals.

Graphing Using Slope

A graph of a line can also be determined if one point and the "steepness" of the line is known. The number that refers to the steepness or inclination of a line is called the **slope** of the line.

From previous math courses, many of you remember slope as the "rise over run," or "the vertical change over the horizontal change" and have often seen it expressed as:

$$\frac{\text{rise}}{\text{run}}, \frac{\text{vertical change}}{\text{horizontal change}}, \frac{\Delta y}{\Delta x} \text{ etc.}$$

We give a precise definition.

Definition: Slope

If (x_1, y_1) and (x_2, y_2) are two different points on a line, the **slope** of the line is

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.2.1}$$

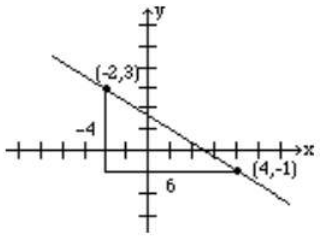
Example 1.2.5

Find the slope of the line passing through points (−2, 3) and (4, −1), and graph the line.

Solution

Let (x_1, y_1) = (−2, 3) and (x_2, y_2) = (4, −1), then the slope (via Equation 1.2.1) is

$$m = \frac{-1-3}{4-(-2)} = \frac{-4}{6} = -\frac{2}{3}$$



To give the reader a better understanding, both the vertical change, -4, and the horizontal change, 6, are shown in the above figure.

When two points are given, it does not matter which point is denoted as (x_1, y_1) and which (x_2, y_2). The value for the slope will be the same.

In Example 1.2.1, if we instead choose (x_1, y_1) = (4, −1) and (x_2, y_2) = (−2, 3), then we will get the same value for the slope as we obtained earlier.

The steps involved are as follows.

$$m = \frac{3-(-1)}{-2-4} = \frac{4}{-6} = -\frac{2}{3}$$

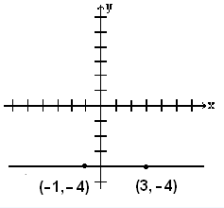
Example 1.2.6

Find the slope of the line that passes through the points (−1, −4) and (3, −4) and graph the line.

Solution

Let (x_1, y_1) = (−1, −4) and (x_2, y_2) = (3, −4), then the slope is

$$m = \frac{-4-(-4)}{3-(-1)} = \frac{0}{4} = 0$$



Note: The slope of any horizontal line is 0

Example 1.2.7

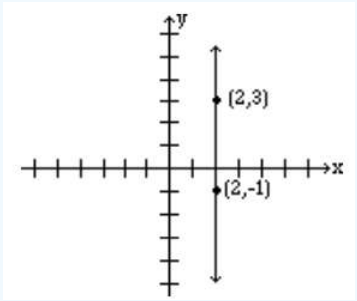
Find the slope of the line that passes through the points (2, 3) and (2, −1), and graph the line.

Solution

Let (x_1, y_1) = (2, 3) and (x_2, y_2) = (2, −1) then the slope is

$$m = \frac{-1-3}{2-2} = \frac{4}{0} = \text{undefined.}$$

NOTE: Remember that zero can NEVER be in the denominator of a fraction. When we encounter this scenario, the value is undefined. It does not exist. This is different than having a value of zero.



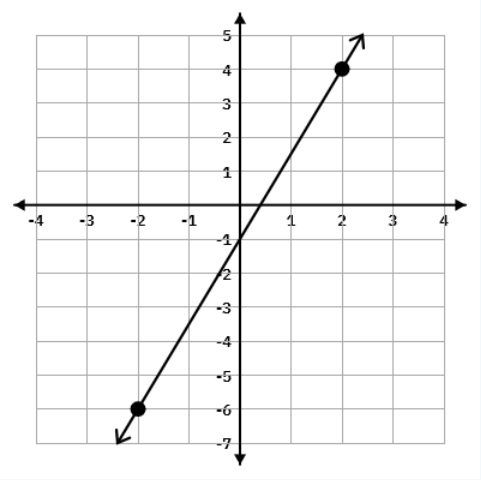
Note: The slope of any vertical line is undefined.

Exercise 1.2.3

Find the slope of the line that passes through the points (-2, -6) and (2, 4), and graph the line.

Click here to check your answer

The slope is $m = \frac{5}{2}$.



Using the previous four examples, students should observe that:

- if a line rises when going from left to right, the line has a positive slope.
- if a line falls going from left to right, the line has a negative slope.
- if a line is horizontal where the value of y never changes, the line has a slope of zero.
- if a line is vertical where the value of x never changes, the line does not have a slope.

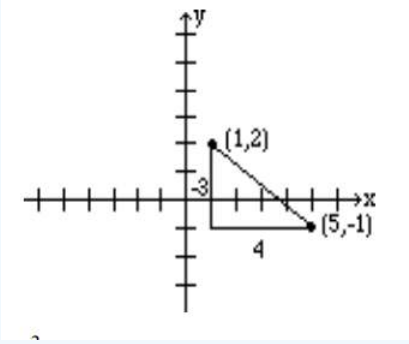
Sometimes we are given the slope and a point on the line and need to graph the entire line. In this case, we can use the given point as a "starting point" and use slope as $\frac{\text{rise}}{\text{run}}$ to graph additional points.

Example 1.2.8

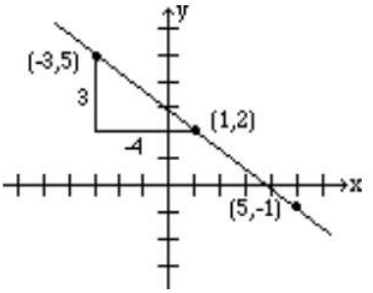
Graph the line that passes through the point (1, 2) and has slope $-\frac{3}{4}$.

Solution

Slope equals $\frac{\text{rise}}{\text{run}}$. The fact that the slope is $\frac{-3}{4}$, means that for every rise of -3 units (fall of 3 units) there is a run of 4. So if from the given point (1, 2) we go down 3 units and go right 4 units, we reach the point (5, -1). The graph is obtained by connecting these two points.



Alternatively, since $\frac{3}{-4}$ represents the same number, the line can be drawn by starting at the point (1,2) and choosing a rise of 3 units followed by a run of -4 units. So from the point (1, 2), we go up 3 units, and to the left 4, thus reaching the point (-3, 5) which is also on the same line. See figure below.



Remember that mathematically, $-\frac{3}{4}$, $\frac{-3}{4}$, and $\frac{3}{-4}$ are all equivalent to one another.

Example 1.2.9

Find the slope of the line $2x+3y=6$ by finding two points on the line.

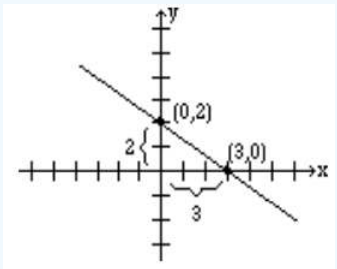
Solution

In order to find the slope of this line, we will choose any two points on this line.

Again, the selection of x and y intercepts seems to be a good choice. The x-intercept is (3, 0), and the y-intercept is (0, 2). Therefore, the slope is

$$m=\frac{2-0}{0-3}=-\frac{2}{3}.$$

The graph below shows the line and the x-intercepts and y-intercepts:



Example 1.2.10

Find the slope and the y-intercept of the line $y=3x+2$.

Solution

We again find two points on the line, e.g., (0, 2) and (1, 5). Therefore, the slope is

$$m=\frac{5-2}{1-0}=\frac{3}{1}=3.$$

Look at the slopes and the y-intercepts of the following lines.

The line	slope	y-intercept
$y=3x+2$	3	2
$y=-2x+5$	-2	5
$y=\frac{3}{2}x-4$	$\frac{3}{2}$	-4

It is no coincidence that when an equation of the line is solved for y, the coefficient of the x term represents the slope, and the constant term represents the y-intercept.

In other words, for the line $y = mx + b$, m is the slope, and b is the y-intercept. We will examine this more in the next section.

Example 1.2.11

Determine the slope and y-intercept of the line $2x + 3y = 6$.

Solution

We solve for y:

$$\begin{aligned} 2x + 3y &= 6 \\ 3y &= -2x + 6 \\ y &= (-2/3)x + 2 \end{aligned}$$

The slope = the coefficient of the x term = − 2/3.

The y-intercept = the constant term = 2.

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1.2E: Exercises - Solving Linear Equations in Two Variables

PROBLEM SET: GRAPHING A LINEAR EQUATION

Work the following problems.

1) Is the point (2, 3) on the line 5x - 2y = 4?	2) Is the point (1, - 2) on the line 6x - y = 4?
3) For the line 3x - y = 12, complete the following ordered pairs. (2,___) (___, 6) (0,___) (___, 0)	4) For the line 4x + 3y = 24, complete the following ordered pairs. (3,___) (___, 4) (0,___) (___, 0)

Graph the following equations using intercepts.

5) Graph 2x + 4 = 0	6) Graph 2y - 6 = 0
7) Graph the following three equations on the same set of coordinate axes. $y = x + 1 \quad y = 2x + 1 \quad y = -x + 1$	8) Graph the following three equations on the same set of coordinate axes. $y = 2x + 1 \quad y = 2x \quad y = 2x - 1$

Find the slope of the line passing through the following pair of points.

9) (2, 3) and (5, 9)	10) (4, 1) and (2, 5)
11) (- 1, 1) and (1, 3)	12) (4, 3) and (- 1, 3)
13) (3, 4) and (3, 7)	14) (- 2, 4) and (- 3, - 2)
15) (- 3, - 5) and (- 1, - 7)	16) (0, 4) and (3, 0)

Determine the slope of the line from the given equation of the line.

17) y = - 2x + 1	18) y = 3x - 2
19) 2x - y = 6	20) x + 3y = 6
21) 3x - 4y = 12	22) What is the slope of the x-axis? What is the slope of the y-axis?

Graph the line that passes through the given point and has the given slope.

23) (1, 2) and m = - 3/4	24) (2, - 1) and m = 2/3
25) (0, 2) and m = - 2	26) (2, 3) and m = 0

Graph the following equations using their slope and y-intercept.

27) Graph y = 2x + 3	28) Graph y = - 3x + 5
29) Graph y = 4x - 3	30) Graph x - 2y = 8
31) Graph 2x + y = 4	32) Graph 2x - 3y = 6

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1.3: Determining a Linear Equation

Learning Objectives

In this section, you will learn to:

- Find an equation of a line if a point and the slope are given.
- Find an equation of a line if two points are given.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Simplify each expression:

a. $3(2x-5)$

b. $-7(4x-2)$

Click here to check your answer

a. $6x-15$

b. $-28x+14$

If you missed any part of this problem, review here. (Note that this will open a different textbook in a new window.)

2. Find the slope of the line containing each given pair of points.

a. (1, 4) and (5, -2)

b. (-2, 5) and (10, -4).

Click here to check your answer

a. $-\frac{3}{2}$

b. $-\frac{3}{4}$

If you missed any part of this problem, review Section 1.2. (Note that this will open in a new window.)

3. Solve: $\frac{1}{3}x+\frac{1}{5}=\frac{1}{5}x-1$.

Click here to check your answer

$x=-9$

If you missed this problem, review Section 1.1. (Note that this will open in a new window.)

So far, we were given an equation of a line and were asked to give information about it. For example, we were asked to find points on the line, find its slope and even find intercepts. Now we are going to reverse the process. That is, we will be given either two points, or a point and the slope of a line, and we will be asked to find its equation.

An equation of a line can be written in three forms, the **slope-intercept form**, the **point-slope form,** or the **standard form**. We will discuss each of them in this section.

A line is completely determined by two points, or by a point and slope. The information we are given about a particular line will influence which form of the equation is most convenient to use. Once we know any form of the equation of a line, it is easy to re-express the equation in the other forms if needed.

THE SLOPE-INTERCEPT FORM OF A LINE: $y = mx + b$

In the last section we learned that the equation of a line whose slope = m and y-intercept = b is

$$y = mx + b. \tag{1.3.1}$$

This is called the **slope-intercept form** of the line and is the most commonly used form.

Example 1.3.1

Find an equation of a line whose slope is 5, and y-intercept is 3.

Solution

Since the slope is $m = 5$, and the y- intercept is $b = 3$, the equation is $y = 5x + 3$.

Example 1.3.2

Find the equation of the line that passes through the point $(2, 7)$ and has slope 3.

Solution

Since $m = 3$, the partial equation is $y = 3x + b$.

Now b can be determined by substituting the point $(2, 7)$ in the equation $y = 3x + b$.

$$\begin{aligned} 7 &= 3(2) + b \\ b &= 1 \end{aligned}$$

Therefore, the equation is $y = 3x + 1$.

Example 1.3.3

Find an equation of the line that passes through the points (-1, 2), and (1, 8).

Solution

We need to find the slope of the line containing these points first. Remember that if (x_1, y_1) and (x_2, y_2) are two different points on a line, the **slope** of the line is

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

$m = \frac{8-2}{1-(-1)} = \frac{6}{2} = 3$. So the partial equation is $y = 3x + b$.

We can use either of the two points (-1, 2) or (1, 8), to find b. Substituting (-1, 2) gives

$$\begin{aligned} 2 &= 3(-1) + b \\ 5 &= b \end{aligned}$$

So the equation is $y = 3x + 5$.

Example 1.3.4

Find an equation of the line that has x-intercept 3, and y-intercept 4.

Solution

x-intercept = 3, and y-intercept = 4 correspond to the points (3, 0), and (0, 4), respectively.

$$m = \frac{4 - 0}{0 - 3} = -\frac{4}{3}$$

We are told the y-intercept is 4; thus b = 4

Therefore, the equation is $y = -\frac{4}{3}x + 4$.

Example 1.3.5

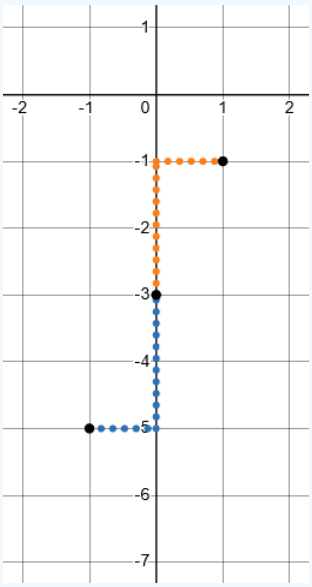
Use slope-intercept form to graph $y = 2x - 3$.

Solution

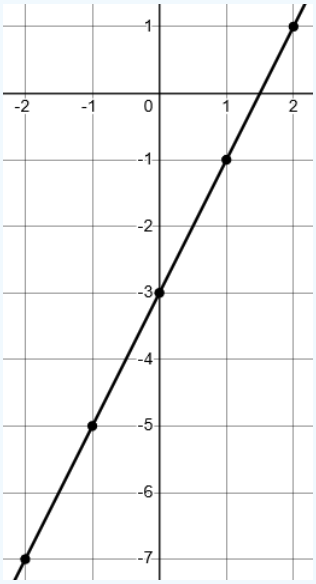
Since this equation is already in $y = mx + b$ form, we can identify the slope as 2 (the coefficient to the x-term) and the y-intercept as -3, or the point $(0, -3)$.

To graph this line, we'll begin with the y-intercept as that is a known point on the graph.

We'll then use the slope to find additional points. The slope is 2, which is more helpful to write in its fractional form of $2/1$. This tells us that the rise (vertical change between points) is 2 and the run (horizontal change between points) is 1. We can find additional points by going up 2 and right 1, as well as down 2 and left 1.



The graph is



THE POINT-SLOPE FORM OF A LINE: $y - y_1 = m(x - x_1)$

The **point-slope** form is useful when we know two points on the line and want to find the equation of the line.

The definition of a slope leads us to the point-slope formula. Using two points (x_1, y_1) and any other (x, y), the slope is $\frac{y-y_1}{x-x_1} = m$.

Multiplying both sides by $(x - x_1)$ gives the point-slope form: $y - y_1 = m(x - x_1)$.

A line with slope m which contains a specific point (x_1, y_1) can be expressed in the form

$$y - y_1 = m(x - x_1). \tag{1.3.2}$$

Example 1.3.6

Find the point-slope form of the equation of the line given in Example 1.3.2. (Find the equation of the line that passes through the point $(2, 7)$ and has slope 3.) Show that the two forms of the equations are equivalent.

Solution

Substituting the point $(x_1, y_1) = (2, 7)$ and $m = 3$ in the point-slope formula, we get

$$\mathbf{y - y_1 = m(x - x_1)}$$

$$y - 7 = 3(x - 2)$$

We can show that the forms are equivalent by solving this equation for y.

$$y - 7 = 3(x - 2)$$

$$y - 7 = 3x - 6$$

$$y = 3x + 1$$

This is the slope-intercept form of the equation that we found in Example 1.3.2.

Example 1.3.7

Find the point-slope form of the equation of a line that has slope $\frac{1}{2}$ and passes through the point (12,4). Then write the equation in slope-intercept form.

Solution

Substituting the point $(x_1, y_1) = (12, 4)$ and $m = \frac{1}{2}$ in the point-slope formula, we get

$$\mathbf{y - y_1 = m(x - x_1)}$$

$$y - 4 = \frac{1}{2}(x - 12)$$

$$y - 4 = \frac{1}{2}x - 6$$

$$y = \frac{1}{2}x - 2$$

THE STANDARD FORM OF A LINE: $Ax + By = C$

Another useful form of the equation of a line is the standard form.

If we know the equation of a line in point-slope form, $y - y_1 = m(x - x_1)$, or if we know the equation of the line in slope-intercept form $y = mx + b$, we can simplify the formula to have all terms for the x and y variables on one side of the equation, and the constant on the other side of the equation.

The result is referred to as the **standard form** of the line: $Ax + By = C$ where A, B, and C are integers and $A > 0$.

We should always be able to convert from one form of an equation to another. For example, if we are given a line in the slope-intercept form, we should be able to express it in the standard form, and vice versa.

Example 1.3.8

Write the equation $y = -\frac{2}{3}x + 3$ in the standard form.

Solution

Multiplying both sides of the equation by 3, we get

$$\begin{aligned} 3y &= -2x + 9 \\ 2x + 3y &= 9 \quad \text{Standard Form} \end{aligned}$$

Example 1.3.9

Write the equation $3x - 4y = 10$ in the slope-intercept form.

Solution

Solving for y, we get

$$\begin{aligned} -4y &= -3x + 10 \\ y &= \frac{3}{4}x - \frac{5}{2} \quad \text{Standard Form} \end{aligned}$$

Example 1.3.10

Using the point-slope formula, find the standard form of an equation of the line that passes through the point (2, 3) and has slope $-3/5$.

Solution: Substituting the point (2, 3) and $m = -\frac{3}{5}$ in the point-slope formula, we get

$$y - 3 = -\frac{3}{5}(x - 2)$$

Multiplying both sides by 5 gives us

$$\begin{aligned} 5(y - 3) &= -3(x - 2) \\ 5y - 15 &= -3x + 6 \\ 3x + 5y &= 21 \quad \text{Standard Form} \end{aligned}$$

Example 1.3.11

Find the standard form of the line that passes through the points (0, -2), and (4, 0).

Solution

First we find the slope: $m = \frac{0-(-2)}{4-0} = \frac{2}{4} = \frac{1}{2}$

Then, the point-slope form is:

$$\begin{aligned} y - (-2) &= \frac{1}{2}(x - 0) \\ y + 2 &= \frac{1}{2}x \end{aligned}$$

Multiplying both sides by 2 gives us

$$\begin{aligned} 2(y + 2) &= 2(\frac{1}{2}x) \\ 2y + 4 &= 1x \\ -1x + 2y + 4 &= 0 \\ -1x + 2y &= -4 \quad \text{Standard Form} \end{aligned}$$

Standard form is useful to find both intercepts for a line.

Example 1.3.12

Use standard form to graph $2x - 4y = 12$.

Solution

To find the x-intercept, we let $y = 0$.

$$\begin{aligned} 2x - 4(0) &= 12 \\ 2x &= 12 \\ x &= 6 \end{aligned}$$

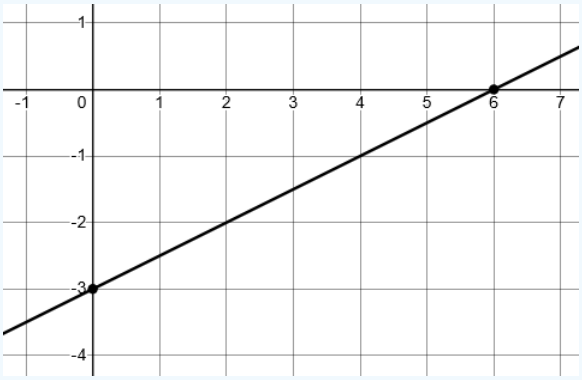
The x-intercept is $(6, 0)$.

To find the y-intercept, we let $x = 0$.

$$\begin{aligned} 2(0) - 4y &= 12 \\ -4y &= 12 \\ y &= -3 \end{aligned}$$

The y-intercept is $(0, -3)$.

We graph the line by graphing both intercepts and connecting the points.



We summarize the forms for equations of a line below:

Slope Intercept form: $\mathbf{y = mx + b}$,
where m = slope, b = y-intercept

Point Slope form: $\mathbf{y - y_1 = m(x - x_1)}$,
where m = slope, (x_1, y_1) is a point on the line

Standard form: $\mathbf{Ax + By = C}$

Horizontal Line: $\mathbf{y = b}$
where b = y-intercept

Vertical Line: $\mathbf{x = a}$
where a = x-intercept

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1.3E: Exercises - Determining a Linear Equation

PROBLEM SET: DETERMINING THE EQUATION OF A LINE

Write an equation of the line satisfying the following conditions.
Write the equation in the form y = mx + b.

1) It passes through the point (3, 10) and has slope = 2.	2) It passes through point (4,5) and has m = 0.
3) It passes through (3, 5) and (2, - 1).	4) It has slope 3, and its y-intercept equals 2.
5) It passes through (5, - 2) and m = 2/5.	6) It passes through (- 5, - 3) and (10, 0).
7) It passes through (4, - 4) and (5, 3).	8) It passes through (7, - 2) ; its y-intercept is 5.
9) It passes through (2, - 5) and its x-intercept is 4.	10) Its a horizontal line through the point (2, - 1).

Write an equation of the line satisfying the following conditions.
Write the equation in the form y = mx + b.

11) It passes through (5, - 4) and (1, - 4).	12) It is a vertical line through the point (3, - 2).
13) It passes through (3, - 4) and (3, 4).	14) It has x-intercept = 3 and y-intercept = 4.

Write an equation of the line satisfying the following conditions.
Write the equation in the form Ax + By = C.

15) It passes through (3, - 1) and m = 2.	16) It passes through (- 2, 1) and m = - 3/2.
17) It passes through (- 4, - 2) and m = 3/4.	18) Its x-intercept equals 3, and m = - 5/3.

Write an equation of the line satisfying the following conditions.
Write the equation in the form Ax + By = C.

19) It passes through (2, - 3) and (5, 1).	20) It passes through (1, - 3) and (- 5, 5).

Write an equation of the line satisfying the following conditions.
Write the equation in point slope form $y-y_1 = m (x-x_1)$

21) It passes through (2, - 3) and (5, 1).	22) It passes through (1, - 3) and (- 5, 2).
23) It passes through (6, -2) and (0, 2).	24) It passes through (8, 2) and (-7, -4).
25) It passes through (-12, 7) and has slope = -1/3.	26) It passes through (8, - 7) and has slope 3/4.

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1.4: Solving Systems of Linear Equations in Two Variables

Learning Objectives

In this section, you will learn to:

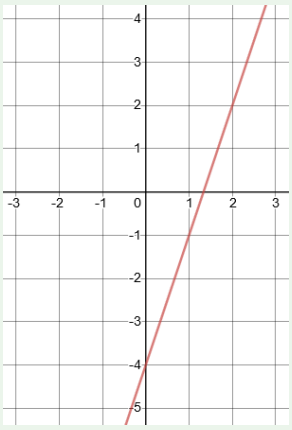
- Determine if a given ordered pair is a solution to a system of equations.
- Solve a system of linear equations in two variables by graphing, substitution, and elimination by addition.
- Find the equilibrium point when a demand and a supply equation are given.
- Find the break-even point when the revenue and the cost functions are given.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Graph all solutions to the equation $y = 3x - 4$.

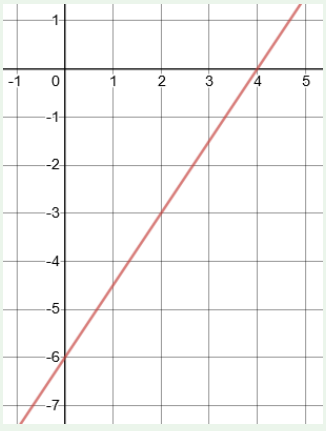
Click here to check your answer



If you missed this problem, review Section 1.2. (Note that this will open in a new window.)

2. Graph all solutions to the equation $3x - 2y = 12$.

Click here to check your answer



If you missed this problem, review Section 1.2. (Note that this will open in a new window.)

3. Is $(2, 5)$ a solution to $x - 5y = 2$?

Click here to check your answer

No, because $-23 \neq 2$.

If you missed this problem, review Section 1.1. (Note that this will open in a new window.)

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.

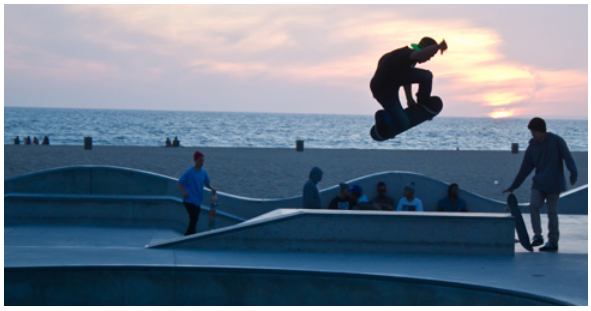


Figure 1.4.1

Solutions to Systems of Linear Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. **A system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously.

To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

$$\begin{aligned} 2x+y &= 15 \\ 3x-y &= 5 \end{aligned}$$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair $(4,7)$ is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

$$\begin{aligned} 2(4)+(7) &= 15 \text{ True} \\ 3(4)-(7) &= 5 \text{ True} \end{aligned}$$

Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

Example 1.4.1: Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5,1)$ is a solution to the given system of equations.

$$\begin{aligned} x+3y &= 8 \\ 2x-9 &= y \end{aligned}$$

Solution

Substitute the ordered pair $(5, 1)$ into both equations.

$$\begin{aligned} (5)+3(1) &= 8 \\ 8 &= 8 \text{ True} \\ 2(5)-9 &= (1) \\ 1 &= 1 \text{ True} \end{aligned}$$

The ordered pair $(5, 1)$ satisfies both equations, so it is is the solution to the system.

Analysis

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See Figure $1.4.2$.

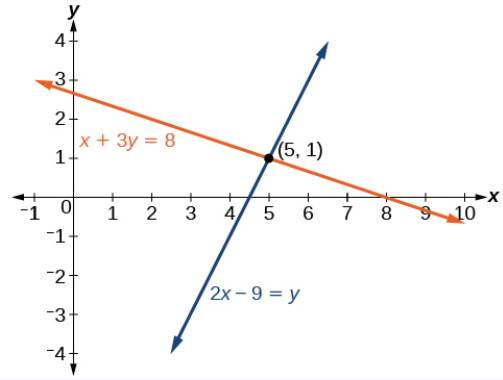


Figure 1.4.2

? Exercise 1.4.1

Determine whether the ordered pair $(8, 5)$ is a solution to the following system.

$$\begin{aligned} 5x-4y &= 20 \\ 2x+1 &= 3y \end{aligned}$$

Answer

Not a solution.

Generally, we aren't given the coordinates of a possible solution. Instead, we need to find it.

There are three main methods for finding a solution to a system of equations:

1. Graphing,
2. Substitution
3. Elimination by addition

We'll work through each method below.

Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

How to: Given a system of two equations in two variables, solve using the graphing method.

1. Carefully and accurately graph each equation on the same set of axes.
2. Identify the point at which the lines intersect.
3. Check the solution in both equations.
4. Write the soluiton as an ordered pair.

Example 1.4.2: Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing.

$$\begin{aligned} 2x+y &= -8 \\ x-y &= -1 \end{aligned}$$

Solution

Solve the first equation for y.

$$\begin{aligned} 2x+y &= -8 \\ y &= -2x-8 \end{aligned}$$

Solve the same equation for y.

$$\begin{aligned} x-y &= -1 \\ &= x+1 \end{aligned}$$

Graph both equations on the same set of axes as in Figure 1.4.3.

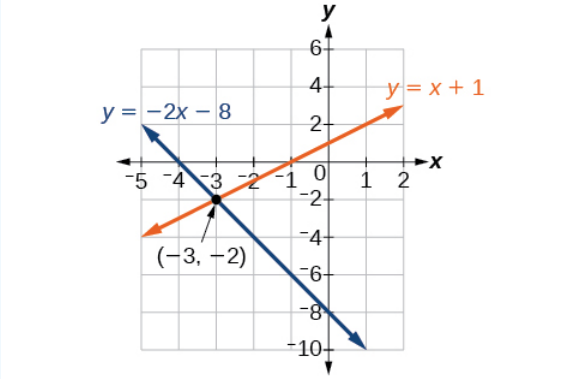


Figure 1.4.3

The lines appear to intersect at the point $(-3,-2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

$$\begin{aligned} 2(-3)+(-2) &= -8 \\ -8-8 \ \text{True} & \\ (-3)-(-2) &= -1 \\ -1 &= -1 \ \text{True} \end{aligned}$$

Exercise 1.4.2

Solve the following system of equations by graphing.

$$\begin{aligned} 2x-5y &= -25 \\ -4x+5y &= 35 \end{aligned}$$

Answer

The solution to the system is the ordered pair $(-5,3)$.

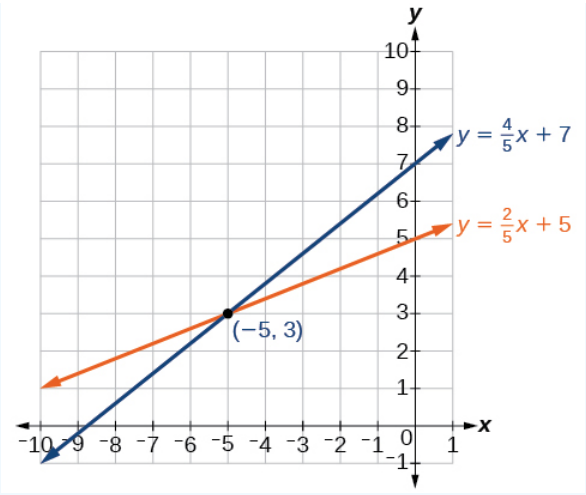


Figure 1.4.4

Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a **system of linear equations** that are more precise than graphing. One such method method is solving a system of equations by the **substitution method**, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

How to: Given a system of two equations in two variables, solve using the substitution method.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the other equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.
5. Write the solution as an ordered pair.

Example 1.4.3: Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.

$$\begin{aligned} -x+y &= -5 \\ 2x-5y &= 1 \end{aligned}$$

Solution

First, we will solve the first equation for y.

$$\begin{aligned} -x+y &= -5 \\ y &= x-5 \end{aligned}$$

Now we can substitute the expression $x-5$ for y in the second equation.

$$\begin{aligned} 2x-5y &= 1 \\ 2x-5(x-5) &= 1 \\ 2x-5x+25 &= 1 \\ -3x &= -24 \\ x &= 8 \end{aligned}$$

Now, we substitute $x = 8$ into the first equation and solve for y.

$$\begin{aligned} -(8)+y &= -5 \\ y &= 3 \end{aligned}$$

Our solution is $(8, 3)$.

Check the solution by substituting $(8, 3)$ into both equations.

$$\begin{aligned} -x+y &= -5 \\ -(8)+(3) &= -5 \text{ True} \\ 2x-5y &= 1 \\ 2(8)-5(3) &= 1 \text{ True} \end{aligned}$$

? Exercise 1.4.3

Solve the following following system of equations by substitution.

$$\begin{aligned} x &= y+3 \\ 4 &= 3x-2y \end{aligned}$$

Answer

$(-2, -5)$

Solving Systems of Equations in Two Variables by the Elimination by Addition Method

A third method of solving systems of linear equations is the Addition by addition method of solving systems of linear equations is the addition method. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

How to: Given a system of equations, solve using the elimination by addition method.

1. Write both equations with *x*- and *y*-variables on the left side of the equal sign and constants on the right. Write one equation above the other, lining up corresponding variables.
2. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.
6. Write the solution as an ordered pair.

✓ Example 1.4.4: Solving a System by the Elimination by Addition Method

Solve the given system of equations by elimination by addition.

$$\begin{aligned} x+2y &= -1 \\ -x+y &= 3 \end{aligned}$$

Solution

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, -1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.

$$\begin{aligned} x+2y &= -1 \\ \underline{-x+y} &= \underline{3} \\ 3y &= 2 \end{aligned}$$

Now that we have eliminated x, we can solve the resulting equation for y.

$$\begin{aligned} 3y &= 2 \\ y &= \frac{2}{3} \end{aligned}$$

Then, we substitute this value for y into one of the original equations and solve for x.

$$\begin{aligned} -x+y &= 3 \\ -x+\frac{2}{3} &= 3 \\ -x &= 3-\frac{2}{3} \\ -x &= \frac{7}{3} \\ x &= -\frac{7}{3} \end{aligned}$$

The solution to this system is $\left(-\frac{7}{3}, \frac{2}{3}\right)$.

Check the solution in the first equation.

$$\begin{aligned} x+2y &= -1 \\ \left(-\frac{7}{3}\right)+2\left(\frac{2}{3}\right) &= -1 \\ -\frac{7}{3}+\frac{4}{3} &= -1 \\ -\frac{3}{3} &= -1 \\ -1 &= -1 \qquad \text{True} \end{aligned}$$

Analysis

We gain an important perspective on systems of equations by looking at the graphical representation. See Figure 1.4.5 to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.

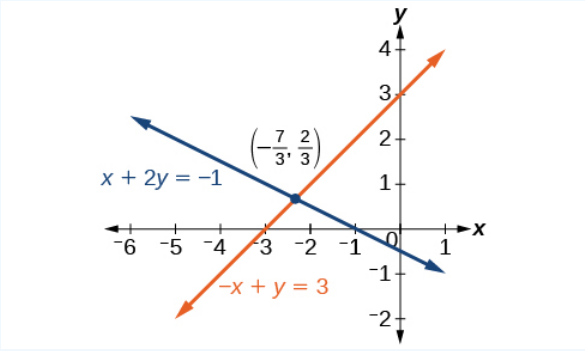


Figure 1.4.5

Example 1.4.5: Using the Elimination by Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the elimination by addition method.

$$3x+5y=-11$$
$$x-2y=11$$

Solution

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So if we multiply the second equation has x. So if we multiply the second equation by -3,the x-terms will add to zero.

$$\begin{aligned} x-2y &= 11 \\ -3(x-2y) &= -3(11) \qquad \text{Multiply both sides by } -3. \\ -3x+6y &= -33 \qquad \text{Use the distributive property.} \end{aligned}$$

Now, let's add them.

$$\begin{aligned} 3x+5y &= -11 \\ \underline{-3x+6y} &= \underline{-33} \\ 11y &= -44 \\ y &= -4 \end{aligned}$$

For the last step, we substitute $y=-4$ into one of the original equations and solve for x.

$$\begin{aligned} 3x+5y &= -11 \\ 3x+5(-4) &= -11 \\ 3x-20 &= -11 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Our solution is the ordered pair $(3,-4)$. See Figure 1.4.6. Check the solution in the original second equation.

$$\begin{aligned} x-2y &= 11 \\ (3)-2(-4) &= 3+8 \\ &= 11 \qquad \text{True} \end{aligned}$$

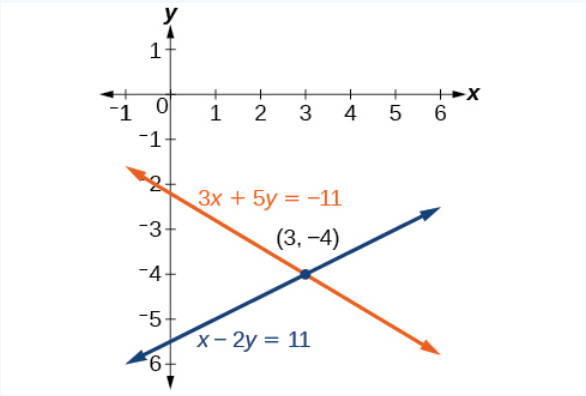


Figure 1.4.6

Exercise 1.4.5

Solve the system of equations by elimination by addition.

$$2x-7y=2$$
$$3x+y=-20$$

Answer

$(-6,-2)$

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Example 1.4.6: Using the Elimination by Addition Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables by elimination by addition.

$$\begin{aligned} 2x+3y &= -16 \\ 5x-10y &= 30 \end{aligned}$$

Solution

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate x by multiplying the first equation by -5 and the second equation by 2.

$$\begin{aligned} -5(2x+3y) &= -5(-16) \\ -10x-15y &= 80 \\ 2(5x-10y) &= 2(30) \\ 10x-20y &= 60 \end{aligned}$$

Then, we add the two equations together.

$$\begin{aligned} -10x-15y &= 80 \\ \underline{10x-20y} &= \underline{60} \\ -35y &= 140 \\ y &= -4 \end{aligned}$$

Substitute $y=-4$ into the original first equation.

$$\begin{aligned} 2x+3(-4) &= -16 \\ 2x-12 &= -16 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

The solution is $(-2,-4)$. Check it in the other equation.

$$\begin{aligned} 5x-10y &= 30 \\ 5(-2)-10(-4) &= 30 \\ -10+40 &= 30 \\ 30 &= 30 \end{aligned}$$

See Figure 1.4.7.

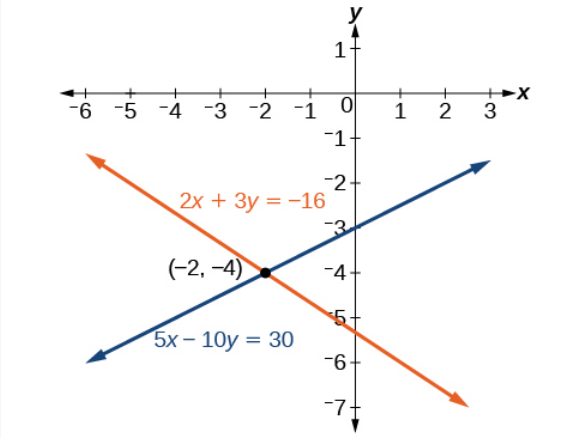


Figure 1.4.7

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SUPPLY, DEMAND AND THE EQUILIBRIUM PRICE

In a free market economy the supply curve for a commodity is the number of items of a product that can be made available at different prices, and the demand curve is the number of items the consumer will buy at different prices.

As the price of a product increases, its demand decreases and supply increases. On the other hand, as the price decreases the demand increases and supply decreases. The **equilibrium price** is reached when the demand equals the supply.

Example 1.4.7

The supply curve for a product is $y = 3.5x - 14$ and the demand curve for the same product is $y = -2.5x + 34$, where x is the price and y the number of items produced. Find the following.

a. How many items will be supplied at a price of $10?
b. How many items will be demanded at a price of $10?
c. Determine the equilibrium price.
d. How many items will be produced at the equilibrium price?

Solution

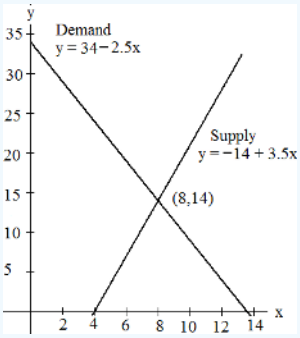
a) We substitute $x = 10$ in the supply equation, $y = 3.5x - 14$; the answer is $y = 3.5(10) - 14 = 21$.

b) We substitute $x = 10$ in the demand equation, $y = -2.5x + 34$; the answer is $y = -2.5(10) + 34 = 9$.

c) By letting the supply equal the demand, we get

$$\begin{aligned} 3.5x - 140 &= -25x + 340 \\ 6x &= 480 \\ x &= \$80 \end{aligned}$$

d) We substitute $x = 8$ in either the supply or the demand equation; we get $y = 14$.



The graph shows the intersection of the supply and the demand functions and their point of intersection, (8,14).

Interpretation: At equilibrium, the price is $8 per item, and 14 items are produced by suppliers and purchased by consumers.

Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's revenue function is the function used to calculate the amount the amount of money that comes into the business. It can be represented by the equation $R = xp$, where x=quantity and p=price. The revenue function is shown in orange in Figure 1.4.11.

The cost function is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in Figure 1.4.11. The x-axis represents quantity in hundreds of units. The y-axis represents either cost or revenue in hundreds of dollars.

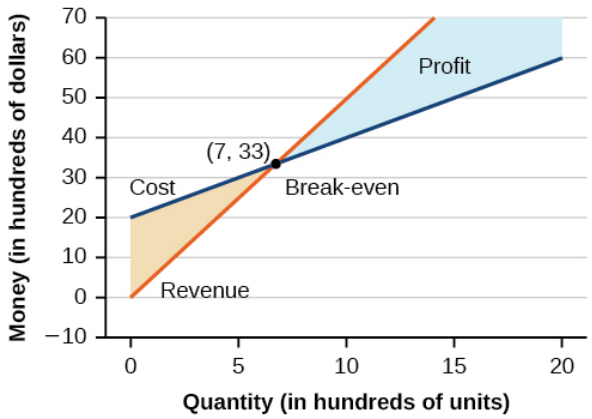


Figure 1.4.11

The point at which the two lines intersect is called the break-even point. We can see from the graph that if 700 units are produced, the cost is $\$3,300$ and the revenue is also $\$3,300$. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to shaded region to the left represents quantities for which the company suffers a loss. The profit function is the revenue function minus the cost function, written as $P = R - C$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

Example 1.4.8: Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C = 0.85x + 35,000$ and the revenue function function $R = 1.55x$, find the break-even point and the profit function.

Solution

Since the break-even point is where $R = C$, these can both be represented just using y.

$$\begin{aligned} y &= 0.85x + 35,000 \\ y &= 1.55x \end{aligned}$$

Substitute the expression $0.85x + 35,000$ from the first equation into the second equation and solve for x.

$$\begin{aligned} 0.85x + 35,000 &= 1.55x \\ 35,000 &= 0.7x \\ 50,000 &= x \end{aligned}$$

Then, we substitute $x = 50,000$ into either the cost function or or the revenue function.

$$1.55(50,000) = 77,500$$

The break-even is $(50,000, 77,500)$

The profit function is found using is formula formula $P = R - C$.

$$\begin{aligned} P &= 1.55x - (0.85x + 35,000) \\ &= 0.7x - 35,000 \end{aligned}$$

The profit function is $P = 0.7x - 35,000$.

Analysis

The cost to produce $50,000$ units is $\$77,500$, and the revenue from the sales of $50,000$ units is also $\$77,500$. To make a profit, the business must produce and sell more than $50,000$ units. See Figure $1.4.12$

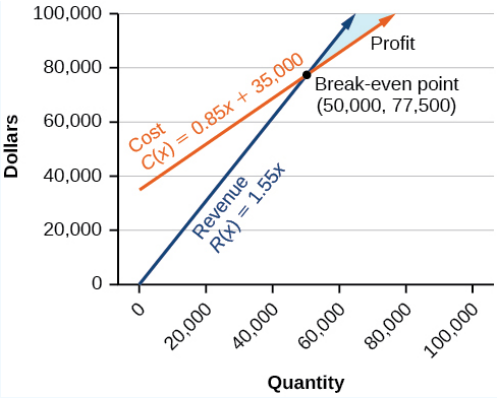


Figure 1.4.12

We see from the graph in Figure 1.4.13 that the profit function has a negative value until $x = 50,000$, when the graph crosses the x-axis. Then, the graph emerges into positive y-values and continues on this path as the profit function is a straight line. This illustrates that function is a straight line. This illustrates that the break-even point for businesses occurs when for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.

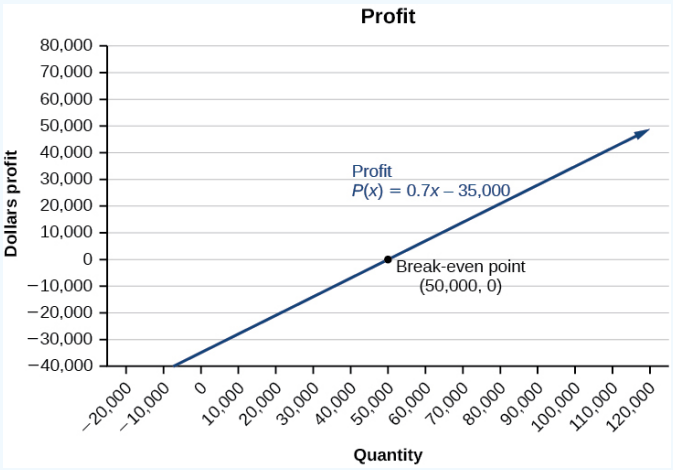


Figure 1.4.13

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1.4E: Exercises - Solving Systems of Linear Equations in Two Variables

Solve the following problems.

1) Determine if (0, 5) is a solution to the system. $6x - 2y = -10$ $3x + y = 8$	2) Determine if (-3, 2) is a solution to the system. $-4x - 3y = 6$ $2x + 5y = 4$
3) Solve this system using graphing. $y = 3x + 4$ $y = 5x - 2$	4) Solve this system using graphing. $x - 2y = -6$ $y = -2x - 7$
5) Solve this system using substitution. $y = 3x - 1$ $2x + y = 4$	6) Solve this system using substitution. $2x + y = 9$ $3x - 2y = -4$
7) Solve this system using elimination by addition. $3x - 7y = -1$ $2x + 7y = -24$	8) Solve this system using elimination by addition. $2x - 3y = 4$ $3x - 4y = 5$
9) The supply and demand curves for a product are: Supply $y = 2000x - 6500$ Demand $y = -1000x + 28000$, where x is price and y is the number of items. At what price will supply equal demand and how many items will be produced at that price?	10) The supply and demand curves for a product are Supply $y = 300x - 18000$ and Demand $y = -100x + 14000$, where x is price and y is the number of items. At what price will supply equal demand, and how many items will be produced at that price?
11) A company's revenue and cost in dollars are given by $R = 225x$ and $C = 75x + 6000$, where x is the number of items. Find the number of items that must be produced to break-even.	12) A break-even point is the intersection of the cost function and the revenue function, that is, where total cost equals revenue, and profit is zero. Mrs. Jones Cookies Store's cost and revenue, in dollars, for x number of cookies is given by $C = .05x + 3000$ and $R = .80x$. Find the number of cookies that must be sold to break even.

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1.5: Chapter 1 Review

PROBLEM SET: CHAPTER 1 REVIEW

Solving Linear Equations in One Variable (1.1)

1. Is $x = -2$ a solution to the equation $-4x - 12 = -2$?
2. Solve $6x - 9 = 18$
3. Solve $5 - 3a = 5$
4. Solve $\frac{2-5y}{6} = -8$
5. Solve $7 - 9y + 12 = 3y + 11 - 11y$
6. Solve $\frac{1}{3}x - \frac{3}{2} + \frac{5}{2}x = \frac{5}{6}x + \frac{1}{4}$
7. Solve $4(4a - 1) = 5(a - 3) + 2(a - 2)$
8. Solve $3(4 - y) - 2(y + 7) = -5y$
9. Twice the sum of a number and 4 is equal to 3 times the sum of the number and 1. Find the number.

Solving Linear Equations in Two Variables (1.2)

1. Is the point $(3, -2)$ on the line $5x - 2y = 11$?
2. Find two points on the line $2x - 6 = 0$.
3. Find the slope of the line whose equation is $2x + 3y = 6$.
4. Find the slope of the line whose equation is $y = -3x + 5$.
5. Graph the line $y = \frac{2}{3}x - 5$ by using slope-intercept form.
6. Graph the line $y = -2x + 3$ by using slope-intercept form.
7. Find both the x and y intercepts of the line $3x - 2y = 12$.
8. Graph the line $5x - 3y = 30$ by using intercepts.
9. Graph the line $2x - 3y + 6 = 0$ by using intercepts.

Determining the Equation of a Line (1.3)

1. Find an equation of the line whose slope is 3 and y-intercept 5.
2. Find an equation of the line whose x-intercept is 2 and y-intercept 3.
3. Find an equation of the line that has slope 3 and passes through the point (2, 15).
4. Find an equation of the line that has slope -3/2 and passes through the point (4, 3).
5. Find an equation of the line that passes through the points (0, 32) and (100, 212).
6. Find an equation of the x-axis.

Solving Systems of Equations in Two Variables (1.4)

Determine whether the given ordered pair is a solution to the system of equations.

1. $\begin{cases} -3x - 5y &= 13 \\ -x + 4y &= 10 \end{cases}$ and $(-6, 1)$

   Answer

   Yes

2. $\begin{cases} 3x + 7y &= 1 \\ 2x + 4y &= 0 \end{cases}$ and $(2, 3)$

Solve these two systems by graphing.

1. $\begin{cases} y &= 2x - 7 \\ y &= -x + 2 \end{cases}$

   Answer

   $(3, -1)$

2. $\begin{cases} y &= -\frac{1}{2}x - 1 \\ y &= 3x + 6 \end{cases}$

Solve these two systems by substitution.

1. $\begin{cases} x + 5y &= 5 \\ 2x + 3y &= 4 \end{cases}$

   Answer

   $(-1, 2)$

2. $\begin{cases} x - 0.2y &= 1 \\ -10x + 2y &= 5 \end{cases}$

Solve these two systems by elimination by addition.

1. $\begin{cases} x + 5y &= 5 \\ 2x + 3y &= 4 \end{cases}$

   Answer

   $(-1, 2)$

2. $\begin{cases} x - 0.2y &= 1 \\ -10x + 2y &= 5 \end{cases}$

3. The supply curve for a product is $y = 250x - 1000$. The demand curve for the same product is $y = -350x + 8,000$, where x is the price and y the number of items produced. Find the following.
   a. At a price of $10, how many items will be in demand?
   b. At what price will 4,000 items be supplied?
   c. What is the equilibrium price for this product?
   d. How many items will be manufactured at the equilibrium price?
4. The supply curve for a product is $y = 625x - 600$ and the demand curve for the same product is $y = -125x + 8,400$, where x is the price and y the number of items produced. Find the equilibrium price and determine the number of items that will be produced at that price.
5. A cell phone factory has a cost of production of $C(x) = 150x + 10,000$ and a revenue function $R(x) = 200x$. What is the break-even point?
6. A musician charges $C(x) = 64x + 20,000$, where x is the total number of attendees at the concert. The venue charges $80 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

   Answer

   $(1250, \$100,000)$

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CHAPTER OVERVIEW

2: Solving Linear Inequalities

2.1: Solving Linear Inequalities in One Variable

2.1E: Exercises - Solving Linear Inequalities in One Variable

2.2: Solving Linear Inequalities in Two Variables

2.2E: Exercises - Solving Linear Inequalities in Two Variables

2.3: Solving Systems of Linear Inequalities in Two Variables

2.3E: Exercises - Solving Systems of Linear Inequalities in Two Variables

2.4: Chapter 2 Review

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2.1: Solving Linear Inequalities in One Variable

Learning Objectives

In this section, you will learn to:

- Verify solutions to linear inequalities.
- Represent infinite solutions using interval notation.
- Solve linear inequalities in one variable.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Verify if each value is a solution to the equation $3x-2(x+4)=6x+7$

a. $x=2$

b. $x=-3$

Click here to check your answer

a. No

b. Yes

If you missed any part of this problem, review Section 1.1. (Note that this will open a in a new window.)

2. Solve each equation.

a. $-2x-3=5x+11$

b. $-4x+2-x=1$

Click here to check your answer

a. $x=-2$

b. $x=\frac{1}{5}$

If you missed any part of this problem, review Section 1.1. (Note that this will open a in a new window.)

3. Determine whether each inequality is true or false.

a. $4<9$

b. $7>12$

c. $-13<-8$

d. $-5\geq -5$

e. $-5>-5$

f. $-100<50$

Click here to check your answer

a. True

b. False

c. True

d. True

e. False

f. True (Any negative is ALWAYS less than any positive.)

If you missed any part of this problem, review HERE. (Note that this will open a different textbook in a new window.)

Linear Inequalities

A **linear inequality**[138] is a mathematical statement that relates a linear expression as either less than or greater than another. The following are some examples of linear inequalities, all of which are solved in this section:

$5x+7<22$	$-2(x+8)+6\geq 20$	$-2(4x-5)<9-2(x-2)$

Table 1.1.5

A **solution to a linear inequality**[139] is a real number that will produce a true statement when substituted for the variable.

Example 2.1.1:

Are $x=-4$ and $x=6$ solutions to $5x+7<22$?

Solution

Substitute the values in for x, simplify, and check to see if we obtain a true statement.

Check $x=-4$	**Check $x=6$**
$5(-4)+7<22$ $-20+7<22$ $-13<22$ ✓	$5(6)+7<22$ $30+7<22$ $37<22$ ✗

Table 1.1.6

Answer:

$x=-4$ is a solution and $x=6$ is not

Linear inequalities have either infinitely many solutions or no solution. If there are infinitely many solutions, we graph the solution set on a number line and/or express the solution using interval notation.

Expressing Solutions to Linear Inequalities

What number would make the inequality $x>3$ true? Are you thinking, "x could be four"? That's correct, but x could be 6, too, or 137, or even 3.0001. Any number greater than three is a solution to the inequality $x>3$. We show all the solutions to the inequality $x>3$ on the number line by shading in all the numbers to the right of three, to show that all numbers greater than three are solutions. Because the number three itself is not a solution, we put an open parenthesis at three.

We can also represent inequalities using **interval notation**. There is no upper end to the solution to this inequality. In interval notation, we express $x>3$ as $(3,\infty)$. The symbol ∞ is read as "**infinity**." It is not an actual number. Figure 2.1.2 shows both the number line and the interval notation.

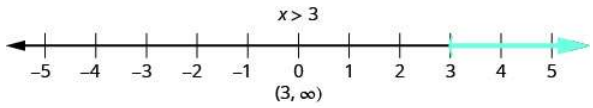


Figure 2.1.2: The inequality $x>3$ is graphed on this number line and written in interval notation.

We use the left parenthesis symbol, (, to show that the endpoint of the inequality is not included. The left bracket symbol, [, would show that the endpoint is included.

The inequality $x\leq 1$ means all numbers less than or equal to one. Here we need to show that one is a solution, too. We do that by putting a bracket at $x=1$. We then shade in all the numbers to the left of one, to show that all numbers less than one are solutions (Figure 2.1.3). There is no lower end to those numbers. We write $x\leq 1$ in interval notation as $(-\infty,1]$. The symbol $-\infty$ is read as "negative infinity."

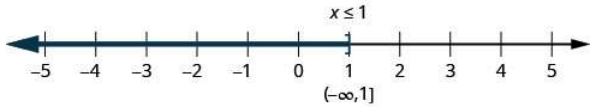
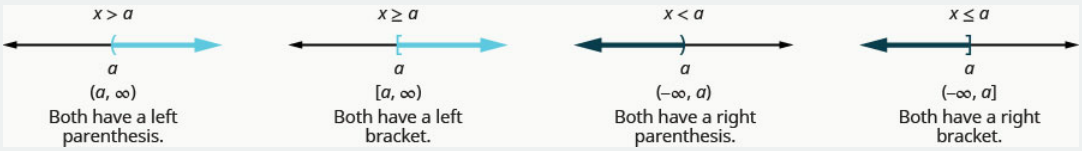


Figure 2.1.3: The inequality $x \leq 1$ is graphed on this number line and written in interval notation.

Figure 2.1.4 shows both the number line and interval notation.

INEQUALITIES, NUMBER LINES, AND INTERVAL NOTATION



The notation for inequalities on a number line and in interval notation use the same symbols to express the endpoints of intervals. Notice that ∞ and $-\infty$ always use parentheses in interval notation, never brackets.

Example 2.1.2

Graph each inequality on the number line and write in interval notation.

a. $x \geq -3$
b. $x < 2.5$
c. $-\frac{3}{5} \geq x$

Answer

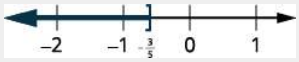
a.

	$x \geq -3$
Shade to the right of -3, and put a bracket at -3.	
Write in interval notation.	$[-3, \infty)$

b.

	$x < 2.5$
Shade to the left of 2.5 and put a parenthesis at 2.5.	
Write in interval notation.	$(-\infty, 2.5)$

c.

	$-\frac{3}{5} \geq x$
Shade to the left of $-\frac{3}{5}$, and put a bracket at $-\frac{3}{5}$.	
Write in interval notation.	$\left(-\infty, \frac{3}{5}\right]$

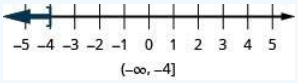
Exercise 2.1.1

Graph each inequality on the number line and write in interval notation:

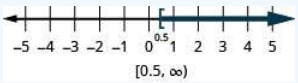
a. $x \leq -4$
b. $x \geq 0.5$
c. $x < -\frac{2}{3}$.

Answer

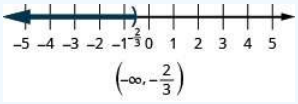
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Solving Linear Inequalities

Recall the following **properties of equality** from Section 1.1. Given algebraic expressions A and B, and real number c:

Addition property of equality:	If $A = B$, then $A+c = B+c$
Subtraction property of equality:	If $A = B$, then $A-c = B-c$
Multiplication property of equality:	If $A = B$, then $cA = cB$
Division property of equality:	If $A = B$, then $\frac{A}{c} = \frac{B}{c}, c \neq 0$

All but one of the techniques learned for solving linear equations apply to solving linear inequalities. You may add or subtract any real number to both sides of an inequality, and you may multiply or divide both sides by any *positive* real number to create equivalent inequalities.

Example 2.1.3:

Solve and graph the solution set: $5x + 7 < 22$.

Solution

$$\begin{aligned} 5x + 7 &< 22 \\ 5x + 7 - 7 &< 22 - 7 \\ 5x &< 15 \\ \frac{5x}{5} &< \frac{15}{5} \\ x &< 3 \end{aligned}$$

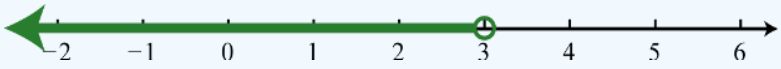


Figure 2.1.5

It is helpful to take a minute and choose a few values in and out of the solution set, substitute them into the original inequality, and then verify the results. As indicated, you should expect $x = 0$ to solve the original inequality and that $x = 5$ should not.

Check $x = 0$	Check $x = 5$
$5(0)+7<22$ $7<22$ ✓	$5(5)+7<22$ $25+7<22$ $32<22$ ✗

Checking in this manner gives us a good indication that we have solved the inequality correctly.

Answer

$(-\infty,\ 3)$

When working with linear inequalities, different rules apply when multiplying or dividing by a positive number vs. a negative number. To illustrate the problem, consider the true statement $10>-5$. We'll divide both sides by 5 and then by -5.

$$\begin{aligned} &10>-5 \quad \checkmark True \\ &\tfrac{10}{5}>\tfrac{-5}{5} \quad Divide\ both\ sides\ by\ 5. \\ &2>-1 \quad \checkmark Still\ True \end{aligned}$$

Dividing both sides of the inequality by 5 kept the statement true. Dividing both sides of any inequality by $c>0$ will be the same process as solving a linear equation.

$$\begin{aligned} &10>-5 \quad \checkmark True \\ &\tfrac{10}{-5}>\tfrac{-5}{-5} \quad Divide\ both\ sides\ by\ -5. \\ &-2>1 \quad \times False \end{aligned}$$

Dividing by -5 results in a false statement since $c<0$. To retain a true statement, the inequality must be reversed.

$$\begin{aligned} &10>-5 \quad \checkmark True \\ &\tfrac{10}{-5}>\tfrac{-5}{-5} \quad Divide\ both\ sides\ by\ -5\ and\ flip\ the\ inequality. \\ &-2<1 \quad \checkmark Still\ True \end{aligned}$$

The same problem occurs when multiplying by a negative number. This leads to the following new rule: *when multiplying or dividing by a negative number, reverse the inequality*. It is easy to forget to do this so take special care to watch for negative coefficients. In general, given algebraic expressions A and B, and real number c, we have the following **properties of inequalities**[140]:

Addition property of inequalities:	If $A<B$ then, $A+c<B+c$
Subtraction property of inequalities:	If $A<B$ then, $A-c<B-c$
Multiplication property of inequalities:	If $A<B$, then $cA<cB$ If $A<B$, then $-cA>-cB$
Division property of inequalities:	If $A<B$, then $\frac{A}{c}<\frac{B}{c}, c\neq 0$ If $A<B$, then $\frac{A}{-c}>\frac{B}{-c}, c\neq 0$

Table 1.1.8

We use these properties to obtain an **equivalent inequality**[141], one with the same solution set, where the variable is isolated. The process is similar to solving linear equations.

Example 2.1.4:

Solve and graph the solution set: $-2(x+8)+6\geq 20$.

Solution

$$
\begin{aligned}
-2(x+8)+6 &\geq 20 && \textit{Distribute.}\\
-2x-16+6 &\geq 20 && \textit{Combine like terms.}\\
-2x-10 &\geq 20 && \textit{Solve for } x.\\
-2x &\geq 30 && \textit{Divide both sides by } -2.\\
\frac{-2x}{-2} &\leq \frac{30}{-2} && \textit{Reverse the inequality.}\\
x &\leq -15
\end{aligned}
$$



Figure 2.1.6

Answer:

Interval notation $(-\infty,\ -15]$

Example 2.1.5:

Solve and graph the solution set: $-2(4x-5)<9-2(x-2)$.

Solution

$$
\begin{aligned}
-2(4x-5) &< 9-2(x-2)\\
-8x+10 &< 9-2x+4\\
-8x+10 &< 13-2x\\
-6x &< 3\\
\tfrac{-6x}{-6} &> \tfrac{3}{-6} \quad \textit{Reverse the inequality.}\\
x &> -\tfrac{1}{2}
\end{aligned}
$$



Figure 2.1.7

Answer:

Interval notation $(-\frac{1}{2},\ \infty)$

Example 2.1.6:

Solve and graph the solution set: $\frac{1}{2}x-2\geq\frac{1}{2}(\frac{7}{4}x-9)+1$.

Solution

$$
\begin{aligned}
\tfrac{1}{2}x-2 &\geq \tfrac{1}{2}\left(\tfrac{7}{4}x-9\right)+1\\
\tfrac{1}{2}x-2 &\geq \tfrac{7}{8}x-\tfrac{9}{2}+1\\
\tfrac{1}{2}x-\tfrac{7}{8}x &\geq -\tfrac{7}{2}+2\\
-\tfrac{3}{8}x &\geq -\tfrac{3}{2}\\
\left(-\tfrac{8}{3}\right)\left(-\tfrac{3}{8}x\right) &\leq \left(-\tfrac{8}{3}\right)\left(-\tfrac{3}{2}\right) \quad \textit{Reverse the inequality.}\\
x &\leq 4
\end{aligned}
$$

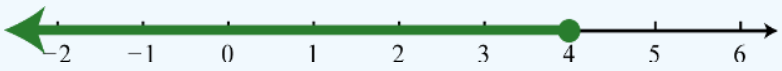


Figure 2.1.8

Answer:

Interval notation: $(-\infty,\ \ 4]$

Exercise 2.1.2

Solve and graph the solution set: $10-5(2x+3)\leq 25$

Answer

$[-3,\infty)$;

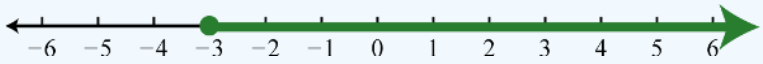


Figure 2.1.9

Video Solution: www.youtube.com/v/COLLNtwYFm8

Translation of Linear Inequalities

Some of the key words and phrases that indicate inequalities are summarized below:

Key Phrases	Translation
A number is **at least** 5.	$x\geq 5$
A number is 5 **or more inclusive.**	$x\geq 5$
A number is **at most** 3.	$x\leq 3$
A number is 3 **or less inclusive.**	$x\leq 3$
A number is **strictly less than** 4.	$x<4$
A number is **less than** 4, **noninclusive.**	$x<4$
A number **is greater than** 7.	$x>7$
A number is **more than** 7, **noninclusive**.	$x>7$
A number is **in between** 2 and 10.	$2<x<10$
A number is at least 5 **and** at most 15.	$5\leq x\leq 15$
A number may **range** from 5 to 15.	$5\leq x\leq 15$

Key Takeaways

- Inequalities typically have infinitely many solutions. The solutions are presented graphically on a number line or using interval notation or both.
- All but one of the rules for solving linear inequalities are the same as solving linear equations. If you divide or multiply an inequality by a negative number, reverse the inequality to obtain an equivalent inequality.

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2.1E: Exercises - Solving Linear Inequalities in One Variable

Exercise 2.1E.1

Determine whether or not the given value is a solution.

1. $5x-1<-2; x=-1$
2. $-3x+1>-10; x=1$
3. $2x-3<-5; x=1$
4. $5x-7<0; x=2$
5. $9y-4\geq 5; y=1$
6. $-6y+1\leq 3; y=-1$

Answer

1. Yes
3. No
5. Yes

Exercise 2.1E.2

Graph all solutions on a number line and provide the corresponding interval notation.

1. $3x+5>-4$
2. $2x+1>-1$
3. $6-a\leq 6$
4. $-2a+5>5$
5. $\frac{5x+6}{3}\leq 7$
6. $\frac{4x+11}{6}\leq \frac{1}{2}$
7. $2(3x+14)<-2$
8. $5(2y+9)>-15$
9. $5x-2(x-3)<3(2x-1)$
10. $3(2x-1)-10>4(3x-2)-5x$

Answer

1. $(-3,\infty)$;



Figure 1.8.13

3. $[0,\infty)$;



Figure 1.8.14

5. $(-\infty,3]$;

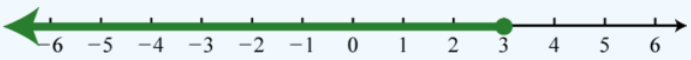


Figure 1.8.15

7. $(-\infty,-5)$;

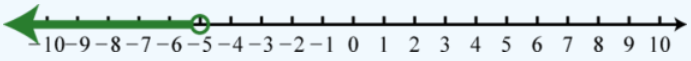


Figure 1.8.16

9. $(3,\infty)$;

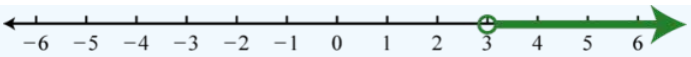


Figure 1.8.17

Footnotes

[138]Linear expressions related with the symbols $\leq, <, \geq$, and $>$.

[139]A real number that produces a true statement when its value is substituted for the variable.

[140]Properties used to obtain equivalent inequalities and used as a means to solve them.

[141]Inequalities that share the same solution set.

[142]Two or more inequalities in one statement joined by the word “and” or by the word “or.”

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2.2: Solving Linear Inequalities in Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Verify solutions to an inequality in two variables.
- Recognize the relation between the solutions of an inequality and its graph.
- Graph linear inequalities in two variables.
- Solve applications using linear inequalities in two variables.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

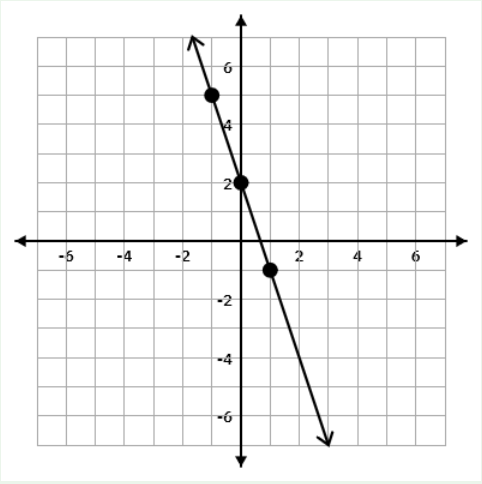
1. Graph each line using the given form.

a. $y = -3x + 2$

b. $3x - y = 6$

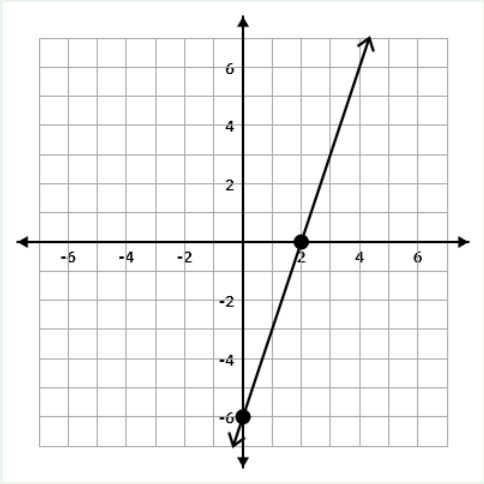
Click here to check your answer

a.



If you missed part a of this problem, review slope-intercept form in Section 1.3. (Note that this will open in a new window.)

b.



If you missed part b of this problem, review standard form in Section 1.3. (Note that this will open in a new window.)

2. Is $(3, 7)$ a solution to $4x - y = 5$?

Click here to check your answer

Yes, because both sides of the equation equal 5.

If you missed this problem, review Section 1.2. (Note that this will open in a new window.)

3. Solve each inequality and write each solution in interval notation.

a. $-3x + 5 < 8$

b. $-3x + 5 \leq 8$

Click here to check your answer

a. $(-1, \infty)$

b. $[-1, \infty)$

If you missed this problem, review Section 2.1. (Note that this will open in a new window.)

Verify Solutions to an Inequality in Two Variables

In Section 2.1 we learned to solve inequalities with only one variable. We will now learn about inequalities containing two variables. In particular we will look at **linear inequalities** in two variables which are very similar to linear equations in two variables.

Linear inequalities in two variables have many applications. If you ran a business, for example, you would want your revenue to be greater than your costs—so that your business made a profit.

LINEAR INEQUALITY

A **linear inequality** is an inequality that can be written in one of the following forms:

$$Ax + By > C \quad Ax + By \geq C \quad Ax + By < C \quad Ax + By \leq C$$

Where *A* and *B* are not both zero.

Recall that an inequality with one variable had many solutions. For example, the solution to the inequality $x > 3$ is any number greater than 3. We showed this on the number line by shading in the number line to the right of 3, and putting an open parenthesis at 3.

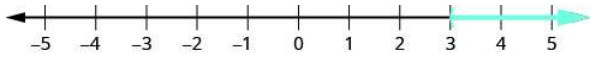


Figure 2.2.1

Similarly, linear inequalities in two variables have many solutions. Any ordered pair (x, y) that makes an inequality true when we substitute in the values is a **solution to a linear inequality**.

Solution TO A LINEAR INEQUALITY

An ordered pair (x, y) is a **solution to a linear inequality** if the inequality is true when we substitute the values of *x* and *y*.

Example 2.2.1

Determine whether each ordered pair is a solution to the inequality y>x+4:

a. (0,0) b. (1,6) c. (2,6) d. (−5,−15) e. (−8,12)

Answer

a.

$(0, 0)$	$y > x + 4$
Substitute 0 for x and 0 for y.	$0 \stackrel{?}{>} 0 + 4$
Simplify.	$0 \not> 4$
	So, $(0, 0)$ is not a solution to $y > x + 4$.

b.

$(1, 6)$	$y > x + 4$
Substitute 1 for x and 6 for y.	$6 \stackrel{?}{>} 1 + 4$
Simplify.	$6 > 5$
	So, $(1, 6)$ is a solution to $y > x + 4$.

c.

$(2, 6)$	$y > x + 4$
Substitute 2 for x and 6 for y.	
Simplify.	$6 \not> 6$
	So, $(2, 6)$ is not a solution to $y > x + 4$.

d.

$(-5, -15)$	$y > x + 4$
	$-15 \stackrel{?}{>} -5 + 4$
Simplify.	
	So, $(-5, -15)$ is not a solution to $y > x + 4$.

e.

$(-8, 12)$	$y > x + 4$
Substitute -8 for x and 12 for y.	$12 \stackrel{?}{>} -8 + 4$
Simplify.	$12 > -4$
	So, $(-8, 12)$ is a solution to $y > x + 4$.

Example 2.2.2

Determine whether each ordered pair is a solution to the inequality $y > x - 3$:

a. $(0, 0)$ b. $(4, 9)$ c. $(-2, 1)$ d. $(-5, -3)$ e. $(5, 1)$

Answer

a. yes b. yes c. yes d. yes e. no

Example 2.2.3

Determine whether each ordered pair is a solution to the inequality $y < x + 1$:

a. $(0, 0)$ b. $(8, 6)$ c. $(-2, -1)$ d. $(3, 4)$ e. $(-1, -4)$

Answer

a. yes b. yes c. no d. no e. yes

Recognize the Relation Between the Solutions of an Inequality and its Graph

Now, we will look at how the solutions of an inequality relate to its graph. Consider the inequality $x > 3$.

Let's think about the number line in shown previously again. The point $x = 3$ separated that number line into two parts. On one side of 3 are all the numbers less than 3, which make the inequality false. On the other side of 3 all the numbers are greater than 3, which make the inequality true.

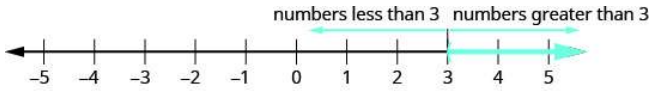


Figure 2.2.2: The solution to $x > 3$ is the shaded part of the number line to the right of $x = 3$.

Similarly, the line $y = x + 4$ separates the plane into two regions. On one side of the line are points with $y < x + 4$. On the other side of the line are the points with $y > x + 4$. We call the line $y = x + 4$ a **boundary line**.

BOUNDARY LINE

The line with equation $Ax + By = C$ is the **boundary line** that separates the region where $Ax + By > C$ from the region where $Ax + By < C$.

For an inequality in one variable, the endpoint is shown with a parenthesis or a bracket depending on whether or not the endpoint is included in the solution:



Similarly, for an inequality in two variables, the boundary line is shown with a solid or dashed line to show whether or not the line is included in the solution.

$Ax + By < C$	$Ax + By \leq C$
$Ax + By > C$	$Ax + By \geq C$
Boundary line is $Ax + By = C$.	Boundary line is $Ax + By = C$.
Boundary line is not included in solution.	Boundary line is included in solution.
Boundary line is dashed.	**Boundary line is solid.**

Now, let's take a look at what we found in Example 4.1.1. We'll start by graphing the line $y = x + 4$, and then we'll plot the five points we tested, as shown in the graph.

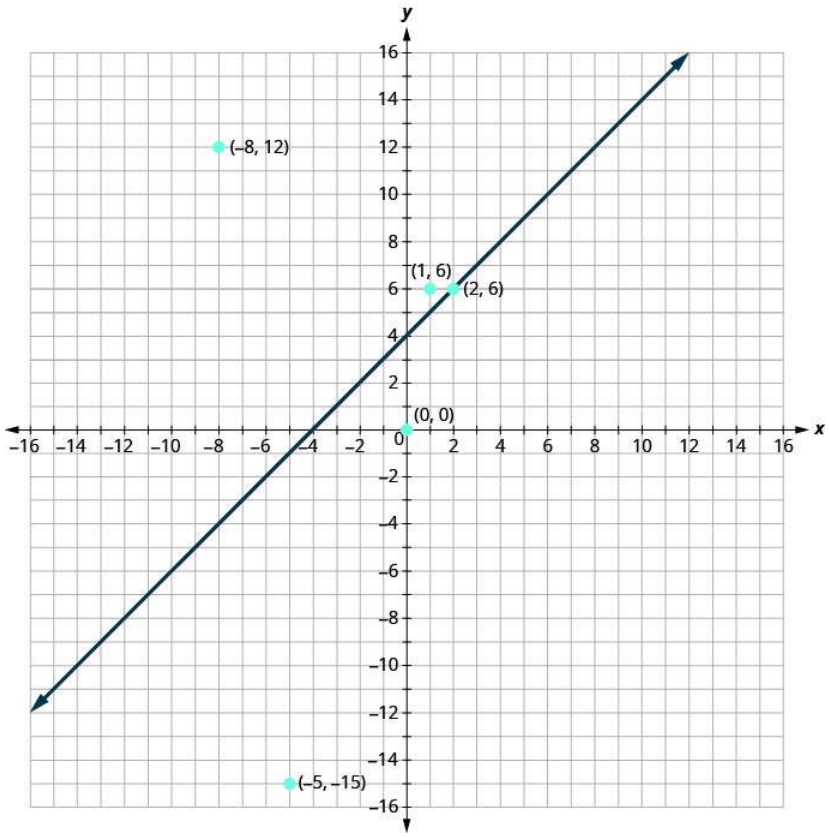


Figure 2.2.3

In Example 4.1.1 we found that some of the points were solutions to the inequality $y > x + 4$ and some were not.

Which of the points we plotted are solutions to the inequality $y > x + 4$?

The points $(1, 6)$ and $(-8, 12)$ are solutions to the inequality $y > x + 4$. Notice that they are both on the same side of the boundary line $y = x + 4$.

The two points $(0, 0)$ and $(-5, -15)$ are on the other side of the boundary line $y = x + 4$, and they are not solutions to the inequality $y > x + 4$. For those two points, $y < x + 4$.

What about the point $(2, 6)$? Because $6 = 2 + 4$, the point is a solution to the equation $y = x + 4$, but not a solution to the inequality $y > x + 4$. So the point $(2, 6)$ is on the boundary line.

Let's take another point above the boundary line and test whether or not it is a solution to the inequality $y > x + 4$. The point $(0, 10)$clearly looks to above the boundary line, doesn't it? Is it a solution to the inequality?

$$\begin{array}{ccc} y & > & x+4 \\ 10 & \stackrel{?}{>} & 0+4 \\ 10 & > & 4 \end{array}$$

So, $(0, 10)$ is a solution to $y > x + 4$.

Any point you choose above the boundary line is a solution to the inequality $y > x + 4$. All points above the boundary line are solutions.

Similarly, all points below the boundary line, the side with $(0, 0)$ and $(-5, -15)$, are not solutions to $y > x + 4$, as shown in the graph below.

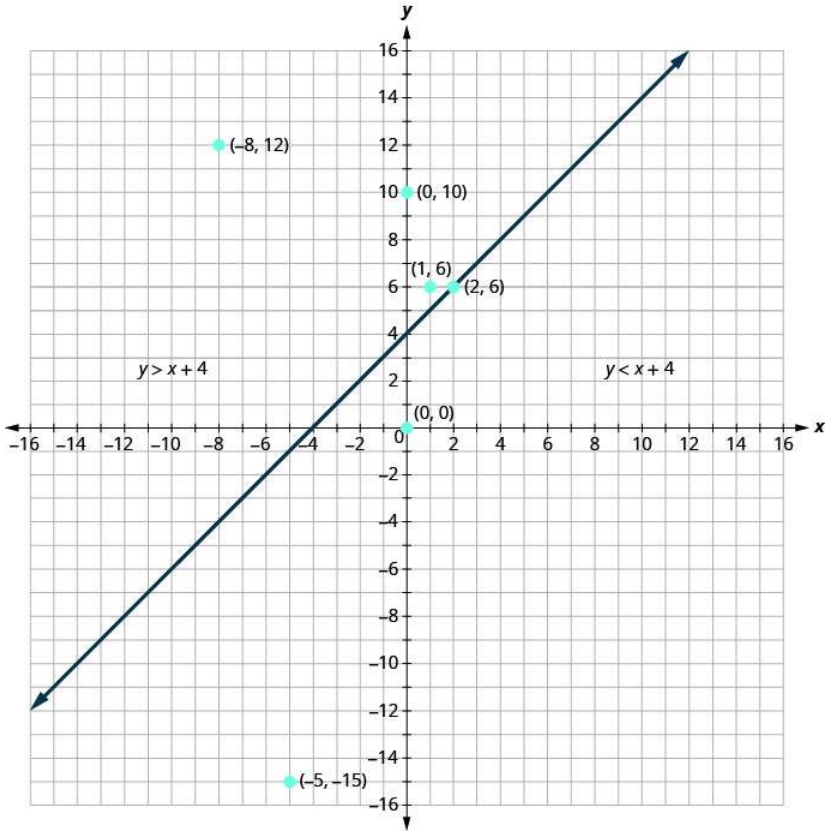
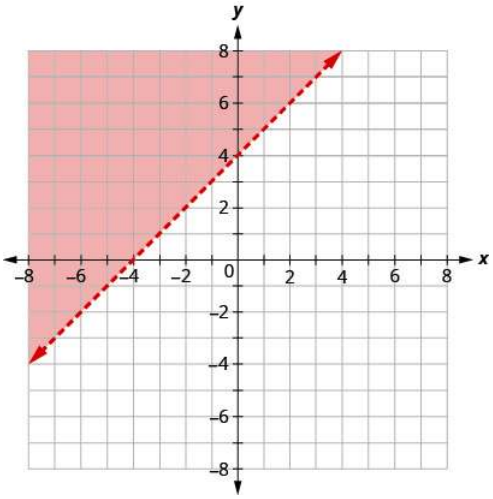


Figure 2.2.4

The graph of the inequality $y > x + 4$ is shown in below.

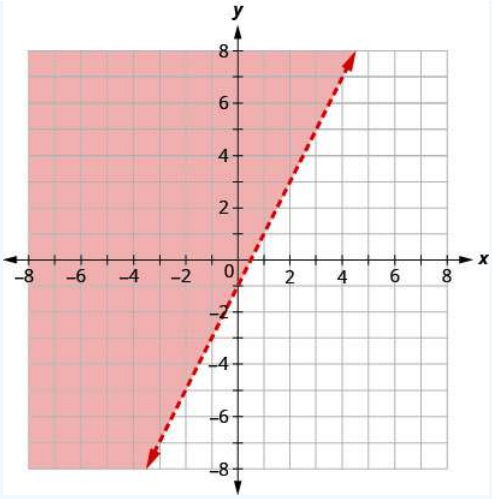
The line $y = x + 4$ divides the plane into two regions. The shaded side shows the solutions to the inequality $y > x + 4$.

The points on the boundary line, those where $y = x + 4$, are not solutions to the inequality $y > x + 4$, so the line itself is not part of the solution. We show that by making the line dashed, not solid.



Example 2.2.4

The boundary line shown in this graph is $y = 2x - 1$. Write the inequality shown by the graph.



Answer

The line $y = 2x - 1$ is the boundary line. On one side of the line are the points with $y > 2x - 1$ and on the other side of the line are the points with $y < 2x - 1$.

Let's test the point $(0, 0)$ and see which inequality describes its position relative to the boundary line.

At $(0, 0)$, which inequality is true: $y > 2x - 1$ or $y < 2x - 1$?

$$\begin{array}{ll} y > 2x - 1 & y < 2x - 1 \\ 0 \overset{?}{>} 2 \cdot 0 - 1 & 0 \overset{?}{<} 2 \cdot 0 - 1 \\ 0 > -1 \text{ True} & 0 < -1 \text{ False} \end{array}$$

Since, $y > 2x - 1$ is true, the side of the line with $(0, 0)$, is the solution. The shaded region shows the solution of the inequality $y > 2x - 1$.

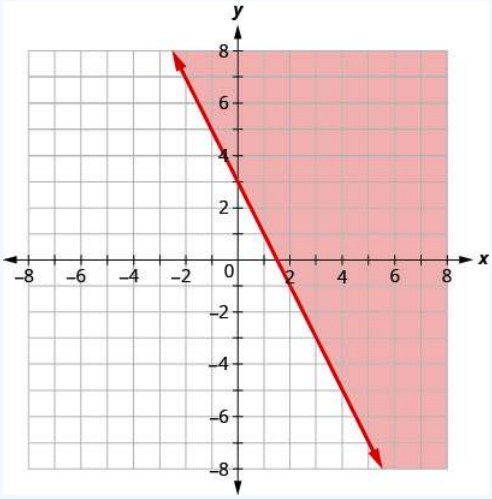
Since the boundary line is graphed with a dotted line, the inequality does not include the equal sign.

The graph shows the inequality $y > 2x - 1$.

We could use any point as a test point, provided it is not on the line. Why did we choose $(0, 0)$? Because it's the easiest to evaluate. You may want to pick a point on the other side of the boundary line and check that $y < 2x - 1$.

Example 2.2.5

Write the inequality shown by the graph with the boundary line $y = -2x + 3$.

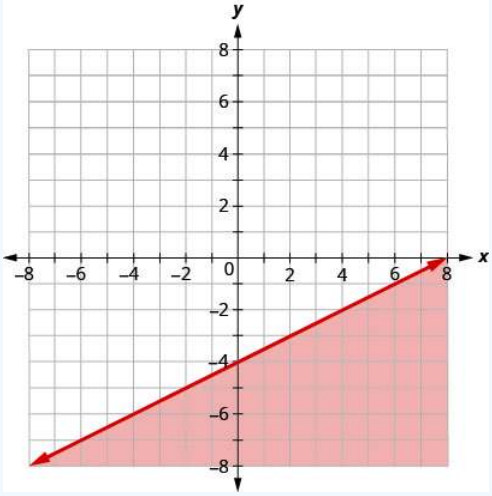


Answer

$y \geq -2x + 3$

Example 2.2.6

Write the inequality shown by the graph with the boundary line $y = \frac{1}{2}x - 4$.

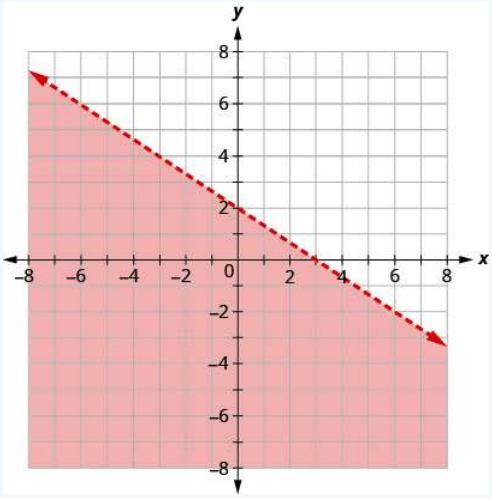


Answer

$y \leq \frac{1}{2}x - 4$

Example 2.2.7

The boundary line shown in this graph is $2x + 3y = 6$. Write the inequality shown by the graph.



Answer

The line $2x + 3y = 6$ is the boundary line. On one side of the line are the points with $2x + 3y > 6$ and on the other side of the line are the points with $2x + 3y < 6$.

Let's test the point $(0, 0)$ and see which inequality describes its side of the boundary line.

At $(0, 0)$, which inequality is true: $2x + 3y > 6$ or $2x + 3y < 6$?

$$\begin{array}{ll} 2x+3y>6 & 2x+3y<6 \\ 2(0)+3(0)\overset{?}{>}6 & 2(0)+3(0)\overset{?}{<}6 \\ 0>6 \text{ False} & 0<6 \text{ True} \end{array}$$

So the side with $(0,0)$ is the side where $2x+3y<6$.

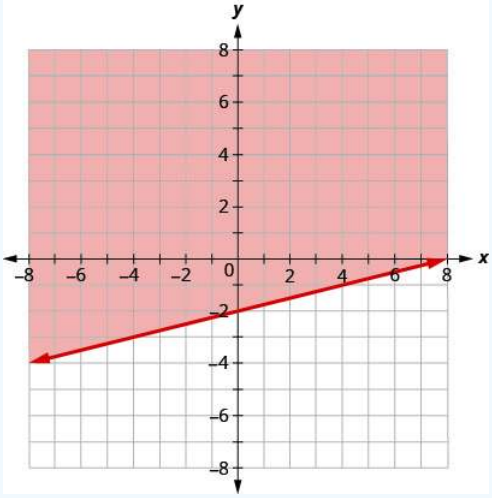
(You may want to pick a point on the other side of the boundary line and check that $2x+3y>6$.)

Since the boundary line is graphed as a dashed line, the inequality does not include an equal sign.

The shaded region shows the solution to the inequality $2x+3y<6$.

Example 2.2.8

Write the inequality shown by the shaded region in the graph with the boundary line $x-4y=8$.

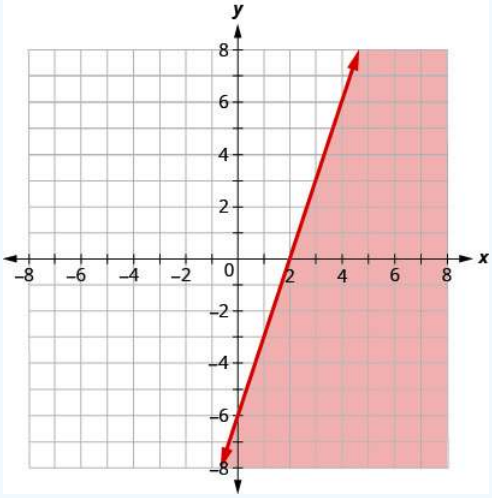


Answer

$x-4y\leq 8$

Example 2.2.9

Write the inequality shown by the shaded region in the graph with the boundary line $3x-y=6$.



Answer

$3x - y \geq 6$

Graph Linear Inequalities in Two Variables

Now that we know what the graph of a linear inequality looks like and how it relates to a boundary equation we can use this knowledge to graph a given linear inequality.

The steps we take to graph a linear inequality are summarized here.

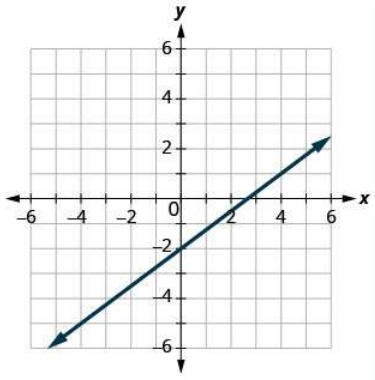
GRAPH A LINEAR INEQUALITY IN TWO VARIABLES.

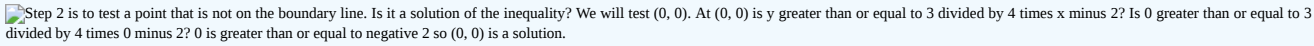
1. Identify and graph the boundary line.
   - If the inequality is $\leq$ or $\geq$, the boundary line is solid.
   - If the inequality is < or >, the boundary line is dashed.
2. Test a point that is not on the boundary line. (If not on the boundary line, $(0, 0)$ is usually a convenient test point.) Is it a solution of the inequality?
3. Shade in one side of the boundary line.
   - If the test point is a solution, shade in the side that includes the point.
   - If the test point is not a solution, shade in the opposite side.

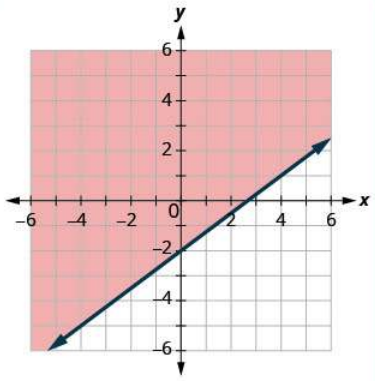
Example 2.2.10: How to Graph a Linear Equation in Two Variables

Graph the linear inequality $y \geq \frac{3}{4}x - 2$.

Solution

Step 1. Identify and graph the boundary line. • If the inequality is ≤ or ≥, the boundary line is solid. • If the inequality is < or >, the boundary line is dashed.	Replace the inequality sign with an equal sign to find the boundary line. Graph the boundary line $y = \frac{3}{4}x - 2$. The inequality sign is ≥, so we draw a solid line.	

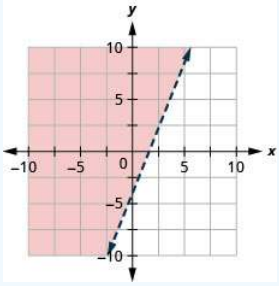

Step 2 is to test a point that is not on the boundary line. Is it a solution of the inequality? We will test (0, 0). At (0, 0) is y greater than or equal to 3 divided by 4 times x minus 2? Is 0 greater than or equal to 3 divided by 4 times 0 minus 2? 0 is greater than or equal to negative 2 so (0, 0) is a solution.

Step 3. Shade in one side of the boundary line. • If the test point is a solution, shade in the side that includes the point. • If the test point is not a solution, shade in the opposite side.	The test point (0, 0) is a solution to $y \geq \frac{3}{4}x - 2$. So we shade in that side.	 All points in the shaded region and on the boundary line represent the solutions to $y \geq \frac{3}{4}x - 2$.

Example 2.2.11

Graph the linear inequality $y > \frac{5}{2}x - 4$.

Answer



All points in the shaded region and on the boundary line, represent the solutions to $y > \frac{5}{2}x - 4$.

Example 2.2.12

Graph the linear inequality $y < \frac{2}{3}x - 5$.

Answer

This figure has the graph of a straight dashed line on the x y-coordinate plane. The x and y axes run from negative 10 to 10. A straight dashed line is drawn through the points (0, negative 5), (3, negative 3), and (5, negative 1). The line divides the x y-coordinate plane into two halves. The top left half is shaded red to indicate that this is where the solutions of the inequality are.

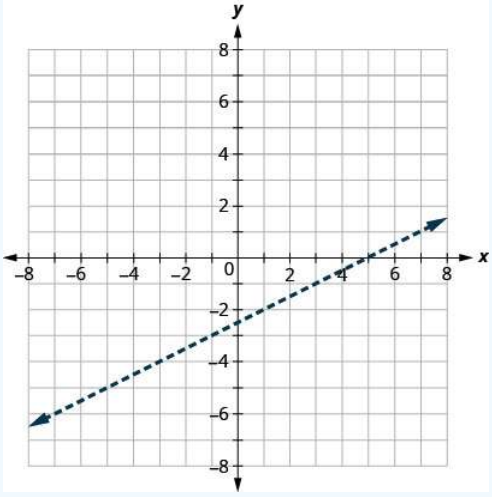
All points in the shaded region, but not those on the boundary line, represent the solutions to $y < \frac{2}{3}x - 5$.

Example 2.2.13

Graph the linear inequality $x - 2y < 5$.

Answer

First, we graph the boundary line $x - 2y = 5$. The inequality is $<$ so we draw a dashed line.



Then, we test a point. We'll use $(0, 0)$ again because it is easy to evaluate and it is not on the boundary line.

Is $(0, 0)$ a solution of $x - 2y < 5$?

$$\begin{aligned} 0-2(0) &\stackrel{?}{<} 5 \\ 0-0 &\stackrel{?}{<} 5 \\ 0 &< 5 \end{aligned}$$

The point $(0,0)$ is a solution of $x-2y<5$, so we shade in that side of the boundary line.

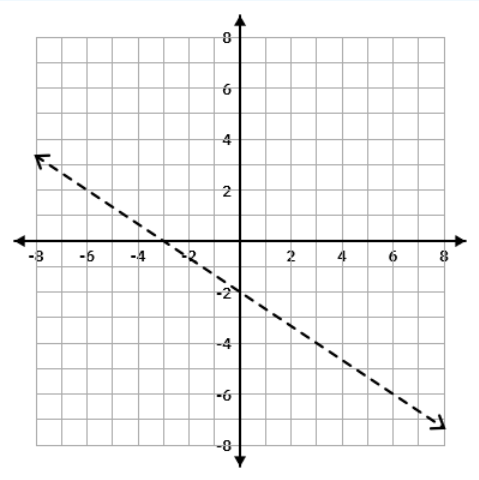
All points in the shaded region, but not those on the boundary line, represent the solutions to $x-2y<5$.

Example 2.2.14

Graph the linear inequality: $2x+3y<-6$.

Answer

First, we graph the boundary line $2x+3y=-6$. The inequality is $<$ so we draw a dashed line.

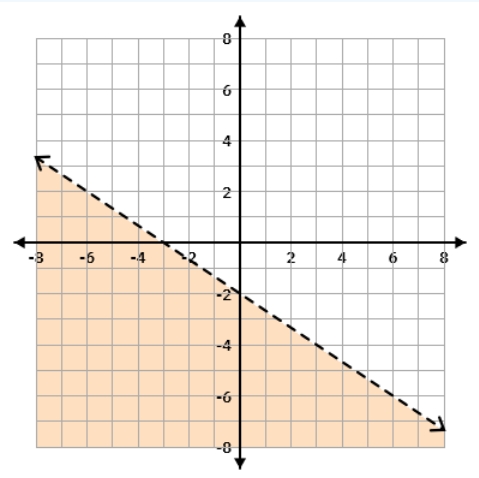


Then, we test a point. We'll use $(0,0)$ again because it is easy to evaluate and it is not on the boundary line.

Is $(0,0)$ a solution of $2x+3y<-6$?

$$\begin{aligned} 2(0)+3(0) &< -6 \\ 0 &< -6 \end{aligned}$$

The point $(0,0)$ is NOT a solution of $2x+3y<-6$, so we shade in the OTHER side of the boundary line.

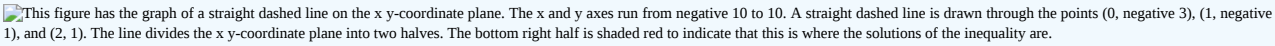


All points in the shaded region, but not those on the boundary line, represent the solutions to $2x+3y<-6$.

Example 2.2.15

Graph the linear inequality: $2x-y>3$.

Answer



All points in the shaded region, but not those on the boundary line, represent the solutions to $2x-y>3$.

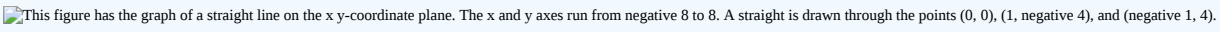
What if the boundary line goes through the origin? Then, we won't be able to use $(0,0)$ as a test point. No problem—we'll just choose some other point that is not on the boundary line.

Example 2.2.16

Graph the linear inequality: $y\leq -4x$.

Answer

First, we graph the boundary line $y=-4x$. It is in slope–intercept form, with $m=-4$ and $b=0$. The inequality is $\leq$ so we draw a solid line.



Now we need a test point. We can see that the point (1,0)(1,0) is not on the boundary line.

Is $(1,0)$ a solution of $y\leq -4x$?

The point $(1,0)$ is not a solution to $y\leq -4x$, so we shade in the opposite side of the boundary line.

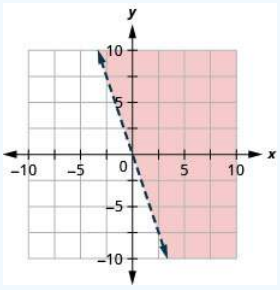


All points in the shaded region and on the boundary line represent the solutions to $y\leq -4x$.

Example 2.2.17

Graph the linear inequality: $y>-3x$.

Answer

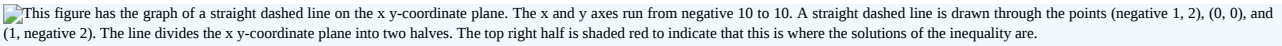


All points in the shaded region, but not those on the boundary line, represent the solutions to $y>-3x$.

Example 2.2.18

Graph the linear inequality: $y\geq -2x$.

Answer


This figure has the graph of a straight dashed line on the x y-coordinate plane. The x and y axes run from negative 10 to 10. A straight dashed line is drawn through the points (negative 1, 2), (0, 0), and (1, negative 2). The line divides the x y-coordinate plane into two halves. The top right half is shaded red to indicate that this is where the solutions of the inequality are.

All points in the shaded region and on the boundary line, represent the solutions to $y \geq -2x$.

Some linear inequalities have only one variable. They may have an *x* but no *y*, or a *y* but no *x*. In these cases, the boundary line will be either a vertical or a horizontal line.

Recall that:

$$\begin{aligned} x &= a \quad \text{vertical line} \\ y &= b \quad \text{horizontal line} \end{aligned}$$

? Example 2.2.19

Graph the linear inequality: $y > 3$.

Answer

First, we graph the boundary line $y = 3$. It is a horizontal line. The inequality is $>$ so we draw a dashed line.

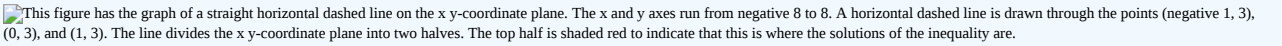
We test the point $(0, 0)$.

$$y > 3$$

$$0 \backslash \text{slashed} > 3$$

So, $(0, 0)$ is not a solution to $y > 3$.

So we shade the side that does not include $(0, 0)$ as shown in this graph.

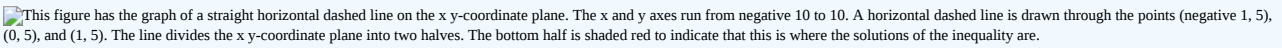

This figure has the graph of a straight horizontal dashed line on the x y-coordinate plane. The x and y axes run from negative 8 to 8. A horizontal dashed line is drawn through the points (negative 1, 3), (0, 3), and (1, 3). The line divides the x y-coordinate plane into two halves. The top half is shaded red to indicate that this is where the solutions of the inequality are.

All points in the shaded region, but not those on the boundary line, represent the solutions to $y > 3$.

? Example 2.2.20

Graph the linear inequality: $y < 5$.

Answer

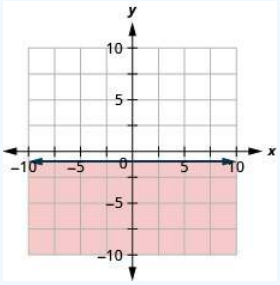

This figure has the graph of a straight horizontal dashed line on the x y-coordinate plane. The x and y axes run from negative 10 to 10. A horizontal dashed line is drawn through the points (negative 1, 5), (0, 5), and (1, 5). The line divides the x y-coordinate plane into two halves. The bottom half is shaded red to indicate that this is where the solutions of the inequality are.

All points in the shaded region, but not those on the boundary line, represent the solutions to $y < 5$.

? Example 2.2.21

Graph the linear inequality: $y \leq -1$.

Answer



All points in the shaded region and on the boundary line represent the solutions to $y \leq -1$.

Solve Applications using Linear Inequalities in Two Variables

Many fields use linear inequalities to model a problem. While our examples may be about simple situations, they give us an opportunity to build our skills and to get a feel for how they might be used.

Example 2.2.22

Hilaria works two part time jobs in order to earn enough money to meet her obligations of at least $240 a week. Her job in food service pays $10 an hour and her tutoring job on campus pays $15 an hour. How many hours does Hilaria need to work at each job to earn at least $240?

a. Let x be the number of hours she works at the job in food service and let *y* be the number of hours she works tutoring. Write an inequality that would model this situation.

b. Graph the inequality.

c. Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Hilaria.

Answer

a. We let *x* be the number of hours she works at the job in food service and let *y* be the number of hours she works tutoring.

She earns $10 per hour at the job in food service and $15 an hour tutoring. At each job, the number of hours multiplied by the hourly wage will gives the amount earned at that job.

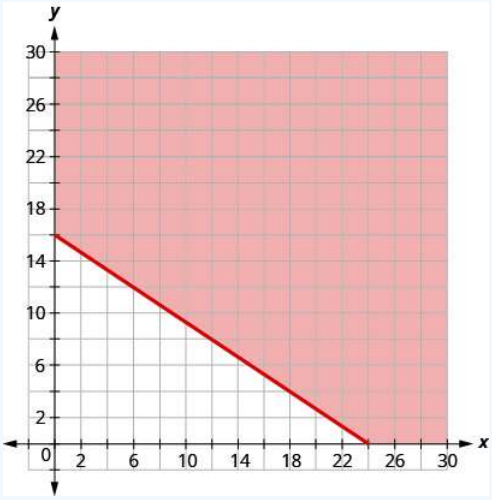
Amount earned at the food service job	plus	the amount earned tutoring	is at least	$240
$10x$	+	$15y$	≥	240

b. To graph the inequality, we put it in slope–intercept form.

$$10x + 15y \geq 240 \quad (2.2.1)$$

$$15y \geq -10x + 240 \quad (2.2.2)$$

$$y \geq -\frac{2}{3}x + 16 \quad (2.2.3)$$



c. From the graph, we see that the ordered pairs $(15, 10)$, $(0, 16)$, $(24, 0)$ represent three of infinitely many solutions. Check the values in the inequality.

First we test the point (15, 10) in the inequality 10 x plus 15 y greater than or equal to 240. Is 10 times 15 plus 15 times 10 greater than or equal to 240? Since 300 is greater than or equal to 240 (15, 10) is a solution. Next we test the point (0, 16) in the inequality 10 x plus 15 y greater than or equal to 240. Is 10 times 0 plus 15 times 16 greater than or equal to 240? Since 240 is greater than or equal to 240 (0, 16) is a solution. Then we test the point (24, 0) in the inequality 10 x plus 15 y greater than or equal to 240. Is 10 times 24 plus 15 times 0 greater than or equal to 240? Since 240 is greater than or equal to 240 (24, 0) is a solution.

For Hilaria, it means that to earn at least $240, she can work 15 hours tutoring and 10 hours at her fast-food job, earn all her money tutoring for 16 hours, or earn all her money while working 24 hours at the job in food service.

Example 2.2.23

Hugh works two part time jobs. One at a grocery store that pays $10 an hour and the other is babysitting for $13 hour. Between the two jobs, Hugh wants to earn at least $260 a week. How many hours does Hugh need to work at each job to earn at least $260?

a. Let x be the number of hours he works at the grocery store and let y be the number of hours he works babysitting. Write an inequality that would model this situation.

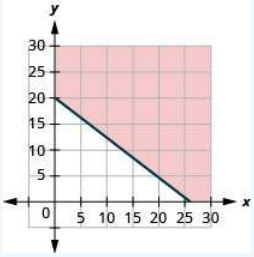
b. Graph the inequality.

c. Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Hugh.

Answer

a. $10x + 13y \geq 260$

b.



c. Answers will vary.

Example 2.2.24

Veronica works two part time jobs in order to earn enough money to meet her obligations of at least $280 a week. Her job at the day spa pays $10 an hour and her administrative assistant job on campus pays $17.50 an hour. How many hours does Veronica need to work at each job to earn at least $280?

a. Let x be the number of hours she works at the day spa and let y be the number of hours she works as administrative assistant. Write an inequality that would model this situation.

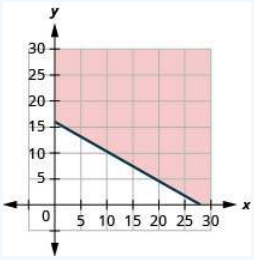
b. Graph the inequality.

c. Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Veronica

Answer

a. $10x + 17.5y \geq 280$

b.



c. Answers will vary.

Access this online resource for additional instruction and practice with graphing linear inequalities in two variables.

- Graphing Linear Inequalities in Two Variables

Key Concepts

- **How to graph a linear inequality in two variables.**
  1. Identify and graph the boundary line.
     If the inequality is $\leq$ or $\geq$, the boundary line is solid.
     If the inequality is $<$ or $>$, the boundary line is dashed.
  2. Test a point that is not on the boundary line. Is it a solution of the inequality?
  3. Shade in one side of the boundary line.
     If the test point is a solution, shade in the side that includes the point.
     If the test point is not a solution, shade in the opposite side.

Glossary

boundary line

The line with equation $Ax + By = C$ is the boundary line that separates the region where $Ax + By > C$ from the region where $Ax + By < C$.

linear inequality

A linear inequality is an inequality that can be written in one of the following forms: $Ax + By > C$, $Ax + By \geq C$, $Ax + By < C$, or $Ax + By \leq C$, where *A* and *B* are not both zero.

solution to a linear inequality

An ordered pair (x, y) is a solution to a linear inequality if the inequality is true when we substitute the values of *x* and *y*.

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2.2E: Exercises - Solving Linear Inequalities in Two Variables

Practice Makes Perfect

Verify Solutions to an Inequality in Two Variables

In the following exercises, determine whether each ordered pair is a solution to the given inequality.

1. Determine whether each ordered pair is a solution to the inequality $y > x - 1$:

a. $(0, 1)$
b. $(-4, -1)$
c. $(4, 2)$
d. $(3, 0)$
e. $(-2, -3)$

Answer

a. yes b. yes c. no d. no e. no

2. Determine whether each ordered pair is a solution to the inequality $y \leq 3x + 2$:

a. $(0, 3)$
b. $(-2, 0)$
c. $(-3, -2)$
d. $(0, 0)$
e. $(-2, -4)$

Answer

a. no b. yes c. no d. yes e. yes

3. Determine whether each ordered pair is a solution to the inequality $3x - 4y > 4$:

a. $(5, 1)$
b. $(-2, 6)$
c. $(3, 2)$
d. $(10, -5)$
e. $(0, 0)$

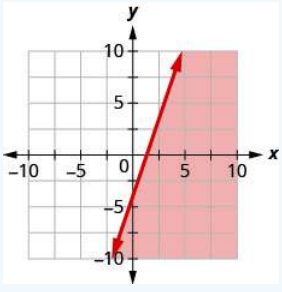
Answer

a. yes b. no c. no d. no e. no

Recognize the Relation Between the Solutions of an Inequality and its Graph

In the following exercises, write the inequality shown by the shaded region.

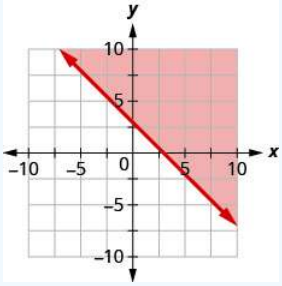
4. Write the inequality shown by the graph with the boundary line $y = 3x - 4$.



Answer

$y \leq 3x - 4$

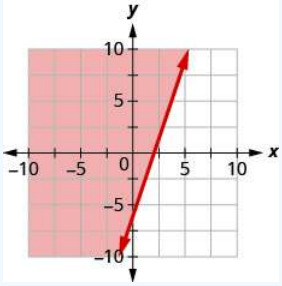
5. Write the inequality shown by the shaded region in the graph with the boundary line $y = -x + 3$.



Answer

$y \geq -x + 3$

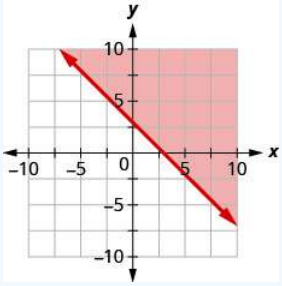
6. Write the inequality shown by the shaded region in the graph with the boundary line $3x - y = 6$.



Answer

$3x - y \leq 6$

7. Write the inequality shown by the shaded region in the graph with the boundary line $x + y = 3$.



Answer

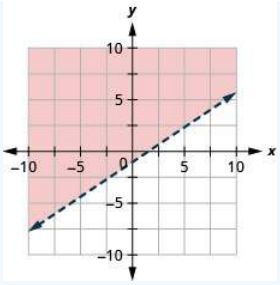
$x + y \geq 3$

Graph Linear Inequalities in Two Variables

In the following exercises, graph each linear inequality.

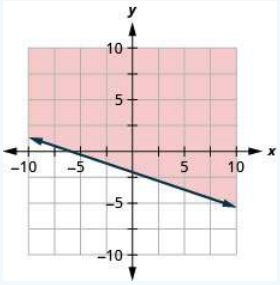
8. Graph the linear inequality: $y > \frac{2}{3}x - 1$.

Answer



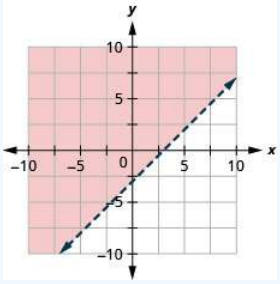
9. Graph the linear inequality: $y \geq -\frac{1}{2}x + 4$.

Answer



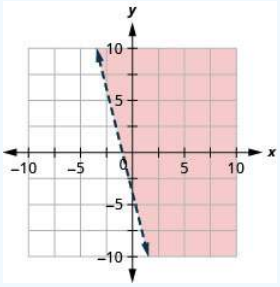
10. Graph the linear inequality: $x - y < 3$.

Answer



11. Graph the linear inequality: $4x + y > -4$.

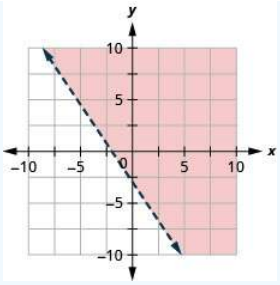
Answer



12. Graph the linear inequality: $3x + 2y > -6$.

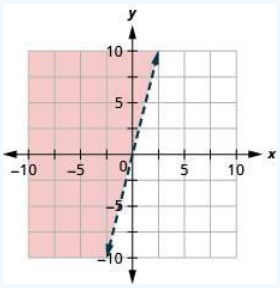
Answer

https://math.libretexts.org/@go/page/147287



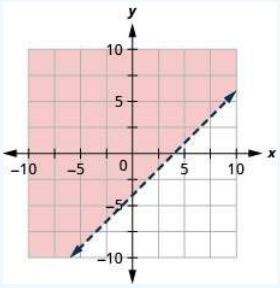
13. Graph the linear inequality: $y > 4x$.

Answer



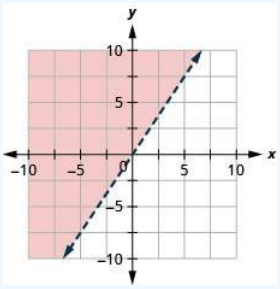
14. Graph the linear inequality: $x - y < 4$.

Answer



15. Graph the linear inequality: $y > \frac{3}{2}x$.

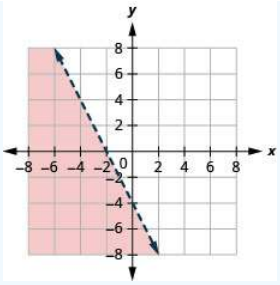
Answer



16. Graph the linear inequality: $2x + y < -4$.

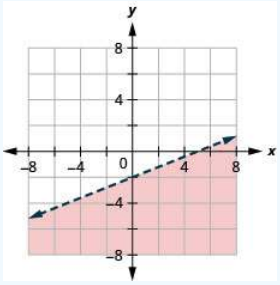
Answer

https://math.libretexts.org/@go/page/147287



17. Graph the linear inequality: $2x - 5y > 10$.

Answer



Solve Applications using Linear Inequalities in Two Variables

18. Harrison works two part time jobs. One at a gas station that pays $11 an hour and the other is IT troubleshooting for $16.50$16.50an hour. Between the two jobs, Harrison wants to earn at least $330 a week. How many hours does Harrison need to work at each job to earn at least $330?

a. Let *x* be the number of hours he works at the gas station and let *y* be the number of hours he works troubleshooting. Write an inequality that would model this situation.

b. Graph the inequality.

c. Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Harrison.

Answer

a. $11x + 16.5y \geq 330$

b.

This figure has the graph of a straight line on the x y-coordinate plane. The x and y axes run from 0 to 35. A line is drawn through the points (0, 20), (15, 10), and (30, 0). The line divides the x y-coordinate plane into two halves. The line and the top right half are shaded red to indicate that this is where the solutions of the inequality are.

c. Answers will vary.

19. The doctor tells Laura she needs to exercise enough to burn 500 calories each day. She prefers to either run or bike and burns 15 calories per minute while running and 10 calories a minute while biking.

a. If *x* is the number of minutes that Laura runs and *y* is the number minutes she bikes, find the inequality that models the situation.

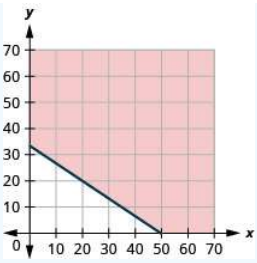
b. Graph the inequality.

c. List three solutions to the inequality. What options do the solutions provide Laura?

Answer

a. $15x + 10y \geq 500$

b.



c. Answers will vary.

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2.3: Solving Systems of Linear Inequalities in Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of linear inequalities.
- Solve a system of linear inequalities by graphing.
- Solve applications of systems of inequalities.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Is $(3, 12)$ a solution to $y > 2x + 3$?

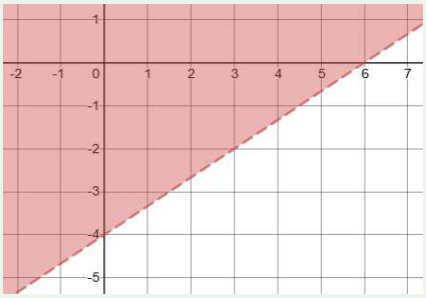
Click here to check your answer

Yes, because $12 > 9$.

If you missed this problem, review Section 2.2. (Note that this will open in a new window.)

2. Graph all solutions to $2x - 3y < 12$.

Click here to check your answer



If you missed this problem, review Section 2.2. (Note that this will open in a new window.)

3. Where do the lines $y = 2x + 1$ and $y = -3x + 6$ intersect?

Click here to check your answer

$(1, 3)$

If you missed this problem, review Section 1.4. (Note that this will open in a new window.)

Determine whether an ordered pair is a solution of a system of linear inequalities

The definition of a **system of linear inequalities** is very similar to the definition of a system of linear equations.

SYSTEM OF LINEAR INEQUALITIES

Two or more linear inequalities grouped together form a system of linear inequalities.

A system of linear inequalities looks like a system of linear equations, but it has inequalities instead of equations. A system of two linear inequalities is shown here.

$$\begin{cases} x+4y \geq 10 \\ 3x-2y < 12 \end{cases}$$

To solve a system of linear inequalities, we will find values of the variables that are solutions to both inequalities. We solve the system by using the graphs of each inequality and show the solution as a graph. We will find the region on the plane that contains all ordered pairs (x, y) that make both inequalities true.

SOLUTIONS OF A SYSTEM OF LINEAR INEQUALITIES

Solutions of a system of linear inequalities are the values of the variables that make all the inequalities true.

The solution of a system of linear inequalities is shown as a shaded region in the *x*, *y* coordinate system that includes all the points whose ordered pairs make the inequalities true.

To determine if an ordered pair is a solution to a system of two inequalities, we substitute the values of the variables into each inequality. If the ordered pair makes both inequalities true, it is a solution to the system.

Example 2.3.1

Determine whether the ordered pair is a solution to the system $\begin{cases} x+4y \geq 10 \\ 3x-2y < 12 \end{cases}$

a. $(-2, 4)$

b. $(3, 1)$

Answer

a. Is the ordered pair $(-2, 4)$ a solution?

We substitute $x = -2$ and $y = 4$ into both inequalities.

$x + 4y \geq 10$	$3x - 2y < 12$
$-2 + 4(4) \stackrel{?}{\geq} 10$	$3(-2) - 2(4) \stackrel{?}{<} 12$
$14 \geq 10$ true	$-14 < 12$ true

The ordered pair $(-2, 4)$ made both inequalities true. Therefore $(-2, 4)$ is a solution to this system.

b. Is the ordered pair $(3, 1)$ a solution?

We substitute $x = 3$ and $y = 1$ into both inequalities.

$x + 4y \geq 10$	$3x - 2y < 12$
$3 + 4(1) \stackrel{?}{\geq} 10$	$3(3) - 2(1) \stackrel{?}{<} 12$
$7 \geq 10$ false	$7 < 12$ true

The ordered pair $(3, 1)$ made one inequality true, but the other one false. Therefore $(3, 1)$ is not a solution to this system.

Example 2.3.2

Determine whether the ordered pair is a solution to the system: $\begin{cases} x-5y > 10 \\ 2x+3y > -2 \end{cases}$

a. $(3, -1)$

b. $(6, -3)$

Answer

a. no b. yes

Example 2.3.3

Determine whether the ordered pair is a solution to the system: $\begin{cases} y > 4x - 2 \\ 4x - y < 20 \end{cases}$

a. $(-2, 1)$

b. $(4, -1)$

Answer

a. yes b. no

Solve a System of Linear Inequalities by Graphing

The solution to a single linear inequality is the region on one side of the boundary line that contains all the points that make the inequality true. The solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. To find this region, we will graph each inequality separately and then locate the region where they are both true. The solution is always shown as a graph.

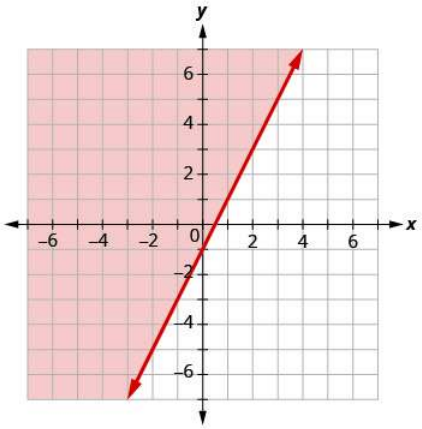
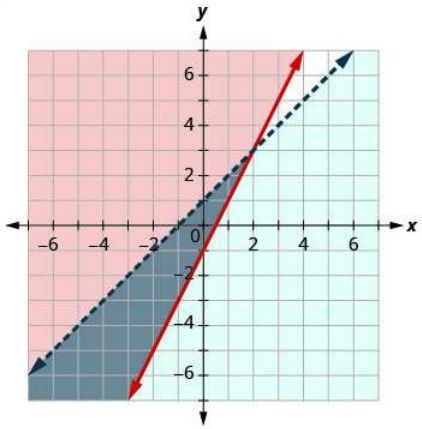
SOLVE A SYSTEM OF LINEAR INEQUALITIES BY GRAPHING.

1. Graph the first inequality.
   - Graph the boundary line.
   - Shade in the side of the boundary line where the inequality is true.
2. On the same grid, graph the second inequality.
   - Graph the boundary line.
   - Shade in the side of that boundary line where the inequality is true.
3. The solution is the region where the shading overlaps.
4. Check by choosing a test point.

Example 2.3.4: How to Solve a System of Linear Inequalities by Graphing

Solve the system by graphing: $\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$

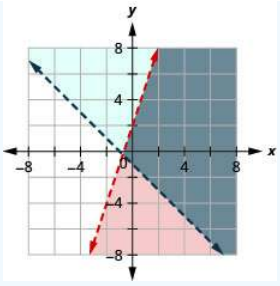
Solution

Step 1. Graph the first inequality. • Graph the boundary line. • Shade in the side of the boundary line where the inequality is true.	We will graph $y \geq 2x - 1$. We graph the line $y = 2x - 1$. It is a solid line because the inequality sign is $\geq$. We choose (0, 0) as a test point. It is a solution to $y \geq 2x - 1$, so we shade in above the boundary line.	$\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$ 
Step 2. On the same grid, graph the second inequality. • Graph the boundary line. • Shade in the side of that boundary line where the inequality is true.	We will graph $y < x + 1$ on the same grid. We graph the line $y = x + 1$. It is a dashed line because the inequality sign is $<$. Again, we use (0, 0) as a test point. It is a solution so we shade in that side of the line $y = x + 1$.	
Step 3. The solution is the region where the shading overlaps.	The point where the boundary lines intersect is not a solution because it is not a solution to $y < x + 1$.	The solution is all points in the area shaded twice—which appears as the darkest shaded region.
Step 4. Check by choosing a test point.	We'll use (–1, –1) as a test point.	Is (–1, –1) a solution to $y \geq 2x - 1$? $-1 \stackrel{?}{\geq} 2(-1) - 1$ $-1 \geq -3$ true Is (–1, –1) a solution to $y < x + 1$? $-1 \stackrel{?}{<} -1 + 1$ $-1 < 0$ true The region containing (–1, –1) is the solution to this system.

Example 2.3.5

Solve the system by graphing: $\begin{cases} y < 3x + 2 \\ y > -x - 1 \end{cases}$

Answer

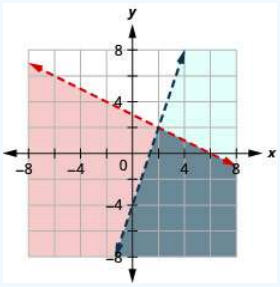


The solution is the darkest shaded region.

Example 2.3.6

Solve the system by graphing: $\begin{cases} y < -\frac{1}{2}x + 3 \\ y < 3x - 4 \end{cases}$

Answer



The solution is the darkest shaded region.

Example 2.3.7

Solve the system by graphing: $\begin{cases} x - y > 3 \\ y < -15x + 4 \end{cases}$

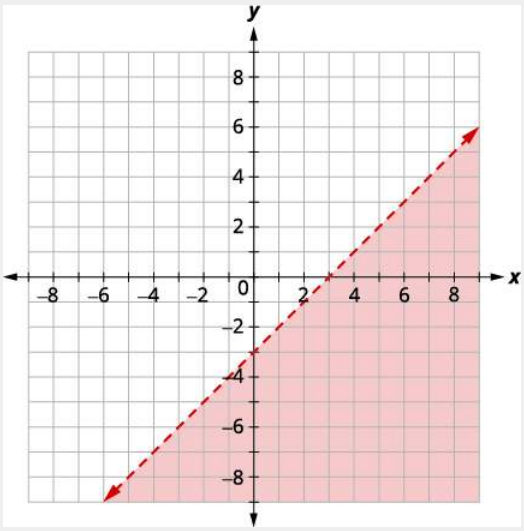
Answer

	$\begin{cases} x - y > 3 \\ y < -15x + 4 \end{cases}$

Graph $x - y > 3$, by graphing $x - y = 3$ and testing a point.

The intercepts are $x = 3$ and $y = -3$ and the boundary line will be dashed.

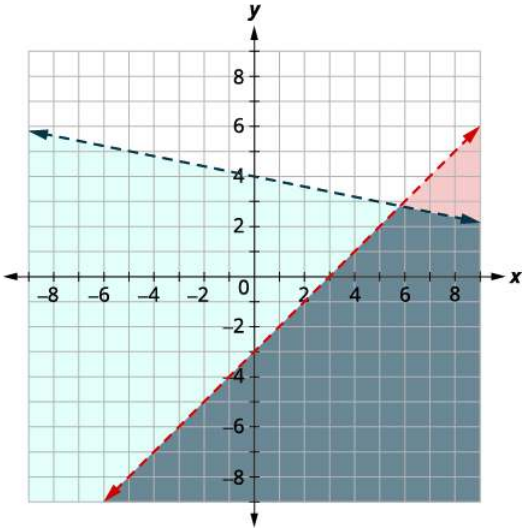
Test (0, 0) which makes the inequality false so shade (red) the side that does not contain (0, 0).



Graph $y < -15x + 4$ by graphing $y = -15x + 4$ using the slope $m = -15$ and y-intercept $b = 4$. The boundary line will be dashed

Test (0, 0) which makes the inequality true, so shade (blue) the side that contains (0, 0).

Choose a test point in the solution and verify that it is a solution to both inequalties.

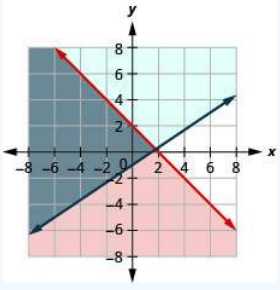


The point of intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twice—which appears as the darkest shaded region.

Example 2.3.8

Solve the system by graphing: $\begin{cases} x + y \leq 2 \\ y \geq \frac{2}{3}x - 1 \end{cases}$

Answer

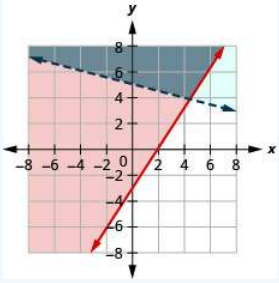


The solution is the darkest shaded region.

Example 2.3.9

Solve the system by graphing: $\begin{cases} 3x-2y \le 6 \\ y > -\frac{1}{4}x+5 \end{cases}$

Answer



The solution is the darkest shaded region.

Example 2.3.10

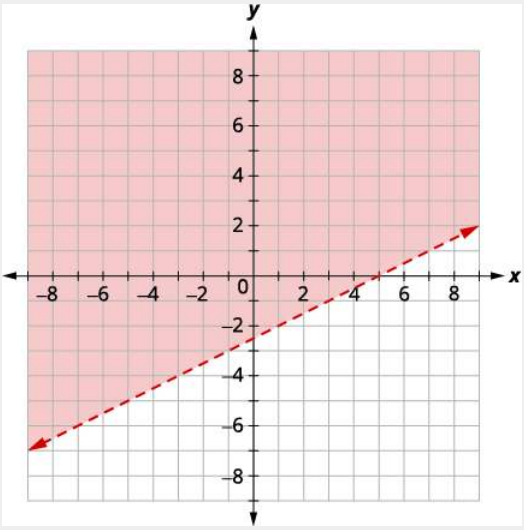
Solve the system by graphing: $\begin{cases} x-2y < 5 \\ y > -4 \end{cases}$

Answer

$\begin{cases} x-2y < 5 \\ y > -4 \end{cases}$

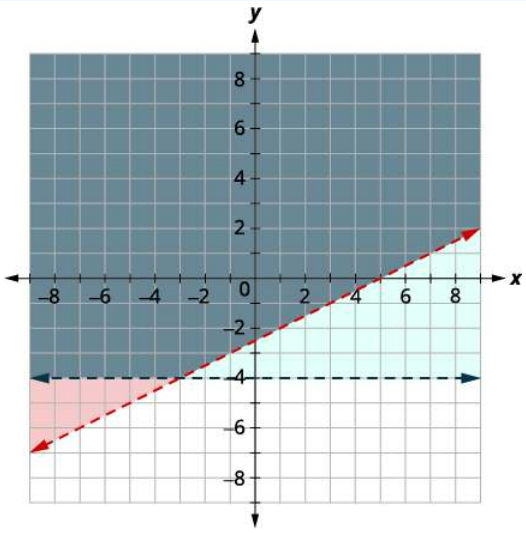
Graph $x-2y<5$, by graphing $x-2y=5$ and testing a point. The intercepts are $x = 5$ and $y = -2.5$ and the boundary line will be dashed.

Test (0, 0) which makes the inequality true, so shade (red) the side that contains (0, 0).



Graph $y > -4$, by graphing $y = -4$ and recognizing that it is a horizontal line through $y = -4$. The boundary line will be dashed.

Test (0, 0) which makes the inequality true so shade (blue) the side that contains (0, 0).



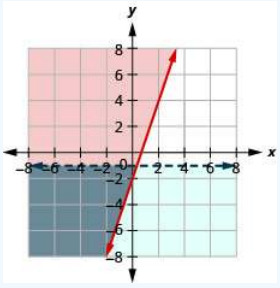
The point $(0, 0)$ is in the solution and we have already found it to be a solution of each inequality. The point of intersection of the two lines is not included as both boundary lines were dashed.

The solution is the area shaded twice—which appears as the darkest shaded region.

Example 2.3.11

Solve the system by graphing: $\begin{cases} y \geq 3x - 2 \\ y < -1 \end{cases}$

Answer

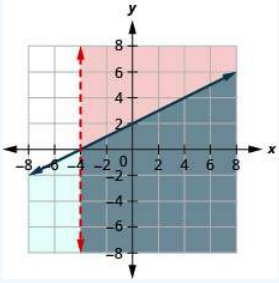


The solution is the darkest shaded region.

Example 2.3.12

Solve the system by graphing: $\begin{cases} x > -4x - 2 \\ y \geq -4 \end{cases}$

Answer



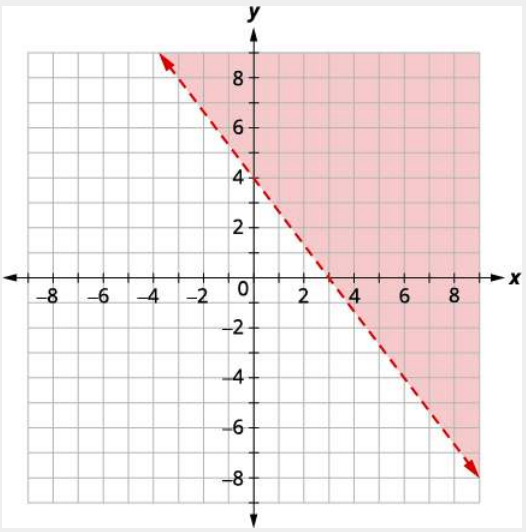
The solution is the darkest shaded region.

Systems of linear inequalities where the boundary lines are parallel might have no solution. We'll see this in the next example.

Example 2.3.13

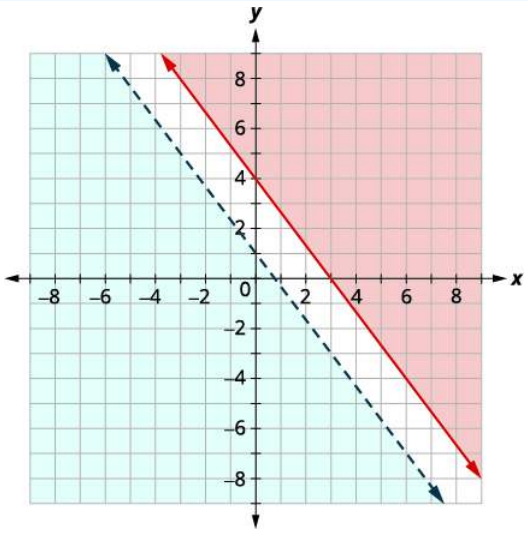
Solve the system by graphing: $\begin{cases} 4x+3y \geq 12 \\ y < -\frac{4}{3}x+1 \end{cases}$

Answer

	$\begin{cases} 4x+3y \geq 12 \\ y < -\frac{4}{3}x+1 \end{cases}$
Graph $4x+3y \geq 12$, by graphing $4x+3y=12$ and testing a point. The intercepts are x = 3 and y = 4 and the boundary line will be solid. Test (0, 0) which makes the inequality false, so shade (red) the side that does not contain (0, 0).	

Graph $y < -\frac{4}{3}x + 1$ by graphing $y = -\frac{4}{3}x + 1$ using the slope $m = -\frac{4}{3}$ and y-intercept $b = 1$. The boundary line will be dashed.

Test (0, 0) which makes the inequality true, so shade (blue) the side that contains (0, 0).

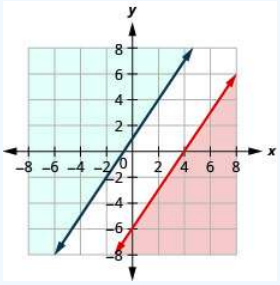


There is no point in both shaded regions, so the system has no solution.

Example 2.3.14

Solve the system by graphing: $\begin{cases} 3x - 2y \geq 12 \\ y \geq \frac{3}{2}x + 1 \end{cases}$

Answer

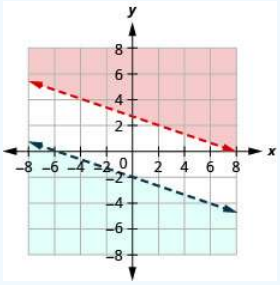


No solution.

Example 2.3.15

Solve the system by graphing: $\begin{cases} x + 3y > 8 \\ y < -\frac{1}{3}x - 2 \end{cases}$

Answer



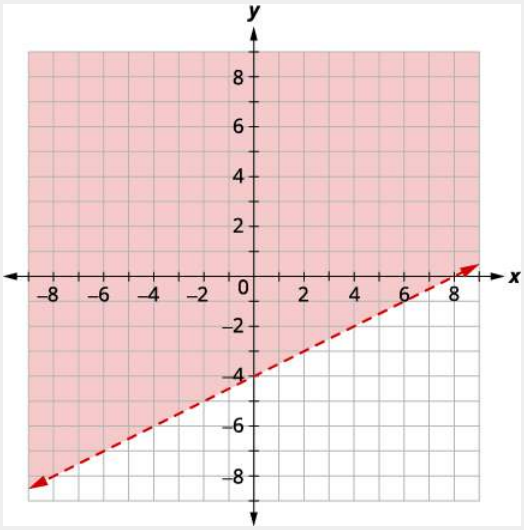
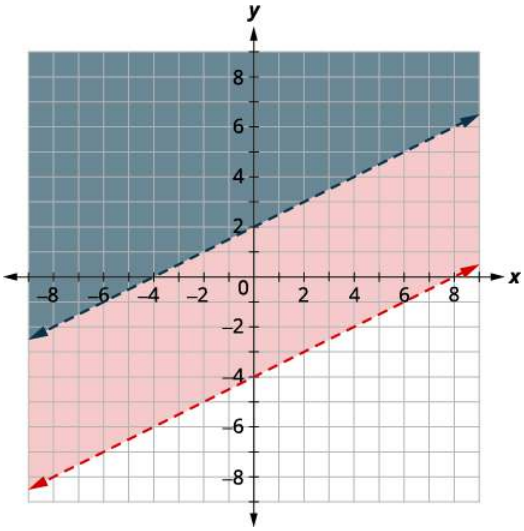
No solution.

Some systems of linear inequalities where the boundary lines are parallel will have a solution. We'll see this in the next example.

Example 2.3.16

Solve the system by graphing: $\begin{cases} y > \frac{1}{2}x - 4 \\ x - 2y < -4 \end{cases}$

Answer

	$\begin{cases} y > \frac{1}{2}x - 4 \\ x - 2y < -4 \end{cases}$
Graph $y > \frac{1}{2}x - 4$ by graphing $y = \frac{1}{2}x - 4$ using the slope $m = \frac{1}{2}$ and the intercept $b = -4$. The boundary line will be dashed. Test (0, 0) which makes the inequality true, so shade (red) the side that contains (0, 0).	
Graph $x - 2y < -4$ by graphing $x - 2y = -4$ and testing a point. The intercepts are $x = -4$ and $y = 2$ and the boundary line will be dashed. Choose a test point in the solution and verify that it is a solution to both inequalties. Test (0, 0) which makes the inequality false, so shade (blue) the side that does not contain (0, 0).	

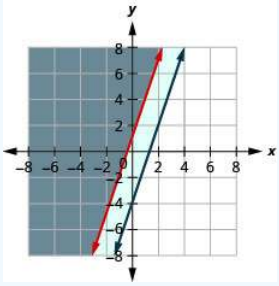
No point on the boundary lines is included in the solution as both lines are dashed.

The solution is the region that is shaded twice which is also the solution to $x - 2y < -4$.

Example 2.3.17

Solve the system by graphing: $\begin{cases} y \geq 3x+1 \\ -3x+y \geq -4 \end{cases}$

Answer

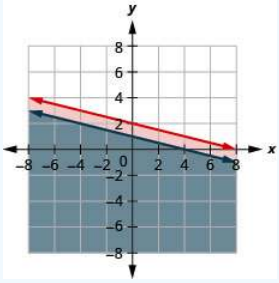


The solution is the darkest shaded region.

Example 2.3.18

Solve the system by graphing: $\begin{cases} y \leq -\frac{1}{4}x+2 \\ x+4y \leq 4 \end{cases}$

Answer



The solution is the darkest shaded region.

Solve Applications of Systems of Inequalities

The first thing we'll need to do to solve applications of systems of inequalities is to translate each condition into an inequality. Then we graph the system, as we did above, to see the region that contains the solutions. Many situations will be realistic only if both variables are positive, so we add inequalities to the system as additional requirements.

Example 2.3.19

Christy sells her photographs at a booth at a street fair. At the start of the day, she wants to have at least 25 photos to display at her booth. Each small photo she displays costs her $4 and each large photo costs her $10. She doesn't want to spend more than $200 on photos to display.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could she display 10 small and 20 large photos?
d. Could she display 20 large and 10 small photos?

Answer

a.

Let $x =$ the number of small photos.
$y =$ the number of large photos

To find the system of equations translate the information.

She wants to have at least 25 photos.
The number of small plus the number of large should be at least 25.

$$x + y \geq 25$$

\$4 for each small and \$10 for each large must be no more than \$200

$$4x + 10y \leq 200$$

The number of small photos must be greater than or equal to 0.

$$x \geq 0$$

The number of large photos must be greater than or equal to 0.

$$y \geq 0$$

We have our system of equations.

$$\begin{cases} x + y \geq 25 \\ 4x + 10y \leq 200 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

b.

Since $x \geq 0$ and $y \geq 0$ (both are greater than or equal to) all solutions will be in the first quadrant. As a result, our graph shows only quadrant one.

To graph $x + y \geq 25$, graph $x + y = 25$ as a solid line. Choose (0, 0) as a test point. Since it does not make the inequality true, shade (red) the side that does not include the point (0, 0). To graph $4x + 10y \leq 200$, graph $4x + 10y = 200$ as a solid line. Choose (0, 0) as a test point. Since it does make the inequality true, shade (blue) the side that include the point (0, 0).	

The solution of the system is the region of the graph that is shaded the darkest. The boundary line sections that border the darkly-shaded section are included in the solution as are the points on the *x*-axis from (25, 0) to (55, 0).

c. To determine if 10 small and 20 large photos would work, we look at the graph to see if the point (10, 20) is in the solution region. We could also test the point to see if it is a solution of both equations.

It is not, Christy would not display 10 small and 20 large photos.

d. To determine if 20 small and 10 large photos would work, we look at the graph to see if the point (20, 10) is in the solution region. We could also test the point to see if it is a solution of both equations.

It is, so Christy could choose to display 20 small and 10 large photos.

Notice that we could also test the possible solutions by substituting the values into each inequality.

Example 2.3.20

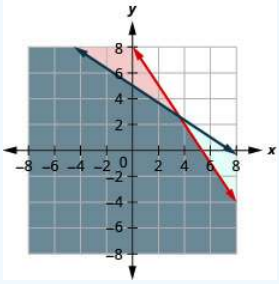
A trailer can carry a maximum weight of 160 pounds and a maximum volume of 15 cubic feet. A microwave oven weighs 30 pounds and has 2 cubic feet of volume, while a printer weighs 20 pounds and has 3 cubic feet of space.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could 4 microwaves and 2 printers be carried on this trailer?
d. Could 7 microwaves and 3 printers be carried on this trailer?

Answer

a. $\begin{cases} 30m+20p \leq 160 \\ 2m+3p \leq 15 \end{cases}$

b.



c. yes

d. no

Example 2.3.21

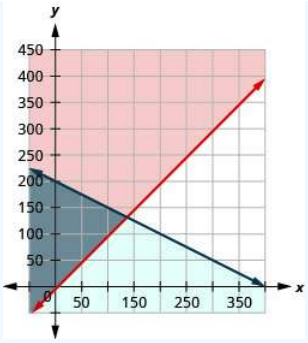
Mary needs to purchase supplies of answer sheets and pencils for a standardized test to be given to the juniors at her high school. The number of the answer sheets needed is at least 5 more than the number of pencils. The pencils cost $2 and the answer sheets cost $1. Mary's budget for these supplies allows for a maximum cost of $400.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could Mary purchase 100 pencils and 100 answer sheets?
d. Could Mary purchase 150 pencils and 150 answer sheets?

Answer

a. $\begin{cases} a \geq p+5 \\ a+2p \leq 400 \end{cases}$

b.



c. no
d. no

When we use variables other than x and y to define an unknown quantity, we must change the names of the axes of the graph as well.

Example 2.3.22

Omar needs to eat at least 800 calories before going to his team practice. All he wants is hamburgers and cookies, and he doesn't want to spend more than $5. At the hamburger restaurant near his college, each hamburger has 240 calories and costs $1.40. Each cookie has 160 calories and costs $0.50.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could he eat 3 hamburgers and 1 cookie?
d. Could he eat 2 hamburgers and 4 cookies?

Answer

a.

\(\begin{array} {ll} \text{Let} &{h=\text{the number of hamburgers.} \\ &{c=\text{the number of cookies}\end{array}\)

To find the system of equations translate the information.

The calories from hamburgers at 240 calories each, plus the calories from cookies at 160 calories each must be more that 800.

$$240h+160c \geq 800$$

The amount spent on hamburgers at $\$1.40$ each, plus the amount spent on cookies at $\$0.50$ each must be no more than $\$5.00$.

$$1.40h+0.50c \leq 5$$

The number of hamburgers must be greater than or equal to 0.

$$h \geq 0$$

The number of cookies must be greater than or equal to 0.

$$c \geq 0$$

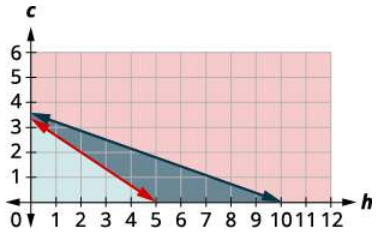
We have our system of equations.

$$\begin{cases} 240h+160c \geq 800 \\ 1.40h+0.50c \leq 5 \\ h \geq 0 \\ c \geq 0 \end{cases}$$

b.
Since $h \geq 0$ and $c \geq 0$ (both are greater than or equal to) all solutions will be in the first quadrant. As a result, our graph shows only quadrant one.

To graph $240h + 160c \geq 800$, graph $240h + 160c = 800$ as a solid line.

Choose (0, 0) as a test point. Since it does not make the inequality true, shade (red) the side that does not include the point (0, 0).



Graph $1.40h + 0.50c \leq 5$. The boundary line is $1.40h + 0.50c = 5$. We test (0, 0) and it makes the inequality true. We shade the side of the line that includes (0, 0).

The solution of the system is the region of the graph that is shaded the darkest. The boundary line sections that border the darkly shaded section are included in the solution as are the points on the *x*-axis from (5, 0) to (10, 0).

c. To determine if 3 hamburgers and 2 cookies would meet Omar's criteria, we see if the point (3, 2) is in the solution region. It is, so Omar might choose to eat 3 hamburgers and 2 cookies.

d. To determine if 2 hamburgers and 4 cookies would meet Omar's criteria, we see if the point (2, 4) is in the solution region. It is, Omar might choose to eat 2 hamburgers and 4 cookies.

We could also test the possible solutions by substituting the values into each inequality.

Example 2.3.23

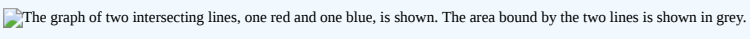
Tension needs to eat at least an extra 1,000 calories a day to prepare for running a marathon. He has only $25 to spend on the extra food he needs and will spend it on $0.75 donuts which have 360 calories each and $2 energy drinks which have 110 calories.

a. Write a system of inequalities that models this situation.
b. Graph the system.
c. Can he buy 8 donuts and 4 energy drinks and satisfy his caloric needs?
d. Can he buy 1 donut and 3 energy drinks and satisfy his caloric needs?

Answer

a. $\begin{cases} 0.75d + 2e \leq 25 \\ 360d + 110e \geq 1000 \end{cases}$

b.


The graph of two intersecting lines, one red and one blue, is shown. The area bound by the two lines is shown in grey.

c. yes
d. no

Example 2.3.24

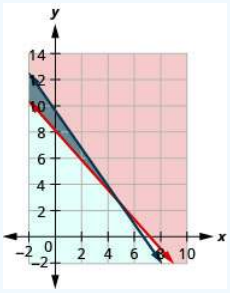
Philip's doctor tells him he should add at least 1,000 more calories per day to his usual diet. Philip wants to buy protein bars that cost $1.80 each and have 140 calories and juice that costs $1.25 per bottle and have 125 calories. He doesn't want to spend more than $12.

a. Write a system of inequalities that models this situation.
b. Graph the system.
c. Can he buy 3 protein bars and 5 bottles of juice?
d. Can he buy 5 protein bars and 3 bottles of juice?

Answer

a. $\begin{cases} 140p + 125j \geq 1000 \\ 1.80p + 1.25j \leq 12 \end{cases}$

b.



c. yes

d. no

Access these online resources for additional instruction and practice with solving systems of linear inequalities by graphing.

- Solving Systems of Linear Inequalities by Graphing
- Systems of Linear Inequalities

Key Concepts

- **Solutions of a System of Linear Inequalities:** Solutions of a system of linear inequalities are the values of the variables that make all the inequalities true. The solution of a system of linear inequalities is shown as a shaded region in the *x*, *y*coordinate system that includes all the points whose ordered pairs make the inequalities true.
- **How to solve a system of linear inequalities by graphing.**
  1. Graph the first inequality.
     Graph the boundary line.
     Shade in the side of the boundary line where the inequality is true.
  2. On the same grid, graph the second inequality.
     Graph the boundary line.
     Shade in the side of that boundary line where the inequality is true.
  3. The solution is the region where the shading overlaps.
  4. Check by choosing a test point.

Glossary

system of linear inequalities

Two or more linear inequalities grouped together form a system of linear inequalities.

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2.3E: Exercises - Solving Systems of Linear Inequalities in Two Variables

Practice Makes Perfect

Determine Whether an Ordered Pair is a Solution of a System of Linear Inequalities

In the following exercises, determine whether each ordered pair is a solution to the system.

1. $\begin{cases} 4x-y<10 \\ -2x+2y>-8 \end{cases}$

ⓐ $(5,-2)$
ⓑ $(-1,3)$

Answer

ⓐ false ⓑ true

2. $\begin{cases} y<\frac{3}{2}x+3 \\ \frac{3}{4}x-2y<5 \end{cases}$

ⓐ $(-4,-1)$
ⓑ $(8,3)$

Answer

ⓐ false ⓑ true

3. $\begin{cases} 6x-5y<20 \\ -2x+7y>-8 \end{cases}$

ⓐ $(1,-3)$
ⓑ $(-4,4)$

Answer

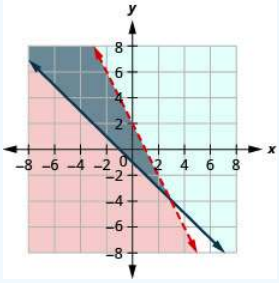
ⓐ false ⓑ true

Solve a System of Linear Inequalities by Graphing

In the following exercises, solve each system by graphing.

4. $\begin{cases} y<-2x+2 \\ y\geq -x-1 \end{cases}$

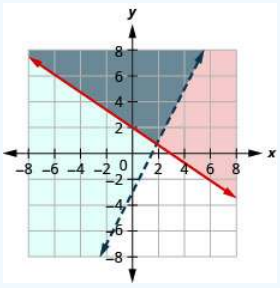
Answer



The solution is the grey region.

5. $\begin{cases} y \geq -\frac{2}{3}x + 2 \\ y > 2x - 3 \end{cases}$

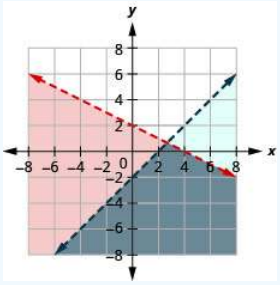
Answer



The solution is the grey region.

6. $\begin{cases} x + 2y < 4 \\ y < x - 2 \end{cases}$

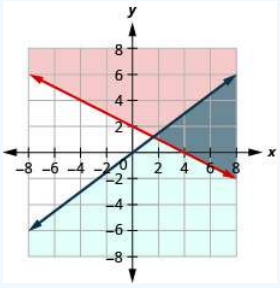
Answer



The solution is the grey region.

7. $\begin{cases} x + 4y \geq 8 \\ y \leq \frac{3}{4}x \end{cases}$

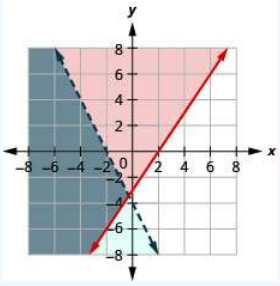
Answer



The solution is the grey region.

8. $\begin{cases} 3x - 2y \leq 6 \\ -4x - 2y > 8 \end{cases}$

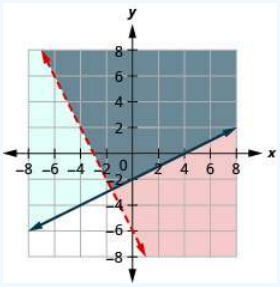
Answer



The solution is the grey region.

9. $\begin{cases} 2x+y>-6 \\ -x+2y \geq -4 \end{cases}$

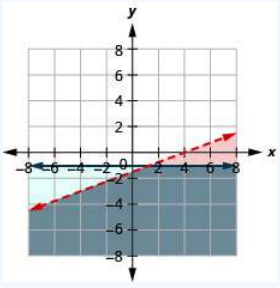
Answer



The solution is the grey region.

10. $\begin{cases} x-3y>4 \\ y \leq -1 \end{cases}$

Answer

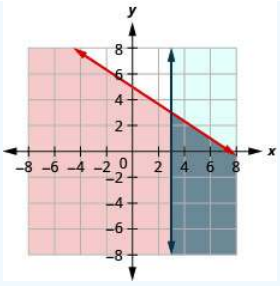


The solution is the grey region.

11. $\begin{cases} y \leq -\frac{2}{3}x+5 \\ x \geq 3 \end{cases}$

Answer

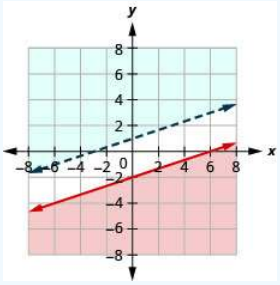
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The solution is the grey region.

12. $\begin{cases} -3x+6y>12 \\ 4y \leq 2x-4 \end{cases}$

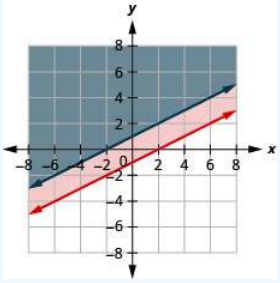
Answer



No solution.

13. $\begin{cases} y \geq \frac{1}{2}x-1 \\ -2x+4y \geq 4 \end{cases}$

Answer



The solution is the grey region.

Solve Applications of Systems of Inequalities

In the following exercises, translate to a system of inequalities and solve.

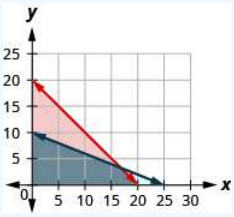
14. Faran does not want to spend more than $50 on bags of fertilizer and peat moss for his garden. Fertilizer costs $2 a bag and peat moss costs $5 a bag. Their van can hold at most 20 bags.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can they buy 15 bags of fertilizer and 4 bags of peat moss?
ⓓ Can they buy 10 bags of fertilizer and 10 bags of peat moss?

Answer

ⓐ $\begin{cases} f \geq 0 \\ p \geq 0 \\ f+p \leq 202 \\ f+5p \leq 50 \end{cases}$

ⓑ



ⓒ yes

ⓓ no

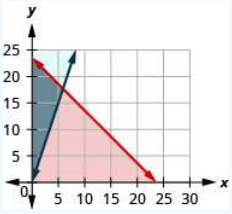
15. Juan is studying for his final exams in chemistry and algebra. he knows he only has 24 hours to study, and it will take him at least three times as long to study for algebra than chemistry.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can he spend 4 hours on chemistry and 20 hours on algebra?
ⓓ Can he spend 6 hours on chemistry and 18 hours on algebra?

Answer

ⓐ $\begin{cases} c \geq 0 \\ a \geq 0 \\ c+a \leq 24 \\ a \geq 3c \end{cases}$

ⓑ



ⓒ yes

ⓓ no

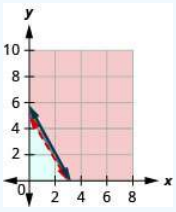
16. Mark is attempting to build muscle mass and so he needs to eat at least an additional 80 grams of protein a day. A bottle of protein water costs $3.20 and a protein bar costs $1.75. The protein water supplies 27 grams of protein and the bar supplies 16 gram. If he has $10 dollars to spend

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Could he buy 3 bottles of protein water and 1 protein bar?
ⓓ Could he buy no bottles of protein water and 5 protein bars?

Answer

ⓐ $\begin{cases} w \geq 0 \\ b \geq 0 \\ 27w + 16b > 80 \\ 3.20w + 1.75b \leq 10 \end{cases}$

ⓑ



ⓒ no

ⓓ yes

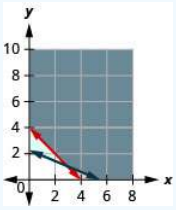
17. Marla is increasing her exercise routine by running and walking at least 4 miles each day. Her goal is to burn a minimum of 1500 calories from this exercise. Walking burns 270 calories/mile and running burns 650 calories.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Could she meet her goal by walking 3 miles and running 1 mile?
ⓓ Could she meet her goal by walking 2 miles and running 2 miles?

Answer

ⓐ $\begin{cases} w \geq 0 \\ r \geq 0 \\ w + r \geq 4 \\ 270w + 650r \geq 1500 \end{cases}$

ⓑ



ⓒ no

ⓓ yes

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2.4: Chapter 2 Review

PROBLEM SET: CHAPTER 2 REVIEW

Solving Linear Inequalities in One Variable (2.1)

Graph all solutions on a number line and provide the corresponding interval notation.

1. $-2a+5>5$
2. $5-2(4+3y)\leq 45$
3. $1-4(3x+7)<-3(x+9)-9x$
4. $5x+11<16-2(x+1)$
5. $-4(2t+5)\leq 11-5t$

Solving Linear Inequalities in Two Variables (2.2)

6. Graph the linear inequality: $y<\frac{3}{5}x+2$.
7. Graph the linear inequality: $y\geq -\frac{1}{3}x-2$.
8. Graph the linear inequality: $2x-4y\geq -12$.
9. Graph the linear inequality: $x+5y<-5$.

Solving Systems of Linear Inequalities in Two Variables (2.3)

10. Determine whether each ordered pair is a solution to the system of inequalities.

$$\begin{cases} 7x+2y>14 \\ 5x-y\leq 8 \end{cases}$$

ⓐ $(2,3)$
ⓑ $(7,-1)$

11. Graph $\begin{cases} y<2x-1 \\ y\leq -\frac{1}{2}x+4 \end{cases}$
12. Graph $\begin{cases} 3x-y\geq 6 \\ y\geq -\frac{1}{2}x \end{cases}$
13. Graph $\begin{cases} 2x+2y>-4 \\ -x+3y\geq 9 \end{cases}$
14. Graph $\begin{cases} y\leq -\frac{1}{4}x-2 \\ x+4y<6 \end{cases}$

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CHAPTER OVERVIEW

3: Applications of Linear Equations and Inequalities

3.1: Linear Applications

3.1.1: Exercises - Linear Applications

3.2: Fitting Linear Models to Data

3.2E: Exercises - Fitting Linear Models to Data

3.3: Linear Programming - Maximization Applications

3.3E: Exercises - Linear Programming Maximization Applications

3.4: Linear Programming - Minimization Applications

3.4.1: Exercises - Linear Programming Minimization Applications

3.5: Chapter 3 Review

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3.1: Linear Applications

Learning Objectives

In this section, you will learn to use linear equations to model real-world applications.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Find the slope of the line containing the points (3, 10) and (5, 40).

Click here to check your answer

15

If you missed this problem, review Section 1.2. (Note that this will open in a new window.)

2. Find the equation of the line containing the points (3, 10) and (5, 40). Write the equation in slope-intercept form.

Click here to check your answer

$y = 15x - 35$

If you missed this problem, review Section 1.3. (Note that this will open in a new window.)

Now that we have learned to solve equations and inequalities, we get to apply these ideas in a variety of real-life situations. Many scenarios follow linear trends. In this section, we'll work on creating a linear equations.

Creating Linear Equations

While each application is different, the following steps represent a general pattern.

1. Read the problem carefully. Highlight important information.
2. Identify the two quantities that change and define the unknowns.
   - One quantity will be independent, meaning it is going to change on its own. In many applications, this quantity is time. Define this quantity as x.
   - Another quantity will be dependent, meaning that its value will depend on how the other quantity changes. In many applications, this is money. Define this quantity as y.
3. Create an equation to represent the relationship between the two variables. When applications have a starting value and a constant rate of change, these will most commonly be written in slope-intercept form, $y = mx + b$. The y-intercept will represent the starting value and the slope will represent the rate of change.

Example 3.1.1

Jadon wants to purchase a new car for $12,500. They also need to budget for expenses of gas, insurance, registration, and repairs, which they estimate to be $450 per month.

a. What will be the total cost of the car if they own it for 10 months?

b. What will be the total cost of the car if they own it for 3 years?

c. What will be the total cost of the car if they don't know how long they will own it?

Solution

a. For only 10 months, they would need to pay the purchase price once and the monthly cost 10 times. The cost would be $\$12,500 + 10(\$450)$ which is $\$17,000$

b. For 3 years, they would need to pay the purchase price once and the monthly cost 36 (12*3) times. The cost would be $\$12,500 + 36(\$450)$ which is $\$28,700$

c. When we don't know the time they'll own the car, we need to use variables to represent the changing quantities.

- The independent variable will be the number of months Jadon owns the car. We'll let x = the number of months.
- The cost of the car depends on the number of months they own it, so we'll let y = the cost.

Since the cost begins at \$12,500 and increases \$350 for every month, the equation can be written as $y = 12500 + 450x$ or $y = 450x + 12500$.

Note that it is also acceptable to choose other variables to represent the quantities. This same equation might be written as $C = 12500 + 450t$, where t represents the number of months and C represents the cost.

In the problem above, \$450 per month is referred to as the **variable cost**, which usually represents the slope in the equation. The flat charge \$12,500 is referred to as the **fixed cost**, and usually represents the y-intercept in the equation.

Since the equation gives the total cost as the output of the equation, this is referred to as the **cost function**.

Example 3.1.2

A taxi service charges \$0.50 per mile plus a \$5 flat fee.

a. What will be the cost of traveling 20 miles?

b. What is the cost function?

Solution

a. The cost of traveling 20 miles is $5 + (0.50)(20)$ which is \$15.

b. To find the cost function of traveling an unknown number of miles,

- Let x = distance traveled in miles
- Let y = cost in dollars

The total cost is \$5 plus \$0.50 for every mile traveled, so the cost function is $y = 5 + 0.50x$.

Example 3.1.3

The variable cost to manufacture a product is \$10 per item and the fixed cost \$2500. Define the unknowns and write the cost function.

Solution

Let x represent the number of items manufactured and let y represents the total cost

The variable cost of \$10 per item tells us that $m = 10$.

The fixed cost represents the y-intercept. So $b = 2500$.

Therefore, the cost equation is $y = 10x + 2500$.

Example 3.1.4

It costs \$750 to manufacture 25 items, and \$1000 to manufacture 50 items.

a. Assuming a linear relationship holds, find the cost equation.

b. Use this function to predict the cost of 100 items.

Solution

a. We let x = the number of items manufactured, and let y = the cost.

Solving this problem is equivalent to finding an equation of a line that passes through the points (25, 750) and (50, 1000).

$$m = \frac{1000 - 750}{50 - 25} = 10$$

Therefore, the partial equation is $y = 10x + b$

By substituting one of the points in the equation, we get $b = 500$

Therefore, the cost equation is $y = 10x + 500$

b. To find the cost of 100 items, substitute $x = 100$ in the equation $y = 10x + 500$

So the cost is

$$y = 10(100) + 500 = 1500$$

It costs $1500 to manufacture 100 items.

Example 3.1.5

The freezing temperature of water in Celsius is 0 degrees and in Fahrenheit 32 degrees. And the boiling temperatures of water in Celsius, and Fahrenheit are 100 degrees, and 212 degrees, respectively.

a. Write a conversion equation from Celsius to Fahrenheit.

b. Use this equation to convert 30 degrees Celsius into Fahrenheit.

Solution

a. Let us look at what is given.

Celsius	Fahrenheit
0	32
100	212

Because the directions ask us to convert from Celsius to Fahrenheit, we want Fahrenheit to be the output of our equation. Therefore, we'll let x represent Celsius and let y represent Fahrenheit.

- Note that if the directions had been written the other way (from Fahrenheit to Celsius), we would have wanted the variables to be switched.

We can now write the information given as two points of data. Solving this problem is equivalent to finding an equation of a line that passes through the points (0, 32) and (100, 212).

Since we are finding a linear relationship, we are looking for an equation $y = mx + b$, or $F = mC + b$ depending on what variables are defined.

$$\text{slope m } = \frac{312 - 32}{100 - 0} = \frac{9}{5}$$

The equation is $F = \frac{9}{5}C + b$

b. Substituting the point (0, 32), we get

$$F = \frac{9}{5}C + 32.$$

To convert 30 degrees Celsius into Fahrenheit, substitute $C = 30$ in the equation

$$\mathrm{F} = \frac{9}{5}\mathrm{C} + 32$$
$$\mathrm{F} = \frac{9}{5}(30) + 32 = 86$$

Example 3.1.6

The population of Canada in the year 1980 was 24.4 million, and in the year 2010 it was 34 million. The population of Canada over that time period can be approximately modeled by a linear function. Let x represent time as the number of years after 1980 and let y represent the size of the population.

a. Write the linear function that gives a relationship between the time and the population.
b. Assuming the population continues to grow linearly in the future, use this equation to predict the population of Canada in the year 2025.

Solution

The problem can be made easier by using 1980 as the base year, that is, we choose the year 1980 as the year zero. This will mean that the year 2010 will correspond to year 30. Now we look at the information we have:

Year	Population
0 (1980)	24.4 million
30 (2010)	34 million

a. Solving this problem is equivalent to finding an equation of a line that passes through the points (0, 24.5) and (30, 34). We use these two points to find the slope:

$$m = \frac{34 - 24.4}{30 - 0} = \frac{9.5}{30} = 0.32$$

The y-intercept occurs when $x = 0$, so $b = 24.4$

$$y = 0.32x + 24.4$$

b. Now to predict the population in the year 2025, we let $x = 2025 - 1980 = 45$

$$y = 0.32x + 24.4$$
$$y = 0.32(45) + 24.4 = 38.8$$

In the year 2025, we predict that the population of Canada will be 38.8 million people.

Note that we assumed the population trend will continue to be linear. Therefore if population trends change and this assumption does not continue to be true in the future, this prediction may not be accurate.

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3.1.1: Exercises - Linear Applications

PROBLEM SET: LINEAR APPLICATIONS

In the following application problems, assume a linear relationship holds.

1) The variable cost to manufacture a product is $25 per item, and the fixed costs are $1200. If x is the number of items manufactured and y is the cost, write the cost function.	2) It costs $90 to rent a car driven 100 miles and $140 for one driven 200 miles. If x is the number of miles driven and y the total cost of the rental, write the cost function.
3) The variable cost to manufacture an item is $20 per item, and it costs a total of $750 to produce 20 items. Define the unknowns and write the cost function.	4) To manufacture 30 items, it costs $2700, and to manufacture 50 items, it costs $3200. Define the unknowns and write the cost function.
5) To manufacture 100 items, it costs $32,000, and to manufacture 200 items, it costs $40,000. Define the unknowns and write the cost function.	6) It costs $1900 to manufacture 60 items, and the fixed costs are $700. Define the unknowns and write the cost function.
7) Mugs Café sells 1000 cups of coffee per week if it does not advertise. For every $50 spent in advertising per week, it sells an additional 150 cups of coffee. a) Find an equation that gives y = number of cups of coffee sold per week x = amount spent on advertising per week. b) How many cups of coffee does Mugs Café expect to sell if $100 per week is spent on advertising?	8) EZ Clean company has determined that if it spends $30,000 on advertising, it can hope to sell 12,000 of its Minivacs a year, but if it spends $50,000, it can sell 16,000. Write an equation that gives a relationship between the number of dollars spent on advertising (x) and the number of minivacs sold (y).
9) The cost of electricity in residential homes is a linear function of the amount of energy used. In Denver a home using 250 kilowatt hours (kwh) of electricity per month pays $55. A home using 600 kwh per month pays $118. Define the unknowns and write the cost of electricity as a function of the amount used. Then use the function to find the cost for a home using 400 kwh of electricity per month.	10) Using the equation from the previous problem, find the level of electricity use that would correspond to a monthly cost of $100.
11) It costs $1,200 to produce 50 pounds of a chemical and it costs $2,200 to produce 150 pounds. The chemical sells for $15 per pound. a. Define the unknowns and find the cost function. b. What is the fixed cost and what is the variable cost? c. How many pounds must be sold to break even? d. Find the cost and revenue at the break-even point.	12) Whackemhard Sports is planning to introduce a new line of tennis rackets. The fixed costs for the new line are $25,000 and the variable cost of producing each racket is $60. a. Define the unknowns and find the cost function. b. How many rackets must be sold to break even? c. Find the cost and revenue at the break-even point.
13) A demand curve for a product is the number of items the consumer will buy at different prices. At a price of $2 a store can sell 2400 of a particular type of toy truck. At a price of $8 the store can sell 600 such trucks. If x represents the price of trucks and y the number of items sold, write an equation for the demand curve.	14) A supply curve for a product is the number of items that can be made available at different prices. A manufacturer of toy trucks can supply 2000 trucks if they are sold for $8 each; it can supply only 400 trucks if they are sold for $4 each. If x is the price and y the number of items, write an equation for the supply curve.
15) The freezing temperatures for water for Celsius and Fahrenheit scales are 0ºC and 32ºF. The boiling temperatures for water are 100 ºC and 212 ºF. Let C denote the temperature in Celsius and F in Fahrenheit. Write the conversion function from Celsius to Fahrenheit. Then use the function to convert 25 ºC into ºF.	16) Using the same data as given in the previous problem, find a conversion function that converts Fahrenheit into Celsius. Then use this conversion function to convert 72 ºF into an equivalent Celsius measure.

17) California's population was 29.8 million in the year 1990, and 37.3 million in 2010. Assume that the population trend was and continues to be linear, write the population function. Use this function to predict the population in 2025. *Hint: Use 1990 as the base year (year 0); then 2010 and 2025 are years 20, and 35, respectively.)*

18) Use the population function for California in the previous problem to find the year in which the population will be 40 million people.

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3.2: Fitting Linear Models to Data

Learning Objectives

- Create and interpret scatter plots.
- Fit a regression line to a set of data and use the linear model to make predictions.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. If $y = -3.215x - 78.2$, solve for y when $x = -21$.

Click here to check your answer

$y = -10.685$

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

2. If $y = -3.215x - 78.2$, solve for x when $y = -46.05$.

Click here to check your answer

$x = -10$

If you missed this problem, review Section 1.1. (Note that this will open in a new window.)

A professor is attempting to identify trends among final exam scores. His class has a mixture of students, so he wonders if there is any relationship between age and final exam scores. One way for him to analyze the scores is by creating a diagram that relates the age of each student to the exam score received. In this section, we will examine one such diagram known as a scatter plot.

Drawing and Interpreting Scatter Plots

A scatter plot is a graph of plotted points that may show a relationship between two sets of data. If the relationship is from a **linear model**, or a model that is nearly linear, the professor can draw conclusions using his knowledge of linear functions. Figure 3.2.1 shows a sample scatter plot.

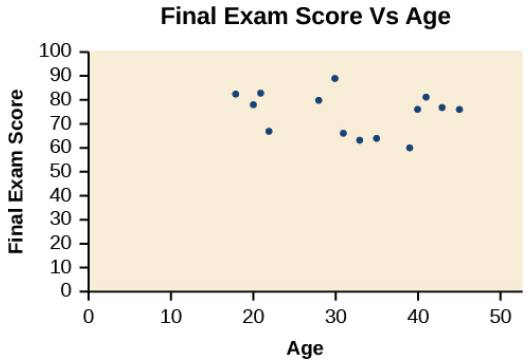


Figure 3.2.1: A scatter plot of age and final exam score variables

Notice this scatter plot does not indicate a **linear relationship**. The points do not appear to follow a trend. In other words, there does not appear to be a relationship between the age of the student and the score on the final exam.

To create a scatter plot on Desmos

1. Open your Desmos app, or go to https://www.desmos.com/calculator.
2. Above the equation list, click the + icon.
3. Click "table."
4. Enter the x-values in the first column and the y-values in the second column.

5. Click the magnifying glass to the left of the table to automatically adjust the viewing window to see the points. You can also manually adjust the window as necessary.

Example 3.2.1: Using a Scatter Plot to Investigate Cricket Chirps

The table below shows the number of cricket chirps in 15 seconds, for several different air temperatures, in degrees Fahrenheit. Plot this data and determine whether the data appears to be linearly related.

Table 3.2.1

Chirps	44	35	20.4	33	31	35	18.5	37	26
Temperature	80.5	70.5	57	66	68	72	52	73.5	53

Solution

Plotting this data, as depicted in Figure 3.2.2 suggests that there may be a trend. We can see from the trend in the data that the number of chirps increases as the temperature increases. The trend appears to be roughly linear, though certainly not perfectly so.

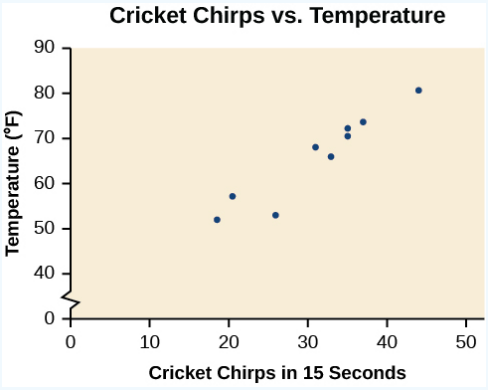


Figure 3.2.2: The line regression is generally positive.

Finding the Line of Best Fit

Once we recognize a need for a linear function to model that data, the natural follow-up question is "what is that linear function?" One way to approximate our linear function is to sketch the line that seems to best fit the data. Then we can extend the line until we can verify the y-intercept. We can approximate the slope of the line by extending it until we can estimate the $\frac{\text{rise}}{\text{run}}$.

Example 3.2.2: "Eyeballing" a Line of Best Fit

Find a linear function that fits the data in Table 3.2.1 by "eyeballing" a line that seems to fit.

Solution

On a graph, we could try sketching a line.

Using the starting and ending points of our hand drawn line, points $(0, 30)$ and $(50, 90)$, this graph has a slope of

$$m = \frac{60}{50} = 1.2$$

and a y-intercept at 30. This gives an equation of

$$T(c) = 1.2c + 30$$

where c is the number of chirps in 15 seconds, and $T(c)$ is the temperature in degrees Fahrenheit. The resulting equation is represented in Figure 3.2.3.

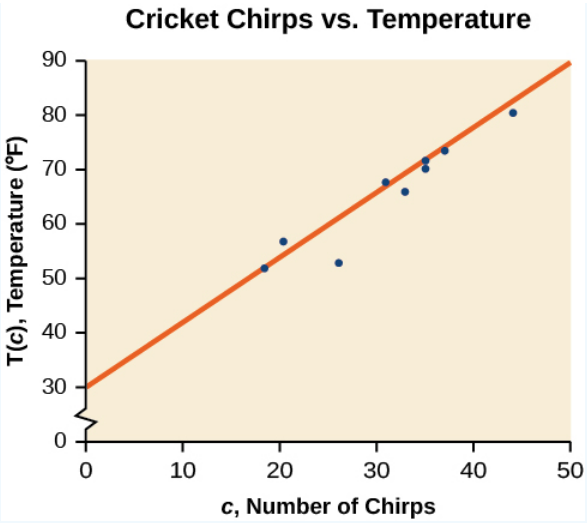


Figure 3.2.3: Scatter plot, showing the line of best fit.

Analysis

This linear equation can then be used to approximate answers to various questions we might ask about the trend.

? Exercise 3.2.1

According to the data from Table 3.2.1, what temperature can we predict it is if we counted 29 chirps in 15 seconds?

Solution

According to the line the temperature would be somewhere in the low 60s, approximately 63°F.

Finding the Line of Best Fit Using a Graphing Program

While eyeballing a line works reasonably well, some statistical techniques exist for fitting a line to data that minimize the differences between the line and data values[2]. One such technique is called **least squares regression** and can be computed by many graphing calculators, spreadsheet software, statistical software, and many web-based calculators[3]. Least squares regression is one means to determine the line that best fits the data, and here we will refer to this method as linear regression.

To find a line of best fit using Desmos

1. Open your Desmos app, or go to https://www.desmos.com/calculator.
2. Create a scatter plot, as listed above.
3. After the xy table, in the space for the next equation, write $y_1 \sim m x_1 + b$. This tells the program to use the values of x_1 and y_1 from the table above.
   - The ~ sign tells the program to approximate the equation, since a linear equation won't fit the data precisely.
   - The x_1 and y_1 tell the program to use the values of x_1 and y_1 from the table above.
   - The m and b represent slope and y-intercept of the line, respectively.
4. Press Enter.
5. The resulting information will display the appropriate slope (m) and y-intercept (b) values of the line of best fit.
   - The information will also display statistics of r^2 and r. An r-value of 1 will indicate an exact fit. The closer the r-value is to 1, the closer the data fits the line of best fit.
6. For a video explanation of this process, click HERE. The first 53 seconds show how to enter a scatter plot. The line of best fit begins at the 0:53 mark.

Example 3.2.3: Finding a Linear Regression Equation

Find the line of best fit using the cricket-chirp data in Table 3.2.1.

Solution

Table 3.2.1

Chirps	44	35	20.4	33	31	35	18.5	37	26
Temperature	80.5	70.5	57	66	68	72	52	73.5	53

Enter the input (chirps) in x_1. Enter the output (temperature) in y_1.

Enter $y_1\ mx_1 + b$ in the next equation space.

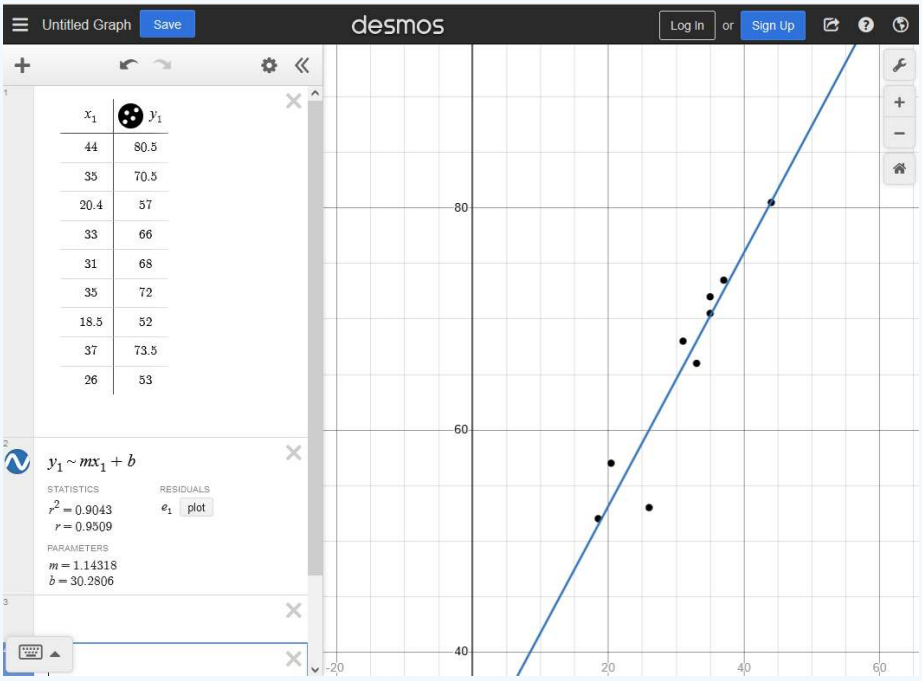
Note the m and b values for the data.

Therefore we obtain the equation:

$$y = 1.143x + 30.281$$

or

$$T = 1.143c + 30.281$$



Will there ever be a case where two different lines will serve as the best fit for the data?

No. There is only one best fit line.

Predicting with a Regression Line

Once we determine that a set of data is linear, we can use the regression line to make predictions. As we learned above, a regression line is a line that is closest to the data in the scatter plot, which means that only one such line is a best fit for the data.

Example 3.2.4: Using a Regression Line to Make Predictions

Gasoline consumption in the United States has been steadily increasing. Consumption data from 1994 to 2004 is shown in Table 3.2.3.

a. Graph the data and determine whether the trend is linear. If so, find a model for the data.

b. Use the model to predict the consumption in 2008.

c. Use the model to predict when oil consumption would reach 160 billion gallons.

Table 3.2.3

Year	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Consumption (billions of gallons)	113	116	118	119	123	125	126	128	131	133	136

a. The scatter plot of the data, including the least squares regression line, is shown in Figure 3.2.5.

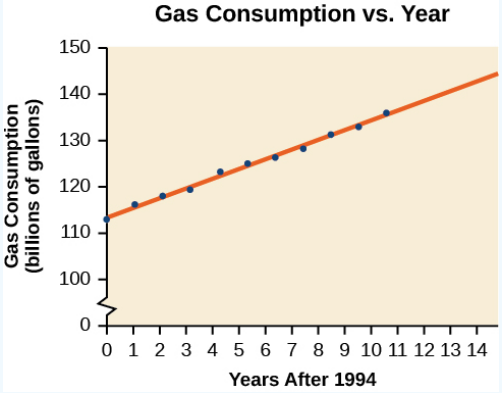


Figure 3.2.5: Scatter plot, showing the line of best fit.

We can introduce new input variable, t,representing years since 1994.

The linear regression equation is:

$$C = 113.318 + 2.209t$$

b. Using this to predict consumption in 2008, set $t = 14$.

$$\begin{aligned} C &= 113.318 + 2.209(14) \\ &= 144.244 \end{aligned}$$

The model predicts 144.244 billion gallons of gasoline consumption in 2008.

c. Using the model to predict time when consumption is 160 billion gallons, set $C = 160$.

$$\begin{aligned} 160 &= 113.318 + 2.209(t) \\ 46.682 &= 2.209(t) \\ 21.133 &= t \end{aligned}$$

21 years after 1994 is the year 2015, so oil consumption is expected to reach 160 billion gallons in 2015.

Exercise 3.2.2

Use the model we created using technology in Example $3.2.4$ to predict the gas consumption in 2011.

Answer

150.871 billion gallons

Key Concepts

- Scatter plots show the relationship between two sets of data.
- Scatter plots may represent linear or non-linear models.
- The line of best fit may be estimated or calculated, using a calculator or statistical software.
- The correlation coefficient, r, indicates the degree of linear relationship between data.
- A regression line best fits the data.
- The least squares regression line may be used to make predictions regarding either of the variables.

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3.2E: Exercises - Fitting Linear Models to Data

Exercise $3.2E.1$

In Europe and Asia, m-commerce is popular. M-commerce users have special mobile phones that work like electronic wallets as well as provide phone and Internet services. Users can do everything from paying for parking to buying a TV set or soda from a machine to banking to checking sports scores on the Internet. For the years 2000 through 2004, what was the relationship between the year and the number of m-commerce users? Construct a scatter plot and find the linear regression model that best fits the data. Let $x =$ the year and let $y =$ the number of m-commerce users, in millions.

Table* $3.2E.1$*: *Table showing the number of m-commerce users (in millions) by year.*

x (year)	y (# of users)
2000	0.5
2002	20.0
2003	33.0
2004	47.0

Exercise $3.2E.2$

A random sample of 11 statistics students produced the following data, where x is the Unit 1 Exam score out of 80, and y is the Final Exam score out of 200.

Table showing the scores on the Final Exam based on scores from the Unit 1 Exam.

x (Unit 1 Exam score)	y (Final Exam score)
65	151
67	133
71	185
71	163
66	126
75	198
67	153
70	163
71	159
69	151
69	159

A random student earned a 58 on the Unit 1 Exam. If that student follows the class trend, what would you expect this student to earn on the final exam?

Exercise $3.2E.3$

SCUBA divers have maximum dive times they cannot exceed when going to different depths. The data in Table show different depths with the maximum dive times in minutes. Use your calculator to find the linear regression line and predict the maximum dive time for 110 feet.

X (depth in feet)	Y (maximum dive time)
50	80
60	55
70	45
80	35
90	25
100	22

Exercise $3.2E.4$

Amelia plays basketball for her high school. She notices that the number of points she scores in a game goes up in response to the number of hours she practices her jump shot each week. She records the following data. She has a goal of playing in college and needs to score 40 points in a game to be considered for the college team. Based on the trend, how many hours each week should she practice her jump shot?

X (hours practicing jump shot)	Y (points scored in a game)
4	13
6	18
7	22
9	27
11	30

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3.3: Linear Programming - Maximization Applications

Learning Objectives

In this section, you will learn to:

- Recognize the typical form of a linear programming problem.
- Formulate maximization linear programming problems.
- Graph feasible regions for maximization linear programming problems.
- Determine optimal solutions for maximization linear programming problems.

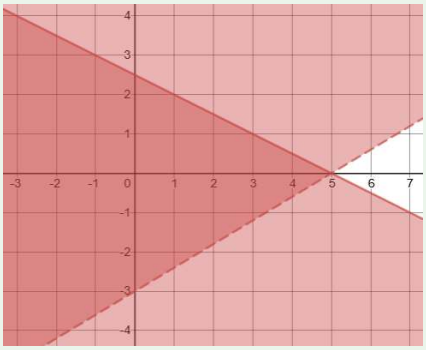
Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Graph this system of inequalities:

$$\begin{cases} 2x + 4y \leq 10 \\ 3x - 5y < 15 \end{cases}$$

Click here to check your answer

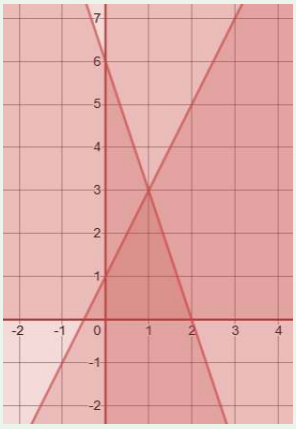


If you missed this problem, review Section 2.3. (Note that this will open in a new window.)

2. Graph this system of inequalities:

$$\begin{cases} y \leq 2x + 1 \\ y \leq -3x + 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Click here to check your answer



(The solution is in the center region.)

If you missed this problem, review Section 2.3. (Note that this will open in a new window.)

Application problems in business, economics, and social and life sciences often ask us to make decisions on the basis of certain conditions. The conditions or constraints often take the form of inequalities. In this section, we will begin to formulate, analyze, and solve such problems, at a simple level, to understand the many components of such a problem.

A typical **linear programming** problem consists of finding an extreme value of a linear equation subject to certain constraints. We are either trying to maximize or minimize the value of this linear equation, such as to maximize profit or revenue, or to minimize cost. That is why these linear programming problems are classified as **maximization** or **minimization problems**, or just **optimization problems**. The value we are trying to optimize is called an **objective function**, and the conditions that must be satisfied are called **constraints**.

When we graph all constraints, the area of the graph that satisfies all constraints is called the **feasible region**. The **Fundamental Theorem of Linear Programming** states that the maximum (or minimum) value of the objective function always takes place at the vertices of the feasible region. We call these vertices **critical points.** These are found using any methods from Section 1.4 as we are looking for the points where any two of the boundary lines intersect.

A typical example is to maximize profit from producing several products, subject to limitations on materials or resources needed for producing these items; the problem requires us to determine the amount of each item produced. Another type of problem involves scheduling; we need to determine how much time to devote to each of several activities in order to maximize income from (or minimize cost of) these activities, subject to limitations on time and other resources available for each activity.

In this chapter, we will work with problems that involve only two variables, and therefore, can be solved by graphing. Here are the steps we'll follow:

The Maximization Linear Programming Problems

1. Define the unknowns.
2. Write the objective function that needs to be maximized.
3. Write the constraints.
   a. For the standard maximization linear programming problems, constraints are of the form: $ax + by \leq c$
   b. When the variables represent quantities that cannot be negative, we include the constraints: $x \geq 0$ and $y \geq 0$.
4. Graph the constraints.
5. Determine the feasible region.
6. Find the corner points.
7. Find the value of the objective function at each corner point to determine the corner point that gives the maximum value.
8. State the answer to the original question.

Example 3.3.1

Niki holds two part-time jobs, Job I and Job II. She never wants to be at work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation.

If Niki makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution

1. We start by defining our unknowns.
   - Let the number of hours per week Niki will work at Job I = x.
   - Let the number of hours per week Niki will work at Job II = y.
2. Now we write the objective function. We're trying to maximize her income, so the equation will represent the calculation for finding her income. Since Niki gets paid $40 an hour at Job I, and $30 an hour at Job II, her total income I is given by the equation

$$I = 40x + 30y$$

3. Our next task is to find the constraints. Each limitation in the problem will give a constraint.
   - The second sentence in the problem states, "She never wants to work more than a total of 12 hours a week." This translates into the following constraint:

$$x + y \leq 12$$

   - The third sentence states, "For every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation." Therefore the next constraint is

$$2x + y \leq 16$$

   - The fact that x and y can never be negative is represented by the following two constraints:

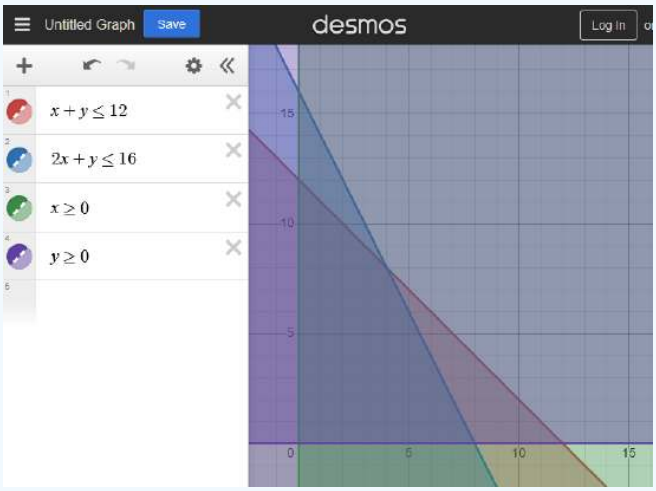
$$x \geq 0, \text{ and } y \geq 0.$$

   - In summary, our entire problem can be written as:

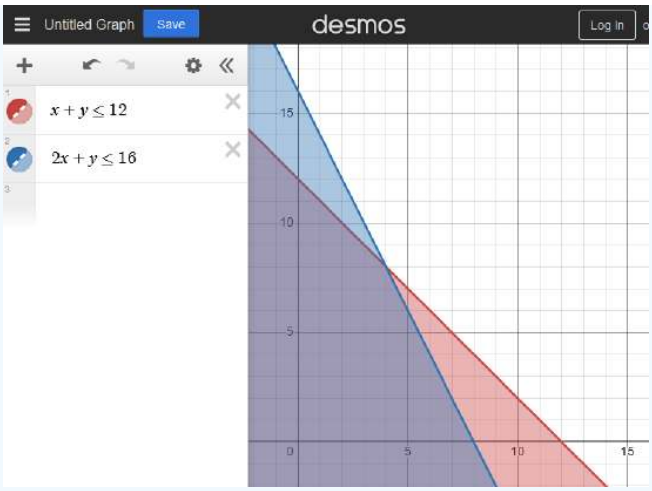
$$\begin{aligned} \textbf{Maximize} \quad & \mathrm{I} = 40\mathrm{x} + 30\mathrm{y} \\ \textbf{Subject to:} \quad & \mathrm{x} + \mathrm{y} \leq 12 \\ & 2\mathrm{x} + \mathrm{y} \leq 16 \\ & \mathrm{x} \geq 0; \mathrm{y} \geq 0 \end{aligned}$$

4. In order to solve the problem, we graph the constraints and shade the region that satisfies **all** the inequality constraints. Any appropriate method can be used to graph the lines for the constraints.
   i. Graph each inequality.
      - When doing this by hand, refer to the strategies in Section 2.2.
      - This may also be done in the Desmos app or on https://www.desmos.com/calculator. On a computer, the $\leq$ symbol can be written by typing the $<$ symbol followed by the $=$ symbol.

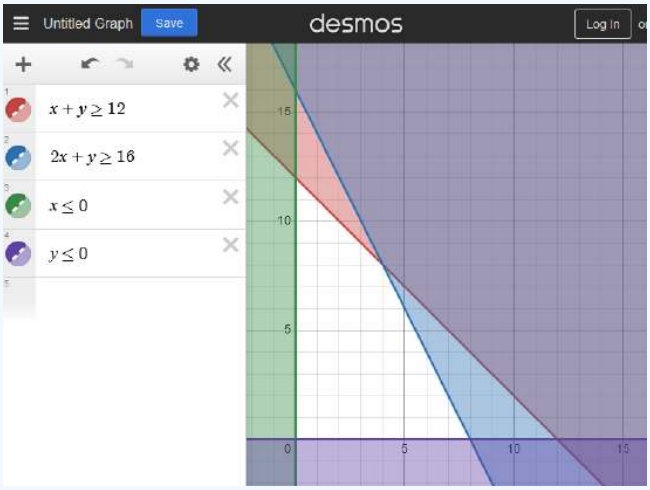
   ii. Determine which direction to shade for each line. There are three main strategies to decide how much shading is put on the graph:
      - Strategy A: Graph all inequalities exactly as written. The negative of this method is that it is often difficult to see exactly which region is covered by all colors. In the graphic below, the feasible region is the darkest gray four-sided figure in the lower left portion of the graph.



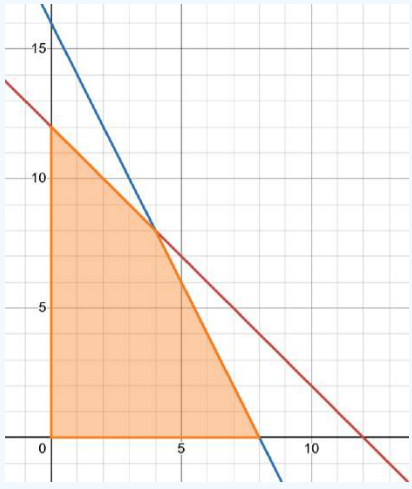
      - Strategy B: Graph ALMOST all inequalities, omitting $x \geq 0$ and $y \geq 0$. The feasible region will be slightly easier to identify, but the negative of this method is that students must remember that the feasible region actually stops at the axes.



- Strategy C: Graph the OPPOSITE of each inequality. The feasible region will be clearly visible as the void or the white space between the graphs. The negative of this strategy is that it can be easy to make a mistake when typing in the inequalities, as they will be different than what is in the problem. Often students switch some inequalities but accidentally don't switch ALL inequalities.

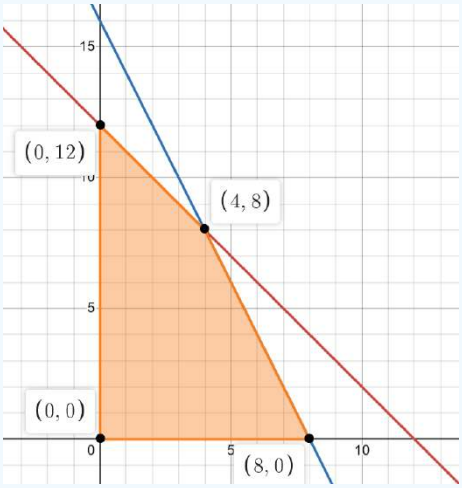


5. Determine the feasible region, which is the region that makes all inequalities true. No matter which strategy is used, the inequalities above should yield the feasible region shown below.



6. Determine the corner points of the feasible region. These are known as the critical points or critical values.

- When graphing by hand, this can be done using any of the methods discussed in Section 1.3.
- When graphing in Desmos, this can be accomplished by clicking on the point of intersection.



7. The Fundamental Theorem of Linear Programming states that the maximum (or minimum) value of the objective function always takes place at the vertices of the feasible region. Therefore, we will substitute these points in the objective function to see which point gives us the maximum value. In this case, we are looking for the highest income per week. We list the results below.

Critical Points	Income = $40x + 30y$
(0, 0)	I = 40(0) + 30(0) = \$0
(0, 12)	I = 40(0) + 30(12) = \$360
(4, 8)	I = 40(4) + 30(8) = \$400
(8, 0)	I = 40(8) + 30(0) = \$320

8. The point (4, 8) gives the most profit: \$400. Therefore, we conclude that Niki should work 4 hours at Job I, and 8 hours at Job II.

Example 3.3.2

A factory manufactures two types of gadgets, regular and premium. Each gadget requires the use of two operations, assembly and finishing, and there are at most 12 hours available for each operation. A regular gadget requires 1 hour of assembly and 2 hours of finishing, while a premium gadget needs 2 hours of assembly and 1 hour of finishing. Due to other restrictions, the company can make at most 7 gadgets a day. If a profit of \$20 is realized for each regular gadget and \$30 for a premium gadget, how many of each should be manufactured to maximize profit?

Solution

1. We start by defining our unknowns:

- Let the number of regular gadgets manufactured each day = x.
- and the number of premium gadgets manufactured each day = y.

2. Now we write the objective function. We are trying to maximize profit, which will be \$20 for however many regular gadgets we manufacture (x and \$30 for however many premium gadgets we manufacture (y. Therefore profit is is given by the equation

$$P = 20x + 30y$$

3. We now write the constraints.

- The fourth sentence states that the company can make at most 7 total gadgets a day. This translates as

$$x + y \leq 7$$

- One limitation given is the hours for the assembly operation. Since the regular gadget requires one hour of assembly and the premium gadget requires two hours of assembly, and there are at most 12 hours available for this operation, we get

$$x + 2y \leq 12$$

- Similarly, we limited by the finishing operation. The regular gadget requires two hours of finishing and the premium gadget one hour. Again, there are at most 12 hours available for finishing. This gives us the following constraint.

$$2x + y \leq 12$$

- We are also limited by reality. The fact that x and y can never be negative is represented by the following two constraints:

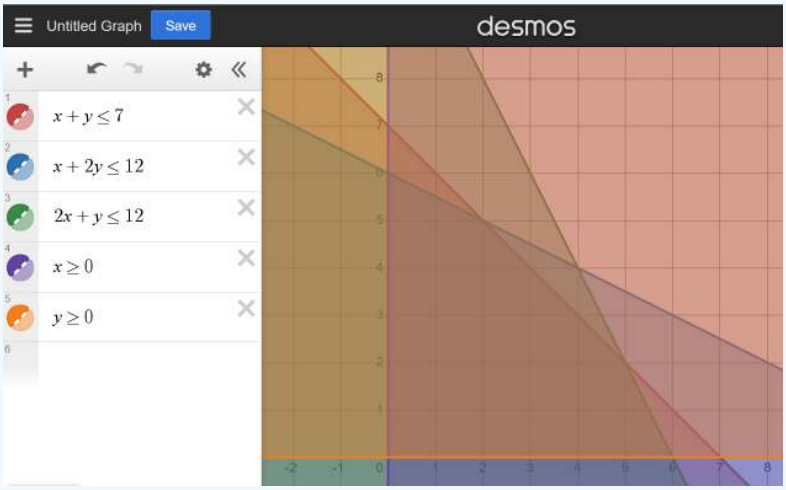
$$x \geq 0, \text{ and } y \geq 0.$$

- In summary, our entire problem can be written as follows:

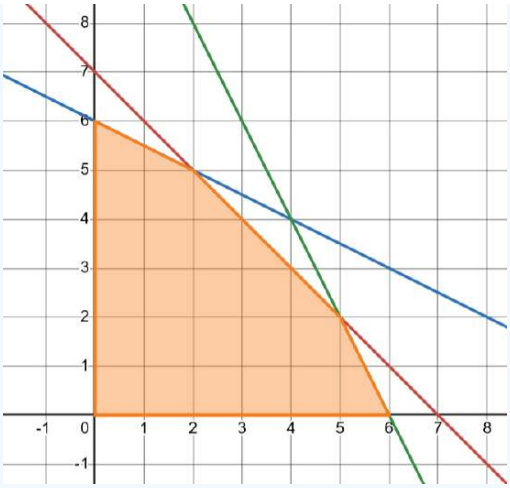
$$\begin{array}{ll} \textbf{Maximize} & \mathrm{P} = 20\mathrm{x} + 30\mathrm{y} \\ \textbf{Subject to:} & \mathrm{x} + \mathrm{y} \leq 7 \\ & \mathrm{x} + 2\mathrm{y} \leq 12 \\ & 2\mathrm{x} + \mathrm{y} \leq 12 \\ & \mathrm{x} \geq 0; \mathrm{y} \geq 0 \end{array}$$

4. In order to solve the problem, we next graph each of the constraints.

   By using Strategy A shown in the example above, we get a graph like this:

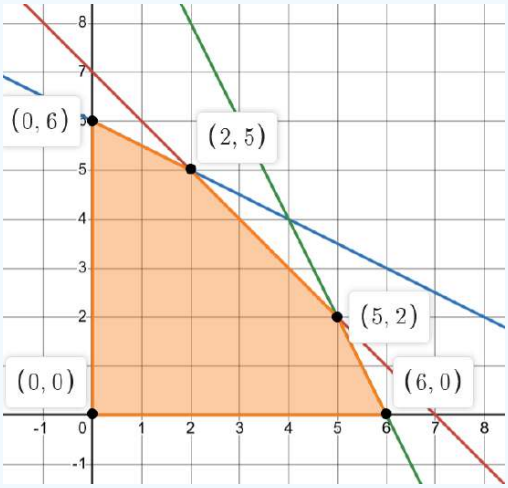


5. Determine the feasible region, which is the region that makes all inequalities true.



6. Determine the corner points of the feasible region.

- When graphing by hand, this can be done using any of the methods discussed in Section 1.3.
- When graphing in Desmos, this can be accomplished by clicking on the point of intersection.



7. We substitute these points in the objective function to see which point gives us the maximum profit each day. The results are listed below.

Critical Point	Profit = $20x + 30y$
(0, 0)	P = 20(0) + 30(0) = $0
(0, 6)	P = 20(0) + 30(6) = $180
(2, 5)	P = 20(2) + 30(5) = $190
(5, 2)	P = 20(5) + 30(2) = $160
(6,0)	P = 20(6) + 30(0) = $120

8. The point (2, 5) gives the most profit, and that profit is $190. Therefore, we conclude that we should manufacture 2 regular gadgets and 5 premium gadgets daily to obtain the maximum profit of $190.

So far we have focused on "**standard maximization problems**" in which

1. The objective function is to be maximized
2. All constraints are of the form $ax + by \leq c$

3. All variables are constrained to by non-negative ($x \geq 0$, $y \geq 0$)

We will next consider an example where that is not the case. Our next problem is said to have "**mixed constraints**", since some of the inequality constraints are of the form $ax + by \leq c$ and some are of the form $ax + by \geq c$. The non-negativity constraints are still an important requirement in any linear program.

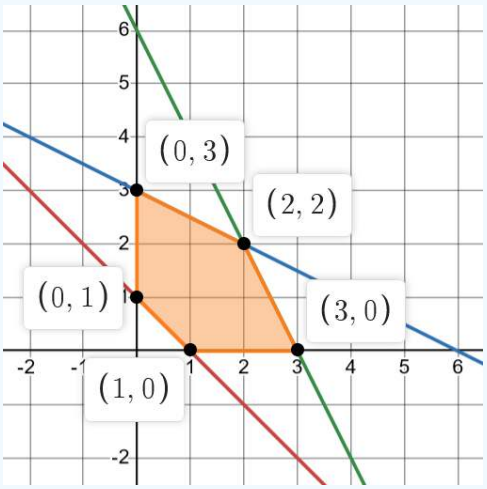
Example 3.3.3

Solve the following maximization problem graphically.

$$\begin{aligned} &\textbf{Maximize} && \mathrm{Z} = 10\mathrm{x} + 15\mathrm{y} \\ &\textbf{Subject to:} && \mathrm{x} + \mathrm{y} \geq 1 \\ &&& \mathrm{x} + 2\mathrm{y} \leq 6 \\ &&& 2\mathrm{x} + \mathrm{y} \leq 6 \\ &&& \mathrm{x} \geq 0; \mathrm{y} \geq 0 \end{aligned}$$

Solution

The graph is shown below.



The five critical points are listed in the above figure.

Critical point	Z = $10x + 15y$
(1, 0)	Z = 10(1) + 15(0) = 10
(3, 0)	Z = 10(3) + 15(0) = 30
(2, 2)	Z = 10(2) + 15(2) = 50
(0, 3)	Z = 10(0) + 15(3) = 45
(0,1)	Z = 10(0) + 15(1) = 15

The point (2, 2) maximizes the objective function to a maximum z-value of 50.

We summarize:

The Maximization Linear Programming Problems

1. Define the unknowns.
2. Write the objective function that needs to be maximized.
3. Write the constraints.
    a. For the standard maximization linear programming problems, constraints are of the form: $ax + by \leq c$

b. Since the variables are non-negative, we include the constraints: $x \geq 0; y \geq 0$.

4. Graph the constraints.
5. Determine the feasible region.
6. Find the corner points.
7. Find the value of the objective function at each corner point to determine the corner point that gives the maximum value.
8. State the answer to the original question.

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3.3E: Exercises - Linear Programming Maximization Applications

For the following maximization problems, choose your variables, write the objective function and the constraints, graph the constraints, shade the feasibility region, label all critical points, and determine the solution that optimizes the objective function.

1) A farmer has 100 acres of land on which she plans to grow wheat and corn. Each acre of wheat requires 4 hours of labor and $20 of capital, and each acre of corn requires 16 hours of labor and $40 of capital. The farmer has at most 800 hours of labor and $2400 of capital available. If the profit from an acre of wheat is $80 and from an acre of corn is $100, how many acres of each crop should she plant to maximize her profit?

2) Mr. Tran has $24,000 to invest, some in bonds and the rest in stocks. He has decided that the money invested in bonds must be at least twice as much as that in stocks. But the money invested in bonds must not be greater than $18,000. If the bonds earn 6%, and the stocks earn 8%, how much money should he invest in each to maximize profit?

3) A factory manufactures chairs and tables, each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 40 hours; the second at most 42 hours; and the third at most 25 hours. A chair requires 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; a table needs 2 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is $20 per unit for a chair and $30 for a table, how many units of each should be manufactured to maximize profit?

4) The Silly Nut Company makes two mixtures of nuts: Mixture A and Mixture B. A pound of Mixture A contains 12 oz of peanuts, 3 oz of almonds and 1 oz of cashews and sells for $4. A pound of Mixture B contains 12 oz of peanuts, 2 oz of almonds and 2 oz of cashews and sells for $5. The company has 1080 lb. of peanuts, 240 lb. of almonds, 160 lb. of cashews. How many pounds of each of mixtures A and B should the company make to maximize profit?
(Hint: Use consistent units. Work the entire problem in pounds by converting all values given in ounces into fractions of pounds).

5)

$$\begin{aligned} \text{Maximize:} \quad & Z = 4x + 10y \\ \text{Subject to:} \quad & x + y \leq 5 \\ & 2x + y \leq 8 \\ & x + 2y \leq 8 \\ & x \geq 0, y \geq 0 \end{aligned}$$

6) This maximization linear programming problem is not in “standard” form. It has mixed constraints, some involving $\leq$ inequalities and some involving $\geq$ inequalities. However with careful graphing, we can solve this using the techniques we have learned in this section.

$$\begin{aligned} \text{Maximize:} \quad & Z = 5x + 7y \\ \text{Subject to:} \quad & x + y \leq 30 \\ & 2x + y \leq 50 \\ & 4x + 3y \geq 60 \\ & 2x \geq y \\ & x \geq 0, y \geq 0 \end{aligned}$$

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3.4: Linear Programming - Minimization Applications

Learning Objectives

In this section, you will learn to:

- Formulate minimization linear programming problems.
- Graph feasible regions for minimization linear programming problems.
- Determine optimal solutions for minimization linear programming problems.

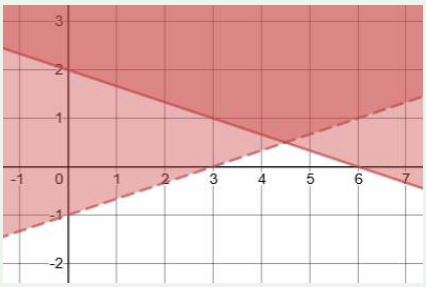
Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Graph this system of inequalities:

$$\begin{cases} x+3y \geq 6 \\ y > \frac{1}{3}x - 1 \end{cases}$$

Click here to check your answer

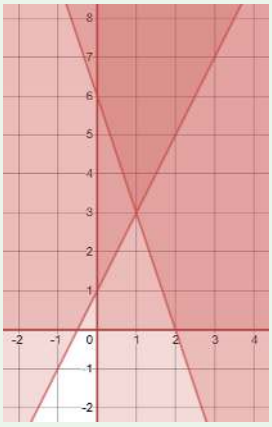


If you missed this problem, review Section 2.3. (Note that this will open a different textbook in a new window.)

2. Graph this system of inequalities:

$$\begin{cases} y \geq 2x+1 \\ y \geq -3x+6 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Click here to check your answer



(The solution is in the upper center region.)

If you missed this problem, review Section 2.3. (Note that this will open a different textbook in a new window.)

Minimization linear programming problems are solved in much the same way as the maximization problems.

For the **standard minimization linear program**, the constraints are of the form $ax + by \geq c$, as opposed to the form $ax + by \leq c$ for the standard maximization problem. As a result, the feasible solution extends indefinitely to the upper right of the first quadrant, and is unbounded. But that is not a concern, since in order to minimize the objective function, the line associated with the objective function is moved towards the origin, and the critical point that minimizes the function is closest to the origin.

However, one should be aware that in the case of an unbounded feasible region, the possibility of no optimal solution exists.

Minimization Linear Programming Problems

1. Define the unknowns.
2. Write the objective function that needs to be minimized.
3. Write the constraints.
   a. For standard minimization linear programming problems, constraints are of the form: $ax + by \geq c$
   b. Since the variables are non-negative, include the constraints: $x \geq 0$; $y \geq 0$.
4. Graph the constraints.
5. Shade the feasible region.
6. Find the corner points.
7. Find the value of the objective function at each corner point to determine the corner point that gives the minimum value.
8. State the answer to the original question.

Example 3.4.1

At a university, Professor Symons wishes to employ two people, John and Mary, to grade papers for his classes. John is a graduate student and can grade 20 papers per hour; John earns $15 per hour for grading papers. Mary is a post-doctoral associate and can grade 30 papers per hour; Mary earns $25 per hour for grading papers. Each must be employed at least one hour a week to justify their employment.

If Prof. Symons has at least 110 papers to be graded each week, how many hours per week should he employ each person to minimize the cost?

Solution

1. We define the unknowns:

- Let the number of hours per week John is employed = x.
- and the number of hours per week Mary is employed = y.

2. Write the objective function. We are trying to minimize cost, so the function should represent the total cost. In this case, cost is given by:

$$C = 15x + 25y$$

3. Find the constraints.

- The fact that each must work at least one hour each week results in the following two constraints:

$$x \geq 1$$
$$y \geq 1$$

  Note that this already includes or covers the usual constraints that x and y are non-negative, so we don't need to include additional inequalities to state this.

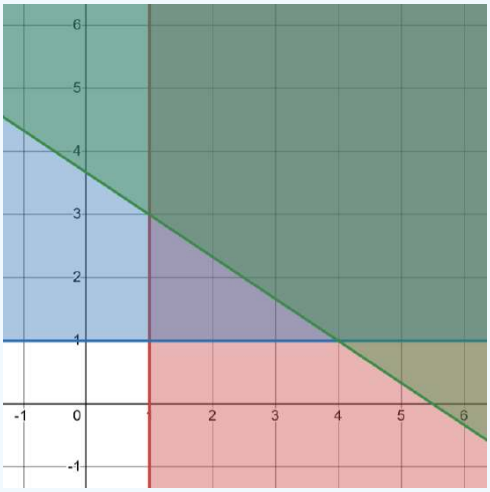
- We are limited by the minimum number of papers that must be graded. Since John can grade 20 papers per hour and Mary 30 papers per hour, and there are at least 110 papers to be graded per week, we get

$$20x + 30y \geq 110$$

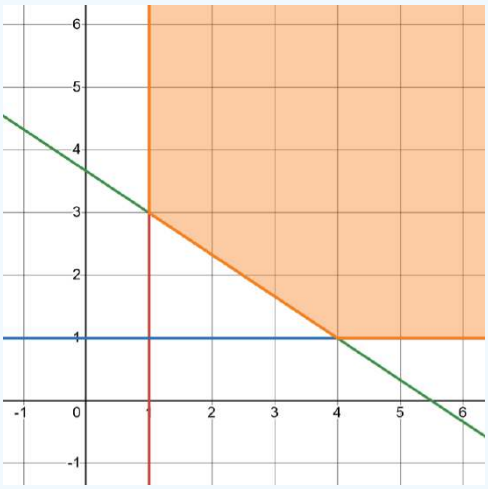
- The problem has been formulated as follows.

$$\begin{aligned} \textbf{Minimize} \quad & C = 15x + 25y \\ \textbf{Subject to:} \quad & x \geq 1 \\ & y \geq 1 \\ & 20x + 30y \geq 110 \end{aligned}$$

4. Next, we graph all constraints:

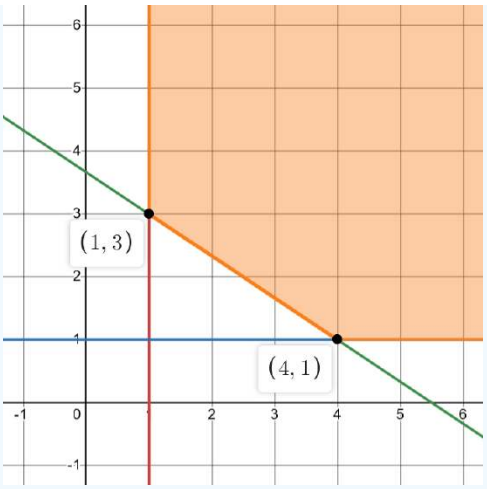


5. Determine the feasible region.



Note that this feasible region extends indefinitely up and to the right. This region is called **unbounded** as it extends without an upper bound. (In practicality, a maximum number of hours for each employee would prevent this, if we had been given this information in this problem.)

6. Determine the corner points of the feasible region.



7. Substitute these points in the objective function to see which point gives us the minimum cost each week. The results are listed below.

Critical points	Cost = $15x + 25y$
(1, 3)	C = 15(1) + 25(3) = $90
(4, 1)	C = 15(4) + 25(1) = $85

8. The point (4, 1) gives the least cost, and that cost is $85. Therefore, we conclude that in order to minimize grading costs, Professor Symons should employ John 4 hours a week, and Mary 1 hour a week at a cost of $85 per week.

Example 3.4.2

Professor Hamer is on a low cholesterol diet. During lunch at the college cafeteria, he always chooses between two meals: pasta or tofu. The table below lists the amount of protein, carbohydrates, and vitamins each meal provides along with the amount of cholesterol he is trying to minimize. Mr. Hamer needs at least 200 grams of protein, 960 grams of carbohydrates, and 40 grams of vitamin C for lunch each month. He plans to eat lunch in the cafeteria at least 20 times. How many days should he have the pasta meal, and how many days the tofu meal so that he gets the adequate amount of protein, carbohydrates, and vitamins and at the same time minimizes his cholesterol intake?

	PASTA	**TOFU**
PROTEIN	8g	16g
CARBOHYDRATES	60g	40g
VITAMIN C	2g	2g
CHOLESTEROL	60mg	50mg

Solution

1. We define the unknowns as follows.

- Let the number of days Mr. Hamer eats pasta = x.
- Let the number of days Mr. Hamer eats tofu = y.

2. The objective function represents the quantity to be minimized. Since he is trying to minimize his cholesterol intake, our objective function represents the total amount of cholesterol C provided by both meals.

$$C = 60x + 50y$$

3. Write all constraints.

- He needs to include a minimum amount of protein. The total amount of protein provided by both meals is

$$8x + 16y \geq 200$$

- Additionally, he needs to include a minimum amount of carbohydrates . The total amount of carbohydrates is

$$60x + 40y \geq 960$$

- Similarly, he needs to include a minimum amount of vitamins. The total amount of vitamin C is

$$2x + 2y \geq 40$$

- Finally, he needs to eat at least 20 meals in the cafeteria. Therefore

$$x + y \geq 20$$

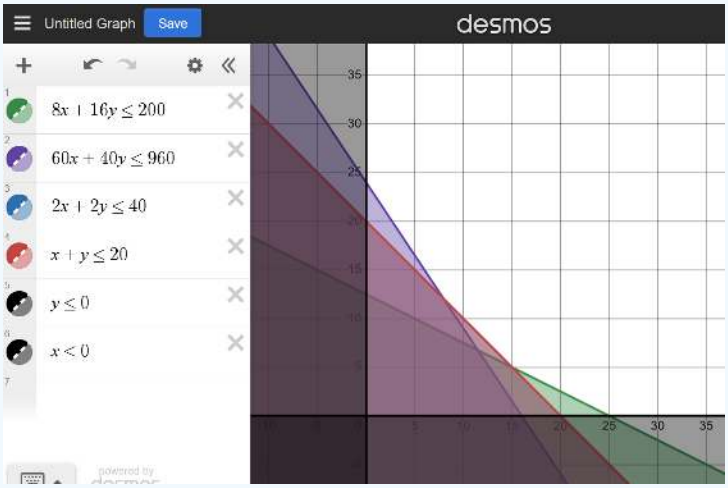
- He cannot eat a negative number of meals, so x and y are both non-negative:

$$x \geq 0, \text{ and } y \geq 0$$

- We summarize all information as follows:

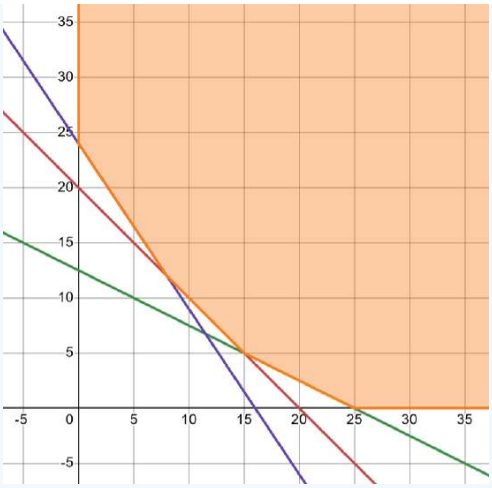
$$\begin{array}{ll} \textbf{Minimize} & \mathrm{C} = 60\mathrm{x} + 50\mathrm{y} \\ \textbf{Subject to:} & 8\mathrm{x} + 16\mathrm{y} \geq 200 \\ & 60\mathrm{x} + 40\mathrm{y} \geq 960 \\ & 2\mathrm{x} + 2\mathrm{y} \geq 40 \\ & \mathrm{x} + \mathrm{y} \geq 20 \\ & \mathrm{x} \geq 0; \mathrm{y} \geq 0 \end{array}$$

4. We graph the constraints. As this particular graph has many inequalities, graphing all of them gets quite cluttered. The graph below shows the OPPOSITE of each inequality, leaving the feasible region as the void or unshaded portion of the graph.

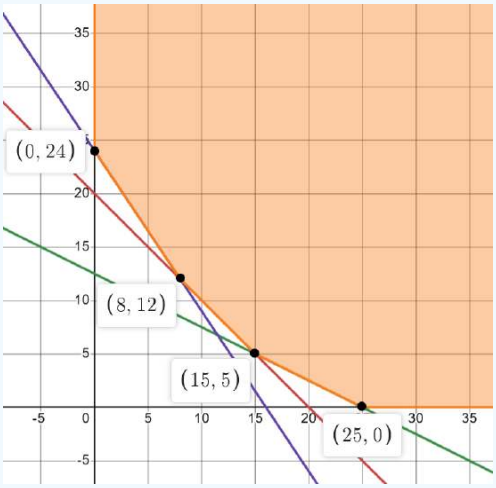


Also note that the line $2x + 2y = 40$ appears to be missing. This is because it is mathematically equivalent to the line $x + y = 20$. In essence, the blue line above is "hidden" underneath the red line.

5. Determine the feasible region.



6. Determine the corner points of the feasible region.



7. Substitute these points in the objective function to see which point gives us the smallest value. The results are listed below.

Critical points	Carbohydrates = $60x + 50y$
(0, 24)	C = 60(0) + 50(24) = 1200
(8, 12)	C = 60(8) + 50(12) = 1080
(15, 5)	C = 60(15) + 50(5) = 1150
(25, 0)	C = 60(25) + 50(0) = 1500

8. The point (8, 12) gives the least cholesterol, which is 1080 mg. This states that for every 20 meals, Professor Hamer should eat the pasta meal 8 days and the tofu meal 12 days.

We must be aware that in some cases, a linear program may not have an optimal solution.

- A linear program can fail to have an optimal solution is if there is not a feasible region. If the inequality constraints are not compatible, there may not be a region in the graph that satisfies **all** the constraints. If the linear program does not have a feasible solution satisfying all constraints, then it can not have an optimal solution.
- A linear program can fail to have an optimal solution if the feasible region is unbounded.
  - The two minimization linear programs we examined had unbounded feasible regions. The feasible region was bounded by constraints on some sides but was not entirely enclosed by the constraints. Both of the minimization problems had optimal solutions.

- However, if we were to consider a maximization problem with a similar unbounded feasible region, the linear program would have no optimal solution. No matter what values of x and y were selected, we could always find other values of x and y that would produce a higher value for the objective function. In other words, if the value of the objective function can be increased without bound in a linear program with an unbounded feasible region, there is no optimal maximum solution.

Although the method of solving minimization problems is similar to that of the maximization problems, we still feel that we should summarize the steps involved.

Minimization Linear Programming Problems

1. Define the unknowns.
2. Write the objective function that needs to be minimized.
3. Write the constraints.
   a. For standard minimization linear programming problems, constraints are of the form: $ax + by \geq c$
   b. Since the variables are non-negative, include the constraints: $x \geq 0$; $y \geq 0$.
4. Graph the constraints.
5. Shade the feasible region.
6. Find the corner points.
7. Find the value of the objective function at each corner point to determine the corner point that gives the minimum value.
8. State the answer to the original question.

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3.4.1: Exercises - Linear Programming Minimization Applications

For each of the following minimization problems, choose your variables, write the objective function and the constraints, graph the constraints, shade the feasibility region, label all critical points, and determine the solution that optimizes the objective function.

1) A diet is to contain at least 2400 units of vitamins, 1800 units of minerals, and 1200 calories. Two foods, Food A and Food B are to be purchased. Each unit of Food A provides 50 units of vitamins, 30 units of minerals, and 10 calories. Each unit of Food B provides 20 units of vitamins, 20 units of minerals, and 40 calories. Food A costs $2 per unit and Food B cost $1 per unit. How many units of each food should be purchased to keep costs at a minimum?

2) A computer store sells two types of computers, laptops and desktops. The supplier demands that at least 150 computers be sold a month. Experience shows that most consumers prefer laptops, but some business customers require desktops. The result is that the number of laptops sold is at least twice of the number of desktops. The store pays its sales staff a $60 commission for each laptop, and a $40 commission for each desktop. Let x = the number of laptops and y = the number of desktop computers. How many of each type must be sold to minimize commission to its sales people?
What is the minimum commission?

3) An oil company has two refineries. Each day, Refinery A produces 200 barrels of high-grade oil, 300 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs $12,000 to operate. Each day, Refinery B produces 100 barrels of high-grade oil, 100 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs $10,000 to operate. The company must produce at least 800 barrels of high-grade oil, 900 barrels of medium-grade oil, and 1,000 barrels of low-grade oil.
How many days should each refinery be operated to meet the goals at a minimum cost?

4) A print shop at a community college in Cupertino, California, employs two different contractors to maintain its copying machines. The print shop needs to have 12 IBM, 18 Xerox, and 20 Canon copying machines serviced. Contractor A can repair 2 IBM, 1 Xerox, and 2 Canon machines at a cost of $800 per month, while Contractor B can repair 1 IBM, 3 Xerox, and 2 Canon machines at a cost of $1000 per month. How many months should each of the two contractors be employed to minimize the cost?

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3.5: Chapter 3 Review

PROBLEM SET: CHAPTER 3 REVIEW

Linear Applications (3.1)

1. The variable cost to manufacture an item is $30 per item; the fixed costs are $2750. Find the cost function.
2. The variable cost to manufacture an item is $10 per item, and it costs $2,500 to produce 100 items. Write the cost function, and use this function to estimate the cost of manufacturing 300 items.
3. In 1990, the average house in Emerald City cost $280,000 and in 2007 the same house cost $365,000. Assuming a linear relationship, write an equation that will give the price of the house in any year, and use this equation to predict the price of a similar house in the year 2020.
4. A supply curve for a product is the number of items of the product that can be made available at different prices. A doll manufacturer can supply 2000 dolls if the dolls are sold for $30 each, but he can supply only 400 dolls if the dolls are sold for $10 each. If x represents the price of dolls and y the number of items, write an equation for the supply curve.

Fitting Linear Models to Data (3.2)

5. A random sample of 8 statistics students produced the following data, where x is the Unit 1 Exam score out of 80, and y is the Final Exam score out of 200. A random student earned a 58 on the Unit 1 Exam. If that student follows the class trend, what would you expect this student to earn on the final exam

Table showing the scores on the Final Exam based on scores from the Unit 1 Exam.

x (Unit 1 Exam score)	y (Final Exam score)
65	151
67	133
71	185
75	198
67	153
70	163
71	159
69	159

6. SCUBA divers have maximum dive times they cannot exceed when going to different depths. The data in Table show different depths with the maximum dive times in minutes. Find the linear regression line and predict the maximum dive time for 110 feet.

X (depth in feet)	Y (maximum dive time)
50	80
60	55
70	45
80	35
90	25
100	22

Maximizing and Minimizing with Linear Programming (3.3 & 3.4)

7. Dr. Lum teaches part-time at two community colleges, Hilltop College and Serra College. Dr. Lum can teach up to 5 classes per semester. For every class he teaches at Hilltop College, he needs to spend 3 hours per week preparing lessons and grading papers. For each class at Serra College, he must do 4 hours of work per week. He has determined that he cannot spend more

than 18 hours per week preparing lessons and grading papers. If he earns $6,000 per class at Hilltop College and $7,500 per class at Serra College, how many classes should he teach at each college to maximize his income, and what will be his income?

8. A company manufactures two types of cell phones, a Basic model and a Pro model. The Basic model generates a profit of $100 per phone and the Pro model has a profit of $150 per phone. On the assembly line the Basic phone requires 7 hours, while the Pro model takes 11 hours. The Basic phone requires one hour and the Pro phone needs 3 hours for finishing, which includes loading software. Both phones require one hour for testing. On a particular production run the company has available 1,540 work hours on the assembly line, 360 work hours for finishing, and 200 work hours in the testing department. How many cell phones of each type should be produced to maximize profit, and what is that maximum profit?
9. Jessica needs at least 60 units of vitamin A, 40 units of vitamin B, and 140 units of vitamin C each week. She can choose between Costless brand or Savemore brand tablets. A Costless tablet costs 5 cents and contains 3 units of vitamin A, 1 unit of vitamin B, and 2 units of vitamin C. A Savemore tablet costs 7 cents and contains 1 unit of A, 1 of B, and 5 of C. How many tablets of each kind should she buy to minimize cost, and what is the minimum cost?
10. A company produces three types of shoes, formal, casual, and athletic, at its two factories, Factory I and Factory II. The company must produce at least 6000 pairs of formal shoes, 8000 pairs of casual shoes, and 9000 pairs of athletic shoes. Daily production of each factory for each type of shoe is given below. Operating Factory I costs $1500 per day and it costs $2000 per day to operate Factory II. How many days should each factory operate to complete the order at a minimum cost, and what is the minimum cost?

	Factory I	Factory II
Formal	100	100
Casual	100	200
Athletic	300	100

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CHAPTER OVERVIEW

4: Applications of Other Functions

Learning Objectives

In this chapter, you will learn to:

- Find domain and range of functions
- Graph seven basic functions
- Evaluate piecewise defined functions
- Use transformations to sketch graphs of functions
- Graph quadratic functions using properties
- Graph polynomial functions using end-behavior
- Identify asymptotes and sketch graphs of rational functions

4.1: Functions and Function Notation

4.1E: Exercises - Functions and Function Notation

4.2: Understanding the Basic Functions

4.2E: Exercises - Understanding the Basic Functions

4.3: Understanding Transformations of Functions

4.3E: Exercises - Understanding Transformations of Functions

4.4: Quadratic Functions and Applications

4.4E: Exercises - Quadratic Functions and Applications

4.5: Chapter 4 Review

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4.1: Functions and Function Notation

Learning Objectives

By the end of this section, you will be able to:

- Find the domain and range of a relation
- Determine if a relation is a function
- Find the value of a function

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Evaluate $2x^2 - 5x + 3$ when $x = 4$.

Click here to check your answer

15

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

2. Evaluate $2x^2 - 5x + 3$ when $x = -4$.

Click here to check your answer

55

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

3. Solve $y^2 = 9$.

Click here to check your answer

$y = 3$ and $y = -3$

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

4. Express these inequalities in interval notation:

a. $x > 3$

b. $x \geq 3$

c. $x \leq 3$

d. $-2 < x \leq 5$

Click here to check your answer

a. $(3, \infty)$

b. $[3, \infty)$

c. $(-\infty, 3]$

d. $(-2, 5]$

If you missed any part of this problem, review Section 2.1. (Note that this will open in a new window.)

Find the Domain and Range of a Relation

As we go about our daily lives, we have many data items or quantities that are paired to our names. Our social security number, student ID number, email address, phone number and our birthday are matched to our name. There is a relationship between our name and each of those items.

When your professor gets her class roster, the names of all the students in the class are listed in one column and then the student ID number is likely to be in the next column. If we think of the correspondence as a set of ordered pairs, where the first element is a student name and the second element is that student's ID number, we call this a **relation**.

$$(\text{Student name},\ \text{Student ID }\#)$$

The set of all the names of the students in the class is called the **domain** of the relation and the set of all student ID numbers paired with these students is the range of the relation.

There are many similar situations where one variable is paired or matched with another. The set of ordered pairs that records this matching is a relation.

Definition: Relation

A **relation** is any set of ordered pairs, (x, y). All the x-values in the ordered pairs together make up the **domain**. All the y-values in the ordered pairs together make up the **range**.

Note that if the relation consists of a finite number of ordered pairs, then the domain and range will be listed in set notation, with {braces} around the list. If the relation consists of an infinite number of ordered pairs, then the domain and range will be expressed in interval notation, described in Section 2.1.

Example 4.1.1

For the relation $\{(1,1),(2,4),(3,9),(4,16),(5,25)\}$

a. Find the domain of the relation.
b. Find the range of the relation.

Answer

$\{(1,1),(2,4),(3,9),(4,16),(5,25)\}$

ⓐ The domain is the set of all x-values of the relation. $\{1,2,3,4,5\}$

ⓑ The range is the set of all y-values of the relation. $\{1,4,9,16,25\}$

Example 4.1.2

For the relation $\{(1,1),(2,8),(3,27),(4,64),(5,125)\}$

a. Find the domain of the relation.
b. Find the range of the relation.

Answer a

$\{1,2,3,4,5\}$

Answer b

$\{1,8,27,64,125\}$

Example 4.1.3

For the relation $\{(1,3),(2,6),(3,9),(4,12),(5,15)\}$

a. Find the domain of the relation.
b. Find the range of the relation.

Answer a

$\{1, 2, 3, 4, 5\}$

Answer b

$\{3, 6, 9, 12, 15\}$

MAPPING

A **mapping** is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

Example 4.1.4

Use the **mapping** of the relation shown to

a. list the ordered pairs of the relation,
b. find the domain of the relation, and
c. find the range of the relation.



Answer

ⓐ The arrow shows the matching of the person to their birthday. We create ordered pairs with the person's name as the x-value and their birthday as the y-value.

{(Alison, April 25), (Penelope, May 23), (June, August 2), (Gregory, September 15), (Geoffrey, January 12), (Lauren, May 10), (Stephen, July 24), (Alice, February 3), (Liz, August 2), (Danny, July 24)}

ⓑ The domain is the set of all x-values of the relation.

{Alison, Penelope, June, Gregory, Geoffrey, Lauren, Stephen, Alice, Liz, Danny}

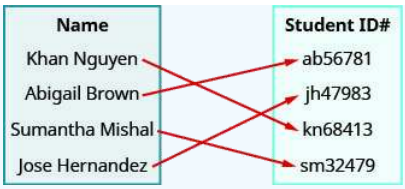
ⓒ The range is the set of all y-values of the relation.

{January 12, February 3, April 25, May 10, May 23, July 24, August 2, September 15}

Example 4.1.5

Use the mapping of the relation shown to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ (Khanh Nguyen, kn68413), (Abigail Brown, ab56781), (Sumantha Mishal, sm32479), (Jose Hernandez, jh47983)

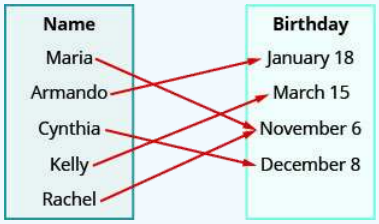
ⓑ {Khanh Nguyen, Abigail Brown, Sumantha Mishal, Jose Hernandez}

ⓒ {kn68413, ab56781, sm32479, jh47983}

Example 4.1.6

Use the mapping of the relation shown to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ (Maria, November 6), (Armando, January 18), (Cynthia, December 8), (Kelly, March 15), (Rachel, November 6)

ⓑ {Maria, Armando, Cynthia, Kelly, Rachel}

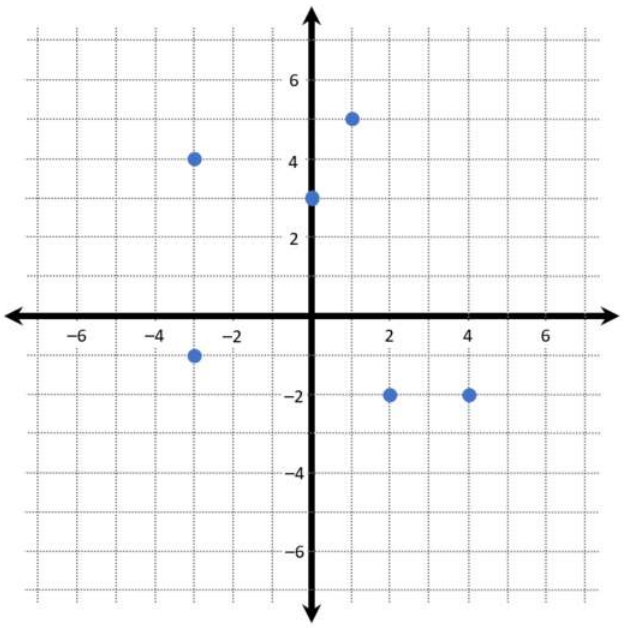
ⓒ{November 6, January 18, December 8, March 15}

A graph is yet another way that a relation can be represented. The set of ordered pairs of all the points plotted is the relation. The set of all x-coordinates is the domain of the relation and the set of all y-coordinates is the range. Generally we write the numbers in ascending order for both the domain and range.

Example 4.1.7

Use the graph of the relation to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ The ordered pairs of the relation are:

$$\{(1,5),(-3,-1),(4,-2),(0,3),(2,-2),(-3,4)\}.$$

ⓑ The domain is the set of all x-values of the relation: $\{-3,0,1,2,4\}$

Notice that while -3 repeats, it is only listed once.

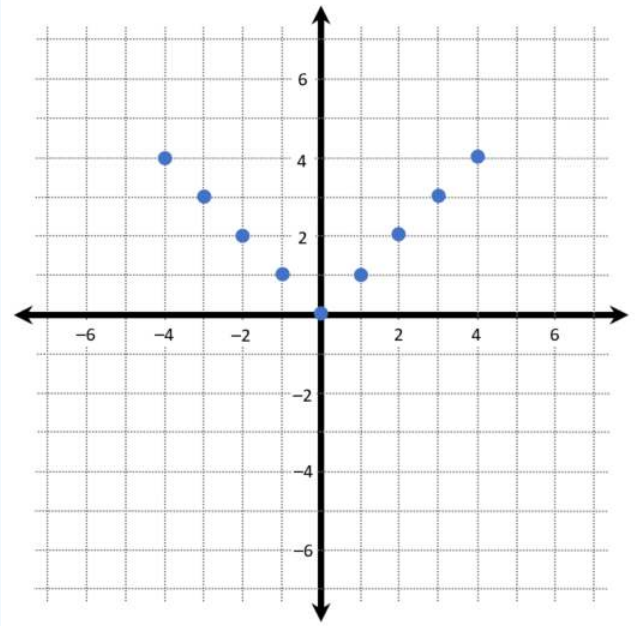
ⓒ The range is the set of all y-values of the relation: $\{-2,-1,3,4,5\}$

Notice that while -2 repeats, it is only listed once.

Example 4.1.8

Use the graph of the relation to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



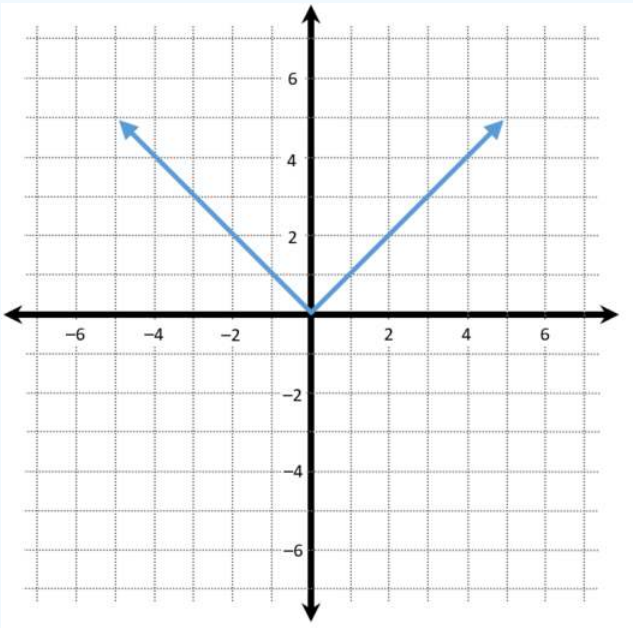
Answer

ⓐ The ordered pairs of the relation are: $(-4,4),(-3,3),(-2,2),(-1,1),(0,0),(1,1),(2,2),(3,3),(4,4)$
ⓑ The domain is the set of all *x*-values of the relation: $\{-4,-3,-2,-1,0,1,2,3,4\}$
ⓒ The range is the set of all *y*-values of the relation: $\{0,1,2,3,4\}$

Example 4.1.9

Use the graph of the relation to

a. find the domain of the relation
b. find the range of the relation.



Answer

ⓐ The domain is the set of all *x*-values of the relation. Since this includes all values from the previous graph and also all the infinite values between them, as well as those farther left and farther right, the domain is all real numbers, or $(-\infty, \infty)$.
ⓑ The range is the set of all *y*-values of the relation. Since this includes all values from the previous graph and also all the infinite values between them, as well as those above them, the range is all real numbers where $y \geq 0$, or $[0, \infty)$.

Determine if a Relation is a Function

A special type of relation, called a **function**, occurs extensively in mathematics. A function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each *x*-value is matched with only one *y*-value.

Definition: Function

A **function** is a relation that assigns to each element in its domain exactly one element in the range.

The birthday example from Example 2.1.6 helps us understand this definition. Every person has a birthday but no one has two birthdays. It is okay for two people to share a birthday. It is okay that Danny and Stephen share July 24th as their birthday and that June and Liz share August 2nd. Since each person has exactly one birthday, the relation in Example 2.1.6 is a function.

The relation shown by the graph in Example 2.1.7 includes the ordered pairs $(-3, -1)$ and $(-3, 4)$. Is that okay in a function? No, as this is like one person having two different birthdays.

Example 4.1.10

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

a. $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$
b. $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

Answer

ⓐ $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$

(i) Each *x*-value is matched with only one *y*-value. So this relation is a function.

(ii) The domain is the set of all *x*-values in the relation.
The domain is: $\{-3, -2, -1, 0, 1, 2, 3\}$

(iii) The range is the set of all *y*-values in the relation. Notice we do not list range values twice.
The range is: $\{27, 8, 1, 0\}$

ⓑ $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

(i) The *x*-value 9 is matched with two *y*-values, both 3 and -3. So this relation is not a function.

(ii) The domain is the set of all *x*-values in the relation. Notice we do not list domain values twice.
The domain is: $\{0, 1, 2, 4, 9\}$

(iii) The range is the set of all *y*-values in the relation.
The range is: $\{-3, -2, -1, 0, 1, 2, 3\}$

Example 4.1.11

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the function.

a. $\{(-3, -6), (-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4), (3, 6)\}$
b. $\{(8, -4), (4, -2), (2, -1), (0, 0), (2, 1), (4, 2), (8, 4)\}$

Answer

ⓐ Yes; $\{-3,-2,-1,0,1,2,3\}$
$\{-6,-4,-2,0,2,4,6\}$
ⓑ No; $\{0,2,4,8\}$
$\{-4,-2,-1,0,1,2,4\}$

Example 4.1.12

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

a. $\{(27,-3),(8,-2),(1,-1),(0,0),(1,1),(8,2),(27,3)\}$
b. $\{(7,-3),(-5,-4),(8,0),(0,0),(-6,4),(-2,2),(-1,3)\}$

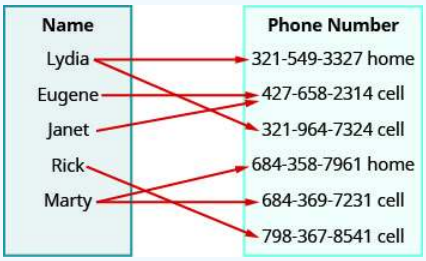
Answer

ⓐ No; $\{0,1,8,27\}$
$\{-3,-2,-1,0,2,2,3\}$
ⓑ Yes; $\{7,-5,8,0,-6,-2,-1\}$
$\{-3,-4,0,4,2,3\}$

Example 4.1.13

Use the mapping to

a. determine whether the relation is a function
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ Both Lydia and Marty have two phone numbers. So each x-value is not matched with only one y-value. So this relation is not a function.

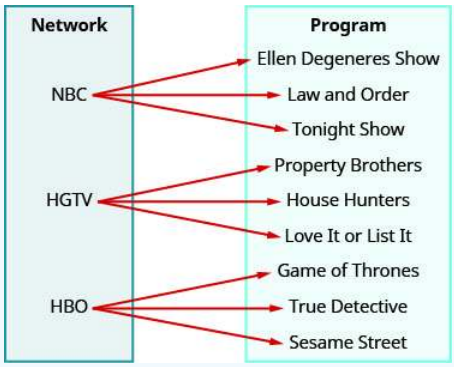
ⓑ The domain is the set of all x-values in the relation. The domain is: {Lydia, Eugene, Janet, Rick, Marty}

ⓒ The range is the set of all y-values in the relation. The range is:

$\{321-549-3327, 427-658-2314, 321-964-7324, 684-358-7961, 684-369-7231, 798-367-8541\}$

Example 4.1.14

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



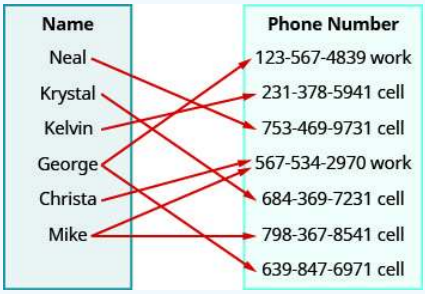
Answer

ⓐ no ⓑ {NBC, HGTV, HBO} ⓒ {Ellen Degeneres Show, Law and Order, Tonight Show, Property Brothers, House Hunters, Love it or List it, Game of Thrones, True Detective, Sesame Street}

Example 4.1.15

Use the mapping to

a. determine whether the relation is a function
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ No ⓑ {Neal, Krystal, Kelvin, George, Christa, Mike} ⓒ {123-567-4839 work, 231-378-5941 cell, 743-469-9731 cell, 567-534-2970 work, 684-369-7231 cell, 798-367-8541 cell, 639-847-6971 cell}

In algebra, more often than not, functions will be represented by an equation. It is easiest to see if the equation is a function when it is solved for y. If each value of x results in only one value of y, then the equation defines a function.

Example 4.1.16

Determine whether each equation is a function. We will explore domain and range of these relations in the upcoming sections.

a. $2x+y=7$
b. $y=x^2+1$
c. $x+y^2=3$

Answer

ⓐ $2x+y=7$

For each value of x, we multiply it by -2 and then add 7 to get the y-value

	$y=-2x+7$

For example, if $x = 3$:	$y = -2 \cdot 3 + 7$
	$y = 1$

We have that when $x = 3$, then $y = 1$. It would work similarly for any value of x. Since each value of x, corresponds to only one value of y the equation defines a function.

ⓑ $y = x^2 + 1$

For each value of x, we square it and then add 1 to get the y-value.

	$y = x^2 + 1$
For example, if $x = 2$:	$y = 2^2 + 1$
	

We have that when $x = 2$, then $y = 5$. It would work similarly for any value of x. Since each value of x, corresponds to only one value of y the equation defines a function.

ⓒ

	$x + y^2 = 3$
Isolate the y term.	$y^2 = -x + 3$
Let's substitute $x = 2$.	$y^2 = -2 + 3$
	$y^2 = 1$
This give us two values for y.	$y = 1\ y = -1$

We have shown that when $x = 2$, then $y = 1$ and $y = -1$. It would work similarly for any value of x. Since each value of x does not corresponds to only one value of y the equation does not define a function.

Example 4.1.17

Determine whether each equation is a function.

a. $4x + y = -3$
b. $x + y^2 = 1$
c. $y - x^2 = 2$

Answer

ⓐ yes ⓑ no ⓒ yes

Example 4.1.18

Determine whether each equation is a function.

a. $x + y^2 = 4$
b. $y = x^2 - 7$
c. $y = 5x - 4$

Answer

ⓐ no ⓑ yes ⓒ yes

Find the Value of a Function

It is very convenient to name a function and most often we name it *f*, *g*, *h*, *F*, *G*, or *H*. In any function, for each *x*-value from the domain we get a corresponding *y*-value in the range. For the function f, we write this range value y as $f(x)$. This is called **function notation** and is read f of x or the value of f at x. In this case the parentheses does not indicate multiplication.

Definition: Function Notation

For the function $y = f(x)$

f is the name of the function
x is the domain value
$f(x)$ is the range value y corresponding to the value x

We read $f(x)$ as f of x or the value of f at x.

We call *x* the independent variable as it can be any value in the domain. We call *y* the dependent variable as its value depends on *x*.

INDEPENDENT AND DEPENDENT VARIABLES

For the function $y = f(x)$,

x is the independent variable as it can be any value in the domain
y the dependent variable as its value depends on x

Much as when you first encountered the variable *x*, function notation may be rather unsettling. It seems strange because it is new. You will feel more comfortable with the notation as you use it.

Let's look at the equation $y = 4x - 5$. To find the value of *y* when $x = 2$, we know to substitute $x = 2$ into the equation and then simplify.

	$y = 4x - 5$
Let x=2.	$y = 4 \cdot 2 - 5$
	$y = 3$

The value of the function at $x = 2$ is 3.

We do the same thing using function notation, the equation $y = 4x - 5$ can be written as $f(x) = 4x - 5$. To find the value when $x = 2$, we write:

	$f(x) = 4x - 5$
Let x=2.	$f(2) = 4 \cdot 2 - 5$
	$f(2) = 3$

The value of the function at $x = 2$ is 3.

This process of finding the value of $f(x)$ for a given value of *x* is called *evaluating the function.*

Example 4.1.19

For the function $f(x) = 2x^2 + 3x - 1$, evaluate the function.

a. $f(3)$
b. $f(-2)$
c. $f(a)$

Answer

ⓐ

	$f(x) = 2x^2 + 3x - 1$
To evaluate $f(3)$, substitute 3 for x.	$f(3) = 2(3)^2 + 3 \cdot 3 - 1$
Simplify.	$f(3) = 2 \cdot 9 + 3 \cdot 3 - 1$
	$f(3) = 18 + 9 - 1$
	$f(3) = 26$

ⓑ

To evaluate $f(-2)$, substitute -2 for x.		$f(-2) = 2(-2)^2 + 3(-2) - 1$
Simplify.		$f(-2) = 2 \cdot 4 + (-6) - 1$
		$f(-2) = 8 + (-6) - 1$
		$f(-2) = 1$

ⓒ

	$f(x) = 2x^2 + 3x - 1$
To evaluate f(a),f(a), substitute a for x.	$f(a) = 2(a)^2 + 3 \cdot a - 1$
Simplify.	

Example 4.1.20

For the function $f(x) = 3x^2 - 2x + 1$, evaluate the function.

a. $f(3)$
b. $f(-1)$
c. $f(t)$

Answer

ⓐ $f(3) = 22$ ⓑ $f(-1) = 6$ ⓒ $f(t) = 3t^2 - 2t + 1$

Example 4.1.21

For the function $f(x) = 2x^2 + 4x - 3$, evaluate the function.

a. $f(2)$
b. $f(-3)$
c. $f(h)$

Answer

ⓐ $(2) = 13$ ⓑ $f(-3) = 3$
ⓒ $f(h) = 2h2 + 4h - 3$

In the last example, we found $f(x)$ for a constant value of x. In the next example, we are asked to find $g(x)$ with values of x that are variables. We still follow the same procedure and substitute the variables in for the x.

Example 4.1.22

For the function $g(x) = 3x - 5$, evaluate the function.

a. $g(h^2)$
b. $g(x+2)$
c. $g(x)+2$
d. $g(x)+g(2)$

Answer

a.

To evaluate $g(h^2)$, substitute h^2 for x.	$g(h^2) = 3(h^2) - 5$
Simplify (if possible).	$g(h^2) = 3h^2 - 5$

b.

To evaluate $g(x+2)$, substitute $x+2$ for x.	$g(x+2) = 3(x+2) - 5$
Simplify.	$g(x+2) = 3x + 6 - 5$
	$g(x+2) = 3x + 1$

c.

To evaluate $g(x)+2$, add 2 to the expression of $g(x)$.	$g(x)+2 = (3x-5)+2$
Simplify.	$g(x)+2 = 3x - 3$

d.

To evaluate $g(x)+g(2)$, substitute 2 for x in the second half of the expression.	$g(x)+g(2) = [g(x)] + [g(2)]$
	$g(x)+g(2) = [3x-5] + [3(2)-5]$
Simplify.	$g(x)+g(2) = (3x-5) + (1)$
	$g(x)+g(2) = 3x - 4$

Notice the difference between parts a, b, and c. Even though the questions may look very similar, the answers are not the same.

Example 4.1.23

For the function $g(x) = 4x - 7$, evaluate the function.

a. $g(m^2)$
b. $g(x-3)$
c. $g(x)-3$
d. $g(x)-g(3)$

Answer

a. $4m^2 - 7$

b. $4x - 19$

c. $4x - 10$

d. $x-12$

Example 4.1.24

For the function $h(x)=2x+1$, evaluate the function.

a. $h(k^2)$
b. $h(x+6)$
c. $h(x)+6$
d. $h(x)+h(6)$

Answer

a. $2k^2+1$

b. $2x+13$

c. $2x+7$

d. $2x+14$

Many everyday situations can be modeled using functions.

Example 4.1.25

The number of unread emails in Sylvia's account is 75. This number grows by 10 unread emails a day. The function $N(t)=75+10t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.
b. Find $N(5)$. Explain what this result means.

Answer

ⓐ The number of unread emails is a function of the number of days. The number of unread emails, *N*, depends on the number of days, *t*. Therefore, the variable *N*, is the dependent variable and the variable tt is the independent variable.

ⓑ Find $N(5)$. Explain what this result means.

	$N(t)=75+5t$
Substitute in t=5.	$N(5)=75+5(5)$
Simplify.	$N(5)=75+25$
	$N(5)=125$

Since 5 is the number of days, $N(5)$, is the number of unread emails after 5 days. After 5 days, there are 125 unread emails in the account.

Example 4.1.26

The number of unread emails in Bryan's account is 100. This number grows by 15 unread emails a day. The function $N(t)=100+15t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.
b. Find $N(7)$. Explain what this result means.

Answer

ⓐ *t* IND; *N* DEP ⓑ 205; the number of unread emails in Bryan's account on the seventh day.

Many of the scenarios from Section 3.1 can be written in function notation.

Example 4.1.27

From Example 3.1.2: A taxi service charges $0.50 per mile plus a $5 flat fee. What is the cost function?

Answer

$C(m) = 0.50m + 5$, where m represents miles driven and C represents the total cost.

Example 4.1.28

From Example 3.1.4: It costs $750 to manufacture 25 items, and $1000 to manufacture 50 items. Assuming a linear relationship holds, find the cost function.

Answer

$C(x) = 10x + 500$, where x represents the number of items manufactured and C represents the total cost.

Access this online resource for additional instruction and practice with relations and functions.

- Introduction to Functions

Key Concepts

- **Function Notation:** For the function $y = f(x)$
  - f is the name of the function
  - x is the domain value
  - $f(x)$ is the range value y corresponding to the value x
    We read $f(x)$ as f of x or the value of f at x.
- **Independent and Dependent Variables:** For the function $y = f(x)$,
  - x is the independent variable as it can be any value in the domain
  - y is the dependent variable as its value depends on x

Glossary

domain of a relation

The domain of a relation is all the x-values in the ordered pairs of the relation.

function

A function is a relation that assigns to each element in its domain exactly one element in the range.

mapping

A mapping is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

range of a relation

The range of a relation is all the y-values in the ordered pairs of the relation.

relation

A relation is any set of ordered pairs,(x,y).(x,y). All the x-values in the ordered pairs together make up the domain. All the y-values in the ordered pairs together make up the range.

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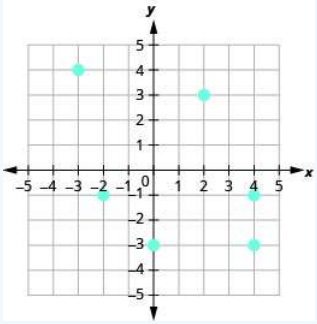
4.1E: Exercises - Functions and Function Notation

Practice Makes Perfect

In the following exercises, use the graph of the relation to:

a. list the ordered pairs of the relation
b. determine if the relation is a function
c. find the domain of the relation
d. find the range of the relation.

1.



Answer

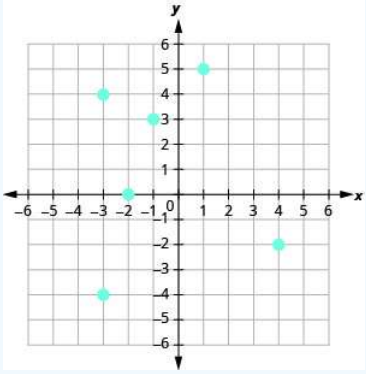
a. $(2,3),(4,-3),(-2,-1),(-3,4),(4,-1),(0,-3)$

b. no

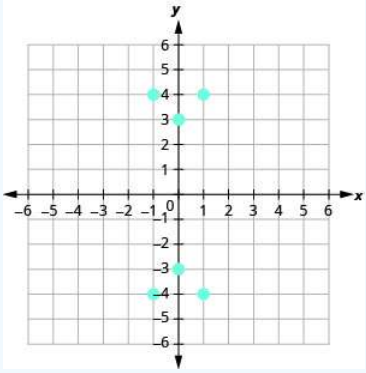
c. $\{-3,-2,0,2,4\}$

d. $\{-3,-1,3,4\}$

2.



3.



Answer

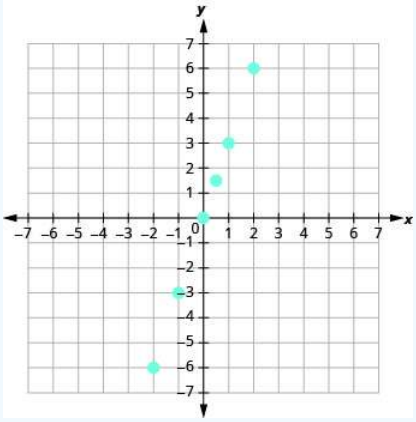
a. $(1,4),(1,-4),(-1,4),(-1,-4),(0,3),(0,-3)$

b. no

c. $\{-1,0,1\}$

d. $\{-4,-3,3,4\}$

4.



Determine if a Relation is a Function

In the following exercises, use the set of ordered pairs to:

a. determine if the relation is a function
b. find the domain of the relation
c. find the range of the relation.

5. $\{(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)\}$

Answer

a. yes

b. $\{-3,-2,-1,0,1,2,3\}$

c. $\{9,4,1,0\}$

6. $\{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$

7. $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

Answer

a. yes

b. $\{-3,-2,-1,0,1,2,3\}$

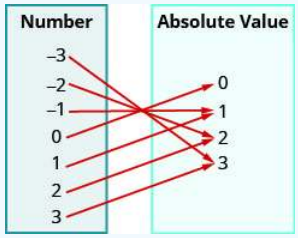
c. $\{0,1,8,27\}$

8. $\{(-3,-27),(-2,-8),(-1,-1),(0,0),(1,1),(2,8),(3,27)\}$

In the following exercises, use the mapping to:

a. determine if the relation is a function
b. find the domain of the relation
c. find the range of the relation.

9.



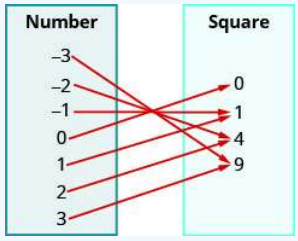
Answer

a. yes

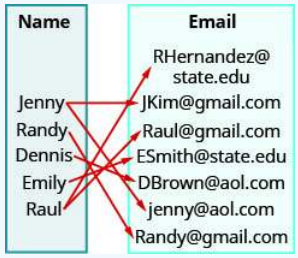
b. $\{-3,-2,-1,0,1,2,3\}$

c. $\{0,1,2,3\}$

10.



11.



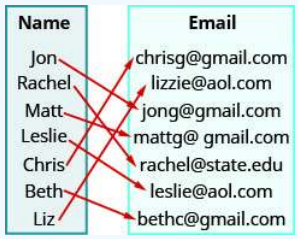
Answer

a. no

b. {Jenny, R and y, Dennis, Emily, Raul}

c. {RHernandez@state.edu, JKim@gmail.com, Raul@gmail.com, ESmith@state.edu, DBrown@aol.com, jenny@aol.cvom, Randy@gmail.com}

12.



In the following exercises, determine whether each equation is a function.

13.

a. $2x+y=-3$
b. $y=x^2$
c. $x+y^2=-5$

Answer

a. yes

b. yes

c. no

14.

a. $y=3x-5$
b. $y=x^3$
c. $2x+y^2=4$

15.

a. $y-3x^3=2$
b. $x+y^2=3$
c. $3x-2y=6$

Answer

a. yes

b. no

c. yes

16.

a. $2x-4y=8$
b. $-4=x^2-y$
c. $y^2=-x+5$

Find the Value of a Function

In the following exercises, evaluate the function: a. $f(2)$ b. $f(-1)$ c. $f(a)$.

17. $f(x)=5x-3$

Answer

a. $f(2)=7$

b. $f(-1)=-8$

c. $f(a)=5a-3$

18. $f(x)=3x+4$

19. $f(x)=-4x+2$

Answer

a. $f(2)=-6$

b. $f(-1)=6$

c. $f(a)=-4a+2$

20. $f(x)=-6x-3$

In the following exercises, evaluate the function: a. $g(h^2)$ b. $g(x+2)$ c. $g(x)+g(2)$.

21. $g(x)=2x+1$

Answer

a. $g(h^2)=2h^2+1$
b. $g(x+2)=4x+5$
c. $g(x)+g(2)=2x+6$

22. $g(x)=5x-8$

23. $g(x)=3-x$

Answer

a. $g(h^2)=3-h^2$
b. $g(x+2)=1-x$
c. $g(x)+g(2)=4-x$

24. $g(x)=7-5x$

In the following exercises, evaluate the function.

25. $f(x)=3x^2-5x$; $f(2)$

Answer

2

26. $g(x)=4x^2-3x$; $g(3)$

27. $f(x)=2x^2-3x+1$; $f(-1)$

Answer

6

28. $g(x) = 3x^2 - 5x + 2$; $g(-2)$

In the following exercises, solve.

29. The number of unwatched shows in Sylvia's DVR is 85. This number grows by 20 unwatched shows per week. The function $N(t) = 85 + 20t$ represents the relation between the number of unwatched shows, *N*, and the time, *t*, measured in weeks.

a. Determine the independent and dependent variable.

b. Find $N(4)$. Explain what this result means

Answer

a. *t* IND; *N* DEP
b. $N(4) = 165$ the number of unwatched shows in Sylvia's DVR at the fourth week.

30. Every day a new puzzle is downloaded into Ken's account. Right now he has 43 puzzles in his account. The function $N(t) = 43 + t$ represents the relation between the number of puzzles, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.

b. Find $N(30)$. Explain what this result means.

31. The daily cost to the printing company to print a book is modeled by the function $C(x) = 3.25x + 1500$ where *C* is the total daily cost and *x* is the number of books printed.

a. Determine the independent and dependent variable.

b. Find $N(0)$. Explain what this result means.

c. Find $N(1000)$. Explain what this result means.

Answer

a. *x* IND; *C* DEP
b. $N(0) = 1500$ the daily cost if no books are printed
c. $N(1000) = 4750$the daily cost of printing 1000 books

32. The daily cost to the manufacturing company is modeled by the function $C(x) = 7.25x + 2500$ where $C(x)$ is the total daily cost and *x* is the number of items manufactured.

a. Determine the independent and dependent variable.

b. Find $C(0)$. Explain what this result means.

c. Find $C(1000)$. Explain what this result means.

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4.2: Understanding the Basic Functions

Learning Objectives

- Define and graph seven basic functions.
- Define and graph piecewise functions.
- Evaluate piecewise defined functions.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Simplify $(-3)^2$ without using a calculator.

Click here to check your answer

9

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

2. Simplify $(-3)^3$ without using a calculator.

Click here to check your answer

-27

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

3. Simplify $|-3|$.

Click here to check your answer

3

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

4. Simplify $\sqrt{4}$.

Click here to check your answer

2

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

5. Simplify $\dfrac{1}{1/3}$.

Click here to check your answer

3

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

Basic Functions

In this section we graph seven basic functions that will be used throughout this course. Each function is graphed by plotting points. Remember that $f(x) = y$ and thus $f(x)$ and y can be used interchangeably.

Any function of the form $f(x) = c$, where c is any real number, is called a **constant function**[43]. Constant functions are linear and can be written $f(x) = 0x + c$. In this form, it is clear that the slope is 0 and the y-intercept is $(0, c)$. Evaluating any value for x, such as $x = 2$, will result in c.

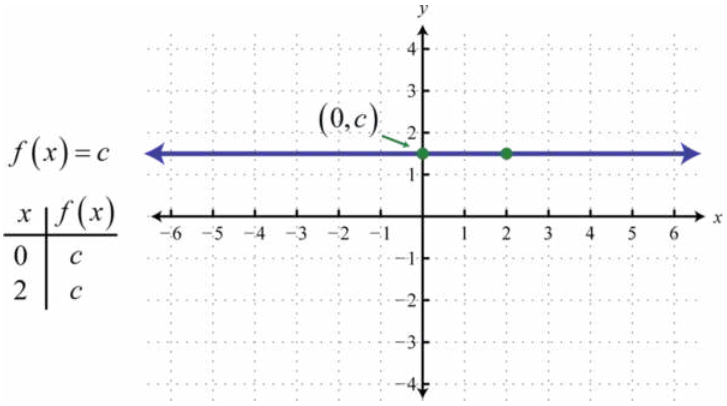


Figure 2.4.1

The graph of a constant function is a horizontal line. The domain is $(-\infty, \infty)$ and the range consists of the single value $\{c\}$.

We next define the **identity function**[44] $f(x) = x$. Evaluating any value for x will result in that same value. For example, $f(0) = 0$ and $f(2) = 2$. The identity function is linear, $f(x) = 1x + 0$, with slope $m = 1$ and y-intercept $(0, 0)$.

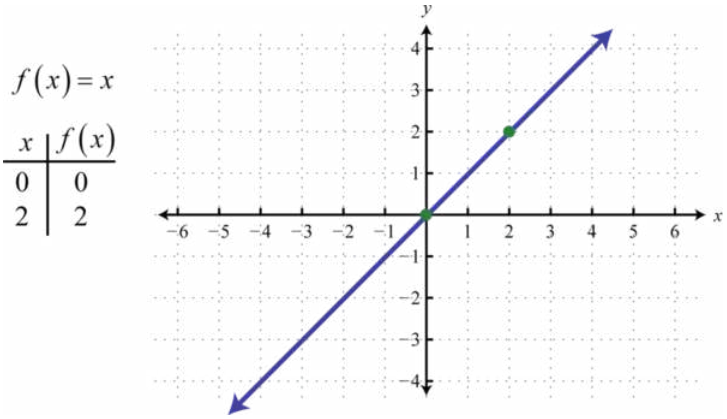


Figure 2.4.2

The domain and range both consist of all real numbers, $(-\infty, \infty)$.

The **squaring function**[45], defined by $f(x) = x^2$, is the function obtained by squaring the values in the domain. For example, $f(2) = (2)^2 = 4$ and $f(-2) = (-2)^2 = 4$.The result of squaring nonzero values in the domain will always be positive.

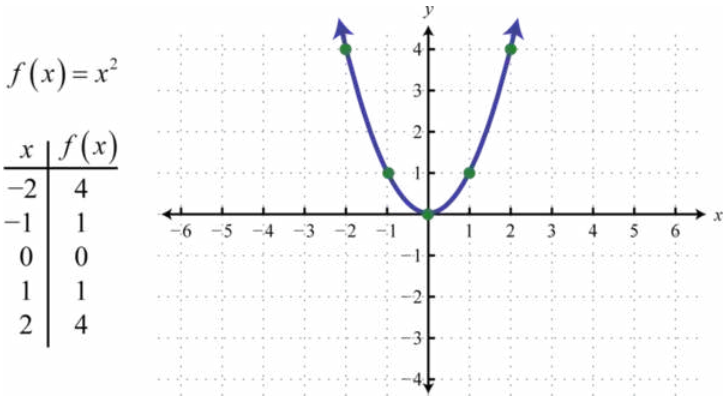


Figure 2.4.3

The resulting curved graph is called a **parabola**[46]. The domain is $(-\infty, \infty)$ and the range is $[0, \infty)$. We will look at parabolas more in depth in Section 2.4.

The **cubing function**[47], defined by $f(x)=x^3$, raises all of the values in the domain to the third power. The results can be either negative, zero, or positive. For example, $f(-1)=(-1)^3=-1$, $f(0)=(0)^3=0$, and $f(1)=(1)^3=1$.

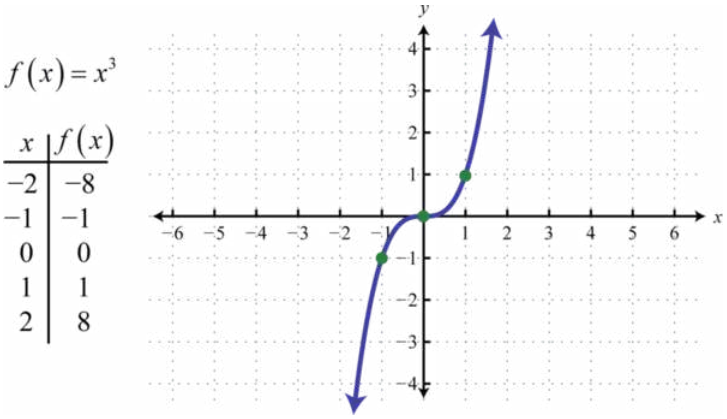


Figure 2.4.4

The domain and range both consist of all real numbers, $(-\infty,\infty)$.

Note that the constant, identity, squaring, and cubing functions are all examples of basic polynomial functions. The next three basic functions are not polynomials.

The **absolute value function**[48], defined by $f(x)=|x|$, is a function where the output represents the distance to the origin on a number line. The result of evaluating the absolute value function for any nonzero value of x will always be positive. For example, $f(-2)=|-2|=2$ and $f(2)=|2|=2$.

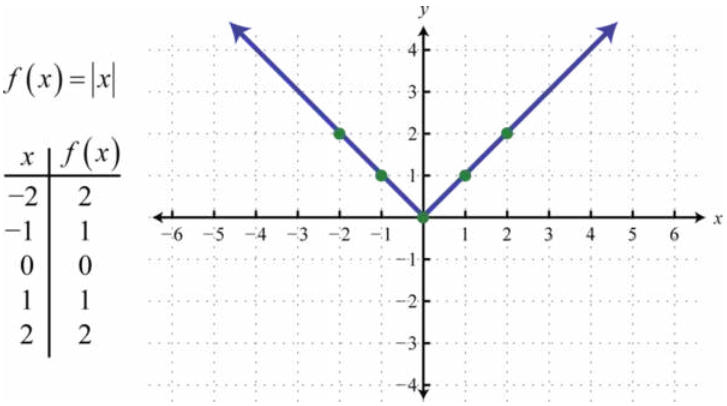


Figure 2.4.5

Like the parabola, the domain is $(-\infty,\infty)$ and the range is $[0,\infty)$.

The **square root function**[49], defined by $f(x)=\sqrt{x}$, is not defined to be a real number if the x-values are negative. Therefore, the smallest value in the domain is zero. For example, $f(0)=\sqrt{0}=0$ and $f(4)=\sqrt{4}=2$.

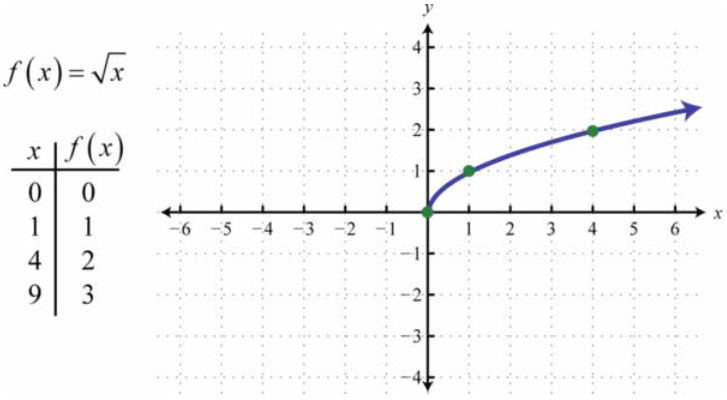


Figure 2.4.6

The domain and range both consist of real numbers greater than or equal to zero $[0, \infty)$.

The **reciprocal function**[50], defined by $f(x) = \frac{1}{x}$, is a rational function with one restriction on the domain, namely $x \neq 0$. The reciprocal of an x-value very close to zero is very large. For example,

$$f(1/10) = \frac{1}{\left(\frac{1}{10}\right)} = 1 \cdot \frac{10}{1} = 10$$
$$f(1/100) = \frac{1}{\left(\frac{1}{100}\right)} = 1 \cdot \frac{100}{1} = 100$$
$$f(1/1,000) = \frac{1}{\left(\frac{1}{1,000}\right)} = 1 \cdot \frac{1,000}{1} = 1,000$$

In other words, as the x-values approach zero their reciprocals will tend toward either positive or negative infinity. This describes a **vertical asymptote**[51] at the y-axis. Furthermore, where the x-values are very large the result of the reciprocal function is very small.

$$f(10) = \frac{1}{10} = 0.1$$
$$f(100) = \frac{1}{100} = 0.01$$
$$f(1000) = \frac{1}{1,000} = 0.001$$

In other words, as the x-values become very large the resulting y-values tend toward zero. This describes a **horizontal asymptote**[52] at the x-axis. After plotting a number of points the general shape of the reciprocal function can be determined.

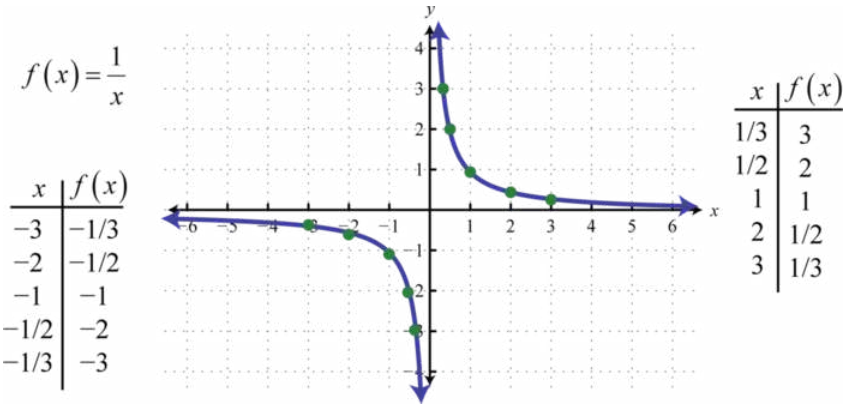


Figure 2.4.7

Both the domain and range of the reciprocal function consists of all real numbers except 0,or $(-\infty, 0) \cup (0, \infty)$. We will look at the reciprocal function more in depth in Section 2.6.

In summary, the basic polynomial functions are:

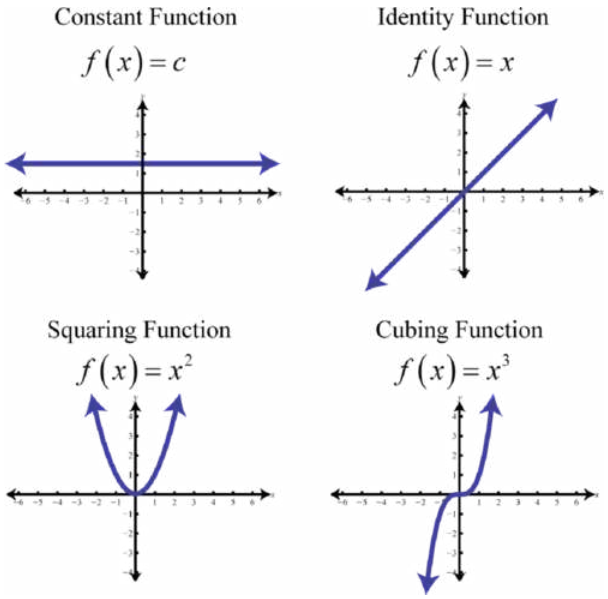


Figure 2.4.8

The basic nonpolynomial functions are:

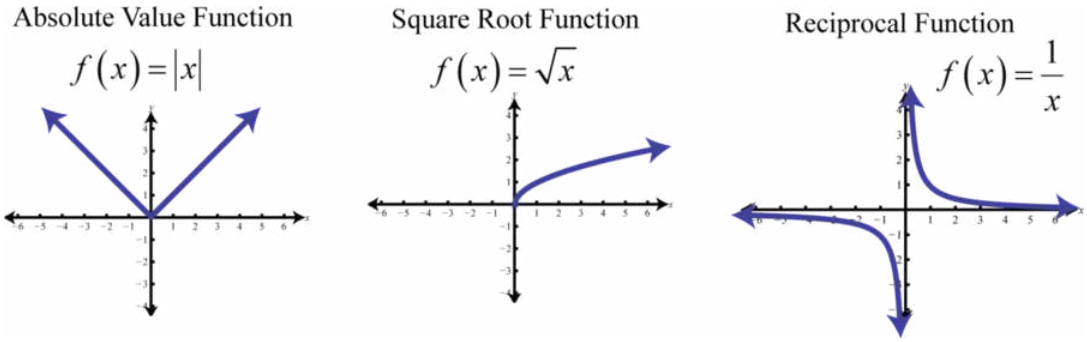


Figure 2.4.9

Piecewise Defined Functions

A **piecewise function**[53], or **split function**[54], is a function whose definition changes depending on the value in the domain. For example, we can write the absolute value function $f(x)=|x|$ as a piecewise function:

$$f(x)=|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$$

In this case, the definition used depends on the sign of the x-value. If the x-value is positive, $x\geq 0$, then the function is defined by $f(x)=x$. And if the x-value is negative, $x<0$, then the function is defined by $f(x)=-x$.

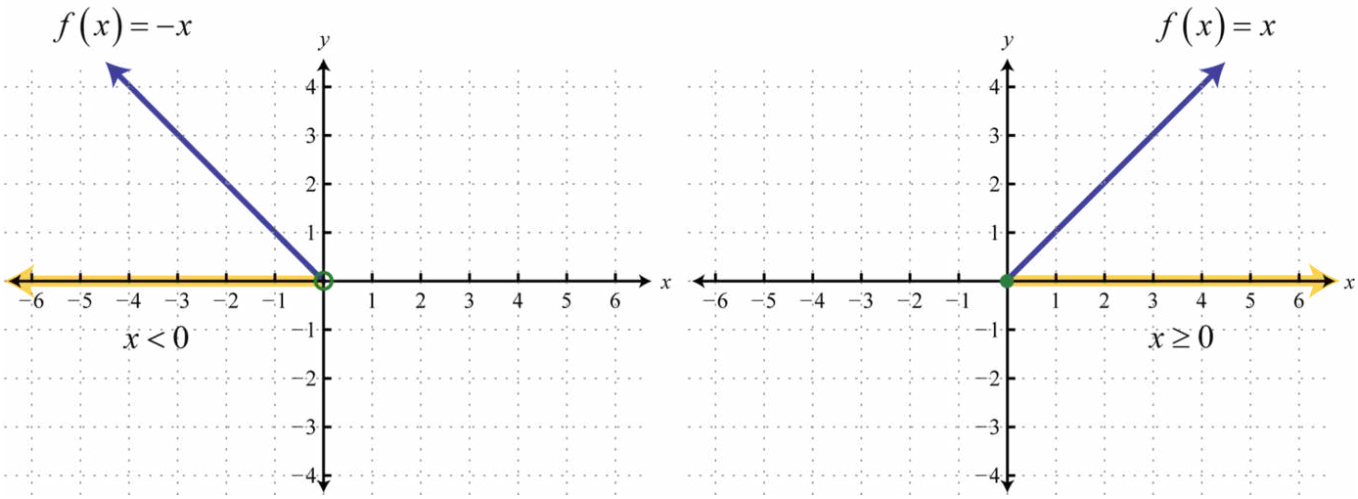


Figure 2.4.10

Following is the graph of the two pieces on the same rectangular coordinate plane:

$$f(x) = |x|$$

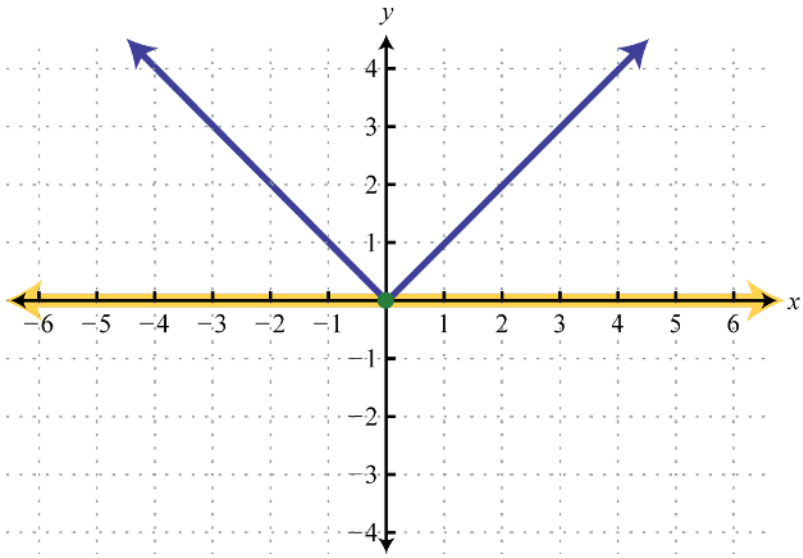


Figure 2.4.11

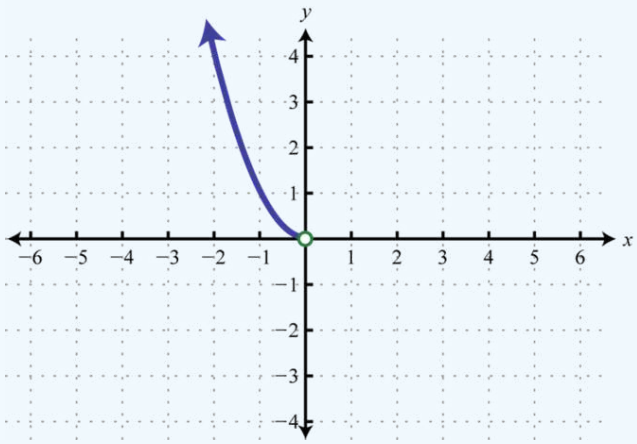
Example 4.2.1:

Graph: $g(x) = \begin{cases} x^2 & \text{if} \quad x < 0 \\ \sqrt{x} & \text{if} \quad x \geq 0 \end{cases}$.

Solution

In this case, we graph the squaring function over negative x-values and the square root function over positive x-values.

$$g(x)=\begin{cases} x^2 & if \quad x<0 \\ \sqrt{x} & if \quad x\geq 0 \end{cases}$$



$$g(x)=\begin{cases} x^2 & if \quad x<0 \\ \sqrt{x} & if \quad x\geq 0 \end{cases}$$

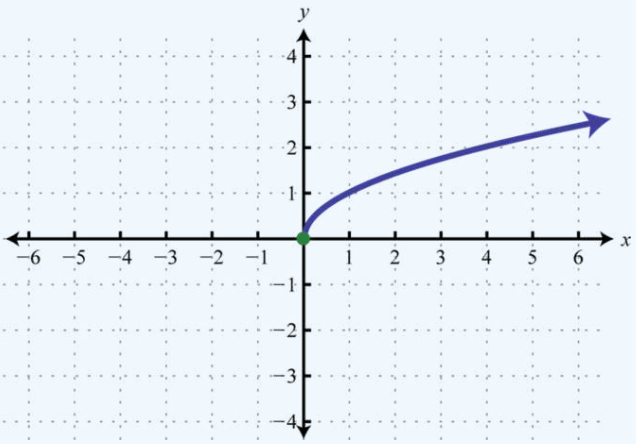


Figure 2.4.12

Notice the open dot used at the origin for the squaring function and the closed dot used for the square root function. This was determined by the inequality that defines the domain of each piece of the function. The entire function consists of each piece graphed on the same coordinate plane.

Answer:

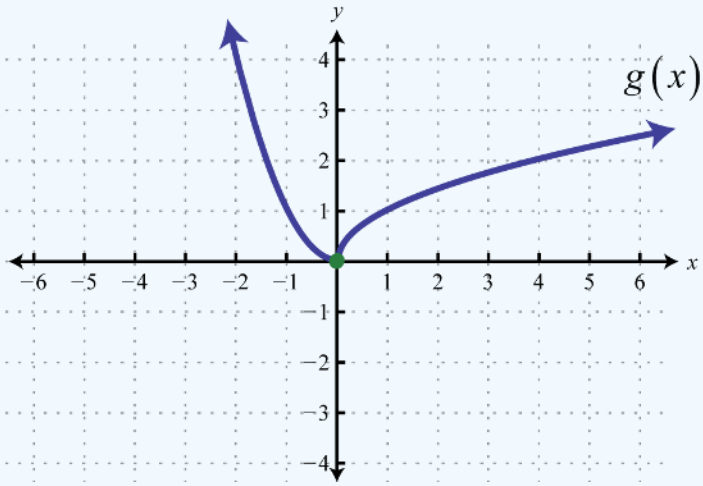


Figure 2.4.13

When evaluating, the value in the domain determines the appropriate definition to use.

Example 4.2.2:

Given the function $h(t)=\begin{cases} 7t+3 & \text{if } t<0 \\ -16t^2+32t & \text{if } t\geq 0 \end{cases}$, find $h(-5), h(0)$, and $h(3)$.

Solution

Use $h(t)=7t+3$ where t is negative, as indicated by $t<0$.

$$\begin{aligned} h(t) &= 7t+5 \\ h(-5) &= 7(-5)+3 \\ &= -35+3 \\ &= -32 \end{aligned}$$

Where t is greater than or equal to zero, use $h(t)=-16t^2+32t$.

$$h(0) = -16(0) + 32(0) \text{ and } h(3) = 16(3)^2 + 32(3)$$
$$= 0 + 0 \qquad\qquad = -144 + 96$$
$$= 0 \qquad\qquad = -48$$

Answer:

$h(-5) = -32, h(0) = 0, \text{ and } h(3) = -48$

Exercise 4.2.1

Graph: $f(x) = \begin{cases} \frac{2}{3}x + 1 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$.

Answer

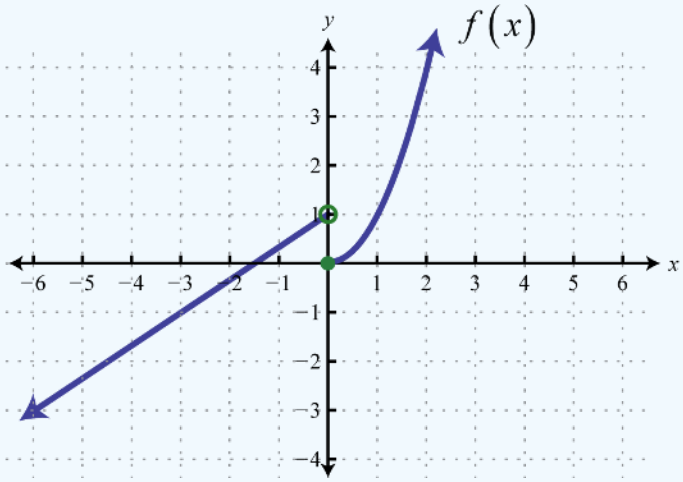


Figure 2.4.14

www.youtube.com/v/0hEUnSN5Blw

The definition of a function may be different over multiple intervals in the domain.

Example 4.2.3:

Graph: $f(x) = \begin{cases} x^3 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 4 \\ 6 & \text{if } x > 4 \end{cases}$.

Solution

In this case, graph the cubing function over the interval $(-\infty, 0)$. Graph the identity function over the interval $[0, 4]$. Finally, graph the constant function $f(x) = 6$ over the interval $(4, \infty)$. And because $f(x) = 6$ where $x > 4$, we use an open dot at the point $(4, 6)$. Where $x = 4$, we use $f(x) = x$ and thus $(4, 4)$ is a point on the graph as indicated by a closed dot.

Answer:

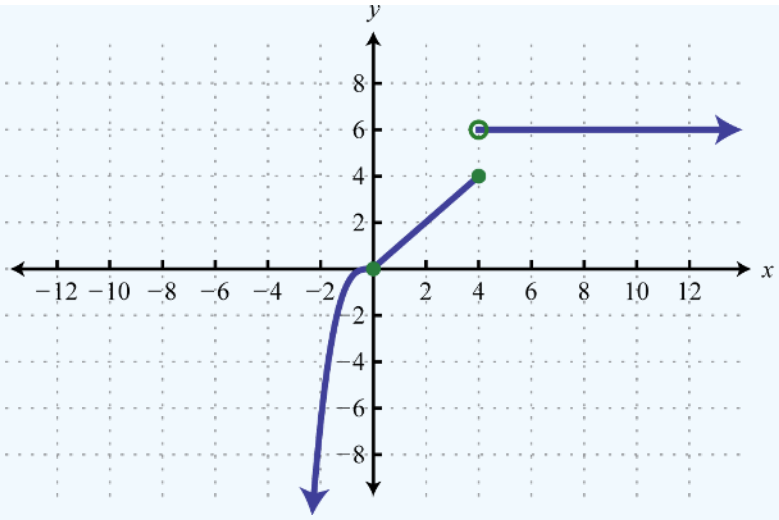


Figure 2.4.15

Many applications can be represented by a piecewise function.

Example 4.2.4:

A drive-in movie theater charges \$3 per person if there are fewer than 4 people in the car, or a flat fee of \$10 if there are 4 or more people in the car.

This can be modeled by $C(p) = \begin{cases} 3p & \text{if } p < 4 \\ 10 & \text{if } p \geq 4 \end{cases}$ where C is the cost for p people. Find and interpret each of the following.

a. $C(5)$

b. $C(4)$

c. $C(3)$

Solution

a. $C(5)$ tells us to find the cost when $p = 5$, or when there are 5 people in the car. Since $p \geq 4$, we will use the bottom equation.

$C(5) = 10$ so the cost will be \$10.

b. $C(4)$ tells us to find the cost when $p = 4$, or when there are 4 people in the car. Again, since $p \geq 4$, we will use the bottom equation.

$C(4) = 10$ so the cost will be \$10.

c. $C(3)$ tells us to find the cost when $p = 3$, or when there are 3 people in the car. Since $p < 4$, we will use the top equation.

$C(3) = 3(3)$ so the cost will be \$9.

Key Takeaways

- Plot points to determine the general shape of the basic functions. The shape, as well as the domain and range, of each should be memorized.
- The basic polynomial functions are: $f(x) = c$, $f(x) = x$, $f(x) = x^2$, and $f(x) = x^3$.
- The basic nonpolynomial functions are: $f(x) = |x|$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$.
- A function whose definition changes depending on the value in the domain is called a piecewise function. The value in the domain determines the appropriate definition to use.

Footnotes

[43]Any function of the form $f(x) = c$ where c is a real number.

[44]The linear function defined by $f(x) = x$.

[45]The quadratic function defined by $f(x) = x^2$.

[46]The curved graph formed by the squaring function.

[47]The cubic function defined by $f(x) = x^3$.

[48]The function defined by $f(x) = |x|$.

[49]The function defined by $f(x) = \sqrt{x}$.

[50]The function defined by $f(x) = \frac{1}{x}$.

[51]A vertical line to which a graph becomes infinitely close.

[52]A horizontal line to which a graph becomes infinitely close where the x-values tend toward $\pm\infty$.

[53]A function whose definition changes depending on the values in the domain.

[54]A term used when referring to a piecewise function.

[55]The function that assigns any real number x to the greatest integer less than or equal to x denoted $f(x) = [x]$.

[56]A term used when referring to the greatest integer function.

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4.2E: Exercises - Understanding the Basic Functions

Exercise 4.2E.1

Match the graph to the function definition.

a.

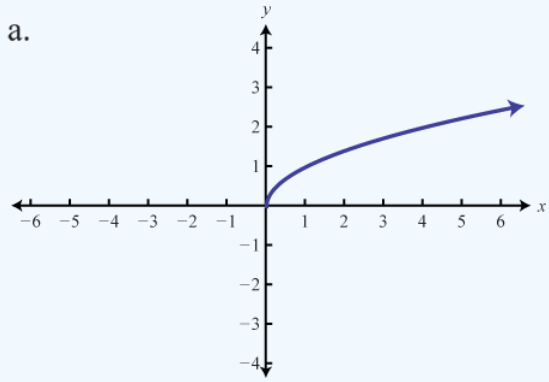


Figure 2.4.17

b.

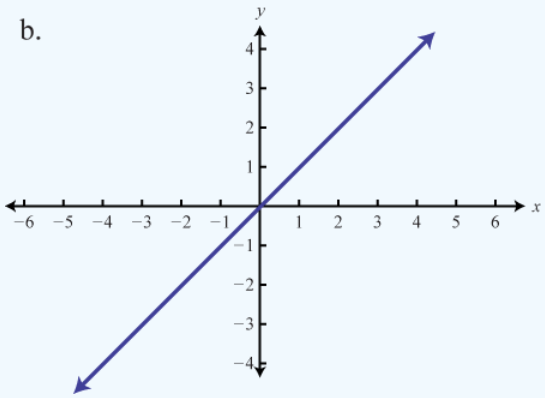


Figure 2.4.18

c.

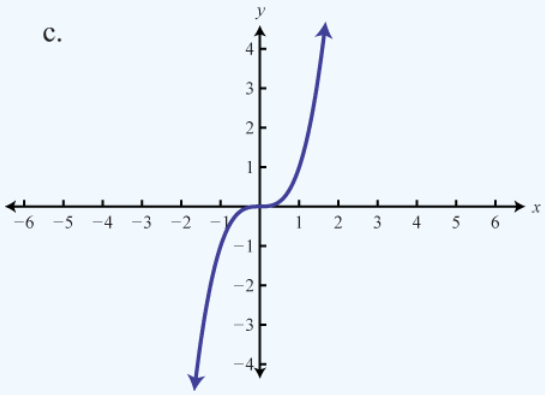


Figure 2.4.19

d.

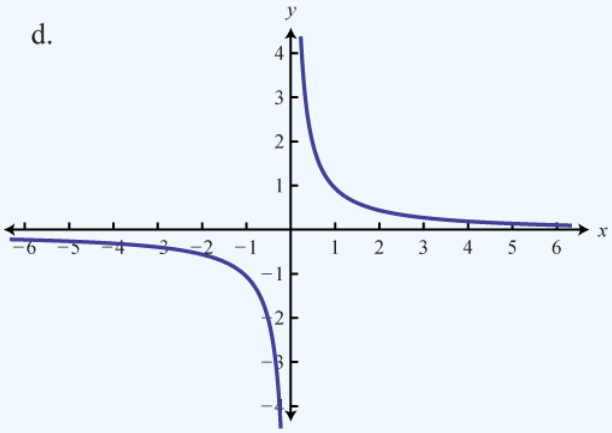


Figure 2.4.20

e.

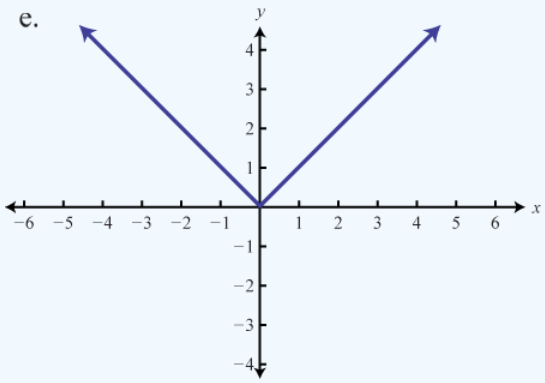


Figure 2.4.21

f.

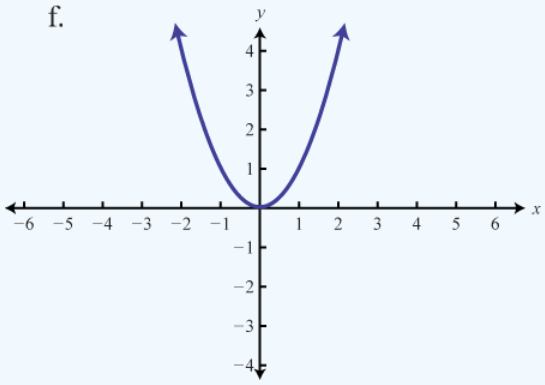


Figure 2.4.22

1. $f(x) = x$
2. $f(x) = x^2$
3. $f(x) = x^3$
4. $f(x) = |x|$
5. $f(x) = \sqrt{x}$
6. $f(x) = \frac{1}{x}$

Answer

1. b
3. c
5. a

Exercise $4.2E.2$

Evaluate.

1. $f(x) = x$; find $f(-10)$, $f(0)$, and $f(a)$.
2. $f(x) = x^2$; find $f(-10)$, $f(0)$, and $f(a)$.
3. $f(x) = x^3$; find $f(-10)$, $f(0)$, and $f(a)$.
4. $f(x) = |x|$; find $f(-10)$, $f(0)$, and $f(a)$.
5. $f(x) = \sqrt{x}$; find $f(25)$, $f(0)$, and $f(a)$ where $a \geq 0$.
6. $f(x) = \frac{1}{x}$; find $f(-10)$, $f(\frac{1}{5})$, and $f(a)$ where $a \neq 0$.
7. $f(x) = 5$; find $f(-10)$, $f(0)$, and $f(a)$.
8. Graph $f(x) = 5$ and state its domain and range.

Answer

1. $f(-10) = -10, f(0) = 0, f(a) = a$
3. $f(-10) = -1,000, f(0) = 0, f(a) = a^3$
5. $f(25) = 5, f(0) = 0, f(a) = \sqrt{a}$
7. $f(-10) = 5, f(0) = 5, f(a) = 5$
8. Domain: $(-\infty, \infty)$; range $\{5\}$

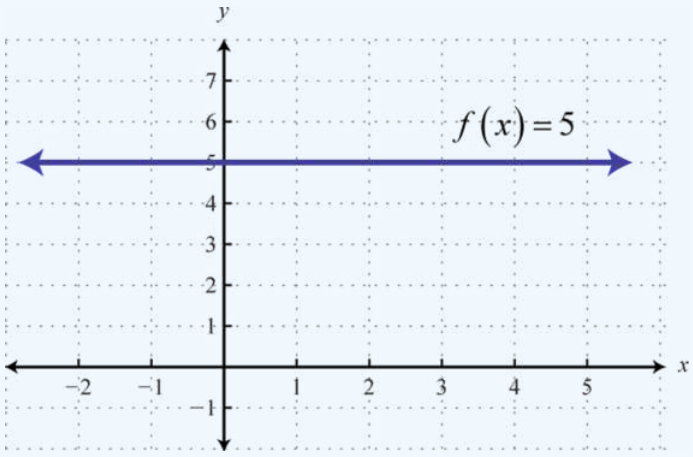


Figure 2.4.23

Exercise $4.2E.3$

Find the ordered pair that specifies the point P.

1.

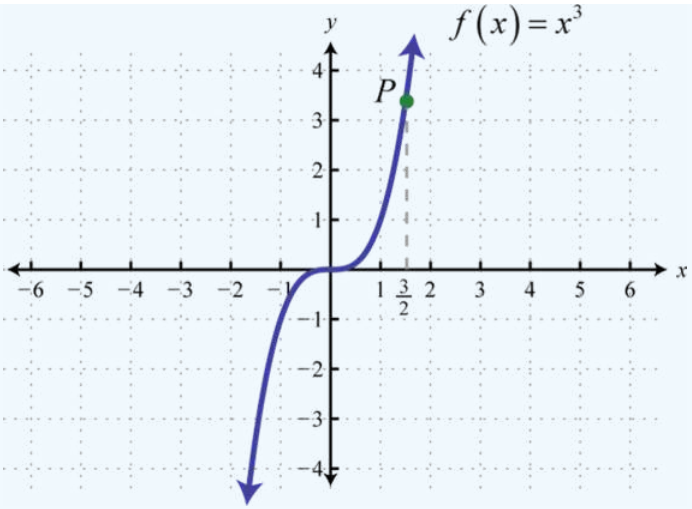


Figure 2.4.25

2.

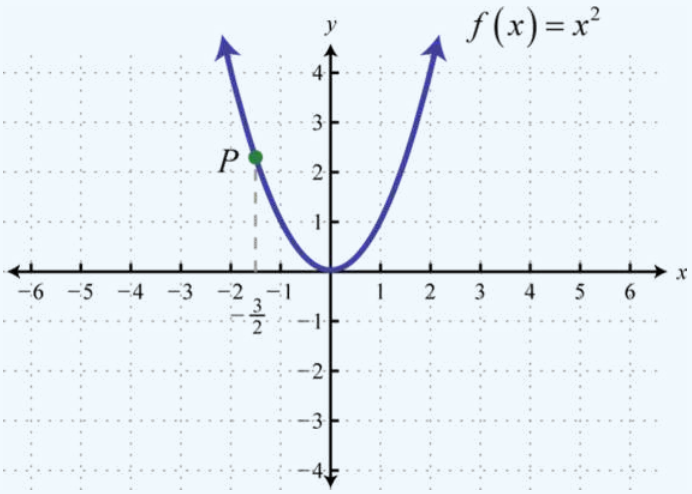


Figure 2.4.26

3.

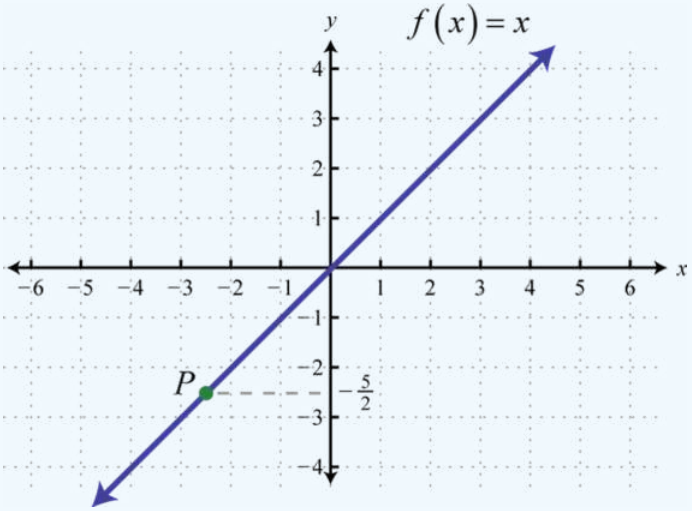


Figure 2.4.27

4.

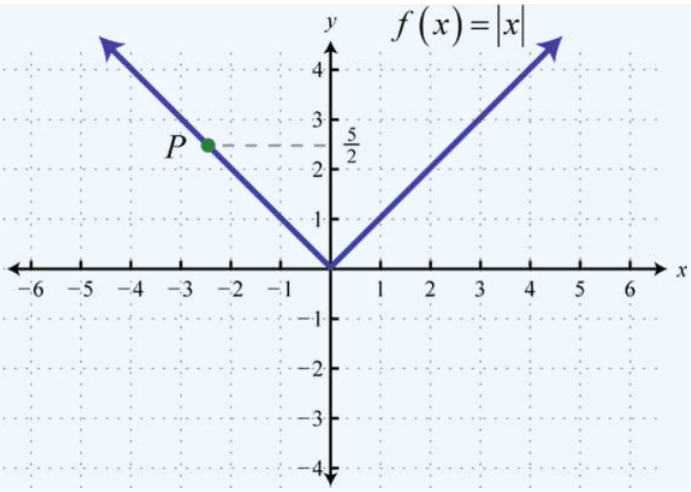


Figure 2.4.28

Answer

1. $\left(\frac{3}{2}, \frac{27}{8}\right)$
3. $\left(-\frac{5}{2}, -\frac{5}{2}\right)$

Exercise $4.2E.4$

Evaluate.

1. $f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ x+2 & \text{if } x > 0 \end{cases}$

   Find $f(-5)$, $f(0)$, and $f(3)$.

2. $g(x) = \begin{cases} x^3 \text{ if } x \leq -2 \\ |x| \text{ if } x > -2 \end{cases}$

   Find $g(-3)$, $g(-2)$, and $g(-1)$.

3. $h(x) = \begin{cases} -5 & \text{if } x < 0 \\ 2x-3 & \text{if } 0 \leq x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$

   Find $h(-2)$, $h(0)$, and $h(4)$.

4. $h(x) = \begin{cases} -3x \text{ if } x \leq 0 \\ x^3 \text{ if } 0 < x \leq 4 \\ \sqrt{x} \text{ if } x > 4 \end{cases}$

   Find $h(-5)$, $h(4)$, and $h(25)$.

Answer

1. $f(-5) = 25$, $f(0) = 0$, and $f(3) = 5$
3. $h(-2) = -5$, $h(0) = -3$, and $h(4) = 16$

Exercise $4.2E.5$

Graph the piecewise functions.

1. $g(x) = \begin{cases} 2 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$
2. $g(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 3 & \text{if } x \geq 0 \end{cases}$
3. $h(x) = \begin{cases} x & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$
4. $h(x) = \begin{cases} |x| \text{ if } x < 0 \\ x^3 \text{ if } x \geq 0 \end{cases}$
5. $f(x) = \begin{cases} |x| \text{ if } x < 2 \\ 4 \text{ if } x \geq 2 \end{cases}$
6. $f(x) = \begin{cases} x & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$
7. $g(x) = \begin{cases} x^2 \text{ if } x \leq -1 \\ x \text{ if } x > -1 \end{cases}$
8. $g(x) = \begin{cases} -3 \text{ if } x \leq -1 \\ x^3 \text{ if } x > -1 \end{cases}$
9. $h(x) = \begin{cases} 0 \text{ if } x \leq 0 \\ \frac{1}{x} \text{ if } x > 0 \end{cases}$
10. $h(x) = \begin{cases} \frac{1}{x} \text{ if } x < 0 \\ x^2 \text{ if } x \geq 0 \end{cases}$
11. $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x & \text{if } 0 \leq x < 2 \\ -2 & \text{if } x \geq 2 \end{cases}$
12. $f(x) = \begin{cases} x & \text{if } x < -1 \\ x^3 & \text{if } -1 \leq x < 1 \\ 3 & \text{if } x \geq 1 \end{cases}$

Answer

1.

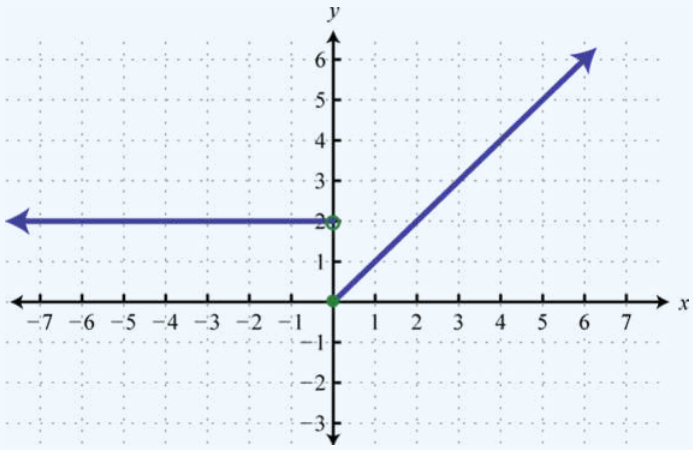


Figure 2.4.29

3.

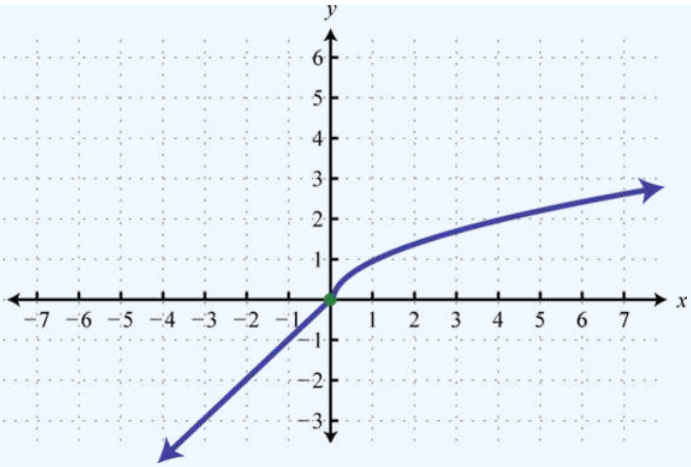


Figure 2.4.50

5.

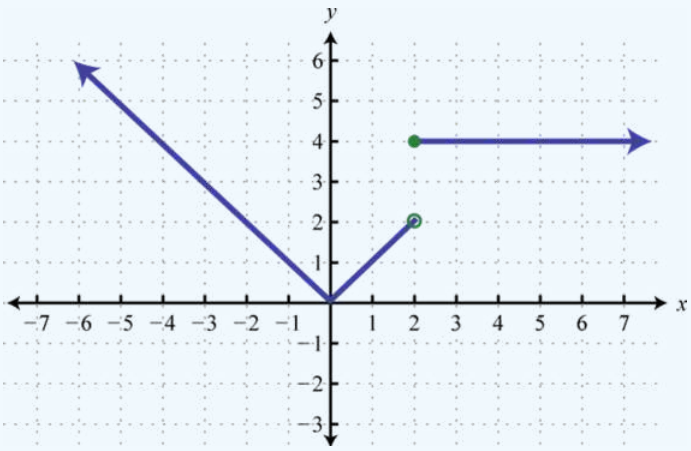


Figure 2.4.51

7.

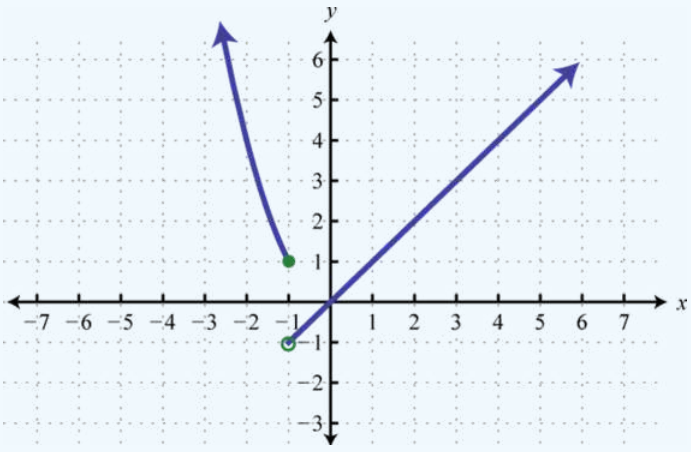


Figure 2.4.52

9.

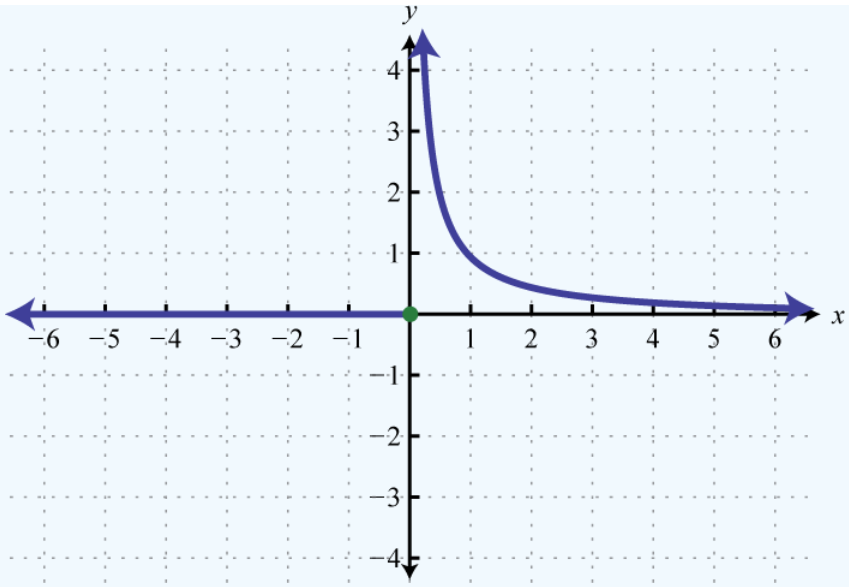


Figure 2.4.53

11.

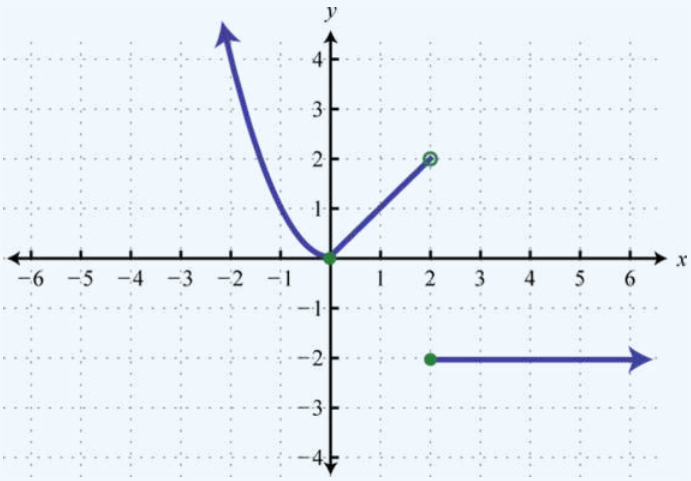


Figure 2.4.54

Exercise $4.2E.6$

Evaluate given the graph of f.

1. Find $f(-4)$, $f(-2)$, and $f(0)$.

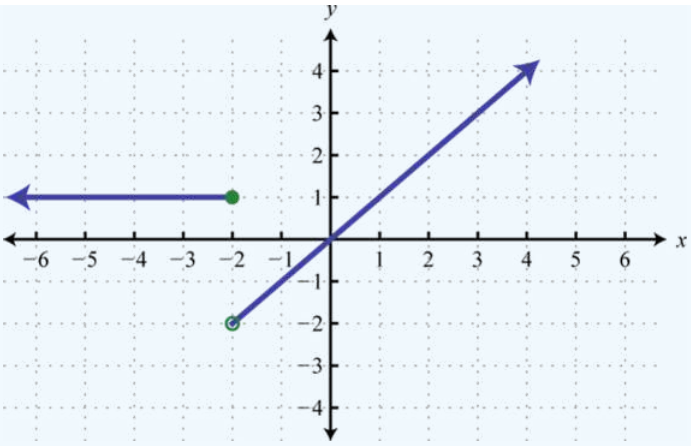


Figure 2.4.59

2. Find $f(-3)$, $f(0)$, and $f(1)$.

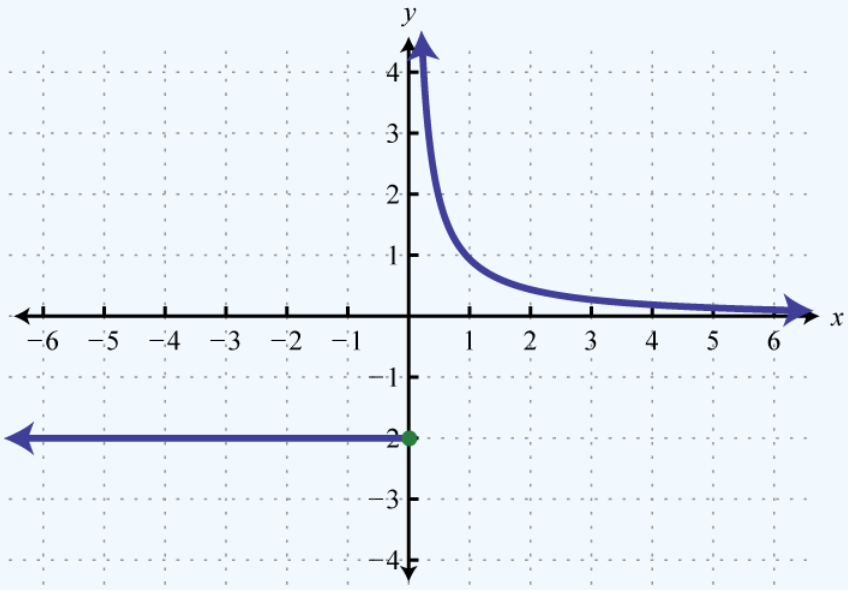


Figure 2.4.60

3. Find $f(0)$, $f(2)$, and $f(4)$.

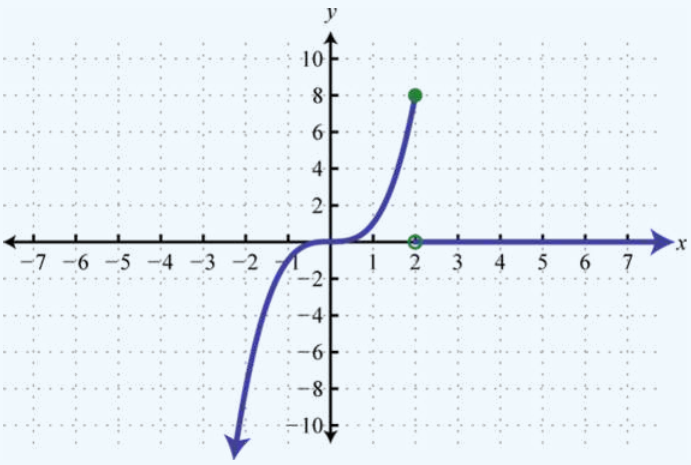


Figure 2.4.61

4. Find $f(-5)$, $f(-2)$, and $f(2)$.

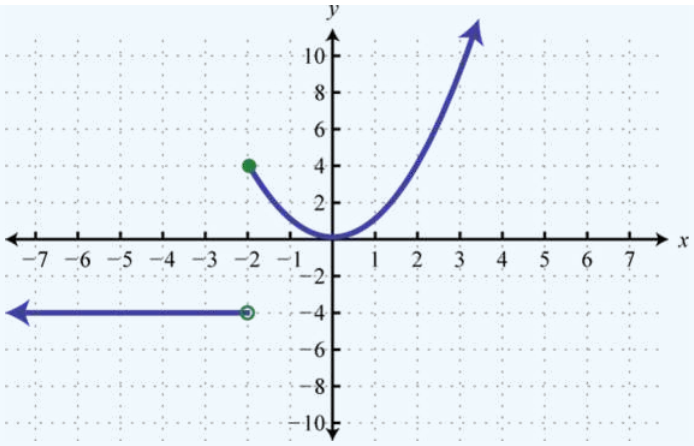


Figure 2.4.62

5. Find $f(-3)$, $f(-2)$, and $f(2)$.

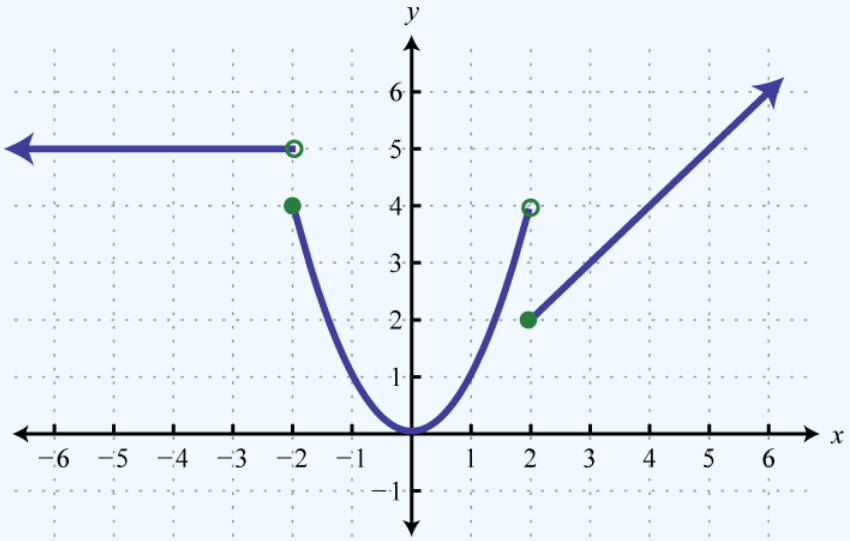


Figure 2.4.63

6. Find $f(-3)$, $f(0)$, and $f(4)$.

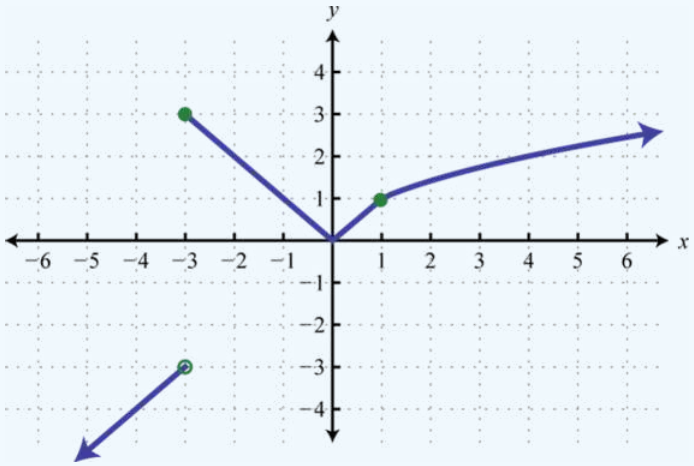


Figure 2.4.64

7. The value of an automobile in dollars is given in terms of the number of years since it was purchased new in 1975:

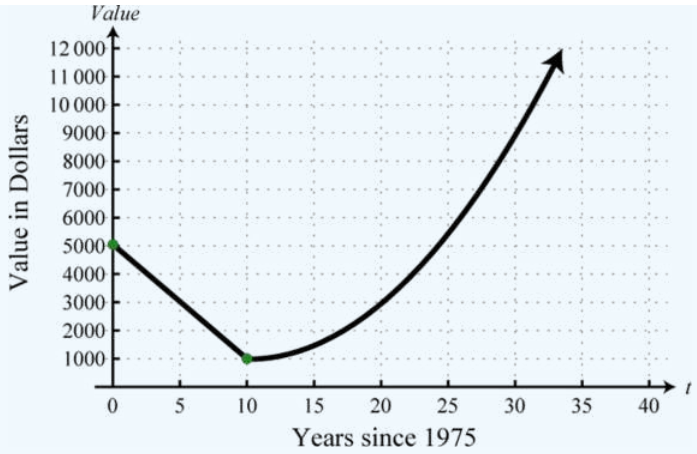


Figure 2.4.67

(1) Determine the value of the automobile in the year 1980.

(2) In what year is the automobile valued at $\$9,000$?

8. The cost per unit in dollars of custom lamps depends on the number of units produced according to the following graph:

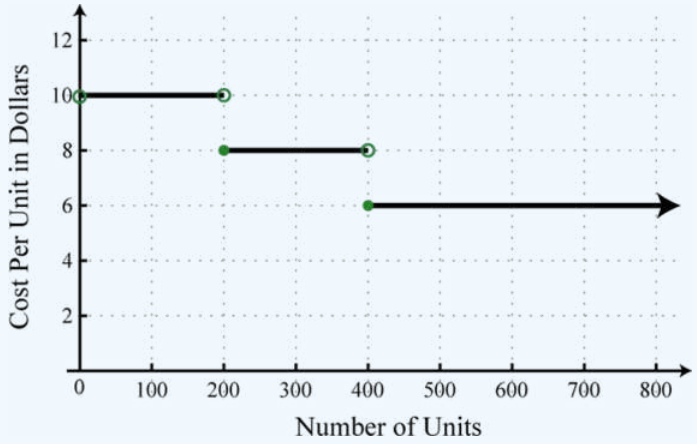


Figure 2.4.68

(1) What is the cost per unit if 250 custom lamps are produced?

(2) What level of production minimizes the cost per unit?

9. An automobile salesperson earns a commission based on total sales each month x according to the function:

$$(x) = \begin{cases} 0.03x & \text{if} \quad 0 \leq x < \$20,000 \\ 0.05x & \text{if} \quad \$20,000 \leq x < \$50,000 \\ 0.07x & \text{if} \quad x \geq \$50,000 \end{cases}$$

(1) If the salesperson's total sales for the month are $\$35,500$, what is her commission according to the function?

(2) To reach the next level in the commission structure, how much more in sales will she need?

10. A rental boat costs $\$32$ for one hour, and each additional hour or partial hour costs $\$8$. Graph the cost of the rental boat and determine the cost to rent the boat for $4\frac{1}{2}$ hours.

Answer

1. $f(-4) = 1, f(-2) = 1$, and $f(0) = 0$
3. $f(0) = 0, f(2) = 8$, and $f(4) = 0$
5. $f(-3) = 5, f(-2) = 4$, and $f(2) = 2$

7. (1) $3,000 (2) 2005

9. (1) $1,775 (2) $14,500

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4.3: Understanding Transformations of Functions

Learning Objectives

- Define the rigid transformations and use them to sketch graphs.
- Define the non-rigid transformations and use them to sketch graphs.

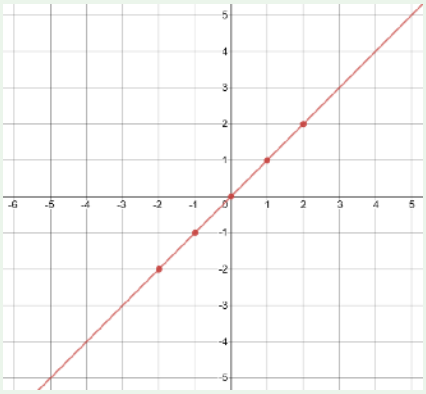
Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Sketch a graph of each of the following functions. Include at least 3 key points on each graph.

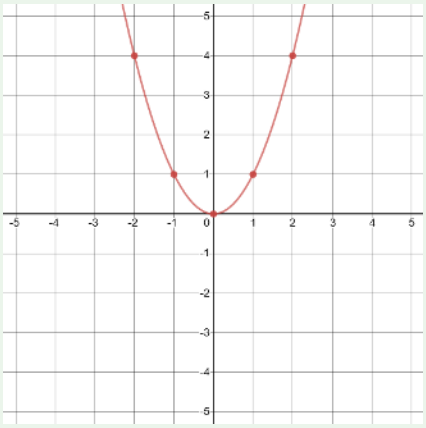
a. $y = x$

Click here to check your answer



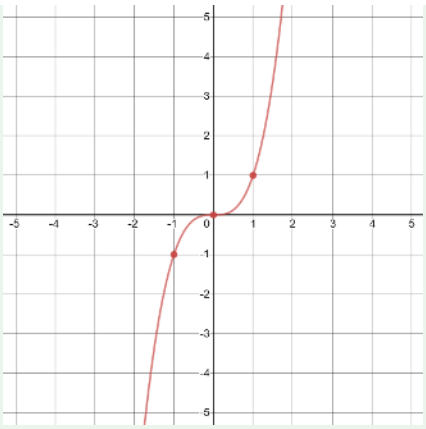
b. $y = x^2$

Click here to check your answer



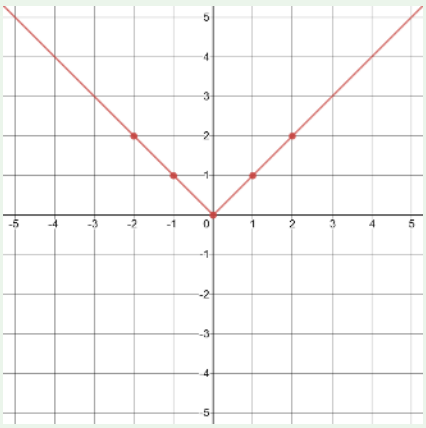
c. $y = x^3$

Click here to check your answer



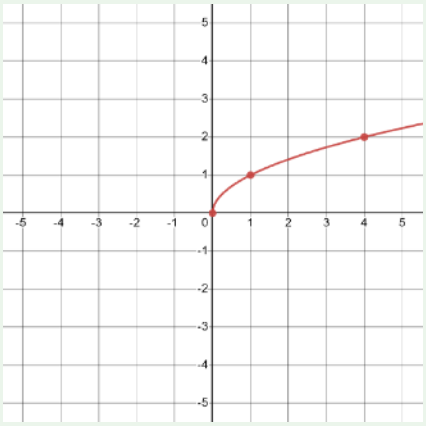
d. $y = |x|$

Click here to check your answer



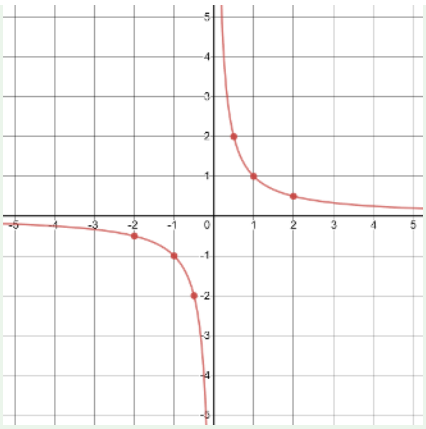
e. $y = \sqrt{x}$

Click here to check your answer



f. $y = \dfrac{1}{x}$

Click here to check your answer



If you missed any part of this problem, review Section 4.2. (Note that this will open in a new window.)

When the graph of a function is changed in appearance and/or location we call it a transformation. There are two types of transformations. A **rigid transformation**[57] changes the location of the function in a coordinate plane, but leaves the size and shape of the graph unchanged. A **non-rigid transformation**[58] changes the size and/or shape of the graph.

Vertical and Horizontal Translations

A **vertical translation**[59] is a rigid transformation that shifts a graph up or down relative to the original graph. This occurs when a constant is added to any function. If we add a positive constant to each y-coordinate, the graph will shift up. If we add a negative constant, the graph will shift down. For example, consider the functions $g(x) = x^2 - 3$ and $h(x) = x^2 + 3$. Begin by evaluating for some values of the independent variable x.

$$g(x) = x^2 - 3 \qquad h(x) = x^2 + 3$$

x	$g(x)$
-2	1
-1	-2
0	-3
1	-2
2	1

x	$h(x)$
-2	7
-1	4
0	3
1	4
2	7

Figure 2.5.1

Now plot the points and compare the graphs of the functions g and h to the basic graph of $f(x) = x^2$, which is shown using a dashed grey curve below.

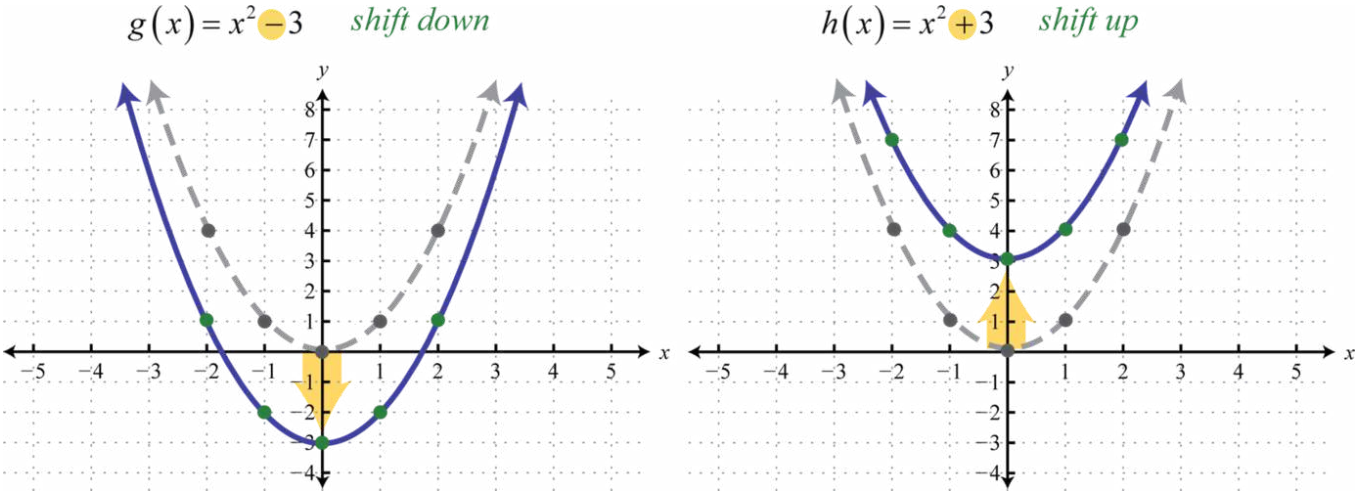


Figure 2.5.2

The function g shifts the basic graph down 3 units and the function h shifts the basic graph up 3 units. In general, this describes the vertical translations; if k is any positive real number:

Vertical shift up k units:	$F(x) = f(x) + k$
Vertical shift down k units:	$F(x) = f(x) - k$

Table 2.5.1

Example 4.3.1:

Identify the basic function and the transformation in $g(x) = \sqrt{x} + 4$. Then sketch the graph.

Solution

Begin with the basic function defined by $f(x) = \sqrt{x}$ and shift the graph up 4 units.

Answer:

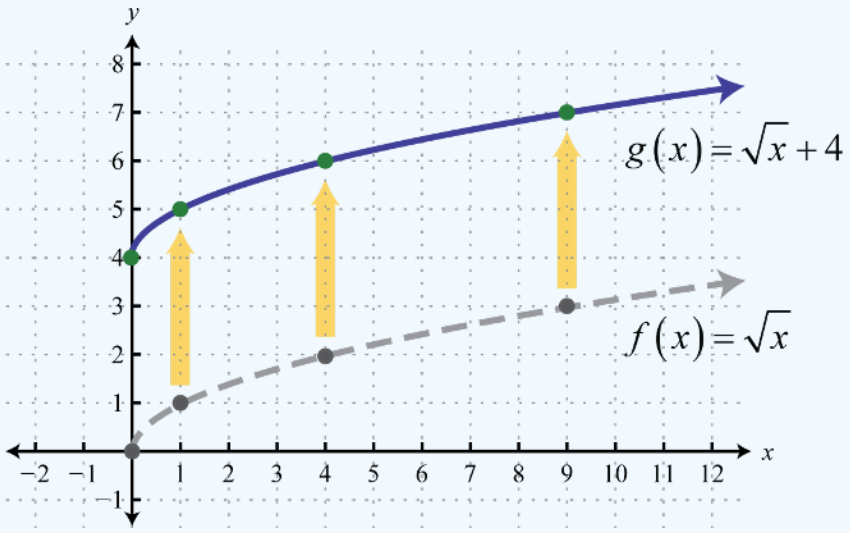


Figure 2.5.3

A **horizontal translation**[60] is a rigid transformation that shifts a graph left or right relative to the original graph. This occurs when we add or subtract constants from the x-coordinate before the function is applied. For example, consider the functions defined by $g(x) = (x + 3)^2$ and $h(x) = (x - 3)^2$ and create the following tables:

$g(x)=(x+3)^2$ $\qquad$ $h(x)=(x-3)^2$

x	$g(x)$
-5	4
-4	1
-3	0
-2	1
-1	4

x	$h(x)$
1	4
2	1
3	0
4	1
5	4

Figure 2.5.4

Here we add and subtract from the *x*-coordinates and then square the result. This produces a horizontal translation.

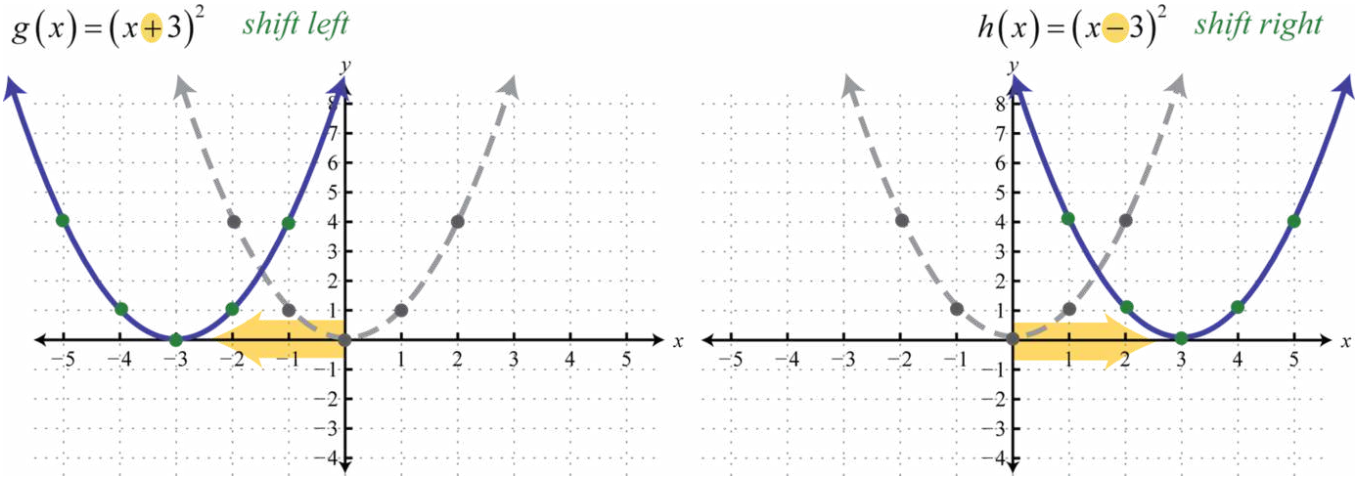


Figure 2.5.5

Note that this is the opposite of what you might expect. In general, this describes the horizontal translations; if h is any positive real number:

Horizontal shift left h units:	$F(x)=f(x+h)$
Horizontal shift right h units:	$F(x)=f(x-h)$

Table 2.5.2

Example 4.3.2:

Identify the basic function and the transformation in $g(x)=(x-4)^3$. Then sketch the graph .

Solution

Begin with a basic cubing function defined by $f(x)=x^3$ and shift the graph 4 units to the right.

Answer:

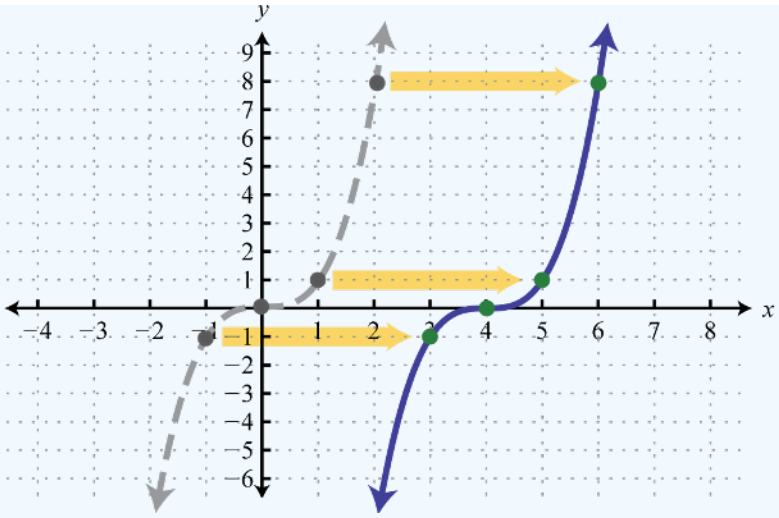


Figure 2.5.6

It is often the case that combinations of translations occur.

Example 4.3.3:

Identify the basic function and the transformation in $g(x) = |x+3| - 5$. Then sketch the graph.

Solution

Start with the absolute value function and apply the following transformations.

$$\begin{aligned} y &= |x| && \textit{Basic function} \\ y &= |x+3| && \textit{Horizontal shift left 3 units} \\ y &= |x+3| - 5 && \textit{Vertical shift down 5 units} \end{aligned}$$

Answer:

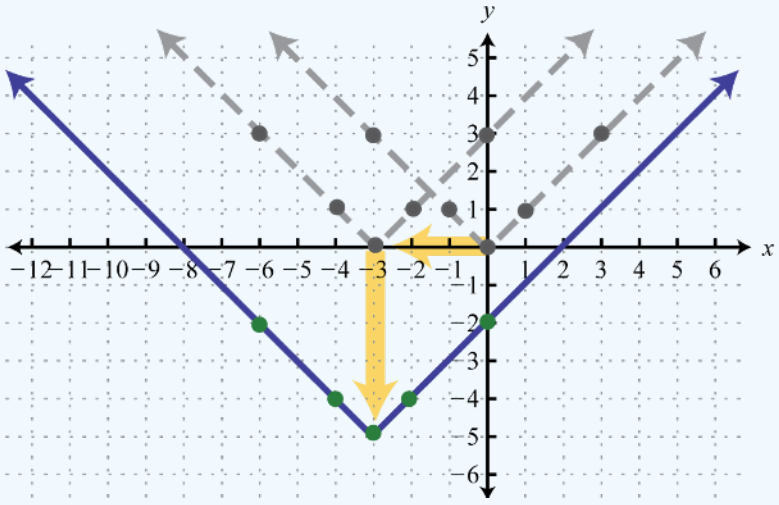


Figure 2.5.7

The order in which we apply horizontal and vertical translations does not affect the final graph.

Example 4.3.4:

Identify the basic function and the transformation in $g(x) = \frac{1}{x-5} + 3$. Then sketch the graph.

Solution

Begin with the reciprocal function and identify the translations.

$$y = \frac{1}{x} \qquad \textit{Basic function}$$
$$y = \frac{1}{x-5} \qquad \textit{Horizontal shift right 5 units}$$
$$y = \frac{1}{x-5} + 3 \qquad \textit{Vertical shift up 3 units}$$

Take care to shift the vertical asymptote from the *y*-axis 5 units to the right and shift the horizontal asymptote from the *x*-axis up 3 units.

Answer:

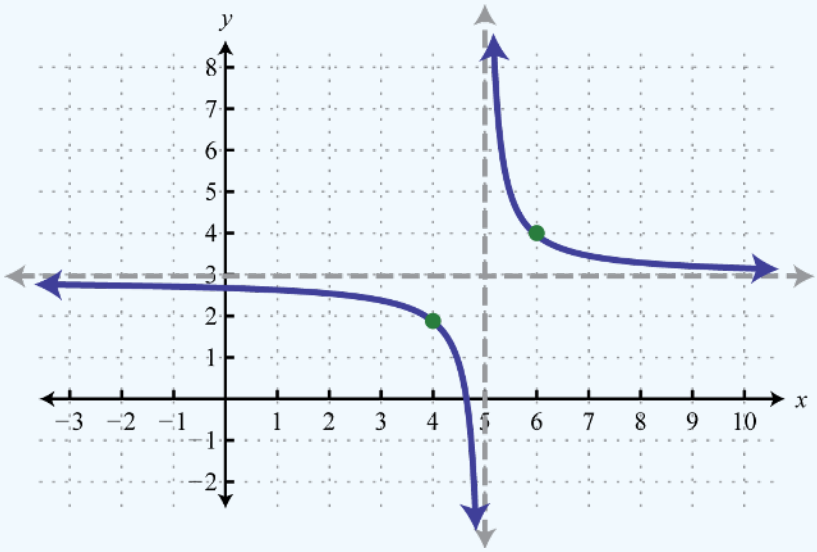


Figure 2.5.8

Exercise 4.3.1

Identify the basic function and the transformation in $g(x) = (x-2)^2 + 1$. Then sketch the graph.

Answer

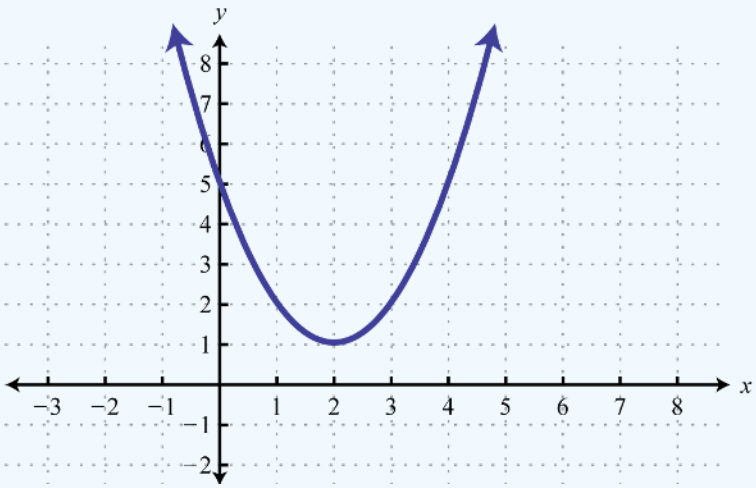


Figure 2.5.9

www.youtube.com/v/6F6zKaogxTE

Reflections

A **reflection**[61] is a transformation in which a mirror image of the graph is produced about an axis. In this section, we will consider reflections about the x- and y-axis. The graph of a function is reflected about the x-axis if each y-coordinate is multiplied by -1. The graph of a function is reflected about the y-axis if each x-coordinate is multiplied by -1 before the function is applied. For example, consider $g(x)=\sqrt{-x}$ and $h(x)=-\sqrt{x}$.

$$g(x)=\sqrt{-x} \qquad h(x)=-\sqrt{x}$$

x	$g(x)$
-4	2
-1	1
0	0

x	$h(x)$
4	-2
1	-1
0	0

Figure 2.5.10

Compare the graph of g and h to the basic square root function defined by $f(x)=\sqrt{x}$, shown dashed in grey below:

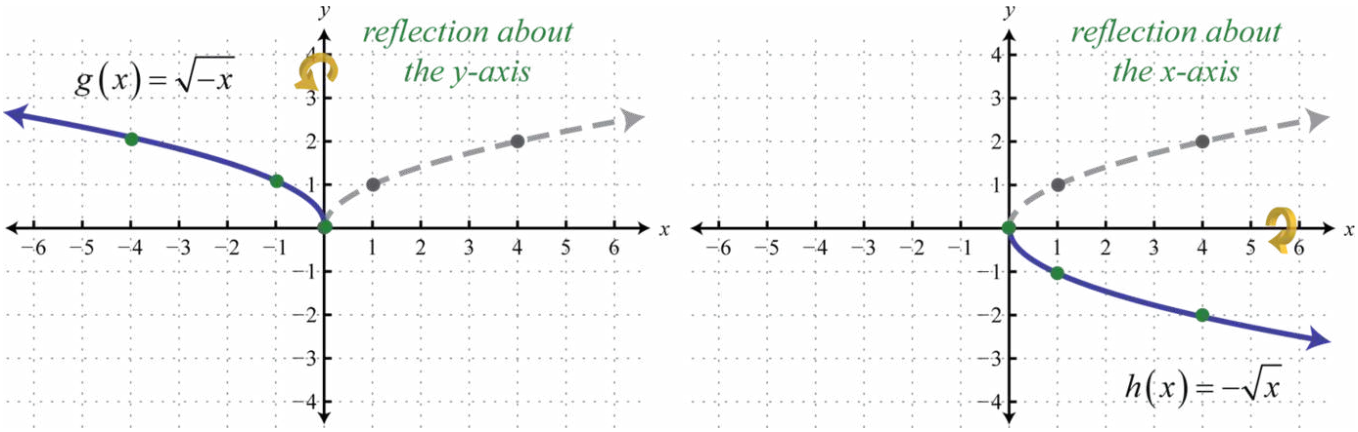


Figure 2.5.11

The first function g has a negative factor that appears "inside" the function; this produces a reflection about the y-axis. The second function h has a negative factor that appears "outside" the function; this produces a reflection about the x-axis. In general, it is true that:

Reflection about the y-axis:	$F(x)=f(-x)$
Reflection about the x-axis:	$F(x)=-f(x)$

Table 2.5.3

When sketching graphs that involve a reflection, consider the reflection first and then apply the vertical and/or horizontal translations.

Example 4.3.5:

Identify the basic function and the transformation in $g(x)=-(x+5)^2+3$. Then sketch the graph.

Solution

Begin with the squaring function and then identify the transformations starting with any reflections.

$$\begin{aligned} y&=x^2 && \textit{Basic function.}\\ y&=-x^2 && \textit{Reflection about the } x-\textit{axis.}\\ y&=-(x+5)^2 && \textit{Horizontal shift left 5 units.}\\ y&=-(x+5)^2+3 && \textit{Vertical shift up 3 units.}\end{aligned}$$

Use these translations to sketch the graph.

Answer:

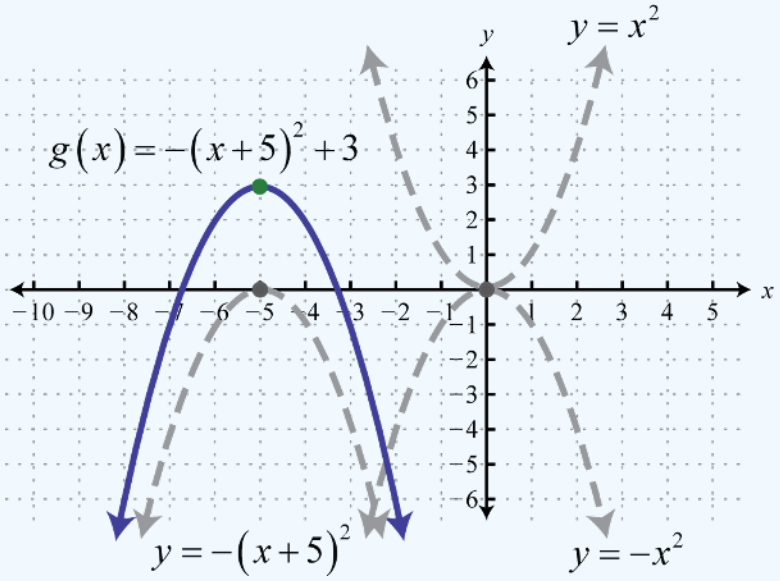


Figure 2.5.12

Exercise 4.3.2

Identify the basic function and the transformation in $g(x) = -|x| + 3$. Then sketch the graph.

Answer

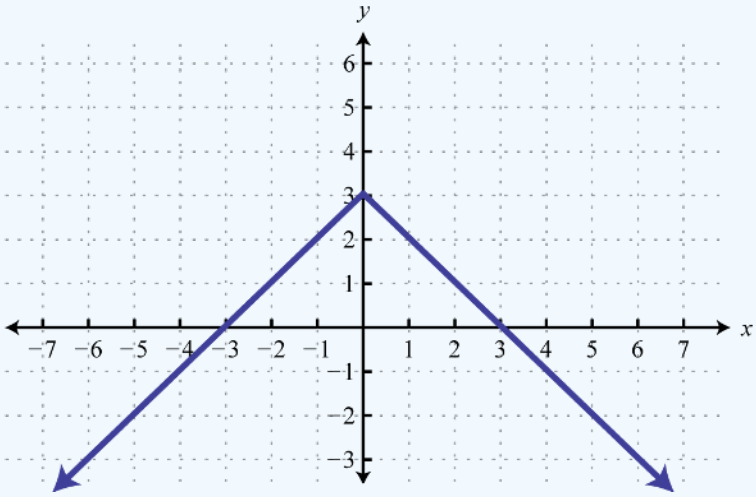


Figure 2.5.13

www.youtube.com/v/XsbLkFWWzBc

Dilations

Horizontal and vertical translations, as well as reflections, are called rigid transformations because the shape of the basic graph is left unchanged, or rigid. Functions that are multiplied by a real number other than 1, depending on the real number, appear to be stretched vertically or stretched horizontally. This type of non-rigid transformation is called a **dilation**[62]. For example, we can multiply the squaring function $f(x) = x^2$ by 4 and $\frac{1}{4}$ to see what happens to the graph.

$g(x) = 4x^2$

x	$g(x)$
-1	4
0	0
1	4

$h(x) = \frac{1}{4}x^2$

x	$h(x)$
-2	1
0	0
2	1

Figure 2.5.14

Compare the graph of g and h to the basic squaring function defined by $f(x) = x^2$, shown dashed in grey below:

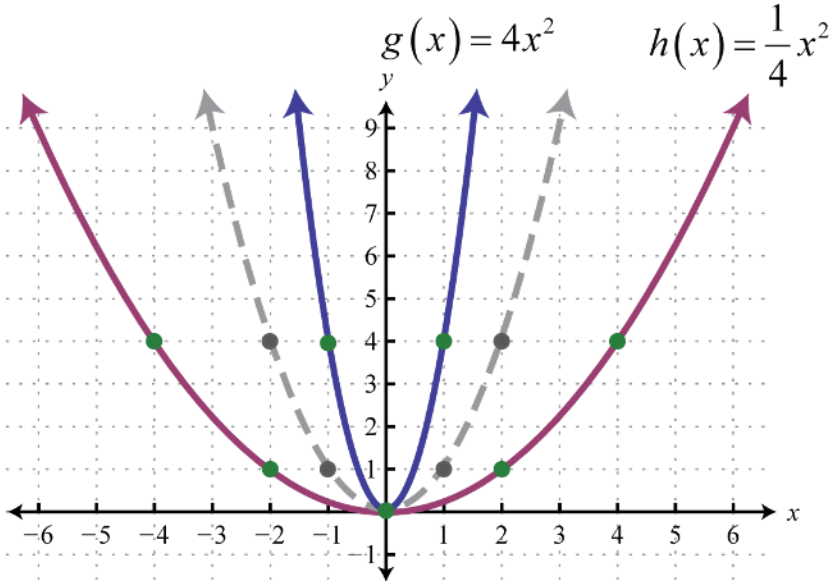


Figure 2.5.15

The function g is steeper than the basic squaring function and its graph appears to have been stretched vertically. The function h is not as steep as the basic squaring function and appears to have been stretched horizontally.

In general, we have:

Dilation:	$F(x) = a \cdot f(x)$

Table 2.5.4

If the factor a is a nonzero fraction between -1 and 1, it will stretch the graph horizontally. Otherwise, the graph will be stretched vertically. If the factor a is negative, then it will produce a reflection as well.

Example 4.3.6:

Identify the basic function and the transformation in $g(x) = -2|x-5| - 3$. Then sketch the graph.

Solution

Here we begin with the product of -2 and the basic absolute value function: $y = -2|x|$.This results in a reflection and a dilation.

x	y	$y = -2\|x\|$ ← *Dilation and reflection*
-1	-2	$y = -2\|-1\| = -2 \cdot 1 = -2$
0	0	$y = -2\|0\| = -2 \cdot 0 = 0$
1	-2	$y = -2\|1\| = -2 \cdot 1 = -2$

Figure 2.5.16

Use the points $\{(-1,-2),(0,0),(1,-2)\}$ to graph the reflected and dilated function $y = -2|x|$. Then translate this graph 5 units to the right and 3 units down.

$y = -2|x|$ *Basic graph with dilation and reflection about the $x-axis$.*
$y = -2|x-5|$ *Shift right 5 units.*
$y = -2|x-5|-3$ *Shift down 3 units.*

Answer:

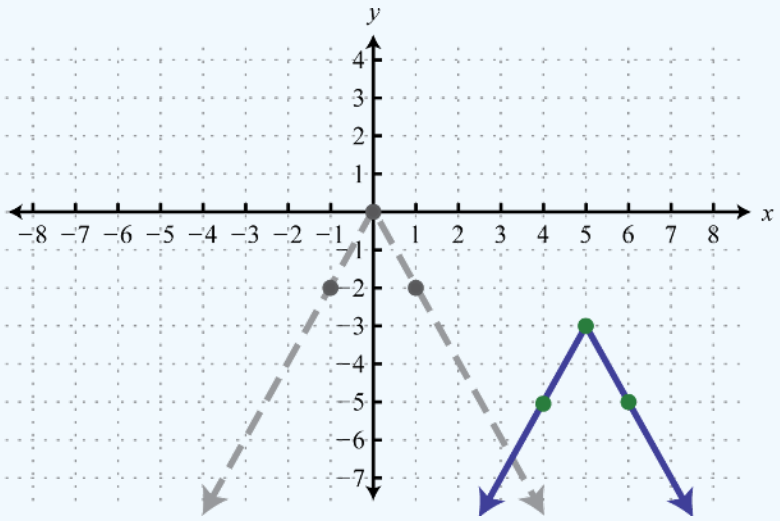


Figure 2.5.17

In summary, given positive real numbers h and k:

Vertical shift up k units:	$F(x) = f(x) + k$
Vertical shift down k units:	$F(x) = f(x) - k$

Table 2.5.1

Horizontal shift left h units:	$F(x) = f(x+h)$
Horizontal shift right h units:	$F(x) = f(x-h)$

Table 2.5.2

Reflection about the y-axis:	$F(x) = f(-x)$
Reflection about the x-axis:	$F(x) = -f(x)$

Table 2.5.3

Dilation:	$F(x) = a \cdot f(x)$

Table 2.5.4

Key Takeaways

- Identifying transformations allows us to quickly sketch the graph of functions. This skill will be useful as we progress in our study of mathematics. Often a geometric understanding of a problem will lead to a more elegant solution.
- If a positive constant is added to a function, $f(x) + k$, the graph will shift up. If a positive constant is subtracted from a function, $f(x) - k$, the graph will shift down. The basic shape of the graph will remain the same.
- If a positive constant is added to the value in the domain before the function is applied, $f(x + h)$, the graph will shift to the left. If a positive constant is subtracted from the value in the domain before the function is applied, $f(x - h)$, the graph will shift right. The basic shape will remain the same.
- Multiplying a function by a negative constant, $-f(x)$, reflects its graph in the x-axis. Multiplying the values in the domain by -1 before applying the function, $f(-x)$, reflects the graph about the y-axis.
- When applying multiple transformations, apply reflections first.
- Multiplying a function by a constant other than 1, $a \cdot f(x)$, produces a dilation. If the constant is a positive number greater than 1, the graph will appear to stretch vertically. If the positive constant is a fraction less than 1, the graph will appear to stretch horizontally.

Footnotes

[57] A set of operations that change the location of a graph in a coordinate plane but leave the size and shape unchanged.

[58] A set of operations that change the size and/or shape of a graph in a coordinate plane.

[59] A rigid transformation that shifts a graph up or down.

[60] A rigid transformation that shifts a graph left or right.

[61] A transformation that produces a mirror image of the graph about an axis.

[62] A non-rigid transformation, produced by multiplying functions by a nonzero real number, which appears to stretch the graph either vertically or horizontally.

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4.3E: Exercises - Understanding Transformations of Functions

Exercise $4.3E.1$

Match the graph to the function definition.

a.

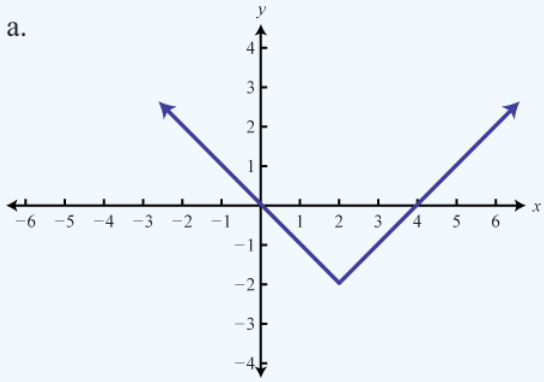


Figure 2.5.18

b.

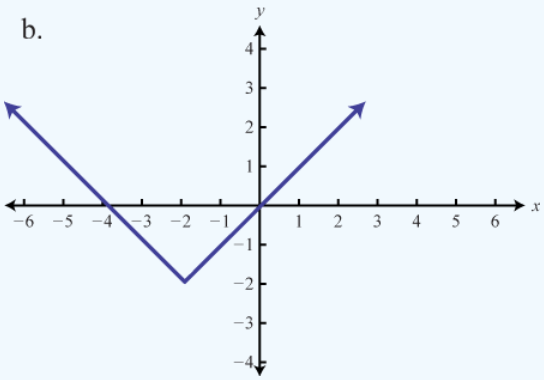


Figure 2.5.19

c.

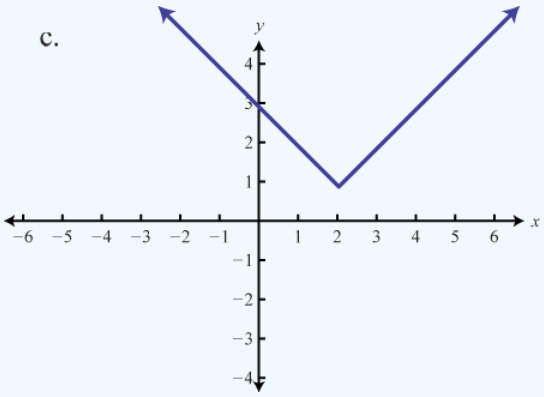


Figure 2.5.20

d.

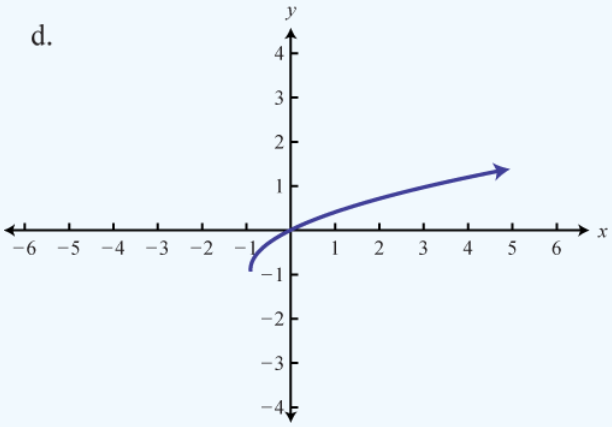


Figure 2.5.21

e.

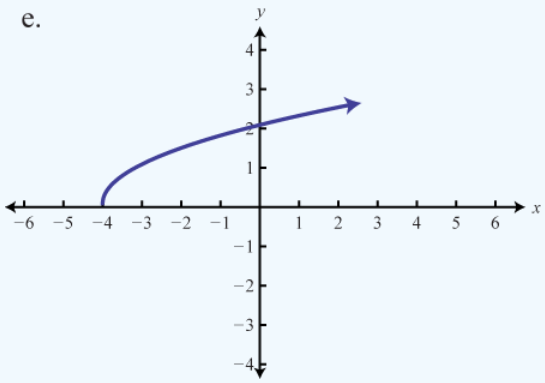


Figure 2.5.22

f.

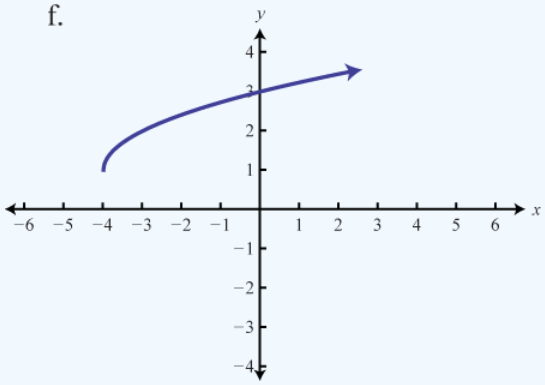


Figure 2.5.23

1. $f(x)=\sqrt{x+4}$
2. $f(x)=|x-2|-2$
3. $f(x)=\sqrt{x+1}-1$
4. $f(x)=|x-2|+1$
5. $f(x)=\sqrt{x+4}+1$
6. $f(x)=|x+2|-2$

Answer

1. e
3. d
5. f

Exercise $4.3E.2$

Graph the given function. Identify the basic function and translations used to sketch the graph. Then state the domain and range.

1. $g(x) = -4$
2. $g(x) = 2$
3. $f(x) = x + 3$
4. $f(x) = x - 2$
5. $g(x) = x^2 + 1$
6. $g(x) = x^2 - 4$
7. $g(x) = (x-5)^2 + 2$
8. $g(x) = (x+2)^2 - 5$
9. $h(x) = |x-1| - 3$
10. $h(x) = |x+2| - 5$
11. $g(x) = \sqrt{x-2} + 1$
12. $g(x) = \sqrt{x+2} + 3$
13. $h(x) = (x-1)^3 - 4$
14. $h(x) = (x+1)^3 + 3$
15. $f(x) = \frac{1}{x+1} - 2$
16. $f(x) = \frac{1}{x-3} + 3$
17. $f(x) = \sqrt[3]{x-2} + 6$
18. $f(x) = \sqrt[3]{x+8} - 4$

Answer

1. Basic graph $y = -4$; domain: $(-\infty, \infty)$; range: $\{-4\}$

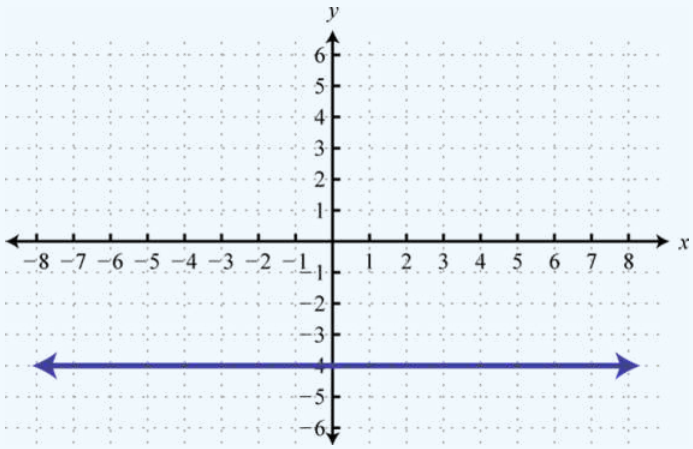


Figure 2.5.37

3. $y = x$; Shift up 3 units; domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

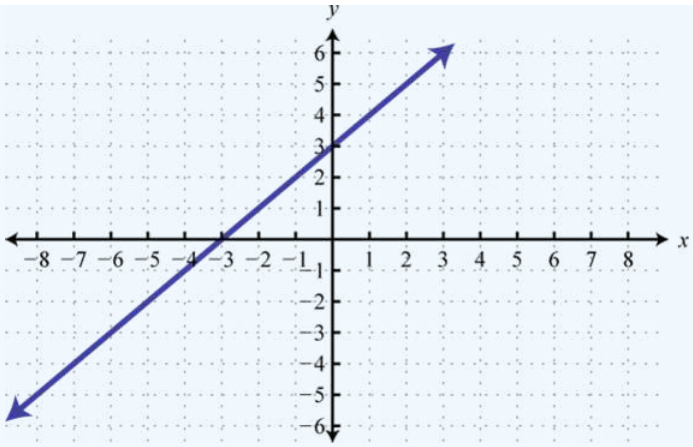


Figure 2.5.24

5. $y = x^2$; Shift up 1 unit; domain: $(-\infty, \infty)$; range: $[1, \infty)$

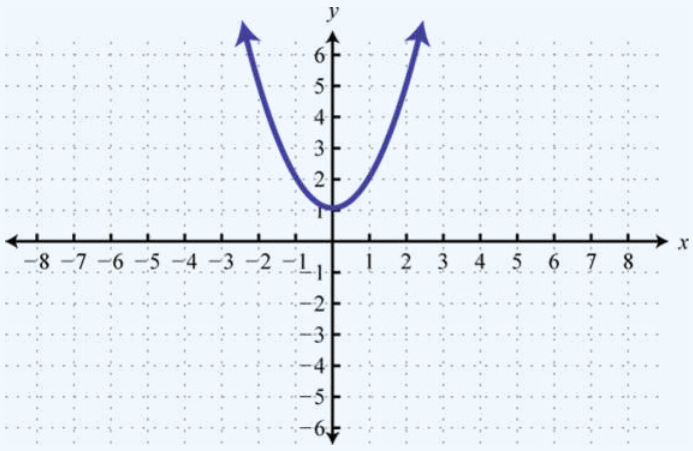


Figure 2.5.25

7. $y = x^2$; Shift right 5 units and up 2 units; domain: $(-\infty, \infty)$; range: $[2, \infty)$

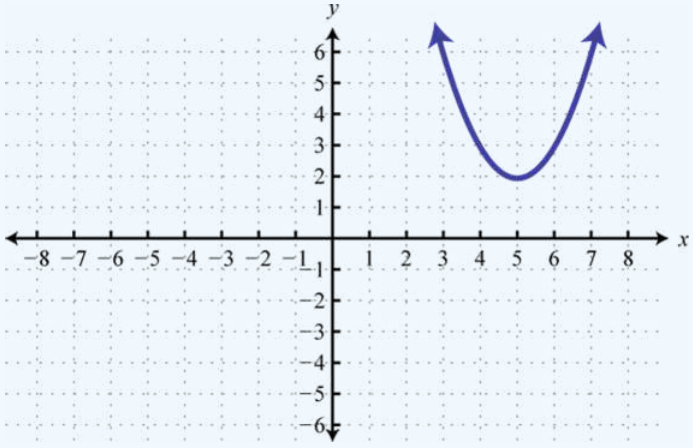


Figure 2.5.27

9. $y = |x|$; Shift right 1 unit and down 3 units; domain: $(-\infty, \infty)$; range: $[-3, \infty)$

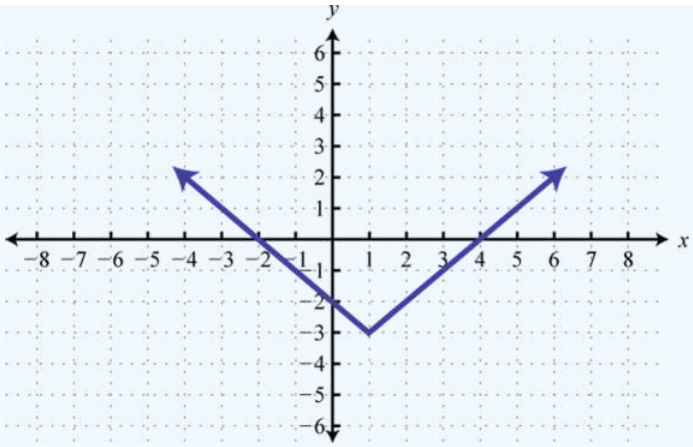


Figure 2.5.29

11. $y = \sqrt{x}$; Shift right 2 units and up 1 unit; domain: $[2, \infty)$; range: $[1, \infty)$

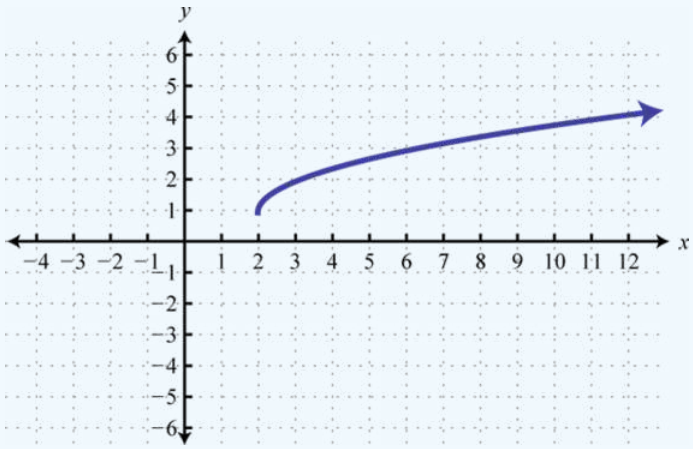


Figure 2.5.31

13. $y = x^3$; Shift right 1 unit and down 4 units; domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

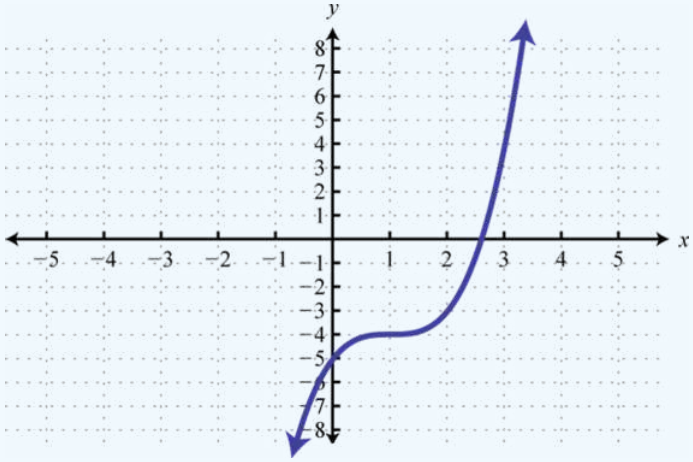


Figure 2.5.33

15. $y = \frac{1}{x}$; Shift left 1 unit and down 2 units; domain: $(-\infty, -1) \cup (-1, \infty)$; range: $(-\infty, -2) \cup (-2, \infty)$

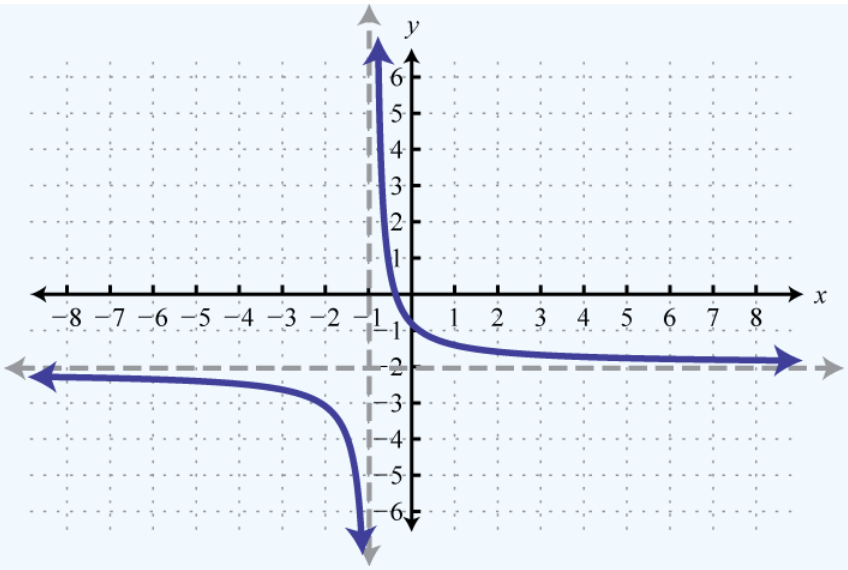


Figure 2.5.36

17. $y = \sqrt[3]{x}$; Shift up 6 units and right 2 units; domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

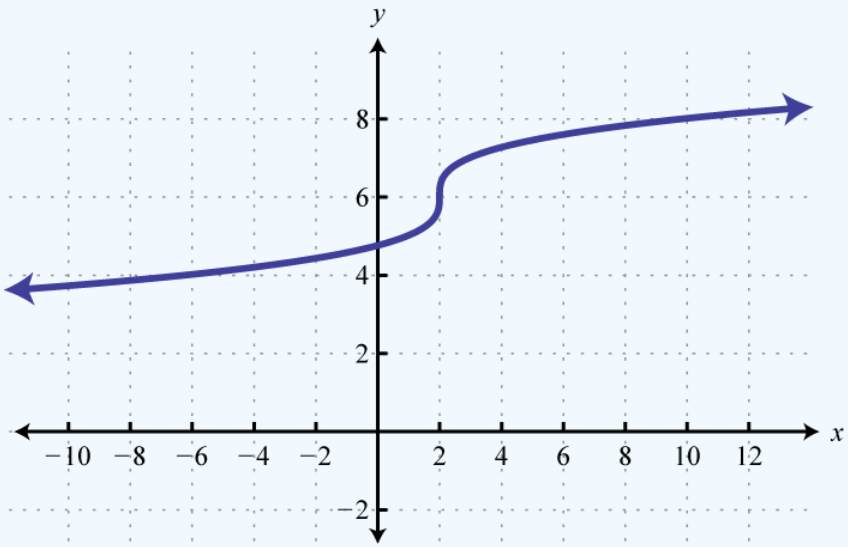


Figure 2.5.38

Exercise $4.3E.3$

Graph the piecewise functions.

1. $h(x) = \begin{cases} x^2 + 2 & \text{if } x < 0 \\ x + 2 & \text{if } x \geq 0 \end{cases}$
2. $h(x) = \begin{cases} x^2 - 3 \text{ if } x < 0 \\ \sqrt{x} - 3 \text{ if } x \geq 0 \end{cases}$
3. $h(x) = \begin{cases} x^3 - 1 & \text{if } x < 0 \\ |x - 3| - 4 & \text{if } x \geq 0 \end{cases}$
4. $h(x) = \begin{cases} x^3 & \text{if } x < 0 \\ (x-1)^2 - 1 & \text{if } x \geq 0 \end{cases}$

Answer

1.

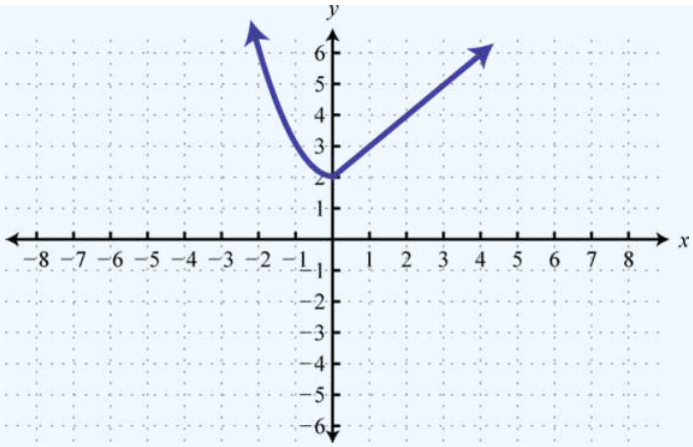


Figure 2.5.39

3.

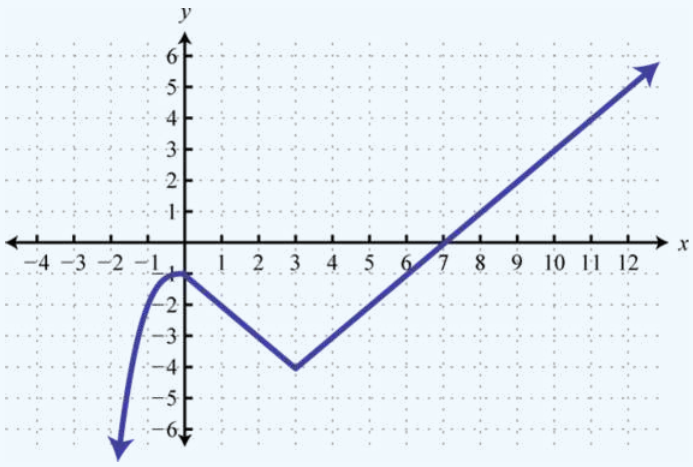


Figure 2.5.40

Exercise $4.3E.4$

Write an equation that represents the function whose graph is given.

1.

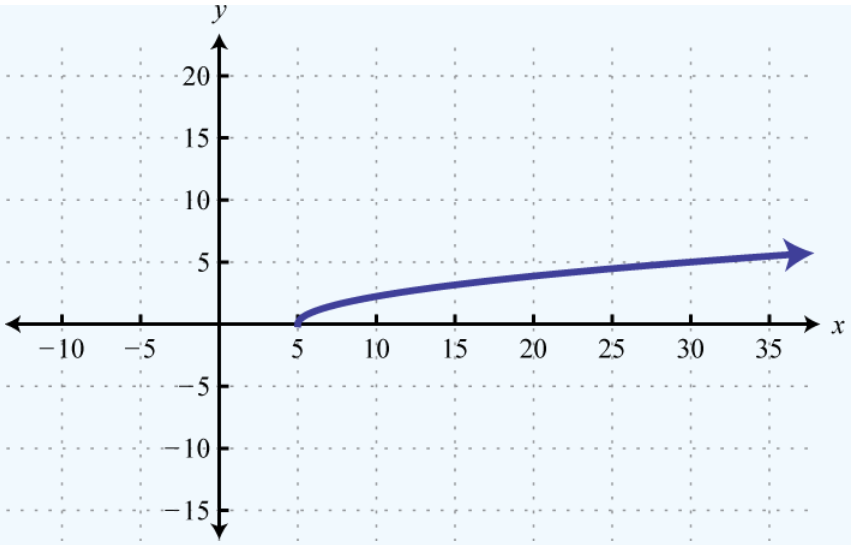


Figure 2.5.43

2.

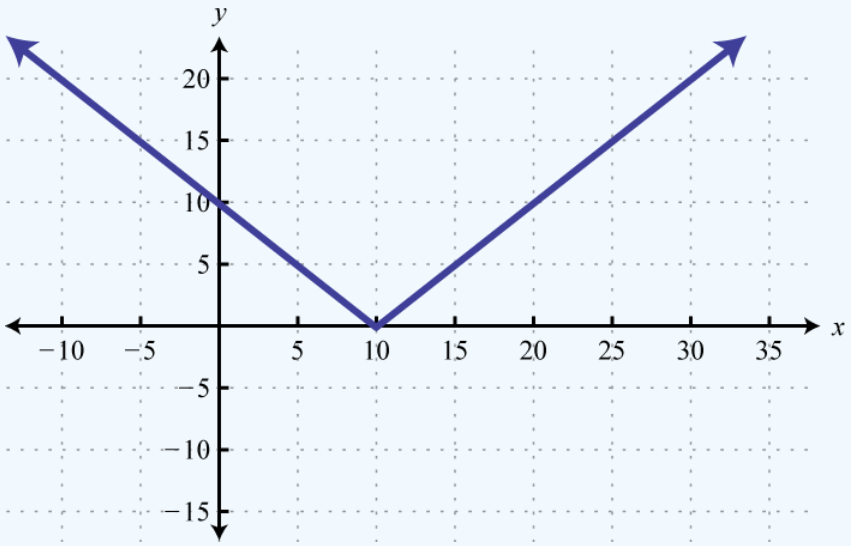


Figure 2.5.44

3.

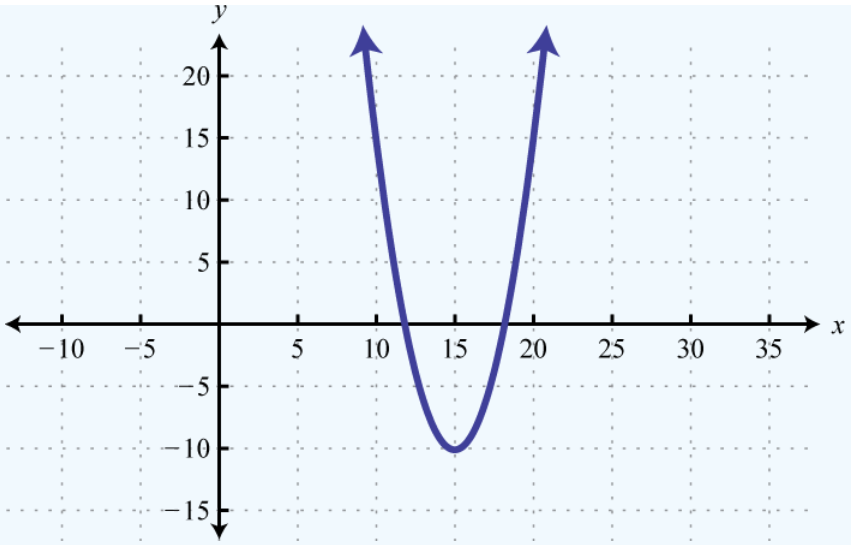


Figure 2.5.45

4.

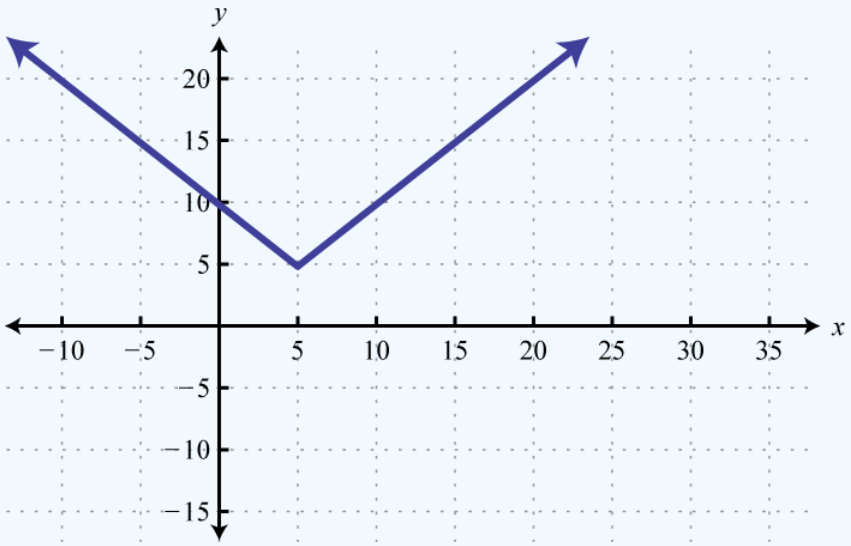


Figure 2.5.46

5.

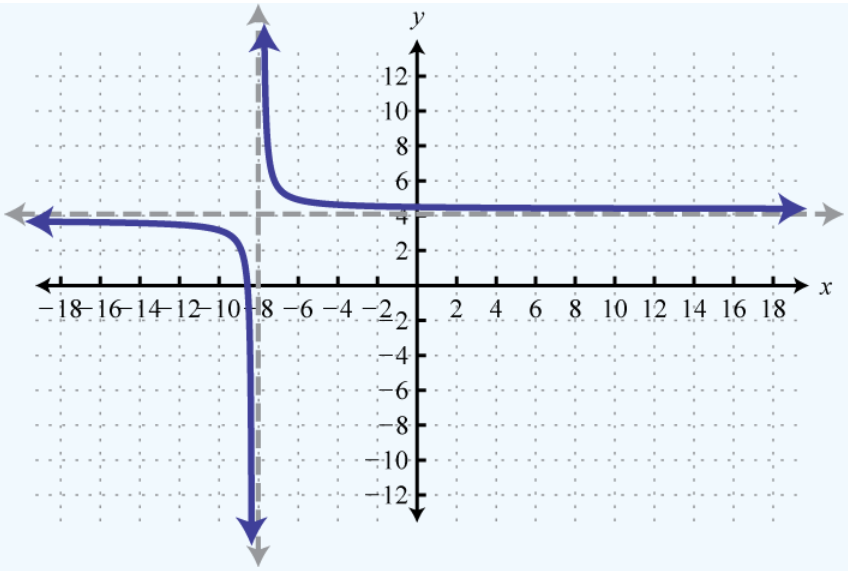


Figure 2.5.47

6.

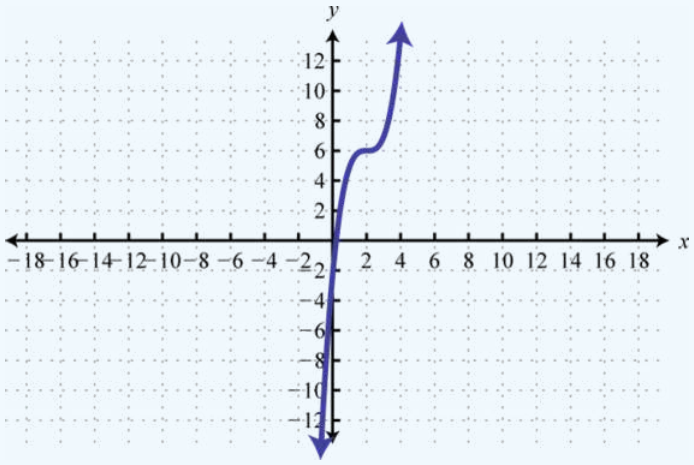


Figure 2.5.48

7.

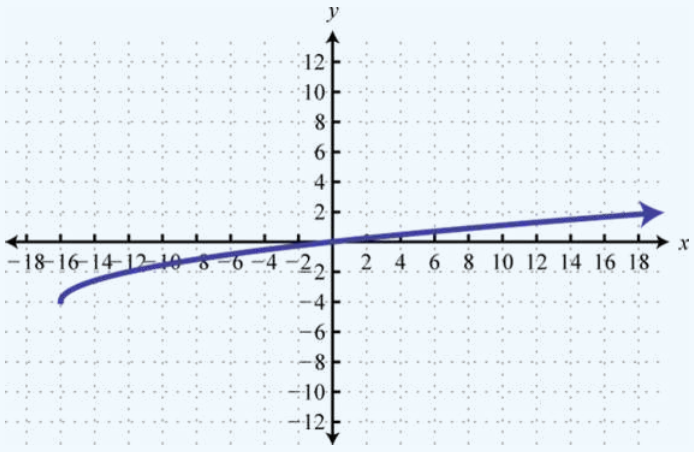


Figure 2.5.49

8.

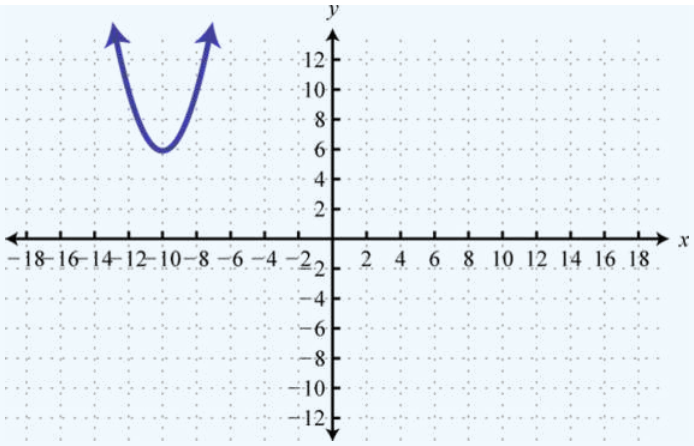


Figure 2.5.50

Answer

1. $f(x) = \sqrt{x-5}$
3. $f(x) = (x-15)^2 - 10$
5. $f(x) = \frac{1}{x+8} + 4$
7. $f(x) = \sqrt{x+16} - 4$

Exercise $4.3E.5$

Match the graph to the given function definition.

a.

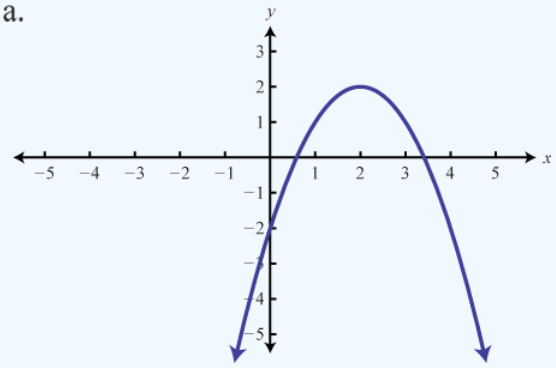


Figure 2.5.51

b.

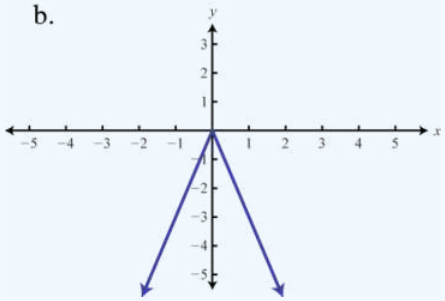


Figure 2.5.52

c.

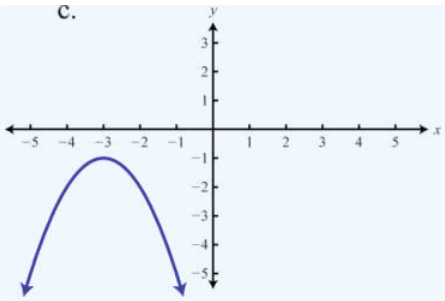


Figure 2.5.53

d.

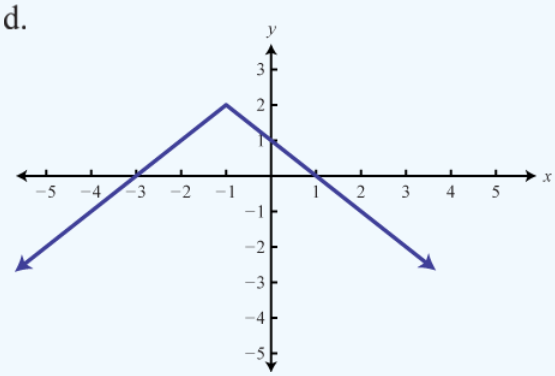


Figure 2.5.54

e.

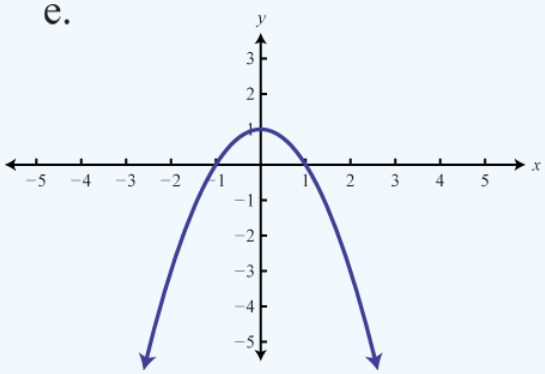


Figure 2.5.55

f.

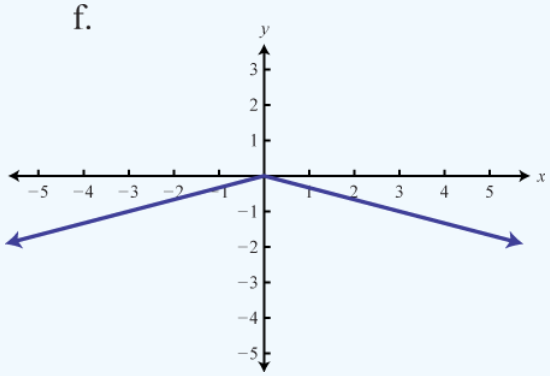


Figure 2.5.56

1. $f(x) = -3|x|$
2. $f(x) = -(x+3)^2 - 1$
3. $f(x) = -|x+1| + 2$
4. $f(x) = -x^2 + 1$
5. $f(x) = -\frac{1}{3}|x|$
6. $f(x) = -(x-2)^2 + 2$

Answer

1. b
3. d
5. f

Exercise $4.3E.6$

Use the transformations to graph the following functions.

1. $f(x) = -x + 5$
2. $f(x) = -|x| - 3$
3. $g(x) = -|x - 1|$
4. $f(x) = -(x + 2)^2$
5. $h(x) = \sqrt{-x} + 2$
6. $h(x) = -\sqrt{x - 2} + 1$
7. $g(x) = -x^3 + 4$
8. $f(x) = -x^2 + 6$

Answer

1.

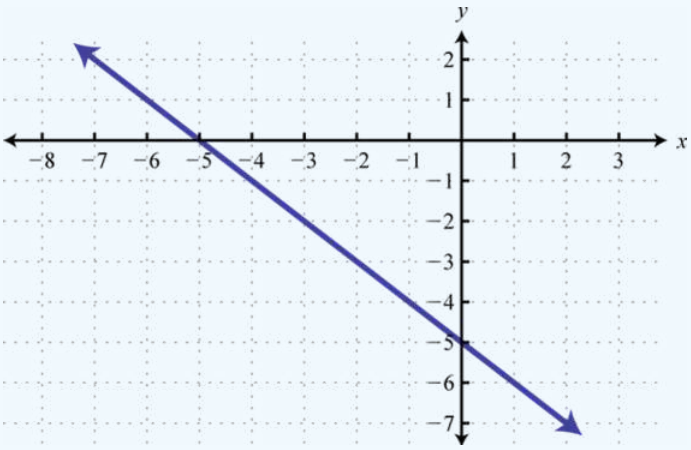


Figure 2.5.57

3.

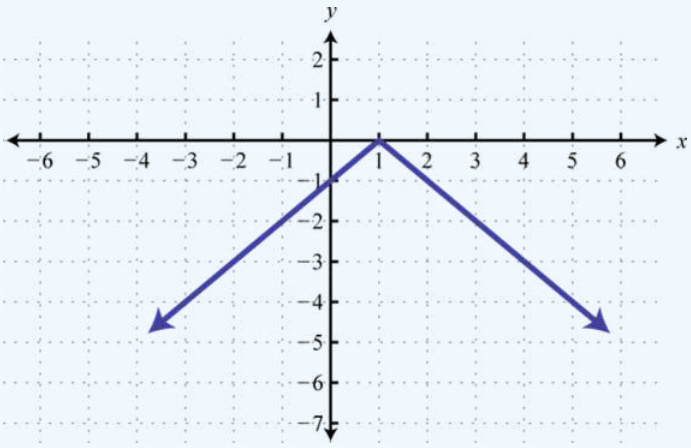


Figure 2.5.58

5.

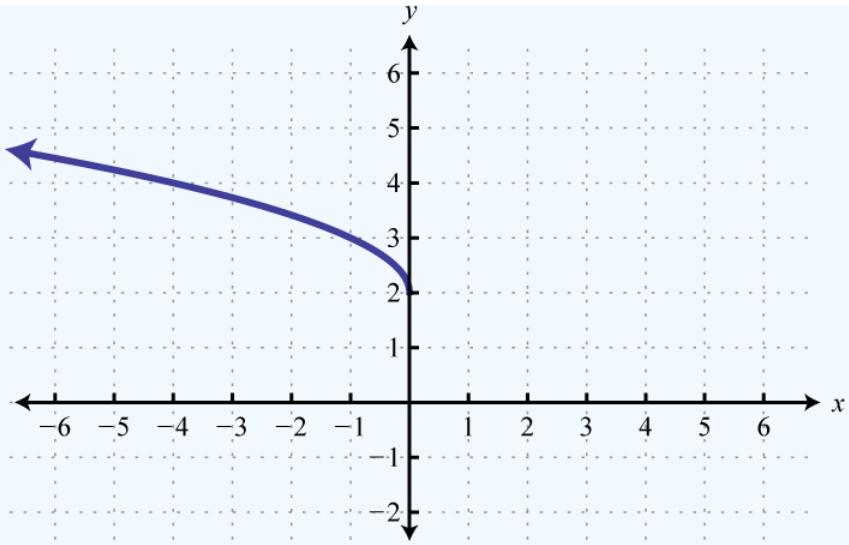


Figure 2.5.59

7.

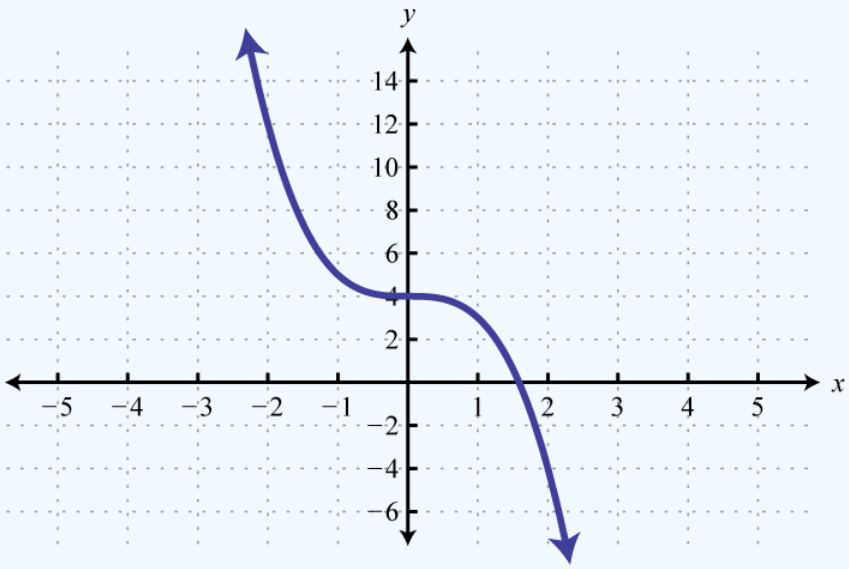


Figure 2.5.61

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4.4: Quadratic Functions and Applications

Learning Objectives

By the end of this section, you will be able to:

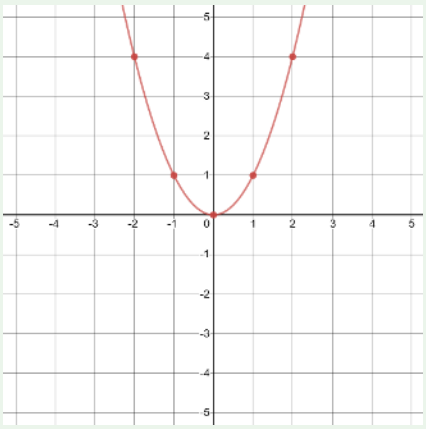
- Recognize the graph of a quadratic function
- Find the axis of symmetry and vertex of a parabola
- Identify domain and range of quadratic functions
- Solve maximum and minimum applications
- Find the intercepts of a parabola
- Find a break-even point in a quadratic application

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Sketch a graph of $y = x^2$. Include at least 3 key points on the graph.

Click here to check your answer



If you missed this problem, review Section 4.2. (Note that this will open in a new window.)

2. If $f(x) = 2x^2 - x + 1$, find each of the following:

a. $f(3)$

b. $f(-2)$

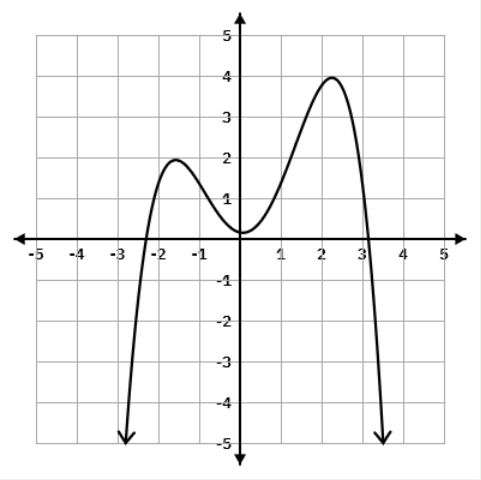
Click here to check your answer

a. 16

b. 11

If you missed any part of this problem, review Section 4.1. (Note that this will open in a new window.)

3. Identify the domain and range of the function graphed here.



Click here to check your answer

Domain: $(-\infty, \infty)$

Range: $(-\infty, 4]$

If you missed any part of this problem, review Section 4.1. (Note that this will open in a new window.)

Recognize the Graph of a Quadratic Function

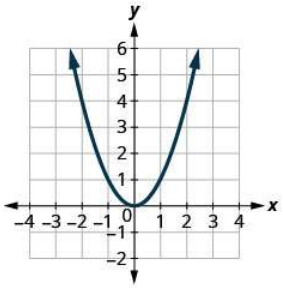
Previously we very briefly looked at the function $f(x) = x^2$, which we called the square function. It was one of the first non-linear functions we looked at. Now we will graph functions of the form $f(x) = ax^2 + bx + c$ if $a \neq 0$. We call this kind of function a quadratic function.

Definition 4.4.1

A **quadratic function**, where a, b, and c are real numbers and $a \neq 0$, is a function of the form

$$f(x) = ax^2 + bx + c$$

We graphed the quadratic function $f(x) = x^2$ by plotting points.



x	$f(x) = x^2$	$(x, f(x))$
–3	9	(–3, 9)
–2	4	(–2, 4)
–1	1	(–1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

Figure 9.6.1

Every quadratic function has a graph that looks like this. We call this figure a **parabola**. Let's practice graphing a parabola by plotting a few points.

Example 4.4.1

Graph: $f(x) = x^2 - 1$.

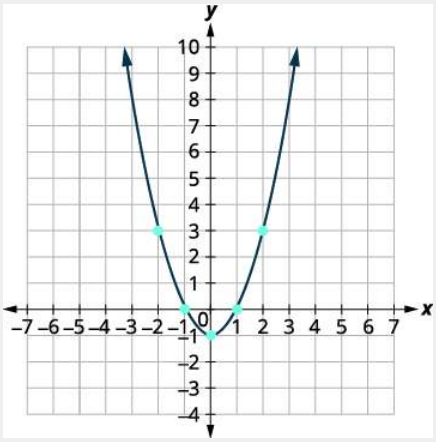
Solution:

We will graph the function by plotting points.

Choose integer values for x,
substitute them into the equation
and simplify to find $f(x)$.
Record the values of the ordered pairs in the chart.

$f(x) = x^2 - 1$	
x	$f(x)$
0	–1
1	0
–1	0
2	3
–2	3

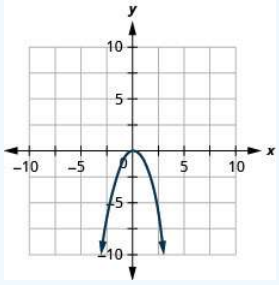
Plot the points, and then connect
them with a smooth curve. The
result will be the graph of the
function $f(x) = x^2 - 1$.



Exercise 4.4.1

Graph $f(x) = -x^2$.

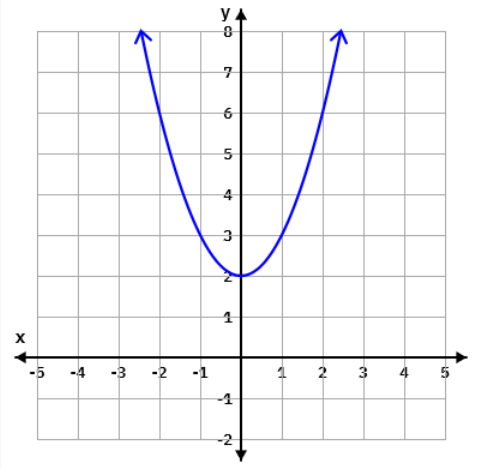
Answer



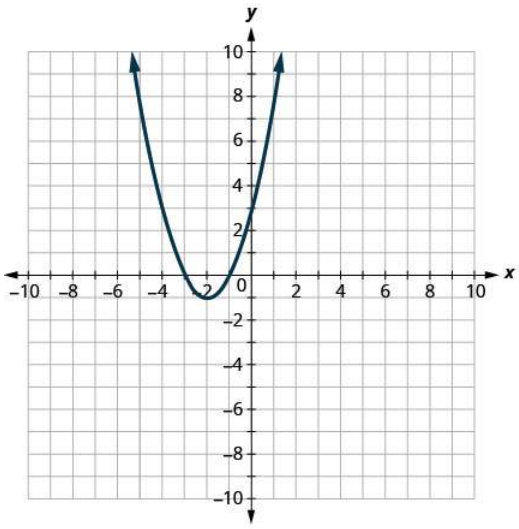
Exercise 4.4.2

Graph $f(x) = x^2 + 2$.

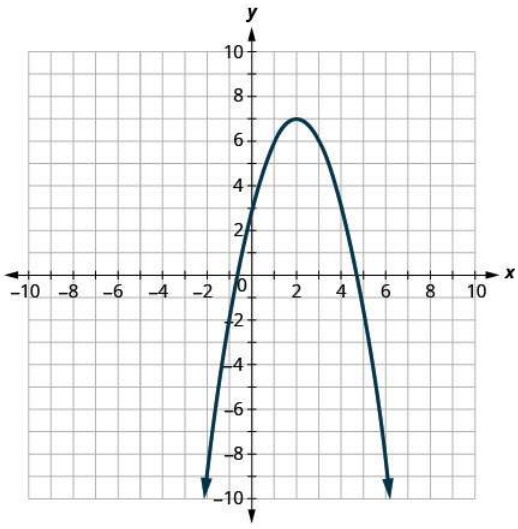
Answer



All graphs of quadratic functions of the form $f(x) = ax^2 + bx + c$ are parabolas that open upward or downward.



$f(x) = a^2 + bx + c$
$f(x) = x^2 + 4x + 3$
$a > 0$
opens upward



$f(x) = a^2 + bx + c$
$f(x) = -x^2 + 4x + 3$
$a < 0$
opens downward

Notice that the only difference in the two equations is the negative sign in the quadratic term ax^2. The negative in front of the ax^2 indicates a vertical reflection and flips the parabola upside-down. Therefore when the quadratic term ax^2 is positive, the parabola opens upward. When the quadratic term ax^2 is negative, the parabola opens downward.

Definition 4.4.2

Parabola Orientation

For the graph of the quadratic function $f(x) = ax^2 + bx + c$:

- If $a > 0$, the parabola opens upward.
- If $a < 0$, the parabola includes a vertical reflection causing the parabola to open downward.

Example 4.4.2

Without graphing, determine whether each parabola opens upward or downward:

a. $f(x) = -3x^2 + 2x - 4$
b. $f(x) = 6x^2 + 7x - 9$

Solution:

a. Find the value of a.

$$f(x) = ax^2 + bx + c$$
$$f(x) = -3x^2 + 2x - 4$$
$$a = -3$$

Since the a is negative, the parabola will open downward.

b. Find the value of a.

$$f(x) = ax^2 + bx + c$$
$$f(x) = 6x^2 + 7x - 9$$
$$a = 6$$

Since the a is positive, the parabola will open upward.

Exercise 4.4.3

Without graphing, determine whether the graph of each function is a parabola that opens upward or downward:

a. $f(x) = 2x^2 + 5x - 2$
b. $f(x) = -3x^2 - 4x + 7$

Answer

a. up
b. down

Exercise 4.4.4

Without graphing, determine whether the graph of each function is a parabola that opens upward or downward:

a. $f(x) = -2x^2 - 2x - 3$
b. $f(x) = 5x^2 - 2x - 1$

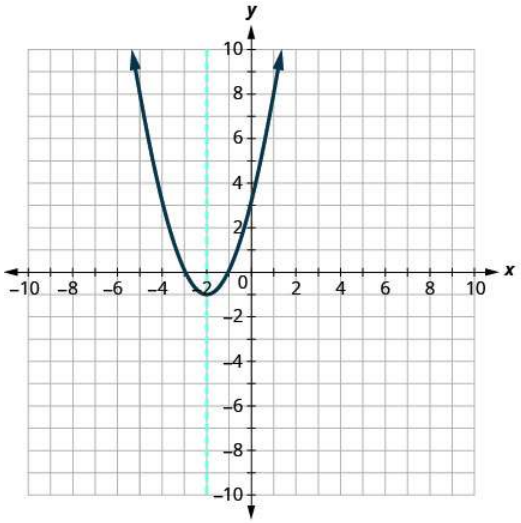
Answer

a. down
b. up

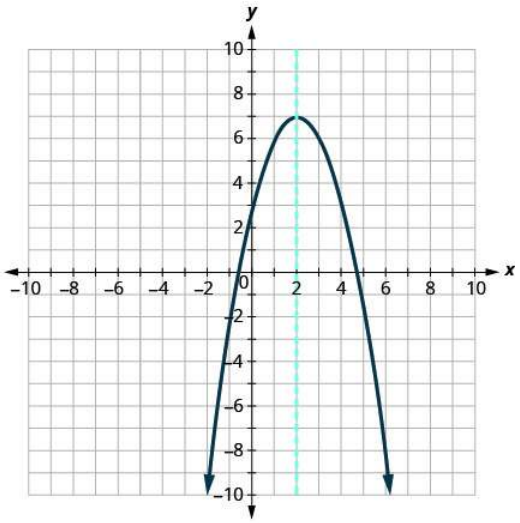
Finding the Axis of Symmetry and Vertex of a Parabola

Look again at *Figure 9.6.10*. Do you see that we could fold each parabola in half and then one side would lie on top of the other? The 'fold line' is a line of symmetry. We call it the **axis of symmetry** of the parabola.

We show the same two graphs again with the axis of symmetry.



$f(x) = x^2 + 4x + 3$



$f(x) = -x^2 + 4x + 3$

The axis of symmetry can be found fairly directly using an algebraic formula. This formula can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result.

The equation of the axis of symmetry of the graph of $f(x) = ax^2 + bx + c$ is $x = -\frac{b}{2a}$. This equation allows us to find the axis of symmetry directly from the equation without even needing to see the graph.

So to find the equation of symmetry of each of the parabolas we graphed above, we will substitute a and b into the formula $x = -\frac{b}{2a}$.

$f(x) = ax^2 + bx + c$	$f(x) = ax^2 + bx + c$
$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
axis of symmetry	axis of symmetry
$x = -\frac{b}{2a}$	$x = -\frac{b}{2a}$
$x = -\frac{4}{2 \cdot 1}$	$x = -\frac{4}{2(-1)}$
$x = -2$	$x = 2$

Notice that these are the equations of the dashed blue lines on the graphs.

The point on the parabola that is the lowest (parabola opens up), or the highest (parabola opens down), lies on the axis of symmetry. This point is called the **vertex** of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its x-coordinate is $-\frac{b}{2a}$. To find the y-coordinate of the vertex we substitute the value of the x-coordinate into the quadratic function.

Definition 4.4.3

The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:

- the **axis of symmetry** is the vertical line $x = -\frac{b}{2a}$.
- the **vertex** is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$
  - the y-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic function.

Example 4.4.3

For the graph of $f(x) = 4x^2 - 16x + 5$ algebraically find:

a. the axis of symmetry
b. the vertex

Solution:

a.

	$f(x)=ax^2+bx+c$ $f(x)=4x^2-16x+5$
The axis of symmetry is the vertical line $x=-\frac{b}{2a}$.	
Substitute the values a, b into the equation.	$x=-\frac{-16}{2\cdot 4}$
Simplify.	$x=2$
	The axis of symmetry is the line $x=2$.

b.

	$f(x)=4x^2-16x+5$
The vertex is a point on the line of symmetry, so its x-coordinate will be $x=2$. Find $f(2)$.	$f(x)=4(2)^2-16(2)+5$
Simplify.	$f(x)=4(4)-16(2)+5$
The result is the y-coordinate.	$f(2)=-11$
	The vertex is $(2,-11)$.

? Exercise 4.4.5

For the graph of $f(x)=2x^2-4x-3$ algebraically find:

a. the axis of symmetry
b. the vertex

Answer

a. $x=1$
b. $(1,-5)$

? Exercise 4.4.6

For the graph of $f(x)=-3x^2-18x-6$ algebraically find:

a. the axis of symmetry
b. the vertex

Answer

a. $x=-3$
b. $(-3,21)$

The axis of symmetry and the vertex can also be found on Desmos.

The vertex will be the turning point of the parabola, which is also the lowest (parabola opens up), or the highest (parabola opens down) point on the function. The axis of symmetry will be the vertical line that includes that vertex.

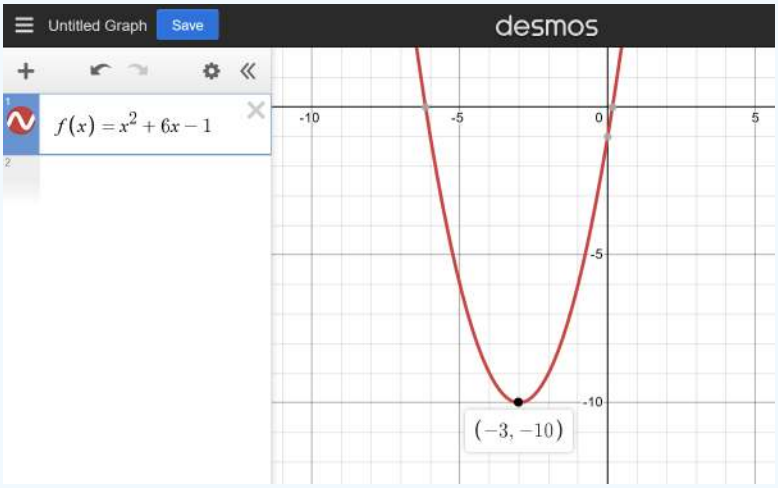
✓ Example 4.4.4

For the graph of $f(x)=x^2+6x-1$ use Desmos to find:

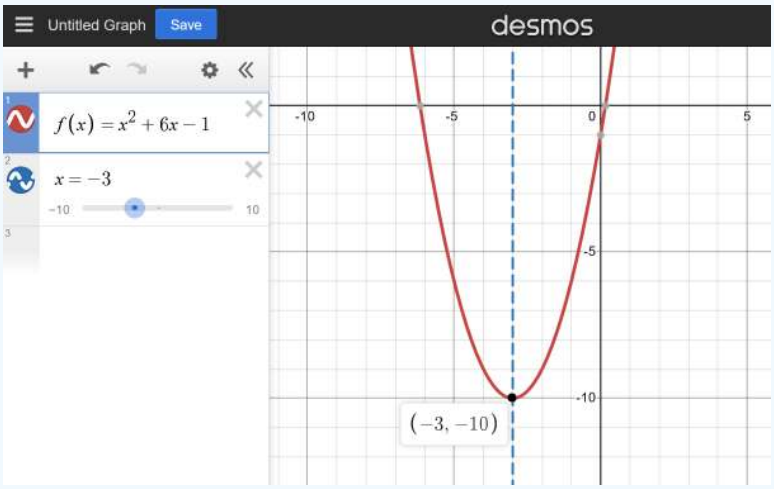
a. the vertex
b. the axis of symmetry

Solution:

a. Begin by graphing the function, and finding the turning point. This turning point is the vertex. The vertex in this problem is (-3,-10).



b. The axis of symmetry is the vertical line that includes the vertex. As every vertical line is written in the form $x=a$, this will always be the x-value of the vertex. In this problem, the axis of symmetry is $x=-3$.



Note that the axis of symmetry won't be shown on Desmos unless you create it as an additional function.

Exercise 4.4.7

For the graph of $f(x)=-2x^2+4x+3$ use Desmos to find:

a. the vertex
b. the axis of symmetry

Answer

a. $(1,5)$
b. $x=1$

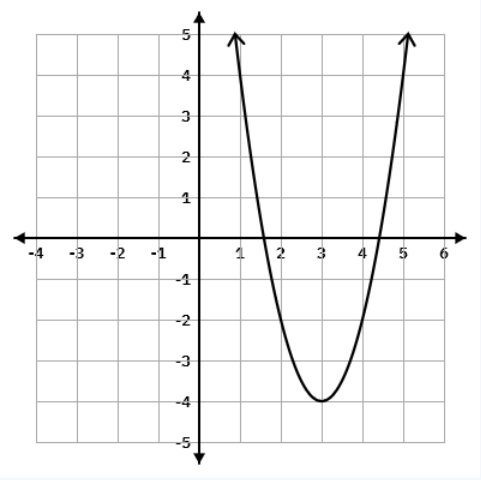
Domain and Range of a Parabola

The graph of each parabola continues forever to the left and right. This means that the domain of any function in the form $f(x)=ax^2+bx+c$ will be $(-\infty,\infty)$.

The range, however, will depend on the a-value and the vertex.

Example 4.4.5

For the graph shown below, identify the domain and range.



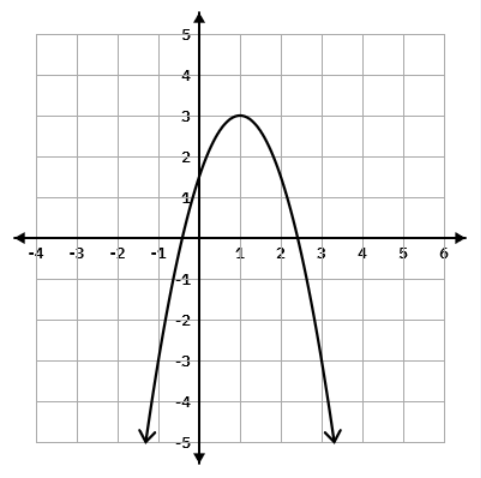
Solution:

The domain of any function is all the possible x-values. Since a parabola continues forever to the left and right, the domain will be $(-\infty, \infty)$.

The range of any function is all the possible y-values. This function includes all y-values above the vertex, but none below the vertex. The vertex includes the minimum y-value of -4. Therefore the range of this function is $[-4, \infty)$. Note the [bracket] on the -4 since that point is included on the graph.

Example 4.4.6

For the graph shown below, identify the domain and range.



Solution:

The domain of any function is all the possible x-values. Since a parabola continues forever to the left and right, the domain will be $(-\infty, \infty)$.

The range of any function is all the possible y-values. This function includes all y-values below the vertex, but none above the vertex. The vertex includes the maximum y-value of 3. Therefore the range of this function is $(-\infty, 3]$. Note the [bracket] on the 3 since that point is included on the graph.

Note that the range of the function depends on the vertex as well as the a-value of the function. If $a > 0$, then the y-values will include the y-value of the vertex and everything higher. It will be expressed as $[f(-\frac{b}{2a}), \infty)$. If $a < 0$, then the y-values will include everything lower than the maximum value, up to and including the maximum value. It will be expressed as $(-\infty, f(-\frac{b}{2a})]$.

Applications of the Vertex, Domain, and Range of a Parabola

The vertex, domain, and range of a parabola have many applications. Since the vertex represents the highest or lowest point of the parabola, this can help us identify maximum or minimum outcomes of a quadratic application. For example, businesses may want to find their maximum profit or minimum cost.

In many applications, negative inputs and/or outputs do not apply. Rather than looking at the algebraic domain and range, we look at **practical domain** and **practical range** as the set of values that make sense in the given scenario.

For example, if x represents the number of items manufactured, it would not be practical to include any negative x-values. Much like linear programming from Section 4.3, we will only allow for x-values where $x \geq 0$.

The output values may or may not be negative. If y represents profit, for example, it IS possible for a company to have a negative profit. If y represents revenue however, it is NOT possible for a company to have a negative revenue.

Example 4.4.7

A company that sells televisions has found their revenue to be determined by $R(t) = -50t^2 + 2000t$, where t is the number of televisions sold.

a. Find the value of t that will produce the maximum revenue.
b. What is the maximum revenue?
c. What is the practical range?
d. What is the practical domain?

Solution:

$$R(t) = -50t^2 + 2000t$$

Note that in this equation:

- t is the input and represents the number of televisions sold. This would be represented by the x-axis of the graph.
- $R(t)$ is the output and represents the revenue. This would be represented by the y-axis of the graph.

a. Find the equation of the axis of symmetry.

$$t = -\frac{b}{2a}$$
$$t = -\frac{2000}{2(-50)}$$
$$t = 20$$

The equation of the axis of symmetry is $t = 20$.

The vertex is on the line $t = 20$.

The maximum revenue occurs when $t = 20$ televisions.

b. To find the maximum revenue, find $R(20)$.

$$R(t) = -50t^2 + 2000t$$
$$R(t) = -50(20)^2 + 2000(20)$$

Use a calculator to simplify.

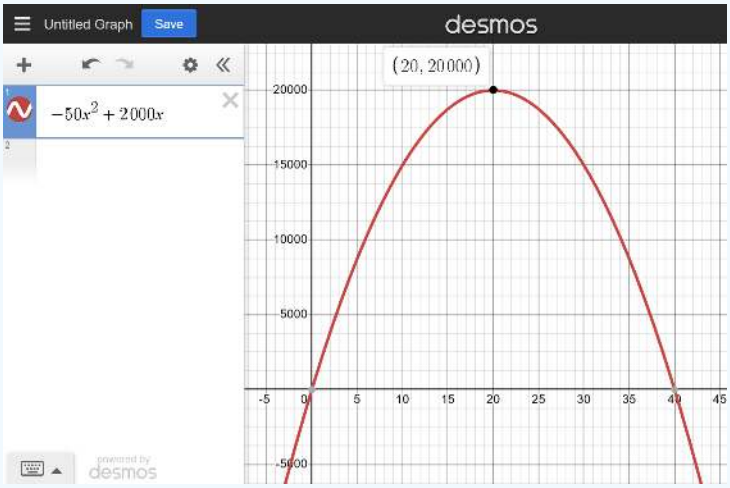
$$R(t) = 20000$$

The vertex is $(20, 20000)$

When the company sells 20 televisions, they will have a revenue of $\$20,000$

c. To find the practical range, we need to consider that the output values represent revenue, which cannot be negative. Therefore the practical range would be all y-values from zero up to the maximum, or $[0, 20000]$

d. The practical domain would be all the input values that are included in the practical range above. We'll look at the algebraic method to find this in the examples below. For now, we can read these values from the graph of the function.



We can see that any x-value over 40 televisions would result in a negative revenue. Therefore the practical domain is $[0, 40]$.

? Exercise 4.4.8

A bakery has found their profit to be determined by $P(c) = -0.5c^2 + 96c - 3047$, where c is the number of cupcakes baked each day.

a. Find the value of c that will produce the maximum profit.
b. What is the maximum profit?

Answer

The bakery will earn a maximum profit of $\$1561$ if 96 cupcakes are baked each day.

Find the Intercepts of a Parabola

When we graphed linear equations, we often used the x- and y-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the y-intercept the value of x is zero. So to find the y-intercept, we substitute $x = 0$ into the function. Likewise, at the x-intercepts the value of y is zero. So to find the x-intercept, we substitute $y = 0$ into the function.

To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$:

- ***y*-intercept:** Let $x = 0$ and solve for $f(x)$.
- ***x*-intercepts:** Let $f(x) = 0$ and solve for x.
  - An x-intercept results when the value of $f(x)$ is zero. To find an x-intercept, we let $f(x) = 0$. In other words, we will need to solve the equation $0 = ax^2 + bx + c$ for x. While several methods exist to solve quadratic equations like this, we will rely on the **quadratic formula** for the purposes of this course.

Definition 4.4.4

The **quadratic formula** states that if $ax^2 + bx + c = 0$ and $a \neq 0$,

$$\text{then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ provided that } b^2 - 4ac \geq 0$$

Example 4.4.8

Find the intercepts of the parabola whose function is $f(x) = x^2 - 2x - 8$.

Solution:

To find the ***y*-intercept**, let $x = 0$ and solve for $f(x)$.	$f(x) = x^2 - 2x - 8$
	$f(0) = 0^2 - 2 \cdot 0 - 8$
	$f(0) = -8$
	When $x = 0$, then $f(0) = -8$. The y-intercept is the point $(0, -8)$.
To find the ***x*-intercept**, let $f(x) = 0$ and solve for x.	$f(x) = x^2 - 2x - 8$
	$0 = x^2 - 2x - 8$
Use the Quadratic Formula to solve for x.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$a = 1, b = -2, c = -8$	$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-8)}}{2(1)}$
Simplify.	$x = \frac{2 \pm \sqrt{36}}{2}$
	$x = \frac{2 \pm 6}{2}$
	$x = 1 \pm 3$
	$x = 4$ or $x = -2$
	When $f(x) = 0$, then $x = 4$ or $x = -2$. The x-intercepts are the points $(4, 0)$ and $(-2, 0)$.

Exercise 4.4.9

Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 12$.

Answer

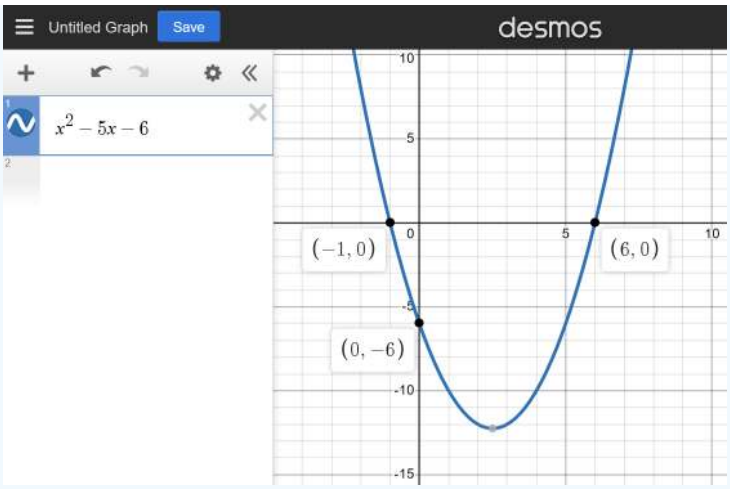
y -intercept: $(0, -12)$ x -intercepts $(-2, 0), (6, 0)$

For the purposes of this course, we can also use Desmos to identify the intercepts of the function.

Example 4.4.9

Find the intercepts of the parabola whose function is $f(x) = x^2 - 5x - 6$.

Solution:



The x-intercepts are $(-1, 0)$ and $(6, 0)$.

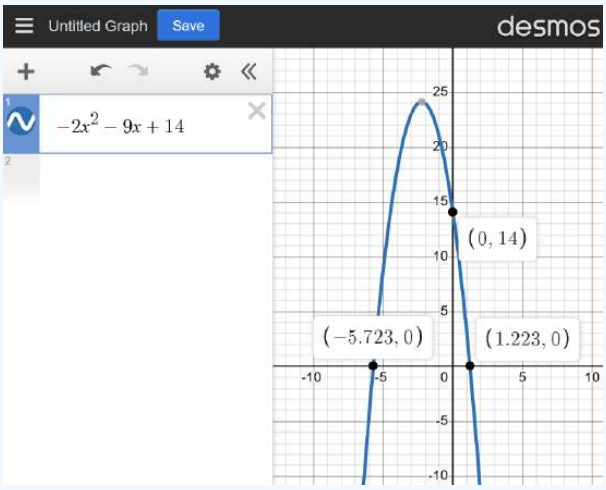
The y-intercept is $0, -6)$.

Note that in many cases, the intercepts are not rational numbers. This is ok!

Example 4.4.10

Find the intercepts of the parabola whose function is $f(x) = -2x^2 - 9x + 14$.

Solution:



The x-intercepts are $(-5.723, 0)$ and $(1.223, 0)$.

The y-intercept is $0, 14)$.

Finding the Break-Even Points

Intercepts also have many business applications. Remember that an *x*-intercept is the value of *x* when the function equals zero. The point at which revenue and cost are equal is known as the **break-even point**. This is also where the profit equals zero.

Example 4.4.11

The television company from example 7 has found their profit to be determined by $P(t) = -50t^2 + 1500t - 4000$, where t is the number of televisions sold.

a. Find the break-even points, rounding to the nearest whole number if necessary.

b. What range of television sales will result in a profit? A loss?
c. What does the y-intercept represent?

Solution:

$$P(t) = -50t^2 + 1500t - 4000$$

a. The break-even points exist when the profit is \$0. On the parabola, this exists when y is zero, or at the x-intercepts. Substitute 0 in for $P(t)$.

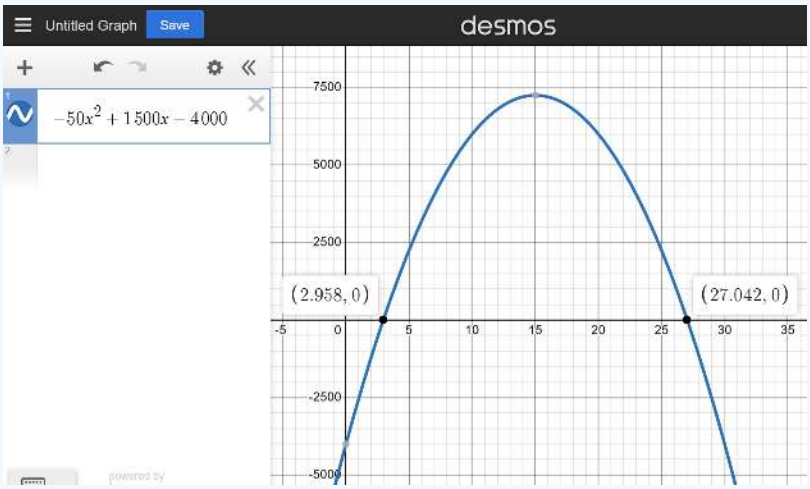
$$P(t) = -50t^2 + 1500t - 4000$$
$$0 = -50t^2 + 1500t - 4000$$

Use the quadratic formula to solve for t.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$t = \frac{-1500 \pm \sqrt{(1500)^2 - 4(-50)(-4000)}}{2(-50)}$$
$$t = \frac{-1500 \pm \sqrt{1450000}}{-100}$$
$$t \approx \frac{-1500 \pm 1204.2}{-100}$$
$$t \approx \frac{-295.8}{-100} \text{ or } t \approx \frac{-2704.2}{-100}$$
$$t \approx 2.958 \text{ or } t \approx 27.042$$

The x-intercepts of the parabola are approximately $(0, 3)$ and $(0, 27)$.

The company will break even, or make a profit of \$0, when either 3 or 27 televisions are sold.



b. Since the a value of the function is negative, the parabola will open *downward*. This means that the profit will be positive between the two x-intercepts and will be negative outside of the x-intercepts.

The company will make a profit by selling more than 3 and less than 27 televisions.

The company have a negative profit, or will lose money, by selling less than than 3 or more than 27 televisions.

c. The y-intercept is $(0, -4000)$. If the company sells 0 televisions, it will have a profit of $-\$4,000$.

Exercise 4.4.10

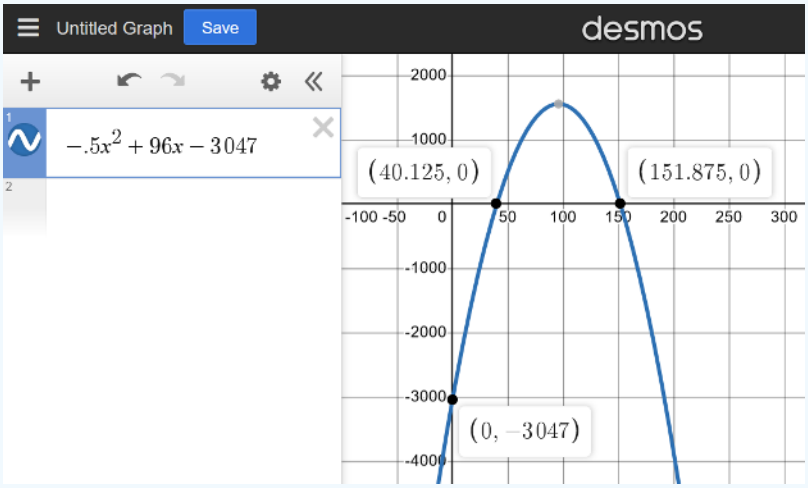
The bakery from exercise 8 has found their profit to be determined by $P(c) = -0.5c^2 + 96c - 3047$, where c is the number of cupcakes baked each day.

a. Find the break-even points, rounding to the nearest whole number if necessary.
b. What range of cupcake production will result in a profit? A loss?

c. What does the y-intercept represent?

Answer

a. The break-even points occur if the bakery makes 40 or 152 cupcakes each day.
b. The bakery will earn a profit if it bakes more than 40 and less than 152 cupcakes each day; it will lose money if it bakes less than 40 or more than 152 cupcakes each day.
c. If the bakery does not bake any cupcakes, it will have a profit of $-\$3,047$ that day.



Key Concepts

- Parabola Orientation
  - For the graph of the quadratic function $f(x)=ax^2+bx+c$, if
    - $a>0$, the parabola opens upward.
    - $a<0$, the parabola opens downward.
- Axis of Symmetry and Vertex of a Parabola The graph of the function $f(x)=ax^2+bx+c$ is a parabola where:
  - the axis of symmetry is the vertical line $x=-\frac{b}{2a}$.
  - the vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$.
  - the y-coordinate of the vertex is found by substituting $x=-\frac{b}{2a}$ into the quadratic equation.
- Find the Intercepts of a Parabola
  - To find the intercepts of a parabola whose function is $f(x)=ax^2+bx+c$:
    - **y-intercept**
      - Let $x=0$ and solve for $f(x)$.
    - **x-intercepts**
      - Let \(f(x)=0\) and solve for x.
- Minimum or Maximum Values of a Quadratic Equation
  - The y-coordinate of the vertex of the graph of a quadratic equation is the
  - *minimum* value of the quadratic equation if the parabola opens *upward*.
  - *maximum* value of the quadratic equation if the parabola opens *downward*.

Glossary

quadratic function

A quadratic function, where a, b, and c are real numbers and $a\neq 0$, is a function of the form $f(x)=ax^2+bx+c$.

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4.4E: Exercises - Quadratic Functions and Applications

Practice Makes Perfect

Exercises 1 - 8: Recognize the Graph of a Quadratic Function

For each of the following exercises, determine if the parabola opens up or down.

1. $f(x) = -2x^2 - 6x - 7$
2. $f(x) = 6x^2 + 2x + 3$
3. $f(x) = 4x^2 + x - 4$
4. $f(x) = -9x^2 - 24x - 16$
5. $f(x) = -3x^2 + 5x - 1$
6. $f(x) = 2x^2 - 4x + 5$
7. $f(x) = x^2 + 3x - 4$
8. $f(x) = -4x^2 - 12x - 9$

Answer

1. down
3. up
5. down
7. up

Exercises 9 - 12: Find the Axis of Symmetry and Vertex of a Parabola

In the following functions, find each of the following. Round answers to the nearest tenth where necessary.

a. The vertex of its graph
b. The equation of the axis of symmetry
c. The x-intercept(s)
d. The y-intercept
e. The domain
f. The range

9. $f(x) = x^2 + 8x - 1$
10. $f(x) = x^2 + 10x + 25$
11. $f(x) = -3x^2 + 2x + 9$
12. $f(x) = -2x^2 - 12x - 3$

Answer

9.

a. Vertex: $(-4, -17)$
b. Axis of symmetry: $x = -4$
c. x-intercepts: $(-8.1, 0), (0.1, 0)$
d. y-intercept: $(0, -1)$
e. Domain: $(-\infty, \infty)$
f. Range: $[-17, \infty)$

11.

a. Vertex: $(0.3, 9.3)$
b. Axis of symmetry: $x = 0.3$

c. x-intercepts: $(-1.4, 0), (2.1, 0)$
d. y-intercept: $(0, 9)$
e. Domain: $(-\infty, \infty)$
f. Range: $(-\infty, 9.3]$

Exercises 13-18: Solve Maximum and Minimum Applications

In the following exercises, find the maximum or minimum value of each function.

13. $f(x) = 2x^2 + x - 1$

14. $y = -4x^2 + 12x - 5$

15. $y = x^2 - 6x + 15$

16. $y = -x^2 + 4x - 5$

17. $y = -9x^2 + 16$

18. $y = 4x^2 - 49$

Answer

13. The minimum value is $-\frac{9}{8}$ when $x = -\frac{1}{4}$.

15. The maximum value is 6 when $x = 3$.

17. The maximum value is 16 when $x = 0$.

Exercises 19 - 23: Solve Maximum and Minimum Applications

In the following exercises, solve. Round answers to the nearest tenth.

19. A computer store owner estimates that by charging x dollars each for a certain computer, he can sell $40 - x$ computers each week. The quadratic function $R(x) = -x^2 + 40x$ is used to find the revenue, R, received when the selling price of a computer is x, Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

20. A cell phone company estimates that by charging x dollars each for a certain cell phone, they can sell $8 - x$ cell phones per day. Use the quadratic function $R(x) = -x^2 + 8x$ to find the revenue received per day when the selling price of a cell phone is x. Find the selling price that will give them the maximum revenue per day, and then find the amount of the maximum revenue.

21. A land owner is planning to build a fenced in rectangular patio behind his garage, using his garage as one of the "walls." He wants to maximize the area using 80 feet of fencing. The quadratic function $A(x) = x(80 - 2x)$ gives the area of the patio, where x is the width of one side. Find the maximum area of the patio.

22. A family of three young children just moved into a house with a yard that is not fenced in. The previous owner gave them 300 feet of fencing to use to enclose part of their backyard. Use the quadratic function $A(x) = x(300 - 2x)$ to determine the maximum area of the fenced in yard.

23. A ball is thrown vertically upward from the ground with an initial velocity of 109 ft/sec. Use the quadratic function $h(t) = -16t^2 + 109t + 0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.

Answer

19. 20 computers will give the maximum of $400 in receipts.

21. The maximum area of the patio is 800 feet.

23. In 3.4 seconds the ball will reach its maximum height of 185.6 feet.

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4.5: Chapter 4 Review

PROBLEM SET: CHAPTER 4 REVIEW

Functions and Function Notation (4.1)

1. Consider the set of points $3,5), (5,4), (2,9), (0,5), (7,9.)$
   a. Find the domain.
   b. Find the range.
   c. Do these points represent a function?
2. For the function $f(x) = -x^2 - 8$, evaluate each of the following.
   a. $f(3)$
   b. $f(-3)$
3. For the function $g(x) = -3x + 4$, evaluate each of the following.
   a. $g(2)$
   b. $g(x) + 2$
   c. $g(x) + 2$

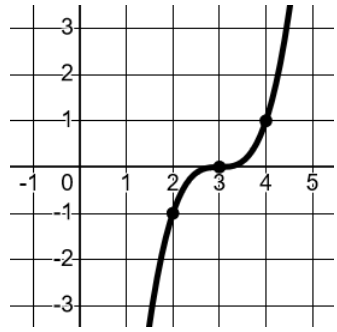
Understanding the Basic Functions (4.2)

4. Without relying on technology, sketch a graph with at least 3 points of each of the following functions.
   a. $f(x) = x^3$
   b. $f(x) = |x|$
   c. $f(x) = x$
   d. $f(x) = \sqrt{x}$
   e. $f(x) = c$
   f. $f(x) = \dfrac{1}{x}$
   g. $f(x) = x^2$
5. Given the function $f(xt) = \begin{cases} 5x - 7 & \text{if } x \leq -1 \\ -2x^2 + 5 & \text{if } x > 0 \end{cases}$, find each of the following values.
   a. $f(-4)$
   b. $f(-1)$
   c. $f(3)$

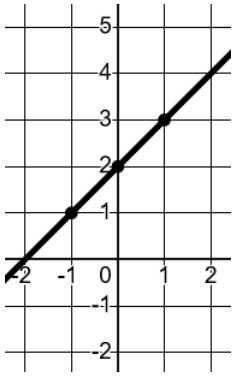
Transformations of Functions (4.3)

6. For each function below, identify the basic function, describe the transformations, and sketch a graph of the transformed function.
   a. $f(x) = \sqrt{x + 9}$
   b. $g(x) = \dfrac{1}{x} - 2$
   c. $h(x) = |x - 4| + 3$
   d. $j(x) = -x^2 - 8$
7. For each function shown below, identify the basic function, describe the transformations, and find an equation for the function.

a.



b.



Quadratic Functions and Their Applications (4.4)

8. How can you determine the direction of the graph of $f(x) = -3x^2 + 5x - 4$ without even graphing it?
9. Use Desmos to find the following characteristics of $g(x) = 2x^2 - 8x + 3$.
    a. Vertex
    b. Axis of symmetry
    c. y-intercept
    d. x-intercepts
    e. Domain
    f. Range
10. The daily profit for a manufacturing company is modeled by the function $P(x) = -0.6x^2 + 168x - 9375$, where x is the number of items manufactured.
    a. What is the company's profit if zero items are manufactured in a day?
    b. What is the company's profit if 90 items are manufactured in a day?
    c. What is the range of items that must be made each day in order to make any profit?
    d. What is the maximum daily profit they can make, and how many items do they need to manufacture in order to make that profit?
    e. What do the x-intercepts of the graph represent in this scenario?

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CHAPTER OVERVIEW

5: Exponential and Logarithmic Functions

Learning Objectives

In this chapter, you will

1. Examine exponential and logarithmic functions and their properties
2. Identify exponential growth and decay functions and use them to model applications
3. Use the natural base e to represent an exponential functions
4. Use logarithmic functions to solve equations involving exponential functions

5.1: Exponential Growth and Decay Models

5.1E: Exercises - Exponential Growth and Decay Models

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5.1: Exponential Growth and Decay Models

Learning Objectives

In this section, you will learn to

- Compare linear and exponential growth.
- Recognize and model exponential growth and decay.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Simplify $3(2)^3$.

Click here to check your answer

24

If you missed this problem, review here and scroll to Example 4. (Note that this will open a different textbook in a new window.)

2. Simplify $-5(3)^2$.

Click here to check your answer

-45

If you missed this problem, review here and scroll to Example 4. (Note that this will open a different textbook in a new window.)

3. Solve $2^x = 16$.

Click here to check your answer

$x = 4$

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

4. Consider the linear function $f(x) = 2x + 3$.

a) Create a table of values including $f(0), f(1), f(2), f(3)$ and $f(4)$.

b) How does the y-intercept of 3 impact the values in the table?

c) How does the slope of 2 impact the values in the table?

Click here to check your answer

a)

x	$f(x)$
0	3
1	5
2	7
3	9
4	11

b) The y-intercept gives the starting y-value of 3, or the y-value when $x = 0$.

c) The slope of 2 makes each subsequent y-value increase by 2.

If you missed this problem, review Section 1.2. (Note that this will open in a new window.)

Comparing Exponential and Linear Growth

Consider two social media sites which are expanding the number of users they have:

- Site A has 10,000 users, and expands by adding 1,500 new users each month
- Site B has 10,000 users, and expands by increasing the number of users by 10% each month.

The number of users for Site A can be modeled as linear growth. The number of users increases by a constant number, 1500, each month. If x = the number of months that have passed and y is the number of users, the number of users after x months is $y = 10000 + 1500x$. For site B, the user base expands by a constant percent each month, rather than by a constant number. Growth that occurs at a constant percent each unit of time is called exponential growth.

We can look at growth for each site to understand the difference. The table compares the number of users for each site for 12 months. The table shows the calculations for the first 4 months only, but uses the same calculation process to complete the rest of the 12 months.

Month	Users at Site A	Users at Site B
0	10000	10000
1	$10000 + 1500 = 11500$	$10000 + 10\%$ of 10000 $= 10000 + 0.10(10000)$ $= 10000(1.10) = 11000$
2	$11500 + 1500 = 13000$	$11000 + 10\%$ of 11000 $= 11000 + 0.10(11000)$ $= 11000(1.10) = 12100$
3	$13000 + 1500 = 14500$	$12100 + 10\%$ of 12100 $= 12100 + 0.10(12100)$ $= 12100(1.10) = 13310$
4	$14500 + 1500 = 16000$	$13310 + 10\%$ of 13310 $= 13310 + 0.10(13310)$ $= 13310(1.10) = 14641$
5	17500	16105
6	19000	17716
7	20500	19487
8	22000	21436
9	23500	23579
10	25000	25937
11	26500	28531
12	28000	31384

For Site B, we can re-express the calculations to help us observe the patterns and develop a formula for the number of users after x months.

- Month 1: $y = 10000(1.1) = 11000$
- Month 2: $y = 11000(1.1) = 10000(1.1)(1.1) = \mathbf{10000(1.1)^2} = 12100$
- Month 3: $y = 12100(1.1) = 10000(1.1)^2(1.1) = \mathbf{10000(1.1)^3} = 13310$
- Month 4: $y = 13310(1.1) = 10000(1.1)^3(1.1) = \mathbf{10000(1.1)^4} = 14641$

By looking at the patterns in the calculations for months 2, 3, and 4, we can generalize the formula. After x months, the number of users y is given by the function $\mathbf{y = 10000(1.1)^x}$

Example 5.1.1

Determine whether each of the following statements represent exponential growth or linear growth.

a. The population of a certain town has been increasing by 2.4% over each of the last 10 years.
b. The population of a certain town has been increasing by 1,000 people over each of the last 10 years.
c. The number of bacterial cells increases by a factor of $\frac{3}{2}$ every 24 hours.

Solution

a. Since the population has been increasing by a constant percent for each unit of time, this is an example of exponential growth.

b. Since the population has been increasing by a constant number for each unit of time, this is an example of linear growth.

c. Since the number of cells has been increasing by a constant percent ($\frac{3}{2} = 150\%$) for each unit of time, this is an example of exponential growth.

Using Exponential Functions to Model Growth and Decay

In exponential growth, the value of the dependent variable y increases at a constant percentage rate as the value of the independent variable (x or t) increases. Examples of exponential growth functions include:

- the number of residents of a city or nation that grows at a constant percent rate.
- the amount of money in a bank account that earns interest if money is deposited at a single point in time and left in the bank to compound without any withdrawals.

In exponential decay, the value of the dependent variable y decreases at a constant percentage rate as the value of the independent variable (x or t) increases. Examples of exponential decay functions include:

- value of a car or equipment that depreciates at a constant percent rate over time
- the amount a drug that still remains in the body as time passes after it is ingested
- the amount of radioactive material remaining over time as a radioactive substance decays.

Exponential functions often model quantities as a function of time; thus we often use the letter t as the independent variable instead of x.

The table compares exponential growth and exponential decay functions:

Exponential Growth	Exponential Decay
Quantity grows by a constant percent per unit of time	Quantity decreases by a constant percent per unit of time
$\mathbf{y = ab^x}$ • a is a positive number representing the initial value of the function when $x = 0$ • b is a real number that is greater than 1: $b > 1$ • the growth rate r is a positive number, $r > 0$ where $b = 1 + r$ (so that $r = b - 1$)	$\mathbf{y = ab^x}$ • a is a positive number representing the initial value of the function when $x = 0$ • b is a real number that is between 0 and 1: $0 < b < 1$ • the decay rate r is a negative number, $r < 0$ where $b = 1 + r$ (so that $r = b - 1$)

In general, the domain of exponential functions is the set of all real numbers. The range of an exponential growth or decay function is the set of all positive real numbers.

In most applications, the independent variable, x or t, represents time. When the independent variable represents time, we may choose to restrict the domain so that independent variable can have only non-negative values in order for the application to make sense. If we restrict the domain, then the range is also restricted as well.

- For an exponential growth function $y = ab^x$ with $b > 1$ and $a > 0$, if we restrict the domain so that $x \geq 0$, then the range is $y \geq a$.
- For an exponential decay function $y = ab^x$ with $0 < b < 1$ and $a > 0$, if we restrict the domain so that $x \geq 0$, then the range is $0 < y \leq a$.

Example 5.1.2

Consider the growth models for social media sites A and B, where x = number of months since the site was started and y = number of users. The number of users for Site A follows the linear growth model:

$$y = 10000 + 1500x.$$

The number of users for Site B follows the exponential growth model:

$$y = 10000(1.1^x)$$

For each site, use the function to calculate the number of users at the end of the first year, to verify the values in the table. Then use the functions to predict the number of users after 30 months.

Solution

Since x is measured in months, then $x = 12$ at the end of one year.

Linear Growth Model:

When $x = 12$ months, then $y = 10000 + 1500(12) = 28,000$ users
When $x = 30$ months, then $y = 10000 + 1500(30) = 55,000$ users

Exponential Growth Model:

When $x = 12$ months, then $y = 10000(1.1^{12}) = 31,384$ users
When $x = 30$ months, then $y = 10000(1.1^{30}) = 174,494$ users

We see that as x, the number of months, gets larger, the exponential growth function grows large faster than the linear function (even though in Example 5.1.2 the linear function initially grew faster). This is an important characteristic of exponential growth: exponential growth functions always grow faster and larger in the long run than linear growth functions.

It is helpful to use function notation, writing $y = f(t) = ab^t$, to specify the value of t at which the function is evaluated.

Example 5.1.3

A forest has a population of 2000 squirrels that is increasing at the rate of 3% per year. Let t = number of years and $y = f(t) =$ number of squirrels at time t.

a. Find the exponential growth function that models the number of squirrels in the forest at the end of t years.
b. Use the function to find the number of squirrels after 5 years and after 10 years

Solution

a. The exponential growth function is $y = f(t) = ab^t$, where $a = 2000$ because the initial population is 2000 squirrels

The annual growth rate is 3% per year, stated in the problem. We will express this in decimal form as $r = 0.03$

Then $b = 1 + r = 1 + 0.03 = 1.03$

Answer: The exponential growth function is $y = f(t) = 2000(1.03)^t$

b. After 5 years, the squirrel population is $y = f(5) = 2000(1.03)^5 \approx 2319$ squirrels

After 10 years, the squirrel population is $y = f(10) = 2000(1.03)^{10} \approx 2688$ squirrels

Example 5.1.4

A large lake has a population of 1000 frogs. Unfortunately the frog population is decreasing at the rate of 5% per year. Let t = number of years and $y = g(t)$ = the number of frogs in the lake at time t.

a. Find the exponential decay function that models the population of frogs.
b. Calculate the size of the frog population after 10 years.

Solution

a. The exponential decay function is $y = g(t) = ab^t$, where $a = 1000$ because the initial population is 1000 frogs

The annual decay rate is 5% per year, stated in the problem. The words decrease and decay indicated that r is negative. We express this as $r = -0.05$ in decimal form.

Then, $b = 1 + r = 1 + (-0.05) = 0.95$

Answer: The exponential decay function is: $y = g(t) = 1000(0.95)^t$

b. After 10 years, the frog population is $y = g(10) = 1000(0.95)^{10} \approx 599$ frogs

Example 5.1.5

A population of bacteria is given by the function $y = f(t) = 100(2)^t$, where t is time measured in hours and y is the number of bacteria in the population.

a. What is the initial population?
b. What happens to the population in the first hour?
c. How long does it take for the population to reach 800 bacteria?

Solution

a. The initial population is 100 bacteria. We know this because $a = 100$ and because at time $t = 0$, then $f(0) = 100(2)^0 = 100(1) = 100$

b. At the end of 1 hour, the population is $y = f(1) = 100(2)^1 = 100(2) = 200$ bacteria.
The population has doubled during the first hour.

c. We need to find the time t at which $f(t) = 800$. Substitute 800 as the value of y:

$$y = f(t) = 100(2)^t$$
$$800 = 100(2)^t$$

Divide both sides by 100 to isolate the exponential expression on the one side

$$8 = 1(2)^t$$

$8 = 2^3$, so it takes $t = 3$ hours for the population to reach 800 bacteria.

Two important notes about Example 5.1.5:

- In solving $8 = 2^t$, we "knew" that t is 3. But we usually can **not** know the value of the variable just by looking at the equation. Later we will use logarithms to solve equations that have the variable in the exponent.
- To solve $800 = 100(2)^t$, we divided both sides by 100 to isolate the exponential expression 2^t. We can not multiply 100 by 2. The exponent applies **only** to the quantity immediately before it, so the exponent t applies only to the base of 2.

Comparing Linear, Exponential and Polynomial Functions

To identify the type of function from its formula, we need to carefully note the position that the variable occupies in the formula.

A linear function can be written in the form $\mathbf{y = ax + b}$

As we studied in chapter 1, there are other forms in which linear equations can be written, but linear functions can all be rearranged to have form $y = mx + b$.

An exponential function has form $\mathbf{y = ab^x}$

The variable x is in the exponent. The base b is a positive number.

- If $b > 1$, the function represents exponential growth.
- If $0 < b < 1$, the function represents exponential decay

A polynomial function has form $\mathbf{y = ax^P + bx^{P-1} + cx^{P-2} + \ldots}$

The variable x is in the base. The exponent p is a non-zero number, and the exponents decrease until the final exponent is zero.

We compare three functions, using increasing x values:

- linear function $y = f(x) = 2x$
- exponential function $y = g(x) = 2^x$
- polynomial function $y = h(x) = x^2$

$\mathbf{x}$	$\mathbf{y = f(x) = 2x}$	$\mathbf{y = g(x) = 2^x}$	$\mathbf{y = h(x) = x^2}$
0	0	1	0
1	2	2	1
2	4	4	4
3	6	8	9
4	8	16	16
5	10	32	25
6	12	64	36
10	20	1024	100
Type of function	Linear $y = mx + b$	Exponential $y = ab^x$	Polynomial$y = cx^P$
How to recognize equation for this type of function.	all terms are first degree; m is slope; b is the y intercept	base is a number $b > 0$; the variable is in the exponent	variable is in the base; exponent is a number $p \neq 0$
	For equal intervals of change in x, y increases by a constant amount	For equal intervals of change in x, y increases by a constant ratio	

For the functions in the previous table: linear function $y = f(x) = 2x$, exponential function $y = g(x) = 2^x$, and polynomial function $y = h(x) = x^2$, if we restrict the domain to $x \geq 0$ only, then all these functions are growth functions. When $x \geq 0$, the value of y increases as the value of x increases.

The exponential growth function grows large faster than the linear and polynomial functions, as x gets large. This is always true of exponential growth functions, as x gets large enough.

Example 5.1.6

Classify the functions below as exponential, linear, or polynomial functions.

a. $y = 10x^3$
b. $y = -200 - 30x$
c. $y = 1000(1.05)^x$
d. $y = 500(0.75)^x$
e. $y = -0.2x^4$
f. $y = 5x - 1$
g. $y = 6x^2 + 3x$

Solution:

The exponential functions are

c. $y = 1000(1.05)^x$ The variable is in the exponent; the base is the number $b = 1.05$

d. $y = 500(0.75)^x$ The variable is in the exponent; the base is the number $b = 0.75$

The linear functions are

b. $y = -200 - 30x$

f. $y = 5x - 1$

The polynomial functions are

a. $y = 10x^3$ The variable is the base; the exponent is a fixed number, $p = 3$.

e. $y = -0.2x^4$ The variable is the base; the exponent is a number, $p = 4$.

g. $y = 6x^2 + 3x$ The variable is the base; the exponent is a number, $p = 2$.

NATURAL BASE: *e*

The number *e* is often used as the base of an exponential function. *e* is called the natural base.

***e* is approximately 2.71828**

e is an irrational number with an infinite number of decimals; the decimal pattern never repeats. Section 8.2 includes an example that shows how the value of *e* is developed and why this number is mathematically important.

When *e* is the base in an exponential growth or decay function, it is referred to as **continuous growth or continuous decay**. We will use *e* in Chapter 8 in financial calculations when we examine interest that compounds continuously.

Any exponential function can be written in the form $y = ae^{kx}$

k is called the continuous growth or decay rate.

- If $k > 0$, the function represents exponential growth
- If $k < 0$, the function represents exponential decay

a is the initial value

We can rewrite the function in the form $y = ab^x$, where $b = e^k$

In general, if we know one form of the equation, we can find the other forms. For now, we have not yet covered the skills to find k when we know b. After we learn about logarithms later in this chapter, we will find k using natural log: $k = \ln b$.

The table below summarizes the forms of exponential growth and decay functions.

	$y = ab^x$	$y = a(1+r)^x$	$y = ae^{kx}$, $k \neq 0$
Initial value	a>0	a>0	a>0
Relationship between b, r, k	b > 0	b=1+ r	b = e^k and k = ln b
Growth	b > 1	r > 0	k > 0
Decay	0 < b < 1	r < 0	k < 0

Example 5.1.7

The value of houses in a city are increasing at a continuous growth rate of 6% per year. For a house that currently costs $400,000:

a. Write the exponential growth function in the form $y = ae^{kx}$.
b. What would be the value of this house 4 years from now?
c. Rewrite the exponential growth function in the form $y = ab^x$.
d. Find and interpret r.

Solution

a. The initial value of the house is a = $400000

The problem states that the **continuous** growth rate is 6% per year, so k = 0.06

The growth function is : $y = 400000e^{0.06x}$

b. After 4 years, the value of the house is $y = 400000e^{0.06(4)}$ = $508,500.

c. To rewrite $y = 400000e^{0.06x}$ in the form $y = ab^x$, we use the fact that $b = e^k$.

$$\begin{aligned} \mathrm{b} &= e^{0.06} \\ \mathrm{b} &= 1.06183657 \approx 1.0618 \\ \mathrm{y} &= 400000(1.0618)^{\mathrm{x}} \end{aligned}$$

d. To find r, we use the fact that $b = 1 + r$

$$\begin{aligned} \mathrm{b} &= 1.0618 \\ 1 + \mathrm{r} &= 1.0618 \\ \mathrm{r} &= 0.0618 \end{aligned}$$

The value of the house is **increasing** at an **annual rate** of 6.18%.

Example 5.1.8

Suppose that the value of a certain model of new car decreases at a continuous decay rate of 8% per year. For a car that costs $20,000 when new:

a. Write the exponential decay function in the form $y = ae^{kx}$.
b. What would be the value of this car 5 years from now?
c. Rewrite the exponential decay function in the form $y = ab^x$.
d. Find and interpret r.

Solution

a. The initial value of the car is a = $20000

The problem states that the **continuous** decay rate is 8% per year, so k = -0.08

The growth function is : $y = 20000e^{-0.08x}$

b. After 5 years, the value of the car is $y = 20000e^{-0.08(5)}$ = $13,406.40.

c. To rewrite $y = 20000e^{-0.08x}$ in the form $y = ab^x$, we use the fact that $b = e^k$.

$$\begin{aligned} \mathrm{b} &= e^{-0.08} \\ \mathrm{b} &= 0.9231163464 \approx 0.9231 \\ \mathrm{y} &= 20000(0.9231)^{\mathrm{x}} \end{aligned}$$

d. To find r, we use the fact that $b = 1 + r$

$$\begin{aligned} \mathrm{b} &= 0.9231 \\ \mathrm{l} + \mathrm{r} &= 0.9231 \\ \mathrm{r} &= 0.9231 - 1 = -0.0769 \end{aligned}$$

The value of the car is **decreasing** at an **annual rate** of 7.69%.

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5.1E: Exercises - Exponential Growth and Decay Models

PROBLEM SET: EXPONENTIAL GROWTH AND DECAY FUNCTIONS

Identify each as an exponential, linear, or polynomial function.

1. $y = 640\,(1.25^x)$	2. $y = 640\,(x^{1.25})$
3. $y = 640(1.25x)$	4. $y = 1.05x - 2.5$
5. $y = 90 - (4/5)x$	6. $y = 42\,(0.92^x)$
7. $y = 37\,(x^{0.25})$	8. $y = 4(1/3)^x$

Indicate if the function represents exponential growth or exponential decay.

9. $y = 127\mathrm{e}^{-0.35\mathrm{t}}$	10. $y = 70\,(0.8^t)$
11. $y = 453\,(1.2^t)$	12. $y = 16e^{0.2t}$

In each of the following, y is an exponential function of t stated in the form $y = ae^{kt}$ where t represents time measured in years.. For each:

a. re-express each function in the form $y = ab^t$ (state the value of b accurate to 4 decimal places)
b. state the annual growth rate or annual decay rate as a percent, accurate to 2 decimal places

13. $y = 127\mathrm{e}^{-0.35\mathrm{t}}$	14. $y = 16e^{0.4t}$
15. $y = 17250\mathrm{e}^{0.24\mathrm{t}}$	16. $y = 4700\mathrm{e}^{-0.07t}$

Identify if the function represents exponential growth, exponential decay, linear growth, or linear decay. In each case write the function and find the value at the indicated time.

17. A house was purchased for $350,000 in the year 2010. The value has been increasing by $7,000 per year. Write the function and find the value of the house after 5 years.	18. A house was purchased for $350,000 in the year 2010. The value has been increasing at the rate of 2% per year. Write the function and find the value of the house after 5 years.
19. A lab purchases new equipment for $50,000. Its value depreciates over time. The value decreases at the rate of 6% annually. Write the function and find the value after 10 years.	20. A lab purchases new equipment for $50,000. Its value depreciates over time. The value decreases by $3000 annually. Write the function and find the value after 10 years
21. A population of bats in a cave has 200 bats. The population is increasing by 10 bats annually. Write the function. How many bats live in the cave after 7 years?	22. A population of bats in a cave has 200 bats. The population is increasing at the rate of 5% annually. Write the function. How many bats live in the cave after 7 years?
23. A population of a certain species of bird in a state park has 300 birds. The population is decreasing at the rate of 7% year. Write the function. How many birds are in the population after 6 years?	24. A population of a certain species of bird in a state park has 300 birds. The population decreases by 20 birds per year. Write the function. How many birds are in the population after 6 years?

In problems 25-28, the problem represents exponential growth or decay and states the CONTINUOUS growth rate or continuous decay rate. Write the exponential growth or decay function and find the value at the indicated time.

Hint: Use the form of the exponential function that is appropriate when the CONTINUOUS growth or decay rate is given.

25. A population of 400 microbes increases at the **continuous** growth rate of 26% per day. Write the function and find the number of microbes in the population at the end of 7 days.	26. The price of a machine needed by a production factory is $28,000. The business expects to replace the machine in 4 years. Due to inflation the price of the machine is increasing at the **continuous** rate of 3.5% per year. Write the function and find the value of the machine 4 years from now.
27. A population of an endangered species consists of 4000 animals of that species. The population is decreasing at the continuous rate of 12% per year. Write the function and find the size of the population at the end of 10 years.	28. A business buys a computer system for $12000. The value of the system is depreciating and decreases at the continuous rate of 20% per year. Write the function and find the value at the end of 3 years.

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5.2: Graphs and Properties of Exponential Growth and Decay Functions

Learning Objectives

In this section, you will:

- Examine properties of exponential functions.
- Examine graphs of exponential functions.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Indicate if the function represents exponential growth or exponential decay.

a. $y = 127\mathrm{e}^{-0.35\mathrm{t}}$

b. $y = 70\left(0.8^{t}\right)$

c. $y = 453\left(1.2^{t}\right)$

d. $y = 16e^{0.2t}$

Click here to check your answer

a. Since $k < 0$, this is exponential decay.

b. Since $1 < b < 1$, this is exponential decay.

c. Since $b > 1$, this is exponential growth.

d. Since $k > 0$, this is exponential growth.

If you missed any part of this problem, review Section 5.1. (Note that this will open in a new window.)

2. Consider the linear function $f(x) = 2x + 3$.

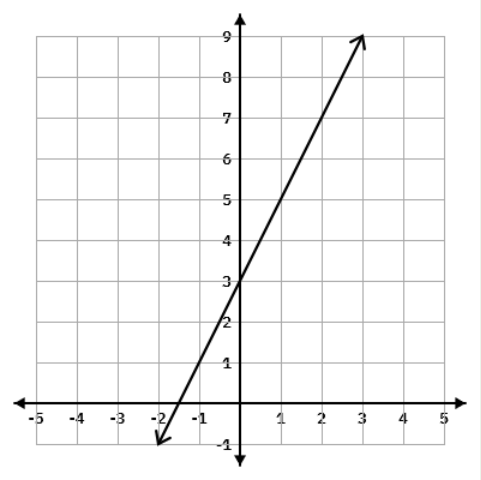
a) Create a graph of $f(x)$.

b) How does the y-intercept of 3 impact the values in the graph?

c) How does the slope of 2 impact the values in the graph?

Click here to check your answer

a)



b) The y-intercept gives the starting y-value of 3, or the y-value when $x = 0$.

c) The slope of 2 makes each subsequent y-value increase by 2.

If you missed this problem, review Section 1.2. (Note that this will open in a new window.)

An exponential function can be written in forms

$$\begin{aligned} f(x) &= ab^x \\ &= a(1+r)^x \\ &= ae^{kx} \end{aligned}$$

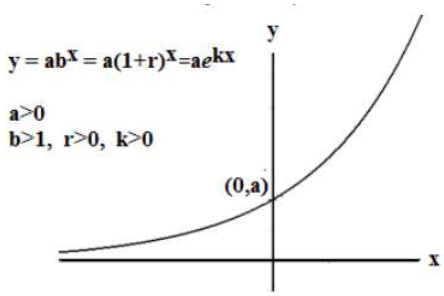
where

- **a is the initial value** because $f(0) = a$. In the growth and decay models that we examine in this finite math textbook, $a > 0$.
- **b is often called the growth factor**. We restrict b to be positive ($b > 0$) because even roots of negative numbers are undefined. We want the function to be defined for all values of x, but b^x would be undefined for some values of x if $b < 0$.
- **r is called the growth or decay rate**. In the formula for the functions, we use r in decimal form, but in the context of a problem we usually state r as a percent.
- **k is called the continuous growth rate or continuous decay rate.**

Properties of Exponential Growth Functions

The function $y = f(x) = ab^x$ where $a > 0$ represents growth if:

- $b > 1$ in the function $y = f(x) = ab^x$.
- Since $b = 1 + r$, $r > 0$ in the function $y = f(x) = a(1+r)^x$.
- $k > 0$ in the function $y = f(x) = ae^{kt}$.

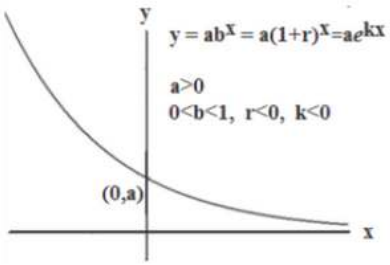


- The function is an increasing function; y increases as x increases.
- Domain: { all real numbers} ; all real numbers can be input to an exponential function.
- Range: {positive real numbers} ; The graph is always above the x axis.
- Horizontal Asymptote: the negative x axis, as x becomes large negative. Using mathematical notation: as x → −∞, then y → 0.
- The y-intercept is the point $(0, a)$ on the y-axis. There is no x-intercept because the function does not ever cross the x-axis.

Properties of Exponential Decay Functions

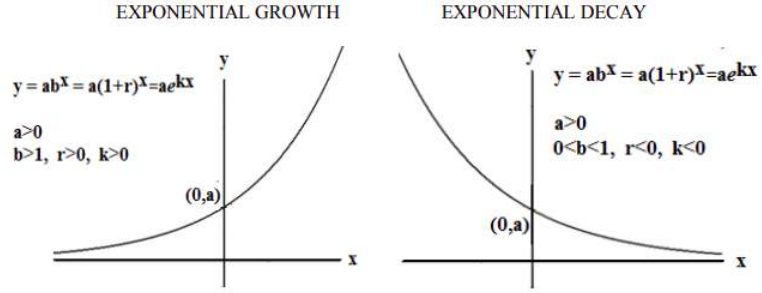
The function $y = f(x) = ab^x$ function where $a > 0$ represents decay if:

- $0 < b < 1$ in the function $y = f(x) = ab^x$.
- Since $b = 1 + r$, $r < 0$ in the function $y = f(x) = a(1+r)^x$.
- $k < 0$ in the function $y = f(x) = ae^{kt}$.



- The function is a decreasing function; y decreases as x increases.
- Domain: { all real numbers} ; all real numbers can be input to an exponential function.
- Range: { positive real numbers } ; The graph is always above the x axis.
- Horizontal Asymptote: the positive x axis as x becomes large positive. Using mathematical notation: as x → ∞, then y → 0.
- The y-intercept is the point $(0, a)$ on the y-axis. There is no x-intercept because the function does not ever cross the x-axis.

The graphs for exponential growth and decay functions are displayed below for comparison.



An Exponential Function is a One-to-One Function

Observe that in the graph of an exponential function, each y value on the graph occurs only once. Therefore, every y value in the range corresponds to only one x value. So, for any particular value of y, you can use the graph to see which value of x is the input to produce that y value as output. This property is called "**one-to-one**".

Because for each value of the output y, you can uniquely determine the value of the corresponding input x, thus every exponential function has an inverse function. The inverse function of an exponential function is a logarithmic function, which we will investigate in the next section.

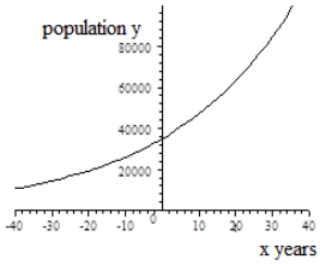
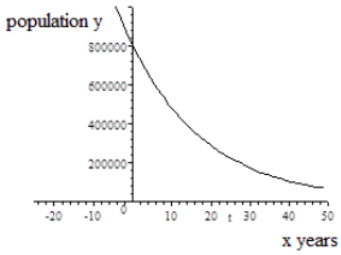
Example 5.2.1

x years after the year 2015, the population of the city of Fulton is given by the function $y = f(x) = 35000(1.03)^x$. x years after the year 2015, the population of the city of Greenville is given by the function $y = g(x) = 80000(0.95)^x$. Use a graphing calculator to compare the graphs of these functions.

Solution

The graphs below were created using computer graphing software. You can also graph these functions using a graphing calculator.

Population of Fulton	Population of Greenville
$y = f(x) = 35000(1.03)^x$	$y = g(x) = 80000(0.95)^x$
Fulton's population is increasing. $b = 1.03 > 1$ and $r = 0.03 > 0$ Exponential Growth	Greenville's population is decreasing. $b = 0.95 < 1$ and $r = -0.05 < 0$ Exponential Decay

Population of Fulton	Population of Greenville
	
y-intercept: (0, 35000) The initial population in 2015 is 35000.	**y-intercept:** (0,80000) The initial population in 2015 is 80000.
Horizontal Asymptote: The negative x axis is the horizontal asymptote. y → 0 as x → − ∞	**Horizontal Asymptote:** The positive x axis is the horizontal asymptote. y → 0 as x → ∞

Domain: In general, the domain of both functions $y = f(x) = 35000(1.03)^x$ and $y = g(x) = 80000(0.95)^x$ is the set of all real numbers. Since x is the years after 2015, negative x-values would represent years before 2015.

Range: The range of both functions is the set of positive real numbers. Both graphs always lie above the x-axis.

Domain and Range in context of this problem:

The functions represent population size as a function of time ***after*** the year 2015 . We restrict the domain in this context, using the "practical domain" as the set of all non-negative real numbers: $x \geq 0$. Then we would consider only the portion of the graph that lies in the first quadrant.

- If we restrict the domain to $x \geq 0$ for the growth function $y = f(x) = 35000(1.03)^x$, then the range for the population of Fulton is $y \geq 35,000$
- If we restrict the domain to $x \geq 0$ for the decay function $y = g(x) = 80000(0.95)^x$, then the range for the population of Greenville is $y \leq 80,000$.

Example 5.2.2

A car was purchased for $24,000 in the year 2015. Its value has been decreasing at a rate of 3% each year.

a. Write the function for the value of the car after *t* years. Use the form $f(x) = ab^x$.

b. Identify the coordinates of the *y*-intercept on the graph of the function.

c. What does the *y*-intercept represent?

Solution

a. The car has an initial value of $24,000, so $a = 24000$. Since the value is decreasing at a rate of 3% each year, $b = 1 - .03 = 0.97$. Therefore the equation is $f(t) = 24000(0.97)^t$.

b. The *y*-intercept is (0, a), or (0, 24000).

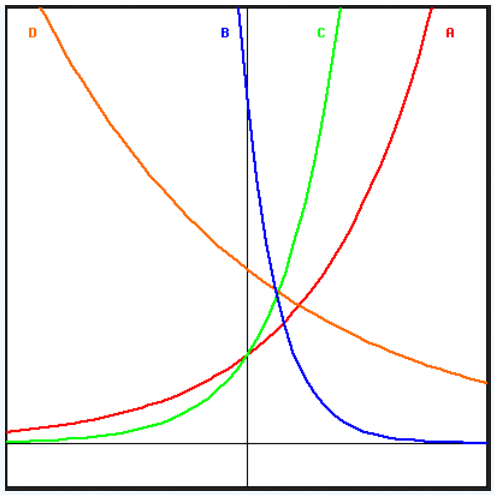
c. The *y*-intercept represents the value of the car, or that when time=0, the value=$24000.

Example 5.2.3

Consider the four functions graphed in the picture below.

a. Using the form $f(x) = ab^x$, which of the four functions has the largest value for a?

b. Using the form $f(x) = ab^x$, which of the four functions has the largest value for b?



Solution

a. The value of a represents the y-intercept. Graph B has the largest value of a since it has the highest y-intercept.

b. The value of b represents the growth or decay factor. A large value of b represents fast growth. Graph C has the largest value of b since it is the steepest graph, or the one growing the fastest.

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5.2E: Exercises - Graphs and Properties of Exponential Growth and Decay Functions

PROBLEM SET: GRAPHS AND PROPERTIES OF EXPONENTIAL GROWTH AND DECAY FUNCTIONS

In questions 1-4, let t = time in years and y = the value at time t or y = the size of the population at time t. The domain is the set of non-negative values for t; $t \geq 0$, because y represents a physical quantity and negative values for time may not make sense. For each question:

a. Write the formula for the function in the form $y = ab^t$
b. Sketch the graph of the function and mark the coordinates of the y-intercept.

1. A house was purchased for $350,000 in the year 2010. The value has been increasing at the rate of 2% per year.	2. A population of a certain species of bird in a state park has 300 birds. The population is decreasing at the rate of 7% year.
3. A lab buys equipment $50,000. Its value depreciates over time. The value decreases at the rate of 6% annually.	4. A population of bats in a cave has 200 bats. The population is increasing at the rate of 5% annually.

In questions 5-8, let t = time in years and y = the value at time t or y = the size of the population at time t. The domain is the set of non-negative values for t; $t \geq 0$, because y represents a physical quantity and negative values for time may not make sense. For each question:

a. Write the formula for the function in the form $y = ae^{kt}$
b. Sketch the graph of the function and mark the coordinates of the y-intercept.

5. A population of 400 microbes increases at the **continuous** growth rate of 26% per day.	6. The price of a machine needed by a production factory is $28,000. Due to inflation the price of the machine is increasing at the **continuous** rate of 3.5% per year.
7. A population of an endangered species consists of 4000 animals of that species. The population is decreasing at the **continuous** rate of 12% per year.	8. A business buys a computer system for $12000. The value of the system is depreciating and decreases at the **continuous** rate of 20% per year.

For questions 9-12

a. Sketch a graph of exponential function.
b. List the coordinates of the y intercept.
c. State the equation of any asymptotes and state the whether the function approaches the asymptote as x $\rightarrow \infty$ or as x $\rightarrow -\infty$.
d. State the domain and range.

9. $y = 10\,(1.5^x)$	10. $y = 10\left(e^{1.2x}\right)$
11. $y = 32\,(0.75^x)$	12. $y = 200\left(e^{-0.5x}\right)$

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5.3: Logarithms and Logarithmic Functions

Learning Objectives

In this section you will learn to:

- Write equivalent logarithmic and exponential expressions.
- Evaluate logs.
- Use properties of logs to solve exponential equations.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Use patterns to simplify each exponent without the use of a calculator.

a. 4^3

b. 4^2

c. 4^1

d. 4^0

e. 4^{-1}

f. 4^{-2}

g. 4^{-3}

Click here to check your answer

a. $4^3 = 64$

b. $4^2 = 16$

c. $4^1 = 4$

d. $4^0 = 1$

e. $4^{-1} = \frac{1}{4}$

f. $4^{-2} = \frac{1}{16}$

g. $4^{-3} = \frac{1}{64}$

If you missed this problem, review here. (Note that this will open a YouTube video in a new window.)

2. Simplify each expression.

a. $4^{\frac{1}{2}}$

b. $9^{\frac{1}{2}}$

Click here to check your answer

a. $4^{\frac{1}{2}} = \sqrt{4} = 2$

b. $9^{\frac{1}{2}} = \sqrt{9} = 3$

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

3. What can "easily" be determined about x in each equation?

a. $4^x = 4$

b. $4^x = 8$

c. $4^x = 16$

Click here to check your answer

a. $x = 1$

b. Since 8 is between 4 and 16, this means x must be between 1 and 2.

c. $x = 2$

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

The Logarithm

Suppose that a population of 50 flies is expected to double every week, leading to a function of the form $f(x) = 50(2)^x$, where x represents the number of weeks that have passed. When will this population reach 500?

Trying to solve this problem leads to

$$500 = 50(2)^x$$

Dividing both sides by 50 to isolate the exponential leads to

$$10 = 2^x.$$

While we have set up exponential models and used them to make predictions, you may have noticed that solving exponential equations has not yet been mentioned. The reason is simple: none of the algebraic tools discussed so far are sufficient to solve exponential equations. Consider the equation $2^x = 10$ above. We know that $2^3 = 8$ and $2^4 = 16$, so it is clear that x must be some value between 3 and 4 since $g(x) = 2^x$ is increasing. We could use technology to create a table of values or graph to better estimate the solution, but we would like to find an algebraic way to solve the equation.

We need an inverse operation to exponentiation in order to solve for the variable if the variable is in the exponent. The inverse function for an exponential function is a logarithmic function.

Logarithm

The logarithm (base b) function, written $\log_b(x)$, is the inverse of the exponential function (base b), b^x.

$$\mathbf{y = \log_b(x) \quad \text{is equivalent to} \quad b^y = x}$$

In general, the statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$.

Note: The base b must be positive: $b > 0$

Inverse Property of Logarithms

Since the logarithm and exponential are inverses, it follows that:

$$\log_b(b^x) \quad \text{and } b^{\log_b(x)} = x$$

Since log is a function, it is most correctly written as $\log_b(c)$, using parentheses to denote function evaluation, just as we would with $f(c)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written as $\log_b c$.

Example 5.3.1

Write these exponential equations as logarithmic equations:

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-3} = \frac{1}{1000}$

Solution

a. $2^3 = 8$ can be written as a logarithmic equation as $\log_2(8) = 3$

b. $5^2 = 25$ can be written as a logarithmic equation as $\log_5(25) = 2$

c. $10^{-3} = \frac{1}{1000}$ can be written as a logarithmic equation as $\log_{10}\left(\frac{1}{1000}\right) = -3$

Example 5.3.2

Write these logarithmic equations as exponential equations:

a. $\log_7(49) = 2$
b. $\log_3(81) = 4$

Solution

a. $\log_7(49) = 2$ can be written as an exponential equation as $7^2 = 49$
b. $\log_3(81) = 4$ can be written as an exponential equation as $3^4 = 81$

By establishing the relationship between exponential and logarithmic functions, we can now solve basic logarithmic and exponential equations by rewriting.

Example 5.3.3

Use the definition of logs to solve $\log_4(x) = 2$ for x.

Solution

By rewriting this expression as an exponential, $4^2 = x$, so $x = 16$

Example 5.3.4

Use the definition of logs to solve $2^x = 10$ for x.

Solution

By rewriting this expression as a logarithm, we get $x = \log_2(10)$

While this does define a solution, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we made earlier. Also, giving an exact expression for a solution is not always useful—often we really need a decimal approximation to the solution. Luckily, this is a task that calculators and computers are quite adept at. Unluckily for us, most calculators and computers will only evaluate logarithms of two bases: base 10 and base e. Happily, this ends up not being a problem, as we'll see soon that we can use a "change of base" formula to evaluate logarithms for other bases.

Common and Natural Logarithms

The **common log** is the logarithm with base 10, and is typically written $\log(x)$ and sometimes like $\log_{10}(x)$. If the base is not indicated in the log function, then the base b used is $b = 10$.

The **natural log** is the logarithm with base e, and is typically written $\ln(x)$.

Note that for any other base b, other than 10, the base must be indicated in the notation $\log_b(x)$.

Example 5.3.5

Evaluate $\log(100)$ using the definition of the common log.

Solution

The table shows values of the common log

number	number as exponential	log(number)
1000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
0.1	10^{-1}	-1
0.01	10^{-2}	-2
0.001	10^{-3}	-3

To evaluate log(100), we can say

$$x = \log(100)$$

Then rewrite the equation in exponential form using the common log base of 10

$$10^x = 100$$

From this, we might recognize that 100 is the square of 10, so

$$x = 2$$

Alternatively, we can use the inverse property of logs to write

$$\log_{10}(10^2) = 2$$

Example 5.3.6

Evaluate $\log\left(\frac{1}{1,000}\right)$ using the definition of the common log.

Solution

To evaluate log(1/1,000), we can say

$$x = \log(1/1,000) = \log\left(1/10^3\right) = \log\left(10^{-3}\right)$$

Then rewrite the equation in exponential form: $10^x = 10^{-3}$

Therefore $x = -3$

Alternatively, we can use the inverse property of logs to find the answer:

$$\log_{10}\left(10^{-3}\right) = -3$$

Example 5.3.7

Evaluate each expression using the definition of the natural log.

a. $\ln e$
b. $\ln e^5$

Solution

a. To evaluate $\ln e$, we can say

$$x = \ln e$$

Then rewrite into exponential form using the natural log base of e

$$e^x = e$$

Notice that there isn't an exponent shown on the right side, so the exponent must be 1.

$$e^x = e^1$$

Therefore $x = 1$.

It is a good idea to remember that $\ln e = 1$.

b. To evaluate $\ln e^5$, we can say

$$x = \ln e^5$$

Then rewrite into exponential form using the natural log base of e

$$e^x = e^5$$

Therefore $x = 5$.

Alternatively, we can use the inverse property of logs to write $\ln\left(e^5\right) = 5$.

Example 5.3.8

Evaluate the following using your calculator:

a. $\log 500$
b. $\ln 500$

Solution

a. Using the LOG key on the calculator to evaluate logarithms in base 10, we evaluate LOG(500)

Answer: $\log 500 \approx 2.69897$

b. Using the LN key on the calculator to evaluate natural logarithms, we evaluate LN(500)

Answer: $\ln 500 \approx 6.214608$

Using Properties of Logarithms to Solve Exponential Equations

We often need to evaluate logarithms using a base other than 10 or e. To find a way to utilize the common or natural logarithm functions to evaluate expressions like $\log_2(10)$, we need some additional properties.

Properties of Logs: Change of Base

$$\log_b(A) = \frac{\log_c(A)}{\log_c(b)} \text{ for any bases } b, c > 0$$

The change of base property allows us to easily convert logarithms into base 10 or base e.

Properties of Logs: One-to-One Property of Logarithms

$$C = D \quad \text{if and only if} \quad \log_b(C) = \log_b(D)$$

The one-to-one property of logs allows us to take the logarithm of both sides of an equation.

Properties of Logs: Power Property

$$\log_b(A^q) = q\log_b(A)$$

The power property allows us to move a variable out of an exponent. When used in conjunction with the one-to-one property of logs, this becomes a convenient way to isolate the variable when it is in the exponent.

Evaluating Logarithms

Example 5.3.9

Evaluate $\log_5(100)$ using the change of base formula.

Solution

We can rewrite this expression using any other base.

Method 1: We can use natural logarithm base e with the change of base formula

$$\log_5(100) = \frac{\ln(100)}{\ln(5)} = \text{LN}(100)/\text{LN}(5) \approx 2.861$$

Method 2: We can use common logarithm base 10 with the change of base formula,

$$\log_5(100) = \frac{\log(100)}{\log(5)} = \text{LOG}(100)/\text{LOG}(5) \approx 2.861$$

With the change of base formula, $\log_b(A) = \frac{\log_c(A)}{\log_c(b)}$ for any bases b, $c > 0$, we can finally find a decimal approximation to our question from the beginning of the section.

Example 5.3.10

Solve $2^x = 10$ for x.

Solution

Method 1: Use the definition of logarithms, followed by the change of base formula.

$$2^x = 10$$

Use the definition of logarithms to rewrite the exponential equation as a logarithmic equation:

$$x = \log_2(10)$$

Using the change of base formula, we can rewrite log base 2 as a logarithm of any other base. Since our calculators can evaluate natural log, one option is to choose to use the natural logarithm, which is the log base e:

Using our calculators to evaluate this, $\frac{\ln(10)}{\ln(2)} = \text{LN}(10)/\text{LN}(2) \approx 3.3219$

Method 2: Use the one-to-one property of logarithms, followed by the power property, to isolate the exponent.

$$2^x = 10$$

Use the one-to-one property of logarithms to take the logarithm of both sides of the equation.

$$\log(2^x) = \log(10)$$

Apply the power property to move the exponent out of the logarithm.

$$x\log(2) = \log(10)$$

Divide both sides by $\log(2)$ to isolate the variable.

$$x = \frac{\log(10)}{\log(2)} \approx 3.3219$$

This finally allows us to answer our original question from the beginning of this section:
For the population of 50 flies that doubles every week, it will take approximately 3.32 weeks to grow to 500 flies.

Example 5.3.11

Solve $2(3)^{5x} = 24$ for x by using the one-to-one property of logs.

Solution

First, isolate the exponential expression by dividing both sides by 2.

$$2(3)^{5x} = 24$$

$$3^{5x} = 12$$

Then use the one-to-one property of logarithms to take the logarithm of both sides of the equation.

$$\log(3^{5x}) = \log(12)$$

Apply the power property to move the exponent out of the logarithm.

$$5x\log(3) = \log(12)$$

The x is multiplied by both the 5 and the $\log(3)$, so divide both sides by $5\log(3)$ to isolate the variable.

$$x = \frac{\log(12)}{5\log(3)} \approx 0.4524$$

We summarize the relationship between exponential and logarithmic functions

Logarithms

The logarithm (base b) function, written $\log_b(x)$, is the inverse of the exponential function (base b), b^x.

$$\mathbf{y = \log_b(x) \quad \text{is equivalent to} \quad b^y = x}$$

In general, the statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$.

Note: The base b must be positive: b>0

Inverse Property of Logarithms

Since the logarithm and exponential are inverses, it follows that:

$$\log_b(b^x) \quad \text{and} \quad b^{\log_b(x)} = x$$

Properties of Logs: Exponential Property: $\log_b(A^q) = q\log_b(A)$

Properties of Logs: Change of Base: $\log_b(A) = \frac{\log_c(A)}{\log_c(b)}$ for any base $b, c > 0$

The inverse, exponential, and change of base properties above will allow us to solve the equations that arise in this course.

For the sake of completeness, we state a few more properties of logarithms here.

Sum of Logs Property: $\log_b(A) + \log_b(C) = \log_b(AC)$

Difference of Logs Property: $\log_b(A) - \log_b(C) = \log_b\left(\frac{A}{C}\right)$

Power Property of Logs: $\log_b(A^n) = n * \log_b(A)$

Logs of Reciprocals: $\log_b\left(\frac{1}{C}\right) = -\log_b(C)$

Reciprocal Bases: $\log_{1/b} C = -\log_b(C)$

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5.3E: Exercises - Logarithms and Logarithmic Functions

PROBLEM SET: LOGARITHMS AND LOGARITHMIC FUNCTIONS

Rewrite each of these exponential expressions in logarithmic form:

1. $3^4 = 81$	2. $10^5 = 100,000$
3. $5^{-2} = 0.04$	4. $4^{-1} = 0.25$
5. $16^{1/4} = 2$	6. $9^{1/2} = 3$

Rewrite each of these logarithmic expressions in exponential form:

7. $\log_5 625 = 4$	8. $\log_2(1/32) = -5$
9. $\log_{11} 1331 = 3$	10. $\log_{10} 0.0001 = -4$
11. $\log_{64} 4 = 1/3$	12. $\ln \sqrt{e} = \frac{1}{2}$

If the expression is in exponential form, rewrite it in logarithmic form.

If the expression is in logarithmic form, rewrite it in exponential form.

13. $5^x = 15625$	14. $x = 9^3$
15. $\log_5 125 = x$	16. $\log_3 x = 5$
17. $\log_{10} y = 4$	18. $e^x = 10$
19. $\ln x = -1$	20. $e^5 = y$

For each equation, rewrite in exponential form and solve for x.

21. $\log_5(x) = 3$	22. $\log_2(x) = -2$
23. $\log_{10}(x) = -3$	24. $\log_3(x) = 6$
25. $\log_{25}(x) = 1/2$	26. $\log_{64}(x) = 1/3$

Evaluate without using your calculator.

27. $\ln \sqrt[3]{e}$	28. $\ln \frac{1}{e^2}$
29. $\ln e^{10}$	30. $\log_{10}(10^e)$

For problems 31 – 38: Evaluate using your calculator. Use the change of base formula if needed

31. $\log 20$	32. $\ln 42$
33. $\ln 2.9$	34. $\log 0.5$
35. $\log_4 36$	36. $\log_7 100$
37. $\log_{105} 3.5$	38. $\log_{1.067} 2$

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5.4: Graphs and Properties of Logarithmic Functions

Learning Objectives

In this section, you will learn to:

- Examine properties of logarithmic functions.
- Examine graphs of logarithmic functions.
- Examine the relationship between graphs of exponential and logarithmic functions.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Consider $f(x) = 2^x$.

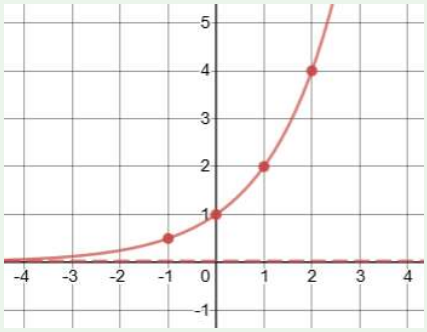
a. Graph $f(x)$.

b. Identify any $x-$ or $y-$ intercepts.

c. Identify any asymptotes.

d. Identify the point (1, ___).

Click here to check your answer

a. 

b. $f(x) = 2^x$ has a $y-$intercept at (0, 1).

c. $f(x) = 2^x$ has an asymptote at $y = 0$.

d. $f(x) = 2^x$ includes the point (1, 2).

If you missed this problem, review Section 5.2. (Note that this will open in a new window.)

2. Consider $g(x) = 5^x$.

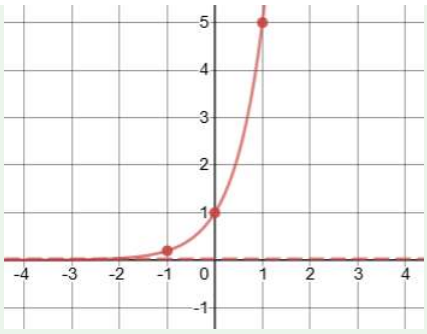
a. Graph $g(x)$.

b. Identify any $x-$ or $y-$ intercepts.

c. Identify any asymptotes.

d. Identify the point (1, ___).

Click here to check your answer

a. 

b. $g(x) = 5^x$ has a $y-$intercept at (0, 1).

c. $g(x) = 5^x$ has an asymptote at $y = 0$.

d. $g(x) = 5^x$ includes the point (1, 5).

If you missed this problem, review Section 5.2. (Note that this will open in a new window.)

3. Consider $h(x) = e^x$.

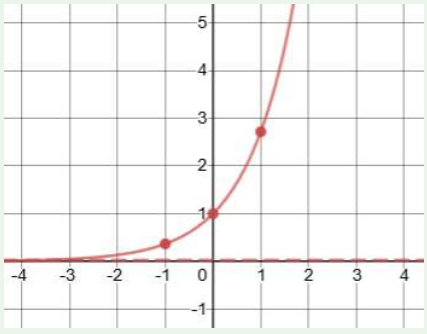
a. Graph $h(x)$.

b. Identify any $x-$ or $y-$ intercepts.

c. Identify any asymptotes.

d. Identify the point (1, ___).

Click here to check your answer

a. 

b. $h(x) = e^x$ has a $y-$intercept at (0, 1).

c. $h(x) = e^x$ has an asymptote at $y = 0$.

d. $h(x) = e^x$ includes the point (1, e), roughly (1, 2.718).

If you missed this problem, review Section 5.2. (Note that this will open in a new window.)

Recall that the exponential function $f(x) = 2^x$ produces this table of values

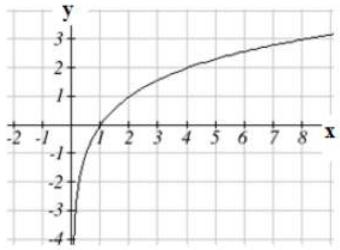
x	-3	-2	-1	0	1	2	3
$f(x)$	1/8	1/4	1/2	1	2	4	8

Since the logarithmic function is an inverse of the exponential, $g(x) = \log_2(x)$ produces the table of values

x	1/8	1/4	1/2	1	2	4	8
$g(x)$	-3	-2	-1	0	1	2	3

In this second table, notice that

1. As the input increases, the output increases.
2. As input increases, the output increases more slowly.
3. Since the exponential function only outputs positive values, the logarithm can only accept positive values as inputs, so the domain of the log function is $(0, \infty)$.
4. Since the exponential function can accept all real numbers as inputs, the logarithm can have any real number as output, so the range is all real numbers or $(-\infty, \infty)$.



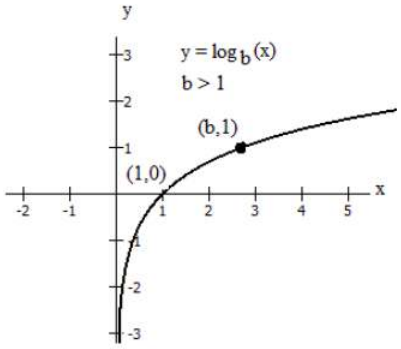
Plotting the graph of $g(x) = \log_2(x)$ from the points in the table, notice that as the input values for x approach zero, the output of the function grows very large in the negative direction, indicating a vertical asymptote at $x = 0$.

In symbolic notation we write

as $x \to 0^+$, $f(x) \to -\infty$

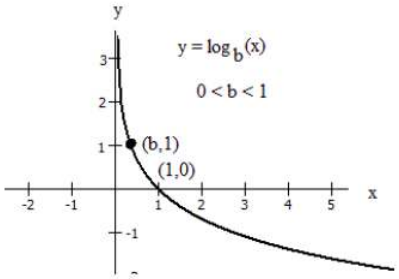
and as $x \to \infty$, $f(x) \to \infty$

Graphically, in the function $g(x) = \log_b(x)$, $b > 1$, we observe the following properties:



- The graph has a horizontal intercept at (1, 0)
- The line x = 0 (the y-axis) is a vertical asymptote; as $x \to 0^+, y \to -\infty$
- The graph is increasing if $b > 1$
- The domain of the function is $x > 0$, or (0, ∞)
- The range of the function is all real numbers, or $(-\infty, \infty)$

However if the base b is less than 1, 0 < b < 1, then the graph appears as below.
This follows from the log property of reciprocal bases : $\log_{1/b} C = -\log_b(C)$



- The graph has a horizontal intercept at (1, 0)
- The line x = 0 (the y-axis) is a vertical asymptote; as $x \to 0^+, y \to \infty$
- The graph is decreasing if $0 < b < 1$
- The domain of the function is $x > 0$, or $(0, \infty)$
- The range of the function is all real numbers, or $(-\infty, \infty)$

When graphing a logarithmic function in the form $f(x) = \log_b(x)$, it can be helpful to remember that the graph will pass through the points (1, 0) and $(b, 1)$.

Example 5.4.1

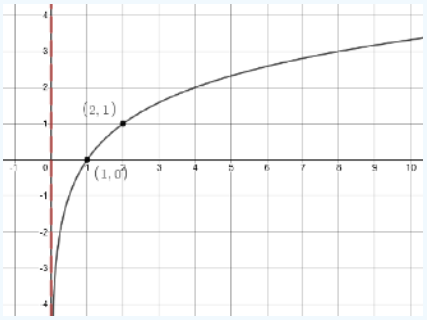
Sketch each of the following functions by graphing the vertical asymptote, the intercept, and the point $(b, 1)$.

a. $f(x) = \log_2(x)$
b. $g(x) = \log_5(x)$
c. $h(x) = \log(x)$
d. $j(x) = \ln(x)$

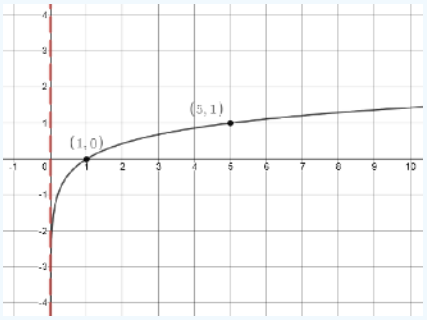
Solution

All logarithmic graphs in the form $f(x) = \log_b(x)$ will have a vertical asymptote on the *y*-axis and an *x*-intercept at (1, 0).

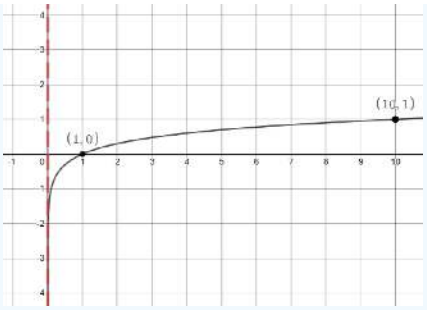
a. $f(x) = \log_2(x)$ will have a point at (2, 1).



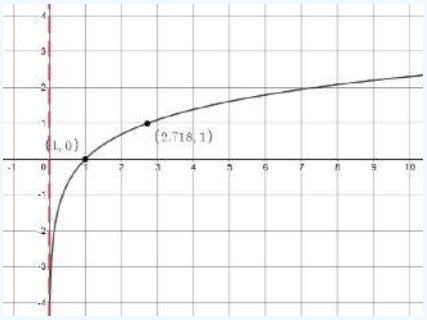
b. $g(x) = \log_5(x)$ will have a point at (5, 1).



c. $h(x) = \log(x)$ represents the common log and has a base of 10, therefore it will have a point at (10, 1).



d. $j(x) = \ln(x)$ represents the natural log and has a base of *e*, therefore it will have a point at (*e*, 1).



Finally, we compare the graphs of $y = b^x$ and $y = \log_b(x)$, shown below on the same axes.

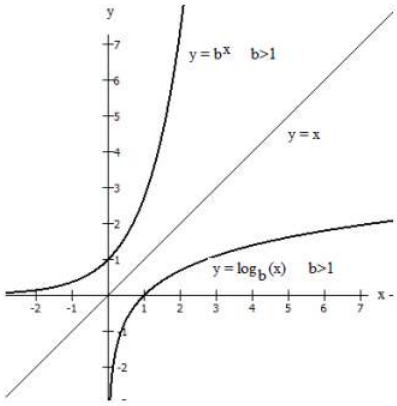
Because the functions are inverse functions of each other, for every specific ordered pair (h, k) on the graph of $y = b^x$, we find the point (k, h) with the coordinates reversed on the graph of $y = \log_b(x)$.

In other words, if the point with $x = h$ and $y = k$ is on the graph of $y = b^x$, then the point with $x = k$ and $y = h$ lies on the graph of $y = \log_b(x)$

The domain of $y = b^x$ is the range of $y = \log_b(x)$

The range of $y = b^x$ is the domain of $y = \log_b(x)$

For this reason, the graphs appear as reflections, or mirror images, of each other across the diagonal line $y = x$. This is a property of graphs of inverse functions that students should recall from their study of inverse functions in their prerequisite algebra class.



	$\mathbf{y = b^x}$, with $\mathbf{b > 0}$	$\mathbf{y = \log_b(x)}$, with $\mathbf{b > 0}$
Domain	all real numbers	all positive real numbers
Range	all positive real numbers	all real numbers
Intercepts	(0,1)	(1,0)

	$y = b^x$, with $b > 0$	$y = \log_b(x)$, with $b > 0$
Asymptotes	Horizontal asymptote is the line y = 0 (the x-axis) As $x \to -\infty,\ y \to 0$	Vertical asymptote is the line x = 0 (the y axis) As $x \to 0^+,\ y \to -\infty$

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5.4E: Exercises - Graphs and Properties of Logarithmic Functions

PROBLEM SET: GRAPHS AND PROPERTIES OF LOGARITHMIC FUNCTIONS

Questions 1 – 3: For each of the following functions

a. Sketch a reasonably accurate graph showing the shape of the graph of the function
b. State the domain
c. State the range
d. State whether the graph has a vertical asymptote or a horizontal asymptote and write the equation of that asymptote
e. Does the graph have an x-intercept or a y-intercept asymptote? Write the coordinates of the x-intercept or the y-intercept.

1. $y = \ln x$ a. Sketch the graph below	b. domain: _____ c. range: _____ d. Is the asymptote horizontal or vertical? _____ Equation of the asymptote: _____ e. Coordinates of x-intercept or y-intercept: _____
2. $y = \log x$ a. Sketch the graph below	b. domain: _____ c. range: _____ d. Is the asymptote horizontal or vertical? _____ Equation of the asymptote: _____ e. Coordinates of x-intercept or y-intercept: _____
3. $y = \log_{0.8} x$ a. Sketch the graph below	b. domain: _____ c. range: _____ d. Is the asymptote horizontal or vertical? _____ Equation of the asymptote: _____ e. Coordinates of x-intercept or y-intercept: _____

Questions 4 - 5: For the pair of inverse functions $y = e^x$ and $y = \ln x$

a. Sketch a reasonably accurate graph showing the shape of the graph of the function
b. State the domain
c. State the range
d. State whether the graph has a vertical asymptote or a horizontal asymptote and write the equation of that asymptote
e. Does the graph have an x-intercept or a y-intercept asymptote? Write the coordinates of the xintercept or the y-intercept.

4. $y = e^x$ a. Sketch the graph below	b. domain: _____ c. range: _____ d. Is the asymptote horizontal or vertical? _____ Equation of the asymptote: _____ e. Coordinates of x-intercept or y-intercept: _____
5. $y = \ln x$ a. Sketch the graph below	b. domain: _____ c. range: _____ d. Is the asymptote horizontal or vertical? _____ Equation of the asymptote: _____ e. Coordinates of x-intercept or y-intercept: _____

Questions 6-11: Match the graph with the function.

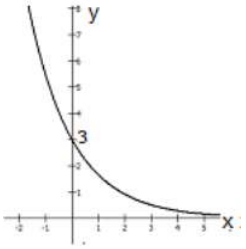
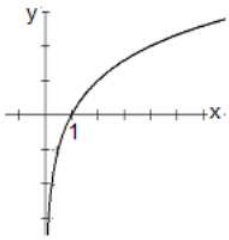
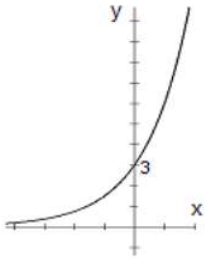
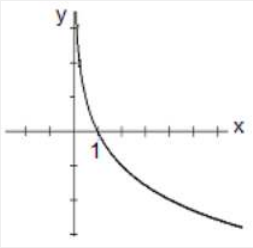
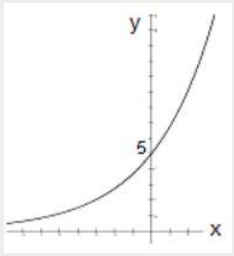
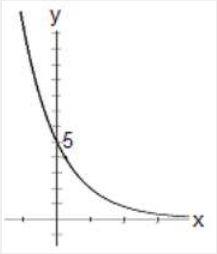
Choose the function from the list below and write it on the line underneath the graph.

Hint: To match the function and the graph, identify these properties of the graph and function

- Is the function increasing decreasing?
- Examine the asymptote

- Determine the x or y intercept

$$y = 3\,(2^x) \quad y = 5\,(0.4^x) \quad y = \log_2(x) \quad y = \log_{1/2}(x) \quad y = 3e^{-0.6x} \quad y = 5e^{0.3x}$$

6. Function: _____ 	7. Function: _____ 	8. Function: _____ 
9. Function: _____ 	10. Function: _____ 	11. Function: _____ 

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5.5: Applications of Exponential and Logarithmic Functions

Learning Objectives

In this section, you will:

1. review strategies for solving equations arising from exponential formulas
2. solve application problems involving exponential functions and logarithmic functions

STRATEGIES FOR SOLVING EQUATIONS THAT CONTAIN EXPONENTS

When solving application problems that involve exponential and logarithmic functions, we need to pay close attention to the position of the variable in the equation to determine the proper way solve the equation we investigate solving equations that contain exponents.

Suppose we have an equation in the form:

$$\textbf{value = coefficient(base)}^{\textbf{exponent}}$$

We consider four strategies for solving the equation:

- **STRATEGY A:** If the coefficient, base, and exponent are all known, we only need to evaluate the expression for coefficient(base)$^{\text{exponent}}$ to evaluate its value.
- **STRATEGY B:** If the variable is the coefficient, evaluate the expression for (base)$^{\text{exponent}}$. Then it becomes a linear equation which we solve by dividing to isolate the variable.
- **STRATEGY C:** If the variable is in the exponent, use logarithms to solve the equation.
- **STRATEGY D:** If the variable is not in the exponent, but is in the base, use roots to solve the equation.

Below we examine each strategy with one or two examples of its use.

STRATEGY A

If the coefficient, base, and exponent are all known, we only need to evaluate the expression for coefficient(base)$^{\text{exponent}}$ to evaluate its value.

Example 5.5.1

Suppose that a stock's price is rising at the rate of 7% per year, and that it continues to increase at this rate. If the value of one share of this stock is $43 now, find the value of one share of this stock three years from now.

Solution

Let y = the value of the stock after t years:

$$y = ab^t$$

The problem tells us that a = 43 and r = 0.07, so

$$b = 1 + r = 1 + 0.07 = 1.07$$

Therefore, function is $y = 43(1.07)^t$.

In this case we know that t = 3 years, and we need to evaluate y when t = 3.

At the end of 3 years, the value of this one share of this stock will be

$$y = 43(1.07)^3 = \$52.68$$

STRATEGY B

If the variable is the coefficient, evaluate the expression for (base)$^{\text{exponent}}$. Then it becomes a linear equation which we solve by dividing to isolate the variable.

Example 5.5.2

The value of a new car depreciates (decreases) after it is purchased. Suppose that the value of the car depreciates according to an exponential decay model. Suppose that the value of the car is $12000 at the end of 5 years and that its value has been decreasing at the rate of 9% per year. Find the value of the car when it was new.

Solution

Let y be the value of the car after t years: $y = ab^t$, r = -0.09 and $b = 1 + r = 1 + (-0.09) = 0.91$

The function is $y = a(0.91)^t$

In this case we know that when t = 5, then y = 12000; substituting these values gives

$$12000 = a(0.91)^5$$

We need to solve for the initial value a, the purchase price of the car when new.

First evaluate $(0.91)^5$; then solve the resulting linear equation to find a.

$$1200 = a(0.624)$$

$a = \frac{12000}{0.624} = \$19,230.77$; The car's value was $19,230.77 when it was new.

STRATEGY C

If the variable is in the exponent, use logarithms to solve the equation.

Example 5.5.3

A national park has a population of 5000 deer in the year 2016. Conservationists are concerned because the deer population is decreasing at the rate of 7% per year. If the population continues to decrease at this rate, how long will it take until the population is only 3000 deer?

Solution

Let y be the number of deer in the national park t years after the year 2016: $y = ab^t$

r = -0.07 and $b = 1 + r = 1 + (-0.07) = 0.93$ and the initial population is a = 5000

The exponential decay function is $y = 5000(0.93)^t$

To find when the population will be 3000, substitute y = 3000

$$3000 = 5000(0.93)^t$$

Next, divide both sides by 5000 to isolate the exponential expression

$$\frac{3000}{5000} = \frac{5000}{5000}(0.93)^2$$
$$0.6 = 0.93^t$$

Rewrite the equation in logarithmic form; then use the change of base formula to evaluate.

$$t = \log_{0.93}(0.6)$$

$t = \frac{\ln(0.6)}{\ln(0.93)} = 7.039$ years; After 7.039 years, there are 3000 deer.

Note: In Example 5.5.3, we needed to state the answer to several decimal places of precision to remain accurate. Evaluating the original function using a rounded value of t = 7 years gives a value that is close to 3000, but not exactly 3000.

$$y = 5000(0.93)^7 = 3008.5 \text{ deer}$$

However using t = 7.039 years produces a value of 3000 for the population of deer

$$y = 5000(0.93)^{7.039} = 3000.0016 \approx 3000 \text{ deer}$$

Example 5.5.4

A video posted on YouTube initially had 80 views as soon as it was posted. The total number of views to date has been increasing exponentially according to the exponential growth function $y = 80e^{0.12t}$, where t represents time measured in days since the video was posted. How many days does it take until 2500 people have viewed this video?

Solution

Let y be the total number of views t days after the video is initially posted.
We are given that the exponential growth function is $y = 80e^{0.2t}$ and we want to find the value of t for which y = 2500. Substitute y = 2500 into the equation and use natural log to solve for t.

$$2500 = 80e^{0.12t}$$

Divide both sides by the coefficient, 80, to isolate the exponential expression.

$$\frac{2500}{80} = \frac{80}{80}e^{0.12t}$$
$$31.25 = e^{0.12t}$$

Rewrite the equation in logarithmic form

$$0.12t = \ln(31.25)$$

Divide both sides by 0.04 to isolate t; then use your calculator and its natural log function to evaluate the expression and solve for t.

$$\mathrm{t} = \frac{\ln(31.25)}{0.12}$$
$$\mathrm{t} = \frac{3.442}{0.12}$$
$$\mathrm{t} \approx 28.7 \text{ days}$$

This video will have 2500 total views approximately 28.7 days after it was posted.

STRATEGY D

If the variable is not in the exponent, but is in the base, we use roots to solve the equation.

> It is important to remember that we only use logarithms when the variable is in the exponent.

Example 5.5.5

A statistician creates a website to analyze sports statistics. His business plan states that his goal is to accumulate 50,000 followers by the end of 2 years (24 months from now). He hopes that if he achieves this goal his site will be purchased by a sports news outlet. The initial user base of people signed up as a result of pre-launch advertising is 400 people. Find the monthly growth rate needed if the user base is to accumulate to 50,000 users at the end of 24 months.

Solution

Let y be the total user base t months after the site is launched.

The growth function for this site is $y = 400(1 + r)^t$;

We don't know the growth rate r. We do know that when t = 24 months, then y = 50000.

Substitute the values of y and t; then we need to solve for r.

$$5000 = 400(1 + r)^{24}$$

Divide both sides by 400 to isolate $(1+r)^{24}$ on one side of the equation

$$\frac{50000}{400} = \frac{400}{400}(1 + r)^{24}$$
$$125 = (1 + r)^{24}$$

Because the variable in this equation is in the base, we use roots:

$$\sqrt[24]{125} = 1 + r$$
$$125^{1/24} = 1 + r$$
$$1.2228 \approx 1 + r$$
$$0.2228 \approx r$$

The website's user base needs to increase at the rate of 22.28% per month in order to accumulate 50,000 users by the end of 24 months.

Example 5.5.6

A fact sheet on caffeine dependence from Johns Hopkins Medical Center states that the half life of caffeine in the body is between 4 and 6 hours. Assuming that the typical half life of caffeine in the body is 5 hours for the average person and that a typical cup of coffee has 120 mg of caffeine.

a. Write the decay function.
b. Find the hourly rate at which caffeine leaves the body.
c. How long does it take until only 20 mg of caffiene is still in the body?
www.hopkinsmedicine.org/psyc...fact_sheet.pdf

Solution

a. Let y be the total amount of caffeine in the body t hours after drinking the coffee.

Exponential decay function $y = ab^t$ models this situation.

The initial amount of caffeine is a = 120.

We don't know b or r, but we know that the half- life of caffeine in the body is 5 hours. This tells us that when t = 5, then there is half the initial amount of caffeine remaining in the body.

$$y = 120b^t$$
$$\tfrac{1}{2}(120) = 120b^5$$
$$60 = 120b^5$$

Divide both sides by 120 to isolate the expression b^5 that contains the variable.

$$\tfrac{60}{120} = \tfrac{120}{120}\mathrm{b}^5$$
$$0.5 = \mathrm{b}^5$$

The variable is in the base and the exponent is a number. Use roots to solve for b:

$$\sqrt[5]{0.5} = \mathrm{b}$$
$$0.5^{1/5} = \mathrm{b}$$
$$0.87 = \mathrm{b}$$

We can now write the decay function for the amount of caffeine (in mg.) remaining in the body t hours after drinking a cup of coffee with 120 mg of caffeine

$$y = f(t) = 120(0.87)^t$$

b. Use $b = 1 + r$ to find the decay rate r. Because $b = 0.87 < 1$ and the amount of caffeine in the body is decreasing over time, the value of r will be negative.

$$0.87 = 1 + r$$
$$r = -0.13$$

The decay rate is 13%; the amount of caffeine in the body decreases by 13% per hour.

c. To find the time at which only 20 mg of caffeine remains in the body, substitute y = 20 and solve for the corresponding value of t.

$$y = 120(.87)^t$$
$$20 = 120(.87)^t$$

Divide both sides by 120 to isolate the exponential expression.

$$\frac{20}{120} = \frac{120}{120}\left(0.87^t\right)$$
$$0.1667 = 0.87^t$$

Rewrite the expression in logarithmic form and use the change of base formula

$$t = \log_{0.87}(0.1667)$$
$$t = \frac{\ln(0.1667)}{\ln(0.87)} \approx 12.9 \text{ hours}$$

After 12.9 hours, 20 mg of caffeine remains in the body.

EXPRESSING EXPONENTIAL FUNCTIONS IN THE FORMS y = ab^t and y = ae^{kt}

Now that we've developed our equation solving skills, we revisit the question of expressing exponential functions equivalently in the forms $y = ab^t$ and $y = ae^{kt}$

We've already determined that if given the form $y = ae^{kt}$, it is straightforward to find b.

Example 5.5.7

For the following examples, assume t is measured in years.

a. Express $y = 3500e^{0.25t}$ in form $y = ab^t$ and find the annual percentage growth rate.
b. Express $y = 28000e^{-0.32t}$ in form $y = ab^t$ and find the annual percentage decay rate.

Solution

a. Express $y = 3500e^{0.25t}$ in the form $y = ab^t$

$$y = ae^{kt} = ab^t$$
$$a\left(e^k\right)^t = ab^t$$

Thus $e^k = b$

In this example $b = e^{0.25} \approx 1.284$

We rewrite the growth function as y = 3500(1.284^t)

To find r, recall that $b = 1 + r$

$$1.284 = 1 + r$$
$$0.284 = \text{r}$$

The continuous growth rate is k = 0.25 and the annual percentage growth rate is 28.4% per year.

b. Express $y = 28000e^{-0.32t}$ in the form $y = ab^t$

$$y = ae^{kt} = ab^t$$
$$a\left(e^k\right)^t = ab^t$$

Thus $e^k = b$

In this example $\text{b} = e^{-0.32} \approx 0.7261$

We rewrite the growth function as y = 28000(0.7261^t)

To find r, recall that $b = 1 + r$

$$0.7261 = 1 + r$$
$$0.2739 = r$$

The continuous decay rate is k = -0.32 and the annual percentage decay rate is 27.39% per year.

In the sentence, we omit the negative sign when stating the annual percentage decay rate because we have used the word “decay” to indicate that r is negative.

Example 5.5.8

a. Express $y = 4200(1.078)^t$ in the form $y = ae^{kt}$
b. Express $y = 150(0.73)^t$ in the form $y = ae^{kt}$

Solution

a. Express $y = 4200(1.078)^t$ in the form $y = ae^{kt}$

$$\begin{aligned} \mathrm{y} = \mathrm{a}e^{\mathrm{k}t} &= \mathrm{ab}^{\mathrm{t}} \\ \mathrm{a}\left(e^{\mathrm{k}}\right)^{\mathrm{t}} &= \mathrm{ab}^{\mathrm{t}} \\ e^{\mathrm{k}} &= \mathrm{b} \\ e^{k} &= 1.078 \end{aligned}$$

Therefore $\mathrm{k} = \ln 1.078 \approx 0.0751$

We rewrite the growth function as $y = 3500e^{0.0751t}$

b. Express $y = 150(0.73)^t$ in the form $y = ae^{kt}$

$$\begin{aligned} y = ae^{kt} &= ab^t \\ a\left(e^{k}\right)^{t} &= ab^t \\ e^{k} &= b \\ e^{k} &= 0.73 \end{aligned}$$

Therefore $\mathrm{k} = \ln 0.73 \approx -0.3147$

We rewrite the growth function as $y = 150e^{-0.3147t}$

AN APPLICATION OF A LOGARITHMIC FUNCTON

Suppose we invest $10,000 today and want to know how long it will take to accumulate to a specified amount, such as $15,000. The time t needed to reach a future value y is a logarithmic function of the future value: $t = g(y)$

Example 5.5.9

Suppose that Vinh invests $10000 in an investment earning 5% per year. He wants to know how long it would take his investment to accumulate to $12000, and how long it would take to accumulate to $15000.

Solution

We start by writing the exponential growth function that models the value of this investment as a function of the time since the $10000 is initially invested

$$y = 10000(1.05)^t$$

We divide both sides by 10000 to isolate the exponential expression on one side.

$$\frac{y}{10000} = 1.05^t$$

Next we rewrite this in logarithmic form to express time as a function of the accumulated future value. We’ll use function notation and call this function $g(y)$.

$$\mathrm{t} = \mathrm{g(y)} = \log_{1.05}\left(\frac{\mathrm{y}}{10000}\right)$$

Use the change of base formula to express t as a function of y using natural logarithm:

$$t = g(y) = \frac{\ln\left(\frac{y}{10000}\right)}{\ln(1.05)}$$

We can now use this function to answer Vinh's questions.

To find the number of years until the value of this investment is \$12,000, we substitute y = \$12,000 into function g and evaluate t:

$$t = g(12000) = \frac{\ln\left(\frac{12000}{10000}\right)}{\ln(1.05)} = \frac{\ln(1.2)}{\ln(1.05)} = 3.74 \text{ years}$$

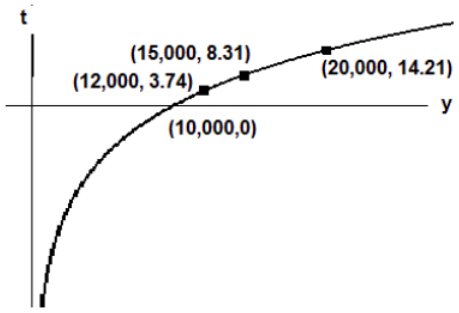
To find the number of years until the value of this investment is \$15,000, we substitute y = \$15,000 into function g and evaluate t:

$$t = g(15000) = \frac{\ln\left(\frac{15000}{10000}\right)}{\ln(1.05)} = \frac{\ln(1.5)}{\ln(1.05)} = 8.31 \text{ years}$$

Before ending this section, we investigate the graph of the function

$$t = g(y) = \frac{\ln\left(\frac{y}{10000}\right)}{\ln(1.05)}.$$

We see that the function has the general shape of logarithmic functions that we examined in section 5.5. From the points plotted on the graph, we see that function g is an increasing function but it increases very slowly.



If we consider just the function

$$t = g(y) = \frac{\ln\left(\frac{y}{10000}\right)}{\ln(1.05)},$$

then the domain of function would be $y > 0$, all positive real numbers, and the range for t would be all real numbers.

In the context of this investment problem, the initial investment at time t = 0 is y =\$10,000. Negative values for time do not make sense. Values of the investment that are lower than the initial amount of \$10,000 also do not make sense for an investment that is increasing in value.

Therefore the function and graph as it pertains to this problem concerning investments has domain $y \geq 10,000$ and range $t \geq 0$. The graph above is restricted to the domain and range that make practical sense for the investment in this problem.

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5.5E: Exercises - Applications of Exponential and Logarithmic Functions

PROBLEM SET: APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

1. An investment's value is rising at the rate of 5% per year. The initial value of the investment is $20,000 in 2016.
   a. Write the function that gives the value of the investment as a function of time t in years after 2016.
   b. Find the value of the investment in 2028
   c. When will the value be $30,000?

2. The population of a city is increasing at the rate of 2.3% per year, since the year 2000. Its population in 2010 was 137,000 people. Find the population of the city in the year 2000.

3. The value of a piece of industrial equipment depreciates after it is purchased. Suppose that the depreciation follows an exponential decay model. The value of the equipment at the end of 8 years is $30,000 and its value has been decreasing at the rate of 7.5% per year. Find the initial value of the equipment when it was purchased.

4. An investment has been losing money. Its value has been decreasing at the rate of 3.2% per year. The initial value of the investment was $75,000 in 2010.
   a. Write the function that gives the value of the investment as a function of time t in years after 2010.
   b. If the investment's value continues to decrease at this rate, find the value of the investment in 2020.

5. A social media site has 275 members initially. The number of members has been increasing exponentially according to the function $y = 275e^{0.21t}$, where t is the number of months since the site's initial launch. How many months does it take until the site has 5000 members? *State answer to the nearest tenth of a month (1 decimal place).*

6. A city has a population of 62000 people in the year 2000. Due to high unemployment, the city's population has been decreasing at the rate of 2% per year. Using this model, find the population of this city in 2016.

7. A city has a population of 87,000 people in the year 2000. The city's population has been increasing at the rate of 1.5% per year. How many years does it take until the population reaches 100,000 people?

8. An investment of $50,000 is in increasing in value at the rate of 6.3% per year. How many years does it take until the investment is worth $70,000?

9. A city has a population of 50,000 people in the year 2000. The city's population increases at a constant percentage rate. Fifteen years later, in 2015, the population of this city was 70,000. Find the annual percentage growth rate.

10. 200 mg of a medication is administered to a patient. After 3 hours, only 100 mg remains in the bloodstream. Using an exponential decay model, find the hourly decay rate.

11. An investment is losing money at a constant percentage rate per year. The investment was initially worth $25,000 but is worth only $20,000 after 4 years. Find the percentage rate at which the investment is losing value each year (that is, find the annual decay rate).

12. Using the information in question 11, how many years does it take until the investment is worth only half of its initial value?

For question 13:

- if the function is given in the form $y = ae^{kt}$, rewrite it in the form $y = ab^t$.
- if the function is given in the form $y = ab^t$, rewrite it in the form $y = ae^{kt}$.

13a. $y = 7900e^{0.472t}$. Write in the form $y = ab^t$

13b. $y = 4567(0.67^t)$. Write in the form $y = ae^{kt}$

13c. $y = 18720(1.47^t)$. Write in the form $y = ae^{kt}$

13d. $y = 1200e^{-0.078t}$. Write in the form $y = ab^t$

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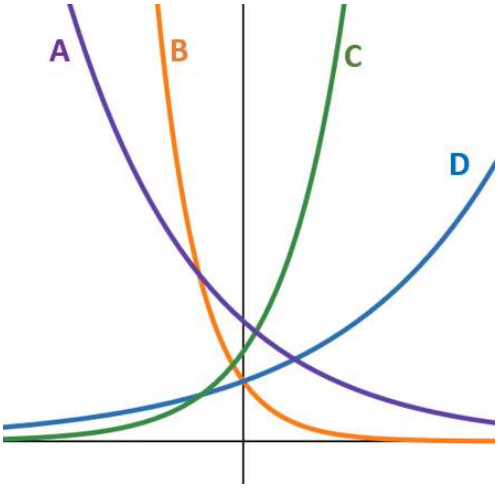
5.6: Chapter 5 Review

Exponential Growth and Decay Models (5.1)

1. Determine whether each of the following statements represents exponential growth or decay or linear growth or decay.
   a. The value of a new boat depreciates at a rate of 8.2% each year after it is purchased.
   b. An investment has been gaining $500 each month.
   c. A 5-year CD has a guaranteed growth rate of 4.5% each year.
2. Mari deposited $2000 into an investment with a guaranteed APY of 3.5% each year. Let t be the number of years for the investment and let $f(t)$ be the value of the investment. Find the exponential function that models the value of their investment, then use the function to find the value after 4 years.
3. The value of a houses in a city are increasing at a continuous growth rate of 7.2% per year. For a house that currently costs $350,000, find the exponential function that models the value of the house. Then use the function to find the value after 10 years.
4. The value of a new vehicle depreciates at a continuous growth rate of 19% per year. Daniel purchased a new truck for $54,000. Find the exponential function that models the value of the truck. Then use the function to find the value of his truck after 8 years.

Graphs and Properties of Exponential Growth and Decay Functions (5.2)

5. A new machine was purchased for $125,000 in the year 2016. Its value has been decreasing at a continuous rate of 5% each year. Find the exponential function that models the value of the machine. Then identify the y-intercept of the graph of the function. What does the y-intercept represent?
6. Consider the graphs shown below.
   a. Using the form $f(x) = ab^x$, which two functions have the same value of a?
   b. Using the form $f(x) = ab^x$, which of the four functions has the largest value of a?
   c. Using the form $f(x) = ab^x$, which of the four functions has the largest value of b?
   d. Using the form $f(x) = ab^x$, which of the four functions has the smallest value of b?



Logarithms and Logarithmic Functions (5.3)

1. Write 5^3 as a logarithmic equation.
2. Write $\log_4 16 = 2$ as an exponential equation.
3. Simplify each logarithm. You should be able to do these without using a calculator.
   a. $\log_2 16$
   b. $\log_5 25$
   c. $\log_3 27$
4. Identify the base in $\log 1000$

5. Identify the base in $\ln 1000$.
6. Use logs to solve each equation.
   a. $3^x = 50$
   b. $5(4)^x = 100$

Graphs and Properties of Logarithmic Functions (5.4)

Sketch each of the following functions by graphing the vertical asymptote, the intercept, and the point $(b, 1)$.

a. $f(x) = \log_3(x)$
b. $g(x) = \log(x)$
c. $h(x) = \ln(x)$

Applications of Exponential and Logarithmic Functions (5.5)

1. The value of a new boat depreciates after it is purchased. The value of the boat 7 years after it was purchased is $25,000 and its value has been decreasing at the rate of 8.2% per year.
   a. Find the initial value of the boat when it was purchased.
   b. How many years after it was purchased will the boat's value be $20,000?
   c. What was its value 3 years after the boat was purchased?
2. Tony invested $40,000 in 2010; unfortunately his investment has been losing value at the rate of 2.7% per year.
   a. Write the function that gives the value of the investment as a function of time t in years after 2010.
   b. Find the value of the investment in 2020, if its value continues to decrease at this rate.
   c. In what year will the investment be worth half its original value?
3. Rosa invested $25,000 in 2005; its value has been increasing at the rate of 6.4% annually.
   a. Write the function that gives the value of the investment as a function of time t in years after 2005.
   b. Find the value of the investment in 2025.
4. The population of a city is increasing at the rate of 3.2% per year, since the year 2000. Its population in 2015 was 235,000 people.
   a. Find the population of the city in the year 2000.
   b. In what year with the population be 250, 000 if it continues to grow at this rate.
   c. What was the population of this city in the year 2008?
5. The population of an endangered species has only 5000 animals now. Its population has been decreasing at the rate of 12% per year.
   a. If the population continues to decrease at this rate, how many animals will be in this population 4 years from now.
   b. In what year will there be only 2000 animals remaining in this population?
6. 300 mg of a medication is administered to a patient. After 5 hours, only 80 mg remains in the bloodstream.
   a. Using an exponential decay model, find the hourly decay rate.
   b. How many hours after the 300 mg dose of medication was administered was there 125 mg in the bloodstream
   c. How much medication remains in the bloodstream after 8 hours?
7. If $y = 240b^t$ and $y = 600$ when $t = 6$ years, find the annual growth rate. State your answer as a percent.

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CHAPTER OVERVIEW

6: Finance Applications

Learning Objectives

In this chapter, you will learn to:

1. Solve financial problems that involve simple interest.
2. Solve problems involving compound interest.
3. Find the future value of an annuity, and the amount of payments to a sinking fund.
4. Find the future value of an annuity, and an installment payment on a loan.

6.1: Simple Interest

6.1E: Exercises - Simple Interest

6.2: Compound Interest

6.2E: Exercises - Compound Interest

6.3: Future Value of Annuities and Sinking Funds

6.3E: Exercises - Annuities and Sinking Funds

6.4: Present Value of Annuities and Installment Payment

6.4E: Exercises - Present Value of an Annuity and Installment Payment

6.5: Classification of Finance Problems

6.5E: Exercises - Classification of Finance Problems

6.6: Additional Application Problems

6.6E: Exercises - Miscellaneous Application Problems

6.7: Chapter 6 Review

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6.1: Simple Interest

Learning Objectives

In this section, you will learn to:

- Use the simple interest formula to find an account balance, principal, rate, or time.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Solve $5 = 2(x + 9)$.

Click here to check your answer

$x = -\frac{13}{2}$

If you missed this problem, review Section 1.1. (Note that this will open in a new window.)

2. Solve $x = -3(7 - 10)$.

Click here to check your answer

$x = 9$

If you missed this problem, review Section 1.1. (Note that this will open in a new window.)

3. Solve $-12 = x(3 + 8)$.

Click here to check your answer

$x = -\frac{12}{11}$

If you missed this problem, review Section 1.1. (Note that this will open in a new window.)

4. Solve $12 = 4(2 - x)$.

Click here to check your answer

$x = -1$

If you missed this problem, review Section 1.1. (Note that this will open in a new window.)

Simple Interest

It costs to borrow money. The rent one pays for the use of money is called the **interest**. The amount of money that is being borrowed or loaned is called the **principal**, also called the present value. Simple interest is paid only on the original amount borrowed. When the money is loaned out, the person who borrows the money generally pays a fixed rate of interest on the principal for the time period he keeps the money. Although the interest rate is most commonly specified for a year, it may be specified for a week, a month, or a quarter, etc. Credit card companies often list their charges as monthly rates, sometimes it is as high as 1.5% a month.

Definition: Simple Interest

If an amount P is borrowed for a time t at an interest rate of r per time period, then the simple interest is given by

$$I = P \cdot r \cdot t \tag{6.1.1}$$

Definition: Accumulated Value

The total amount A, also called the **accumulated value** or the future value, is given by

$$\begin{aligned} A &= P + I \\ &= P + Prt \end{aligned}$$

or

$$A = P(1 + rt) \tag{6.1.2}$$

where interest rate r is expressed in decimals.

Example 6.1.1

Ursula borrows \$600 for 5 months at a simple interest rate of 15% per year. Find the interest, and the total amount she is obligated to pay?

Solution

The interest is computed by multiplying the principal with the interest rate and the time.

$$\begin{aligned} \mathrm{I} &= \mathrm{Prt} \\ &= \$600(0.15)\frac{5}{12} \\ &= \$37.50 \end{aligned}$$

The total amount is

$$\begin{aligned} \mathrm{A} &= \mathrm{P} + \mathrm{I} \\ &= \$600 + \$37.50 \\ &= \$637.50 \end{aligned}$$

Incidentally, the total amount can be computed directly via Equation 6.1.2 as

$$\begin{aligned} A &= P(1 + rt) \\ &= \$600[1 + (0.15)(5/12)] \\ &= \$600(1 + 0.0625) \\ &= \$637.50 \end{aligned}$$

Example 6.1.2

Jose deposited \$2500 in an account that pays 6% simple interest. How much money will he have at the end of 3 years?

Solution

The total amount or the future value is given by Equation 6.1.2.

$$\begin{aligned} A &= P(1 + rt) \\ &= \$2500[1 + (.06)(3)] \\ \mathrm{A} &= \$2950 \end{aligned}$$

Example 6.1.3

Darnel owes a total of \$3060 which includes 12% simple interest for the three years he borrowed the money. How much did he originally borrow?

Solution

This time we are asked to compute the principal P via Equation 6.1.2.

$$\begin{aligned} \$3060 &= \mathrm{P}[1+(0.12)(3)] \\ \$3060 &= \mathrm{P}(1.36) \\ \frac{\$3060}{1.36} &= \mathrm{P} \\ \$2250 &= \mathrm{P} \quad \text{Darnel originally borrowed } \$2250. \end{aligned}$$

Example 6.1.4

A Visa credit card company charges a 1.5% simple interest finance charge each month on the unpaid balance. If Martha owed $2350 and has not paid her bill for three months, how much does she owe now?

Solution

Before we attempt the problem, the reader should note that in this problem the rate of finance charge is given per month and not per year.

The total amount Martha owes is the previous unpaid balance plus the finance charge.

$$A = \$2350 + \$2350(.015)(3) = \$2350 + \$105.75 = \$2455.75$$

Alternatively, again, we can compute the amount directly by using formula $A = P(1+rt)$

$$A = \$2350[1+(.015)(3)] = \$2350(1.045) = \$2455.75$$

Summary

Below is a summary of the formulas we developed for calculations involving simple interest:

Simple interest

If an amount P is borrowed for a time t at an interest rate of r per time period, then the simple interest is given by

$$I = P \cdot r \cdot t$$

The total amount A, also called the accumulated value or the future value, is given by

$$A = P + I = P + Prt$$

or

$$A = P(1+rt)$$

where interest rate r is expressed in decimals.

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6.1E: Exercises - Simple Interest

PROBLEM SET: SIMPLE INTEREST

Do the following simple interest problems.

1) If an amount of $2,000 is borrowed at a simple interest rate of 10% for 3 years, how much is the interest?	2) You borrow $4,500 for six months at a simple interest rate of 8%. How much is the interest?
3) John borrows $2400 for 3 years at 9% simple interest. How much will he owe at the end of 3 years?	4) Jessica takes a loan of $800 for 4 months at 12% simple interest. How much does she owe at the end of the 4-month period?
5) If an amount of $2,160, which includes a 10% simple interest for 2 years, is paid back, how much was borrowed 2 years earlier?	6) Jamie just paid off a loan of $2,544, the principal and simple interest. If he took out the loan six months ago at 12% simple interest, what was the amount borrowed?
7) Shanti charged $800 on her charge card and did not make a payment for six months. If there is a monthly charge of 1.5%, how much does she owe?	8) A credit card company charges 18% interest on the unpaid balance. If you owed $2000 three months ago and have been delinquent since, how much do you owe?
9) An amount of $2000 is borrowed for 3 years. At the end of the three years, $2660 is paid back. What was the simple interest rate?	10) Nancy borrowed $1,800 and paid back $1,920, four months later. What was the simple interest rate?

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6.2: Compound Interest

Learning Objectives

In this section, you will learn to:

- Find the future value of a lump-sum.
- Find the present value of a lump-sum.
- Find the effective interest rate.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Simplify each expression.

a. $100(3+2^2)$

b. $100(3+2)^2$

Click here to check your answer

a. 700

b. 2500

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

2. If an amount of $2,000 is borrowed at a simple interest rate of 10% for 3 years, how much is the interest?

Click here to check your answer

$600

If you missed this problem, review Section 6.1. (Note that this will open in a new window.)

3. You borrow $4,500 for six months at a simple interest rate of 8%. How much is the interest?

Click here to check your answer

$180

If you missed this problem, review Section 6.1. (Note that this will open in a new window.)

4. John borrows $2400 for 3 years at 9% simple interest. How much will he owe at the end of 3 years?

Click here to check your answer

$3048

If you missed this problem, review Section 6.1. (Note that this will open in a new window.)

Compound Interest

In the last section, we examined problems involving simple interest. Simple interest is generally charged when the lending period is short and often less than a year. When the money is loaned or borrowed for a longer time period, if the interest is paid (or charged) not only on the principal, but also on the past interest, then we say the interest is **compounded.**

Suppose we deposit $200 in an account that pays 8% interest each year. At the end of one year, we will have $200 + $200(.08) = $200(1 + .08) = $216.

Now suppose we put this amount, $216, in the same account. After another year, we will have $216 + $216(.08) = $216(1 + .08) = $233.28.

So an initial deposit of $200 has accumulated to $233.28 in two years. Further note that had it been simple interest, this amount would have accumulated to only $232. The reason the amount is slightly higher is because the interest ($16) we earned the first year, was put back into the account. And this $16 amount itself earned for one year an interest of $16(.08) = $1.28, thus resulting in the increase. So we have earned interest on the principal as well as on the past interest, and that is why we call it compound interest.

Now suppose we leave this amount, $233.28, in the bank for another year, the final amount will be $233.28 + $233.28(.08) = $233.28(1 + .08) = $251.94.

Now let us look at the mathematical part of this problem so that we can devise an easier way to solve these problems.

After one year, we had $200(1 + .08) = $216

After two years, we had $216(1 + .08)

But $216 = $200(1 + .08), therefore, the above expression becomes

$$\$200(1+.08)(1+.08) = \$200(1+.08)^2 = \$233.28$$

After three years, we get

$$\$233.28(1+.08) = \$200(1+.08)(1+.08)(1+.08)$$

which can be written as

$$\$200(1+.08)^3 = \$251.94$$

Suppose we are asked to find the total amount at the end of 5 years, we will get

$$200(1+.08)^5 = \$293.87$$

We summarize as follows:

The original amount	$200	= $200
The amount after one year	$200(1 + .08)	= $216
The amount after two years	$200(1 + .08)^2	= $233.28
The amount after three years	$200(1 + .08)^3	= $251.94
The amount after five years	$200(1 + .08)^5	= $293.87
The amount after t years	$200(1 + .08)^t	

Periodically Compounded Interest

Banks often compound interest more than one time a year. Consider a bank that pays 8% interest but compounds it four times a year, or quarterly. This means that every quarter the bank will pay an interest equal to one-fourth of 8%, or 2%.

Now if we deposit $200 in the bank, after one quarter we will have $\$200\left(1+\frac{.08}{4}\right)$ or $204.

After two quarters, we will have $\$200\left(1+\frac{.08}{4}\right)^2$ or $208.08.

After one year, we will have $\$200\left(1+\frac{.08}{4}\right)^4$ or $216.49.

After three years, we will have $\$200\left(1+\frac{.08}{4}\right)^{12}$ or $253.65, etc.

The original amount	$200	= $200
The amount after one quarter	$\$200\left(1+\frac{.08}{4}\right)$	= $204

The amount after two quarters	$\$200\left(1+\frac{.08}{4}\right)^{2}$	= \$208.08
The amount after one year	$\$200\left(1+\frac{.08}{4}\right)^{4}$	= \$216.49
The amount after two years	$\$200\left(1+\frac{.08}{4}\right)^{8}$	= \$234.31
The amount after three years	$\$200\left(1+\frac{.08}{4}\right)^{12}$	= \$253.65
The amount after five years	$\$200\left(1+\frac{.08}{4}\right)^{20}$	= \$297.19
The amount after t years	$\$200\left(1+\frac{.08}{4}\right)^{4t}$	

We can see the formula for compound interest emerge.

Definition: Compound Interest, n times per year

If a lump-sum amount of P dollars is invested at an interest rate r, compounded n times a year, then after t years the final amount is given by

$$A = P\left(1+\frac{r}{n}\right)^{nt} \tag{6.2.1}$$

P is called the **principal** and is also called **the present value**.

Example 6.2.1

If \$3500 is invested at 9% compounded monthly, what will the future value be in four years?

Solution

Clearly an interest of .09/12 is paid every month for four years. The interest is compounded $4 \times 12 = 48$ times over the four-year period. We get

$$\mathrm{A} = \$3500\left(1+\frac{.09}{12}\right)^{48} = \$3500(1.0075)^{48} = \$5009.92$$

\$3500 invested at 9% compounded monthly will accumulate to \$5009.92 in four years.

Example 6.2.2

How much should be invested in an account paying 9% compounded daily for it to accumulate to \$5,000 in five years?

Solution

We know the future value, but need to find the principal.

$$\begin{aligned}\$5000 &= P\left(1+\tfrac{.09}{365}\right)^{365\times 5}\\ \$5000 &= P(1.568225)\\ \$3188.32 &= P\end{aligned}$$

\$3188,32 invested into an account paying 9% compounded daily will accumulate to \$5,000 in five years.

Example 6.2.3

If \$4,000 is invested at 4% compounded annually, how long will it take to accumulate to \$6,000?

Solution

$n = 1$ because annual compounding means compounding only once per year. The formula simplifies to $A = (1 + r)^t$ when $n = 1$.

$$\begin{aligned} \$6000 &= 4000(1 + .04)^t \\ \frac{6000}{4000} &= 1.04^t \\ 1.5 &= 1.04^t \end{aligned}$$

We use logarithms to solve for the value of t because the variable t is in the exponent.

$$t = \log_{1.04}(1.5)$$

Using the change of base formula we can solve for t:

$$t = \frac{\ln(1.5)}{\ln(1.04)} = 10.33 \text{ years}$$

It takes 10.33 years for \$4000 to accumulate to \$6000 if invested at 4% interest, compounded annually

Example 6.2.4

If \$5,000 is invested now for 6 years what interest rate compounded quarterly is needed to obtain an accumulated value of \$8000.

Solution

We have $n = 4$ for quarterly compounding.

$$\begin{aligned} \$8000 &= \$5000\left(1 + \frac{r}{4}\right)^{4\times 6} \\ \frac{\$8000}{\$5000} &= \left(1 + \frac{r}{4}\right)^{24} \\ 1.6 &= \left(1 + \frac{r}{4}\right)^{24} \end{aligned}$$

We use roots to solve for t because the variable r is in the base, whereas the exponent is a known number.

$$\sqrt[24]{1.6} = 1 + \frac{\mathrm{r}}{4}$$

Many calculators have a built in “nth root” key or function. In the TI-84 calculator, this is found in the Math menu. Roots can also be calculated as fractional exponents; if necessary, the previous step can be rewritten as

$$1.6^{1/24} = 1 + \frac{\mathrm{r}}{4}$$

Evaluating the left side of the equation gives

$$\begin{aligned} 1.0197765 &= 1 + \tfrac{\mathrm{r}}{4} \\ 0.0197765 &= \tfrac{\mathrm{r}}{4} \\ \mathrm{r} &= 4(0.0197765) = 0.0791 \end{aligned}$$

An interest rate of 7.91% is needed in order for \$5000 invested now to accumulate to \$8000 at the end of 6 years, with interest compounded quarterly.

Effective Interest Rate

Banks are required to state their interest rate in terms of an **“effective yield”** or **“effective interest rate”**, for comparison purposes. The effective rate is also called the Annual Percentage Yield (APY) or Annual Percentage Rate (APR).

The effective rate is the interest rate compounded annually would be equivalent to the stated rate and compounding periods. We often suppose we invest \$1 over the course of one year to determine the effective rate, as is shown in the next example.

To examine several investments to see which has the best rate, we find and compare the effective rate for each investment.

Example 6.2.5

If Bank A pays 7.2% interest compounded monthly, what is the effective interest rate?
If Bank B pays 7.2% interest compounded semiannually, what is the effective interest rate?

Which bank pays more interest?

Solution

Bank A: Suppose we deposit \$1 in this bank and leave it for a year, we will get

$$1\left(1+\tfrac{0.072}{12}\right)^{12} = 1.0744$$
$$\mathrm{r_{EFF}} = 1.0744 - 1 = 0.0744$$

We earned interest of \$1.0744 - \$1.00 = \$.0744 on an investment of \$1.

The effective interest rate is 7.44%, often referred to as the APY.

Bank B: The effective rate is calculated as

$$\mathbf{r}_{\mathrm{EFF}} = 1\left(1+\frac{0.072}{2}\right)^{2} - 1 = .0733$$

The effective interest rate is 7.33%.

Bank A pays slightly higher interest, with an effective rate of 7.44%, compared to Bank B with effective rate 7.33%.

Definition: Effective Interest Rate, compounded n times per year

If a bank pays an interest rate r per year, compounded n times a year, then **the effective interest rate** is given by

$$\mathbf{r}_{\mathrm{EFF}} = \left(1+\frac{r}{n}\right)^{n} - 1 \tag{6.2.2}$$

This is also reffered to as the **annual percentage yield**, or **APY**.

Continuously Compounded Interest

Interest can be compounded yearly, semiannually, quarterly, monthly, and daily. Using the same calculation methods, we could compound every hour, every minute, and even every second. As the compounding period gets shorter and shorter, we move toward the concept of continuous compounding.

But what do we mean when we say the interest is compounded continuously, and how do we compute such amounts? When interest is compounded "infinitely many times", we say that the interest is **compounded continuously**. Our next objective is to derive a formula to model continuous compounding.

Suppose we put \$1 in an account that pays 100% interest. If the interest is compounded once a year, the total amount after one year will be $\$1(1+1) = \$2$.

- If the interest is compounded semiannually, in one year we will have $\$1(1+1/2)^2 = \$2.25$
- If the interest is compounded quarterly, in one year we will have $\$1(1+1/4)^4 = \$2.44$
- If the interest is compounded monthly, in one year we will have $\$1(1+1/12)^{12} = \$2.61$
- If the interest is compounded daily, in one year we will have $\$1(1+1/365)^{365} = \$2.71$

We show the results as follows:

Frequency of compounding	Formula	Total amount
Annually	$\$1(1+1)$	\$2
Semiannually	$\$1(1+1/2)^2$	\$2.25

Frequency of compounding	Formula	Total amount
Quarterly	$\$1(1+1/4)^4 = \$2.44$	$2.44140625
Monthly	$\$1(1+1/12)^{12}$	$2.61303529
Daily	$\$1(1+1/365)^{365}$	$2.71456748
Hourly	$\$1(1+1/8760)^{8760}$	$2.71812699
Every minute	$\$1(1+1/525600)^{525600}$	$2.71827922
Every Second	$\$1(1+1/31536000)^{31536000}$	$2.71828247
Continuously	$\$1(2.718281828\ldots)$	$2.718281828...

We have noticed that the $1 we invested does not grow without bound. It starts to stabilize to an irrational number 2.718281828... given the name "*e*" after the great mathematician Euler.

In mathematics, we say that as n becomes infinitely large the expression equals $\left(1+\frac{1}{n}\right)^n = e$.

Therefore, it is natural that the number *e* play a part in continuous compounding.
It can be shown that as n becomes infinitely large the expression $\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$

Therefore, it follows that if we invest $\$P$ at an interest rate r per year, compounded continuously, after t years the final amount will be given by

$$A = P \cdot e^{rt}$$

Definition: Continuously Compounded Interest

If an amount P is invested for t years at an interest rate r per year, **compounded continuously**, then the future value is given by

$$\mathrm{A} = \mathrm{P}e^{rt} \tag{6.2.3}$$

Example 6.2.6

$3500 is invested at 9% compounded continuously. Find the future value in 4 years.

Solution

Using the formula for the continuous compounding, we get $A = Pe^{rt}$.

$$A = \$3500e^{0.09\times 4}$$
$$A = \$3500e^{0.36}$$
$$A = \$5016.65$$

Example 6.2.7

If an amount is invested at 7.2% compounded continuously, what is the effective interest rate?

Solution

If we deposit $1 in the bank at 7.2% compounded continuously for one year, and subtract that $1 from the final amount, we get the effective interest rate in decimals.

$$\mathrm{r_{EFF}} = 1\mathrm{e}^{0.072} - 1$$
$$\mathrm{r_{EFF}} = 1.07466 - 1$$
$$\mathrm{r_{EFF}} = .07466 \text{ or } 7.466\%$$

Definition: Effective Interest Rate, compounded continously

If a bank pays an interest rate r per year, compounded continuously, then the effective interest rate is given by

$$\mathrm{r_{EFF}} = e^{\mathrm{r}} - 1 \tag{6.2.4}$$

Example 6.2.8

If an amount is invested at 7% compounded continuously, how long will it take to double?

Solution

We don't know the initial value of the principal but we do know that the accumulated value is double (twice) the principal.

$$\mathrm{P} \cdot e^{0.07t} = 2\mathrm{P}$$

We divide both sides by P

$$e^{.07t} = 2$$

Using natural logarithm:

$$.07\mathrm{t} = \ln(2)$$
$$\mathrm{t} = \ln(2)/.07 = 9.9 \text{ years}$$

It takes 9.9 years for money to double if invested at 7% continuous interest.

Example 6.2.9

a. At the peak growth rate in the 1960's the world's population had a doubling time of 35 years. At that time, approximately what was the growth rate?

b. As of 2015, the world population's annual growth rate was approximately 1.14%. Based on that rate, find the approximate doubling time.

Solution

We expect the world's population to grow continuously, not in discrete intervals such as years or months. Therefore, we will use the formula $A = Pe^{rt}$.

a. Substituting $2\mathrm{P}$ for A and 35 for t gives us the equation

$$2\mathrm{P} = \mathrm{P} \cdot e^{r(35)}$$

We divide both sides by P:

$$2 = e^{r(35)}$$

Using natural logarithm:

$$\ln(2) = r(35)$$

Dividing both sides by 35:

$$\frac{\ln(2)}{35} = r$$
$$0.0198 = r$$

The growth rate was approximately 1.98%

b. Substituting $2\mathrm{P}$ for A and $0.0114$for r gives us the equation

$$2\mathrm{P} = \mathrm{P} \cdot e^{0.0114t}$$

We divide both sides by P:

$$2 = e^{0.0114t}$$

Using natural logarithm:

$$\ln(2) = 0.0114t$$

Dividing both sides by 0.014:

$$\frac{\ln(2)}{0.0114} = t$$
$$60.8 = t$$

If the world population were to continue to grow at the annual growth rate of 1.14% , it would take approximately 60.8 years for the population to double.

SECTION 8.2 SUMMARY

Below is a summary of the formulas we developed for calculations involving compound interest:

COMPOUND INTEREST n times per year

1. If an amount P is invested for t years at an interest rate r per year, compounded n times a year, then the future value is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

   $\mathbf{P}$ is called the principal and is also called the present value.
2. If a bank pays an interest rate r per year, compounded n times a year, then the effective interest rate is given by

$$\mathbf{r}_{\mathrm{EFF}} = \left(1 + \frac{r}{n}\right)^{n} - 1$$

CONTINUOUSLY COMPOUNDED INTEREST

3. If an amount P is invested for t years at an interest rate r per year, compounded continuously, then the future value is given by

$$\mathrm{A} = \mathrm{P}e^{rt}$$

4. If a bank pays an interest rate r per year, compounded continuously, then the effective interest rate is given by

$$\mathrm{r}_{\mathrm{EFF}} = e^{\mathbf{r}} - 1$$

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6.2E: Exercises - Compound Interest

PROBLEM SET: COMPOUND INTEREST

Do the following compound interest problems involving a lump-sum amount.

1) What will the final amount be in 4 years if $8,000 is invested at 9.2% compounded monthly?	2) How much should be invested at 10.3% compounded quarterly for it to amount to $10,000 in 6 years?
3) Lydia's aunt Rose left her $5,000. Lydia spent $1,000 on her wardrobe and deposited the rest in an account that pays 6.9% compounded daily. How much money will she have in 5 years?	4) Thuy needs $1,850 in eight months for her college tuition. How much money should she deposit lump sum in an account paying 8.2% compounded monthly to achieve that goal?
5) Bank A pays 5% compounded daily, while Bank B pays 5.12% compounded monthly. Which bank pays more? Explain.	6) EZ Photo Company needs five copying machines in 2 1/2 years for a total cost of $15,000. How much money should be deposited now to pay for these machines, if the interest rate is 8% compounded semiannually?
7) Jon's grandfather was planning to give him $12,000 in 10 years. Jon has convinced his grandfather to pay him $6,000 now, instead. If Jon invests this $6,000 at 7.5% compounded continuously, how much money will he have in 10 years?	8) What will be the price of a $20,000 car in 5 years if the inflation rate is 6%?

COMPOUND INTEREST

Do the following compound interest problems.

9) At an interest rate of 8% compounded continuously, how many years will it take to double your money?	10) If an investment earns 10% compounded continuously, in how many years will it triple? .
11) The City Library ordered a new computer system costing $158,000; it will be delivered in 6 months, and the full amount will be due 30 days after delivery. How much must be deposited today into an account paying 7.5% compounded monthly to have $158,000 in 7 months?	12) Mr. and Mrs. Tran are expecting a baby girl in a few days. They want to put away money for her college education now. How much money should they deposit in an account paying 10.2% so they will have $100,000 in 18 years to pay for their daughter's educational expenses?
13) Find the effective interest rate for an account paying 7.2% compounded quarterly.	14) If a bank pays 5.75% compounded monthly, what is the effective interest rate?
15) The population of the African nation of Cameroon was 12 million people in the year 2015; it has been growing at the rate of 2.5% per year. If the population continues to grow that rate,what will the population be in 2030? (http://databank.worldbank.org/data on 4/26/2016)	16) According to the Law of 70, if an amount grows at an annual rate of 1%, then it doubles every seventy years. Suppose a bank pays 5% interest, how long will it take for you to double your money? How about at 15%?

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6.3: Future Value of Annuities and Sinking Funds

Learning Objectives

In this section, you will learn to:

- Find the future value of an annuity.
- Find the amount of payments to a sinking fund.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Solve $24 = 6 \cdot x^3$.

Click here to check your answer

$x = \sqrt[3]{4} \approx 1.5874$

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

2. Solve $24 = 6 \cdot 3^x$.

Click here to check your answer

$x = \log_3(4) \approx 1.2619$

If you missed this problem, review Section 5.3. (Note that this will open in a new window.)

3. Simplify $\dfrac{500(2 - \frac{5}{6})^{5 \cdot 4} - 1}{\frac{2}{3}}$ in your calculator.

Click here to check your answer

$16,366.55426$

If you missed this problem, review here. (Note that this will open a YouTube video in a new window.)

Ordinary Annuities

In the first two sections of this chapter, we examined problems where an amount of money was deposited as a lump sum in an account and was left there for the entire time period. Now we will do problems where timely payments are made in an account. When a sequence of payments of some fixed amount are made in an account at equal intervals of time, we call that an **annuity**. And this is the subject of this section.

Example 6.3.1

If at the end of each month a deposit of \$500 is made in an account that pays 8% compounded monthly, what will the final amount be after five years?

Solution

There are 60 deposits made in this account. The first payment stays in the account for 59 months, the second payment for 58 months, the third for 57 months, and so on. We can use the formula for compound interest to examine each payment.

- The first payment of \$500 will accumulate to an amount of $\$500(1 + 0.08/12)^{59}$.

- The second payment of \$500 will accumulate to an amount of $\$500(1 + 0.08/12)^{58}$.
- The third payment will accumulate to $\$500(1 + 0.08/12)^{57}$.
- The fourth payment will accumulate to $\$500(1 + 0.08/12)^{56}$.

And so on . . .

Finally the next to last (59^{th}) payment will accumulate to $\$500(1+0.08/12)^1$.

The last payment is taken out the same time it is made, and will not earn any interest.

To find the total amount in five years, we need to add the accumulated value of these sixty payments.

In other words, we need to find the sum of the following series.

$$\$500(1+0.08/12)^{59}+\$500(1+0.08/12)^{58}+\$500(1+0.08/12)^{57}+\ldots+\$500$$

Written backwards, we have

$$\$500+\$500(1+0.08/12)+\$500(1+0.08/12)^2+\ldots+\$500(1+0.08/12)^{59}$$

This is a geometric series with $a=\$500$, $r=(1+0.08/12)$, and $n=59$. The sum is

$$\begin{aligned}\text{sum} &= \frac{\$500\left[(1+0.08/12)^{60}-1\right]}{0.08/12}\\ &= \$500(73.47686)\\ &= \$36,738.43\end{aligned}$$

When the payments are made at the end of each period rather than at the beginning, we call it an **ordinary annuity.**

Future Value of an Ordinary Annuity

If a payment of m dollars is made in an account n times a year at an interest r, then the final amount A after t years is

$$\mathbf{A=\frac{m\left[(1+r/n)^{nt}-1\right]}{r/n}} \tag{6.3.1}$$

The final amount is also called the future value or the accumulated value.

Example 6.3.2

Tanya deposits \$300 at the end of each quarter in her savings account. If the account earns 5.75% compounded quarterly, how much money will she have in 4 years?

Solution

The future value of this annuity can be found using the above formula.

$$\begin{aligned}A &= \frac{\$300\left[(1+.0575/4)^{16}-1\right]}{0.0575/4}\\ &= \$300(17.8463)\\ &= \$5353.89\end{aligned}$$

If Tanya deposits \$300 into a savings account earning 5.75% compounded quarterly for 4 years, then at the end of 4 years she will have \$5,353.89

Example 6.3.3

Robert needs \$5,000 in three years. How much should he deposit each month in an account that pays 8% compounded monthly in order to achieve his goal?

Solution

If Robert saves m dollars per month, after three years he will have

$$\frac{m\left[(1+.08/12)^{36}-1\right]}{0.08/12}$$

But we'd like this amount to be \$5,000. Therefore,

$$\begin{aligned}\frac{\mathrm{m}\left[(1+.08/12)^{36}-1\right]}{.08/12} &= \$5000\\ \mathrm{m}(40.5356) &= \$5000\\ \mathrm{m} &= \frac{5000}{40.5356}\\ &= \$123.35\end{aligned}$$

Robert needs to deposit \$123.35 at the end of each month for 3 years into an account paying 8% compounded monthly in order to have \$5,000 at the end of 5 years.

Example 6.3.4

Suppose you contribute \$500 each month into an ordinary annuity earning 6.25% compounded monthly. How long do you need to continue making contributions so that you will have at least \$1,000,000?

Solution

Contributing \$500 each month, $m = 500$. Interest is 6.25% compounded monthly, so $r = 0.0625$ and $n = 12$. You want a future value of \$1,000,000 so $A = 1,000,000$.

$$1,000,000 = \frac{500\left[(1+0.0625/12)^{12t}-1\right]}{0.0625/12}$$

Multiply both sides by $(0.0625/12)$to clear the denominator:

$$1,000,000 \cdot (0.0625/12) = 500\left[(1+0.0625/12)^{12t}-1\right]$$

Divide both sides by 500:

$$\frac{1,000,000 \cdot (0.0625/12)}{500} = (1+0.0625/12)^{12t}-1$$

Add 1 to both sides:

$$\frac{1,000,000 \cdot (0.0625/12)}{500}+1 = (1+0.0625/12)^{12t}$$

Convert to logarithmic form to isolate the exponent:

$$\log_{1+0.0625/12}\left(\frac{1,000,000 \cdot (0.0625/12)}{500}+1\right) = 12t$$

Use change of base formula:

$$\frac{\log\left(\frac{1,000,000 \cdot (0.0625/12)}{500}+1\right)}{\log(1+0.0625/12)} = 12t$$

Enter into calculator:

$$468.75 = t$$

Divide both sides by 12:

$$39.06 = t$$

It will take approximately 39 years for you to have at least \$1,000,000.

Sinking Fund

When a business deposits money at regular intervals into an account in order to save for a future purchase of equipment, the savings fund is referred to as a "**sinking fund**". Calculating the sinking fund deposit uses the same method as the previous problem.

Example 6.3.5

A business needs $450,000 in five years. How much should be deposited each quarter in a sinking fund that earns 9% compounded quarterly to have this amount in five years?

Solution

Again, suppose that m dollars are deposited each quarter in the sinking fund. After five years, the future value of the fund should be $450,000. This suggests the following relationship:

$$\begin{aligned} \frac{\mathrm{m}\left[(1+0.09/4)^{20}-1\right]}{0.09/4} &= \$450,000 \\ \mathrm{m}(24.9115) &= 450,000 \\ \mathrm{m} &= \frac{450000}{24.9115} \\ &= \$18,063.93 \end{aligned}$$

The business needs to deposit $18,063.93 at the end of each quarter for 5 years into an sinking fund earning interest of 9% compounded quarterly in order to have $450,000 at the end of 5 years.

Annuity Due

If the payment is made at the beginning of each period, rather than at the end, we call it an **annuity due**. The formula for the annuity due can be derived in a similar manner. Reconsider Example 1, with the change that the deposits are made at the beginning of each month.

Example 6.3.6

If at the beginning of each month a deposit of $500 is made in an account that pays 8% compounded monthly, what will the final amount be after five years?

Solution

There are 60 deposits made in this account. The first payment stays in the account for 60 months, the second payment for 59 months, the third for 58 months, and so on.

- The first payment of $500 will accumulate to an amount of $\$500(1+0.08/12)^{60}$.
- The second payment of $500 will accumulate to an amount of $\$500(1+.08/12)^{59}$.
- The third payment will accumulate to $\$500(1+0.08/12)^{58}$.

And so on . . .

The last payment is in the account for a month and accumulates to $\$500(1+0.08/12)$

To find the total amount in five years, we need to find the sum of the series:

$$\$500(1+0.08/12)^{60}+\$500(1+0.08/12)^{59}+\$500(1+0.08/12)^{58}+\ldots+\$500(1+0.08/12)$$

Written backwards, we have

$$\$500(1+0.08/12)+\$500(1+0.08/12)^{2}+\ldots+\$500(1+0.08/12)^{60}$$

If we add $500 to this series, and later subtract that $500, the value will not change. We get

$$\mathbf{\$500}+\$500(1+0.08/12)+\$500(1+0.08/12)^{2}+\ldots+\$500(1+0.08/12)^{60}-\mathbf{\$500}$$

Except for the last term, we have a geometric series with a = \$500, r = (1 + .08/12), and n = 60. Therefore the sum is

$$\begin{aligned} A &= \frac{\$500\left[(1+0.08/12)^{61}-1\right]}{0.08/12} - \$500 \\ &= \$500(74.9667) - \$500 \\ &= \$37483.35 - \$500 \\ &= \$36983.35 \end{aligned}$$

So, in the case of an annuity due, to find the future value, we increase the number of periods n by 1, and subtract one payment.

Future Value of an "Annuity Due"

$$A = \frac{m\left[(1+r/n)^{nt+1}-1\right]}{r/n} - m$$

Most of the problems we are going to do in this chapter involve ordinary annuities, therefore, we will down play the significance of the last formula for the annuity due. We mentioned the formula for the annuity due only for completeness.

Summary

Finally, it is the author's wish that the student learn the concepts in a way that he or she will not have to memorize every formula. It is for this reason formulas are kept at a minimum. But before we conclude this section we will once again mention one single equation that will help us find the future value, as well as the sinking fund payment.

If a payment of m dollars is made in an account n times a year at an interest r, then the future value A after t years is

$$\mathbf{A = \frac{m\left[(1+r/n)^{nt}-1\right]}{r/n}}$$

Note that the formula assumes that the payment period is the same as the compounding period. If these are not the same, then this formula does not apply.

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6.3E: Exercises - Annuities and Sinking Funds

PROBLEM SET:ANNUITIES AND SINKING FUNDS

Each of the following problems involve an annuity - a sequence of payments.

1) Find the future value of an annuity of $200 per month for 5 years at 6% compounded monthly.	2) How much money should be deposited at the end of each month in an account paying 7.5% for it to amount to $10,000 in 5 years?
3) At the end of each month Rita deposits $300 in an account that pays 5%. What will the final amount be in 4 years?	4) Mr. Chang wants to retire in 10 years and can save $650 every three months. If the interest rate is 7.8%, how much will he have (a) at the end of 5 years? (b) at the end of 10 years?
5) A firm needs to replace most of its machinery in five years at a cost of $500,000. The company wishes to create a sinking fund to have this money available in five years. How much should the quarterly deposits be if the fund earns 8%?	6) Mrs. Brown needs $5,000 in three years. If the interest rate is 9%, how much should she save at the end of each month to have that amount in three years?
7) A company has a $120,000 note due in 4 years. How much should be deposited at the end of each quarter in a sinking fund to payoff the note in four years if the interest rate is 8%?	8) You are now 20 years of age and decide to save $100 at the end of each month until you are 65. If the interest rate is 9.2%, how much money will you have when you are 65?
9) Is it better to receive $400 at the beginning of each month for six years, or a lump sum of $25,000 today if the interest rate is 7%? Explain.	10) To save money for a vacation, Jill decided to save $125 at the beginning of each month for the next 8 months. If the interest rate is 7%, how much money will she have at the end of 8 months?
11) Mrs. Gill puts $2200 at the end of each year in her IRA account that earns 9% per year. How much total money will she have in this account after 20 years?	12) If the inflation rate stays at 6% per year for the next five years, how much will the price be of a $15,000 car in five years? How much must you save at the end of each month at an interest rate of 7.3% to buy that car in 5 years?

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6.4: Present Value of Annuities and Installment Payment

Learning Objectives

In this section, you will learn to:

- Find the present value of an annuity.
- Find the amount of installment payment on a loan.
- Find the outstanding balance on a loan.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. What will the final amount be in 4 years if $8,000 is invested at 9.2% compounded monthly?

Click here to check your answer

$\$11,542.53$

If you missed this problem, review Section 6.2. (Note that this will open in a new window.)

2. How much should be invested at 10.3% compounded quarterly for it to amount to $10,000 in 6 years?

Click here to check your answer

$\$5,432.55$

If you missed this problem, review Section 6.2. (Note that this will open in a new window.)

3. Find the future value of an annuity of $200 per month for 5 years at 6% compounded monthly.

Click here to check your answer

$\$13,954.01$

If you missed this problem, review Section 6.3. (Note that this will open in a new window.)

4. How much money should be deposited at the end of each month in an account paying 7.5% for it to amount to $10,000 in 5 years?

Click here to check your answer

$\$137.88$

If you missed this problem, review Section 6.3. (Note that this will open in a new window.)

PRESENT VALUE OF AN ANNUITY

In the previous two sections, we learned to find the future value of a lump sum and the future value of an annuity. With these two concepts in hand, we will now learn to amortize a loan, and to find the present value of an annuity.

The **present value** of an annuity is the amount of money we would need now in order to be able to make the payments in the annuity in the future. In other words, the present value is the value now of a future stream of payments.

We start by breaking this down step by step to understand the concept of the present value of an annuity. After that, the examples provide a more efficient way to do the calculations by working with concepts and calculations we have already explored in the last two sections.

Suppose Carlos owns a small business and employs an assistant manager to help him run the business. Assume it is January 1 now. Carlos plans to pay his assistant manager a $1000 bonus at the end of this year and another $1000 bonus at the end of the following year. Carlos' business had good profits this year so he wants to put the money for his assistant's future bonuses into a savings account now. The money he puts in now will earn interest at the rate of 4% per year compounded annually while in the savings

account.

How much money should Carlos put into the savings account now so that he will be able to withdraw $1000 one year from now and another $1000 two years from now?

At first, this sounds like a sinking fund. But it is different. In a sinking fund, we put money into the fund with periodic payments to save to accumulate to a specified lump sum that is the future value at the end of a specified time period.

In this case we want to put a lump sum into the savings account now, so that lump sum is our principal, P. Then we want to withdraw that amount as a series of period payments; in this case the withdrawals are an annuity with $1000 payments at the end of each of two years.

We need to determine the amount we need in the account now, the present value, to be able to make withdraw the periodic payments later.



We use the compound interest formula from Section 6.2 with r = 0.04 and n = 1 for annual compounding to determine the present value of each payment of $1000.

Consider the first payment of $1000 at the end of year 1. Let P_1 be its present value

$$\$1000 = P_1(1.04)^1 \text{ so } P_1 = \$961.54$$

Now consider the second payment of $1000 at the end of year 2. Let P_2 is its present value

$$\$1000 = P_2(1.04)^2 \text{ so } P_2 = \$924.56$$

To make the $1000 payments at the specified times in the future, the amount that Carlos needs to deposit now is the present value
$P = P_1 + P_2 = \$961.54 + \$924.56 = \$1886.10$

The calculation above was useful to illustrate the meaning of the present value of an annuity.
But it is not an efficient way to calculate the present value. If we were to have a large number of annuity payments, the step by step calculation would be long and tedious.

Example 6.4.1 investigates and develops an efficient way to calculate the present value of an annuity, by relating the future (accumulated) value of an annuity and its present value.

Example 6.4.1

Suppose you have won a lottery that pays $1,000 per month for the next 20 years. But, you prefer to have the entire amount now. If the interest rate is 8%, how much will you accept?

Solution

This classic present value problem needs our complete attention because the rationalization we use to solve this problem will be used again in the problems to follow.

Consider, for argument purposes, that two people Mr. Cash, and Mr. Credit have won the same lottery of $1,000 per month for the next 20 years. Mr. Credit is happy with his $1,000 monthly payment, but Mr. Cash wants to have the entire amount now.

Our job is to determine how much Mr. Cash should get. We reason as follows:

If Mr. Cash accepts P dollars, then the P dollars deposited at 8% for 20 years should yield the same amount as the $1,000 monthly payments for 20 years. In other words, we are comparing the future values for both Mr. Cash and Mr. Credit, and we would like the future values to equal.

Since Mr. Cash is receiving a lump sum of x dollars, its future value is given by the lump sum formula, and it is

$$A = P(1 + .08/12)^{240}$$

Since Mr. Credit is receiving a sequence of payments, or an annuity, of $1,000 per month, its future value is given by the annuity formula This value is

$$A = \frac{\$1000\left[(1 + .08/12)^{240} - 1\right]}{.08/12}$$

The only way Mr. Cash will agree to the amount he receives is if these two future values are equal. So we set them equal and solve for the unknown.

$$P(1 + .08/12)^{240} = \frac{\$1000\left[(1+.08/12)^{240} - 1\right]}{.08/12}$$

$$P = \frac{\frac{\$1000\left[(1+.08/12)^{240} - 1\right]}{.08/12}}{(1+.08/12)^{240}}$$

$$P = \$119,554.36$$

The present value of an ordinary annuity of $1,000 each month for 20 years at 8% is $119,554.36

The reader should also note that if Mr. Cash takes his lump sum of P = $119,554.36 and invests it at 8% compounded monthly, he will have an accumulated value of A=$589,020.41 in 20 years.

If a person or business needs to buy or pay for something now (a car, a home, college tuition, equipment for a business) but does not have the money, they can borrow the money as a loan.

They receive the loan amount called the principal (or present value) now and are obligated to pay back the principal in the future over a stated amount of time (term of the loan), as regular periodic payments with interest.

We can see the formula begin to develop in the example above.

Definition: Present Value of an Annuity

If a payment of m dollars is made in an account n times a year at an interest r, then the **present value P of the annuity** after t years is

$$\mathbf{P(1 + r/n)^{nt} = \frac{m\left[(1 + r/n)^{nt} - 1\right]}{r/n}} \tag{6.4.1}$$

When used for a loan, the amount P is the **loan amount**, and m is the periodic payment needed to repay the loan over a term of t years with n payments per year.

Note that the formula assumes that the payment period is the same as the compounding period. If these are not the same, then this formula does not apply.

Example 6.4.2

You determine that you can afford $550 per month for a car. At the dealership, the finance department offers you a 3-year loan with a 7% interest rate, compounded monthly.

a. What is the highest priced car you can afford in this scenario?
b. How much will you pay over the 3 years?
c. What is the difference between the answers for part a and part b, and what does it represent?

Solution

a. Using the formula above, we are looking for the total amount of the loan, or P. We know that $m = 550$, $r = 0.07$, $n = 12$, and $t = 3$.

$$\mathbf{P(1 + 0.07/12)^{(12 \cdot 3)} = \frac{550\left[(1+0.07/12)^{(12 \cdot 3)} - 1\right]}{0.07/12}}$$

$$\mathbf{P}=\frac{\frac{550\left[(1+0.07/12)^{(12\cdot 3)}-1\right]}{0.07/12}}{(1+0.07/12)^{(12\cdot 3)}}$$

$$P=17812.55545$$

You can take a loan of $17,812.55 to pay off within 3 years.

b. Over the course of 3 years, you would pay $550 every month, 12 months each year, for 3 years.

Total paid $= \$550 \cdot 12 \cdot 3 = \$19,800.$

c. The difference between $17,812.55 as the purchase price and $19,800 as the repayment price is $\$19800 - \$17812.55 = \$1987.45$ This means that you'll pay $1,987.45 in interest over the 3 years.

Alternate Method to find Present Value of an Annuity

We should note that many finite mathematics and finance books develop the formula for the present value of an annuity differently.

Instead of using the formula:

$$\mathrm{P}(1+\mathrm{r}/\mathrm{n})^{\mathrm{nt}}=\frac{\mathrm{m}\left[(1+\mathrm{r}/\mathrm{n})^{\mathrm{nt}}-1\right]}{\mathrm{r}/\mathrm{n}} \tag{6.4.2}$$

and solving for the present value P after substituting the numerical values for the other items in the formula, many textbooks first solve the formula for P in order to develop a new formula for the present value. Then the numerical information can be substituted into the present value formula and evaluated, without needing to solve algebraically for P.

Starting with formula 6.4.2: $\mathrm{P}(1+\mathrm{r}/\mathrm{n})^{\mathrm{nt}}=\frac{\mathrm{m}\left[(1+\mathrm{r}/\mathrm{n})^{\mathrm{nt}}-1\right]}{\mathrm{r}/\mathrm{n}}$

Divide both sides by $(1+r/n)^{nt}$ to isolate P, and simplify

$$P=\frac{m\left[(1+r/n)^{nt}-1\right]}{r/n}\cdot\frac{1}{(1+r/n)^{nt}}$$

$$P=\frac{m\left[1-(1+r/n)^{-nt}\right]}{r/n} \tag{6.4.3}$$

The authors of this book believe that it is easier to use formula 6.4.2 at the top of this page and solve for P or m as needed. In this approach there are fewer formulas to understand, and many students find it easier to learn. In the problems the rest of this chapter, when a problem requires the calculation of the present value of an annuity, formula 6.4.2 will be used.

However, formula 6.4.3 is ideal when used to solve for t, and some people prefer to use this formula to find present value. It is a mathematically correct option to do so. Note that if you choose to use formula 6.4.3, you need to be careful with the negative exponents in the formula.

INSTALLMENT PAYMENTS ON A LOAN

Example 6.4.3 examines how to calculate the loan payment, using reasoning similar to Example 6.4.1.

Example 6.4.3

Daria purchased a truck costing $45,000 and accepted a 60-month loan at an interest rate of 9%, compounded monthly.

a. Find the monthly installment payments.
b. How much will she pay over the course of the loan?
c. How much interest will she pay over the course of the loan?

Solution

a. The monthly installment payments can be found in a similar process to finding the principal in Example 6.4.2. Note that 60 months is the same as 5 years. We can use $t=5$ or we can use $nt=60$.

$$\$45,000(1+.09/12)^{60} = \frac{m\left[(1+.09/12)^{60}-1\right]}{.09/12}$$

We then multiply both sides by the denominator:

$$(\$45,000(1+.09/12)^{60})(\tfrac{.09}{12}) = m\left[(1+.09/12)^{60}-1\right]$$

Finally dividing by the entire quantity that is multiplied to the m leaves:

$$\frac{(\$45,000(1+.09/12)^{60})(\frac{.09}{12})}{\left[(1+.09/12)^{60}-1\right]} = m$$
$$\$934.13 = m$$

Therefore, the monthly payment needed to repay the loan is $934.13 for five years.

b. Over the course of the loan, she would pay $934.13 every month for 60 months.

Total paid $= \$934.13 \cdot 60 = \$56047.80$

c. To find interest, we find the difference between purchase price and repayment price. $\$56047.80 - \$45000 = \$16047.80$ so she will pay $\$16047.80$in interest over the course of the loan.

OUTSTANDING BALANCE ON A LOAN

One of the most common problems deals with finding the balance owed at a given time during the life of a loan. Suppose a person buys a house and amortizes the loan over 30 years, but decides to sell the house a few years later. At the time of the sale, he is obligated to pay off his lender, therefore, he needs to know the balance he owes. Since most long term loans are paid off prematurely, we are often confronted with this problem.

To find the outstanding balance of a loan at a specified time, we need to find the present value P of all future payments that have not yet been paid. In this case t does not represent the entire term of the loan. Instead:

- t represents the time that still remains on the loan
- nt represents the total number of future payments.

If the problem does not directly state the amount of time still remaining in the term of the loan, then it must be calculated FIRST using t = [original term of loan] - [time already passed since the start date of the loan].

Example 6.4.4

Mr. Jackson bought his house in 1995, and financed the loan for 30 years at an interest rate of 7.8%. His monthly payment was $1260. In 2015, Mr. Jackson decides to pay off the loan. Find the balance of the loan he still owes.

Solution

The reader should note that the original amount of the loan is not mentioned in the problem. That is because we don't need to know that to find the balance.

The original loan was for 30 years. 20 years have passed so there are 10 years still remaining. 12(10) = 120 payments still remain to be paid on this loan.

As for the bank or lender is concerned, Mr. Jackson is obligated to pay $1260 each month for 10 more years; he still owes a total of 120 payments. But since Mr. Jackson wants to pay it all off now, we need to find the present value P at the time of repayment of the remaining 10 years of payments of $1260 each month. Using the formula we get for the present value of an annuity, we get

$$\mathrm{P}(1+.078/12)^{120} = \frac{\$1260\left[(1+.078/12)^{120}-1)\right]}{(.078/12)}$$
$$\mathrm{P}(2.17597) = \$227957.85$$
$$\mathrm{P} = \$104761.48$$

To find the outstanding balance of a loan

If a loan has a payment of m dollars made n times a year at an interest r, then the outstanding value of the loan when there are t years still remaining on the loan is given by P:

$$\mathbf{P(1+r/n)^{nt}=\frac{m\left[(1+r/n)^{nt}-1\right|}{r/n}} \tag{6.4.4}$$

IMPORTANT: Note that t is not the original term of the loan but instead t is the amount of time still remaining in the future; nt is the number of payments still remaining in the future

If the problem does not directly state the amount of time still remaining in the term of the loan, then it must be calculated BEFORE using the above formula as t = [original term of loan] - [time already passed since the start date of the loan].

Note that there are other methods to find the outstanding balance on a loan, but the method illustrated above is generally considered the easiest.

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6.4E: Exercises - Present Value of an Annuity and Installment Payment

PROBLEM SET: PRESENT VALUE OF AN ANNUITY AND INSTALLMENT PAYMENT

For the following problems, show all work.

1) Shawn has won a lottery paying him $10,000 per month for the next 20 years. He'd rather have the whole amount in one lump sum today. If the current interest rate is 8.2%, how much money can he hope to get?	2) Sonya bought a car for $15,000. Find the monthly payment if the loan is to be amortized over 5 years at a rate of 10.1%.
3) You determine that you can afford $250 per month for a car. What is the maximum amount you can afford to pay for a car if the interest rate is 9% and you want to repay the loan in 5 years?	4) Compute the monthly payment for a house loan of $200,000 to be financed over 30 years at an interest rate of 10%.
5) If the $200,000 loan in the previous problem is financed over 15 years rather than 30 years at 10%, what will the monthly payment be?	6) Friendly Auto offers Jennifer a car for $2000 down and $300 per month for 5 years. Jason wants to buy the same car but wants to pay cash. How much must Jason pay if the interest rate is 9.4%?
7) The Gomez family bought a house for $450,000. They paid 20% down and amortized the rest at 5.2% over a 30-year period. Find their monthly payment.	8) Mr. and Mrs. Wong purchased their new house for $350,000. They made a down payment of 15%, and amortized the rest over 30 years. If the interest rate is 5.8%, find their monthly payment.
9) A firm needs a piece of machinery that has a useful life of 5 years. It has an option of leasing it for $10,000 a year, or buying it for $40,000 cash. If the interest rate is 10%, which choice is better?	10) Jackie wants to buy a $19,000 car, but she can afford to pay only $300 per month for 5 years. If the interest rate is 6%, how much does she need to put down?
11) Vijay's tuition at college for the next year is $32,000. His parents have decided to pay the tuition by making nine monthly payments. If the interest rate is 6%, what is the monthly payment?	12) Glen borrowed $10,000 for his college education at 8% compounded quarterly. Three years later, after graduating and finding a job, he decided to start paying off his loan. If the loan is amortized over five years at 9%, find his monthly payment for the next five years.

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6.5: Classification of Finance Problems

Learning Objectives

In this section, you will learn to:

- Re-examine the types of financial problems and classify them.
- Re-examine the vocabulary words used in describing financial calculations

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Describe the differences between solving $50 = x^9$ vs solving $50 = 9^x$.

Click here to check your answer

If the exponent is known, you'll need to take that root of each side of the equation. If the exponent is unknown, you'll need to take the logarithm of each side of the equation.

If you missed this problem, review Section 5.1. (Note that this will open in a new window.)

2. Describe the differences between simple interest vs compound interest.

Click here to check your answer

Simple interest is calculated once, based on the agreed timeframe of the loan. Compound interest is calculated multiple times and added to the account balance before it is calculated again.

If you missed this problem, review Section 6.2. (Note that this will open in a new window.)

3. Describe the differences between periodically compounded interest vs continuously compounded interest.

Click here to check your answer

Periodically compounded interest is calculated at regular time intervals, such as every quarter, month, day, etc. The n-value tells how many times that time interval occurs in a year. Continuously compounded interest is calculated at an infinitely small time interval, using e instead of an n-value.

If you missed this problem, review Section 6.3. (Note that this will open in a new window.)

4. Describe the differences between a lump-sum vs an annuity.

Click here to check your answer

In a lump-sum, money is added to the account once. The only thing that changes the account balance after that is the interest. In an annuity, money is added (and\or removed) from the account at regular time intervals.

If you missed this problem, review Section 6.3. (Note that this will open in a new window.)

5. Describe the differences between the future value of an annuity vs the present value of an annuity. (Be careful because either formula can be used to find what should be done now (in the present) and what would be the result in the future!)

Click here to check your answer

The future value formula is used when interest and regular payments both add to the account balance. The present value formula is used when interest and regular payments work in opposite directions, one increasing the balance and one decreasing the balance.

If you missed this problem, review Section 6.3. (Note that this will open in a new window.)

We'd like to remind the reader that the hardest part of solving a finance problem is determining the category it falls into. So in this section, we will emphasize the classification of problems rather than finding the actual solution. We suggest that the student read each problem carefully and look for the word or words that may give clues to the kind of problem that is presented.

For instance, students often fail to distinguish a lump-sum problem from an annuity. Since the payments are made each period, an annuity problem contains words such as each, every, per etc. One should also be aware that in the case of a lump-sum, only a single deposit is made, while in an annuity numerous deposits are made at equal spaced time intervals. To help interpret the vocabulary used in the problems, we include a glossary at the end of this section.

Students often confuse the present value of an annuity with the future value of an annuity. Don't get too caught up on the words, as either formula can be used to find what should be done now (in the present) and what would be the result in the future.

- The future value formula will be used in the context of investments, where the regular deposits and interest both work together to add to the account balance.
- The present value formula will be used in the context of loans or bank withdraws, where the regular payments and interest work against each other. In a loan, the monthly payments lower the account balance while the interest payments raise the account balance.

CLASSIFICATION OF PROBLEMS AND EQUATIONS

We now list eight problems that form a basis for all finance problems. Read each question and determine which formula would be used to find the missing information.

Example 6.5.1

Dillon takes out a simple interest loan for $5,000 at 10.5% interest with an agreement to pay it back in 10 months. What is the total amount he will repay at the end of the 10 months?

Classification: Since the loan is a simple interest loan and we are looking for the total amount he will repay, we will use the **Accumulated Value of Simple Interest Formula**.

Equation:

$$\mathrm{A} = 5,000(1 + 0.105(\frac{10}{12})) \tag{6.5.1}$$

Example 6.5.2

If $2,000 is invested at 7% compounded quarterly, what will the final amount be in 5 years?

Classification: Since the $2000 is invested once, this is a lump-sum. The interest is compounded each quarter, so this will use the **Accumulated Value of Periodic Compounding Formula**.

Equation:

$$\mathrm{A} = \$2000(1 + .07/4)^{20} \tag{6.5.2}$$

Example 6.5.3

If $2,000 is invested at 7% compounded continuously, what will the final amount be in 5 years?

Classification: Since the $2000 is invested once, this is a lump-sum. The interest is compounded continuously, so this will use the **Accumulated Value of Continuous Compounding Formula**.

Equation:

$$\mathrm{A} = \$2000e^{0.07*5} \tag{6.5.3}$$

Example 6.5.4

How much should be invested at 8% compounded yearly, for the final amount to be $5,000 in five years?

Classification: Since this question is asking for a one-time investment, this is a lump-sum. The interest is compounded each year, so this will use the **Accumulated Value of Periodic Compounding Formula**. Note that we are given the accumulated value at the end of the 5 years and are looking for the principal.

Equation:

$$\$5,000 = \mathrm{P}(1+.08)^{5} \tag{6.5.4}$$

Example 6.5.5

If $200 is invested *each* month at 8.5% compounded monthly, what will the final amount be in 4 years?

Classification: Since the money is invested each month, this is an example of an annuity. This is an investment where the monthly payments and the interest both add to the value, so this will use the **Future Value of an Annuity Formula.**

Equation:

$$\mathrm{A} = \frac{\$200\left[(1+.085/12)^{12*4}-1\right]}{.085/12} \tag{6.5.5}$$

Example 6.5.6

How much should be invested *each* month at 9% for it to accumulate to $8,000 in three years?

Classification: Since the money is invested each month, this is an example of an annuity. This is an investment where the monthly payments and the interest both add to the value, so this will use the **Future Value of an Annuity Formula.** Note that we are given the accumulated value at the end of the 3 years and are looking for the monthly payment.

Equation:

$$\$8,000 = \frac{m\left[(1+.09/12)^{36}-1\right]}{.09/12} \tag{6.5.6}$$

Example 6.5.7

Keith wants to buy a new car worth $30,000 and needs to apply for an auto loan. The dealership is offering him financing at 9.25% compounded monthly over the next 5 years. How much will each monthly payment be?

Classification: Since the money will be paid each month, this is an example of an annuity. This is a loan where the monthly payments lower the account balance and the interest raises the account balance, so this will use the **Present Value of an Annuity Formula.**

Equation:

$$\$30,000(1+.0925/12)^{12*5} = \frac{m\left[(1+.0925/12)^{12*5}-1\right]}{.0925/12} \tag{6.5.7}$$

Example 6.5.8

Maria is saving money for her retirement. The retirement fund earns 8% interest compounded monthly, and she plans to withdraw $1,500 each month for 25 years after her retirement. How much does she need in this fund before she retires?

Classification: Since the money will be withdrawn each month, this is an example of an annuity. This is a fund where the monthly withdraws lower the account balance and the interest raises the account balance, so this will use the **Present Value of an Annuity Formula.**

Equation:

$$\mathrm{P}(1+.08/12)^{12*25}=\frac{\$1,500\left[(1+.08/12)^{12*25}-1\right]}{.08/12} \tag{6.5.8}$$

GLOSSARY: VOCABULARY AND SYMBOLS USED IN FINANCIAL CALCULATIONS

As we've seen in these examples, it's important to read the problems carefully to correctly identify the situation. It is essential to understand to vocabulary for financial problems. Many of the vocabulary words used are listed in the glossary below for easy reference.

t	Term	Time period for a loan or investment. In this book t is represented in years and should be converted into years when it is stated in months or other units.
P	Principal	Principal is the amount of money borrowed in a loan. If a sum of money is invested for a period of time, the sum invested at the start is the Principal.
P	Present Value	Value of money at the beginning of the time period.
A	Accumulated Value Future Value	Value of money at the end of the time period
D	Discount	In loans involving simple interest, a discount occurs if the interest is deducted from the loan amount at the beginning of the loan period, rather than being repaid at the end of the loan period.
m	Periodic Payment	The amount of a constant periodic payment that occurs at regular intervals during the time period under consideration (examples: periodic payments made to repay a loan, regular periodic payments into a bank account as savings, regular periodic payment to a retired person as an annuity,)
n	Number of payment periods and compounding periods per year	In this book, when we consider periodic payments, we will always have the compounding period be the same as the payment period. In general the compounding and payment periods do not have to be the same, but the calculations are more complicated if they are different. If the periods differ, formulas for the calculations can be found in finance textbooks or various online resources. Calculations can easily be done using technology such as an online financial calculator, or financial functions in a spreadsheet, or a financial pocket calculator.
nt	Number of periods	nt = (number of periods per year)$\times$(number of years) nt gives the total number of payment and compounding periods In some situations we will calculate nt as the multiplication shown above. In other situations the problem may state nt, such as a problem describing an investment of 18 months duration compounded monthly. In this example: nt = 18 months and n = 12; then t = 1.5 years but t is not stated explicitly in the problem. The TI-84+ calculators built in TVM solver uses $N=nt$.
r	Annual interest rate Nominal rate	The stated annual interest rate. This is stated as a percent but converted to decimal form when using financial calculation formulas. If a bank account pays 3% interest compounded quarterly, then 3% is the nominal rate, and it is included in the financial formulas as r = 0.03
r/n	Interest rate per compounding period	If a bank account pays 3% interest compounded quarterly, then r/n = 0.03/4 = 0. 075, corresponding to a rate of 0.75% per quarter. Some Finite Math books use the symbol i to represent r/n

r_{EFF}	Effective Rate Effective Annual Interest Rate APY Annual Percentage Yield APR Annual Percentage Rate	The effective rate is the interest rate compounded annually that would give the same interest rate as the compounded rate stated for the investment. The effective rate provides a uniform way for investors or borrowers to compare different interest rates with different compounding periods.
I	Interest	Money paid by a borrower for the use of money borrowed as a loan. Money earned over time when depositing money into a savings account, certificate of deposit, or money market account. When a person deposits money in a bank account, the person depositing the funds is essentially temporarily lending the money to the bank and the bank pays interest to the depositor.
	Sinking Fund	A fund set up by making payments over a period of time into a savings or investment account in order to save to fund a future purchase. Businesses use sinking funds to save for a future purchase of equipment at the end of the savings period by making periodic installment payments into a sinking fund.
	Annuity	An annuity is a stream of periodic payments. In this book it refers to a stream of constant periodic payments made at the end of each compounding period for a specific amount of time. In common use the term annuity generally refers to a constant stream of periodic payments received by a person as retirement income, such as from a pension. Annuity payments in general may be made at the end of each payment period (ordinary annuity) or at the start of each period (annuity due). The compounding periods and payment periods do not need to be equal, but in this textbook we only consider situations when these periods are equal.
	Lump Sum	A single sum of money paid or deposited at one time, rather than being spread out over time. An example is lottery winnings if the recipient chooses to receive a single “lump sum” one-time payment, instead of periodic payments over a period of time or as. Use of the word lump sum indicates that this is a one time transaction and is not a stream of periodic payments.
	Loan	An amount of money that is borrowed with the understanding that the borrower needs to repay the loan to the lender in the future by the end of a period of time that is called the term of the loan. The repayment is most often accomplished through periodic payments until the loan has been completely repaid over the term of the loan. However there are also loans that can be repaid as a single sum at the end of the term of the loan, with interest paid either periodically over the term or in a lump sum at the end of the loan or as a discount at the start of the loan.

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6.5E: Exercises - Classification of Finance Problems

PROBLEM SET: CLASSIFICATION OF FINANCE PROBLEMS

Let the letters A, B, C, D, E and F be represented as follows:

$A = FV$ of a lump-sum $\quad C = FV$ of an annuity $\quad E =$ Installment payment

$B = PV$ of a lump-sum $\quad D =$ sinking fund payment $\quad F = PV$ of an annuity

Classify each by writing the appropriate letter in the box, and write an equation for solution.

1) What monthly deposits made to an account paying 9% will grow to $10,000 in 4 years?

2) An amount of $4000 is invested at 6% compounded daily. What will the final amount be in 5 years?

3) David has won a lottery paying him $10,000 per month for the next 20 years. He'd rather have the whole amount in one lump sum now. If the current interest rate is 7%, how much money can he hope to get?

4) Each month Linda deposits $250 in an account that pays 9%. How much money will she have in 4 years?

5) Find the monthly payment for a $15,000 car if the loan is amortized over 4 years at a rate of 10%.

6) What lump-sum deposited in an account paying 7% compounded daily will grow to $10,000 in 5 years?

7) What amount of quarterly payments will amount to $250,000 in 5 years at a rate of 8%?

8) The Chang family bought their house 25 years ago. They had their loan financed for 30 years at an interest rate of 11% resulting in a payment of $1350 a month. Find the balance of the loan.

A 10-year $1000 bond pays $35 every six months. If the current interest rate is 8%, in order to find the fair market value of the bond, we need to find the following.

9) The present value of $1000.

10) The present value of the $35 per six month payments.

11) What lump-sum deposit made today is equal to 33 monthly deposits of $500 if the interest rate is 8%?

12) What monthly deposits made to an account paying 10% will accumulated to $10,000 in six years?

13) A department store charges a finance charge of 1.5% per month on the outstanding balance.
If Ned charged $400 three months ago and has not paid his bill, how much does he owe?

14) What will the value of $300 monthly deposits be in 10 years if the account pays 12% compounded monthly?

15) What lump-sum deposited at 6% compounded daily will grow to $2000 in three years?

16) A company buys an apartment complex for $5,000,000 and amortizes the loan over 10 years.
What is the yearly payment if the interest rate is 14%?

17) In 2002, a house in Rock City cost $300,000. Real estate in Rock City has been increasing in value at the annual rate of 5.3%.. Find the price of that house in 2016.

18) You determine that you can afford to pay $400 per month for a car. What is the maximum price you can pay for a car if the interest rate is 11% and you want to repay the loan in 4 years?

19) A business needs $350,000 in 5 years. How much lump-sum should be put aside in an account that pays 9% so that five years from now the company will have $350,000?

20) A person wishes to have $500,000 in a pension fund 20 years from now. How much should he deposit each month in an account paying 9% compounded monthly?

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6.6: Additional Application Problems

Learning Objectives

In this section, you will learn to:

- Perform financial calculations in situations involving several stages of savings and/or annuities.
- Find the fair market value of a bond.
- Construct an amortization schedule for a loan.

We have already developed the tools to solve most finance problems. Now we use these tools to solve some additional applications.

PROBLEMS INVOLVING MULTIPLE STAGES OF SAVINGS AND/OR ANNUITIES

Consider the following situations:

a. Suppose a baby, Aisha, is born and her grandparents invest $5000 in a college fund. The money remains invested for 18 years until Aisha enters college, and then is withdrawn in equal semiannual payments over the 4 years that Aisha expects to need to finish college. The college investment fund earns 5% interest compounded semiannually. How much money can Aisha withdraw from the account every six months while she is in college?

b. Aisha graduates college and starts a job. She saves $1000 each quarter, depositing it into a retirement savings account. Suppose that Aisha saves for 30 years and then retires. At retirement she wants to withdraw money as an annuity that pays a constant amount every month for 25 years. During the savings phase, the retirement account earns 6% interest compounded quarterly. During the annuity payout phase, the retirement account earns 4.8% interest compounded monthly. Calculate Aisha's monthly retirement annuity payout.

These problems appear complicated. But each can be broken down into two smaller problems involving compound interest on savings or involving annuities. Often the problem involves a savings period followed by an annuity period. ; the accumulated value from first part of the problem may become a present value in the second part. Read each problem carefully to determine what is needed.

Example 6.6.2

Suppose a baby, Aisha, is born and her grandparents invest $8000 in a college fund. The money remains invested for 18 years until Aisha enters college, and then is withdrawn in equal semiannual payments over the 4 years that Aisha expects to attend college. The college investment fund earns 5% interest compounded semiannually. How much money can Aisha withdraw from the account every six months while she is in college?

Solution

Part 1: Accumulation of College Savings: Find the accumulated value at the end of 18 years of a sum of $8000 invested at 5% compounded semiannually.

$$A = \$8000(1 + .05/2)^{(2\times 18)} = \$8000(1.025)^{36} = \$8000(2.432535)$$
$$A = \$19460.28$$

Part 2: Seminannual annuity payout from savings to put toward college expenses. Find the amount of the semiannual payout for four years using the accumulated savings from part 1 of the problem with an interest rate of 5% compounded semiannually.

A= $19460.28 in Part 1 is the accumulated value at the end of the savings period. This becomes the present value P=$19460.28 when calculating the semiannual payments in Part 2.

$$\$19460.28\left(1+\tfrac{.05}{2}\right)^{2\times 4} = \frac{m\left[\left(1+\frac{.05}{2}\right)^{2\times 4}-1\right]}{(.05/2)}$$
$$\$23710.46 = m(8.73612)$$
$$m = \$2714.07$$

Aisha will be able to withdraw $2714.07 semiannually for her college expenses.

Example 6.6.3

Aisha graduates college and starts a job. She saves $1000 each quarter, depositing it into a retirement savings account. Suppose that Aisha saves for 30 years and then retires. At retirement she wants to withdraw money as an annuity that pays a constant amount every month for 25 years. During the savings phase, the retirement account earns 6% interest compounded quarterly. During the annuity payout phase, the retirement account earns 4.8% interest compounded monthly. Calculate Aisha's monthly retirement annuity payout.

Solution

Part 1: Accumulation of Retirement Savings: Find the accumulated value at the end of 30 years of $1000 deposited at the end of each quarter into a retirement savings account earning 6% interest compounded quarterly.

$$A = \frac{\$1000\left[(1+.06/4)^{4\times 30}-1\right]}{(.06/4)}$$
$$A = \$331288.19$$

Part 2: Monthly retirement annuity payout: Find the amount of the monthly annuity payments for 25 years using the accumulated savings from part 1 of the problem with an interest rate of 4.8% compounded monthly.

A= $331288.19 in Part 1 is the accumulated value at the end of the savings period. This amount will become the present value P =$331288.19 when calculating the monthly retirement annuity payments in Part 2.

$$\$331288.19(1+.048/12)^{12\times 25} = \frac{m\left[(1+.048/12)^{12\times 25}-1\right]}{(.048/12)}$$
$$\$1097285.90 = m(578.04483)$$
$$m = \$1898.27$$

Aisha will have a monthly retirement annuity income of $1898.27 when she retires.

FAIR MARKET VALUE OF A BOND

Whenever a business, and for that matter the U. S. government, needs to raise money it does it by selling bonds. A **bond** is a certificate of promise that states the terms of the agreement. Usually the business sells bonds for the **face amount** of $1,000 each for a stated **term**, a period of time ending at a specified **maturity** date.

The person who buys the bond, the **bondholder**, pays $1,000 to buy the bond.

The bondholder is promised two things: First that he will get his $1,000 back at the maturity date, and second that he will receive a fixed amount of interest every six months.

As the market interest rates change, the price of the bond starts to fluctuate. The bonds are bought and sold in the market at their **fair market value**.

The interest rate a bond pays is fixed, but if the market interest rate goes up, the value of the bond drops since the money invested in the bond could earn more if invested elsewhere. When the value of the bond drops, we say it is trading at a **discount**.

On the other hand, if the market interest rate drops, the value of the bond goes up since the bond now yields a higher return than the market interest rate, and we say it is trading at a **premium**.

Example 6.6.4

The Orange Computer Company needs to raise money to expand. It issues a 10-year $1,000 bond that pays $30 every six months. If the current market interest rate is 7%, what is the fair market value of the bond?

Solution

The bond certificate promises us two things - An amount of $1,000 to be paid in 10 years, and a semi-annual payment of $30 for ten years. Therefore, to find the fair market value of the bond, we need to find the present value of the lump sum of $1,000 we are to receive in 10 years, as well as, the present value of the $30 semi-annual payments for the 10 years.

We will let P_1 = the present value of the face amount of $1,000

$$P_1(1+.07/2)^{20} = \$1,000$$

Since the interest is paid twice a year, the interest is compounded twice a year and nt = 2(10)=20

$$P_1(1.9898) = \$1,000$$
$$P_1 = \$502.56$$

We will let P_2 = the present value of the \$30 semi-annual payments is

$$P_2(1+.07/2)^{20} = \frac{\$30\left[(1+.07/2)^{20}-1\right]}{(.07/2)}$$
$$P_2(1.9898) = 848.39$$
$$P_2 = \$426.37$$

The present value of the lump-sum \$1,000 = \$502.56

The present value of the \$30 semi-annual payments = \$426.37

The fair market value of the bond is $P = P_{1+} P_2$ = \$502.56 + \$426.37 = \$928.93

Note that because the market interest rate of 7% is higher than the bond's implied interest rate of 6% implied by the semiannual payments, the bond is selling at a discount; its fair market value of \$928.93 is less than its face value of \$1000.

Example 6.6.5

A state issues a 15 year \$1000 bond that pays \$25 every six months. If the current market interest rate is 4%, what is the fair market value of the bond?

Solution

The bond certificate promises two things - an amount of \$1,000 to be paid in 15 years, and semi-annual payments of \$25 for 15 years. To find the fair market value of the bond, we find the present value of the \$1,000 face value we are to receive in 15 years and add it to the present value of the \$25 semi-annual payments for the 15 years. In this example, $nt = 2(15) = 30$.

We will let P_1 = the present value of the lump-sum \$1,000

$$P_1(1+.04/2)^{30} = \$1,000$$
$$P_1 = \$552.07$$

We will let P_2 = the present value of the \$25 semi-annual payments is

$$P_2(1+.04/2)^{30} = \frac{\$25\left[(1+.04/2)^{30}-1\right]}{(.04/2)}$$
$$P_2(1.18114) = \$1014.20$$
$$P_2 = \$559.90$$

The present value of the lump-sum \$1,000 = \$552.07

The present value of the \$30 semi-annual payments = \$559.90

Therefore, the fair market value of the bond is

$$P = P_1 + P_2 = \$552.07 + \$559.90 = \$1111.97$$

Because the market interest rate of 4% is lower than the interest rate of 5% implied by the semiannual payments, the bond is selling at a premium: the fair market value of \$1,111.97 is more than the face value of \$1,000.

To summarize:

To find the Fair Market Value of a Bond

Find the present value of the face amount A that is payable at the maturity date:

$$\mathbf{A} = \mathbf{P}_1(\mathbf{1}+\mathbf{r}/\mathbf{n})^{\mathbf{nt}};\ \text{solve to find}\ \mathrm{P}_1 \tag{6.6.1}$$

Find the present value of the semiannually payments of $\$m$ over the term of the bond:

$$\mathbf{P}_2(\mathbf{1}+\mathbf{r}/\mathbf{n})^{\mathbf{nt}} = \frac{\mathbf{m}\left[(\mathbf{1}+\mathbf{r}/\mathbf{n})^{\mathbf{nt}}-\mathbf{1}\right]}{\mathbf{r}/\mathbf{n}} \quad ;\ \text{solve to find}\ \mathbf{P}_2 \tag{6.6.2}$$

The fair market value (or present value or price or current value) of the bond is the sum of the present values calculated above:

$$\mathrm{P} = \mathbf{P}_1 + \mathbf{P}_2 \tag{6.6.3}$$

AMORTIZATION SCHEDULE FOR A LOAN

An amortization schedule is a table that lists all payments on a loan, splits them into the portion devoted to interest and the portion that is applied to repay principal, and calculates the outstanding balance on the loan after each payment is made.

Example 6.6.6

An amount of $500 is borrowed for 6 months at a rate of 12%. Make an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the portion of the payment contributing toward reducing the debt, and the outstanding balance.

Solution

The reader can verify that the monthly payment is $86.27.

The first month, the outstanding balance is $500, and therefore, the monthly interest on the outstanding balance is

(outstanding balance)(the monthly interest rate) = ($500)(.12/12) = $5

This means, the first month, out of the $86.27 payment, $5 goes toward the interest and the remaining $81.27 toward the balance leaving a new balance of $500 - $81.27 = $418.73.

Similarly, the second month, the outstanding balance is $418.73, and the monthly interest on the outstanding balance is ($418.73)(.12/12) = $4.19. Again, out of the $86.27 payment, $4.19 goes toward the interest and the remaining $82.08 toward the balance leaving a new balance of $418.73 - $82.08 = $336.65. The process continues in the table below.

Payment #	Payment	Interest	Debt Payment	Balance
1	$86.27	$5	$81.27	$418.73
2	$86.27	$4.19	$82.08	$336.65
3	$86.27	$3.37	$82.90	$253.75
4	$86.27	$2.54	$83.73	$170.02
5	$86.27	$1.70	$84.57	$85.45
6	$86.27	$0.85	$85.42	$0.03

Note that the last balance of 3 cents is due to error in rounding off.

An amortization schedule is usually lengthy and tedious to calculate by hand. For example, an amortization schedule for a 30 year mortgage loan with monthly payments would have (12)(30)=360 rows of calculations in the amortization schedule table. A car loan with 5 years of monthly payments would have 12(5)=60 rows of calculations in the amortization schedule table. However it would be straightforward to use a spreadsheet application on a computer to do these repetitive calculations by inputting and copying formulas for the calculations into the cells.

Most of the other applications in this section's problem set are reasonably straightforward, and can be solved by taking a little extra care in interpreting them. And remember, there is often more than one way to solve a problem.

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6.6E: Exercises - Miscellaneous Application Problems

PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS

For problems 1 - 4, assume a $200,000 house loan is amortized over 30 years at an interest rate of 5.4%.

1) Find the monthly payment.	2) Find the balance owed after 20 years.
3) Find the balance of the loan after 100 payments.	4) Find the monthly payment if the original loan were amortized over 15 years.

MISCELLANEOUS APPLICATION PROBLEMS

5) Mr. Patel wants to pay off his car loan. The monthly payment for his car is $365, and he has 16 payments left. If the loan was financed at 6.5%, how much does he owe?	6) An amount of $2000 is borrowed for a year at a rate of 7%. Make an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the portion of the payment going toward reducing the debt, and the balance.
7) Fourteen months after Dan bought his new car he lost his job. His car was repossessed by his lender after he made only 14 monthly payments of $376 each. If the loan was financed over a 4-year period at an interest rate of 6.3%, how much did the car cost the lender? In other words, how much did Dan still owe on the car?	8) You have a choice of either receiving $5,000 at the end of each year for the next 5 years or receiving $3000 per year for the next 10 years. If the current interest rate is 9%, which is better?
9) Mr. Smith is planning to retire in 25 years and would like to have $250,000 then. What monthly payment made at the end of each month to an account that pays 6.5% will achieve his objective?	10) Assume Mr. Smith has reached retirement and has $250,000 in an account which is earning 6.5%. He would now like to make equal monthly withdrawals for the next 15 years to completely deplete this account. Find the withdrawal payment.
11) Mrs. Garcia is planning to retire in 20 years. She starts to save for retirement by depositing $2000 each quarter into a retirement investment account that earns 6% interest compounded quarterly. Find the accumulated value of her retirement savings at the end of 20 years.	12) Assume Mrs. Garcia has reached retirement and has accumulated the amount found in question 13 in a retirement savings account. She would now like to make equal monthly withdrawals for the next 15 years to completely deplete this account. Find the withdrawal payment. Assume the account now pays 5.4% compounded monthly.

13) A ten-year $1,000 bond pays $35 every six months. If the current interest rate is 8.2%, find the fair market value of the bond. Hint: You must do the following. a) Find the present value of $1000. b) Find the present value of the $35 payments. c) The fair market value of the bond = a + b	14) Find the fair market value of the ten-year $1,000 bond that pays $35 every six months, if the current interest rate has dropped to 6%. Hint: You must do the following. a) Find the present value of $1000. b) Find the present value of the $35 payments. c) The fair market value of the bond = a + b
15) A twenty-year $1,000 bond pays $30 every six months. If the current interest rate is 4.2%, find the fair market value of the bond. Hint: You must do the following. a) Find the present value of $1000. b) Find the present value of the $30 payments. c) The fair market value of the bond = a + b	16) Find the fair market value of the twenty-year $1,000 bond that pays $30 every six months, if the current interest rate has increased to 7.5%.

17) Mr. and Mrs. Nguyen deposit $10,000 into a college investment account when their new baby grandchild is born. The account earns 6.25% interest compounded quarterly.
a) When their grandchild reaches the age of 18, what is the accumulated value of the college investment account?
b) The Nguyen's grandchild has just reached the age of 18 and started college. If she is to withdraw the money in the college savings account n equal monthly payments over the next 4 years, how much money will be withdrawn each month?

18) Mr. Singh is 38 and plans to retire at age 65. He opens a retirement savings account.
a) Mr. Singh wants to save enough money to accumulate $500,000 by the time he retires. The retirement investment account pays 7% interest compounded monthly. How much does he need to deposit each month to achieve this goal?
b) Mr. Singh has now reached at 65 and retires. How much money can he withdraw each month for 25 years if the retirement investment account now pays 5.2% interest, compounded monthly?

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6.7: Chapter 6 Review

PROBLEM SET: CHAPTER REVIEW

Simple Interest (6.1)

1. Manuel borrows $800 for 6 months at 18% simple interest. How much does he owe at the end of 6 months?
2. An amount of $2300 is borrowed for 7 months at a simple interest rate of 16%. Find the total that will need to be paid back.

Compound Interest (6.2)

3. In the year 2000, an average house in Star City cost $250,000. If the average annual inflation rate for the past years has been about 4.7%, what was the price of that house in 2015?
4. A Visa credit card company has a finance charge of 1.5% per month (18% per year) on the outstanding balance. John owed $3200 and has been delinquent for 5 months. How much total does he owe, now?
5. A sum of $5000 is deposited in a bank today. What will the final amount be in 20 months if the bank pays 9% and the interest is compounded monthly?
6. City Bank pays an interest rate of 6%, while Western Bank pays 5.8% compounded continuously. Which one is a better deal?
7. If a bank pays 6.8% compounded continuously, how long will it take to double your money?

Future Value of Annuities and Sinking Funds (6.3)

8. A corporation estimates it will need $300,000 in 8 years to replace its existing machinery. How much should it deposit each quarter in a sinking fund earning 8.4% compounded quarterly to meet this obligation?
9. A business must raise $400,000 in 10 years. What should be the size of the owners' monthly payments to a sinking fund paying 6.5% compounded monthly?
10. A mutual fund claims a growth rate of 8.3% per year. If $500 per month is invested, what will the final amount be in 15 years?

Present Value of Annuities and Installment Payments (6.4)

11. You look at your budget and decide that you can afford $250 per month for a car. What is the maximum amount you can afford to pay for the car if the interest rate is 8.6% and you want to finance the loan over 5 years?
12. Lisa buys a car for $16,500, and receives $2400 for her old car as a trade-in value. Find the monthly payment for the balance if the loan is amortized over 5 years at 8.5%.
13. You want to purchase a home for $200,000 with a 30-year mortgage at 9.24% interest. Find
    a. the monthly payment
    b. the balance owed after 20 years.
14. Ali has inherited $20,000 and is planning to invest this amount at 7.9% interest. At the same time she wishes to make equal monthly withdrawals to use up the entire sum in 5 years. How much can she withdraw each month?
15. Mr. Albers borrowed $425,000 from the bank for his new house at an interest rate of 4.7%. He will make equal monthly payments for the next 30 years. How much money will he end up paying the bank over the life of the loan, and how much is the interest?

Classification of Finance Problems (6.5)

16. Determine which formula would be used to answer the question, then find the answer.
    a. The United States paid about 4 cents an acre for the Louisiana Purchase in 1803. Suppose the value of this property grew at a rate of 5.5% annually. What would an acre be worth in the year 2000?
    b. When Jose bought his car, he amortized his loan over 6 years at a rate of 9.2%, and his monthly payment came out to be $350 per month. He has been making these payments for the past 40 months and now wants to pay off the remaining balance. How much does he owe?
    c. What amount should be invested per month at 9.1% compounded monthly so that it will become $5000 in 17 months?
    d. How much should Mr. Shackley deposit in a trust account so that his daughter can withdraw $400 per month for 4 years if the interest rate is 8%?

Additional Application Problems (6.6)

17. Mr. Nakahama bought his house in the year 1998. He had his loan financed for 30 years at an interest rate of 6.2% resulting in a monthly payment of $1500. In 2015, 17 years later, he paid off the balance of the loan. How much did he pay?

18. Find the 'fair market' value of a ten-year $1000 bond which pays $30 every six months if the current interest rate is 7%. What if the current interest rate is 5%?
19. Mrs. Tong puts away $500 per month for 10 years in an account that earns 9.3%. After 10 years, she decides to withdraw $1,000 per month. If the interest rate stays the same, how long will it take Mrs. Tong to deplete the account?
20. An amount of $5000 is borrowed for 15 months at an interest rate of 9%. Find the monthly payment and construct an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the amount of payment contributing towards debt, and the outstanding debt.

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CHAPTER OVERVIEW

7: Sets and Counting

Learning Objectives

In this chapter, you will learn to:

- Use set theory and Venn diagrams to solve counting problems.
- Use the Multiplication Axiom to solve counting problems.
- Use Permutations to solve counting problems.
- Use Combinations to solve counting problems.
- Use the Binomial Theorem to expand $(x+y)^n$

7.1: Sets and Counting

7.1E: Exercises - Sets and Counting

7.2: Tree Diagrams and the Multiplication Axiom

7.2E: Exercises - Tree Diagrams and the Multiplication Axiom

7.3: Permutations

7.3E: Exercises - Permutations

7.4: Combinations

7.4E: Exercises - Combinations

7.5: Binomial Theorem

7.5E: Exercises - Binomial Theorem

7.6: Chapter 7 Review

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7.1: Sets and Counting

Learning Objectives

In this section, you will learn to:

- Use set notation to represent unions, intersections, and complements of sets.
- Use Venn diagrams to solve counting problems.

Introduction to Sets

In this section, we will familiarize ourselves with set operations and notations, so that we can apply these concepts to both counting and probability problems. We begin by defining some terms.

Definition: Set and Elements

A **set** is a collection of objects, and its members are called the **elements** of the set.

We name the set by using capital letters, and enclose its members in braces. Suppose we need to list the members of the chess club. We use the following set notation.

C = { Ken, Bob, Tran, Shanti, Eric }

Definition: Empty Set

A set that has no members is called an **empty set**. The empty set is denoted by the symbol $\varnothing$.

Definition: Set Equality

Two sets are **equal** if they have the same elements.

Definition: Subset

Set A is a **subset** of a set B if every member of A is also a member of B.

Suppose C = { Al, Bob, Chris, David, Ed } and A = { Bob, David }. Then A is a subset of C, written as A $\subseteq$ C.

Every set is a subset of itself, and the empty set is a subset of every set.

Definition: Union of Two Sets

Let A and B be two sets, then the **union** of A and B, written as $A \cup B$, is the set of all elements that are either in A or in B, or in both A and B.

Definition: Intersection of Two Sets

Let A and B be two sets, then the **intersection** of A and B, written as $A \cap B$, is the set of all elements that are common to both sets A and B.

Definition: Universal Set

A **universal set** U is the set consisting of all elements under consideration.

Definition: Complement of a Set and Disjoint Sets

Let A be any set, then the **complement** of set A, written as $\overline{\text{A}}$, is the set consisting of elements in the universal set U that are not in A.

Note that different texts may use $\overline{\text{A}}$, A^c, or A', all representing the complement of set A.

Two sets A and B are called **disjoint sets** if their intersection is an empty set. Clearly, a set and its complement are disjoint; however two sets can be disjoint and not be complements.

Example 7.1.1

List all the subsets of the set of primary colors { red, yellow, blue }.

Solution

The subsets are ∅, {red}, {yellow}, {blue}, {red, yellow}, {red, blue}, {yellow, blue}, {red, yellow, blue}

Note that the empty set is a subset of every set, and a set is a subset of itself.

Example 7.1.2

Let F = { Aikman, Jackson, Rice, Sanders, Young }, and let B = { Griffey, Jackson, Sanders, Thomas }.
Find the intersection of the sets F and B.

Solution

The intersection of the two sets is the set whose elements belong to both sets. Therefore, F ∩ B = { Jackson, Sanders }

Example 7.1.3

Find the union of the sets F and B given as follows.

- F = { Aikman, Jackson, Rice, Sanders, Young }
- B = { Griffey, Jackson, Sanders, Thomas }

Solution

The union of two sets is the set whose elements are either in A or in B or in both A and B. Observe that when writing the union of two sets, the repetitions are avoided.

$$F \cup B = \{ \text{Aikman, Griffey, Jackson, Rice, Sanders, Thomas, Young} \}$$

Example 7.1.4

Let the universal set U = { red, orange, yellow, green, blue, indigo, violet }, and P = { red, yellow, blue }. Find the complement of P.

Solution

The complement of a set P is the set consisting of elements in the universal set U that are not in P. Therefore,

$$\overline{P} = \{ \text{orange, green, indigo, violet} \}$$

To achieve a better understanding, let us suppose that the universal set U represents the colors of the spectrum, and P the primary colors, then $\overline{P}$ represents those colors of the spectrum that are not primary colors.

Example 7.1.5

Let the universal set U = { red, orange, yellow, green, blue, indigo, violet }, and P = { red, yellow, blue } . Find a set R so that R is not the complement of P but R and P are disjoint.

Solution

R = { orange, green } and P = { red, yellow, blue } are disjoint because the intersection of the two sets is the empty set. The sets have no elements in common. However they are not complements because their union P ∪ R = { red, yellow, blue, orange, green } is not equal to the universal set U.

Example 7.1.6

Let U = { red, orange, yellow, green, blue, indigo, violet }, P = { red, yellow, blue }, Q = { red, green }, and R = { orange, green, indigo }. Find $\overline{P \cup Q} \cap \overline{R}$.

Solution

We do the problems in steps:

P ∪ Q = { red, yellow, blue, green }
$\overline{P \cup Q}$ = { orange, indigo, violet }
$\overline{R}$ = { red, yellow, blue, violet }
$\overline{P \cup Q} \cap \overline{R}$ = {violet}

Venn Diagrams

We now use **Venn diagrams** to illustrate the relations between sets. In the late 1800s, an English logician named John Venn developed a method to represent relationship between sets. He represented these relationships using diagrams, which are now known as Venn diagrams.

A Venn diagram represents a set as the interior of a circle. Often two or more circles are enclosed in a rectangle where the rectangle represents the universal set. An overlapping area of the circles represents the intersection of those sets. In this section, we will mainly use Venn diagrams to sort various populations and count objects.

Most often, a Venn Diagram represents too many objects to list specific elements. Instead, it will show the number of elements within each set. The notation n(set) is used to identify the number of items within each set. Using U = { red, orange, yellow, green, blue, indigo, violet } as an example, n(U)=7 since there are 7 elements within set U.

Example 7.1.7

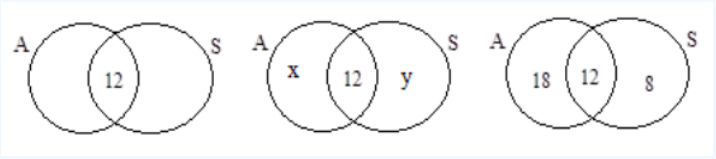
Suppose a survey of car enthusiasts showed that over a certain time period, 30 drove cars with automatic transmissions, 20 drove cars with standard transmissions, and 12 drove cars of both types. Draw a Venn diagram to represent this scenario. If everyone in the survey drove cars with one of these transmissions, how many people participated in the survey?

Solution

We will use Venn diagrams to solve this problem.

Let the set A represent those car enthusiasts who drove cars with automatic transmissions, and set S represent the car enthusiasts who drove the cars with standard transmissions. Now we use Venn diagrams to sort out the information given in this problem.

Since 12 people drove both cars, we place the number 12 in the region common to both sets. $n(A \cap S) = 12$.



Because 30 people drove cars with automatic transmissions, the circle A must contain 30 elements. This means that

x + 12 = 30, or x = 18. $n(A \cap \overline{S}) = 18$.

Similarly, since 20 people drove cars with standard transmissions, the circle B must contain 20 elements.

Thus, y + 12 = 20 which in turn makes y = 8. $n(S \cap \overline{A}) = 8$.

Now that all the information is sorted out, it is easy to read from the diagram that 18 people drove cars with automatic transmissions only, 12 people drove both types of cars, and 8 drove cars with standard transmissions only.

Therefore, 18 + 12 + 8 = 38 people took part in the survey. $n(A \cup S) = 38$.

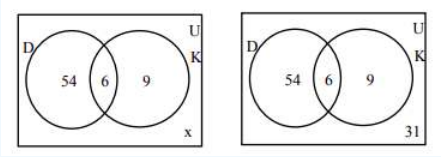
Example 7.1.8

A survey of 100 people in California indicates that 60 people have visited Disneyland, 15 have visited Knott's Berry Farm, and 6 have visited both. Draw a Venn diagram to represent this scenario. How many people have visited neither place?

Solution

The problem is similar to the one in the previous example.

Let the set D represent the people who have visited Disneyland, and K the set of people who have visited Knott's Berry Farm.



We fill the three regions associated with the sets D and K in the same manner as before. Since 100 people participated in the survey, the rectangle representing the universal set U must contain 100 objects, or $n(U) = 100$. Let x represent those people in the universal set that are neither in the set D nor in K. This means 54 + 6 + 9 + x = 100, or x = 31.

Therefore, there are 31 people in the survey who have visited neither place. In set notation, $n(\overline{D \cup K}) = 31$.

Example 7.1.9

A survey of 100 exercise conscious people resulted in the following information:

- 50 jog, 30 swim, and 35 cycle
- 14 jog and swim
- 7 swim and cycle
- 9 jog and cycle
- 3 people take part in all three activities

a. Represent each of the above statements in set notation.
b. Draw a Venn diagram to represent this scenario.
c. How many jog but do not swim or cycle?
d. How many take part in only one of the activities?
e. How many do not take part in any of these activities?

Solution

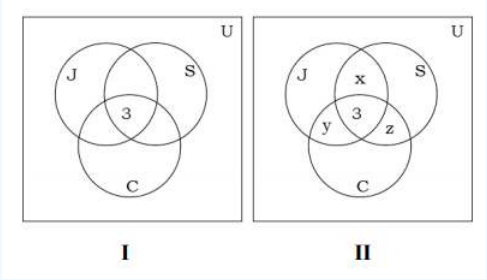
Let J represent the set of people who jog, S the set of people who swim, and C who cycle.

a. Set notation:

- 50 jog, 30 swim, and 35 cycle: $n(J) = 50, n(S) = 30, n(C) = 35$
- 14 jog and swim: $n(J \cap S) = 14$
- 7 swim and cycle: $n(S \cap C) = 7$
- 9 jog and cycle: $n(J \cap C) = 9$
- 3 people take part in all three activities: $n(J \cap S \cap C) = 3$

b. In using Venn diagrams, our ultimate aim is to assign a number to each region. We always begin by first assigning the number to the innermost region and then working our way out.

We'll show the solution step by step. As you practice working out such problems, you will find that with practice you will not need to draw multiple copies of the diagram.



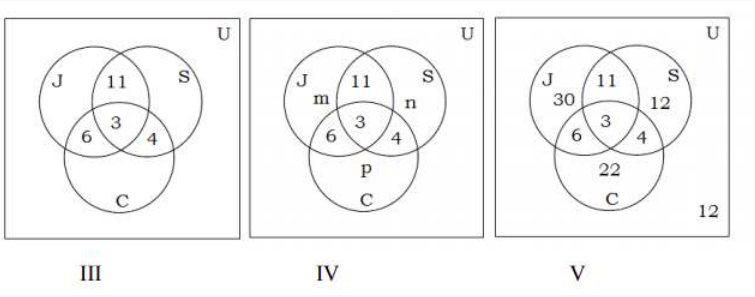
We place a 3 in the innermost region of figure I because it represents the number of people who participate in all three activities. Next we use figure II to compute x, y and z.

Since 14 people jog and swim, x + 3 = 14, or x = 11.

The fact that 9 people jog and cycle results in y + 3 = 9, or y = 6.

Since 7 people swim and cycle, z + 3 = 7, or z = 4.

This information is depicted in figure III.



Now we proceed to find the unknowns m, n and p, as shown in figure IV

Since 50 people jog, m + 11 + 6 + 3 = 50, or m = 30.

30 people swim, therefore, n + 11 + 4 + 3 = 30, or n = 12.

35 people cycle, therefore, p + 6 + 4 + 3 = 35, or p = 22.

By adding all the entries in all three sets, we get a sum of 88.
Since 100 people were surveyed, the number inside the universal set but outside of all three sets is 100 - 88, or 12.

In figure V, all the information is sorted out, and the questions can readily be answered.

c. The number of people who jog but do not swim or cycle is 30.

d. The number who take part in only one of these activities is 30 + 12 + 22 = 64.

e. The number of people who do not take part in any of these activities is 12.

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7.1E: Exercises - Sets and Counting

PROBLEM SET: SETS AND COUNTING

Find the indicated sets.

1. List all subsets of the following set. { Al, Bob }	2. List all subsets of the following set. { Al, Bob, Chris }
3. List the elements of the following set. { Al, Bob, Chris, Dave } ∩ { Bob, Chris, Dave, Ed }	4. List the elements of the following set. { Al, Bob, Chris, Dave } ∪ {Bob, Chris, Dave, Ed }

Problems 5 – 8: Let Universal set U = { a, b, c, d, e, f, g, h, i, j }, sets V = { a, e, f, h, i }, W = { a, c, e, g, i }. List the members of the following sets.

5. V ∪ W	6. V ∩ W
7. $\overline{V \cup W}$	8. $\overline{V} \cap \overline{W}$

Problems 9 – 12: Let Universal set U = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 } and sets A = { 1, 2, 3, 4, 5 }, B = { 1, 3, 4, 6 }, and C = { 2, 4, 6 }.

List the members of the following sets.

9. A ∪ B	10. A ∩ C
11. $\overline{A \cup B} \cap C$	12. $\bar{A} \cup \overline{B \cap C}$

Use Venn Diagrams to find the number of elements in the following sets.

13. In Mrs. Yamamoto's class of 35 students, 12 students are taking history, 18 are taking English, and 4 are taking both. Draw a Venn diagram and use it determine how many students are taking neither history nor English.	14. In a survey of 1200 college students, 700 used Spotify to listen to music and 400 used iTunes to listen to music; of these, 100 used both. a. Draw a Venn Diagram and find the number of people in each region of the diagram. b. How many used either Spotify or iTunes?
15. A survey of athletes revealed that for their minor aches and pains, 30 used aspirin, 50 used ibuprofen, and 15 used both. How many athletes were surveyed?	16. In 2016, 80 college students were surveyed about what video services they subscribed to. Suppose the survey showed that 50 use Amazon Prime, 30 use Netflix, 20 use Hulu. Of those, 13 use Amazon Prime and Netflix, 9 use Amazon Prime and Hulu, 7 use Netflix and Hulu. 3 students use all three services. a. Draw a Venn Diagram and use it to determine the number of people in each region of the diagram. b. How many use at least one of these? c. How many use none of these?

17. A survey of 100 students at a college finds that 50 take math, 40 take English, and 30 take history. Of these 15 take English and math, 10 take English and history, 10 take math and history, and 5 take all three subjects. Draw a Venn diagram and find the numbers in each region. Use the diagram to answer the questions below.
    a. Find the number of students taking math but not the other two subjects.
    b. The number of students taking English or math but not history.
    c. The number of students taking none of these subjects.

18. In a survey of investors it was found that 100 invested in stocks, 60 in mutual funds, and 50 in bonds. Of these, 35 invested in stocks and mutual funds, 30 in mutual funds and bonds, 28 in stocks and bonds, and 20 in all three. Draw a Venn diagram and find the numbers in each region. Use the diagram to answer the questions below.
    a. Find the number of investors that participated in the survey.
    b. How many invested in stocks or mutual funds but not in bonds?
    c. How many invested in exactly one type of investment?

19. A survey of 100 students at a college finds that 50 take math, 40 take English, and 30 take history. Of these 15 take English and math, 10 take English and history, 10 take math and history, and 5 take all three subjects. (This question relates back to question #17.) For each of the following draw a Venn Diagram and shade the indicated sets and determine the number of students in the set.

a. Students who take at least one of these classes	b. Students who take exactly one of these classes
c. Students who take at least two of these classes	d. Students who take exactly two of these classes
e. Students who take at most two of these classes	f. Students who take English or Math but not both
g. Students who take Math or History but not English	h. Students who take all of these classes

20. In a survey of investors it was found that 100 invested in stocks, 60 in mutual funds, and 50 in bonds. Of these, 35 invested in stocks and mutual funds, 30 in mutual funds and bonds, 28 in stocks and bonds, and 20 in all three. (This question relates back to question #18.) For each of the following draw a Venn Diagram and shade the indicated sets and determine the number of students in the set.

a. Investors who invested in mutual funds only	b. Investors who invested in stocks and bonds but not mutual funds
c. Investors who invested in exactly one of these investments	d. Investors who invested in exactly two of these investments
e. Investors who invested in at least two of these investments	f. Investors who invested in at most two of these investments
g. Investors who did not invest in bonds	h. Investors who invested in all three investments

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7.2: Tree Diagrams and the Multiplication Axiom

Learning Objectives

In this section you will learn to:

- Use trees to count possible outcomes in a multi-step process.
- Use the multiplication axiom to count possible outcomes in a multi-stop process.

In this chapter, we are trying to develop counting techniques that will be used in the next chapter to study probability. One of the most fundamental of such techniques is called the Multiplication Axiom. Before we introduce the multiplication axiom, we first look at some examples using tree diagrams.

A **tree diagram** is a visual way to show all possible outcomes of a certain event. Each event represents a split in the branches with each possible outcome listed at the end of a branch.

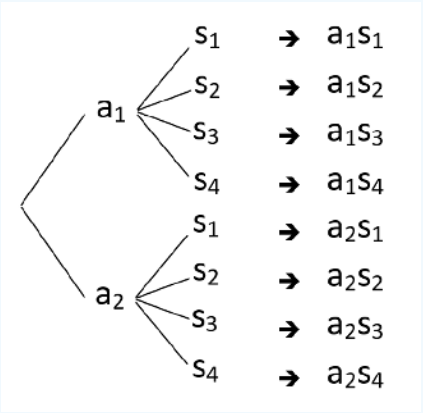
Example 7.2.1

If a food truck sells 2 types of appetizers and 4 types of sandwiches, how many different meal options can be ordered? Construct a tree diagram to represent this scenario.

Solution

Suppose we call the appetizers a_1 and a_2, and sandwiches s_1, s_2, s_3, and s_4.

We can draw a tree diagram:



The tree diagram shows us all eight possible meals:

$$a_1s_1, a_1s_2, a_1s_3, a_1s_4, a_2s_1, a_2s_2, a_2s_3, a_2s_4$$

The method involves two events: choosing an appetizer and choosing a sandwich. Each events is represented by a group of branches, with each possible outcome listed at the end of a branch. The tree diagram helps us visualize these possibilities.

If, in the previous example, we add the desserts to the meal, we have the following problem.

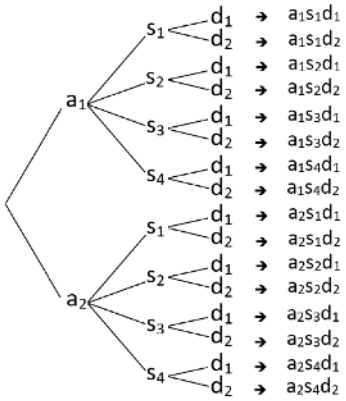
Example 7.2.2

If a food truck sells 2 types of appetizers, 4 types of sandwiches, and 2 types of desserts, how many different meal options can be ordered? Construct a tree diagram to represent this scenario.

Solution

Suppose we call the appetizers a_1 and a_2, the sandwiches s_1, s_2, s_3, and s_4, and the desserts d_1 and d_2.

The following tree diagram results.



We count the number of branches in the tree, and see that there are 16 different possibilities.

The method involves three events: choosing an appetizer, choosing a sandwich, and choosing a dessert. Each eventnis represented by a group of branches, with each possible outcome listed at the end of a branch.

All in all, we have $2 \cdot 4 \cdot 2 = 16$ different possibilities.

Tree diagrams help us visualize the different possibilities, but they are not practical when the possibilities are numerous. Besides, we are mostly interested in finding the number of elements in the set and not the actual list of all possibilities; once the problem is envisioned, we can solve it without a tree diagram. The two examples we just solved may have given us a clue to do just that.

Let us now try to solve Example $7.2.2$ without a tree diagram. The problem involves three steps: choosing an appetizer, choosing a sandwich, and choosing a dessert. The number of ways of choosing each are listed below. By multiplying these three numbers we get 16, which is what we found when we did the problem using a tree diagram.

The number of ways of choosing an appetizer	The number of ways of choosing a sandwich	The number of ways of choosing a dessert
2	4	2

The procedure we just employed is called the multiplication axiom.

If a task can be done in m ways, and a second task can be done in n ways, then the operation involving the first task followed by the second can be performed in $m \cdot n$ ways.

The general multiplication axiom is not limited to just two tasks and can be used for any number of tasks.

Example 7.2.3

A truck license plate consists of a letter followed by four digits. How many such license plates are possible?

Solution

Since there are 26 letters and 10 digits, we have the following choices for each.

Letter	Digit	Digit	Digit	Digit
26	10	10	10	10

Therefore, the number of possible license plates is $26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 260,000$.

Example 7.2.4

In how many different ways can a 3-question true-false test be answered?

Solution

Since there are two choices for each question, we have

Question 1	Question 2	Question 3
2	2	2

Applying the multiplication axiom, we get $2 \cdot 2 \cdot 2 = 8$ different ways.

We list all eight possibilities: TTT, TTF, TFT, TFF, FTT, FTF, FFT, FFF

The reader should note that the first letter in each possibility is the answer corresponding to the first question, the second letter corresponds to the answer to the second question, and so on. For example, TFF, says that the answer to the first question is given as true, and the answers to the second and third questions false.

Example 7.2.5

In how many different ways can four people be seated in a row?

Solution

Suppose we put four chairs in a row, and proceed to put four people in these seats.

There are four choices for the first chair we choose. Once a person sits down in that chair, there are only three choices for the second chair, and so on. We list as shown below.

4	3	2	1

So there are altogether $4 \cdot 3 \cdot 2 \cdot 1 = 24$ different ways.

Example 7.2.6

How many three-letter word sequences can be formed using the letters { A, B, C } if no letter is to be repeated?

Solution

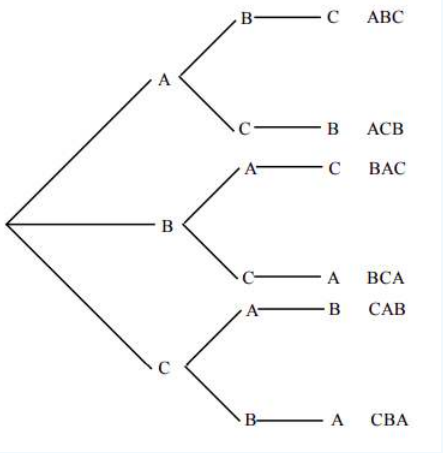
The problem is very similar to the previous example.

Imagine a child having three building blocks labeled A, B, and C. Suppose he puts these blocks on top of each other to make word sequences. For the first letter he has three choices, namely A, B, or C. Let us suppose he chooses the first letter to be a B, then for the second block which must go on top of the first, he has only two choices: A or C. And for the last letter he has only one choice. We list the choices below.

3	2	1

Therefore, 6 different word sequences can be formed.

Finally, we'd like to illustrate this with a tree diagram showing all six possibilities.



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7.2E: Exercises - Tree Diagrams and the Multiplication Axiom

Do the following problems using a tree diagram or the multiplication axiom.

1. A man has 3 shirts, and 2 pairs of pants. Use a tree diagram to determine the number of possible outfits.	2. In a city election, there are 2 candidates for mayor, and 3 for supervisor. Use a tree diagram to find the number of ways to fill the two offices.
3. There are 4 roads from Town A to Town B, 2 roads from Town B to Town C. Use a tree diagram to find the number of ways one can travel from Town A to Town C.	4. Brown Home Construction offers a selection of 3 floor plans, 2 roof types, and 2 exterior wall types. Use a tree diagram to determine the number of possible homes available.
5. For lunch, a small restaurant offers 2 types of soups, three kinds of sandwiches, and two types of soft drinks. Use a tree diagram to determine the number of possible meals consisting of a soup, sandwich, and a soft drink.	6. A California license plate consists of a number from 1 to 5, then three letters followed by any three digits. Repetition is possible. How many such plates are possible?

Do the following problems using the Multiplication Axiom

7. A license plate consists of three letters followed by three digits. How many license plates are possible if no letter may be repeated?	8. How many different 4-letter radio station call letters can be made if the first letter must be K or W and no letters can be repeated?
9. How many seven-digit telephone numbers are possible if the first two digits cannot be ones or zeros?	10. How many 3-letter word sequences can be formed using the letters {a, b, c, d} if no letter is to be repeated?

Use a tree diagram for questions 11 and 12:

11. A family has two children, use a tree diagram to determine all four possibilities of outcomes by gender.	12. A coin is tossed three times and the sequence of heads and tails is recorded. Use a tree diagram to list all the possible outcomes.

Do the following problems using the Multiplication Axiom

13. In how many ways can a 4-question true-false test be answered?	14. In how many ways can three people be arranged to stand in a straight line?
15. A combination lock is opened by first turning to the left, then to the right, and then to the left again. If there are 30 digits on the dial, how many possible combinations are there?	16. How many different answers are possible for a multiple-choice test with 10 questions and five possible answers for each question?
17. In the past, a college required students to use a 4 digit PIN (Personal Identification Number) as their password for its registration system. How many different PINs are possible if each must have 4 digits with no restrictions on selection or arrangement of the digits used?	18. The college decided that a more secure password system is needed. New passwords must have 3 numerical digits followed by 6 letters. There are no restrictions on the selection of the numerical digits. However, the letters I and O are not permitted. How many different passwords are possible?

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7.3: Permutations

Learning Objectives

In this section you will learn to:

- Perform calculations using factorials.
- Count the number of possible permutations (ordered arrangement) of n items taken r at a time.
- Count the number of possible permutations when there are conditions imposed on the arrangements.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. How many three-letter word sequences can be formed using the letters { A, B, C } if no letter is to be repeated?

Click here to check your answer

6 sequences can be formed.

If you missed this problem, review Section 7.2. (Note that this will open in a new window.)

2. A California license plate consists of a number from 1 to 5, then three letters followed by any three digits. Repetition is possible. How many such plates are possible?

Click here to check your answer

$87,880,000$different plates are possible.

If you missed this problem, review Section 7.2. (Note that this will open in a new window.)

3. How many different 4-letter radio station call letters can be made if the first letter must be K or W and no letters can be repeated?

Click here to check your answer

$27,600$station names can be made.

If you missed this problem, review Section 7.2. (Note that this will open in a new window.)

Factorials

When working with the multiplication axiom, we will often need to multiply sequential, descending numbers as we did in Example 5.2.5. We have a special notation for that calculation, which we will use a great deal in this as well as in the next chapter.

Definition: Factorial

$n!$ is read as "n factorial."

$$n! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1 \tag{7.3.1}$$

where n is a natural number.

$0! = 1$

Example 7.3.1

Calculate 5!

Solution

Using the definition of a factorial, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. Multiplying these numbers gives a value of 120.

Permutations

In Example 5.2.6, we were asked to find the word sequences formed by using the letters { A, B, C } if no letter is to be repeated. The tree diagram gave us the following six arrangements.

ABC, ACB, BAC, BCA, CAB, and CBA.

Arrangements like these, where order is important and no element is repeated, are called permutations.

Definition: Permutations

A permutation of a set of elements is an ordered arrangement where each element is used once.

Example 7.3.2

Use the multiplication axiom to determine how many three-letter word sequences can be formed using the letters { A, B, C, D }.

Solution

There are four choices for the first letter of our word, three choices for the second letter, and two choices for the third.

4	3	2

Applying the multiplication axiom, we get $4 \cdot 3 \cdot 2 = 24$ different arrangements.

Example 7.3.3

Use the multiplication axiom to determine how many permutations of the letters of the word ARTICLE have consonants in the first and last positions.

Solution

In the word ARTICLE, there are 4 consonants.

Since the first letter must be a consonant, we have four choices for the first position, and once we use up a consonant, there are only three consonants left for the last spot. We show as follows:

4						3

Since there are no more restrictions, we can go ahead and make the choices for the rest of the positions.

So far we have used up 2 letters, therefore, five remain. So for the next position there are five choices, for the position after that there are four choices, and so on. We get

4	5	4	3	2	1	3

So the total permutations are $4 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 = 1440$.

Example 7.3.4a

Given five letters { A, B, C, D, E }. Find the following:

a. The number of four-letter word sequences.
b. The number of three-letter word sequences.
c. The number of two-letter word sequences.

Solution

The problem is easily solved by the multiplication axiom, and answers are as follows:

a. The number of four-letter word sequences is $5 \cdot 4 \cdot 3 \cdot 2 = 120$.
b. The number of three-letter word sequences is $5 \cdot 4 \cdot 3 = 60$.
c. The number of two-letter word sequences is $5 \cdot 4 = 20$.

We often encounter situations where we have a set of n objects and we are selecting r objects to form permutations. We refer to this as **permutations of n objects taken r at a time**, and we usually write it as **nPr**.

Definition: nPr

The Number of Permutations of n Objects Taken r at a Time

$$\mathrm{nPr} = \frac{\mathrm{n!}}{(\mathrm{n}-\mathrm{r})!} \tag{7.3.2}$$

or

$$\mathrm{nPr} = \mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)(\mathrm{n}-3)\cdots(\mathrm{n}-\mathrm{r}+1) \tag{7.3.3}$$

where n and r are natural numbers.

Note that different texts may use different notation for permutations. $nPr, P(n,r)$, or P_r^n all represent permutations.

Example 7.3.4b

Given five letters { A, B, C, D, E }. Use the permutation formula to find the following:

a. The number of four-letter word sequences.
b. The number of three-letter word sequences.
c. The number of two-letter word sequences.

Solution

Since we are selecting from 5 different letters in the set, $n = 5$. The value for r is the number of letters we use in each sequence.

a. The number of four-letter word sequences is 5P4 = $\frac{5!}{(5-4)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} = 5 \cdot 4 \cdot 3 \cdot 2 = 120$
b. The number of three-letter word sequences is 5P3 = $\frac{5!}{(5-3)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 5 \cdot 4 \cdot 3 = 60$
c. The number of two-letter word sequences is 5P2 = $\frac{5!}{(5-2)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 5 \cdot 4 = 20$

Example 7.3.5

An auto service station has 6 employees and 3 bays. How many different ways can the employees be placed at the three bays if each bay only gets one employee?

Solution

We will identify n and r in each case and solve using the formulas provided.

Since we have 6 employees to select from, $n = 6$. Since we are placing only 3 of them at a time, $r = 3$. Therefore we are trying to calculate 6P3.

$$6\text{P}3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1} = 6\cdot5\cdot4 = 120$$

Next we consider some more permutation problems to get further insight into these concepts.

Example 7.3.6

In how many different ways can 4 people be seated in a straight line if two of them insist on sitting next to each other?

Solution

Let us suppose we have four people A, B, C, and D. Further suppose that A and B want to sit together. For the sake of argument, we tie A and B together and treat them as one person.

The four people are $\boxed{AB}$ CD. Since $\boxed{AB}$ is treated as one person, we have the following possible arrangements.

$$\boxed{AB}CD, \boxed{AB}DC, C\boxed{AB}D, D\boxed{AB}C, CD\boxed{AB}, DC\boxed{AB}$$

Note that there are six more such permutations because A and B could also be tied in the order BA. And they are

$$\boxed{BA}CD, \boxed{BA}DC, C\boxed{BA}D, D\boxed{BA}C, CD\boxed{BA}, DC\boxed{BA}$$

So altogether there are 12 different permutations.

Let us now do the problem using the multiplication axiom.

After we tie two of the people together and treat them as one person, we can say we have only three people. The multiplication axiom tells us that three people can be seated in 3! ways. Since two people can be tied together 2! ways, there are 3! 2! = 12 different arrangements

Example 7.3.7

You have 4 math books and 5 history books to put on a shelf that has 5 slots. In how many ways can the books be shelved if the first three slots are filled with math books and the next two slots are filled with history books?

Solution

We first do the problem using the multiplication axiom.

Since the math books go in the first three slots, there are 4 choices for the first slot,
3 choices for the second and 2 choices for the third.

The fourth slot requires a history book, and has five choices. Once that choice is made, there are 4 history books left, and therefore, 4 choices for the last slot. The choices are shown below.

4	3	2	5	4

Therefore, the number of permutations are $4\cdot3\cdot2\cdot5\cdot4 = 480$.

Alternately, we can see that $4\cdot3\cdot2$ is really same as 4P3, and $5\cdot4$ is 5P2.

So the answer can be written as (4P3) (5P2) = 480.

Clearly, this makes sense. For every permutation of three math books placed in the first three slots, there are 5P2 permutations of history books that can be placed in the last two slots. Hence the multiplication axiom applies, and we have the answer (4P3) (5P2).

We summarize the concepts of this section:

Note

1. Factorial

$$n! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$$

Where n is a natural number.

$$0! = 1$$

2. Permutations

A permutation of a set of elements is an ordered arrangement where each element is used once.

3. Permutations of n Objects Taken r at a Time

$$nPr = n(n-1)(n-2)(n-3)\cdots(n-r+1)$$

or

$$nPr = \frac{n!}{(n-r)!}$$

where n and r are natural numbers.

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7.3E: Exercises - Permutations

Do the following problems using permutations.

1. How many three-letter words can be made using the letters { a, b, c, d, e } if no repetitions are allowed?	2. A grocery store has five checkout counters, and seven clerks. How many different ways can the 7 clerks be assigned to the 5 counters?
3. A group of fifteen people who are members of an investment club wish to choose a president, and a secretary. How many different ways can this be done?	4. Compute the following. a. 9P2 b. 6P4 c. 8P3 d. 7P4
5. In how many ways can the letters of the word CUPERTINO be arranged if each letter is used only once in each arrangement?	6. How many permutations of the letters of the word PROBLEM end in a vowel?
7. How many permutations of the letters of the word SECURITY end in a consonant?	8. How many permutations of the letters PRODUCT have consonants in the second and third positions?
9. How many three-digit numbers are there?	10. How many three-digit odd numbers are there?
11. In how many different ways can five people be seated in a row if two of them insist on sitting next to each other?	12. In how many different ways can five people be seated in a row if two of them insist on not sitting next to each other?
13. In how many ways can 3 English, 3 history, and 2 math books be set on a shelf, if the English books are set on the left, history books in the middle, and math books on the right?	14. In how many ways can 3 English, 3 history, and 2 math books be set on a shelf, if they are grouped by subject?
15. You have 5 math books and 6 history books to put on a shelf with five slots. In how many ways can you put the books on the shelf if the first two slots are to be filled with math books and the next three with history books?	16. You have 5 math books and 6 history books to put on a shelf with five slots. In how many ways can you put the books on the shelf if the first two slots are to be filled with the books of one subject and the next three slots are to be filled with the books of the other subject?
17. A bakery has 9 different fancy cakes. In how many ways can 5 of the 9 fancy cakes be lined up in a row in the bakery display case?	18. A landscaper has 6 different flowering plants. She needs to plant 4 of them in a row in a garden. How many different ways can 4 of the 6 plants be arranged in a row?
19. At an auction of used construction vehicles, there are 7 different vehicles for sale. In how many orders could these 7 vehicles be listed in the auction program?	20. A landscaper has 6 different flowering plants and 4 different non-flowering bushes. She needs to plant a row of 6 plants in a garden. There must be a bush at each end, and four flowering plants in a row in between the bushes. How many different arrangements in a row are possible?
21. In how many ways can all 7 letters of the word QUIETLY be arranged if the letters Q and U must be next to each other in the order QU?	22. a. In how many ways can the letters ABCDEXY be arranged if the X and Y must be next to each other in either order XY or YX? b. In how many ways can the letters ABCDEXY be arranged if the X and Y can not be next to each other?

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7.4: Combinations

Learning Objectives

In this section you will learn to:

- Count the number of combinations of r out of n items (selections without regard to arrangement).
- Use factorials to perform calculations involving combinations.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Calculate each expression without the use of a calculator:

a. $4!$

b. $\frac{5!}{3!}$

Click here to check your answer

a. $4! = 24$

b. $\frac{5!}{3!} = 20$

If you missed this problem, review Section 7.3. (Note that this will open in a new window.)

2. Calculate each expression:

a. $5P2$

b. $12P3$

Click here to check your answer

a. $5P2 = 20$

b. $12P3 = 1320$

If you missed this problem, review Section 7.3. (Note that this will open in a new window.)

Suppose we have a set of three letters { A, B, C }, and we are asked to make two-letter word sequences. We have the following six permutations.

AB BA BC CB AC CA

Now suppose we have a group of three people { A, B, C } as Al, Bob, and Chris, respectively, and we are asked to form committees of two people each. This time we have only three committees, namely,

AB BC AC

When forming committees, the order is not important, because the committee that has Al and Bob is no different than the committee that has Bob and Al. As a result, we have only three committees and not six.

Forming word sequences is an example of permutations, while forming committees is an example of **combinations** - the topic of this section.

Permutations are those arrangements where order is important, while combinations are those arrangements where order is not significant. From now on, this is how we will tell permutations and combinations apart.

In the above example, there were six permutations, but only three combinations.

Just as the symbol nPr represents the number of permutations of n objects taken r at a time, nCr represents the number of combinations of n objects taken r at a time.

So in the above example, 3P2 = 6, and 3C2 = 3.

Our next goal is to determine the relationship between the number of combinations and the number of permutations in a given situation.

In the above example, if we knew that there were three combinations, we could have found the number of permutations by multiplying this number by 2!. That is because each combination consists of two letters, and that makes 2! permutations.

Example 7.4.1

Given the set of letters { A, B, C, D }. Write the number of combinations of three letters, and then from these combinations determine the number of permutations.

Solution

We have the following four combinations.

ABC BCD CDA BDA

Since every combination has three letters, there are 3! permutations for every combination. We list them below.

ABC	BCD	CDA	BDA
ACB	BDC	CAD	BAD
BAC	CDB	DAC	DAB
BCA	CBD	DCA	DBA
CAB	DCB	ACD	ADB
CBA	DBC	ADC	ABD

The number of permutations are 3! times the number of combinations; that is

4P3 = 3! · 4C3

or

$$4C3 = \frac{4P3}{3!}$$

In general,

$$nCr = \frac{nPr}{r!}$$

Since

$$nPr = \frac{n!}{(n-r)!}$$

We have,

$$nCr = \frac{n!}{(n-r)!r!}$$

Summarizing,

Definition: Combinations

A **combination** of a set of elements is an arrangement where each element is used once, and order is not important.

The Number of Combinations of n Objects Taken r at a Time

$$nCr = \frac{n!}{(n-r)!r!} \tag{7.4.1}$$

where n and r are natural numbers.

Example 7.4.2

Compute:

a. 5C3
b. 7C3

Solution

We use the above formula.

$$5C3 = \frac{5!}{(5-3)!3!} = \frac{5!}{2!3!} = 10$$

$$7C3 = \frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = 35$$

Example 7.4.3

In how many different ways can a student select to answer five questions from a test that has seven questions, if the order of the selection is not important?

Solution

Since the order is not important, it is a combination problem, and the answer is

7C5 = 21

Example 7.4.4

How many line segments can be drawn by connecting any two of the six points that lie on the circumference of a circle?

Solution

Since the line that goes from point A to point B is same as the one that goes from B to A, this is a combination problem.

It is a combination of 6 objects taken 2 at a time. Therefore, the answer is

$$6C2 = \frac{6!}{4!2!} = 15$$

Example 7.4.5

There are ten people at a party. If they all shake hands, how many hand-shakes are possible?

Solution

Note that between any two people there is only one hand shake. Therefore, we have

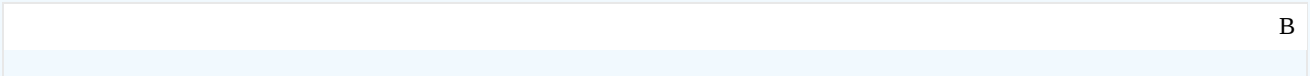
10C2 = 45 hand-shakes.

Example 7.4.6

The shopping area of a town is in the shape of square that is 5 blocks by 5 blocks. How many different routes can a taxi driver take to go from one corner of the shopping area to the opposite cater-corner?

Solution

Let us suppose the taxi driver drives from the point A, the lower left hand corner, to the point B, the upper right hand corner as shown in the figure below.



A				

To reach his destination, he has to travel ten blocks; five horizontal, and five vertical. So if out of the ten blocks he chooses any five horizontal, the other five will have to be the vertical blocks, and vice versa.

Therefore, all he has to do is to choose 5 out of ten to be the horizontal blocks

The answer is 10C5, or 252.

Alternately, the problem can be solved by permutations with similar elements.

The taxi driver's route consists of five horizontal and five vertical blocks. If we call a horizontal block H, and a vertical block a V, then one possible route may be as follows.

HHHHHVVVVV

Clearly there are $\frac{10!}{5!5!} = 252$ permutations.

Further note that by definition 10C5 = $\frac{10!}{5!5!}$.

Example 7.4.7

If a coin is tossed six times, in how many ways can it fall four heads and two tails?

Solution

First we solve this problem using section 6.5 technique-permutations with similar elements.

We need 4 heads and 2 tails, that is

HHHHTT

There are $\frac{6!}{4!2!} = 15$ permutations.

Now we solve this problem using combinations.

Suppose we have six spots to put the coins on. If we choose any four spots for heads, the other two will automatically be tails. So the problem is simply

6C4 = 15.

Incidentally, we could have easily chosen the two tails, instead. In that case, we would have gotten

6C2 = 15.

Further observe that by definition

$$6\text{C}4 = \frac{6!}{2!4!}$$

and

$$6\text{C}2 = \frac{6!}{4!2!}$$

Which implies 6C4 = 6C2.

So far we have solved the basic combination problem of r objects chosen from n different objects. Now we will consider certain variations of this problem.

Example 7.4.8

How many five-people committees consisting of 2 men and 3 women can be chosen from a total of 4 men and 4 women?

Solution

We list 4 men and 4 women as follows:

$$M_1 M_2 M_3 M_4 W_1 W_2 W_3 W_4$$

Since we want 5-people committees consisting of 2 men and 3 women, we'll first form all possible two-man committees and all possible three-woman committees. Clearly there are 4C2 = 6 two-man committees, and 4C3 = 4 three-woman committees, we list them as follows:

2-Man Committees	3-Woman Committees
M_1M_2	
M_1M_3	$W_1W_2W_3$
M_1M_4	$W_1W_2W_4$
M_2M_3	$W_1W_3W_4$
M_2M_4	$W_2W_3W_4$
M_3M_4	

For every 2-man committee there are four 3-woman committees that can be chosen to make a 5-person committee. If we choose M_1M_2 as our 2-man committee, then we can choose any of $W_1W_2W_3$, $W_1W_2W_4$, $W_1W_3W_4$, or $W_2W_3W_4$ as our 3-woman committees. As a result, we get

$$\boxed{M_1M_2}W_1W_2W_3, \boxed{M_1M_2}W_1W_2W_4, \boxed{M_1M_2}W_1W_3W_4, \boxed{M_1M_2}W_2W_3W_4$$

Similarly, if we choose M_1M_3 as our 2-man committee, then, again, we can choose any of $W_1W_2W_3$, $W_1W_2W_4$, $W_1W_3W_4$, or $W_2W_3W_4$ as our 3-woman committees.

$$\boxed{M_1M_3}W_1W_2W_3, \boxed{M_1M_3}W_1W_2W_4, \boxed{M_1M_3}W_1W_3W_4, \boxed{M_1M_3}W_2W_3W_4$$

And so on.

Since there are six 2-man committees, and for every 2-man committee there are four 3-woman committees, there are altogether $6 \cdot 4 = 24$ five-people committees.

In essence, we are applying the multiplication axiom to the different combinations.

Example 7.4.9

A high school club consists of 4 freshmen, 5 sophomores, 5 juniors, and 6 seniors. How many ways can a committee of 4 people be chosen that includes

a. One student from each class?
b. All juniors?
c. Two freshmen and 2 seniors?
d. No freshmen?
e. At least three seniors?

Solution

a. Applying the multiplication axiom to the combinations involved, we get

$$(\ 4C1\ )\ (\ 5C1\ )\ (\ 5C1\ )\ (\ 6C1\ ) = 600$$

b. We are choosing all 4 members from the 5 juniors, and none from the others.

$$5C4 = 5$$

c. $4C2 \cdot 6C2 = 90$

d. Since we don't want any freshmen on the committee, we need to choose all members from the remaining 16. That is

$$16C4 = 1820$$

e. Of the 4 people on the committee, we want at least three seniors. This can be done in two ways. We could have three seniors, and one non-senior, or all four seniors.

$$(6C3) (14C1) + 6C4 = 295$$

Example 7.4.10

How many five-letter word sequences consisting of 2 vowels and 3 consonants can be formed from the letters of the word INTRODUCE?

Solution

First we select a group of five letters consisting of 2 vowels and 3 consonants.
Since there are 4 vowels and 5 consonants, we have

$$(4C2) (5C3)$$

Since our next task is to make word sequences out of these letters, we multiply these by 5!.

$$(4C2) (5C3) (5!) = 7200.$$

Example 7.4.11

A standard deck of playing cards has 52 cards consisting of 4 suits each with 13 cards. In how many different ways can a 5-card hand consisting of four cards of one suit and one card of another suit be drawn?

Solution

We will do the problem using the following steps.
Step 1. Select a suit.
Step 2. Select four cards from this suit.
Step 3. Select another suit.
Step 4. Select a card from that suit.

Applying the multiplication axiom, we have

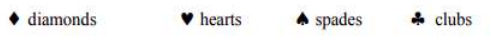
Ways of selecting the first suit	Ways of selecting 4 cards from this suit	Ways of selecting the next suit	Ways of selecting a card from that suit
4C1	13C4	3C1	13C1

$(4C1) (13C4) (3C1) (13C1) = 111,540.$

A STANDARD DECK OF 52 PLAYING CARDS

As in the previous example, many examples and homework problems in this book refer to a standard deck of 52 playing cards. Before we end this section, we take a minute to describe a standard deck of playing cards, as some readers may not be familiar with this.

A standard deck of 52 playing cards has 4 suits with 13 cards in each suit.



Each suit is associated with a color, either black (spades, clubs) or red (diamonds, hearts).

Each suit contains 13 denominations (or values) for cards:

an Ace (A), nine numbers 2, 3, 4, …., 10, and Jack(J), Queen (Q), King (K).

The Jack, Queen and King are called “face cards” because they have pictures on them. Therefore a standard deck has 12 face cards: (3 values J, Q, K) x (4 suits ♦, ♥, ♠, ♣)

We can visualize the 52 cards by the following display

Suit	Color	Values (Denominations)
♦ Diamonds	Red	A 2 3 4 5 6 7 8 9 10 J Q K
♥ Hearts	Red	A 2 3 4 5 6 7 8 9 10 J Q K
♠ Spades	Black	A 2 3 4 5 6 7 8 9 10 J Q K
♣ Clubs	Black	A 2 3 4 5 6 7 8 9 10 J Q K

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7.4E: Exercises - Combinations

Do the following problems using combinations.

1. How many different 3-people committees can be chosen from ten people?	2. How many different 5-player teams can be chosen from eight players?
3. In how many ways can a person chose to vote for three out of five candidates on a ballot for a school board election?	4. Compute the following: a. 9C2 b. 6C4 c. 8C3 d. 7C4
5. How many 5-card hands can be chosen from a deck of cards?	6. How many 13-card bridge hands can be chosen from a deck of cards?
7. There are twelve people at a party. If they all shake hands, how many different hand-shakes are there?	8. In how many ways can a student choose to do four questions out of five on a test?
9. There are five teams in a league. How many games are played if every team plays each other twice?	10. A team plays 15 games a season. In how many ways can it have 8 wins and 7 losses?
11. In how many different ways can a 4-child family have 2 boys and 2 girls?	12. A coin is tossed five times. In how many ways can it fall three heads and two tails?
13. A team of 7 workers is assigned to a project. In how many ways can 3 of the 7 workers be selected to make a presentation to the management about their progress on the project?	14. A real estate company has 12 houses listed for sale by their clients. In how many ways can 5 of the 12 houses be selected to be featured in an advertising brochures?

The following problems involve combinations from several different sets.

15. How many 5-people committees consisting of three boys and two girls can be chosen from a group of four boys and four girls?	16. A club has 4 men, 5 women, 8 boys and 10 girls as members. In how many ways can a group of 2 men, 3 women, 4 boys and 4 girls be chosen?
17. How many 4-people committees chosen from 4 men and 6 women will have at least 3 men?	18. A batch contains 10 transistors of which three are defective. If three are chosen, in how many ways can they be selected with two defective?
19. In how many ways can five counters labeled A, B, C, D and E at a store be staffed by two men and three women chosen from a group of four men and six women?	20. How many 4-letter word sequences consisting of two vowels and two consonants can be made from the letters of the word PHOENIX if no letter is repeated?

Three marbles are chosen from an urn that contains 5 red, 4 white, and 3 blue marbles. How many samples of the following type are possible?

21. All three white.	22. Two blue and one white
23. One of each color.	24. All three of the same color.
25. At least two red.	26. None red.

The following problems involve combinations from several different sets.

Five coins are chosen from a bag that contains 4 dimes, 5 nickels, and 6 pennies. How many samples of five coins of the following types are possible?

27. At least four nickels.	28. No pennies.
29. Five of a kind.	30. Four of a kind.
31. Two of one kind and two of another kind.	32. Three of one kind and two of another kind.

Find the number of different ways draw a 5-card hand from a deck to have the following combinations.

33. Three face cards.	34. A heart flush (all hearts).
35. Two hearts and three diamonds	36. Two cards of one suit, and three of another suit.
37. Two kings and three queens.	38. 2 cards of one value and 3 of another value

The party affiliation of the 100 United States Senators in the 114th Congress, January 2015, was:

44 Democrats, 54 Republicans, and 2 Independents.

39. In how many ways could a 10 person committee be selected if it is to contain 4 Democrats, 5 Republicans, and 1 Independent?	40. In how many different ways could a 10 person committee be selected with 6 or 7 Republicans and the Democrats (with no Independents)?

The 100 United States Senators in the 114th Congress, January 2015, included 80 men and 20 women. Suppose a committee of senators is working on legislation about wage discrimination by gender.

41. In how many ways could a 12 person committee be selected to contain equal numbers of men and women.	42. In how many ways could a 6 person committee be selected to contain fewer women than men?

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7.5: Binomial Theorem

We end this chapter with one more application of combinations. Combinations are used in determining the coefficients of a binomial expansion such as $(x+y)^n$. Expanding a binomial expression by multiplying it out is a very tedious task, and is not practiced. Instead, a formula known as the Binomial Theorem is utilized to determine such an expansion. Before we introduce the Binomial Theorem, however, consider the following expansions.

$$\begin{aligned}(x+y)^2 &= x^2+2xy+y^2\\ (x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\ (x+y)^4 &= x^4+4x^2y+6x^2y^2+4xy^3+y^4\\ (x+y)^5 &= x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5\\ (x+y)^6 &= x6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6\end{aligned}$$

We make the following observations.

1. There are $n+1$ terms in the expansion $(x+y)^n$
2. The sum of the powers of x and y is n.
3. The powers of x begin with n and decrease by one with each successive term.
   The powers of y begin with 0 and increase by one with each successive term.

Suppose we want to expand $(x+y)^3$. We first write the expansion without the coefficients. We temporarily substitute a blank in place of the coefficients.

$$(x+y)^3 = \square x^3+\square x^2y+\square xy^2+\square y^3 \tag{7.5.1}$$

Our next job is to replace each of the blanks in equation (7.5.1) with the corresponding coefficients that belong to this expansion. Clearly,

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

If we multiply the right side and do not collect terms, we get the following.

$$xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy$$

Each product in the above expansion is the result of multiplying three variables by picking one from each of the factors $(x+y)(x+y)(x+y)$. For example, the product xxy is gotten by choosing x from the first factor, x from the second factor, and y from the third factor. There are three such products that simplify to x^2y, namely xxy, xyx, and yxx. These products take place when we choose an x from two of the factors and choose a y from the other factor. Clearly this can be done in 3C2, or 3 ways. Therefore, the coefficient of the term x^2y is 3. The coefficients of the other terms are obtained in a similar manner.

We now replace the blanks with the coefficients in equation (7.5.1), and we get

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

Example 7.5.1

Find the coefficient of the term x^2y^5 in the expansion $(x+y)^7$.

Solution

The expansion $(x+y)^7 = (x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$

In multiplying the right side, each product is gotten by picking an x or y from each of the seven factors $(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$.

The term x^2y^5 is obtained by choosing an x from two of the factors and a y from the other five factors. This can be done in 7C2, or 21 ways.

Therefore, the coefficient of the term x^2y^5 is 21.

Example 7.5.2

Expand $(x+y)^7$

Solution

We first write the expansion without the coefficients.

$$(x+y)^7 = \square x^7 + \square x^6 y + \square x^5 y^2 + \square x^4 y^3 + \square x^3 y^4 + \square x^2 y^5 + \square x y^6 + \square y^7$$

Now we determine the coefficient of each term as we did in Example 7.5.1.

The coefficient of the term x^7 is 7C7 or 7CO which equals 1.

The coefficient of the term $x^6 y$ is 7C6 or 7C1 which equals 7.

The coefficient of the term $x^5 y^2$ is 7C5 or 7C2 which equals 21.

The coefficient of the term $x^4 y^3$ is 7C4 or 7C3 which equals 35,

and so on.

Substituting, we get: $(x+y)^7 = x^7 + 7x^6 y + 21x^5 y^2 + 35x^4 y^3 + 35x^3 y^4 + 21x^2 y^5 + 7xy^6 + y^7$

We generalize the result.

Binomial Theorem

$$(x+y)^n = {}_n C_0 x^n + {}_n C_1 x^{n-1} y + {}_n C_2 x^{n-2} y^2 + \cdots + {}_n C_{n-1} x y_n^{n-1+} C_n y^n$$

Example 7.5.3

Expand $(3a-2b)^4$

Solution

If we let $x = 3a$ and $y = -2b$, and apply the Binomial Theorem, we get

$$\begin{aligned}(3a-2b)^4 &= 4\,\mathrm{Co}(3a)^4 + 4Cl(3a)^3(-2b) + 4C2(3a)^2(-2b)^2 + 4C3(3a)(-2b)^3 + 4C4(-2b)^4 \\ &= 1\left(81a^4\right) + 4\left(27a^3\right)(-2b) + 6\left(9a^2\right)\left(4b^2\right) + 4(3a)\left(-8b^3\right) + 1\left(16b^3\right) \\ &= 81a^4 - 216a^3 b + 216a^2 b^2 - 96ab^3 + 16b^4\end{aligned}$$

Example 7.5.4

Find the fifth term of the expansion $(3a-2b)^7$.

Solution

The Binomial theorem tells us that in the r-th term of an expansion, the exponent of the y term is always one less than r, and, the coefficient of the term is ${}_{\mathrm{n}}\mathrm{C}_{\mathrm{r}-1}$.

$n = 7$ and $r - 1 = 5 - 1 = 4$, so the coefficient is $7\mathrm{C}4 = 35$

Thus, the fifth term is $(7\mathrm{C}4)(3\mathrm{a})^3(-2\mathrm{b})^4 = 35\left(27\mathrm{a}^3\right)\left(16\mathrm{b}^4\right) = 15120\mathrm{a}^3\mathrm{b}^4$

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7.5E: Exercises - Binomial Theorem

Use the Binomial Theorem to do the following problems.

1. Expand $(a+b)^5$.	2. Expand $(a-b)^6$.
3. Expand $(x-2y)^5$.	4. Expand $(2x-3y)^4$.
5. Find the third term of $(2x-3y)^6$.	6. Find the sixth term of $(5x+y)^8$.
7. Find the coefficient of the x^3y^4 term in the expansion of $(2x+y)^7$.	8. Find the coefficient of the a^4b^6 term in the expansion of $(3a-b)^{10}$.
9. A coin is tossed 5 times, in how many ways is it possible to get three heads and two tails?	10. A coin is tossed 10 times, in how many ways is it possible to get seven heads and three tails?
11. How many subsets are there of a set that has 6 elements?	12. How many subsets are there of a set that has n elements?

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7.6: Chapter 7 Review

PROBLEM SET: CHAPTER REVIEW

Sets and Counting (7.1)

1. Suppose of the 4,000 freshmen at a college everyone is enrolled in a mathematics or an English class during a given quarter. If 2,000 are enrolled in a mathematics class, and 3,000 in an English class, how many are enrolled in both a mathematics class and an English class?
2. In a survey of 250 people, it was found that 125 had read Time magazine, 175 had read Newsweek, 100 had read U. S. News, 75 had read Time and Newsweek, 60 had read Newsweek and U. S. News, 55 had read Time and U. S. News, and 25 had read all three.
    a. How many had read Time but not the other two?
    b. How many had read Time or Newsweek but not the U. S. News And World Report?
    c. How many had read none of these three magazines?

Tree Diagrams and the Multiplication Axiom (7.2)

3. At a manufacturing plant, a product goes through assembly, testing, and packing. If a plant has three assembly stations, two testing stations, and two packing stations, in how many different ways can a product achieve its completion?
4. Six people are to line up for a photograph. How many different lineups are possible if three of them insist on standing next to each other?
5. How many four-letter word sequences can be made from the letters of the word OXYGEN?
6. In how many different ways can a 20-question multiple choice test be designed so that its answers contain 2 A's, 4 B's, 9 C's, 3 D's, and 2 E's?

Permutations and Combinations (7.3 & 7.4)

7. How many three digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if no repetitions are allowed?
8. Compute:
    a. $9C4$
    b. $8P3$
    c. $\frac{10!}{4!(10-4)!}$
9. From a group of 6 people, 3 are assigned to cleaning, 2 to hauling and one to garbage collecting. How many different ways can this be done?
10. In how many ways can 3 books be selected from 4 English and 2 History books if at least one English book must be chosen?
11. A club consists of six men and nine women. In how many ways can a president, a vice president and a treasurer be chosen if the two of the officers must be women?
12. Of its 12 sales people, a company wants to assign 4 to its Western territory, 5 to its Northern territory, and 3 to its Southern territory. How many ways can this be done?
13. How many five-card poker hands consisting of the following distribution are there?
    a. A flush(all five cards of a single suit)
    b. Three of a kind(e.g. three aces and two other cards)
    c. Two pairs(e.g. two aces, two kings and one other card)
    d. A straight(all five cards in a sequence)
14. Company stocks on an exchange are given symbols consisting of three letters. How many different three-letter symbols are possible?
15. A United Nations delegation consists of 6 Americans, 5 Russians, and 4 Chinese. Answer the following questions.
    a. How many committees of five people are there?
    b. How many committees of three people consisting of at least one American are there?
    c. How many committees of four people having no Russians are there?
    d. How many committees of three people have more Americans than Russians?
    e. How many committees of three people do not have all three Americans?
16. If a coin is flipped five times, in how many different ways can it show up three heads?

Binomial Theorem (7.5)

17. Find the fourth term of the expansion $(2x-3y)^8$.
18. Find the coefficient of the a^5b^4 term in the expansion of $(a-2b)^9$.

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CHAPTER OVERVIEW

8: Probability

Learning Objectives

In this chapter, you will learn to:

1. Write sample spaces.
2. Determine whether two events are mutually exclusive.
3. Use the Addition Rule.
4. Calculate probabilities using both tree diagrams and combinations.
5. Do problems involving conditional probability.
6. Determine whether two events are independent.

8.1: Sample Spaces and Probability

8.1E: Exercises - Sample Spaces and Probability

8.2: Mutually Exclusive Events and the Addition Rule

8.2E: Exercises - Mutually Exclusive Events and the Addition Rule

8.3: Probability Using Tree Diagrams and Combinations

8.3E: Exercises - Probability Using Tree Diagrams and Combinations

8.4: Conditional Probability

8.4E: Exercises - Conditional Probability

8.5: Bayes' Theorem

8.5E: Exercises - Bayes' Formula

8.6: Independent Events

8.6E: Exercises - Independent Events

8.7: Expected Value

8.7E: Exercises - Expected Value

8.8: Chapter 8 Review

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8.1: Sample Spaces and Probability

Learning Objectives

In this section, you will learn to:

- Write sample spaces.
- Calculate probabilities by examining simple events in sample spaces.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. In a city election, there are 2 candidates for mayor, and 3 for supervisor. Use a tree diagram to find the number of ways to fill the two offices.

Click here to check your answer

The offices can be filled in 6 different ways.

If you missed this problem, review Section 7.2. (Note that this will open in a new window.)

2. For lunch, a small restaurant offers 2 types of soups, three kinds of sandwiches, and two types of soft drinks. Use a tree diagram to determine the number of possible meals consisting of a soup, sandwich, and a soft drink.

Click here to check your answer

There are 12 different possible meals.

If you missed this problem, review Section 7.2. (Note that this will open in a new window.)

If two coins are tossed, what is the probability that both coins will fall heads? The problem seems simple enough, but it is not uncommon to hear the incorrect answer 1/3. A student may incorrectly reason that if two coins are tossed there are three possibilities, one head, two heads, or no heads. Therefore, the probability of two heads is one out of three. The answer is wrong because if we toss two coins there are four possibilities and not three. For clarity, assume that one coin is a penny and the other a nickel. Then we have the following four possibilities.

HH HT TH TT

The possibility HT, for example, indicates a head on the penny and a tail on the nickel, while TH represents a tail on the penny and a head on the nickel. It is for this reason, we emphasize the need for understanding sample spaces.

Sample Spaces

An act of flipping coins, rolling dice, drawing cards, or surveying people are referred to as a probability **experiment**. A **sample space** of an experiment is the set of all possible outcomes.

Example 8.1.1

A single die is rolled. Write the sample space.

Solution

A die has six faces each having an equally likely chance of appearing. Therefore, the set of all possible outcomes S is

$$\{ 1, 2, 3, 4, 5, 6 \}.$$

Example 8.1.2

A family has three children. Write the sample space showing the birth order with respect to gender. (This means that one possibility is having a boy, then a boy, then a girl. A different possibility is having a girl, then a boy, then a boy.)

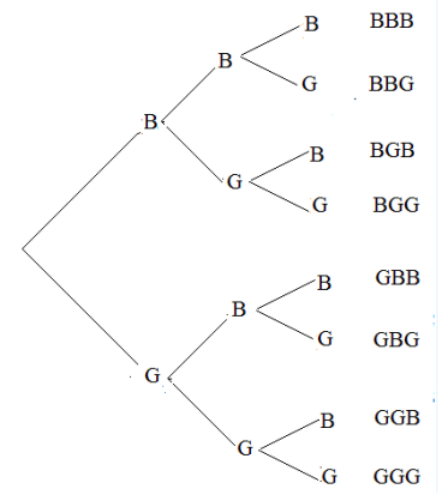
Solution

The sample space consists of eight possibilities.

{ BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG }

The possibility BGB, for example, indicates that the first born is a boy, the second born a girl, and the third a boy.

We illustrate these possibilities with a tree diagram.



Example 8.1.3

Two dice are rolled. Write the sample space.

Solution

Let's suppose one of the dice is red, and the other green. We have the following 36 possibilities.

Green						
Red	**1**	**2**	**3**	**4**	**5**	**6**
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The entry (2, 5), for example, indicates that the red die shows a 2, and the green a 5.

Probability

Now that we understand the concept of a sample space, we will define probability.

An **event** is a subset of a sample space. For example, in the experiment of rolling two dice, an event might be rolling a sum of 5.

Definition: Probability

The **probability of an event** describes the chance or likelihood of that event occurring.

For a sample space S, and an event A,

- $P(A) = \dfrac{\text{number of ways A appears in S}}{\text{total number of outcomes in S}}$
- $0 \leq P(A) \leq 1$
- The sum of the probabilities of all the outcomes in S equals 1.

The probability $P(A)$ of an event A describes the chance or likelihood of that event occurring.

- If $P(A) = 0$, event A is certain not to occur. If $P(A) = 1$, event A is certain to occur.
- If $P(A) = 0.5$, then event A is equally likely to occur or not occur.
- If we toss a fair coin that is equally likely to land on heads or tails, then P(Head) = 0.50.
- If the weather forecast says there is a 70% chance of rain today, then P(Rain) = 0.70, indicating is it more likely to rain than to not rain.

Example 8.1.4

If two dice, one red and one green, are rolled, find the probability that the red die shows a 3 and the green shows a six.

Solution

Since two dice are rolled, there are 36 possibilities. The probability of each outcome, listed in Example 8.1.3, is equally likely.

Since (3, 6) is one such outcome, the probability of obtaining (3, 6) is 1/36.

The example we just considered consisted of only one outcome of the sample space. If an event consists of only one outcome, it is called a **simple event.**

We are often interested in finding probabilities of several outcomes represented by an event.

Example 8.1.5

If two dice are rolled, find the probability that the sum of the faces of the dice is 7.

Solution

Let E represent the event that the sum of the faces of two dice is 7.

The possible cases for the sum to be equal to 7 are: (1, 6), (2,5), (3, 4), (4, 3), (5, 2), and (6, 1), so event E is

E = {(1, 6), (2,5), (3, 4), (4, 3), (5, 2), (6, 1)}

The probability of the event E is

P(E) = 6/36 or 1/6.

Example 8.1.6

One 6 sided die is rolled once. Find the probability that the result is greater than 4.

Solution

The sample space consists of the following six possibilities in set S: $\mathrm{S} = 1, 2, 3, 4, 5, 6$

Let E be the event that the number rolled is greater than four: $\mathrm{E} = 5, 6$

Therefore, the probability of E is: $\mathrm{P(E)} = 2/6$ or $1/3$.

Example 8.1.7

A jar contains 3 red, 4 white, and 3 blue marbles. If a marble is chosen at random, what is the probability that the marble is a red marble or a blue marble?

Solution

We assume the marbles are r_1, r_2, r_3, w_1, w_2, w_3, w_4, b_1, b_2, b_3. Let the event C represent that the marble is red or blue.

The sample space $\mathrm{S} = \{\mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3, \mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3, \mathrm{w}_4, \mathrm{b}_1, \mathrm{b}_2, \mathrm{b}_3\}$.

And the event $\mathrm{C} = \{\mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3, \mathrm{b}_1, \mathrm{b}_2, \mathrm{b}_3\}$

Therefore, the probability of C,

$$\mathrm{P(C)} = 6/10 \text{ or } 3/5$$

Example 8.1.8

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **without replacement**, what is the probability that the sum of the numbers is 5?

Note: The two marbles in this example are drawn consecutively **without replacement**. That means that after a marble is drawn it is not replaced in the jar, and therefore is no longer available to select on the second draw.

Solution

Since two marbles are drawn without replacement, the sample space consists of the following six possibilities.

$$\mathrm{S} = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$$

Note that (1,1), (2,2) and (3,3) are not listed in the sample space. These outcomes are not possible when drawing without replacement, because once the first marble is drawn but not replaced into the jar, that marble is not available in the jar to be selected again on the second draw.

Let the event E represent that the sum of the numbers is five. Then

$$\mathrm{E} = \{(2,3), (3,2)\}$$

Therefore, the probability of E is

$$\mathrm{P(E)} = 2/6 \text{ or } 1/3.$$

Example 8.1.9

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **without replacement**, what is the probability that the sum of the numbers is *at least* 4?

Solution

The sample space, as in Example 8.1.7, consists of the following six possibilities.

$$\mathrm{S} = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$$

Let the event F represent that the sum of the numbers is at least four. Then

$$\mathrm{F} = \{(1,3), (3,1), (2,3), (3,2)\}$$

Therefore, the probability of F is

$$\mathrm{P(F)} = 4/6 \text{ or } 2/3$$

Example 8.1.10

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **with replacement**, what is the probability that the sum of the numbers is 5?

Note: The two marbles in this example are drawn consecutively **with replacement**. That means that after a marble is drawn it IS replaced in the jar, and therefore is available to select again on the second draw.

Solution

When two marbles are drawn with replacement, the sample space consists of the following nine possibilities.

$$S = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

Note that (1,1), (2,2) and (3,3) are listed in the sample space. These outcomes are possible when drawing with replacement, because once the first marble is drawn and replaced, that marble is not available in the jar to be drawn again.

Let the event E represent that the sum of the numbers is four. Then

$$E = (2,3), (3,2)$$

Therefore, the probability of F is $P(E) = 2/9$

Note that in Example 8.1.9 when we selected marbles with replacement, the probability has changed from Example 8.1.7 where we selected marbles without replacement.

Example 8.1.11

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **with replacement**, what is the probability that the sum of the numbers is *at least* 4?

Solution

The sample space when drawing with replacement consists of the following nine possibilities.

$$S = (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)$$

Let the event F represent that the sum of the numbers is at least four. Then

$$F = (1,3), (3,1), (2,3), (3,2), (2,2), (3,3)$$

Therefore, the probability of F is

$$P(F) = 6/9 \text{ or } 2/3.$$

Note that in Example 8.1.10 when we selected marbles with replacement, the probability is the same as in Example 8.1.8 where we selected marbles without replacement.

Thus sampling with or without replacement MAY change the probabilities, but may not, depending on the situation in the particular problem under consideration. We'll re-examine the concepts of sampling with and without replacement in Section 6.3.

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8.1E: Exercises - Sample Spaces and Probability

PROBLEM SET: SAMPLE SPACES AND PROBABILITY

In problems 1 - 6, write a sample space for the given experiment.

1) A die is rolled.	2) A penny and a nickel are tossed.
3) A die is rolled, and a coin is tossed.	4) Three coins are tossed.
5) Two dice are rolled.	6) A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn.

In problems 7 - 12, one card is randomly selected from a deck. Find the following probabilities.

7) P(an ace)	8) P(a red card)
9) P(a club)	10) P(a face card)
11) P(a jack or a spade)	12) P(a jack and a spade)

For problems 13 - 16: A jar contains 6 red, 7 white, and 7 blue marbles. If one marble is chosen at random, find the following probabilities.

13) P(red)	14) P(white)
15) P(red or blue)	16) P(red and blue)

For problems 17 - 22: Consider a family of three children. Find the following probabilities.

17) P(two boys and a girl)	18) P(at least one boy)
19) P(children of both sexes)	20) P(at most one girl)
21) P(first and third children are male)	22) P(all children are of the same gender)

For problems 23 - 27: Two dice are rolled. Find the following probabilities.

23) P(the sum of the dice is 5)	24) P(the sum of the dice is 8)
25) P(the sum is 3 or 6)	26) P(the sum is more than 10)
27) P(the result is a double) (Hint: a double means that both dice show the same value)	

For problems 28-31: A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn randomly WITHOUT REPLACEMENT. That means that after a marble is drawn it is NOT replaced in the jar before the second marble is selected. Find the following probabilities.

28) P(the sum of the numbers is 5)	29) P(the sum of the numbers is odd)
30) P(the sum of the numbers is 9)	31) P(one of the numbers is 3)

For problems 32-33: A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn randomly WITH REPLACEMENT. That means that after a marble is drawn it is replaced in the jar before the second marble is selected. Find the following probabilities.

32) P(the sum of the numbers is 5)	33) P(the sum of the numbers is 2)

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8.2: Mutually Exclusive Events and the Addition Rule

Learning Objectives

In this section, you will learn to:

- Define compound events using union, intersection, and complement.
- Identify mutually exclusive events.
- Use the Addition Rule to calculate probability for unions of events.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Let Universal set U = { a, b, c, d, e, f, g, h, i, j }, sets V = { a, e, i, f, h }, W = { a, c, e, g, i }. List the members of the following sets:

a. $V \cup W$

b. $V \cap W$

c. $\overline{V \cup W}$

d. $\overline{V} \cap \overline{W}$

Click here to check your answer

a. $V \cup W$ = {a, c, e, f, g, h, i}

b. $V \cap W$ = {a, e, i}

c. $\overline{V \cup W}$ = {b, d, j}

d. $\overline{V} \cap \overline{W}$ = {b, d, j}

If you missed this problem, review Section 7.1. (Note that this will open in a new window.)

2. A survey of athletes revealed that for their minor aches and pains, 30 used aspirin, 50 used ibuprofen, 15 used both, and 5 didn't use either. Create a Venn Diagram to determine how many athletes were surveyed.

Click here to check your answer

70 athletes were surveyed.

If you missed this problem, review Section 7.1. (Note that this will open in a new window.)

In Section 5.1, we learned to find the union, intersection, and complement of a set. We will now use these set operations to describe events.

- The **union** of two events E and F, E ∪ F, is the set of outcomes that are in E or in F or in both.
- The **intersection** of two events E and F, E ∩ F, is the set of outcomes that are in both E and F.
- The **complement** of an event E, denoted by $\overline{E}$, is the set of outcomes in the sample space S that are not in E.

Note that since the probability of all events in a sample space have a sum of 1, it follows that $P(\overline{E}) = 1 - P(E)$. This is known as the **complement rule** of probability.

Of particular interest to us are the events whose outcomes do not overlap. We call these events mutually exclusive.

Two events E and F are said to be **mutually exclusive** if they do not intersect: E ∩ F = ∅ .

Next we'll determine whether a given pair of events are mutually exclusive.

Example 8.2.1

A card is drawn from a standard deck. Determine whether the pair of events given below is mutually exclusive.

$$E = \{\text{The card drawn is an Ace}\}$$

$$F = \{\text{The card drawn is a heart}\}$$

Solution

Clearly the ace of hearts belongs to both sets. That is

$$E \cap F = \{\text{ Ace of hearts }\} \neq \varnothing$$

Therefore, the events E and F are not mutually exclusive.

Example 8.2.2

Two dice are rolled. Determine whether the pair of events given below is mutually exclusive.

$$G = \{\text{The sum of the faces is six}\}$$

$$H = \{\text{One die shows a four}\}$$

Solution

For clarity, we list the elements of both sets.

G = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)} and H = {(2, 4), (4, 2)}

Clearly, $G \cap H = \{(2, 4), (4, 2)\} \neq \varnothing$.

Therefore, the two sets are not mutually exclusive.

Example 8.2.3

A family has three children. Determine whether the following pair of events are mutually exclusive.

$$M = \{\text{The family has at least one boy}\}$$

$$N = \{\text{The family has all girls}\}$$

Solution

Although the answer may be clear, we list both the sets.

$$M = \{\text{ BBB, BBG, BGB, BGG, GBB, GBG, GGB }\} \text{ and } N = \{\text{ GGG }\}$$

Clearly, $M \cap N = \varnothing$

Therefore, events M and N are mutually exclusive.

We will now consider problems that involve the union of two events.

Given two events, E, F, then finding the probability of $E \cup F$, is the same as finding the probability that E will happen, or F will happen, or both will happen.

Example 8.2.4

If a die is rolled, what is the probability of obtaining an even number or a number greater than four?

Solution

Let E be the event that the number shown on the die is an even number, and let F be the event that the number shown is greater than four.

The sample space S = { 1, 2, 3, 4, 5, 6 }. The event E = { 2, 4, 6 }, and event F = { 5, 6 }

We need to find $P(E \cup F)$.

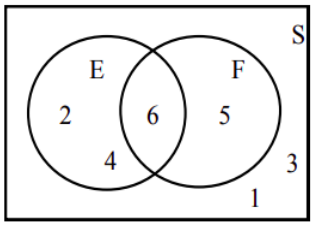
Since P(E) = 3/6, and P(F) = 2/6, a student may say P(E ∪ F) = 3/6 + 2/6. This will be incorrect because the element 6, which is in both E and F has been counted twice, once as an element of E and once as an element of F. In other words, the set E ∪ F has only four elements and not five: set E ∪ F = { 2,4,5,6 }

Therefore, P(E ∪ F) = 4/6 and not 5/6.

This can be illustrated by a Venn diagram. We'll use the Venn Diagram to re-examine Example 8.2.4 and derive a probability rule that we can use to calculate probabilities for unions of events.

The sample space S, the events E and F, and E ∩ F are listed below.

$$S = \{1, 2, 3, 4, 5, 6\}, E = \{2, 4, 6\}, F = \{5, 6\}, \text{ and } E \cap F = \{6\}.$$



The above figure shows S, E, F, and E ∩ F.

Finding the probability of E ∪ F, is the same as finding the probability that E will happen, or F will happen, or both will happen.

If we count the number of elements n(E) in E, and add to it the number of elements n(F) in F, the points in both E and F are counted twice, once as elements of E and once as elements of F. Now if we subtract from the sum, n(E) + n(F), the number n(E ∩ F), we remove the duplicity and get the correct answer. So as a rule,

$$n(E \cup F) = n(E) + n(F) - n(E \cap F)$$

By dividing the entire equation by n(S), we get

$$\frac{n(E \cup F)}{n(S)} = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(E \cap F)}{n(S)}$$

Since the probability of an event is the number of elements in that event divided by the number of all possible outcomes, we have

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Applying the above for Example 8.2.4, we get

$$P(E \cup F) = 3/6 + 2/6 - 1/6 = 4/6$$

This is because, when we add P(E) and P(F), we have added P(E ∩ F) twice. Therefore, we must subtract P(E ∩ F), once.

This gives us the general formula, called **the Addition Rule**, for finding the probability of the union of two events. Because event E ∪ F is the event that E will happen, OR F will happen, OR both will happen, we sometimes call this the **Addition Rule for OR Events.** It states

Addition Rule

$$\mathbf{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$$

If, and only if, two events E and F are mutually exclusive, then $E \cap F = \varnothing$ and $P(E \cap F) = 0$, and we get $P(E \cup F) = P(E) + P(F)$

Example 8.2.5

If a card is drawn from a deck, use the addition rule to find the probability of obtaining an ace or a heart.

Solution

Let A be the event that the card is an ace, and H the event that it is a heart.

Since there are four aces, and thirteen hearts in the deck,

$$P(A) = 4/52 \text{ and } P(H) = 13/52.$$

Furthermore, since the intersection of two events consists of only one card, the ace of hearts, we now have:

$$P(A \cap H) = 1/52$$

We need to find $P(A \cup H)$:

$$\begin{aligned} P(A \cup H) &= P(A) + P(H) - P(A \cap H) \\ &= 4/52 + 13/52 - 1/52 = 16/52 \end{aligned}$$

Example 8.2.6

Two dice are rolled, and the events F and T are as follows:

F = {The sum of the dice is four} and T = {At least one die shows a three}

Find $P(F \cup T)$.

Solution

We list F and T, and $F \cap T$ as follows:

F = {(1, 3), (2, 2), (3, 1)}

T = {(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (1, 3), (2, 3), (4, 3), (5, 3), (6, 3)}

$F \cap T$ = {(1, 3), (3, 1)}

Since $P(F \cup T) = P(F) + P(T) - P(F \cap T)$

We have $P(F \cup T) = 3/36 + 11/36 - 2/36 = 12/36$.

Example 8.2.7

Mr. Washington is seeking a mathematics instructor's position at his favorite community college in Cupertino. His employment depends on two conditions: whether the board approves the position, and whether the hiring committee selects him. There is a 80% chance that the board will approve the position, and there is a 70% chance that the hiring committee will select him. If there is a 90% chance that at least one of the two conditions, the board approval or his selection, will be met, what is the probability that Mr. Washington will be hired?

Solution

Let A be the event that the board approves the position, and S be the event that Mr. Washington gets selected. We have,

$P(A) = .80$, $P(S) = .70$, and $P(A \cup S) = .90$.

We need to find, $P(A \cap S)$.

The addition formula states that,

$$P(A \cup S) = P(A) + P(S) - P(A \cap S)$$

Substituting the known values, we get

$$.90 = .80 + .70 - P(A \cap S)$$

Therefore, $P(A \cap S) = .60$.

Example 8.2.8

The probability that this weekend will be cold is .6, the probability that it will be rainy is .7, and probability that it will be both cold and rainy is .5. What is the probability that it will be neither cold nor rainy?

Solution

Let C be the event that the weekend will be cold, and R be event that it will be rainy. We are given that

$$\mathrm{P(C)} = .6, \quad \mathrm{P(R)} = .7, \quad \mathrm{P(C \cap R)} = .5$$

First we find P(C ∪ R) using the Addition Rule.

$$P(C \cup R) = P(C) + P(R) - P(C \cap R) = .6 + .7 - .5 = .8$$

Then we find $P(\overline{C \cup R})$ using the Complement Rule.

$$P(\overline{C \cup R}) = 1 - P(C \cup R) = 1 - .8 = .2$$

We summarize this section by listing the important rules.

Summary

The Addition Rule

For Two Events E and F, $\mathrm{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$

The Addition Rule for Mutually Exclusive Events

If Two Events E and F are Mutually Exclusive, then $\mathrm{P(E \cup F) = P(E) + P(F)}$

The Complement Rule

If $\overline{E}$ is the Complement of Event E, then $\mathrm{P}(\overline{E}) = 1 - \mathrm{P(E)}$

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8.2E: Exercises - Mutually Exclusive Events and the Addition Rule

PROBLEM SET: MUTUALLY EXCLUSIVE EVENTS AND THE ADDITION RULE

Determine whether the following pair of events are mutually exclusive.

1) A = {A person earns more than $25,000} B = {A person earns less than $20,000}	2) A card is drawn from a deck. C = {It is a King} D = {It is a heart}.
3) A die is rolled. E = {An even number shows} F = {A number greater than 3 shows}	4) Two dice are rolled. G = {The sum of dice is 8} H = {One die shows a 6}
5) Three coins are tossed. I = {Two heads come up} J = {At least one tail comes up}	6) A family has three children. K = {First born is a boy} L = {The family has children of both sexes}

Use the Addition Rule to find the following probabilities.

7) A card is drawn from a deck. Events C and D are: C = {It is a king} D = {It is a heart} Find P(C ∪ D).	8) A die is rolled. The events E and F are: E = {An even number shows} F = {A number greater than 3 shows} Find P(E ∪ F).
9) Two dice are rolled. Events G and H are: G = {The sum of dice is 8} H ={Exactly one die shows a 6} Find P(G ∪ H).	10) Three coins are tossed. Events I and J are: I = {Two heads come up} J = {At least one tail comes up} Find P(I ∪ J).
11) At a college, 20% of the students take Finite Mathematics, 30% take Statistics and 10% take both. What percent of students take Finite Mathematics or Statistics?	12) This quarter, there is a 50% chance that Jason will pass Accounting, a 60% chance that he will pass English, and 80% chance that he will pass at least one of these two courses. What is the probability that he will pass both Accounting and English?

Questions 13 - 20 refer to the following: The table shows the distribution of Democratic and Republican U.S by gender in the 114th Congress as of January 2015.

	MALE(M)	FEMALE(F)	TOTAL
DEMOCRATS (D)	30	14	44
REPUBLICANS(R)	48	6	54
OTHER (T)	2	0	2
TOTALS	80	20	100

Use this table to determine the following probabilities.

13) P(M ∩ D)	14) P(F ∩ R)
15) P(M ∪ D)	16) P(F ∪ R)
17) P(M ∪ R)	18) P(M ∪ F)
19) Are the events F, R mutually exclusive? Use probabilities to support your conclusions.	20) Are the events F, T mutually exclusive? Use probabilities to support your conclusion.

Use the Addition Rule to find the following probabilities.

21) If P(E) = .5 , P(F) = .4 , E and F are mutually exclusive, find P(E ∩ F).	22) If P(E) = .4 , P(F) = .2 , E and F are mutually exclusive, find P(E ∪ F).
23) If P(E) = .3, P(E ∪ F) = .6 , P(E ∩ F) = .2, find P(F).	24) If P(E) = .4, P(F) = .5 , P(E ∪ F) = .7, find P(E ∩ F).

25) In a box of assorted cookies, 36% of cookies contain chocolate and 12% of cookies contain nuts. 8% of cookies have both chocolats and nuts. Sean is allergic to chocolate and nuts. Find the probability that a cookie has chocolate chips **or** nuts (he can't eat it).	26) At a college, 72% of courses have final exams and 46% of courses require research papers. 32% of courses have both a research paper and a final exam. Let F be the event that a course has a final exam and R be the event that a course requires a research paper. Find the probability that a course requires a final exam **or** a research paper.

Questions 25 and 26 are adapted from Introductory Statistics from OpenStax under a creative Commons Attribution 3.0 Unported License, available for download free athttp://cnx.org/content/col11562/latest u

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8.3: Probability Using Tree Diagrams and Combinations

Learning Objectives

In this section, you will learn to:

- Use probability tree diagrams to calculate probabilities.
- Use combinations to calculate probabilities.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Brown Home Construction offers a selection of 3 floor plans, 2 roof types, and 2 exterior wall types. Use a tree diagram to determine the number of possible homes available.

Click here to check your answer

There are 12 possible homes available.

If you missed this problem, review Section 7.2. (Note that this will open in a new window.)

2. Calculate each expression:

a. $10\text{P}2$

b. $15\text{P}6$

Click here to check your answer

a. $10\text{P}2 = 90$

b. $15\text{P}6 = 3,603,600$

If you missed this problem, review Section 7.3. (Note that this will open in a new window.)

3. In how many ways can the letters of the word MAJESTIC be arranged if each letter is used only once in each arrangement?

Click here to check your answer

The letters can be arranged 40,320 different ways.

If you missed this problem, review Section 7.3. (Note that this will open in a new window.)

4. A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn randomly WITHOUT REPLACEMENT. That means that after a marble is drawn it is NOT replaced in the jar before the second marble is selected. Find the following probabilities.

a. P(the sum of the numbers is 5)

b. P(the sum of the numbers is odd)

Click here to check your answer

a. $\frac{1}{3}$

b. $\frac{2}{3}$

If you missed this problem, review Section 8.1. (Note that this will open in a new window.)

In this section, we will apply previously learnt counting techniques in calculating probabilities, and use tree diagrams to help us gain a better understanding of what is involved.

USING TREE DIAGRAMS TO CALCULATE PROBABILITIES

We already used tree diagrams to list events in a sample space in Section 5.2. Tree diagrams can be helpful in organizing information in probability problems; they help provide a structure for understanding probability. In this section we expand our previous use of tree diagrams to situations in which the events in the sample space are not all equally likely.

We assign the appropriate probabilities to the events shown on the branches of the tree. By multiplying probabilities along a path through the tree, we can find probabilities for “and” events, which are intersections of events.

We begin with an example.

Example 8.3.1

Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn with replacement, what is the probability that both marbles are red?

Solution

Let E be the event that the first marble drawn is red, and let F be the event that the second marble drawn is red.

We need to find $\mathrm{P}(\mathrm{E} \cap \mathrm{F})$

By the statement, "two marbles are drawn with replacement," we mean that the first marble is replaced before the second marble is drawn.

There are 7 choices for the first draw. And since the first marble is replaced before the second is drawn, there are, again, seven choices for the second draw. Using the multiplication axiom, we conclude that the sample space S consists of 49 ordered pairs. Of the 49 ordered pairs, there are $3 \times 3 = 9$ ordered pairs that show red on the first draw and, also, red on the second draw. Therefore,

$$P(E \cap F) = \frac{9}{49}$$

Further note that in this particular case

$$P(E \cap F) = \frac{9}{49} = \frac{3}{7} \cdot \frac{3}{7}$$

giving us the result that in this example: $\mathbf{P}(\mathbf{E} \cap \mathbf{F}) = \mathbf{P}(\mathbf{E}) \cdot \mathbf{P}(\mathbf{F})$

Example 8.3.2

If in Example $8.3.1$, the two marbles are drawn without replacement, then what is the probability that both marbles are red?

Solution

By the statement, "two marbles are drawn without replacement," we mean that the first marble is not replaced before the second marble is drawn.

Again, we need to find $\mathrm{P}(\mathrm{E} \cap \mathrm{F})$.

There are, again, 7 choices for the first draw. And since the first marble is not replaced before the second is drawn, there are only six choices for the second draw. Using the multiplication axiom, we conclude that the sample space S consists of 42 ordered pairs. Of the 42 ordered pairs, there are $3 \times 2 = 6$ ordered pairs that show red on the first draw and red on the second draw. Therefore,

$$P(E \cap F) = \frac{6}{42}$$

Note that we can break this calculation down as

$$P(E \cap F) = \frac{6}{42} = \frac{3}{7} \cdot \frac{2}{6}$$

Here 3/7 represents $\mathrm{P(E)}$, and 2/6 represents the probability of drawing a red on the second draw, given that the first draw resulted in a red.

We write the latter as P(red on the second | red on first) or $\mathrm{P(F|E)}$. The "|" represents the word "given" or “if”. This leads to the result that:

$$\mathbf{P(E \cap F) = P(E) \cdot P(F|E)}$$

The is an important result, called the **Multiplication Rule,** which will appear again in later sections.

We now demonstrate the above results with a tree diagram.

Example 8.3.3

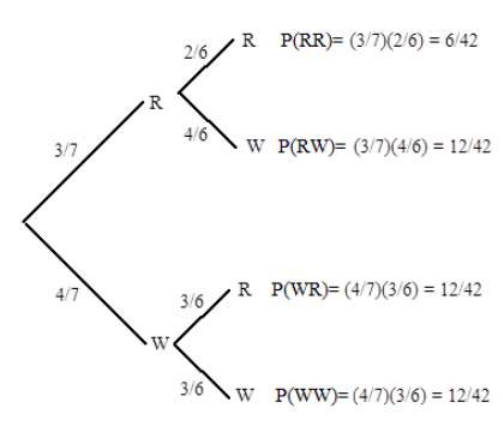
Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn without replacement, find the following probabilities using a tree diagram.

a. The probability that both marbles are red.
b. The probability that the first marble is red and the second white.
c. The probability that one marble is red and the other white.

Solution

Let R be the event that the marble drawn is red, and let W be the event that the marble drawn is white.

We draw the following tree diagram.



a. The probability that both marbles are red is $\mathrm{P(RR)} = 6/42$
b. The probability that the first marble is red and the second is white is $\mathrm{P(RW)} = 12/42$
c. For the probability that one marble is red and the other is white, we observe that this can be satisfied if the first is red and the second is white, **or** if the first is white and the second is red. The “or” tells us we’ll be using the Addition Rule from Section 6.2.

Furthermore events RW and WR are mutually exclusive events, so we use the form of the Addition Rule that applies to mutually exclusive events.

Therefore

P(one marble is red and the other marble is white)

$$\begin{aligned} &= \mathrm{P(RW \text{ or } WR)} \\ &= \mathrm{P(RW)} + \mathrm{P(WR)} \\ &= 12/42 + 12/42 = 24/42 \end{aligned}$$

USING COMBINATIONS TO FIND PROBABILITIES

Although the tree diagrams give us better insight into a problem, they are not practical for problems where more than two or three things are chosen. In such cases, we use the concept of combinations that we learned in the last chapter. This method is best suited for problems where the order in which the objects are chosen is not important, and the objects are chosen without replacement.

Example 8.3.4

Suppose a jar contains 3 red, 2 white, and 3 blue marbles. If three marbles are drawn without replacement, find the following probabilities.

a. P(Two red and one white)
b. P(One of each color)
c. P(None blue)
d. P(At least one blue)

Solution

Let us suppose the marbles are labeled as $R_1, R_2, R_3, W_1, W_2, B_1, B_2, B_3$.

a. P(Two red and one white)

Since we are choosing 3 marbles from a total of 8, there are 8C3 = 56 possible combinations. Of these 56 combinations, there are $3\mathrm{C}2 \times 2\mathrm{C}1 = 6$ combinations consisting of 2 red and one white. Therefore,

$$P(\text{ Two red and one white }) = \frac{3\mathrm{C}2 \times 2\mathrm{C}1}{8\mathrm{C}3} = \frac{6}{56} = \frac{3}{28}.$$

b. P(One of each color)

Again, there are 8C3 = 56 possible combinations. Of these 56 combinations, there are $3\mathrm{Cl} \times 2\mathrm{Cl} \times 3\mathrm{Cl} = 18$ combinations consisting of one red, one white, and one blue. Therefore,

$$P(\text{ One of each color }) = \frac{3\mathrm{C}1 \times 2\mathrm{C}1 \times 3\mathrm{C}1}{8\mathrm{C}3} = \frac{18}{56} = \frac{9}{28}$$

c. P(None blue)

There are 5 non-blue marbles, therefore

$$\mathrm{P}(\text{ None blue }) = \frac{5\mathrm{C}3}{8\mathrm{C}3} = \frac{10}{56} = \frac{5}{28}$$

d. P(At least one blue)

By "at least one blue marble," we mean the following: one blue marble and two non-blue marbles, *OR* two blue marbles and one non-blue marble, *OR* all three blue marbles. So we have to find the sum of the probabilities of all three cases.

$$\mathrm{P}(\text{At least one blue }) = \mathrm{P}(1 \text{ blue, } 2 \text{ non-blue}) + \mathrm{P}(2 \text{ blue, } \mathrm{l} \text{ non-blue}) + \mathrm{P}(3 \text{ blue})$$

$$P(\text{ At least one blue }) = \frac{3\mathrm{C}1 \times 5\mathrm{C}2}{8\mathrm{C}3} + \frac{3\mathrm{C}2 \times 5\mathrm{C}1}{8\mathrm{C}3} + \frac{3\mathrm{C}3}{8\mathrm{C}3}$$

$$P(\text{ At least one blue }) = 30/56 + 15/56 + 1/56 = 46/56 = 23/28$$

Alternately, we can use the fact that $\mathrm{P}(\mathrm{E}) = 1 - \mathrm{P}(\overline{\mathrm{E}})$. If the event E = At least one blue, then $\overline{E}$ = None blue.

But from part c of this example, we have $(\overline{\mathrm{E}}) = 5/28$, so $\mathrm{P}(\mathrm{E}) = 1 - 5/28 = 23/28$.

Example 8.3.5

Five cards are drawn from a deck. Find the probability of obtaining two pairs, that is, two cards of one value, two of another value, and one other card.

Solution

Let us first do an easier problem-the probability of obtaining a pair of kings and queens.

Since there are four kings, and four queens in the deck, the probability of obtaining two kings, two queens and one other card is

$$\mathrm{P}(\text{A pair of kings and queens}) = \frac{4\mathrm{C}2 \times 4\mathrm{C}2 \times 44\mathrm{C}1}{52\mathrm{C}5}$$

To find the probability of obtaining two pairs, we have to consider all possible pairs.

Since there are altogether 13 values, that is, aces, deuces, and so on, there are $13\mathrm{C}2$ different combinations of pairs.

$$P(\text{ Two pairs }) = 13\mathrm{C}2 \cdot \frac{4\mathrm{C}2 \times 4\mathrm{C}2 \times 44\mathrm{C}1}{52\mathrm{C}5} = .04754$$

Example 8.3.6

A cell phone store receives a shipment of 15 cell phones that contains 8 iPhones and 7 Android phones. Suppose that 6 cell phones are randomly selected from this shipment. Find the probability that a randomly selected set of 6 cell phones consists of 2 iPhones and 4 Android phones.

Solution

There are $8\mathrm{C}2$ ways of selecting 2 out of the 8 iPhones.

and $7\mathrm{C}4$ ways of selecting 4 out of the 7 Android phones

But altogether there are $15\mathrm{C}6$ ways of selecting 6 out of 15 cell phones.

Therefore we have

$$P(2 \text{ iPhones and } 4 \text{ Android phones }) = \frac{8\mathrm{C}2 \times 7\mathrm{C}4}{15\mathrm{C}6} = \frac{(28)(35)}{5005} = \frac{980}{5005} = 0.1958$$

Example 8.3.7

One afternoon, a bagel store still has 53 bagels remaining: 20 plain, 15 poppyseed, and 18 sesame seed bagels. Suppose that the store owner packages up a bag of 9 bagels to bring home for tomorrow's breakfast, and selects the bagels randomly. Find the probability that the bag contains 4 plain, 3 poppyseed, and 2 sesame seed.

Solution

There are $20\mathrm{C}4$ ways of selecting 4 out of the 20 plain bagels,

and $15\mathrm{C}3$ ways of selecting 3 out of the 15 poppyseed bagels,

and $18\mathrm{C}2$ ways of selecting 2 out of the 18 sesame seed bagels.

But altogether there are $53\mathrm{C}9$ ways of selecting 9 out of the 53 bagels.

$$\begin{aligned}\mathrm{P}(4 \text{ plain, } 3 \text{ poppyseed, and } 2 \text{ sesame seed}) &= \frac{20\mathrm{C}4 \times 15\mathrm{C}3 \times 18\mathrm{C}2}{53\mathrm{C}9} \\ &= \frac{(4845)(455)(153)}{4431613550} \\ &= 0.0761\end{aligned}$$

We end the section by solving a famous problem called the **Birthday Problem**.

Example 8.3.8: Birthday Problem

If there are 25 people in a room, what is the probability that at least two people have the same birthday?

Solution

Let event E represent that at least two people have the same birthday.

We first find the probability that no two people have the same birthday.

We analyze as follows.

Suppose there are 365 days to every year. According to the multiplication axiom, there are 365^{25} possible birthdays for 25 people. Therefore, the sample space has 365^{25} elements. We are interested in the probability that no two people have the same birthday. There are 365 possible choices for the first person and since the second person must have a different birthday, there are 364 choices for the second, 363 for the third, and so on. Therefore,

$$\mathrm{P}(\text{No two have the same birthday }) = \frac{365 \cdot 364 \cdot 363 \cdots 341}{365^{25}} = \frac{365\mathrm{P}25}{365^{25}}$$

Since P(at least two people have the same birthday) = 1 - P(No two have the same birthday),

$$\mathrm{P}(\text{ at least two people have the same birthday }) = 1 - \frac{365\mathrm{P}25}{365^{25}} = .5687$$

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8.3E: Exercises - Probability Using Tree Diagrams and Combinations

PROBLEM SET: PROBABILITIES USING TREE DIAGRAMS AND COMBINATIONS

Two apples are chosen from a basket containing five red and three yellow apples.
Draw a tree diagram below, and find the following probabilities.

1) P(both red)	2) P(one red, one yellow)
3) P(both yellow)	4) P(First red and second yellow)

A basket contains six red and four blue marbles. Three marbles are drawn at random.
Find the following probabilities using the method shown in Example 8.3.2. Do not use combinations.

5) P(All three red)	6) P(two red, one blue)
7) P(one red, two blue)	8) P(first red, second blue, third red)

Three marbles are drawn from a jar containing five red, four white, and three blue marbles.
Find the following probabilities using combinations.

9) P(all three red)	10) P(two white and 1 blue)
11) P(none white)	12) P(at least one red)

A committee of four is selected from a total of 4 freshmen, 5 sophomores, and 6 juniors. Find the probabilities for the following events.

13) At least three freshmen.	14) No sophomores.
15) All four of the same class.	16) Not all four from the same class.
17) Exactly three of the same class.	18) More juniors than freshmen and sophomores combined.

Five cards are drawn from a deck. Find the probabilities for the following events.

19) Two hearts, two spades, and one club.	20) A flush of any suit (*all cards of a single suit*).
21) A full house of nines and tens (*3 nines and 2 tens*).	22) Any full house.
23) A pair of nines and a pair of tens (*and the fifth card is not a nine or ten*).	24) Any two pairs (*two cards of one value, two more cards of another value, and the fifth card does not have the same value as either pair*).

Jorge has 6 rock songs, 7 rap songs and 4 country songs that he likes to listen to while he exercises.
He randomly selects six (6) of these songs to create a playlist to listen to today while he exercises.

Find the following probabilities:

25) P(playlist has 2 songs of each type)	26) P(playlist has no country songs)
27) P(playlist has 3 rock, 2 rap, and 1 country song)	28) P(playlist has 3 or 4 rock songs and the rest are rap songs)

A project is staffed 12 people: 5 engineers, 4 salespeople, and 3 customer service representatives.
A committee of 5 people is selected to make a presentation to senior management.

Find the probabilities of the following events.

29) The committee has 2 engineers, 2 salespeople, and 1 customer service representative.	30) The committee contains 3 engineer and 2 salespeople.
31) The committee has no engineers.	32) The committee has all salespeople.

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8.4: Conditional Probability

Learning Objectives

In this section, you will learn to:

- Recognize situations involving conditional probability.
- Calculate conditional probabilities.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. Three coins are tossed. Write a sample space for the experiment:

a. If all three coins are pennies.

b. If the three coins are a penny, a nickel, and a dime.

Click here to check your answer

a. {3H, 2H1T, 1H2T, 3T} or {HHH, HHT, HTT, TTT}

b. {HHH, HHT, HTH, HTT, TTT, TTH, THT, THH}

If you missed this problem, review Section 8.1. (Note that this will open in a new window.)

2. Refer to the following table, which shows the distribution of Democratic and Republican U.S by gender in the 114th Congress as of January 2015.

	MALE(M)	FEMALE(F)	TOTAL
DEMOCRATS (D)	30	14	44
REPUBLICANS(R)	48	6	54
OTHER (T)	2	0	2
TOTALS	80	20	100

Use this table to determine the following probabilities.

a. $P(M \cap D)$

b. $P(F \cap R)$

Click here to check your answer

a. $P(M \cap D) = \frac{3}{10}$

b. $P(F \cap R) = \frac{3}{50}$

If you missed this problem, review Section 8.2. (Note that this will open in a new window.)

3. Calculate each expression:

a. $10P2$

b. $15P6$

Click here to check your answer

a. $10P2 = 90$

b. $15P6 = 3,603,600$

If you missed this problem, review Section 7.3. (Note that this will open in a new window.)

4. Without using a calculator, simplify $\frac{a}{b}$ if $a = \frac{2}{15}$ and $b = \frac{4}{15}$.

Click here to check your answer

$b = \frac{1}{2}$

If you missed this problem, review here. (Note that this will open a different textbook in a new window.)

Suppose a friend, an avid traveler, asks you for the probability that they will encounter snow today.

If your friend is in Alaska today, the probability of snow today might be quite substantial. If they are in Florida though, the probability of snow today is likely very tiny.

Let:

- S = the event that it will snow today
- A = the event that your friend is in Alaska today
- F = the event that your friend is in Florida today

Because the probability of snow is affected by the location, we can't just write $\mathrm{P}(\mathrm{S})$ for the probability of snow. We need to indicate the other information we know. We need to use **conditional probability**.

The event we are interested in is event S for snow. The other event is called the condition, representing the location in this case.

We represent conditional probability using a vertical line | that means "if", or "given that", or "if we know that". The event of interest appears on the left of the |. The condition appears on the right side of the |. This will appear as $\mathbf{P}(\mathbf{event}|\mathbf{condition})$

The probability it will snow given that (if) your friend is in Alaska is represented by $\mathbf{P}(\mathbf{S}|\mathbf{A})$. In this case, the condition is A.

The probability that it will snow given that (if) your friend is in Florida is represented by $\mathbf{P}(\mathbf{S}|\mathbf{F})$. In this case, the condition is F.

Now, let's examine a situation where we can calculate some probabilities.

Suppose you and a friend play a game that involves choosing a single card from a well-shuffled deck. Your friend deals you one card, face down, from the deck and offers you the following deal: If the card is a king, he will pay you $5, otherwise, you pay him $1. Should you play the game?

You reason in the following manner. Since there are four kings in the deck, the probability of obtaining a king is 4/52 or 1/13. So, probability of not obtaining a king is 12/13. This implies that the ratio of your winning to losing is 1 to 12, while the payoff ratio is only $1 to $5. Therefore, you determine that you should not play.

But consider the following scenario. While your friend was dealing the card, you happened to get a glance of it and noticed that the card was a face card. Should you, now, play the game?

Since there are 12 face cards in the deck, the total elements in the sample space are no longer 52, but just 12. This means the chance of obtaining a king is 4/12 or 1/3. So your chance of winning is 1/3 and of losing 2/3. This makes your winning to losing ratio 1 to 2 which fares much better with the payoff ratio of $1 to $5. This time, you determine that you should play.

In the second part of the above example, we were finding the probability of obtaining a king knowing that a face card had shown. This is an example of **conditional probability**. Whenever we are finding the probability of an event E under the condition that another event F has happened, we are finding conditional probability.

The symbol $\mathrm{P}(\mathrm{E}|\mathrm{F})$ denotes the problem of finding the probability of E given that F has occurred. We read $\mathrm{P}(\mathrm{E}|\mathrm{F})$ as "the probability of E, given F."

Example 8.4.1

A family has three children.

a. Find the probability of having two boys and one girl.
b. Find the probability of having two boys and one girl given that the first born is a boy.

Solution

First, we the sample space for a family of three children as follows.

$$S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$$

Let event E be that the family has two boys and a girl, and F that the first born is a boy.

a. $\mathrm{P}(\mathrm{E}) = 3/8$ because 3 of the 8 options in the sample space include two boys and one girl.

b. Given that the first born is a boy though, our possibilities narrow down to four outcomes: BBB, BBG, BGB, and BGG.

Among the four, BBG and BGB represent two boys and a girl.

Therefore, $\mathrm{P}(\mathrm{E}|\mathrm{F})$ = 2/4 or 1/2.

Example 8.4.2

One six sided die is rolled once.

a. Find the probability that the result is even.
b. Find the probability that the result is even given that the result is greater than three.

Solution

The sample space is $\mathrm{S} = 1, 2, 3, 4, 5, 6$

Let event E be that the result is even and T be that the result is greater than 3.

a. $\mathrm{P}(\mathrm{E})$ = 3/6 because $\mathrm{E} = 2, 4, 6$

b. Because $T = 4, 5, 6$, we know that 1, 2, 3 cannot occur; only outcomes 4, 5, 6 are possible. Therefore of the values in E, only 4, 6 are possible.

Therefore, $\mathrm{P}(\mathrm{E}|\mathrm{T})$ = 2/3

Example 8.4.3

A fair coin is tossed twice.

a. Find the probability that the result is is two heads.
b. Find the probability that the result is two heads given that at least one head is obtained.

Solution

The sample space is $S = HH, HT, TH, TT$

Let event E be that the two heads are obtained and F be at least one head is obtained

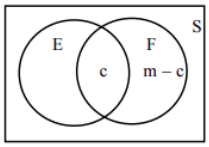
a. $\mathrm{P}(\mathrm{E})$ = 1/4 because $\mathrm{E} = HH$ and the sample space S has 4 outcomes.

b. $\mathrm{F} = HH, HT, TH$. Since at least one head was obtained, TT did not occur.
We are interested in the probability event $\mathrm{E} = HH$ out of the 3 outcomes in the reduced sample space F.

Therefore, $\mathrm{P}(\mathrm{E}|\mathrm{F})$ = 1/3

Let us now develop a formula for the conditional probability $\mathrm{P}(\mathrm{E}|\mathrm{F})$.

Suppose an experiment consists of n equally likely events. Further suppose that there are m elements in F, and c elements in $\mathrm{E} \cap \mathrm{F}$, as shown in the following Venn diagram.



If the event F has occurred, the set of all possible outcomes is no longer the entire sample space, but instead, the subset F. Therefore, we only look at the set F and at nothing outside of F. Since F has m elements, the denominator in the calculation of $\mathrm{P(E|F)}$ is m. We may think that the numerator for our conditional probability is the number of elements in E. But clearly we cannot consider the elements of E that are not in F. We can only count the elements of E that are in F, that is, the elements in $\mathrm{E \cap F}$. Therefore,

$$\mathrm{P(E|F) = \frac{c}{m}}$$

Dividing both the numerator and the denominator by n, we get

$$\mathrm{P(E|F) = \frac{c/n}{m/n}}$$

But $c/n = \mathrm{P(E \cap F)}$, and $m/n = \mathrm{P(F)}$.

Substituting, we derive the following formula for $\mathrm{P(E|F)}$.

Conditional Probability Rule

For two events E and F, the probability of "E Given F" is

$$\mathbf{P(E|F) = \frac{P(E \cap F)}{P(F)}} \tag{8.4.1}$$

Example 8.4.4

A single die is rolled. Use the above formula to find the conditional probability of obtaining an even number given that a number greater than three has shown.

Solution

Let E be the event that an even number shows, and F be the event that a number greater than three shows. We want $\mathrm{P(E|F)}$.

$\mathrm{E} = 2, 4, 6$ and $\mathrm{F} = 4, 5, 6$. Which implies, $\mathrm{E \cap F} = 4, 6$

Therefore, $\mathrm{P(F)}$ = 3/6, and $\mathrm{P(E \cap F)}$ = 2/6

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{2/6}{3/6} = \frac{2}{3}.$$

Example 8.4.5

The following table shows the distribution by gender of students at a community college who take public transportation and the ones who drive to school.

	Male(M)	Female(F)	Non-Binary(N)	Total
Public Transportation(T)	8	10	3	21
Drive(D)	37	41	1	79
Total	45	51	4	100

The events M, F, N, T, and D are labeled. Find the following probabilities.

a. $\mathrm{P(D|N)}$

b. $P(F|D)$
c. $P(M|T)$

Solution 1

Conditional probabilities can often be found directly from a contingency table. If the condition corresponds to only one row or only one column in the table, then you can ignore the rest of the table and read the conditional probability directly from the row or column indicated by the condition.

a. The condition is event N; we can look at only the "Non-Binary" column of the table and ignore the rest of the table: $P(D|N) = \frac{1}{4}$.
b. The condition is event D; we can look at only the "Drive" row of the table and ignore the rest of the table: $P(F|D) = \frac{41}{79}$.
c. The condition is event T; we can look at only the "Public Transportation" row of the table and ignore the rest of the table: $P(M|T) = \frac{8}{21}$.

Solution 2

We use the conditional probability formula $P(E|F) = \frac{P(E \cap F)}{P(F)}$.

a. $$P(D|N) = \frac{P(D \cap N)}{P(N)} = \frac{1/100}{4/100} = \frac{1}{4}.$$

b. $$P(F|D) = \frac{P(F \cap D)}{P(D)} = \frac{41/100}{79/100} = \frac{41}{79}.$$

c. $$P(M|T) = \frac{P(M \cap T)}{P(T)} = \frac{8/100}{21/100} = \frac{8}{21}$$

Example 8.4.6

Given $P(E) = .5$, $P(F) = .7$, and $P(E \cap F) = .3$. Find the following:

a. $P(E|F)$
b. $P(F|E)$

Solution

We use the conditional probability formula.

a. $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{3}{7} = \frac{3}{7}$
b. $P(F|E) = \frac{P(E \cap F)}{P(E)} = .3/.5 = 3/5$

Example 8.4.7

E and F are mutually exclusive events such that $P(E) = .4$, $P(F) = .9$. Find $P(E|F)$.

Solution

E and F are mutually exclusive, so $P(E \cap F) = 0$.
Therefore $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{0}{9} = 0$.

Example 8.4.8

Given $P(F|E) = .5$, and $P(E \cap F) = .3$. Find $P(E)$.

Solution

Using the conditional probability formula $P(E|F) = \frac{P(E \cap F)}{P(F)}$, we get

$$P(F|E) = \frac{P(E \cap F)}{P(E)}$$

Substituting and solving:

$$.5 = \frac{.3}{P(E)} \quad \text{or} \quad P(E) = 3/5$$

Example 8.4.9

In a family of three children, find the conditional probability of having two boys and a girl, given that the family has at least two boys.

Solution

Let event E be that the family has two boys and a girl, and let F be the probability that the family has at least two boys. We want $P(E|F)$.

We list the sample space along with the events E and F.

$$S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$$
$$E = \{BBG, BGB, GBB\} \text{ and } F = \{BBB, BBG, BGB, GBB\}$$
$$E \cap F = \{BBG, BGB, GBB\}$$

Therefore, $P(F)$ = 4/8, and $P(E \cap F)$ = 3/8, and

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{3/8}{4/8} = \frac{3}{4}.$$

Example 8.4.10

At a community college 65% of the students subscribe to Amazon Prime, 50% subscribe to Netflix, and 20% subscribe to both. If a student is chosen at random, find the following probabilities:

a. the student subscribes to Amazon Prime given that they subscribe to Netflix
b. the student subscribes to Netflix given that they subscribe to Amazon Prime

Solution

Let A be the event that the student subscribes to Amazon Prime, and N be the event that the student subscribes to Netflix.

First identify the probabilities and events given in the problem.

P(student subscribes to Amazon Prime) = $P(A)$ = 0.65

P(student subscribes to Netflix) = $P(N)$ = 0.50

P(student subscribes to both Amazon Prime and Netflix) = $P(A \cap N)$ = 0.20

Then use the conditional probability rule:

a. $P(A|N) = \frac{P(A \cap N)}{P(N)} = \frac{.20}{.50} = \frac{2}{5}$
b. $P(N|A) = \frac{P(A \cap N)}{P(A)} = \frac{.20}{.65} = \frac{4}{13}$

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8.4E: Exercises - Conditional Probability

PROBLEM SET: CONDITIONAL PROBABILITY

Questions 1 - 4: Do these problems using the conditional probability formula: $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

1. A card is drawn from a deck. Find the conditional probability of P(a queen | a face card).	2. A card is drawn from a deck. Find the conditional probability of P(a queen | a club).
3. A die is rolled. Find the conditional probability that it shows a three if it is known that an odd number has shown.	4. If $P(A)$ = .3 , $P(B)$ = .4, $P(A \cap B)$ = .12, find: a. $P(A\|B)$ b. $P(B\|A)$

Questions 5 - 8 refer to the following: The table shows the distribution of Democratic and Republican U.S. Senators by gender in the 114th Congress as of January 2015.

	MALE(M)	FEMALE(F)	TOTAL
DEMOCRATS (D)	30	14	44
REPUBLICANS(R)	48	6	54
OTHER (T)	2	0	2
TOTALS	80	20	100

Use this table to determine the following probabilities:

5. $P(M\|D)$	6. $P(D\|M)$
7. $P(F\|R)$	8. $P(R\|F)$

Do the following conditional probability problems.

9. At a college, 20% of the students take Finite Math, 30% take History, and 5% take both Finite Math and History. If a student is chosen at random, find the following conditional probabilities. a. He is taking Finite Math given that he is taking History. b. He is taking History assuming that he is taking Finite Math.	10. At a college, 60% of the students pass Accounting, 70% pass English, and 30% pass both of these courses. If a student is selected at random, find the following conditional probabilities. a. He passes Accounting given that he passed English. b. He passes English assuming that he passed Accounting.
11. If $P(F) = .4$, $P(E\|F) = .3$, find $P(E \cap F)$.	12. $P(E) = .3$, $P(F) = .3$; $E \cap F$ are mutually exclusive. Find $P(E\|F)$.
13. If $P(E) = .6$, $P(E \cap F)$ = .24, find $P(F\|E)$.	14. If $P(E \cap F)$ = .04 , $P(E\|F) = .1$, find $P(F)$.

At a college, 72% of courses have final exams and 46% of courses require research papers. 32% of courses have both a research paper and a final exam. Let F be the event that a course has a final exam and R be the event that a course requires a research paper.

15. Find the probability that a course has a final exam given that it has a research paper.	16. Find the probability that a course has a research paper if it has a final exam.

Consider a family of three children. Find the following probabilities.

17. P(two boys | first born is a boy)	18. P(all girls | at least one girl is born)

19. P(children of both sexes | first born is a boy)	20. P(all boys | there are children of both sexes)

Questions 21 - 26 refer to the following:
The table shows highest attained educational status for a sample of US residents age 25 or over:

	(D) Did not Complete High School	(H) High School Graduate	(C) Some College	(A) Associate Degree	(B) Bachelor Degree	(G) Graduate Degree	TOTAL
25-44 (R)	95	228	143	81	188	61	796
45-64 (S)	83	256	136	80	150	67	772
65+ (T)	96	191	84	36	80	41	528
Total	274	675	363	197	418	169	2096

Use this table to determine the following probabilities:

21. $P(C\|T)$	22. $P(S\|A)$	23. $P(C \cap T)$
24. $P(R\|B)$	25. $P(B\|R)$	26. $P(G\|S)$

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8.5: Bayes' Theorem

Learning Objectives

In this section, you will learn to:

- Find probabilities using Bayes' theorem.
- Use a probability tree to find and represent values needed when using Bayes' theorem.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. If $P(F) = .4$, $P(E|F) = .3$, find $P(E \cap F)$.

Click here to check your answer

$.12$

If you missed this problem, review Section 8.4. (Note that this will open in a new window.)

2. $P(E) = .3$, $P(F) = .3$; $E \cap F$ are mutually exclusive. Find $P(E|F)$.

Click here to check your answer

0

If you missed this problem, review Section 8.4. (Note that this will open in a new window.)

3. If $P(E) = .6$, $P(E \cap F) = .24$, find $P(F|E)$.

Click here to check your answer

$.4$

If you missed this problem, review Section 8.4. (Note that this will open in a new window.)

4. If $P(E \cap F) = .04$, $P(E|F) = .1$, find $P(F)$.

Click here to check your answer

$.4$

If you missed this problem, review Section 8.4. (Note that this will open in a new window.)

In this section, we will develop and use Bayes' theorem to solve an important type of probability problem. Bayes' theorem is a method of calculating the conditional probability $P(F|E)$ from $P(E|F)$. The ideas involved here are not new, and most of these problems can be solved using a tree diagram. However, Bayes' theorem does provide us with a tool with which we can solve these problems without a tree diagram.

We begin with an example.

Example 8.5.1

Suppose a certain disease afflicts 1% of the population. A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it), but the false positive rate is 5% (that is, even among those without the disease, 5% who take the test will test positive).

Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease?

Solution

Let's begin by interpreting the information we have.

Let

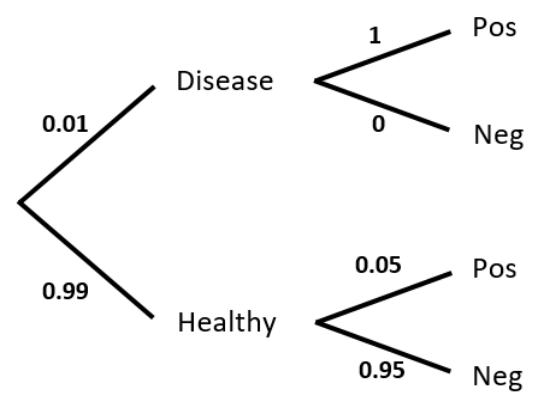
- Disease = the event that the person has the disease
- Healthy= the event that the person is healthy (does not have the disease)
- Pos = the event that the test is positive
- Neg = the event that the test is negative

Then

- P(Disease) = 0.01
- P(Healthy) = 0.99
- P(Pos | Healthy) = 0.05 (5% of healthy people will test positive.)
- P(Neg | Healthy) = 0.95 (The other 95% of healthy people will test negative.)
- P(Pos | Disease) = 1 (All who have the disease will test positive.)
- P(Neg | Disease) = 0 (None who have the disease will test negative.)

To find the probability that a person who tests positive actually has the disease, we are trying to find P(Disease | Pos). Note that this is the opposite order of the information we are given.

We can construct a tree diagram and use our knowledge about conditional probability to find this.



To find P(Disease | Pos), we will need to know P(Pos). The probability of a positive test is the probability of a healthy person testing positive plus the probability of a diseased person testing positive.

From the tree diagram, we can see:

- P(Disease ∩ Pos) = (0.01)(1) = 0.01
- P(Healthy ∩ Pos) = (0.99)(0.05) = 0.0495

Therefore, P(Pos) = 0.01 + 0.0495 = 0.0595. For the purpose of understanding Bayes' theorem, it might be helpful to rewrite this in its unsimplified form, of P(Pos) = (0.01)(1) + (0.99)(0.05).

Using the formula for conditional probability,

$$\begin{aligned} P(\text{Disease}|\text{Pos}) &= \frac{P(\text{Disease} \cap \text{Pos})}{P(\text{Pos})} \\ &= \frac{(0.01)(1)}{(0.01)(1)+(0.99)(0.05)} \\ &= 0.168 \end{aligned}$$

This means that only 16.8% of those who test positive actually have the disease!

In the above example, we found that the $P(\text{Disease}|\text{Pos}) = \frac{P(\text{Disease} \cdot P(\text{Pos}|\text{Disease})}{P(\text{Pos})}$. Note that the denominator included the sum of all the branches that ended in a positive test.

This leads us to Bayes' theorem.

Bayes' Theorem

For any events E and F,

$$P(E|F) = \frac{P(E)P(F|E)}{P(F)}$$

We begin with the following example.

Example 8.5.2

A department store buys 50% of its appliances from Manufacturer A, 30% from Manufacturer B, and 20% from Manufacturer C. It is estimated that 6% of Manufacturer A's appliances, 5% of Manufacturer B's appliances, and 4% of Manufacturer C's appliances need repair before the warranty expires. An appliance is chosen at random. If the appliance chosen needed repair before the warranty expired, what is the probability that the appliance was manufactured by Manufacturer A? Manufacturer B? Manufacturer C?

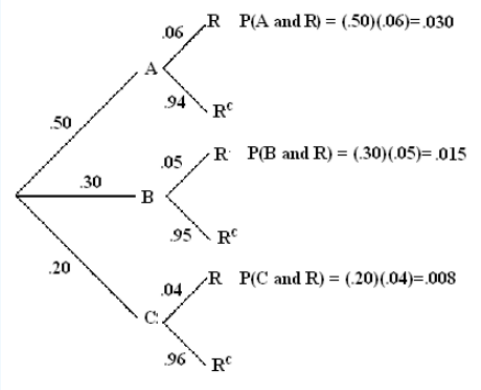
Solution

Let A, B and C be the events that the appliance is manufactured by Manufacturer A, Manufacturer B, and Manufacturer C, respectively. Further, suppose that the event R denotes that the appliance needs repair before the warranty expires.

We need to find P(A | R), P(B | R) and P(C | R).

We will do this problem both by using a tree diagram and also by using Bayes' theorem.

We draw a tree diagram.



The probability P(A | R), for example, is a fraction whose denominator is the sum of all probabilities of all branches of the tree that result in an appliance that needs repair before the warranty expires, and the numerator is the branch that is associated with Manufacturer A. P(B | R) and P(C | R) are found in the same way.

$$P(A|R) = \frac{.030}{(.030)+(.015)+(.008)} = \frac{.030}{.053} = .566$$
$$P(B|R) = \frac{.015}{.053} = .283 \text{ and } P(C|R) = \frac{.008}{.053} = .151$$

Alternatively, using Bayes' theorem,

$$P(A|R) = \frac{P(R|A)P(A)}{P(R|A)P(A)+P(R|B)P(B)+P(R|C)P(C)}$$
$$= \frac{.030}{(.030)+(.015)+(.008)} = \frac{.030}{.053} = .566$$

P(B | R) and P(C | R) can be determined in the same manner.

P(B | R) = $\frac{.015}{.053} = .283$ and P(C | R) = $\frac{.008}{.053} = .151$.

Example 8.5.3

There are five American Furniture Warehouse stores in the Denver area. The percent of employees over the age of 50 is given in the table below.

Store Number	Number of Employees	Percent of 50+ Employees
1	300	.40
2	150	.65
3	200	.60
4	250	.50
5	100	.70
	Total = 1000	

If an employee chosen at random is over 50 years old, what is the probability that the employee works at store III?

Solution

Let k = 1, 2, . . . , 5 be the event that the employee worked at store k, and A be the event that the employee is over 50 years old. Since there are a total of 1000 employees at the five stores,

$$P(1) = .30 \quad P(2) = .15 \quad P(3) = .20 \quad P(4) = .25 \quad P(5) = .10$$

Using Bayes' theorem,

$$P(3|A) = \frac{P(A|3)P(3)}{P(A|1)P(1)+P(A|2)P(2)+P(A|3)P(3)+P(A|4)P(4)+P(A|5)P(5)}$$
$$= \frac{(.60)(.20)}{(.40)(.30)+(.65)(.15)+(.60)(.20)+(.50)(.25)+(.70)(.10)}$$
$$= .2254$$

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8.5E: Exercises - Bayes' Formula

PROBLEM SET: BAYES' FORMULA

1. Jar I contains five red and three white marbles, and Jar II contains four red and two white marbles. A jar is picked at random and a marble is drawn. Draw a tree diagram below, and find the following probabilities.
   a. P(marble is red)
   b. P(It came from Jar II | marble is white)
   c. P(Red | Jar I)

2. In Mr. Symons' class, if a student does homework most days, the chance of passing the course is 90%. On the other hand, if a student does not do homework most days, the chance of passing the course is only 20%.
   H = event that the student did homework
   C = event that the student passed the course
   Mr. Symons claims that 80% of his students do homework on a regular basis. If a student is chosen at random from Mr. Symons' class, find the following probabilities.
   a. P(C)
   b. P(H|C)
   c. P(C|H)

3. A city has 60% Democrats, and 40% Republicans. In the last mayoral election, 60% of the Democrats voted for their Democratic candidate while 95% of the Republicans voted for their candidate. Which party's mayor runs city hall?

4. In a certain population of 48% men and 52% women, 56% of the men and 8% of the women are color-blind.
   a. What percent of the people are color-blind?
   b. If a person is found to be color-blind, what is the probability that the person is a male?

5. A test for a certain disease gives a positive result 95% of the time if the person actually carries the disease. However, the test also gives a positive result 3% of the time when the individual is not carrying the disease. It is known that 10% of the population carries the disease. If a person tests positive, what is the probability that he or she has the disease?

6. A person has two coins: a fair coin and a two-headed coin. A coin is selected at random, and tossed. If the coin shows a head, what is the probability that the coin is fair?

7. A computer company buys its chips from three different manufacturers. Manufacturer I provides 60% of the chips and is known to produce 5% defective; Manufacturer II supplies 30% of the chips and makes 4% defective; while the rest are supplied by Manufacturer III with 3% defective chips. If a chip is chosen at random, find the following probabilities:
   a. P(the chip is defective)
   b. P(chip is from Manufacturer II | defective)
   c. P(defective |chip is from manufacturer III)

8. Lincoln Union High School District is made up of three high schools: Monterey, Fremont, and Kennedy, with an enrollment of 500, 300, and 200, respectively. On a given day, the percentage of students absent at Monterey High School is 6%, at Fremont 4%, and at Kennedy 5%. If a student is chosen at random, find the probabilities below: *Hint: Convert the enrollments into percentages.*
   a. P(the student is absent)
   b. P(student is from Kennedy | student is absent)
   c. P(student is absent | student is from Fremont)

9. At a retail store, 20% of customers use the store's online app to assist them when shopping in the store ; 80% of store shoppers don't use the app.
Of those customers that use the online app while in the store, 50% are very satisfied with their purchases, 40% are moderately satisfied, and 10% are dissatisfied.
Of those customers that do not use the online app while in the store, 30% are very satisfied with their purchases, 50% are moderately satisfied and 20% are dissatisfied.
Indicate the events by the following:
A = shopper uses the app in the store
N = shopper does not use the app in the store
V = very satisfied with purchase
M = moderately satisfied
D = dissatisfied
a. Find P(A and D), the probability that a store customer uses the app and is dissatisfied
b. Find P(A|D), the probability that a store customer uses the app if the customer is dissatisfied.

10. A medical clinic uses a pregnancy test to confirm pregnancy in patients who suspect they are pregnant. Historically data has shown that overall, 70% of the women at this clinic who are given the pregnancy test are pregnant, but 30% are not.
The test's manufacturer indicates that if a woman is pregnant, the test will be positive 92% of the time.
But if a woman is not pregnant, the test will be positive only 2% of the time and will be negative 98% of the time.
a. Find the probability that a woman at this clinic is pregnant and tests positive.
b. Find the probability that a woman at this clinic is actually pregnant given that she tests positive.

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8.6: Independent Events

Learning Objectives

In this section, you will learn to:

- Define independent events.
- Identify whether two events are independent or dependent.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. A card is drawn from a deck. Find the conditional probability of P(a queen | a face card).

Click here to check your answer

$\frac{1}{3}$

If you missed this problem, review Section 8.4 . (Note that this will open in a new window.)

2. A card is drawn from a deck. Find the conditional probability of P(a queen | a club).

Click here to check your answer

$\frac{1}{13}$

If you missed this problem, review Section 8.4. (Note that this will open in a new window.)

3. A die is rolled. Find the conditional probability that it shows a three if it is known that an odd number has shown.

Click here to check your answer

$\frac{1}{3}$

If you missed this problem, review Section 8.4. (Note that this will open in a new window.)

4. If $P(A)$ = .3 , $P(B)$ = .4, $P(A \cap B)$ = .12, find:

a. $P(A|B)$

b. $P(B|A)$

Click here to check your answer

a. .3

b. .4

If you missed this problem, review Section 8.4. (Note that this will open in a new window.)

In the last section, we considered conditional probabilities. In some examples, the probability of an event changed when additional information was provided. This is not always the case. The additional information may or may not alter the probability of the event.

In Example 8.6.1 we revisit the discussion at the beginning of the previous section and then contrast that with Example 8.6.2.

Example 8.6.1

A card is drawn from a deck. Find the following probabilities.

a. The card is a king.
b. The card is a king given that the card is a face card.

Solution

a. Clearly, P(The card is a king) = 4/52 = 1/13.

b. To find P(The card is a king | The card is a face card), we reason as follows:

There are 12 face cards in a deck of cards. There are 4 kings in a deck of cards.

$$P(\text{The card is a king} \mid \text{The card is a face card}) = 4/12 = 1/3.$$

The reader should observe that in the above example,

$$P(\text{The card is a king} \mid \text{The card is a face card}) \neq P(\text{The card is a king})$$

In other words, the additional information, knowing that the card selected is a face card changed the probability of obtaining a king.

Example 8.6.2

A card is drawn from a deck. Find the following probabilities.

a. The card is a king.
b. The card is a king given that a red card has shown.

Solution

a. Clearly, P(The card is a king) = 4/52 = 1/13.

b. To find P(The card is a king | A red card has shown), we reason as follows:

Since a red card has shown, there are only twenty six possibilities. Of the 26 red cards, there are two kings. Therefore,

$$P(\text{The card is a king} \mid \text{A red card has shown}) = 2/26 = 1/13.$$

The reader should observe that in the above example,

$$P(\text{The card is a king} \mid \text{A red card has shown}) = P(\text{The card is a king})$$

In other words, the additional information, a red card has shown, did not affect the probability of obtaining a king.

Whenever the probability of an event E is not affected by the occurrence of another event F, and vice versa, we say that the two events E and F are **independent.** This leads to the following definition.

Definition: Independent

Two Events E and F are **independent** if and only if at least one of the following two conditions is true.

1. $\mathbf{P(E|F) = P(E)}$ or
2. $\mathbf{P(F|E) = P(F)}$

If one of these conditions is true, then both are true.

If the events are not independent, then they are dependent.

We can use the definition of independence to determine if two events are independent.

We can use that definition to develop another way to test whether two events are independent.

Recall the conditional probability formula:

$$\mathrm{P(E|F)} = \frac{\mathrm{P(E \cap F)}}{\mathrm{P(F)}}$$

Multiplying both sides by $\mathrm{P(F)}$, we get

$$\mathrm{P(E \cap F) = P(E|F)P(F)}$$

Now if the two events are independent, then by definition

$$\mathrm{P(E|F) = P(E)}$$

Substituting, $P(E \cap F) = P(E)P(F)$

We state it formally as follows.

Test For Independence

Two events E and F are independent if and only if

$$\mathbf{P(E \cap F) = P(E)P(F)}$$

In the Examples $8.6.3$ and $8.6.4$, we'll examine how to check for independence using both methods:

- Examine the probability of intersection of events to check whether $P(E \cap F) = P(E)P(F)$
- Examine conditional probabilities to check whether $P(E|F) = P(E)$ or $P(F|E) = P(F)$

We need to use only **one** of these methods. Both methods, if used properly, will always give results that are consistent with each other.

Use the method that seems easier based on the information given in the problem.

Example 8.6.3

The table below shows the distribution of color-blind people by gender, based on 100 survey participants.

	Male(M)	Female(F)	Total
Color-Blind(C)	6	1	7
Not Color-Blind(N)	46	47	93
Total	52	48	100

where M represents male, F represents female, C represents color-blind, and N represents not color-blind. Are the events color-blind and male independent?

Solution 1: According to the test for independence, C and M are independent if and only if $\mathrm{P(C \cap M) = P(C)P(M)}$.

From the table: $P(C)$ = 7/100, $P(M)$ = 52/100 and $P(C \cap M)$ = 6/100

So $P(C)P(M)$ = (7/100)(52/100) = .0364

which is **not** equal to $P(C \cap M)$ = 6/100 = .06

Therefore, the two events are not independent. We may say they are dependent.

Solution 2: C and M are independent if and only if $P(C|M) = P(C)$.

From the total column $P(C)$ = 7/100 = 0.07

From the male column $P(C|M)$ = 6/52= 0.1154

Therefore $P(C|M) \neq P(C)$, indicating that the two events are not independent.

Example 8.6.4

In a city with two airports, 100 flights were surveyed. 20 of those flights departed late.

- 45 flights in the survey departed from airport A; 9 of those flights departed late.
- 55 flights in the survey departed from airport B; 11 flights departed late.

Are the events "depart from airport A" and "departed late" independent?

Solution 1

Let A be the event that a flight departs from airport A, and L the event that a flight departs late. We have

$$P(A \cap L) = 9/100,\ P(A) = 45/100 \text{ and } P(L) = 20/100$$

In order for two events to be independent, we must have $P(A \cap L) = P(A)P(L)$

Since $P(A \cap L) = 9/100 = 0.09$

and $P(A)P(L) = (45/100)(20/100) = 900/10000 = 0.09$

the two events "departing from airport A" and "departing late" are independent.

Solution 2

The definition of independent events states that two events are independent if $P(E|F) = P(E)$.

In this problem we are given that

$$P(L|A) = 9/45 = 0.2 \text{ and } P(L) = 20/100 = 0.2$$

$P(L|A) = P(L)$, so events "departing from airport A" and "departing late" are independent.

Example 8.6.5

A coin is tossed three times, and the events E, F and G are defined as follows:

E: The coin shows a head on the first toss.

F: At least two heads appear.

G: Heads appear in two successive tosses.

Determine whether the following events are independent.

a. E and F
b. F and G
c. E and G

Solution

We list the sample space, the events, their intersections and the probabilities.

$$\begin{aligned}
&\mathrm{S} = \{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{HTT}, \mathrm{THH}, \mathrm{THT}, \mathrm{TTH}, \mathrm{TTT}\\
&\mathrm{E} = \{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{HTT}\}, \quad \mathrm{P(E)} = 4/8 \text{ or } 1/2\\
&\mathrm{F} = \{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{THH}\}, \quad \mathrm{P(F)} = 4/8 \text{ or } 1/2\\
&\mathrm{G} = \{\mathrm{HHT}, \mathrm{THH}\}, \quad \mathrm{P(G)} = 2/8 \text{ or } 1/4\\
&\mathrm{E} \cap \mathrm{F} = \{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}\}, \quad \mathrm{P(E} \cap \mathrm{F)} = 3/8\\
&\mathrm{F} \cap \mathrm{G} = \{\mathrm{HHT}, \mathrm{THH}\}, \quad \mathrm{P(F} \cap \mathrm{G)} = 2/8 \text{ or } 1/4\\
&\mathrm{E} \cap \mathrm{G} = \{\mathrm{HHT}\} \quad \mathrm{P(E} \cap \mathrm{G)} = 1/8
\end{aligned}$$

a. E and F will be independent if and only if $P(E \cap F) = P(E)P(F)$

$$P(E \cap F) = 3/8 \text{ and } P(E)P(F) = 1/2 \cdot 1/2 = 1/4.$$

Since 3/8 ≠ 1/4, we have $P(E \cap F) \neq P(E)P(F)$.

Events E and F are not independent.

b. F and G will be independent if and only if $P(F \cap G) = P(F)P(G)$.

$$P(F \cap G) = 1/4 \text{ and } P(F)P(G) = 1/2 \cdot 1/4 = 1/8.$$

Since 3/8 ≠ 1/4, we have $P(F \cap G) \neq P(F)P(G)$.

Events F and G are not independent.

c. E and G will be independent if $P(E \cap G) = P(E)P(G)$

$$P(E \cap G) = 1/8 \text{ and } P(E)P(G) = 1/2 \cdot 1/4 = 1/8$$

Events E and G are independent events because $P(E \cap G) = P(E)P(G)$

Example 8.6.6

The probability that Jaime will visit his aunt in Baltimore this year is .30, and the probability that he will go river rafting on the Colorado river is .50. If the two events are independent, what is the probability that Jaime will do both?

Solution

Let A be the event that Jaime will visit his aunt this year, and R be the event that he will go river rafting.

We are given $P(A)$ = .30 and $P(R)$ = .50, and we want to find $P(A \cap R)$.

Since we are told that the events A and R are independent,

$$P(A \cap R) = P(A)P(R) = (.30)(.50) = .15$$

Example 8.6.7

Given $P(B|A) = .4$. If A and B are independent, find $P(B)$.

Solution

If A and B are independent, then by definition $P(B|A) = P(B)$

Therefore, $P(B) = .4$

Example 8.6.8

Mark's probability of passing physics is 80%. David's probability of passing the same course is 70%. If the two events are independent, find the following probabilities.

a. P(Both of them will pass physics)

b. P(At least one of them will pass physics)

Solution

Let M be the event that Mark passes physics and let D be the event that David passes physics.

a. The probability that they both pass physics is represented by $P(M \cap D)$.

Since the events are independent,

$P(M \cap D) = P(M)P(D) = (.80)(.70) = 0.56$

The probability they will both pass physics is 56%.

b. The probability that at least one of them will pass physics is represented by $P(M \cup D)$.

Using the Addition Rule,

$P(M \cup D) = P(M) + P(D) - P(M \cap D)$

$P(M \cup D) = .80 + .70 - .56 = 0.94$

> The probability that at least one of them will pass physics is 94%.

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8.6E: Exercises - Independent Events

PROBLEM SET: INDEPENDENT EVENTS

The distribution of the number of fiction and non-fiction books checked out at a city's main library and at a smaller branch on a given day is as follows.

	MAIN (M)	BRANCH (B)	TOTAL
FICTION (F)	300	100	400
NON-FICTION (N)	150	50	200
TOTALS	450	150	600

Use this table to determine the following probabilities:

1. $P(F)$	2. $P(M\|F)$
3. $P(N\|B)$	4. Is the fact that a person checks out a fiction book independent of the main library? Use probabilities to justify your conclusion.

For a two-child family, let the events E, F, and G be as follows.

E: The family has at least one boy
F: The family has children of both sexes
G: The family's first born is a boy

5. Find the following. a. $P(E)$ b. $P(F)$ c. $P(E \cap F)$ d. Are E and F independent? Use probabilities to justify your conclusion.	6. Find the following. a. $P(F)$ b. $P(G)$ c. $P(F \cap G)$ d. Are F and G independent? Use probabilities to justify your conclusion.

Do the following problems involving independence.

7. If $P(E) = .6$, $P(F) = .2$, and E and F are independent, find $P(E$ and $F)$.	8. If $P(E) = .6$, $P(F) = .2$, and E and F are independent, find $P(E$ or $F)$.
9. If $P(E) = .9$, $P(F\|E) = .36$, and E and F are independent, find $P(F)$.	10. If $P(E) = .6$, $P(E$ or $F)$ = .8, and E and F are independent, find $P(F)$.
11. In a survey of 100 people, 40 were casual drinkers, and 60 did not drink. Of the ones who drank, 6 had minor headaches. Of the non-drinkers, 9 had minor headaches. Are the events "drinkers" and "had headaches" independent?	12. It is known that 80% of the people wear seat belts, and 5% of the people quit smoking last year. If 4% of the people who wear seat belts quit smoking, are the events, wearing a seat belt and quitting smoking, independent?
13. John's probability of passing statistics is 40%, and Linda's probability of passing the same course is 70%. If the two events are independent, find the following probabilities. a. $P($ both of them will pass statistics) b. $P($at least one of them will pass statistics)	14. Jane is flying home for the Christmas holidays. She has to change planes twice. There is an 80% chance that she will make the first connection, and a 90% chance that she will make the second connection. If the two events are independent, find the probabilities: a. $P($ Jane will make both connections) b. $P($Jane will make at least one connection)

For a three-child family, let the events E, F, and G be as follows.

E: The family has at least one boy
F: The family has children of both sexes
G: The family's first born is a boy

15. Find the following.
    a. $P(E)$
    b. $P(F)$
    c. $P(E \cap F)$
    d. Are E and F independent?

16. Find the following.
    a. $P(F)$
    b. $P(G)$
    c. $P(F \cap G)$
    d. Are F and G independent?

17. $P(K|D) = 0.7$, $P(D) = 0.25$ and $P(K) = 0.7$
    a. Are events K and D independent? Use probabilities to justify your conclusion.
    b. Find $P(K \cap D)$

18. $P(R|S) = 0.4$, $P(S) = 0.2$ and $P(R) = 0.3$
    a. Are events R and S independent? Use probabilities to justify your conclusion.
    b. Find $P(R \cap S)$

19. At a college:
    54% of students are female
    25% of students are majoring in engineering.
    15% of female students are majoring in engineering.
    Event E = student is majoring in engineering
    Event F = student is female
    a. Are events E and F independent? Use probabilities to justify your conclusion.
    b. Find $P(E \cap F)$

20. At a college:
    54% of all students are female
    60% of all students receive financial aid.
    60% of female students receive financial aid.
    Event A = student receives financial aid
    Event F = student is female
    a. Are events A and F independent? Use probabilities to justify your conclusion.
    b. Find $P(A \cap F)$

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8.7: Expected Value

Learning Objectives

In this section, you will learn to:

- Find the expected value of a discrete probability distribution.
- Interpret expected value as a long-run average.

Prerequisite Skills

Before you get started, take this prerequisite quiz.

1. A jar contains 6 red, 7 white, and 7 blue marbles. If one marble is chosen at random, find the following probabilities.

a. P(red)

b. P(white)

Click here to check your answer

a. $\frac{3}{10}$

b. $\frac{7}{20}$

If you missed this problem, review Section 8.1. (Note that this will open in a new window.)

2. Two dice are rolled. Find the following probabilities.

a. P(the sum of the dice is 5)

b. P(the sum of the dice is 8)

Click here to check your answer

a. $\frac{1}{9}$

b. $\frac{5}{36}$

If you missed this problem, review Section 8.1. (Note that this will open in a new window.)

An expected gain or loss in a game of chance is called **Expected Value**. The concept of expected value is closely related to a *weighted average*. Consider the following situations.

1. Suppose you and your friend play a game that consists of rolling a die. Your friend offers you the following deal: If the die shows any number from 1 to 5, he will pay you the face value of the die in dollars, that is, if the die shows a 4, he will pay you $4. But if the die shows a 6, you will have to pay him $18.

Before you play the game you decide to find the expected value. You analyze as follows.

Since a die will show a number from 1 to 6, with an equal probability of 1/6, your chance of winning $1 is 1/6, winning $2 is 1/6, and so on up to the face value of 5. But if the die shows a 6, you will lose $18. You write the expected value.

$$\mathrm{E} = \$1(1/6) + \$2(1/6) + \$3(1/6) + \$4(1/6) + \$5(1/6) - \$18(1/6) = -\$.50$$

This means that every time you play this game, you can expect to lose 50 cents. In other words, if you play this game 100 times, theoretically you will lose $50. Obviously, it is not to your interest to play.

2. Suppose of the ten quizzes you took in a course, on eight quizzes you scored 80, and on two you scored 90. You wish to find the average of the ten quizzes.

The average is

$$\mathrm{A} = \frac{(80)(8)+(90)(2)}{10} = (80)\frac{8}{10} + (90)\frac{2}{10} = 82$$

It should be observed that it would be incorrect to take the average of 80 and 90 because you scored 80 on eight quizzes, and 90 on only two of them. Therefore, you take a "weighted average" of 80 and 90. That is, the average of 8 parts of 80 and 2 parts of 90, which is 82.

In the first situation, to find the expected value, we multiplied each payoff by the probability of its occurrence, and then added up the amounts calculated for all possible cases. In the second example, if we consider our test score a payoff, we did the same. This leads us to the following definition.

Expected Value

If an experiment has the following probability distribution,

Payoff	x_1	x_2	x_3	$\ldots$	x_n
Probability	$P(x_1)$	$P(x_2)$	$P(x_3)$	$\ldots$	$P(x_n)$

then the expected value of the experiment is

$$\text{Expected Value } = x_1 P(x_1) + x_2 P(x_2) + x_3 P(x_3) + \cdots + x_n P(x_n)$$

Example 8.7.1

In a town, 10% of the families have three children, 60% of the families have two children, 20% of the families have one child, and the rest of the families have no children. What is the expected number of children in a family?

Solution

We list the information in the following table.

Number of children	3	

Probability	0.10

1-(0.10+0.60+0.20)=0.10

Expected Value = $x_1P(x_1)+x_2P(x_2)+x_3P(x_3)+x_4P(x_4)$

$E=3(.10)+2(.60)+1(.20)+0(.10)=1.7$

So on average, there are 1.7 children to a family.

Example 8.7.2

To sell an average house, a real estate broker spends $1200 for advertisement expenses. If the house sells in three months, the broker makes $8,000. Otherwise, the broker loses the listing. If there is a 40% chance that the house will sell in three months, what is the expected payoff for the real estate broker?

Solution

The broker makes $8,000 with a probability of .40, but he loses $1200 whether the house sells or not.

$$E=(\$8000)(.40)-(\$1200)=\$2,000.$$

Alternatively, the broker makes $(8000 - 1200) with a probability of .40, but loses $1200 with a probability of .60. Therefore,

$$E=(\$6800)(.40)-(\$1200)(.60)=\$2,000.$$

Example 8.7.3

In a town, the attendance at a football game depends on the weather. On a sunny day the attendance is 60,000, on a cold day the attendance is 40,000, and on a stormy day the attendance is 30,000. If for the next football season, the weatherman has predicted that 30% of the days will be sunny, 50% of the days will be cold, and 20% days will be stormy, what is the expected attendance for a single game?

Solution

Using the expected value formula, we get

$$E=(60,000)(.30)+(40,000)(.50)+(30,000)(.20)=44,000.$$

Example 8.7.4

A lottery consists of choosing 6 numbers from a total of 51 numbers. The person who matches all six numbers wins $2 million. If the lottery ticket costs $1, what is the expected payoff?

Solution

Since there are $51C6 = 18,009,460$combinations of six numbers from a total of 51 numbers, the chance of choosing the winning number is 1 out of 18,009,460.

So the expected payoff is: $\mathrm{E} = (\$2 \text{ million }) \left(\frac{1}{18009460}\right) - \$1 = -\$0.89$

This means that every time a person spends $1 to buy a ticket, he or she can expect to lose 89 cents.

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8.7E: Exercises - Expected Value

PROBLEM SET: EXPECTED VALUE

Do the following problems using the expected value concepts learned in this section,

1. You are about to make an investment which gives you a 30% chance of making $60,000 and 70% chance of losing $ 30,000. Should you invest? Explain.
2. In a town, 40% of the men and 30% of the women are overweight. If the town has 46% men and 54% women, what percent of the people are overweight?
3. A game involves rolling a Korean die (4 faces). If a one, two, or three shows, the player receives the face value of the die in dollars, but if a four shows, the player is obligated to pay $4. What is the expected value of the game?
4. A game involves rolling a single die. One receives the face value of the die in dollars. How much should one be willing to pay to roll the die to make the game fair?
5. In a European country, 20% of the families have three children, 40% have two children, 30% have one child, and 10% have no children. On average, how many children are there to a family?
6. A game involves drawing a single card from a standard deck. One receives 60 cents for an ace, 30 cents for a king, and 5 cents for a red card that is neither an ace nor a king. If the cost of each draw is 10 cents, should one play? Explain.
7. Hillview Church plans to raise money by raffling a television worth $500. A total of 3000 tickets are sold at $1 each. Find the expected value of the winnings for a person who buys a ticket in the raffle.
8. During her four years at college, Niki received A's in 30% of her courses, B's in 60% of her courses, and C's in the remaining 10%. If A = 4, B = 3, and C = 2, find her grade point average.
9. Attendance at a Stanford football game depends upon which team Stanford is playing against. If the game is against U. C. Berkeley,attendance will be 70,000; if it is against another California team, it will be 40,000; and if it is against an out of state team, it will be 30,000. If the probability of playing against U. C. Berkeley is 10%, against a California team 50% and against an out of state team 40%, how many fans are expected to attend a game?
10. A Texas oil drilling company has determined that it costs $25,000 to sink a test well. If oil is hit, the revenue for the company will be $500,000. If natural gas is found, the revenue will be $150,000. If the probability of hitting oil is 3% and of hitting gas is 6%, find the expected value of sinking a test well.
11. A $1 lottery ticket offers a grand prize of $10,000; 10 runner-up prizes each pay $1000; 100 third-place prizes each pay $100; and 1,000 fourth-place prizes each pay $10. Find the expected value of entering this contest if 1 million tickets are sold.
12. Assume that for the next heavyweight fight the odds of current champion winning are 15 to 2. A gambler bets $10 that the current champion will lose. If current champion loses, how much can the gambler hope to receive?
13. In a housing development, 35% of households have no school age children, 20% of households have 1 school age child, 25% of households have 2 school age children, 15% have 3, and 5% have 4 school age children.
    a. Find the average number of children per household
    b. If there are 300 homes in this housing development, what is the total number of children expected to attend school?
14. At a large community college, 30% of students take one course, 15% take two courses, 25% take three courses and 20% take four courses. The rest of the students take five courses.
    a. What percent of students take 5 courses?
    b. Find the average number of courses that students take.

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8.8: Chapter 8 Review

SECTION 8.8 PROBLEM SET: CHAPTER REVIEW

Sample Spaces and Probability (8.1)

1. Two dice are rolled. Find the probability that the sum of the dice is
   a. four
   b. five
2. A card is drawn from a standard deck. Find the following probabilities:
   a. P(a jack or a king)
   b. P(a jack or a spade)
3. A jar contains 3 red, 4 white, and 5 blue marbles. If a marble is chosen at random, find the following probabilities:
   a. P(red or blue)
   b. P(not blue)
4. A family has four children. Find the following probabilities:
   a. P(All boys)
   b. P(1 boy and 3 girls)

Mutually Exclusive Events and the Addition Rule (8.2)

1. At a college, twenty percent of the students take history, thirty percent take math, and ten percent take both. What percent of the students take at least one of these two courses?
2. A student feels that her probability of passing accounting is .62, of passing mathematics is .45, and her passing accounting or mathematics is .85. Find the probability that the student passes both accounting and math.
3. If $P(E) = .6$ and $P(F) = .4$ and E and F are mutually exclusive, find P(E and F).

Probability Using Tree Diagrams and Combinations (8.3)

1. A basket contains 3 red and 2 yellow apples. Two apples are chosen at random. Find the following probabilities:
   a. P(one red, one yellow)
   b. P(at least one red)
2. Jar I contains 1 red and 3 white, and Jar II contains 2 red and 3 white marbles. A marble is drawn from Jar I and put in Jar II. Now if one marble is drawn from Jar II, what is the probability that it is a red marble?
3. Five cards are drawn from a deck. Find the probability of obtaining
   a. four cards of a single suit
   b. two cards of one suit, two of another suit, and one from the remaining
   c. a pair(e.g. two aces and three other cards)
   d. a straight flush(five in a row of a single suit but not a royal flush)
4. A trade delegation consists of four Americans, three Japanese and two Germans. Three people are chosen at random. Find the following probabilities:
   a. P(two Americans and one Japanese)
   b. P(at least one American)
   c. P(One of each nationality)
   d. P(no German)
5. Suppose that 60% of the voters in California intend to vote Democratic in the next election. If we choose five people at random, what is the probability that at least four will vote Democratic?
6. A basketball player has a .70 chance of sinking a basket on a free throw. What is the probability that he will sink at least 4 baskets in six shots?

Conditional Probability (8.4)

1. If $P(E) = .4$ and $P(F|E) = .5$, find P(E and F).
2. If $P(E) = .6$ and P(E and F) = .3, find $P(F|E)$.

3. Consider a family of three children. Find the following:
   a. P(children of both sexes | first born is a boy)
   b. P(all girls | children of both sexes)
4. A college has found that 20% of its students take advanced math courses, 40% take advanced English courses and 15% take both advanced math and advanced English courses. If a student is selected at random, what is the probability that
   a. the student is taking English given that they are taking math?
   b. the student is taking math or English?
5. At a clothing outlet 20% of the clothes are irregular, 10% have at least a button missing and 4% are both irregular and have a button missing. If Martha found a dress that has a button missing, what is the probability that it is irregular?

Bayes Formula (8.5)

1. Suppose a test is given to determine if a person is infected with HIV. If a person is infected with HIV, the test will detect it in 90% of the cases; and if the person is not infected with HIV, the test will show a positive result 3% of the time. If we assume that 2% of the population is actually infected with HIV, what is the probability that a person obtaining a positive result is actually infected with HIV?
2. Two machines make all the products in a factory, with the first machine making 30% of the products and the second 70%. The first machine makes defective products 3% of the time and the second machine 5% of the time.
   a. Overall what percent of the products made are defective?
   b. If a defective product is found, what is the probability that it was made on the second machine?
3. An instructor in a finite math course estimates that a student who does his homework has a 90% of chance of passing the course, while a student who does not do the homework has only a 20% chance of passing the course. It has been determined that 60% of the students in a large class do their homework.
   a. What percent of all the students will pass?
   b. If a student passes, what is the probability that he did the homework?

Independent Events (8.6)

1. If $P(E) = .5$ and $P(F) = .3$ and E and F are independent, find $P(E \cup F)$.
2. If $P(F) = .9$ and $P(E|F) = .36$ and E and F are independent, find $P(E)$.
3. If $P(E) = .4$ and P(E or F) =.9 and E and F are independent, find $P(F)$.
4. If $P(E) = .3$ and $P(F) = .4$ and E and F are independent, find $P(E|F)$.
5. The following table shows a distribution of drink preferences by gender.

	Coke(C)	Pepsi(P)	Seven Up(S)	TOTALS
Male(M)	60	50	22	132
Female(F)	50	40	18	108
TOTALS	110	90	40	240

The events M, F, C, P and S are defined as Male, Female, Coca Cola, Pepsi, and Seven Up, respectively. Find the following:

   a. P(F | S)
   b. P(P | F)
   c. P(C | M)
   d. P(M | P $\cup$ C)
   e. Are the events F and S mutually exclusive?
   f. Are the events F and S independent?

Expected Value (8.7)

1. A stock has a 50% chance of a 10% gain, a 30% chance of no gain, and otherwise it will lose 8%. Find the expected return.
2. In an evening finite math class of 30 students, it was discovered that 5 students were of age 20, 8 students were about 25 years old, 10 students were close to 30, 4 students were 35, 2 students were 40 and one student 55. What is the average age of a student in this class?

3. Jar I contains 4 marbles of which one is red, and Jar II contains 6 marbles of which 3 are red. Katy selects a jar and then chooses a marble. If the marble is red, she gets paid 3 dollars, otherwise she loses a dollar. If she plays this game ten times, what is her expected payoff?

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