MATH 373: TRIGONOMETRY FOR CALCULUS



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Math 373: Trigonometry for Calculus (Simpson)

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Foreword to the Instructor

The current approach to teaching mathematics at the undergraduate level is akin to taking the student to a vast ocean that is only three inches deep. The breadth of the water represents the number of topics, and the depth of the water is how much we delve into each topic. Now, come with me.

Welcome to my 30-foot-deep pond.

Sources for this textbook

The content for this textbook is pulled from, and influenced by, three major sources.

Yoshiwara's *Trigonometry*

At its foundation, the textbook you are currently reading has been built from Katherine Yoshiwara's *Trigonometry* textbook. She has graciously offered her text to the education community as an OER (Open Educational Resource) to be used and modified in any fashion the educator sees fit. I chose to start with her textbook as a basis because of the measured approach and her wonderful descriptions.

Unfortunately, there were many decisions in the design of the Yoshiwara text for which I was not comfortable, and unraveling these design choices without rewriting the textbook was impossible. Some examples include:

- Waiting until halfway through the textbook to introduce radian measure
- Graphing trigonometric functions using *only* degree measure
- Waiting until the middle of the textbook to introduce proofs in Trigonometry
- Introducing the Laws of Sines and Cosines immediately after the students' first experience with Trigonometry (via right triangle trigonometry)
- Never introducing half-angle identities, power reduction formulas, or the sum-to-product and product-to-sum identities

Despite these shortcomings, I truly like the *style* the Yoshiwara text brings to the mathematical landscape and, as such, it is the basis of this textbook.

OpenStax *Algebra and Trigonometry*

The folks over at OpenStax have done an admirable job at playing a leadership role in the OER community. I chose not to use their *Algebra and Trigonometry* textbook as a basis for this book due to voice and rigor (I have used their text in the past for Trigonometry and found that the levels and expectations were too low); however, I *did* supplement this text with some of their examples when it came to the trigonometric identities material—the Yoshiwara text was lacking in this area (literally not covering some of the major identities).

Stitz and Zeager's *Precalculus*

The Stitz and Zeager *Precalculus* is an OER textbook that treats mathematics instruction in a very classic, and rigorous, style. Since the Stitz and Zeager is a Precalculus textbook, the only content from that textbook used to augment the textbook you are currently reading comes from the final two chapters on Trigonometry. I could not use those two chapters as a basis of a single-course Trigonometry textbook due to the terseness of the text and the assumption of a pre-exposure to Trigonometry; however, I did incorporate some of their examples into this textbook. Additionally, I used their textbook to fill in the gaps that were missing from the Yoshiwara textbook (namely for sinusoidal applications).

McKeague and Turner's *Trigonometry*

Trigonometry by McKeague and Turner is completely copyrighted and, as such, there is nothing in this textbook directly from their textbook; however, of all the texts I have read covering Trigonometry, the McKeague and Turner textbook is the best (by far). As

such, this textbook is heavily influenced by their style and the *flow* of this text is close to their approach.

The Result

If you have read the previous information, then you might be concerned about the "voice" of the text sounding as if it has been written by five authors—this is not the case. I have rewritten the vast majority of the text, aligning notations, pedagogical styles, voice, and approach to my own. Moreover, at least a dozen sections that were missing (or not adequate) from the Yoshiwara text have been written solely by me.

Textbook Level

What are you looking for?

- Are you looking for a textbook that reads like a "how-to" book for solving math problems?
- Do you search long hours into the night for that one textbook that contains all the "recipes" so that students don't have to understand the math?
- Are you satisfied with the vast majority of algebra textbooks at the higher education level that are watered down?

Then, go no further! Literally... **GO NO FURTHER**... at least, not with this textbook. Close it now and move on to anything else. If I still have your interest, then please read on.

The decline of modern mathematics education

In modern day undergraduate math textbooks, it has become an increasing issue that content is being watered down and treated superficially so that more students can "get it." Unfortunately, this is having the opposite effect. Students have become professionals at mimicking the shallow procedures given them and are subsequently losing out on the logic-building skills mathematics has to offer. Content has been made to be rapid fire, with little-to-no depth of investigation. The effect of this shallow-depth treatment is that students give up quickly and are less likely to focus on truly understanding the process. Contrary to building a society of critical thinkers, we are engineering a culture of followers insistent upon knowing the shortcuts. Philosophical questions of why have been replaced with utilitarian queries of how. This is not to say that the application of mathematics is secondary to theoretical mathematics; however, instead of an integrated focus between the two approaches, many authors have chosen to throw out the theory for fear of overwhelming the student.

Adding fuel to this inferno of the uneducated is the lack of time we, as professors, have in the classroom with our students. It seems that every few years an hour or so is lost from our schedules while another topic is introduced to the curriculum. It has become too easy to give students the "recipe" for solving a problem and move on to the next topic all-the-while hoping that they can figure out *why* it works on their own.

A proposed reinvention—the 30-foot-deep pond

If the United States is to compete in the global market, we must somehow increase the level of creativity of our students. Rewarding perseverance and reconsidering how we have approached the treatment of mathematics is paramount. The days of holding back on number theory because it is "too abstract" must fall by the wayside. The frenetic pace of shoveling one chapter per week into a student's brain needs to be slowed; thus, allowing the student to begin a meaningful meander through the beautiful realm of mathematics. The development of concepts can no longer be piecemeal and staccato. Topics must be introduced in a logical fashion and a thorough exploration of the subject should be allowed. Chapters should not be designed to be "consumed" in a certain, short period of time. Instead, they should organically continue until their focus has been thoroughly explored.

We stand at a point in history where information flows to us in immense quantities and the possibilities for an educated society are staggering. However, in these days of "big-box store education," students are treated more like cattle than the possible leaders of our future. The days of sitting in the round and discussing topics from a philosophical standpoint are gone and the term "professor" has lost its meaning. We no longer profess—we regurgitate. Professing is left for late night YouTube channels dedicated to "educating" in 10-minute videos. The Socrates of today is a one-way conversation brought to you in high definition with limited commercial interruption. The word "educate" in media circles is synonymous with "entertain."

We must invest time in honoring the theory and history of mathematics while satiating a student's desire to see math in action. We should treat mathematics as an exploration in understanding rather than just a lesson in computation. We must do what we ask our students to do and think critically about why particular topics have been taught in certain courses. Only after detailed consideration of such issues should we redesign our curriculum in an intelligent manner that properly conveys the importance of mathematics in our daily lives.

The core concept behind the design of this textbook is encapsulated in the opening statement of this Foreword—"*welcome to my 30-foot-deep pond*." I firmly believe that the "deep pond" approach is the absolute best method to develop true understanding of the theory and usefulness, as well as the beauty, of mathematics. In this text, I treat each topic thoroughly and with such great detail that the student could be trusted to know not only how to compute trivial "math textbook" problems, but also how to apply their knowledge to the real world.

Prerequisites

Most sections have a list of prerequisite mathematical skills necessary for the student to understand and have mastered *prior* to understanding the material from that section. To keep the text concise, once a prerequisite is mentioned (say, in Section 1.2), it is no longer mentioned in future sections. As such, if you choose to cover the material in this text in an order that is different than what is presented, you're on your own when it comes to determining what material the students likely need as prerequisites for a given section.

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Templates

Core Prerequisites

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

Operational Prerequisites

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

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1: TRIANGLES AND CIRCLES

Circles and triangles are the most straightforward geometric figures. A circle is the simplest curve, and a triangle is the simplest polygon - the one with the fewest sides. Each has properties that make it useful in many fields of endeavor. A triangle is the most stable polygon because once its sides are fixed in length, its angles cannot change. A triangular truss bridge is a stable structure. A three-legged stool will not wobble.

Figure 1





A circle encloses more area than any figure of the same length or perimeter, and a sphere encloses more space. A geodesic dome is a portion of a sphere constructed with triangles. It has been called "the strongest, lightest, and most efficient means of enclosing space known to man."

Geodesic domes may also help us learn how to live on another planet. In 2017, the United Arab Emirates began construction on Mars Science City, a series of interconnecting geodesic domes designed to be a realistic model for living on Mars. The city will cover 1.9 million (1.9×10^6) square feet, and its walls will be 3D-printed using sand from the desert.

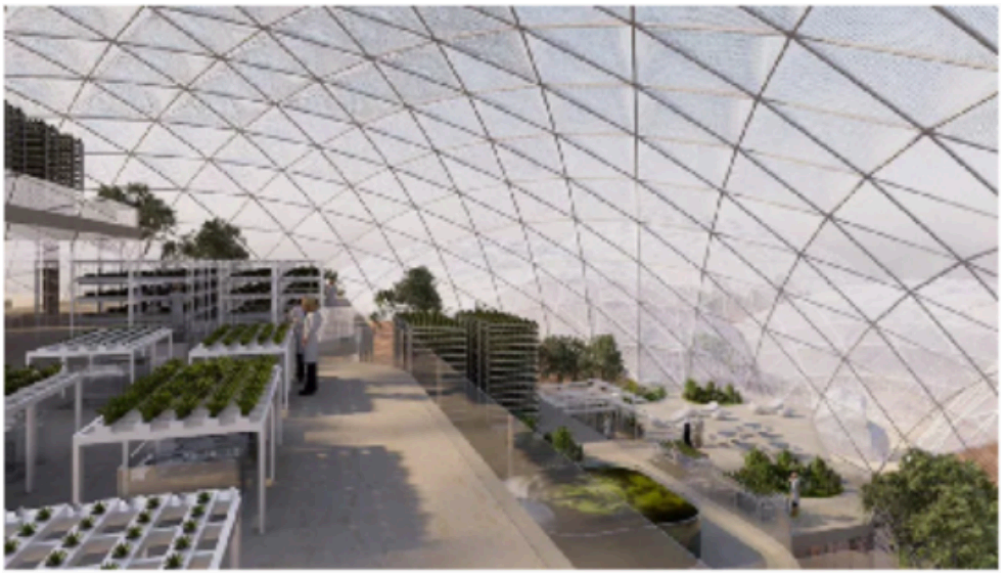
Figure 2



The city will contain laboratories to address food, sustainability, and energy issues worldwide. Finally, the project will implement

an experiment where a team will spend a year living on the simulated planet.

Figure 3



Activity: Properties of Triangles

Materials available at the end of this section.

A. What do we know about the sides of a triangle?
   1. a. Can you make a triangle with sides of length 2 inches, 3 inches, and 6 inches?
      b. Use the pieces of length 2 inches and 3 inches to form two sides of a triangle. What happens to the length of the third side as you increase the angle between the first two sides?
      c. What is the longest that the third side could be? What is the smallest?
   2. Two sides of a triangle are 6 centimeters and 8 centimeters long. What are the possible lengths of the third side?
   3. Two sides of a triangle are p units and q units long. What are the possible lengths of the third side?

B. What do we know about the angles of a triangle?
   1. Use a protractor to measure the three angles of the paper triangle in degrees. Now, add up the three angles. What is their sum?
   2. Tear off the three corners of the triangle. Place them side-by-side with their vertices (tips) at the same point. What do you find?
   3. How are your answers to parts (1) and (2) related?

C. How are the sides and angles of a triangle related?
   1. A standard way to label a triangle is to call the angles A, B, and C. The side opposite angle A is called a, the side opposite angle B is called b, and the side opposite angle C is called c. Sketch a triangle and label it with standard notation.
   2. Using a ruler, carefully draw a triangle and label it with standard notation so that $a > b > c$. Now, use a protractor to measure the angles and list them from largest to smallest. What do you observe?
   3. Using a ruler, carefully draw another triangle and label it with standard notation so that $A > B > C$. Now, use a ruler to measure the sides and list them in order from largest to smallest. What do you observe?

D. What do we know about right triangles?
   1. The **hypotenuse** is the side opposite the 90° angle in a right triangle. Why is the hypotenuse always the longest side of a right triangle?
   2. The **Pythagorean Theorem** states the following:
      IF: a, b, and c are the sides of a right triangle, and c is the hypotenuse $a^2 + b^2 = c^2$
      THEN: $a^2 + b^2 = c^2$
      The "if" part of the theorem is called the **hypothesis**, and the "then" part is called the **conclusion**. The **converse** of a theorem is the new statement you obtain when you interchange the hypothesis and the conclusion. Write the converse of the Pythagorean Theorem.

3. The converse of the Pythagorean Theorem is also true. Use the converse to decide whether each of the following triangles is a right triangle. Support your conclusions with calculations.
   a. $a = 9, b = 16, c = 25$
   b. $a = 12, b = 16, c = 20$
   c. $a = \sqrt{8}, b = \sqrt{5}, c = \sqrt{13}$
   d. $a = \frac{\sqrt{3}}{2}, b = \frac{1}{2}, c = 1$

1.1: Angles and Basic Geometry

1.1.1: Resources and Key Concepts

1.1.2: Homework

1.2: Triangles

1.2.1: Resources and Key Concepts

1.2.2: Homework

1.3: Similar Triangles

1.3.1: Resources and Key Concepts

1.3.2: Homework

1.4: Circles and Angles in the Rectangular Coordinate System

1.4.1: Resources and Key Concepts

1.4.2: Homework

Answers

Activity: Properties of Triangles

You will need paper and pencil, scissors, a ruler and compass, and two plastic straws.

Cut the straw into pieces of length 2 inches, 3 inches, and 6 inches. Use the ruler to draw a large triangle and cut it out.

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1.1: Angles and Basic Geometry

Section Preview

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Angles

Before we can jump into Trigonometry, we must define angles; however, before defining angles, we need to define what makes up the components of an angle.

Definition: Ray

A **ray** is a line segment that has one endpoint and extends infinitely in one direction. The point from which the ray originates is called the **initial point** of the ray. A ray can also be described as a **half-line**.



Figure 1.1.1**:** *A ray with initial point* P

Note that we label the initial point P. The choice of letter is arbitrary, so we could just have easily chosen Q, R, or S. The choice of using capital letters for points on a line or ray, however, is tradition and we stick with that convention in this text. We now move on to defining an angle.

Definition: Angle

When two rays share a common initial point, they form an **angle**, and the shared initial point is called the angle's **vertex**.

Figure 1.1.2 shows two examples of what are commonly considered as angles.



Figure 1.1.2: *An angle with vertex P (left) and an angle with vertex Q (right)*

Figure 1.1.3 also depicts angles—albeit these are, in some sense, extreme cases. In the first case, the two rays are directly opposite each other, forming what is known as a **straight angle**; in the second, the rays are identical, so the "angle" is indistinguishable from the ray itself.



Figure 1.1.3: *A straight angle (left) and a "closed" angle (right)*

Definition: Straight Angle

A **straight angle** is defined as an angle that is formed when the two rays forming the angle extend in opposite directions.

Now that we have an idea of what makes an angle, and a "beginner's notation," it's time we define the **measure of an angle**.

Definition: Angle Measure

The **measure of an angle** is a number that indicates the amount of rotation that separates the rays of the angle.

There is one immediate problem with this definition, as should be evident from Figure 1.1.4 below.



Figure 1.1.4: *The ambiguity of angles*

Which amount of rotation are we attempting to quantify?

We have just discovered that this diagram describes at least two angles. Clearly, these two angles have different measures because one appears to represent a larger rotation than the other, so we must label them differently.

In Trigonometry, we often use lowercase Greek letters to represent the measure of the angle. For example, we use the Greek letters α (alpha) and β (beta) as representations of the unknown angle measures in Figure 1.1.5.

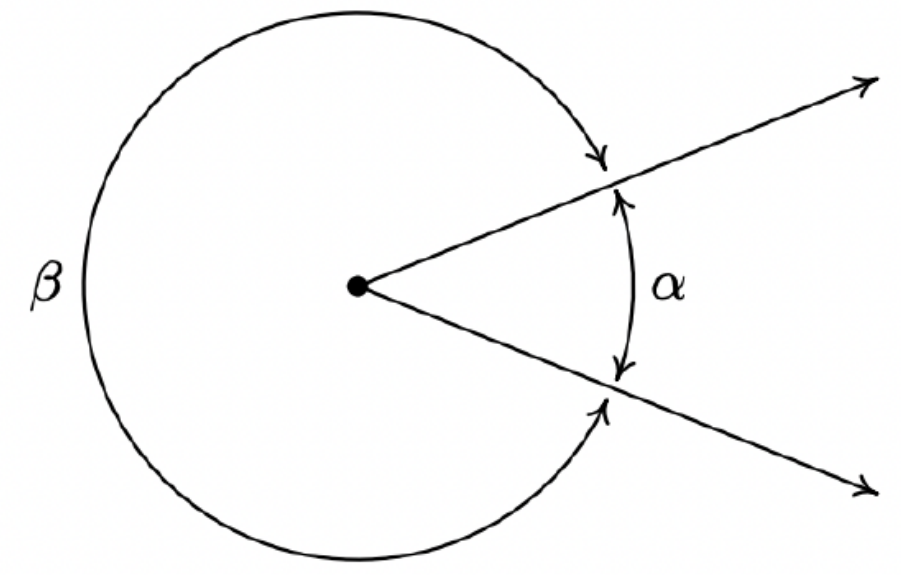


Figure 1.1.5

Identifying the lowercase Greek letters will be beneficial in this course. Table 1.1.1 summarizes the (lowercase) Greek alphabet. The letters in **bold** are the ones most commonly used within Trigonometry.

Table 1.1.1**:** *Lowercase Letters in the Greek Alphabet*

Greek Alphabet
[redacted]

Caution

The lowercase Greek letter π is *never* used as a symbolic representation of an *unknown* angle. π is reserved for the universal constant $\pi \approx 3.141592654$ (which is an **irrational number**).

Another ambiguity that sneaks into angular measure is which direction the angle is "opening." We need to extend our notion of "angle" from merely measuring an extent of rotation to quantities associated with real numbers. To that end, we introduce the concept of an **oriented angle**.

Definition: Oriented Angle

An **oriented angle** refers to the measure of rotation between two rays with a common vertex, taking into account the direction of rotation (clockwise or counterclockwise). The angle is swept out starting from an **initial side** and ending at a **terminal side**. When the rotation is counterclockwise from the initial side to the terminal side, we say that the angle is **positive**; when the rotation is clockwise, we say that the angle is **negative**.

As its name suggests, in an oriented angle, the direction of the rotation is essential.



Figure 1.1.6**:** *A positive angle (left) and a negative angle (right)*

Alternative Angle Notations

We have already mentioned that we will frequently use lowercase Greek letters when referencing angles; however, this is not the only convention that exists when referencing an angle. Angles may be labeled with a single letter at the vertex as long as it is clear that there is only one such angle. For example, in Figure 1.1.7 (below), it is clear that angle A is the vertex. There is no ambiguity about this fact. Therefore, an acceptable notation to reference this angle is $\angle A$ (some texts might use the notation $m\angle A$, meaning "the measure of angle A").

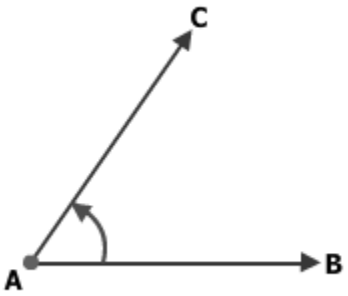


Figure 1.1.7

However, if there is the possibility for confusion, angles are labeled by specifying 3 points, with the center point being the angle's vertex. Therefore, we could reference the angle in Figure 1.1.7 as $\angle BAC$. The first letter indicates the initial side of the angle, the middle is the vertex, and the last letter indicates the terminal side. It is crucial that the center letter in this notation is the angle's vertex.

Common Angle Notations

The common notations used for angles are as follows:

- Lowercase Greek letters (most commonly α, β, and θ)
- $\angle A$ or $m\angle A$, where A is the vertex of the rays where the angle is formed
- $\angle BAC$, where A is the vertex where the rays emanating from A and going through B and C meet.

Degree Measure

One commonly used system to measure angles is **degree measure**. Quantities measured in degrees are denoted by the familiar "°" symbol.

Definition: Degree Measure

We define **degree measure** by dividing the full rotation around the circle into 360 segments. The angular measure of one of these segments is defined to be one degree, denoted 1°. That is, 1° represents the measure of an angle which constitutes $\frac{1}{360}$ of a revolution.

Figure $1.1.8$ shows the degree measures for a full, half, and quarter revolution. Since we define a full revolution as being 360°, half of a revolution measures $\frac{1}{2}(360^\circ) = 180^\circ$, a quarter of a revolution measures $\frac{1}{4}(360^\circ) = 90^\circ$, and so on.

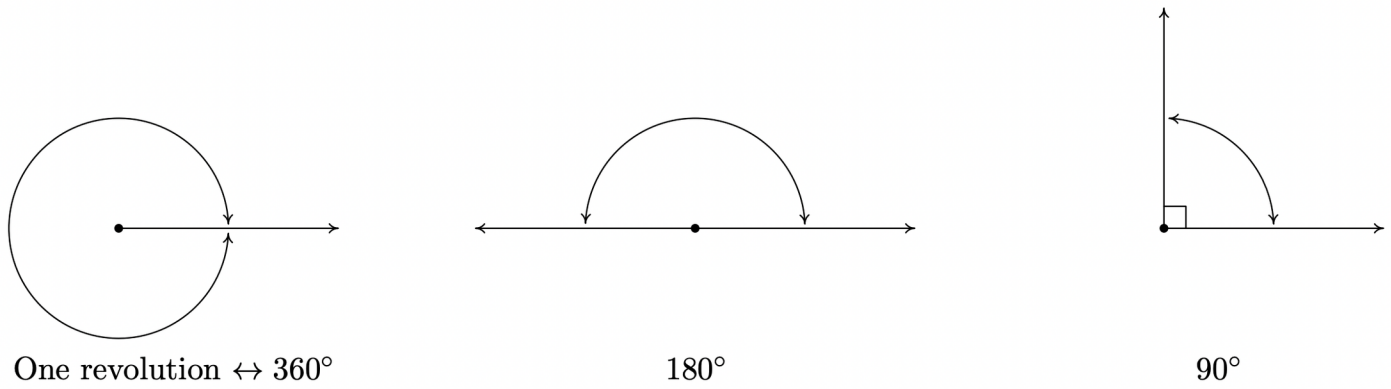


Figure $1.1.8$**:** *A full revolution (left), one-half a revolution (middle), and one-quarter of a revolution (right)*

The following theorem is a direct result of Figure $1.1.8$.

Theorem: Degree Measure of a Straight Angle

The degree measure of a straight angle is 180°.

In Figure $1.1.8$, we have used the small square "□" to denote a very special angle—the **right angle**.

Definition: Right Angle

A **right angle** is defined to be the angle measure of a quarter-circle, which is 90°.

The right angle is a very special angle in Trigonometry. As such, we use it as a sort of "measuring stick" against which we judge other angles. One such judgment is whether the measure of an angle is smaller or larger than the measure of a right angle.

Definition: Acute Angle

An angle that measures strictly between 0° and 90° is called an **acute angle**.

Definition: Obtuse Angle

An angle that measures strictly between 90° and 180° is called an **obtuse angle**.

It is important to note that, theoretically, we can know the measure of any angle as long as we know the proportion it represents of the entire revolution. For instance, the measure of an angle which represents a rotation of $\frac{2}{3}$ of a revolution would measure $\frac{2}{3}(360^\circ) = 240^\circ$, the measure of an angle which constitutes only $\frac{1}{12}$ of a revolution measures $\frac{1}{12}(360^\circ) = 30^\circ$, and an angle which indicates no rotation at all is measured as 0° (see Figure $1.1.9$).

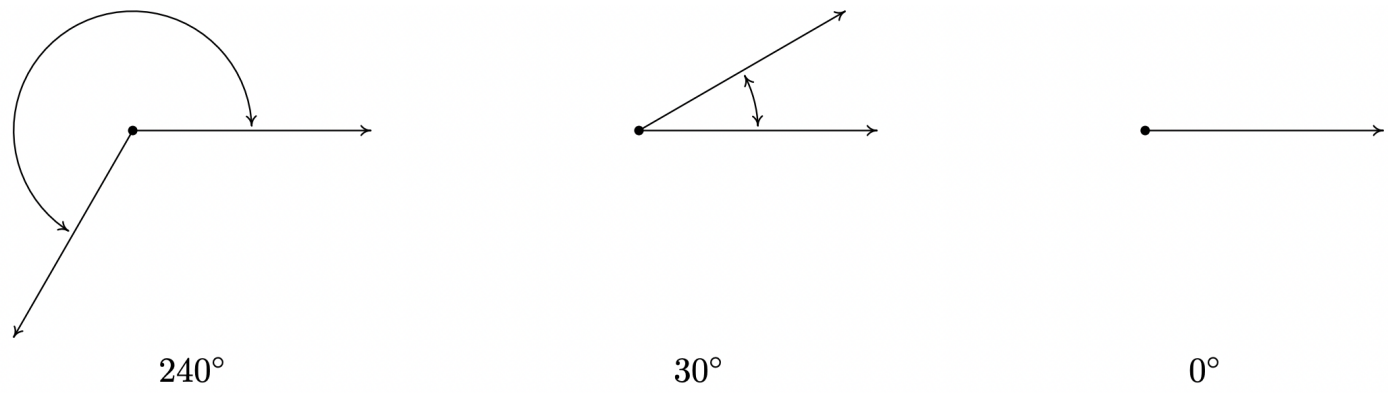


Figure 1.1.9**:** *An obtuse angle (left), an acute angle (middle), and a "closed" angle (right)*

Interactive Element: Angle as an Amount of Rotation

The following Interactive Element might help you visualize how an angle's measure is just the amount its terminal ray has rotated from the initial ray.

Interact: Drag the orange point to swing the terminal ray around the vertex.

Observation: The measure grows smoothly as you rotate. Watch it pass 90° (right), climb through the obtuse range, and reach 180° (straight)—and notice that sweeping the "long way" gives a very different measure for the same terminal ray.

We now investigate the relationship between two angles whose measures add to 90° or 180°.

Definition: Complementary Angles

Two acute angles are called **complementary angles** if their measures add to 90°.

Definition: Supplementary Angles

Two angles, either a pair of right angles or one acute angle and one obtuse angle, are called **supplementary angles** if their measures add to 180°.

In the diagram below, the angles α and β are supplementary while the pair γ and θ are complementary angles.

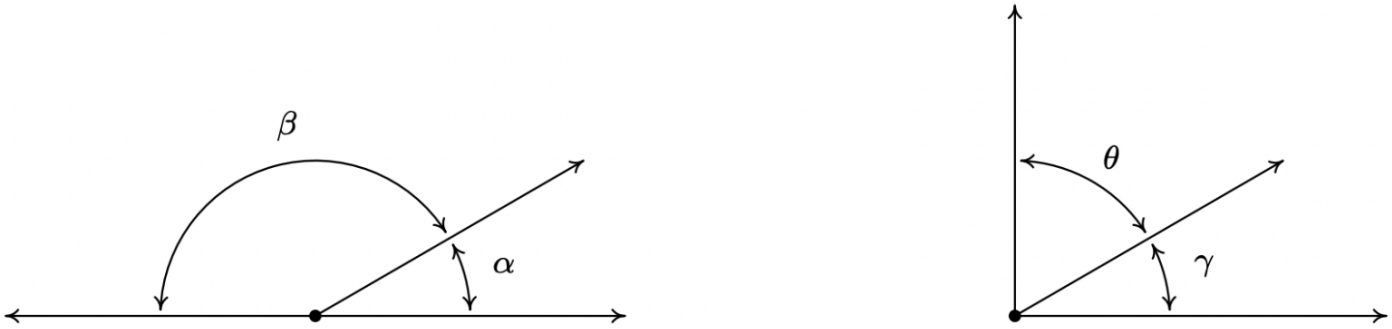


Figure 1.1.10**:** *Supplementary angles, α and β (left) and complementary angles, γ and θ (right)*

Caution

In this text, we will restrict the definition of complementary and supplementary angles to positive angles only. That is, despite their sum being 90°, we will *not* say that $\alpha = -20^\circ$ and $\beta = 110^\circ$ are complementary. Other textbooks allow for this silliness, but in truth there is no need to expand these definitions to include negative angles.

Interactive Element: Supplementary Angles

The following Interactive Element might help you visualize why two angles that sit on a straight line are supplementary.

Interact: Drag the red point to tilt the ray standing on the line.

Observation: Both α and β change as you drag, but $\alpha + \beta$ is always 180°. The same idea over a right angle gives complementary angles, which sum to 90°.

Finally, in practice, the distinction between the angle itself and its measure is blurred so that the sentence "α is an angle measuring 42°" is often abbreviated as "$\alpha = 42^\circ$."

Example 1.1.1

Consider Figure 1.1.11 below.

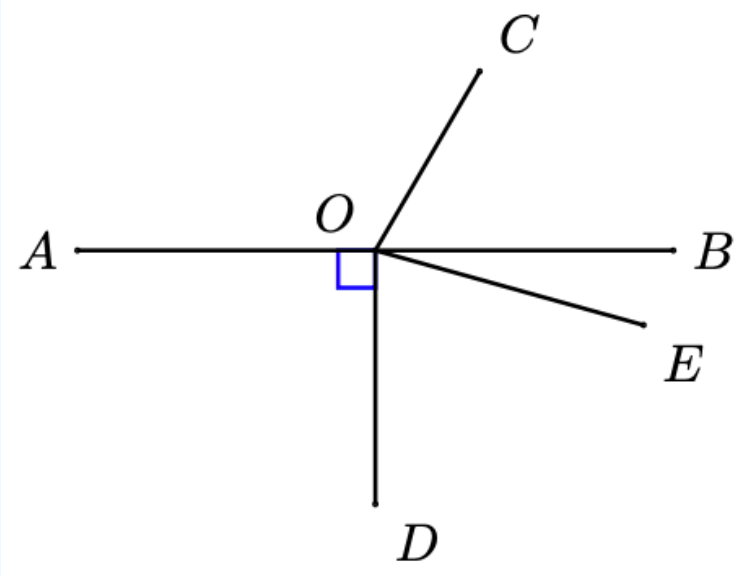


Figure 1.1.11

a. Which angle pairs are supplementary?
b. Which angle pairs are complementary?
c. Which angles are obtuse?
d. Which angles are acute?

Solutions available at the end of this section.

It could be argued in Example 1.1.1d that $\angle COE$ is acute; however, there is no valid reference for the measure of $\angle COB$ and $\angle BOE$—only a visual cue. Therefore, we avoid assuming that $\angle COE$ is acute.

Checkpoint 1.1.1

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View this Interactive Content online

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The measuring "system" we use to measure angles is formally called the **Decimal Degree** (**DD**) system.

Definition: Decimal Degree

An angle written as a decimal is said to belong to the **decimal degree (DD)** system of measuring angles.

Therefore, angles like 38°, 23.1°, and 123.456° belong to the decimal degree system. There are other angle measure systems:

- **Degree-Minute-Second** (**DMS**)**:** While we will eventually use this angular measurement system (specifically, in some applications), it is not necessary to introduce it at this time.
- **Radians:** We will introduce this "better" angular measurement system later. Without this system, we could not perform Calculus.

Angles in Rotations

Now that we have some basic information about angles and have reintroduced ourselves to a measurement system for angles, it's time to introduce an application of angular measure. Specifically, we will learn to use angles to describe rotation. For example, think of the minute hand on a clock. The minute hand moves through one complete rotation every hour, or 360°. In two hours, the minute hand rotates through 720°.

Example 1.1.2

Through how many degrees does the minute hand rotate in an hour and a half? In forty minutes?

Solution available at the end of this section.

Checkpoint 1.1.2

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Geometry Necessary for Trigonometry

Trigonometry requires a decent grasp of basic Geometry. As such, let's review some pertinent topics.

Definition: Vertical Angles

When two lines intersect, they form four angles. The two pairs of nonadjacent angles—those lying opposite each other at the point of intersection—are called **vertical angles** (or **vertically opposite angles**).

For example, $\angle 1$ and $\angle 3$ in Figure $1.1.13$ are vertical angles, as are the angles labeled $\angle 2$ and $\angle 4$.

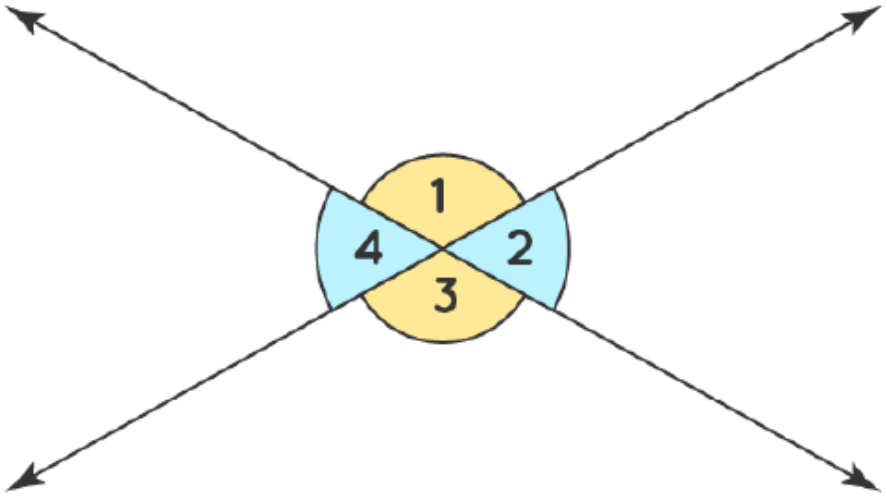


Figure $1.1.13$

Before introducing a theorem relating vertical angles, we need to define what it means for angles to be **equal**.

Definition: Equality of Angles

Two angles are defined to be **equal** if their measures are equal.

Having clarified this, we introduce a theorem that is very useful for deriving other concepts from Geometry.

Theorem: Congruency of Vertical Angles

Vertical angles are equal (also known in Geometry as **congruent**).

Proof available at the end of this section.

A line intersecting two parallel lines forms eight angles, as shown in Figure $1.1.14$.

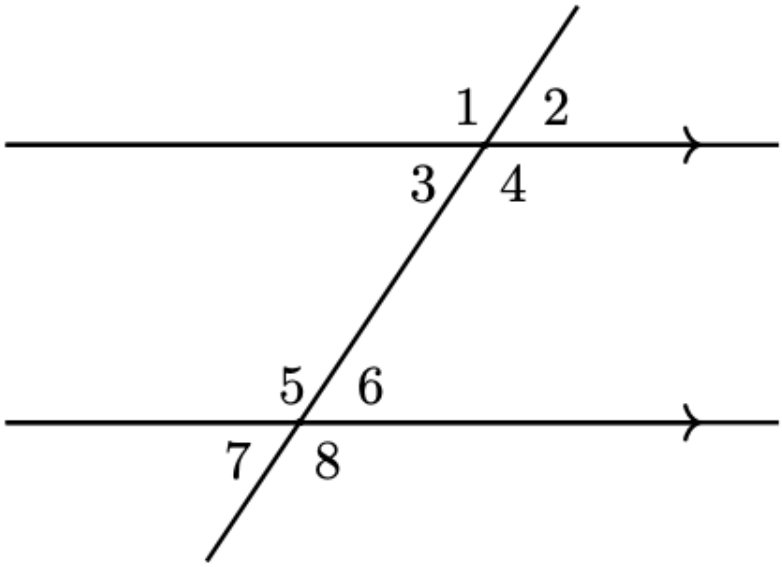


Figure 1.1.14**:** *Two parallel lines "cut" by a line*

The line cutting through the parallel lines in Figure 1.1.14 gets a special name.

Definition: Transversal

A **transversal** is a line that intersects two or more lines at distinct points.

It's important to note that transversals need not cut through parallel lines. That is, a line cutting through any other two lines (whether those lines are parallel or not) is also called a transversal.

Figure 1.1.14 shows that there are four pairs of vertical angles formed by a transversal—$\angle 1$ and $\angle 4$, $\angle 2$ and $\angle 3$, $\angle 5$ and $\angle 8$, and finally $\angle 6$ and $\angle 7$; however, there are other relationships that can be seen in Figure 1.1.14.

Definition: Corresponding Angles

Corresponding angles are a pair of angles, one at each intersection, that lie on the same side of the transversal with one angle interior and the other exterior. Equivalently, they occupy the same relative position at the two intersections.

For example, $\angle 1$ and $\angle 5$ are corresponding angles, as are $\angle 4$ and $\angle 8$. When the lines the transversal is cutting through are parallel, we get the following theorem.

Theorem: Corresponding Angles Theorem

Two lines cut by a transversal are parallel if and only if each pair of corresponding angles is congruent.

The "if and only if" part of this theorem is very important. It states that, "If two lines cut by a transversal are parallel, then each pair of corresponding angles is congruent." Moreover, it states that, "If each pair of corresponding angles is congruent, then the two lines cut by a transversal are parallel."

Our final definition concerning the transversals is one of the most popular from Geometry.

Definition: Alternate Interior Angles

Alternate interior angles are a pair of interior angles that lie on opposite sides of the transversal, with one angle at each intersection.

In Figure 1.1.14, $\angle 3$ and $\angle 6$ are alternate interior angles, as are $\angle 4$ and $\angle 5$.

Theorem: Alternate Angles Theorem

If two parallel lines are cut by a transversal, then each pair of alternate interior angles is congruent, and each pair of alternate exterior angles is congruent.

Proof available at the end of this section.

Interactive Element: Angles Formed by a Transversal

The following Interactive Element might help you visualize the angle relationships formed when a transversal cuts two parallel lines.

Interact: Drag the orange point to tilt the transversal.

Observation: However you tilt it, the two corresponding angles stay equal, and only two values ever appear among all eight angles: θ and its supplement $180^\circ - \theta$. Alternate interior angles are equal for the same reason.

Example 1.1.3

The parallelogram $ABCD$ shown below is formed by the intersection of two sets of parallel lines. Show that the opposite angles of the parallelogram are equal.

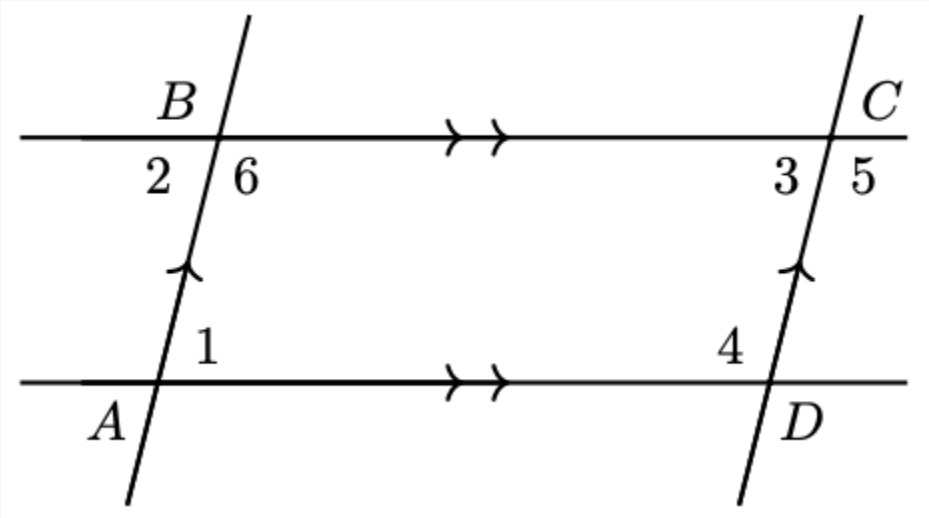


Figure 1.1.15

Solution available at the end of this section.

Checkpoint 1.1.3

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Circles

We end this section with one of the defining geometric structures in all of Mathematics—the **circle**.

Definition: Circle

The shape created from all points in a plane that are equidistant from a given point is called a **circle**. The given point is called the **center** of the circle, and the distance from the center to any point along the circle is called the **radius** of the circle.

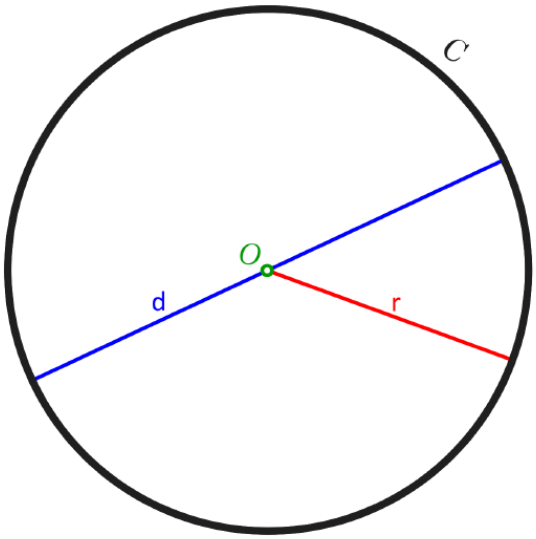


Figure 1.1.16: *A circle with center O and radius r*

Figure 1.1.16 shows a circle of radius r centered at a point O. In fact, the word "radius" plays two roles in Geometry. By definition, the radius is the distance between the center of a circle and its edge; however, *any* line segment joining the center of a circle with any single point on the circle itself is called a radius. Therefore, the word radius simultaneously refers to a distance and a line segment.

Definition: Radius (as a line segment)

A line segment with one endpoint at the center of a circle and the other on the circle is called a **radius** of the circle.

Just like the radius, the **diameter** of a circle, denoted as d in Figure 1.1.16, has two interpretations. Our first is in terms of distance.

Definition: Diameter (as a distance)

The **diameter** of a circle is the length of any line segment going through the center of a circle and whose endpoints are on the circle.

This definition naturally leads to the following theorem.

Theorem: Diameter in Terms of Radius

Given a circle of radius r, the diameter, d, of the circle satisfies

$$d = 2r.$$

As with the radius of a circle, the second interpretation of the diameter is in terms of a line segment.

Definition: Diameter (as a line segment)

A line segment going through the center of a circle and whose endpoints are on the circle is called a **diameter** of the circle.

The terminology from Geometry related to circles extends beyond center, radius, and diameter (e.g., arc, chord, sector, and tangent); however, we will introduce these only as needed.

Circumference

Now that we have the basic definition of a circle, let's dive into some measurements.

Definition: Circumference

The **circumference** is the distance around a circle.

In Figure $1.1.16$, the circumference is denoted using the letter C.

Theorem: Circumference

A circle's circumference, C, is proportional to its radius, r. That is,

$$C = kr.$$

The constant of proportionality is 2π. Hence,

$$C = 2\pi r.$$

A proof of this fundamental theorem in Mathematics is beyond the scope of this course.

In the theorem presented for the circumference, it would be more impactful to state the circumference is proportional to the diameter. That is,

$$C = \text{ some number } \times d.$$

It turns out that the value of "some number" (which is commonly called the **constant of proportionality**) is the universal constant π. That is,

$$C = \pi d \implies \pi = \frac{C}{d}.$$

This fascinating fact tells us that no matter the size of the circle, the ratio of its circumference to its diameter is always this magical number that we call π.

Language Note: "Circumference of a Circle" is Redundant

It is common to hear someone say, "the circumference of a circle." However, the word circumference *only applies to circles*. Therefore, saying "the circumference of a circle" is redundant. For non-circular two-dimensional shapes, we use the word **perimeter** instead of circumference.

Example 1.1.4

Loi buys a ring light for her YouTube videos. The light is circular and has a radius of 7 inches. What is the circumference of her ring light?

Solution available at the end of this section.

Checkpoint 1.1.4

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Area of a Circle

The area of a circle is proportional to the square of its radius.

Theorem: Area of a Circle

The area of a circle of radius r is given by

$$A = \pi r^2.$$

Again, the proof of this theorem is beyond the scope of this course.

Interactive Element: Circumference vs. Area

The following Interactive Element might help you visualize how a circle's circumference and area both depend on its radius.

Interact: Drag the red point along the axis to change the radius.

Observation: Both C and A grow as the radius increases, but the area grows faster—$A = \pi r^2$ depends on the square of r, while $C = 2\pi r$ depends on r directly.

Example 1.1.5

Phuong is making a circular tablecloth for her grandfather's antique oak table. If she wants the tablecloth to have a diameter of at least 7 feet (the bare minimum to cover the table), but no more than 9 feet (allowing a little bit of tablecloth to hang over the edge of the table), between what two values must the area of the tablecloth be? Round your answers to the nearest tenth of a square foot.

Solution available at the end of this section.

Checkpoint 1.1.5

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Hidden Resources: Borrowing Technology

Having a computer is a requirement for any college class these days; however, what do you do if you don't have a computer? Cosumnes River College (and the Los Rios Community College District) has pulled together a great list of resources for students needing a computer. Whether you want to borrow a Chromebook from the CRC library for the semester or you prefer to purchase a discounted computer, the CRC Library has all the information you need!

MyOpenMath Advice: "Message Instructor" Within Questions

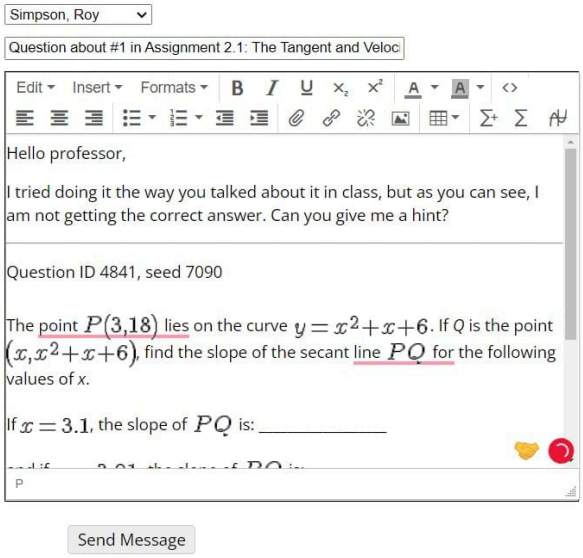
What is it?

If you are stuck on a homework question, and if your professor has set up a discussion area on Canvas, post your question to that area first; however, not all professors have enabled a discussions area. In this case, you can ask your professor for help directly by clicking on the "Message instructor" link at the bottom of the question.



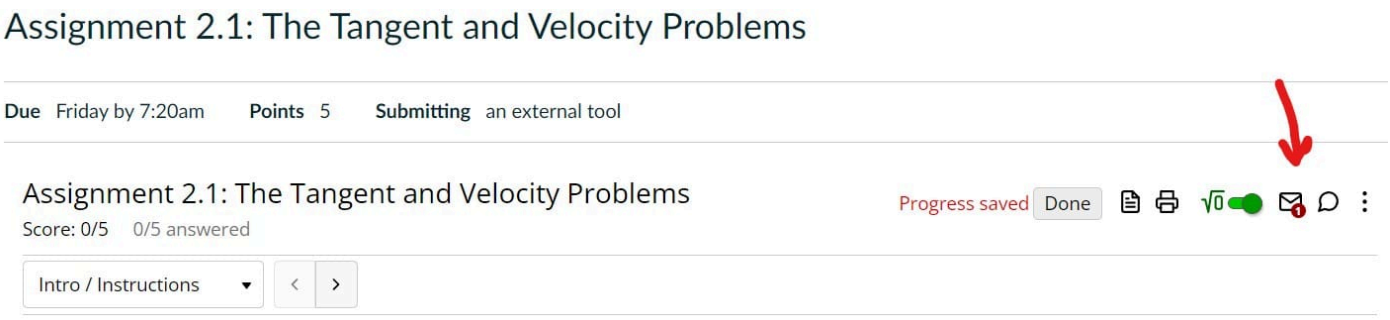
Why use it?

The nice thing about using the "Message instructor" link is that it sends *your version* of the question you are asking about directly to your professor. This helps them better understand your question. Be sure to include some information as to what might be causing your confusion.



How do you know when your professor has responded?

This is the only tricky part. When your professor responds, you will *not* get an email notification. This is a weakness of the MyOpenMath embed into Canvas (the messaging systems do not work well together); however, when your professor responds, you will see a message flag within *any* homework assignment you open.



Just click on the envelope and you will see your professor's response.

Answers

Section Preview

- **Sets and Numbers**
  - The Language of Algebra (focus: definitions and theorems): This section is definition- and theorem-heavy from the very first paragraph; students need to understand what a definition and theorem are in order to engage with the material as presented.

Section Preview

- **Arithmetic**
  - Mathematical Operations - Addition and Subtraction: Computing complements/supplements requires addition/subtraction with signed numbers.
  - Mathematical Operations - Powers: Finding circumference and area requires the multiplication of powers since $A=\pi r^2$ involves squaring.
  - Multiplying and Dividing Fractions: Computing partial rotations (e.g., $\frac{2}{3}\times 360^\circ$) requires multiplying and dividing fractions.

Section Preview

- Identify acute, obtuse, right, and straight angles.
- Compute the complement and supplement of an angle.
- Determine the number of degrees in a partial rotation.
- Use vertical, alternate interior, or corresponding angles to solve for an angle.
- Find the circumference and area of a circle.

Example 1.1.1

a. $\angle AOC$ and $\angle BOC$ are supplementary, $\angle AOD$ and $\angle BOD$ are supplementary, and $\angle AOE$ and $\angle BOE$ are supplementary.
b. $\angle DOE$ and $\angle BOE$ are complementary.

c. $\angle AOC$, $\angle AOE$, and $\angle DOC$ are each obtuse angles.
d. $\angle BOC$, $\angle BOE$, and $\angle DOE$ are each acute angles.

Example 1.1.2

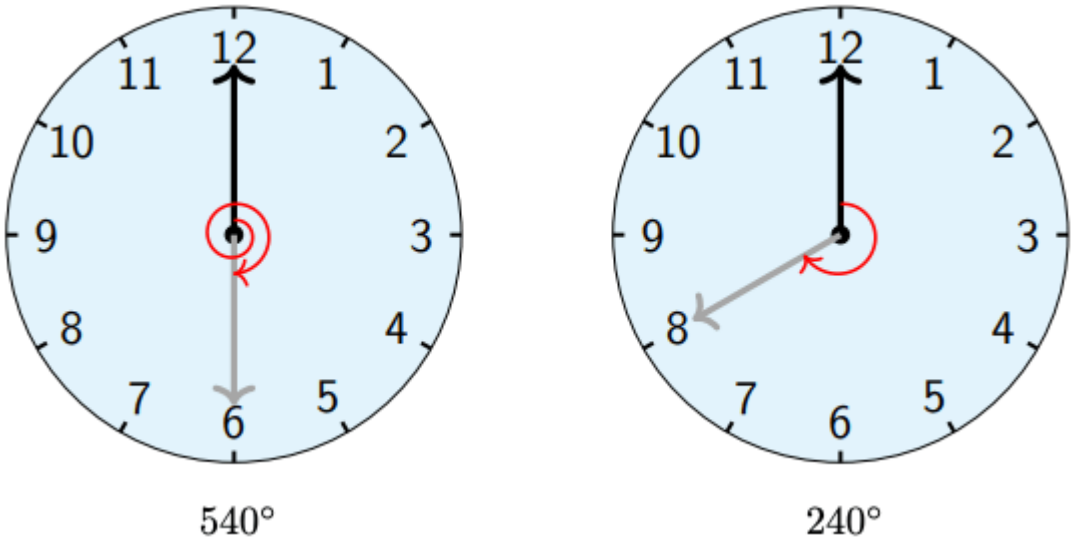


Figure 1.1.12

Look at Figure 1.1.12. An hour and a half represents 1.5 complete rotations, or

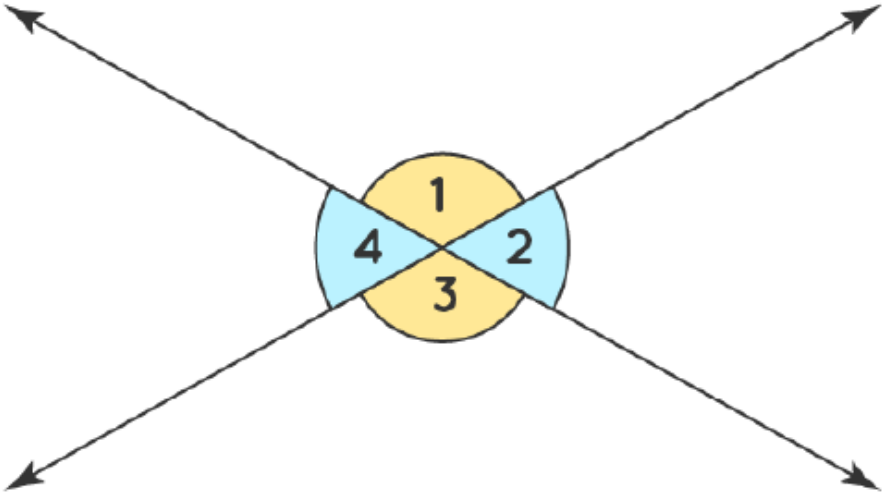
$$1.5(360^\circ) = 540^\circ.$$

Forty minutes is two-thirds of an hour, so the minute hand rotates through

$$\frac{2}{3}(360^\circ) = 240^\circ.$$

Theorem: Congruency of Vertical Angles

Suppose two lines intersect, as shown in the figure below.



Since $\angle 1 + \angle 2$ forms a straight angle,

$$\angle 1 + \angle 2 = 180^\circ.$$

By a similar argument,

$$\angle 1 + \angle 4 = 180^\circ.$$

Therefore,

$$\angle 1 + \angle 2 = 180^\circ = \angle 1 + \angle 4.$$

That is,

$$\angle 2 = \angle 4.$$

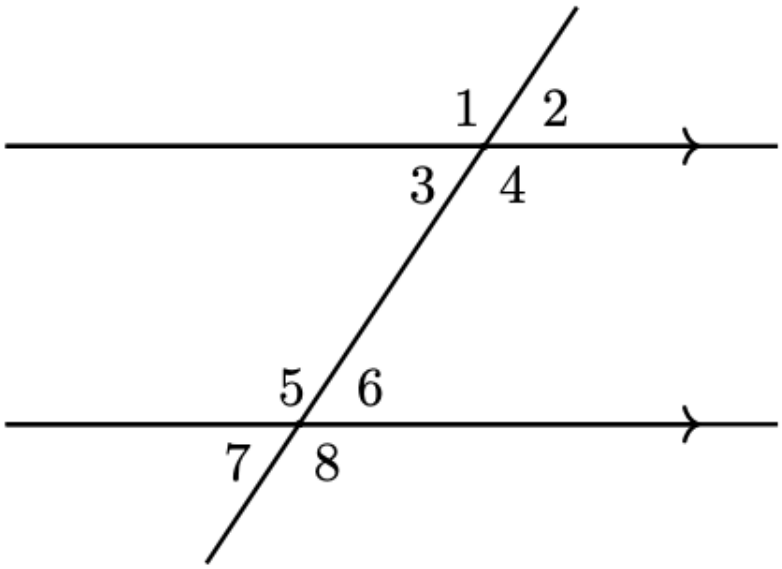
By a similar set of arguments, we can show that

$$\angle 1 = \angle 3.$$

Hence, it must always be the case that vertical angles are congruent.

Theorem: Alternate Angles Theorem

Suppose a transversal cuts two parallel lines as shown below.



We know that the corresponding and vertical angles are congruent. Therefore,

$$\angle 1 = \angle 5$$

and

$$\angle 1 = \angle 4.$$

Hence,

$$\angle 5 = \angle 4.$$

Since our choice of angles was arbitrary, this shows that alternate interior angles are always congruent. A similar argument shows the alternate exterior angles are congruent.

Example 1.1.3

Angles 1 and 2 are equal because they are alternate interior angles, and angles 2 and 3 are equal because they are corresponding angles. Therefore, angles 1 and 3, the opposite angles of the parallelogram, are equal. A similar argument shows that angles 4, 5, and 6 are equal.

Example 1.1.4

We are given the fact that the radius is 7 inches. Therefore, $r = 7$ (inches). Using the formula for circumference, we find that Loi's ring light has a circumference of

$$C = 2\pi r = 2\pi(7 \text{ inches}) = 14\pi \text{ inches} \approx 44 \text{ inches}.$$

Example 1.1.5

To barely cover the table, Phuong needs a tablecloth with radius 3.5 feet. Therefore, the minimum area of the tablecloth should be

$$A_{\text{minimum}} = \pi r^2_{\text{minimum}} = \pi(3.5)^2 \approx 38.5 \text{ square feet}.$$

At most, if she wants a little bit of the tablecloth to hang over the edge, Phuong will want the radius to be 4.5 feet. This leads to an area of

$$A_{\text{maximum}} = \pi r^2_{\text{maximum}} = \pi(4.5)^2 \approx 63.6 \text{ square feet}.$$

Any amount of circular tablecloth between 38.5 and 63.6 square feet should work nicely for Phuong's project.

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1.1.1: Resources and Key Concepts

Resources

Videos

- Lines, Rays, Vertices, Angles, Oriented Angles, Initial Side, Terminal Side
  - Angles - Creating Angles
- Degree Measure
  - Angles - Angle Measure: Degrees
- Circles and Their Properties
  - Intermediate Algebra - Circles: The Concept

Key Concepts

Definitions

- **Ray:** A line segment with one starting point (the initial point) that extends infinitely in one direction. Also called a half-line.
- **Angle:** A figure formed when two rays share a common initial point.
- **Vertex:** The shared initial point of the two rays that form an angle.
- **Straight Angle:** An angle formed when its two rays extend in opposite directions, forming a straight line.
- **Angle Measure:** A numerical value indicating the amount of rotation that separates the two rays of an angle.
- **Oriented Angle:** An angle whose measure considers the direction of rotation from an initial side to a terminal side.
- **Initial Side:** The starting ray of an oriented angle.
- **Terminal Side:** The ending ray of an oriented angle after rotation.
- **Positive Angle:** An angle formed by a counterclockwise rotation from the initial side to the terminal side.
- **Negative Angle:** An angle formed by a clockwise rotation from the initial side to the terminal side.
- **Degree Measure:** A system for measuring angles where a full rotation is divided into 360 equal parts, with each part being one degree (1°).
- **Right Angle:** An angle that measures exactly 90°, representing a quarter-circle rotation.
- **Acute Angle:** An angle whose measure is strictly between 0° and 90°.
- **Obtuse Angle:** An angle whose measure is strictly between 90° and 180°.
- **Complementary Angles:** Two positive acute angles whose measures sum to 90°.
- **Supplementary Angles:** Two positive angles whose measures sum to 180°.
- **Decimal Degree (DD) system:** A system of measuring angles where the angle is written as a decimal number.
- **Vertical Angles:** The non-adjacent angles that are formed by the intersection of two straight lines.
- **Equality of Angles:** A condition where two angles are considered equal if their measures are equal.
- **Transversal:** A line that intersects two parallel lines.
- **Corresponding Angles:** Angles that are in the same relative position at each intersection where a transversal crosses two lines.
- **Alternate Interior Angles:** A pair of angles on opposite sides of the transversal and inside the two parallel lines.
- **Circle:** The set of all points in a plane that are the same distance (equidistant) from a central point.
- **Center:** The given point from which all points on a circle are equidistant.
- **Radius:** The distance from the center to any point on the circle. The term also refers to any line segment connecting the center to a point on the circle.
- **Diameter:** The length of a line segment that passes through the center of a circle and has both endpoints on the circle. The term also refers to the line segment itself.
- **Circumference:** The distance around a circle.

Theorems

- **Degree Measure of a Straight Angle:** The degree measure of a straight angle is 180°.

- **Congruency of Vertical Angles:** Vertical angles are equal (congruent).
- **Congruency of Corresponding Angles:** Corresponding angles are congruent.
- **Alternate Interior Angles:** If a transversal intersects parallel lines, the alternate interior angles are congruent.
- **Diameter in Terms of Radius:** For a circle with radius r and diameter d, the relationship is $d = 2r$.
- **Circumference:** The circumference C of a circle with radius r is given by the formula $C = 2\pi r$.
- **Area of a Circle:** The area A of a circle with radius r is given by the formula $A = \pi r^2$.

Common Mistakes

- **Using π for an angle:** The lowercase Greek letter π is reserved for the constant approximately equal to 3.14159 and should never be used to represent an unknown angle.
- **Using negative angles for complements/supplements:** In this textbook, complementary and supplementary angles are restricted to be positive angles only.
- **Relying on visual appearance:** Do not assume angles are acute, obtuse, or have a specific measure based on a diagram alone. Conclusions must be supported by definitions or theorems. For example, in Example 1.1.1, one cannot conclude $\angle COE$ is acute without more information.
- **Ambiguous angle notation:** Using a single letter to name an angle can be ambiguous if multiple angles share the same vertex. Use the three-point notation (e.g., $\angle BAC$) to be clear, ensuring the middle letter is the vertex.
- **Redundant language:** The word "circumference" applies only to circles, so saying "the circumference of a circle" is redundant. For other shapes, the term "perimeter" is used.

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1.1.2: Homework

Skills Refresher

The following is a set of review exercises you will need for this section.

Skills Refresher

For the following exercises, solve the equation.

1. $x - 8 = 19 - 2x$
2. $2x - 9 = 12 - x$
3. $13x + 5 = 2x - 28$
4. $4 + 9x = -7 + x$

For the following exercises, solve the system.

5. $\begin{cases} 5x - 2y &= -13 \\ 2x + 3y &= -9 \end{cases}$
6. $\begin{cases} 4x + 3y &= 9 \\ 3x + 2y &= 8 \end{cases}$

Answers available at the end of this section.

Homework

Reading Questions

1. What is a "ray"?
2. How is the "vertex" of an angle defined?
3. What distinguishes a "positive angle" from a "negative angle"?
4. How many degrees constitute a full revolution?
5. What is an "obtuse angle"?
6. What are "supplementary angles"?
7. According to the theorem presented, what is true about "vertical angles"?
8. What does a "transversal" line do?
9. What is the formula for the circumference of a circle in terms of its radius, r?
10. If a circle has a diameter d, how is its radius r related to d?

Vocabulary Check

11. (MyOpenMath) When two rays share a common initial point they form a(n) ___.
12. (MyOpenMath) Trigonometry focuses on oriented angles. An oriented angle starts from an ___ side and ends at a ___ side.
13. (MyOpenMath) ___ angles start at an initial side and rotate clockwise to the terminal side.
14. (MyOpenMath) A positive angle opens with a ___ rotation.
15. We define ___ measure by dividing the full rotation around the circle into 360 segments.
16. One-quarter of a complete revolution around a circle forms a ___ angle, which measures ___ degrees.
17. ___ angles measure between 0° and 90°, while ___ angles measure between 90° and 180°.
18. The degree system for measuring the size of an angle is also known as the ___ system.
19. When two lines intersect, the non-adjacent angles formed from this intersection are called ___ angles.
20. A line intersecting two parallel lines is called a ___.
21. The set of all points equidistant to a single point on a plane is called a ___.
22. The ___ of a circle is simultaneously the largest distance between any two points on a circle and is the name of the line segment that passes through the center of the circle whose endpoints lie on the circle.
23. π is a(n) ___ number (unlike 3, which is a natural number).

Concept Check

24. Greek letters like α, β, and θ are often used to represent angles in Trigonometry. Name one Greek letter that is *never* used as a label for an angle.
25. If a positive angle rotates from its initial to its terminal side, what direction is the rotation?
26. If a negative angle rotates from its initial to its terminal side, what direction is the rotation?
27. Suppose you have a straight angle and you subtract a right angle. What type of angle is the result?
28. Can a right angle be acute? How about obtuse?
29. Suppose the vertex of an angle is at the point P, and the points R and S lie on the initial and terminal sides of the angle, respectively. Which of the following are correct notations for the angle?
    i. $\angle P$
    ii. P
    iii. $m\angle P$
    iv. $\triangle P$
    v. $\angle PRS$
    vi. $\angle RPS$
    vii. $\angle SPR$
30. Quantities measured in degrees use the ___ symbol.

31. According to this text, can 107° have a complement? If so, what is the value of this complement?

32. According to this text, can 107° have a supplement? If so, what is the value of this supplement?

33. Can two acute angles be supplementary?

34. Choose two of the eight angles formed by a pair of parallel lines cut by a transversal. Those two angles are either equal or ___.

35. Write two formulas for the circumference of a circle - one involving the radius of the circle and the other involving the diameter.

36. The formula for the circumference of a circle is ___.

37. The formula for the area of a circle is ___.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

38. If α and β are vertical angles, then $\alpha + \beta = 90^\circ$.

39. (MyOpenMath) The complement to 47° is 43°.

40. The supplement to 190° is -10°.

41. Alternate interior angles sum to 180°.

42. The distance from the center to the edge of a circle is called the radius of the circle.

43. The line connecting the center of a circle to its edge is called a radius.

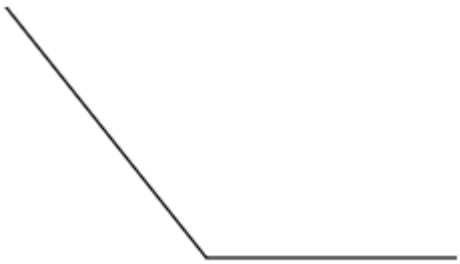
44. The circumference of a square is the sum of the lengths of its sides.

45. $\pi = 3.14$

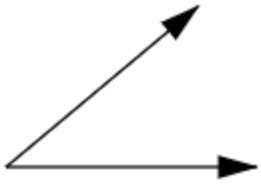
46. $\pi = \frac{22}{7}$

Basic Skills

47. (MyOpenMath) State (if possible) if the angle is acute or obtuse.



48. (MyOpenMath) Which is the closest measure to the angle shown below: 40°, 65°, 95°, or 120^{ \circ }?



For the following exercises, state (if possible) which angles are acute and which are obtuse. Give the complement and the supplement of each angle, if applicable.

49. (MyOpenMath) 30°

50. 45°

51. 60°

52. 43°

53. 90°

54. 120°

55. 135°

56. 143°

57. 150°

For the following exercises, give the complement of each angle.

58. 60°

59. 80°

60. 25°

61. 18°

62. 64°

63. 47°

For the following exercises, give the supplement of each angle.

64. 30°

65. 45°

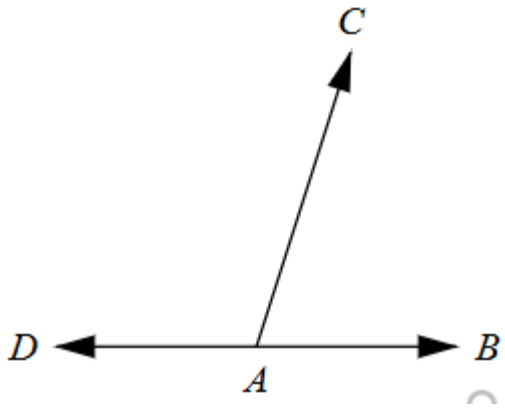
66. 120°

67. 25°

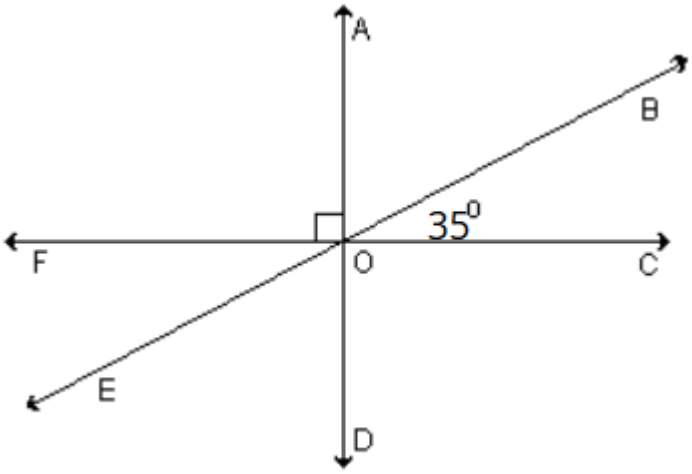
68. 165°

69. 110°

70. (MyOpenMath) If the measure of $\angle CAB = 62^\circ$, then find the measure of $\angle CAD$.



71. (MyOpenMath)



a. Which angle is supplementary to $\angle BOC$?

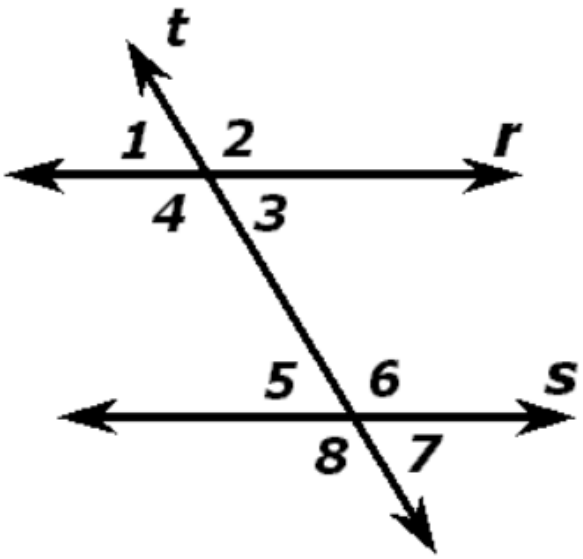
b. Which angle is complementary to $\angle BOC$?

c. What is the measure of $\angle EOF$?

d. What is the measure of $\angle AOE$?

e. What is the measure of $\angle BOF$?

For the following exercises, use the following graph to answer the question. Assume lines r and s are parallel.



72. (MyOpenMath) $\angle 3$ is alternate interior to which other angle(s)?

73. (MyOpenMath) $\angle 5$ is a corresponding angle to which other angle(s)?

74. (MyOpenMath) $\angle 7$ and what other angle(s) are vertical angles?

75. (MyOpenMath) $\angle 8$ is alternate exterior to which other angle(s)?

76. (MyOpenMath) Find $m\angle 5$ given that $m\angle 1 = 46^\circ$.

77. (MyOpenMath) Find $m\angle 8$ given that $m\angle 6 = 154^\circ$.

78. (MyOpenMath) Find $m\angle 5$ given that $m\angle 3 = 82^\circ$.

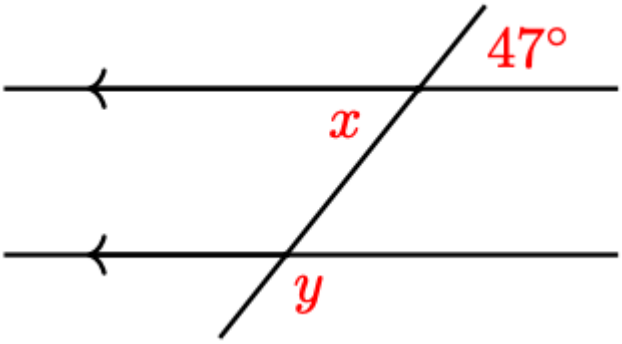
79. (MyOpenMath) Find $m\angle 3$ given that $m\angle 6 = 105^\circ$.

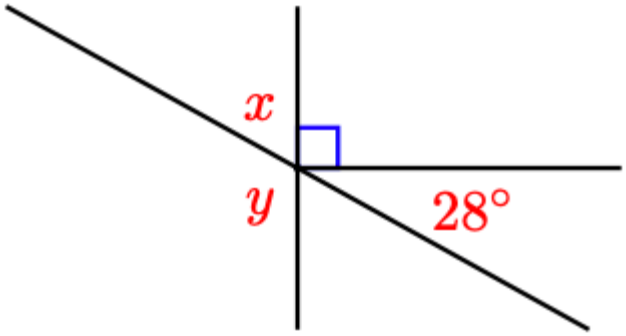
80. (MyOpenMath) Find $m\angle 8$ given that $m\angle 3 = 76^\circ$.

81. (MyOpenMath) Find the remaining angles given that $m\angle 1 = 15^\circ$.

For the following exercises, arrows on a pair of lines indicate that they are parallel. Find x and y.

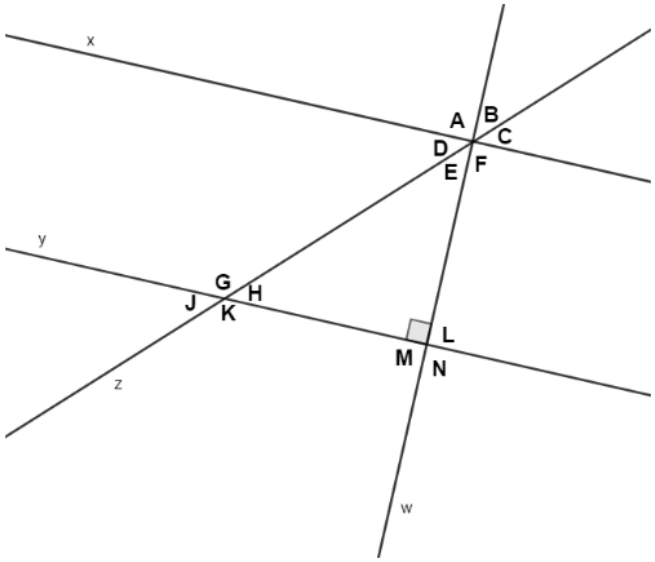
82.



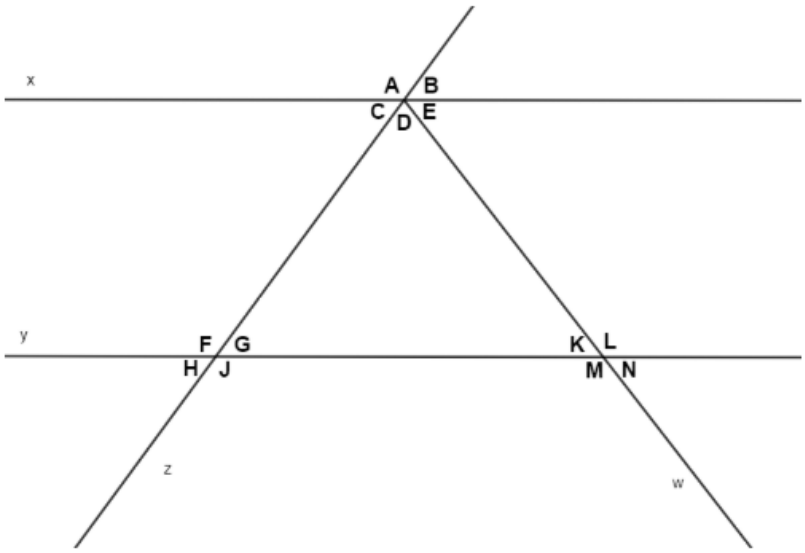


33.

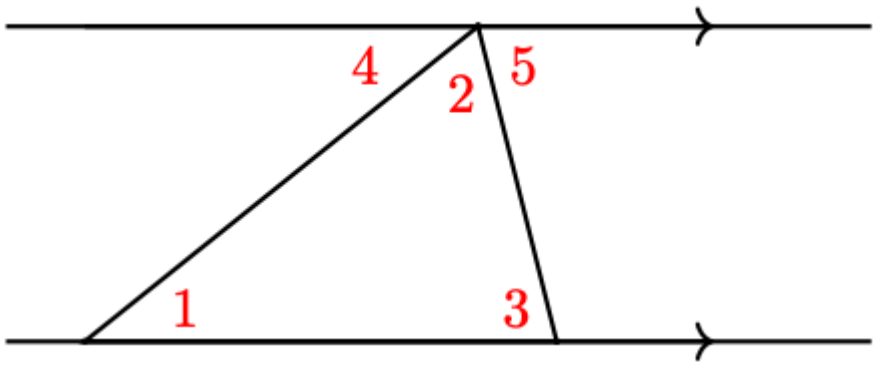
34. (MyOpenMath) The lines x and y are parallel, and are cut by transversals z and w. Given $\angle E = 57°$, find the measure of all remaining angles.



35. (MyOpenMath) The lines x and y are parallel, and are cut by transversals z and w. Given $\angle H = 36°$ and $\angle E = 40°$, find the measure of all remaining angles.



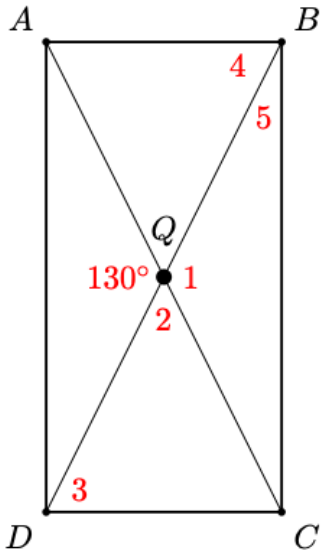
36. a. Among the angles labeled 1 through 5 in the figure below, find two pairs of equal angles.



b. $\angle 4 + \angle 2 + \angle 5 =$_____.

c. Use parts (a) and (b) to explain why the sum of the angles of a triangle is 180°.

37. $ABCD$ is a rectangle. The diagonals of a rectangle bisect each other. In the figure, $\angle AQD = 130^\circ$. Find the angles labeled 1 through 5 in order, and give a reason for each answer.

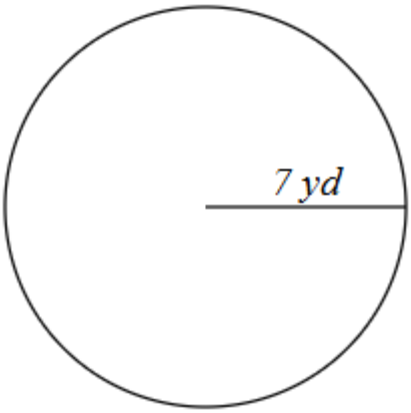


For the following exercises, give an exact answer, and then round your answer to hundredths.

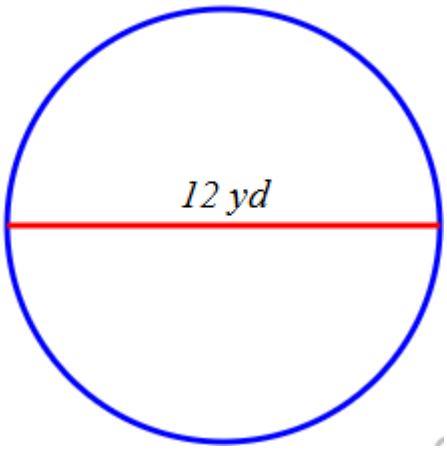
38. What is the area of a circle whose radius is 5 inches?

39. What is the circumference of a circle whose radius is 5 meters?

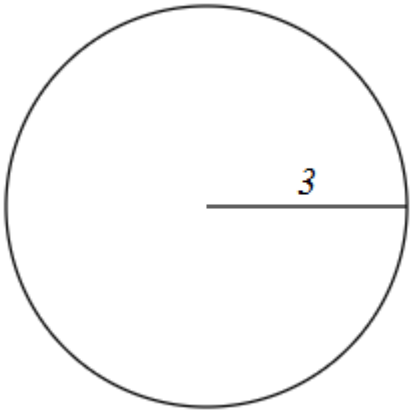
40. (MyOpenMath) What is the circumference of the following circle?



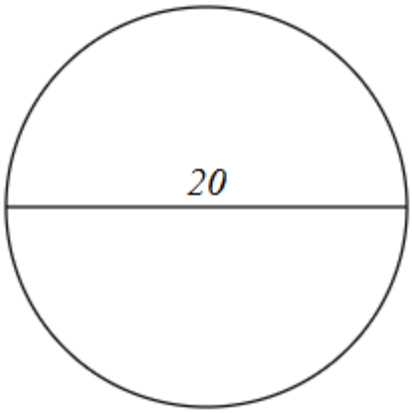
91. (MyOpenMath) What is the circumference of the following circle?



92. (MyOpenMath) What is the area of the following circle?



93. (MyOpenMath) What is the area of the following circle?



For the following exercises, find the circumference and area for each circle having the given radius r or diameter d.

94. (MyOpenMath) $r = 7$ feet

95. $r = 8.4$ centimeters

96. $d = 16$ yards

97. $d = 12$ meters

98. Find the diameter of a circle having a circumference of 4π kilometers.

99. (MyOpenMath) Find the radius of a circle having a circumference of 10 inches.

100. (MyOpenMath) Find the radius of a circle having an area of 24π square meters.

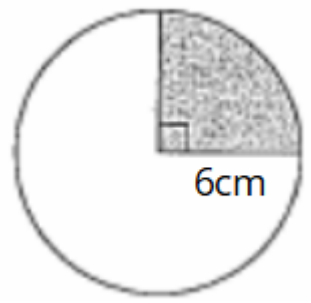
101. (MyOpenMath) Find the radius of a circle having an area of 272 square meters.

102. (MyOpenMath) Find the diameter of a circle having an area of 64 square feet.

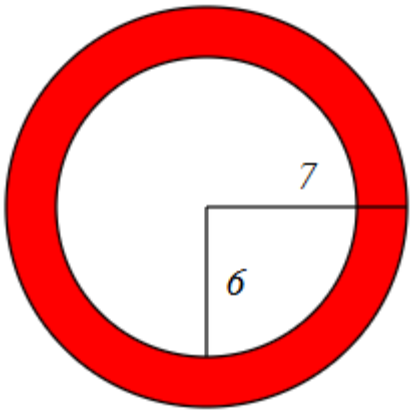
103. Find the circumference of a circle having an area of 81π square miles.

Synthesis Questions

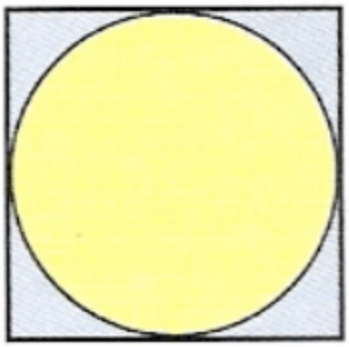
104. (MyOpenMath) Find the area of the shaded region. Round your answer to the nearest tenth.



105. (MyOpenMath) Find the area of the shaded region. Round your answer to the nearest tenth.



96. (MyOpenMath) Suppose the sides of the square are 8 miles in length.



a. What is the area of the square?

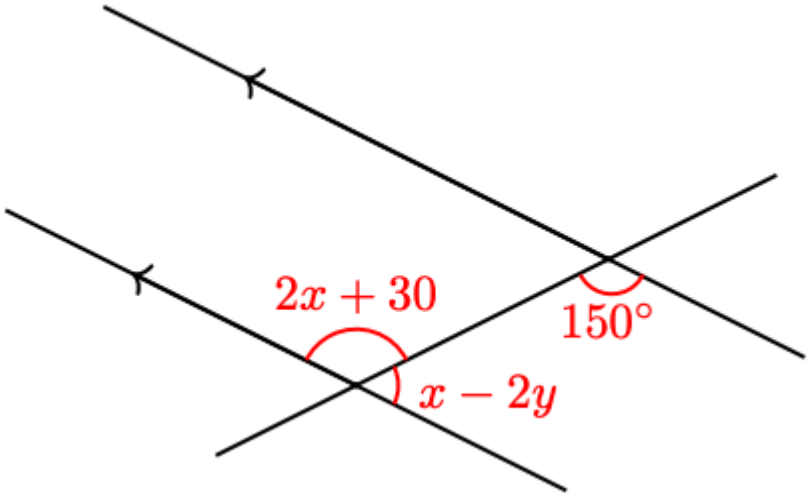
b. What is the area of the circle? Round your answer to the nearest tenth.

c. What is the area of the blue shaded region? Round your answer to the nearest tenth.

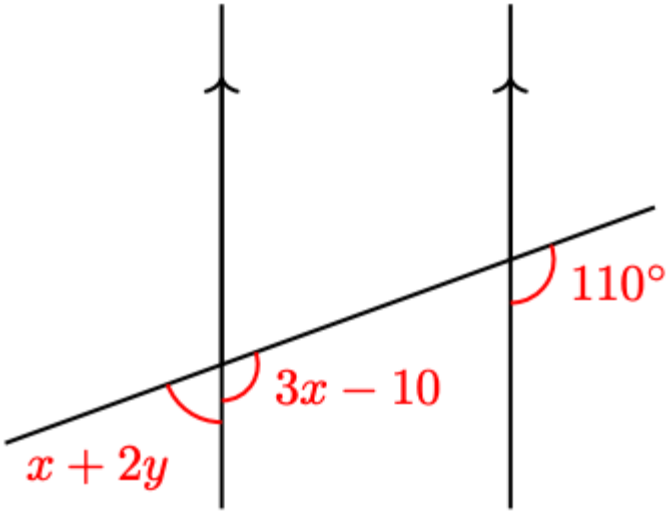
97. State the condition(s) for which α is an acute angle, and for which α is an obtuse angle. In each case, give the complement and the supplement of α.

For the following exercises, arrows on a pair of lines indicate that they are parallel. Find x and y.

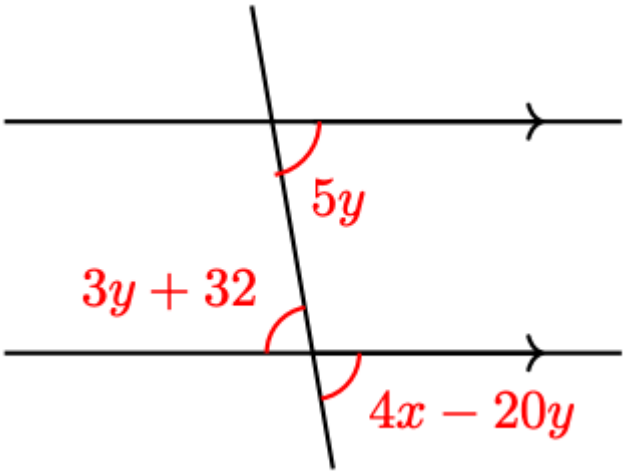
98.



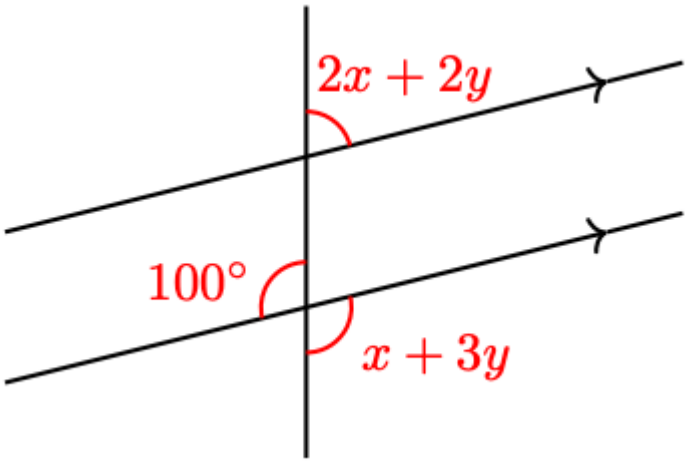
9.



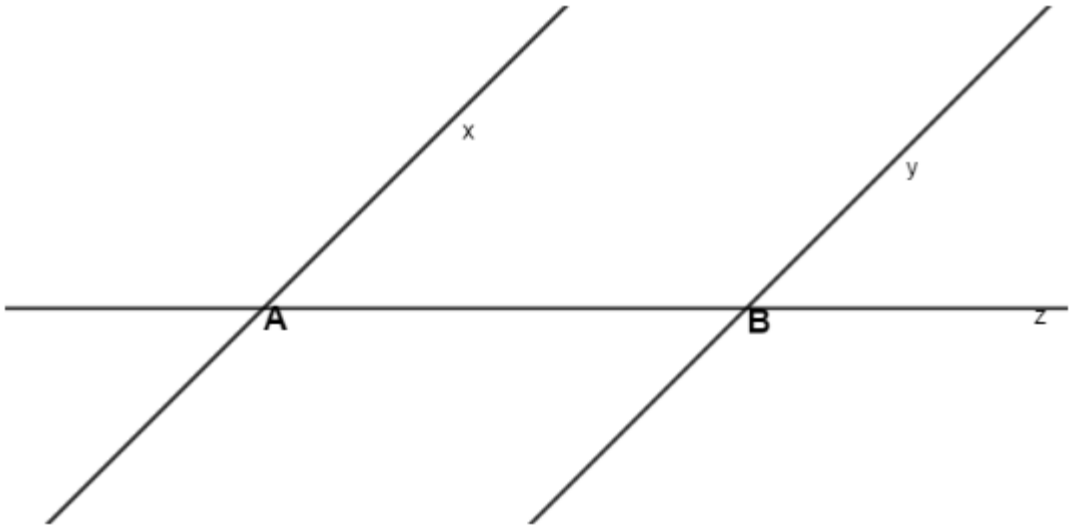
10.



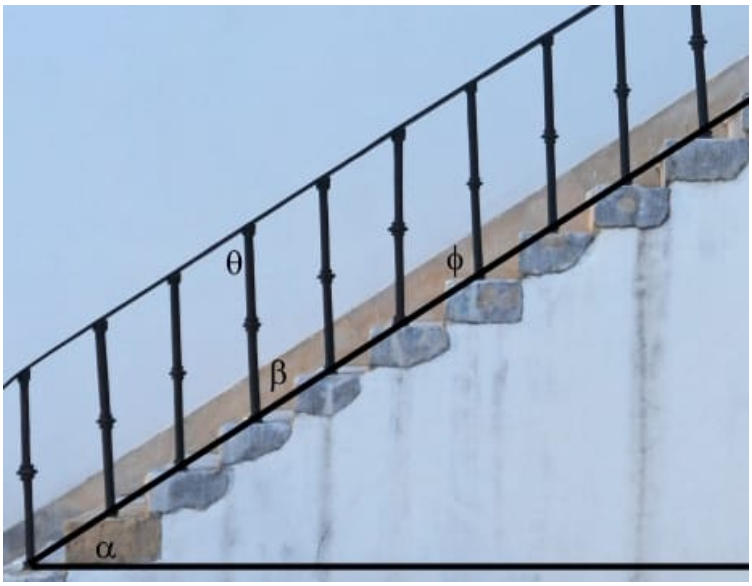
11.



12. (MyOpenMath) The lines x and y are parallel and cut by a transversal z. Given that $\angle A = k^2 + 22k$ and $\angle B = 14k + 48$, find k.



13. **Climbing Stairs.** In each of the following problems, refer to the following image of a staircase and rails. Assume the rails are perpendicular to the ground, they are parallel to each other, and that the top rail (handrail) is parallel to the bottom rail (the black line along the steps).



a. What is the relationship between α and β?

b. What is the relationship between β and θ?

c. What is the relationship between β and ϕ?

d. What is the relationship between θ and ϕ?

e. Find α if $\beta = 48^\circ$.

f. Find θ if $\alpha = 30^\circ$.

Applications

14. **Emergency Light.** An ambulance has a rotating light on its roof. The light rotates through one complete revolution every 2 seconds. How long does it take the light to rotate through 90°?

15. **Rotation of the Earth (MyOpenMath).** The Earth goes through a complete circuit around the sun in approximately 365.25 days. Through how many degrees does the Earth move in one week?

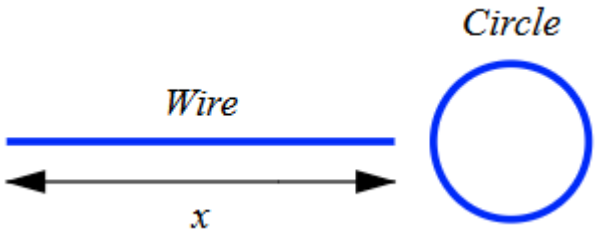
16. **Earth's Rotation About the Sun.** The Earth takes 24 hours to make a complete rotation on its axis. If your math class is 2 hours and 20 minutes, through how many degrees does the Earth turn during your math class?

17. **Circumference of the Earth.** The radius of the Earth is approximately 3,960 miles. Using this approximation, find the circumference of the Earth.

18. **Earth's Rotation About the Sun.** The Earth has a nearly circular orbit about the sun. The radius of this near-circular orbit is approximately 93 million miles. Find the distance the Earth travels when it completes one full orbit about the sun.

Challenge Problems

19. (MyOpenMath) A straight piece of wire, x centimeters long, is bent into a perfect circle. Express the area of the circle, A, in terms of x.



Answers

Skills Refresher

1. 9
2. 2
3. -3
4. -2
5. $x = -3, y = -1$
6. $x = 6, y = -5$

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1.2: Triangles

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Triangle Basics

The study of Trigonometry inevitably involves understanding everything we possibly can about **triangles**.

Definition: Triangle

When three line segments bound a portion of the plane, the resulting shape is called a **triangle** (also known as a **three-sided polygon**). The line segments are called the **sides** of the triangle. A point where two sides meet is called a **vertex** of the triangle. Finally, the angle formed in the triangle's interior where two sides meet is called an **angle** of the triangle.

In Figure $1.2.1$, the sides are denoted $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. The vertices (plural of vertex) are A, B, and C. The angles are denoted $\angle A$, $\angle B$, and $\angle C$. Using the notation $\triangle ABC$, we refer to the triangle by its vertices.

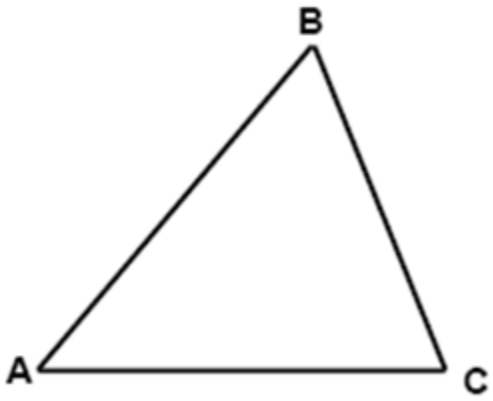


Figure $1.2.1$: $\triangle ABC$

If one tears off the corners of any triangle and lines them up, as shown in Figure $1.2.2$, they will always form a straight angle.

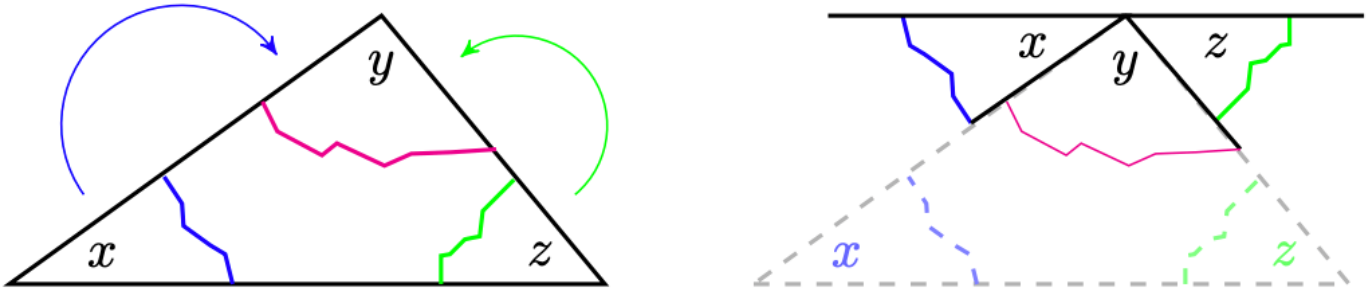


Figure 1.2.2**:** *Tearing corners from a triangle to form a straight angle*

Theorem: Triangle Sum

The sum of the interior angles in a triangle is 180°.

Proof available at the end of this section.

Example 1.2.1

Two of the angles in the triangle shown below are 25° and 115°. Find the third angle.

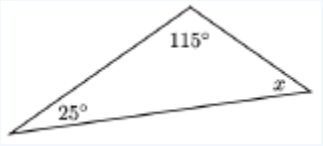


Figure 1.2.3

Solution available at the end of this section.

What do Arithmetic and ~~Arithmetic~~ mean?

To get you used to the Mathematical Mantra, every once in a while I will include the "thought processes" during solutions. Just to review, we perform Mathematics in the order we learned it—Arithmetic, Algebra, Trigonometry, ... At each step during a "mechanical" solution process, we should pause and ask ourselves if there is some simple Arithmetic to be done. If not, we move on to any Algebra that can be done. If there is no Algebra to be done, we then move on to Trigonometry.

Thus,

- "$\text{Arithmetic} \implies$ " in a solution means that there is some Arithmetic we could do to clean up the previous expression.
- "$\cancel{\text{Arithmetic}} \to \text{Algebra} \implies$ " means there isn't any Arithmetic but there *is* some Algebra we could use to clean up the previous expression or equation.
- "$\cancel{\text{Arithmetic}} \to \cancel{\text{Algebra}} \to \text{Trigonometry} \implies$ " means there isn't any Arithmetic nor is there any Algebra that can be done to clean up the previous step, so we need to start looking at doing something from Trigonometry.

Checkpoint 1.2.1

Find all the unknown angles in each figure.

a.

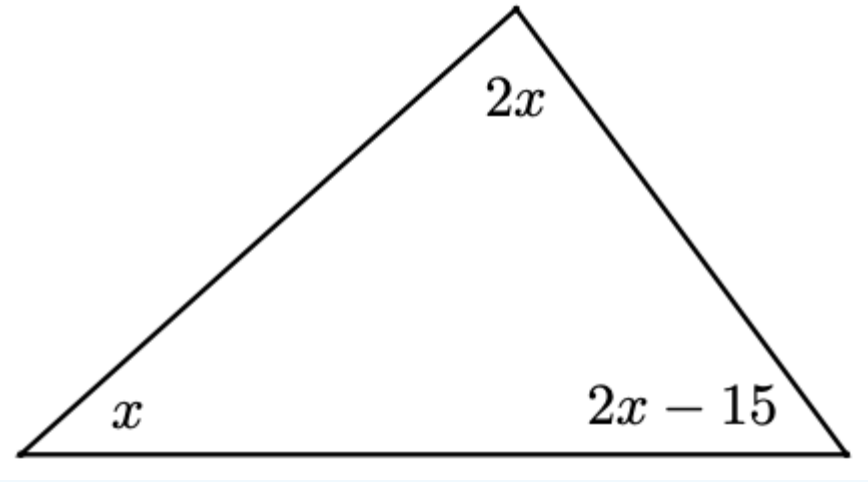


Figure 1.2.4

b.

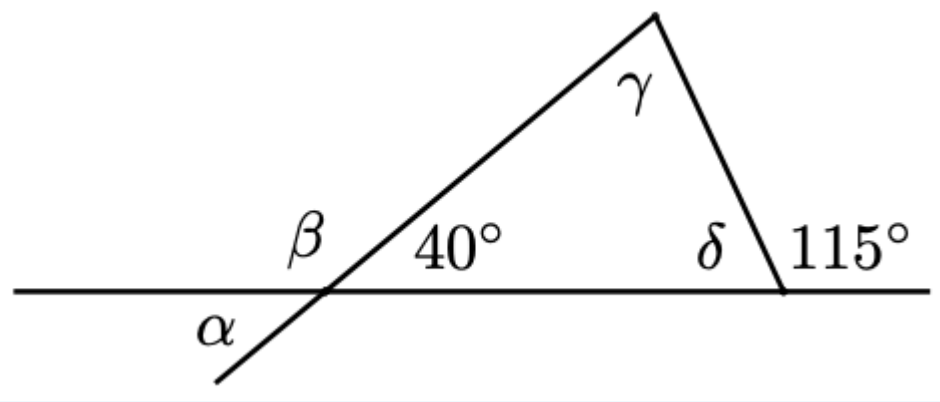
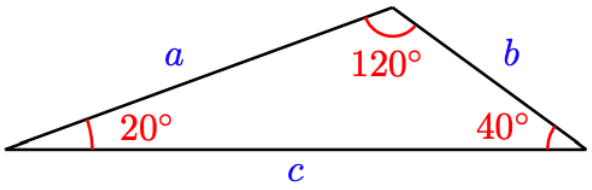


Figure 1.2.5

Answer available at the end of this section.

In any triangle, the longest side is always opposite the largest angle, and the shortest side is opposite the smallest, as illustrated in Figure 1.2.6. Although obvious to most people, this fact will become invaluable as we move through Trigonometry.



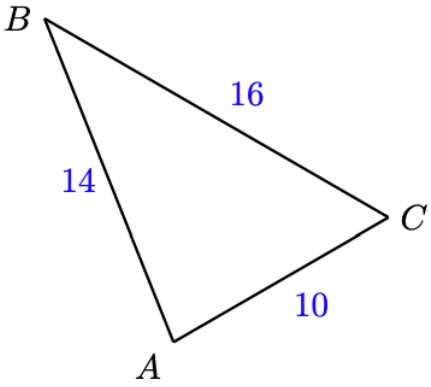


Figure 1.2.6

Notation: Angles and Sides of a Triangle

It is common (though not necessary) to label the angles of a triangle with capital letters and the side opposite each angle with the corresponding lowercase letter, as shown in Figure 1.2.7. We will follow this practice unless indicated otherwise.

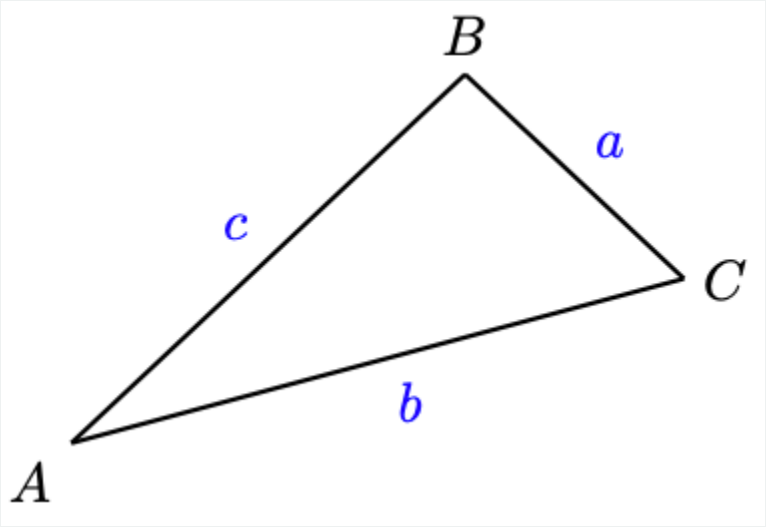


Figure 1.2.7

Example 1.2.2

In $\triangle FGH$, $\angle F = 48^\circ$, and $\angle G$ is obtuse. Side f is 6 feet long. What can be concluded about the other sides?

Solution available at the end of this section.

Checkpoint 1.2.2

In $\triangle RST$, $\angle S$ is 72°. The remaining two angles have the same size. Which side is longer, s or t?

Answer available at the end of this section.

The Triangle Inequality

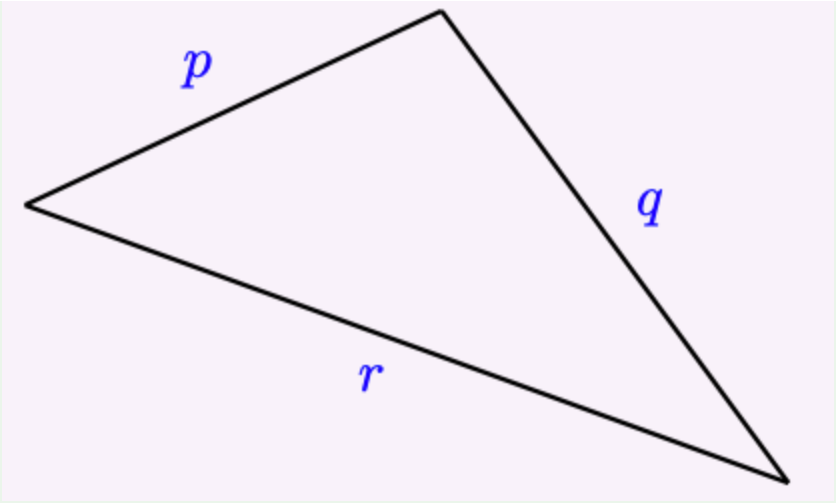
The sum of the lengths of any two sides of a triangle must be greater than the third side; otherwise, the two sides will not meet to form a triangle. This fact is known as the **triangle inequality**. The theorem is presented without proof (however, you can always find a suitable proof online).

Theorem: Triangle Inequality

In any triangle, we must have that

$$p + q > r,$$

where p, q, and r are the lengths of the sides of the triangle (see the figure below).



That is, the sum of the lengths of any two sides of a triangle is greater than the length of the other side.

We cannot use the triangle inequality to find the *exact* lengths of a triangle's sides, but we can find the largest and smallest possible values for the length.

Example 1.2.3

Two sides of a triangle have lengths 7 inches and 10 inches, as shown below. What can be said about the length of the third side?

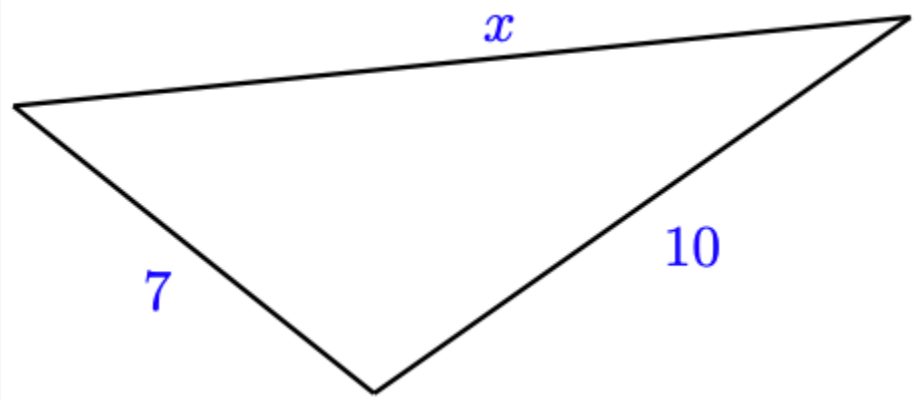


Figure 1.2.8

Solution available at the end of this section.

Interactive Element: The Triangle Inequality

The following Interactive Element might help you visualize why the third side of a triangle cannot be too long or too short.

Interact: Drag the large red vertex left and right to change the length of the third side.

Observation: The two fixed sides form a triangle only while the third side stays between about 3 and 17; beyond either limit, the two sides can no longer reach each other.

Checkpoint 1.2.3

Can a triangle be made with three wooden sticks of lengths 14 feet, 26 feet, and 10 feet? Sketch a picture, and explain why or why not.

Answer available at the end of this section.

Types of Triangles

There are three types of triangles classified by whether they have three equal side lengths, two equal side lengths, or no side lengths that are equal. These are called the equilateral, isosceles, and scalene triangles, respectively.

Equilateral Triangles

Definition: Equilateral Triangle

A triangle in which all three sides have the same length is called an **equilateral triangle**.

Since all three sides of an equilateral triangle have the same length, there is no smallest side for which the smallest angle will be opposite, and there is no largest side for which the largest angle will be opposite. This gives us the following theorem.

Theorem: Angles of an Equilateral Triangle

A triangle is equilateral if and only if it is equiangular, in which case each interior angle measures 60°.

Example 1.2.4

All three sides of a triangle are 4 feet long. Find the angles.

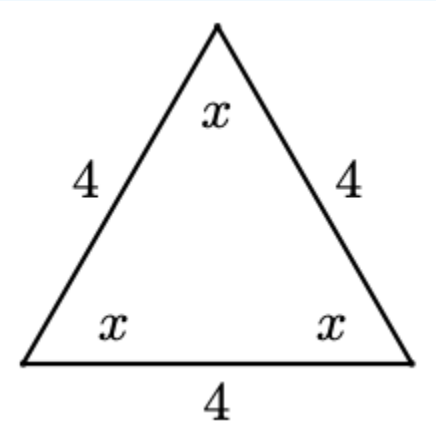


Figure 1.2.9

Solution available at the end of this section.

Checkpoint 1.2.4

Find x, y, and z in the triangle.

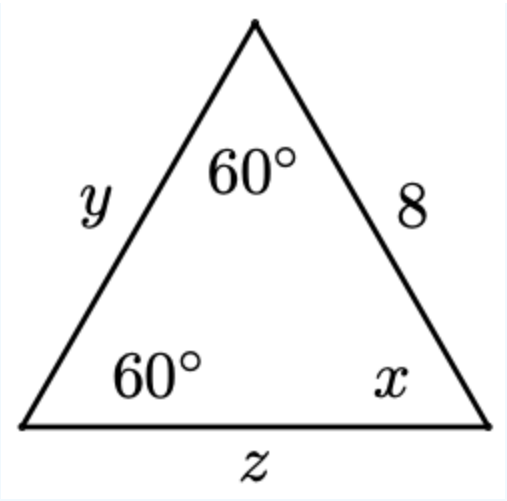


Figure 1.2.10

Answer available at the end of this section.

Isosceles Triangles

Definition: Isosceles Triangle

A triangle in which at least two sides are of equal length is called an **isosceles triangle**. The angle between the equal sides is called the **vertex angle**, and the other two angles are called the **base angles**.

Theorem: Angles of an Isosceles Triangle

The base angles of an isosceles triangle are equal.

In Example 1.2.5, we introduce another common notation used throughout Geometry—the **tick mark**. We use tick marks on line segments to denote they have equal length. Similarly, tick marks on angles denote that the angles have the same measure.

Example 1.2.5

Explain why $\alpha = \beta$ in the triangle below.

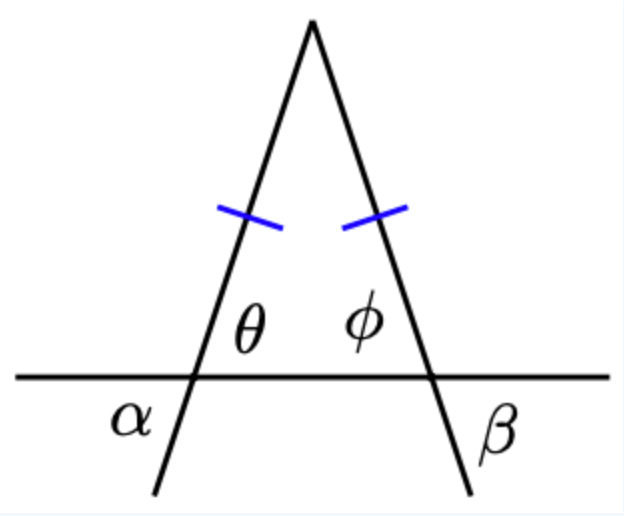


Figure 1.2.11

Solution available at the end of this section.

Example 1.2.6

Find x and y in the triangle.

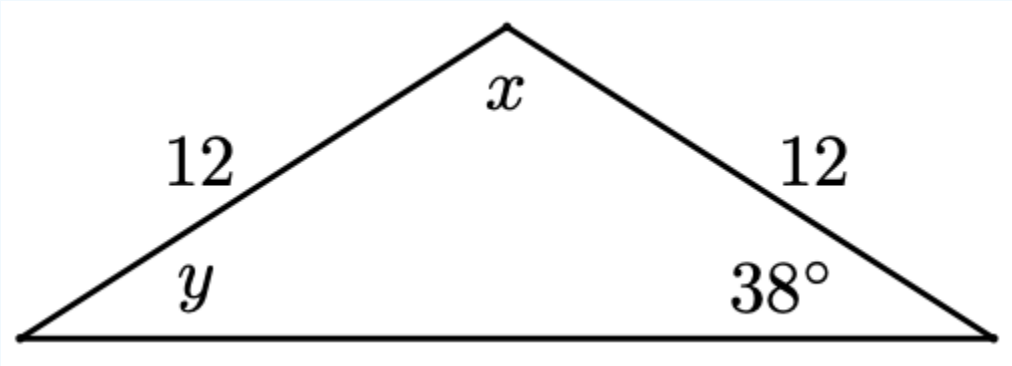


Figure 1.2.12

Solution available at the end of this section.

Checkpoint 1.2.6

Find x and y.

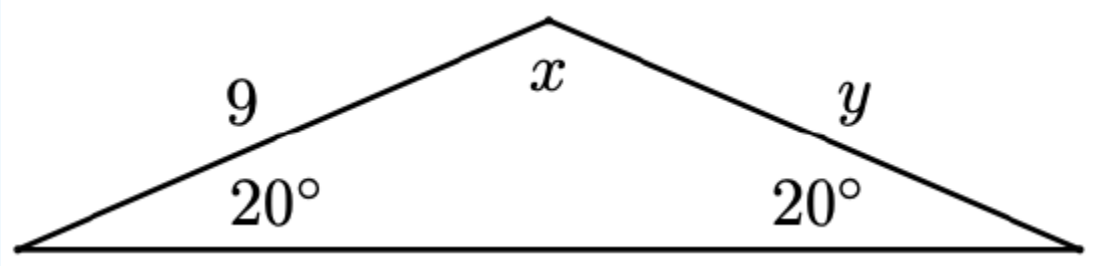


Figure 1.2.13

Answer available at the end of this section.

Scalene Triangles

Finally, we have the **scalene** triangles.

Definition: Scalene Triangle

A triangle in which none of the sides have equal length (and, therefore, none of the angles are equal) is called a **scalene triangle**.

Figure $1.2.14$ shows a general scalene triangle. The different "hash marks" on the sides indicate the side lengths are different. The same is true for the angles.

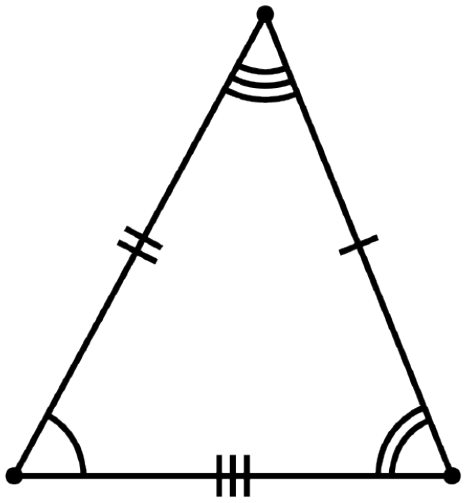


Figure $1.2.14$: *A general scalene triangle*

Scalene triangles become important in this course when we learn the Law of Sines and the Law of Cosines.

Classification of Triangles by Largest Angle

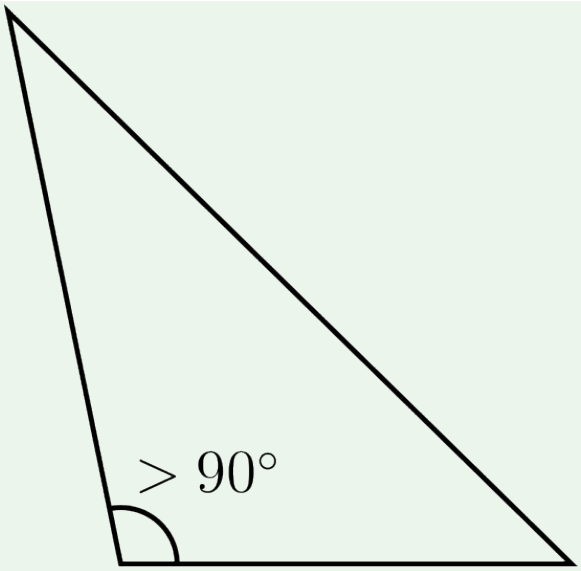
Most people are not likely to refer to a triangle by its type (equilateral, isosceles, or scalene); however, knowing the features of each type of triangle is important in Trigonometry. Another classification system for triangles, and one that will be referenced constantly in Trigonometry, is based on the size of the triangle's largest angle. Three classifications exist: right, obtuse, and acute triangles.

Obtuse Triangles

We start with the class of triangles having their largest angle greater than 90°.

Definition: Obtuse Triangle

An **obtuse triangle** is a triangle that has one interior angle measuring more than 90°.



Recall that an obtuse angle has measure strictly between $90°$ and $180°$ (meaning that we exclude $90°$ and $180°$). Since the sum of the interior angles of a triangle must be $180°$, and one of the angles in an obtuse triangle is greater than $90°$, the two smaller angles in an obtuse triangle must sum to strictly less than $90°$.

Theorem: Angles of an Obtuse Triangle

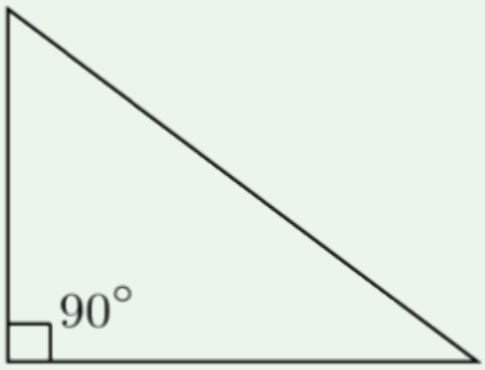
In an obtuse triangle, one angle must be greater than $90°$, and the remaining two angles must each be less than $90°$.

Right Triangles

Reducing the largest possible angle to *exactly* $90°$, we arrive at arguably the most essential triangle classification in Trigonometry—the **right triangles**.

Definition: Right Triangle

A **right triangle** is a triangle that has one interior angle measuring exactly $90°$. The side opposite the right angle is the **hypotenuse**. The two remaining sides are the **legs**.



Example 1.2.7

One of the smaller angles of a right triangle is $34°$. What is the third angle?

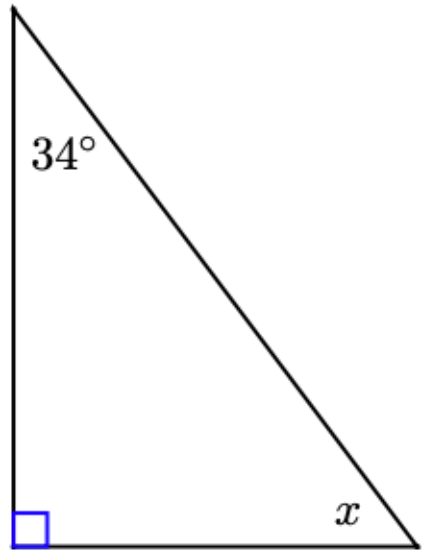


Figure 1.2.15

Solution available at the end of this section.

Checkpoint 1.2.7

Two angles of a triangle are 35° and 45°. Can it be a right triangle?

Answer available at the end of this section.

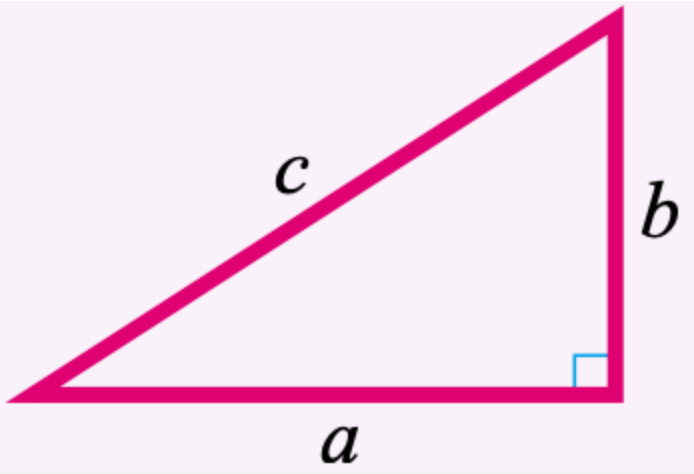
Right Triangles: The Pythagorean Theorem

Arguably, the most important theorem in Trigonometry is the **Pythagorean Theorem**, mainly because without it, Trigonometry would not exist.

Theorem: Pythagorean Theorem

In a right triangle, if c is the length of the hypotenuse, and the lengths of the two legs are denoted by a and b (as shown in the figure below), then

$$a^2 + b^2 = c^2.$$



Interactive Element: The Pythagorean Theorem as Areas

The following Interactive Element might help you visualize what $a^2 + b^2 = c^2$ means in terms of area.

Interact: Drag either red endpoint to change the lengths of the two legs.

Observation: The squares on the two legs always add up in area to the square on the hypotenuse, no matter how the right triangle is reshaped — that is $a^2 + b^2 = c^2$.

The proof of the Pythagorean Theorem is left as a guided exercise in the homework.

Example 1.2.8

A 25-foot ladder is placed against a wall so that its foot is 7 feet from the base of the wall. How far up the wall does the ladder reach?

Solution available at the end of this section.

Checkpoint 1.2.8

A baseball diamond is a square with sides that are 90 feet long. When the catcher sees a runner on first trying to steal second, he throws the ball to the second baseman. Find the straight-line distance from home plate to second base.

Answer available at the end of this section.

Caution

The Pythagorean Theorem applies only to right triangles, and c must always be the hypotenuse—the side opposite the right angle, which is also the longest side.

Most people do not realize that the Pythagorean Theorem is a biconditional. That is, the converse of the theorem is also true. If the sides of a triangle satisfy the relationship $a^2 + b^2 = c^2$, then the triangle must be a right triangle with hypotenuse c. We can use this fact to test whether a given triangle has a right angle.

Example 1.2.9

Mike is paving a patio in his backyard and would like to know if the corner at C is a right angle. He measures 20 cm along one side from the corner and 48 cm along the other, placing pegs P and Q at each position, as shown below. The line joining those two pegs is 52 cm long. Is the corner a right angle?

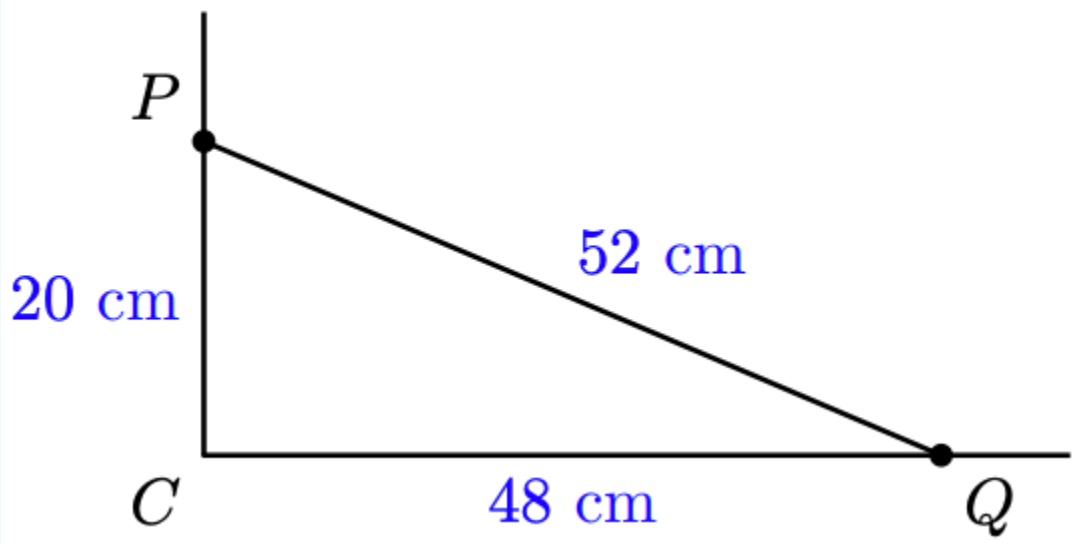


Figure 1.2.17

Solution available at the end of this section.

Checkpoint 1.2.9

The sides of a triangle measure 15, 25, and 30 inches long. Is the triangle a right triangle?

Answer available at the end of this section.

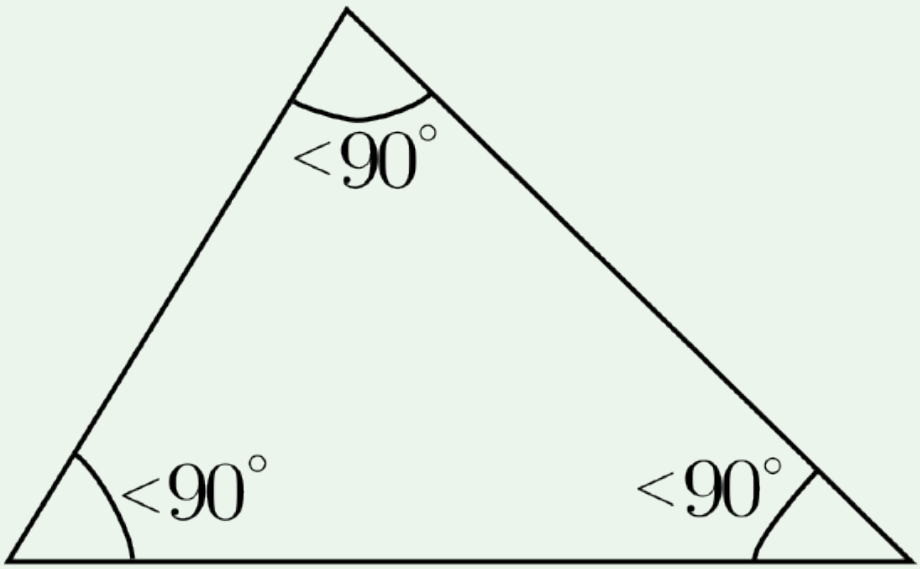
The Pythagorean Theorem relates the sides of *right* triangles. However, for information about the sides of other triangles, the best we can do (without Trigonometry) is the triangle inequality. Moreover, the Pythagorean Theorem doesn't help us find the *angles* in a triangle—that comes later.

Acute Triangles

Finally, we reduce the largest possible angle within a triangle to strictly less than 90° and obtain the **acute triangles**.

Definition: Acute Triangle

An **acute triangle** is a triangle whose three interior angles each measure less than 90°.



Special Triangles

We finally arrive at what could be considered the most essential triangles in elementary Trigonometry—the **special triangles**. These triangles are revered in Trigonometry because they allow the learner (read as "you") to perform computations without needing a calculator. This, in turn, allows you to focus on concepts rather than computations.

In order to derive one of the theorems we are about to introduce, we need a quick definition.

Definition: Altitude of a Triangle

The **altitude** of a triangle is a line segment drawn from a vertex of the triangle perpendicular to the opposite side (or the line containing the opposite side).

The altitude of a triangle introduces a right angle, as the following theorem states.

Theorem: Altitudes and Right Angles

The altitude of a triangle represents the shortest distance from the vertex to the opposite side and forms a 90° angle with that side.

The 30°-60°-90° Triangle

The first of our two special triangles is the 30°-60°-90° **triangle**.

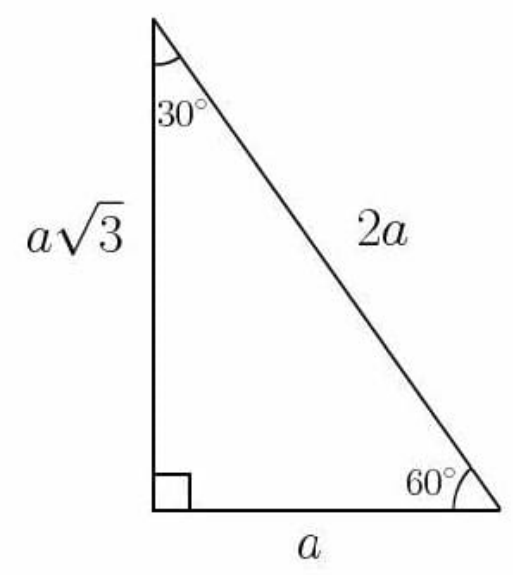
Definition: 30°-60°-90° Triangle

A right triangle in which one angle is 30° and another angle is 60° is called a 30°-60°-90° **triangle**.

Working with this triangle allows us to know the length of all three sides if only given the length of one side.

Theorem: Side Relationships for a $30°$-$60°$-$90°$ Triangle

In any right triangle in which the two acute angles are $30°$ and $60°$, the hypotenuse is always twice the length of the shortest side (the side opposite the $30°$ angle), and the remaining side (opposite the $60°$ angle) is always $\sqrt{3}$ times the shortest side.



Proof available at the end of this section.

Before we go any further, notice in the proof of this theorem the convention of writing $x = a\sqrt{3}$ instead of $x = \sqrt{3}a$. This is purposeful. The latter often leads the reader to think that the a is "under" the square root. It's good form always to write the radical at the end of a term to avoid this visual ambiguity.

Example $1.2.10$

Matterhorn Chocolate bars are sold in boxes shaped like triangular prisms. The two triangular ends of a box are equilateral triangles with 8-centimeter sides. When stacking the boxes on a shelf, the second level will be a triangle's altitude above the bottom level. Find the exact length of the triangle's altitude, h.

Solution available at the end of this section.

Checkpoint $1.2.10$

A ladder is leaning against a house so that the top is 12 feet above the ground, and the bottom makes a $60°$ angle with the ground. Assuming the wall of the house is perpendicular to the ground (and the ground is level), how long is the ladder? State your answer in exact form and round to two decimal places.

Answer available at the end of this section.

Note that the exact form of the answer in Checkpoint $1.2.10$ was left as $\frac{24}{\sqrt{3}}$. While it is true that you could rationalize the denominator to get

$$\frac{24}{\sqrt{3}} = \frac{24}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{24\sqrt{3}}{3} = 8\sqrt{3},$$

we will commonly leave radicals in the denominators of purely numeric expressions.

The $45°$-$45°$-$90°$ Triangle

The second of our two special triangles is the $45°$-$45°$-$90°$ **triangle**. This is also known as an **isosceles right triangle**.

Definition: Isosceles Right Triangle

An isosceles triangle in which the vertex angle is $90°$ is called an **isosceles right triangle**.

Since the right angle takes $90°$, the remaining two angles must share the remaining $90°$ evenly. That is, the remaining two angles must be $45°$ each.

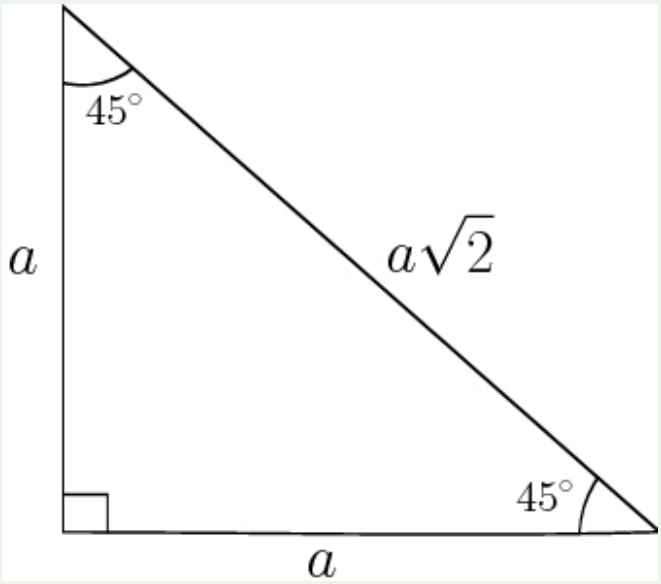
Definition: $45°$-$45°$-$90°$ Triangle

Since an isosceles right triangle has two angles that are $45°$, it is commonly called a $45°$-$45°$-$90°$ **triangle**.

Just as with our first special triangle, knowing one side of a $45°$-$45°$-$90°$ triangle allows us to easily compute the lengths of the remaining sides.

Theorem: Side Relationships for a $45°$-$45°$-$90°$ Triangle

In any right triangle in which the two acute angles are $45°$, the hypotenuse is always $\sqrt{2}$ times the side length.



Proof available at the end of this section.

Example 1.2.11

You are hired to build a truss to support a balcony. Figure $1.2.19$ shows the design plans you have submitted for the project. What is the length of the beam needed from point A to point D?

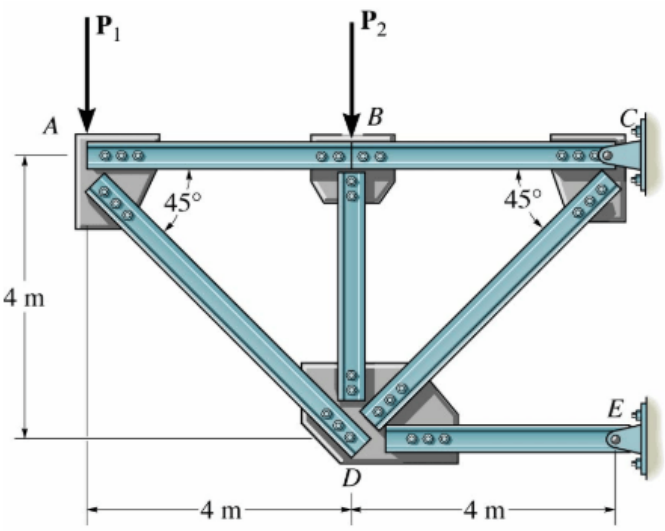


Figure 1.2.19

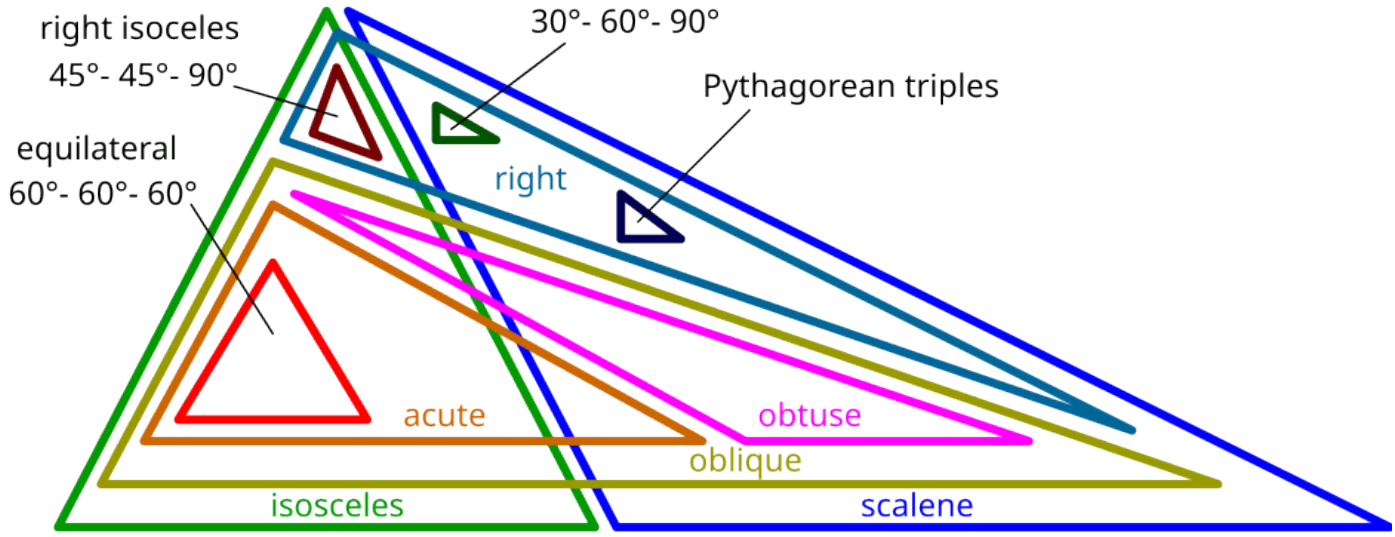
Solution available at the end of this section.

Checkpoint 1.2.11

Suppose, instead of knowing the 4-meter lengths in Example 1.2.11, we knew that the length of the beam from A to D was 12 meters. How long is the beam from B to D?

Answer available at the end of this section.

Now that we have successfully covered all of the types of triangles, we have the ability to appreciate the Euler diagram of the types of triangles (don't worry—you are not responsible for memorizing this).



Area of a Triangle

Our final review topic for triangles in this section is something necessary at multiple points throughout Trigonometry and

Calculus—the area of a triangle.

Theorem: Area of a Triangle

Let b be the length of one side of a triangle and h be the altitude height from that side to its opposite vertex. Then the area of the triangle is given by

$$A = \frac{1}{2}bh.$$

Proof available at the end of this section.

Interactive Element: Base, Height, and Area

The following Interactive Element might help you visualize why a triangle's area depends only on its base and height.

Interact: Drag the red apex left and right along the dashed rail.

Observation: The triangle changes shape, but with the base and height unchanged, the area $\frac{1}{2}bh = \frac{1}{2}(8)(4) = 16$ stays the same.

Example 1.2.12

Compute the area of the triangle in Figure 1.2.20.

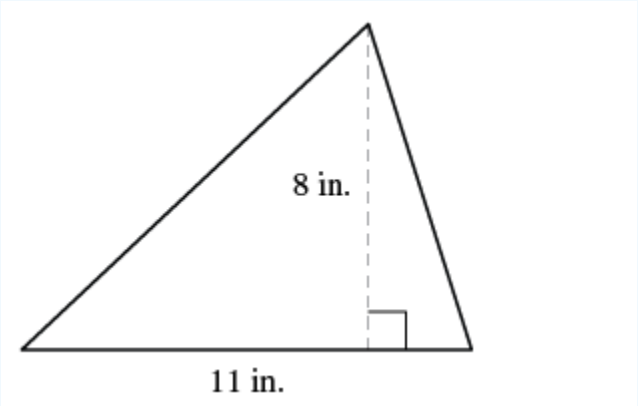


Figure 1.2.20

Solution available at the end of this section.

Checkpoint 1.2.12

Compute the area of the triangle in Figure 1.2.21 if we know that $c = 5.24$ and $h = 1.25$.

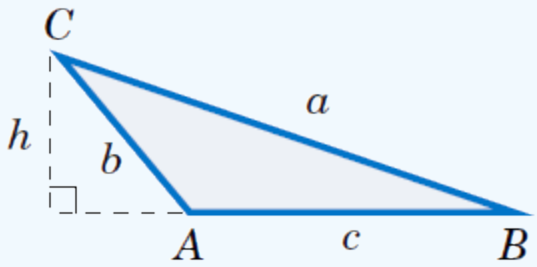


Figure 1.2.21

Answer available at the end of this section.

Time-Sensitive Advice: HawkConnect

What is HawkConnect?

In the past, navigating through all the student resources (EOPS, TRIO, MI CASA, etc.) required you to go to individual sites and apply individually for each program. HawkConnect now unifies the entire process into a single, guided application process.

To learn more about HawkConnect and to apply, go to the CRC Support Services website.

Success in Mathematics: Show Your Work

Many mathematics exams include the statement, "Show Your Work" (if it's not explicitly written, then it is implied). The exact meaning of these words can change from one professor to another; however, they usually mean that you need to show more than just the answer to a problem. You need to explain or show how you found the answer. Your responsibility is to learn what "Show Your Work" means to your instructor.

Why do you need to show your work?

Your *authentic* work (e.g., without use of A.I., tutors, or any other external help) demonstrates what you know. It shows what you understand and what you do not understand. By looking at your work and the work of others, an instructor can help you improve your understanding.

Everyone makes mistakes. To learn from these mistakes, you need to be able to find them. If you show your work, you can go back and find mistakes. This helps you identify misunderstandings and bad habits. Without your work, all you can do is start over.

Finally, although you may be able to do a problem "in your head," this will not always be possible. The problems get more challenging and involve more detail as we move forward. Showing your work now develops the habits and procedures you will need to do math in the future.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. A student did a homework exercise. She checked her answer with the back of the book. It was wrong. So, she erased her work and started over. Why should she not erase her work?
2. A student did a homework exercise. He checked his answer with the back of the book. It was wrong. He marked the problem on his homework with an asterisk. He went on with the assignment. He never went back to the problem. Why is it essential for this student to finish the problem?

If you are unsure about the answers to either of these questions, feel free to ask your instructor - that's what they are there for!

Answers

Section Preview

Triangles, being the focus of Trigonometry, deserve their own full section. Focus here is on types of triangles, the Pythagorean Theorem, the special triangles, and the area of a triangle (not requiring trigonometric functions). Many exercises involving the 30°-60°-90° and 45°-45°-90° triangles that would normally be presented in Right Triangle Trigonometry are found in this section and its homework (but without reference to the trigonometric functions). Similar triangles are held off until next section.

Section Preview

- **Sets and Numbers**
  - Inequalities and Inequality Notation: For understanding the Triangle Inequality.
- **Simplifying Expressions**
  - Simplifying Products of Radical Expressions: For following the special triangle theorems and their proofs.

Section Preview

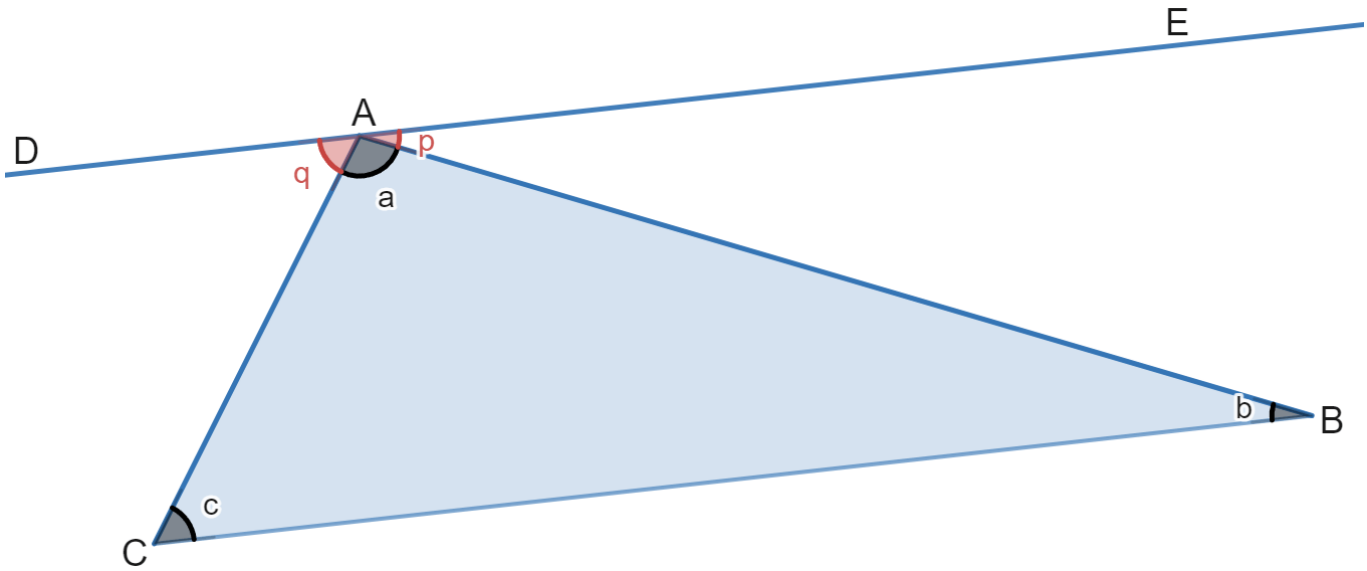
- **Arithmetic**
  - Mathematical Operations - Square Roots: For applying the Pythagorean Theorem numerically.
- **Geometry**
  - Volumes and Surface Areas of Common Shapes: For specific homework applications.
- **Solving Equations**
  - Solving Quadratic Equations Using Extraction of Roots: For solving $x^2 = k$.
- **Solving Inequalities**
  - Solving Linear Inequalities: For the Triangle Inequality problems.
  - Solving Compound Inequalities: For the Triangle Inequality problems.

Section Preview

- Find the missing angle within a triangle using the Triangle Sum.
- Use the Triangle Inequality to determine constraints on an unknown side length of a triangle.
- Use the Pythagorean Theorem to find the third side of a right triangle.
- Find the missing sides of a 30°-60°-90° or 45°-45°-90° triangle given one side.
- Solve real-world applications requiring the Pythagorean Theorem or the special triangles.

Theorem: Triangle Sum

Given the triangle, $\triangle ABC$, draw a line parallel to side $\overline{BC}$ going through vertex A (see the figure below).



Since $\angle q + \angle a + \angle p$ forms a straight angle,

$$\angle q + \angle a + \angle p = 180^\circ.$$

Moreover, the line segment $\overline{AC}$ is a transversal for the parallel lines $\overline{BC}$ and $\overline{DE}$. Therefore, $\angle c = \angle p$. By a similar argument, $\angle b = \angle q$. Hence,

$$\angle b + \angle a + \angle c = 180^\circ.$$

Thus, the sum of the angles of a triangle is 180°.

Example 1.2.1

To find the third angle, we write an equation.

$$\begin{aligned} & & x + 25 + 115 &= 180 && \text{(Triangle Sum)} \\ \xrightarrow{\text{Arithmetic}} & & x + 140 &= 180 && \text{(add)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & & x &= 40 && \text{(subtract 140 from both sides)} \end{aligned}$$

The third angle is 40°.

Checkpoint 1.2.1

a. $x = 39^\circ, 2x = 78^\circ, 2x - 15 = 63^\circ$
b. $\alpha = 40^\circ, \beta = 140^\circ, \gamma = 75^\circ, \delta = 65^\circ$

Example 1.2.2

Because $\angle G$ is greater than 90°, we know that $\angle F + \angle G$ is greater than $90^\circ + 48^\circ = 138^\circ$, so $\angle H$ is less than $180^\circ - 138^\circ = 42^\circ$. Thus, $\angle H < \angle F < \angle G$, and consequently \(h6\) feet long.

Checkpoint 1.2.2

s is longer

Example 1.2.3

We let x represent the length of the third side of the triangle. By looking at each side in turn, we can apply the triangle inequality in three different ways to get\[7-3\) because x must be positive, but the other two inequalities do give us new information. The third side must be greater than 3 inches but less than 17 inches long.

Checkpoint 1.2.3

No, $10 + 14$ is not greater than 26.

Example 1.2.4

The triangle is equilateral, so all of its angles are equal. Thus

$$\begin{aligned} & & 3x &= 180 && \text{(Triangle Sum)} \\ \xRightarrow{\text{Arithmetic} \to \text{Algebra}} & & x &= 60 && \text{(divide both sides by 3)} \end{aligned}$$

Each of the angles is 60°.

Checkpoint 1.2.4

$x = 60^\circ, y = 8, z = 8$

Example 1.2.5

Because they are the base angles of an isosceles triangle, θ (theta) and ϕ (phi) are equal. Also, $\alpha = \theta$ because they are vertical angles, and similarly $\beta = \phi$. Therefore, $\alpha = \beta$ because they are equal to equal quantities.

Example 1.2.6

The triangle is isosceles, so the base angles are equal. Therefore, $y = 38^\circ$. To find the vertex angle, we solve

$$\begin{aligned} & & x + 38 + 38 &= 180 && \text{(Triangle Sum)} \\ \xRightarrow{\text{Arithmetic}} & & x + 76 &= 180 && \text{(add)} \\ \xRightarrow{\text{Arithmetic} \to \text{Algebra}} & & x &= 104 && \text{(subtract 76 from both sides)} \end{aligned}$$

The vertex angle is 104°.

Checkpoint 1.2.6

$x = 140^\circ, y = 9$

Example 1.2.7

The sum of the two smaller angles in a right triangle is 90°. So

$$\begin{aligned} & x + 34 &= 90 \\ \xRightarrow{\text{Arithmetic} \to \text{Algebra}} & x &= 56 && \text{(subtract 34 from both sides)} \end{aligned}$$

The unknown angle must be 56°.

Checkpoint 1.2.7

No.

Example 1.2.8

We sketch the situation, as shown below, and label any known dimensions. We will call the unknown height h.

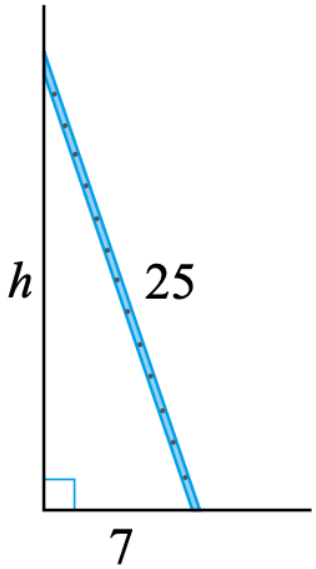


Figure 1.2.16

The ladder forms the hypotenuse of a right triangle, so we can apply the Pythagorean Theorem. Substituting 25 for c, 7 for b, and h for a, we get the following:

$$\begin{array}{rcrcll} & & a^2+b^2 & = & c^2 & \\ \xcancel{\text{Arithmetic}}\to\text{Algebra} & \Longrightarrow & h^2+7^2 & = & 25^2 & \text{(substitution)} \\ \text{Arithmetic} & \Longrightarrow & h^2+49 & = & 625 & \text{(apply powers)} \\ \xcancel{\text{Arithmetic}}\to\text{Algebra} & \Longrightarrow & h^2 & = & 576 & \text{(subtract 49 from both sides)} \\ \xcancel{\text{Arithmetic}}\to\text{Algebra} & \Longrightarrow & h & = & \pm\sqrt{576} & \text{(Extraction of Roots)} \\ \xcancel{\text{Arithmetic}}\to\text{Algebra} & \Longrightarrow & h & = & \pm 24 & \text{(simplify the radical)} \end{array}$$

The height must be a positive number, so the solution -24 does not make sense for this problem. The ladder reaches 24 feet up the wall.

Checkpoint 1.2.8

$90\sqrt{2} \approx 127.3$ feet

Example 1.2.9

If it is a right triangle, its sides must satisfy $p^2+q^2=c^2$. We find

$$\begin{aligned} p^2+q^2 &= 20^2+48^2 \\ &= 400+2304 \\ &= 2704 \end{aligned}$$

and

$$\begin{aligned} c^2 &= 52^2 \\ &= 2704 \end{aligned}$$

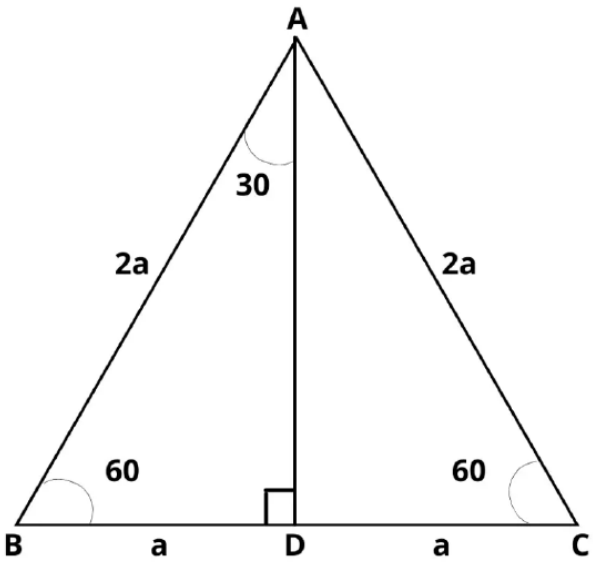
Therefore, yes, because $p^2+q^2=c^2$, the corner at C is a right angle.

Checkpoint 1.2.9

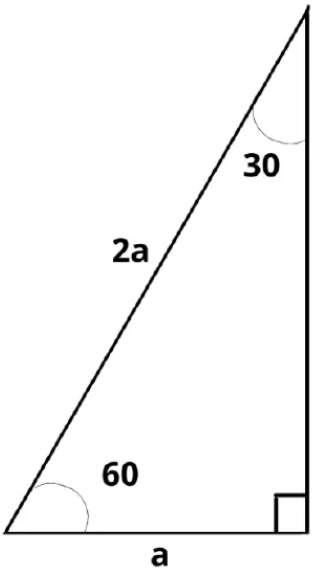
No

Theorem: Side Relationships for a 30°-60°-90° Triangle

Consider an equilateral triangle with side lengths $2a$, as shown in the figure below.



Since this triangle is equilateral, each angle is 60°. Drawing an altitude divides the triangle into two smaller triangles. Since the altitude is perpendicular to the base, each of these smaller triangles is a right triangle. Moreover, since one angle in each of these smaller right triangles is 60°, the remaining angle must be 30°. Furthermore, the side opposite the 30° angle must have length a (in the original equilateral triangle, the side opposite 60° had length $2a$, so it should make sense that the side opposite 30° is half of this). We now consider one of these smaller right triangles (see the figure below).



The missing side length, which we will call x for now, can be found using the Pythagorean Theorem.

$$\begin{aligned} & x^2 + a^2 &=& (2a)^2 \\ \implies & x^2 + a^2 &=& 4a^2 \\ \implies & x^2 &=& 3a^2 \\ \implies & x &=& a\sqrt{3} \end{aligned}$$

Example 1.2.10

The altitude divides the triangle into two $30^\circ - 60^\circ - 90^\circ$ right triangles, as shown in Figure 1.2.18 below.

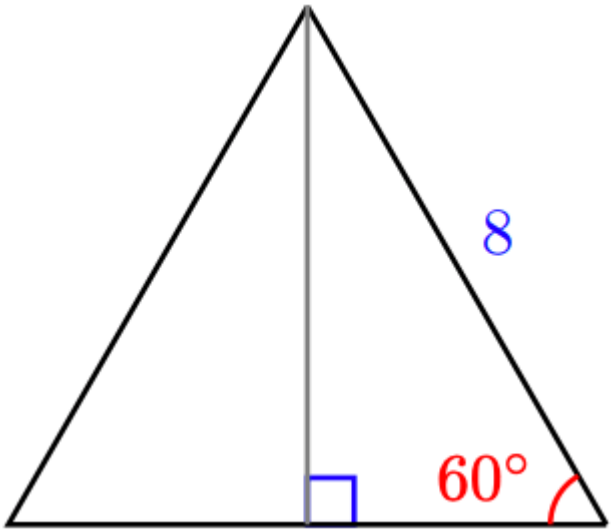


Figure 1.2.18

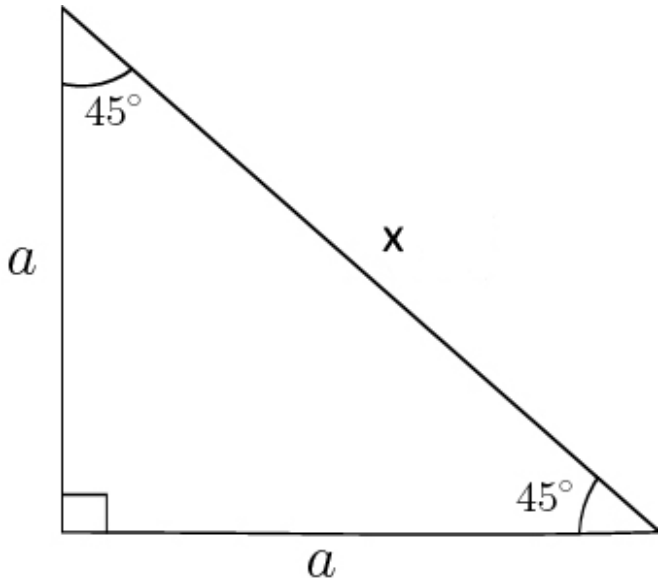
We see that the hypotenuse is $2a = 8$, which implies the shortest side (opposite the 30° angle) is $a = 4$. Hence, the side opposite the 60° angle is $a\sqrt{3} = 4\sqrt{3}$ ≈ 6.9282 centimeters long.

Checkpoint 1.2.10

$\frac{24}{\sqrt{3}} \approx 13.86$ feet

Theorem: Side Relationships for a 45°-45°-90° Triangle

The sides opposite the 45° angles in a 45°-45°-90° triangle must have the same length. Let a be the lengths of these sides and x be the length of the hypotenuse, as shown in the figure below.



Since this is a right triangle, we can use the Pythagorean Theorem to find the length of the hypotenuse in terms of a.

$$\begin{aligned} a^2 + a^2 &= x^2 \\ \Longrightarrow \quad 2a^2 &= x^2 \\ \Longrightarrow \quad a\sqrt{2} &= x \end{aligned}$$

Example 1.2.11

$\triangle ABD$ is a 45°-45°-90° triangle. Therefore, if the leg length (opposite the 45° angle) is $a = 4$ meters, then the beam between points A and D must measure $a\sqrt{2} = 4\sqrt{2} \approx 5.66$ meters.

Checkpoint 1.2.11

$\frac{12}{\sqrt{2}} \approx 8.49$ meters

Theorem: Area of a Triangle

To be done in homework.

Example 1.2.12

We are given the length of one leg of this triangle as 11 inches. Let's call this leg the base. Thus, $b = 11$. The altitude from the base to its opposite vertex has length 8 inches. Hence, $h = 8$. Therefore, the area of this triangle is

$$A = \frac{1}{2}bh = \frac{1}{2}(11)(8) = 44 \,\text{square inches}.$$

Checkpoint 1.2.12

3.275

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1.2.1: Resources and Key Concepts

Resources

Videos

- Triangles, their types, and the Pythagorean Theorem
  - Right Triangle Trigonometry - Triangles
- Special Triangles
  - Right Triangle Trigonometry - Special Triangles in Trigonometry

Key Concepts

Definitions

- **Triangle:** A three-sided polygon formed by three line segments bounding a portion of a plane. The line segments are its sides, the points where sides meet are its vertices, and the interior angles are its angles.
- **Equilateral Triangle:** A triangle where all three sides have the same length.
- **Isosceles Triangle:** A triangle where at least two sides have equal length. The angle between the two equal sides is the vertex angle, and the other two angles are the base angles.
- **Scalene Triangle:** A triangle where no sides have equal length.
- **Obtuse Triangle:** A triangle that contains one obtuse angle (an angle greater than 90°).
- **Right Triangle:** A triangle that contains one right angle (an angle of exactly 90°).
- **Acute Triangle:** A triangle in which all three angles are acute (less than 90°).
- **30°-60°-90° Triangle:** A right triangle whose acute angles measure 30° and 60°.
- **Isosceles Right Triangle:** An isosceles triangle where the vertex angle is 90°.
- **45°-45°-90° Triangle:** A common name for an isosceles right triangle, since its two other angles must be 45°.

Theorems

- **Triangle Sum:** The sum of the interior angles in a triangle is 180°.
- **Triangle Inequality:** In any triangle with side lengths p, q, and r, the sum of the lengths of any two sides is greater than the length of the third side ($p + q > r$).
- **Angles of an Equilateral Triangle:** All the angles of an equilateral triangle are equal.
- **Angles of an Isosceles Triangle:** The base angles of an isosceles triangle are equal.
- **Angles of an Obtuse Triangle:** In an obtuse triangle, one angle must be greater than 90°, and the remaining two angles must each be less than 90°.
- **Pythagorean Theorem:** In a right triangle, if c is the length of the hypotenuse and the lengths of the two legs are a and b, then $a^2 + b^2 = c^2$.
- **Converse of the Pythagorean Theorem:** If the sides of a triangle satisfy the relationship $a^2 + b^2 = c^2$, then the triangle must be a right triangle with hypotenuse c.
- **Side Relationships for a 30°-60°-90° Triangle:** In any 30°-60°-90° triangle, the hypotenuse is twice the length of the shortest side (the side opposite the 30° angle), and the side opposite the 60° angle is $\sqrt{3}$ times the length of the shortest side.
- **Side Relationship for a 45°-45°-90° Triangle:** In any 45°-45°-90° triangle, the hypotenuse is $\sqrt{2}$ times the length of a leg.

Common Mistakes

- **Misapplying the Pythagorean Theorem:** The Pythagorean Theorem is only valid for right triangles. It cannot be used to find side lengths in acute or obtuse triangles.
- **Confusing the Triangle Inequality with the Pythagorean Theorem:** The Triangle Inequality theorem determines if a

triangle can be formed from three side lengths. The Converse of the Pythagorean Theorem determines if a triangle is a right triangle. A triangle can exist without being a right triangle.

- **Radical Notation Ambiguity:** When a term includes a variable and a radical, it is best to write the radical last to avoid confusion. For example, write $a\sqrt{3}$ instead of $\sqrt{3}a$, as the latter might be misinterpreted as $\sqrt{3a}$.
- **Confusing Isosceles Triangle Angles:** The vertex angle of an isosceles triangle is the angle between the two equal sides. The base angles are opposite the equal sides. The base angles are equal to each other, but not necessarily equal to the vertex angle.

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1.2.2: Homework

Skills Refresher

The following is a set of review exercises you will need for this section.

Skills Refresher

For the following exercises, solve the inequality.

1. $6 - x > 3$
2. $\frac{-3x}{4} \geq -6$
3. $3x - 7 \leq -10$
4. $4 - 3x < 2x + 9$

For the following exercises, assume that $x < 0$. If this is the case, is the expression positive or negative?

5. $-x$
6. $-(-x)$
7. $|x|$
8. $-|x|$
9. $-|-x|$
10. x^{-1}

Answers available at the end of this section.

Homework

Reading Questions

1. What is the sum of the interior angles in any triangle?
2. What does the Triangle Inequality Theorem state about the side lengths of a triangle?
3. What is the definition of an equilateral triangle?
4. In an isosceles triangle, which angles are referred to as the "base angles," and what is true about their measures?
5. What is the name for the side opposite the right angle in a right triangle?
6. State the Pythagorean Theorem using the terms hypotenuse and legs.
7. If you are given the three side lengths of a triangle, how can you test if it is a right triangle?
8. Is it possible for a triangle to contain two right angles? Why or why not?

9. In a 30°-60°-90° triangle, which side is the longest?
10. What is the relationship between the hypotenuse and the shortest side in a 30°-60°-90° triangle?
11. What is another name for a 45°-45°-90° triangle?
12. If the two legs of a 45°-45°-90° triangle each have length a, what is the length of the hypotenuse?

Vocabulary Check

13. A three-sided polygon is another way of saying ___.
14. (MyOpenMath) If given the triangle $\triangle ABC$, it is common notation to label the side opposite $\angle A$ with the letter ___.
15. (MyOpenMath) A triangle where all three sides have the same length is called a ___ triangle.
16. An isosceles triangle has two ___ angles.
17. The ___ angle of an isosceles triangle lies between the two sides of equal length.
18. (MyOpenMath) A(n) ___ triangle must have one angle ___ 90°.
19. A(n) ___ triangle must have all angles ___ 90°.
20. The two special triangles are the ___-90° and ___-90° triangles.
21. Another name for the 45°-45°-90° triangle is a(n) ___ triangle.

Concept Check

22. Is it always true that the hypotenuse is the longest side in a right triangle? Why or why not?
23. In $\triangle DEF$, is it possible that $d + e > f$ and $e + f > d$ are both true? Explain your answer.
24. In a right triangle with hypotenuse c, we know that $a^2 + b^2 = c^2$. Is it also true that $a + b = c$? Why or why not?
25. Is having more than one obtuse angle in a triangle possible? Why or why not?
26. Draw any quadrilateral (a four-sided polygon) and divide it into two triangles by connecting two opposite vertices by a diagonal. What is the sum of the angles in your quadrilateral?
27. What is the difference between a vertex and vertical angles?
28. In any triangle, the sum of the three angles is ___.
29. The longest side in a triangle is opposite the ___ angle, and the shortest side is opposite the ___ angle.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

30. A right triangle always has one angle of 90°.
31. Given a triangle with side lengths a, b, and c, where c is the longest side,
$$a^2 + b^2 = c^2.$$
32. All of the angles of an equilateral triangle are equal.
33. The vertex angles of an isosceles triangle are equal.

34. If the sum of the lengths of any two sides is greater than the length of the third side, then the triangle must be a right triangle.

35. In $\triangle ABC$, $a^2 + b^2 = c^2$.

36. If the sides of a triangle satisfy the relationship $a^2 + b^2 = c^2$, then the triangle is a right triangle.

Basic Skills

For the following exercises, sketch and label a triangle with the given properties.

37. An isosceles triangle with vertex angle 30°

38. A scalene triangle with one obtuse angle

39. (MyOpenMath) A right triangle with legs 4 and 7

40. (MyOpenMath) A right triangle with hypotenuse 12, where one leg is of length 8.

41. An isosceles right triangle

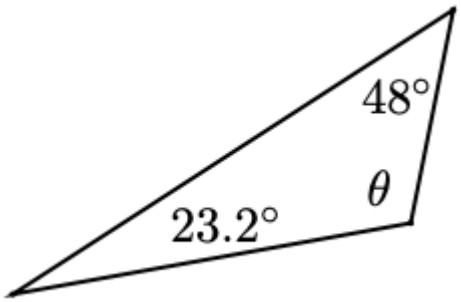
42. An isosceles triangle with one obtuse angle

43. A right triangle with one angle 20°

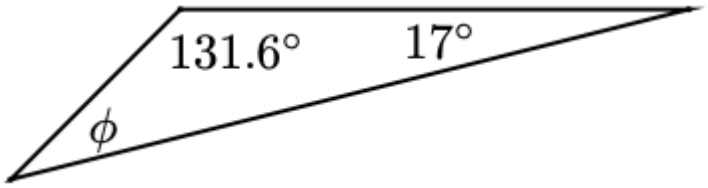
44. (MyOpenMath) The triangle $\triangle ABC$ has $m\angle A = 88^\circ$ and $m\angle B = 87^\circ$, find $m\angle C$.

For the following exercises, find the unknown angle.

45. (MyOpenMath)

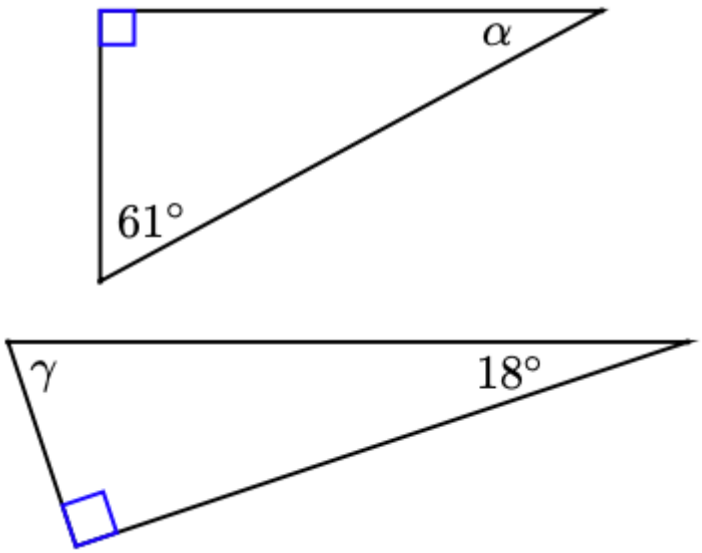


46.

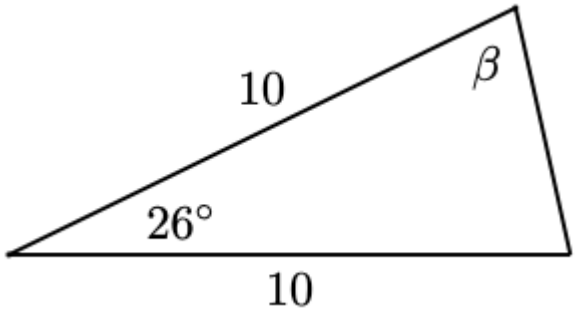


47. (MyOpenMath)

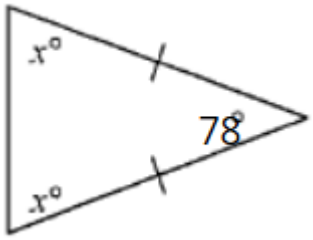
18.



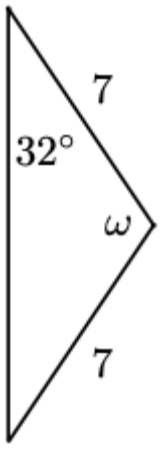
19.



50. (MyOpenMath)



51. (MyOpenMath)



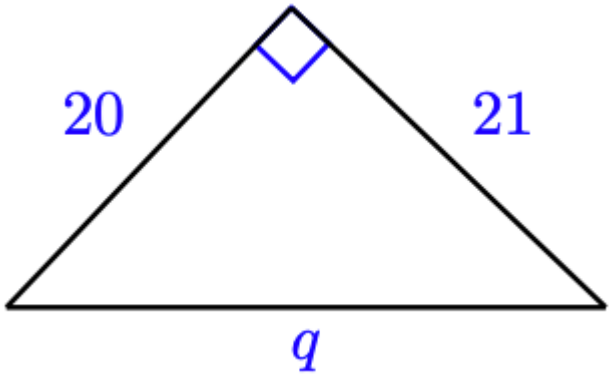
52. If two sides of a triangle are 6 feet and 10 feet long, what are the largest and smallest possible values for the length of the third side?

53. Two adjacent sides of a parallelogram are 3 cm and 4 cm long. What are the largest and smallest possible values for the length of the diagonal?

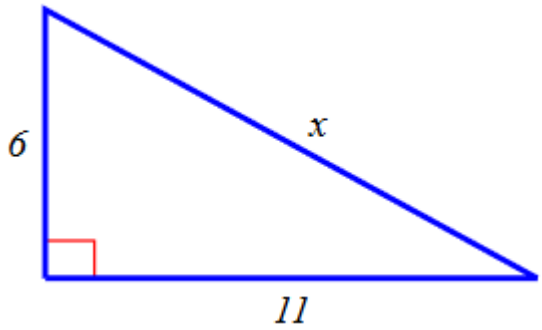
54. If one of the equal sides of an isosceles triangle is 8 millimeters long, what are the largest and smallest possible values for the length of the base?

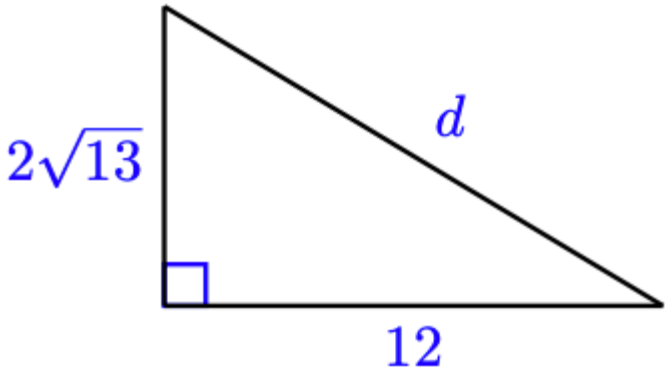
For the following exercises, find the unknown side of the triangle.

55.



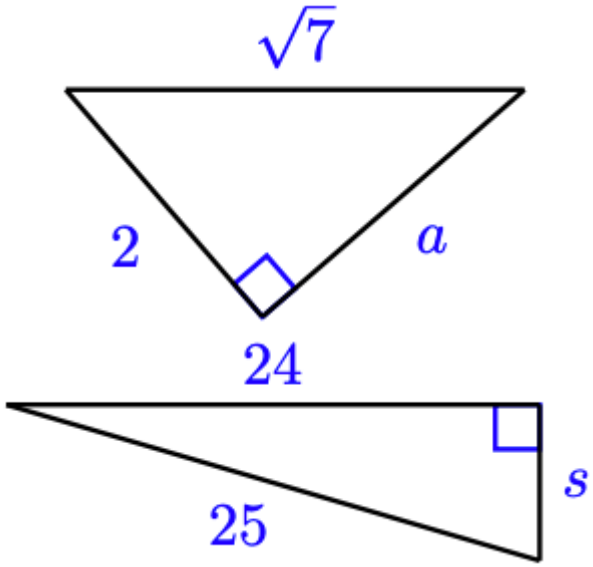
56. (MyOpenMath)





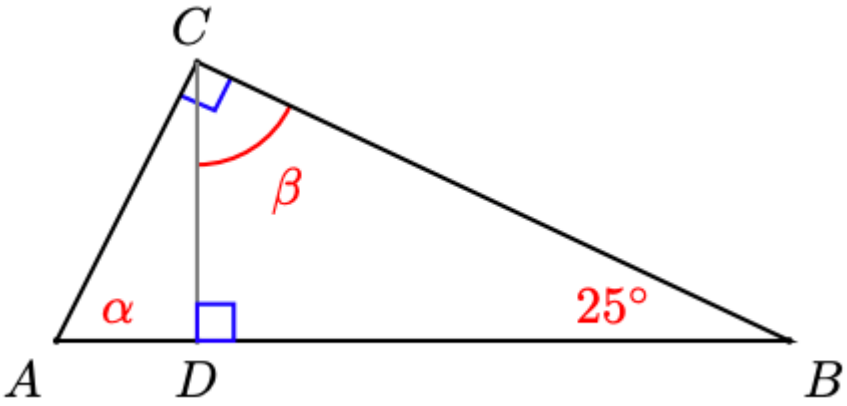
57.

58. (MyOpenMath)

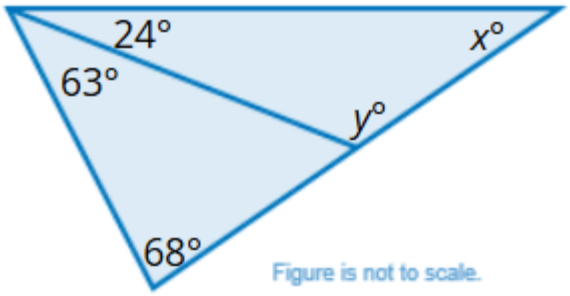


59.

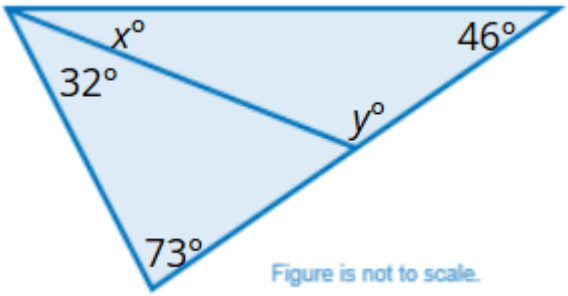
60. Find α and β.



61. (MyOpenMath) Find x and y.



52. (MyOpenMath) Find x and y.



For the following exercises, decide whether a triangle with the given sides is a right triangle.

53. (MyOpenMath) 9 in, 16 in, 25 in

54. 12 m, 16 m, 20 m

55. 5 m, 12 m, 13 m

56. 5 ft, 8 ft, 13 ft

57. 5^2 ft, 8^2 ft, 13^2 ft

58. $\sqrt{5}$ ft, $\sqrt{8}$ ft, $\sqrt{13}$ ft

59. (MyOpenMath) Compute the exact value of the altitude h in an equilateral triangle, where all sides are 14 inches long.

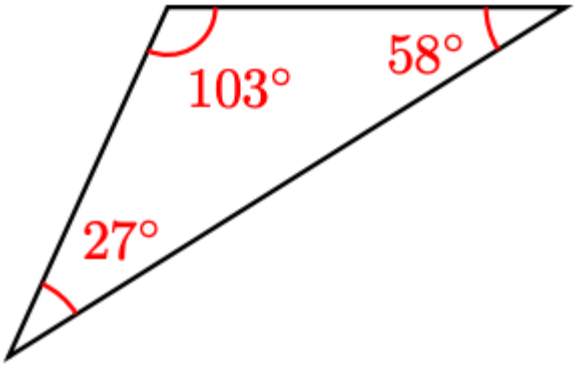
Synthesis Questions

70. (MyOpenMath) In a certain triangle, angle A is three times as big as angle C and angle B is 55° more than angle C. What is the degree measure of angle B?

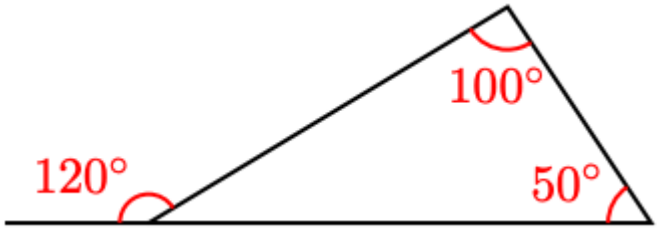
71. The two shorter sides of an obtuse triangle are 3 in and 4 in. What are the possible lengths for the third side?

For the following exercises, explain why the measurements shown cannot be accurate.

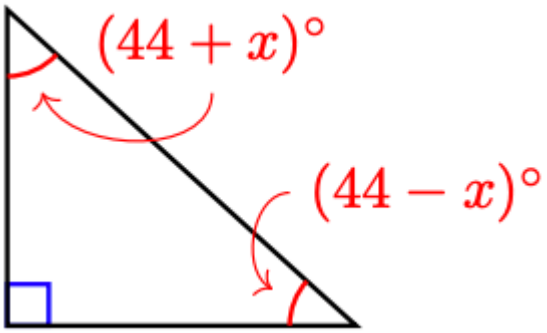
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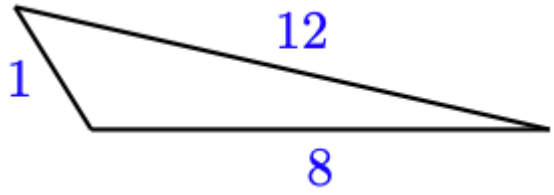
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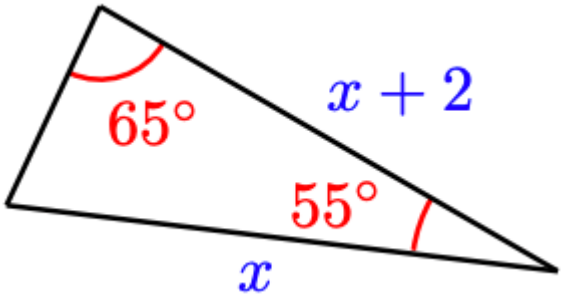
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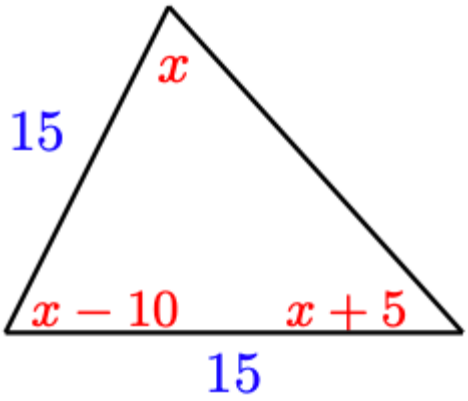
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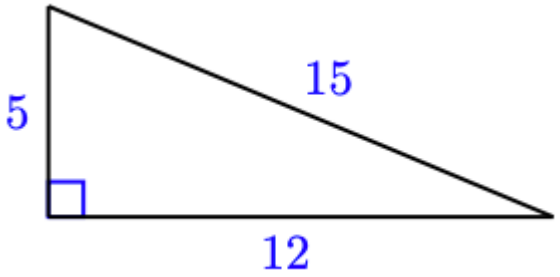
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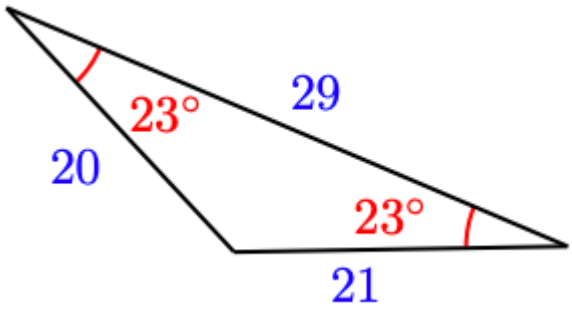
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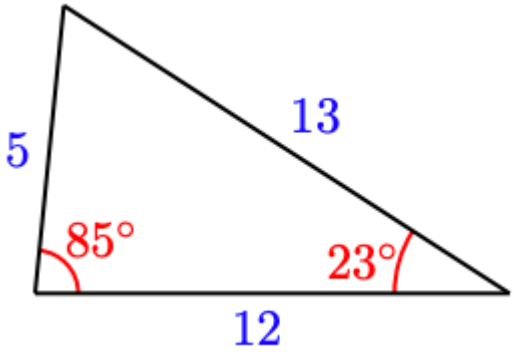
78.



79.

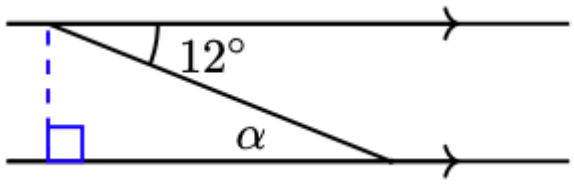


80.

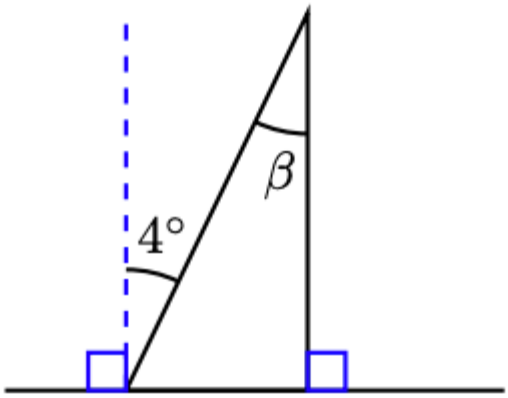


For the following exercises, find the unknown angle. Arrows on a pair of lines indicate that they are parallel.

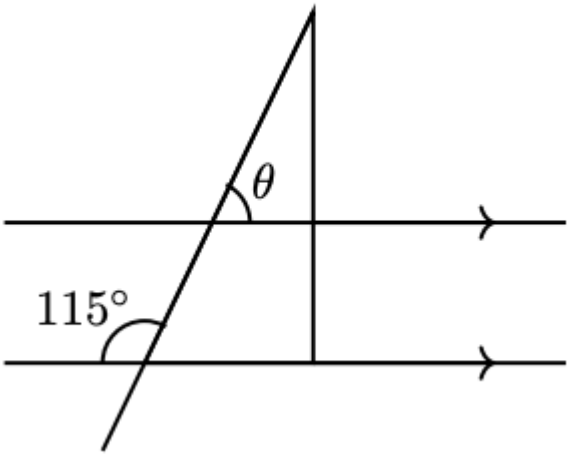
81.



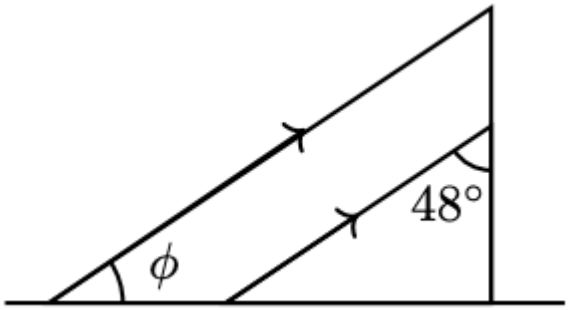
32.



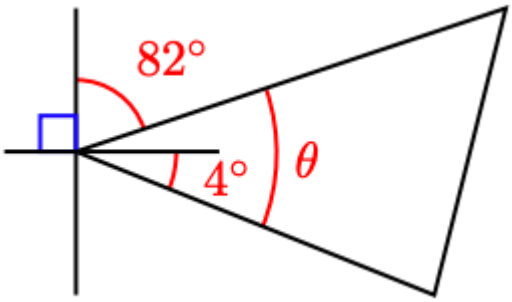
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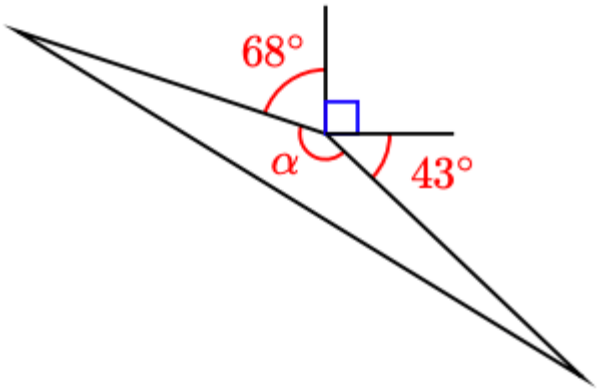


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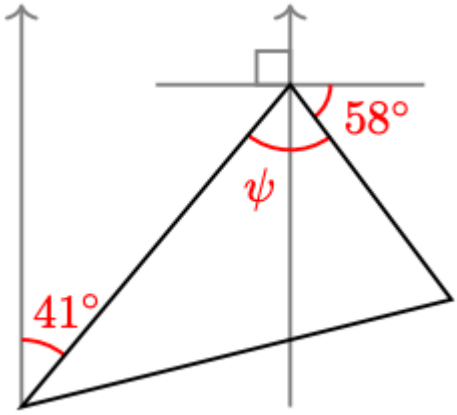


35.

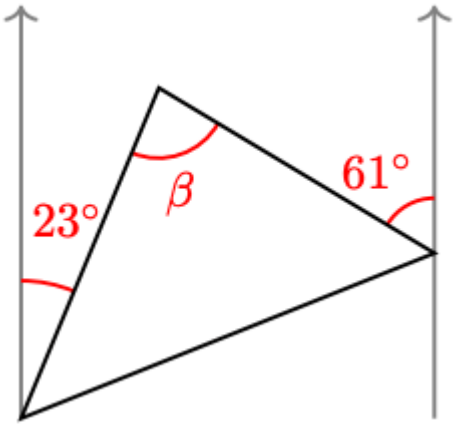




36.



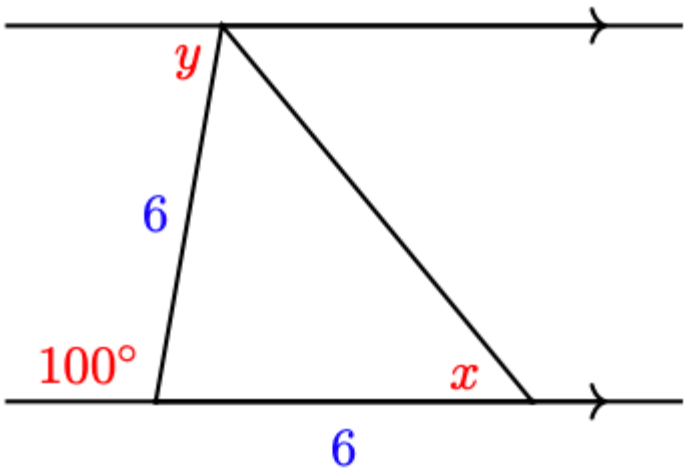
37.



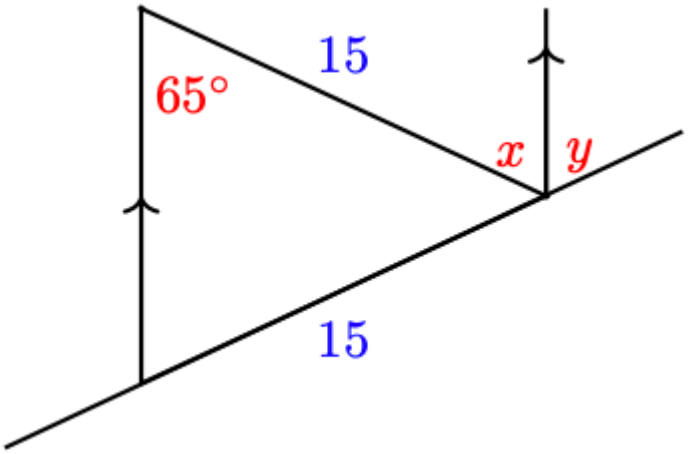
38.

For the following exercises, arrows on a pair of lines indicate that they are parallel. Find x and y.

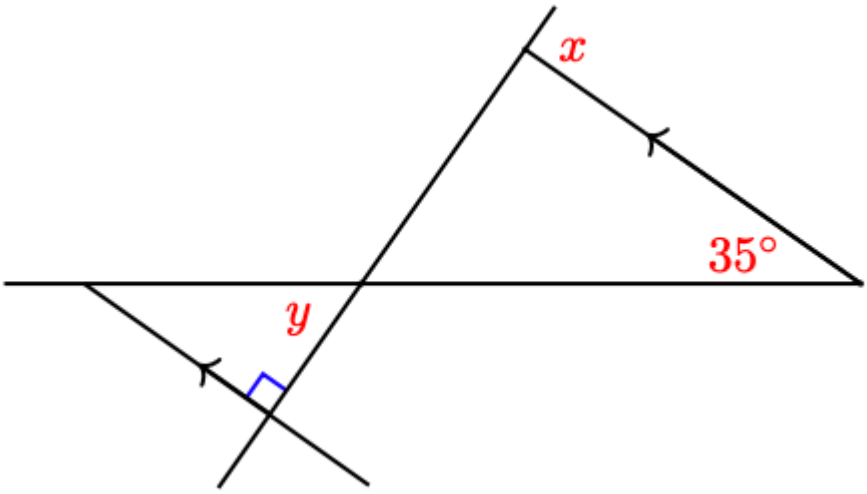
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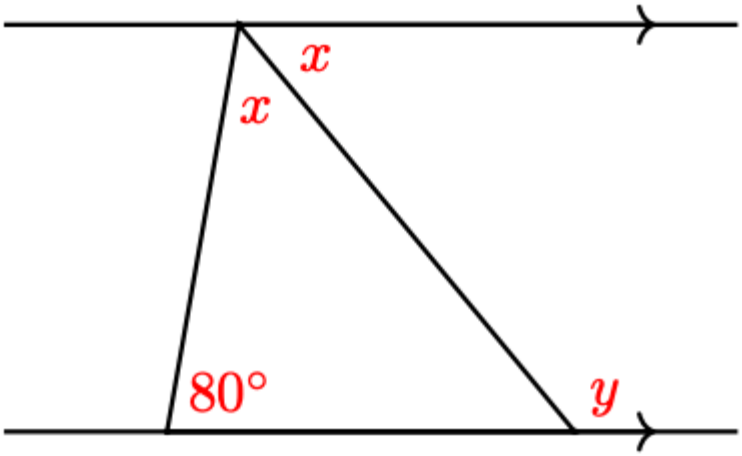


40.



41.





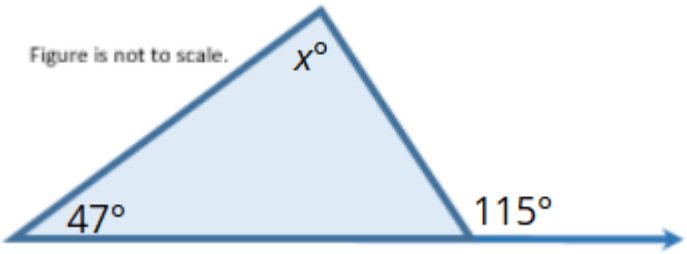
92.

93. How long is the diagonal of a square whose side is 6 centimeters?

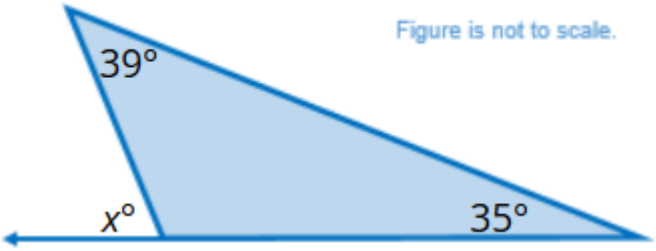
94. How long is the side of a cube whose volume is 80 cubic feet?

For the following exercises, find the value of x.

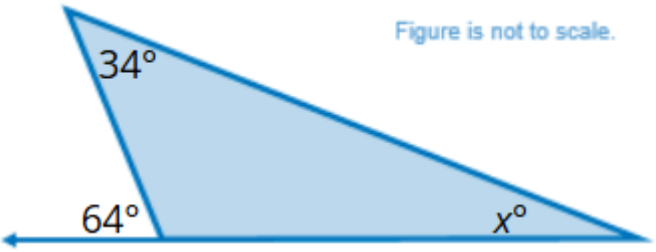
95. (MyOpenMath)



96. (MyOpenMath)

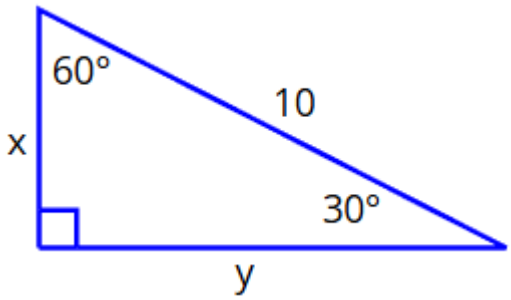


97. (MyOpenMath)

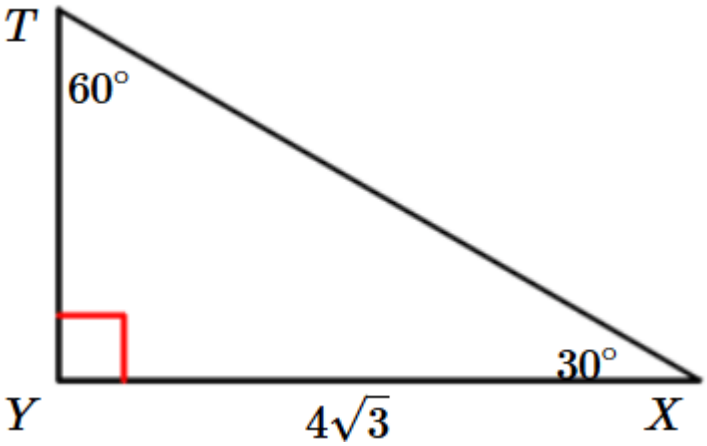


For the following exercises, find the length of the indicated side(s).

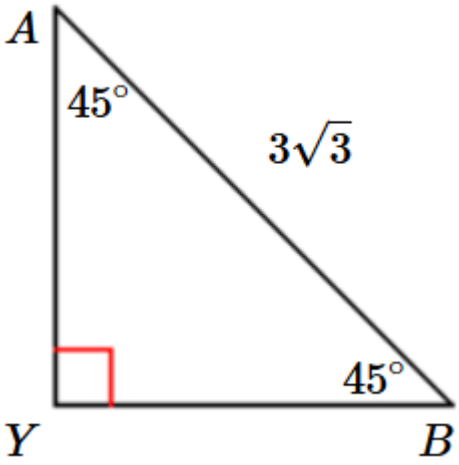
28. (MyOpenMath) x and y



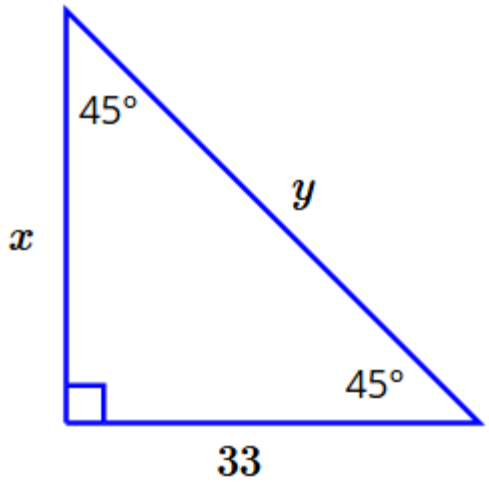
29. (MyOpenMath) $\overline{TY}$ and $\overline{TX}$



30. (MyOpenMath) $\overline{AY}$ and $\overline{YB}$

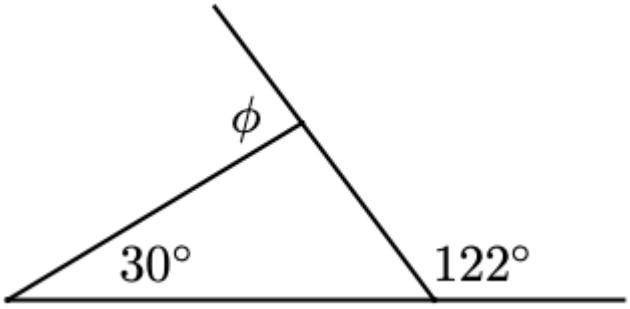


31. (MyOpenMath) x and y

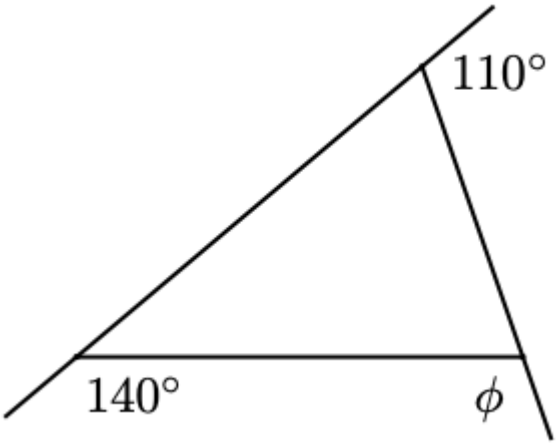


For the following exercises, the angle labeled ϕ is called an exterior angle of the triangle, formed by one side and the extension of an adjacent side. Find ϕ.

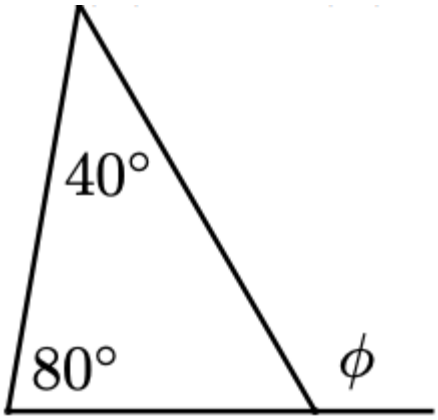
92.



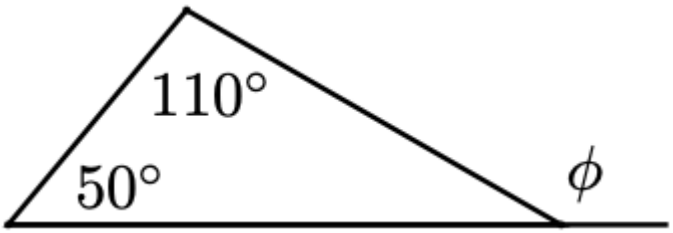
93.



94.



95.



For the following exercises, make a sketch of the situation described and label a right triangle. Then use the Pythagorean Theorem to solve the problem.

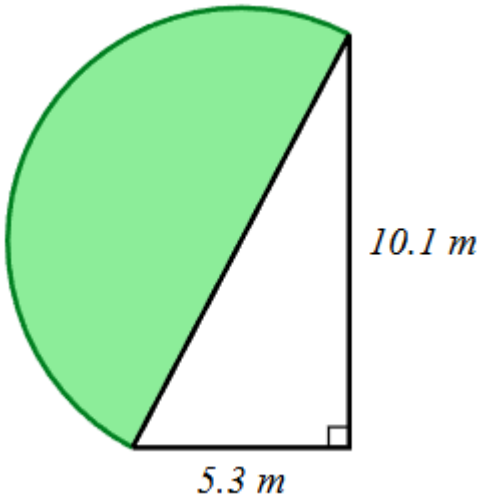
96. The diagonal of a square is 12 inches long. How long is the side of the square?

97. The length of a rectangle is twice its width, and its diagonal is meters long. Find the dimensions of the rectangle.

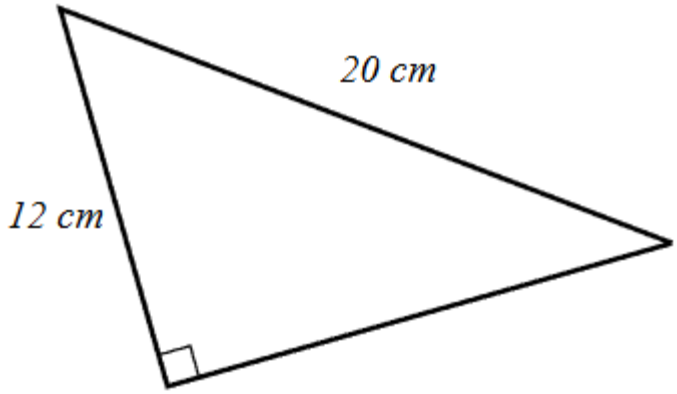
98. (MyOpenMath) The hypotenuse of right triangle is 146 centimeters long. The difference between the other two sides is 14 centimeters. Find the exact values of the missing sides.

99. (MyOpenMath) The shortest side of a right triangle is 9 centimeters long. The difference between the lengths of the other two sides is 1 centimeter. Find the exact values of the missing side and hypotenuse.

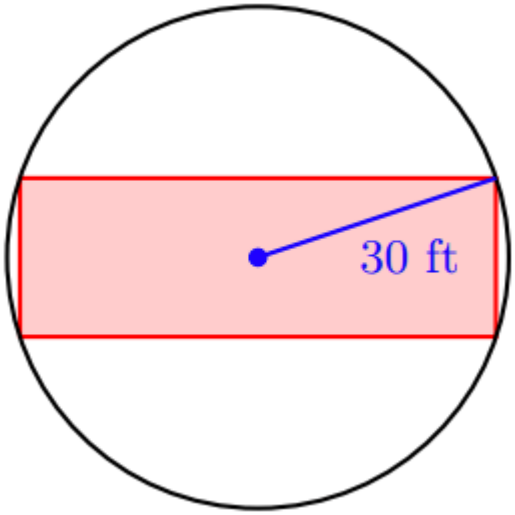
100. (MyOpenMath) The shaded, half circular disk is drawn on the side of a right triangle, whose legs are labeled 5.3 meters and 10.1 meters. Find the area of this half disk. Round your answer to one decimal place.



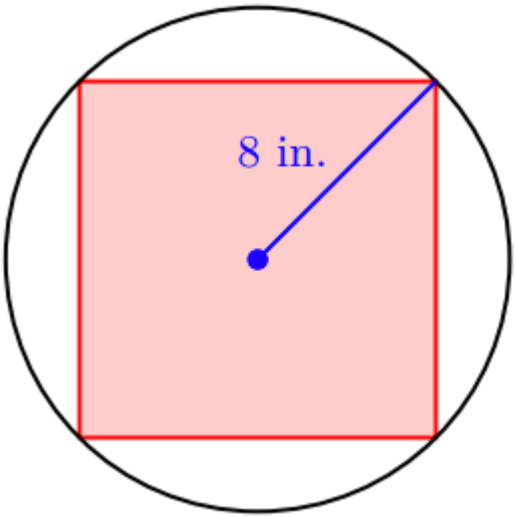
101. (MyOpenMath) Imagine that for the right triangle shown below, each of its three sides is bent into a perfect circle. What would be the *combined total area* of all three circles? Round your answer to one decimal place.



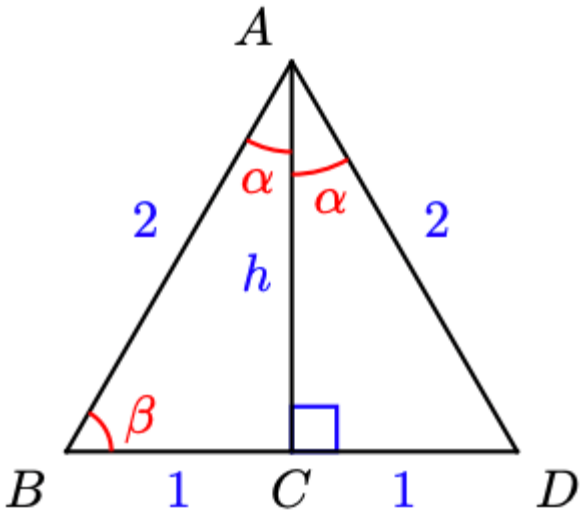
12. What size rectangle can be inscribed in a circle of radius 30 feet if the length of the rectangle must be three times its width?



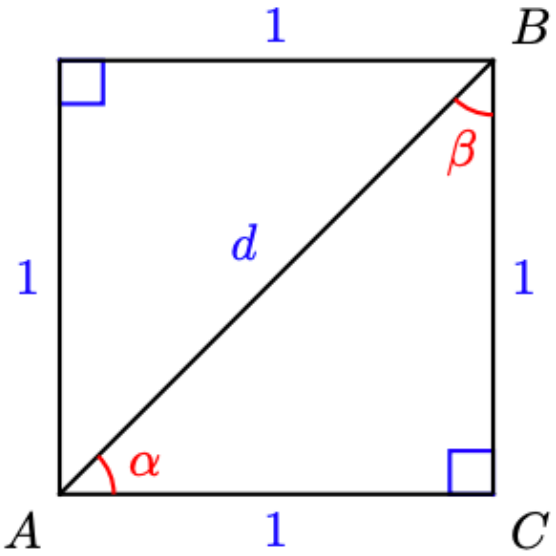
13. What size square can be inscribed inside a circle of radius 8 inches, so that its vertices just touch the circle?



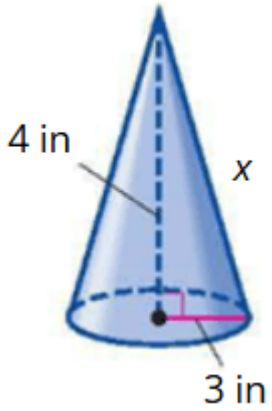
14. Find α, β and h.



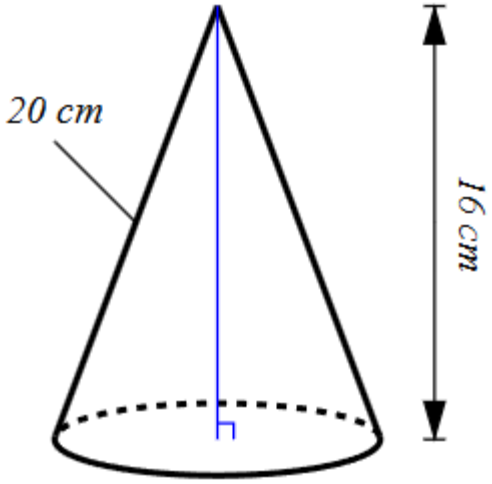
15. Find α, β and d.



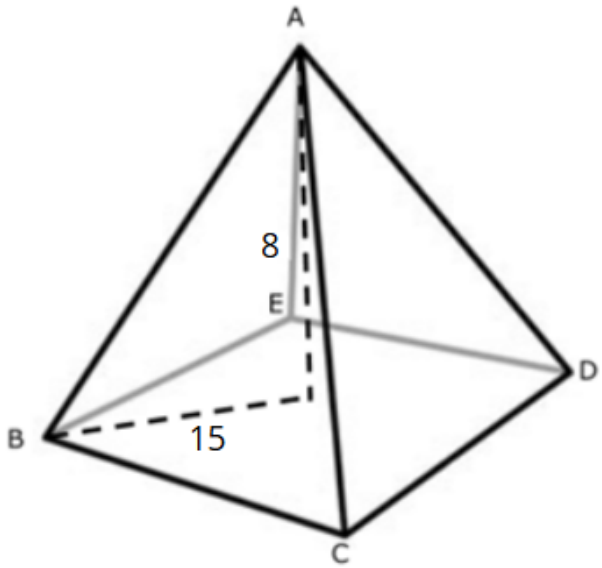
16. (MyOpenMath) Find the slant height (labeled x) of the right circular cone.



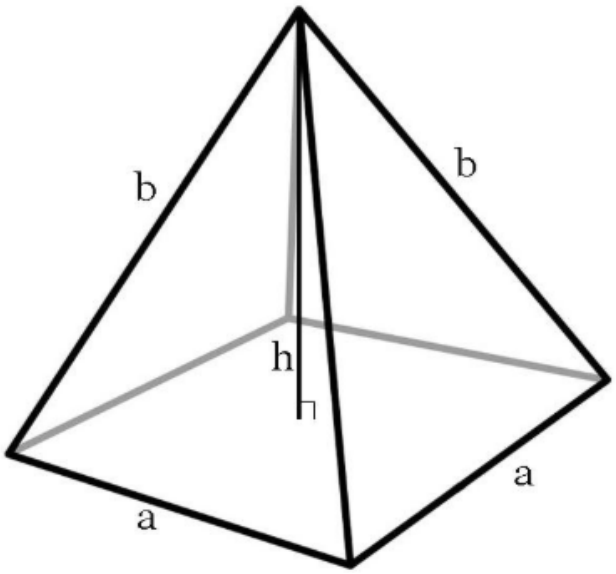
17. (MyOpenMath) Compute the radius of the base of the right circular cone if given the slant height of 20 cm and the cone height of 16 cm.



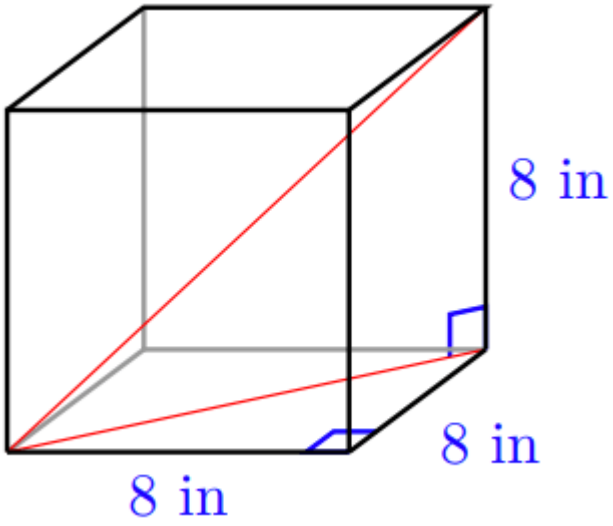
18. (MyOpenMath) A right square pyramid is shown. The height of the pyramid is 8 units. The distance from the center of the base of the pyramid to vertex B is 15 units, as shown. Find the length of segment $\overline{AB}$.



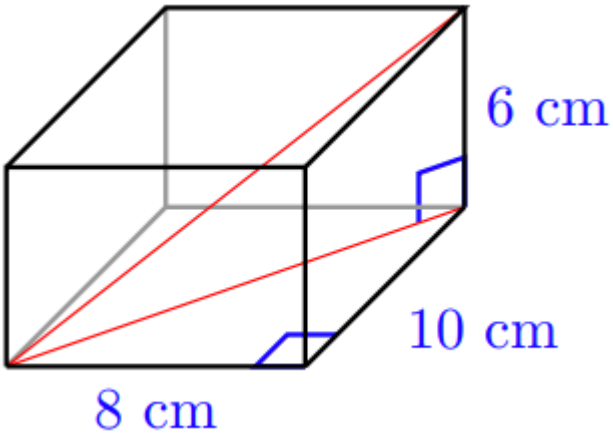
19. (MyOpenMath) Compute the exact value of the height h of the square-based straight pyramid, given that the base is a square with sides $a = 36$ inches long, and all other edges are $b = 54$ inches long.



20. Find the diagonal of a cube of side 8 inches. Hint: Find the diagonal of the base first.

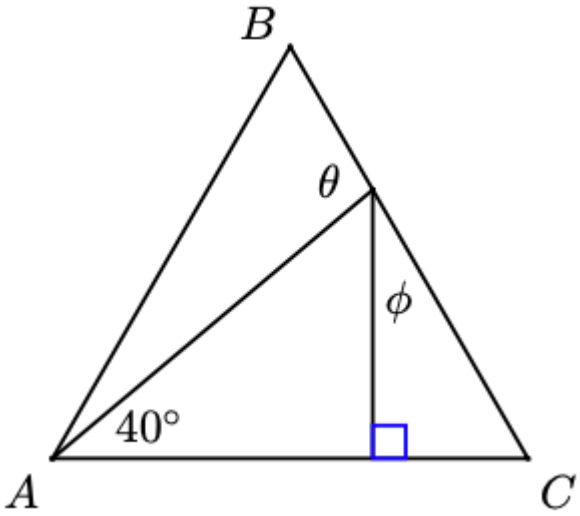


21. Find the diagonal of a rectangular box whose sides are 6 cm by 8 cm by 10 cm. Hint: Find the diagonal of the base first.

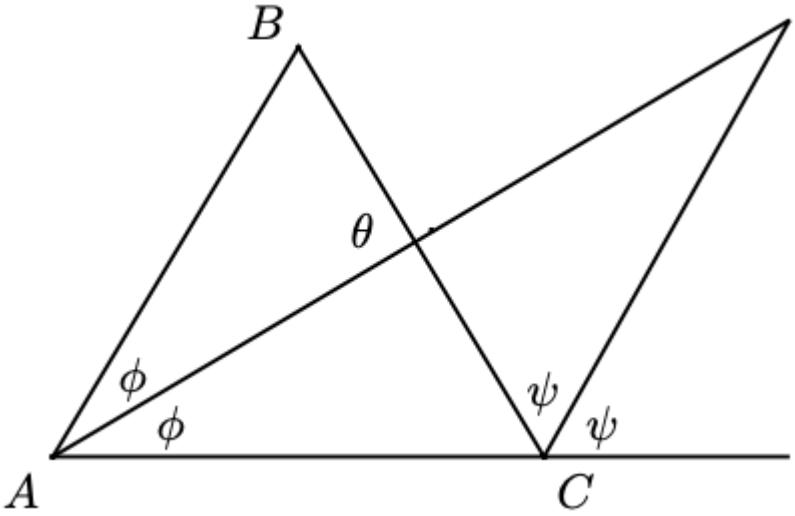


For the following exercises, ΔABC is equilateral. Find the unknown angles.

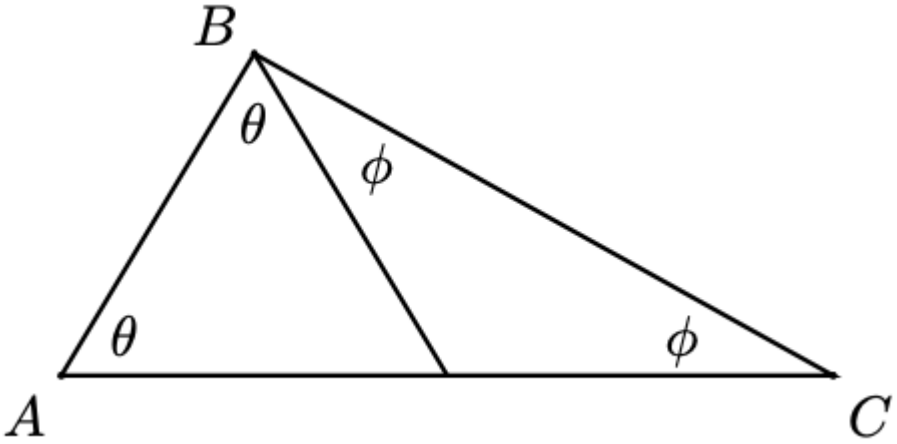
22.



23.



24. Consider the following figure.

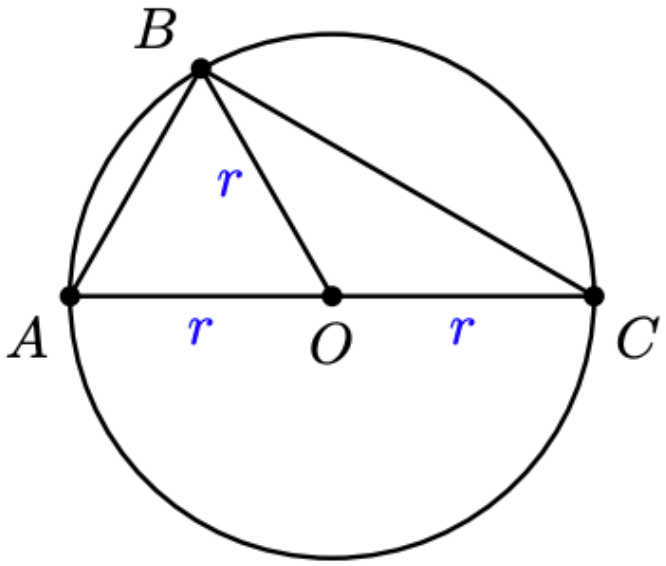


a. Compute $2\theta + 2\phi$.

b. Compute $\theta + \phi$.

c. What must be true about $\triangle ABC$?

25. Consider the following figure.

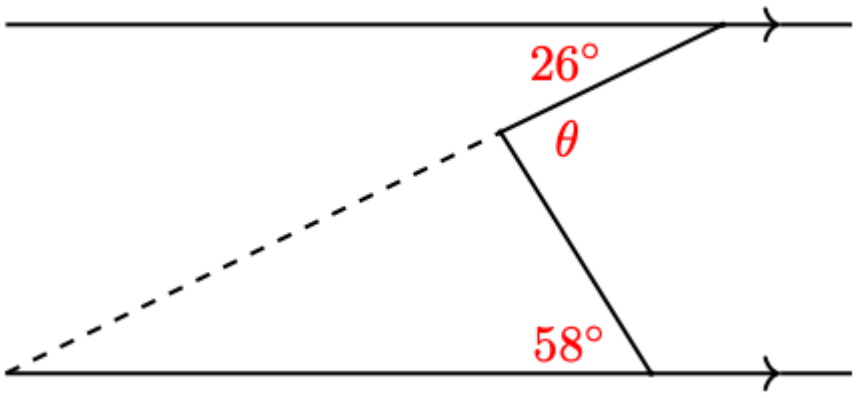


a. Explain why $\angle OAB$ and $\angle ABO$ are equal in measure.

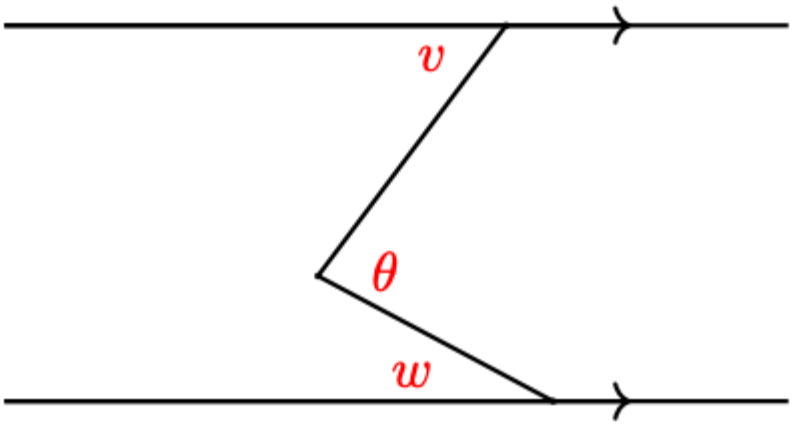
b. Explain why $\angle OBC$ and $\angle BCO$ are equal in measure.

c. Explain why $\angle ABC$ is a right angle.

26. a. In the figure below, find θ, and justify your answer.



b. Write an algebraic expression for θ in the figure below.



Applications

27. **Bounding Distances.** The town of Madison is 15 miles from Newton, and 20 miles from Lewis. What are the possible values for the distance from Lewis to Newton?

For the following exercises, make a sketch of the situation described and label a right triangle. Then use the Pythagorean Theorem to solve the problem.

28. **TV Dimensions.** The size of a TV screen is the length of its diagonal. If the width of a 35-inch TV screen is 28 inches, what is its height?

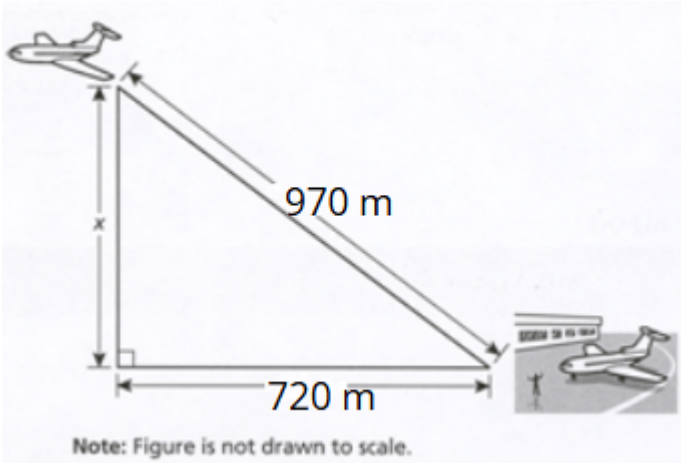
29. **TV Dimensions (MyOpenMath).** Piyali goes to the store to buy a new TV. She tells the salesperson that she needs a TV that has a maximum width of 37 inches and a maximum height of 28 inches. Assuming that she can get a TV with these maximum dimensions, and knowing that TVs are measured by the diagonal of the screen, what size TV can Piyali buy?

30. **Tree Shadows.** If a 30-meter pine tree casts a shadow of 30 meters, how far is the tip of the shadow from the top of the tree?

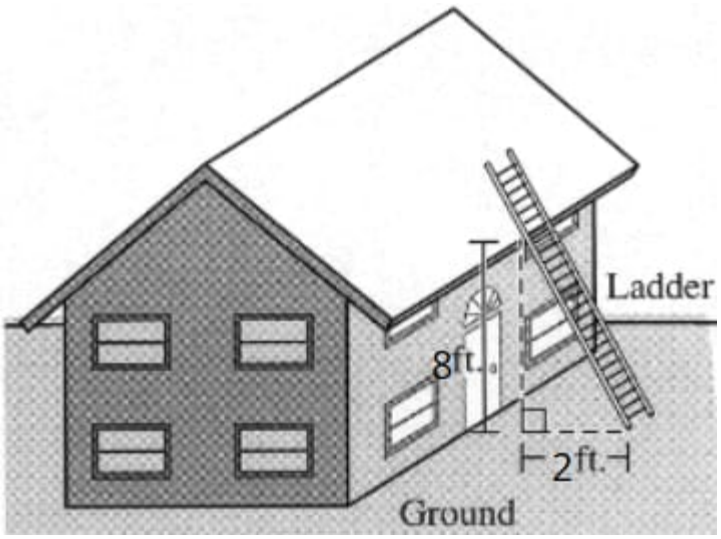
31. **Utility Pole (MyOpenMath).** A utility pole is 9 m high. A cable is stretched from the top of the pole to a point in the ground that is 8 m from the bottom of the pole. How long is the cable?

32. **Distance Between Towns (MyOpenMath).** If Nhat is 22 miles due west of Sacramento and Loi is 30 miles due north of Sacramento, what is the shortest distance from Loi to Nhat?

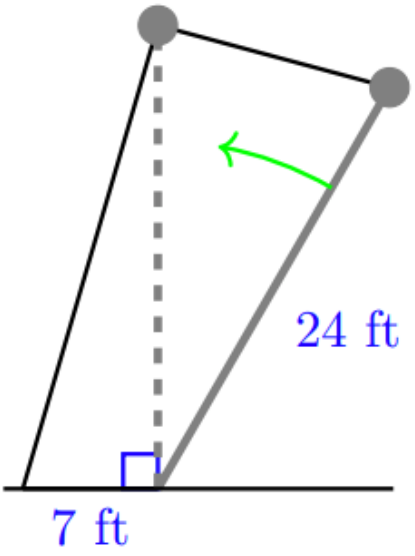
33. **Altitude (MyOpenMath).** What is the altitude of the plane?



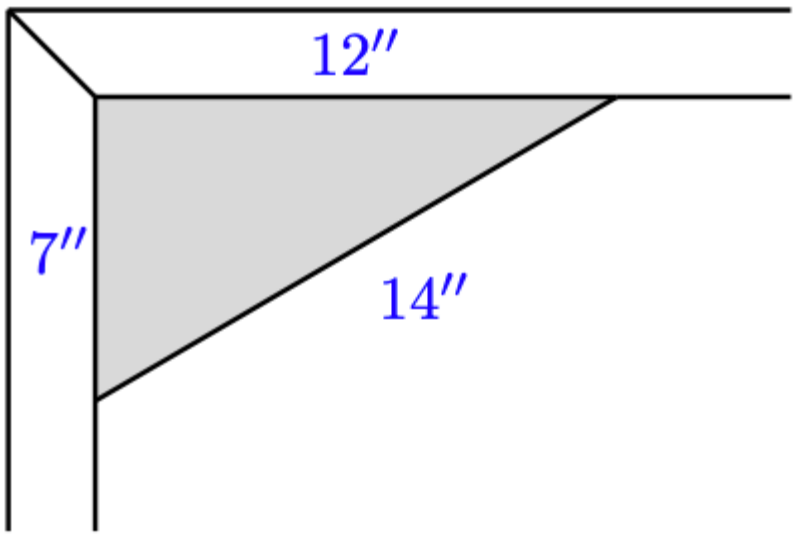
34. **Ladder Length (MyOpenMath).** What is the minimum length the ladder must be to reach the roof?



35. A 24-foot flagpole is being raised by a rope and pulley, as shown in the figure. The loose end of the rope can be secured to a ring on the ground 7 feet from the base of the pole. From the ring to the top of the pulley, how long should the rope be when the flagpole is vertical?



36. To check whether the corners of a frame are square, carpenters sometimes measure the sides of a triangle, with two sides meeting at the join of the boards. Is the corner shown in the figure square?



For the following exercises, make a sketch and solve.

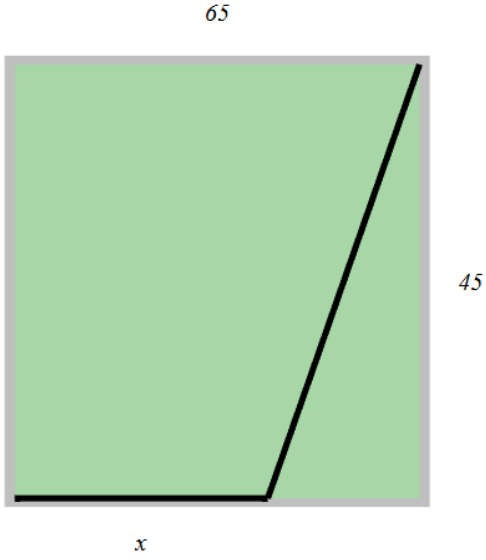
37. **Pesky Pipe.** The back of Ron's pickup truck is five feet wide and seven feet long. He wants to bring home a 9-foot length of copper pipe. Will it lie flat on the floor of the truck?

38. **Pesky Pipe.** Mary has a pickup truck with a camper shell. The camper shell floor is five feet wide and seven feet long, but the shell is only three feet tall. Will a 9-foot copper pipe fit diagonally across the back of the truck?

39. What is the longest curtain rod that will fit inside a box 60 inches long by 10 inches wide by 4 inches tall?

Challenge Problems

40. (MyOpenMath) A sidewalk goes all the way around a rectangular field, as shown below (not drawn to scale). The field is 65 feet long and 45 feet wide. You need to get from the bottom-left corner to the top-right corner. Your walking speed is 7 feet per second along the sidewalk and 4 feet per second through the field.



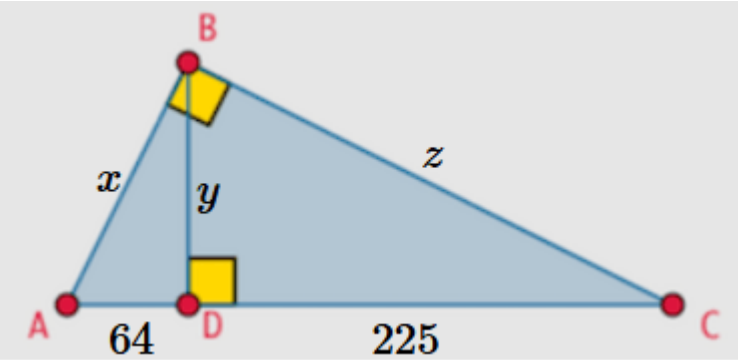
a. How many seconds will it take you if you walk from the bottom-left corner to the top-right corner directly across the field?

b. How many seconds will it take you if you walk all 65 feet along the sidewalk (from bottom-left to bottom-right) and then

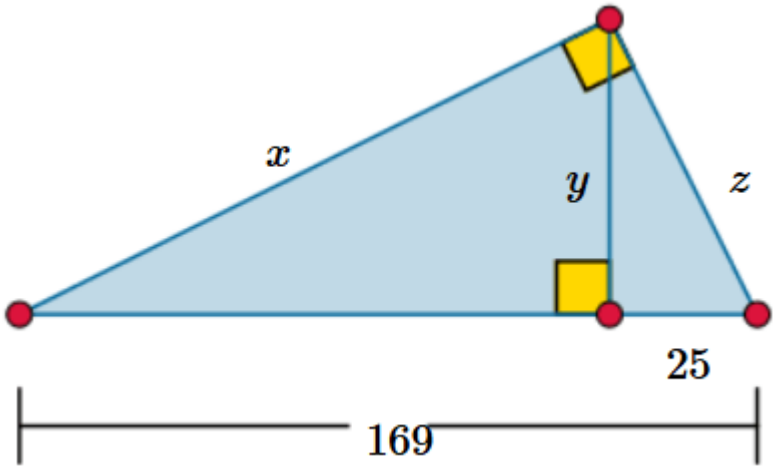
45 feet along the sidewalk (from bottom-right to top-right)?

c. Suppose you walk x feet along the sidewalk and then turn to walk the remaining distance through the field directly to the top-right corner. Write an expression in x representing the number of seconds it would take you to walk this path.

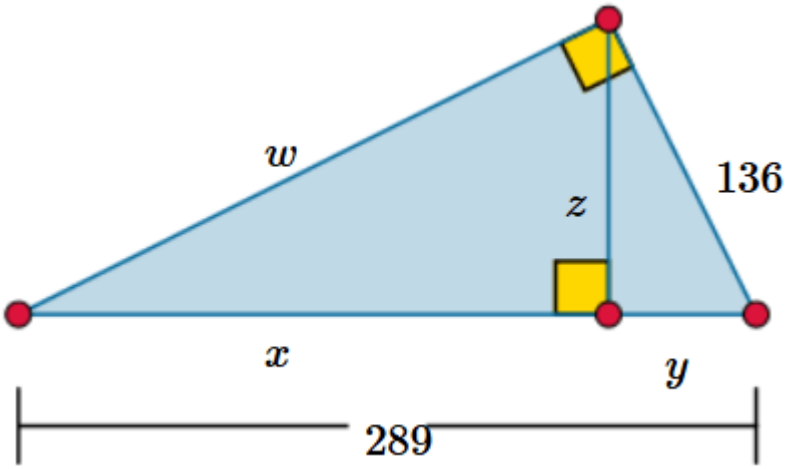
41. (MyOpenMath) Find the values of x, y, and z to two decimal places.



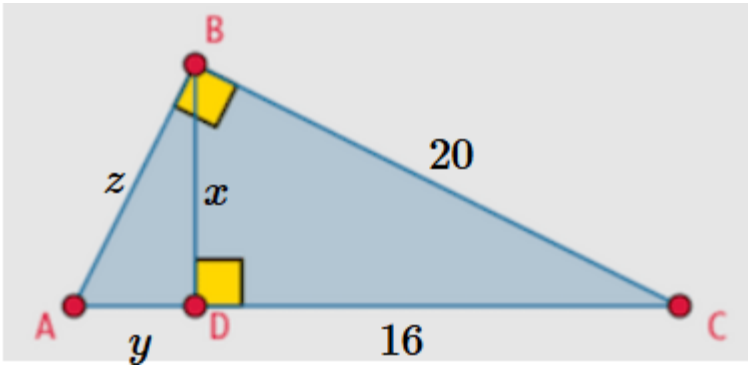
42. (MyOpenMath) Find the values of x, y, and z to two decimal places.



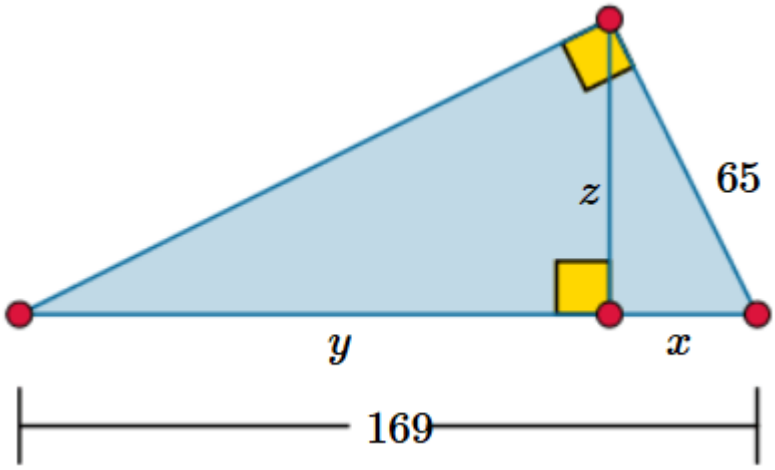
43. (MyOpenMath) Find the values of x, y, and z to two decimal places.



44. (MyOpenMath) Find the values of x, y, and z to two decimal places.

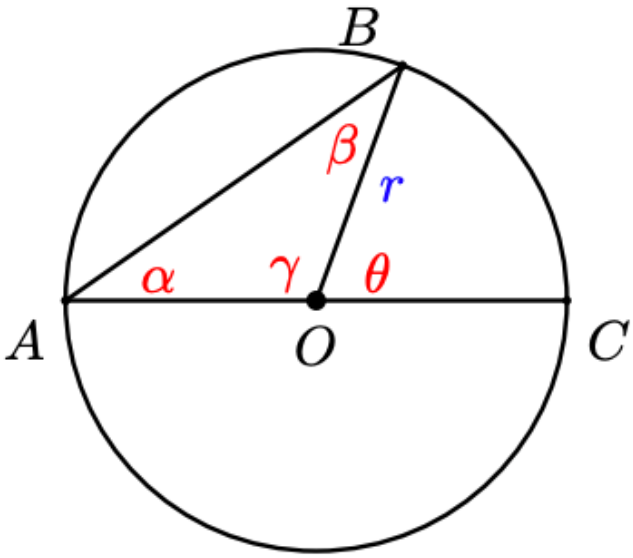


15. (MyOpenMath) Find the values of x, y, and z to two decimal places.



16. (MyOpenMath) An arch is in the shape of a semicircle. At a point along the base 4 feet from an end of the arch, the height of the arch is 8 feet. Find the exact value of the maximum height of the arch.

17. Consider the following figure.

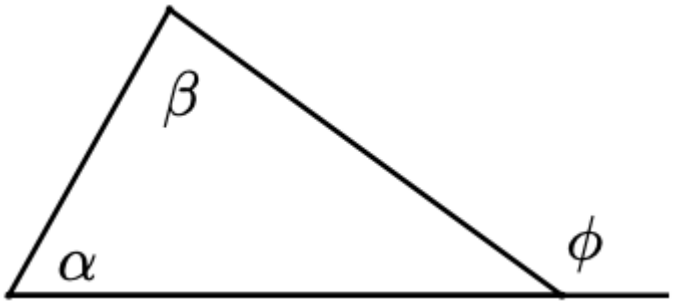


a. Compare θ with $\alpha + \beta$. (Hint: What do you know about supplementary angles and the sum of angles in a triangle?

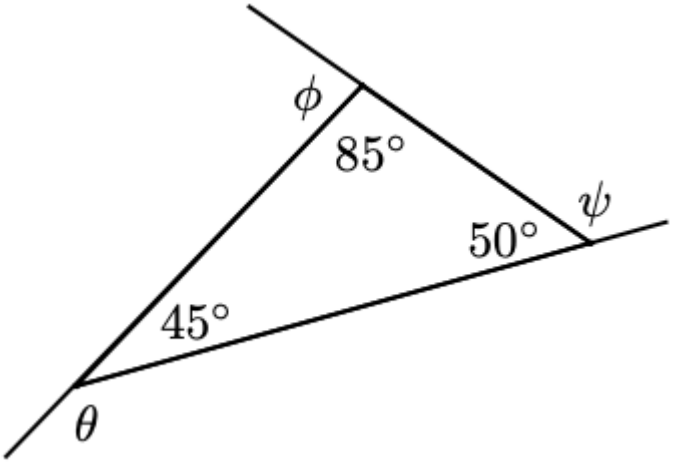
b. Compare α and β.

c. Explain why the inscribed angle $\angle BAO$ is half the size of the central angle $\angle BOC$.

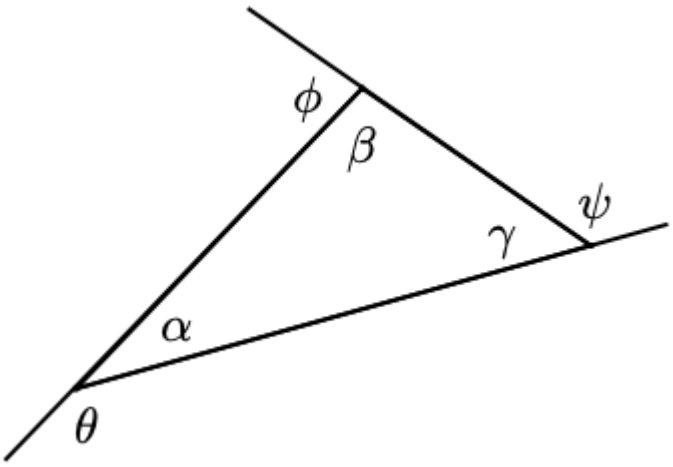
18. Find an algebraic expression for ϕ and use your answer to write a rule for finding an exterior angle of a triangle.



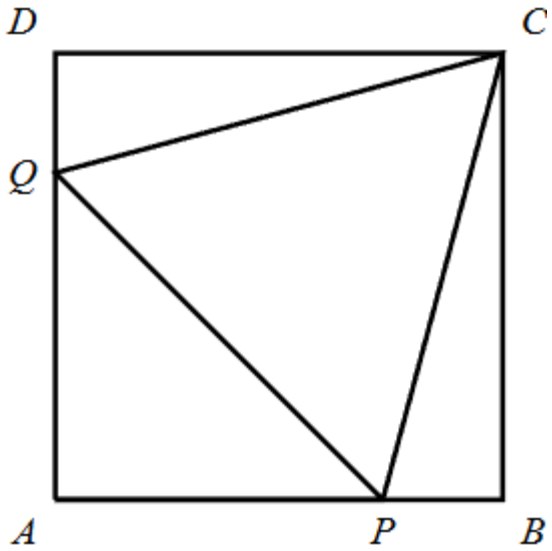
19. a. Find the three exterior angles of the triangle. What is the sum of the exterior angles?



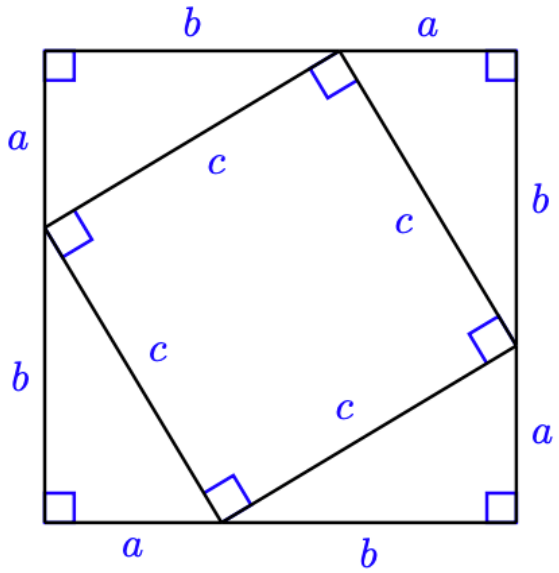
b. Write an algebraic expression for each exterior angle in terms of one of the angles of the triangle. What is the sum of the exterior angles?



20. (MyOpenMath) The area of square $ABCD$ is one square unit. Triangle $\triangle PCQ$ is equilateral. What are the exact lengths of the sides of the triangle?



51. **Proving the Pythagorean Theorem.** There are many proofs of the Pythagorean Theorem. Here is a simple visual argument.



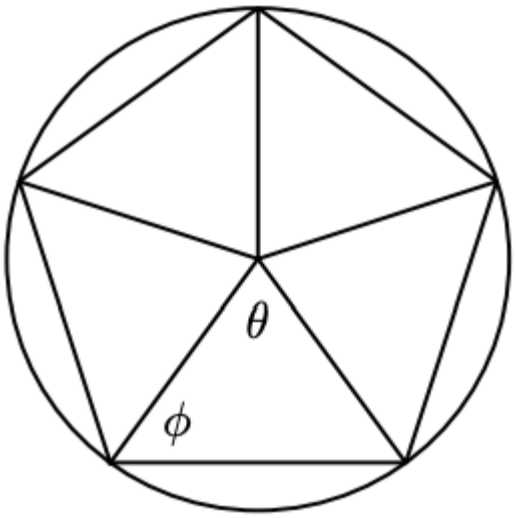
a. What is the length of the side of the large square in the figure? Write an expression for its area.

b. Write another expression for the area of the large square by adding the areas of the four right triangles and the smaller central square.

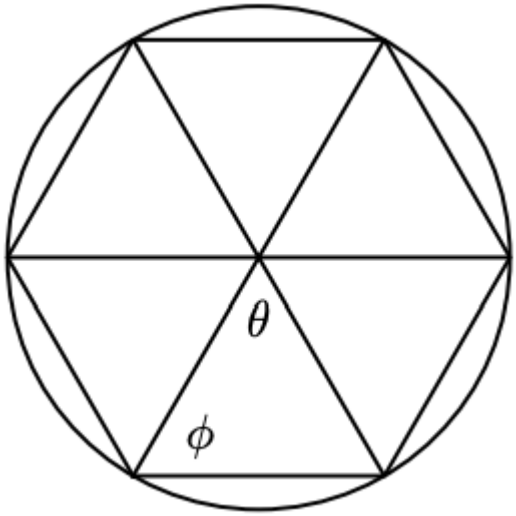
c. Equate your two expressions for the area of the large square, and deduce the Pythagorean Theorem.

For the following exercises, the figures inscribed in each problem are regular polygons, which means that all their sides are the same length, and all the angles have the same measure. Find the angles θ and ϕ.

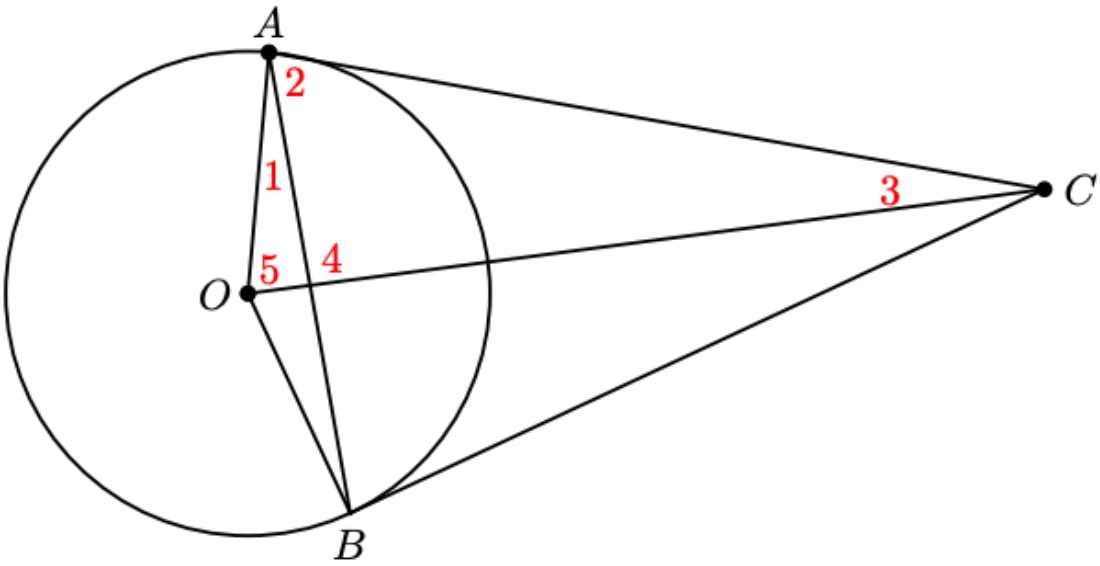
52.



53.



54. A tangent meets the radius of a circle at a right angle. In the figure, $\angle AOB = 140^\circ$. Find the angles labeled 1 through 5 in order, and give a reason for each answer.



Answers

Skills Refresher

1. $x < 3$
2. $x \leq 8$
3. $x \leq -1$
4. $x > -1$
5. Positive
6. Negative
7. Positive
8. Negative
9. Negative
10. Negative

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1.3: Similar Triangles

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Congruent Triangles

We begin this section by identifying what it means for two triangles to be **congruent**.

Definition: Congruent Triangles

Two triangles are said to be **congruent** if their corresponding angles are equal, and their corresponding sides have the same lengths. Symbolically, we use the congruency symbol $\cong$ to state that two triangles are congruent.

In simple terms, two triangles are considered congruent if they have the same size and shape. For example, it looks as though $\triangle ABC$ and $\triangle DEF$ in Figure $1.3.1$ are identical in both shape and size ($\triangle DEF$ has been rotated from $\triangle ABC$), so they are congruent. We write $\triangle ABC \cong \triangle DEF$.

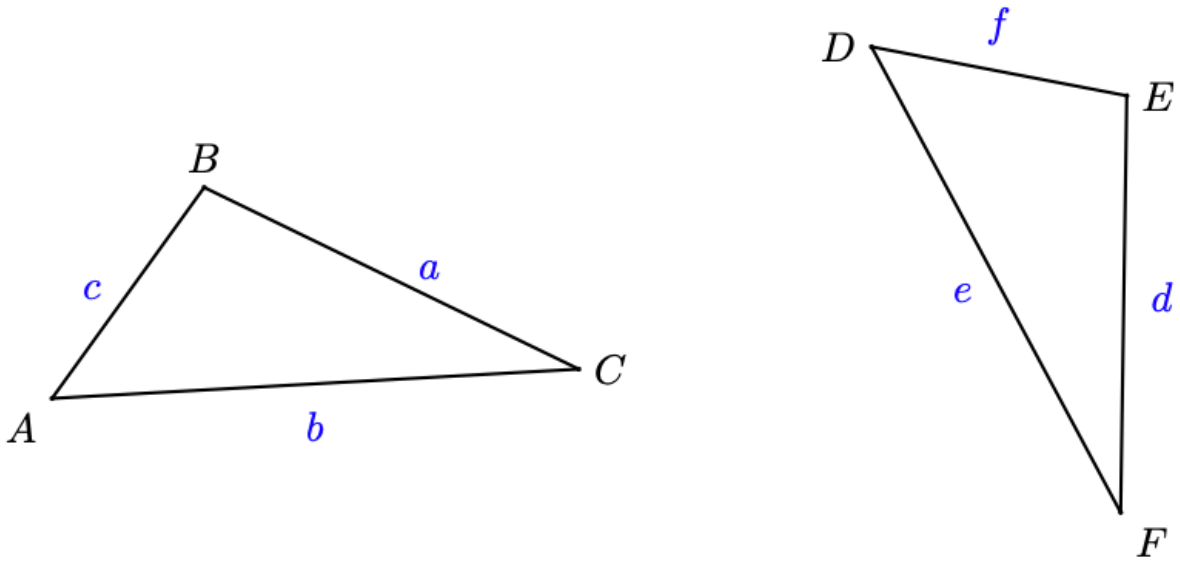


Figure $1.3.1$**:** *Congruent triangles* $\triangle ABC$ *and* $\triangle DEF$

Example 1.3.1

The two triangles in Figure 1.3.2 below are congruent. List the corresponding parts, and find the angles θ, ϕ, and χ, and side z.

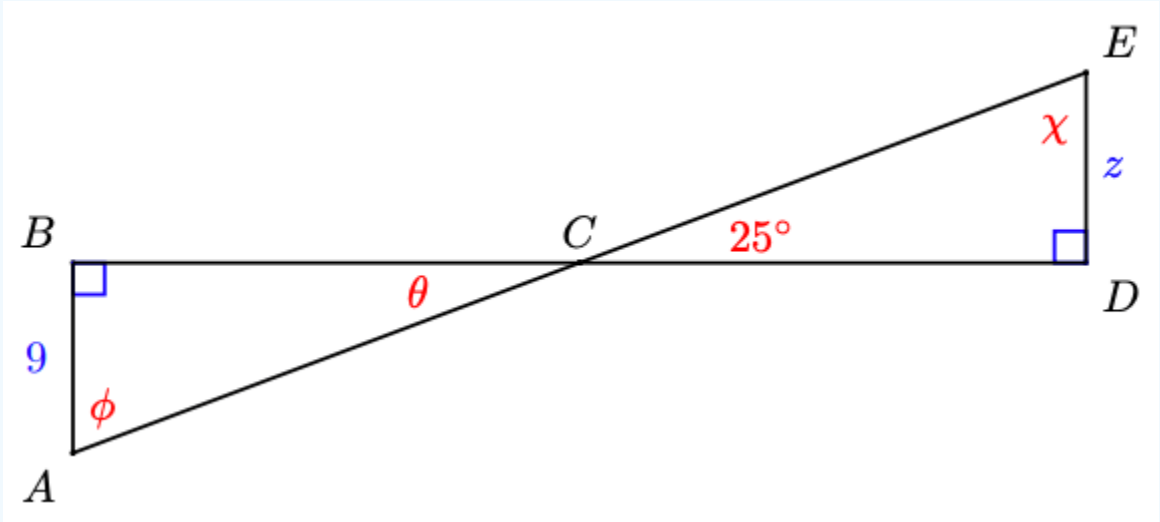


Figure 1.3.2

Solution available at the end of this section.

Checkpoint 1.3.1

The two triangles in Figure 1.3.3 below are congruent. Find the values of α, β, and γ.

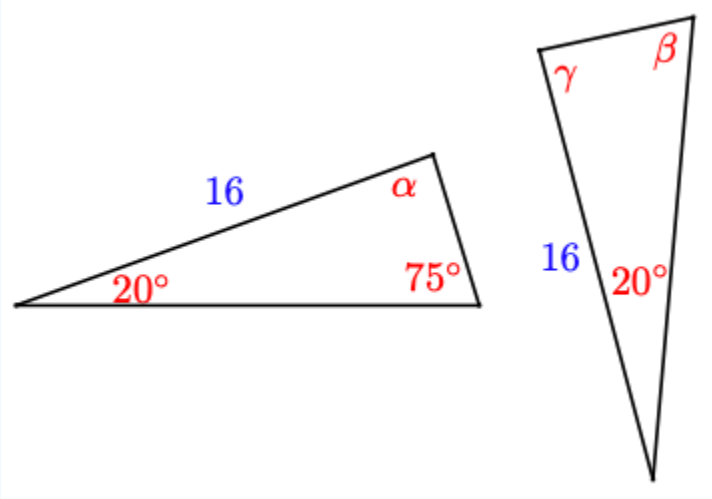


Figure 1.3.3

Answer available at the end of this section.

Recall that the altitude of a triangle is the segment from one vertex of the triangle perpendicular to the opposite side.

Example 1.3.2

Show that the altitude of an equilateral triangle divides it into two congruent right triangles.

Solution available at the end of this section.

For the following Checkpoint, you should recall the following definition from your Geometry class.

Definition: Parallelogram

A **parallelogram** is a quadrilateral with two pairs of parallel sides. The opposite or facing sides of a parallelogram are of equal length and the opposite angles of a parallelogram are of equal measure.

Checkpoint 1.3.2

The diagonal of a parallelogram divides it into two congruent triangles, as shown in Figure 1.3.5 below. List the corresponding parts of the two triangles and explain why each pair is equal.

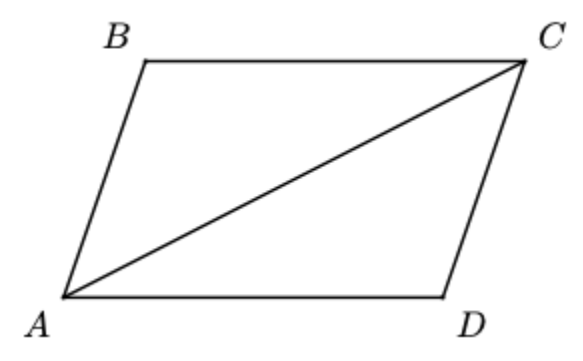


Figure 1.3.5

Answer available at the end of this section.

Similar Triangles

The most common comparison of triangles in both Trigonometry and Calculus will be with **similar triangles**.

Definition: Similar Triangles

Two triangles are **similar** if their corresponding angles are congruent and the lengths of their corresponding sides are proportional. If $\triangle ABC$ is similar to $\triangle DEF$, you write $\triangle ABC \sim \triangle DEF$, where the order of the vertices specifies the correspondence: A with D, B with E, and C with F.

The definition of similar triangles looks almost like the definition of congruent triangles; however, similarity between triangles is a weaker form of congruence—the triangles have the same shape but might be sized differently. This flexibility grants us much power in mathematics.

Fun Fact!

Congruent triangles are similar, but similar triangles need not be congruent.

The corresponding angles between similar triangles are equal, and the corresponding sides are proportional. We can think of one similar triangle as an enlargement or a reduction of the other (see Figure 1.3.6 below).

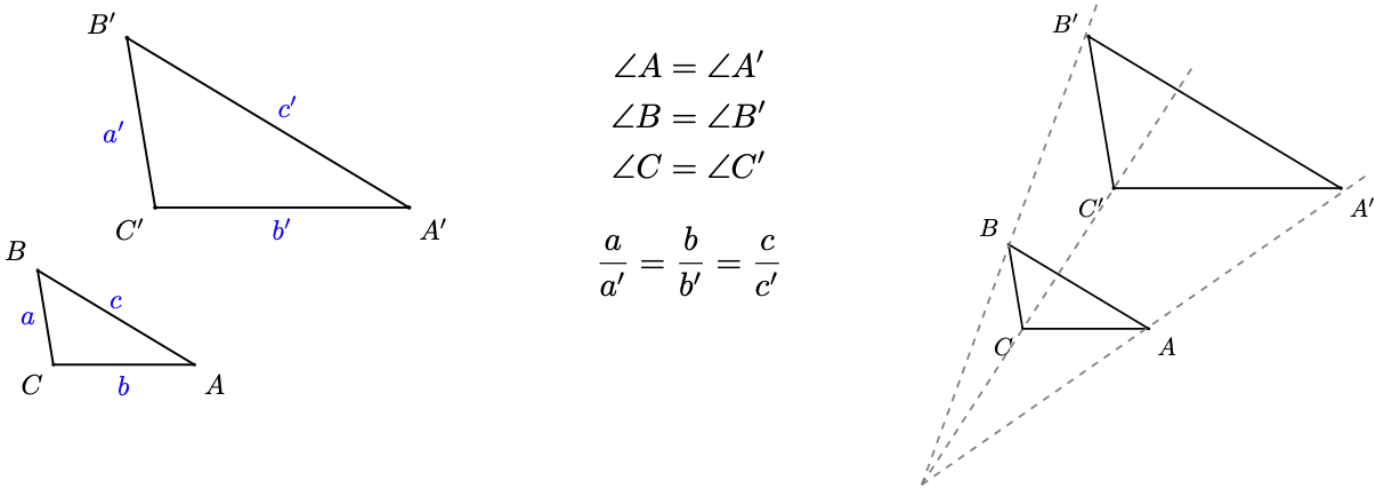


Figure 1.3.6: *Proportional relationships between similar triangles*

Interactive Element: Similar Triangles as Scaled Copies

The following Interactive Element might help you visualize how one similar triangle is just an enlargement or reduction of another.

Interact: Drag the green apex point of the right-hand triangle up and down to change the scale factor k.

Observation: The triangle changes size but keeps its shape — corresponding angles stay equal and every corresponding side ratio equals k. When $k > 1$ it is an enlargement; when $k < 1$, a reduction.

To decide whether two triangles are similar, we must verify *only one* of the two similarity conditions (the other condition will be true automatically).

Theorem: Similarity Conditions

Two triangles are similar whenever any one of the following conditions holds:

- **(AA)** Two angles of one triangle are congruent to two angles of the other. The third pair of angles is then automatically congruent.
- **(SSS)** All three pairs of corresponding sides are proportional.
- **(SAS)** Two pairs of corresponding sides are proportional, and the angles included between those sides are congruent.

Example 1.3.3

Which of the pairs of triangles shown below are similar?

a.

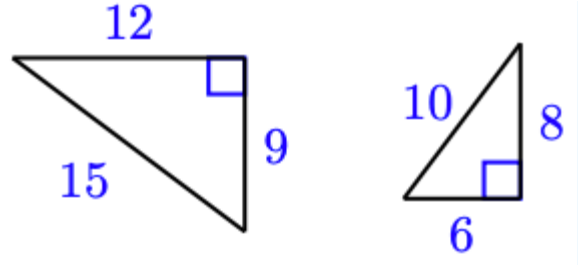


Figure 1.3.7

b.

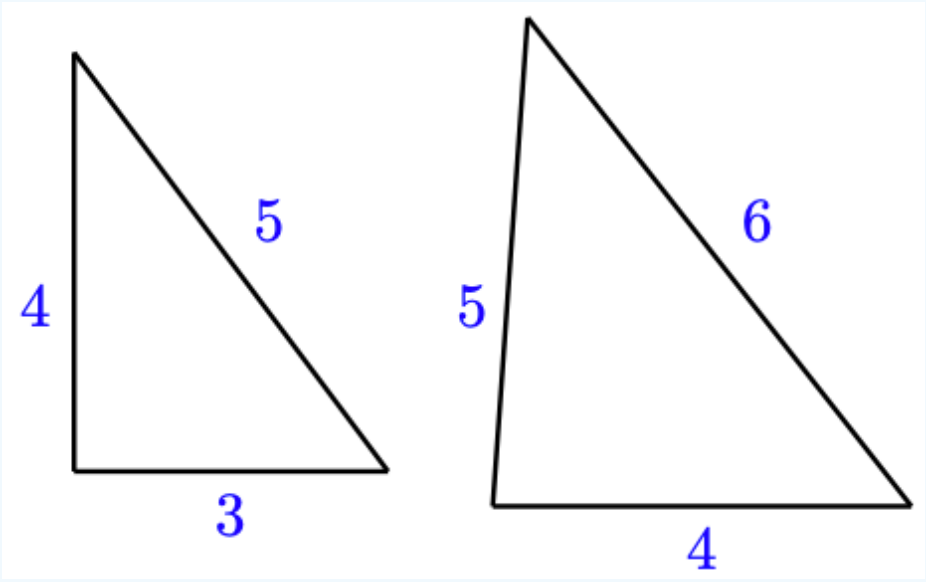


Figure 1.3.8

c.

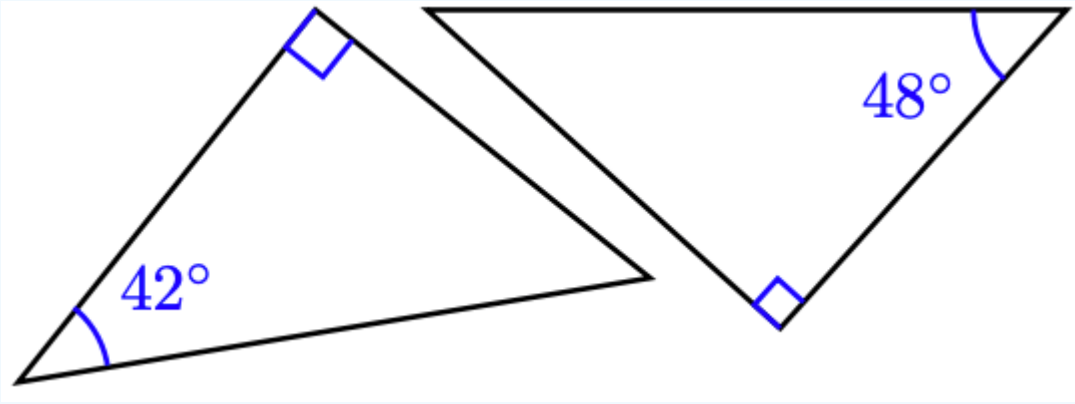


Figure 1.3.9

Solutions available at the end of this section.

Checkpoint 1.3.3

Are the triangles below similar? Explain why or why not in each case.

a.

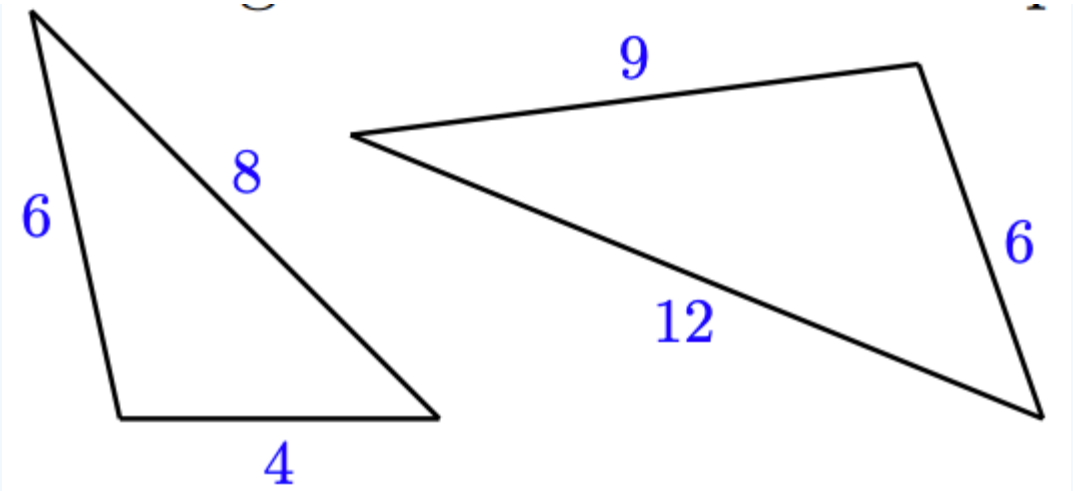


Figure 1.3.10

b.

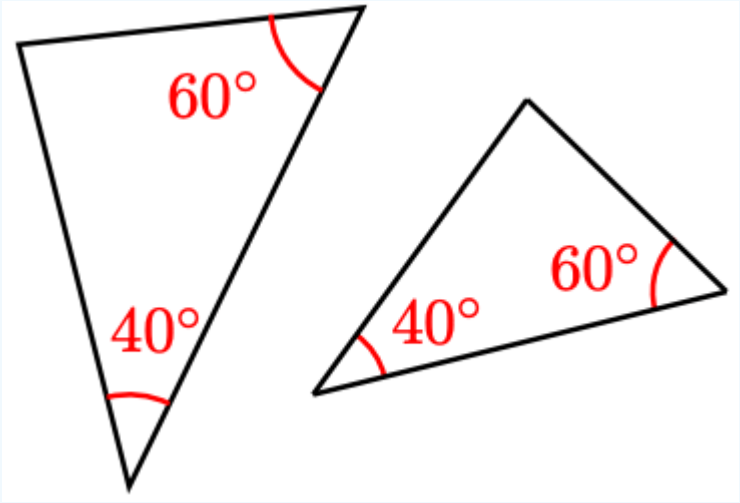


Figure 1.3.11

Answers available at the end of this section.

In part (b) of Checkpoint 1.3.3, the third angle in each triangle must be 80° because the sum of the angles is 180°. Thus, we only need to show that *two* pairs of angles are equal to show that two triangles are similar.

Using Proportions with Similar Triangles

Figure 1.3.12 shows a parallelogram $ABCD$ and two triangles, $\triangle ABE$ and $\triangle FCE$. Can we find the unknown lengths x and y in the larger triangle?

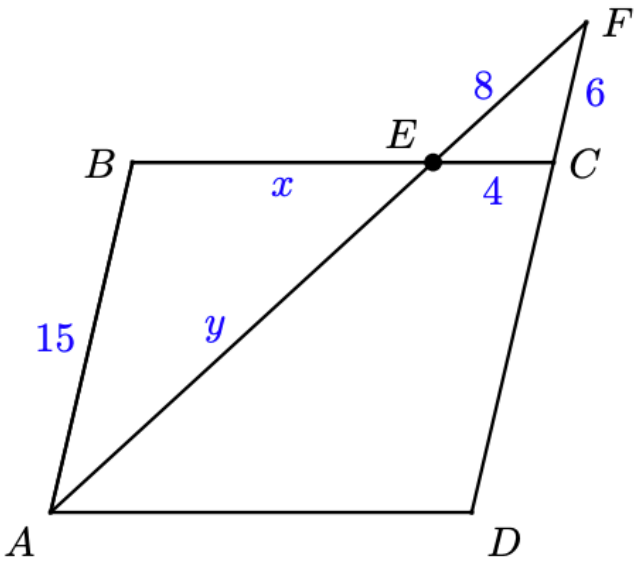


Figure 1.3.12

First, note that two pairs of corresponding angles in the triangles are equal: $\angle BEA$ and $\angle FEC$ are vertical angles, and $\angle EFC$ and $\angle BAE$ are alternate interior angles. However, if two pairs of corresponding angles are equal, the third pair must also be equal. This means that the two triangles are similar, and we can use the fact that their corresponding sides are proportional to find x and y.

Example 1.3.4

Find the value of x in Figure 1.3.12.

Solution available at the end of this section.

What do Arithmetic and $\cancel{\text{Arithmetic}}$ mean?

To get you used to the Mathematical Mantra, every once in a while I will include the "thought processes" during solutions. Just to review, we perform Mathematics in the order we learned it—Arithmetic, Algebra, Trigonometry, ... At each step during a "mechanical" solution process, we should pause and ask ourselves if there is some simple Arithmetic to be done. If not, we move on to any Algebra that can be done. If there is no Algebra to be done, we then move on to Trigonometry.

Thus,

- "$\text{Arithmetic} \implies$" in a solution means that there is some Arithmetic we could do to clean up the previous expression.
- "$\cancel{\text{Arithmetic}} \to \text{Algebra} \implies$" means there isn't any Arithmetic but there *is* some Algebra we could use to clean up the previous expression or equation.
- "$\cancel{\text{Arithmetic}} \to \cancel{\text{Algebra}} \to \text{Trigonometry} \implies$" means there isn't any Arithmetic nor is there any Algebra that can be done to clean up the previous step, so we need to start looking at doing something from Trigonometry.

Checkpoint 1.3.4

Find the value of y in the previous example.

Answer available at the end of this section.

Similar Right Triangles

If two *right* triangles have one pair of corresponding acute angles with the same measure, they are similar. We can use this fact about right triangles to make indirect measurements.

Example 1.3.5

Ivan wants to know the height of a particular building. He asks Kim to hold up a 5-foot pole near the building and measure the length of its shadow. The shadow of the pole is 3 feet long, and the shadow of the building is 12 feet long.

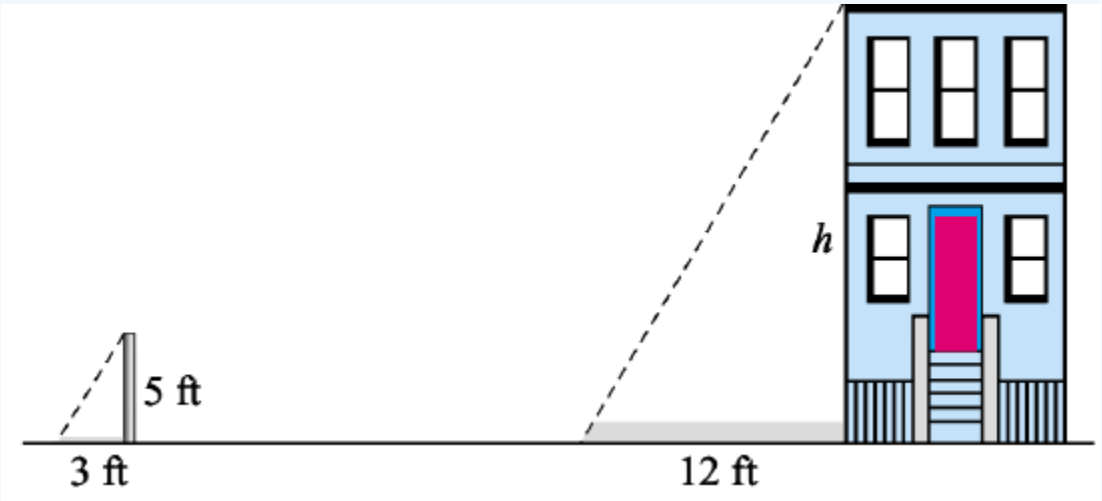


Figure 1.3.13

a. Use similar triangles to write a proportion involving the height of the building.
b. Solve the proportion to find the height of the building.

Solutions available at the end of this section.

Interactive Element: Shadows and Similar Right Triangles

The following Interactive Element might help you visualize why shadows let you measure the height of an object you can't reach directly.

Interact: Drag the green point — the tip of the pole's shadow — left and right to change the sun's angle.

Observation: Both shadows grow and shrink together with the sun rays parallel, so the two right triangles stay similar. The building's shadow stays 4 times the pole's, equal to the ratio of their heights $\left(\frac{20}{5}\right)$ — which is exactly what lets you solve for the unknown height.

Checkpoint 1.3.5

In Example 1.3.2, we created a $30°$-$60°$-$90°$ triangle in which the shorter leg was 4 inches and the hypotenuse was 8 inches. The hypotenuse of another $30°$-$60°$-$90°$ triangle is 5 feet. What is the length of the side opposite the $30°$ angle?

Answer available at the end of this section.

Overlapping Triangles

In some applications, similar triangles may share a side or an angle.

Example 1.3.6

Identify two similar triangles in Figure 1.3.14 below, and write a proportion to find H.

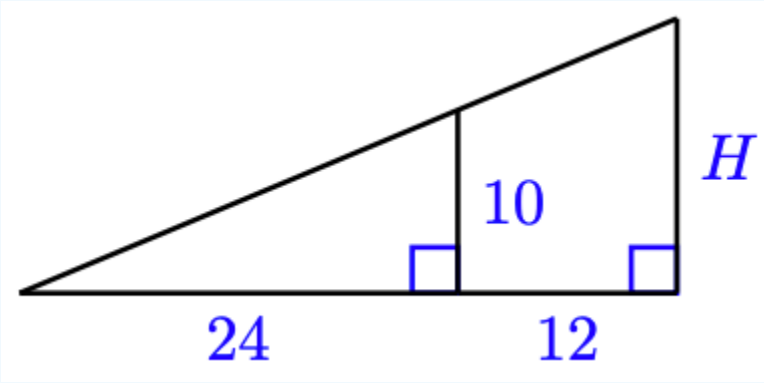


Figure 1.3.14

Solution available at the end of this section.

Checkpoint 1.3.6

Min wants to know the height of a street lamp. She discovers that when she is 12 feet from the lamp, her shadow is 6 feet long. Find the height of the street lamp.

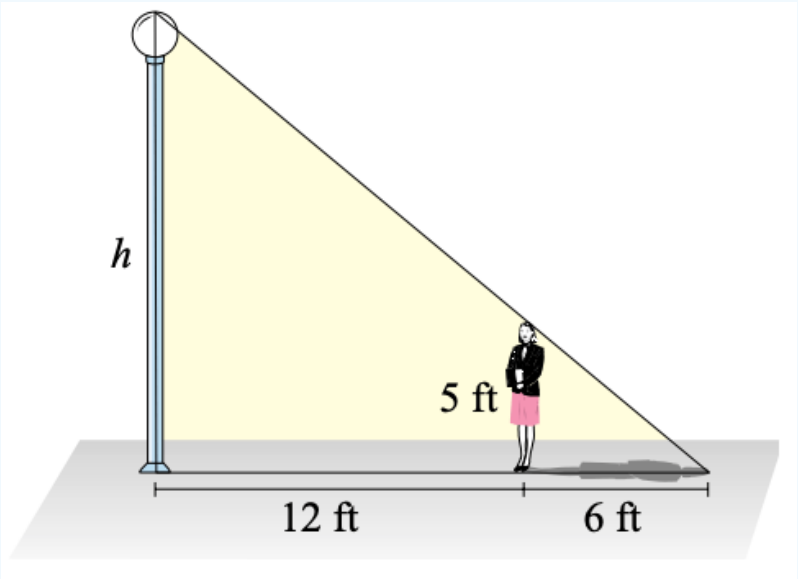


Figure 1.3.16

Answer available at the end of this section.

Time-Sensitive Advice: "Full Refund" and "No Notation" Drop Deadlines

"Full Refund" Drop Deadline

Within the Los Rios Community College District, the last day you can drop a class for a full refund is the Friday of the second week of class. Any drops processed after that point will only receive a *partial* refund.

"No Notation" Drop Deadline

The Sunday following the "full refund" drop deadline is the last day to drop without a "W" (withdrawal) notation on your transcript. You can only take a course, and receive a notation, a maximum of three times within a given community college district in California - this includes notations of D, F, *and* W.

Once you withdraw (W) or fail (D or F) a course three times within the Los Rios Community College District (ARC, CRC, FLC, or SCC), you are *never* allowed to take that course again at any of the colleges within the District. If you find yourself in that unfortunate position, you would need to take this course outside the Los Rios Community College District.

Take a moment of self-reflection to determine if you feel ready to continue this journey for the rest of this long semester. If not, consider dropping by this course by the "full refund" deadline or the "no notation" deadline, at the latest.

Success in College: What is Tutoring?

Tutoring is an age-old practice. The dictionary definition describes a tutor as a person who gives individual or, in some cases, small group instruction. The purpose of tutoring is to help students help themselves or assist or guide them to the point at which they become independent learners and thus no longer need a tutor.

Content knowledge is an essential ingredient for a tutor; however, to be truly effective, a tutor must combine content knowledge with empathy, honesty, and humor. Empathy requires tutors to "read" their students' emotional states, attitudes, and perceptions. Empathy is the ability to see others from their frame of reference and communicate this understanding to the person involved. To establish a supportive relationship with their students, tutors must be open and honest. Students are often reluctant to talk with a stranger about their academic problems. If a tutor is perceived as genuine and strongly desires to listen, students will be more willing to open up and discuss their problems. Humor can also play an essential part in a tutoring session. Humor can reduce tension. Shared laughter is a powerful way to reinforce learning, and humor can set students at ease and increase rapport. Tutors can also use humor to compliment, guide, or positively provide feedback.

In addition, a successful tutor demonstrates a caring attitude. Caring consists of being punctual and organized for the tutoring session, establishing a learning relationship with the student, developing unique teaching strategies, and becoming familiar with the learning process. Ultimately, tutoring is sharing yourself with another student in a way that makes a difference in both your lives.

There are many benefits to tutoring.

- Heightens a sense of competency/adequacy in conforming to a new role.
- Encourages higher levels of thinking.
- Permits more advanced students to study below-level material without embarrassment.
- Increases motivation to learn to maintain the new role.
- Increases ability to manage own learning and study strategies.
- Increases subject-specific knowledge.
- Increases related general knowledge.
- Increases understanding of the subject area.
- Improves attitude toward the subject area.
- Provides more empathy with students.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Does your college have a place such as a tutoring center where you can get help?
2. If so, what are the hours that this tutoring center is open?
3. If you are unsure about the answers to either of these questions, feel free to ask your professor - that's what they are there for!

Answers

Section Preview

This entire section focuses solely on similar triangles and their applications.

Section Preview

- **Arithmetic**
  - Ratios, Proportions, and Proportional Reasoning: Understanding what a proportion *is* conceptually (ratios and proportional reasoning) is the prerequisite for grasping why similar triangles have proportional sides. Proportions are also used within the examples.

Section Preview

- **Arithmetic**
  - Simplifying Fractions and Mixed Numbers: For verifying equal ratios.
- **Solving Equations**
  - Solving Rational Equations: For solving proportions algebraically.

Section Preview

- Determine when two triangles are congruent.
- Use proportion to find a side given two similar triangles.
- Solve application problems involving similar right triangles.

Example 1.3.1

In these triangles, $\angle B = \angle D$ because they are both right angles and $\angle BCA = \angle DCE$ because they are vertical angles, so $\theta = 25^\circ$. The third angles, $\angle A$ and $\angle E$, must also be equal, so $\phi = \chi = 65^\circ$. (Do you see why?) The sides opposite each pair of corresponding angles are equal, so $AB = DE, BC = CD$ and $AC = CE$. In particular, we find that $z = 9$.

Checkpoint 1.3.1

$\alpha = 85^\circ, \beta = 75^\circ, \gamma = 85^\circ$

Example 1.3.2

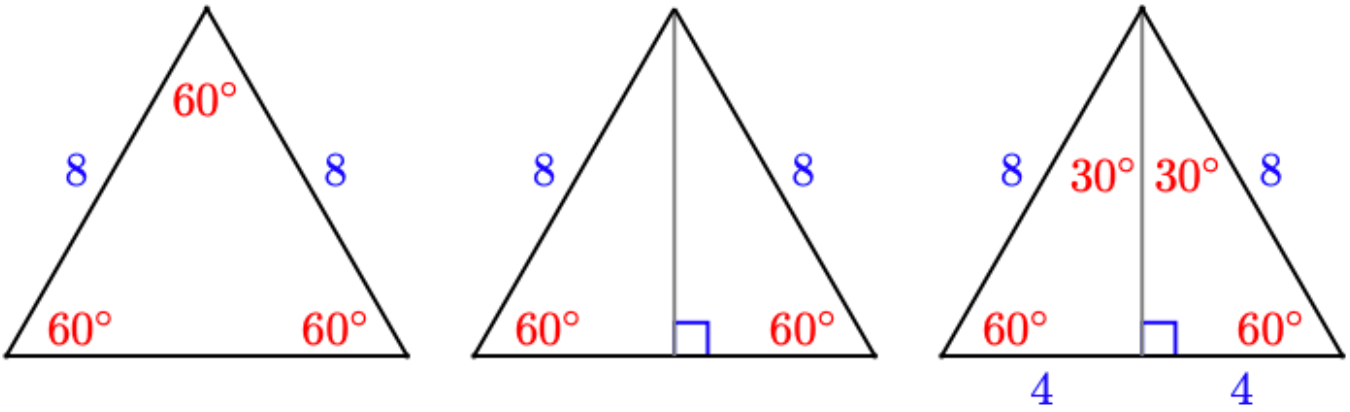


Figure 1.3.4

Consider, for example, an equilateral triangle of side 8 inches, as shown above. The altitude is perpendicular to the base, so each half of the original triangle is a right triangle. Because each right triangle contains a 60° angle, the remaining angle in each triangle must be $90^\circ - 60^\circ = 30^\circ$. Both triangles have a side of length 8 between the angles of 30° and 60°, so they are congruent. (Consequently, the short sides of the congruent triangles are equal, so each is half the original base.)

Checkpoint 1.3.2

$\angle BCA = \angle CAD$ and $\angle BAC = \angle ACD$ because they are alternate interior angles. If two angles in a triangle are equal, so is the third pair, so $\angle B = \angle D$. $BC = AD$ and $AB = CD$ because they are opposite sides of a parallelogram, and $AC = AC$.

Example 1.3.3

a. We will check whether the corresponding sides are proportional. We compute the ratios of the corresponding sides, making sure to write each ratio in the same order, $\frac{\text{larger triangle}}{\text{smaller triangle}}$. (The other order, $\frac{\text{smaller triangle}}{\text{larger triangle}}$, would also work, as long as we use the same order for all the ratios.)

$$\text{shorter legs: } \frac{9}{6} \quad \text{longer legs: } \frac{12}{8} \quad \text{hypotenuses: } \frac{15}{10}$$

Because all of these ratios are equal to 1.5, the triangles are similar.

b. The ratios of corresponding sides are not equal: the ratio of the longest sides is $\frac{6}{5}$, but the ratio of the smallest sides is $\frac{4}{3}$. The triangles are not similar.

c. The missing angle of the first right triangle is 48°, and the missing angle in the second right triangle is 42°, so three pairs of angles match. The triangles are similar.

Checkpoint 1.3.3

a. The triangles are similar because $\frac{4}{6} = \frac{6}{9} = \frac{8}{12}$, so the sides are proportional.

b. The third angle in both triangles is $80°$, so the triangles are similar because their corresponding angles are equal.

Example 1.3.4

We see that x is the length of the shortest side in $\triangle ABE$. We know the short side in $\triangle FCE$ and the lengths of the medium sides in each triangle. If we form the ratios of the short and medium sides, we obtain the following proportion:

$$\frac{\text{larger triangle}}{\text{smaller triangle}} : \frac{x}{4} = \frac{15}{6}$$

To solve the proportion, we multiply both sides by the LCD to get

$$\begin{aligned} & & 6x &= 4(15) & \\ \text{Arithmetic} \implies & & 6x &= 60 & \text{(multiply)} \\ \cancel{\text{Arithmetic}}\text{Algebra} \implies & & x &= 10 & \text{(divide both sides by 6)} \end{aligned}$$

Checkpoint 1.3.4

$y = 20$

Example 1.3.5

a. In Figure 1.3.13, we see two right triangles: One triangle is formed by the building and its shadow, and the pole and its shadow form the other. Because the light rays from the sun are parallel, the two angles at the tips of the shadows are equal. Thus, the two right triangles are similar, and their corresponding sides are proportional. The ratios of heights and bases in the two triangles yield the proportion:

$$\frac{\text{larger triangle}}{\text{smaller triangle}} : \frac{h}{5} = \frac{12}{3}$$

b. To solve the proportion, we multiply both sides by the LCD to get

$$\begin{aligned} \cancel{\text{Arithmetic}}\text{Algebra} \implies & 3h = 5(12) & \text{(multiply both sides by the LCD of all terms in the equation)} \\ \text{Arithmetic} \implies & 3h = 60 & \text{(multiply)} \\ \cancel{\text{Arithmetic}}\text{Algebra} \implies & h = 20 & \text{(divide both sides by 3)} \end{aligned}$$

The building is 20 feet tall.

Checkpoint 1.3.5

2.5 feet

Example 1.3.6

The two triangles overlap, sharing the marked angle, as shown below. Because each triangle also has a right angle, they are similar.

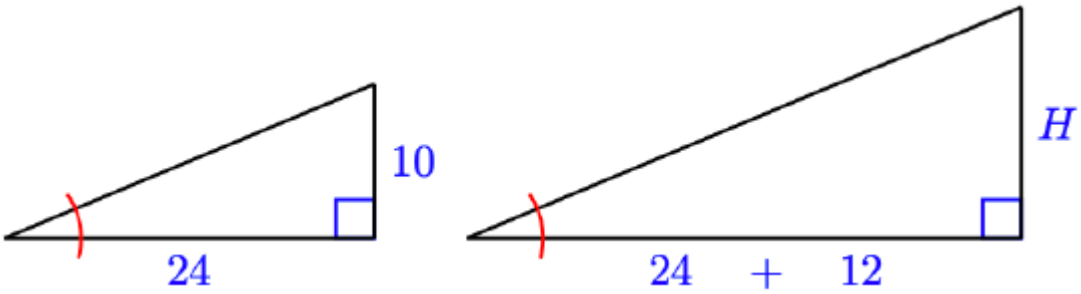


Figure 1.3.15

Note that the base of the larger triangle is $24 + 12 = 36$. The ratio of the heights and the ratio of the bases must be equal, so we write the following proportion:

$$\begin{aligned} & \frac{H}{10} = \frac{36}{24} & \text{(start with the proportion)} \\ \cancel{\text{Arithmetic}}\text{Algebra} \implies & 24H = 360 & \text{(multiply both sides by the LCD)} \\ \cancel{\text{Arithmetic}}\text{Algebra} \implies & H = \frac{360}{24} & \text{(divide both sides by 24)} \\ \text{Arithmetic} \implies & H = 15 & \text{(division)} \end{aligned}$$

Checkpoint 1.3.6

15 feet

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1.3.1: Resources and Key Concepts

Key Concepts

Definitions

- **Congruent Triangles:** Two triangles that have equal corresponding angles and identical corresponding side lengths. They are the same shape and size.
- **Parallelogram:** A four-sided figure (quadrilateral) in which both pairs of opposite sides are parallel and have equal length.
- **Similar Triangles:** Two triangles whose corresponding angles are equal. This implies they have the same shape but may be different sizes.

Theorems

- **Similarity Conditions:** Two triangles are determined to be similar if one of two conditions is met: either their corresponding angles are equal, or their corresponding sides are proportional. If one condition is true, the other is automatically true.

Common Mistakes

- **Confusing Congruent and Similar:** While congruent triangles are always similar, similar triangles are not necessarily congruent. Similarity requires only the same shape (equal angles), not necessarily the same size.
- **Incorrectly Setting Up Proportions:** When creating ratios of corresponding sides, the order must be consistent. For example, if the first ratio is $\frac{\text{larger triangle side}}{\text{smaller triangle side}}$, all subsequent ratios in the proportion must follow the same order.
- **Using Incorrect Side Lengths in Overlapping Triangles:** When triangles overlap, be careful to identify the full side length of the larger triangle. For instance, in Example 6, the base of the larger triangle is the sum of two segments $(24 + 12)$, not just one of the individual segments.

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1.3.2: Homework

Reading Questions

1. What two conditions must be met for two triangles to be congruent?
2. What is the symbol used to indicate that two triangles are congruent?
3. What is the key difference between congruent triangles and similar triangles?
4. Are all congruent triangles similar? Are all similar triangles congruent?
5. What does it mean for the corresponding sides of two triangles to be "proportional"?
6. According to the Similarity Conditions theorem, what are the two ways to verify that two triangles are similar?
7. If two right triangles share one pair of equal acute angles, are the triangles similar?
8. In Example 2, how is the altitude of an equilateral triangle used to create two congruent right triangles?
9. In Example 5, what assumption is made about the sun's light rays that allows us to conclude the triangles are similar?
10. When triangles overlap, as in Example 6, what must you be careful about when determining the side lengths for your proportion?

Skills Refresher

The following is a set of review exercises you will need for this section.

Skills Refresher

For each of the following exercises, solve the equation.

1. $\frac{x}{12} = \frac{3}{x}$
2. $1 + \frac{x}{2} = \frac{2x}{5}$
3. $\frac{x}{6} = \frac{1}{6} + \frac{x}{6}$
4. $x + \frac{6}{x} = -5$
5. $\frac{5}{2x} = \frac{17}{18} - \frac{1}{3x}$

Answers available at the end of this section.

Homework

Vocabulary Check

1. If given two triangles that are the same size and shape, we say they are ___; however, if their size is different but their shape remains the same, we call them ___ triangles.

2. The ___ of a triangle is the segment from one vertex of the triangle perpendicular to the ___ side.

Concept Check

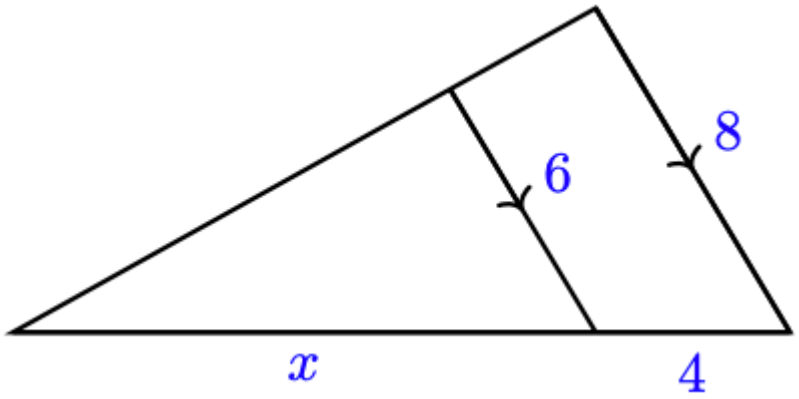
3. What is the difference between congruent triangles and similar triangles?
4. What is the name of the short-cut method for solving proportions? Why does the method work?
5. In two triangles, are the triangles similar if two corresponding pairs of angles are equal? How do you know?

True or False? For each of the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

6. The diagonal of a parallelogram splits the shape into two congruent triangles.
7. In a $30°$-$60°$-$90°$ triangle, the leg opposite the $60°$ angle is half the length of the hypotenuse.
8. If two right triangles have one pair of angles with the same measure, then the triangles are similar.

Basic Skills

9. For the triangles shown, which of the following equations is true? Explain why.



a. $\frac{4}{x} = \frac{6}{8}$

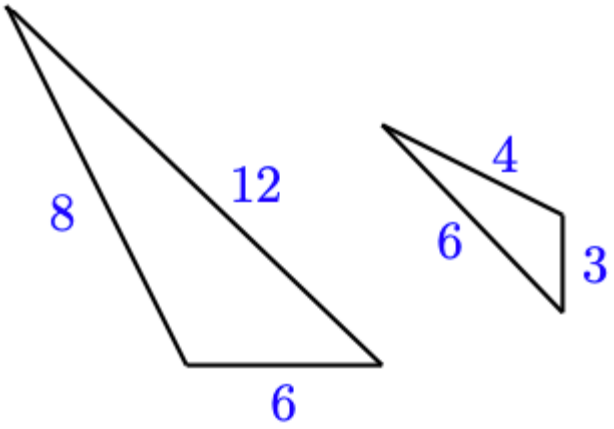
b. $\frac{x}{4} = \frac{6}{8}$

c. $\frac{x}{x+4} = \frac{6}{8}$

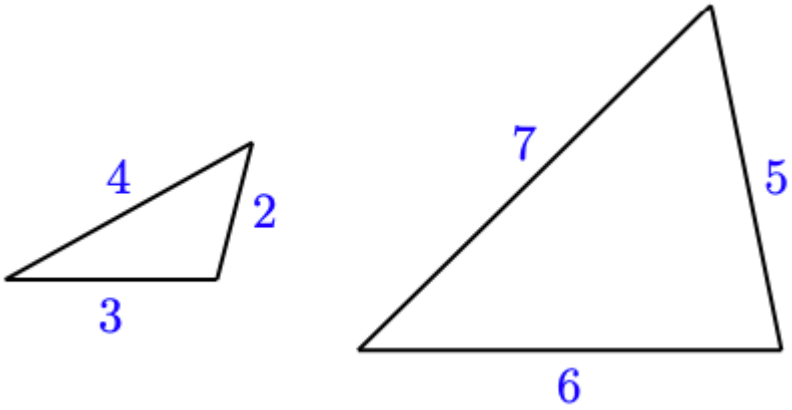
d. $\frac{x}{x+4} = \frac{6}{14}$

For each of the following exercises, decide whether the triangles are similar and explain why or why not.

10.

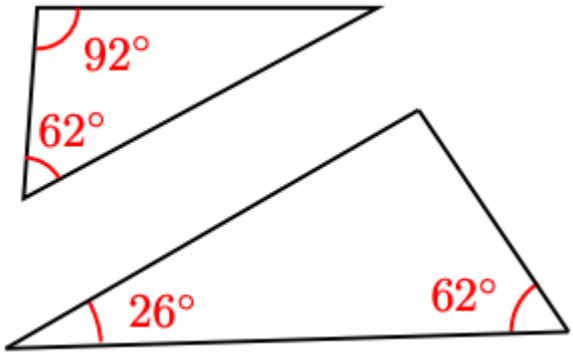


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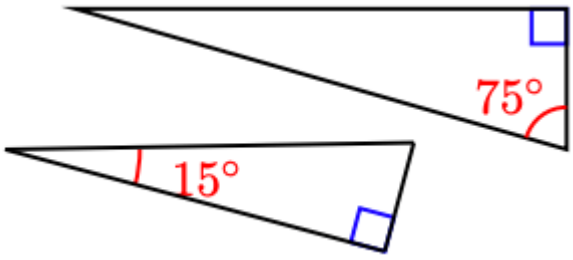


12.

MyOpenMath

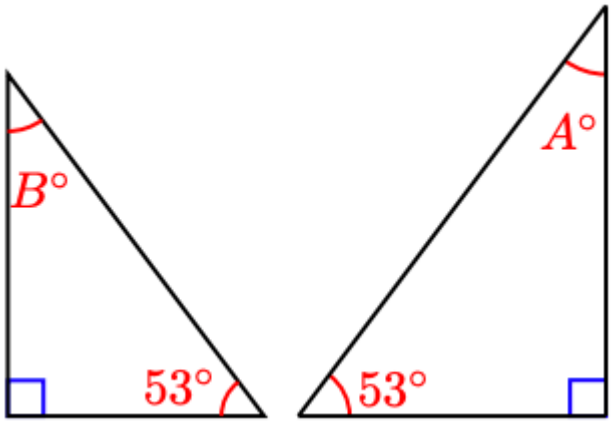


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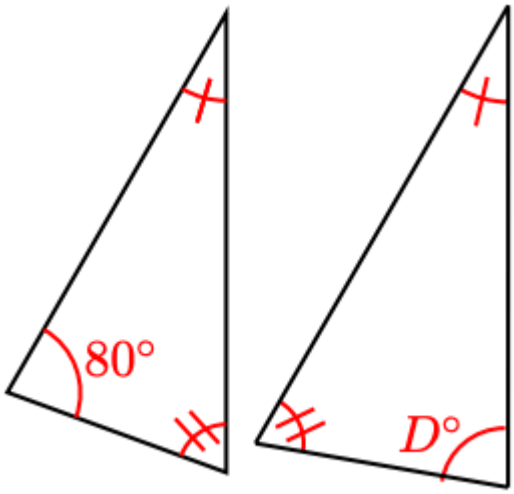


For each of the following exercises, assume the triangles are similar. Solve for the variables. (Figures are not drawn to scale.)

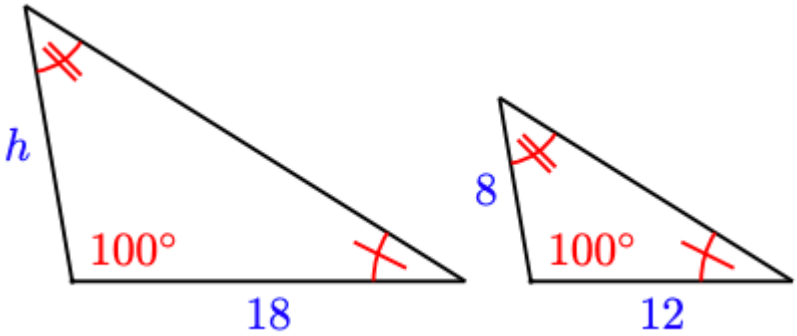
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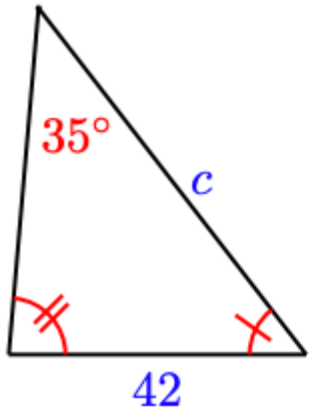
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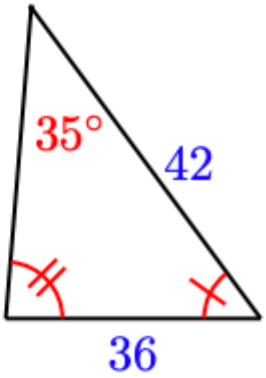


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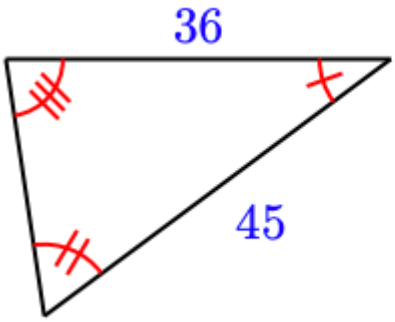
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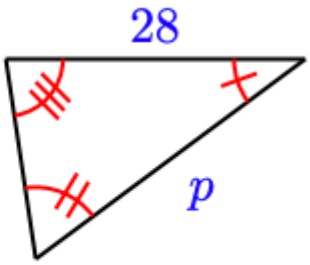




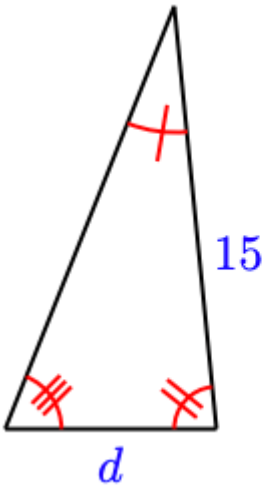
For each of the following exercises, use properties of similar triangles to solve for the variable.

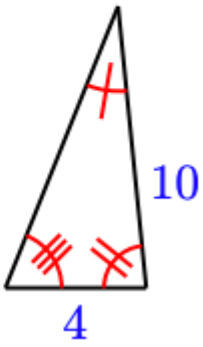
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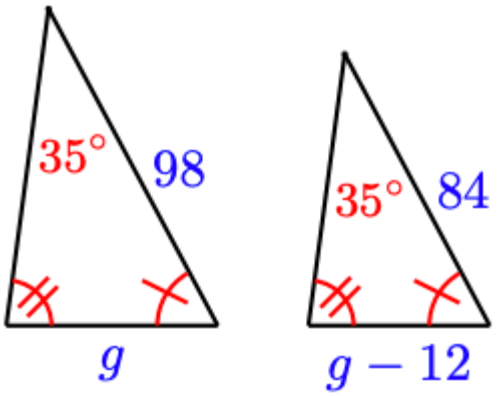
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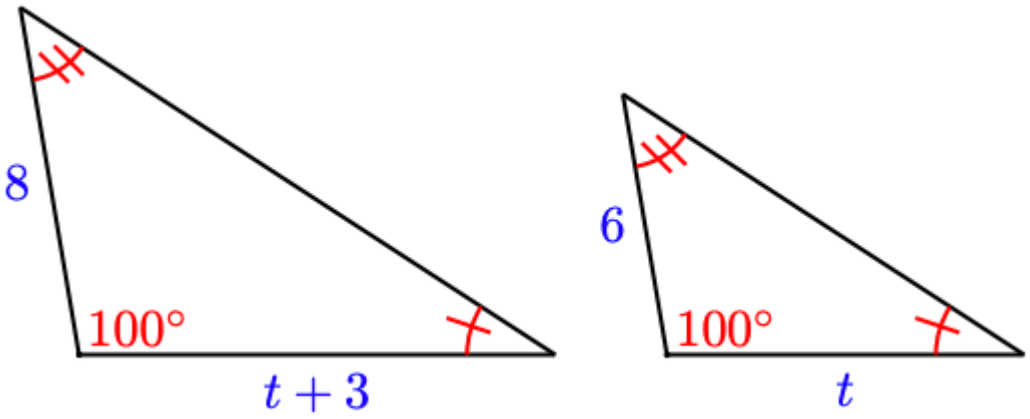


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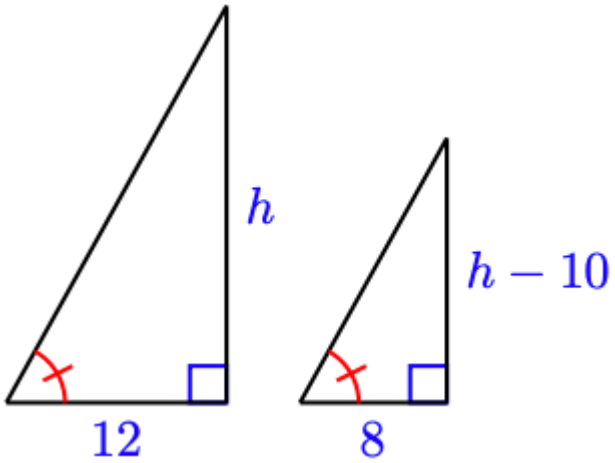
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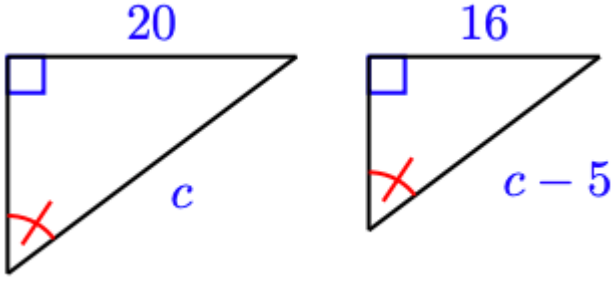


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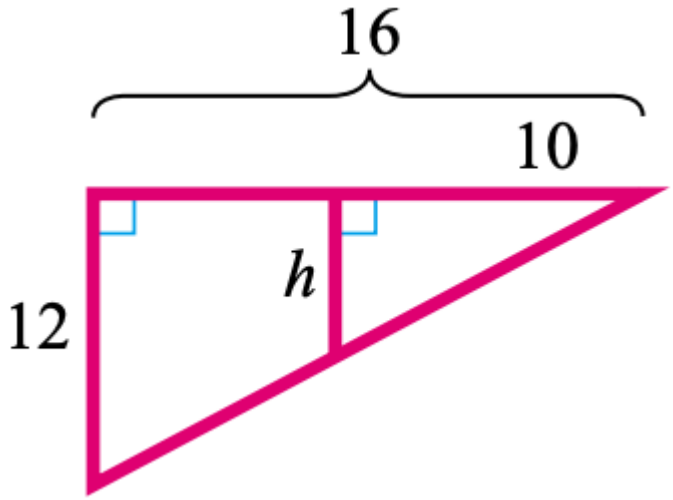
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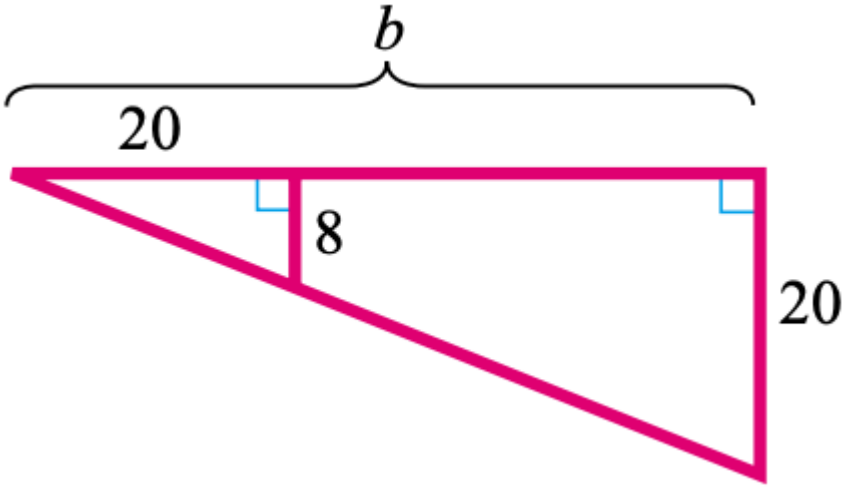


For each of the following exercises, use properties of similar triangles to solve for the variable.

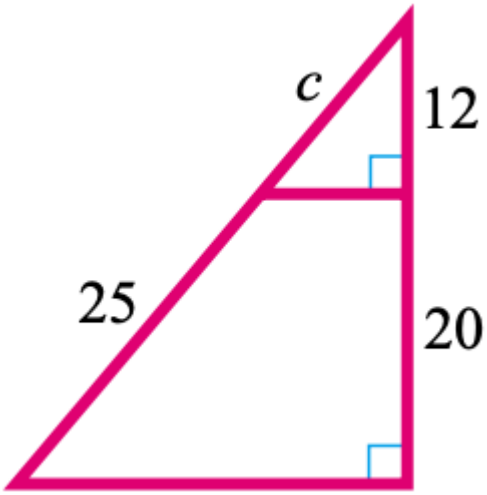
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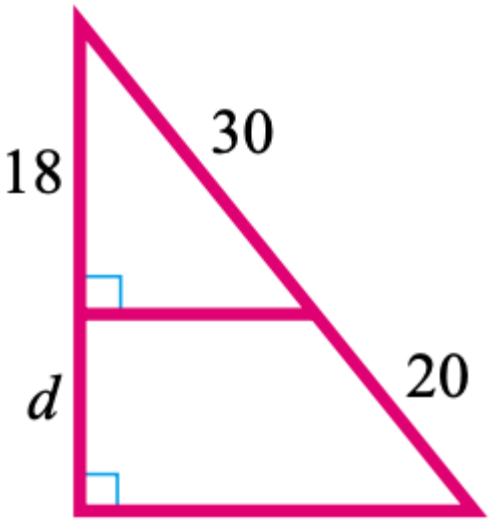


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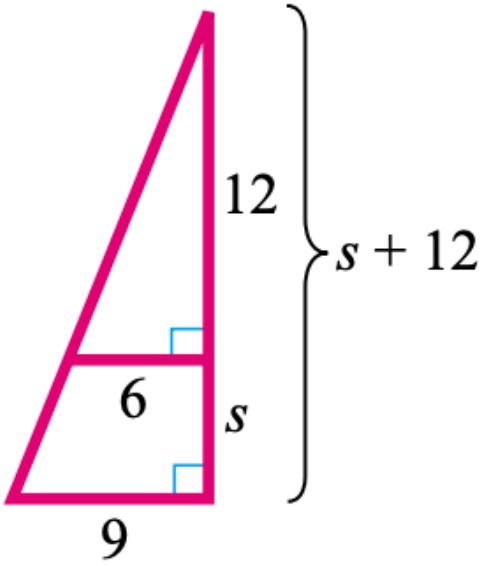


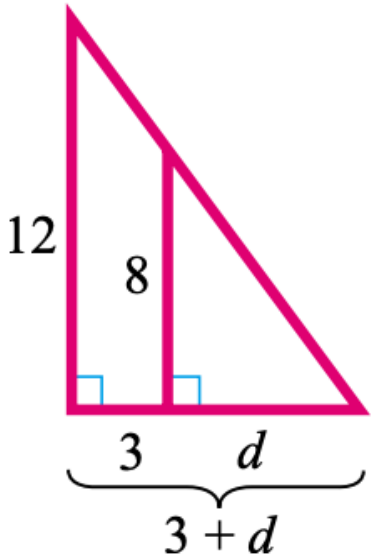
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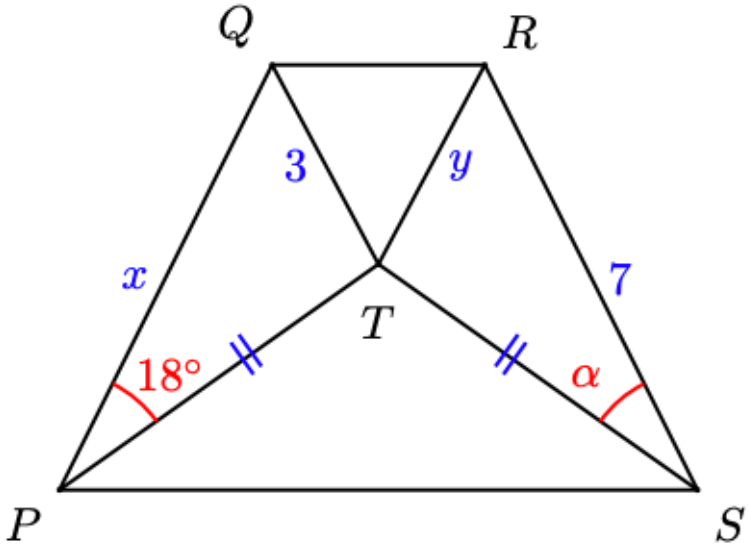




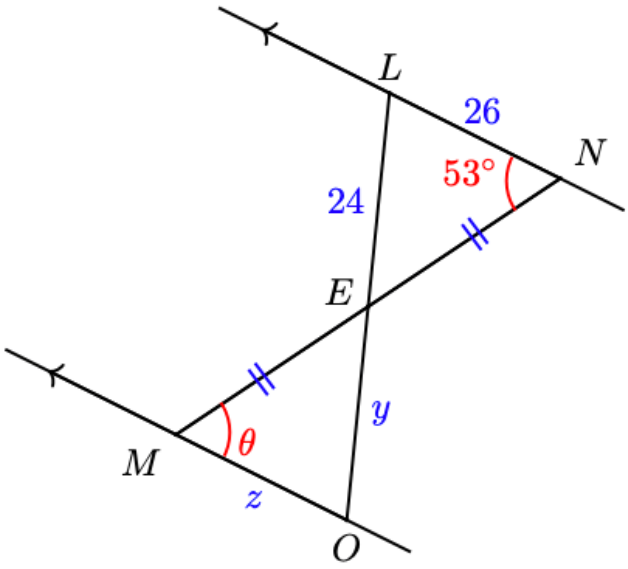
Synthesis Questions

For each of the following exercises, name two congruent triangles and find the unknown quantities.

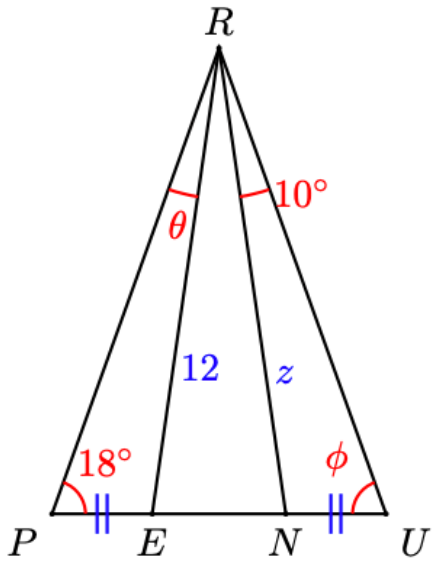
30. $PQRS$ is an isosceles trapezoid.



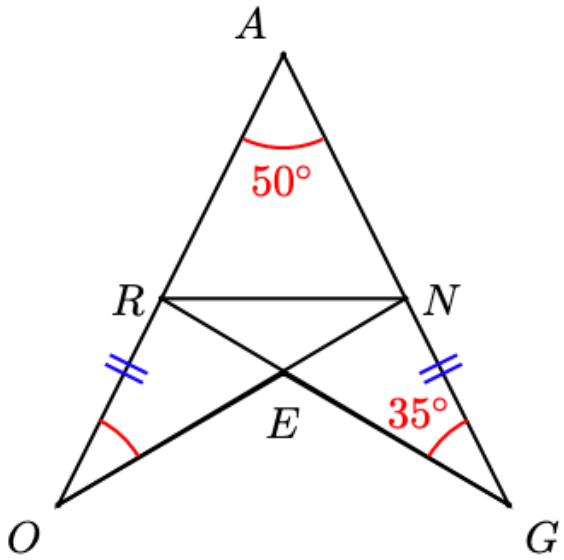
31.



32. ΔPRU is isosceles.

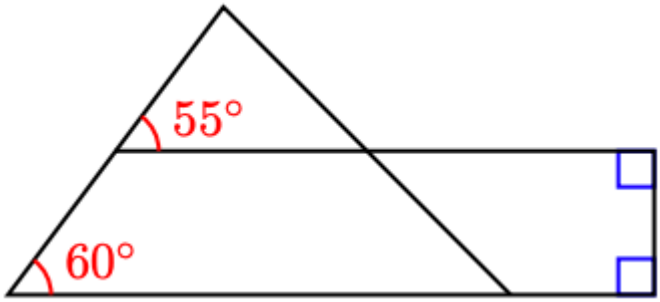


33. ΔARN is isosceles and $OR = NG$. Find $\angle RNG$ and $\angle RNO$.

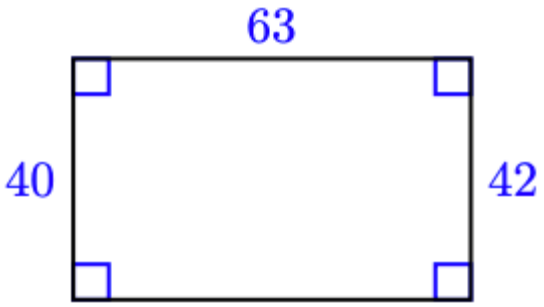


For each of the following exercises, explain why the measurements shown are inaccurate.

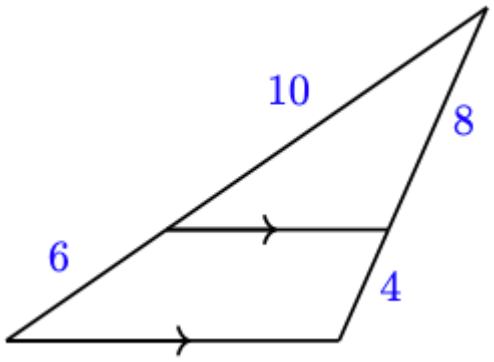
34.



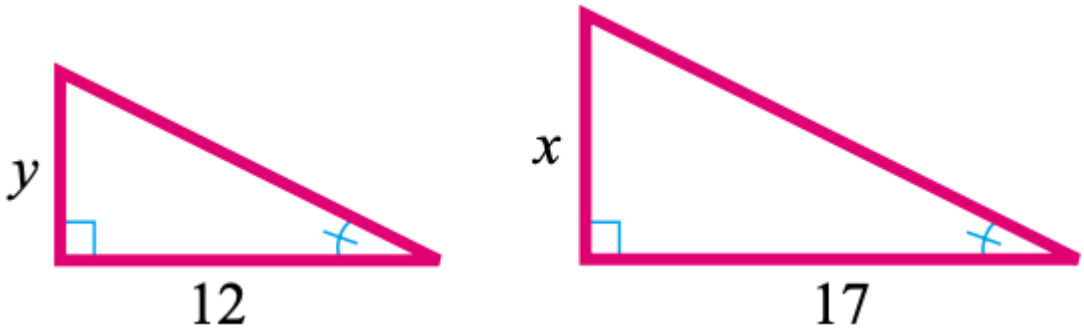
35.



36.

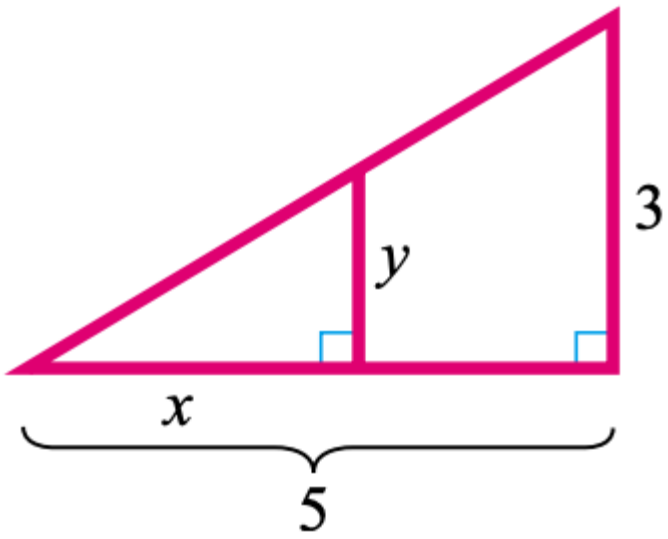


37. The follow pairs of triangles are similar. Solve for y in terms of x.

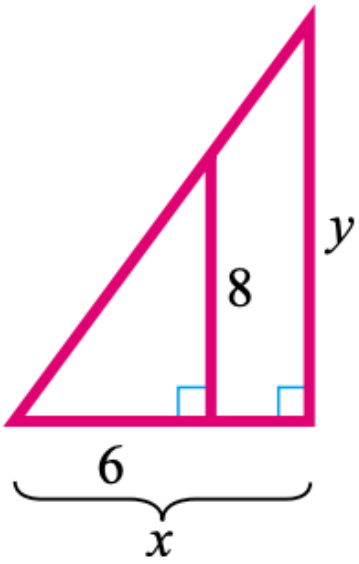


For each of the following exercises, solve for y in terms of x.

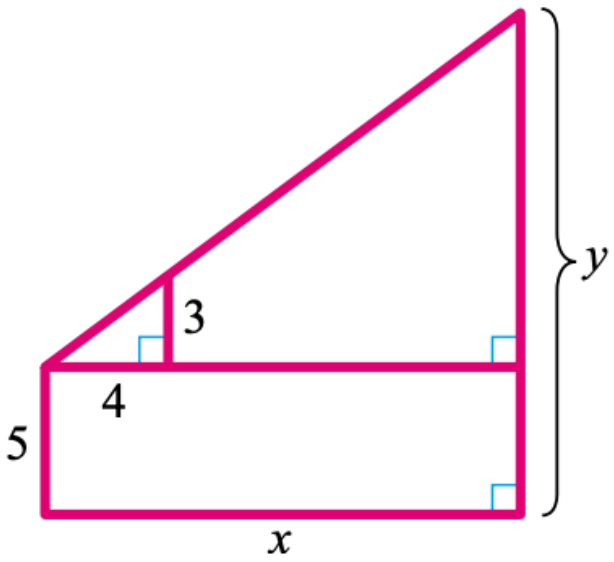
38.



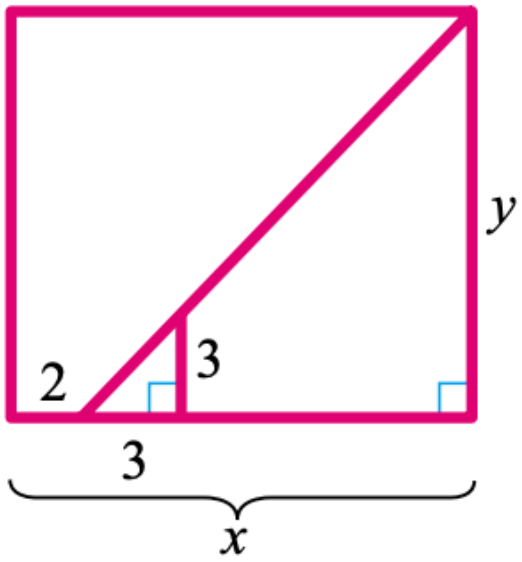
39.



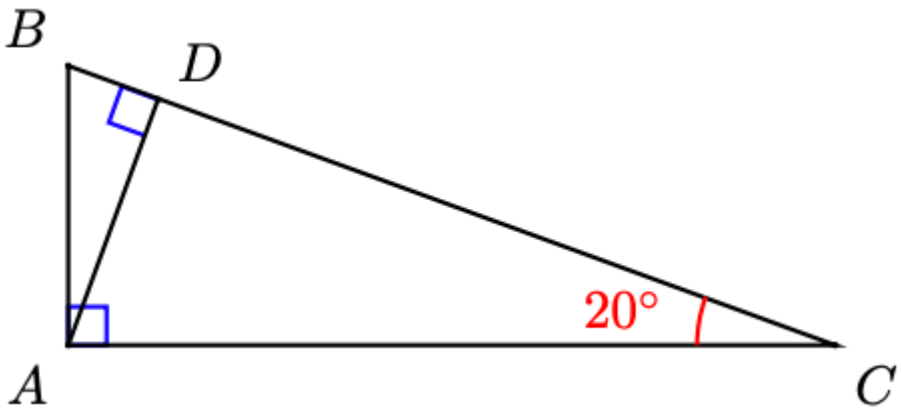
40.



41.



42. Triangle ABC is a right triangle, and AD meets the hypotenuse BC at a right angle.



a. If $\angle ACB = 20^\circ$, find $\angle B$, $\angle CAD$, and $\angle DAB$.

b. Find two triangles similar to ΔABC. List the corresponding sides in each of the triangles.

Applications

43. **Measuring Distances.** Ivan and Kim want to measure the distance across a stream. They mark point A directly across the stream from a tree at point T on the opposite bank. Ivan walks from point A down the bank a short distance to point B and sights the tree. He measures the angle between his line of sight and the stream bank.

a. Draw a figure showing the stream, the tree, and the right triangle $\triangle ABT$.

b. Meanwhile, Kim, still standing at point A, walks away from the stream at right angles to Ivan's path. Ivan watches her progress and tells her to stop at point C when the angle between the stream bank and his line of sight to Kim is the same as the angle from the stream bank to the tree. Add triangle $\triangle ABC$ to your figure.

c. Ivan now measures the distance from point A to Kim at point C. Explain why this distance is the same as the distance across the stream.

44. **Measuring Distances.** If you have a baseball cap, here is another way to measure the distance across a river. Stand at point

A directly across the river from a convenient landmark, say a large rock, on the other side. Tilt your head down so that the brim of the cap points directly at the base of the rock, R.

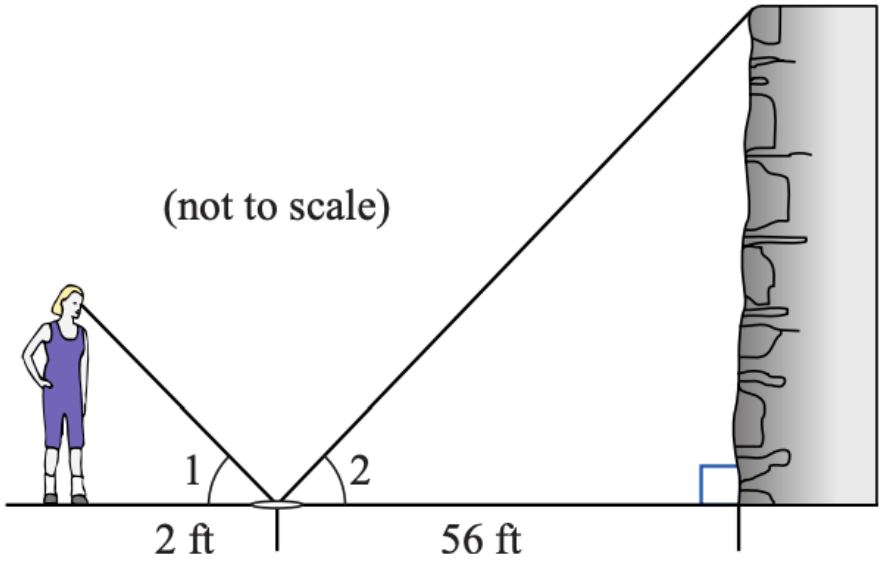
a. Draw a figure showing the river, the rock, and the right triangle $\triangle ABR$, where B is the location of your baseball cap on your head.

b. Now, without changing the angle of your head, rotate 90° and sight along the bank on your side of the river. Have a friend mark the spot C on the ground where the brim of your cap points. Add triangle $\triangle ABC$ to your figure.

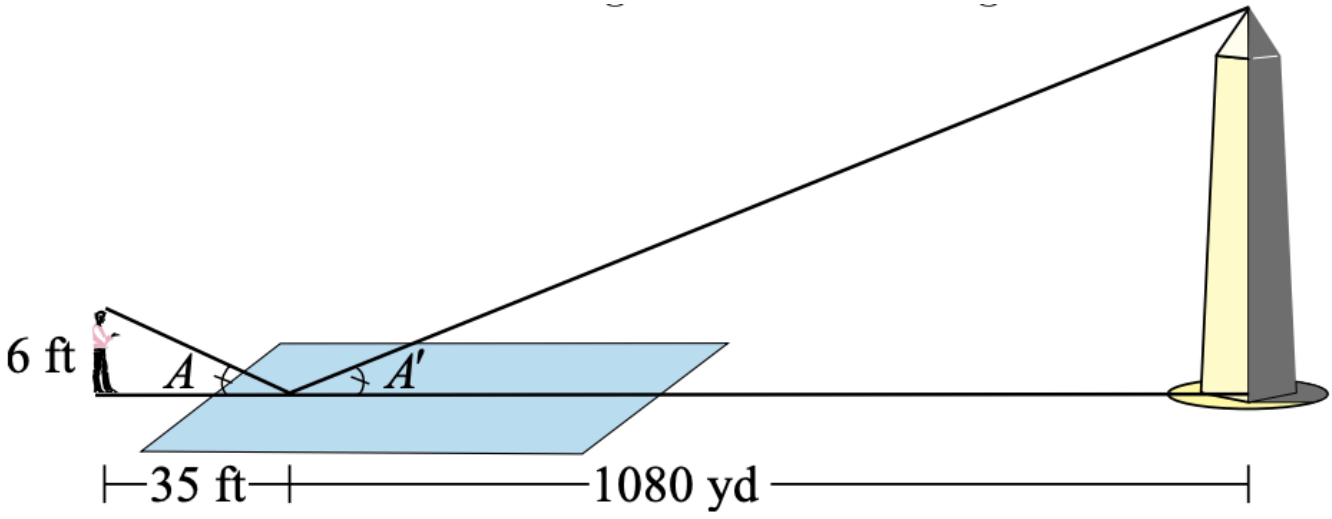
c. Finally, you can measure the distance from point A to point C. Explain why this distance is the same as the distance across the river.

For each of the following exercises, use properties of similar triangles to solve.

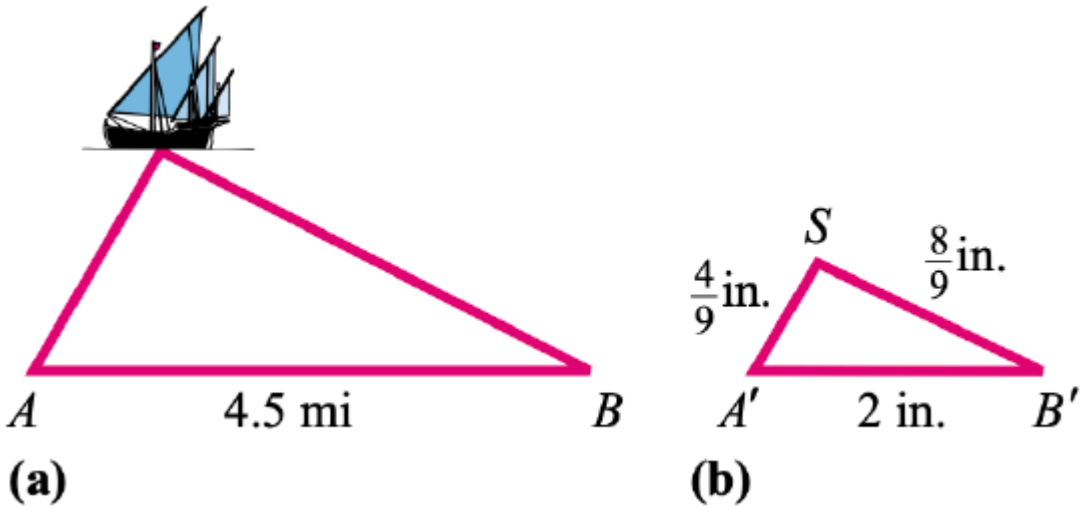
15. **Heights.** A rock climber estimates the height of a cliff she plans to scale. She places a mirror on the ground to see the top of the cliff in the mirror while she stands straight. The angles 1 and 2 formed by the light rays are equal, as shown in the figure. She then measures the distance to the mirror (2 feet) and the distance from the mirror to the base of the cliff (56 feet). How high is the cliff if she is 5 feet 6 inches tall?



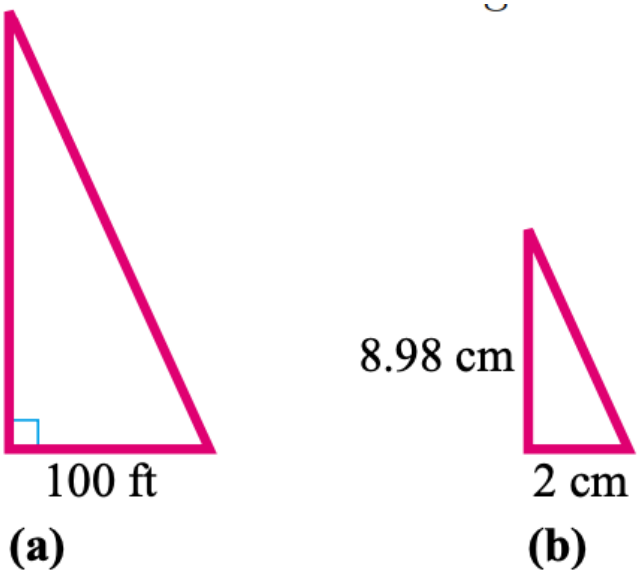
16. **Heights.** Lap wants to estimate the height of the Washington Monument. He notices that he can see the reflection of the top of the monument in the reflecting pool. He is 35 feet from the tip of the reflection, and that point is 1080 yards from the base of the monument, as shown below. From his physics class, Lap knows that the angles marked (in the image below) are equal. If Lap is 6 feet tall, what is his estimate for the height of the Washington Monument?



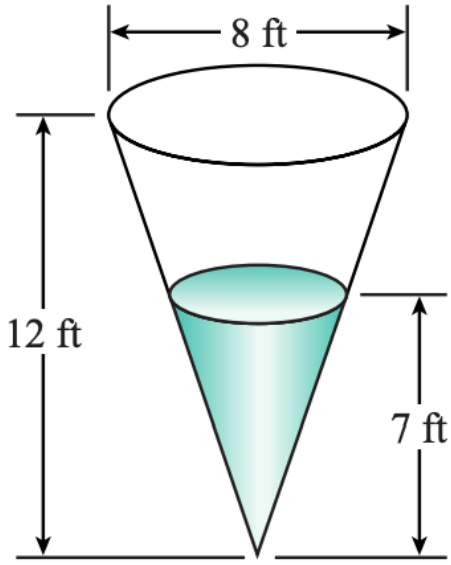
17. **Distances.** In the sixth century BC, the Greek philosopher and mathematician Thales used similar triangles to measure the distance to a ship at sea. Two observers on the shore at points A and B would sight the ship and measure the angles formed, as shown in Figure (a). They would then construct a similar triangle, as shown in Figure (b), with the same angles at A and B , and measure its sides. (This method is called *triangulation*.) Use the lengths given in the figures to find the distance from the observer at location B to the ship.



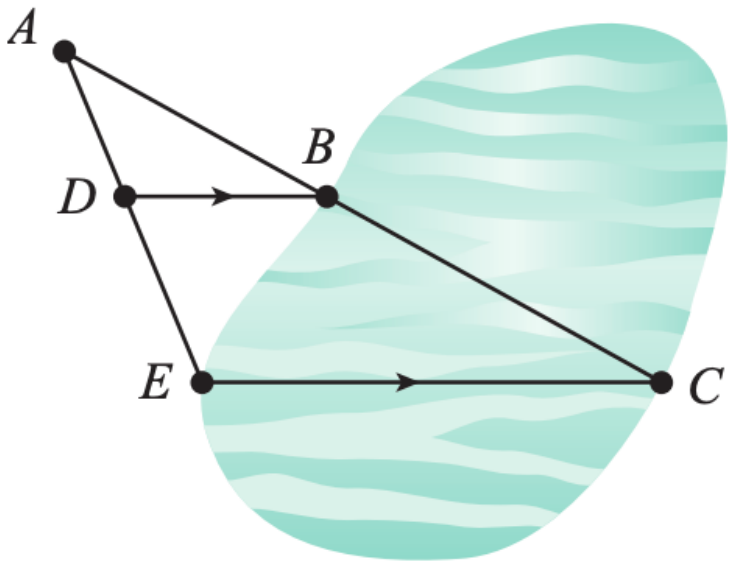
18. **Distances.** The Capilano Suspension Bridge is a footbridge that spans a 230-foot gorge north of Vancouver, British Columbia. Before crossing the bridge, you decide to estimate its length. You walk 100 feet downstream from the bridge and sight its far end, noting the angle formed by your line of sight, as shown in Figure (a). You then construct a similar right triangle with a two-centimeter base, as shown in Figure (b). You find that the height of your triangle is 8.98 centimeters. How long is the Capilano Suspension Bridge?



49. **Surface Area.** A conical tank is 12 feet deep, and the top's diameter is 8 feet. If the tank is filled with water to a depth of 7 feet, as shown in the figure at right, what is the area of the exposed surface of the water?

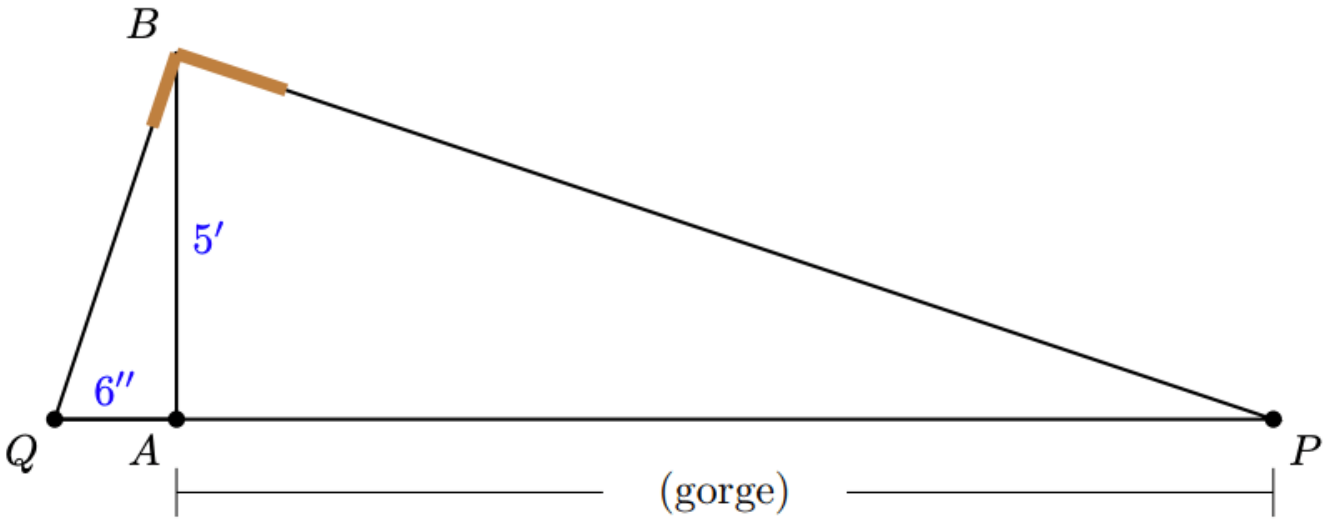


50. **Distances.** To measure the distance EC across the lake shown in the figure at right, stand at A and sight point C across the lake, then mark point B. Then sight to point E and mark point D so that DB is parallel to EC. If $AD = 25$ yards, $AE = 60$ yards, and $BD = 30$ yards, how wide is the lake?



Challenge Problems

51. **Distances.** Here is a way to find the distance across a gorge using a carpenter's square and a five-foot pole. Plant the pole vertically on one side of the gorge at point A and place the angle of the carpenter's square on top of the pole at point B, as shown in the figure. Sight along one side of the square so that it points to the opposite side of the gorge at point P. Without moving the square, sight along the other side and mark point Q. If the distance from Q to A is six inches, calculate the width of the gorge. Explain your method.



Answers

Skills Refresher

1. $x = \pm 6$
2. $x = -10$
3. $x = 4$
4. $x = -3$ and $x = -2$
5. $x = 3$

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1.4: Circles and Angles in the Rectangular Coordinate System

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The Distance Formula

Matt is hiking in the Santa Monica mountains. He would like to know the distance from the Sycamore Canyon trailhead, located at $12-\mathrm{C}$ on his map, to the Coyote Trail junction, located at $8-\mathrm{F}$, as shown below.

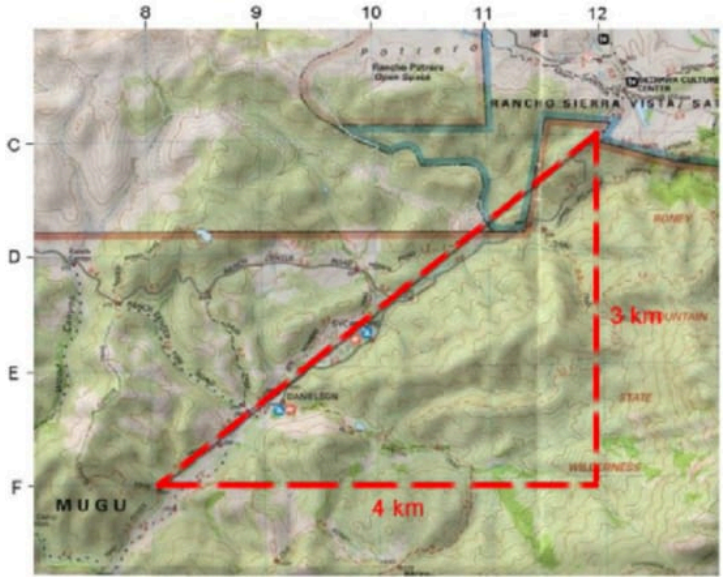


Figure 1.4.1

Each interval on the map represents one kilometer. Matt remembers the Pythagorean Theorem and uses the map coordinates to label the sides of a right triangle. The distance he wants is the hypotenuse of the triangle, so

$$\begin{aligned} & d^2 &=& 4^2+3^2 \\ \Longrightarrow\ & d^2 &=& 16+9 \\ \Longrightarrow\ & d^2 &=& 25 \\ \Longrightarrow\ & d &=& \sqrt{25} \\ \Longrightarrow\ & d &=& 5 \end{aligned}$$

The straight-line distance to Coyote junction is about 5 kilometers. The formula for the distance between two points is obtained similarly.

We first label a right triangle with points P_1 and P_2 on opposite ends of the hypotenuse (see Figure 1.4.2). The sides of the triangle have lengths $|x_2 - x_1|$ and $|y_2 - y_1|$. We can use the Pythagorean Theorem to calculate the distance between P_1 and P_2: $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

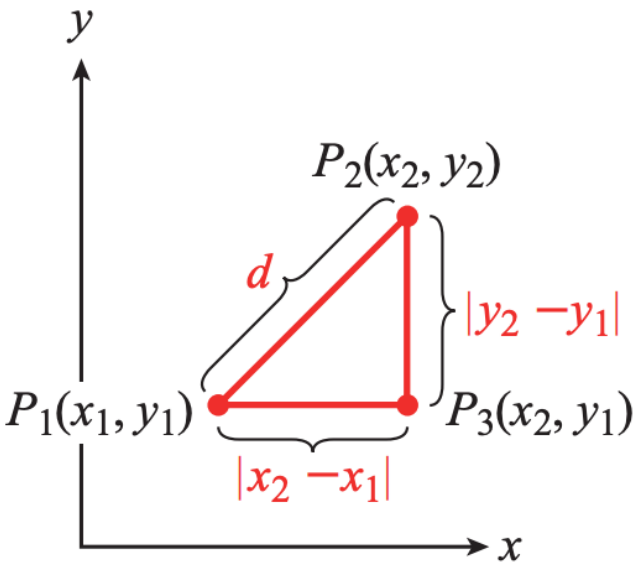


Figure 1.4.2

Taking the (positive) square root of each side of this equation gives us the **Distance Formula**.

Theorem: Distance Formula

The distance d between two points $P_1\,(x_1, y_1)$ and $P_2\,(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 1.4.1

Find the distance between $(2, -1)$ and $(4, 3)$.

Solution available at the end of this section.

Critical Mistake

In Example 1.4.1, the radical $\sqrt{4+16}$ cannot be simplified to $\sqrt{4}+\sqrt{16}$. (Do you remember why not?)

Checkpoint 1.4.1

a. Find the distance between the points $(-5, 3)$ and $(3, -9)$.
b. Plot the points on a Cartesian coordinate system and show how the Pythagorean Theorem is used to calculate the distance.

Answers available at the end of this section.

Equation for a Circle

The Distance Formula is required for us to find the equation of a circle.

Theorem: Equation of a Circle Centered at the Origin

The equation for a circle of radius r centered at the origin is

$$x^2 + y^2 = r^2.$$

Proof available at the end of this section.

Example 1.4.2

Find two points on the circle $x^2 + y^2 = 4$ with an x-coordinate of -1.

Solution available at the end of this section.

Checkpoint 1.4.2

Find the coordinates of two points on the circle $x^2 + y^2 = 1$ with y-coordinate $\frac{1}{2}$.

Answer available at the end of this section.

The circle in Checkpoint 1.4.2 holds a special place in Trigonometry.

Definition: Unit Circle

The circle centered at the origin and having radius 1, is called the **unit circle**. The unit circle has equation

$$x^2 + y^2 = 1.$$

While the definition of the unit circle is only mentioned in passing here, it will become one of the foundational pieces of Trigonometry. We will revisit this later.

Later in this course, we will need to work with circles that do not have centers at the origin.

Theorem: Equation of a Circle Centered Off-Origin

The equation for a circle of radius r centered at the point (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2.$$

This is also know as the **standard form** of the equation of a circle.

Proof available at the end of this section.

Example 1.4.3

Write the standard equation of the circle with center $(-2, 3)$ and radius 5.

Solution available at the end of this section.

Checkpoint 1.4.3

Graph $(x+2)^2 + (y-1)^2 = 4$. Find the center and radius.

Answer available at the end of this section.

Interactive Element: Circle in Standard Form

The following Interactive Element might help you visualize how the numbers h, k, and r in $(x-h)^2 + (y-k)^2 = r^2$ control a circle's center and size.

Interact: Drag the red center point to move the circle; drag the green handle to change the radius r.

Observation: The center sits exactly at (h, k) — so $(x+2)^2 + (y-3)^2 = 25$ is centered at $(-2, 3)$, not $(2, 3)$. The radius is r, and $r = 1$ at the origin gives the unit circle.

Angles in the Cartesian Coordinate System

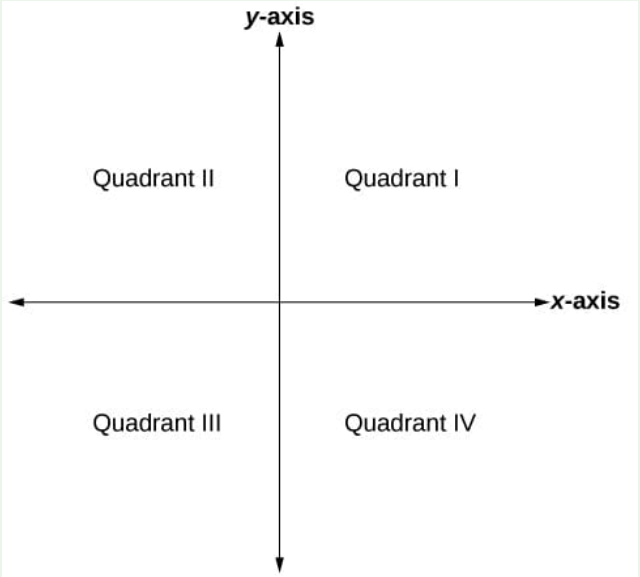
Now that we have reacquainted ourselves with angles, circles, and the Cartesian coordinate system, the next natural evolution toward Trigonometry is introducing angles to the Cartesian coordinate plane. Before we begin, however, it will be instrumental in this course to divide the Cartesian coordinate system into four sections, called **quadrants**.

Definition: Quadrants

In the Cartesian coordinate system, the section of the plane where

- x and y are both positive is called **Quadrant I** (**QI**),
- $x < 0$ and $y > 0$ is called **Quadrant II** (**QII**),
- x and y are both negative is called **Quadrant III** (**QIII**), and
- $x > 0$ and $y < 0$ is called **Quadrant IV** (**QIV**).

The four quadrants of the Cartesian coordinate system



Knowing this definition and understanding how the signs of x and y relate to these quadrants will be critical as we move forward.

Angles in Standard Position

In general, and without worrying about a coordinate system, the degree measure of an angle depends only on the fraction of a whole rotation between its sides, not on the angle's location or position. To compare and analyze angles, we place them in **standard position**.

Definition: Standard Position of an Angle

An angle is said to be in **standard position** if the vertex of the angle is at the origin and the initial side of the angle is on the positive x-axis.

Figure 1.4.6 below shows several angles moved into standard position.

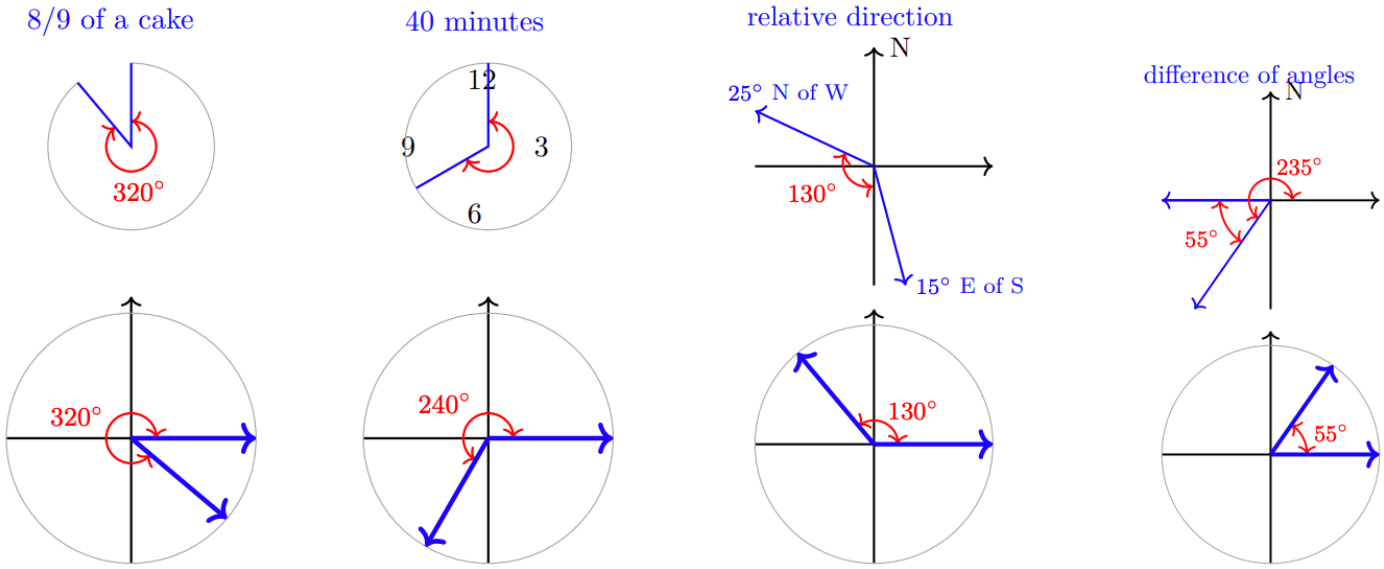


Figure 1.4.6**:** *Angles in nonstandard position (top row) and their equivalent angles in standard position (bottom row)*

The 320° in Figure 1.4.6 has its terminal side in quadrant IV. We denote this as $320^\circ \in \text{QIV}$. Likewise, the remaining angles in Figure 1.4.6 are listed as $240^\circ \in \text{QIII}$, $130^\circ \in \text{QII}$, and $55^\circ \in \text{QI}$.

Example 1.4.4

Find the degree measure of each angle shown below, state the quadrant where the angle terminates, and sketch the angle in standard position.



Figure 1.4.7

Solutions available at the end of this section.

Checkpoint 1.4.4

Find the degree measure of each angle below, sketch the angle in standard position, and state the quadrant in which the angle terminates.

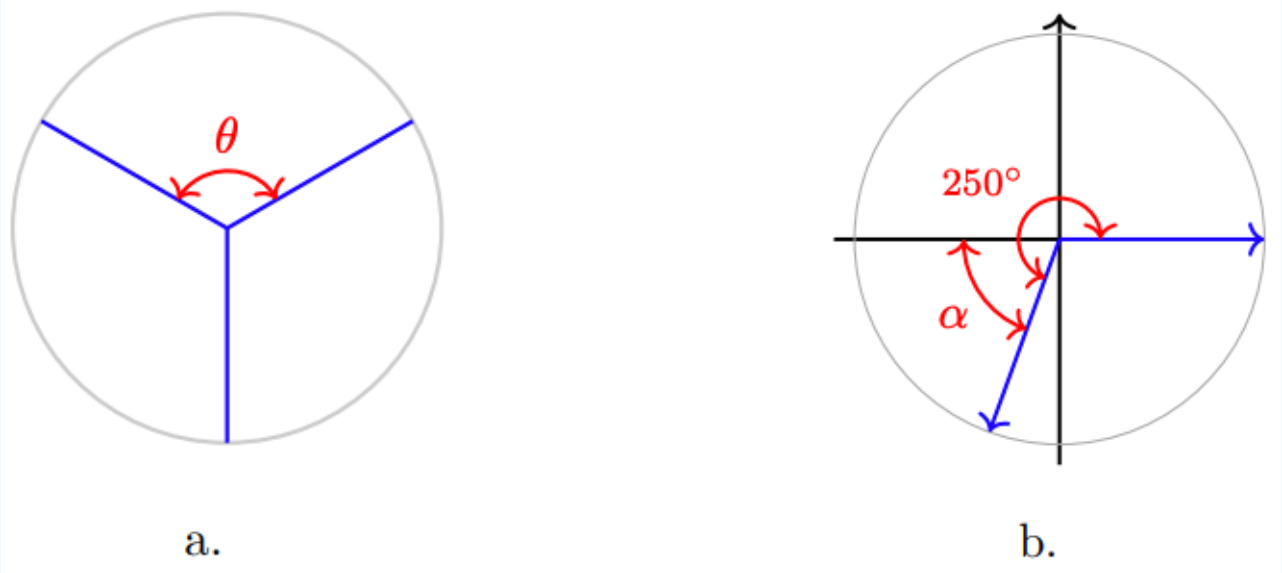


Figure 1.4.9

Answers available at the end of this section.

Quadrantal Angles

Now that we are acquainted with angles in standard position, it's time we start classifying these angles by type. First up are the **quadrantal angles**.

Definition: Quadrantal Angle

An angle in standard position having terminal side on an axis is called a **quadrantal angle**.

For example, 270° has a terminal side on the negative y-axis. Therefore, we would call 270° a quadrantal angle. Figure $1.4.12$ illustrates a few quadrantal angles; however, as we will learn shortly, there are infinite quadrantal angles.

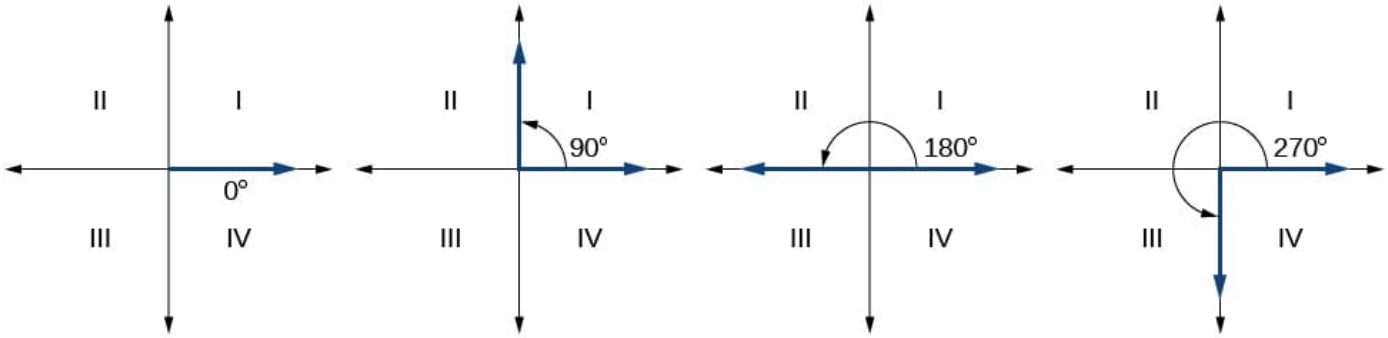


Figure $1.4.12$**:** *Some quadrantal angles*

Negative Angles

Aligning with what we said earlier in this textbook, an angle is positive if it opens counterclockwise. If an angle opens clockwise, the angle is considered negative. For example, Figure $1.4.13$ shows the angle $\alpha = -135^\circ$, which terminates in QIII.

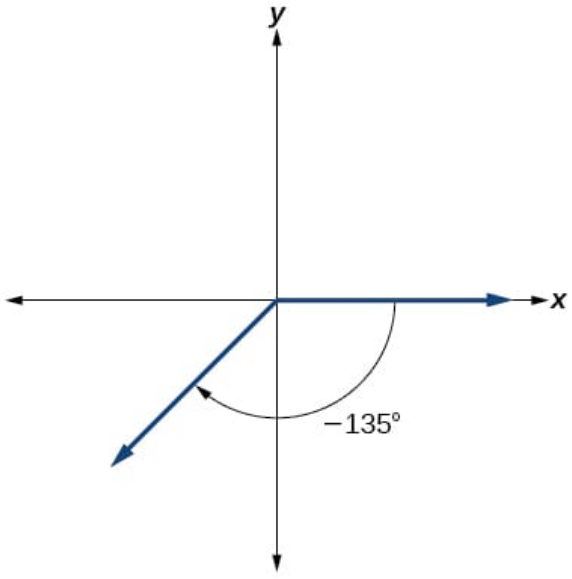


Figure $1.4.13$**:** *A negative angle*

Graphing positive and negative angles in standard position will be a critical skill in Trigonometry.

Coterminal Angles

Because 360° represents one complete revolution, we can add or subtract a multiple of 360° to any angle and the terminal side of the resulting angle will arrive at the same position. For example, the angles 70° and 430° have the same terminal side because $430^\circ = 70^\circ + 360^\circ$. Such angles are called **coterminal**.

Definition: Coterminal Angles

Angles in standard position that share a terminal side are called **coterminal angles**.

The angle 790° is also coterminal with 70°, because $790^\circ = 70^\circ + 2\,(360^\circ)$ (see Figure $1.4.14$ below).



Figure $1.4.14$**:** *Graphical representations of coterminal angles*

Coterminal angles need not be positive. For example, we could have been given $\alpha = 140^\circ$ and chosen to *subtract* 360° to get the coterminal angle -220° as shown in Figure $1.4.15$.

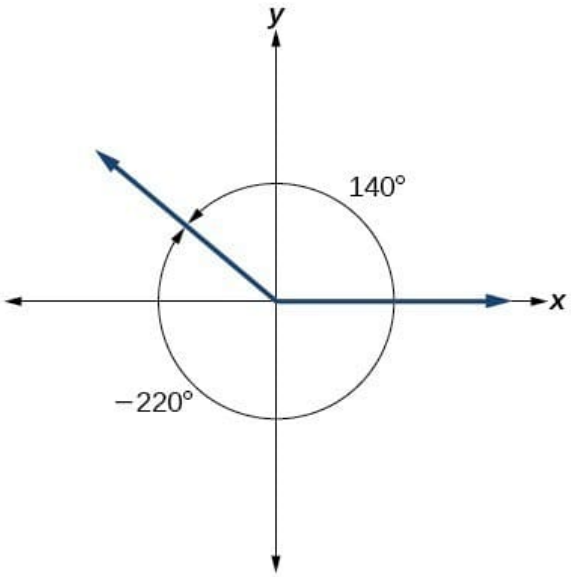


Figure $1.4.15$

In fact, we could denote all angles coterminal to $\alpha = 140^\circ$ using the notation

$$140^\circ + 360^\circ k, \text{ where } k \in \mathbb{Z}.$$

The symbol, $\mathbb{Z}$, denotes the set of all integers (the set of numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$). This notation gets used frequently in Mathematics from this point forward, so it is best to get used to it quickly.

Interactive Element: Standard-Position and Coterminal Angles

The following Interactive Element might help you visualize how a single terminal side can be named by infinitely many angle measures.

Interact: Drag the red point around the circle — spin it past a full turn, then back the other way.

Observation: The measure changes as you rotate, but every time you add or subtract 360° you land on the exact same terminal side — those are coterminal angles. Notice which quadrant the terminal side falls in.

Theorem: Coterminal Angle Formula

All angles coterminal to the angle α (in standard position) have the form

$$\alpha + 360^\circ k, \text{ where } k \in \mathbb{Z}.$$

Previously, we mentioned that the angles illustrated in Figure $1.4.12$ were only a "few quadrantal angles," implying there were more than what we listed. This was because, for example, given the quadrantal angle 90°, we could generate an infinite number of other quadrantal angles with the same terminal side by considering

$$90^\circ + 360^\circ k, \text{ where } k \in \mathbb{Z}.$$

Moreover, since the quadrantal angles only differ from each other by 90°, we could generate *all* quadrantal angles with the single statement

$$0^\circ + 90^\circ k, \text{ where } k \in \mathbb{Z}.$$

There is no need for the 0° angle in that formula, so we could shorten it up to just

$$90^\circ k, \text{ where } k \in \mathbb{Z}.$$

Example $1.4.5$

Graph each of the angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

a. $\alpha = 60^\circ$
b. $\beta = -225^\circ$
c. $\gamma = 540^\circ$
d. $\phi = -750^\circ$

Solutions available at the end of this section.

Checkpoint $1.4.5$

Find two angles coterminal with 102°, one positive and one negative.

Answer available at the end of this section.

MyOpenMath Advice: Check For Feedback

When grading your uploaded (handwritten) work, your professor might have left feedback. Sometimes this feedback can be as

simple as, "Please read the written solution," but it can be as robust as a breakdown of your mistake and how you can avoid such a mistake in the future. To check a past due assignment for any feedback, go to the Grades area in Canvas, click on the assignment in question, and you should be able to see any feedback there. If you cannot see feedback, please let your professor know.

Success in College: Meeting With Your Professor

Meeting with a faculty member might be intimidating. Students are encouraged to use the following guidelines to make the faculty meeting productive. These strategies were developed by the Retention Office at U. of Toledo (OH) and first appeared in National On-Campus Report's sister publication, Recruitment and Retention in Higher Education.

Step 1: Make an appointment

- Review the syllabus for the professor's office hours and make an appointment.
- If you're uncertain of your professor's office hours, ask.
- If your schedule conflicts with the office hours, tell your professor why you can't make the scheduled office hours. Then tell them about your specific concern.
- Set up a specific time to meet with your professor. Be sure to indicate the amount of time you will need.

Step 2: Establish rapport

- Be on time for your appointment.
- When you arrive, be pleasant, smile, introduce yourself again (include your name and class), and shake hands if appropriate.

Step 3: Present your concern

- Focus on the specific questions you've identified as problematic.
- Have your problem(s) written out so that the professor can see where your difficulties are.

Step 4: Provide background information

- Briefly tell your professor about your high school background and preparation for the course – if relevant to solving the problem.
- Explain the study strategies you've used to understand the material.

Step 5: Redirect for clarification

- If the professor's explanation isn't clear, redirect their attention to the specific point where you became confused.
- "Talk through" your problem.

Step 6: Summarize Resolution of the Problem

- "I was missing this step in... I need to apply this formula..."

Step 7: Thank your Professor

- Ask to come back if necessary.

Armed with this information, think about the following questions:

1. How would you make an appointment to talk to your professor if your schedule prevents you from visiting during office hours? If it is not possible to make such an appointment or your professor does not have office hours, think about another way you can get extra help.
2. Some professors prefer you talk to them using their title and last name. For example, you might say Dr. Rodriguez or Ms.

Smith. Others prefer that you use just their first name. When asking your professor a question, what name or title do they prefer that you use?

If you are unsure about the answers to either of these questions, feel free to ask your professor - that's what they are there for!

Answers

Section Preview

Up to this point, all your instruction should have omitted the Cartesian coordinate system and equations depicting solutions on the coordinate plane. We now glue our beloved coordinate system to the Geometry reviewed up to this point. Much of the content here cannot be done without mastering the previous sections.

Section Preview

- **Sets and Numbers**
  - Sets of Numbers: For understanding and properly using the notations for sets of numbers (e.g., $\mathbb{Z}$ for the integers).
- **Graphs**
  - The Cartesian Coordinate System and the Quadrants: Students must arrive already knowing how to work in the coordinate plane to engage with any of this content.
  - Ordered Pairs and Graphing Relations by Point-Plotting: Students must arrive already knowing how to work in the coordinate plane to engage with any of this content.

Section Preview

- **Arithmetic**
  - Mathematical Operations - Multiplication: Computing coterminal angles requires multiplying $360^\circ \times k$.

Section Preview

- Compute the distance between two points on the Cartesian plane.
- Verify a point lies on the unit circle.
- Graph a circle given its equation.
- Determine in which quadrant an angle in standard position terminates.
- Sketch an angle in standard position.
- Find an angle coterminal with a given angle.

Example 1.4.1

We substitute $(2, -1)$ for (x_1, y_1) and $(4, 3)$ for (x_2, y_2) in the Distance Formula to obtain

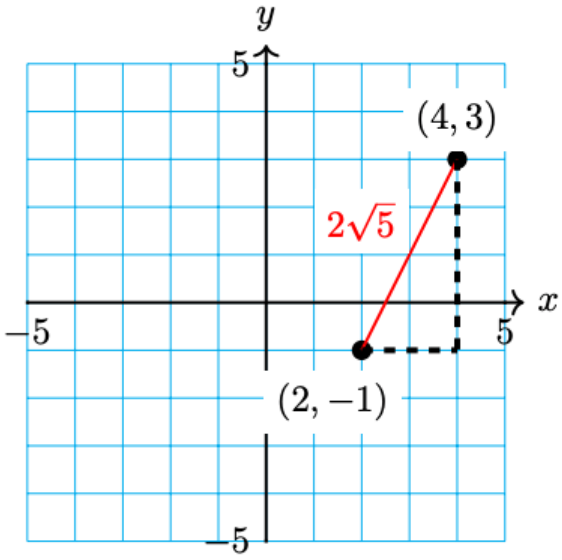


Figure 1.4.3

$$
\begin{aligned}
& & d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{(Distance Formula)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & &= \sqrt{(4-2)^2 + [3-(-1)]^2} && \text{(substitution)} \\
\xRightarrow{\text{Arithmetic}} & & &= \sqrt{2^2 + 4^2} && \text{(addition and subtraction)} \\
\xRightarrow{\text{Arithmetic}} & & &= \sqrt{4 + 16} && \text{(applying powers)} \\
\xRightarrow{\text{Arithmetic}} & & &= \sqrt{20} && \text{(addition)} \\
\xRightarrow{\text{Arithmetic}} & & &= 2\sqrt{5} && \text{(simplify the radical)}
\end{aligned}
$$

The exact value of the distance is $2\sqrt{5}$ units. We obtain the same answer if we use $(4, 3)$ for P_1 and $(2, -1)$ for P_2:

$$\begin{aligned} d &= \sqrt{(2-4)^2+[(-1)-3]^2} \\ &= \sqrt{4+16} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

We can use a calculator to obtain an approximation for this value and find $2\sqrt{5} \approx 2(2.236) = 4.472$.

Checkpoint 1.4.1

a. $4\sqrt{13}$

b. The points $(-5, 3)$ and $(3, -9)$ are two vertices of a right triangle with one horizontal leg of length 8 and one vertical leg of length 12. The distance between $(-5, 3)$ and $(3, -9)$ is the hypotenuse c.

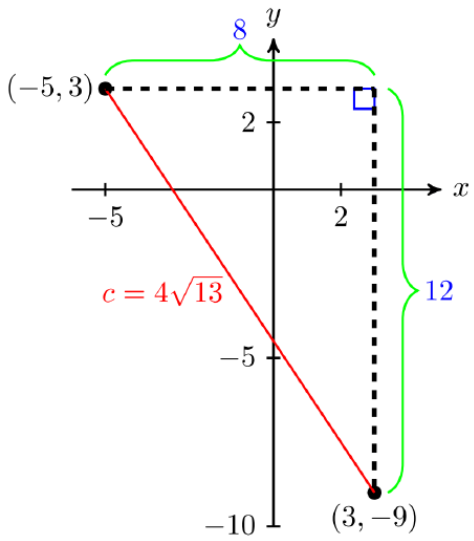
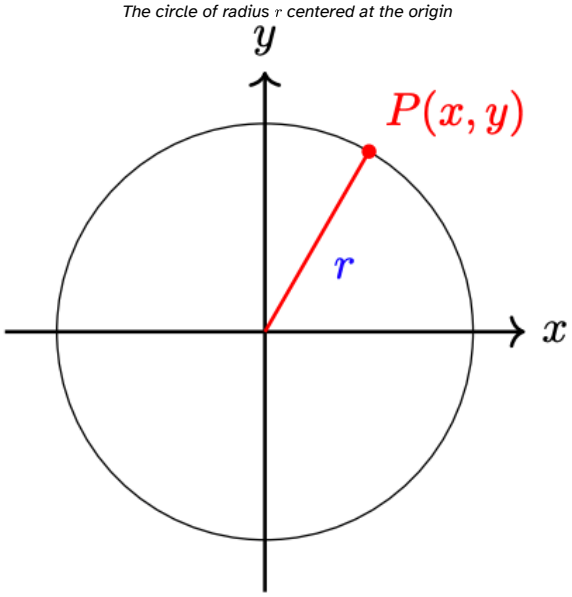


Figure 1.4.4

Theorem: Equation of a Circle Centered at the Origin

Consider a circle of radius r centered at the origin, as shown in the following figure.

The circle of radius r centered at the origin



The distance from the origin to any point $P(x, y)$ on the circle is r. Therefore, by the Distance Formula,

$$\sqrt{(x-0)^2+(y-0)^2} = r$$

or, squaring both sides,

$$(x-0)^2+(y-0)^2 = r^2.$$

Because every point on the circle must satisfy this equation, we have found an equation for the circle.

Example 1.4.2

We substitute $x = -1$ into the equation for the circle and solve for y.

$$\begin{aligned} \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad (-1)^2 + y^2 &= 4 && \text{(substitution)} \\ \xRightarrow{\text{Arithmetic}} \quad 1 + y^2 &= 4 && \text{(applying the power)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad y^2 &= 3 && \text{(subtract 1 from both sides)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad y &= \pm\sqrt{3} && \text{(Extraction of Roots)} \end{aligned}$$

The points are $(-1, \sqrt{3})$ and $(-1, -\sqrt{3})$. Note that $\sqrt{3} \approx 1.732$.

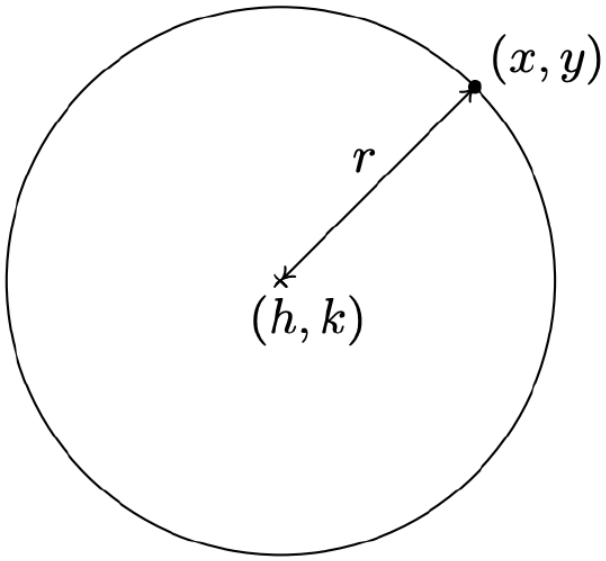
Checkpoint 1.4.2

$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Theorem: Equation of a Circle Centered Off-Origin

Consider a circle of radius r centered at the point (h, k), as shown in the following figure.

A circle of radius r centered at the point (h, k)



Since this is a circle of radius r, the point (x, y) is on the circle if and only if its distance to (h, k) is r. By the Distance Formula, this means

$$r = \sqrt{(x-h)^2 + (y-k)^2}.$$

By squaring both sides of this equation, we get

$$r^2 = (x-h)^2 + (y-k)^2.$$

Example 1.4.3

Here, $(h, k) = (-2, 3)$ and $r = 5$, so we get

$$\begin{aligned} (x-(-2))^2 + (y-3)^2 &= (5)^2 \\ \Longrightarrow \quad (x+2)^2 + (y-3)^2 &= 25 \end{aligned}$$

Checkpoint 1.4.3

From the standard form of a circle we have that $x + 2$ is $x - h$, so $h = -2$ and $y - 1$ is $y - k$ so $k = 1$. This tells us that our center is $(-2, 1)$. Furthermore, $r^2 = 4$, so $r = 2$. Thus we have a circle centered at $(-2, 1)$ with a radius of 2. Graphing gives us

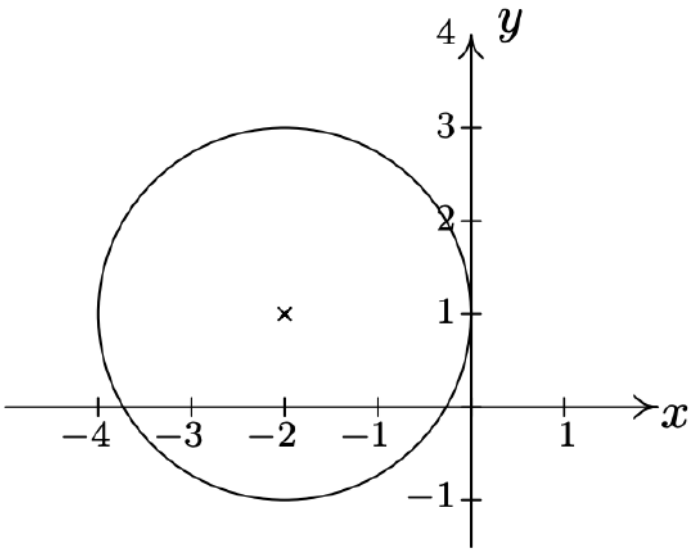


Figure 1.4.5

Example 1.4.4

a. The angle α is one-fifth of a complete revolution, or

$$\frac{1}{5}(360^\circ) = 72^\circ.$$

In standard position, $\alpha = 72^\circ \in$ QI, as shown in Figure $1.4.8(a)$ below.

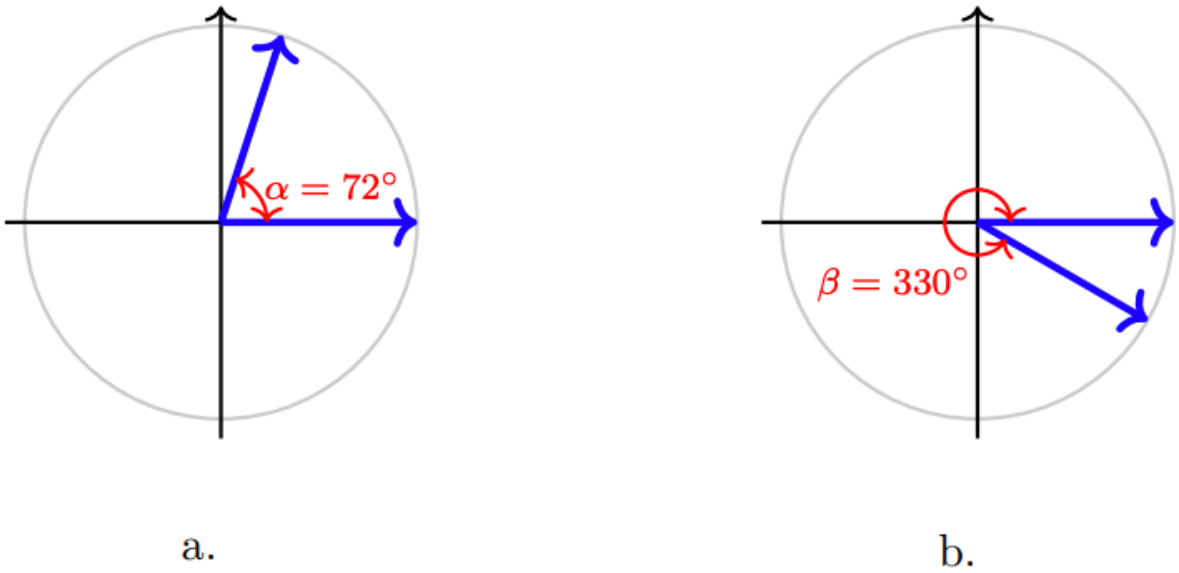


Figure 1.4.8

b. The angle β is $\frac{11}{12}$ of a complete revolution, or

$$\frac{11}{12}(360^\circ) = 330^\circ.$$

In standard position, $\beta = 330^\circ \in$ QIV. (See Figure $1.4.8(b)$.)

Checkpoint 1.4.4

a. $\theta = 120^\circ \in$ QII

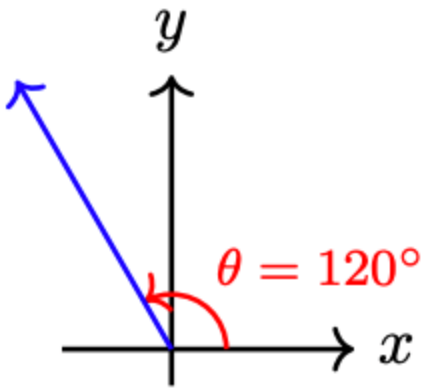


Figure 1.4.10

b. $\alpha = 250^\circ - 180^\circ = 70^\circ \in \text{QI}$

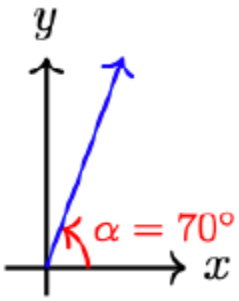
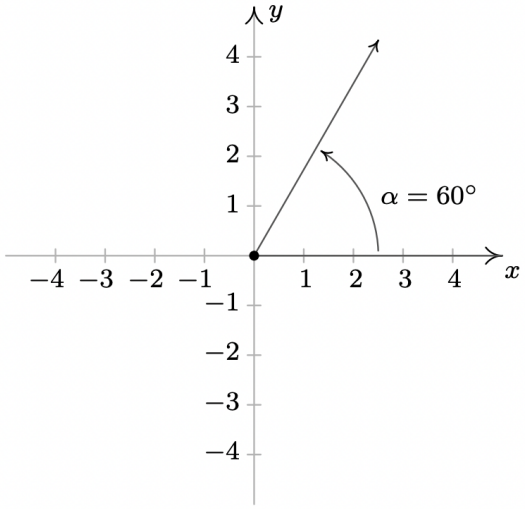


Figure 1.4.11

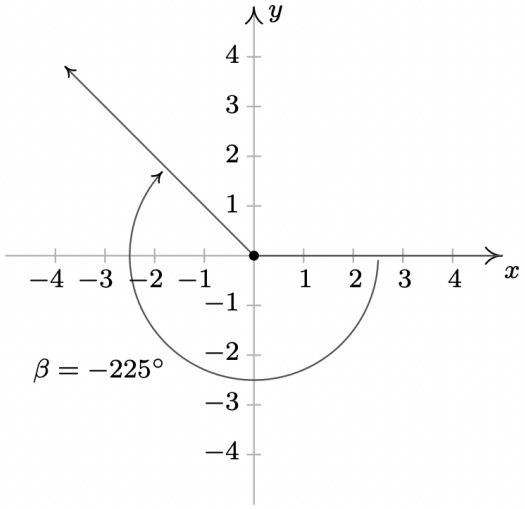
Example 1.4.5

a. To graph $\alpha = 60^\circ$, we draw an angle with its initial side on the positive x-axis and rotate counterclockwise $\frac{60^\circ}{360^\circ} = \frac{1}{6}$ of a revolution. We see that α is a Quadrant I angle. To find coterminal angles, we look for angles θ of the form $\theta = \alpha + 360^\circ \cdot k$, for some integer k. When $k = 1$, we get $\theta = 60^\circ + 360^\circ = 420^\circ$. Substituting $k = -1$ gives $\theta = 60^\circ - 360^\circ = -300^\circ$. Finally, if we let $k = 2$, we get $\theta = 60^\circ + 720^\circ = 780^\circ$.

b. Since $\beta = -225^\circ$ is negative, we start at the positive x-axis and rotate *clockwise* $\frac{225^\circ}{360^\circ} = \frac{5}{8}$ of a revolution. We see that β is a Quadrant II angle. To find coterminal angles, we proceed as before and compute $\theta = -225^\circ + 360^\circ \cdot k$ for integer values of k. We find 135°, -585° and 495° are all coterminal with -225°.



$\alpha = 60^\circ$ in standard position.

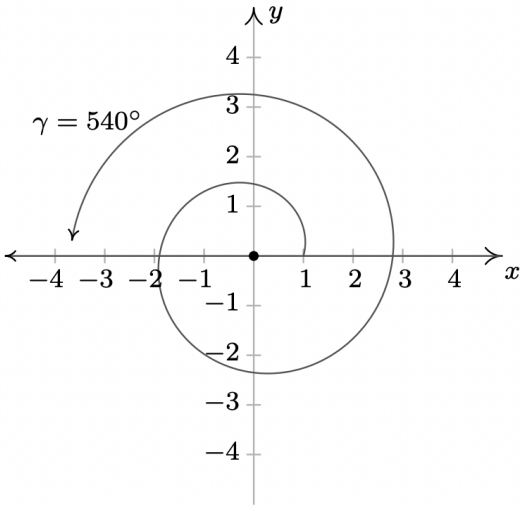


$\beta = -225^\circ$ in standard position.

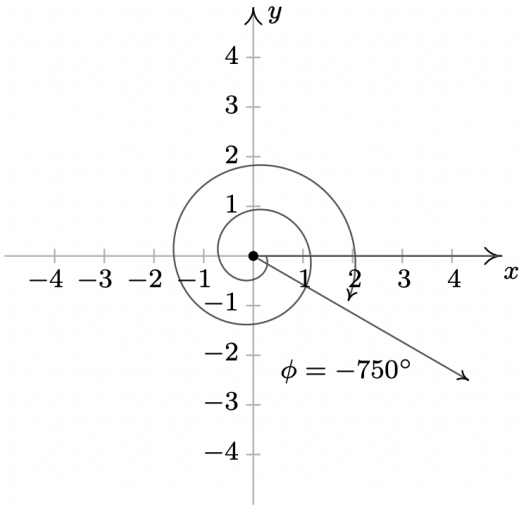
Figure 1.4.16

c. Since $\gamma = 540^\circ$ is positive, we rotate counterclockwise from the positive x-axis. One full revolution accounts for 360°, with 180°, or $\frac{1}{2}$ of a revolution remaining. Since the terminal side of γ lies on the negative x-axis, γ is a quadrantal angle. All angles coterminal with γ are of the form $\theta = 540^\circ + 360^\circ \cdot k$, where k is an integer. Working through the arithmetic, we find three such angles: 180°, -180° and 900°.

d. The Greek letter ϕ is pronounced "fee" or "fie," and since ϕ is negative, we begin our rotation clockwise from the positive x-axis. Two full revolutions account for 720°, with just 30° or $\frac{1}{12}$ of a revolution to go. We find that ϕ is a Quadrant IV angle. To find coterminal angles, we compute $\theta = -750^\circ + 360^\circ \cdot k$ for a few integers k and obtain -390°, -30° and 330°.



$\gamma = 540°$ in standard position.



$\phi = -750°$ in standard position.

Figure 1.4.17

Checkpoint 1.4.5

$462°, -258°$

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1.4.1: Resources and Key Concepts

Resources

Videos

- Distance Formula
  - Intermediate Algebra - Distance Formula: The Concept
  - Intermediate Algebra - Distance Formula: General and Advanced Uses
- Equation of a Circle
  - Intermediate Algebra - Circles: The Concept
  - Intermediate Algebra - Circles: Equations and Graphs

Key Concepts

Definitions

- **Unit Circle:** The circle with a radius of 1 that is centered at the origin of the Cartesian plane. Its equation is $x^2 + y^2 = 1$.
- **Quadrants:** The four regions into which the Cartesian coordinate system is divided by the x- and y-axes. They are numbered I, II, III, and IV, starting from the top right and moving counterclockwise.
- **Standard Position of an Angle:** An angle is in standard position if its vertex is located at the origin and its initial side lies along the positive x-axis.
- **Quadrantal Angle:** An angle in standard position whose terminal side lies on either the x-axis or the y-axis.
- **Coterminal Angles:** Angles that are in standard position and share the same terminal side.

Theorems

- **Distance Formula:** The distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- **Equation of a Circle Centered at the Origin:** The equation for a circle with radius r centered at the origin is $x^2 + y^2 = r^2$.
- **Equation of a Circle Centered Off-Origin:** The standard form for the equation of a circle with radius r centered at the point (h, k) is $(x - h)^2 + (y - k)^2 = r^2$.
- **Coterminal Angle Formula:** All angles that are coterminal with an angle α can be expressed in the form $\alpha + 360°k$, where k is any integer.

Common Mistakes

- **Distributing a Radical over a Sum:** A square root cannot be distributed across terms that are added or subtracted. For example, $\sqrt{a^2 + b^2}$ is not equal to $a + b$, and $\sqrt{4 + 16}$ is not equal to $\sqrt{4} + \sqrt{16}$. This is a critical mistake when using the Distance Formula.
- **Direction of Rotation for Angles:** Positive angles are generated by a counterclockwise rotation from the initial side. Negative angles are generated by a clockwise rotation.

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1.4.2: Homework

Reading Questions

1. What fundamental theorem is used as the basis for deriving the Distance Formula?
2. Write the equation for a circle that has its center at the origin and a radius of r.
3. What is a "unit circle"?
4. What is the "standard form" of the equation for a circle with center (h, k) and radius r?
5. In which quadrant are the x-coordinates negative and the y-coordinates positive?
6. What two conditions define an angle in "standard position"?
7. Where is the terminal side of a 90° angle in standard position?
8. What is a "quadrantal angle"?
9. What does it mean for two angles to be "coterminal"?
10. Describe the direction of rotation (clockwise or counterclockwise) for a negative angle.
11. If you are given an angle of 50°, how would you find a negative angle that is coterminal with it?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, find the slope and y-intercept of the graph for the given equation (do not graph).

1. $y = 5.2x - 1.9$
2. $y = x$
3. $y = -8$
4. $x = 7$
5. $12y - 10x = 5$

For the following exercises, write the equation of the line that satisfies the given conditions.

6. The line goes through the points $(-2, 6)$ and $(3, -4)$.
7. The line goes through the points $(5, -7)$ and $(5, -5)$.

For the following exercises, determine if the statement is true or false.

8. $\sqrt{a^2 + b^2} = a + b$
9. $\sqrt{36 + 64} = 6 + 8$
10. $\sqrt{16x^4} = 4x^2$
11. $\sqrt{2x}\sqrt{3y} = \sqrt{6xy}$
12. $\sqrt{5x} + \sqrt{3x} = \sqrt{8x}$
13. $\sqrt{4 + N} = 2 + \sqrt{N}$

14. $\sqrt{\frac{x}{4}} = \frac{\sqrt{x}}{2}$

15. $\sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$

For the following exercises, solve the equation. If no solution exists, state why.

16. $x^2 - 81 = 0$

17. $2y^2 - 98 = 0$

18. $x^2 + 12 = 8$

19. $-3x^2 + 19 = 10$

Answers available at the end of this section.

Homework

Concept Check

1. Explain why the Distance Formula,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

cannot be simplified to

$$(x_2 - x_1) + (y_2 - y_1).$$

2. What is a unit circle, and what is its equation?

3. In which quadrant are the x-values for points negative but the y-values are positive?

4. Explain what it means for angles to be coterminal. Demonstrate by creating three coterminal angles, one of which is a negative angle.

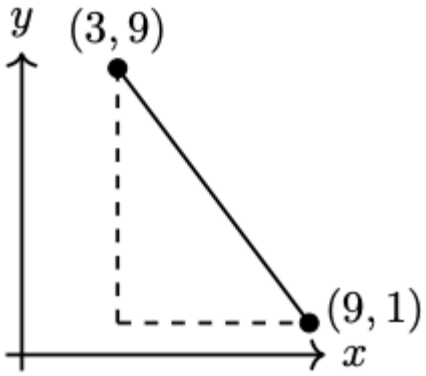
True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

5. Positive angles open clockwise.

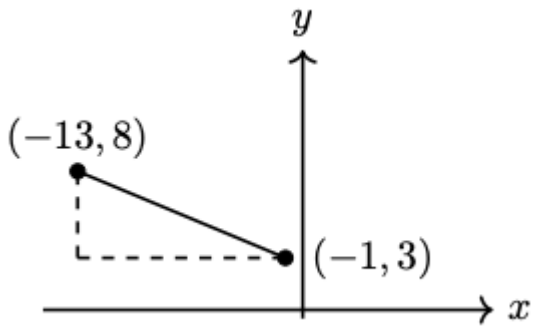
Basic Skills

For the following exercises, find the distance between the points. Give your answer as an exact value, then as a decimal rounded to hundredths.

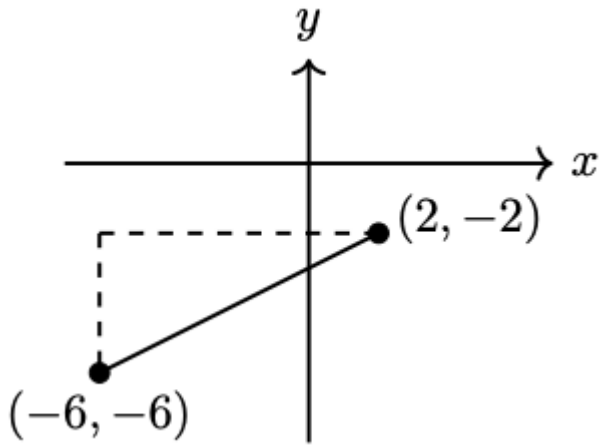
6.



7.



8. (MyOpenMath)



For the following exercises, find the distance between the points.

9. (1, 1), (4, 5)
10. (−1, 1), (5, 9)
11. (MyOpenMath) (2, −3), (−2, −1)
12. (5, −4), (−1, 1)
13. (3, 5), (−2, 5)
14. (−2, −5), (−2, 3)

15. a. Write an expression for the distance between the points (x, y) and $(-3, 4)$.

    b. Write an equation that says "The distance between the points (x, y) and $(-3, 4)$ is 5 units."

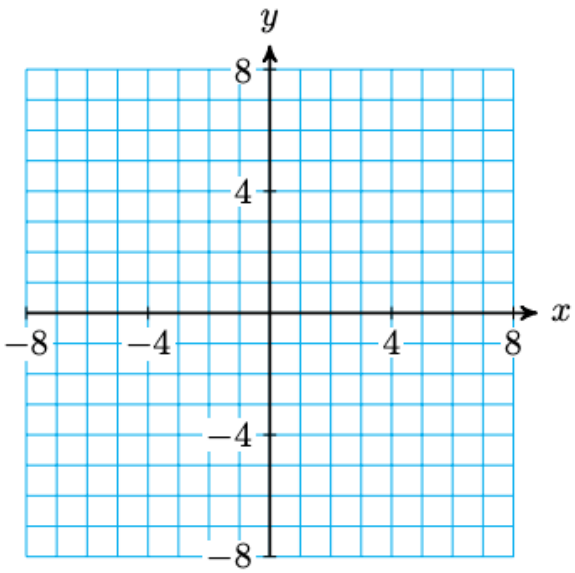
16. a. Write an expression for the distance between the points $(-6, -1)$ and (h, k).

    b. Write an equation that says, "The point $(-6, -1)$ is 3 units from the point (h, k)."

17. a. Complete the table of values for the equation $x^2 + y^2 = 25$.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y											

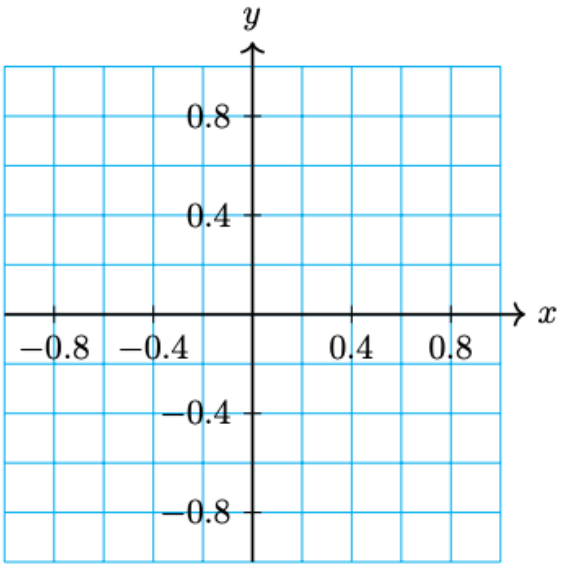
    b. Plot the points and graph the equation.



18. a. Complete the table of values for the equation $x^2 + y^2 = 1$.

x	-1	-0.8	-0.6	0	0.6	0.8	2
y							

    b. Plot the points and graph the equation.



19. a. Sketch a graph of all points in the plane that lie 6 units from the origin.

    b. Write an equation for your graph.

20. a. Sketch a graph of all points in the plane that lie $\sqrt{12}$ units from the origin.

    b. Write an equation for your graph.

21. a. Sketch a graph of all points in the plane that lie less than 3 units from the origin.

    b. Write an inequality for your graph.

22. a. Sketch a graph of all points in the plane that lie more than 5 units from the origin.

    b. Write an inequality for your graph.

23. a. Explain why the solutions of the equation $x^2 + y^2 = 16$ must have $-4 \leq x \leq 4$.

    b. What does part (a) tell you about the graph of the equation?

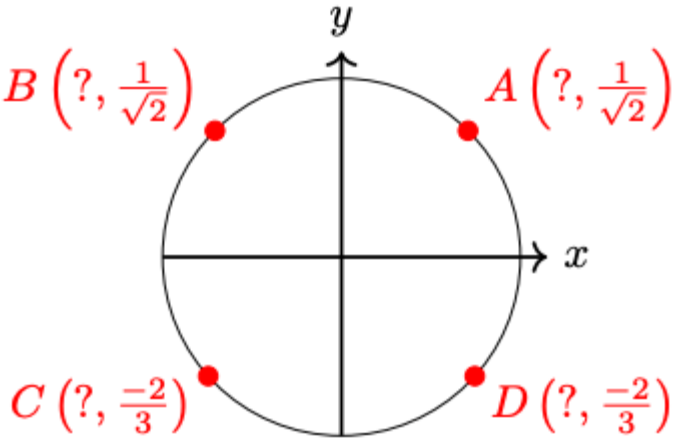
24. a. Explain why the solutions of the equation $x^2 + y^2 = 100$ must have $-10 \leq x \leq 10$

    b. What does part (a) tell you about the graph of the equation?

25. The point $(1, -3)$ lies on a circle centered at the origin. What is the radius of the circle?

26. The point $(-2, \sqrt{6})$ lies on a circle centered at the origin. What is the radius of the circle?

27. Find the coordinates of the points on the unit circle.

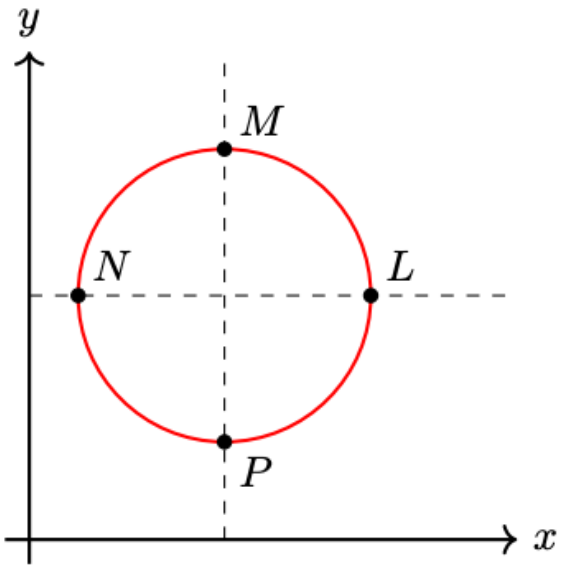


28. a. Use the Distance Formula to write an equation for the circle of radius 6 centered at the point $(3, -2)$.

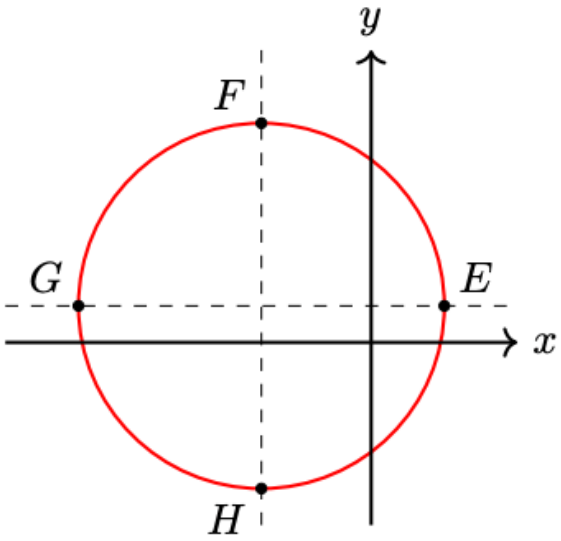
b. Use the Distance Formula to derive an equation for the circle of radius r centered at the point (h, k).

For the following exercises, find the coordinates of the indicated points on each circle.

29. $(x-4)^2 + (y-5)^2 = 9$



30. $(x+3)^2 + (y-1)^2 = 25$



31. How many degrees are in each angle?

    a. $\frac{3}{5}$ of one rotation

    b. $\frac{3}{10}$ of one rotation

    c. $\frac{4}{3}$ of one rotation

    d. $\frac{8}{3}$ of one rotation

32. How many degrees are in each angle?

    a. $\frac{5}{6}$ of one rotation

    b. $\frac{3}{8}$ of one rotation

    c. $\frac{7}{4}$ of one rotation

    d. $\frac{7}{12}$ of one rotation

33. What fraction of a complete rotation is represented by each angle?

    a. 45°

    b. 300°

    c. 540°

    d. 420°

34. What fraction of a complete rotation is represented by each angle?

    a. 60°

    b. 240°

    c. 450°

    d. 150°

For the following exercises, find two angles, one positive and one negative, that are coterminal with the given angle.

35. 40°

36. 160°

37. 215°

38. 250°

39. 305°

40. 340°

For the following exercises, find a positive angle between 0° and 360° that is coterminal with the given angle.

41. -65°

42. -140°

43. -290°

44. -325°

45. (MyOpenMath) 655°

46. (MyOpenMath) -750°

47. (MyOpenMath) Write an expression describing all the angles that are coterminal with 252°.

Synthesis Questions

47. Show that the triangle with vertices $(0,0)$, $(6,0)$, and $(3,3)$ is an isosceles right triangle, that is, a right triangle with two sides of the same length.

48. Two opposite vertices of a square are $A(-9,-5)$ and $C(3,3)$.

    a. Find the length of a diagonal of the square.

    b. Find the length of the side of the square.

49. Find all of the points on the line $y = 2x + 1$ which are 4 units from the point $(-1,3)$.

50. Sketch a triangle with vertices $(10,1)$, $(3,1)$, and $(5,9)$, and find its perimeter. Round your answer to tenths.

51. Sketch a triangle with vertices $(-1,5)$, $(8,-7)$, and $(4,1)$, and find its perimeter. Round your answer to tenths.

For the following exercises, interpret the equations as statements about distance.

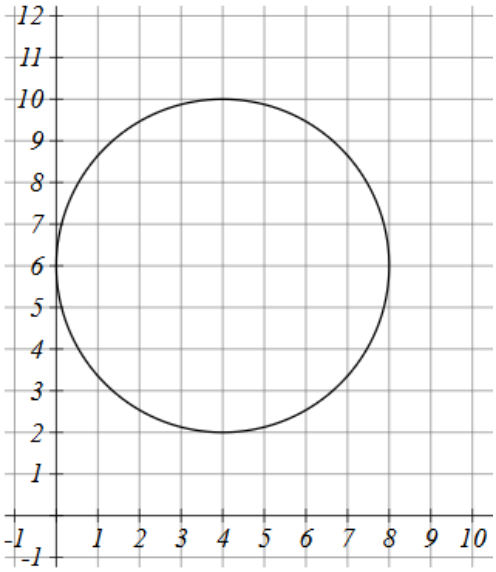
52. $\sqrt{(x-4)^2+(y+1)^2} = 3$

53. $\sqrt{(-2-h)^2+(5-k)^2} = l$

For the following exercises, use the graph to answer the question.

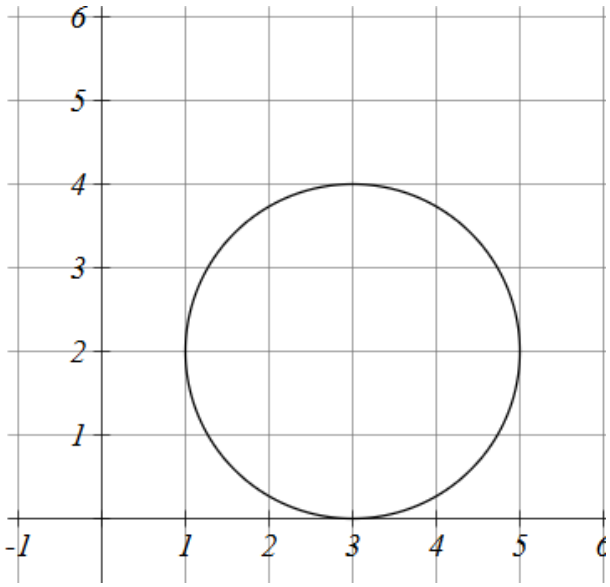
54. Find the circumference of the circle.

MyOpenMath



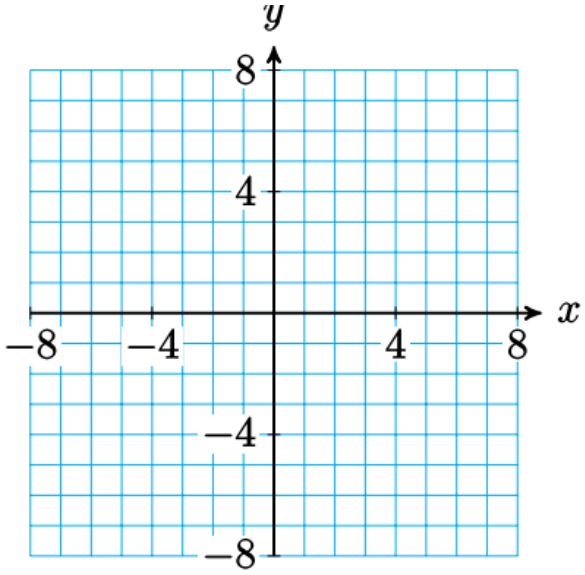
55. Find the area of the circle.

MyOpenMath

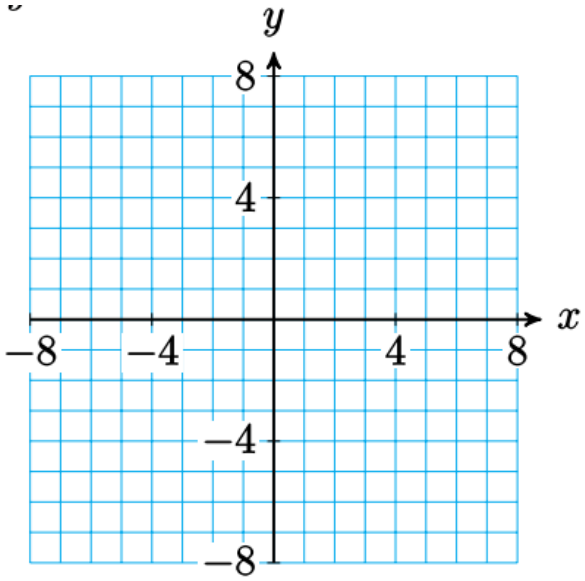


For the following exercises, graph the equation and find the circumference of the circle.

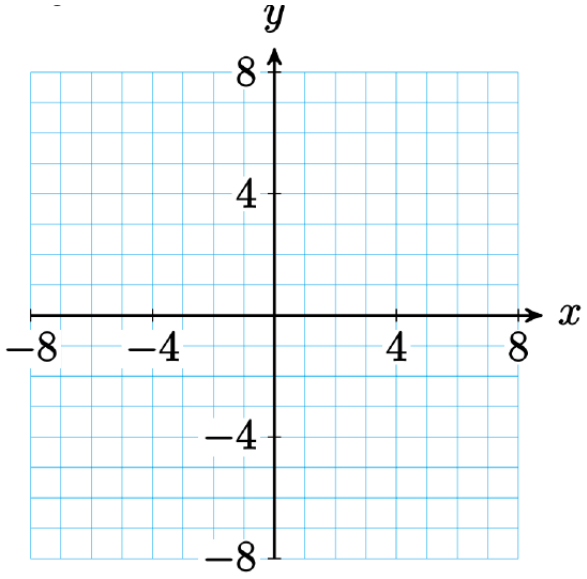
56. $x^2 + y^2 = 36$



57. $x^2 + y^2 = 16$

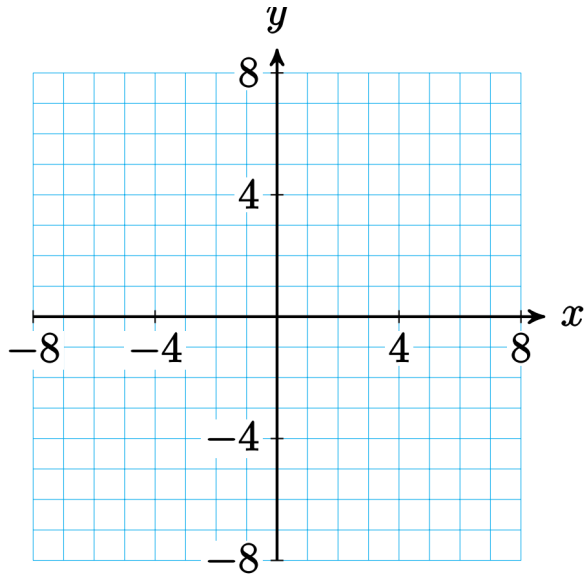


58. $4x^2 + 4y^2 = 16$



59. Give the coordinates of two points on the circle in the previous exercise that have $y = -4$. Plot those points on your graph.

60. $2x^2 + 2y^2 = 18$



61. Give the coordinates of two points on the circle in the previous exercise that have $x = -2$. Plot those points on your graph.

Applications

62. **Distance.** Paige is sailing and is currently 3 miles west and 5 miles south of the harbor. She heads directly towards an island 8 miles west and 7 miles north of the harbor. How far is Paige from the island?

63. **Distance.** Nam is 100 meters east and 250 meters north of Kristen. He walks directly towards a tree 220 meters east and 90 meters north of Kristen. How far is Nam from the tree?

64. **Computing Angles (MyOpenMath).**

    a. Through what angle does the hour hand of a clock rotate between 2 pm and 10 pm?

    b. Through what angle does the hour hand of a clock rotate between 2 am and 10 pm?

35. **Computing Angles.**

    a. Through what angle does the hour hand of a clock rotate between 3:25 am and 3:50 am?

    b. Through what angle does the hour hand of a clock rotate between 4:10 pm and 6:25 pm?

36. **Astronomy.** The Distance Formula can be modified to compute distances between objects in three-dimensional space. To do so, we must expand the Cartesian coordinate system to include one more spatial dimension. In three-space, coordinates are listed in **ordered triples**, such as (x, y, z). If we have two points, $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, in three-dimensional space, the distance between these points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

    Suppose our Sun is given the very special coordinate of $(0, 0, 0)$ (thereby affirming that we are the center of the universe). Compute the distance between each star (these computed distances are in light-years). Round answers to the nearest hundredth of a light-year.

    a. Our Sun, $(0, 0, 0)$, and Alpha Centauri, $(-1.8, 0, 3.9)$.

    b. Our Sun, $(0, 0, 0)$, and Sirius, $(-3.4, -3.1, 7.3)$.

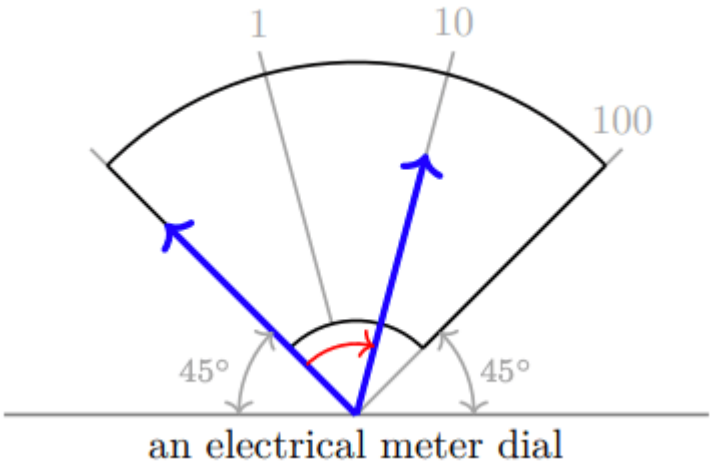
    c. Our Sun, $(0, 0, 0)$, and Polaris, $(99.6, 28.2, 376.0)$.

    d. Alpha Centauri, $(-1.8, 0, 3.9)$, and Sirius, $(-3.4, -3.1, 7.3)$.

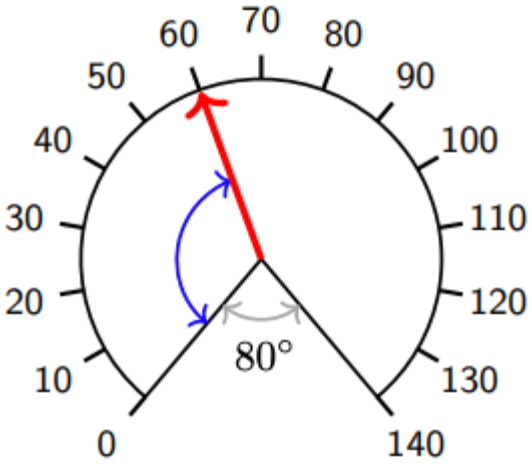
    e. Zubenelgenubi, $(64.6, -22.0, 23.0)$, and Polaris, $(99.6, 28.2, 376.0)$.

Computing Angles. For the following exercises, calculate the degree measure of the unknown angle and sketch the angle in standard position.

37.

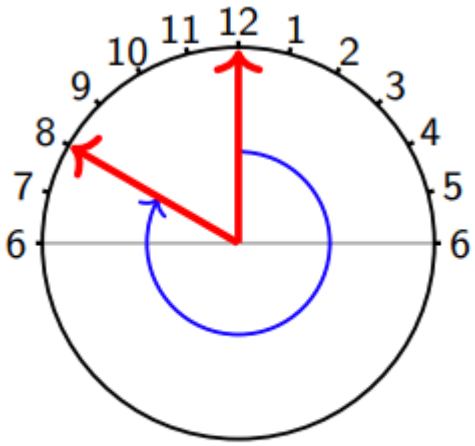


38.



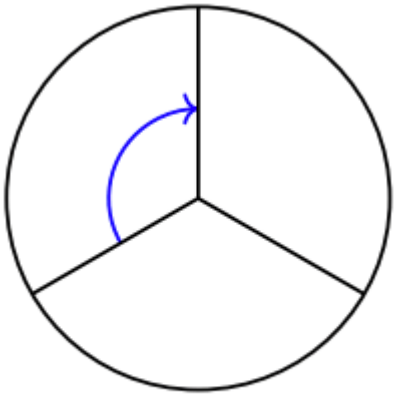
a speedometer

69.



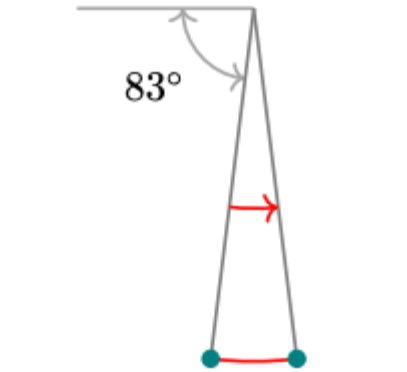
a sundial

70.



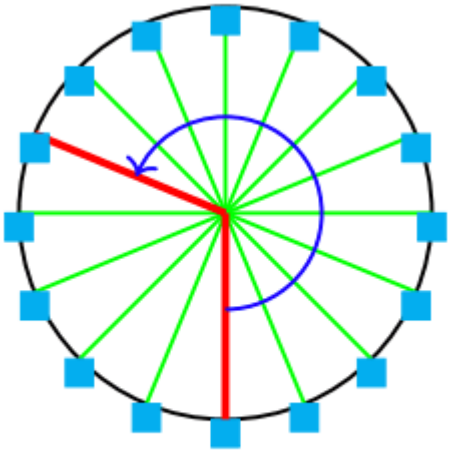
a revolving door

71.



a swinging pendulum

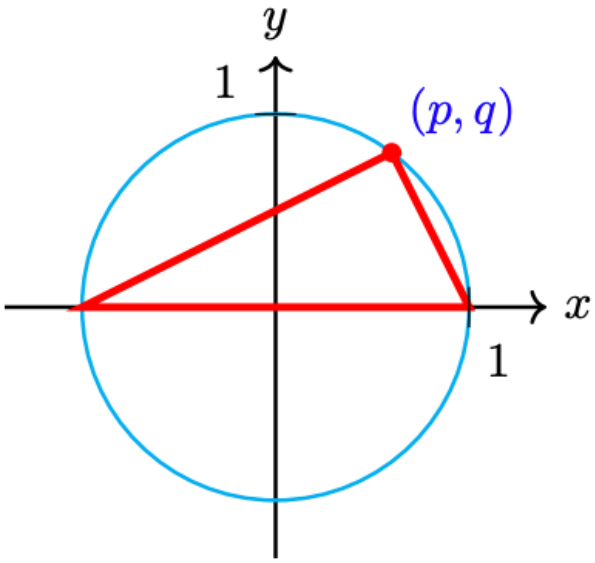
72.



a Ferris wheel

Challenge Problems

73. In this problem, we'll show that any angle inscribed in a semi-circle must be a right angle. The figure shows a triangle inscribed in a unit circle, one side lying on the diameter of the circle and the opposite vertex at point (p, q) on the circle.

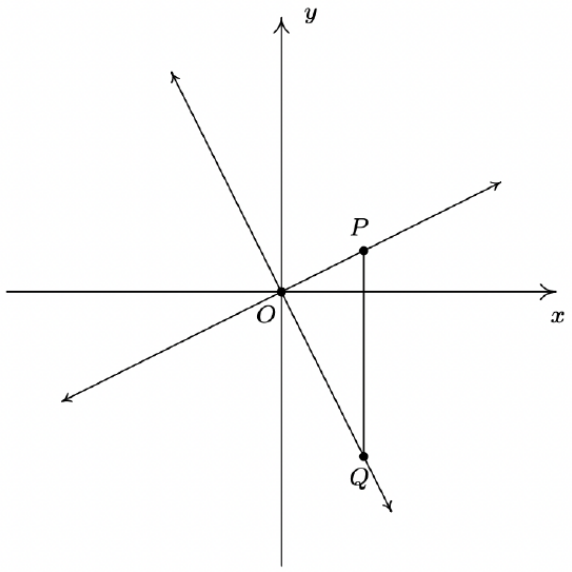


a. What are the coordinates of the other two vertices of the triangle? What is the length of the side joining those vertices?

b. Use the Distance Formula to compute the lengths of the other two sides of the triangle.

c. Show that the sides of the triangle satisfy the Pythagorean Theorem, $a^2 + b^2 = c^2$.

74. We shall now prove that $y = m_1x + b_1$ is perpendicular to $y = m_2x + b_2$ if and only if $m_1 \cdot m_2 = -1$. To make our lives easier we shall assume that $m_1 > 0$ and $m_2 < 0$. We can also "move" the lines so that their point of intersection is the origin without messing things up, so we'll assume $b_1 = b_2 = 0$. (Take a moment with your classmates to discuss why this is okay.) Graphing the lines and plotting the points $O(0, 0)$, $P(1, m_1)$ and $Q(1, m_2)$ gives us the following set up.



The line $y = m_1x$ will be perpendicular to the line $y = m_2x$ if and only if $\triangle OPQ$ is a right triangle. Let d_1 be the distance from O to P, let d_2 be the distance from O to Q and let d_3 be the distance from P to Q. Use the Pythagorean Theorem to show that $\triangle OPQ$ is a right triangle if and only if $m_1 \cdot m_2 = -1$ by showing $d_1^2 + d_2^2 = d_3^2$ if and only if $m_1 \cdot m_2 = -1$.

Answers

Skills Refresher

1. $m = 5.2 \quad (0, -1.9)$
2. $m = 1 \quad (0, 0)$

3. $m = 0 \quad (0, -8)$
4. The slope is undefined and there is not a y-intercept.
5. $m = \frac{5}{6} \quad \left(0, \frac{5}{12}\right)$
6. $y = -2x + 2$
7. $x = 5$
8. False
9. False
10. True
11. True
12. False
13. False
14. True
15. True
16. $x = \pm 9$
17. $y = \pm 7$
18. No solution because x^2 should never be negative, but when solving, we get $x^2 = -4$.
19. $x = \pm\sqrt{3}$

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2: COORDINATE TRIGONOMETRY

2.1: The Trigonometric Functions - Coordinate Definition

2.1.1: Resources and Key Concepts

2.1.2: Homework

2.2: Algebra with Trigonometric Functions

2.2.1: Resources and Key Concepts

2.2.2: Homework

2.3: Introduction to Trigonometric Identities

2.3.1: Resources and Key Concepts

2.3.2: Homework

2.4: Introduction to Proofs in Trigonometry

2.4.1: Resources and Key Concepts

2.4.2: Homework

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2.1: The Trigonometric Functions - Coordinate Definition

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Defining Trigonometric Functions in Terms of Coordinates

Armed with our knowledge from the previous chapter, we now define the **trigonometric functions**. We will be viewing the trigonometric functions from three different perspectives in this course—**coordinate trigonometry**, right triangle trigonometry, and unit circle trigonometry (since we are not covering right triangle trigonometry and unit circle trigonometry in this chapter, I chose to only bold coordinate trigonometry.). For the most part, each perspective is mathematically equivalent to the others; however, each has an inherent weakness as a standalone perspective of the trigonometric functions. While you can find a favorite perspective and stick with that one for most of the course, understanding all three is the only way to succeed in Trigonometry.

Definition: Trigonometric Functions (Coordinate Definition)

Let θ be an angle in standard position with the point $P(x, y)$ on the terminal side of θ, where P is not the origin. Define $r = \sqrt{x^2 + y^2}$. The trigonometric functions of the angle θ are defined as follows:



A few notes are in order. First, the trigonometric functions are simply the ratios of x, y, and r, where $P(x, y)$ is a point on the terminal side of θ and r is the distance from the origin to the point P (see Figure $2.1.1$ below).

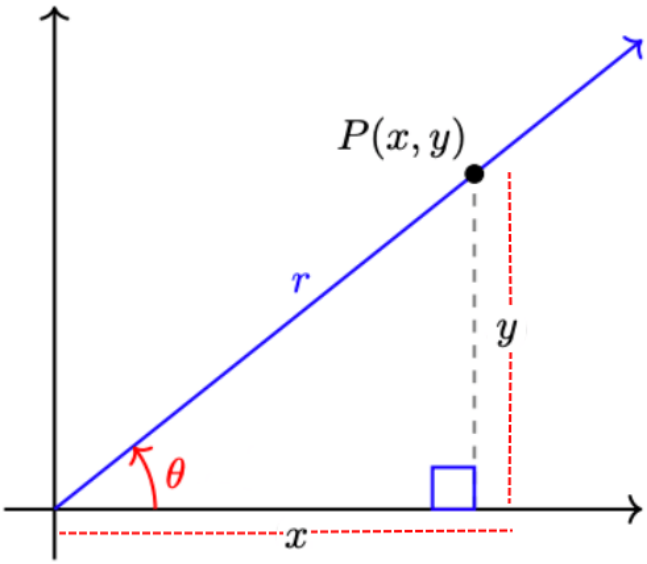


Figure 2.1.1

Second, because it's the distance from the origin to P, r is always positive. However, x and y can be positive, negative, or zero (but not both zero), depending on the angle θ. For example, in the second quadrant, x is negative, but y is positive, so the cosine and the tangent of angles between 90° and 180° are negative, but their sines are positive.

Third, referencing Figure 2.1.1, since the tangent function is defined to be $\frac{y}{x}$ and the terminal side of θ is always a ray emanating from the origin, we get that

$$\frac{y}{x} = \frac{\text{rise}}{\text{run}} = \text{slope of the terminal side of } \theta.$$

The interpretation of the value of the tangent function at θ being the slope of the ray that is the terminal side of θ grants us a potent conceptual understanding of the tangent function. Specifically, since a vertical line has an undefined slope, it makes sense that the tangent function is undefined when $x = 0$. Any ray emanating from the origin and going through a point $P\,(0, y)$, where $y \neq 0$, will be vertical. While the secant function is also undefined when $x = 0$ (and the cosecant and cotangent functions are undefined when $y = 0$), we will hold off on gaining a conceptual understanding of why until we familiarize ourselves with more of the basics of the trigonometric functions.

Interactive Element: The Tangent Is the Slope of the Terminal Side

The following Interactive Element might help you visualize how the tangent of an angle is just the slope of its terminal side—and why it is undefined precisely when that side is vertical.

Interact: Drag the red point P; the blue ray is the terminal side of the angle, and the green and orange legs show the run x and the rise y.

Observation: The slope of the terminal side is $\text{rise}/\text{run} = y/x = \tan\theta$. As you drag P toward the y-axis, the run shrinks to 0, the ray stands vertical, and $\tan\theta$ grows without bound and is undefined; along the x-axis the slope is 0.

Note: Angles are in Standard Position

Unless otherwise stated, all angles from this point forward will be considered angles in standard position.

Example 2.1.1

Find the values of the six trigonometric functions if $P(-3,-8)$ is on the terminal side of θ.

Solution available at the end of this section.

Note: Rationalizing Denominators

In Algebra, we learned to rationalize denominators so we could "simplify" expressions like $-\frac{8}{\sqrt{73}}$; however, in Trigonometry, such manipulations often are pointless. The rule of thumb I will adhere to in this textbook is to rationalize denominators *if* the denominator involves a variable (e.g., $\frac{3}{\sqrt{x}-2}$). In all other cases (specifically, when the numerator and denominator do *not* contain variables), I will only rationalize denominators if there is an "obvious" benefit.

In the case of $-\frac{8}{\sqrt{73}}$, if we tried to rationalize the denominator, we would get

$$-\frac{8}{\sqrt{73}}\cdot\frac{\sqrt{73}}{\sqrt{73}}=-\frac{8\sqrt{73}}{73},$$

which is no better than the original fraction.

Checkpoint 2.1.1

a. Find the equation of the terminal side of the angle in the previous example. (Hint: The terminal side lies on a line that goes through the origin and the point $(-3,-8)$).
b. Show that the point $P'(-6,-16)$ also lies on the terminal side of the angle.
c. Compute the trigonometric ratios for θ using the point P' instead of P.

Answers available at the end of this section.

Example 2.1.2

Find the values of $\cos(\theta)$ and $\tan(\theta)$ if θ is an obtuse angle with $\sin(\theta)=\frac{1}{3}$.

Solution available at the end of this section.

Example 2.1.2 illustrates a common theme in Trigonometry—we will often use the quadrant of the terminal side of an angle to inform our decision on the sign of x- and y-values.

If given a negative trigonometric ratio, as in Checkpoint 2.1.2, it is important to remember that the value of r is always non-negative (at least, prior to introducing something called the polar coordinate system).

Checkpoint 2.1.2

a. Sketch an obtuse angle θ whose cosine is $\frac{-8}{17}$.
b. Find the cosecant and the tangent of θ.

Answers available at the end of this section.

Our use of function notation (e.g., the parentheses in $\sin(\theta)=\frac{y}{r}$) is purposeful. This is because, a given angle θ (which is considered the independent variable) returns a single value for the associated trigonometric ratio (which we consider to be the dependent variable). Thus, by definition, the trigonometric functions are... functions (of angles).

Caution: Notational Abuse

We will sometimes write $\cos\theta$ when we really mean $\cos(\theta)$. This is a common, but accepted, abuse of notation reserved specifically for the trigonometric functions (and, at times, the logarithmic functions).

A common mistake is to think of $\cos\theta$ or $\cos(\theta)$ as a product, cos *times* θ, but this makes no sense, because "cos" by itself has no meaning. Remember that $\cos\theta$ represents a single number, namely the output of the cosine function.

Trigonometric Values at Special Angles

As was mentioned earlier in this textbook, special triangles play a crucial role in Trigonometry. In fact, the angles 30°, 45°, and 60° are just a few of the so-called **special angles** because we can express the exact values of their trigonometric functions in terms of radicals. In the following example, we explore the values of the trigonometric functions at one of the special angles.

Example 2.1.3

Find the values of all six trigonometric functions of 30°.

Solution available at the end of this section.

At this point, you should be asking yourself why we arbitrarily chose $a = 1$ in Example 2.1.3 to "lock in" the value of the point $P\left(\sqrt{3}, 1\right)$. Had we chosen, for example, $a = 5$, we would have gotten

$$x = 5\sqrt{3}, \quad y = 5, \quad \text{and} \quad r = 10.$$

Therefore,

$$\sin(30^\circ) = \frac{y}{r} = \frac{5}{10} = \frac{1}{2} \qquad \csc(30^\circ) = \frac{r}{y} = \frac{10}{5} = 2$$

$$\cos(30^\circ) = \frac{x}{r} = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2} \qquad \sec(30^\circ) = \frac{r}{x} = \frac{10}{5\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$\tan(30^\circ) = \frac{y}{x} = \frac{5}{5\sqrt{3}} = \frac{1}{\sqrt{3}} \qquad \cot(30^\circ) = \frac{x}{y} = \frac{5\sqrt{3}}{5} = \sqrt{3}$$

Notice that the values of the trigonometric functions didn't change? In fact, if we had chosen to leave a "as is," we would have arrived at

$$x = a\sqrt{3}, \quad y = a, \quad \text{and} \quad r = 2a.$$

Therefore,

$$\sin(30^\circ) = \frac{y}{r} = \frac{a}{2a} = \frac{1}{2} \qquad \csc(30^\circ) = \frac{r}{y} = \frac{2a}{a} = 2$$

$$\cos(30^\circ) = \frac{x}{r} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2} \qquad \sec(30^\circ) = \frac{r}{x} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$\tan(30^\circ) = \frac{y}{x} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}} \qquad \cot(30^\circ) = \frac{x}{y} = \frac{a\sqrt{3}}{a} = \sqrt{3}$$

The choice of a doesn't change the values of the trigonometric functions at all! This demonstration *implies* (but does not prove) that the values of the trigonometric functions do not depend on which point you choose on the terminal side of 30°—just as long as that point is *not* the origin.

Note

The result of this last statement is beneficial. If we have angles from our special triangles, we can just set $a = 1$ in those special triangles to get the x- and y-values of the related point.

Checkpoint 2.1.3

Find the values of the six trigonometric functions at $\theta = 45^\circ$.

Answer available at the end of this section.

We do not need to limit our use of the special angles and special triangles to angles given in the first quadrant. The following example illustrates our tactics if given an angle terminating in a quadrant other than the first, where we can use our knowledge of the special triangles.

Example 2.1.4

Find exact values for all six trigonometric functions of 240°.

Solution available at the end of this section.

Checkpoint 2.1.4

Find exact values for the sine, cosine, and tangent of 300°.

Answer available at the end of this section.

All special angles between 0° and 360° are shown in Figure 2.1.12 below.

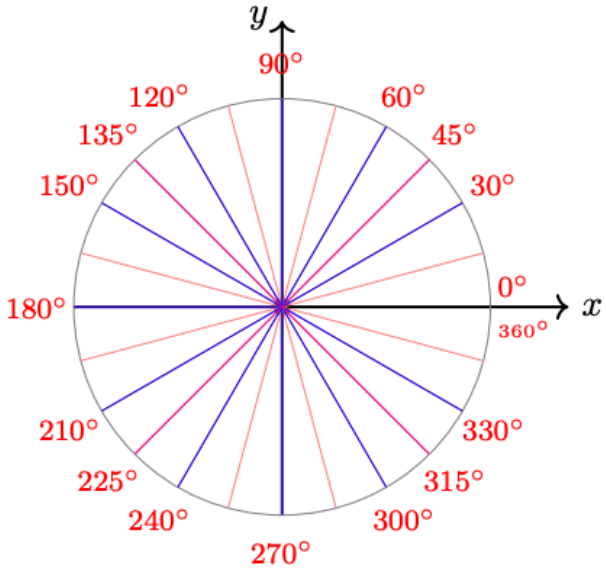


Figure 2.1.12

The astute student will notice that these can be succinctly written using the following definition.

Definition: Special Angles

The **special angles** are defined to be any angle that can be written as either $\pm 30^\circ k$ or $\pm 45^\circ k$, where $k \in \mathbb{Z}$.

It would be best if you memorized these values or be able to calculate them quickly.

Trigonometric Values at Quadrantal Angles

Evaluating trigonometric functions at quadrantal angles is much easier than at non-quadrantal angles. However, we have to watch out for division by zero.

Example 2.1.5

Find the values of the six trigonometric functions at -270°.

Solution available at the end of this section.

Example 2.1.5 illustrates the fact that there are angles for which some of the trigonometric functions are not defined. In terms of functions, we would say that these functions have domain restrictions. We will discuss this in more detail later in this textbook. It suffices to know that the tangent, cotangent, secant, and cosecant functions each have angles for which the functions are undefined. You can see from the definitions of these trigonometric functions where these domain issues occur (either when $x = 0$ or when $y = 0$, depending on the trigonometric function).

Checkpoint 2.1.5

Find the values of the six trigonometric functions at 540°.

Answer available at the end of this section.

Caution: Picky Language

Very technically, "$\cot(540^\circ) = \text{ UNDEFINED}$" is not correct. It not only sounds odd (read the following sentence out loud, and you will understand what we mean, "This value equals undefined"), but it also is not mathematically accurate. Once you use the equals sign ($=$), you are assigning a value; however, "UNDEFINED" is not a value.

A mathematically accurate response would be, "$\cot(540^\circ)$ is undefined."

Comparing Values of the Trigonometric Functions

Understanding how to compare the values of a trigonometric function for different angles can be beneficial in future math courses. For example, in Calculus, you will need to have an innate understanding that $\sin(50^\circ) < \sin(80^\circ)$. The following example illustrates the thought process.

Example 2.1.6

Without reaching for technology, can you determine which is greater, $\sin(50^\circ)$ or $\sin(80^\circ)$?

Solution available at the end of this section.

Checkpoint 2.1.6

Without reaching for technology, can you determine which is greater, $\cos(100^\circ)$ or $\cos(170^\circ)$?

Answer available at the end of this section.

Quadrants and the Signs of the Trigonometric Functions

Example 2.1.7

Give the sign of the sine, cosine, and tangent of the angle.

a. 200°
b. 300°

Solutions available at the end of this section.

Example 2.1.7 demonstrates a common need in Trigonometry—to determine the sign of a trigonometric function. Luckily, our task is fairly easy.

Theorem: Signs of the Trigonometric Functions

The signs of the trigonometric functions depend on the trigonometric function in question and the quadrant where the terminal side of θ lies.

All trigonometric functions are positive when the terminal side of θ is in the first quadrant.

Sine and the cosecant functions are positive when the terminal side of θ is in the second quadrant.

Tangent and the cotangent functions are positive when the terminal side of θ is in the third quadrant.

Cosine and the secant functions are positive when the terminal side of θ is in the fourth quadrant.

Proof available at the end of this section.

Figure 2.1.17 summarizes these results.

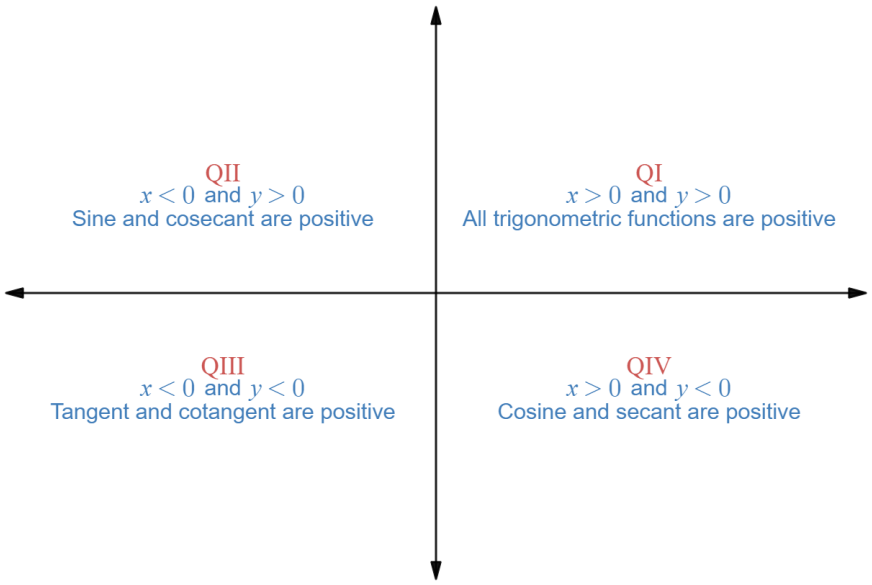


Figure 2.1.17

The bold lettering in the previous theorem is purposeful. Figure 2.1.18 gives the lovely mnemonic, "All Student Take Calculus," to remember that **a**ll trigonometric functions are positive in the first quadrant, the **s**ine and cosecant functions are positive in the second quadrant, the **t**angent and cotangent functions are positive in the third quadrant, and the **c**osine and secant functions are positive in the fourth quadrant.

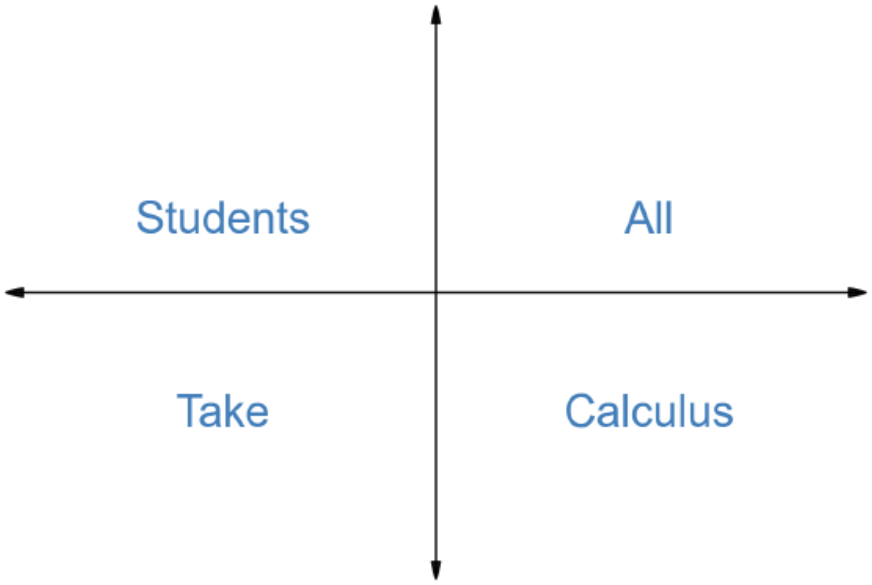


Figure 2.1.18

Example 2.1.8

Given that $\cos(\theta) = -\frac{3}{5}$ and θ terminates in QIII, find the values of the remaining trigonometric functions of θ.

Solution available at the end of this section.

Checkpoint 2.1.8

In which quadrant(s) is the sine negative but the cosine positive?

Answer available at the end of this section.

Proving that Values of the Trigonometric Functions are Independent of P

Up to this point, we have avoided *proving* that the choice of P on the terminal side of θ makes no difference (other than choosing the origin—which is not allowed). We now officially prove the result so we can use it from this point forward.This proves the following theorem.

Theorem: Trigonometric Ratios are Independent of P

Let $P(x,y)$ and $\bar{P}(\bar{x},\bar{y})$ be two distinct points on the terminal side of θ, neither of which are the origin, and let $r=\sqrt{x^2+y^2}$ and $\bar{r}=\sqrt{\bar{x}^2+\bar{y}^2}$. Then

$$\begin{aligned}
\sin(\theta) &= \frac{y}{r} = \frac{\bar{y}}{\bar{r}} & \csc(\theta) &= \frac{r}{y} = \frac{\bar{r}}{\bar{y}}\\
\cos(\theta) &= \frac{x}{r} = \frac{\bar{x}}{\bar{r}} & \sec(\theta) &= \frac{r}{x} = \frac{\bar{r}}{\bar{x}}\\
\tan(\theta) &= \frac{y}{x} = \frac{\bar{y}}{\bar{x}} & \cot(\theta) &= \frac{x}{y} = \frac{\bar{x}}{\bar{y}}
\end{aligned}$$

Proof available at the end of this section.

Hidden Resources: Tutoring Services

Technically, this is not a "hidden resource," per se, but the tutoring services at CRC are a bit spread out between programs. We have the Tutoring Center, Math Center at CRC, Math Center at Elk Grove, Reading and Writing Center, Accounting Lab, Embedded Tutoring (Supplemental Instruction), and NetTutor. If you're an athlete, you might have additional access to tutors through the athletics program. Needless to say, being overwhelmed by the number of overlapping tutoring services is completely understandable. Luckily, CRC has a single webpage with all of the links to these services. Check out the details on this CRC webpage.

Hidden Resources: Student Health and Wellness Services

Cosumnes River College has amazing support services for students in need of emotional and/or mental health services. Our Student Health and Wellness Services include the following:

Physical Health Services (no cost)

- Health and illness assessments
- First aid for minor injuries & illness
- Over-the-counter medications
- Party Safe Kits - Narcan, fentanyl & drink spike test strips
- Sexual health education
- Condoms (internal & external) & lubricant
- Opill (birth control)
- Emergency contraception (Plan B)
- STI testing
- Pregnancy testing
- Flu A/B & Covid testing
- Flu vaccine
- Strep throat testing

- TB risk assessments
- Health requirements for academic programs

Physical Health Services (low cost)

- Immunizations/titers
- TB skin test

Mental Health Services (no cost)

- **Togetherall:** This is a safe online space to connect anonymously with other students to get and give support.
- **Therapy:** Meet with a Los Rios Student Health and Wellness Center (SHWC) therapist or advocate. Support is available in person and virtually.
- **BetterMynd:** Online Therapy: Talk to an online licensed therapist during the day, evenings or weekends. Or call 24/7 support line: (844) BTR-MYND.

For more information, please see this page on the CRC website.

Answers

Section Preview

The definition of the trigonometric functions introduced in this section focuses on trigonometric functions of *any* angle (in degrees)—not just acute, obtuse, or positive angles; however, if given an angle (rather than a coordinate) and asked to evaluate a trigonometric function, the angle will strictly be a *special angle*. Therefore, using technology (e.g., calculators) to evaluate any trigonometric function in this section is *strictly avoided* so as to get the student familiar with these functions at a fundamental level. Simply stated, **students should not require a calculator to evaluate any of the trigonometric functions within this entire chapter!**

Section Preview

- **An Overview of Functions**
  - The Definition of a Function: Trig functions are defined as ratios y/r, x/r, etc., where $r = \sqrt{x^2 + y^2}$—so students need to already understand what a function is before the section can define trig functions *as* functions.
  - The Domain and Range of a Function: Domain and range are mentioned when discussing undefined values at quadrantal angles.

Section Preview

- **Simplifying Expressions**
  - Rationalizing Denominators and Numerators; Simplifying Quotients of Radical Expressions: Ratios involving radicals appears in this material, which may involve radical denominators requiring rationalization.
- **An Overview of Functions**
  - Evaluating Functions: Required to understand the notation, and resulting values, of computations in this section.

Section Preview

- Find the value of a trigonometric function of an angle given a point on the terminal side of the angle.
- Use the coordinate definition of the trigonometric functions to answer a conceptual question about a trigonometric function.
- Determine the quadrants in which an angle could terminate.
- Evaluate the trigonometric functions at the special angles and the quadrantal angles using the coordinate definition of the trigonometric functions.
- Find the value of a trigonometric function given one of the other trigonometric values.
- Compare the values of trigonometric functions at "non-special" angles without using technology.
- Use the quadrant in which an angle terminates to restrict the sign of a trigonometric function.

Example 2.1.1

It's a good habit in Trigonometry to start by sketching a graph of the situation.

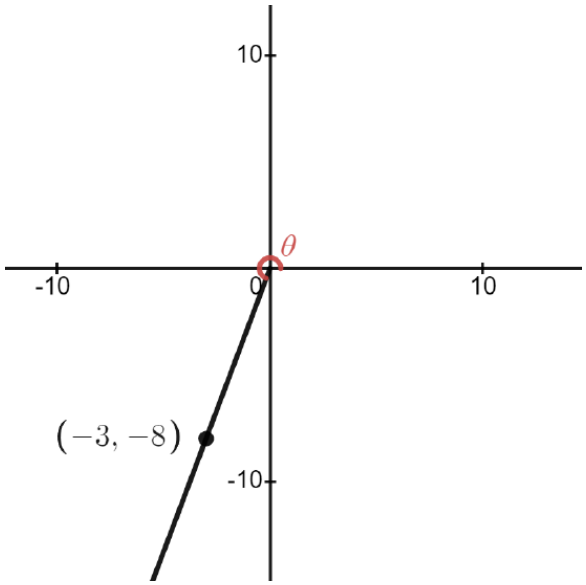


Figure 2.1.2

We have $x=-3$, $y=-8$, and $r=\sqrt{(-3)^2+(-8)^2}=\sqrt{9+64}=\sqrt{73}$. Therefore,

$$\sin(\theta) = \frac{y}{r} = \frac{-8}{\sqrt{73}} = -\frac{8}{\sqrt{73}} \qquad \csc(\theta) = \frac{r}{y} = \frac{\sqrt{73}}{-8} = -\frac{\sqrt{73}}{8}$$

$$\cos(\theta) = \frac{x}{r} = \frac{-3}{\sqrt{73}} = -\frac{3}{\sqrt{73}} \qquad \sec(\theta) = \frac{r}{x} = \frac{\sqrt{73}}{-3} = -\frac{\sqrt{73}}{3}$$

$$\tan(\theta) = \frac{y}{x} = \frac{-8}{-3} = \frac{8}{3} \qquad \cot(\theta) = \frac{x}{y} = \frac{-3}{-8} = \frac{3}{8}$$

Checkpoint 2.1.1

a. $y=\frac{8}{3}x$

b. $(-6,-16)$ satisfies $y=\frac{8}{3}x$. That is, the equation $-16=\frac{8}{3}(-6)$ is true.

c. $r^2=(-6)^2+(-16)^2=292$, so $r=\sqrt{292}=2\sqrt{73}$. Then

$$\sin(\theta) = \frac{y}{r} = \frac{-16}{2\sqrt{73}} = -\frac{8}{\sqrt{73}} \qquad \csc(\theta) = \frac{r}{y} = \frac{2\sqrt{73}}{-16} = -\frac{\sqrt{73}}{8}$$

$$\cos(\theta) = \frac{x}{r} = \frac{-6}{2\sqrt{73}} = -\frac{3}{\sqrt{73}} \qquad \sec(\theta) = \frac{r}{x} = \frac{2\sqrt{73}}{-6} = -\frac{\sqrt{73}}{3}$$

$$\tan(\theta) = \frac{y}{x} = \frac{-16}{-6} = \frac{8}{3} \qquad \cot(\theta) = \frac{x}{y} = \frac{-6}{-16} = \frac{3}{8}$$

Example 2.1.2

Because θ is obtuse, the terminal side of the angle lies in the second quadrant, as shown in Figure 2.1.3.

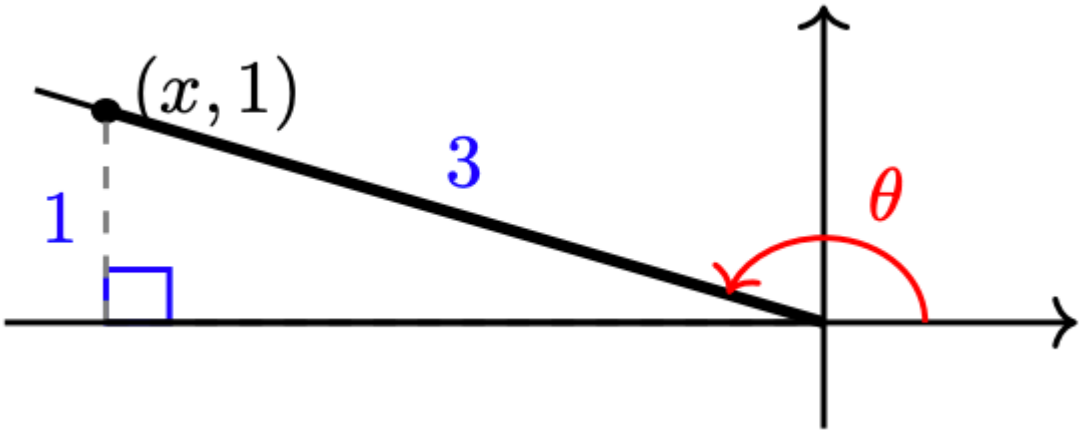


Figure 2.1.3

Because $\sin(\theta)=\frac{1}{3}$, we know that $\frac{y}{r}=\frac{1}{3}$, so we can choose a point P with $y=1$ and $r=3$. To find $\cos(\theta)$ and $\tan(\theta)$, we need to know the value of x. From the Pythagorean Theorem,

$$\begin{array}{rcrcll} & & x^2+1^2 & = & 3^2 & \text{(substitute known values)} \\ \xrightarrow{\text{Arithmetic}} & \Longrightarrow & x^2+1 & = & 9 & \text{(Laws of Exponents)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & \Longrightarrow & x^2 & = & 8 & \text{(subtract 1 from both sides)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & \Longrightarrow & x & = & \pm\sqrt{8} & \text{(Extraction of Roots)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & \Longrightarrow & x & = & \pm 2\sqrt{2} & \text{(simplifying radicals)} \end{array}$$

Remembering that x is negative in the second quadrant, we get that $x = -2\sqrt{2}$. Thus,

$$\cos(\theta) = \frac{x}{r} = \frac{-2\sqrt{2}}{3} \quad \text{and} \quad \tan(\theta) = \frac{y}{x} = \frac{-1}{2\sqrt{2}}.$$

Checkpoint 2.1.2

a.

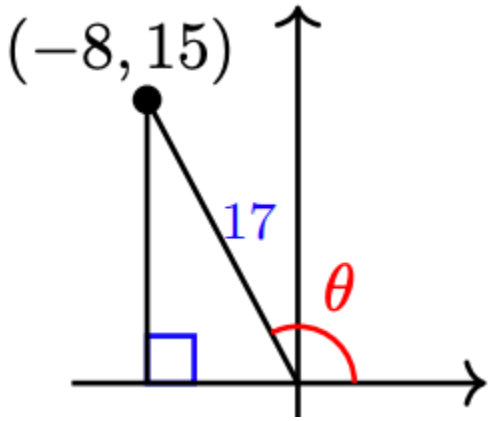


Figure 2.1.4

b. $\csc(\theta) = \frac{17}{15}, \tan(\theta) = \frac{-15}{8}$

Example 2.1.3

Like any good Trigonometry student, we start with a sketch.

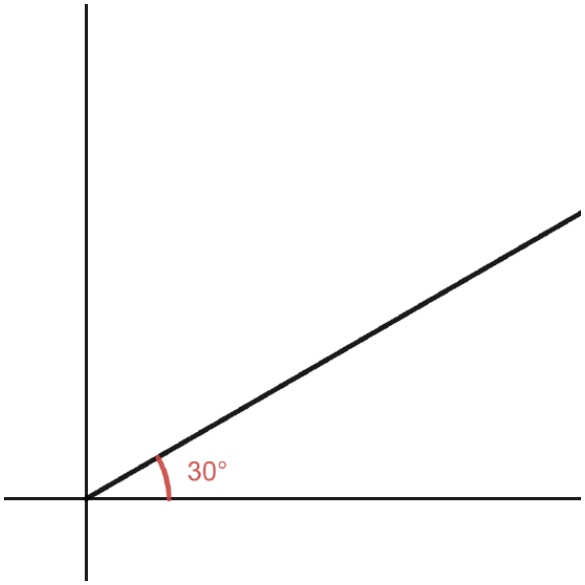


Figure 2.1.5

Unfortunately, we were not given a value for a point on the terminal side of $\theta = 30°$; however, recall the following figure from our exploration of the special triangles last chapter.

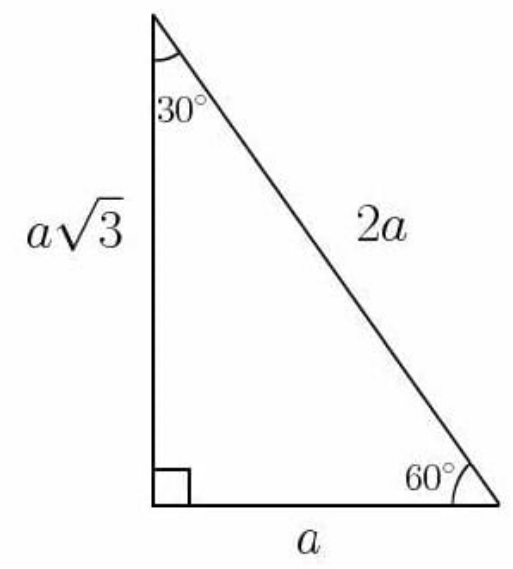


Figure 2.1.6

Rotating this special triangle counterclockwise by 90°, we can visualize fitting it to our first sketch, almost like a jigsaw puzzle.

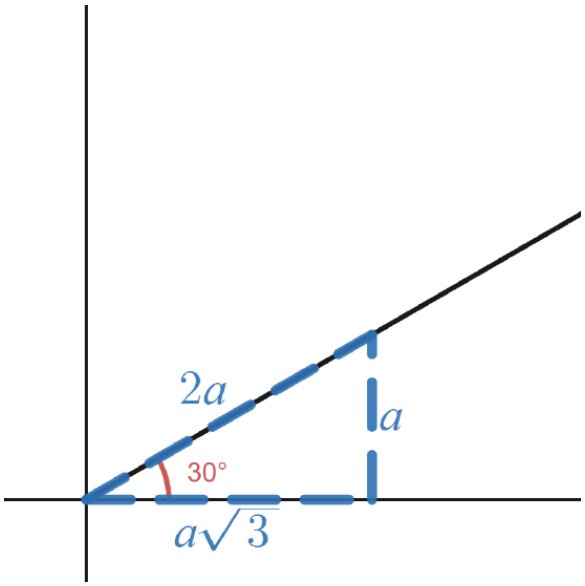


Figure 2.1.7

If we let $a = 1$ (a completely arbitrary choice), we get

$$x = \sqrt{3}, \quad y = 1, \quad \text{and} \quad r = 2.$$

Therefore,

$$\sin(30^\circ) = \frac{y}{r} = \frac{1}{2} \qquad \csc(30^\circ) = \frac{r}{y} = 2$$

$$\cos(30^\circ) = \frac{x}{r} = \frac{\sqrt{3}}{2} \qquad \sec(30^\circ) = \frac{r}{x} = \frac{2}{\sqrt{3}}$$

$$\tan(30^\circ) = \frac{y}{x} = \frac{1}{\sqrt{3}} \qquad \cot(30^\circ) = \frac{x}{y} = \sqrt{3}$$

Checkpoint 2.1.3

$$\sin(45^\circ) = \frac{1}{\sqrt{2}} \qquad \csc(45^\circ) = \sqrt{2}$$

$$\cos(45^\circ) = \frac{1}{\sqrt{2}} \qquad \sec(45^\circ) = \sqrt{2}$$

$$\tan(45^\circ) = 1 \qquad \cot(45^\circ) = 1$$

Example 2.1.4

An angle of 240° lies in the third quadrant, as seen in Figure 2.1.8.

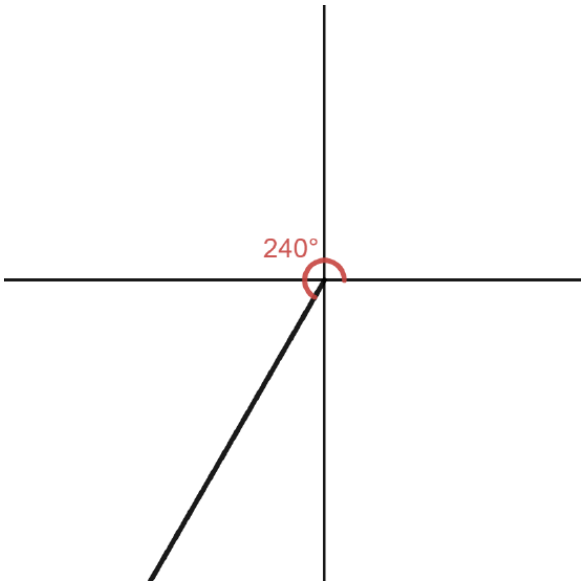


Figure 2.1.8

If we want to work with a smaller angle, we consider the angle between the terminal side of 240° and the negative x-axis. This can be found by taking the larger angle (240°) and subtracting off the smaller angle (the 180° angle that gets to the negative x-axis).

$$240^\circ - 180^\circ = 60^\circ.$$

This action is illustrated in Figure 2.1.9.

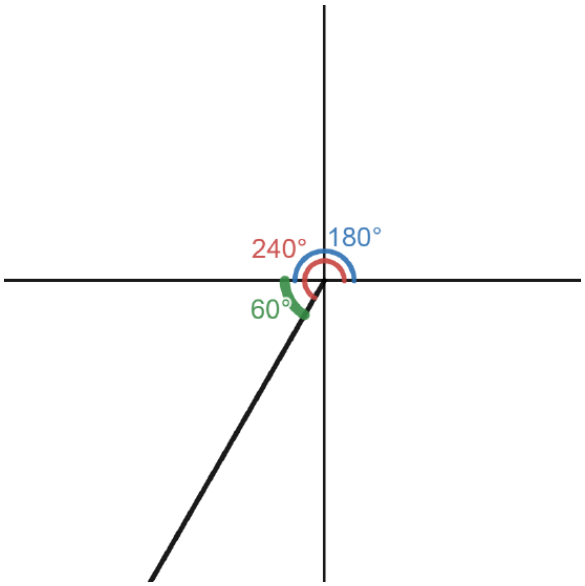


Figure 2.1.9

That 60° angle between 240° and 180° should be familiar.

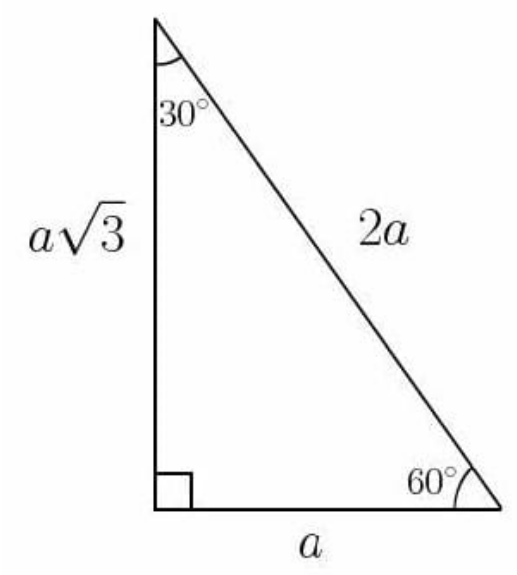


Figure 2.1.10

Superimposing this $30°$-$60°$-$90°$ special triangle into our graph, we get Figure 2.1.11.

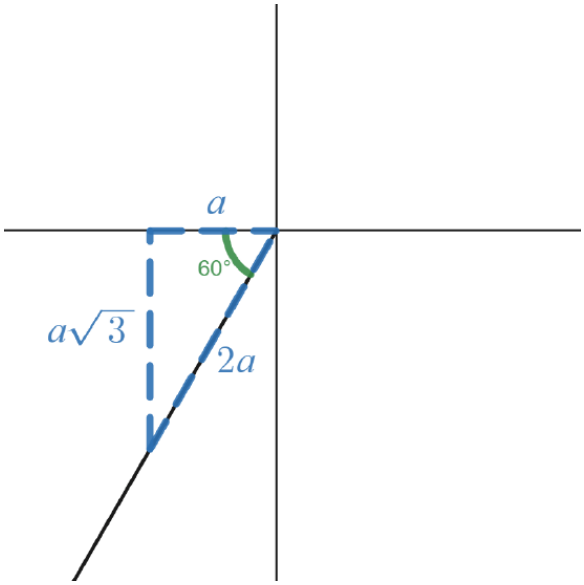


Figure 2.1.11

Using our new knowledge that we can let a be any convenient number, we choose $a = 1$. **However**, here is where we need to be careful! If $a = 1$, it is not true that $x = a = 1$. This is because the point $P(x, y)$ on the terminal side of $240°$ must be in QIII. Therefore, $x = -1$. Likewise, $y = -\sqrt{3}$. No matter what, however, $r > 0$. Thus, $r = 2a = 2$. We now compute the values of the trigonometric functions.

$$\sin(240°) = \frac{y}{r} = -\frac{\sqrt{3}}{2} \qquad \csc(240°) = \frac{r}{y} = -\frac{2}{\sqrt{3}}$$

$$\cos(240°) = \frac{x}{r} = -\frac{1}{2} \qquad \sec(240°) = \frac{r}{x} = -2$$

$$\tan(240°) = \frac{y}{x} = \sqrt{3} \qquad \cot(240°) = \frac{x}{y} = \frac{1}{\sqrt{3}}$$

Checkpoint 2.1.4

$\sin(300°) = \dfrac{-\sqrt{3}}{2}$, $\cos(300°) = \dfrac{1}{2}$, $\tan(300°) = -\sqrt{3}$

Example 2.1.5

We get Figure 2.1.13 by creating a quick sketch for this situation and remembering that negative angles open clockwise.

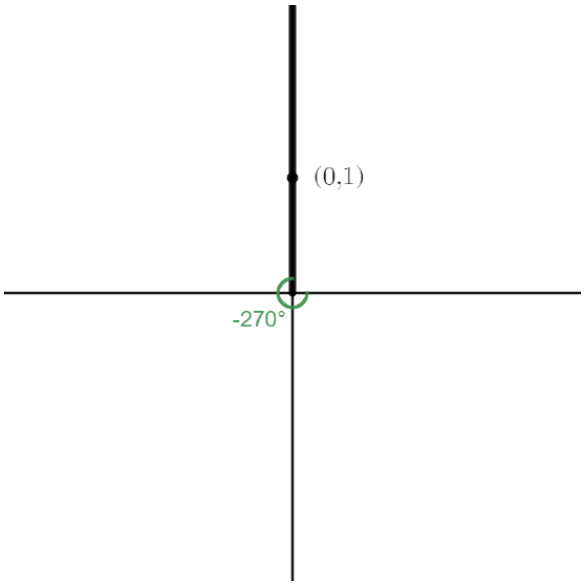


Figure 2.1.13

We took the liberty to label a point on the terminal side of -270°. In this case, $x = 0$, $y = 1$, and $r = 1$. Therefore,

$$\sin(-270^\circ) = \frac{y}{r} = \frac{1}{1} = 1 \qquad \csc(-270^\circ) = \frac{r}{y} = \frac{1}{1} = 1$$

$$\cos(-270^\circ) = \frac{x}{r} = \frac{0}{1} = 0 \qquad \sec(-270^\circ) = \frac{r}{x} = \frac{1}{0} = \text{UNDEFINED}$$

$$\tan(-270^\circ) = \frac{y}{x} = \frac{1}{0} = \text{UNDEFINED} \qquad \cot(-270^\circ) = \frac{x}{y} = \frac{0}{1} = 0$$

Checkpoint 2.1.5

$$\sin(540^\circ) = 0 \qquad \csc(540^\circ) = \text{UNDEFINED}$$
$$\cos(540^\circ) = -1 \qquad \sec(540^\circ) = -1$$
$$\tan(540^\circ) = 0 \qquad \cot(540^\circ) = \text{UNDEFINED}$$

Example 2.1.6

Quickly sketching two angles, $\alpha = 50^\circ$ and $\beta = 80^\circ$, we get the following graph.

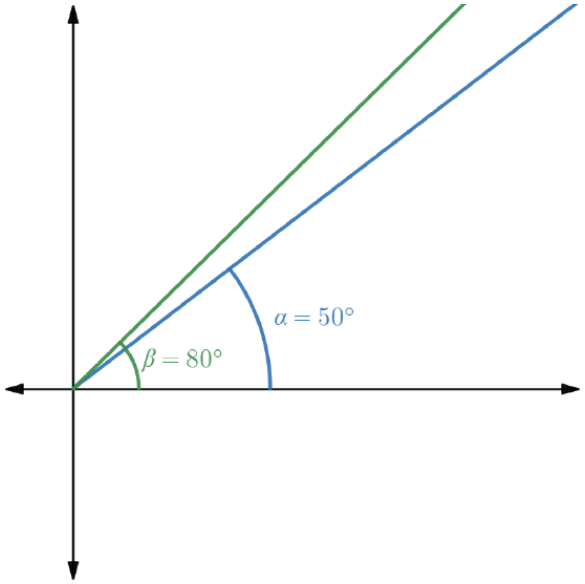


Figure 2.1.14

By itself, this graph is not helpful; however, if we recall that the sine of an angle is defined to be $\frac{y}{r}$, where (x, y) is *any* point on the terminal side of the angle and r is the distance from the origin to the point, then we could choose points on the terminal sides of $\alpha = 50^\circ$ and $\beta = 80^\circ$ such that $r = 1$. This gives us $\sin(50^\circ) = \frac{y_1}{r} = \frac{y_1}{1} = y_1$ and $\sin(80^\circ) = \frac{y_2}{r} = \frac{y_2}{1} = y_2$ (see Figure 2.1.15).

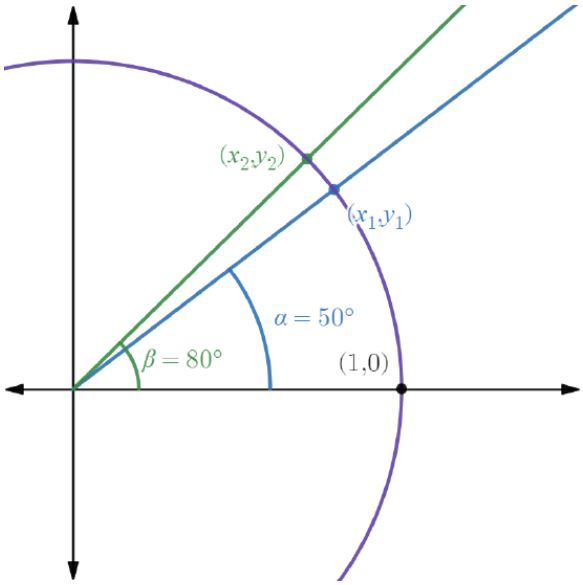


Figure 2.1.15

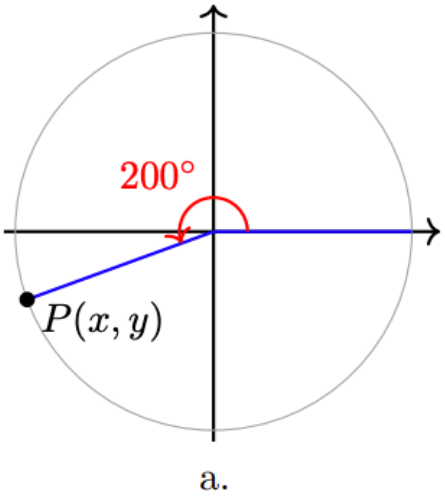
From Figure 2.1.15, we can see that $y_1 < y_2$. Therefore, $\sin(50°) < \sin(80°)$.

Checkpoint 2.1.6

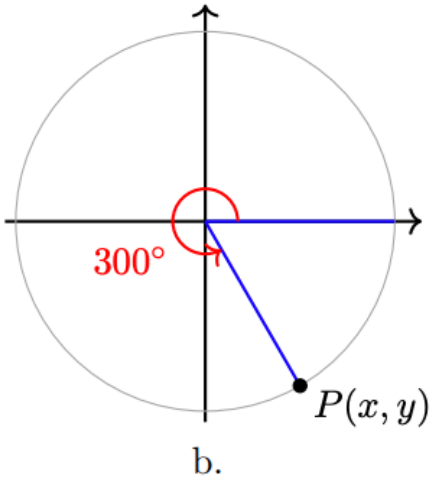
$\cos(100°)$ is greater than $\cos(170°)$

Example 2.1.7

a. In standard position, the terminal side of an angle of $200°$ lies in QIII (see Figure $2.1.16(a)$ below). In the third quadrant, $x < 0$ and $y < 0$, but $r > 0$. Thus, $\sin(200°)$ is negative, $\cos(200°)$ is negative, and $\tan(200°)$ is positive.



a.



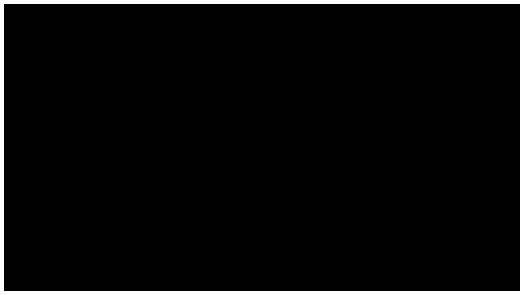
b.

Figure 2.1.16

b. $300° \in$ QIV, so $x > 0$ and $y < 0$, and $r > 0$. Thus, $\sin(300°)$ is negative, $\cos(300°)$ is positive, and $\tan(300°)$ is negative.

Theorem: Signs of the Trigonometric Functions

Since the value of r is always positive (unless the point is the origin), the signs of the trigonometric functions are completely dependent on the signs of the x- and y-coordinates of the point on the terminal side of θ.



Therefore, the signs of the sine and cosecant functions completely depend on the signs of the y-value of the point on the terminal side of θ. Hence, the sine and

cosecant are positive in QI and QII. On the other hand, these functions are negative when the angle terminates in either QIII or QIV.

Likewise, the signs of the cosine and secant functions completely depend on the signs of the x-value of the point on the terminal side of θ. Therefore, the cosine and secant functions are positive in QI and QIV - they are negative in QII and QIII.

Finally, since the tangent and cotangent functions rely on the values of both x and y, we must consider what would make these ratios positive and negative. If the values of x and y are both positive or both negative, their ratios will be positive. This occurs in the first and third quadrants. Thus, the tangent and cotangent functions are positive in QI and QIII - they are negative in QII and QIV.

Example 2.1.8

Since we are given $\cos(\theta) = -\frac{3}{5}$, we know the ratio of x to r is $-\frac{3}{5}$. Letting $x = -3$ and $r = 5$, we find y by solving

$$\begin{array}{rrcll} & r & = & \sqrt{x^2+y^2} & \text{(definition of } r\text{)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & 5 & = & \sqrt{(-3)^2+y^2} & \text{(substitute values)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & 25 & = & 9+y^2 & \text{(Laws of Exponents and squaring both sides)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & 16 & = & y^2 & \text{(subtracting 9 from both sides)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & \pm 4 & = & y & \text{(Extraction of Roots)} \end{array}$$

Using the fact that $\theta \in$ QIII, we throw out the positive possibility and set $y = -4$. We now have everything we need!

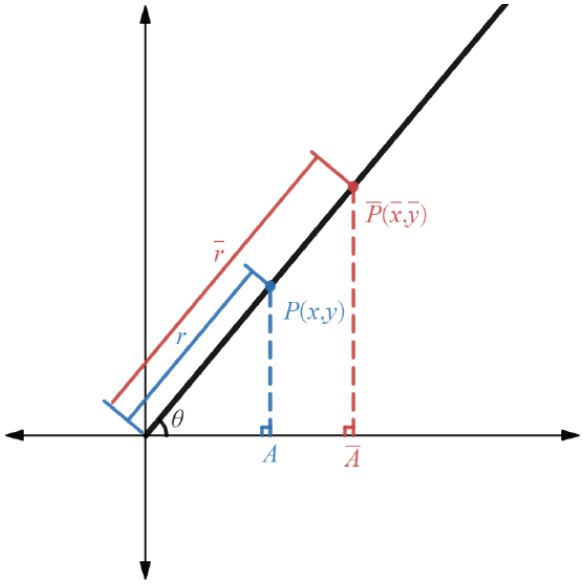
$$\begin{array}{rclrcl} \sin(\theta) & = & -\frac{4}{5} & \csc(\theta) & = & -\frac{5}{4} \\ \cos(\theta) & = & -\frac{3}{5} & \sec(\theta) & = & -\frac{5}{3} \\ \tan(\theta) & = & \frac{4}{3} & \cot(\theta) & = & \frac{3}{4} \end{array}$$

Checkpoint 2.1.8

QIV

Theorem: Trigonometric Ratios are Independent of P

Let $P(x,y)$ and $\bar{P}(\bar{x},\bar{y})$ be two distinct points on the terminal side of θ, neither of which are the origin, and let $r = \sqrt{x^2+y^2}$ and $\bar{r} = \sqrt{\bar{x}^2+\bar{y}^2}$ (see the following figure).



Since $\triangle AOP$ and $\triangle \bar{A}O\bar{P}$ are similar,

$$\begin{array}{rclrcl} \sin(\theta) & = & \frac{y}{r} = \frac{\bar{y}}{\bar{r}} & \csc(\theta) & = & \frac{r}{y} = \frac{\bar{r}}{\bar{y}} \\ \cos(\theta) & = & \frac{x}{r} = \frac{\bar{x}}{\bar{r}} & \sec(\theta) & = & \frac{r}{x} = \frac{\bar{r}}{\bar{x}} \\ \tan(\theta) & = & \frac{y}{x} = \frac{\bar{y}}{\bar{x}} & \cot(\theta) & = & \frac{x}{y} = \frac{\bar{x}}{\bar{y}} \end{array}$$

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2.1.1: Resources and Key Concepts

Resources

Videos

- Trigonometric Functions (Coordinate Definition)
  - Trigonometric Functions of Angles - Review of Trigonometric Functions - The Point Definition

Key Concepts

Definitions

- **Trigonometric Functions (Coordinate Definition):** Let θ be an angle in standard position and let $P(x, y)$ be a point on its terminal side. If $r = \sqrt{x^2 + y^2}$ is the distance from the origin to P, the six trigonometric functions are defined as the ratios of x, y, and r. For example, $\sin(\theta) = \frac{y}{r}$ and $\cos(\theta) = \frac{x}{r}$.
- **Special Angles:** Angles that can be expressed in the form $\pm 30^\circ k$ or $\pm 45^\circ k$, where k is an integer. The exact values of their trigonometric functions can be found using special triangles.

Theorems

- **Signs of the Trigonometric Functions:** The signs of the trigonometric function values depend on the quadrant in which the angle's terminal side lies.
  - **QI:** All functions are positive.
  - **QII:** Sine and cosecant are positive.
  - **QIII:** Tangent and cotangent are positive.
  - **QIV:** Cosine and secant are positive.
- **Trigonometric Ratios are Independent of P:** The values of the trigonometric functions of an angle θ depend only on the angle itself, not on the specific point $P(x, y)$ chosen on the terminal side of θ (provided the point is not the origin).

Common Mistakes

- **Confusing Function Notation with Multiplication:** The notation $\cos(\theta)$ represents a single value, which is the output of the cosine function for the input angle θ. It is not a product of "cos" and "θ". The term "cos" by itself has no mathematical meaning.
- **Incorrectly Stating Undefined Values:** Mathematically, it is more accurate to state that a function "is undefined" at a certain value rather than saying it "equals UNDEFINED". For example, the statement "$\cot(540^\circ)$ is undefined" is preferred over "$\cot(540^\circ) = \text{UNDEFINED}$".
- **Assuming r Can Be Negative:** In the coordinate definition of trigonometric functions, r represents a distance and is always non-negative.
- **Ignoring the Quadrant for Signs:** When finding trigonometric values, the quadrant of the angle is crucial for determining the correct signs of the x and y coordinates. For example, for an obtuse angle (in QII), the x-coordinate must be negative.

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2.1.2: Homework

Reading Questions

1. Using the variables x, y, and r, what are the definitions for $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$?
2. What is the formula for r in the coordinate definition of the trigonometric functions?
3. The value of $\tan(\theta)$ is equivalent to what geometric property of the angle's terminal side?
4. Why are the values of the trigonometric functions independent of the specific point chosen on the terminal side?
5. Why are some trigonometric functions, like $\tan(90^\circ)$, undefined?
6. What makes an angle a "special angle"?
7. In which quadrant are the sine and cosine functions both negative?
8. What does the mnemonic "All Students Take Calculus" help you remember?
9. If you are given that $\sin(\theta) > 0$ and $\cos(\theta) < 0$, in which quadrant must the terminal side of θ lie?
10. Explain the "rule of thumb" this textbook will use for deciding when to rationalize denominators.

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, evaluate the function.

1. $f(x) = x^2 - 2x$
   a. $f(-3)$
   b. $f(a - 3)$
   c. $f(a) - 5$
   d. $f(a) - f(5)$
2. $g(x) = \sqrt{x + 4}$
   a. $g(9)$
   b. $g(4h)$
   c. $g(0) + g(1)$
   d. $g(c^2)$
3. $F(x) = \dfrac{2}{x}$
   a. $F\left(\dfrac{-1}{2}\right)$
   b. $F\left(\dfrac{w}{2}\right)$
   c. $F(w + 2)$
   d. $F(w) + F(2)$

4. $G(x) = 2^x$
   a. $G(-3)$
   b. $G(a+3)$
   c. $G(a) + G(3)$
   d. $G\left(\frac{3}{2}\right)$

For the following exercises, rationalize the denominator.

5. $\frac{7}{\sqrt{5}}$
6. $\sqrt{\frac{2}{7}}$
7. $\frac{\sqrt{3}}{\sqrt{32}}$
8. $\frac{3}{2\sqrt{5x}}$
9. $\frac{11}{6-\sqrt{3}}$
10. $\frac{\sqrt{3}}{3-\sqrt{5}}$
11. $\frac{\sqrt{5}-2}{\sqrt{3}-\sqrt{5}}$
12. $\frac{-4}{\sqrt[4]{3x^3}}$

For the following exercises, rationalize the numerator.

13. $\frac{11-\sqrt{2}}{6}$
14. $\frac{\sqrt{x}-\sqrt{h}}{\sqrt{x}}$

Answers available at the end of this section.

Homework

Vocabulary Check

1. ___ defines new functions in terms of the coordinates of points on the Cartesian Plane and their distance to the origin.
2. In the function notation $f(x)$, x is called the ___ variable.
3. 30°, 45°, and 60° are examples of ___ angles.
4. A helpful way to remember in which quadrants the trigonometric functions are positive is ASTC. This stands for **A**___ **S**___

T__ C__.

Concept Check

5. We define the trigonometric ratios of any angle by placing the angle in standard position and choosing a point on the terminal side, with $r = \sqrt{x^2 + y^2}$.

6. We use the notation $y = f(x)$ to indicate that y is a function of x, that is, x is the input variable and y is the output variable.

7. The trigonometric ratios $\sin\theta$ and $\cos\theta$ are functions of the angle θ.

8. If θ is an angle in standard position, and (x, y) is a point on its terminal side, with $r = \sqrt{x^2 + y^2}$, then $\sin(\theta) = \frac{y}{r}$ $\cos(\theta) = \frac{x}{r}$ $\tan(\theta) = \frac{y}{x}$ $\csc(\theta) = \frac{r}{y}$ $\sec(\theta) = \frac{r}{x}$ $\cot(\theta) = \frac{x}{y}$

9. There are always two angles between 0° and 360° (except for the quadrantal angles) with a given trigonometric ratio.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

10. The angle used in the coordinate definition of the trigonometric functions is measured from the positive y-axis.

11. The value, $\frac{1}{\sqrt{2}}$, is "more correct" than its rationalized form $\frac{\sqrt{2}}{2}$.

12. $\cos\theta$ is another notation for $\cos(\theta)$.

13. $3\sin(30^\circ) = \sin(3 \cdot 30^\circ) = \sin(90^\circ) = 1$.

14. $\sin(\theta)$ can be undefined.

15. $\tan(\theta)$ can be undefined.

16. All students take Calculus.

17. If P and Q are points on the terminal side of θ, where neither are the origin, and Q is three times the distance from the origin as P, the values of the trigonometric functions defined by Q are three times the values of the same trigonometric functions defined by P.

Basic Skills

For the following exercises, find the exact values of all six trigonometric functions of θ if the given point is on the terminal side of θ.

18. $(3, 4)$

19. $\left(-\sqrt{3}, 1\right)$

20. $(-5, -5)$

21. $(15, -8)$

22. $(2, 3)$

23. $\left(-\sqrt{5}, 1\right)$

24. $(-6, 0)$

25. $(0, -2)$

26. (A, B)

27. $(-6t, 9t)$ (assume $t > 0$)

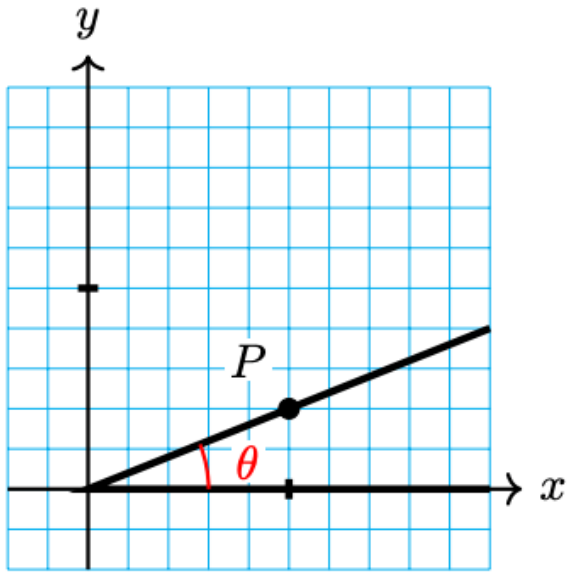
For the following exercises, use technology to find the approximate values of all six trigonometric functions of θ if the given point is on the terminal side of θ.

28. $(1.27, 5.03)$

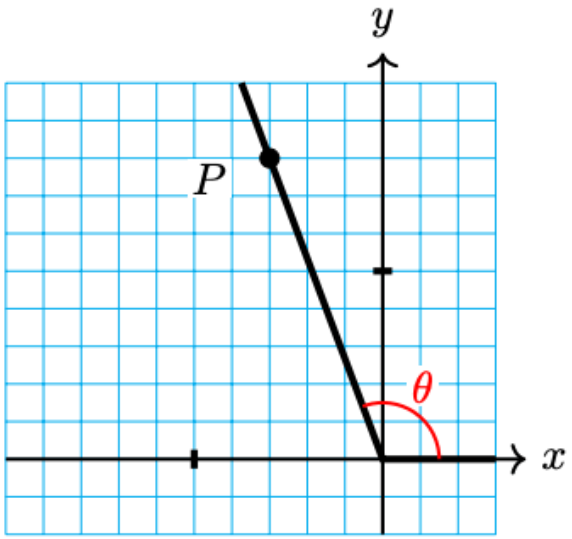
29. $(3.39, -7.46)$

For the following exercises,
(a) Give the coordinates of point P on the terminal side of the angle.
(b) Find the distance from the origin to point P.
(c) Find $\cos\theta, \sin\theta$, and $\tan\theta$.

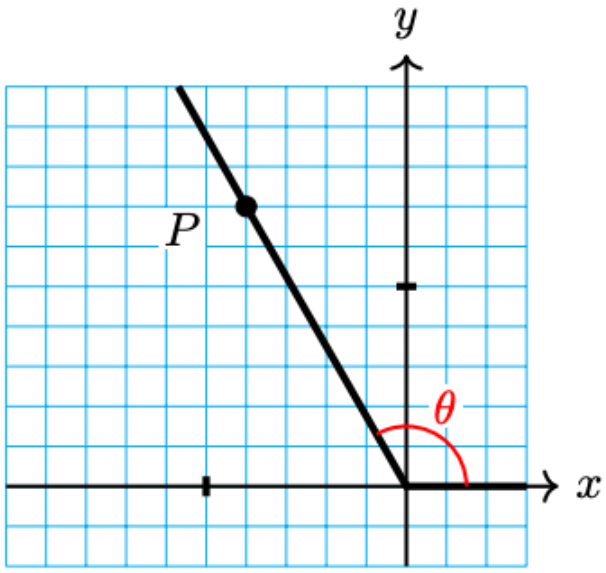
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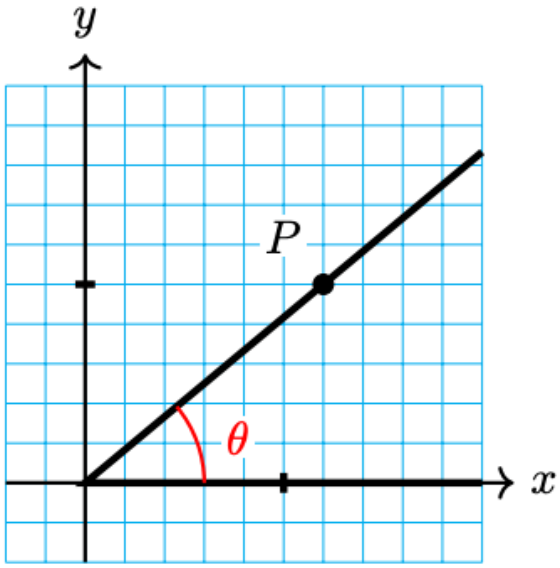
31.



32.



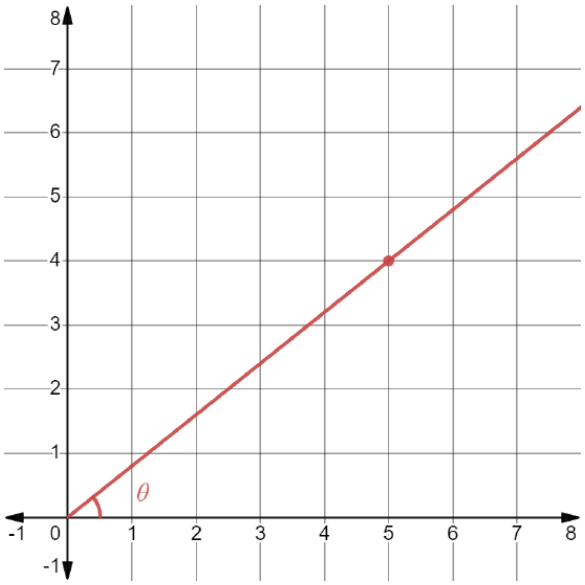
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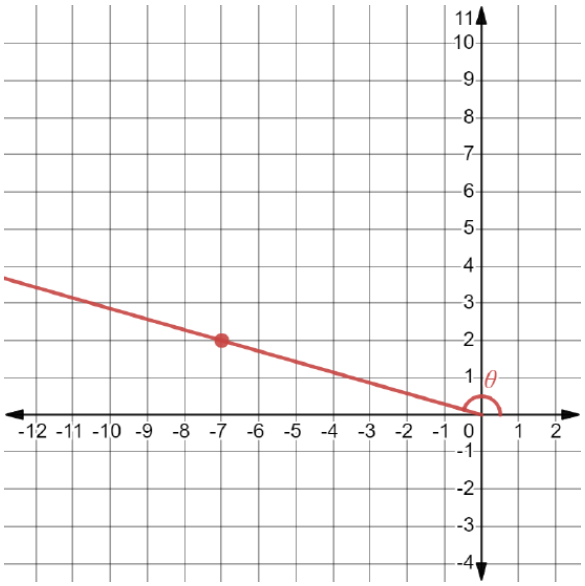
For the following exercises, determine the $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ for the θ in the given graph. Assume θ is in standard position.

34.

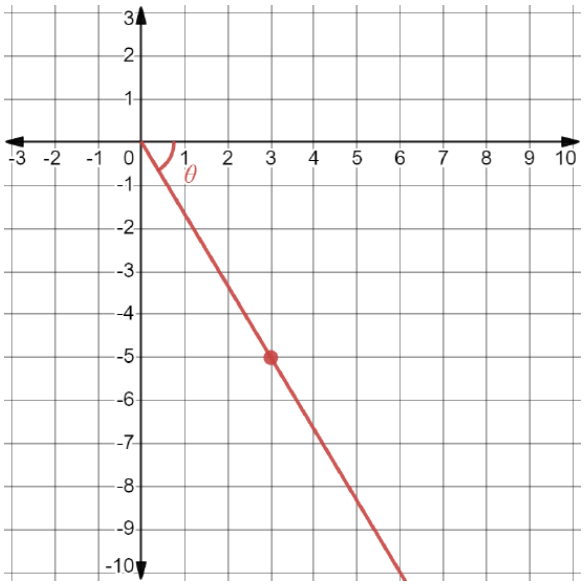
35.

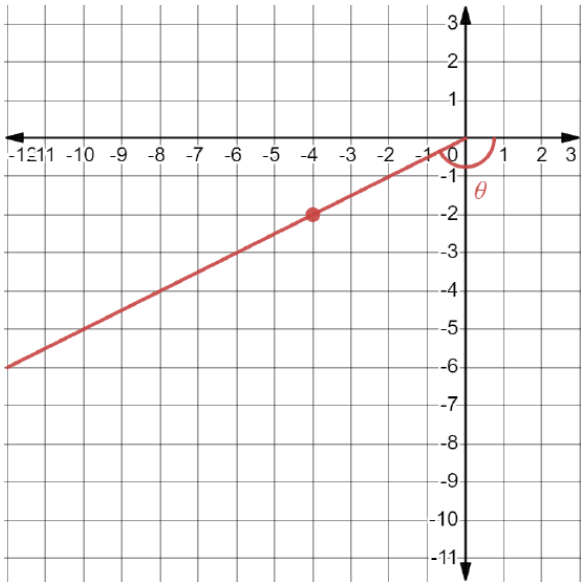


36.



37.





38. Complete the table with exact values.

θ	30°	45°	60°	120°	135°	150°	210°	225°	240°	300°	315°	330°
$\cos(\theta)$												
$\sin(\theta)$												
$\tan(\theta)$												

39. Fill in the table from memory with exact values. Do you notice any patterns that might help you memorize the values?

θ	0°	30°	45°	60°	90°
$\sin(\theta)$					
$\cos(\theta)$					
$\tan(\theta)$					

40. Fill in the table from memory with decimal approximations to four places.

θ	0°	30°	45°	60°	90°
$\sin(\theta)$					
$\cos(\theta)$					
$\tan(\theta)$					

41. Complete the table with exact values.

θ	0	30°	45°	60°
$\sec(\theta)$				
$\csc(\theta)$				
$\cot(\theta)$				

For the following exercises, choose all values from the list below that are exactly equal to, or decimal approximations for, the given trigonometric ratio. (Try not to use a calculator!)



42. $\cos 30^\circ$

43. $\sin 45^\circ$

44. $\tan 30^\circ$

45. $\cos 60^\circ$

46. $\sin 90^\circ$

47. $\cos 0^\circ$

For the following exercises, evaluate. Give exact values.

48. $\sin(30^\circ)$

49. $\cos(0^\circ)$

50. $\tan(45^\circ)$

51. $\sin(60^\circ)$

52. $\csc(30^\circ)$

53. $\sec(0^\circ)$

54. $\cot(45^\circ)$

55. $\csc(60^\circ)$

For the following exercises, draw the angle in standard position, find a point on the terminal side of the angle, and find the value of all six trigonometric functions of the angle.

56. 150°

57. -90°

58. 225°

59. 180°

60. -300°

61. 90°

62. 315°

63. Explain why it makes sense that $\sin 0^\circ = 0$ and $\sin 90^\circ = 1$. Use a figure to illustrate your explanation.

64. Explain why it makes sense that $\cos 0^\circ = 1$ and $\cos 90^\circ = 0$. Use a figure to illustrate your explanation.

For the following exercises, state two possible quadrants where θ could terminate.

65. $\sin(\theta) = \frac{7}{12}$

66. $\cos(\theta) = -\frac{\sqrt{2}}{9}$

67. $\tan(\theta) = -1.2$

68. $\csc(\theta) = -3.87$

69. $\sec(\theta) = \sqrt{2}$

70. $\cot(\theta) = \frac{3}{2}$

For the following exercises, determine in which two quadrants is the statement true.

71. The sine is negative.

72. The cosine is negative.

73. The tangent is positive.

For the following exercises, state the quadrant(s) where θ must terminate.

74. $\sin(\theta) < 0$ and $\cos(\theta) > 0$

75. $\sec(\theta) > 0$ and $\tan(\theta) < 0$

76. $\csc(\theta) < 0$ and $\cot(\theta) < 0$

77. $\sin(\theta)$ and $\cot(\theta)$ have the same sign

For the following exercises, find the values of the remaining trigonometric functions of θ based on the given information.

78. $\sin(\theta) = \frac{21}{29}$, where $\theta \in \text{QI}$

79. $\cos(\theta) = -\frac{3}{5}$, where $\theta \in \text{QIII}$

80. $\tan(\theta) = \frac{15}{8}$, where $\theta \in \text{QIII}$

81. $\csc(\theta) = -\frac{2}{\sqrt{3}}$, where $\tan(\theta) > 0$

82. $\sec(\theta) = \sqrt{2}$, where $\theta \in \text{QIV}$

83. $\cot(\theta) = -\frac{1}{\sqrt{3}}$, where $\theta \in \text{QII}$

84. $\sin(\theta) = -\frac{1}{4}$, where $\cos(\theta) < 0$

85. $\tan(\theta) = \frac{a}{b}$, where both a and b are positive

For the following exercises, find all angles between 0° and 360° for which the statement is true.

86. $\cos(\theta) = -1$

87. $\sin(\theta) = -1$

88. $\tan(\theta) = -1$

89. Find two angles, $0 \le \theta \le 360^\circ$, with $\sin(\theta) = 0$.

90. Find two angles, $0 \le \theta \le 360^\circ$, with $\cos(\theta) = 0$.

91. Find all angles, $0 \le \theta \le 360^\circ$, with $\sin(\theta) = -1$.

92. Find all angles, $0 \le \theta \le 360^\circ$, with $\cos(\theta) = -1$.

Synthesis Questions

For the following exercises, determine if the statement is true or false. Justify your response.

93. If $\alpha < \beta$, then $\sin(\alpha) < \sin(\beta)$.

94. If $0 \le \alpha < \beta \le 90^\circ$, then $\sin(\alpha) < \sin(\beta)$.

95. If $0 \le \alpha < \beta \le 90^\circ$, then $\cos(\alpha) < \cos(\beta)$.

96. If $0 \le \alpha < \beta \le 90^\circ$, then $\csc(\alpha) < \csc(\beta)$.

97. Is there an angle θ such that $\cos(\theta) = 3$? Why or why not?

98. Is there an angle θ such that $\csc(\theta) = \frac{1}{5}$? Why or why not?

99. Explain why $|\cos(\theta)| \le 1$ for any angle θ.

100. Is it true that $\tan(\theta) \le 1$ for any angle θ? Explain.

101. Use a sketch to explain why $\cos 90^\circ = 0$.

102. Use a sketch to explain why $\cos 180^\circ = 1$.

For the following exercises, compare the given value with the trigonometric ratios of the special angles to answer the questions.

103. Is the acute angle larger or smaller than 45°?

a. $\sin\alpha = 0.7$

b. $\tan\beta = 1.2$

c. $\cos\gamma = 0.65$

94. Is the acute angle larger or smaller than 60°?

a. $\cos\theta = 0.75$

b. $\tan\phi = 1.5$

c. $\sin\psi = 0.72$

For the following exercises, describe what happens to the function values as θ increases from 0° to 90°.

95. $\sin(\theta)$

96. $\cos(\theta)$

97. $\tan(\theta)$

98. $\csc(\theta)$

99. Find two angles, $0 \le \theta \le 360^\circ$, with $\sin(\theta) = \cos(\theta)$.

100. Find two angles, $0 \le \theta \le 360^\circ$, with $\sin(\theta) = -\cos(\theta)$.

101. Explain why the definitions of the trigonometric ratios for a third-quadrant angle (between 180° and 270°) are independent of the point P chosen on the terminal side. Illustrate with a figure.

102. Explain why the definitions of the trigonometric ratios for a fourth-quadrant angle (between 270° and 360°) are independent of the point P chosen on the terminal side. Illustrate with a figure.

103. Find the values of all the trigonometric functions of θ if the terminal side of θ lies on the line $y = 3x$ in QI.

104. Find the values of all the trigonometric functions of θ if the terminal side of θ lies on the line $y = -\frac{1}{3}x$ in QII.

105. Find the values of all the trigonometric functions of θ if the terminal side of θ lies on the line $y = \frac{4}{5}x$ in QIII.

106. Find the values of all the trigonometric functions of θ if the terminal side of θ lies on the line $y = -6x$ in QIV.

For the following exercises, evaluate the expression for $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$.

107. $3 + f(30^\circ)$

108. $3f(30^\circ)$

109. $4g(225^\circ) - 1$

110. $-4 + 2g(225^\circ) - 1$

111. $-2f(3\theta)$, for $\theta = 90^\circ$

112. $6f\left(\frac{\theta}{2}\right)$, for $\theta = 90^\circ$

113. $8 - 5g\left(\frac{\theta}{3}\right)$, for $\theta = 360^\circ$

114. $1 - 4g(4\theta)$, for $\theta = 135^\circ$

For the following exercises,
(a) Sketch an angle θ in standard position, $0^\circ \le \theta \le 180^\circ$, with the given properties.
(b) Find expressions for $\cos\theta, \sin\theta$, and $\tan\theta$ in terms of the given variable.

25. $\cos\theta = \dfrac{x}{3}, x < 0$

26. $\tan\theta = \dfrac{4}{\alpha}, \alpha < 0$

27. θ is obtuse and $\sin\theta = \dfrac{y}{2}$

28. θ is obtuse and $\tan\theta = \dfrac{q}{-7}$

29. θ is obtuse and $\tan\theta = m$

30. $\cos\theta = h$

Answers

Skills Refresher

1. a. 15
   b. $a^2 - 8a + 15$
   c. $a^2 - 2a - 5$
   d. $a^2 - 2a - 15$
2. a. $\sqrt{13}$
   b. $2\sqrt{h+1}$
   c. $2 + \sqrt{5}$
   d. $\sqrt{c^2+4}$
3. a. -4
   b. $\dfrac{4}{w}$
   c. $\dfrac{2}{w+2}$
   d. $\dfrac{2}{w} + 1$
4. a. $\dfrac{1}{8}$
   b. $8(2^a)$
   c. $2^a + 8$
   d. $2\sqrt{2}$
5. $\dfrac{7\sqrt{5}}{5}$
6. $\dfrac{\sqrt{14}}{7}$
7. $\dfrac{\sqrt{6}}{8}$
8. $\dfrac{3\sqrt{5x}}{10x}$
9. $\dfrac{6+\sqrt{3}}{3}$
10. $\dfrac{3\sqrt{3}+\sqrt{15}}{4}$
11. $\dfrac{-\sqrt{15}+2\sqrt{3}-5+2\sqrt{5}}{2}$
12. $\dfrac{-4\sqrt[4]{27x^9}}{3x^3}$
13. $\dfrac{3}{22+2\sqrt{2}}$
14. $\dfrac{x-h}{x+\sqrt{hx}}$

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2.2: Algebra with Trigonometric Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

As we move forward, we will need to apply some techniques from Algebra to analyze more complicated trigonometric expressions. Therefore, let's review some algebraic terminology.

Definition: Algebraic Expression

An **algebraic expression** is a well-formed mathematical phrase that combines numbers, variables, and mathematical operations. It does not include an equals sign.

For example, the height of a golf ball is given in feet by the expression $-16t^2 + 64t$, where t is the number of seconds after the ball is hit.

Definition: Evaluate (an algebraic expression)

To **evaluate** an algebraic expression means to substitute a specific value for the variable or variables involved.

Thus, after 1 second, the height of the golf ball is $-16(1)^2 + 64(1) = -16 + 64 = 48$ feet, and after 2 seconds, the height is $-16(2)^2 + 64(2) = -64 + 128 = 64$ feet, and so on.

Evaluating Trigonometric Expressions

The value of a trigonometric function represents a number, and it may appear as part of an algebraic expression. Expressions containing trigonometric functions can be simplified or evaluated like other algebraic expressions.

Example 2.2.1

Evaluate each expression for $X = 30^\circ$ and $Y = 240^\circ$.

a. $\frac{1}{\sqrt{3}}\tan(Y) + 3\sin(X)$
b. $6\tan(X)\cos(Y)$

Solutions available at the end of this section.

Checkpoint 2.2.1

Evaluate each expression for $X = 30^\circ, Y = 60^\circ$.

a. $4\sin(3X + 45^\circ)$
b. $1 - \cos(4Y)$

Answers available at the end of this section.

Caution: Do NOT "Split" Arguments of Functions

In Checkpoint 2.2.1, $\sin(3X + 45^\circ)$ is *not* equal to $\sin(3X) + \sin(45^\circ)$. That is,

$$\sin(90^\circ + 45^\circ) \neq \sin(90^\circ) + \sin(45^\circ)$$

(You can check this for yourself.) Before applying the sine function, we must follow the Order of Operations and evaluate the expression $3X + 45^\circ$ inside parentheses.

Powers of Trigonometric Functions

Compare the two expressions

$$(\cos(\theta))^2 \quad \text{and} \quad \cos(\theta^2).$$

They are not the same. The first expression, $(\cos(\theta))^2$, says to compute $\cos(\theta)$ and then square the result. $\cos(\theta^2)$, on the other hand, says to square the angle first and then compute the cosine.

For example, if $\theta = 30^\circ$, then

$$(\cos(30^\circ))^2 = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$$

but

$$\cos\left((30^\circ)^2\right) = \cos(900^\circ)$$

We have not delved into the details of how to compute the cosine of such a large angle (yet). Still, it should make sense that if the point $P(x, y)$ is on the terminal side of $\theta = 900^\circ$, then P must be on the terminal side of any angle coterminal with $\theta = 900^\circ$. Specifically, P must be on the terminal side of $900^\circ - 360^\circ = 540^\circ$. Still, this angle is too unwieldy to deal with, so we find another, smaller angle coterminal to 540°. Luckily, $540^\circ - 360^\circ = 180^\circ$ fits that bill! Hence, if $P(x, y)$ is on the terminal side of 900°, it must also be on the terminal side of 180°. Since the point $(-1, 0)$ is on the terminal side of 180°, we can now compute the cosine of 900°.

$$\cos\left((30^\circ)^2\right) = \cos(900^\circ) = \cos(180^\circ) = \frac{x}{r} = \frac{-1}{1} = -1.$$

Notice this is not the same value as $\cos(30^\circ)$.

This brings us to a new notation reserved specifically for trigonometric functions.

Notation: Powers of Trigonometric Functions

$$\sin^2(\theta) \quad \text{means} \quad (\sin(\theta))^2 \qquad \csc^2(\theta) \quad \text{means} \quad (\csc(\theta))^2$$
$$\cos^2(\theta) \quad \text{means} \quad (\cos(\theta))^2 \qquad \sec^2(\theta) \quad \text{means} \quad (\sec(\theta))^2$$
$$\tan^2(\theta) \quad \text{means} \quad (\tan(\theta))^2 \qquad \cot^2(\theta) \quad \text{means} \quad (\cot(\theta))^2$$

Example 2.2.2

Evaluate $\sin^2(45^\circ)$.

Solution available at the end of this section.

Other powers are written in the same fashion. Thus, for example, $\sin^3(\theta) = (\sin(\theta))^3$.

Checkpoint 2.2.2

Evaluate $\tan^4(60^\circ)$

Answer available at the end of this section.

Caution: Negative Exponents

We have already stated that $[\cos(\theta)]^2 = \cos^2(\theta)$ and $[\sin(\theta)]^5 = \sin^5(\theta)$. **However**, we will **not** adopt such notation for negative exponents. That is,

$$[\cos(\theta)]^{-1} \neq \cos^{-1}(\theta).$$

The reason for excluding the negative exponents from this "shorthand" notation will become apparent when we discuss inverse trigonometric functions.

Products of Trigonometric Functions

We can multiply trigonometric expressions just as we multiply algebraic expressions. Recall that we use the Distributive Law in computing products such as $x(3x-2) = 3x^2 - 2x$ and $(x-3)(x+5) = x^2 + 2x - 15$.

Example 2.2.3

Using the Distributive Law, multiply

$$\cos(t)(3\cos(t) - 2).$$

Solution available at the end of this section.

Checkpoint 2.2.3

Multiply $2\tan(\beta)(4\tan^2(\beta) + \tan(\alpha))$

Answer available at the end of this section.

We can also use the Distributive Law to multiply binomials that include trigonometric functions.

Example 2.2.4

Multiply $(4\sin(C)-1)(3\sin(C)+2)$.

Solution available at the end of this section.

Checkpoint 2.2.4

Expand $(4\cos(\alpha)+3)^2$

Answer available at the end of this section.

Factoring Expressions Involving Trigonometric Functions

We can factor trigonometric expressions with the same techniques we use for algebraic expressions. The next few subsections of this topic quickly review all our previous factorization techniques. It is important to note that each of these factoring techniques will be needed at some point during Trigonometry.

Factoring Out a Common Expression

Likely, the first technique you encountered when learning to factor was factoring out the greatest common factor (GCF). I call this factoring out a common expression.

In the next two examples, compare the familiar algebraic factoring with a similar trigonometric expression.

Example 2.2.5

Factor.

a. $6w^2-9w$
b. $6\sin^2(\theta)-9\sin(\theta)$

Solutions available at the end of this section.

Checkpoint 2.2.5

Factor.

a. $2a^2-ab$
b. $2\cos^2(\phi)-\cos(\phi)\sin(\phi)$

Answers available at the end of this section.

Factoring by Grouping

Once you mastered factoring out common expressions from a collection of terms, you moved to factoring by grouping. We only review via example here as we did in the previous topic.

Example 2.2.6

Factor.

a. $10x^2 + 2x - 15xy - 3y$
b. $10\sin^2(\theta) + 2\sin(\theta) - 15\sin(\theta)\cos(\theta) - 3\cos(\theta)$

Solutions available at the end of this section.

Example $2.2.6b$ illustrates a common tactic we will use as we move through mathematics. Namely, when an expression (or equation) looks overly complicated, we make it easier to work with by performing substitutions. In Calculus, this will be tied to a technique called u**-Substitutions** (where we use the variable u instead of x).

Checkpoint 2.2.6

Factor.

a. $12rt + 9r - 20t - 15$
b. $12\sec(\alpha)\tan(\beta) + 9\sec(\alpha) - 20\tan(\beta) - 15$

Answers available at the end of this section.

Factoring Trinomials

Despite being more complex than factoring a difference of squares (which we will talk about next), the next factorization technique you learned was factoring trinomials.

Example 2.2.7

Factor.

a. $t^2 - 3t - 10$
b. $\tan^2(\alpha) - 3\tan(\alpha) - 10$

Solutions available at the end of this section.

Checkpoint 2.2.7

Factor.

a. $3z^2 - 2z - 1$
b. $3\sin^2(\beta) - 2\sin(\beta) - 1$

Answers available at the end of this section.

Factoring Differences of Squares

In addition to factoring out common expressions and factoring trinomials, special factoring formulas for the Difference of Squares and the Sums and Differences of Cubes will be needed. Recall the following theorem from Algebra.

Theorem: Difference of Squares

For any algebraic expressions F and L,

$$F^2 - L^2 = (F - L)(F + L).$$

Proof available at the end of this section.

Example 2.2.8

Factor completely.

$$16\cos^4(\theta) - 1$$

Solution available at the end of this section.

A couple of notes are in order for Example 2.2.8. First and foremost, while we took our time with substitutions and being meticulous with our work, you need to reach a level of proficiency in mathematics where you do not need to perform those substitutions to do the factoring. That is, you need to reach the point (and very soon) where you see $16\cos^4(\theta) - 1$ and immediately rewrite it (without substitutions) as

$$\left(4\cos^2(\theta) - 1\right)\left(4\cos^2(\theta) + 1\right) = (2\cos(\theta) - 1)(2\cos(\theta) + 1)\left(4\cos^2(\theta) + 1\right).$$

I cannot express how important it is for you, as the student, to reach that level of mathematical fluency as quickly as possible (truthfully, in a college-level mathematics course, you should already have this level of fluency).

The second note is better mentioned as a cautionary statement.

Caution: $F^2 + L^2$ is NOT Factorable

A common mistake of students in lower levels of Algebra is to think that $F^2 + L^2$ is factorable—this is **not** true (unless we are factoring over the complex number system). That is,

$$4x^2 + 1$$

cannot be factored (over the real number system).

Checkpoint 2.2.8

Factor completely.

$$\sin^4(\alpha) - 81\cos^4(\beta)$$

Answer available at the end of this section.

Factoring Sums and Differences of Cubes

We have now reached the final factorization method, which we must recall from Algebra.

Theorem: Sums and Differences of Cubes

For any algebraic expressions F and L,

$$\begin{aligned} F^3 + L^3 &= (F + L)(F^2 - FL + L^2) \\ F^3 - L^3 &= (F - L)(F^2 + FL + L^2) \end{aligned}$$

Proof available at the end of this section.

Example 2.2.9

Factor.

$$27\sin^3(A) - 8$$

Solution available at the end of this section.

Checkpoint 2.2.9

Factor.

$$125\csc^3(\theta) + 64\cot^3(\phi)$$

Answer available at the end of this section.

Believe it or not, other factorization methods exist beyond the basic techniques reviewed in this section. For example, in College Algebra (and in Precalculus), you learn about factoring high-degree polynomials with integer coefficients using the Rational Roots Theorem. Moreover, there are critically important methods for factoring expressions involving negative rational exponents—these are encountered frequently in Calculus.

Hidden Resources: The Hawk Spot Food Pantry

The Hawk Spot Food Pantry is a program at CRC that gives food once per week to CRC students.

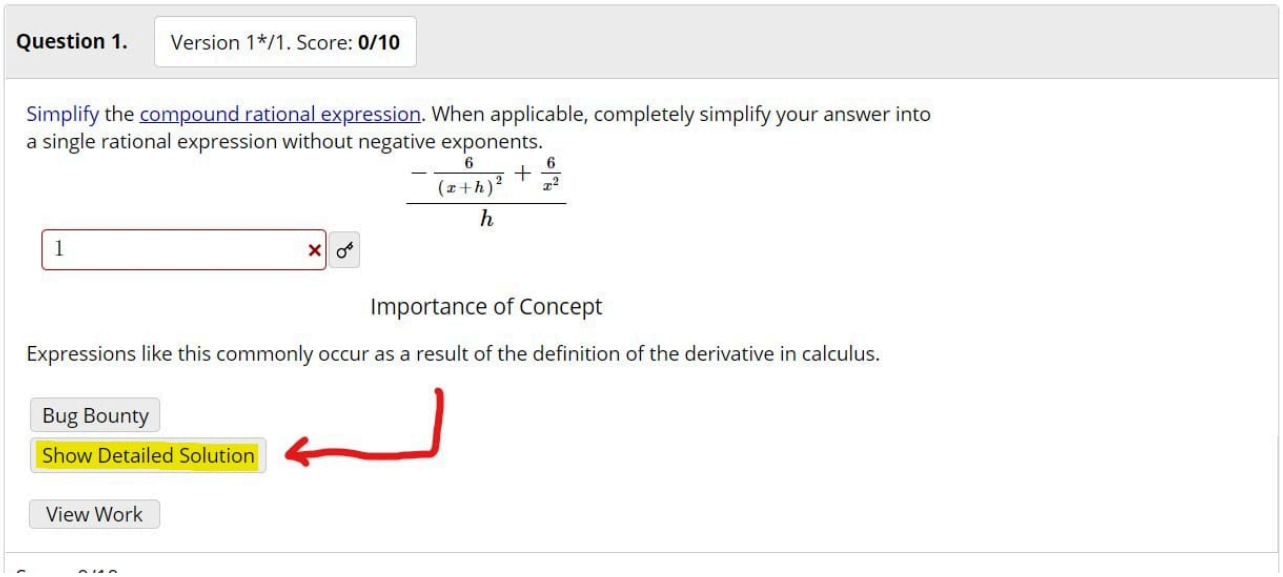
How Does CalFresh Work?

Details may change, so use the link below to verify what is stated here; however, you show your Student ID, fill out the intake form (on first visit), and pick-up your premade bag. Each student is allowed to be served once per week.

For more information, please see this page on the CRC website.

MyOpenMath Advice: Reading Solutions to Exercises

Some problems have complete worked-out solutions. If your feedback has a comment like, "Please see the solutions," then you should probably look through these solutions. You can see solutions by going into the Grades area, finding the assignment, and clicking "Show Detailed Solution" on an exercise with a solution. Solutions are generally not shown until after the due date has passed.



I usually leave a comment of the form, "Please see the written solutions," if there were too many mistakes for me to cover by typing or if I could not follow your logic.

- Professor Simpson, Cosumnes River College

Answers

Section Preview

This section focuses on performing our old Algebra (factoring, distribution, powers, etc.) on expressions involving trigonometric functions. The results will not require any trigonometric identities. The point here is for the student to realize that Algebra still works, even though we are not using x.

Section Preview

- **Simplifying Expressions**
  - Laws of Exponents — Product and Quotient Rules: Understanding that trig functions behave as algebraic objects requires knowing the Product and Quotient Rules from the Laws of Exponents—without them, a student cannot follow the conceptual framework of the section.
  - Laws of Exponents — Power Rules: Understanding that trig functions behave as algebraic objects requires knowing the Power Rules from the Laws of Exponents—without them, a student cannot follow the conceptual framework of the section.
  - Multiplying Polynomials and the Distributive Property: Understanding that trig functions behave as algebraic objects requires knowing the Distributive Property—without this, a student cannot follow the conceptual framework of the section.

Section Preview

- **Arithmetic**
  - The Order of Operations: This skill is explicitly called out in the cautionary statement regarding not splitting functions across arguments.
- **Factoring Techniques**
  - Greatest Common Factor: This mechanical skill is needed to complete the homework problems.
  - Factoring by Grouping: This mechanical skill is needed to complete the homework problems.
  - Factoring Trinomials: This mechanical skill is needed to complete the homework problems.
  - Factoring Differences of Squares: This mechanical skill is needed to complete the homework problems.
  - Factoring Sums and Differences of Cubes: This mechanical skill is needed to complete the homework problems.

Section Preview

- Evaluate expressions involving trigonometric functions.
- Compute powers of trigonometric functions.
- Evaluate products of trigonometric functions.
- Perform factorization on expressions involving trigonometric functions.

Example 2.2.1

a. Substituting the values for X and Y, we get the following:

$$\begin{aligned}
\xrightarrow{\text{Arithmetic}\to\text{Algebra}} \quad \Longrightarrow \quad \tfrac{1}{\sqrt{3}}\tan(Y)+3\sin(X) &= \tfrac{1}{\sqrt{3}}\tan(240^\circ)+3(\sin(30^\circ)) && \text{(substitute)}\\
\xrightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} \quad \Longrightarrow \quad &= \tfrac{1}{\sqrt{3}}(\sqrt{3})+3\left(\tfrac{1}{2}\right) && \text{(evaluating the trigonometric functions)}\\
\xrightarrow{\text{Arithmetic}} \quad \Longrightarrow \quad &= 1+\tfrac{3}{2} && \text{(canceling like factors)}\\
\xrightarrow{\text{Arithmetic}} \quad \Longrightarrow \quad &= \tfrac{5}{2} && \text{(addition)}
\end{aligned}$$

b. This expression includes the product of two trigonometric functions.

$$\begin{aligned}
\cancel{\text{Arithmetic}}\to\text{Algebra} \implies 6\tan(X)\cos(Y) &= 6(\tan(30^\circ))(\cos(240^\circ)) && \text{(substitute)}\\
\cancel{\text{Arithmetic}}\to\cancel{\text{Algebra}}\to\text{Trigonometry} \implies &= 6\left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{2}\right) && \text{(evaluating the trigonometric functions)}\\
\text{Arithmetic} \implies &= -\frac{3}{\sqrt{3}} && \text{(simplify)}\\
\cancel{\text{Arithmetic}}\to\text{Algebra} \implies &= -\sqrt{3} && \text{(rationalizing the denominator)}
\end{aligned}$$

Checkpoint 2.2.1

a. $2\sqrt{2}$

b. $\frac{3}{2}$

Example 2.2.2

$$\begin{aligned}
\sin^2(45^\circ) &= (\sin(45^\circ))^2 && \text{(notational definition)}\\
\cancel{\text{Arithmetic}}\to\cancel{\text{Algebra}}\to\text{Trigonometry} \implies &= \left(\frac{1}{\sqrt{2}}\right)^2 && \text{(evaluating the trigonometric function)}\\
\cancel{\text{Arithmetic}}\to\text{Algebra} \implies &= \frac{1}{2} && \text{(Laws of Exponents)}
\end{aligned}$$

Checkpoint 2.2.2

9

Example 2.2.3

Think of $\cos(t)$ as a single variable, and multiply by each term inside parentheses. (The algebraic form of the calculation is shown on the right in bold).

$$\begin{aligned}
\cancel{\text{Arithmetic}}\to\text{Algebra} \implies \cos(t)(3\cos(t)-2) &= (\cos(t))(3\cos(t)) - (\cos(t)\cdot 2) && \text{(distribution: } \boldsymbol{x(3x-2) = x\cdot 3x - x\cdot 2}\text{)}\\
\cancel{\text{Arithmetic}}\to\text{Algebra} \implies &= 3\cos^2(t) - 2\cos(t) && \text{(Commutative Property of Multiplication)}
\end{aligned}$$

Notice that we write $(\cos(t))^2$ as $\cos^2(t)$.

Checkpoint 2.2.3

$8\tan^3(\beta) + 2\tan(\beta)\tan(\alpha)$

Example 2.2.4

This calculation is similar to the product $(4x-1)(3x+2)$, except that the variable x has been replaced by $\sin(C)$. Compare the calculations for the two products; first the familiar algebraic product:

$$\begin{aligned}
(4x-1)(3x+2) &= 4x\cdot 3x + 4x\cdot 2 - 1\cdot 3x - 1\cdot 2\\
&= 12x^2 + 8x - 3x - 2 = 12x^2 + 5x - 2
\end{aligned}$$

We compute the product in this example in the same way, but replacing x by $\sin(C)$.

$$\begin{aligned}
(4\sin(C)-1)(\sin(C)+2) &= (4\sin(C))(3\sin(C)) + (4\sin(C)\cdot 2) - 1(3\sin(C)) - 1\cdot 2\\
&= 12\sin^2(C) + 5\sin(C) - 2
\end{aligned}$$

Checkpoint 2.2.4

$16\cos^2(\alpha) + 24\cos(\alpha) + 9$

Example 2.2.5

a. We factor out the common factor, $3w$.

$$6w^2 - 9w = 3w(2w-3).$$

b. We factor out the common factor, $3\sin(\theta)$.

$$6\sin^2(\theta) - 9\sin(\theta) = 3\sin(\theta)(2\sin(\theta) - 3).$$

Checkpoint 2.2.5

a. $a(2a-b)$

b. $\cos(\phi)(2\cos(\phi) - \sin(\phi))$

Example 2.2.6

a. Grouping terms, we get

$$\begin{aligned}
10x^2 + 2x - 15xy - 3y &= (10x^2 + 2x) + (-15xy - 3y) && \text{(Grouping)}\\
&= 2x(5x+1) - 3y(5x+1) && \text{(Factoring out the GCF from each group)}\\
&= 2x(\mathbf{5x+1}) - 3y(\mathbf{5x+1}) && \text{(Noticing the result has two common factors)}\\
&= (5x+1)(2x-3y) && \text{(Factoring out the common factor of } (5x+1)\text{)}
\end{aligned}$$

b. The structure of this expression is almost identical to that of the previous problem. Let $x = \sin(\theta)$ and $y = \cos(\theta)$ to get

$$\begin{aligned}
10\sin^2(\theta) + 2\sin(\theta) - 15\sin(\theta)\cos(\theta) - 3\cos(\theta) &= 10x^2 + 2x - 15xy - 3y && \text{(Substitution: } x = \sin(\theta) \text{ and } y = \cos(\theta)\text{)}\\
&= (5x+1)(2x-3y) && \text{(From part (a))}\\
&= (5\sin(\theta)+1)(2\sin(\theta) - 3\cos(\theta)) && \text{("Back-substitute:" } \sin(\theta) = x \text{ and } \cos(\theta) = y\text{)}
\end{aligned}$$

Checkpoint 2.2.6

a. $(3r-5)(4t+3)$
b. $(3\sec(\alpha)-5)(4\tan(\beta)+3)$

Example 2.2.7

a. We look for numbers p and q so that $(t+p)(t+q)=t^2-3t-10$. Their product is $pq=-10$, and their sum is $p+q=-3$. By checking the factors of -10 for the correct sum, we find $p=-5$ and $q=2$. Thus,

$$t^2-3t-10=(t-5)(t+2).$$

b. Now replace t by $\tan(\alpha)$ to find

$$\tan^2(\alpha)-3\tan(\alpha)-10=(\tan(\alpha)-5)(\tan(\alpha)+2).$$

Checkpoint 2.2.7

a. $(3z+1)(z-1)$
b. $(3\sin(\beta)+1)(\sin(\beta)-1)$

Theorem: Difference of Squares

$$\begin{aligned}(F-L)(F+L) &= F^2+FL-FL-L^2 && \text{(distributing)}\\ &= F^2-L^2 && \text{(simplifying)}\end{aligned}$$

Example 2.2.8

By this point, you (hopefully) understand what we will do.

$$\begin{aligned}16\cos^4(\theta)-1 &= 16x^4-1 && \text{(Substitute: } x=\cos(\theta)\text{)}\\ &= (4x^2)^2-1^2 && \text{(Laws of Exponents)}\\ &= F^2-L^2 && \text{(Substitute: } F=4x^2 \text{ and } L=1\text{)}\\ &= (F-L)(F+L) && \text{(Difference of Squares)}\\ &= (4x^2-1)(4x^2+1) && \text{(Resubstitute: } 4x^2=F \text{ and } 1=L\text{)}\\ &= ((2x)^2-1^2)(4x^2+1) && \text{(Laws of Exponents)}\\ &= \left(\overline{F}^2-\overline{L}^2\right)(4x^2+1) && \left(\text{Substitute: } \overline{F}=2x \text{ and } \overline{L}=1\right)\\ &= \left(\overline{F}-\overline{L}\right)\left(\overline{F}+\overline{L}\right)(4x^2+1) && \text{(Difference of Squares)}\\ &= (2x-1)(2x+1)(4x^2+1) && \left(\text{Resubstitute: } 2x=\overline{F} \text{ and } 1=\overline{L}\right)\\ &= (2\cos(\theta)-1)(2\cos(\theta)+1)(4\cos^2(\theta)+1) && \text{(Resubstitute: } \cos(\theta)=x\text{)}\end{aligned}$$

Checkpoint 2.2.8

$(\sin(\alpha)-3\cos(\beta))(\sin(\alpha)+3\cos(\beta))(\sin^2(\alpha)+9\cos^2(\beta))$

Theorem: Sums and Differences of Cubes

$$\begin{aligned}(F+L)(F^2-FL+L^2) &= F^3-F^2L+FL^2+F^2L-FL^2+L^3 && \text{(distributing)}\\ &= F^3+L^3 && \text{(simplifying)}\end{aligned}$$

and

$$\begin{aligned}(F-L)(F^2+FL+L^2) &= F^3+F^2L+FL^2-F^2L-FL^2-L^3 && \text{(distributing)}\\ &= F^3-L^3 && \text{(simplifying)}\end{aligned}$$

Example 2.2.9

This time, we will speed things up a bit.

$$\begin{aligned}27\sin^3(A)-8 &= 27x^3-8 && \text{(Substitute: } x=\sin(A)\text{)}\\ &= (3x)^3-2^3 && \text{(Laws of Exponents)}\\ &= (3x-2)((3x)^2+(3x)(2)+2^2) && \text{(Difference of Cubes)}\\ &= (3x-2)(9x^2+6x+4)\\ &= (3\sin(A)-2)(9\sin^2(A)+6\sin(A)+4) && \text{(Resubstitute: } \sin(A)=x\text{)}\end{aligned}$$

Checkpoint 2.2.9

$(5\csc(\theta)+4\cot(\phi))(25\csc^2(\theta)-20\csc(\theta)\cot(\phi)+16\cot^2(\phi))$

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2.2.1: Resources and Key Concepts

Resources

Key Concepts

Definitions

- **Algebraic Expression:** A mathematical phrase that is a combination of numbers, variables, and mathematical operations, but does not contain an equals sign.
- **Evaluate (an algebraic expression):** The process of substituting a given value for the variable(s) in an algebraic expression and simplifying the result.

Theorems

- **Difference of Squares:** For any algebraic expressions F and L, the formula for factoring a difference of squares is $F^2 - L^2 = (F - L)(F + L)$.
- **Sums and Differences of Cubes:** For any algebraic expressions F and L, the formulas for factoring are:
  - Sum of Cubes: $F^3 + L^3 = (F + L)(F^2 - FL + L^2)$
  - Difference of Cubes: $F^3 - L^3 = (F - L)(F^2 + FL + L^2)$

Common Mistakes

- **Do NOT "Split" Arguments of Functions:** A function cannot be distributed across terms inside its argument. For example, $\sin(A + B)$ is not equal to $\sin(A) + \sin(B)$. The expression inside the parentheses must be evaluated first.
- **Incorrect Notation for Negative Exponents:** The shorthand notation $\cos^2(\theta)$ for $(\cos(\theta))^2$ does not apply to negative exponents. Specifically, $[\cos(\theta)]^{-1}$ is not the same as $\cos^{-1}(\theta)$, which represents the inverse cosine function.
- - **Factoring a Sum of Squares:** A sum of two squares, such as $F^2 + L^2$, is not factorable over the real numbers. Attempting to factor it as $(F + L)(F + L)$ or in any other way is incorrect.

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2.2.2: Homework

Reading Questions

1. What is the difference between an algebraic expression and an equation?
2. Explain the difference between evaluating $(\cos(30^\circ))^2$ and $\cos((30^\circ)^2)$.
3. What does the notation $\sin^2(\theta)$ mean?
4. Why is the notation $\cos^{-1}(\theta)$ not used to represent $\frac{1}{\cos(\theta)}$?
5. When multiplying expressions like $\cos(t)(3\cos(t)-2)$, what algebraic property is used?
6. What is the purpose of using a substitution, like letting $x=\sin(\theta)$, when factoring a complex trigonometric expression?
7. What is the formula for factoring a difference of squares, F^2-L^2?
8. Is an expression like $\sin^2(\theta)+9$ factorable over the real numbers? Why or why not?
9. What is the formula for factoring a sum of cubes, F^3+L^3?
10. What is the formula for factoring a difference of cubes, F^3-L^3?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, factor completely.

1. $2x^2+5x-12$
2. $3x^2-2x-8$
3. $12x^2+x-1$
4. $6x^2-13x-15$
5. $6x^2-x-12$
6. $24x^2+10x-21$
7. $9y^2-16$
8. $12m^2-75n^2$
9. $81x^4-1$
10. x^8-100y^4
11. x^3+27
12. x^6-64y^3
13. y^9-1
14. $16m^3-54y^3$

Answers available at the end of this section.

Homework

Concept Check

1. To evaluate $\cos^2(30^\circ)$, Wyatt used the keystrokes

$$\text{COS} \quad 30 \quad \wedge \quad 2 \quad \text{ENTER}$$

and got the answer -1. Were his keystrokes correct? Why or why not?

2. Make up an example to show that $\tan(\theta+\phi) \neq \tan(\theta)+\tan(\phi)$.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

3. Expressions containing trigonometric functions can be simplified or evaluated like other algebraic expressions.

4. We can *solve* expressions like $3\sin(x)+2\sin(x)$.

5. $\sin(X+Y)=\sin(X)+\sin(Y)$

6. $\cos^2(\theta)=(\cos(\theta))^2$

7. $\sin^{-1}(\theta)=\dfrac{1}{\sin(\theta)}$

8. $64x^2-9=(8x-3)(8x+3)$

9. $64x^2+9=(8x+3)(8x+3)=(8x+3)^2$

Basic Skills

For the following exercises, evaluate the expressions, using exact values for the trigonometric ratios.

10. $5\tan 135^\circ+6\cos 60^\circ$

11. $3\tan 240^\circ+8\sin 300^\circ$

12. $\sin(15^\circ+30^\circ)$

13. $\cos(2\cdot 75^\circ)$

14. $8\cos^2(30^\circ)$

15. $12\sin^2(315^\circ)$

16. $3\tan^2(150^\circ)-\sin^2(45^\circ)$

17. $1+\tan^2(120^\circ)$

For the following exercises, evaluate the expressions for $x=30^\circ, y=45^\circ$, and $z=60^\circ$. Give exact values for your answers.

18. $3\sin x+5\cos y$

19. $4\tan y+6\cos y$

20. $-2\tan 3y$

21. $\sin(3x-2x)$

22. $\cos^2 x + \sin^2 x$

23. $7\sin^2 y + 7\cos^2 y$

24. $\cos x \cos x - \sin x \sin z$

25. $\tan(180^\circ - x)\tan x$

For the following exercises, combine like terms.

26. a. $3x^2 - x - 5x^2$

    b. $3\cos^2\theta - \cos\theta - 5\cos^2\theta$

27. a. $4x + 5y - y$

    b. $4\cos\theta + 5\sin\theta - \sin\theta$

28. a. $-3SC + 7SC$

    b. $-3\sin\theta\cos\theta + 7\sin\theta\cos\theta$

29. a. $4SC + 11SC - 17SC$

    b. $4\sin\theta\cos\theta + 11\sin\theta\cos\theta - 17\sin\theta\cos\theta$

30. a. $-C^2S^3 + 6C^2S^3$

    b. $-\cos^2\theta\sin^3\theta + 6\cos^2\theta\sin^3\theta$

31. a. $7C^2S^2 - (2CS)^2$

    b. $7\cos^2\theta\sin^2\theta - (2\cos\theta\sin\theta)^2$

For the following exercises, simplify the expression, then evaluate.

32. $\cos t + 2\cos t\sin t - 3\cos t$, for $t = 30^\circ$

33. $11\tan w - 4\tan w + 6\tan w\cos w$, for $w = 60^\circ$

34. $7\tan\theta - 4\tan\phi + 3\tan\phi - 6\tan\theta$, for $\theta = 45^\circ, \phi = 120^\circ$

35. $5\sin A + \sin B - 6\sin B + 6\sin A$, for $A = 210^\circ, B = 135^\circ$

36. $-\sin x\cos x - \sin(2x) + 3\sin x\cos x$, for $x = 45^\circ$

37. $4\cos u\sin u + 3\sin 2u - 10\sin u\cos u$, for $u = 270^\circ$

For the following exercises, multiply or expand.

38. a. $x(2x - 1)$

    b. $\sin A(2\sin A - 1)$

39. a. $q(5r - q)$

    b. $\cos\theta(5\sin\theta - \cos\theta)$

40. a. $a(b - 3a)$

    b. $\tan A(\tan B - 3\tan A)$

41. a. $3w(2w - z)$

    b. $3\sin\alpha(2\sin\alpha - \sin\beta)$

42. a. $(C + 1)(2C - 1)$

    b. $(\cos\phi + 1)(2\cos\phi - 1)$

43. a. $(3S - 2)(S + 1)$

    b. $(3\sin B - 2)(\sin B + 1)$

44. a. $(a + b)(a - b)$

    b. $(\cos\theta + \cos\phi)(\cos\theta - \cos\phi)$

45. a. $(t - w)(t + 4w)$

    b. $(\tan\alpha - \tan\beta)(\tan\alpha + 4\tan\beta)$

46. a. $(1 - T)^2$

    b. $(1 - \tan\theta)^2$

47. a. $(2 + 3S)^2$

    b. $(2 + 3\sin\theta)^2$

48. a. $\left(T^2 + 2\right)\left(T^2 - 2\right)$

    b. $\left(\tan^2\theta + 2\right)\left(\tan^2\theta - 2\right)$

49. a. $\left(2c^2 - 3\right)\left(2c^2 + 3\right)$

    b. $\left(2\cos^2\phi - 3\right)\left(2\cos^2\phi + 3\right)$

For the following exercises, decide whether or not the expressions are equivalent. Justify your response by citing a theorem or providing a counterexample.

50. $\cos(x + y), \quad \cos x + \cos y$

51. $\sin(\theta - \phi), \quad \sin\theta - \sin\phi$

52. $\tan(2A), \quad 2\tan A$

53. $\cos\left(\frac{1}{2}\beta\right), \quad \frac{1}{2}\cos\beta$

54. $(\sin\alpha)^2, \quad \sin^2\alpha$

55. $(\tan B)^2, \quad \tan\left(B^2\right)$

56. $\sin 3t + \sin 5t, \quad \sin 8t$

57. $3\cos 2x + 5\cos 2x, \quad 8\cos 2x$

58. $\tan(\theta + 45^\circ) - \tan\theta, \quad \tan 45^\circ$

59. $\sin(30^\circ + z) + \sin(30^\circ - z), \quad \sin 60^\circ$

60. Factor each expression, if possible.

a. $x^2 - 4x$

b. $x^2 - 4$

c. $x^2 + 4$

d. $x^2 + 4x + 4$

e. $x^2 - 4x + 4$

f. $x^2 + 4x$

g. $x^2 - 4x - 4$

h. $-x^2 + 4$

For the following exercises, factor.

61. a. $9m + 15n$

b. $9\cos\alpha + 15\cos\beta$

62. a. $12p - 20q$

b. $12\sin\theta - 20\sin\phi$

63. a. $5r^2 - 10qr$

b. $5\tan^2 C - 10\tan B\tan C$

64. a. $2x^2 - 6xy$

b. $2\sin^2 A - 6\sin A\cos A$

65. a. $9C^2 - 1$

b. $9\cos^2\beta - 1$

66. a. $25S^2 - 16S$

b. $25\sin^2\beta - 16\sin\beta$

67. a. $6T^3 - 8T^2$

b. $6\tan^3 A - 8\tan^2 A$

68. a. $9T^2 - 15T^3$

b. $9\tan^2 B - 15\tan^3 B$

69. a. $t^2 - t - 20$

b. $\tan^2\theta - \tan\theta - 20$

70. a. $s^2 + 5s - 6$

b. $\sin^2 A + 5\sin A - 6$

71. a. $3c^2 + 2c - 1$

b. $3\cos^2 B + 2\cos B - 1$

72. a. $8S^2 - 6S + 1$

    b. $8\sin^2\phi - 6\sin\phi + 1$

73. a. $6S^2 - 5S - 1$

    b. $6\sin^2\alpha - 5\sin\alpha - 1$

74. a. $T^2 - 4T - 12$

    b. $\tan^2\alpha - 4\tan\alpha - 12$

75. a. $9C^2 - 1$

    b. $9\cos^2(\theta) - 1$

76. a. $16 - 9S^2$

    b. $16 - 9\sin^2(\alpha)$

77. a. $81T^2 - 16C^2$

    b. $81\tan^2(A) - 16\cos^2(B)$

78. a. $81T^4 - 16C^4$

    b. $81\tan^4(A) - 16\cos^4(B)$

79. a. $\frac{1}{9}S^2 - \frac{4}{25}C^2$

    b. $\frac{1}{9}\sin^2(\phi) - \frac{4}{25}\cos^2(\lambda)$

80. a. $27C^3 - 1$

    b. $27\cos^3(\theta) - 1$

81. a. $64 + 27S^3$

    b. $64 + 27\sin^3(\alpha)$

82. a. $729T^3 + 64C^3$

    b. $729\tan^3(A) + 64\cos^3(B)$

83. a. $729T^6 - 64C^6$

    b. $729\tan^6(A) - 64\cos^6(B)$

84. a. $\frac{1}{27}S^3 - \frac{8}{125}C^3$

    b. $\frac{1}{27}\sin^3(\phi) - \frac{8}{125}\cos^3(\lambda)$

Answers

Skills Refresher

1. $(2x - 3)(x + 4)$
2. $(3x + 4)(x - 2)$
3. $(4x - 1)(3x + 1)$

4. $(6x+5)(x-3)$
5. $(2x-3)(3x+4)$
6. $(6x+7)(4x-3)$
7. $(3y-4)\,(3y+4)$
8. $3\,(2m-5n)\,(2m+5n)$
9. $(3x-1)\,(3x+1)\,(9x^2+1)$
10. $(x^4-10y^2)\,(x^4+10y^2)$
11. $(x+3)\,(x^2-3x+9)$
12. $(x^2-4y)\,(x^4+4x^2y+16y^2)$
13. $(y-1)\,(y^2+y+1)\,(y^6+y^3+1)$
14. $2\,(2m-3y)\,(4m^2+6my+9y^2)$

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2.3: Introduction to Trigonometric Identities

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

What is an Identity?

Recall that an equation may be true or false, depending on the values of any variables involved. For example, the equation $x^2 + 3x = 10$ is true only if $x = 2$ or $x = -5$. You might have forgotten a bit of language from the past, so let's quickly redefine **conditional equations** and **identities**.

Definition: Conditional Equation

An equation that is true only for certain variable values (and, therefore, is false for others) is called a **conditional equation**.

When you solve a conditional equation, you find the variable values that make the equation true.

Definition: Identity

An equation that is true for all allowable variable values is called an **identity**.

Here are some examples of identities:

$$\begin{aligned} 3(x+y) &= 3x+3y \\ (x+1)^2 &= x^2+2x+1 \end{aligned}$$

In an identity, the statements on either side of the equal sign are called **equivalent expressions**.

Definition: Equivalent Expressions

Two algebraic expressions are called **equivalent** if they have the same value for all allowable variable values.

Example 2.3.1

Which of the following equations are identities?

a. $3s + 7s = 10s$
b. $5c(c - 2s) = 5c^2 - 10cs$

c. $2t - 1 = 3$

Solutions available at the end of this section.

Checkpoint 2.3.1

Which of the following equations are identities?

a. $(c - s)(c + s) = c^2 - s^2$
b. $3t^2 = 1$
c. $(2c + 1) + (s - 3) = 2c + s - 2$

Answers available at the end of this section.

Of course, you wouldn't be asked to *solve* an identity because all values of the variable are solutions. Instead, we use identities to replace one form of an expression with a more useful form. You do this when you solve a quadratic equation by factoring. For example,

$$\begin{aligned} 2x^2 - x - 1 &= 0 \\ \implies (2x + 1)(x - 1) &= 0 \qquad \text{(factoring)} \end{aligned}$$

Because $(2x + 1)(x - 1)$ is equivalent to $2x^2 - x - 1$, we have not changed the equation or its solutions; however, we now can apply the Zero Factor Principle and solve the equation.

Using identities is especially useful when we are working with trigonometric ratios.

Using Trigonometric Ratios in Identities

Because the identity $2x^2 - x - 1 = (2x + 1)(x - 1)$ is true for any value of x, it is true when x is replaced, for instance, by $\cos(\theta)$. This gives us a new identity:

$$2\cos^2(\theta) - \cos(\theta) - 1 = (2\cos(\theta) + 1)(\cos(\theta) - 1).$$

Expressions involving the trigonometric functions can be manipulated by the same rules that we use with simple variables (such as the Distributive Law or the Laws of Exponents).

Example 2.3.2

Which of the following equations are identities?

a. $3\sin(\theta) + 7\sin(\theta) = 10\sin(\theta)$
b. $5\cos(\theta)(\cos(\theta) - 2\sin(\theta)) = 5\cos^2(\theta) - 10\cos(\theta)\sin(\theta)$
c. $2\tan(\theta) - 1 = 3$

Solutions available at the end of this section.

Checkpoint 2.3.2

Which of the following equations are identities?

a. $(\cos(\theta) - \sin(\theta))(\cos(\theta) + \sin(\theta)) = \cos^2(\theta) - \sin^2(\theta)$
b. $3\tan^2(\theta) = 1$
c. $(2\cos(\theta) + 1) + (\sin(\theta) - 3) = 2\cos(\theta) + \sin(\theta) - 2$

Answers available at the end of this section.

Caution: Examples are Not Proofs

To show that an equation is *not* an identity, we need only find one value of the variable for which the equation is false (this would create what we call a **contradiction**). However, to show that a given equation *is* an identity, it is not enough to show that the equation is valid for one, or even a dozen, values of the variable. The equation must be true for *all* legitimate values of the variable. Because we cannot check all values of the variable numerically, we must use algebraic methods to *prove* that an equation is an identity.

Ratio Identities

All of the trigonometric functions are related. We begin by considering the relationship between the tangent and the sine and cosine functions (we will find a similar relationship for the cotangent). Complete the following table with exact values.

θ	$\cos(\theta)$	$\sin(\theta)$	$\frac{\sin(\theta)}{\cos(\theta)}$	$\tan(\theta)$
0°				
30°				
45°				
60°				

For the angles in the table, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$. The question you should have is, "Is this true for all angles?" The short answer is no, but before we describe why, let's understand why this relationship exists between the tangent, sine, and cosine.

The sine function for an angle θ is defined as $\sin(\theta) = \frac{y}{r}$, where (x, y) is any point on the terminal side of θ (other than the origin) and $r = \sqrt{x^2 + y^2}$. Likewise, the cosine is defined to be $\cos(\theta) = \frac{x}{r}$. If we look at the ratio of the sine to the cosine, we get

$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{y/r}{x/r} = \frac{(y/r)}{(x/r)} \cdot \frac{r}{r} = \frac{(y/\cancel{r})}{(x/\cancel{r})} \cdot \frac{\cancel{r}}{\cancel{r}} = \frac{y}{x} = \tan(\theta).$$

A similar argument shows that $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$.

One essential item to note, however, is that because $\cos(90^\circ) = 0$, the quotient $\frac{\sin(\theta)}{\cos(\theta)}$ is undefined for $\theta = 90^\circ$, and, as such, $\tan(90^\circ)$ is undefined. The same applies for $\theta = 270^\circ$. For all other angles between 0° and $360^\circ, \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$. A similar argument shows that $\cot(0^\circ)$ and $\cot(180^\circ)$ are undefined.

We are now ready to formalize these statements as a theorem.

Theorem: Ratio Identities

For any angle not coterminal with 90° or 270°,

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$

For any angle not coterminal with 0° or 180°,

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}.$$

Advice: Memorize These Identities

The Ratio Identities are needed throughout the rest of Mathematics, Science, and Engineering. The need to commit them to memory and their importance cannot be overstated.

A cute little problem:

$$\frac{\sin(\text{gerine})}{\cos(\text{gerine})} = ???$$

Example 2.3.3

If $\sin(\theta) = \frac{12}{13}$ and $\cos(\theta) = -\frac{5}{13}$, find the values of $\tan(\theta)$ and $\cot(\theta)$.

Solution available at the end of this section.

Checkpoint 2.3.3

If $\sin(\theta) = -\frac{1}{5}$ and $\cos(\theta) = -\frac{2\sqrt{6}}{5}$, find the values of $\tan(\theta)$ and $\cot(\theta)$.

Answer available at the end of this section.

Reciprocal Identities

We now state one of the more obvious identities. We find the following relationships by comparing the definitions of cosecant, secant, and cotangent to the three fundamental trigonometric functions (sine, cosine, and tangent).

Theorem: Reciprocal Identities

$$\sin(\theta) = \frac{1}{\csc(\theta)} \quad (\text{provided } \csc(\theta) \neq 0) \qquad \csc(\theta) = \frac{1}{\sin(\theta)} \quad (\text{provided } \sin(\theta) \neq 0)$$

$$\cos(\theta) = \frac{1}{\sec(\theta)} \quad (\text{provided } \sec(\theta) \neq 0) \qquad \sec(\theta) = \frac{1}{\cos(\theta)} \quad (\text{provided } \cos(\theta) \neq 0)$$

$$\tan(\theta) = \frac{1}{\cot(\theta)} \quad (\text{provided } \cot(\theta) \neq 0) \qquad \cot(\theta) = \frac{1}{\tan(\theta)} \quad (\text{provided } \tan(\theta) \neq 0)$$

You likely already figured out the Reciprocal Identities because, for example, $\sin(\theta) = \frac{y}{r}$ and $\csc(\theta) = \frac{r}{y}$ are obvious reciprocals.

Advice: Memorize These Identities

The Reciprocal Identities are needed throughout the rest of Mathematics, Science, and Engineering. The need to commit them to memory and their importance cannot be overstated.

Example 2.3.4

Compute $\cos(\theta)$ given $\sec(\theta) = -\frac{10}{3}$.

Solution available at the end of this section.

Checkpoint 2.3.4

If $\cot(\theta) = \frac{8}{5}$, find $\tan(\theta)$.

Answer available at the end of this section.

Caution: Cosecant, Secant, and Cotangent are NOT Inverses

Refrain from calling the cosecant, secant, and cotangent inverse trigonometric ratios. The word "inverse" is very special in Mathematics. While it is true that $\sin(\theta)$ and $\csc(\theta)$ are **multiplicative inverses**, they are not *functional* inverses.

Pythagorean Identities

All of the trigonometric functions are related. We begin by considering the relationship between $\sin(\theta)$ and $\cos(\theta)$. Complete the following table with exact values.

θ	$\cos(\theta)$	$\sin(\theta)$	$\cos^2(\theta)$	$\sin^2(\theta)$	$\cos^2(\theta)+\sin^2(\theta)$
0°					
30°					
45°					
60°					
90°					

For all the angles in the table, $\cos^2(\theta)+\sin^2(\theta)=1$. *This observation leads to the most important result in all of Trigonometry.*

Theorem: The Pythagorean Identity

For any angle θ,

$$\cos^2(\theta)+\sin^2(\theta)=1.$$

Proof available at the end of this section.

Interactive Element: The Pythagorean Identity on the Unit Circle

The following Interactive Element might help you visualize why $\cos^2(\theta)+\sin^2(\theta)=1$ holds for *every* angle, not just the ones in the table above.

Interact: Drag the orange point around the circle to change the angle θ.

Observation: The green leg is $\cos(\theta)$ and the red leg is $\sin(\theta)$; the hypotenuse is the radius, which is always 1. Since (leg)^2 +(leg)^2 =(hypotenuse)^2, we get $\cos^2(\theta)+\sin^2(\theta)=1$ at every position — even when $\cos(\theta)$ or $\sin(\theta)$ is negative.

From its name, the Pythagorean Identity is intimately related to the Pythagorean Theorem.

Advice: Memorize This Identity

The Pythagorean Identity (also known as the Fundamental Pythagorean Identity) is needed throughout the rest of Mathematics. The need to commit this to memory and its importance cannot be overstated.

Example 2.3.5

Evaluate $\cos^2(27^\circ) + \sin^2(27^\circ)$ without using a calculator.

Solution available at the end of this section.

Checkpoint 2.3.5

Evaluate $5\left[\cos^2(12^\circ) + \sin^2(12^\circ)\right]$ without using a calculator.

Answer available at the end of this section.

When we solve more complicated trigonometric equations in later chapters, we will need to simplify trigonometric expressions so that they involve only one of the trigonometric functions. The Pythagorean Identity is useful when we wish to write an equivalent expression for either $\cos^2(\theta)$ or for $\sin^2(\theta)$. The following equivalent forms to the Pythagorean Identity will be helpful throughout Trigonometry.

Corollary: Alternate Forms of the Pythagorean Identity

$$\sin^2(\theta) = 1 - \cos^2(\theta) \implies \sin(\theta) = \pm\sqrt{1 - \cos^2(\theta)}$$

$$\cos^2(\theta) = 1 - \sin^2(\theta) \implies \cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$$

The Pythagorean Identity, along with the Ratio and Reciprocal Identities, show us that all trigonometric functions of an angle are related. If we know the value of one trigonometric function at θ and where θ terminates, we can calculate the remaining trigonometric functions. We do *not* need to find the angle itself to do this. We only need to know which quadrant the angle lies in to determine the correct sign for the trigonometric ratios.

Advice: Memorize These Identities

The alternate forms of the Pythagorean Identity are needed throughout the rest of Mathematics, Science, and Engineering. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated.

Example 2.3.6 shows that we do not need to memorize the alternate forms of the Pythagorean Identity as long as we are confident manipulating mathematical expressions.

Example 2.3.6

If $\sin(\theta) = \frac{7}{25}$ and $90^\circ < \theta < 180^\circ$, find $\cos(\theta)$ and $\tan(\theta)$.

Solution available at the end of this section.

We could have solved Example 2.3.6 by creating a quick sketch of θ terminating in QII. Because $\sin(\theta) = \frac{7}{25}$, we can choose a point on the terminal side to have y-coordinate 7 and $r = 25$ (see Figure 2.3.1 below).

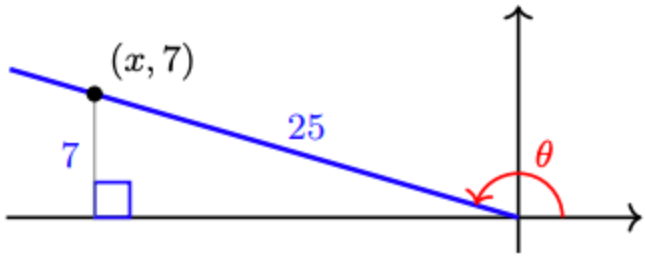


Figure 2.3.1

In order to calculate $\cos(\theta)$ and $\tan(\theta)$, we must find the x-coordinate of the point. By the Pythagorean Theorem,

$$\begin{aligned} x^2 + 7^2 &= 25^2 \\ \Longrightarrow \quad x^2 &= 576 && \text{(isolating } x^2\text{)} \\ \Longrightarrow \quad x &= -24 && \text{(taking the square root of both sides)} \end{aligned}$$

Note that x is negative because the point $(-24, 7)$ is in the second quadrant. Then

$$\cos(\theta) = \frac{x}{r} = -\frac{24}{25} \quad \text{and} \quad \tan(\theta) = \frac{y}{x} = \frac{7}{-24} = -\frac{7}{24}.$$

Checkpoint 2.3.6

a. If $\cos(\theta) = \frac{5}{13}$ and $-90^\circ < \theta < 0^\circ$, find $\sin(\theta)$ and $\tan(\theta)$.
b. Find $\sin(\theta)$ and $\tan(\theta)$ by using an appropriate sketch.

Answers available at the end of this section.

You might have noticed that this subsection is "Pythagorean Identities" (plural). Yet, we only named a single Pythagorean Identity (purposefully titled **The** Pythagorean Identity). We now introduce two additional Pythagorean Identities that will be used frequently throughout Trigonometry.

$$\begin{aligned} \cos^2(\theta) + \sin^2(\theta) &= 1 && \text{(the Pythagorean Identity)} \\ \Longrightarrow \quad \frac{\cos^2(\theta)}{\cos^2(\theta)} + \frac{\sin^2(\theta)}{\cos^2(\theta)} &= \frac{1}{\cos^2(\theta)} && \text{(dividing both sides by } \cos^2(\theta)\text{)} \\ \Longrightarrow \quad 1 + \left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2 &= \left(\frac{1}{\cos(\theta)}\right)^2 && \text{(cancelling like factors and Laws of Exponents)} \\ \Longrightarrow \quad 1 + (\tan(\theta))^2 &= (\sec(\theta))^2 && \text{(Ratio and Reciprocal Identities)} \\ \Longrightarrow \quad 1 + \tan^2(\theta) &= \sec^2(\theta) && \text{(Laws of Exponents)} \end{aligned}$$

A similar process shows

$$1 + \cot^2(\theta) = \csc^2(\theta).$$

We formalize these statements in a summary theorem.

Theorem: Pythagorean Identities

For any angle θ where the functions are defined,

$$\begin{aligned} \cos^2(\theta) + \sin^2(\theta) &= 1 \\ 1 + \tan^2(\theta) &= \sec^2(\theta) \\ \cot^2(\theta) + 1 &= \csc^2(\theta) \end{aligned}$$

Advice: Memorize These Identities

Each of the Pythagorean Identities are needed throughout the rest of Mathematics. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated.

Success in College: Study Time

Believe it or not, there is a direct link between the number of units a class is worth and how many hours you should be studying (outside of class). For each hour of class time, the California state education code suggests at least two hours of studying outside of class. If your math background is weak, you may need to spend more time studying.

At Cosumnes River College, each unit of a math course corresponds to one hour of in-class time per week. Thus, a five-unit math course requires five hours of in-class interaction per week and *ten* hours of study time outside of class per week.

One way to study is to do what you're doing now - read! Class time is an excellent opportunity to introduce new concepts and see how they fit with those previously learned; however, there is usually not enough time to address details with the depth that a textbook can. In this way, your professor and the textbook should be considered as a combined primary source of information in your course.

Here are some questions to consider regarding how you study.

1. Do you study at least two hours for every one hour of class time? If not, why?
2. How many hours do you dedicate to TV, YouTube, and social media every week (on average)?
3. How many hours do you dedicate to leisure activities per week (e.g., hanging out with friends, going to the movies, gaming)?
4. Would it be too difficult to trade a few hours for study time?

Success in College: I Understood It... Until I Left Class

You understand a concept when the professor does it in class, but you're lost when trying it on your own - this is very common. There are two general remedies to this problem.

The first revolves around how the information is delivered and how you engage with it. We tend to retain the least when only *listening* (maybe this is why those who passively sit in the back of a classroom and zone out end up having the highest failure rate). We gain a little more retention when we *see* someone do something (e.g., watching your professor, but not taking notes). A substantial leap in retention and understanding occurs when we *write* concepts down (e.g., taking notes during lectures). Finally, the biggest gains in understanding and retention occur when we *do* work on our own (e.g., practicing, doing homework, independently investigating a concept).

This kind of makes sense, right? This is why you see the most successful students in class attempting the examples before the professor starts them. It forces the information into their memory.

It gets even better if we involve "real-world" examples with manipulatives. Imagine learning about mathematical models that deal with finance (exponential and simple interest formulas) by *actually* investing. How cool would it be to build a fence that borders a river by using minimization techniques learned in Algebra? The problem with math and how it is taught is that we hear about the applications but nobody wants to go out and do them. If you had more physical or engaging examples, your chance of success would skyrocket. Unfortunately, there is only a limited amount of time in class, and this kind of teaching would require students to allot several more hours a day to their math class (however, the benefit would be enormous).

The second remedy is squarely on your shoulders. You will never master a skill by putting only half of your effort into it. To excel and enjoy mathematics, you must immerse yourself in it. Imagine yourself as a mathematician and look at the world differently. As you drive your car to work, make a mental graph of the distance you have traveled versus the time you have been driving (if you're in Calculus, then interpret the slope of the tangent line to this graph at any given point as the derivative). When you purchase items, do the math in your head... yes, even the tax! When you buy a car, make a function that describes the cost of owning the car! You'd be surprised at how helpful mathematics is.

In college, class time is limited. Often, class is over before students feel confident about new concepts. They leave class worried they won't be able to do the homework. These are just some suggestions, but others in your class might have some wonderful additions to this list. Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. How often does this happen to you (understanding in class but completely lost out of class)? When it does, what do you do about it?
2. How often do you think this happens to the other class members?

Answers

Section Preview

The purpose of this section is to *introduce* the student to identities and contradictions. We officially introduce the Reciprocal, Ratio, and Pythagorean Identities to allow the student a different way to *compute* the value of a trigonometric function given the value of a different trigonometric function at the same angle—***proofs are not done by the student here***!

Section Preview

- **Simplifying Expressions**
  - Multiplying and Dividing Rational Expressions: Used when simplifying the compound fraction in the Ratio Identity derivation, as well as in the homework.
  - Adding and Subtracting Rational Expressions: This skill is needed in the homework.
  - Simplifying Compound Rational Expressions: The derivation of the Ratio Identities involves simplifying the compound fraction $\frac{y/r}{x/r}$; a student must understand compound fractions to follow this.
- **Solving Equations**
  - Solving Literal Equations: The concept of an identity versus a conditional equation requires understanding what it means for an equation to be "true for all values"—while not an obvious placement, this appears in the Solving Literal Equations material.

Section Preview

- Determine if an algebraic equation represents an identity.
- Find the value of a trigonometric function using the Reciprocal, Ratio, and Pythagorean Identities.

Example $2.3.1$

Many of the algebraic operations you have already learned, such as combining like terms or applying the Distributive Law, produce equivalent expressions.

a. Equation (a) is an identity obtained by combining like terms on the left side.
b. Equation (b) is an identity obtained by applying the Distributive Law on the left side.
c. Equation (c) is not an identity because the equation is valid only for $t = 2$.

Checkpoint $2.3.1$

(a) and (c)

Example $2.3.2$

These equations are obtained from the equations in Example $2.3.1$ by replacing the variable s with $\sin(\theta)$, the variable c with $\cos(\theta)$, and the variable t with $\tan(\theta)$. Equations (a) and (b) are identities, and equation (c) is not, for the same reasons as before.

Checkpoint $2.3.2$

(a) and (c)

Example $2.3.3$

We apply the Ratio Identities to obtain

$$\begin{aligned} \tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} \\ &= \frac{12/13}{-5/13} \\ &= \frac{12/13}{-5/13} \cdot \frac{13}{13} \\ &= \frac{12}{-5} \\ &= -\frac{12}{5} \end{aligned}$$

and

$$\begin{aligned}\cot(\theta) &= \frac{\cos(\theta)}{\sin(\theta)}\\ &= \frac{-5/13}{12/13}\\ &= \frac{-5/13}{12/13}\cdot\frac{13}{13}\\ &= \frac{-5}{12}\\ &= -\frac{5}{12}\end{aligned}$$

Checkpoint 2.3.3

$\tan(\theta) = \frac{1}{2\sqrt{6}}$ and $\cot(\theta) = 2\sqrt{6}$

Example 2.3.4

$$\cos(\theta) = \frac{1}{\sec(\theta)} = \frac{1}{-10/3} = -\frac{3}{10}$$

Checkpoint 2.3.4

$\tan(\theta) = \frac{5}{8}$

Theorem: The Pythagorean Identity

From the coordinate definition of the trigonometric functions, we know that $r = \sqrt{x^2+y^2}$. Thus,

$$\begin{aligned} & r = \sqrt{x^2+y^2} & \\ \Longrightarrow\quad & r^2 = x^2+y^2 & \text{(squaring both sides)}\\ \Longrightarrow\quad & 1 = \frac{x^2}{r^2}+\frac{y^2}{r^2} & \text{(dividing both sides by } r^2)\\ \Longrightarrow\quad & 1 = \left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2 & \text{(Laws of Exponents)}\\ \Longrightarrow\quad & 1 = (\cos(\theta))^2+(\sin(\theta))^2 & \text{(definitions of the cosine and sine functions)}\\ \Longrightarrow\quad & 1 = \cos^2(\theta)+\sin^2(\theta) & \text{(using a better notation)}\end{aligned}$$

Example 2.3.5

Because the Pythagorean Identity, $\cos^2(\theta)+\sin^2(\theta) = 1$, is true for any value of θ, it is true in particular for $\theta = 27^\circ$. So $\cos^2(27^\circ)+\sin^2(27^\circ) = 1$.

Checkpoint 2.3.5

5

Example 2.3.6

We substitute $\sin(\theta) = \frac{7}{25}$ into the Pythagorean Identity $\cos^2(\theta)+\sin^2(\theta) = 1$ to get

$$\begin{aligned} \cos^2(\theta)+\left(\frac{7}{25}\right)^2 &= 1 && \text{(substituting)}\\ \Longrightarrow\quad \cos^2(\theta) &= 1-\left(\frac{7}{25}\right)^2 && \text{(isolating } \cos^2(\theta))\\ \Longrightarrow\quad \cos^2(\theta) &= 1-\frac{49}{625} &&\\ \Longrightarrow\quad \cos^2(\theta) &= \frac{576}{625} &&\\ \Longrightarrow\quad \cos(\theta) &= \pm\sqrt{\frac{576}{625}} && \text{(taking the square root of both sides)}\\ \Longrightarrow\quad \cos(\theta) &= \pm\frac{24}{25} &&\end{aligned}$$

Because θ is between 90° and 180°, $\cos(\theta)$ must be negative. Therefore, $\cos(\theta) = -\frac{24}{25}$. To find $\tan(\theta)$, we use the Ratio Identities to get

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{7/25}{-24/25} = -\frac{7}{24}.$$

Checkpoint 2.3.6

$\sin(\theta) = -\frac{12}{13}$, $\quad \tan(\theta) = -\frac{12}{5}$

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2.3.1: Resources and Key Concepts

Resources

Videos

- Pythagorean Identities
  - Fundamental Identities - The Pythagorean Identities: Theorem, Proof, and Examples

Key Concepts

Definitions

- **Conditional Equation:** An equation that is true for only a specific set of variable values and is false for others.
- **Identity:** An equation that holds true for all permissible values of the variables involved.
- **Equivalent Expressions:** Two algebraic expressions that produce the same value for all allowable inputs. They form the two sides of an identity.

Theorems

- **Ratio Identities:**
  - For any angle θ not coterminal with 90° or 270°, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.
  - For any angle θ not coterminal with 0° or 180°, $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$.
- **Reciprocal Identities:**
  - $\sin(\theta) = \frac{1}{\csc(\theta)}$ and $\csc(\theta) = \frac{1}{\sin(\theta)}$
  - $\cos(\theta) = \frac{1}{\sec(\theta)}$ and $\sec(\theta) = \frac{1}{\cos(\theta)}$
  - $\tan(\theta) = \frac{1}{\cot(\theta)}$ and $\cot(\theta) = \frac{1}{\tan(\theta)}$
- **The Pythagorean Identities:** For any angle θ where the functions are defined:
  - $\cos^2(\theta) + \sin^2(\theta) = 1$
  - $1 + \tan^2(\theta) = \sec^2(\theta)$
  - $\cot^2(\theta) + 1 = \csc^2(\theta)$
- **Alternate Forms of the Pythagorean Identity:**
  - $\sin^2(\theta) = 1 - \cos^2(\theta) \implies \sin(\theta) = \pm\sqrt{1 - \cos^2(\theta)}$
  - $\cos^2(\theta) = 1 - \sin^2(\theta) \implies \cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$

Common Mistakes

- **Using Examples as Proofs:** To prove an equation is an identity, you must use algebraic methods. Showing that an equation works for a few numerical examples is not sufficient, as it does not prove it is true for all possible values.
- **Incorrectly Taking Square Roots:** Taking the square root of both sides of an equation like $\cos^2(\theta) + \sin^2(\theta) = 1$ does not result in $\cos(\theta) + \sin(\theta) = 1$. The square root operation does not distribute over addition.
- **Confusing Reciprocals with Inverses:** The cosecant, secant, and cotangent functions are the multiplicative reciprocals of the sine, cosine, and tangent functions, respectively. They are not functional inverses. The term "inverse" has a specific meaning for functions that will be discussed later.

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2.3.2: Homework

Reading Questions

1. What is the difference between a conditional equation and an identity?
2. To prove that an equation is an identity, is it enough to show that it is true for several different values of the variable? Why or why not?
3. State the Ratio Identity for $\tan(\theta)$.
4. For which angles is the identity for $\tan(\theta)$ not defined?
5. State the three Reciprocal Identities that relate the six trigonometric functions.
6. What is the Fundamental Pythagorean Identity?
7. If you know that $\sin(\theta) = \frac{7}{25}$, why is it also necessary to know the quadrant of θ to find the exact value of $\cos(\theta)$?
8. What are the two Pythagorean Identities that involve the tangent, cotangent, secant, and cosecant functions?
9. Are $\csc(\theta)$ and $\sin(\theta)$ multiplicative inverses or functional inverses?
10. How can you use the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ to derive the identity involving $\tan^2(\theta)$ and $\sec^2(\theta)$?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, write the expression as a single fraction in simplest form.

1. $\frac{1}{x} + \frac{1}{y}$
2. $1 - \frac{1}{x}$
3. $\frac{1}{x-1} + \frac{1}{x+1}$
4. $x - \frac{x}{x+1}$
5. $\frac{\frac{y}{x} - \frac{x}{y}}{\frac{y}{x} + 1}$
6. $\frac{1 - \frac{x^2}{y^2}}{1 + \frac{x^2}{y^2}}$
7. $\frac{\frac{2a}{x}}{1 - \frac{a^2}{x^2}}$
8. $\frac{\frac{a}{x} + \frac{b}{y}}{1 - \frac{ab}{xy}}$

Answers available at the end of this section.

Homework

Vocabulary Check

1. A ___ equation is true only for certain variable values and false for others.
2. An equation that is true for all legitimate values of the variables is called a(n) ___.
3. The expressions on either side of the equal sign in an identity are called ___ because they have the same value for all values of the variable.

Concept Check

4. What is the difference between a conditional equation and an identity? Give an example of each.
5. What happens when you try to "solve" an identity?
6. Nhat claims that since $\cos^2(\theta) + \sin^2(\theta) = 1$, we can take the square root of both sides to get the simpler form $\cos(\theta) + \sin(\theta) = 1$. Is she correct? Why or why not?
7. If $\frac{a}{b} = \frac{2}{5}$, is it necessarily true that $a = 2$ and $b = 5$? Explain.
8. If $\tan(\theta) = \frac{3}{11}$, is it true that $\sin(\theta) = 3$ and $\cos(\theta) = 11$? Explain.
9. Explain how you would arrive at the identity
$$1 + \cot^2(\theta) = \csc^2(\theta)$$
given that you know $\cos^2(\theta) + \sin^2(\theta) = 1$.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

10. The reciprocal of the secant is the cosecant.
11. For any angle θ,
$$\cos^2(\theta) + \sin^2(\theta) = 1.$$
12. For any angle θ,
$$\cos(\theta) + \sin(\theta) = 1.$$
13. For any angle θ,
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$
14. To show that an equation is an identity, we just need to find one value of the unknown that makes the statement true.

Basic Skills

For the following exercises, decide which of the following equations are identities. Explain your reasoning.

15. $(\sqrt{a} + \sqrt{b})^2 = a + b$
16. $\sqrt{a^2 - b^2} = a - b$
17. $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$

18. $\dfrac{a+b}{a} = b$

19. $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

20. $\dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

21. $(1+\tan\theta)^2 = 1+\tan^2\theta$

22. $\sqrt{1-\sin^2\phi} = 1-\sin\phi$

For the following exercises, evaluate (without using a calculator).

23. $3\cos^2 1.7^\circ + 3\sin^2 1.7^\circ$

24. $4-\cos^2 338^\circ - \sin^2 338^\circ$

25. $\left(\cos^2 20^\circ + \sin^2 20^\circ\right)^4$

26. $\dfrac{18}{\cos^2 17^\circ + \sin^2 17^\circ}$

27. $\dfrac{6}{\cos^2 53^\circ} - 6\tan^2 53^\circ$

28. $\dfrac{1}{\sin^2 102^\circ} - \dfrac{\cos^2 102^\circ}{\sin^2 102^\circ}$

For the following exercises, find exact values for the other trigonometric functions of the given angle.

29. $\cos A = \dfrac{12}{13}$ and \(270^{\circ}

30. $\sin B = \dfrac{-3}{5}$ and \(180^{\circ}

31. $\sin\phi = \dfrac{1}{7}$ and $90^\circ < \phi < 180^\circ$

32. $\cos t = \dfrac{-2}{3}$ and \(180^{\circ}

33. $\tan\theta = -\dfrac{1}{2}$ and $270^\circ < \theta < 360^\circ$

34. $\tan\theta = 2$ and $180^\circ < \theta < 270^\circ$

35. $\tan\theta = \dfrac{3}{4}$ and $180^\circ < \theta < 270^\circ$

36. $\tan\theta = -3$ and $90^\circ < \theta < 180^\circ$

37. $2\cos A + 9 = 8$ and \(90^{\circ}

38. $25\sin B + 8 = -12$ and \(180^{\circ}

39. $8\tan\beta + 5 = -11$ and $90^\circ < \beta < 180^\circ$

40. $6(\tan\beta - 4) = -24$ and $90^\circ < \beta < 270^\circ$

41. $\tan^2 C - 3 = 0$ and \(90^{\circ}

42. $4\cos^2 A - \cos A = 0$ and \(0^{\circ}

Synthesis Questions

13. **Proving the Pythagorean Identity.** Prove the Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$ by carrying out the following steps. Sketch an angle θ in standard position, and label a point (x, y) on the terminal side, at a distance r from the vertex.

    a. Begin with the equation $\sqrt{x^2 + y^2} = r$, and square both sides.

    b. Divide both sides of your equation from part (a) by r^2.

    c. Write the left side of the equation as the sum of the squares of two fractions.

    d. Substitute the appropriate trigonometric ratio for each fraction.

14. **Proving the Ratio Identity.** Prove the Ratio Identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ by carrying out the following steps. Sketch an angle θ in standard position, and label a point (x, y) on the terminal side, at a distance r from the vertex.

    a. Write $\sin\theta$ in terms of y and r, and solve for y.

    b. Write $\cos\theta$ in terms of x and r, and solve for x.

    c. Write $\tan\theta$ in terms of x and y, then substitute your results from parts (a) and (b).

    d. Simplify your fraction in part (c).

Answers

Skills Refresher

1. $\frac{x+y}{xy}$
2. $\frac{x-1}{x}$
3. $\frac{2x}{x^2-1}$
4. $\frac{x^2}{x+1}$
5. $\frac{y-x}{y}$
6. $\frac{y^2-x^2}{y^2+x^2}$
7. $\frac{2ax}{x^2-a^2}$
8. $\frac{ay+bx}{xy-ab}$

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2.4: Introduction to Proofs in Trigonometry

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We have already been introduced to several fundamental identities in Trigonometry. Namely, the Ratio, Reciprocal, and Pythagorean Identities. We then spent most of that section using those identities to help us evaluate trigonometric functions. In this section, we learn how to use those identities to simplify expressions involving trigonometric functions. We then start dipping our toes into proving identities.

Simplifying Expressions Involving Trigonometric Functions

Recall that an algebraic expression (or mathematical expression) is a combination of symbols that are mathematically "well-formed." The mathematical symbols can include numbers (constants), variables, operations (e.g., addition, subtraction, multiplication, etc.), functions, brackets, and other grouping symbols to help determine the order of operations. The one symbol that is always missing from an expression is the equals sign ($=$).

Reminder: Equations versus Expressions

Equations have equals signs—expressions do not.

You **solve** equations, but you **simplify** expressions.

When we simplify an algebraic expression, we obtain a new expression that has the same values as the old one, but is easier to work with. For example, we can apply the Distributive Law and combine like terms to simplify

$$\begin{aligned} 2x(x-6)+3(x+2) &= 2x^2-12x+3x+6 \\ &= 2x^2-9x+6 \end{aligned}$$

The new expression is equivalent to the old one, that is, the expressions have the same value when we evaluate them at any value of x. For instance, you can check that, at $x=3$, the expressions $2x(x-6)+3(x+2)$ and $2x^2-9x+6$ become

$$\begin{aligned} 2(3)(3-6)+3(3+2) &= 6(-3)+3(5) &= -3 \\ 2(3)^2-9(3)+6 &= 18-27+6 &= -3 \end{aligned}$$

To simplify an expression containing trigonometric functions, we treat each function as a single variable. Compare the two calculations below:

$$\begin{array}{ccccc} 8xy & - & 6xy & = & 2xy \\ 8\cos(\theta)\sin(\theta) & - & 6\cos(\theta)\sin(\theta) & = & 2\cos(\theta)\sin(\theta) \end{array}$$

Both calculations are examples of combining like terms. In the second calculation, we treat $\cos(\theta)$ and $\sin(\theta)$ as variables, just as we treat x and y in the first calculation.

In Trigonometry, we are often tasked with simplifying expressions involving trigonometric functions. In doing so, we will use many of our skills from Algebra (e.g., simplifying compound rational expressions, factoring, distributing, etc.) in combination with the identities we have recently discovered (along with those we will soon discover). One strategy for simplifying a trigonometric expression is to reduce the number of different trigonometric functions involved. The following example showcases this process using the Ratio Identities.

Example 2.4.1

a. Simplify: $\cos(\theta)\tan(\theta)+\sin(\theta)$
b. Multiply: $(\cos(\theta)+\sin(\theta))^2$
c. Add: $\frac{\cos(\theta)}{\sin(\theta)}+\frac{\sin(\theta)}{\cos(\theta)}$
d. Simplify: $3\tan(A)+4\tan(A)-2\cos(A)$
e. Simplify: $2-\sin(B)+2\sin(B)$

Solutions available at the end of this section.

It is often easier to know how a **trigonometric expression** (an expression involving trigonometric functions) will simplify once you try simplification techniques. In Example 2.4.1*a*, most students new to Trigonometry would likely never have looked at $\cos(\theta)\tan(\theta)+\sin(\theta)$ and thought, "Hey, I bet that simplifies down to something nice... like $2\sin(\theta)$." Luckily, as you move forward in Trigonometry (and mathematics), you develop an intuition for when an expression can be simplified; however, cultivating this intuition takes time and experimentation.

Checkpoint 2.4.1

Simplify each expression.

a. $(1-\cos(\theta))(1+\cos(\theta))$
b. $2\cos(t)-4\cos(w)\sin(w)+3\cos(t)-2\cos(w)$

Answers available at the end of this section.

Caution

In Checkpoint 2.4.1*b*, note that $\cos(t)$ and $\cos(w)$ are not like terms. (We can choose values for t and w so that $\cos(t)$ and $\cos(w)$ have different values.)

Rewriting Trigonometric Expressions

It is often necessary, especially in Calculus, to rewrite a trigonometric expression in terms of a single trigonometric function. To do so, we must use identities.

Example 2.4.2

a. Rewrite $\sin(\theta)\cos^2(\theta)$ as an expression involving only sums or differences of powers of $\sin(\theta)$.
b. Rewrite $\cot(\theta)$ in terms of only $\cos(\theta)$.

Solutions available at the end of this section.

If we graph the original expression from Example $2.4.2a$ as $y_1 = \sin(x)\cos^2(x)$ and our resulting equivalent expression as $y_2 = \sin(x) - \sin^3(x)$, we see that they have the same graph, as shown in Figure $2.4.1$ below.

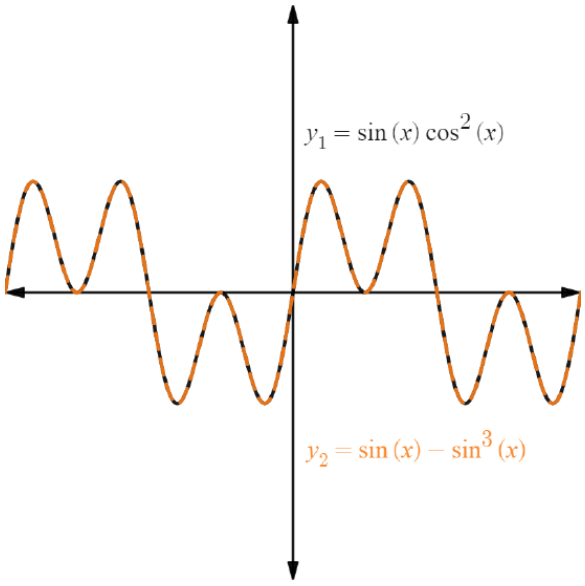


Figure $2.4.1$

This should convince us that $\sin(\theta)\cos^2(\theta)$ truly is equivalent to $\sin(x) - \sin^3(x)$.

Caution

If you tried to graph the functions above using your graphing calculator or another graphing technology and didn't get graphs similar to those in Figure $2.4.1$, it is likely because your graphing device is in degree mode. The graphs we create in Trigonometry require a mode called radian mode. The meaning of these modes will be explained later.

The results of Example $2.4.2$ can be thought of as two new identities:

$$\sin(\theta)\cos^2(\theta) = \sin(\theta) - \sin^3(\theta) \quad \text{and} \quad \cot(\theta) = \pm\frac{\cos(\theta)}{\sqrt{1-\cos^2(\theta)}},$$

however, before you get too concerned with having to memorize these as two more identities, let's be clear:

> Unless formally stated as a theorem, there is no need to memorize the hundreds of identities we will create, prove, or encounter in Trigonometry.

This means that the only identities you are responsible for memorizing (so far) are the Reciprocal, Ratio, and Pythagorean Identities.

Checkpoint $2.4.2$

a. Rewrite $\sin^2(\alpha)\cos^2(\alpha)$ as an expression involving only sums or differences of powers of $\cos(\alpha)$.
b. Verify your identity by graphing.

Answers available at the end of this section.

The following example is extremely useful in Calculus II (Integral Calculus).

Example 2.4.3

Simplify the expression

$$\frac{3}{\sqrt{x^2-4}}$$

as much as possible by substituting $2\sec(\theta)$ for x.

Solution available at the end of this section.

Our solution to Example 2.4.3 needs some clarification to ensure you understand what happened. Most of the work should be understandable; however, two steps might throw you off.

First, from the Pythagorean Identities, we used the fact that

$$1+\tan^2(\theta)=\sec^2(\theta);$$

however, we modified this identity slightly by subtracting 1 from both sides to get

$$\tan^2(\theta)=\sec^2(\theta)-1.$$

The implication of that subtle modification cannot be overstated.

> Being comfortable with the available identities and willing to manipulate them as needed will play a critical role in your success in Trigonometry.

The second item that needs our attention is the mathematical equivalence

$$\sqrt{\tan^2(\theta)}=|\tan(\theta)|.$$

A lot of students forget about the absolute values. Let's focus on what is happening here.

Suppose a friend of yours is claiming that the equation $\sqrt{x^2}=x$ is an identity (this is the same as someone saying $\sqrt{\tan^2(\theta)}=\tan(\theta)$). As an astute mathematics student, you know that even though the equation is true for all *positive* values of x, it is false for negative values of x. For example, if $x=-3$, then

$$\sqrt{x^2}=\sqrt{(-3)^2}=\sqrt{9}=3$$

so $\sqrt{x^2}\neq x$. The radical symbol $\sqrt{\ }$ stands for the **principal square root**—that is, the nonnegative square root. Therefore, the left side of the equation, $\sqrt{x^2}$, is never negative. Thus, $\sqrt{x^2}$ cannot equal x when x is a negative number. The equation is false for $x<0$.

One way to see that $\sqrt{x^2}$ and x are not equivalent is to compare the graphs of $Y_1=\sqrt{x^2}$ and $Y_2=x$, shown in Figure 2.4.2 below. You can see that $\sqrt{x^2}$ and x do not have the same value for $x<0$.

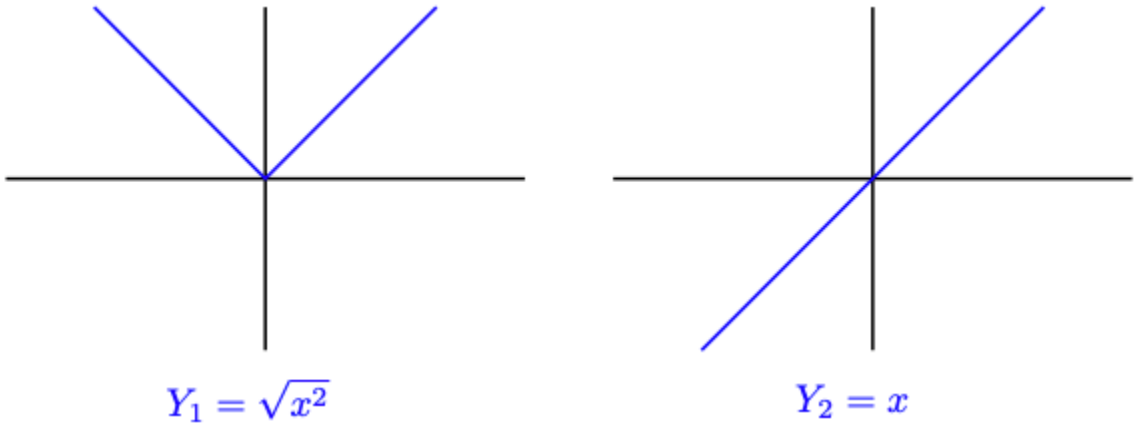


Figure 2.4.2

Interactive Element: $\sqrt{x^2}$ Is $|x|$, Not x

The following Interactive Element might help you visualize why $\sqrt{x^2} = |x|$ rather than x.

Interact: Drag the red point along the x-axis. Two markers ride the curves $y = \sqrt{x^2}$ and $y = x$; watch the segment between them.

Observation: For $x \geq 0$ the two markers coincide, but for $x < 0$ they split apart—$\sqrt{x^2}$ stays positive while x is negative. So $\sqrt{x^2} = |x|$, and it equals x only when $x \geq 0$.

Coming back to the last step in the solution of Example $2.4.3$, we should now feel comfortable saying that $\sqrt{\tan^2(\theta)} \neq \tan(\theta)$ and we should be okay with saying

$$\sqrt{\tan^2(\theta)} = |\tan(\theta)|.$$

Checking the Plausibility of Identities Graphically

Before jumping into how to rigorously prove a trigonometric identity, let's focus on ways to show that a *claimed* identity is *not* an identity.

From the discussion after Example $2.4.3$, we can see that, to check whether an equation *might be* an identity, we can compare graphs of $Y_1 =$(left side of the equation) and $Y_2 =$ (right side of the equation). If the two graphs are identical, it is plausible that the equation is an identity. If the two graphs differ, the equation is *not* an identity.

Read that last paragraph again.

> You cannot use a graph to prove an equation is an identity; however, you can use a graph to demonstrate it is *not* an identity.

This is crucial to understand. The following example provides some clarity.

Example $2.4.4$

Which of the following equations *might* be identities?

a. $\sin(2\alpha) = 2\sin(\alpha)$
b. $\cos\left(x + \frac{\pi}{180}\right) = \cos(x)$
c. $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos(\theta)}{2}$

Solutions available at the end of this section.

Example $2.4.4$ has a few cautionary tales.

Caution: The Trouble with Technology

1. Example $2.4.4b$ illustrates that graphs can be deceiving: even if two graphs look identical, it is always a good idea to zoom in or check some numerical values.
2. The related equation is *not* an identity if the two graphs are different.
3. Graphs cannot be used to prove that an equation is an identity.

Checkpoint 2.4.4

Use graphs to decide which of the following equations *might* be identities.

a. $\cos(2\theta) = 2\cos(\theta)$
b. $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
c. $\cos(\theta^2) = \cos^2(\theta)$

Answers available at the end of this section.

Proving Identities

We have proved a trigonometric identity when we show that one trigonometric expression is equivalent to another. In Example 2.4.1a we proved that the equation

$$\cos(\theta)\tan(\theta) + \sin(\theta) = 2\sin(\theta)$$

is an identity; it is valid for all values of θ (as long as the tangent function is defined).

A common strategy for proving an identity is to transform one side of the equation using equivalent expressions until it is identical to the other side. To help us choose the transformations at each step of the proof, we try to match the algebraic form of the final expression.

Example 2.4.5

Prove the identity:

$$1 + \tan^2(t) = \frac{1}{\cos^2(t)}.$$

Solution available at the end of this section.

When you write out the proof of an identity, your goal is to transform the expression on one side of the identity into the expression on the other, showing one step of the calculation on each line of your proof. You can justify each step to the right of the calculation. The proof of the identity in the previous example would look like this:

$$\begin{aligned}
\text{LHS} &= 1 + \tan^2(t) \\
&= 1 + \left(\frac{\sin(t)}{\cos(t)}\right)^2 && \text{(Ratio Identities)} \\
&= 1 + \frac{\sin^2(t)}{\cos^2(t)} && \text{(Laws of Exponents)} \\
&= \frac{\cos^2(t)}{\cos^2(t)} + \frac{\sin^2(t)}{\cos^2(t)} && \text{(Get a common denominator to combine fractions)} \\
&= \frac{\cos^2(t) + \sin^2(t)}{\cos^2(t)} && \text{(adding fractions)} \\
&= \frac{1}{\cos^2(t)} && \text{(Pythagorean Identity)} \\
&= \text{RHS}
\end{aligned}$$

For now, we will focus on transforming the left side of an equation into the right side; however, as we move forward in Trigonometry, we will adopt the rule of thumb of transforming the more "complicated" side of the equation into the other side.

Example 2.4.6

Prove the identity:

$$(\sin(\theta) - \cos(\theta))^2 - 1 = -2\sin(\theta)\cos(\theta)$$

Solution available at the end of this section.

Checkpoint 2.4.6

Prove the identity: $2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

Answer available at the end of this section.

Before we leave this section, it's important to note that you will be proving many identities in Trigonometry. It takes PRACTICE! We have introduced the idea of a proof early so that we spend a good deal of time sharpening those skills and giving advice on tactics as we move forward. For now, the best advice is to do all the proofs in the homework section—even if your instructor does not assign them.

MyOpenMath Advice: Latepasses and File Uploads

Sometimes, life can get overwhelmingly busy, and missing a due date happens. In those instances, you can use a Latepass to extend the original due date of the homework (see the syllabus for how long a single Latepass extends the original due date by).

Warning: Hand-Written Solutions

Read the syllabus to find out what happens to the questions that must be graded by hand (file uploads) - most professors are not likely to grade the late submitted hand-written homework solutions due to time constraints and fairness to other students.

Success in Mathematics: Holding Onto Old, Broken Techniques

It's common for students to learn a problem-solving technique years ago and not want to learn a different method because they are comfortable with their old habits. Unfortunately, we often fail to recognize that if our old, comfortable technique worked well, we should be much further along in our mathematical journey.

This adherence to comfort is widespread for all human beings in various situations. Whether it's a dead-end job, an abusive relationship, or a bad habit, we tend to ignore signals—remaining in a known situation rather than taking the chance to improve and evolve.

In general, when your professor gives you advice or a different approach to doing something, they are doing so from the lens of years of experience seeing students repeat old, broken techniques or habits that tend to lead students to make mistakes. Paying attention to these words of wisdom is critical.

EXAMPLE

Simplify the compound rational expression

$$\frac{-\frac{2}{(x+h)^2} + \frac{2}{x^2}}{h}$$

Students stuck in old habits tend to find the common denominator of the fractions in the numerator, make equivalent fractions, add fractions, divide the result by h, then simplify. This entire process looks like the following:

$$\begin{aligned}
\frac{-\frac{2}{(x+h)^2}+\frac{2}{x^2}}{h} &= \frac{-\frac{2x^2}{x^2(x+h)^2}+\frac{2(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \frac{\frac{-2x^2+2(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \frac{-2x^2+2(x+h)^2}{x^2(x+h)^2} \div h \\
&= \frac{-2x^2+2(x+h)^2}{x^2(x+h)^2} \cdot \frac{1}{h} \\
&= \frac{-2x^2+2(x+h)^2}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2(x^2+2hx+h^2)}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2x^2+4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{h(4x+2h)}{hx^2(x+h)^2} \\
&= \frac{\cancel{h}(4x+2h)}{\cancel{h}x^2(x+h)^2} \\
&= \frac{(4x+2h)}{x^2(x+h)^2}
\end{aligned}$$

Compare this to the advice given by your math professor to begin by multiplying both numerator and denominator of the original expression by the LCD of all minor fractions within the compound rational expression.

$$\begin{aligned}
\frac{-\frac{2}{(x+h)^2}+\frac{2}{x^2}}{h} &= \frac{-2x^2+2(x+h)^2}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2(x^2+2hx+h^2)}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2x^2+4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{h(4x+2h)}{hx^2(x+h)^2} \\
&= \frac{\cancel{h}(4x+2h)}{\cancel{h}x^2(x+h)^2} \\
&= \frac{(4x+2h)}{x^2(x+h)^2}
\end{aligned}$$

As you can see, the second method is cleaner and less complex.

The moral of the story is, when learning, try to be open to... learn.

Answers

Section Preview

The proofs done by the student in this section only require the Ratio, Reciprocal, and Pythagorean Identities. The material here is also meant to give the student experience in simplifying complex-looking expressions involving trigonometric functions into much cleaner expressions. Algebra involving trigonometric functions (e.g., multiplying expressions or adding rational expressions involving trigonometric functions) is treated. Finally, the student should be able to write a trigonometric function in terms of another (using the Pythagorean Identities).

Section Preview

- **Sets and Numbers**
  - The Language of Algebra (focus: proofs): The proof-writing framework requires understanding what a mathematical proof is and the logical structure of an argument.
- **Graphs**
  - Reading Values from Graphs: Examples within the text and lecture use graphs to check plausibility of identities.

Section Preview

- **Arithmetic**
  - The Absolute Value of a Number: The section makes a notable point that $\sqrt{\tan^2(\theta)} = |\tan(\theta)|$, explicitly discussing the principle square root and why $\sqrt{x^2} \neq x$ for negative x. Students must already understand numerical absolute value to follow this argument.
- **Simplifying Expressions**
  - Simplifying Ratios of Polynomials: The proofs in several examples (and homework questions) require simplifying ratios of polynomial-like expressions.

Section Preview

- Simplify a trigonometric expression.
- Rewrite a trigonometric expression in terms of sines and cosines.
- Simplify an algebraic expression by performing a trigonometric substitution.
- Determine the plausibility of an identity graphically and understand this does not prove an identity (but *can* disprove an identity).
- Prove basic trigonometric identities.

Example 2.4.1

a. We apply the Ratio Identities to replace $\tan(\theta)$ by $\frac{\sin(\theta)}{\cos(\theta)}$ and obtain

$$\begin{aligned}\cos(\theta)\tan(\theta)+\sin(\theta) &= \cos(\theta)\left(\tfrac{\sin(\theta)}{\cos(\theta)}\right)+\sin(\theta)\\ &= \sin(\theta)+\sin(\theta)\\ &= 2\sin(\theta)\end{aligned}$$

b. The instruction to multiply implies we need to square the binomial and simplify.

$$\begin{aligned}(\cos(\theta)+\sin(\theta))^2 &= \cos^2(\theta)+2\cos(\theta)\sin(\theta)+\sin^2(\theta) && \text{(distributing)}\\ &= \cos^2(\theta)+\sin^2(\theta)+2\cos(\theta)\sin(\theta) && \text{(Commutative Property of Addition)}\\ &= 1+2\cos(\theta)\sin(\theta) && \text{(Pythagorean Identity)}\end{aligned}$$

c. While we *could* use the Ratio Identities to make an equivalent expression, the result would be $\cot(\theta)+\tan(\theta)$. At that point, we would be stuck. Instead, let's use the Mathematical Mantra and perform some arithmetic before trying our new Trigonometry skills. Specifically, let's get a common denominator and perform the subtraction.

$$\begin{aligned}\frac{\cos(\theta)}{\sin(\theta)}+\frac{\sin(\theta)}{\cos(\theta)} &= \frac{\cos(\theta)}{\sin(\theta)}\cdot\frac{\cos(\theta)}{\cos(\theta)}+\frac{\sin(\theta)}{\cos(\theta)}\cdot\frac{\sin(\theta)}{\sin(\theta)} && \text{(multiplying each fraction by an expression}\\ & && \text{equivalent to 1 to get common denominators)}\\ &= \frac{\cos^2(\theta)}{\cos(\theta)\sin(\theta)}+\frac{\sin^2(\theta)}{\cos(\theta)\sin(\theta)}\\ &= \frac{\cos^2(\theta)+\sin^2(\theta)}{\cos(\theta)\sin(\theta)} && \text{(adding fractions with like denominators)}\\ &= \frac{1}{\cos(\theta)\sin(\theta)} && \text{(Pythagorean Identity)}\end{aligned}$$

d. Combine like terms:

$$3\tan(A)+4\tan(A)-2\cos(A)=7\tan(A)-2\cos(A)$$

Note that $\tan(A)$ and $\cos(A)$ are not like terms.

e. Combine like terms:

$$2-\sin(B)+2\sin(B)=2+\sin(B)$$

Note that $-\sin(B)$ means $-1\cdot\sin(B)$.

Checkpoint 2.4.1

a. $\sin^2(\theta)$

b. $5\cos(t)-4\cos(w)\sin(w)-2\cos(w)$

Example 2.4.2

a. Using one of the alternate forms of the Pythagorean Identity, we replace $\cos^2(\theta)$ with $1-\sin^2(\theta)$ to get

$$\begin{aligned}\sin(\theta)\cos^2(\theta) &= \sin(\theta)\left(1-\sin^2(\theta)\right) && \text{(Pythagorean Identity)}\\ &= \sin(\theta)-\sin^3(\theta) && \text{(Distributive Law)}\end{aligned}$$

b.

$$\begin{aligned}\cot(\theta) &= \frac{\cos(\theta)}{\sin(\theta)} && \text{(Ratio Identities)}\\ &= \frac{\cos(\theta)}{\pm\sqrt{1-\cos^2(\theta)}} && \text{(Pythagorean Identities (alternate form))}\\ &= \pm\frac{\cos(\theta)}{\sqrt{1-\cos^2(\theta)}}\end{aligned}$$

Checkpoint 2.4.2

$\cos^2(\alpha)-\cos^4(\alpha)$

Example 2.4.3

$$\begin{aligned}\frac{3}{\sqrt{x^2-4}} &= \frac{3}{\sqrt{(2\sec(\theta))^2-4}} && \text{(substituting } 2\sec(\theta) \text{ for } x)\\ &= \frac{3}{\sqrt{4\sec^2(\theta)-4}} && \text{(Laws of Exponents)}\\ &= \frac{3}{\sqrt{4\left(\sec^2(\theta)-1\right)}} && \text{(factoring)}\\ &= \frac{3}{2\sqrt{\sec^2(\theta)-1}} && \text{(simplifying)}\\ &= \frac{3}{2\sqrt{\tan^2(\theta)}} && \text{(Pythagorean Identity: } \tan^2(\theta)=\sec^2(\theta)-1)\\ &= \frac{3}{2\left|\tan(\theta)\right|} && \text{(see comments below)}\end{aligned}$$

Example 2.4.4

a. Compare the graphs of $y_1=\sin(2x)$ and $y_2=2\sin(x)$. The Desmos graphs for both equations are shown in the figure below.

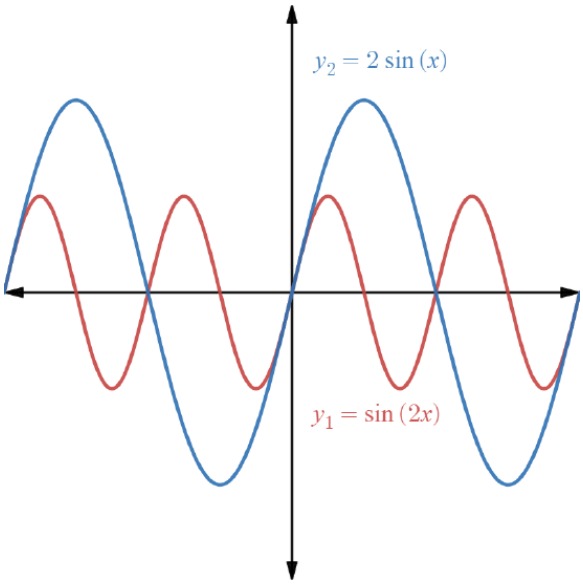


Figure 2.4.3

Because there are two distinct graphs, the expressions $\sin(2x)$ and $2\sin(x)$ are not equivalent, and consequently, $\sin(2\alpha)=2\sin(\alpha)$ is not an identity.

b. This time we graph $y_1=\cos\left(x+\frac{\pi}{180}\right)$ and $y_2=\cos(x)$.

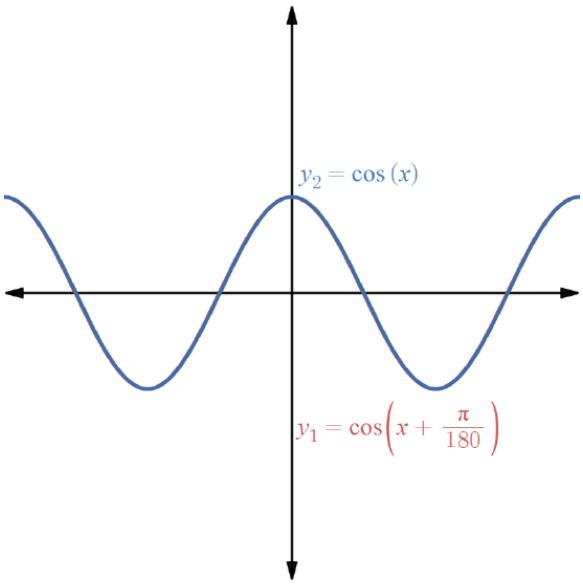


Figure 2.4.4

Although the graphs appear identical, when we zoom in, we see that the graphs are, indeed, not the same.

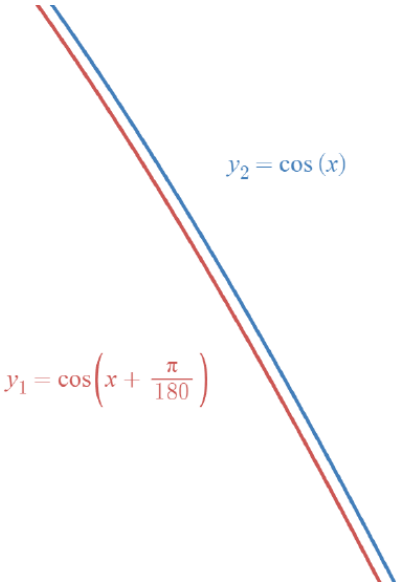


Figure 2.4.5

The graphs are so close together that Desmos' resolution does not distinguish them, but zooming in reveals that they are not identical. Because the two graphs differ, the equation $\cos\left(x + \frac{\pi}{180}\right) = \cos(x)$ is not an identity.

c. Letting $y_1 = \cos^2\left(\frac{x}{2}\right)$ and $y_2 = \frac{1+\cos(x)}{2}$, we get the following graph from Desmos.

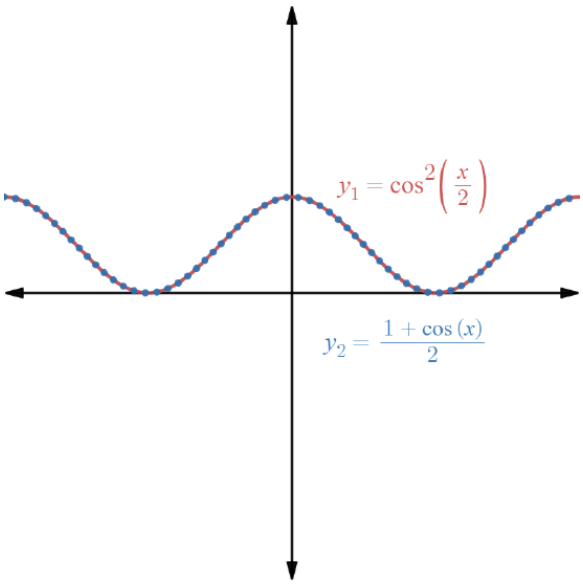


Figure 2.4.6

These two graphs *look* identical; however, it is best to zoom in to double-check.

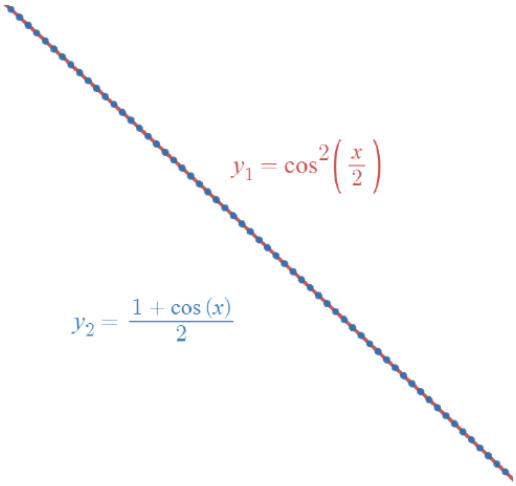


Figure 2.4.7

No matter how much we zoom in, the two graphs *appear* identical. Therefore, we can say that it is *plausible* that $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos(\theta)}{2}$ is an identity.

Checkpoint 2.4.4

(b)

Example 2.4.5

By manipulating the *left side* of the equation, we will show that the expression $1+\tan^2(t)$ is equivalent to $\frac{1}{\cos^2(t)}$. First, we use the Ratio Identities to write the expression in terms of sines and cosines:

$$1+\tan^2(t) = 1+\left(\frac{\sin(t)}{\cos(t)}\right)^2 = 1+\frac{\sin^2(t)}{\cos^2(t)}$$

Next, we notice that the right side of the proposed identity has only one term, so we combine the terms on the *left side*. So that the fractions have the same denominator, we write 1 as $\frac{\cos^2(t)}{\cos^2(t)}$.

$$1+\frac{\sin^2(t)}{\cos^2(t)} = \frac{\cos^2(t)}{\cos^2(t)}+\frac{\sin^2(t)}{\cos^2(t)} = \frac{\cos^2(t)+\sin^2(t)}{\cos^2(t)}$$

Finally, we apply the Pythagorean Identity to the numerator.

$$\frac{\cos^2(t)+\sin^2(t)}{\cos^2(t)} = \frac{1}{\cos^2(t)}$$

Thus, $1+\tan^2(t) = \frac{1}{\cos^2(t)}$, and the identity is proved.

Example 2.4.6

$$\begin{aligned}
\text{LHS} &= (\sin(\theta) - \cos(\theta))^2 - 1 \\
&= \sin^2(\theta) - 2\cos(\theta)\sin(\theta) + \cos^2(\theta) - 1 && \text{(distributing)} \\
&= \sin^2(\theta) + \cos^2(\theta) - 1 - 2\cos(\theta)\sin(\theta) && \text{(Commutative Property of Addition)} \\
&= 1 - 1 - 2\cos(\theta)\sin(\theta) && \text{(Pythagorean Identity)} \\
&= -2\cos(\theta)\sin(\theta) && \text{(simplifying)} \\
&= \text{RHS}
\end{aligned}$$

Checkpoint 2.4.6

$2\cos^2(x) - 1 = 2(1 - \sin^2(x)) - 1 = 2 - 2\sin^2(x) - 1 = 1 - 2\sin^2(x)$

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2.4.1: Resources and Key Concepts

Resources

Videos

- Proving Identities
  - Fundamental Identities - Tactics for Proving Trigonometric Identities
- Trigonometric Substitution and Simplifying Trigonometric Expressions
  - Fundamental Identities - The Pythagorean Identities: Theorem, Proof, and Examples

Key Concepts

Common Mistakes

- **Confusing Expressions with Equations:** Expressions do not have equals signs and are simplified, whereas equations have equals signs and are solved.
- **Treating Different Angle Variables as Like Terms:** Trigonometric functions with different angle variables, such as $\cos(t)$ and $\cos(w)$, are not like terms and cannot be combined.
- **Forgetting Absolute Value with Square Roots:** The square root of a squared expression is its absolute value, not just the expression itself. For example, $\sqrt{\tan^2(\theta)} = |\tan(\theta)|$, not just $\tan(\theta)$.
- **The Trouble with Technology:**
  - Graphs can be deceiving; two different graphs may appear identical unless you zoom in.
  - If the graphs of the two sides of an equation are different, the equation is not an identity.
  - You cannot use a graph to prove that an equation is an identity, only to show it is plausible or to disprove it.

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2.4.2: Homework

Reading Questions

1. What is the difference between simplifying an expression and solving an equation?
2. What is a common first strategy for simplifying a trigonometric expression that involves multiple trigonometric functions?
3. In the expression $5\cos(t) - 2\cos(w)$, can the two terms be combined? Why or why not?
4. Explain the difference between $\sqrt{x^2}$ and x. Why is this important when simplifying expressions like $\sqrt{\tan^2(\theta)}$?
5. Can you use a graph to prove that an equation is an identity? Explain your answer.
6. How can you use a graph to prove that an equation is *not* an identity?
7. What does it mean if the graphs of the two sides of an equation appear to be identical?
8. What is the general strategy described in this section for proving a trigonometric identity?
9. In Example 1b, what algebraic process is used to multiply $(\cos(\theta) + \sin(\theta))^2$?
10. What fundamental identity is used to simplify $\cos^2(\theta) + \sin^2(\theta)$ to 1 in Example 1c?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, simplify the expression.

1. $\dfrac{\frac{1}{x}}{\frac{1}{y}}$
2. $\dfrac{1}{x} - \dfrac{1}{y}$
3. $\dfrac{1}{x} - x$
4. $\dfrac{y}{x} + \dfrac{1}{y}$
5. $\dfrac{x}{y} - \dfrac{y}{x}$

For the following exercises, multiply.

6. $(x - y)^2$
7. $(x - 1)(x + 1)$
8. $(x - 3)(x + 5)$

Answer available at the end of this section.

Homework

Vocabulary Check

1. A(n) ___ is a combination of symbols that are mathematically "well-formed."
2. You ___ equations and ___ expressions.

Concept Check

3. Explain how to use graphs to disprove an identity.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

4. To check whether an equation is an identity, we can compare graphs of the left side of the equation and the right side of the equation. If the two graphs agree, the equation is an identity.
5. To prove an identity, we write one side of the equation in equivalent forms until it is identical to the other side of the equation.

Basic Skills

For the following exercises, use identities to rewrite each expression.

6. $\cos(\theta)$ as an expression in $\sin(\theta)$ only
7. $\sin(\theta)$ as an expression in $\cos(\theta)$ only
8. $\cot(\theta)$ as an expression in $\sin(\theta)$ only
9. $\csc(\theta)$ as an expression in $\cos(\theta)$ only

For the following exercises, use graphs to decide which of the following equations *might be* identities.

10. $\sin 2t = 2\sin t$
11. $\cos\theta + \sin\theta = 1$
12. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$
13. $\dfrac{\tan^2 x}{1+\tan^2 x} = \sin^2 x$
14. $\tan x + \dfrac{1}{\tan x} = \sin x \cos x$

For the following exercises, write each expression in terms of $\sin(\theta)$ and $\cos(\theta)$ and simplify, if possible.

15. $\csc(\theta)\tan(\theta)$
16. $\dfrac{\sec(\theta)}{\csc(\theta)}$
17. $\dfrac{\sec(\theta)}{\tan(\theta)}$
18. $\dfrac{\cot(\theta)}{\tan(\theta)}$

19. $\sec(\theta) + \tan(\theta)$

20. $\sin(\theta)\tan(\theta) + \sec(\theta)$

21. $\csc(\theta) - \cot(\theta)\cos(\theta)$

For the following exercises, add or subtract the expressions and simplify completely. Leave your answers in terms of sine or cosine.

22. $\cos(\theta) - \dfrac{1}{\sin(\theta)}$

23. $\dfrac{1}{\sin(\theta)} - \dfrac{1}{\cos(\theta)}$

24. $\tan(\theta) + \csc(\theta)$

25. $\dfrac{1}{\cos^2\beta} - \dfrac{\sin^2\beta}{\cos^2\beta}$

26. $\dfrac{1}{\sin^2\phi} - \dfrac{1}{\tan^2\phi}$

For the following exercises, multiply and simplify completely.

27. $(\sec(\theta) - 1)(\sec(\theta) + 1)$

28. $(\sin(\theta) + 3)(\sin(\theta) + 5)$

29. $(\cos(\theta) + 1)(\cos(\theta) - 1)$

30. $(\sin(\theta) - \cos(\theta))^2$

31. $(\sin(\theta) - 9)^2$

For the following exercises, simplify the expression completely. Write your answers in terms of a single trigonometric function or a constant.

32. $\cos^2\alpha(1 + \tan^2\alpha)$

33. $\cos^3\phi + \sin^2\phi\cos\phi$

34. $\tan^2 A - \tan^2 A\sin^2 A$

35. $\cos^2 B\tan^2 B + \cos^2 B$

36. $\dfrac{1 - \cos^2 z}{\cos^2 z}$

37. $\dfrac{\sin t}{\cos t\tan t}$

For the following exercises, use identities to rewrite each expression.

38. $2 - \cos^2\theta + 2\sin\theta$ as an expression in $\sin\theta$ only

39. $3\sin^2 B + 2\cos B - 4$ as an expression in $\cos B$ only

40. $\cos^2\phi - 2\sin^2\phi$ as an expression in $\cos\phi$ only

41. $\cos^2\phi\sin^2\phi$ as an expression in $\sin\phi$ only

For the following exercises, use the given substitution and completely simplify the result.

42. $\sqrt{25-x^2}$; substituting $5\sin(\theta)$ for x

43. $\sqrt{x^2+1}$; substituting $\tan(\theta)$ for x

44. $\sqrt{x^2-9}$; substituting $3\sec(\theta)$ for x

45. $\sqrt{81-9x^2}$; substituting $3\sin(\theta)$ for x

46. $\sqrt{36x^2+1296}$; substituting $6\tan(\theta)$ for x

For the following exercises, show that the equation is an identity by transforming the left side into the right side.

47. $\cos(\theta)\csc(\theta)\tan(\theta)=1$

48. $\dfrac{\cos(\theta)}{\sec(\theta)}=\cos^2(\theta)$

49. $\dfrac{\csc(\theta)\tan(\theta)}{\sec(\theta)}=1$

50. $\cos(\theta)\cot(\theta)+\sin(\theta)=\csc(\theta)$

51. $\tan^2(\theta)+1=\sec^2(\theta)$

52. $\csc(\theta)\tan(\theta)-\cos(\theta)=\dfrac{\sin^2(\theta)}{\cos(\theta)}$

53. $(1+\sin w)(1-\sin w)=\cos^2 w$

54. $(\cos\theta-1)(\cos\theta+1)=-\sin^2\theta$

55. $(\cos\theta-\sin\theta)^2=1-2\sin\theta\cos\theta$

56. $\sin^2 x-\cos^2 x=1-2\cos^2 x$

57. $\sec(\theta)(\sin(\theta)+\cos(\theta))=\tan(\theta)+1$

58. $\tan\theta\cos\theta=\sin\theta$

59. $\dfrac{\sin\mu}{\tan\mu}=\cos\mu$

60. $\cos^4 x-\sin^4 x=\cos^2 x-\sin^2 x$

61. $1-2\cos^2 v+\cos^4 v=\sin^4 v$

62. $\dfrac{\sin u}{1+\cos u}=\dfrac{1-\cos u}{\sin u}$
Hint. Multiply the numerator and denominator of the left side by $1-\cos u$.

63. $\dfrac{\sin v}{1-\sin v}=\dfrac{\tan v(1+\sin v)}{\cos v}$
Hint. Multiply the numerator and denominator of the left side by $1+\sin v$.

For the following exercises, rewrite tangents in terms of sines and cosines, then simplify each side separately (without adding, subtracting, multiplying, or dividing between sides). Finally, show the equation is an identity by transforming the left side into the right side. (These problems involve simplifying complex fractions. See the Algebra Refresher to review this skill.)

54. $\dfrac{\tan\alpha}{1+\tan\alpha} = \dfrac{\sin\alpha}{\sin\alpha+\cos\alpha}$

55. $\dfrac{1-\tan u}{1+\tan u} = \dfrac{\cos u-\sin u}{\cos u+\sin u}$

56. $\dfrac{1+\tan^2\beta}{1-\tan^2\beta} = \dfrac{1}{\cos^2\beta-\sin^2\beta}$

57. $\tan^2 v-\sin^2 v=\tan^2 v\sin^2 v$

Answers

Skills Refresher

1. $\dfrac{y}{x}$
2. $\dfrac{y-x}{xy}$
3. $\dfrac{(1-x)(1+x)}{x}$
4. $\dfrac{y^2+x}{xy}$
5. $\dfrac{(x-y)(x+y)}{xy}$
6. $x^2-2xy+y^2$
7. x^2-1
8. $x^2+2x-15$

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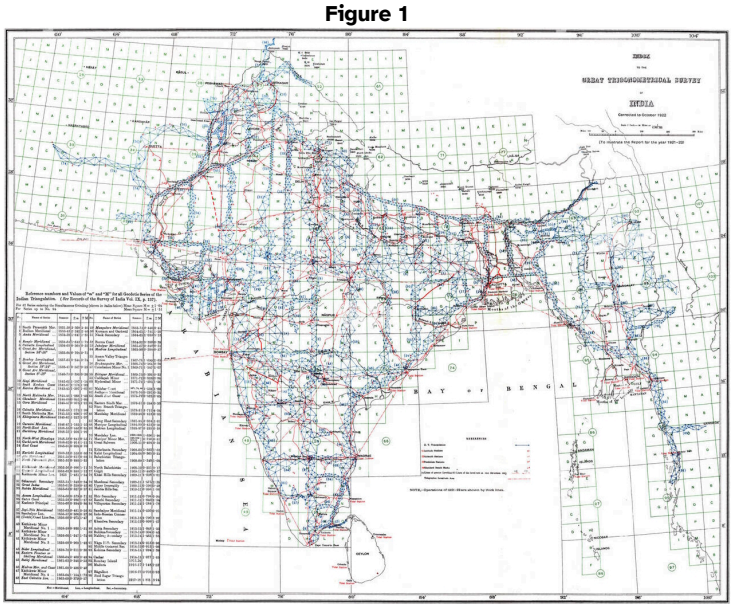
3: RIGHT TRIANGLE TRIGONOMETRY

How would one measure the distance to an inaccessible object, such as a ship at sea? In the 6th century BC, the Greek philosopher Thales estimated the distances to ships at sea using **triangulation**, a method for calculating distances by forming triangles. Using Trigonometry and the measured length of just one side, the lengths of the other sides can be calculated.

Triangulation has been used to compute distances ever since. In the 16th century, mapmakers used triangulation to position distant places accurately. As new methods in navigation and astronomy required greater precision, the idea of a survey using chains of triangles was developed.

In 1802, the East India Company embarked on the Great Trigonometrical Survey of India, which aimed to measure the entire Indian subcontinent with scientific precision.

Figure 1



The surveyors began by measuring a baseline near Madras. The baseline was the only distance they measured; all other distances were calculated from it using measured angles. Each calculated distance became the base side of another triangle used to calculate the distance to another point, which in turn started another triangle. Eventually this process formed a chain of triangles connecting the origin point to other locations.

Because of the size of the area to be surveyed, the surveyors did not triangulate the whole of India. Instead, they created a "gridiron" of triangulation chains running from north to south and east to west, as seen on the survey map above.

The survey, completed in 1871, calculated the heights of the Himalayan giants: Everest, K2, and Kanchenjunga. It also provided one of the first accurate measurements of a section of an arc of longitude.

Today, triangulation is used for many purposes, including surveying, navigation, metrology, astrometry, binocular vision, and location of earthquakes.

3.1: The Trigonometric Functions - Right Triangle Definition

3.1.1: Resources and Key Concepts

3.1.2: Homework

3.2: Technology and Right Triangle Trigonometry

3.2.1: Resources and Key Concepts
3.2.2: Homework

3.3: Solving Right Triangles

3.3.1: Resources and Key Concepts
3.3.2: Homework

3.4: Applications Involving Right Triangles

3.4.1: Resources and Key Concepts
3.4.2: Homework

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3.1: The Trigonometric Functions - Right Triangle Definition

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

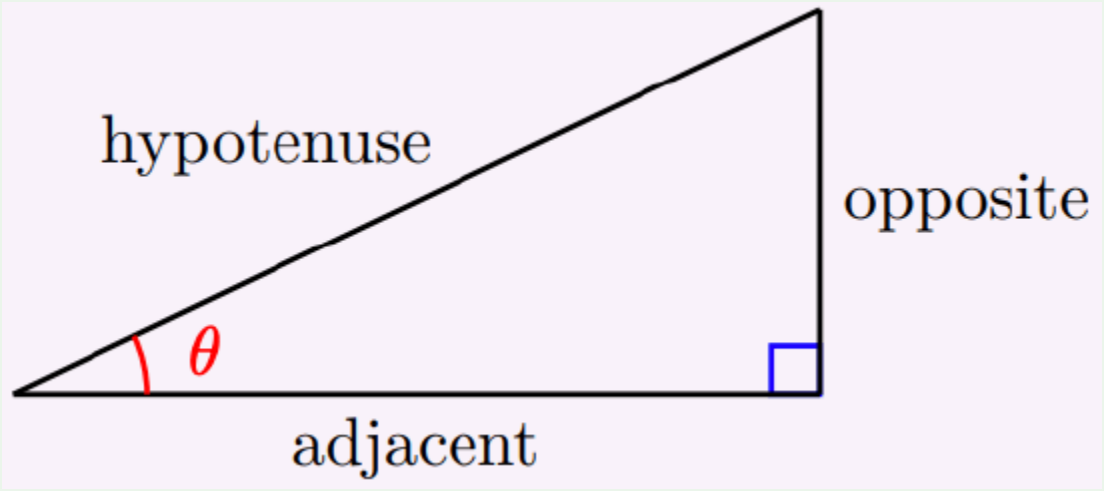
Learning Objectives (click to expand) available at the end of this section.

Introduction to Right Triangle Trigonometry

Our initial foray into Trigonometry involved defining the trigonometric functions in terms of points on the Cartesian plane and the distances from those points to the origin. In this chapter, we focus on Trigonometry outside of a coordinate system. Specifically, we redefine the trigonometric functions in terms of ratios of sides of a right triangle.

Definition: Trigonometric Functions (Right Triangle Definition)

If θ is one of the angles in a right triangle, where the side opposite θ is called "opposite," the side adjacent to θ is called "adjacent," and the hypotenuse is named as such, we define the trigonometric functions of θ as follows:



While this definition looks different than our coordinate definition of the trigonometric functions, it is, in fact, the same definition

if we restrict the coordinates in our original definition to the first quadrant. While it might seem limiting for us to create a new definition of the trigonometric functions by restricting our original definition, this new definition serves a greater purpose—it simplifies Trigonometry immensely. It allows us to separate the triangle from the Cartesian coordinate system.

Example 3.1.1

Find all trigonometric ratios of the labeled angle in each triangle.

a.

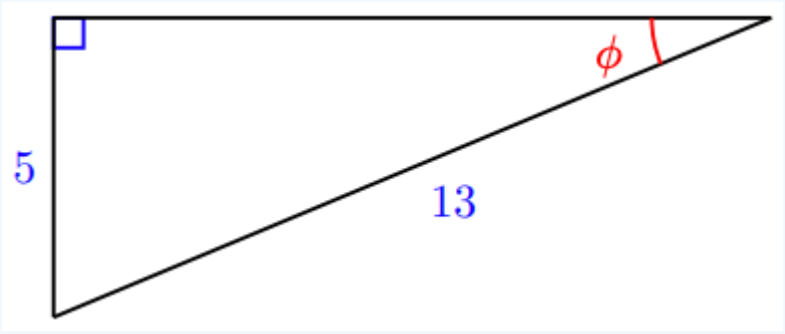


Figure 3.1.1

b.

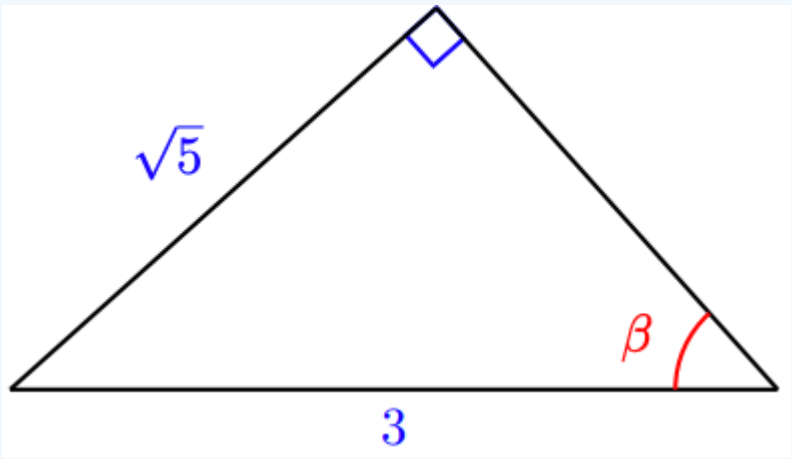


Figure 3.1.2

Solutions available at the end of this section.

The result of the computation to find the missing side in Example 3.1.1 part (a) reveals one of the **Pythagorean triples**, $(5, 12, 13)$.

Definition: Pythagorean Triple

A **Pythagorean triple** consists of three positive integers a, b, and c, such that $a^2 + b^2 = c^2$.

There are an infinite number of Pythagorean triples, but the common ones found in Trigonometry textbooks are $(3, 4, 5)$, $(5, 12, 13)$, and $(7, 24, 25)$. Triangles having sides lengths matching Pythagorean triples get a special name.

Definition: Pythagorean Triangle

A right triangle whose sides form a Pythagorean triple is called a **Pythagorean triangle**.

It can be proved that every Pythagorean triangle must be a right triangle. For more information, see this Wikipedia page.

Checkpoint 3.1.1

Find the sine of the labeled angle in the triangle below. Round the answer to 4 decimal places.

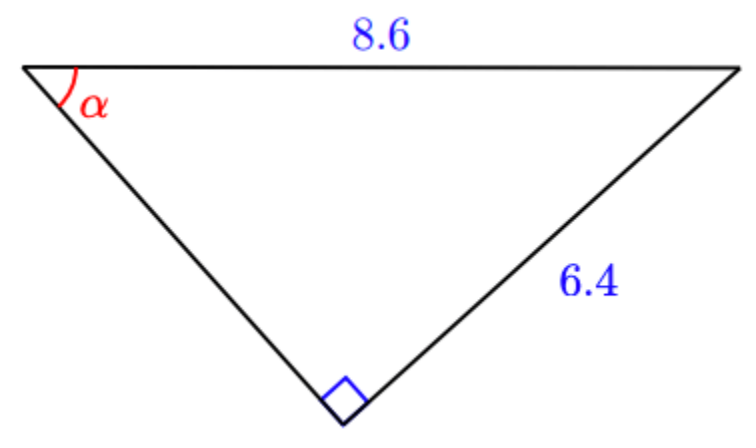


Figure 3.1.3

Answer available at the end of this section.

The following example illustrates a necessary skill for us as we move forward in Trigonometry.

Example 3.1.2

Write algebraic expressions for the six trigonometric ratios of the angle θ if the hypotenuse has length 2 and the side adjacent to θ has length D.

Solution available at the end of this section.

Checkpoint 3.1.2

If the hypotenuse of a right triangle is h and the side opposite θ is L, write expressions in h and L for all six trigonometric functions of θ.

Answer available at the end of this section.

Caution: Right Triangle Trigonometry Works For... Right Triangles

To calculate the trigonometric function of an angle within a triangle, it is crucial that the triangle be a *right* triangle. For example, in the triangle below, $\sin(\alpha) = \frac{4}{7}$, because $\triangle ABC$ is a right triangle. It is *not* true that $\sin(\alpha) = \frac{5}{7}$, or $\sin(\alpha) = \frac{6}{7}$.

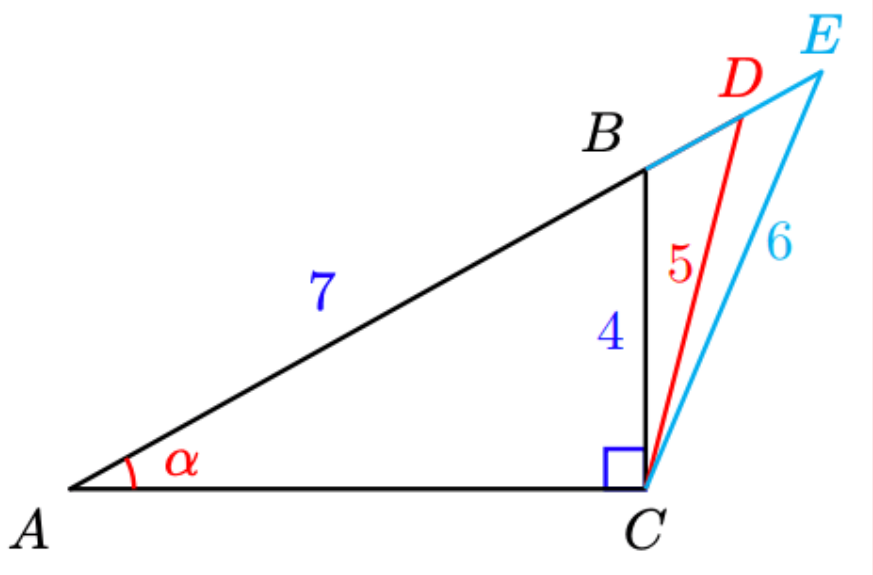


Figure 3.1.4

Revisiting the Fundamental Identities

Our previous identities (the Ratio, Reciprocal, and Pythagorean Identities) still hold when using the right triangle definition of the trigonometric functions. For example, notice that:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\frac{\text{hypotenuse}}{\text{opposite}}} = \frac{1}{\csc(\theta)}.$$

We leave it to the reader to show that all other Reciprocal Identities also hold.

Likewise:

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\frac{\text{opposite}}{\text{hypotenuse}}}{\frac{\text{adjacent}}{\text{hypotenuse}}} = \frac{\sin(\theta)}{\cos(\theta)}.$$

Again, we leave it to the reader to finish showing that both Ratio Identities still hold.

Finally, it is left as a homework exercise to show that the Pythagorean Identity still holds with this new definition.

Example 3.1.3

If θ is an acute angle in a right triangle and $\tan(\theta) = \frac{15}{8}$, compute the remaining trigonometric ratios of θ.

Solution available at the end of this section.

Checkpoint 3.1.3

If β is an acute angle in a right triangle and $\csc(\beta) = \frac{2}{\sqrt{3}}$, compute the remaining trigonometric ratios of β.

Answer available at the end of this section.

Example 3.1.4

If $\sin(\theta) = \frac{3}{7}$ and $\cos(\theta) = \frac{2\sqrt{10}}{7}$, where θ is an acute angle within a right triangle, find the value of $\cot(\theta)$ without drawing a triangle.

Solution available at the end of this section.

Checkpoint 3.1.4

If λ is an acute angle in a right triangle, $\tan(\lambda) = \frac{2}{3}$, and $\cos(\lambda) = \frac{3}{\sqrt{13}}$, find $\sin(\lambda)$ using only the Ratio Identities.

Answer available at the end of this section.

Right Triangle Trigonometry and the Special Angles

Now that we have a right triangle definition for the trigonometric functions, we can use our special triangles to help us quickly evaluate the trigonometric functions at 30°, 45°, and 60° angles. These angles are foundations upon which we build the rest of Trigonometry.

Table 3.1.1: *Fundamental Trigonometric Ratios for the Special Angles*



The exact values for these trigonometric ratios should be committed to memory. A great way to remember them is to know the two special triangles in Figure 3.1.6 below. Writing down the three trigonometric ratios for the special angles from these triangles is simple.



Figure 3.1.6: *Left: The 45°-45°-90° special triangle, Right: The 30°-60°-90° special triangle*

For example, we can quickly compute $\tan(60^\circ)$ and $\sec(30^\circ)$ by sketching the 30°-60°-90° triangle and finding the ratios of the relevant sides:

$$\sec(30^\circ) = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{2}{\sqrt{3}} \qquad \tan(60^\circ) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3}. \tag{3.1.1}$$

These are *exact* values for the trigonometric ratios, but we can also find decimal approximations. Use a calculator to verify the following approximate values.

		exact value		approximation
$\sec(30^\circ)$	$=$	$\frac{2}{\sqrt{3}}$	$\approx$	1.1547
$\tan(60^\circ)$	$=$	$\sqrt{3}$	$\approx$	1.7321

Caution: Approximations are NOT Exact

It is essential to understand the difference between *exact* and *approximate* values. The decimal approximations above are rounded off. Even if a calculator shows ten or twelve digits, the values are not precisely correct—although they are adequate for most practical calculations.

Computational Note: To Rationalize or Not To Rationalize... That is the Question

Did you notice that we left $\sec(30^\circ) = \frac{2}{\sqrt{3}}$ with a radical in its denominator? This is quite common in Trigonometry. While one can rationalize the denominator to get

$$\sec(30^\circ) = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3},$$

the need to do so with numerical expressions (expressions *not* involving variables) is an antiquated "throwback" to the days when calculators did not exist. As such, unless it truly simplifies an expression, we will be okay with leaving radicals in the denominators of numerical expressions for the rest of mathematics.

Example 3.1.5

Matterhorn Chocolate bars are sold in boxes shaped like triangular prisms. The two triangular ends of a box are equilateral triangles with 8-centimeter sides. When stacking the boxes on a shelf, the second level will be a triangle's altitude above the bottom level. Find the exact length of the triangle's altitude, h.

Solution available at the end of this section.

Checkpoint 3.1.5

Show that $\cos^2(60^\circ) + \sin^2(60^\circ) = 1$.

Answer available at the end of this section.

Example 3.1.6

Let $x = 60^\circ$ and $y = 45^\circ$. Evaluate:

$$\sqrt{3}\csc(x) - \frac{5}{\sqrt{2}}\sec(y) + \sqrt{6}\tan(90^\circ - x)\sin(y)$$

Solution available at the end of this section.

Checkpoint 3.1.6

Evaluate and write as a single expression:

$$3\cos(45^\circ) - 2\cot(30^\circ)$$

Answer available at the end of this section.

Cofunction Identities

The following definition may answer a burning question you have had since being introduced to trigonometric functions.

Definition: Cofunction

The sine and cosine are called **cofunctions**. Likewise, the tangent and cotangent, as well as the secant and cosecant, are called cofunctions.

We say the sine is the cofunction of the cosine, and the cosine is the cofunction of the sine. The same holds for tangent and cotangent, and secant and cosecant. Now that we have a definition for cofunctions, what's the big deal?

The right triangle definition of the trigonometric functions reveals a new identity that was not so obvious when working with the coordinate definition of trigonometric functions. Consider the right triangle in Figure 3.1.8 below.

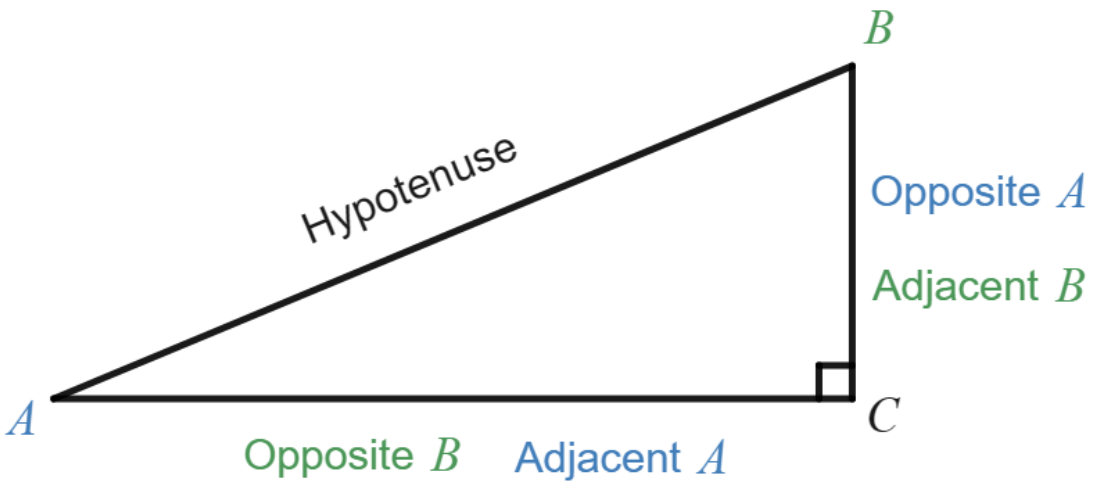


Figure 3.1.8

By the Triangle Sum, $\angle A + \angle B + \angle C = 180^\circ$; however, $\angle C = 90^\circ$. Therefore, $\angle A + \angle B = 90^\circ$. That is, $\angle A = 90^\circ - \angle B$. By the right triangle definition,

$$\sin(\angle A) = \frac{\text{Opposite } A}{\text{Hypotenuse}} = \frac{\text{Adjacent } B}{\text{Hypotenuse}} = \cos(\angle B) = \cos(90^\circ - \angle A)$$

$$\sec(\angle A) = \frac{\text{Hypotenuse}}{\text{Adjacent } A} = \frac{\text{Hypotenuse}}{\text{Opposite } B} = \csc(\angle B) = \csc(90^\circ - \angle A)$$

$$\tan(\angle A) = \frac{\text{Opposite } A}{\text{Adjacent } A} = \frac{\text{Adjacent } B}{\text{Opposite } B} = \cot(\angle B) = \cot(90^\circ - \angle A)$$

At this point, I hope you can see a budding relationship between a function and its cofunction. The following theorem summarizes the results (the proof of which will be held off until we have more mathematics to work with).

Theorem: Cofunction Identities

The trigonometric function of an angle is equal to the cofunction evaluated at the complement of the angle. That is,

$$\begin{aligned} \sin(\theta) &= \cos(90^\circ - \theta) \quad \text{and} \quad \cos(\theta) = \sin(90^\circ - \theta) \\ \sec(\theta) &= \csc(90^\circ - \theta) \quad \text{and} \quad \csc(\theta) = \sec(90^\circ - \theta) \\ \tan(\theta) &= \cot(90^\circ - \theta) \quad \text{and} \quad \cot(\theta) = \tan(90^\circ - \theta) \end{aligned}$$

Example 3.1.7

Evaluate (by hand):

$$\sin^2(13^\circ) + \sin^2(77^\circ)$$

Solution available at the end of this section.

Checkpoint 3.1.7

Write the function in terms of its cofunction.

a. $\sin(13^\circ)$
b. $\cot(60^\circ)$
c. $\sec(y)$

Answers available at the end of this section.

Observations of Values for Trigonometric Functions Using Right Triangle Trigonometry

We have now spent a decent amount of time with the trigonometric functions—both defined in terms of coordinates on the Cartesian plane and in terms of ratios of sides of right triangles. It's now time to formalize some observations.

Trigonometric Functions are Invariant When Scaling Right Triangles

The trigonometric function of an angle is the same for any right triangle, no matter the size or orientation of the triangle. Another way to say this is the values of the trigonometric functions are **invariant** under scaling of the right triangle. For example, Figure 3.1.9 (below) shows three right triangles, each with a 50° angle. Although the sides of the triangle may be bigger or smaller, the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ is always the same for that angle because the triangles are similar. Therefore, $\sin(50^\circ)$ is the same for all three right triangles.

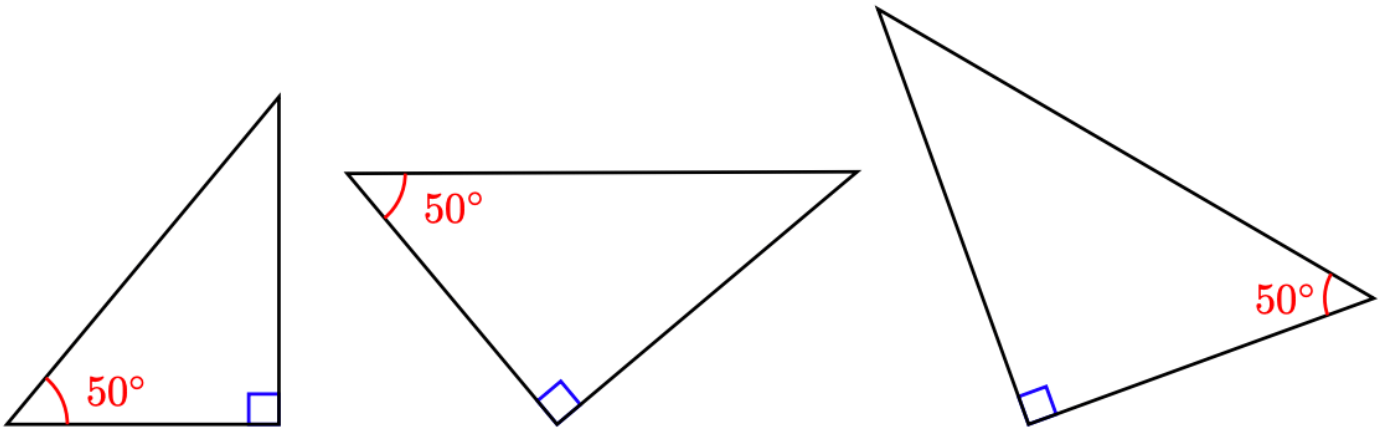


Figure 3.1.9

Similarly, the values of all the other trigonometric functions at 50° remain constant when we scale the triangles to be larger or smaller.

Interactive Element: Trig Ratios Are Invariant Under Scaling

The following Interactive Element might help you visualize why the trigonometric ratios of a fixed angle do not depend on the size of the right triangle.

Interact: Drag the red point along the horizontal base to grow or shrink the triangle; the 50° angle at the corner stays fixed.

Observation: Every side length changes as you resize the triangle, yet the ratio $\frac{\text{opposite}}{\text{hypotenuse}} = \sin(50^\circ)$ stays the same—similar triangles share the same trigonometric ratios.

Some Trigonometric Functions Have Limited Ranges

Consider the right triangle definition of the trigonometric functions. Since the hypotenuse is the longest side of a right triangle, the ratios

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{and} \quad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

will always be less than 1.

Moreover, if we think about the coordinate definition of the trigonometric functions,

$$\sin(\theta) = \frac{y}{r} \quad \text{and} \quad \cos(\theta) = \frac{x}{r}.$$

Since $r = \sqrt{x^2+y^2} \geq \sqrt{y^2} = |y|$ and $r = \sqrt{x^2+y^2} \geq \sqrt{x^2} = |x|$,

$$|y| \leq r \implies \frac{|y|}{r} \leq 1 \implies \left|\frac{y}{r}\right| \leq 1 \implies |\sin(\theta)| \leq 1 \implies -1 \leq \sin(\theta) \leq 1$$

and

$$|x| \leq r \implies \frac{|x|}{r} \leq 1 \implies \left|\frac{x}{r}\right| \leq 1 \implies |\cos(\theta)| \leq 1 \implies -1 \leq \cos(\theta) \leq 1$$

The only times the sine or cosine become 1 or -1 is when the given angle is a quadrantal angle.

Example 3.1.8

If θ is an angle within a right triangle, is it possible for $\sec(\theta) = \frac{1}{2}$? Explain.

Solution available at the end of this section.

Checkpoint 3.1.8

Can $\csc(\theta)$ ever be 2?

Answer available at the end of this section.

Using Special Angles to Approximate Trigonometric Functions at Other Angles

In a right triangle, as the angle θ increases, $\sin(\theta)$ increases but $\cos(\theta)$ decreases. Figure 3.1.10 (below) demonstrates why this is true. In each right triangle, the hypotenuse has the same length. However, as the angle increases, the opposite side gets longer, and the adjacent side gets shorter.

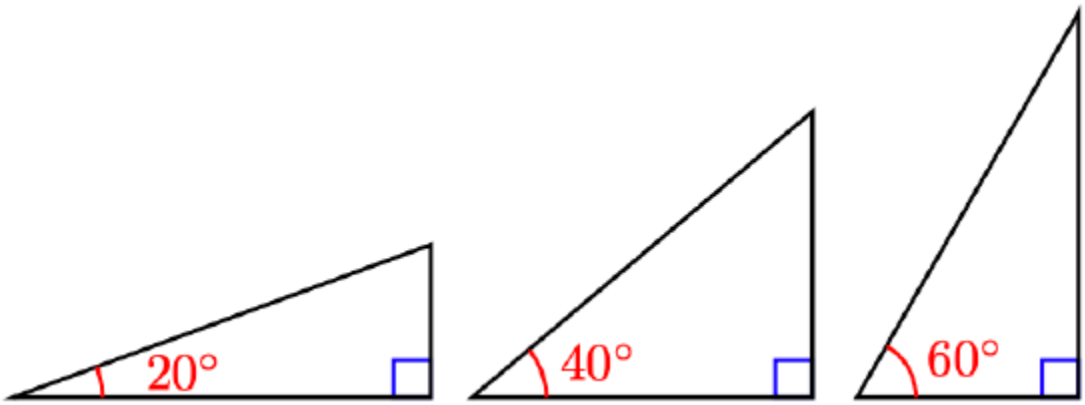


Figure 3.1.10

Interactive Element: As the Angle Opens, Sine Rises and Cosine Falls

The following Interactive Element might help you visualize how sine and cosine change as the acute angle grows while the hypotenuse is held fixed.

Interact: Drag the red point along the base; the top corner glides along the arc of the fixed-length hypotenuse.

Observation: As the angle increases, the opposite side lengthens and the adjacent side shortens, so $\sin$ increases toward 1 while $\cos$ decreases toward 0.

We can use the special angles as benchmarks for estimating and mental calculation. For example, we know that $\sin(60°) \approx 0.8660$, so if $\sin(\theta) = 0.95$ for some unknown angle θ within a right triangle, we know that $\theta > 60°$, because as θ increases, the sine of θ increases as well.

Example 3.1.9

If $\cos(\alpha) > \frac{\sqrt{3}}{2}$, where α is an angle within a right triangle, what can we say about α?

Solution available at the end of this section.

Checkpoint 3.1.9

If $1 < \tan(\beta) < \sqrt{3}$, where β is one of the angles in a right triangle, what can we say about β?

Answer available at the end of this section.

Hidden Resources: CalFresh

CalFresh (formerly known as Food Stamps) is a nutrition program that helps students and their families buy healthy foods.

How Does CalFresh Work?

CalFresh benefits are issued on an electronic benefit transfer (EBT) card that works like a debit card and can be used at most grocery stores and many farmers' markets to buy food. You can even use CalFresh to purchase food online through Amazon and Walmart.com.

If you are on a limited income and are struggling to buy the healthy foods that you need, then CalFresh may be able to help! Plus, CalFresh benefits do not impact your financial aid, and you do not need your parents' tax return information to apply.

For more information, please see this page on the CRC website.

Success in Mathematics: Why Study Mathematics? (Part 1)

Until modern times, only a tiny percent of people had the opportunity to be "liberally educated." These people were not studying to learn a job or trade - they wanted to learn what other humans had thought and created. They were also trying to develop new knowledge. Menaechmus (350 B.C.E) was this kind of student. As he studied geometry, he discovered the properties of conic sections. Others built on his knowledge. Apollonius (350 B.C.E) wrote an eight-chapter book about conic sections.

Today we use our knowledge of conic sections to predict orbits, build telescopes, locate ships at sea, make "eyes" for robots, and collect solar energy. But we also study conic sections as an accomplishment of our human ancestors. In developing their equations, we also develop logic and creative problem-solving skills. Think about the football coach who has his team doing agility drills and running through tires. Why? When has a football team ever had to run through tires during a game? Similarly, with the right attitude, you can develop your ability to reason and think in your life outside of college by doing mathematics. You are responsible for recognizing that learning and studying may be valuable even when it is not immediately practical.

> One problem with teaching mathematics in the K-12 system - and I see it as a major difficulty - is that there is virtually nothing the pupils learn that has a non-trivial application in today's world. The most a teacher can tell a student who inquires, entirely reasonably, "How is this useful?" is that almost all mathematics finds uses, in many cases important ones, and that what they learn in school leads on to mathematics that definitely is used.
>
> - Keith Devlin, Standford University

By "non-trivial," Dr. Devlin means that many real-life applications require mathematics beyond what is studied in high school.

1. Younger students often ask, " When will I use this math?" Their teachers often reply, "You need to learn this math to go on to the next course. In college, you can take math that is very powerful for doing all kinds of practical things." Do you think this is the best reply to give to the student? Explain.
2. I've often pondered the best mixture of application and theory in a mathematics course. My love of mathematics lies in theory; however, my research in mathematics is in applications. As a student, which do you prefer? Explain.

Answers

Section Preview

This section introduces the student to the Right Triangle definition of the trigonometric functions. As such, all angles will be acute (and in degrees); however, if given an angle (rather than the sides of a triangle) and asked to evaluate a trigonometric function, the angle will strictly be an acute *special angle*. Therefore, using technology (e.g., calculators) to evaluate any trigonometric function in this section is *strictly avoided* so as to get the student familiar with right triangle trigonometry at a fundamental level.

Section Preview

- **Geometry**
  - The Pythagorean Theorem, Distance Formula, and Midpoint Formula: Students must understand the Pythagorean Theorem before the right triangle definition can be applied.

Section Preview

- Find the value of a trigonometric function of an unknown angle in a right triangle given at least two sides of the triangle.
- Compute the values of the remaining trigonometric functions of an unknown angle within a right triangle given one trigonometric ratio.
- Use the Ratio, Reciprocal, and Pythagorean Identities, along with Right Triangle Trigonometry, to compute trigonometric values.
- Find the exact value of a trigonometric function for a special angle.
- Use the Cofunction Identities to find the value of a trigonometric function.
- Use exact values to simplify an expression involving trigonometric functions.
- Recognize and understand the constraints on the values of certain trigonometric functions.

Example 3.1.1

a. The side opposite angle ϕ is 5 and the hypotenuse is 13. Let x be the length of the side adjacent to ϕ. To find x, we invoke the Pythagorean Theorem.

$$\begin{aligned} & x^2+5^2 = 13^2 && \text{(Pythagorean Theorem)} \\ \Longrightarrow\ & x^2+25 = 169 && \text{(simplifying)} \\ \Longrightarrow\ & x^2 = 144 && \text{(subtracting 25 from both sides)} \\ \Longrightarrow\ & x = \pm 12 && \text{(Extraction of Roots)} \\ \Longrightarrow\ & x = 12 && (x \text{ must be positive---it's a side length}) \end{aligned}$$

Therefore,

$$\begin{aligned} \sin(\phi) &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5}{13} & \csc(\phi) &= \frac{\text{hypotenuse}}{\text{opposite}} = \frac{13}{5} \\ \cos(\phi) &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{12}{13} & \sec(\phi) &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{13}{12} \\ \tan(\phi) &= \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{12} & \cot(\phi) &= \frac{\text{adjacent}}{\text{opposite}} = \frac{12}{5} \end{aligned}$$

b. The side opposite angle β is $\sqrt{5}$ and the hypotenuse is 3. Labeling the side adjacent to β as x, we get

$$\begin{aligned} & x^2+\left(\sqrt{5}\right)^2 = 3^2 && \text{(Pythagorean Theorem)} \\ \Longrightarrow\ & x^2+5 = 9 && \text{(simplifying)} \\ \Longrightarrow\ & x^2 = 4 && \text{(subtracting 5 from both sides)} \\ \Longrightarrow\ & x = 2 && \text{(Extraction of Roots)} \end{aligned}$$

Therefore,

$$\begin{aligned} \sin(\beta) &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{5}}{3} & \csc(\beta) &= \frac{\text{hypotenuse}}{\text{opposite}} = \frac{3}{\sqrt{5}} \\ \cos(\beta) &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2}{3} & \sec(\beta) &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{3}{2} \\ \tan(\beta) &= \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{5}}{2} & \cot(\phi) &= \frac{\text{adjacent}}{\text{opposite}} = \frac{2}{\sqrt{5}} \end{aligned}$$

Checkpoint 3.1.1

$\sin(\alpha) \approx 0.7442$

Example 3.1.2

We begin by finding the missing side, which we shall call x.

$$\begin{aligned} & D^2+x^2 = 2^2 && \text{(Pythagorean Theorem)} \\ \Longrightarrow\ & D^2+x^2 = 4 \\ \Longrightarrow\ & x^2 = 4-D^2 \\ \Longrightarrow\ & x = \sqrt{4-D^2} \end{aligned}$$

We can now fill in the values of the trigonometric ratios.

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{4-D^2}}{2} \qquad \csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{2}{\sqrt{4-D^2}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{D}{2} \qquad \sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{2}{D}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{4-D^2}}{D} \qquad \cot(\theta) = \frac{\text{adjacent}}{\text{opposite}} = \frac{D}{\sqrt{4-D^2}}$$

Checkpoint 3.1.2

$$\sin(\theta) = \frac{L}{h} \qquad \csc(\theta) = \frac{h}{L}$$

$$\cos(\theta) = \frac{\sqrt{h^2-L^2}}{h} \qquad \sec(\theta) = \frac{h}{\sqrt{h^2-L^2}}$$

$$\tan(\theta) = \frac{L}{\sqrt{h^2-L^2}} \qquad \cot(\theta) = \frac{\sqrt{h^2-L^2}}{L}$$

Example 3.1.3

It is best to sketch the right triangle with θ and where the side adjacent to θ has length 8 and the side opposite θ has length 15.

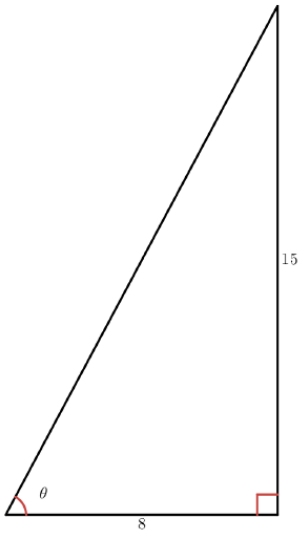


Figure 3.1.5

Using the Pythagorean Theorem, we get

$$\begin{aligned} 8^2 + 15^2 &= h^2 \\ \Longrightarrow \quad 289 &= h^2 \\ \Longrightarrow \quad 17 &= h \end{aligned}$$

Hence,

$$\sin(\theta) = \frac{15}{17} \Longrightarrow \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{15/17} = \frac{17}{15}$$

$$\cos(\theta) = \frac{8}{17} \Longrightarrow \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{8/17} = \frac{17}{8}$$

$$\tan(\theta) = \frac{15}{8} \Longrightarrow \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{1}{15/8} = \frac{8}{15}$$

Checkpoint 3.1.3

$$\sin(\beta) = \frac{\sqrt{3}}{2} \qquad \csc(\beta) = \frac{2}{\sqrt{3}}$$

$$\cos(\beta) = \frac{1}{2} \qquad \sec(\beta) = 2$$

$$\tan(\beta) = \sqrt{3} \qquad \cot(\beta) = \frac{1}{\sqrt{3}}$$

Example 3.1.4

$$\begin{aligned}\cot(\theta) &= \frac{\cos(\theta)}{\sin(\theta)}\\ &= \frac{2\sqrt{10}/7}{3/7}\\ &= \frac{2\sqrt{10}}{3}\end{aligned}$$

Checkpoint 3.1.4

$\sin(\lambda) = \frac{2}{\sqrt{13}}$

Example 3.1.5

We have seen this question before (near the beginning of the textbook). The altitude divides the triangle into two $30°$-$60°$-$90°$ right triangles, as shown in Figure 3.1.7 below.

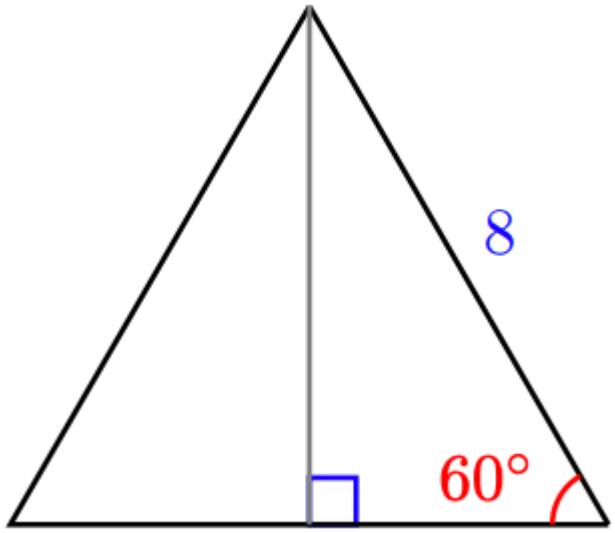


Figure 3.1.7

The altitude is adjacent to the $30°$ angle and the hypotenuse of the right triangle is 8 centimeters. Thus, using the fact that we already know $\cos(30°) = \frac{\sqrt{3}}{2}$, we get

$$\begin{aligned}\cos(30°) &= \frac{\text{adjacent}}{\text{hypotenuse}} && \text{(Fill in the values.)}\\ \Rightarrow \quad \frac{\sqrt{3}}{2} &= \frac{h}{8}\\ \Rightarrow \quad h &= 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3} && \text{(multiplying both sides by 8)}\end{aligned}$$

The altitude is *exactly* $4\sqrt{3}$ centimeters long. From this exact answer, we can find approximations to any degree of accuracy we like. You can check that $4\sqrt{3} \approx 6.9282$, so the altitude is approximately 6.9 centimeters long.

Checkpoint 3.1.5

$\cos^2(60°) + \sin^2(60°) = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1.$

Example 3.1.6

Using our two special triangles in Figure 3.1.6, we get

$$\begin{aligned}\sqrt{3}\csc(x) - \frac{5}{\sqrt{2}}\sec(y) + \sqrt{6}\tan(90° - x)\sin(y) &= \sqrt{3}\csc(60°) - \frac{5}{\sqrt{2}}\sec(45°) + \sqrt{6}\tan(90° - 60°)\sin(45°)\\ &= \sqrt{3}\csc(60°) - \frac{5}{\sqrt{2}}\sec(45°) + \sqrt{6}\tan(30°)\sin(45°)\\ &= \cancel{\sqrt{3}}\left(\frac{2}{\cancel{\sqrt{3}}}\right) - \left(\frac{5}{\cancel{\sqrt{2}}}\right)\left(\frac{\cancel{\sqrt{2}}}{1}\right) + \left(\cancel{\sqrt{6}}\right)\left(\frac{1}{\cancel{\sqrt{3}}}\right)\left(\frac{1}{\cancel{\sqrt{2}}}\right)\\ &= 2 - 5 + 1\\ &= -2\end{aligned}$$

Checkpoint 3.1.6

$\frac{3 - 2\sqrt{6}}{\sqrt{2}}$

Example 3.1.7

$$\begin{aligned}\sin^2(13^\circ)+\sin^2(77^\circ) &= \sin^2(13^\circ)+(\sin(77^\circ))^2\\ &= \sin^2(13^\circ)+(\cos(90^\circ-77^\circ))^2\\ &= \sin^2(13^\circ)+(\cos(13^\circ))^2\\ &= \sin^2(13^\circ)+\cos^2(13^\circ)\\ &= 1.\end{aligned}$$

Checkpoint 3.1.7

a. $\sin(13^\circ)=\cos(90^\circ-13^\circ)=\cos(77^\circ)$
b. $\cot(60^\circ)=\tan(90^\circ-60^\circ)=\tan(30^\circ)$
c. $\sec(y)=\csc(90^\circ-y)$

Example 3.1.8

$\sec(\theta)$ is the ratio of the hypotenuse to the adjacent side of θ; however, the hypotenuse of a right triangle is always longer than the other sides. Therefore, the ratio $\frac{\text{hypotenuse}}{\text{adjacent}}$ must be greater than 1 (by the discussion before this example, the secant can attain a value of 1 only if the given angle is a quadrantal angle).

Checkpoint 3.1.8

Yes!

Example 3.1.9

As an angle increases from 0° to 90°, its cosine decreases (why?). Now, $\cos(30^\circ)=\dfrac{\sqrt{3}}{2}$, so if $\cos(\alpha)>\frac{\sqrt{3}}{2}$, then α must be less than 30°.

Checkpoint 3.1.9

$45^\circ<\beta<60^\circ$

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3.1.1: Resources and Key Concepts

Resources

Videos

- Right Triangle Trigonometry
  - Right Triangle Trigonometry - Trigonometric Functions – A Right Triangle Approach
  - Trigonometry of Right Triangles - Review of Trigonometric Functions - A Right Triangle Approach
- Special Angles
  - Right Triangle Trigonometry - Evaluating Trigonometric Functions for the Special Acute Angles
  - Trigonometry of Right Triangles - Review of Special Triangles in Trigonometry
- Cofunction Identities
  - Right Triangle Trigonometry - The Cofunction Identities

Key Concepts

Definitions

- **Trigonometric Functions (Right Triangle Definition):** For an angle θ in a right triangle, the trigonometric functions are defined as ratios of the side opposite θ, the side adjacent to θ, and the hypotenuse. For example, $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$.
- **Pythagorean Triple:** A set of three positive integers, a, b, and c, such that $a^2 + b^2 = c^2$.
- **Pythagorean Triangle:** A right triangle whose side lengths form a Pythagorean triple.
- **Cofunction:** Pairs of trigonometric functions consisting of sine and cosine, tangent and cotangent, and secant and cosecant. For example, the cofunction of sine is cosine.

Theorems

- **Cofunction Identities:** The value of a trigonometric function of an angle is equal to the value of its cofunction at the complement of the angle. For example, $\sin(\theta) = \cos(90^\circ - \theta)$ and $\tan(\theta) = \cot(90^\circ - \theta)$.

Common Mistakes

- **Applying Right Triangle Trigonometry to Non-Right Triangles:** The right triangle definitions of trigonometric functions are only valid for angles within a right triangle. They cannot be applied to obtuse or acute triangles in general.
- **Confusing Exact and Approximate Values:** An exact value, like $\frac{\sqrt{3}}{2}$, is precise. A decimal approximation, like 0.8660, is a rounded value. For theoretical work, exact values are required.
- **Rationalizing Numerical Denominators Unnecessarily:** While rationalizing is sometimes useful, expressions like $\frac{2}{\sqrt{3}}$ are perfectly acceptable in trigonometry, and rationalizing them to $\frac{2\sqrt{3}}{3}$ is often an unnecessary step.

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3.1.2: Homework

Reading Questions

1. In a right triangle, what are the definitions of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ in terms of the opposite side, adjacent side, and hypotenuse?
2. What is a Pythagorean triple?
3. Can the right triangle definitions of the trigonometric functions be applied to an obtuse triangle?
4. If you are given the lengths of the opposite and adjacent sides in a right triangle, what theorem must you use to find the length of the hypotenuse?
5. What is the cofunction of the tangent function?
6. What does the Cofunction Identity theorem state about $\sin(20^\circ)$?
7. Why is the value of $\cos(\theta)$ always less than or equal to 1 for any acute angle θ?
8. Is it possible for $\sec(\theta)$ to equal $\frac{1}{2}$ for an angle θ in a right triangle? Explain why or why not.
9. What happens to the value of $\sin(\theta)$ as θ increases from 0° to 90°?
10. What is the difference between an exact value, like $\sqrt{2}$, and an approximate value, like 1.414?

Skills Refresher

Review the following skills you will need for this section.

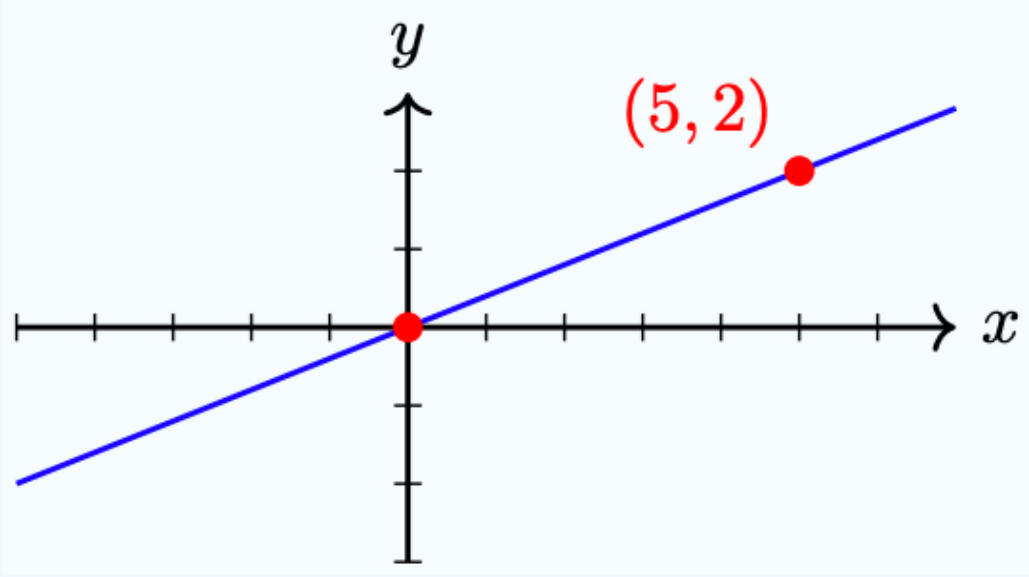
Skills Refresher

For the following exercises, write two more ratios equivalent to the given fraction.

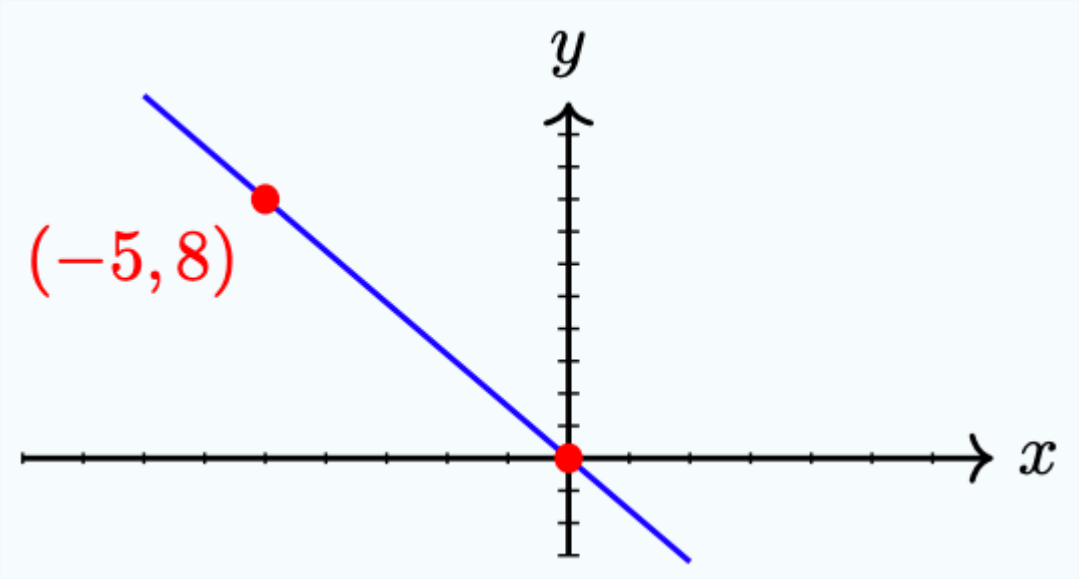
1. $\frac{10}{4}$
2. $\frac{6}{8}$
3. 0.6
4. 1.5

For the following exercises, compute the slope of the line.

5.

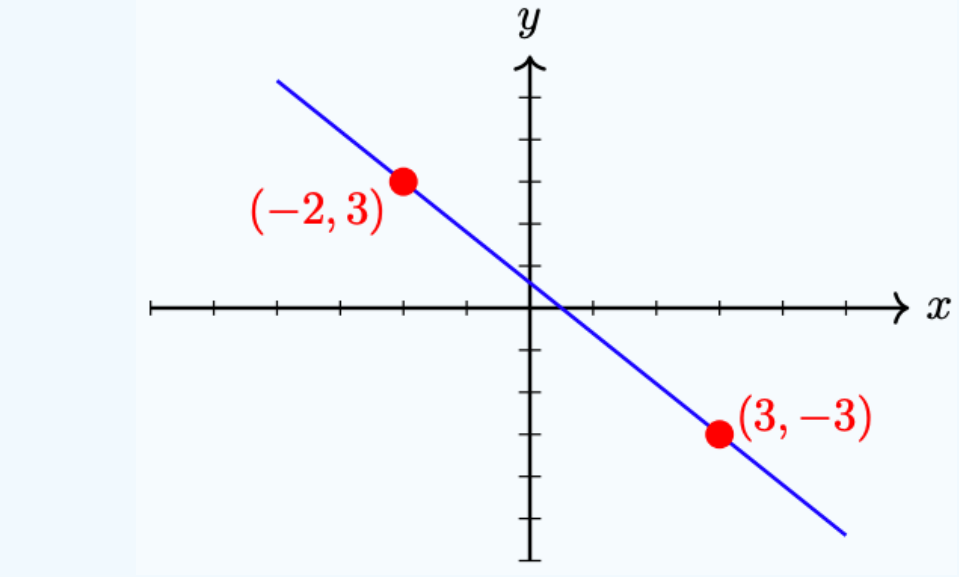


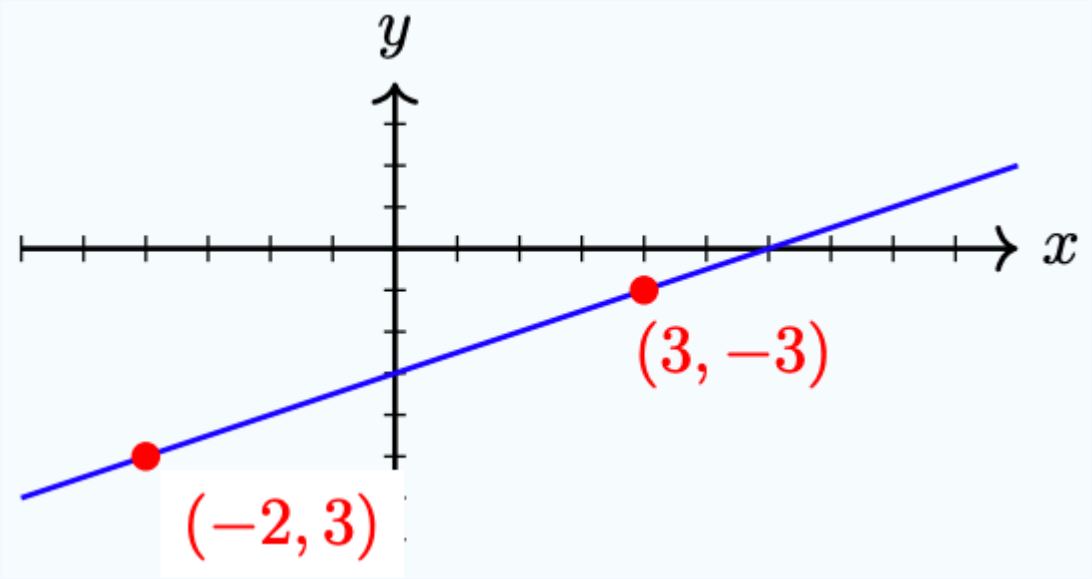
6.



7.

8.





For the following exercises, solve.

9. $\frac{12}{x} = 48$

10. $\frac{60}{x} = 80$

For the following exercises, solve the formula for the specified variable.

11. $C = 2\pi r$ for r (Circumference of a Circle)

12. $r = \frac{d}{t}$ for t

13. $P = 2a + b - c$ for a

14. $x = \frac{2}{3}y + 10$ for y

15. $S = 2\pi rh + 2\pi r^2$ for h (Surface Area of a Right Circular Cylinder)

Answers available at the end of this section.

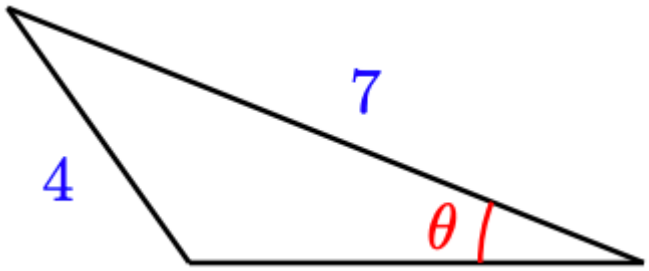
Homework

Vocabulary Check

1. The sequences of natural numbers, (3, 4, 5) and (7, 24, 25), are both examples of ___.
2. Just like the tangent and cotangent, the secant and cosecant are called ___.

Concept Check

3. Sketch a figure that illustrates why $\cos 25^\circ$ is the same for every right triangle with a 25° angle.
4. Sketch a figure that illustrates why $\cos\theta$ decreases as θ increases from 0° to 90°.
5. Linda says that $\sin\theta = \frac{4}{7}$ in the figure. Is she correct? Why or why not?



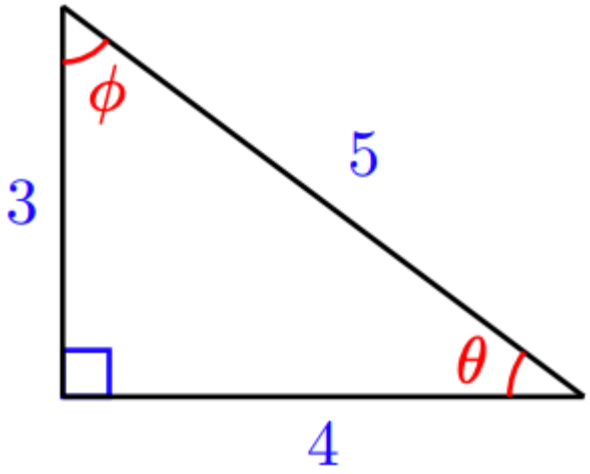
True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

6. Given a triangle having θ as one of its interior angles, the tangent of the angle θ is the ratio of the side opposite θ to the side adjacent to θ.
7. $\sin(60^\circ) = 0.8660$
8. The value of a trigonometric function of an angle is equal to the value of its cofunction at the complement of the angle.

Basic Skills

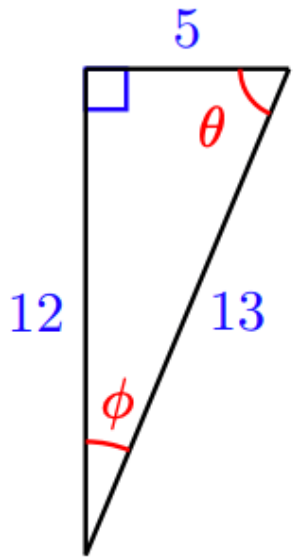
For the following exercises, fill in the table.

9.



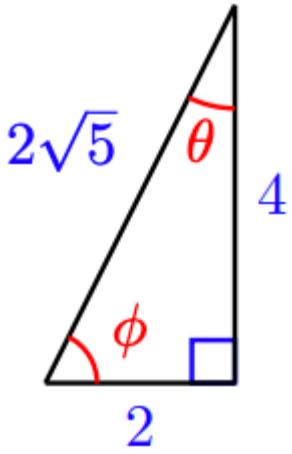
	sin	cos	tan
θ			
ϕ			

10.



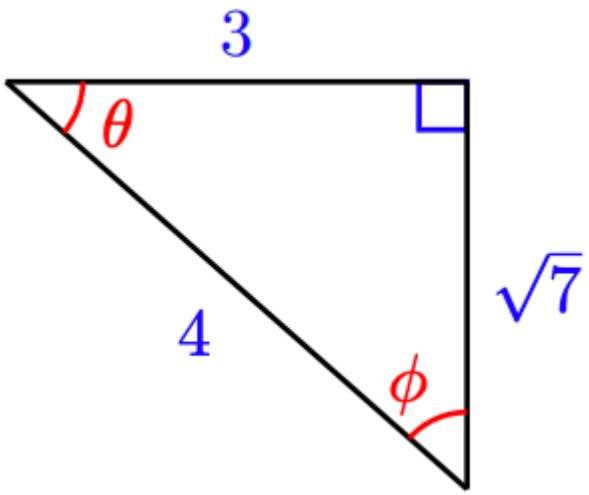
	sin	cos	tan
θ			
ϕ			

11.



	sin	cos	tan
θ			
ϕ			

12.



	sin	cos	tan
θ			
ϕ			

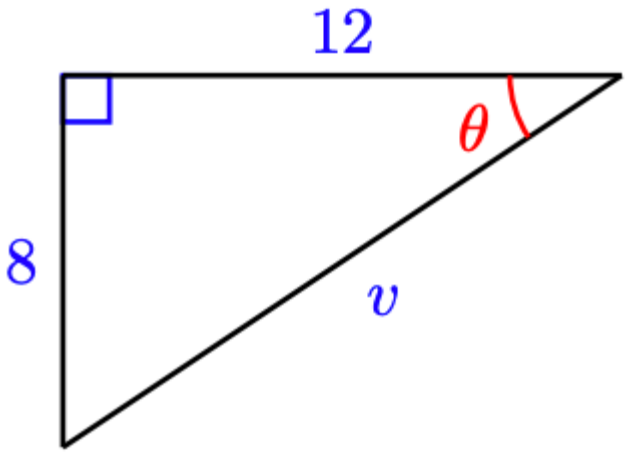
13. In each of the figures for the previous exercises, what is the relationship between the angles θ and ϕ?

14. Study the tables for the previous problems. What do you notice about the values of the sine and cosine for the angles θ and ϕ? Explain why this is true.

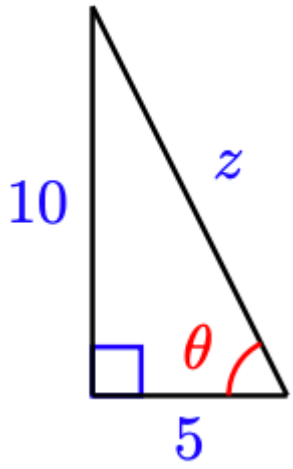
15. There is a relationship between the tangent, the sine, and the cosine of any angle. Study the tables for the previous problems to discover this relationship. Write your answer as an equation.

For the following exercises,
(a) Find the length of the unknown side.
(b) Find the sine, cosine, and tangent of θ. Round your answers to four decimal places.

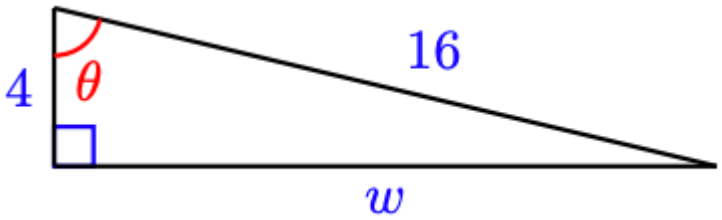
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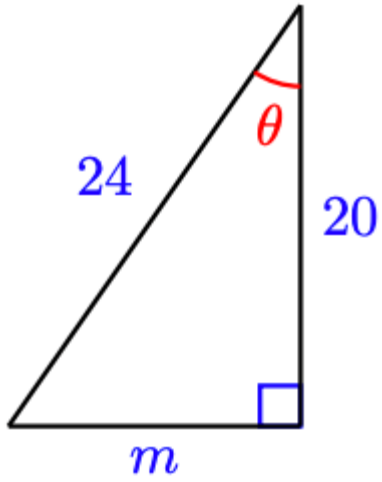
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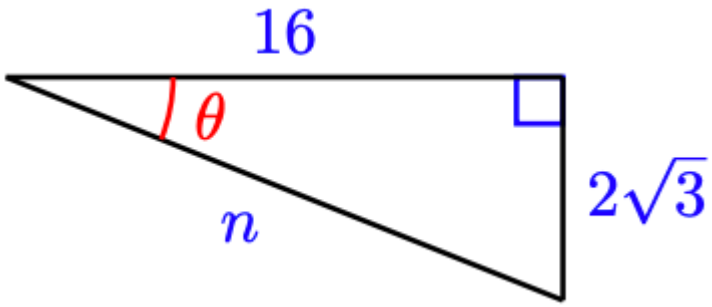
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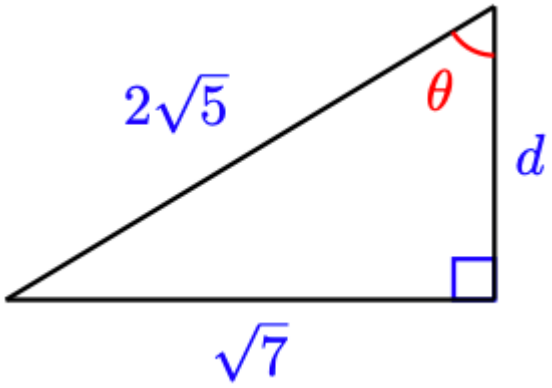
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20.

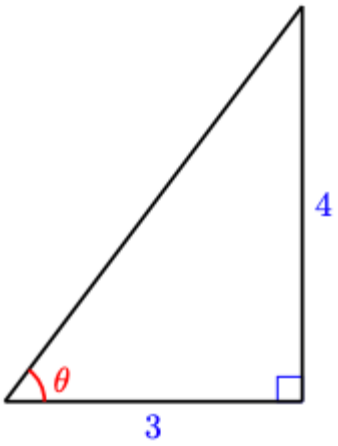


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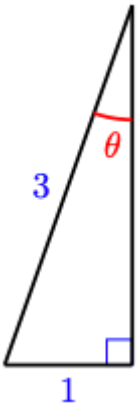


For the following exercises, find exact values for the six trigonometric ratios of the angle θ.

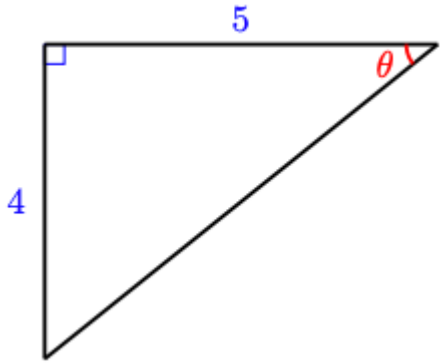
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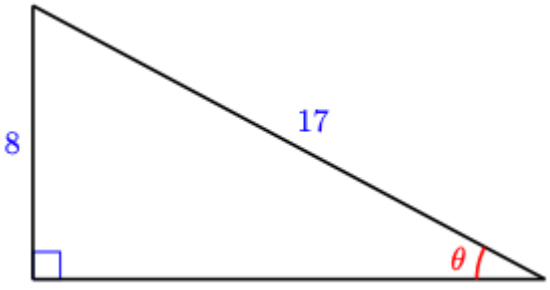
23.



24.



25.



For the following exercises,
(a) Sketch and label the sides of a right triangle with angle θ.

(b) Sketch and label another right triangle with angle θ and longer sides.

26. $\cos\theta = \frac{3}{5}$

27. $\tan\theta = \frac{7}{2}$

28. $\tan\theta = \frac{11}{4}$

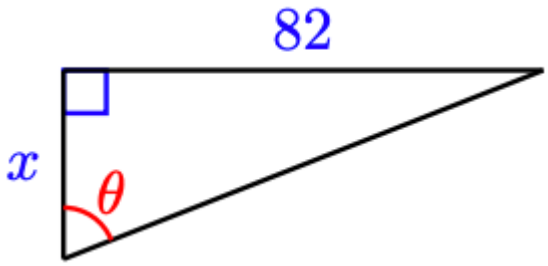
29. $\sin\theta = \frac{4}{9}$

30. $\sin\theta = \frac{1}{9}$

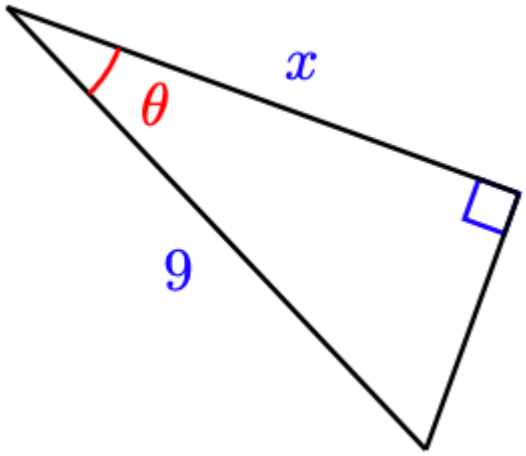
31. $\cos\theta = \frac{7}{8}$

For the following exercises, use a trigonometric ratio to write an equation for x in terms of θ.

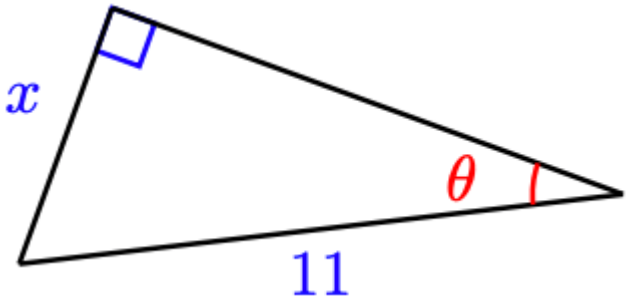
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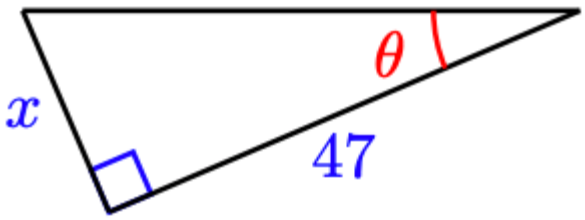


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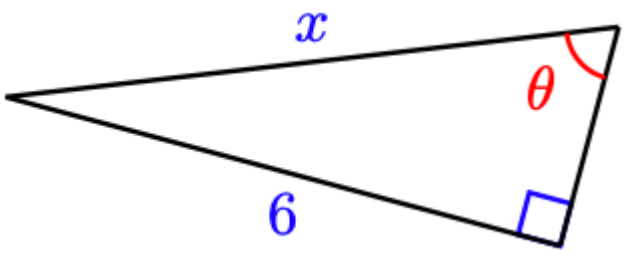


35.

36.



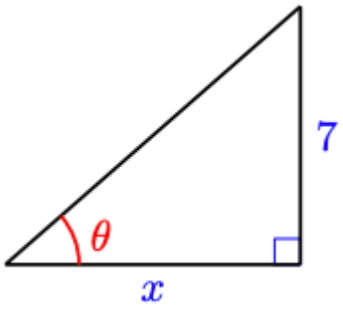
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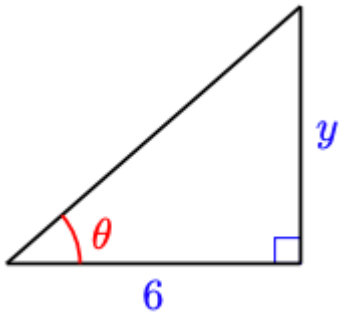
For the following exercises, write algebraic expressions for the six trigonometric ratios of the angle θ.

38.

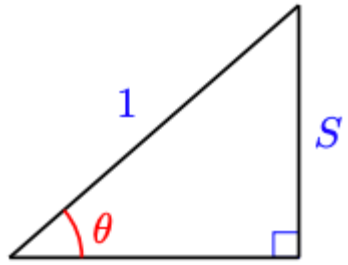
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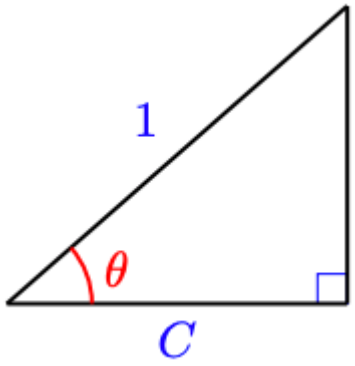


40.



41.





For the following exercises, write the expression in terms of sine and cosine, and simplify.

42. $\dfrac{\csc t}{\cot t}$

43. $\dfrac{\tan v}{\sec v}$

44. $\sec \beta - \tan \beta$

45. $\cot \alpha + \csc \alpha$

For the following exercises, choose all values from the list below that are exactly equal to, or decimal approximations for, the given trigonometric ratio. (Try not to use a calculator!)



46. $\cos 30^\circ$

47. $\sin 45^\circ$

48. $\tan 30^\circ$

49. $\cos 60^\circ$

For the following exercises, evaluate. Give exact values.

50. $\sin(30^\circ)$

51. $\tan(45^\circ)$

52. $\sin(60^\circ)$

53. $\csc(30^\circ)$

54. $\cot(45^\circ)$

55. $\csc(60^\circ)$

Synthesis Questions

For the following exercises, evaluate.

56. $4\cot(60^\circ) + 2\sec(45^\circ)$

57. $\frac{1}{2}\csc(30^\circ) - \frac{1}{4}\cot(30^\circ)$

For the following exercises, solve the equation assuming that θ is an acute angle in a right triangle.

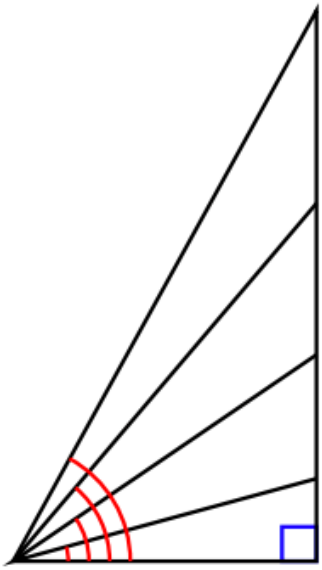
58. $3\csc(\theta) + 2 = 8$

59. $-2\sec(\theta) + 8 = 6$

60. $\sqrt{2}\sec(\theta) = 2$

61. $\sqrt{3}\cot(\theta) = 1$

62.

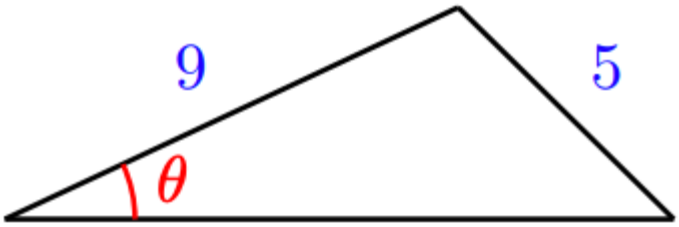


a. Use the figure to explain what happens to $\tan\theta$ as θ increases, and why.

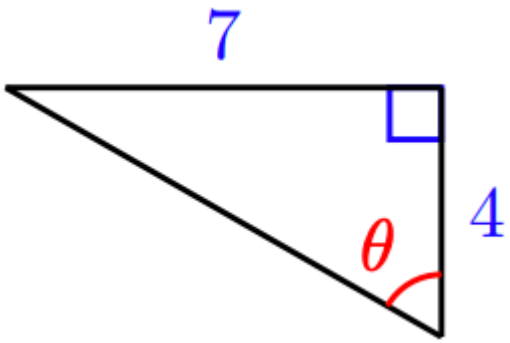
b. Use the figure to explain what happens to $\cos\theta$ as θ increases, and why.

For the following exercises, explain why the trigonometric ratio is *not* correct.

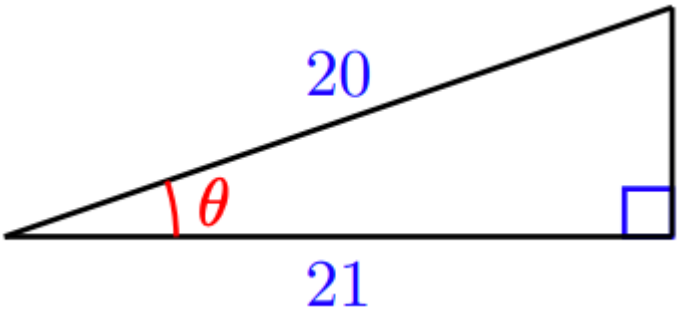
63. $\sin\theta = \frac{5}{9}$



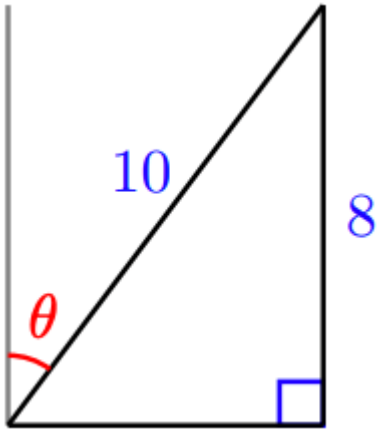
54. $\tan\theta = \frac{4}{7}$



55. $\cos\theta = \frac{21}{20}$



56. $\sin\theta = \frac{8}{10}$



For the following exercises, sketch and label a right triangle, then fill in the blank.

57. a. If $\sin\theta = 0.2358$, then $\cos(90^\circ - \theta) =$_____.

b. If $\cos\alpha = \frac{3}{11}$, then _____ $(90^\circ - \alpha) = \frac{3}{11}$.

c. If $\sin 42^\circ = n$, then $\cos$_____ $= n$.

d. If $\cos 13^\circ = z$, then $\sin$_____ $= z$.

58. a. If $\cos\beta = \frac{2}{\sqrt{7}}$, then $\sin(90^\circ - \beta) =$_____.

b. If $\sin\phi = 0.693$, then _____$-(90^\circ - \phi) = 0.693$.

c. If $\cos 87^\circ = p$, then $\sin$_____ $= p$.

d. If $\sin 59^\circ = w$, then $\cos$_____ $= w$.

59. a. If $\sin\phi = \frac{5}{13}$ and $\cos\phi = \frac{12}{13}$, then $\tan\phi =$_____.

b. If $\cos\beta = \frac{1}{\sqrt{10}}$, and $\sin\beta = \frac{3}{\sqrt{10}}$, then $\tan\beta =$_____.

c. If $\tan B = \frac{2}{\sqrt{5}}$ and $\cos B = \frac{\sqrt{5}}{3}$, then $\sin B =$_____.

d. If $\sin W = \sqrt{\frac{3}{7}}$ and $\tan W = \frac{\sqrt{3}}{2}$, then $\cos W =$_____.

60. a. If $\cos\theta = \frac{2}{\sqrt{10}}$ and $\sin\theta = \sqrt{\frac{3}{5}}$, then $\tan\theta =$_____.

b. If $\sin\alpha = \frac{\sqrt{2}}{4}$, and $\cos\alpha = \frac{\sqrt{14}}{4}$, then $\tan\alpha =$_____.

c. If $\tan A = \frac{\sqrt{7}}{3}$ and $\cos A = \frac{3}{4}$, then $\sin A =$_____.

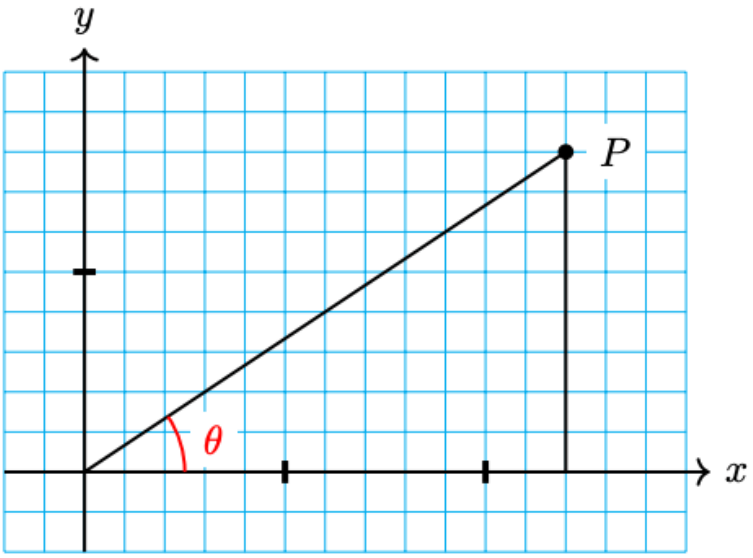
d. If $\sin V = \sqrt{\frac{10}{5}}$ and $\tan V = \frac{2}{5}$, then $\cos V =$_____.

71. Explain why the cosine of a 73° angle is always the same, no matter what size triangle the angle is in. Illustrate your explanation with a sketch.

72. a. What is the slope of the line through the origin and point P?

b. What is the tangent of the angle θ?

c. On the same grid, sketch an angle whose tangent is $\frac{8}{5}$



73. Which of the following numbers are equal to $\cos 45^\circ$?

a. $\frac{\sqrt{2}}{2}$

b. $\frac{1}{\sqrt{2}}$

c. $\frac{2}{\sqrt{2}}$

d. $\sqrt{2}$

74. Which of the following numbers are equal to $\tan 30^\circ$?

a. $\sqrt{3}$

b. $\frac{1}{\sqrt{3}}$

c. $\frac{\sqrt{3}}{3}$

d. $\frac{3}{\sqrt{3}}$

75. Which of the following numbers are equal to $\tan 60^\circ$?

a. $\sqrt{3}$

b. $\frac{1}{\sqrt{3}}$

c. $\frac{\sqrt{3}}{3}$

d. $\frac{3}{\sqrt{3}}$

76. Which of the following numbers are equal to $\sin 60^\circ$?

a. $\frac{3}{\sqrt{2}}$

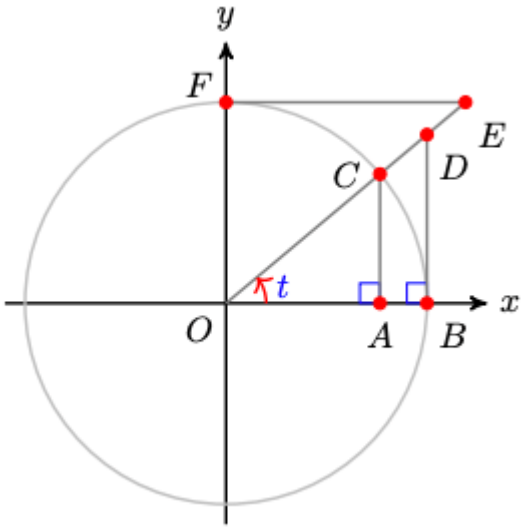
b. $\frac{\sqrt{3}}{2}$

c. $\frac{\sqrt{2}}{3}$

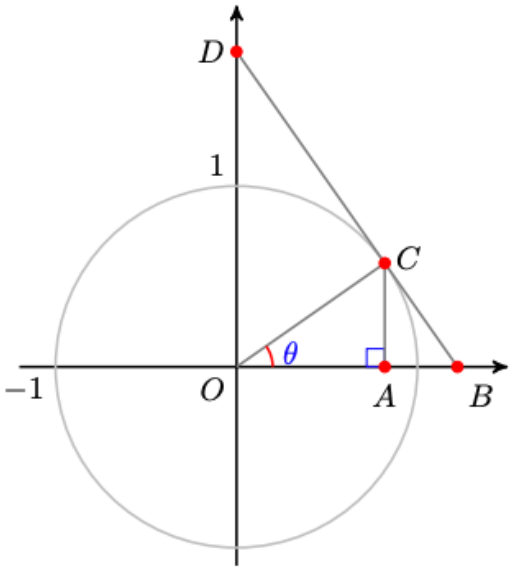
d. $\frac{2}{\sqrt{3}}$

Challenge Problems

77. The diagram shows a unit circle. Find six line segments whose lengths are, respectively, $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$.



78. The figure shows a unit circle and an angle θ in standard position. Each of the six trigonometric ratios for θ is represented by the length of a line segment in the figure. Find the line segment for each ratio, and explain your choice.



Answers

Skills Refresher

(Many answers are possible for 1-4.)

1. $\frac{5}{2}, \frac{20}{8}$
2. $\frac{3}{4}, \frac{12}{16}$
3. $\frac{3}{5}, \frac{12}{20}$
4. $\frac{6}{4}, \frac{12}{8}$
5. $\frac{2}{5}$
6. $\frac{-8}{5}$
7. $\frac{-6}{5}$
8. $\frac{1}{2}$
9. $\frac{1}{4}$
10. $\frac{3}{4}$
11. $r = \frac{C}{2\pi}$
12. $t = \frac{d}{r}$
13. $a = \frac{P-b+c}{2}$
14. $y = \frac{3}{2}(x - 10)$
15. $h = \frac{S-2\pi r^2}{2\pi r}$

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3.2: Technology and Right Triangle Trigonometry

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Using a Calculator to Compute the Values of Trigonometric Functions

The trigonometric values for non-special angles were originally collected into tables and are available on scientific calculators (and other, more recent, technology). For example, let's find the sine of 50°.

Note: Two Calculator Modes

Before evaluating any trigonometric function, ensure your technology is set to the correct "mode." In Trigonometry, the two standard modes are **degree mode** and **radian mode**. When first learning Trigonometry and the trigonometric functions, you will only be working with angles measured in degrees, so you must only concern yourself with degree mode. If using a scientific or graphing calculator, refer to your instruction manual on setting the calculator to degree mode. The scientific calculator in Desmos is set to degree mode by default, so you do not have to make any adjustments if using that technology; however, the *graphing* calculator in Desmos defaults to radian mode. To change Desmos' graphing calculator to degree mode, open the "Graph Settings" and select "Degrees."

Radian mode will be used almost excusively from Calculus onward.

Example 3.2.1

Approximate $\sin(50^\circ)$.

Scientific and Graphing Calculators available at the end of this section.

It is important to note that the values returned by the technology used in Example $3.2.1$ are not exact values; it turns out that the sine of 50° is an irrational number, and your calculator shows as many digits as its display will allow (not all sine values are as "nice" as the sine of 30°). Unless otherwise instructed, we round to four decimal places, so we write

$$\sin(50^\circ) \approx 0.7660.$$

Take note of using the approximation sign, $\approx$, instead of the equal sign—this is important!

Checkpoint 3.2.1

a. Use a calculator to complete the table, rounding answers to four decimal places.

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	89°
$\tan(\theta)$										

b. What do you notice about the values of $\tan(\theta)$ as θ increases from 0° to 89°? If the values of $\tan(\theta)$ are plotted against the values of θ, will the graph be a straight line? Why or why not?

Answers available at the end of this section.

Example 3.2.2

Use technology to compute the value of each trigonometric function.

a. $\cos(23.7^\circ)$
b. $\tan(72.13^\circ)$
c. $\sin^2(10^\circ)$

Solutions available at the end of this section.

Caution: Save Rounding for the Final Answer

It is important to note if your instructor hasn't said it already - **rounding is for display purposes only!** Never use a rounded result for other computations.

For example, if asked to compute $\sin^2(10^\circ)$ rounded to four decimal places, it would be incorrect to take the rounded approximation of $\sin(10^\circ)$ (which is 0.1736 and then square it to get approximately 0.0301. Instead, you would compute $\sin(10^\circ)$ and, without rounding, immediately square the unrounded result to arrive at the more accurate result of 0.0304.

While the difference between these two computations may seem insignificant, these rounding errors can propagate in such a way as to make massive differences between answers.

Checkpoint 3.2.2

a. Use a calculator to complete the table. Round the values of sine and cosine to four decimal places.

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
$\sin(\theta)$										
$\cos(\theta)$										

b. Is there a pattern to the values of sine and cosine? Why this is true? (Hint: If one (non-right) angle of a right triangle measures x degrees, how big is the other angle? Sketch the triangle and label each angle's opposite and adjacent sides.)

Answers available at the end of this section.

Now that we have access to technology to approximate the value of trigonometric functions at non-special angles, we can revisit simplifying and evaluating trigonometric expressions.

Example 3.2.3

Simplify, and evaluate for $z = 40^\circ$.

$$3\sin(z) - \sin(z)\tan(z) + 3\sin(z)$$

Solution available at the end of this section.

Checkpoint 3.2.3

Simplify, and evaluate for $x = 25^\circ, y = 70^\circ$.

$$3\cos(x) + \cos(y) - 2\cos(y) + \cos(x)$$

Answer available at the end of this section.

Unlike the fundamental trigonometric functions, calculators do not have keys for the cosecant, secant, and cotangent functions. Instead, we calculate their values as reciprocals.

Example 3.2.4

Use technology to approximate $\sec(47^\circ)$ to three decimal places.

Solution available at the end of this section.

Checkpoint 3.2.4

Use technology to approximate $\csc(82^\circ)$ to three decimal places.

Answer available at the end of this section.

Small Angles: Degrees, Minutes, and Seconds

The traditional angular measurement system we have covered is called the **Decimal Degree** (**DD**) system. Decimal Degrees offer simplicity and ease of use, making them a preferred choice for everyday applications like GPS navigation. However, it's essential to recognize that they are not a one-size-fits-all solution. In scenarios where precision is paramount, such as offshore navigation or land surveying, a slightly modified **Degree-Minute-Second** (**DMS**) system shines.

The DMS system divides degrees into smaller units called **minutes** and **seconds**. One minute is $\frac{1}{60}$ of a degree, and 1 second is $\frac{1}{60}$ of a minute, or $\frac{1}{3600}$ of a degree. We use the following notation for minutes and seconds.

Definition: Minutes and Seconds

- One minute: $1' = \dfrac{1^\circ}{60}$
- One second: $1'' = \dfrac{1'}{60} = \dfrac{1^\circ}{3600}$

To convert a measure of 42.125° to the DMS system, we start by noting that $42.125^\circ = 42^\circ + 0.125^\circ$. Converting the partial amount of degrees to minutes, we find $0.125^\circ\left(\frac{60'}{1^\circ}\right) = 7.5' = 7' + 0.5'$. Converting the partial amount of minutes to seconds gives $0.5'\left(\frac{60''}{1'}\right) = 30''$. Putting it all together yields

$$\begin{aligned} 42.125^\circ &= 42^\circ + 0.125^\circ \\ &= 42^\circ + 7.5' \\ &= 42^\circ + 7' + 0.5' \\ &= 42^\circ + 7' + 30'' \\ &= 42^\circ\ 7'\ 30'' \end{aligned}$$

On the other hand, to convert $117^\circ\ 15'\ 45''$ to decimal degrees, we first compute $15'\left(\frac{1^\circ}{60'}\right) = \frac{1}{4}^\circ$ and $45''\left(\frac{1^\circ}{3600''}\right) = \frac{1}{80}^\circ$. Then we find

$$\begin{aligned} 117^\circ 15' 45'' &= 117^\circ + 15' + 45'' \\ &= 117^\circ + \frac{1}{4}^\circ + \frac{1}{80}^\circ \\ &= \frac{9381}{80}^\circ \\ &= 117.2625^\circ \end{aligned}$$

Example 3.2.5

Let $\alpha = 111.371^\circ$ and $\beta = 37^\circ\ 28'\ 17''$.

a. Convert α to the DMS system. Round your answer to the nearest second.
b. Convert β to Decimal Degrees. Round your answer to the nearest thousandth of a degree.
c. Sketch α and β.
d. Find a supplementary angle for α.
e. Find a complementary angle for β.

Solutions available at the end of this section.

Checkpoint 3.2.5

a. Convert $10^\circ\ 45'$ to DD.
b. Convert 27.25° to DMS.

Answers available at the end of this section.

Calculator Note

Most scientific and graphing calculators have keys to enter angles in DMS format. You may also have keys or commands that let you convert from one format to another. Consult your calculator's manual to see how this is done with your particular model.

Some argue that DMS coordinates belong to a bygone era, overshadowed by the convenience of Decimal Degrees and digital mapping tools. However, this couldn't be further from the truth. DMS coordinates remain a cornerstone in various industries, including maritime, aviation, surveying, and astronomy, where precision and tradition go hand in hand. These coordinates have stood the test of time, adapting to modern technology while retaining their significance.

Another misconception is that DMS coordinates have limited utility, primarily confined to specialized fields. While DMS coordinates are prevalent in aviation, maritime, and surveying, they also have broader applications in geocaching, emergency services, and outdoor recreational activities. Their precision and familiarity make them a versatile choice in various contexts.

Sometimes, you need to add or subtract angles in the DMS system.

Example 3.2.6

Perform the arithmetic.

a. $23^\circ\ 16'\ 47'' + 17^\circ\ 50'\ 24''$
b. $23^\circ\ 16'\ 47'' - 17^\circ\ 50'\ 24''$

Solutions available at the end of this section.

Checkpoint 3.2.6

a. Add $53^\circ\ 34'\ 12''$ and $10^\circ\ 25'\ 55''$.
b. Subtract $23^\circ\ 12'$ from 77°.

Answers available at the end of this section.

Computing the value of a trigonometric function where the angle has been given in the DMS system is as simple as first converting the angle to the DD system.

Example 3.2.7

Evaluate $\cot(10^\circ\ 45')$.

Solution available at the end of this section.

Checkpoint 3.2.7

Evaluate $\sec(23^\circ\ 16'\ 47'')$.

Answer available at the end of this section.

Finding an Acute Angle Using a Calculator

While watching her niece at the playground, Nhat wonders how steep the slide is. She takes out her measuring tape and finds the slide to be 77 inches high over a horizontal distance of 136 inches, as shown below.

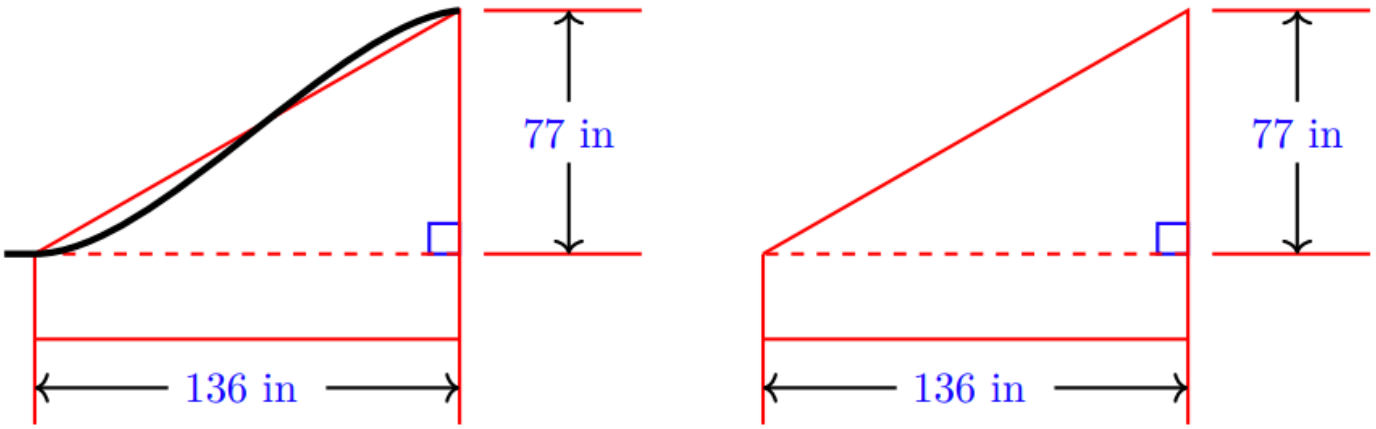


Figure 3.2.2

Nhat knows that one way to describe the steepness of an incline is to calculate its slope, which, in this case, is

$$\frac{\Delta y}{\Delta x} = \frac{77}{136} \approx 0.5662.$$

However, she wants to know what angle the slide makes with the horizontal. Thinking about it for a moment, she realizes that the slope she had just calculated is also the tangent of the angle she wants (can you see this from the figures above?). This begs the question: If we know the tangent of an angle, can we find the angle? The answer is yes!

Locate the key labeled TAN^{-1} on your calculator; it is probably the second function above the TAN key. Enter

$$\mathrm{2nd} \quad \mathrm{TAN} \quad 0.5662$$

to find that

$$\tan^{-1}(0.5662) \approx 29.5185^\circ.$$

This means that 29.5185° is the approximate angle whose tangent is 0.5662. We read the notation, $\tan^{-1}(0.5662) \approx 29.5185^\circ$ as "**inverse tangent** of 0.5662 is approximately 29.5185 degrees."

Note: Use Degree Mode

Until later in this course, you should make sure your calculator is set to "degree" mode. If you do not do this, your answers, while correct in a different angular measure system, will be incorrect.

Caution: Do Not Use Rounded Values for Calculations

Before describing the inverse tangent, we must note the flaw in our work. We *approximated* the value of the fraction, $\frac{77}{136}$, and then used that approximation to approximate the angle—this is very bad form. In truth, we should not have approximated $\frac{77}{136}$, but instead just computed

$$\tan^{-1}\left(\frac{77}{136}\right) \approx 29.5175^\circ.$$

Do you see that the returned angle from the inverse tangent is slightly different from our calculations using the rounded approximation of the fraction?

The moral of the story is that, whenever possible, **do not use rounded or approximate values to get yet another approximation**—this leads to a propagation of rounding errors.

Interactive Element: From Ratio to Angle

The following Interactive Element might help you visualize how the inverse tangent turns a rise-over-run ratio into an angle.

Interact: Drag the red point at the top of the ramp up and down. The run stays fixed at 1, so the rise you set is the ratio you would feed to $\tan^{-1}$; watch the angle $\theta = \tan^{-1}(\mathrm{rise}/\mathrm{run})$ respond.

Observation: The angle is completely determined by the ratio—but not proportionally. As the ramp gets steeper, ever-larger increases in the ratio nudge the angle only slightly closer to 90°. Drag down to a ratio near 0.57 to recover the $\approx 29.5^\circ$ of Nhat's slide.

When we find $\tan^{-1}$ of a number, we find an angle whose tangent is that number. Similarly, $\sin^{-1}$ and $\cos^{-1}$ are read as "**inverse sine**" and "**inverse cosine**." They find an angle with the given sine or cosine.

Example 3.2.8

Find the angle whose sine is 0.6834.

Solution available at the end of this section.

Note: Equivalent Statements

In Example 3.2.8, the two equations

$$\sin(43.11^\circ) \approx 0.6834 \quad \text{and} \quad \sin^{-1}(0.6834) \approx 43.11^\circ$$

say the same thing in different ways.

Caution: Inverse Trigonometric Functions are NOT Reciprocals

The notation $\sin^{-1}(x)$ does *not* mean $\frac{1}{\sin(x)}$!

We indeed use negative exponents to indicate reciprocals of numbers, for example, $a^{-1} = \frac{1}{a}$ and $3^{-1} = \frac{1}{3}$. However, "sin" by itself is not a variable.

- $\sin^{-1}(x)$ means "the angle whose sine is x"
- $\frac{1}{\sin(x)}$ means "the reciprocal of the sine of angle x"

(You may recall that $f^{-1}(x)$ denotes the inverse function for $f(x)$. We will study inverse trigonometric functions in a later chapter.)

Checkpoint 3.2.8

Write the following fact in two different ways: 68° is the angle whose cosine is 0.3746.

Answer available at the end of this section.

Note: Inputs and Outputs of Inverse Trigonometric Functions

Without going into too much detail (yet), it is essential to note that an inverse trigonometric function's input (a.k.a., the argument) is a ratio. The output is an angle.

Example 3.2.9

Find the acute angle for which the equation is true. Round answers to the nearest tenth.

a. $\cos(\beta) = 0.28$
b. $\tan(\kappa) = 1.93$

Solutions available at the end of this section.

Checkpoint 3.2.9

Find the acute angle for which $\tan(\theta) = 1.23$. Round your answer to the nearest tenth.

Answer available at the end of this section.

Finding acute angles is slightly more complex when the equation involves the cosecant, secant, or cotangent. A good rule of thumb is to rewrite the equation in terms of the fundamental trigonometric functions using the Reciprocal Identities.

Example 3.2.10

Suppose x is an acute angle within a right triangle. Solve each equation. Round answers to two decimal places.

a. $\sec(x) = 8.104$
b. $3\cot(x) - 2 = 7$

Solutions available at the end of this section.

Checkpoint 3.2.10

Let ω be an acute angle within a right triangle. Solve $2\csc(\omega) + 3 = 11$. Round the answer to two decimal places.

Answer available at the end of this section.

Activities

Activity: Trigonometric Ratios

A. Using Ratios and Proportions
Recall that two related quantities or variables are proportional if their ratio is always the same.

1. a. On any given day, the cost of filling up your car's gas tank is proportional to the number of gallons of gas you buy. For each purchase below, compute the ratio

$$\frac{\text{total cost of gasoline}}{\text{number of gallons}}$$

Gallons of Gas Purchased	Total Cost	$\frac{\text{Dollars}}{\text{Gallon}}$
5	\$14.45	
12	\$34.68	
18	\$52.02	

   b. Write an equation that you can solve to answer the question: How much does 21 gallons of gas cost? Use the ratio $\frac{\text{Dollars}}{\text{Gallon}}$ in your equation.
   c. Write an equation that you can solve to answer the question: How many gallons of gas can you buy for $\$46.24$? Use the ratio $\frac{\text{Dollars}}{\text{Gallon}}$ in your equation.
2. A recipe for coffee cake calls for $\frac{3}{4}$ cup of sugar and $1\frac{3}{4}$ cup of flour.
   a. What is the ratio of sugar to flour? Write your answer as a common fraction, then give a decimal approximation rounded to four places.

   For parts (b) and (c) below, write an equation that you can solve to answer the question. Use the ratio $\frac{\text{Amount of sugar}}{\text{Amount of flour}}$ your equation.
   b. How much sugar should you use if you use 4 cups of flour? Compute your answer in two ways: write the ratio as a common fraction and then write the ratio as a decimal approximation. Are your answers the same?
   c. How much flour should you use if you use 4 cups of sugar? Compute your answer in two ways: write the ratio as a common fraction and then write the ratio as a decimal approximation. Are your answers the same?
3. You are making a scale model of the Eiffel Tower, which is 324 meters tall and 125 meters wide at its base.

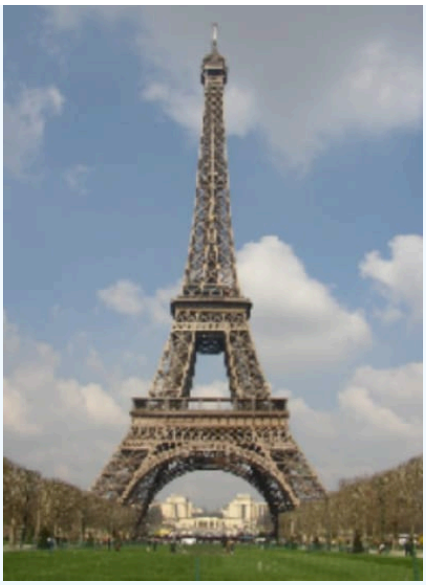


Figure 3.2.3

a. Compute the ratio of the base width to the tower's height. Round your answer to four decimal places.

Use your ratio to write equations and answer the questions below:

b. If the base of your model is 8 inches wide, how tall should the model be?
c. If you make a larger model that is 5 feet tall, how wide will the base be?

B. Similar Triangles

1. Recall that two triangles are similar if their corresponding sides are proportional. The corresponding angles of similar triangles are equal.
   a. What is the ratio of the two given sides in each triangle? Are the corresponding sides of the three triangles proportional? How do we know that $\alpha = \beta = \gamma$?

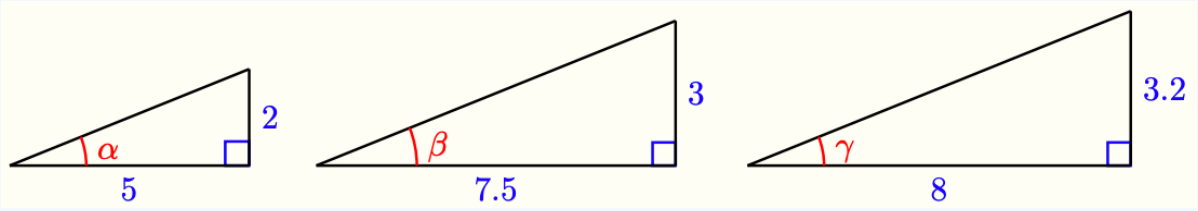


Figure 3.2.4

   b. Find the hypotenuse of each right triangle.
   c. Use the sides of the appropriate triangle to compute $\sin(\alpha), \sin(\beta)$, and $\sin(\gamma)$. Round your answers to four decimal places. Does the sine of an angle depend on the lengths of its sides?
   d. How do you know that the triangle below is similar to the three triangles in part (a)? Write an equation using the ratio from part (c) to find x.

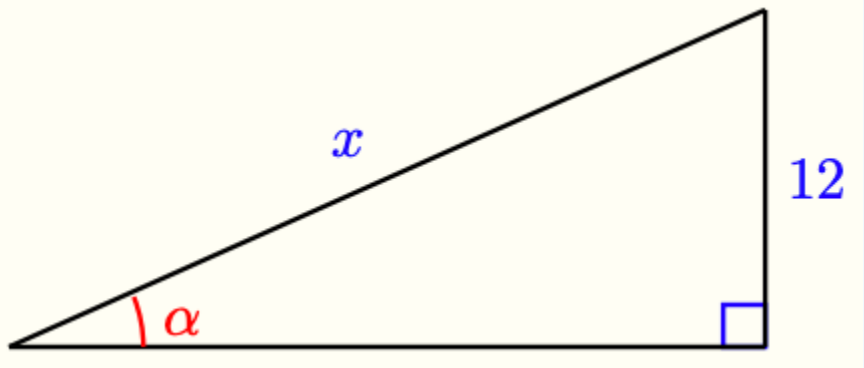


Figure 3.2.5

2. In the three right triangles below, the angle θ is the same size.

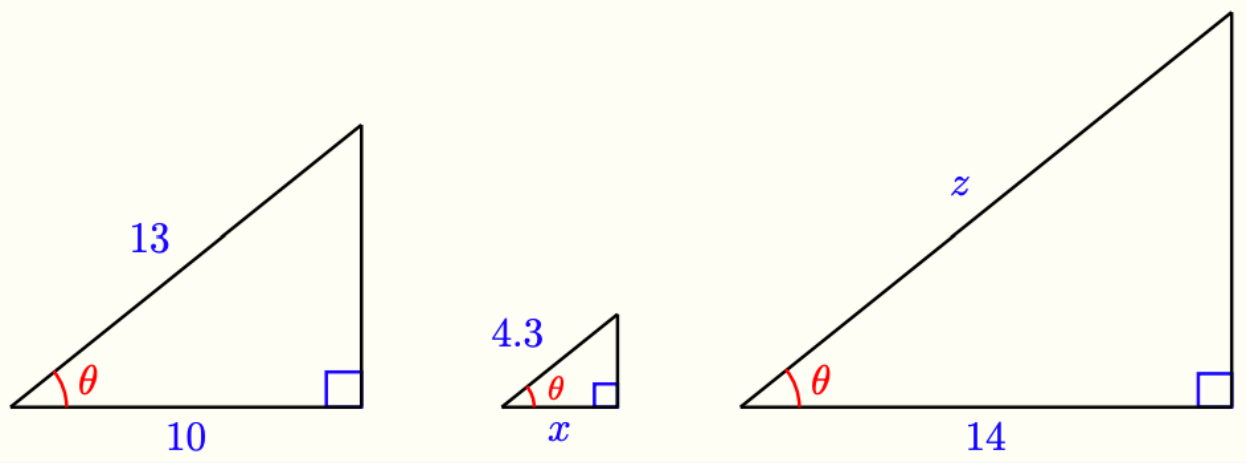


Figure 3.2.6

   a. Use the first triangle to calculate $\cos(\theta)$. Round your answer to four decimal places.
   b. In the second triangle, explain why $\frac{x}{4.3} = \frac{10}{13}$. Write an equation using your answer to part (a) and solve it to find x.
   c. Write and solve an equation to find z in the third triangle.

3. a. Use your calculator to find the value of $\frac{h}{2.4}$. (Hint: Which trigonometric ratio should you use?) What is the length of side h?

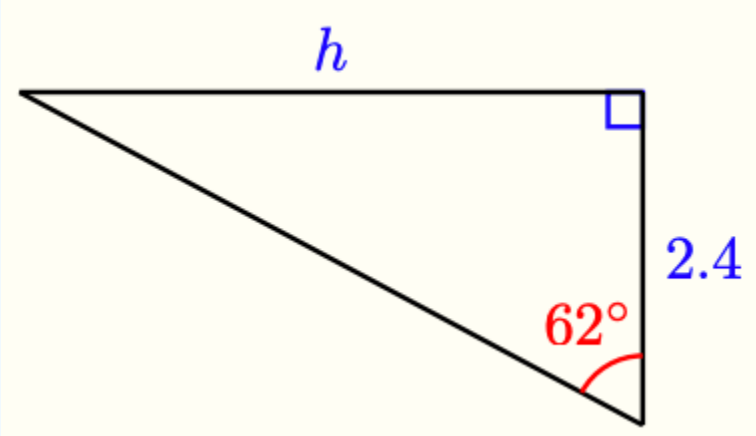


Figure 3.2.7

   b. What is the value of $\frac{6}{w}$ for the triangle below? Write an equation and solve for w.

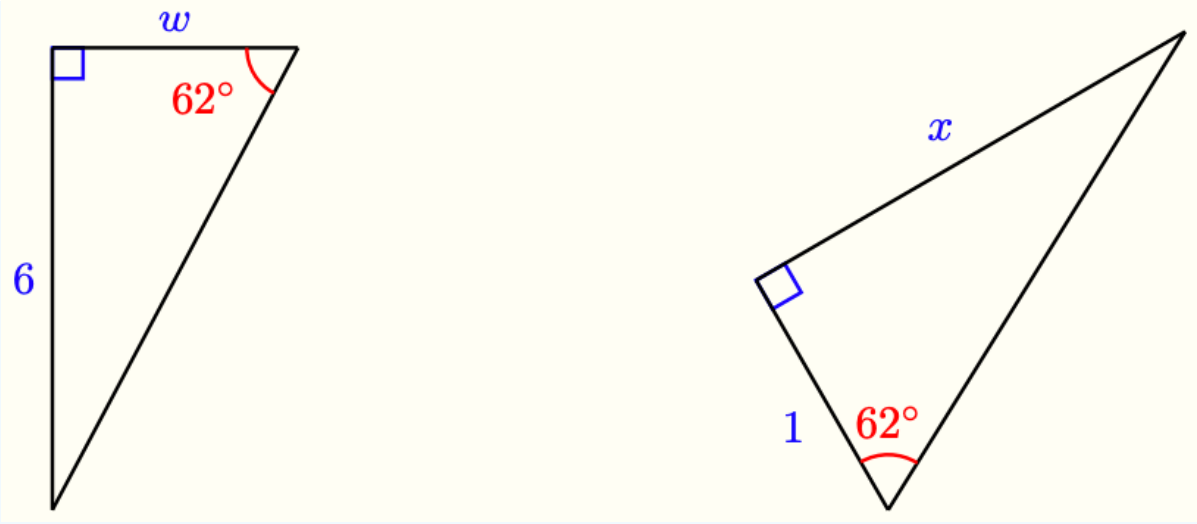


Figure 3.2.8

c. Write an equation and solve it to find x in the triangle above.

Activity: Obtuse Angles

An obtuse angle has measure between 90° and 180°, exclusively. In this activity, we will investigate the trigonometric functions of an obtuse angle.

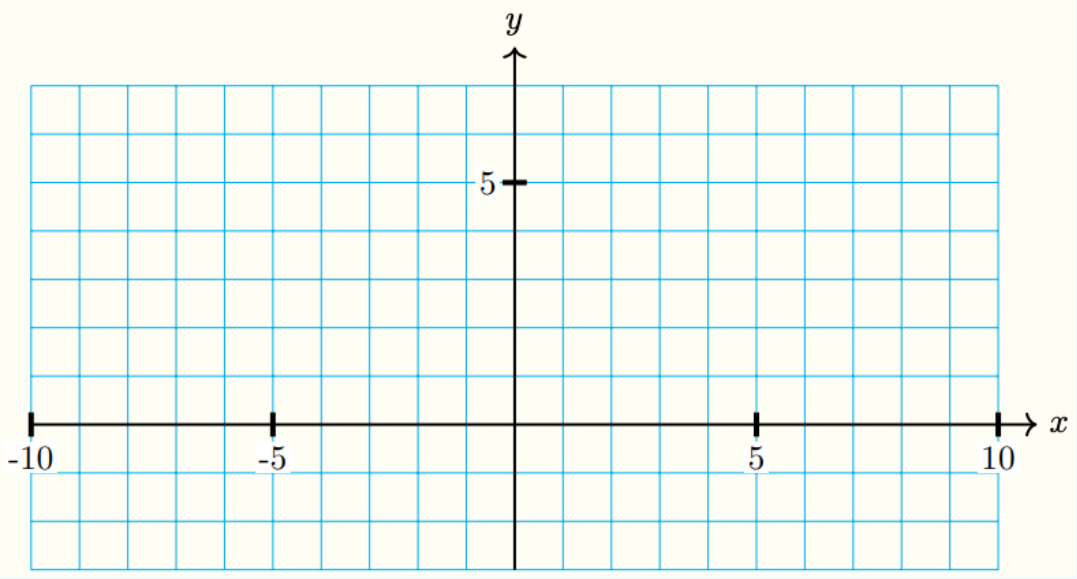


Figure 3.2.9

a. Draw an angle θ in standard position with the point $P(6, 4)$ on its terminal side.
b. Find r, the distance from the origin to P.
c. Calculate all six trigonometric functions of θ. Give both exact answers and decimal approximations rounded to four places.
d. Use the inverse cosine key on your calculator to find θ. Use your calculator to verify the values of $\sin(\theta), \cos(\theta)$, and $\tan(\theta)$ that you found in part (c).
e. Draw another angle ϕ in standard position with the point $Q(-6, 4)$ on its terminal side. Explain why ϕ is the supplement of θ. (Hint: Consider the right triangles formed by drawing vertical lines from P and Q).
f. Can you use the right triangle definition of the trigonometric functions to compute the sine and cosine of ϕ? Why or why not?

g. Calculate all six trigonometric functions of ϕ. How are these trigonometric values of ϕ related to the trigonometric values of θ?
h. Explain why θ and ϕ have the same sine but different cosines.
i. Use the inverse cosine key on your calculator to find ϕ. Use your calculator to verify the values of $\sin(\phi), \cos(\phi)$, and $\tan(\phi)$ that you found in part (g).
j. Compute $180^\circ - \phi$. What answer should you expect to get?

Success in College: Dealing With Failure (Part 1)

People sometimes do not achieve their goals. Athletes lose games. Students fail tests or classes. A job interview does not go well. It is your responsibility to learn how to cope with failure.

Former Chicago Bears football coach, Mike Ditka, said, "Success isn't permanent, and failure isn't fatal." This idea applies beyond the game of football. Failure on one test or even failing a class does not mean you cannot earn your degree. However, a continuing pattern of failures can prevent success.

When failure happens, you need to take action. Change the way you study. Redo the problems you missed on a test. Make an appointment with your professor to learn what you don't understand. If you are failing a course, meet with your advisor. Perhaps you should consider withdrawing. Make sure you know the effect of withdrawing or failing on any financial aid you receive.

Failure is not fun. It is tempting to skip class the day after a nasty test. It is easy to avoid talking to your professor about it. However, it is your responsibility to be honest about failure. Ignoring failure does not make it go away.

1. A class has four tests. A student taking this class fails one test with a score of 45%. The student's other test scores were 72%, 72%, and 79%. Find the student's test average for this class.
2. The effect of a bad test score depends on the number of tests in the class. It also depends on how much test scores contribute to the final course grade. If you had the same test average in this class as the student in the last question, could you still pass? Explain.

Success in College: Identifying Your Learning Style(s)

Visual/Verbal Learners

These people learn best when information is presented visually and in a written form. In a classroom setting, they prefer professors who use visual aids (i.e., whiteboard or PowerPoint presentation) to list the essential points of a lecture to provide them with an outline to follow. They benefit from information obtained from textbooks and class notes. These learners like to study by themselves in quiet environments. They visualize information in their "minds' eyes" to remember something. The online environment is especially appropriate for visual/verbal learners because most of the information for a course is presented in written form.

Visual/Nonverbal Learners

These people learn best when information is presented visually and in a picture or design format. In a classroom setting, they benefit from professors who supplement their lectures with materials such as film, video, maps, and diagrams. They relate well to information obtained from the images and charts in textbooks. They tend to prefer to work alone in quiet environments. They visualize an image of something in their mind when trying to remember it. These learners may also be artistic and enjoy visual art and design. The online environment is well suited for this learner because graphical representations of information can help them remember concepts and ideas. Graphical information can be presented using charts, tables, graphs, and images.

Auditory/Verbal Learners

These people learn best when information is presented aurally. They benefit from listening to lectures and participating in group

discussions in a classroom setting. They also benefit from obtaining information from audio recordings. When trying to remember something, they often repeat it aloud and mentally "hear" how the professor explained the information to them. They learn best when interacting with others in a listening/speaking activity. Online learning environments can complement these learners' styles. Although most information is presented visually (either written or graphically), group participation and collaborative activities are accomplished well online. In addition, streaming audio and computer conferencing can be incorporated into an online course to best meet the learning style of these students.

Tactile/Kinesthetic Learners

These people learn best when doing a physical "hands-on" activity. In the classroom, they prefer to learn new materials in a lab setting where they can touch and manipulate materials. They learn best in physically active learning situations. They benefit from professors who use in-class demonstrations, hands-on learning experiences, and fieldwork outside the classroom. Online environments can provide learning opportunities for tactile/kinesthetic learners. Simulations with 3-Dimensional graphics can replicate physical demonstrations. Lab sessions can be conducted at predetermined locations or homes and then discussed online. Also, outside fieldwork can be incorporated into the coursework, with ample online discussion preceding and following the experience. Finally, the online environment is well suited for the presentation and discussion of either group or individual projects and activities.

Take the learning style survey at Learning Styles.

1. What was your most extreme score, and what was it about?
2. Of all the results, which surprised you the most?

Answers

Section Preview

We finally reach the moment where we are evaluating trigonometric functions at non-special *acute* angles. As such, technology (e.g., Desmos or a calculator) will be necessary. This will also be needed to convert a Decimal Degree angle to DMS (and vice versa), as well as to find the *acute* angle within a right triangle given the sides of the triangle.

Section Preview

- **Arithmetic**
  - Unit Analysis and Unit Conversion: The DMS conversion work conceptually requires understanding that $1^\circ = 60'$ and $1' = 60''$ as a unit relationship.

Section Preview

- **Arithmetic**
  - Measurement, Approximation, Rounding, and Significant Figures: Rounding to specified decimal places requires the rounding skill.
- **Solving Equations**
  - Solving Linear Equations — Multistep with Variables on One Side: Solving equations like $3\cot(x) - 2 = 7$ for the trig function value before using the inverse key requires multi-step linear equation solving.

Section Preview

- Use a calculator to approximate the value of a trigonometric function.
- Perform addition and subtraction on angles expressed in DMS.
- Convert between DMS form and DD form.
- Use a calculator to approximate an acute angle given the value of a trigonometric function.

Example 3.2.1

Press the SIN button. Your calculator will display $\sin($ with an open parenthesis as the prompt to enter an angle. Type 50 and press ENTER (since your calculator should already be in degree mode, it will interpret the 50 as 50°). The calculator returns 0.7660444431 (please verify).

Using the scientific calculator in Desmos, type $\sin(50)$ and hit ENTER (since the scientific calculator in Desmos defaults to degree mode, it will interpret the 50 as 50°). Desmos returns 0.7660444431 (please verify).

Checkpoint 3.2.1

a.

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
$\tan(\theta)$	0	0.1763	0.3640	0.5774	0.8391	1.1918	1.7321	2.7475	5.6713	57.2900

b. The values of $\tan(\theta)$ increase seemingly without bound as θ increases from 0° to 89°. The graph will not be a line because the slopes between successive points are not constant.

Example 3.2.2

a. $\cos(23.7^\circ) \approx 0.9156625933 \approx 0.9157$
b. $\tan(72.13^\circ) \approx 3.101611189 \approx 3.1016$
c. To compute $\sin^2(10^\circ)$, we need to recall that

$$\sin^2(10^\circ) = [\sin(10^\circ)]^2.$$

Therefore, we start by getting the approximation $\sin(10^\circ) \approx 0.1736481777$. Using as many of these digits as possible, we then square the result to get

$$\sin^2(10^\circ) \approx (0.1736481777)^2 \approx 0.03015368961 \approx 0.0304.$$

Checkpoint 3.2.2

a.

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
$\sin(\theta)$	0	0.1737	0.3420	0.5	0.6428	0.7660	0.8660	0.9397	0.9848	1
$\cos(\theta)$	1	0.9849	0.9397	0.8660	0.7660	0.6428	0.5	0.3420	0.1737	0

b. The cosine of θ is equal to the sine of the complement of θ, or $\cos(\theta) = \sin(90^\circ - \theta)$. This result confirms the Cofunction Identity from the previous section.

Example 3.2.3

We can combine like terms to get

$$6\sin(z) - \sin(z)\tan(z).$$

Because 40° is not one of the angles for which we know exact trig values, we use a calculator to evaluate the expression. It is *very* important to make sure we save any rounding until the very end of the computation. Therefore, we enter the following keystrokes into our calculator (or other technology):

$$6 \times \text{SIN} \quad 40 \quad) \quad - \quad \text{SIN} \quad 40 \quad) \quad \times \quad \text{TAN} \quad 40 \quad \text{ENTER}$$

Doing so, we get approximately 3.3174.

Checkpoint 3.2.3

3.2832

Example 3.2.4

Enter

$$1 \div \cos 47) \text{ ENTER}$$

to obtain $\sec(47^\circ) \approx 1.466$. Or we can calculate $\cos(47^\circ)$ first, and then use the reciprocal key:

$$\cos 47) \text{ ENTER } x^{-1} \text{ ENTER}$$

Checkpoint 3.2.4

1.010

Example 3.2.5

a. To convert α to the DMS system, we start with $111.371^\circ = 111^\circ + 0.371^\circ$. Next we convert $0.371^\circ\left(\frac{60'}{1^\circ}\right) = 22.26'$. Writing $22.26' = 22' + 0.26'$, we convert $0.26'\left(\frac{60''}{1'}\right) = 15.6''$. Hence,

$$\begin{aligned} 111.371^\circ &= 111^\circ + 0.371^\circ \\ &= 111^\circ + 22.26' \\ &= 111^\circ + 22' + 0.26' \\ &= 111^\circ + 22' + 15.6'' \\ &= 111^\circ\, 22'\, 15.6'' \end{aligned}$$

Rounding to seconds, we obtain $\alpha \approx 111^\circ\ 22'\ 16''$.

b. To convert β to decimal degrees, we convert $28'\left(\frac{1^\circ}{60'}\right) = \frac{7}{15}^\circ$ and $17''\left(\frac{1^\circ}{3600''}\right) = \frac{17}{3600}^\circ$. Putting it all together, we have

$$\begin{aligned} 37^\circ\, 28'\, 17'' &= 37^\circ + 28' + 17'' \\ &= 37^\circ + \frac{7}{15}^\circ + \frac{17}{3600}^\circ \\ &= \frac{134897}{3600}^\circ \\ &\approx 37.471^\circ \end{aligned}$$

c. To sketch α, we first note that $90^\circ < \alpha < 180^\circ$. If we divide this range in half, we get $90^\circ < \alpha < 135^\circ$, and once more, we have $90^\circ < \alpha < 112.5^\circ$. This gives us a pretty good estimate for α, as shown below. Proceeding similarly for β, we find $0^\circ < \beta < 90^\circ$, then $0^\circ < \beta < 45^\circ$, $22.5^\circ < \beta < 45^\circ$, and lastly, $33.75^\circ < \beta < 45^\circ$.

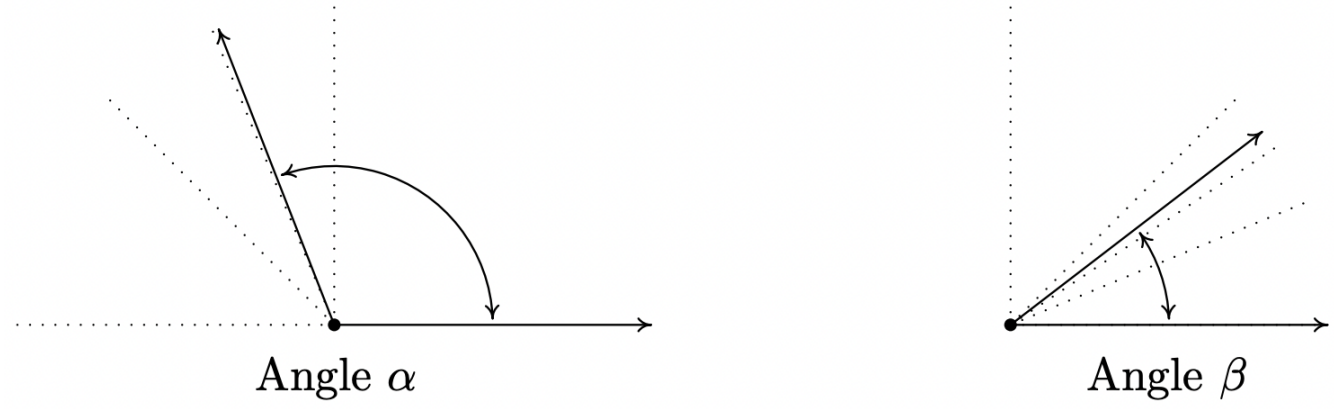


Figure 3.2.1

d. To find a supplementary angle for α, we seek an angle θ so that $\alpha + \theta = 180^\circ$. We get $\theta = 180^\circ - \alpha = 180^\circ - 111.371^\circ = 68.629^\circ$.
e. To find a complementary angle for β, we seek an angle γ so that $\beta + \gamma = 90^\circ$. We get $\gamma = 90^\circ - \beta = 90^\circ - 37^\circ\ 28'\ 17''$. While we could reach for the calculator to obtain an approximate answer, let's do something more mathematically engaging. We first rewrite $90^\circ = 90^\circ\ 0'\ 0'' = 89^\circ\ 60'\ 0'' = 89^\circ\ 59'\ 60''$. In essence, we are "borrowing" $1^\circ = 60'$ from the degree place, and then "borrowing" $1' = 60''$ from the minutes place. This yields, $\gamma = 90^\circ - 37^\circ\ 28'\ 17'' = 89^\circ\ 59'\ 60'' - 37^\circ\ 28'\ 17'' = 52^\circ\ 31'\ 43''$.

Checkpoint 3.2.5

a. 10.75°
b. $27^\circ\ 15'$

Example 3.2.6

a. When adding angles in the DMS system, it is best to perform the arithmetic as we did in elementary school. This is because we will be "carrying" between measures (i.e., carrying $60''$ to $1'$, and $60'$ to 1°).

$$\begin{array}{rrrr} & 23^\circ & 16' & 47'' \\ + & 17^\circ & 50' & 24'' \\ \hline & & & 71'' \end{array} \quad \Longrightarrow \quad \begin{array}{rrrr} & & & 1' \\ & 23^\circ & 16' & 47'' \\ + & 17^\circ & 50' & 24'' \\ \hline & & & 11'' \end{array} \qquad \text{(Carrying the } 60'' \text{ to } 1'\text{)}$$

$$\Longrightarrow \quad \begin{array}{rrrr} & & & 1' \\ & 23^\circ & 16' & 47'' \\ + & 17^\circ & 50' & 24'' \\ \hline & & 67' & 11'' \end{array}$$

$$\Longrightarrow \quad \begin{array}{rrrr} & 1^\circ & 1' & \\ & 23^\circ & 16' & 47'' \\ + & 17^\circ & 50' & 24'' \\ \hline & & 7' & 11'' \end{array} \qquad \text{(Carrying the } 60' \text{ to } 1^\circ\text{)}$$

$$\Longrightarrow \quad \begin{array}{rrrr} & 1^\circ & 1' & \\ & 23^\circ & 16' & 47'' \\ + & 17^\circ & 50' & 24'' \\ \hline & 41^\circ & 7' & 11'' \end{array}$$

Therefore, the sum is $41^\circ\ 7'\ 11''$.

b. When subtracting angles in the DMS system, it is best to perform the arithmetic like we did in elementary school. This is because we will be "borrowing" between measures (i.e., borrowing $1'$ to $60''$, and 1° to $60'$).

$$\begin{array}{rrrr} & 23^\circ & 16' & 47'' \\ - & 17^\circ & 50' & 24'' \\ \hline & & & 23'' \end{array} \quad \Longrightarrow \quad \begin{array}{rrrr} & 22^\circ & 76' & 47'' \\ - & 17^\circ & 50' & 24'' \\ \hline & & & 23'' \end{array} \qquad \text{(Borrowing the } 1^\circ \text{ to } 60'\text{)}$$

$$\Longrightarrow \quad \begin{array}{rrrr} & 22^\circ & 76' & 47'' \\ - & 17^\circ & 50' & 24'' \\ \hline & 5^\circ & 26' & 23'' \end{array}$$

Therefore, the difference is $5^\circ\ 26'\ 23''$.

Checkpoint 3.2.6

a. $64^\circ\ 0'\ 7''$

b. $53^\circ\ 48'$

Example 3.2.7

In Checkpoint 3.2.5, you found $10^\circ\ 45' = 10.75^\circ$. Thus,

$$\cot(10^\circ 45') = \cot(10.75^\circ) = \frac{1}{\tan(10.75^\circ)} \approx 5.2672.$$

Checkpoint 3.2.7

≈ 1.0886

Example 3.2.8

Enter

$$\text{2nd} \quad \text{SIN} \quad 0.6834$$

into your calculator to find

$$\sin^{-1}(0.6834) \approx 43.11^\circ.$$

So 43.11° is the angle whose sine is *approximately* 0.6834. Alternatively, we can say that

$$\sin(43.11^\circ) \approx 0.6834.$$

You can check the last equation on your calculator.

Checkpoint 3.2.8

$\cos(68^\circ) = 0.3746$ or $\cos^{-1}(0.3746) = 68^\circ$

Example 3.2.9

a. We use the following button sequence:

$$\text{2nd} \quad \text{COS} \quad 0.28 \quad \text{ENTER}.$$

This returns 73.73979529. Hence, $\beta \approx 73.7^\circ$.

b. Using the sequence

$$\text{2nd} \quad \text{TAN} \quad 1.93 \quad \text{ENTER},$$

we get 62.60975803. Hence, $\kappa \approx 62.6^\circ$.

Checkpoint 3.2.9

$\theta \approx 50.9^\circ$

Example 3.2.10

a. We first rewrite the equation as

$$\frac{1}{\cos(x)} = 8.104 \implies \cos(x) = \frac{1}{8.104}$$

We then enter

$$\text{2nd} \quad \text{COS} \quad 1 \quad \div \quad 8.104$$

into the calculator to find

$$\cos^{-1}\left(\frac{1}{8.104}\right) \approx 82.91^\circ.$$

Therefore, the solution to $\sec(x) = 8.104$ is $x \approx 82.91^\circ$.

b. The equation handed to us needs to be simplified into the form $\cot(\text{angle}) = \text{number}$ before we can find the angle.

$$\begin{aligned} & 3\cot(x) - 2 = 7 \\ \implies \quad & 3\cot(x) = 9 \\ \implies \quad & \cot(x) = 3 \end{aligned}$$

We now rewrite the cotangent as the tangent using the Reciprocal Identities.

$$\cot(x) = 3 \implies \frac{1}{\tan(x)} = 3 \implies \tan(x) = \frac{1}{3}$$

Finally, we grab our technology and enter

$$\text{2nd} \quad \text{TAN} \quad 1 \quad \div \quad 3$$

into the calculator to find

$$\tan^{-1}\left(\frac{1}{3}\right) \approx 18.43^\circ$$

Therefore, the solution to $3\cot(x) - 2 = 7$ is $x \approx 18.43^\circ$.

Checkpoint 3.2.10

$\omega \approx 14.48^\circ$

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3.2.1: Resources and Key Concepts

Key Concepts

Definitions

- **Minutes and Seconds:** A system for measuring fractions of a degree. One minute ($1'$) is defined as $\frac{1}{60}$ of a degree. One second ($1''$) is defined as $\frac{1}{60}$ of a minute, or $\frac{1}{3600}$ of a degree.

Common Mistakes

- **Using the Wrong Calculator Mode:** Before evaluating any trigonometric function, you must ensure your calculator is set to the correct mode (degree mode for this section). Using the wrong mode will result in incorrect answers.
- **Rounding Intermediate Steps:** Rounding should only be done for the final answer. If you round a value and then use that rounded value in another calculation, the rounding error can propagate and lead to a significantly different and less accurate final result.
- **Confusing Inverse Functions with Reciprocals:** The notation $\sin^{-1}(x)$ represents the inverse sine, which finds the angle whose sine is x. It does not mean the reciprocal, $\frac{1}{\sin(x)}$, which is $\csc(x)$.
- **Using Rounded Values in Inverse Functions:** When finding an angle, you should use the exact ratio if possible. For example, to find the angle whose tangent is $\frac{77}{136}$, you should compute $\tan^{-1}\left(\frac{77}{136}\right)$ directly rather than first approximating $\frac{77}{136}$ as a decimal and then taking the inverse tangent of the rounded decimal.

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3.2.2: Homework

Reading Questions

1. What calculator mode must you be in to evaluate trigonometric functions of angles given in degrees?
2. When a calculator displays a value for $\sin(50^\circ)$, is this an exact value or an approximation?
3. What does the text mean by "rounding is for display purposes only"?
4. How many minutes are in one degree? How many seconds are in one minute?
5. Explain the process of converting an angle from Decimal Degree (DD) form to Degrees-Minutes-Seconds (DMS) form.
6. How do you calculate the value of $\sec(47^\circ)$ on a calculator that does not have a SEC button?
7. What does the notation $\tan^{-1}(0.5662)$ represent?
8. What is the input for an inverse trigonometric function (an angle or a ratio)? What is the output?
9. Explain the difference in meaning between $\sin^{-1}(x)$ and $(\sin(x))^{-1}$.
10. When performing addition of angles in DMS form, what value do you "carry" when the sum of the seconds is 60 or more?

Skills Refresher

Review the following skills you will need for this section.

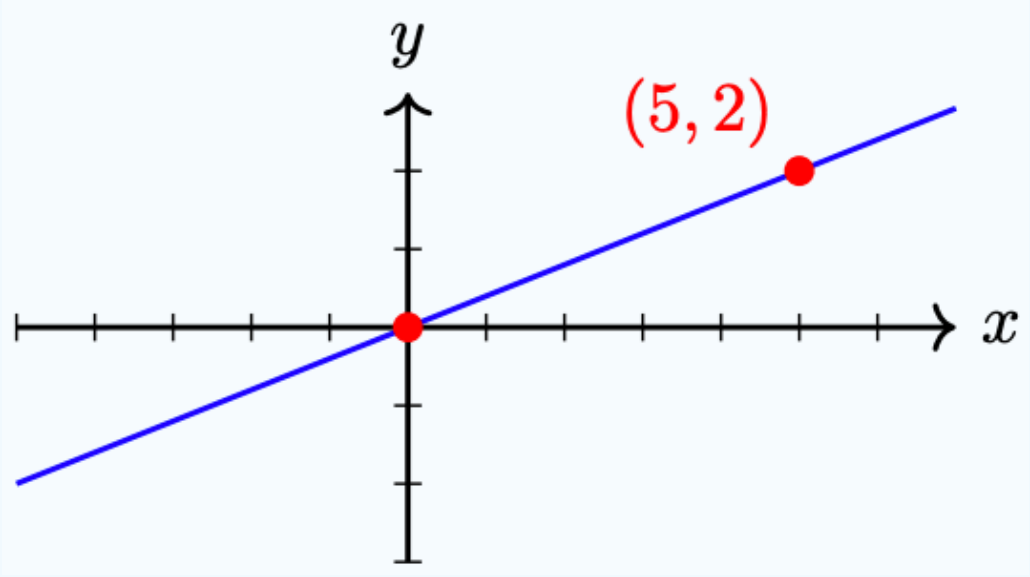
Skills Refresher

For the following exercises, write two more ratios equivalent to the given fraction.

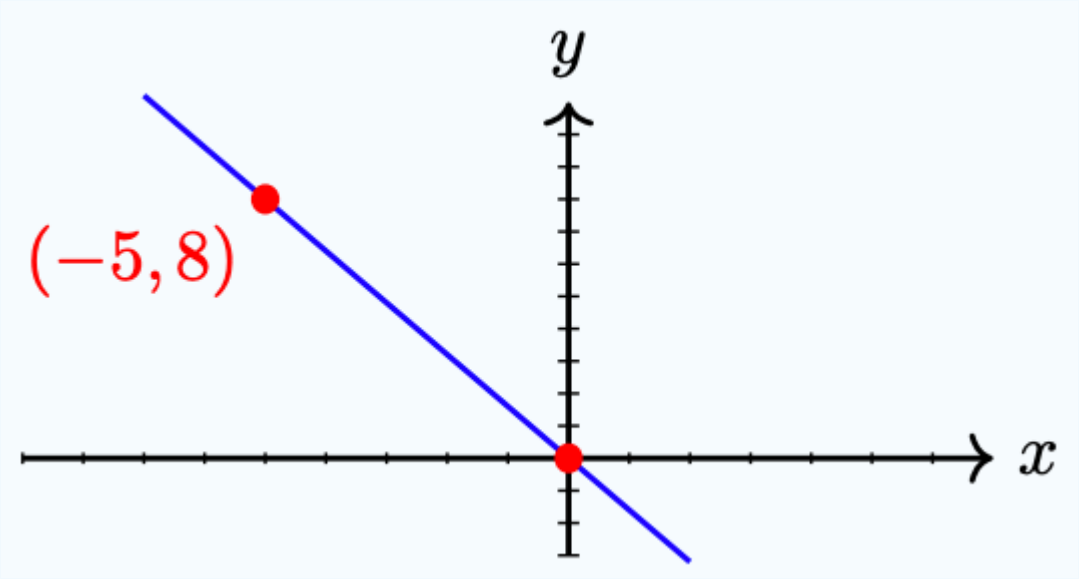
1. $\frac{10}{4}$
2. $\frac{6}{8}$
3. 0.6
4. 1.5

For the following exercises, compute the slope of the line.

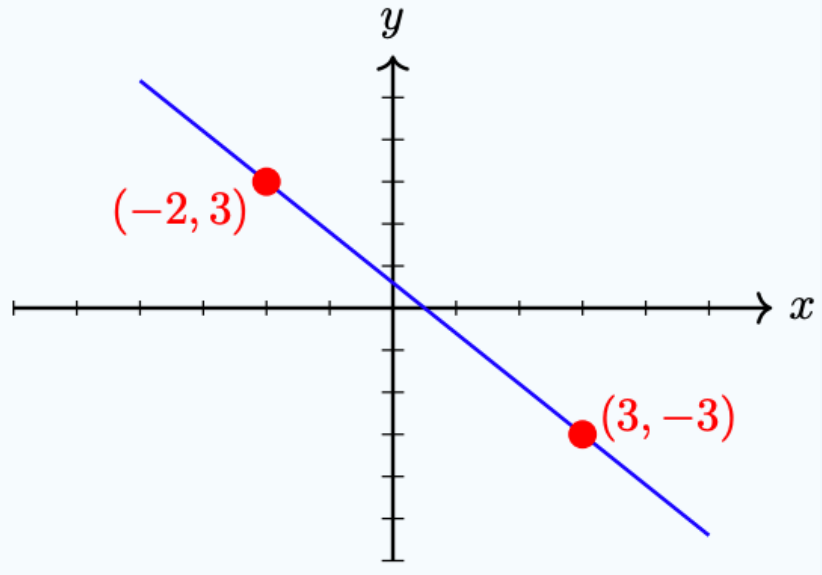
5.



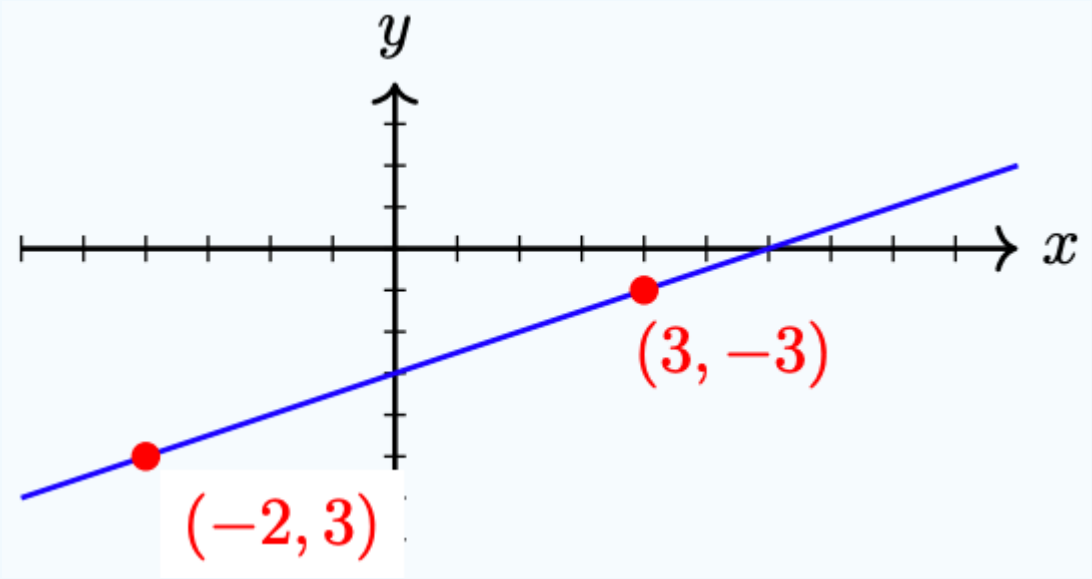
6.



7.



8.



For the following exercises, solve.

9. $\frac{12}{x} = 48$

10. $\frac{60}{x} = 80$

Answers available at the end of this section.

Homework

Vocabulary Check

1. Calculators and other technology have two modes that deal with the values of trigonometric functions - ___ mode and ___ mode. At this point, we only use ___ mode.

Concept Check

2. Explain how you would convert an angle given in Decimal Degrees (DD) to Degrees-Minutes-Seconds (DMS).
3. Does your calculator give you the exact decimal values for the trigonometric ratios of acute angles?
4. 1° is equal to ___ minutes(s).
5. 1° is equal to ___ second(s).
6. $1'$ is equal to ___ degree(s).
7. $1'$ is equal to ___ seconds(s).
8. $1''$ is equal to ___ degree(s).
9. $1''$ is equal to ___ minutes(s).
10. The SIN, COS, and TAN keys on a calculator (or other technology) returns

    $$\text{the value of the trigonometric function} \qquad \text{an angle}$$

    (circle one) when you input

    $$\text{the value of the trigonometric function} \qquad \text{an angle}$$

    (circle one).
11. The SIN^{-1}, COS^{-1}, and TAN^{-1} keys on a calculator (or other technology) returns

    $$\text{the value of the trigonometric function} \qquad \text{an angle}$$

    (circle one) when you input

    $$\text{the value of the trigonometric function} \qquad \text{an angle}$$

    (circle one).
12. Min and Paige try to approximate $2.1\tan(37^\circ)$. Min types the following into her calculator

    $$2.1 \quad \times \quad \mathrm{TAN} \quad 37 \quad \mathrm{ENTER},$$

    and the calculator returns approximately 1.5825; however, Paige types

    $$\mathrm{TAN} \quad 37 \quad \times \quad 2.1 \quad \mathrm{ENTER}$$

    into her calculator and gets an approximation of 4.5864. They each have their calculators set to Degree Mode, so that's not the issue. Who has the wrong approximation? If she cannot change the order of what she types into her calculator, but *can* add or remove a button press, what can she do to fix her error?

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

13. $\cot^{-1}(\theta) = \dfrac{1}{\cot(\theta)} = \tan(\theta)$.
14. $\cos^{-1}(\theta) = \sin(\theta)$

Basic Skills

For the following exercises, evaluate. Round answers to 3 decimal places.

15. $\sin(53^\circ)$
16. $\cos(18^\circ)$

17. $\tan(5^\circ)$

18. $\sin(85^\circ)$

19. $\csc(27^\circ)$

20. $\sec(8^\circ)$

21. $\cot(65^\circ)$

22. $\csc(11^\circ)$

23. a. Fill in the table for values of $\tan\theta$. Round your answers to four decimal places.

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°
$\tan\theta$									

b. Fill in the table for values of $\tan\theta$. Round your answers to three decimal places.

θ	81°	82°	83°	84°	85°	86°	87°	88°	89°
$\tan\theta$									

c. What happens to $\tan\theta$ as θ increases?

d. What value does your calculator give for $\tan 90^\circ$? Why?

24. a. Use your calculator to fill in a table of values for $\cos\theta$, rounded to hundredths.

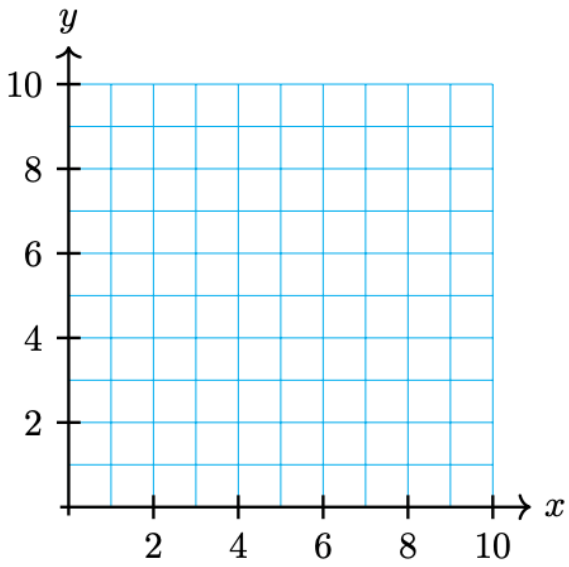
θ	0°	15°	30°	45°	60°	75°	90°
$\cos\theta$							

b. If you plotted the points in your table, would they lie on a straight line? Why or why not?

25. a. Use your calculator to complete the table. Rounded your answers to hundredths

θ	14°	22°	35°	42°	58°	78°
$\tan\theta$						

b. Use the values of $\tan\theta$ to sketch all the angles listed in the table. Locate the vertex of each angle at the origin, and the initial side along the positive x-axis.



For the following exercises, evaluate the expressions using a calculator.

26. a. $\sin(10^\circ + 40^\circ)$

    b. $\sin 10^\circ + \sin 40^\circ$

    c. $\sin 10^\circ \cos 40^\circ + \cos 10^\circ \sin 40^\circ$

27. a. $\cos(20^\circ + 50^\circ)$

    b. $\cos 20^\circ + \cos 50^\circ$

    c. $\cos 20^\circ \cos 50^\circ - \sin 20^\circ \sin 50^\circ$

28. a. $\cos(2 \cdot 24^\circ)$

    b. $2\cos 24^\circ$

    c. $2\cos^2(24^\circ) - 1$

29. a. $\cos(3 \cdot 19^\circ)$

    b. $3\cos 19^\circ$

    c. $4\cos^3(19^\circ) - 3\cos 19^\circ$

30. a. $\cos^2(17^\circ) + \sin^2(17^\circ)$

    b. $\sec^2(86^\circ) - \tan^2(86^\circ)$

    c. $\csc^2(32.1^\circ) - \cot^2(32.1^\circ)$

31. a. $\sin^2(25^\circ) + \cos^2(25^\circ)$

    b. $\dfrac{1}{\cos^2(80^\circ)} - \tan^2(80^\circ)$

    c. $\dfrac{1}{\sin^2(8^\circ)} - \cot^2(8^\circ)$

For the following exercises, convert the angles into the DMS system. Round each of your answers to the nearest second.

32. 18.9°

33. 6.78°

34. 55.17°

35. 31.425°

36. 70.214°

37. 159.84°

38. 220.43°

39. 116.323°

For the following exercises, convert the angles into decimal degrees. Round each of your answers to three decimal places.

40. $25^\circ\ 15'$

41. $13^\circ\ 50'$

42. $89^\circ\ 12'\ 9''$

43. $32^\circ\ 10'\ 12''$

44. $502^\circ\ 35''$

45. $237^\circ\ 58'\ 43''$

For the following exercises, find the complement of the given angle.

46. $27^\circ\ 45'\ 24''$

47. $88^\circ\ 10'\ 36''$

48. $2^\circ\ 27'\ 13''$

For the following exercises, find the supplement of the given angle.

49. $27^\circ\ 45'\ 24''$

50. $88^\circ\ 10'\ 36''$

51. $113^\circ\ 7''$

For the following exercises, perform the arithmetic.

52. $88^\circ\ 10'\ 36'' - 27^\circ\ 45'\ 24''$

53. $88^\circ\ 10'\ 36'' + 27^\circ\ 45'\ 24''$

54. $113^\circ\ 7'' - 2^\circ\ 27'\ 13''$

55. $113^\circ\ 7'' + 2^\circ\ 27'\ 13''$

For the following exercises, evaluate the trigonometric function.

56. $\sin\left(27^\circ\, 45'\, 24''\right)$

57. $\cos(55^\circ\, 12')$

58. $\tan(32^\circ\, 10'\, 40'')$

59. $\csc(7^\circ\, 7'\, 50'')$

60. $\sec(19^\circ\, 57'')$

61. $\cot(27^\circ\, 55')$

62. $\sin(5^\circ\, 55')$

63. $\tan(89^\circ\, 40'\, 1'')$

For the following exercises, evaluate the expression.

64. $\sin^{-1} 0.2$

65. $\cos^{-1} 0.8$

66. $\tan^{-1} 1.5$

67. $\tan^{-1} 2.5$

68. $\cos^{-1} 0.2839$

69. $\sin^{-1} 0.4127$

70. Evaluate the expressions, and explain what each means.

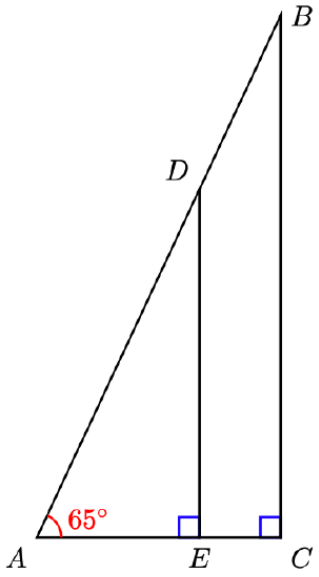
$$\sin^{-1}(0.6), \quad (\sin 6^\circ)^{-1}$$

71. Evaluate the expressions, and explain what each means.

$$\cos^{-1}(0.36), \quad (\cos 36^\circ)^{-1}$$

Synthesis Questions

72. Here are two right triangles with a 65° angle.



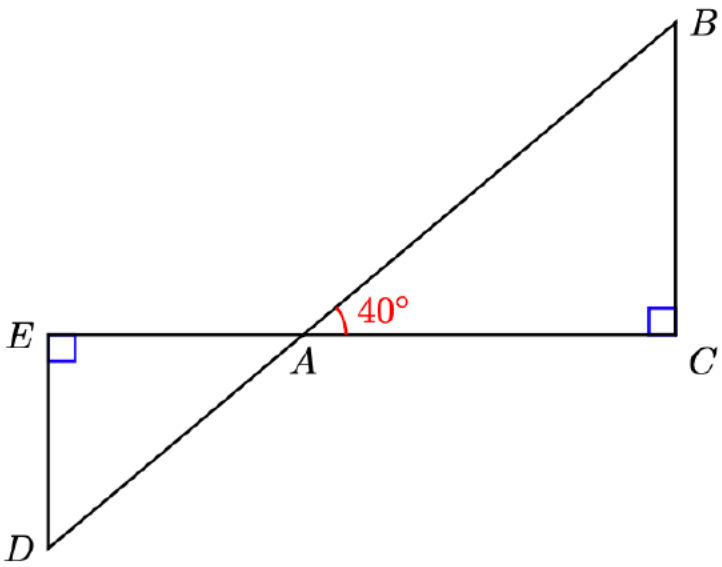
a. Measure the sides AB and BC with a ruler. Use the lengths to estimate $\sin 65^\circ$.

b. Measure the sides AD and DE with a ruler. Use the lengths to estimate $\sin 65^\circ$.

c. Use your calculator to look up $\sin 65^\circ$. Compare your answers. How close were your estimates?

73. Use the figure in the previous problem to calculate two estimates each for the cosine and tangent of 65°. Compare your estimates to your calculator's values for $\cos 65^\circ$ and $\tan 65^\circ$.

74. Here are two right triangles with a 40° angle.



a. Measure the sides AB and AC with a ruler. Use the lengths to estimate $\cos 40^\circ$.

b. Measure the sides AD and AE with a ruler. Use the lengths to estimate $\cos 40^\circ$.

c. Use your calculator to look up $\cos 40^\circ$. Compare your answers. How close were your estimates?

75. Use the figure in Problem 74 to calculate two estimates each for the cosine and tangent of 40°. Compare your estimates to your calculator's values for $\sin 40^\circ$ and $\tan 40^\circ$.

For the following exercises, write two different equations for the statement.

76. The cosine of 15° is 0.9659.

77. The sine of 70° is 0.9397.

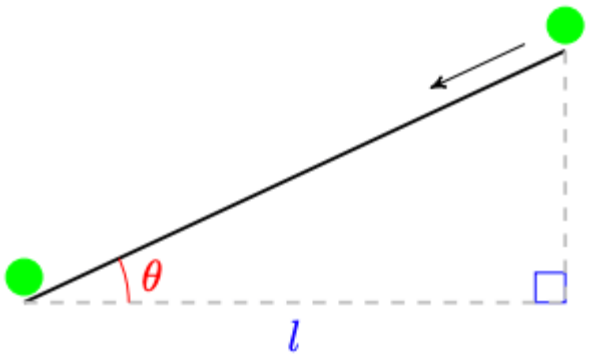
78. The angle whose tangent is 3.1445 is 65°.

79. The angle whose cosine is 0.0872 is 85°.

Applications

80. When a plane is tilted by an angle θ from the horizontal, the time required for a ball starting from rest to roll a horizontal distance of l feet on the plane is

$$t = \sqrt{\frac{l}{8}\csc(2\theta)} \text{ seconds}$$



a. How long, to the nearest 0.01 second, will it take the ball to roll 2 feet horizontally on a plane tilted by 12°?

b. Solve the formula for l in terms of t and θ.

81. After a heavy rainfall, the depth, D, of the runoff flow at a distance x feet from the watershed down a slope at angle α is given by

$$D = (kx)^{0.6}(\cot(\alpha))^{0.3} \text{ inches}$$

where k is a constant determined by the surface roughness and the intensity of the runoff.

a. How deep, to the nearest 0.01 inch, is the runoff 100 feet down a slope of 10° if $k = 0.0006$?

b. Solve the formula for x in terms of D and α.

Answers

Skills Refresher

(Many answers are possible for 1-4.)

1. $\frac{5}{2}, \frac{20}{8}$
2. $\frac{3}{4}, \frac{12}{16}$
3. $\frac{3}{5}, \frac{12}{20}$
4. $\frac{6}{4}, \frac{12}{8}$
5. $\frac{2}{5}$
6. $\frac{-8}{5}$
7. $\frac{-6}{5}$
8. $\frac{1}{2}$
9. $\frac{1}{4}$
10. $\frac{3}{4}$

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3.3: Solving Right Triangles

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

With the Pythagorean Theorem, we can find one side of a right triangle if we know the other two sides. For example, the missing side in Figure $3.3.1$ can be found as follows:

$$\begin{aligned} & b^2 + 5^2 &&= 12^2 \\ \implies & b^2 + 25 &&= 144 \\ \implies & b^2 &&= 119 \\ \implies & b &&= \sqrt{119} \end{aligned}$$

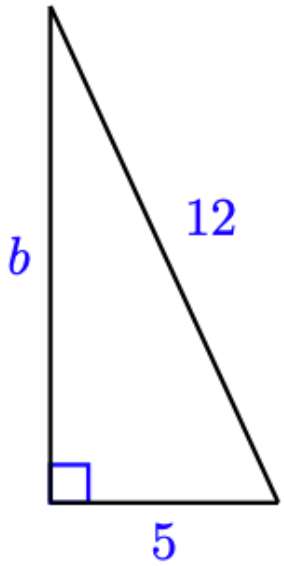


Figure $3.3.1$

However, is there a way to find the length of side b in Figure $3.3.2$, given that we only know the length of one side of the right triangle and one of the non-right angles?

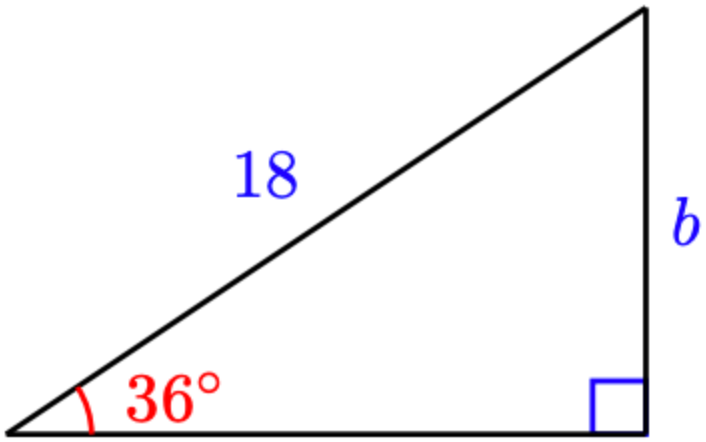


Figure 3.3.2

Using a Trigonometric Function to Find an Unknown Side

When finding the length of a side of a right triangle, we must be given at least two pieces of information. If we are given two side lengths (as in Figure 3.3.1), we use the Pythagorean Theorem. If, on the other hand, we are given an angle and a side length, we determine the trigonometric function relating the missing side and the provided information.

Example 3.3.1

Find the side length opposite the 50° angle in Figure 3.3.3.

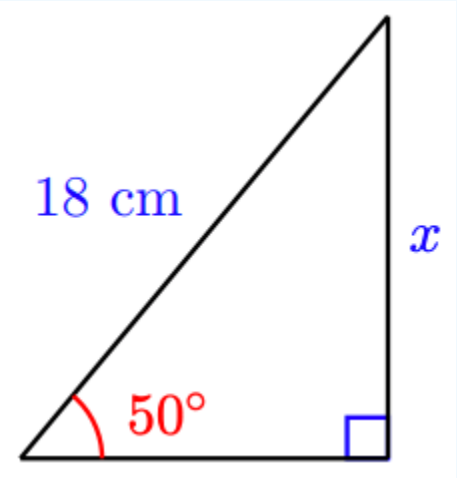


Figure 3.3.3

Solution available at the end of this section.

Caution: Save Rounding for Display Purposes Only

In the previous example, even though we showed only four decimal places in $\sin(50^\circ)$, you should not round off intermediate steps in a calculation—the answer loses accuracy with each rounding. The following keystrokes can be used to avoid entering an extended approximation for $\sin(50^\circ)$:

$$18 \quad \times \quad \text{SIN} \quad 50 \quad \text{ENTER}$$

The calculator returns 13.78879998.

As an additional warning, recall that *most* scientific and graphing calculators return $\text{SIN}($ when you press the SIN key. If you intend to perform the multiplication in the opposite order (i.e., you intend to compute $\sin(50^\circ) \times 18$), you must close the parentheses on the argument of the sine. That is, you would use the following keystrokes in your calculator:

$$\text{SIN} \quad 50 \quad) \quad \times \quad 18.$$

Without the closing parenthesis, the calculator computes $\sin(50 \times 18^\circ)$, which is not equivalent to $\sin(50^\circ) \times 18$.

Checkpoint 3.3.1

a. Use the tangent ratio to find x in Figure 3.3.4.

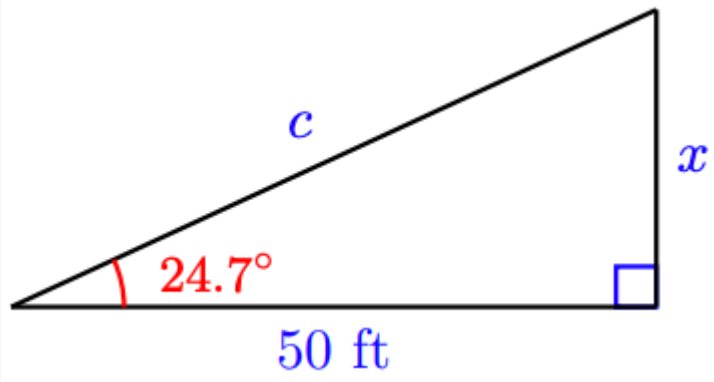


Figure 3.3.4

b. Use the sine ratio to find the hypotenuse, c, of the triangle.
c. Use the Pythagorean Theorem to find the hypotenuse of the triangle. Is the answer the same with both methods? Explain why the calculations might give (slightly) different answers.

Answers available at the end of this section.

Solving Triangles

A triangle has six parts: three sides and three angles. In a right triangle, we know that one of the angles is 90°. If given enough information, we can find the measures of the remaining angles within the right triangle and the lengths of the sides. This is called **solving the triangle**.

Definition: Solving a Triangle

Solving a triangle means finding all the missing side lengths and angle measures, given some initial information about the triangle.

Example 3.3.2

In the right triangle $\triangle ABC$, $\angle A = 37^\circ$ and (non-hypotenuse) side $b = 11$. Solve the triangle.

Solution available at the end of this section.

Traditional Notation: The Right Angle of $\triangle ABC$

Given the right triangle $\triangle ABC$, we adopt the convention used throughout most math textbooks that $\angle C$ is the 90° angle and side c is the hypotenuse.

Checkpoint 3.3.2

Sketch a right triangle with one angle of 37° and the side adjacent to that angle of length 5 centimeters. Without doing the calculations, list the steps used to solve the triangle.

Answer available at the end of this section.

In Example 3.3.2 and Checkpoint 3.3.2, we were given an angle and a side. Using those two pieces of information (along with the fact that one of the other angles in the right triangle is 90°), we were able to solve the triangles completely. What if we were not given any angles (other than the right angle) but two side lengths instead?

Example 3.3.3

Solve $\triangle ABC$ given that $a = 3$ and $c = 5$.

Solution available at the end of this section.

Checkpoint 3.3.3

Solve the right triangle $\triangle ABC$ given $b = 1.12$ and $c = 3.25$. Round your *displayed* answers to the nearest hundredth.

Answer available at the end of this section.

The power of Trigonometry comes from its usefulness in helping us solve physical applications.

Example 3.3.4

A 10-foot ladder is leaning against a wall. If the ladder's base is too far from the wall, the base can slide out. If the ladder base is too close to the wall, there's a risk that the ladder could tip over backward. The leaning ladder should make a 75° angle with the ground for safety reasons.

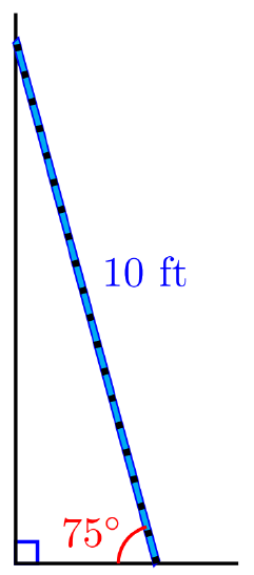


Figure 3.3.7

a. How far should the base of the ladder be from the wall?

b. How far up the wall will the top of the ladder reach?

Solutions available at the end of this section.

Checkpoint 3.3.4

The length, L, of the shadow cast by a flagpole on a sunny day depends on the height, h, of the flagpole and the angle, θ, that the sun's rays make with the ground (see Figure 3.3.8).

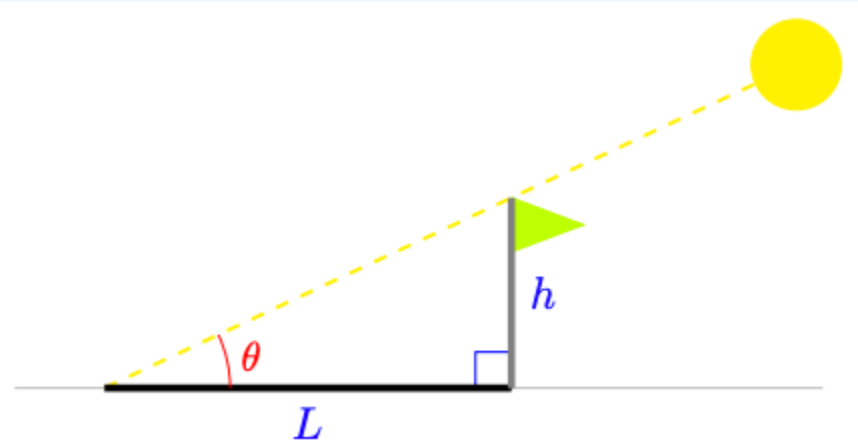


Figure 3.3.8

a. Write an expression for the length, L, of the shadow cast by a flagpole of height h when the sun makes an angle of θ from the ground.
b. Find the length (to the nearest 0.01 meter) of the shadow cast by a 3-meter flagpole when the sun makes an angle of 20° from the ground.

Answers available at the end of this section.

Interactive Element: Sun Angle and Shadow Length

The following Interactive Element might help you visualize how a flagpole's shadow length depends on the sun's angle, θ, above the ground.

Interact: Drag the bright point at the tip of the shadow left and right along the ground.

Observation: The lower the sun sits (smaller θ), the longer the shadow — and near a low sun a tiny drop in angle stretches the shadow dramatically, matching $L = h\cot(\theta)$.

Introducing Right Triangles to Non-Right Triangles

While the trigonometric functions are defined in terms of ratios of sides of a right triangle, they are useful on many occasions when a non-right triangle is involved.

Example 3.3.5

The vertex angle of an isosceles triangle is 34°, and the equal sides are 16 meters long. Find the altitude of the triangle.

Solution available at the end of this section.

Checkpoint 3.3.5

Another isosceles triangle has base angles of 72° and equal sides of length 6.8 centimeters. Find the size of the base.

Answer available at the end of this section.

Thought-Provoking Right Triangle Geometry

Part of your excursion into Trigonometry (and higher-level mathematics) is to think critically and problem-solve. One way to engage your imagination and thinking skills is through abstract geometry problems that are often fun!

Example 3.3.6

Figure 3.3.10 shows a circle of radius r and a right triangle with side a along the radius of the circle and side b tangent to the circle.

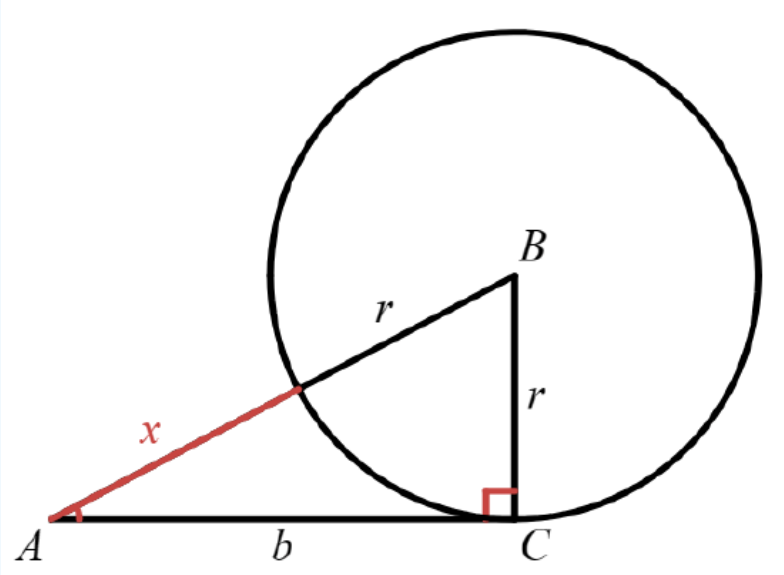


Figure 3.3.10

If $\angle B = 50^\circ$ and $b = 10$ inches, find the radius of the circle and the partial length of the hypotenuse, x, that sits outside the circle. Round your answers to the nearest tenth.

Solution available at the end of this section.

We continually harp on this but notice that we used the *exact* value of r when doing our final computation in Example 3.3.6. This is because we always want to avoid using rounded values for subsequent computations.

Checkpoint 3.3.6

Let $r = 5$ and $\angle A = 24^\circ$ in Figure 3.3.10 (Example 3.3.6), find x. Round your answer to the nearest tenth.

Answer available at the end of this section.

Example 3.3.7

In Figure 3.3.11, $x = 5$, $\angle A = 25°$, and $\angle BDC = 48°$. Find y.

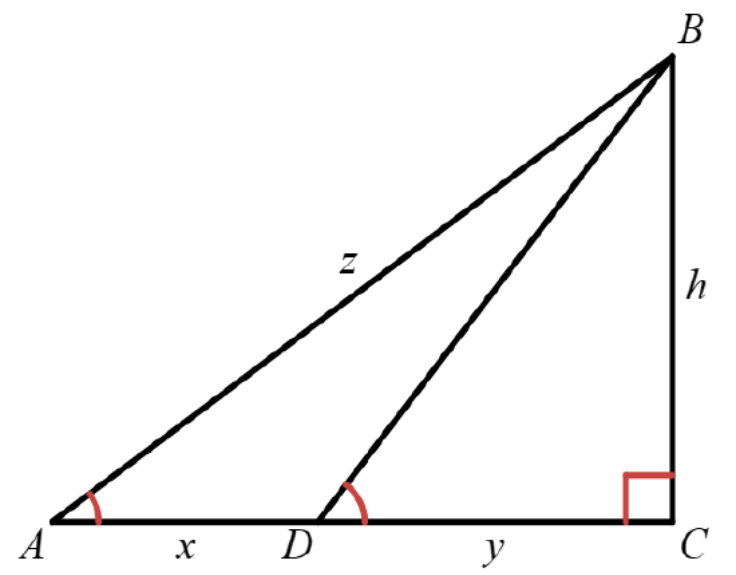


Figure 3.3.11

Solution available at the end of this section.

Checkpoint 3.3.7

Let $\angle A = 24°$, $y = 12$, and $\angle BDC = 62°$ in Figure 3.3.11 (Example 3.3.7). Find x. Round your answer to the nearest tenth.

Answer available at the end of this section.

Success in College: Learning From Tests (Part 1)

1. A student scored 75% on their last math test. They tried to redo the questions and problems that they missed. Now they need to know if their new answers are correct. For your current math class, how could they find this out?
2. Look over your last math test. Did you miss any points because you did not show enough work? If so, explain why you didn't show the work on the test.

Success in Mathematics: Math Anxiety (Part 1)

The following is an excerpt of an article written at MathAcademy.com.

> A famous stage actress was once asked if she had ever suffered from stage fright and, if so, how she had gotten over it. She laughed at the interviewer's naive assumption that she must not feel that kind of anxiety since she was an accomplished actress. She assured him that she had always had stage fright and had never gotten over it. Instead, despite it, she had learned to walk on stage and perform.
>
> Like stage fright, math anxiety can be a disabling condition, causing humiliation, resentment, and even panic.
>
> Even though we all suffer from math anxiety to some degree – just as anyone feels at least a little nervous when speaking

to an audience – for some of us, it is a severe problem. This burden interferes with our lives, preventing us from achieving our goals. The first step, without which no further progress is possible, is to recognize that math anxiety is an emotional response. (In fact, severe math anxiety is a learned emotional response.) As with any strong emotional reaction, there are constructive and unconstructive ways to manage math anxiety. Unconstructive (and damaging) ways include rationalization, suppression, and denial.

By "rationalization," we mean finding reasons why it is okay and perhaps even inevitable – and therefore justified – for you to have this reaction. The myths of "men are better at math than women" or "math is for logical people and not creative people" are examples of rationalizations. While they may make you feel better (or at least less bad) about having math anxiety, they will do nothing to lessen it or to help you get it under control. Therefore, rationalization is unconstructive.

"Suppression" means being aware of the anxiety – but trying very hard not to feel it. I have found that students very commonly attempt this, and some severe self-criticism accompanies it. Students feel that they shouldn't feel this anxiety, that it's a weakness they should overcome by brute force if necessary. When this effort doesn't succeed (as invariably it doesn't), the self-criticism becomes ever harsher, leading to a deep sense of frustration and often a severe loss of self-esteem – mainly if the stakes for a student are high, as when their career or personal goals are riding on a successful outcome in a math class, or when parental disapproval is a factor. Consequently, suppression of math anxiety is not only unconstructive but can be damaging.

Finally, there is denial. People using this approach probably aren't likely to see this essay, much less read it, for they carefully construct their lives to avoid all mathematics as much as possible. They choose college majors and careers that don't require math and let the bank or their spouse balance the checkbook. This approach has the advantage that feelings of frustration and anxiety about math are mostly avoided. However, their lives are drastically constrained, for, in our society, fewer than 25% of all careers are, so to speak, "math-free." Thus, their choices of personal and professional goals are severely limited. (Incidentally, most math-free jobs are low-status and low-pay.)

People in denial about mathematics miss out on something else, too, for the student of mathematics learns to see aspects of the structure and beauty of our world that can be seen in no other way and to which the "innumerate" necessarily remain forever blind. It would be a lot like never hearing music or never seeing colors. (I understand that some people have these disabilities but didn't choose to have them.)

Okay, so what is the constructive way to manage math anxiety? I call it "taking possession." It involves identifying the sources of math anxiety in one's life, accepting those feelings without self-criticism, and then learning strategies for disarming math anxiety's influence on one's future study of mathematics.

Begin by understanding that your feelings of math anxiety are not uncommon and do not indicate that there is anything wrong with you or inferior about your ability to learn math. For some, this can be hard to accept, but it is worth trying to accept – since, after all, it happens to be true. This can be made more accessible by exploring your own "math history." Think back across your career as a math student, and identify those experiences which have contributed most to your feelings of frustration about math. For some, this will be a memory of a humiliating experience in school, such as being made to stand at the blackboard and embarrassed in front of one's peers. For others, it may involve interaction with a parent. Whatever the principle episodes are, recall them as vividly as you can. Then, write them down. This is important. After you have written the episode on a sheet(s) of paper, write down your reaction to the episode, both at the time and how it makes you feel to recall it now.

After completing this exercise, take a fresh sheet of paper and try to sum up in a few words what your feelings about math are at this point in your life, together with the reason or reasons you wish to succeed at math. This, too, is important. Not until after we lay out for ourselves consciously and deliberately what our feelings and desires are towards mathematics will it become possible to take possession of our feelings of math anxiety and become free to implement strategies for coping with those feelings.

At this point, sharing your memories, feelings, and goals with others can be enormously helpful. In a math class I teach for art majors, I hand out a questionnaire early in the semester asking students to do precisely what is described above. After spending about twenty minutes writing down their recollections and goals, I lead them in a classroom discussion on math anxiety. This process of dialogue and sharing – though it may seem just a bit on the goopy side – invariably brings out of each student their barriers to math, often helping them become completely conscious of these barriers for the first time. Just as important, it helps all my students understand that the negative experiences they have had, and their reactions to them, are shared one way or another by almost everyone else in the room.

If you do not have the opportunity to engage in a group discussion in a classroom setting, find friends or relatives you trust to respect your feelings and induce them to talk about their own math anxiety experiences and listen to yours.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Are there counselors at your school that are available to help students who are struggling with math anxiety? If so, where are they located, and how to make an appointment to see one?
2. Estimate the percent of students in your class you think are negatively affected by math anxiety.

If you are unsure about the answers to either of these questions, feel free to ask your instructor - that's what they are there for!

Answers

Section Preview

This section focuses on solving right triangles. As such, all angles will be acute. In addition, applications and concept exercises require the student to find missing angles and/or sides within a right triangle; however, more robust applications involving angles of elevation or depression, navigation, and problems involving more than a single right triangle in a three-dimensional environment are reserved for the next section.

Section Preview

- **Simplifying Expressions**
  - Simplifying Sums and Differences of Radical Expressions: For the Skills Refresher homework.
- **Solving Equations**
  - Solving Linear Equations — Multistep with Variables on Both Sides: For Example 7's algebraic chain.

Section Preview

- Solve a right triangle for a missing side.
- Solve a right triangle for a missing angle.
- Solve an application using Right Triangle Trigonometry.
- Use Right Triangle Trigonometry in a robust geometry problem.

Example $3.3.1$

In this triangle, the trigonometric function that relates the missing side to the given information is the sine:

$$\sin(50^\circ) = \frac{\text{opposite}}{\text{hypotenuse}}$$

We use a calculator to find an approximate value for the sine of 50°, filling in the lengths of the hypotenuse and the opposite side to get

$$0.7660 \approx \frac{x}{18}.$$

We solve for x to find

$$x \approx 18(0.7660) = 13.788.$$

To two decimal places, the length of the opposite side is approximately 13.79 centimeters.

Checkpoint $3.3.1$

a. $x \approx 23$ ft
b. $c \approx 55$ ft
c. The answers agree when rounded to units. Rounding during calculation can cause the results to differ.

Example $3.3.2$

Since $\angle A = 37^\circ$, side a is not the hypotenuse of the right triangle. Moreover, we were told that side b is not the hypotenuse. Thus, c must be the hypotenuse and $\angle C = 90^\circ$. Creating a rough sketch of $\triangle ABC$, we get

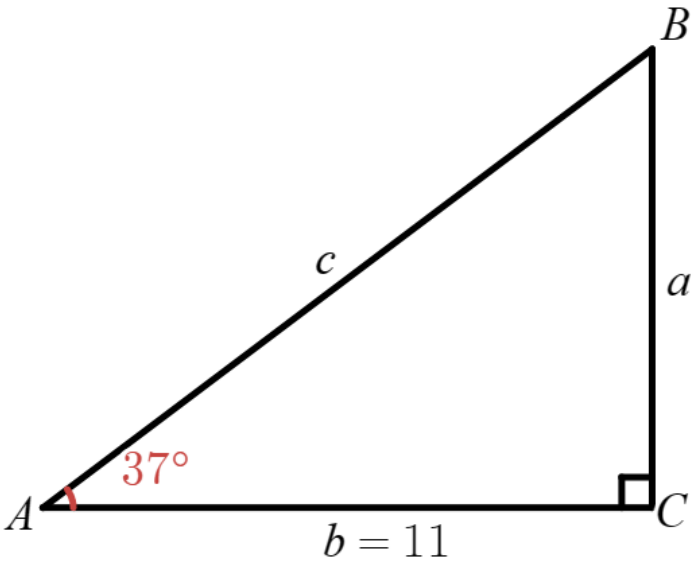


Figure 3.3.5

Using the Triangle Sum, we quickly compute $\angle B = 90^\circ - 37^\circ = 53^\circ$. We now concentrate on finding the missing sides by using the trigonometric functions and the given information. The tangent relates 37° to sides a and b. Therefore,

$$\tan(37^\circ) = \frac{a}{11} \implies a = 11\tan(37^\circ) \approx 8.2891.$$

Finally, we use the cosine to relate 37° to the given side b and the hypotenuse c.

$$\cos(37^\circ) = \frac{11}{c} \implies c = \frac{11}{\cos(37^\circ)} \approx 13.7735.$$

Checkpoint 3.3.2

Use $\tan(37^\circ)$ to find the opposite side. Use $\cos(37^\circ)$ to find the hypotenuse. Subtract 37° from 90° to find the third angle.

Example 3.3.3

It's always best to start by drawing a triangle.

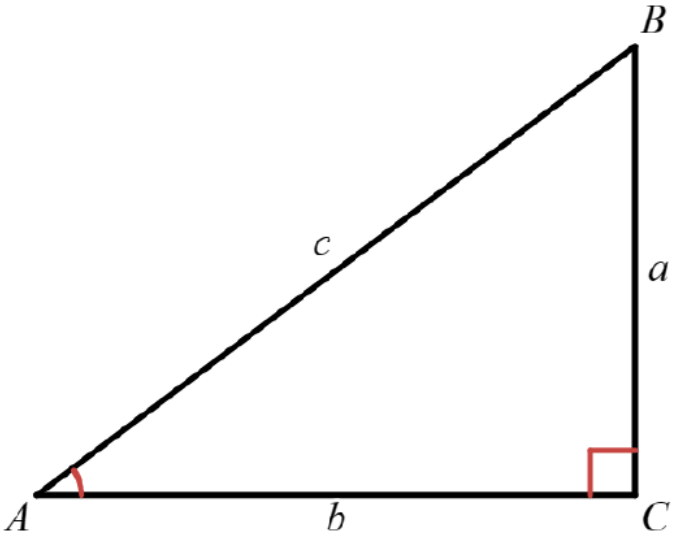


Figure 3.3.6

We can use the Pythagorean Theorem to compute the length of side b, or we could recognize that this is a Pythagorean Triple. No matter how we compute it, $b = 4$. Now, we need to find the missing angles. Luckily, we were given that this is a right triangle, so once we find either $\angle A$ or $\angle B$, we can use the Triangle Sum to find the other angle. Let's go ahead and find $\angle A$. Since $\sin(\angle A) = \frac{a}{c} = \frac{3}{5}$, we use a calculator to compute $\angle A = \sin^{-1}\left(\frac{3}{5}\right) \approx 36.8699^\circ$. Thus,

$$\angle B = 180^\circ - \angle C - \angle A \approx 180^\circ - 90^\circ - 36.8699^\circ \approx 53.1301^\circ.$$

It's often best to organize all of this information in a table like so:

$$\begin{array}{lcllcl} \angle A & \approx & 36.8699^\circ & a & = & 3 \\ \angle B & \approx & 53.1301^\circ & b & = & 4 \\ \angle C & = & 90^\circ & c & = & 5 \end{array}$$

Checkpoint 3.3.3

$$\begin{aligned} \angle A &\approx 69.84^\circ & a &\approx 3.05 \\ \angle B &\approx 20.16^\circ & b &= 1.12 \\ \angle C &= 90^\circ & c &= 3.25 \end{aligned}$$

Example 3.3.4

a. The distance between the ladder's base and the wall is the side adjacent to the 75° angle. We can find the side adjacent to the 75° angle using the cosine ratio.

$$\begin{aligned} \cos(75^\circ) &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ \implies 0.2588 &\approx \frac{b}{10} \\ \implies 10(0.2588) &\approx b \\ \implies b &\approx 2.588 \end{aligned}$$

The ladder's base should be about 2.6 feet from the wall.

b. We could use the Pythagorean Theorem to find how far up the wall the ladder will reach. However, using the given information rather than the calculated values to find the other unknown parts is better. We will use the sine ratio.

$$\begin{aligned} \sin(75^\circ) &= \frac{\text{opposite}}{\text{hypotenuse}} \\ \implies 0.9659 &\approx \frac{a}{10} \\ \implies 10(0.9659) &\approx a \\ \implies a &\approx 9.659 \end{aligned}$$

The ladder will reach about 9.7 feet.

Checkpoint 3.3.4

a. $L = h\cot(\theta)$

b. The shadow is about 8.24 meters long.

Example 3.3.5

The triangle described is *not* a right triangle. However, the altitude of an isosceles triangle bisects the vertex angle. It divides the triangle into two congruent right triangles, as shown in Figure 3.3.9. The 16-meter side becomes the hypotenuse of the right triangle. The altitude, h, of the original triangle is the side adjacent to the 17° angle.

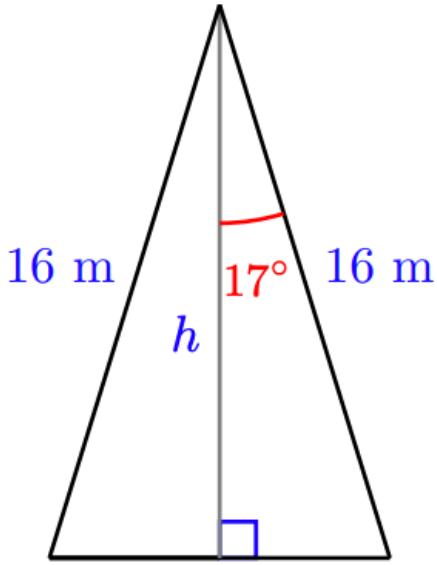


Figure 3.3.9

Which of the three trigonometric ratios is helpful in this problem? The cosine is the ratio that relates the hypotenuse and the adjacent side. Therefore, we will begin with the equation:

$$\cos(17^\circ) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

We use a calculator to find $\cos(17^\circ)$ and fill in the lengths of the sides to get

$$0.9563 \approx \frac{h}{16}.$$

Solving for h gives

$$h \approx 16(0.9563) = 15.3008.$$

The altitude of the triangle is about 15.3 meters long.

Checkpoint 3.3.5

4 cm

Example 3.3.6

Throughout Trigonometry, you will be faced with thought-provoking questions. The beautiful thing about right triangle trigonometry is that you typically need to find a trigonometric function that relates the given information to the missing information. In this case, we are given $\angle B$ and side b. We are initially asked for the radius of the circle, which is the same as the side length a of the right triangle. The trigonometric function relating $\angle B$ to sides a and b is the tangent:

$$\tan(\angle B) = \frac{b}{a} = \frac{b}{r}.$$

Therefore,

$$\tan(50^\circ) = \frac{10}{r} \implies r = \frac{10}{\tan(50^\circ)}.$$

Hence, the radius of the circle is $\frac{10}{\tan(50^\circ)} \approx 8.4$ inches.

Now we need to find the length x. Again, we want to find a trigonometric function that relates the information to what we seek. Unfortunately, we are looking for only *part* of the hypotenuse of the right triangle; however, the *entire* hypotenuse is $x + r$, and we just found r. The sine is the trigonometric function that relates $\angle B$ to side b and the hypotenuse:

$$\sin(\angle B) = \frac{b}{x+r} \implies x + r = \frac{b}{\sin(\angle B)} \implies x = \frac{b}{\sin(\angle B)} - r.$$

Substituting the given values for $\angle B$ and side b, along with the **exact** value for r, we get

$$x = \frac{10}{\sin(50^\circ)} - \frac{10}{\tan(50^\circ)} \approx 4.66 \text{ inches}.$$

Checkpoint 3.3.6

$x \approx 7.3$

Example 3.3.7

It is helpful to redraw only $\triangle BCD$ and then $\triangle ABC$, noticing that they share the common side h. For $\triangle BCD$, we only have one piece of given information, $\angle BDC$; however, we need to find y, and it will be useful to work with h since both triangles share that side (I call this the "link" between the triangles). The trigonometric function relating these is the tangent:

$$\tan(\angle BDC) = \frac{h}{y} \implies y\tan(\angle BDC) = h.$$

Hence,

$$h = y\tan(48^\circ).$$

Moving to $\triangle ABC$, since we were given x and $\angle A$, it seems best to use the tangent again to get h involved. Therefore,

$$\tan(\angle A) = \frac{h}{x+y} \implies (x+y)\tan(\angle A) = h.$$

That is,

$$h = (5+y)\tan(25^\circ).$$

We now equate these two versions of h to each other (since they both equal h, they are both equal to each other).

$$\begin{aligned} y\tan(48^\circ) = (5+y)\tan(25^\circ) &\implies y\tan(48^\circ) = 5\tan(25^\circ) + y\tan(25^\circ) \\ &\implies y\tan(48^\circ) - y\tan(25^\circ) = 5\tan(25^\circ) \\ &\implies y(\tan(48^\circ) - \tan(25^\circ)) = 5\tan(25^\circ) \\ &\implies y = \frac{5\tan(25^\circ)}{\tan(48^\circ) - \tan(25^\circ)} \\ &\implies y \approx 3.62 \end{aligned}$$

Checkpoint 3.3.7

$x \approx 38.7$

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3.3.1: Resources and Key Concepts

Resources

Videos

- Trigonometric Applications
  - Trigonometry of Right Triangles - Review of Trigonometric Functions - A Right Triangle Approach
  - Right Triangle Trigonometry - Applications of Right Triangle Trigonometry (Part 1)
  - Trigonometry of Right Triangles - Review of Applications of Right Triangle Trigonometry (Part 1)
  - Inverse Trigonometric Functions - Applications of Right Triangle Trigonometry (Part 2)

Key Concepts

Definitions

- **Solving a Triangle:** The process of finding the measures of all unknown side lengths and angles of a triangle, based on some given initial information.

Common Mistakes

- **Save Rounding for Display Purposes Only:** You should not round off intermediate steps in a calculation. Using a rounded value in a subsequent calculation introduces and propagates errors, leading to a less accurate final answer. Use your calculator's stored values or the exact expressions whenever possible.
- **Using Calculated Values Instead of Given Values:** When solving a triangle, it is better to use the original given information for each calculation rather than a value you calculated in a previous step. This helps avoid carrying forward any rounding errors.
- **Incorrect Calculator Entry:** Be careful with the order of operations on your calculator. For example, to compute $\sin(50^\circ) \times 18$, you may need to close the parenthesis after the 50, as in `SIN(50) * 18`. Without the parenthesis, the calculator might compute $\sin(50^\circ \times 18)$, which is incorrect.

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3.3.2: Homework

Reading Questions

1. What does it mean to "solve a triangle"?
2. If you are given one acute angle and the length of the hypotenuse in a right triangle, which trigonometric function would you use to find the length of the side opposite the given angle?
3. If you are given the lengths of the two legs of a right triangle, how do you find the measure of one of the acute angles?
4. Why is it generally better to use the given information for each calculation, rather than using values you have already calculated?
5. What is the first step you should always take when starting to solve a triangle, especially in an application problem?
6. How can right triangle trigonometry be used to find the altitude of an isosceles triangle that is not a right triangle?
7. In a right triangle $\triangle ABC$ with right angle at C, if you are given sides a and c, what inverse trigonometric function would you use to find $\angle A$?
8. The text warns that when calculating $\sin(50^\circ) \times 18$, you might need to enter `SIN(50) * 18` into your calculator. What might the calculator compute if you enter `SIN(50 * 18)` instead?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, simplify.

1. $\sqrt{2}\sqrt{2}$
2. $\frac{3}{\sqrt{3}}$
3. $\sqrt{8}$
4. $\sqrt{\frac{3}{4}}$

For the following exercises, rationalize the denominator.

5. $\dfrac{1}{\sqrt{2}}$
6. $\dfrac{2}{\sqrt{3}}$
7. $\dfrac{6}{\sqrt{3}}$
8. $\dfrac{4}{\sqrt{8}}$

Answers available at the end of this section.

Homework

Vocabulary Check

1. Given enough initial information about a triangle, we can find the lengths of all remaining sides and angles. This process is known as ___.

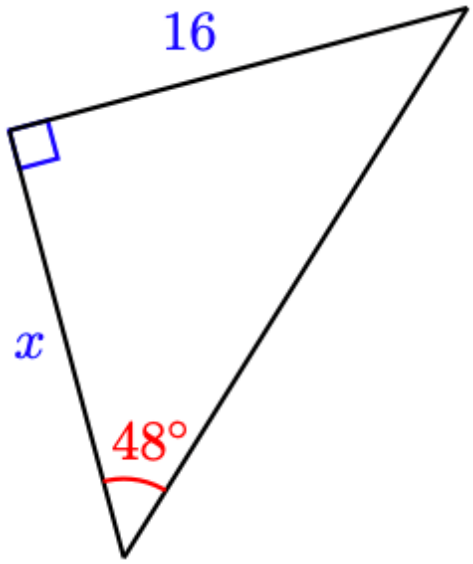
Concept Check

2. How many parts of a right triangle (including the right angle) do you need to know to solve it?
3. Why is it better to use the given values when solving a triangle than the ones you calculated?
4. What is the $\sin^{-1}$ (or $\cos^{-1}$ or $\tan^{-1}$) button on the calculator used for?
5. If you knew one acute angle and the side opposite that angle, which trigonometric ratio would you use to find the hypotenuse of a right triangle?

Basic Skills

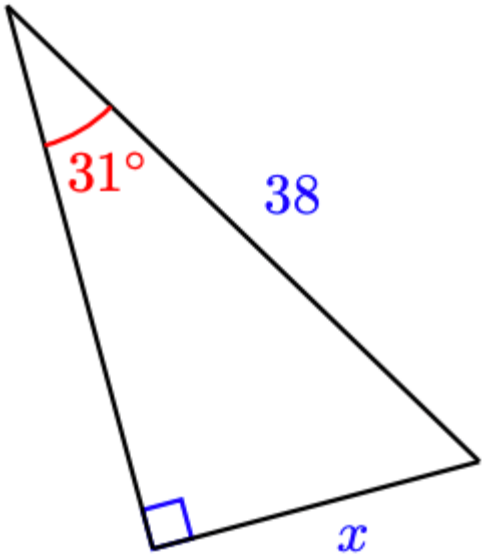
For the following exercises, use one of the three trigonometric ratios to find the unknown side of the triangle. Round your answer to hundredths.

6.

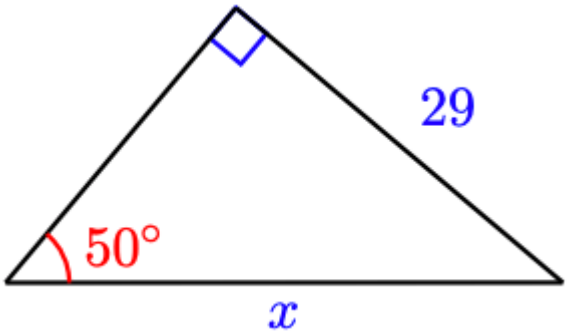


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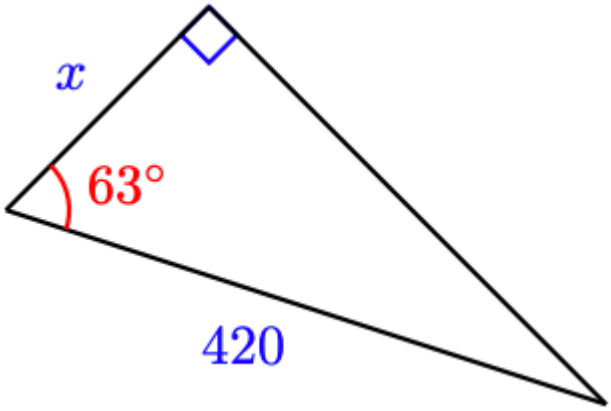
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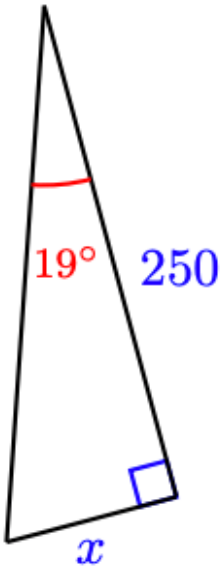
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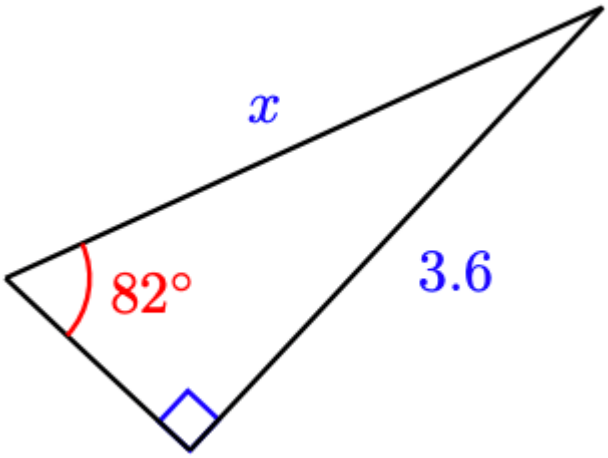


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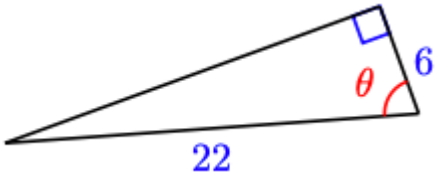
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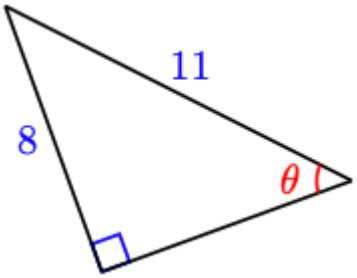


For the following exercises, find the labeled angle. Round your answer to tenths of a degree.

12.

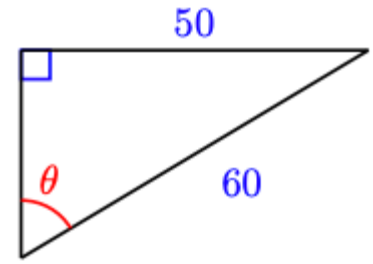


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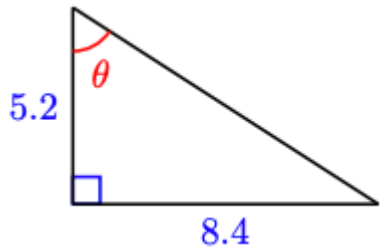


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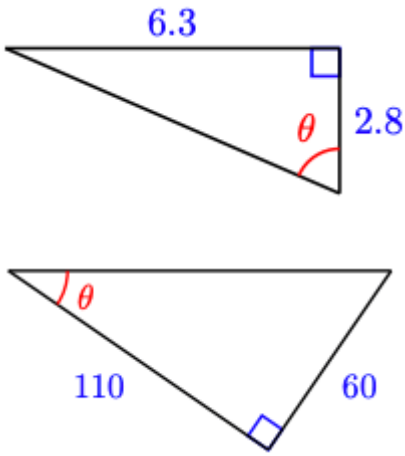
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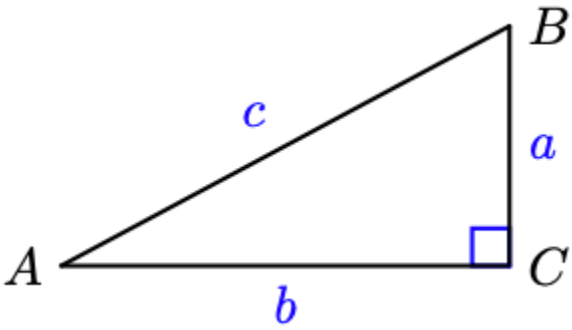
16.



17.



In these Homework Problems, we use the following standard notation for a right triangle: in $\triangle ABC, \angle C$ is a right angle. The side opposite $\angle C$ has length c, and so on. (See the figure at below.)



For the following exercises,
(a) Sketch the right triangle described.
(b) List the steps you would use to solve the triangle without doing the calculations.

18. $B = 53.7^\circ, b = 8.2$

19. $B = 80^\circ, a = 250$

20. $A = 25^\circ, b = 40$

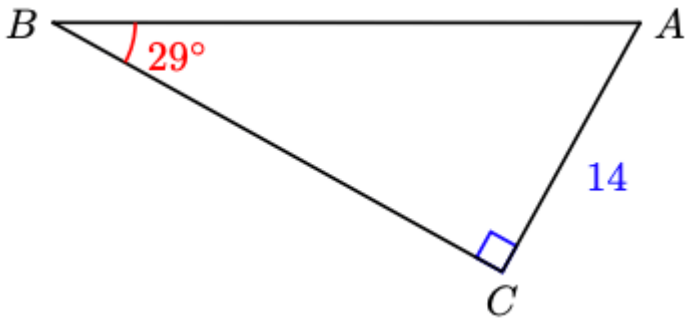
21. $A = 15^\circ, c = 62$

22. $A = 64.5^\circ, c = 24$

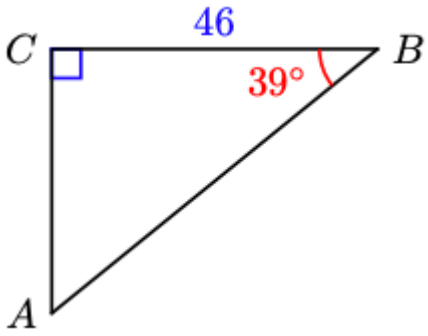
23. $B = 44^\circ, b = 0.6$

For the following exercises, solve the triangle. Round answers to hundredths.

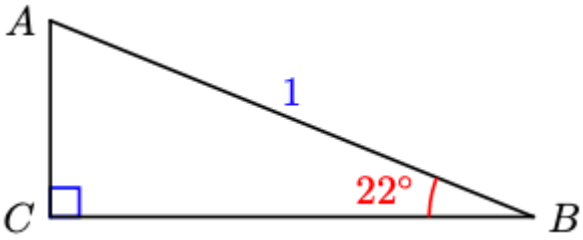
24.



25.



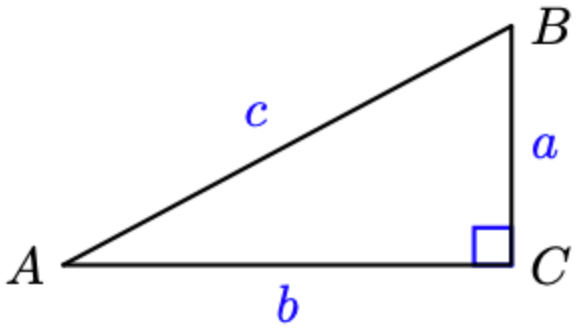
26.



27.



In these Homework Problems, we use the following standard notation for a right triangle: in $\triangle ABC, \angle C$ is a right angle. The side opposite $\angle C$ has length c, and so on. (See the figure at below.)



For the following exercises,
(a) Sketch the right triangle described.
(b) Solve the triangle.

28. $A = 42^\circ, c = 26$

29. $B = 28^\circ, c = 6.8$

30. $B = 33^\circ, a = 300$

31. $B = 79^\circ, a = 116$

32. $A = 12^\circ, a = 4$

33. $A = 50^\circ, a = 10$

For the following exercises, sketch and label a right triangle with the given properties. Solve the triangle.

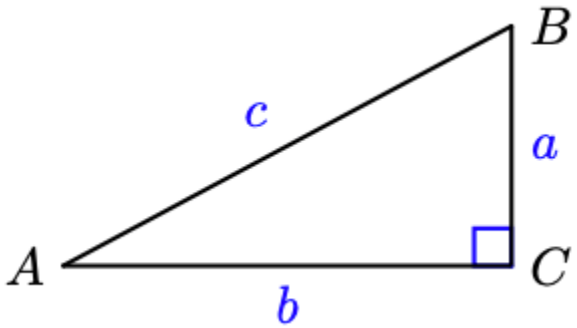
34. One angle is 40°, and the side opposite that angle is 8 inches

35. One angle is 65°, and the side adjacent to that angle is 30 yards

36. One angle is 28°, and the hypotenuse is 56 feet

37. One leg is 15 meters, and the hypotenuse is 18 meters

In these Homework Problems, we use the following standard notation for a right triangle: in $\triangle ABC, \angle C$ is a right angle. The side opposite $\angle C$ has length c, and so on. (See the figure at below.)



For the following exercises,
(a) Sketch the right triangle described.
(b) Solve the triangle.

38. $a = 18, b = 26$

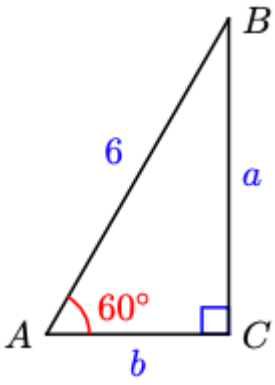
39. $a = 35, b = 27$

10. $b = 10.6, c = 19.2$

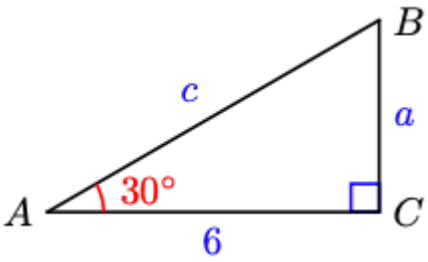
11. $a = 88, c = 132$

For the following exercises, solve the triangle. Give your answers as exact values.

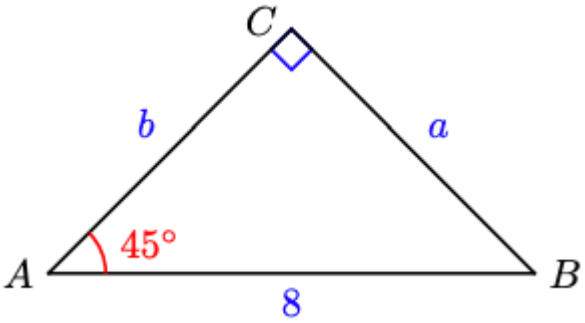
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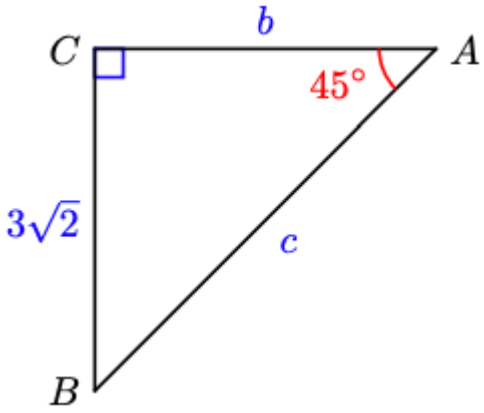
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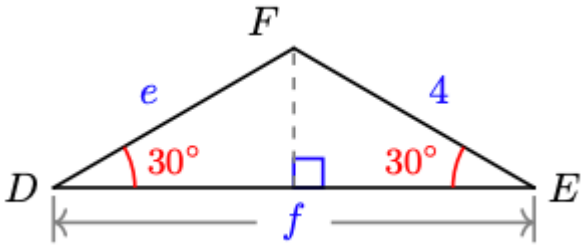
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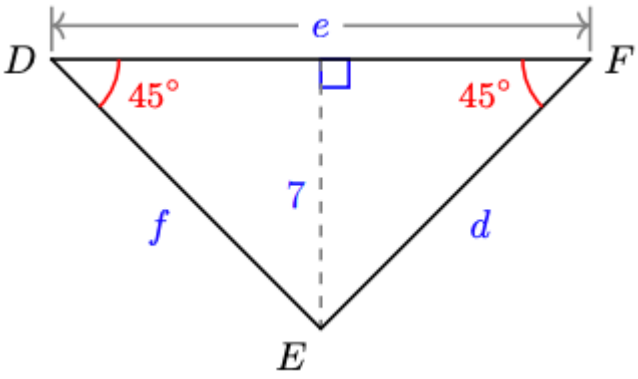
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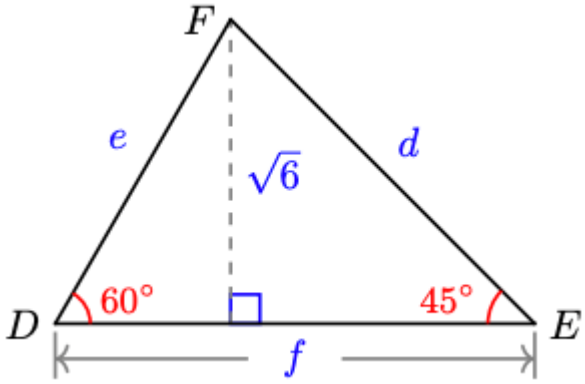
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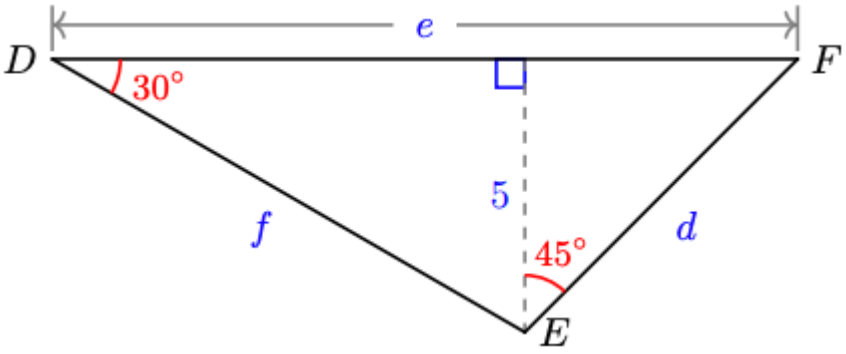
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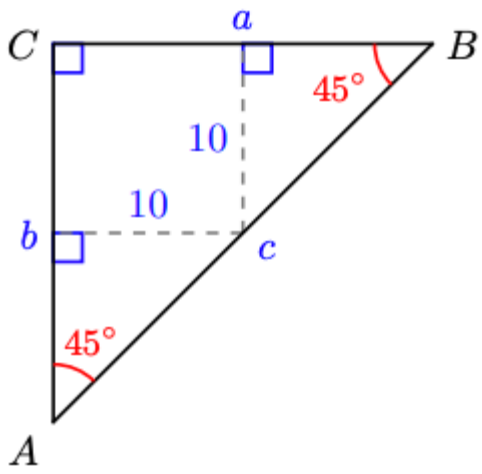
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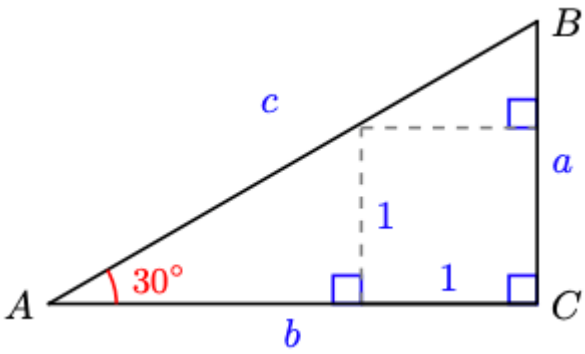
19.



20.



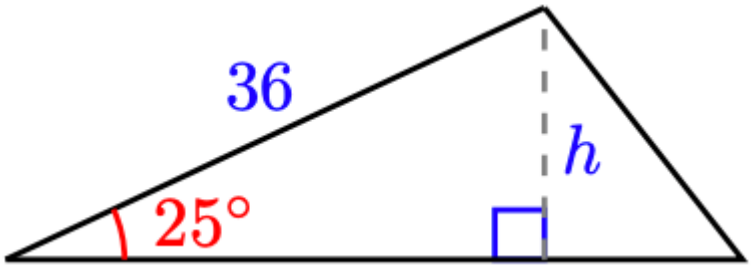
51.



Synthesis Questions

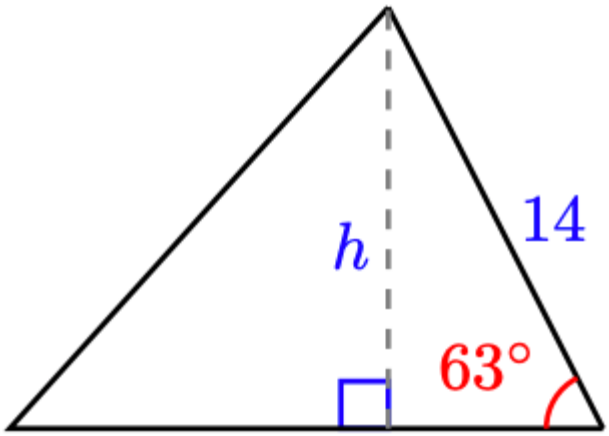
For the following exercises, find the triangle's altitude. Round your answer to two decimal places.

52.

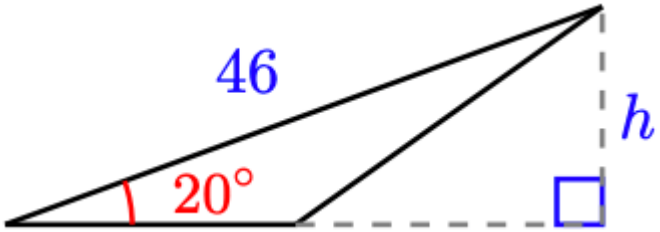


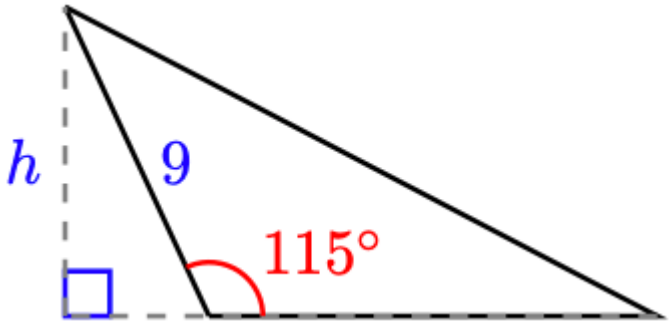
53.

54.



55.

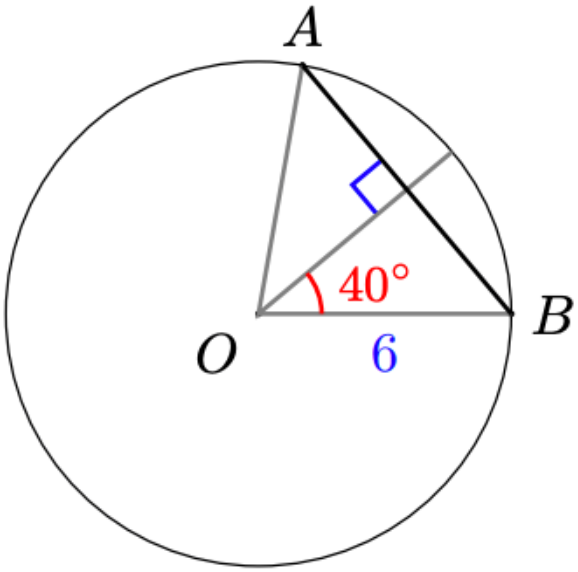


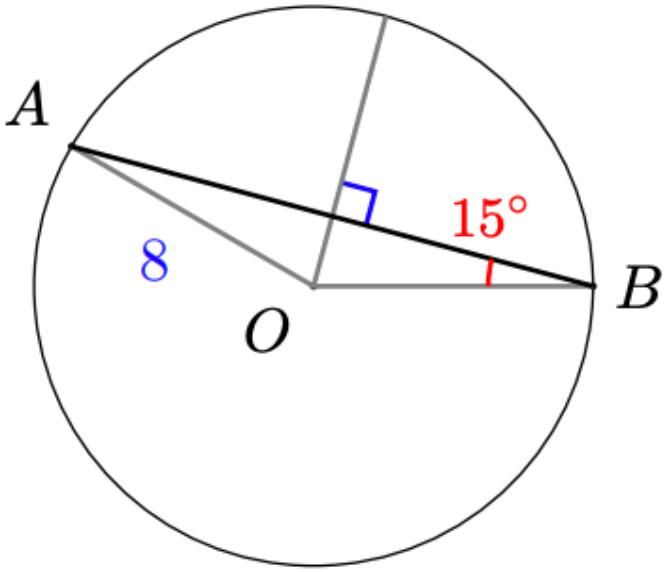


For the following exercises, find the chord length AB. Round your answer to two decimal places.

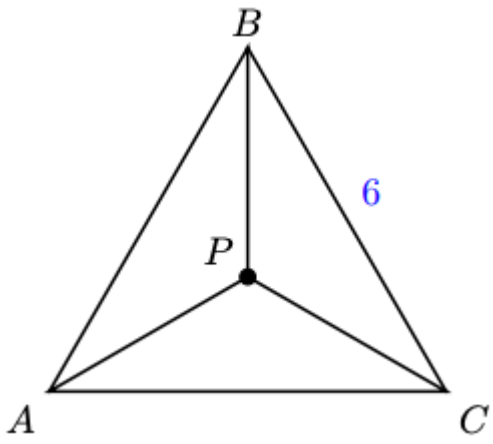
56.

57.



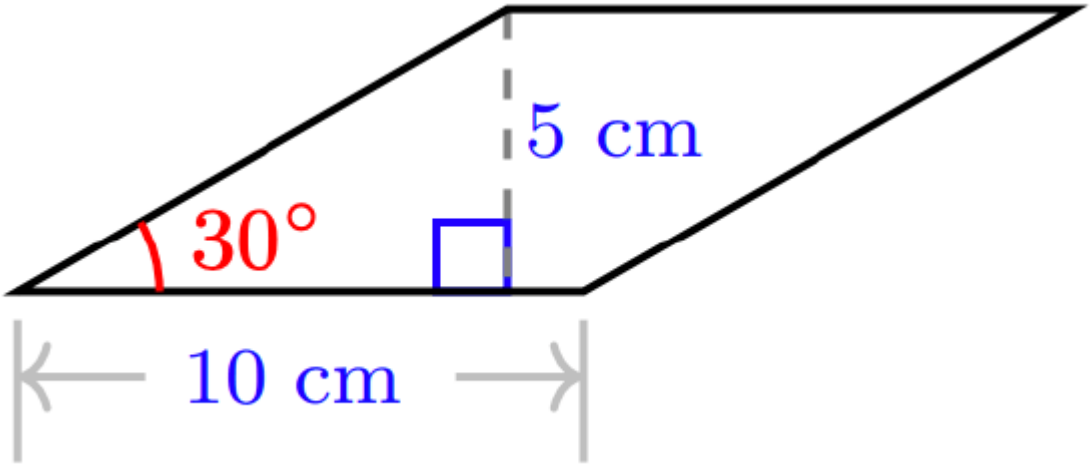


58. Triangle *ABC* is equilateral, and its angle bisectors meet at point *P*. The sides of $\triangle ABC$ are 6 inches long. Find the length of AP.

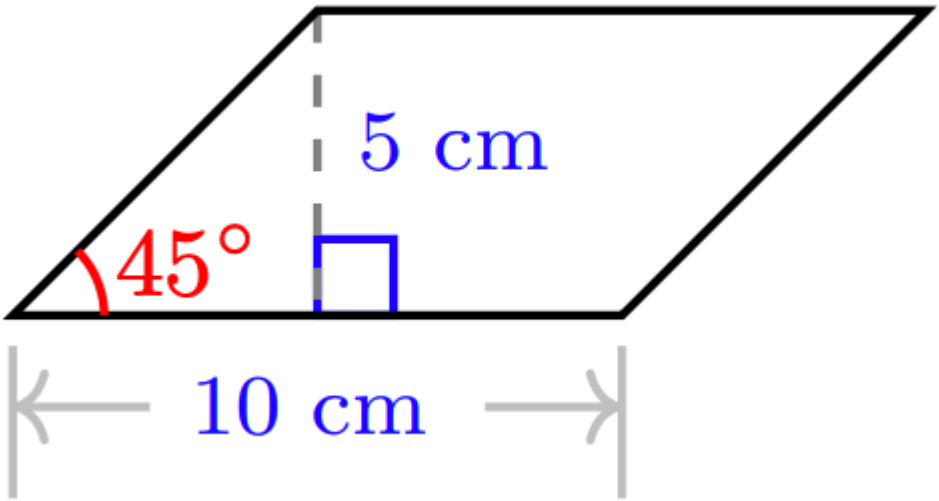


59. Find an exact value for the perimeter of each parallelogram.

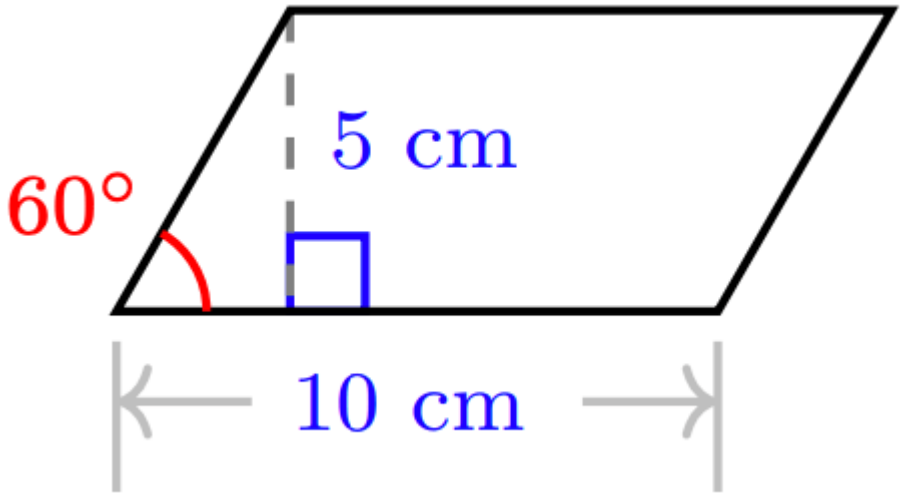
a.



b.



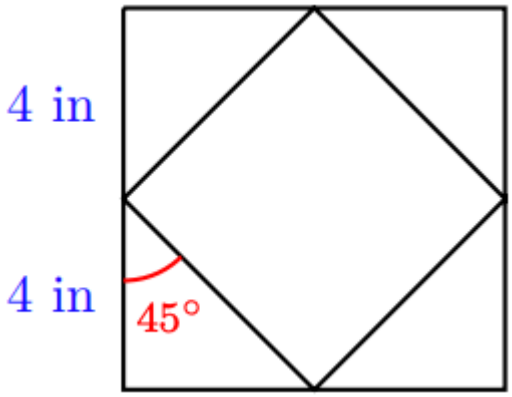
c.



60. a. Find the area of the outer square.

b. Find the dimensions and the area of the inner square.

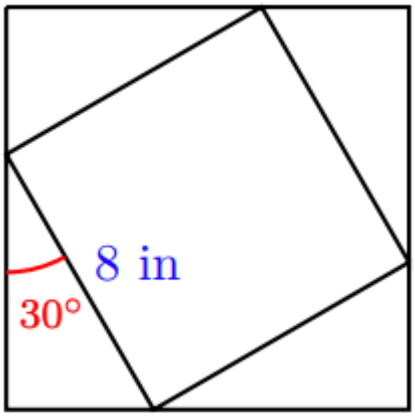
c. What is the ratio of the area of the outer square to the area of the inner square?



61. a. Find the area of the inner square.

b. Find the dimensions and the area of the outer square.

c. What is the ratio of the area of the outer square to the area of the inner square?



Applications

For the following exercises,
(a) Sketch a right triangle that illustrates the situation. Label your sketch with the given information.
(b) Choose the appropriate trigonometric ratio, write an equation, and solve the problem.

62. **Distance Across a Canyon.** To measure the distance across a canyon, Roy first sights an interesting rock directly opposite on the other side. He then walks 200 yards down the canyon's rim and sights the rock again, this time at an angle of 18.5° from the canyon rim. What is the width of the canyon?

63. **Engineering Accessibility.** Ramps for wheelchairs should be no steeper than an angle of 6°. How much horizontal distance should a ramp that rises 5 feet in height be allowed?

64. **Cable Length.** A cable car on a ski lift traverses a horizontal distance of 1800 meters at an angle of 38°. How long is the cable?

65. **Cable Angle.** The gondola cable for the ski lift at Snowy Peak is 2458 yards long and climbs 1860 feet. What angle with the horizontal does the cable make?

66. **Construction.** Loi is building the loft on her summer cottage. At its central point, the height of the loft is 8 feet, and the roof's

pitch should be 24°. How long should the rafters be?

67. **Leaning Tower of Pisa.** The Leaning Tower of Pisa, if standing perfectly vertical, would have a height of 55 meters - unfortunately, the tower is leaning (thus, its famous name). An object dropped from the top of the tower lands 4.8 meters from the base of the tower. At what angle from the horizontal does the tower lean?

68. **Mining.** A mining company locates a vein of minerals at a depth of 32 meters. However, a layer of granite is directly above the minerals, so they decide to drill at an angle, starting 10 meters from their original location. At what angle from the horizontal should they drill?

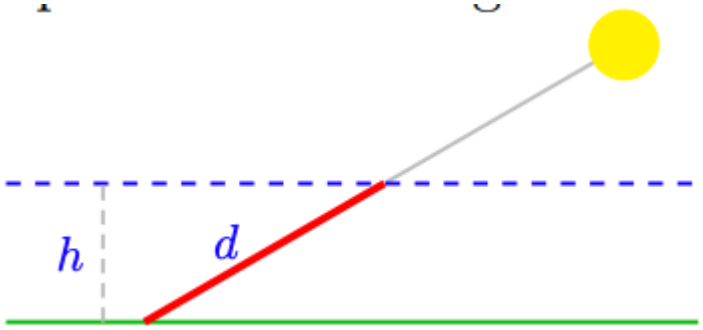
69. **Chichen Itza.** The Mayan pyramid of El Castillo in Chichen Itza, Mexico, has 91 steps, each 26 cm high and 30 cm deep.

    a. What angle does the side of the pyramid make with the horizontal?

    b. What is the distance up the face of the pyramid, from the base to the top platform?

70. **Ferris Wheel.** The first Ferris Wheel was built for the 1893 Chicago World's Fair. It had a diameter of 250 feet, and the boarding platform at the wheel's base was 14 feet above the ground. What would be your height above the ground if you boarded the wheel and rotated through an angle of 50°?

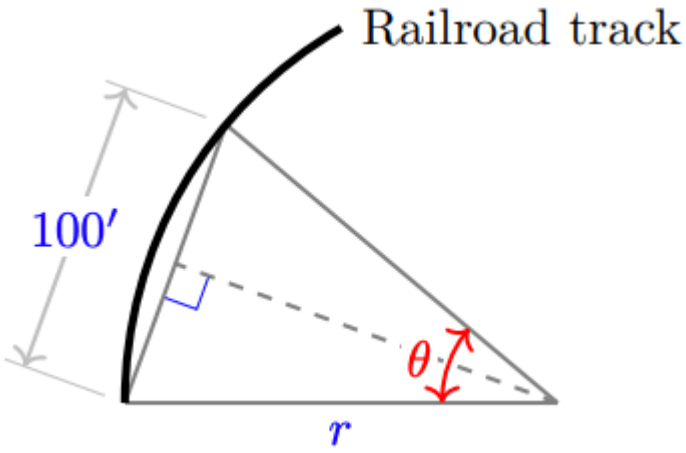
71. **The Physics of Light.** The distance that sunlight must travel to pass through a layer of Earth's atmosphere depends on the thickness of the atmosphere and the sun's angle.



    a. Write an expression for the distance, d, that sunlight travels through a layer of atmosphere of thickness h.

    b. Find the distance (to the nearest mile) that sunlight travels through a 100-mile layer of atmosphere when the sun is 40° above the horizon.

72. **Railroad Engineering.** In railroad design, the degree of curvature of a section of track is the angle subtended by a chord 100 feet long.



a. Use the figure to write an expression for the radius, r, of a curve whose degree of curvature is θ. (Hint: The bisector of the angle θ is perpendicular to the chord.)

b. Find the radius of a curve whose degree of curvature is 43°.

Answers

Skills Refresher

1. 2
2. $\sqrt{3}$
3. $2\sqrt{2}$
4. $\frac{\sqrt{3}}{2}$
5. $\frac{\sqrt{2}}{2}$
6. $\frac{2\sqrt{3}}{3}$
7. $2\sqrt{3}$
8. $\sqrt{2}$

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3.4: Applications Involving Right Triangles

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We now venture into practical applications of right triangle trigonometry. The concepts introduced in this section are used throughout the rest of this text and are the most common applications (although not the only applications) within Trigonometry.

Angles of Elevation or Depression

Suppose you want to measure the height of a tall tree without climbing it (see the figure below).

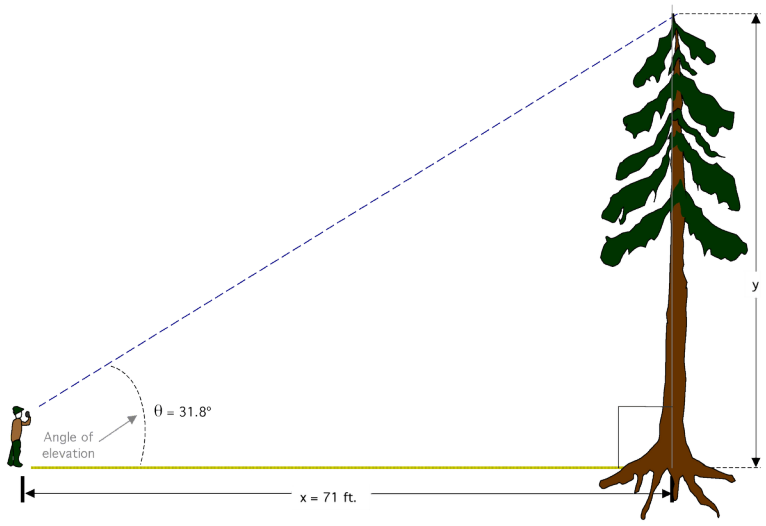


Figure 3.4.1

We can measure the distance to the tree's base and use a surveying tool called a theodolite to measure the angle between the ground and our line of sight to the top of the tree. This angle is called the **angle of elevation**.

Definition: Angle of Elevation

The **angle of elevation** (or **angle of inclination**) is the angle formed between a horizontal line and the line of sight when an observer looks upwards at an object. It represents the amount an observer would need to elevate their gaze (or line of sight) from a horizontal plane to see an object above them.

Since the height we want is the side opposite the angle of elevation, and we already measured the distance to the base of the tree, we could relate the quantities using the tangent function.

Example 3.4.1

Find the height of a cliff if the angle of elevation to the top of the cliff is 58° at a distance of 300 feet from the base of the cliff.

Solution available at the end of this section.

Checkpoint 3.4.1

The tallest living tree is a coastal redwood named Hyperion, measuring 378.1 feet tall. From a distance of 100 feet from the base of the tree, what is the angle of elevation to the top of the tree? Round the answer to the nearest degree.

Answer available at the end of this section.

The angle of elevation has a "sister" concept called the **angle of depression**.

Definition: Angle of Depression

The **angle of depression** (or **angle of declination**) is the angle formed between a horizontal line and the line of sight when an observer looks downwards at an object. It represents the amount an observer would need to lower their gaze (or line of sight) from a horizontal plane to see an object below them.

The two concepts are illustrated in Figure 3.4.3.

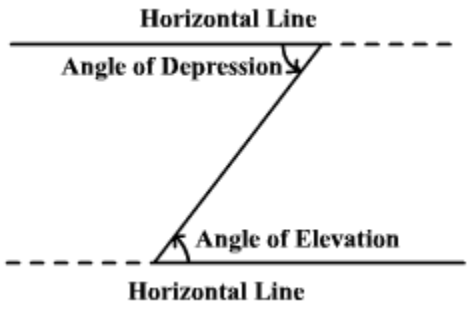


Figure 3.4.3

Example 3.4.2

Phuong takes a helicopter tour through the Grand Canyon. At one point, the pilot hovers 500 feet from the canyon wall. Phuong, being a prepared mathematician, takes out her protractor and measures the angle of elevation to Pima Point (the top of the canyon wall) as 67° and the angle of depression to the canyon floor as 82°. To the nearest foot, what is the elevation gain from the canyon floor to Pima Point?

Solution available at the end of this section.

Checkpoint 3.4.2

On August 7th, 1974, French street performer Philippe Petit began the most famous high-wire walk in history, calmly traversing the space between the former Twin Towers (in New York City) at an elevation of 1,350 feet without any safety gear. Not long after stepping out onto the thin wire, the angle of depression to the base of one building was 88°, and the angle of depression to the base of the other building was 86°. How many feet was his tightrope walk?

Answer available at the end of this section.

Angles of Lines

As a rare moment in Mathematics, we have an alternate definition of **angle of inclination** when it comes to lines.

Definition: Angle of Inclination of a Line

The **angle of inclination of a line** is the measure of the angle in the positive direction from the positive x-axis to the line.

Figure 3.4.5 demonstrates the angle of inclination of a line.

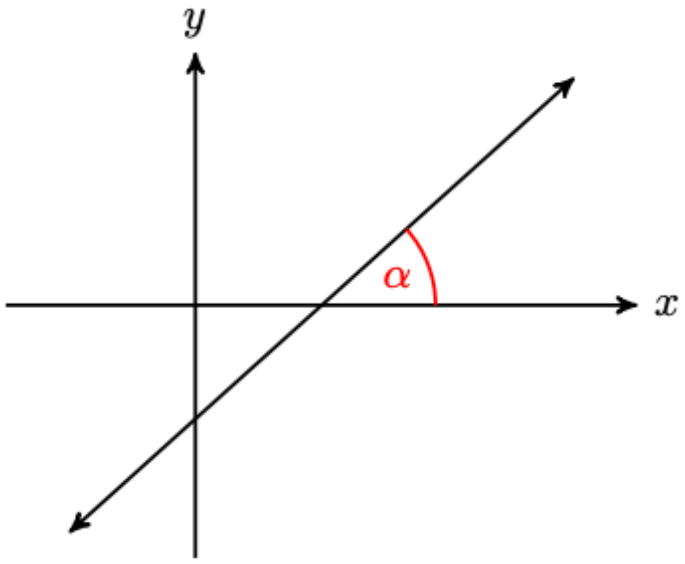


Figure 3.4.5

It is important to note that the angle of inclination to the line is measured from where the line crosses the x-axis. For example, the angle of inclination in Figure 3.4.6 is the *positive* angle, β, between the x-axis (in the positive direction) and the line, and this happens to be an obtuse angle.

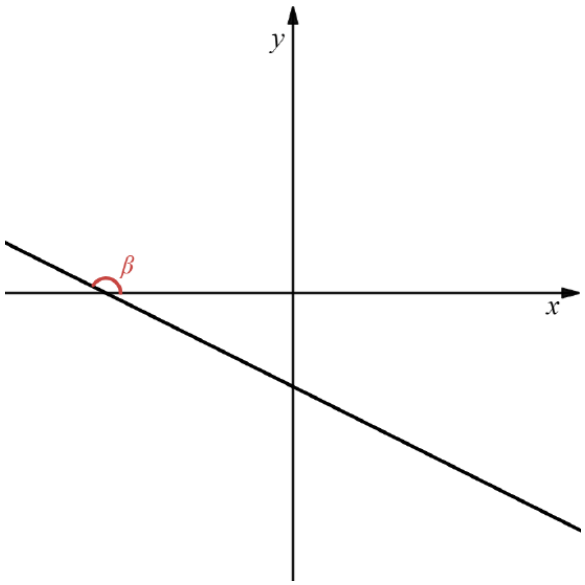


Figure 3.4.6

Recall that the slope of a line is given by the ratio

$$\frac{\text{change in } y}{\text{change in } x}$$

as we move from one point to another on the line. So, if we create a right triangle by dropping a perpendicular segment from the line to the x-axis, the ratio of sides $\frac{\text{opposite}}{\text{adjacent}}$ gives the slope of the line.

Theorem: The Tangent and the Angle of Inclination of a Line

If the slope of the line is m, then the angle of inclination of the line, α, is related to the slope by the formula

$$\tan(\alpha) = m$$

where $0^\circ \leq \alpha \leq 180^\circ$.

Interactive Element: Angle of Inclination and Slope

The following Interactive Element might help you visualize how a line's angle of inclination depends on its slope.

Interact: Drag the large red point to tilt the line, and watch the angle of inclination at the origin, the reference right triangle, and the slope readout update together.

Observation: A positive slope gives an *acute* angle of inclination equal to $\tan^{-1}(\text{slope})$; dragging the point below the x-axis makes the slope negative and the angle of inclination *obtuse*. A vertical line has an inclination of 90° and no defined slope.

Example 3.4.3

Find the angle of inclination of the line $y = \frac{3}{4}x - 3$.

Solution available at the end of this section.

Checkpoint 3.4.3

Find the angle of inclination of the line in Figure 3.4.8.

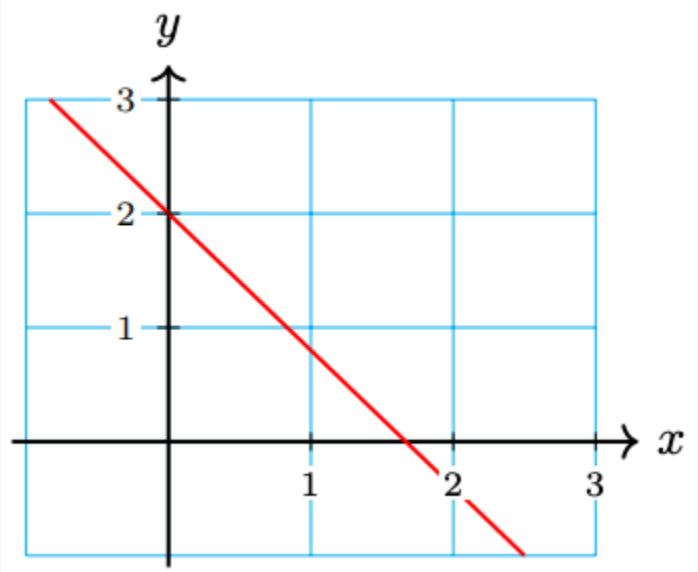


Figure 3.4.8: $y = -\frac{6}{5}x + 2$

Answer available at the end of this section.

Navigation Problems

Up to this point, we have presented two definitions for the trigonometric functions. Coordinate trigonometry allows us to work with any angle (positive, negative, acute, or obtuse) as long as it is in standard position. Right triangle trigonometry will enable us to work with acute angles within a right triangle without a coordinate system. With navigation problems, we officially begin to marry the two definitions into a unified approach to Trigonometry.

Heading

Navigational directions for vessels or aircraft are often given as a **heading**.

Definition: Heading (navigational)

In aviation and nautical navigation, the **heading** of a vessel or aircraft is the compass direction measured clockwise from due north in which the craft's bow or nose is pointed.

For example, a heading of 110° is shown in Figure 3.4.9.

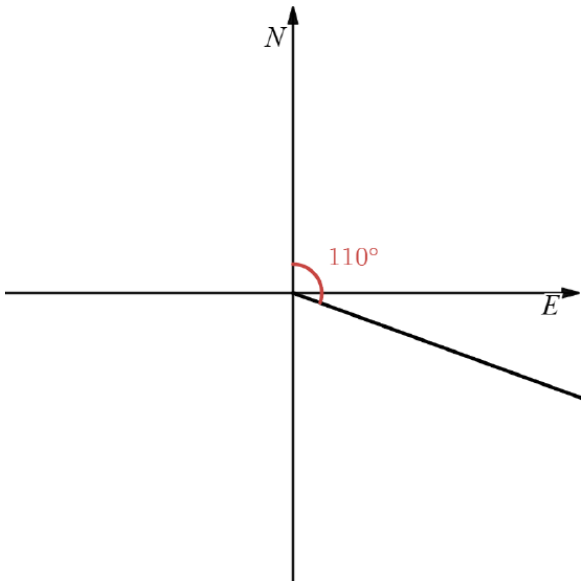


Figure 3.4.9

However, as shown in Figure 3.4.10, this heading generates a terminal side shared with the angle -20° in standard position.

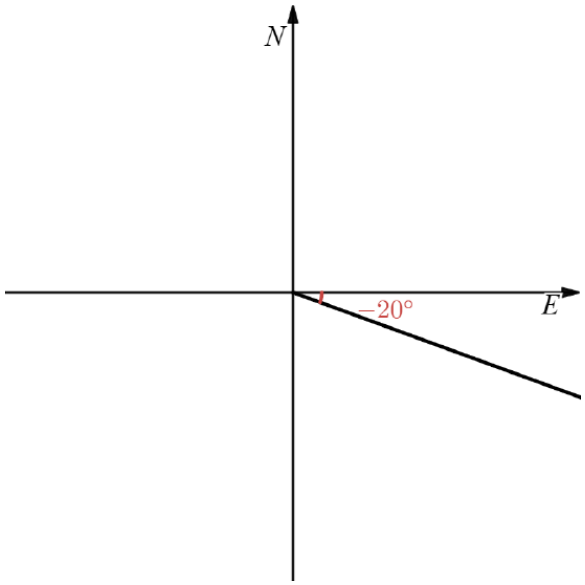


Figure 3.4.10

While there is a method for converting a heading to an angle in standard position, it is much better to get used to making a sketch of the given situation and superimposing a right triangle with hypotenuse along the terminal side of the given heading angle and adjacent side along the x-axis, as shown in Figure 3.4.11.

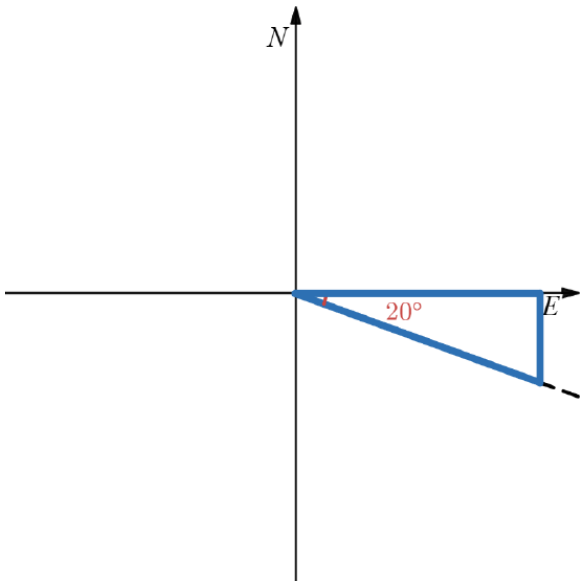


Figure 3.4.11

Once we have the superimposed right triangle, we can use right triangle trigonometry to answer all questions about this situation.

Example 3.4.4

Piyali leaves the airport at a heading of 245° and flies 60 miles. How far south of the airport is she at that time?

Solution available at the end of this section.

Checkpoint 3.4.4

Lap leaves the airport and flies 150 miles at a heading of 132°. How far east of the airport is he at that time?

Answer available at the end of this section.

Bearing

Land-based navigation often uses **bearing** instead of heading.

Definition: Bearing (navigational)

In land-based navigation, the **bearing** (also called the **quadrant bearing**) of an object (person, vehicle, etc.) represents the direction that object is facing or moving. The notation for bearing is a letter followed by an acute angle followed by a letter. These are known as **Meridian - Angle - Direction**.

- **Meridian:** The first letter indicates whether the angle is measured from North (N) or South (S), whichever is closer to the direction.
- **Angle:** This is the acute angle measured from the chosen meridian to the direction of interest.
- **Direction:** The second letter indicates whether the angle is measured towards the East (E) or West (W).

An example of a bearing is $\mathrm{S}\ 20^\circ\ \mathrm{W}$ (read as, "south by 20 degrees west"). Conceptually, this means the following: initially face south, then turn by 20° in the westerly direction. Thus, a bearing describes a line as heading north or south and deflected some degrees toward the east or west. Since a bearing always starts with a north or south facing and then a turn toward east or west, bearings will always have an acute angle.

Figure 3.4.15 illustrates some directions given in bearings.

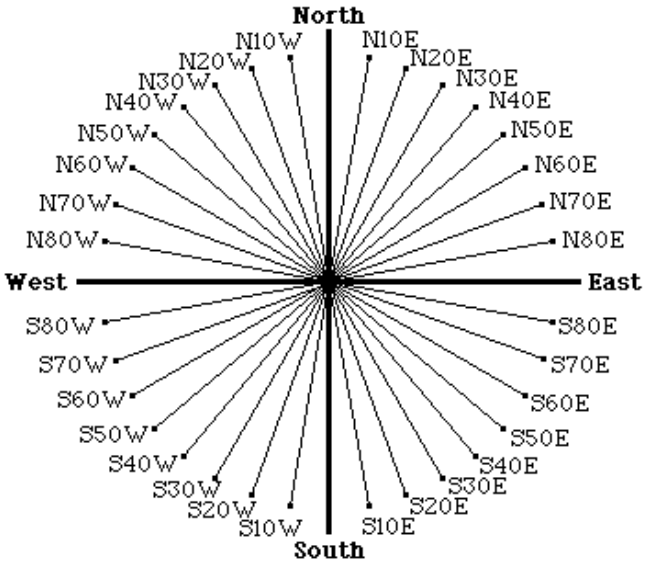


Figure 3.4.15

When working with directions given in bearings, it is crucial that you understand the angle given will always be between the y-axis (the north-south line) and the terminal side of the angle. This is the only time in Trigonometry you will see such a convention. As long as you keep your thinking cap on and do not try to memorize your way through Trigonometry, you should be fine (especially if you get in the habit of superimposing right triangles on your sketches as we did with headings).

Example 3.4.5

Roy is hiking on the Pacific Crest Trail. He leaves his campsite at 6:00 A.M., walking at a bearing of $\mathrm{N}\ 37^\circ\ \mathrm{W}$. His average hiking speed is 3.4 miles per hour. At 9:00 A.M., he decides to eat breakfast at Scissor's Crossing. At this time, how far north is he from the campsite?

Solution available at the end of this section.

Caution: Avoid Creating Reference Triangles Along the y-axis

You would not be incorrect if you superimposed a right triangle with a side along the y-axis. If you do this, you will use the cosine of 37° to determine how far north Roy is from his campsite; however, superimposing right triangles with sides along the y-axis will eventually lead to some confusion and mistakes. It's best to nip that bad habit in the bud and always superimpose our right triangles with sides along the x-axis.

Checkpoint 3.4.5

Ivan leaves the airport and flies 150 miles at a bearing of $\mathrm{S}\ 48^\circ\ \mathrm{E}$. How far east of the airport is he at that time?

Answer available at the end of this section.

Example 3.4.6

Mount Whitney is 5.86 miles due east of the Pacific Crest Trail, labeled as point P. The John Muir Trail, labeled as point J, is 1.32 miles due south of Mount Whitney. You hike from Mount Whitney to point J on the John Muir Trail. Once at point J, you decide to hike to point P on the Pacific Crest Trail. What is the bearing in which you must hike *from* point J on the John Muir Trail *to* point P on the Pacific Crest Trail, and how many miles will your hike be? (For simplicity, we will ignore the fact that there are significant elevation gains and losses between each of these points.)

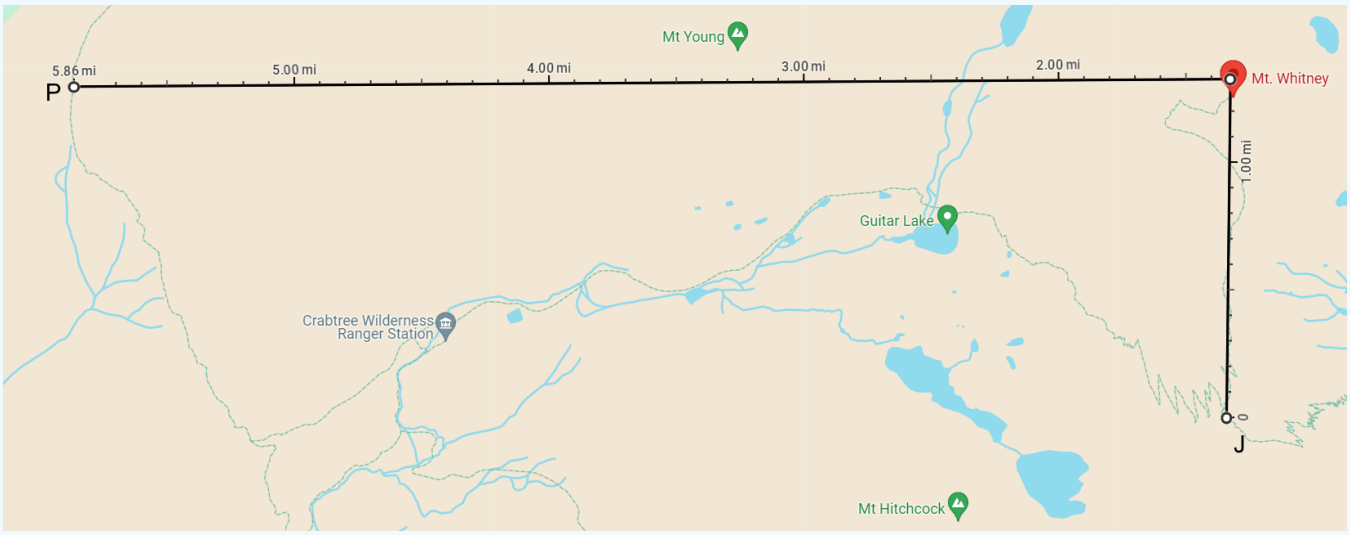


Figure 3.4.18

Solution available at the end of this section.

Checkpoint 3.4.6

Sutter County Airport is 28.4 miles due north of Sacramento International Airport. If Placerville Airport is 45.3 miles due east of Sacramento International Airport, what is the bearing from Sutter County Airport to Placerville Airport, and what is the distance between these two airports?

Answer available at the end of this section.

In Examples 3.4.5 and 3.4.6, we were given a single destination for a bearing. We were essentially told, "Go this direction for so many miles." However, an important application within navigation occurs when you have been given "stop-over points" where the bearing is changed. The following example illustrates, in great detail, how to manage these problems.

Example 3.4.7

Roy has made significant progress along the Pacific Crest Trail and has reached the Sierra Nevada range. Last night, he made camp at Junction A. In the morning, he breaks camp at 6:00 A.M. and hikes at a bearing of $\mathrm{N}\ 42^\circ\ \mathrm{E}$. After two hours, he reaches Junction B, where the trail turns 90° in the bearing of $\mathrm{N}\ 48^\circ\ \mathrm{W}$. After another hour, he runs into Junction C, which is a "shortcut" trail that directly connects his current location back to his original campsite at Junction A.

Normally, Roy would never consider "back-tracking" as it's a complete waste of time and energy; however, seeing the "shortcut" trail sparks his memory that he left his trash bag back at his campsite (Junction A)! Being a conscientious hiker, he decides to burn the time and return to ensure he "leaves no trace" on the trail.

If Roy hikes at an average pace of 3.4 miles per hour, what is the distance between Junction C and Junction A, and what is his bearing from Junction C to Junction A?

Solution available at the end of this section.

Note: Place Small Sets of Axes at Each Stop-Over Point

As Example 3.4.7 illustrates, bearing problems where direction changes occur are more manageable if you get into the good habit of drawing small "north-south, east-west" axes at each stop-over point.

Checkpoint 3.4.7

A ship leaves port bearing $\mathrm{S}\ 32^\circ\ \mathrm{E}$. It travels in this direction for 1 mile and changes its bearing to $\mathrm{S}\ 58^\circ\mathrm{W}$. It then travels an additional 10 miles. At this point, the captain remembers he left his phone back at port (and he wanted to post a ton of videos of this trip on his social media platforms). Find the direct distance and the bearing from this point back to port so that the "social-influencer" captain can get his phone. (Round answers to the nearest tenth.)

Answer available at the end of this section.

Compound Right Triangle Applications

An application that requires you to use more than a single right triangle is called a **compound right triangle application** (don't bother looking for that phrase elsewhere, we just made it up to label applications that fit into this category). These entertaining applications often require you to think in three dimensions, and they help build your ability to break larger problems into small pieces.

Example 3.4.8

Camille is worried about her neighbor's tree eventually falling and hitting her house. Unfortunately, her neighbor has refused to allow her to pay someone to measure the exact height of the tree, so Camille resorts to her top-notch math skills. Since she cannot step onto her neighbor's property, Camille stands on her property, facing the tree, and finds the angle of elevation to the top of the tree to be 38°. She then turns 90° and walks for 32 feet (staying on her property the entire time). At this point, she turns to face her neighbor's tree and measures the angle between the path she just walked and her neighbor's tree to be 50°. Use this information and Figure 3.4.25 to find the tree's height.

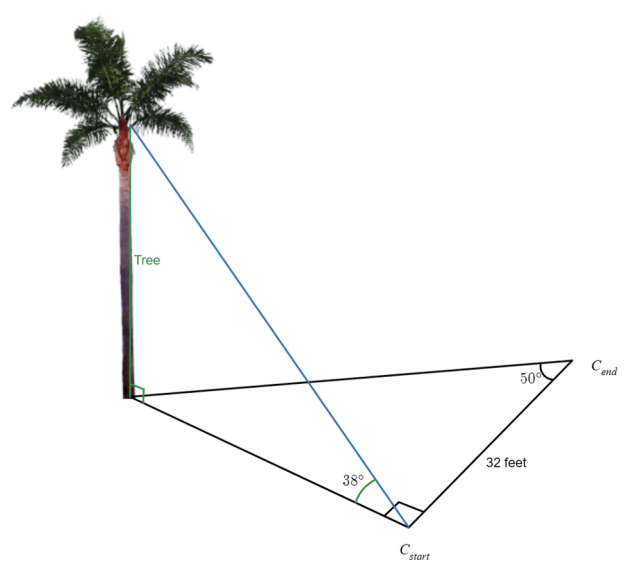


Figure 3.4.25

Solution available at the end of this section.

Checkpoint 3.4.8

Sang's boat loses power while on a boating trip out on the ocean. She radios her position to a pair of radar stations 43 miles apart along the coastline. The radio stations are directly north and south of each other. The bearing of her boat from the southern radio station is $\mathrm{N}\ 54^\circ\ \mathrm{W}$, while the bearing from the northern radio station is $\mathrm{S}\ 72^\circ\ \mathrm{W}$. To the nearest mile, how far from the coastline is Sang's boat?

Answer available at the end of this section.

Success in College: Learning From Tests (Part 2)

1. A student earned 75% of the points on her last math test. On the first class day after the test, the class started a new chapter with new material. She does not understand several vital procedures from the last test. What do you think she should do about this situation?
2. A student usually finishes tests right as class ends. He doesn't have time to check for careless mistakes. He looks carefully at his corrected tests. He realizes that many of his errors are arithmetic or algebra mistakes. He knew how to do the problems; he didn't do them well enough. What do you think he should do?

Success in College: Cheating and Consequences

The following is an excerpt from an article at WorldNetDaily.com by Dennis Prager.

> A report by Donald McCabe, professor of organization management at Rutgers University and founding president of the Center for Academic Integrity at Duke University, describes a pretty bad ethical situation in America's high schools. Seventy-five percent of students engage in "serious cheating" (such as repeated cheating and significant plagiarism).
>
> It gets worse.
>
> First, according to the professor, this epidemic is the culmination of three decades of increased cheating in high schools.
>
> Second, about half of all students (not just half of all cheaters) see nothing wrong with cheating.
>
> As one who has written and lectured on ethics for 30 years, during which time I, too, have spoken at high schools across America, I am certain that these statistics are accurate. I also believe that I know the major reasons.
>
> One reason is that parents emphasize many things over character.
>
> A second reason is the absurd emphasis on getting into the best college. This preoccupation on the part of parents (which often begins in preschool) has utterly distorted our children's (and high schools') priorities. Because students are taught that getting into the right college is the most important thing in their lives, they readily compromise their character to achieve this goal. And they could not care less about learning. Since the purpose of school is to get good grades, learning is incidental, and loving learning is a non-issue.
>
> A third and final reason is the death of the concept of sin, a product of the thoroughly secular education American high schools offer. As noted, far worse than the number of students who cheat is the number of students who think that there is nothing wrong with cheating. I can testify from my talks at high schools that the vast majority of students believe what they have been taught - that right and wrong - are entirely subjective concepts. There is no morality higher than one's own; therefore, right and wrong are defined by the individual, who can rationalize any behavior.

When students do not understand a concept, they need help. Instead, some students in this situation choose to cheat. In college, any form of cheating is "academic dishonesty."

1. Has your professor discussed or posted their cheating policy?
2. What happens if you cheat in this math class?

   To be clear, cheating - in any form - results in more than just a "zero" on that assignment. This includes, but is not limited to, using notes on exams, using a device that is not allowed during an exam, or using unapproved technology to help during an exam. If I suspect a student of cheating, I will have them meet me outside of class time in my office to discuss what happened and why it is considered cheating. They will receive a zero on the assignment/exam and will be reported to the Student Discipline Officer for further discussions. The Student Discipline Officer is involved for two reasons: to give the student tools to help them break the "cheating habit," and to create an official record within the Los Rios Community College District to track that student's cheating offenses - if multiple offenses have occurred, the student could like be placed on academic probation or, in severe cases, completely expelled from the District.

   - Professor Simpson, Cosumnes River College

Answers

Section Preview

This section expands on the applications and methods introduced in the previous section. The applications treated here cover angles of elevation and depression, navigation, and right triangle applications in three dimensions. As is the case with everything in this chapter, all angles are acute.

Section Preview

- **Linear Functions**
  - The Slope of a Line: To understand the angles of inclination and declination, students must have a strong understanding of slope.

Section Preview

- **Linear Functions**
  - Equations of Lines — Slope-Intercept Form: Students will be asked to find the equation of a line given its angle of inclination.

Section Preview

- Solve an application problem involving an angle of elevation or depression.
- Determine the angle of inclination (or declination) of a line.
- Solve a navigation problem involving bearing or heading.
- Solve a compound right triangle application.

Example 3.4.1

As always, we begin by sketching the situation.



Figure 3.4.2

We know the angle of elevation, 58°, and the side length adjacent to that angle, 300 feet. We use the tangent function to find the cliff's height (which is opposite the given angle). Letting h be the height of the cliff, we get

$$\tan(58^\circ) = \frac{\text{opposite}}{\text{adjacent}} = \frac{h}{300} \implies h = 300\tan(58^\circ) \approx 480 \text{ feet.}$$

Checkpoint 3.4.1

75°

Example 3.4.2

We start by sketching the situation.

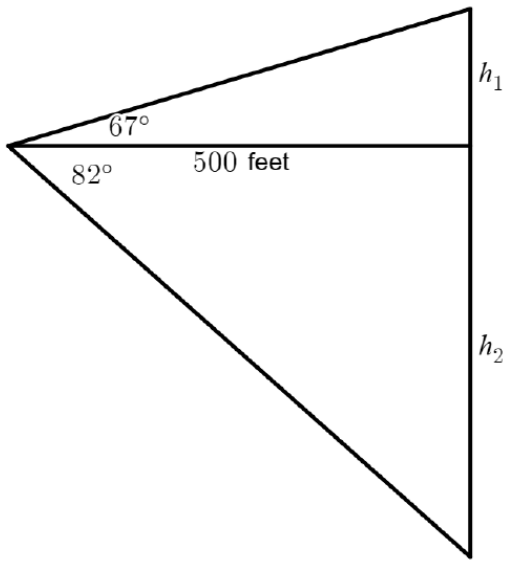


Figure 3.4.4

We must find the value of $h_1 + h_2$. The trigonometric function relating 67° to h_1 and 500 is the tangent. Thus,

$$\tan(67^\circ) = \frac{h_1}{500} \implies 500\tan(67^\circ) = h_1.$$

Similarly,

$$\tan(82^\circ) = \frac{h_2}{500} \implies 500\tan(82^\circ) = h_2.$$

Therefore, the elevation gain from the canyon floor to Pima Point is

$$h_1 + h_2 = 500\tan(67^\circ) + 500\tan(82^\circ) \approx 4,736 \text{ feet}.$$

Checkpoint 3.4.2

Roughly 141.5 feet.

Example 3.4.3

The slope of the line is $\frac{3}{4}$. Therefore,

$$\begin{aligned} \tan(\alpha) &= \frac{3}{4} \\ \implies \alpha &= \tan^{-1}\left(\frac{3}{4}\right) \approx 36.9^\circ \end{aligned}$$

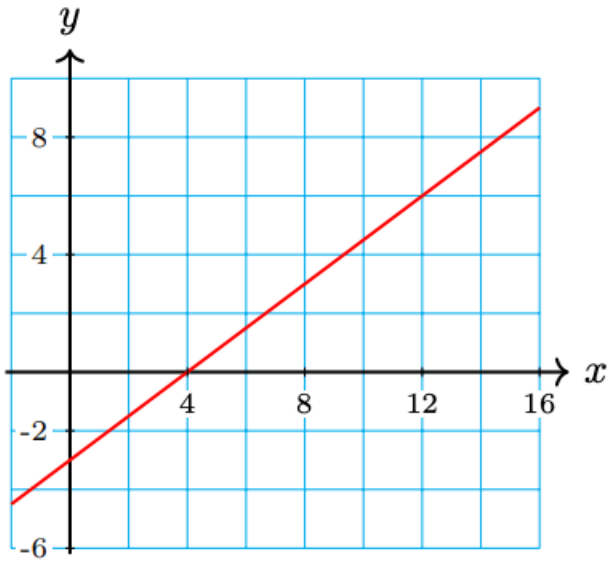


Figure 3.4.7

The angle of inclination is approximately 36.9°.

Checkpoint 3.4.3

$\approx 129.8^\circ$

Example 3.4.4

Sketching a heading angle of 245°, we get

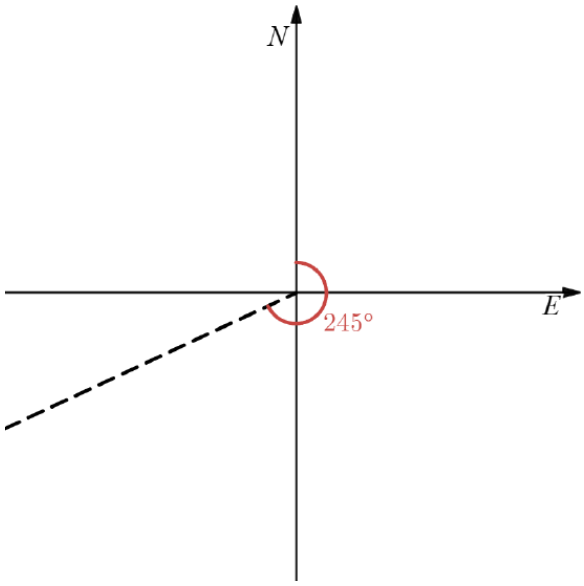


Figure 3.4.12

We now superimpose a right triangle on this graph, where the hypotenuse is on the terminal side of the heading angle and the adjacent side is on the x-axis.

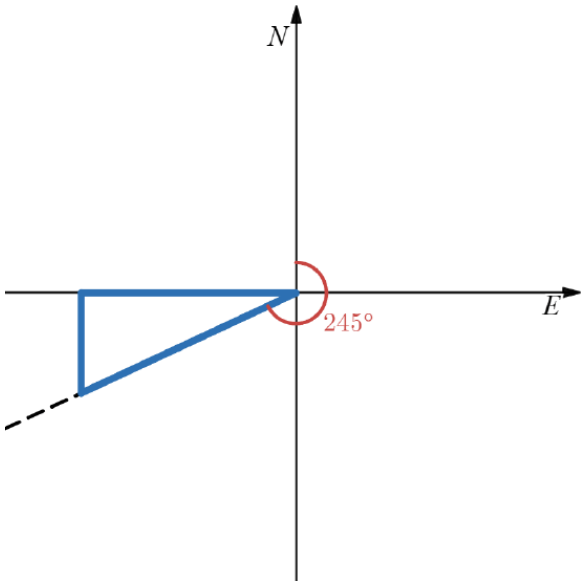


Figure 3.4.13

The key to using this superimposed right triangle is to use our *deductive* skills to determine the angle between the hypotenuse (terminal side of the heading angle) and the adjacent side (x-axis). Had the heading angle been 270°, Piyali's plane would be heading directly west along the negative x-axis (the adjacent side of the right triangle). This means there must be a small angle of 25° between the adjacent side of the right triangle and the hypotenuse. Said differently, to get to the 245° heading, we could have turned clockwise from due north by 270° and then backed off by 25°. Figure 3.4.14 shows the resulting angle within the superimposed right triangle.

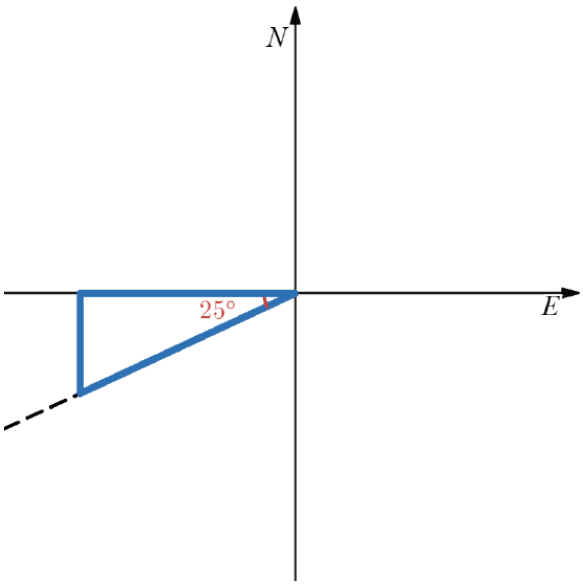


Figure 3.4.14

Piyali's plane flew along the hypotenuse of this right triangle for 60 miles. To determine how far south she is from the airport (located at the origin), we need to compute the length of the side opposite the 25° angle. The trigonometric function relating this angle, the opposite side, and the hypotenuse is the sine.

$$\sin(25^\circ) = \frac{\text{opposite}}{60} \implies \text{opposite} = 60\sin(25^\circ) \approx 25.36.$$

Hence, Piyali is about 25.4 miles south of the airport.

Checkpoint 3.4.4

111.5 miles

Example 3.4.5

Quickly sketching the situation, we get Figure 3.4.16.

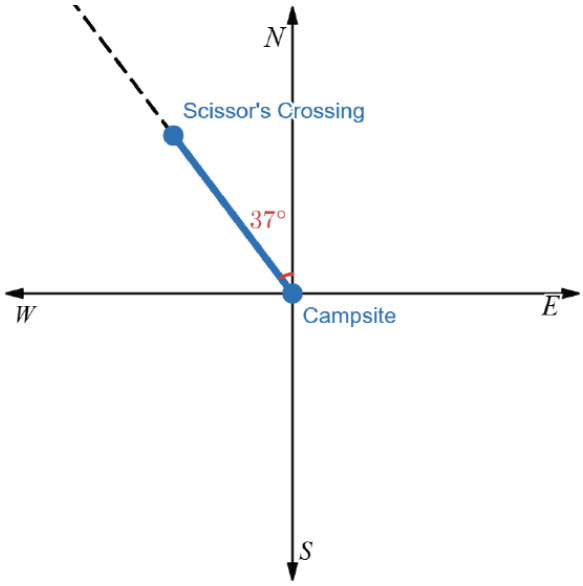


Figure 3.4.16

We superimpose a right triangle on this figure, where the hypotenuse is along the terminal side of the bearing angle, and one side is on the x-axis. The result is Figure 3.4.17.

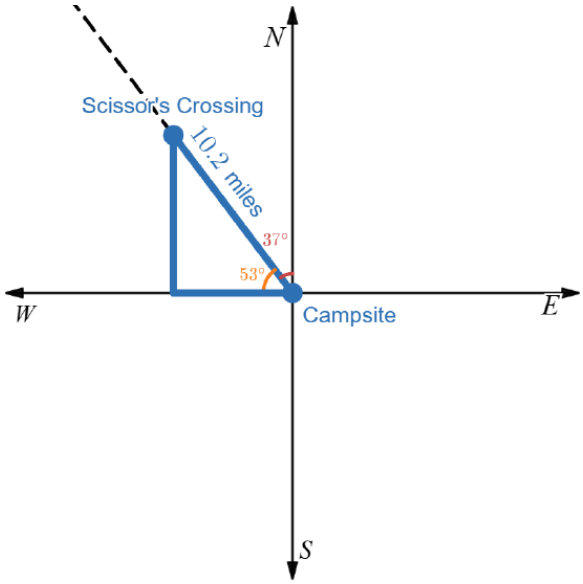


Figure 3.4.17

Notice that we already included that Roy has hiked a total of $3 \times 3.4 = 10.2$ miles by 9:00 A.M. Also, from the superimposed right triangle, we can see that the trigonometric function relating the hypotenuse to the vertical distance and the 53° angle (which is the complement of the given 37° angle) is the sine function. Hence,

$$\sin(53^\circ) = \frac{y}{10.2} \implies y = 10.2\sin(53^\circ) \approx 8.1461.$$

Therefore, Roy is approximately 8 miles north of his campsite at 9:00 A.M.

Checkpoint 3.4.5

111.5 miles

Example 3.4.6

With bearing problems, it is beneficial to draw a small set of axes at the point we are moving *from*, as seen in Figure 3.4.19.

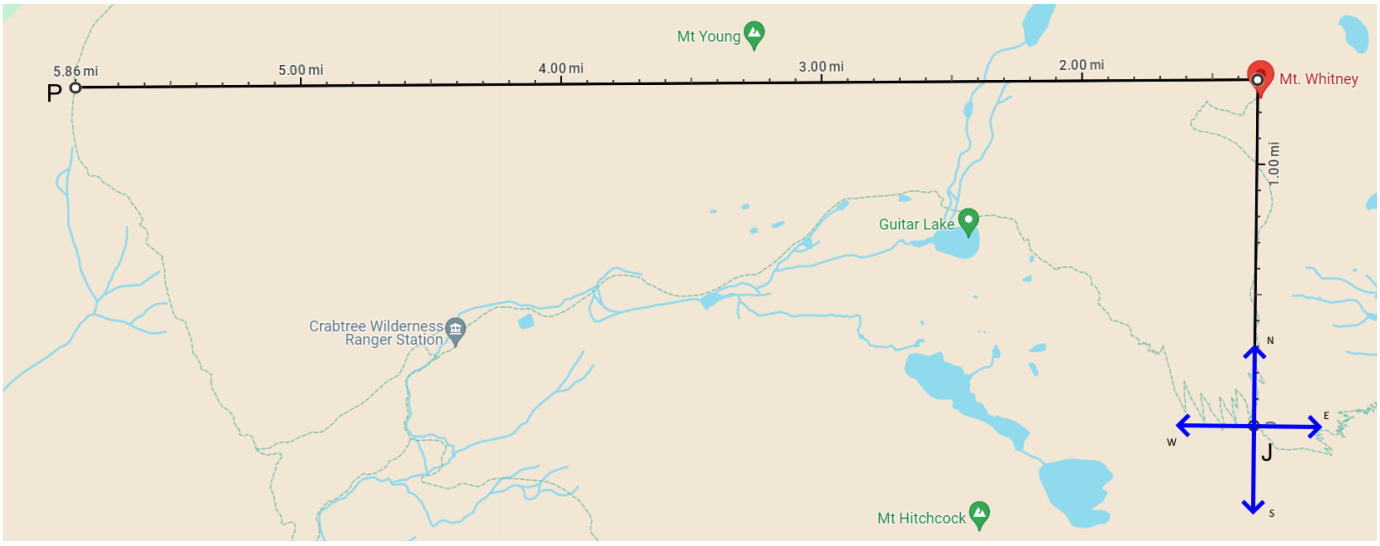


Figure 3.4.19

We then draw a line connecting points J and P, as follows.

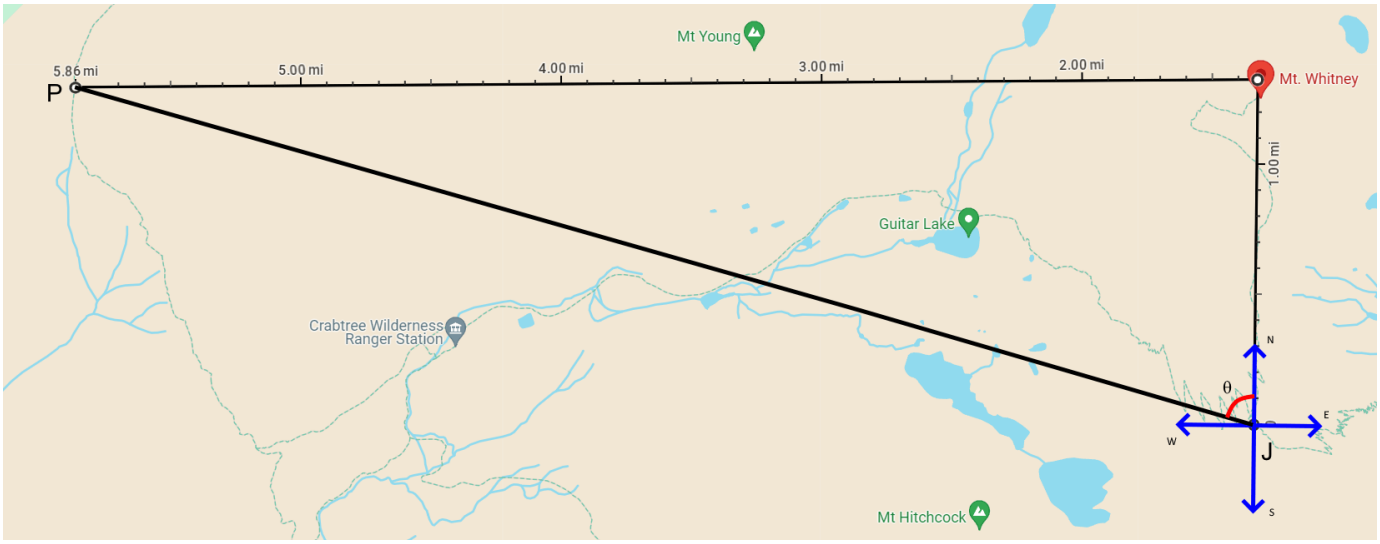


Figure 3.4.20

From the given image, the bearing *from* point J *to* point P has the form $\mathrm{N}\ \theta^\circ\ \mathrm{W}$. The tangent is the trigonometric function that relates the given sides of the right triangle and the angle θ. Hence,

$$\tan(\theta) = \frac{5.86}{1.32} \implies \theta = \tan^{-1}\left(\frac{5.86}{1.32}\right) \approx 77.3^\circ.$$

Hence, to hike from point J on the John Muir Trail to point P on the Pacific Crest Trail, we must maintain an approximate bearing of $\mathrm{N}\ 77.3^\circ\ \mathrm{W}$. The distance covered during that hike (if there was a direct trail between points J and P) is the hypotenuse of the right triangle.

$$d = \sqrt{5.86^2 + 1.32^2} \approx 6 \text{ miles}.$$

Checkpoint 3.4.6

$\mathrm{S}\ 57.9^\circ\ \mathrm{E}$, and 53.5 miles.

Example 3.4.7

This is an excellent example for several reasons. First, in the "real world," the application of mathematics often occurs when you have a lot of extraneous information and "word noise" in a problem. It would be best to get used to simplifying the language in a situation and removing extraneous bits. Here's a great rewording:

"The bearing from A to B is $\mathrm{N}\ 42^\circ\ \mathrm{E}$ and the distance is 6.8. The bearing from B to C is $\mathrm{N}\ 48^\circ\mathrm{W}$ and the distance is 3.4. Find the distance between A and C, and the bearing from C to A."

Now that we have removed as much extraneous wording as possible, let's draw a picture of the situation. We begin with a simple sketch of the bearing from A, including that Roy hiked for 2 hours at 3.4 miles per hour.

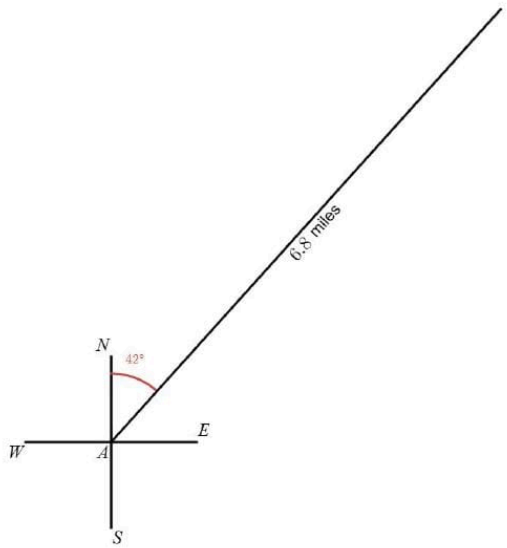


Figure 3.4.21

It's important to note that we drew the north-south and east-west axes at the initial point in Figure 3.4.21. Once we have drawn the angle and its terminal side, we place B at the tip of that terminal side (see Figure 3.4.22. We also draw in another set of north-south and east-west axes.

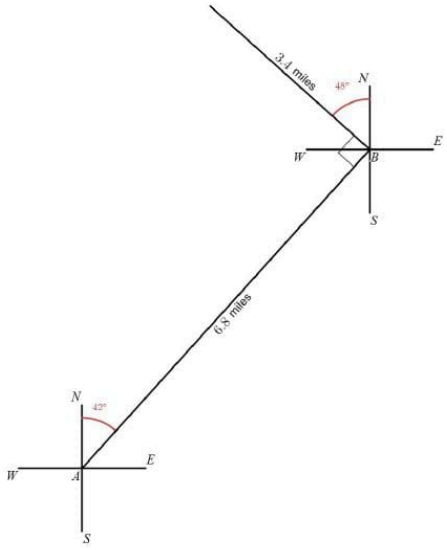


Figure 3.4.22

The next "stop-over" point, at the end of that last 3.4-mile hike, is C. Plotting that point and drawing in a final set of north-south and east-west axes, we arrive at Figure 3.4.23 below.

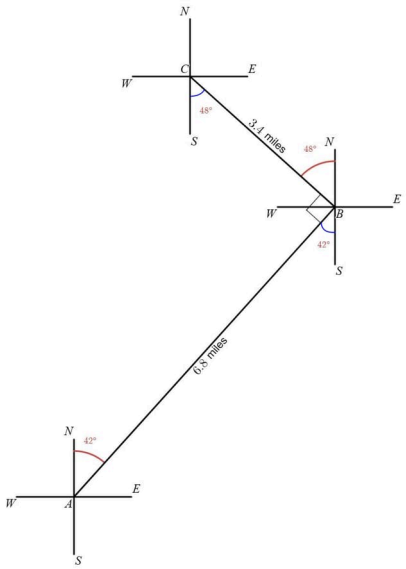


Figure 3.4.23

Take a moment to compare Figures 3.4.22 and 3.4.23. The latter figure has more information than just an extra set of axes—we have included alternate interior angles! As you will soon see, this information will be critical in helping us finish the problem. Speaking of which, let's draw a line connecting C to A.

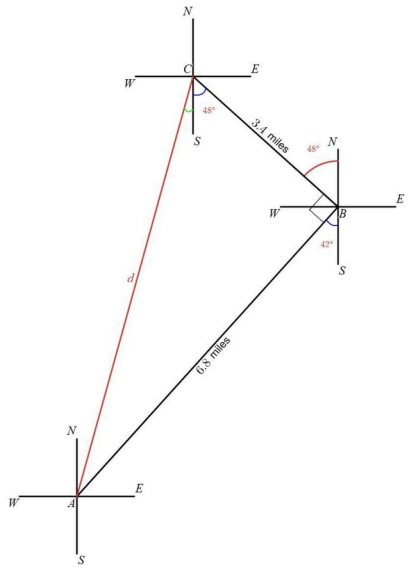


Figure 3.4.24

Figure 3.4.24 is the most important for you to be able to draw. It would be best if you had it to complete this problem successfully. The triangle we have drawn is a right triangle. We should immediately know this from the clue that Roy turns 90° once he arrives at Junction B ($\angle ABC + 42^\circ + 48^\circ = 180^\circ \implies \angle ABC = 90^\circ$). The distance back to A from C is the length, d, of the hypotenuse of the right triangle. Hence,

$$d = \sqrt{6.8^2 + 3.4^2} \approx 7.6 \,\text{miles}.$$

To find the bearing, we first must recognize that, from C, Roy must face south and turn through a small angle in the westerly direction to face A. We need that little sliver of a green angle. Focusing only on the right triangle, we can see that

$$\tan(\angle ACB) = \frac{6.8}{3.4} = 2 \implies \angle ACB = \tan^{-1}(2) \approx 63.4^\circ.$$

However, if we call that little sliver of an angle α, then $\angle ACB = \alpha + 48^\circ$. Therefore,

$$63.4^\circ \approx \alpha + 48^\circ \implies \alpha \approx 15.4^\circ.$$

Thus, Roy's bearing from C to A is $\text{S } 15.4^\circ \text{ W}$.

Checkpoint 3.4.7

10.0 miles at a bearing of $\text{N } 52.3^\circ \text{ E}$.

Example 3.4.8

This is a compound right triangle problem because the situation requires *at least* two right triangles. The key to solving such problems is to recognize that these triangles share a common side. If we call this shared side s, we can "separate" the triangles, as seen in Figure 3.4.26.

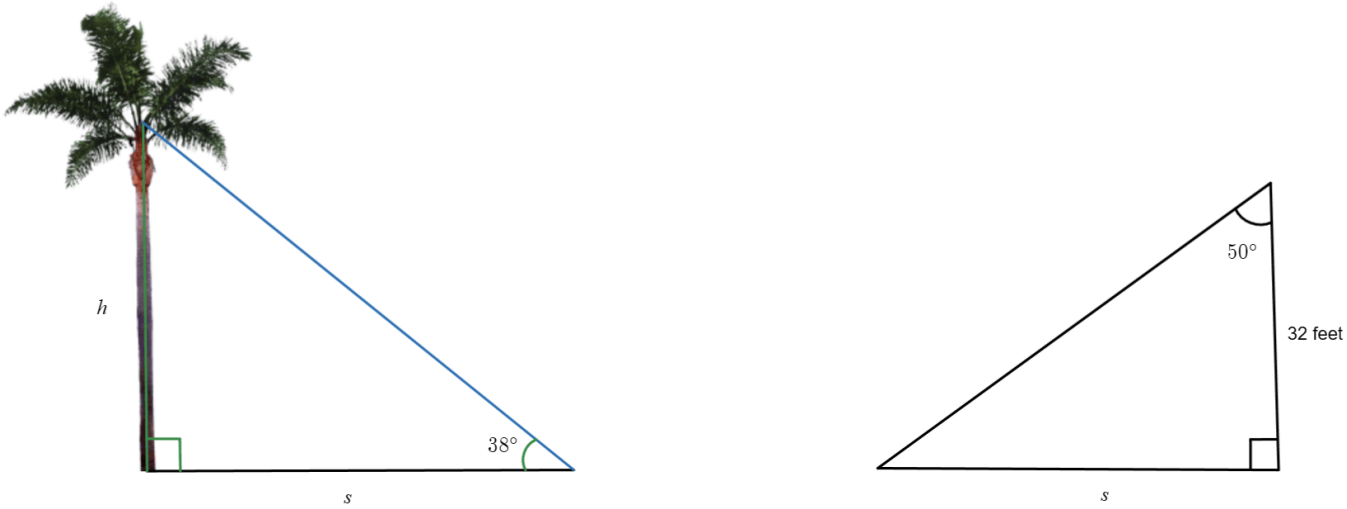


Figure 3.4.26

In both cases, we can use the tangent function to relate the given angle and the two sides so that

$$\tan(38^\circ) = \frac{h}{s} \quad \text{and} \quad \tan(50^\circ) = \frac{s}{32}.$$

We now use the shared side, s, as the common link between the triangles. That is, solve both equations for s.

$$s = \frac{h}{\tan(38^\circ)} \quad \text{and} \quad s = 32\tan(50^\circ).$$

Therefore,

$$\frac{h}{\tan(38^\circ)} = 32\tan(50^\circ).$$

We solve this final equation for h to find that the height of the tree is

$$h = 32\tan(50^\circ)\tan(38^\circ) \approx 29.8 \text{ feet}.$$

Checkpoint 3.4.8

41 miles

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3.4.1: Resources and Key Concepts

Resources

Videos

- Angle of Elevation and Depression
  - Trigonometry of Right Triangles - Review of Applications of Right Triangle Trigonometry (Part 1)
- Compound Right Triangle
  - Right Triangle Trigonometry - Applications of Right Triangle Trigonometry (Part 1)

Key Concepts

Definitions

- **Angle of Elevation (or Angle of Inclination):** The angle formed between a horizontal line and the line of sight when an observer looks upwards at an object.
- **Angle of Depression (or Angle of Declination):** The angle formed between a horizontal line and the line of sight when an observer looks downwards at an object.
- **Angle of Inclination of a Line:** The measure of the angle in the positive direction from the positive x-axis to the line.
- **Heading (navigation):** The compass direction in which a vessel or aircraft is pointed, measured clockwise from due north.
- **Bearing (navigational):** The direction of an object, given as an acute angle measured from a north-south line toward either east or west. It is also called a quadrant bearing.

Theorems

- **The Tangent and the Angle of Inclination of a Line:** If a line has a slope of m, then its angle of inclination, α, is related to the slope by the formula $\tan(\alpha) = m$, where $0^\circ \leq \alpha < 180^\circ$.

Common Mistakes

- **Confusing Heading and Bearing:** A heading is a single angle measured clockwise from due north. A bearing is an acute angle measured from the north-south line toward the east or west.
- **Incorrectly Orienting Triangles in Navigation Problems:** When superimposing a right triangle onto a navigation sketch, it is best to always place one of the legs on a horizontal (east-west) axis to avoid confusion, rather than on a vertical (north-south) axis.
- **Forgetting to Place Axes at Stop-Over Points:** In navigation problems with multiple parts or changes in direction, it is much easier to manage the geometry if you draw a new set of small north-south, east-west axes at each point where a turn is made.

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3.4.2: Homework

Reading Questions

1. What is the difference between an angle of elevation and an angle of depression?
2. How is the angle of inclination of a line defined?
3. How is the slope of a line, m, related to its angle of inclination, α?
4. In navigation, what is a "heading"? From what direction and in what manner is it measured?
5. What are the three components of a navigational "bearing"?
6. What is the primary difference between how a heading is measured and how a bearing is measured?
7. In a multi-part navigation problem, what is a helpful drawing strategy to use at each "stop-over" point?
8. What defines a "compound right triangle application"?
9. In a compound right triangle problem, what is the key to solving the problem?

Homework

Vocabulary Check

1. The angle between the horizon and the line of sight above the horizon is called the angle of ___, also known as the angle of ___.
2. A(n) ___ is an angle measured clockwise from due north.
3. A(n) ___ is a direction given in the form north or south by a certain number of degrees east or west (denoted, for example, by $\mathrm{S}\ 13^\circ\ \mathrm{W}$).

Concept Check

4. If the angle of inclination of a line is greater than 45°, what can you say about its slope?
5. What is the bearing that shares a terminal side with the standard position angle $\theta = 45^\circ$?
6. What is the bearing that shares a terminal side with the standard position angle $\theta = 120^\circ$?
7. What is the bearing that shares a terminal side with the standard position angle $\theta = -150^\circ$?

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

8. An angle of depression is measured from due south to an individual's line of sight below the horizon.
9. The angle given in a bearing must be acute.
10. The angle given in a heading must be acute.
11. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle 120°.
12. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle -120°.
13. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle 60°.
14. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle -60°.
15. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle 240°.

16. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle -240°.

Basic Skills

For the following exercises, sketch the situation using a right triangle and label each angle.

17. The angle of elevation from the ground to the top of a power pole is 50°.

18. The angle of depression from the eye of a hiker on a cliff to the bottom of the canyon is 85°.

19. Two people are looking up at a hot air balloon. They are positioned on opposite sides of the ballon. The angle of elevation to the ballon for one person is 57°, and the angle of elevation to the ballon for the other person is 30°.

For the following exercises, sketch a diagram illustrating the given bearing.

20. The bearing from Sacramento to South Lake Tahoe is $\text{N } 72^\circ \text{ E}$.

21. The bearing from New York City to Philadelphia is $\text{S } 48^\circ \text{ W}$.

22. The bearing from the Southern Terminus of the Pacific Crest Trail to the Northern Terminus is $\text{N } 10^\circ \text{ W}$.

23. The bearing from Lincoln Airport to Placerville Airport is $\text{S } 69^\circ \text{ E}$.

For the following exercises, find the angle of inclination of the line.

24. $y = \frac{5}{4}x - 3$

25. $y = 6 + \frac{2}{9}x$

26. $y = -2 - \frac{3}{8}x$

27. $y = \frac{-7}{2}x + 1$

For the following exercises, find an equation for the line passing through the given point with an angle of inclination α.

28. $(3, -5), \alpha = 28^\circ$

29. $(-2, 6), \alpha = 67^\circ$

30. $(-8, 12), \alpha = 112^\circ$

31. $(-4, -1), \alpha = 154^\circ$

Synthesis Questions

32. The slope of a line is a function of its angle of inclination, $m = f(\alpha)$. Complete the table and sketch a graph of the function.

α	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
m													

   a. What happens to the slope of the line as α increases toward 90°?

   b. What happens to the slope of the line as α decreases toward 90°?

33. The angle of inclination of a line is a function of its slope, $\alpha = g(m)$. Complete the table and sketch a graph of the function.

m	-20	-10	-5	-2	-1	-0.75	-0.5	-0.25	0
α									

m	0.25	0.5	0.75	1	2	5	10	20
α								

a. What happens to the angle of inclination as the slope increases toward infinity?

b. What happens to the angle of inclination as the slope decreases toward negative infinity?

Applications

Angles of Elevation and Angles of Depression

For the following exercises,
(a) Sketch a right triangle that illustrates the situation. Label your sketch with the given information.
(b) Choose the appropriate trigonometric ratio, write an equation, and solve the problem.

34. **Estimating Height.** The birdhouse in Phuong's front yard is 12 feet tall, and its shadow at 4 P.M. is 15 feet 4 inches long. What is the angle of elevation of the sun at 4 P.M.?

35. **Estimating the Angle of Elevation.** If a 65.2-foot sculpture casts a shadow 42.1 feet long, find the sun's angle of elevation to the nearest tenth of a degree.

36. **Air Traffic Control.** To measure the height of cloud cover, airport controllers fix a searchlight to shine a vertical beam on the clouds. The searchlight is 120 yards from the office. A technician in the office measures the angle of elevation to the light on the cloud cover at 54.8°. What is the height of the cloud cover?

37. **Salvage Crew.** A salvage ship is searching for the wreck of a pirate vessel on the ocean floor. Using sonar, they locate the wreck at an angle of depression of 36.2°. The depth of the ocean at their location is 260 feet. How far should they move to be directly above the wrecked vessel?

38. **Volcanology.** Cinder cones are the most common type of volcano in the world. They may look like an idealized depiction of a volcano as they are steep, conical hills that usually have a prominent crater at the top. Maha decides to hike to the top of Capulin Volcano, a cinder cone in New Mexico. Once at the rim, she measures the angle of depression to the surrounding plain to be 26.3°. Her pedometer shows that her hike up to the top of Capulin Volcano covered 2,939 feet. Assuming the hike up the volcano was along a straight path, what was her elevation gain?

39. **Meteorology.** The radio signal from a weather balloon indicates that it is 1500 meters from a meteorologist on the ground. The angle of elevation to the balloon is 48°. What is the balloon's altitude?

40. **Ancient Warfare.** According to Chinese legend, around 200 BC, the general Han Xin used a kite to determine the distance from his location to an enemy palace. He then dug a secret tunnel which emerged inside the palace. When the kite was directly above the palace, its angle of elevation was 27° and the string to the kite was 1850 feet long. How far did Han Xin's troops have to dig?

41. **Ocean Rescue.** A hot air balloon was forced far off course, and the occupants had to make a forced landing at sea. They sent out a distress signal before taking the balloon down, and, very luckily, a United States Lifesaving Association boat was nearby. The distress signal the boat receives gives the balloon's altitude before descending as 1,128 feet. Immediately, an observer on the boat could see the balloon and measure the angle of elevation to the balloon as 22.1°. How far would the boat travel to reach any survivors if the balloon went directly (vertically) down and crashed into the water?

42. **Airplane Navigation.** An airplane begins its descent when its altitude is 10 kilometers. The angle of descent should be 3° from horizontal.

a. How far from the airport (measured along the ground) should the airplane begin its descent?

b. How far will the airplane travel on its descent to the airport?

13. **Estimating Height.** Linda is on the upper floor of a tall building. She measures the angle of elevation to the building across the street to be 38° and the angle of depression to the base of that building as 65°. If she knows the two buildings are 50 feet apart, how tall is the building across the street?

14. **Height of a Flagpole.** To estimate the height of a flagpole, Kim positions herself due south of the pole, and Matt stands due east of the pole. If they are the same distance from the pole and 50 feet from each other, find the height of the pole if the angle of elevation from the ground to the top of the pole at each person's position is 49°.

15. **Estimating Height - D&D Edition.** Deitrich Wyvernfang bravely stands in defense against a massive stone giant. He is waiting for the precise moment when the creature steps within reach of his trusty glaive; however, as with most stone giants, the beast is being measured in its approach and purposefully keeps its distance. In the intervening moments, Hoot de'La Temp, Deitrich's trusty but enigmatic companion, is overwhelmed with a momentary curiosity. In his studies of all creatures magical and curious, Hoot had never read of a stone giant this tall.

    Always taking advantage of the moment, Hoot places his hand on Deitrich's shoulder, casually saying, "Hold the giant off for me. This is a rare opportunity to measure the stature of such a creature in its non-deceased form." Deitrich's surprised, wide-eyed glance at Hoot is all the wizard needs to know that his friend must share his excitement about such an opportunity. Kneeling, Hoot quickly utters the arcane words for his *calculate* spell and measures the angle of elevation to the stone giant's head to be $52^\circ\ 34'$. Standing, he nods at Deitrich (who, as Hoot thinks about it, now looks disgusted for some reason) and says, "Keep him still for just a few moments longer."

    Hoot walks directly away from the stone giant, keeping Deitrich between himself and the dangerous creature. Each step he takes is a measured 3-foot distance. He smiles warmly as his friend decides to help him by barking "But... But... But..." in time with his steps - what a great companion. After ten such steps, Hoot turns, kneels, and casts his *calculate* spell again. This time, the angle of elevation to the now confused giant's head measures $23^\circ\ 31'$. Standing, the wizard shouts victoriously, "Got it! That beast is precisely ___ feet tall! Well, to the nearest tenth of a foot, that is. You may slay him now."

    What is the number of feet that Hoot estimated the height of this stone giant to be?

16. **Neighborly Dispute.** Nam's neighbors have built a massive pile of junk in their yard topped with a standard door frame. He would like to report them to the county for the eyesore, but the county will only give the neighbor a citation if the structure is more than 30 feet tall. Unfortunately, Nam's neighbors are not allowing him to measure the height of the junk pile on their property, so he resorts to his mathematics training. From a point on the ground (level with the base of the junk pile), Nam measures the angle of elevation to the top of the entire pile to be 37°. From that same point, he measures the angle of elevation to the base of the door frame (on the top of the junk pile) to be 32°. Assuming the door frame is made to industry standards, it should be 80 inches tall. How tall is the entire pile of junk (in feet)? Can the county cite the neighbor?

17. **Estimating Distance.** To find the distance across a ravine, Roy takes some measurements from a small airplane. When he is a short distance from the ravine at an altitude of 500 feet, he finds that the angle of depression to the near side of the ravine is 56°, and the angle of depression to the far side is 32°. What is the width of the ravine? (Hint: First find the horizontal distance from Roy to the near side of the ravine.)

18. **Radius of the Earth.** A communications satellite is in a low earth orbit (LOE) at an altitude of 400 km. From the satellite, the angle of depression to the earth's horizon is 19.728°. Use this information to calculate the radius of the planet.

19. **Sunlit Area.** The window in Nhat's office is 4 feet wide and 5 feet tall. The bottom of the window is 3 feet from the floor. When the sun is at an angle of elevation of 64°, what is the area of the sunny spot on the floor?

Heading Problems

For the following exercises, a ship sails from the seaport on the given heading for the given distance.
(a) Make a sketch showing the ship's current location relative to the seaport.
(b) How far east or west of the seaport is the ship's present location? How far north or south?

50. $36^\circ, 26$ miles

51. $124^\circ, 80$ km

52. $230^\circ, 120$ km

53. $318^\circ, 75$ miles

54. $285^\circ, 32$ km

55. $192^\circ, 260$ miles

56. A private plane flies 2.5 hours at 110 mph on a heading of 45°. It then turns and continues another 1.5 hours at the same speed but on a bearing of 135°.

    a. How far is the plane from its starting point at the end of this time?

    b. What is the heading from the pilot's starting location to her ending location?

Bearing Problems

57. **Ship Navigation.** A ship travels on a course of bearing $\mathrm{N}\ 62^\circ\ \mathrm{E}$ for 83.1 miles. How many miles north and how many miles east has the boat traveled?

58. **Ship Navigation.** A ship leaves port and travels 20 miles in the direction $\mathrm{S}\ 29^\circ\ \mathrm{W}$. It then turns 90° and travels another 14.4 miles in the direction $\mathrm{N}\ 61^\circ\ \mathrm{W}$. At that time, how far is the ship from port, and what is its bearing back to the port?

59. **Hiking Navigation.** Mike parks his car on a road that runs east-west. From this road, he sets out on a straight trail at a bearing of $\mathrm{N}\ 65^\circ\ \mathrm{E}$ and hikes for 25 miles to a water cache. After a short break, Mike turns onto another straight trail with a bearing of $\mathrm{S}\ 25^\circ\ \mathrm{E}$ for an additional 5 miles to a lunch spot.

    a. To the nearest tenth of a mile, how far was Mike from the east-west road when he reached the water cache?

    b. By the time Mike reaches the lunch spot, what is the bearing from the lunch spot back to his car?

    c. How far away (in terms of direct distance) is Mike's car from the lunch spot?

60. ***Locate Object* - D&D Edition.** The Bandits, as the heroes are known, are searching for a *wand of water control* that has been lost to the overgrowth of the Amedio Jungle. Hoot de'La Temp, the group's enigmatic wizard, casts a *locate object* spell multiple times to find the missing wand. Unfortunately, this spell only reveals the *direction* of an object - not its distance (another issue is that the spell can only find an object once the object is within 1,000 feet of the wizard). Frustrated and tired, the Bandits begin trekking back to town; however, after a short time, Hoot decides to give the spell one more try.

    "Wait! I sense the wand - it's that way," Hoot exclaims, pointing deeper into the jungle.

    Anlie, the Bandit's gruff ranger quickly snaps, "How far? Not all the way back to that group of gnolls, I hope."

    "I don't know. This magic only gives me direction - not distance," Hoot responds, frustratedly shaking his head. "This particular issue concerns the bending of the Gygaxian weaves with those of the natural weaves present in the surrounding region. I have often thought there might be a better..."

    Cutting the wizard off, Anlie tersely states, "I don't need a magic lesson, Hoot. A simple, 'I don't know' works."

    The ranger looks up through the jungle canopy, searching for signs only someone with her skill could observe. "Okay, you're pointing roughly north by 45° east." She looks around, spotting a path with less vegetation, "How much time do you have left on that spell?"

    Hoot, unruffled by the interruption, shrugs, "Seven minutes." Pausing, he adds, "And 36 seconds?"

    Rolling her eyes, Anlie quickly moves down the trail. "Follow me."

After a few minutes of hiking through the jungle, Anlie turns to Hoot. "Now point toward the wand."

Furrowing his brow and giving the ranger an appraising look, Hoot absentmindedly raises his arm to the side, pointing his index finger in the perceived direction of the magic item.

Anlie looks at the wizard accusingly, "Are you sure?"

Feigning a look of innocence, the wizard responds, "Nobody can be absolutely certain about these things, but I have the utmost confidence that it's roughly out there," wiggling his finger.

Sighing, Anlie looks around, spending a minute looking for those hidden signs only she seems to see. "Okay, you're pointing north by 45° west. It's 1,050 feet back to where we started in the direction north by 78° west."

"Are *you* sure," the wizard asks with a grin.

Giving him a flat stare, Anlie sighs again, "Yes, and I know how far away the wand is."

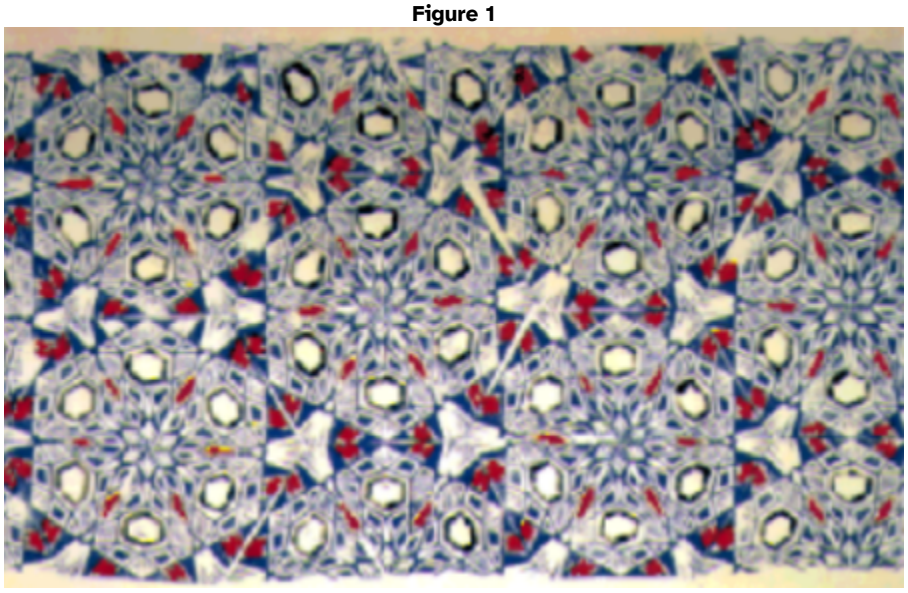
How far away is the *wand of water control* from their current position?

61. **Distance.** From two observation points 25 miles apart on a straight shoreline running north and south, the bearings to a boat at sea are $\mathrm{N}\ 26^\circ\ \mathrm{W}$ and $\mathrm{S}\ 56^\circ\ \mathrm{W}$. What is the shortest distance from the boat to the shore?

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4: RADIAN MEASURE AND THE CIRCULAR FUNCTIONS

Figure 1



Have you ever wondered why we divide the circle into 360 degrees? Nobody knows the answer, but it may have started around 600 BCE with the Babylonians.

The Babylonians lived between the Tigris and Euphrates rivers in present-day Turkey and Syria. They kept written records using a stylus to press cuneiform, or wedge-shaped, symbols into wet clay tablets, which were then baked in the sun. Thousands of these tablets have survived and give us detailed information about the mathematical practices of the time.

Figure 2

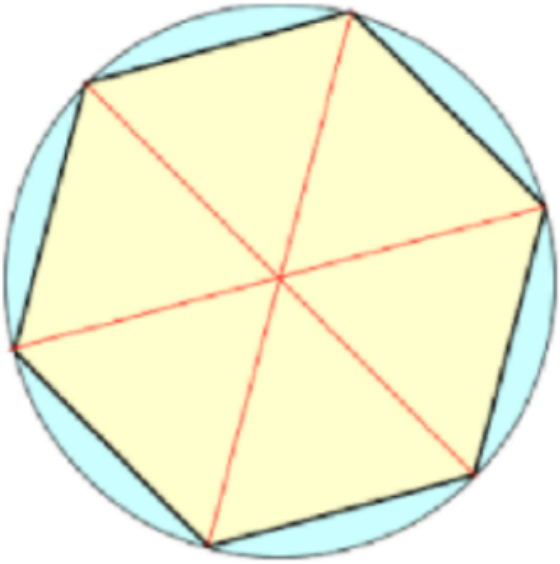


The Babylonians used a base 60 number system because the number 60 has many factors. They did not invent decimal fractions, so they found it challenging to deal with remainders when dividing. But 60 can be divided evenly by 2, 3, 4, 5, and 6, which made calculations with common fractions much more manageable.

We still have traces of their base 60 system: there are 60 seconds in a minute and 60 minutes in an hour.

In Geometry, Babylonian mathematicians used the corner of an equilateral triangle as their basic unit of angular measure. Naturally, they divided that angle into 60 smaller angles. Now, suppose the corners of six equilateral triangles are placed together. In that case, they form a complete circle, so there are six times 60, or 360, degrees of arc in a circle.

Figure 3

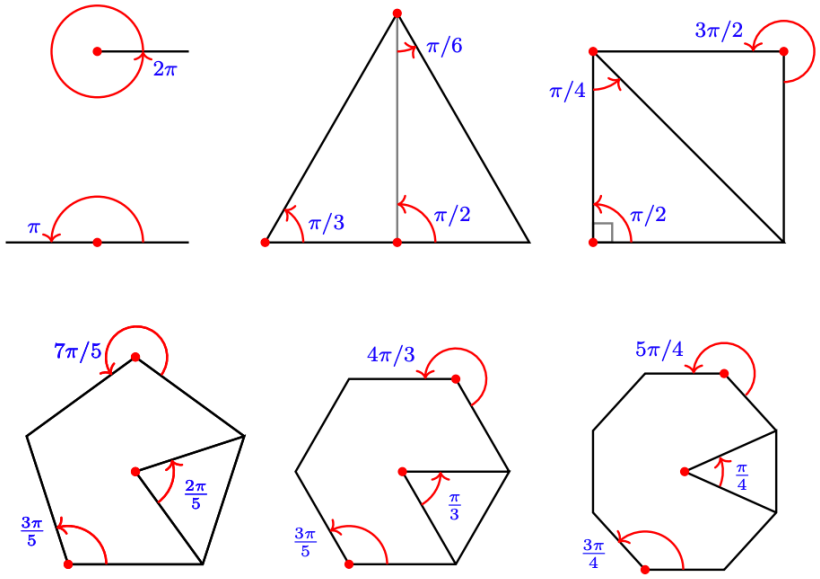


During the reign of Nebuchadnezzar, using the tools and technology available to them, Babylonian astronomers calculated that a complete year numbered 360 days. This made dividing the circle into 360 degrees even more helpful.

The number 360 is not fundamental to the nature of a circle. If ancient civilizations had defined the whole circle as some other number of degrees, we'd probably be using that number today.

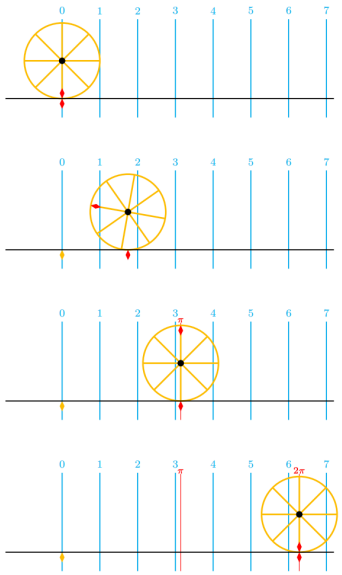
But why do we need another, different way to measure angles? In this chapter, we'll study **radian measure**, which initially may seem awkward and unnatural. As a hint, consider that although 360 is not fundamental to circles, the number π is.

Figure 4



Radians connect the measure of an angle with the arc length it cuts out on a circle.

Figure 5



Imagine a circle with a radius of 1 unit rolling along a straight line. The circumference of a circle is 2π times the length of its radius, so when the circle has turned through one complete revolution, it has traveled a distance of 2π units. When the circle has turned through half a revolution, it has traveled a distance of π units.

In radian measure, the angle through which the circle has turned is equal to the distance the circle has traveled or, put another way, the length of the arc that has unrolled on the line.

Roger Cotes, an English mathematician who worked closely with Isaac Newton, developed the idea of radian measure. He described the radian in everything but name and recognized its naturalness as a unit of angular measure.

The term **radian** first appeared in print on June 5, 1873, on an exam written by James Thomson, the brother of Lord Kelvin, at Queen's College in Belfast.

In Calculus and most other branches of mathematics beyond practical Geometry, angles are nearly always measured in radians. Because radians arise naturally when dealing with circles, essential relationships are expressed more concisely in radians. In particular, results involving trigonometric functions are more straightforward when the variables are expressed in radians.

4.1: Reference Angles

4.1.1: Resources and Key Concepts

4.1.2: Homework

4.2: Radian Measure and Arc Length

4.2.1: Resources and Key Concepts

4.2.2: Homework

4.3: The Trigonometric Functions - Unit Circle Definition

4.3.1: Resources and Key Concepts

4.3.2: Homework

4.4: Velocities and Areas of Sectors

4.4.1: Resources and Key Concepts

4.4.2: Homework

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4.1: Reference Angles

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The town of Avery lies 48 miles due east of Baker, and Clio is 34 miles in the bearing $\mathrm{N}\ 35^\circ\ \mathrm{W}$ from Baker. How far is it from Avery to Clio?

We know how to solve right triangles using the trigonometric ratios; however, the triangle formed by the three towns is *not* a right triangle because it includes an obtuse angle of 125° at B, as shown in Figure $4.1.1$.

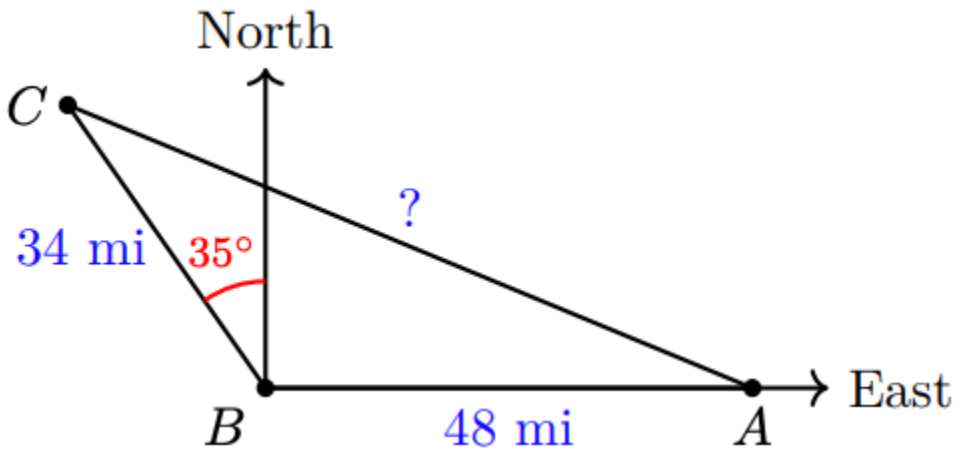


Figure $4.1.1$

Recall that we call this type of triangle an oblique triangle. Later in this course, we learn how to solve oblique triangles, but first, we must be able to find the values of the trigonometric functions for obtuse angles.

Reference Angles

We now introduce an essential tool for evaluating trigonometric functions and solving trigonometric equations involving non-acute angles.

Definition: Reference Angle

The **reference angle** for an angle θ given in standard form is the acute angle formed between the terminal side of θ and the x-axis. By convention, the reference angle is always considered positive (even if it opens in a clockwise direction). It is denoted with the "hat" notation, $\hat{\theta}$.

Any acute angle θ is the reference angle for four angles between 0° and 360°, one in each quadrant. The figure below shows the

four angles in standard position whose reference angle is 35°. Note that each angle is found by measuring 35° from the x-axis in the appropriate quadrant and that the four angles together make a "bow-tie" shape.

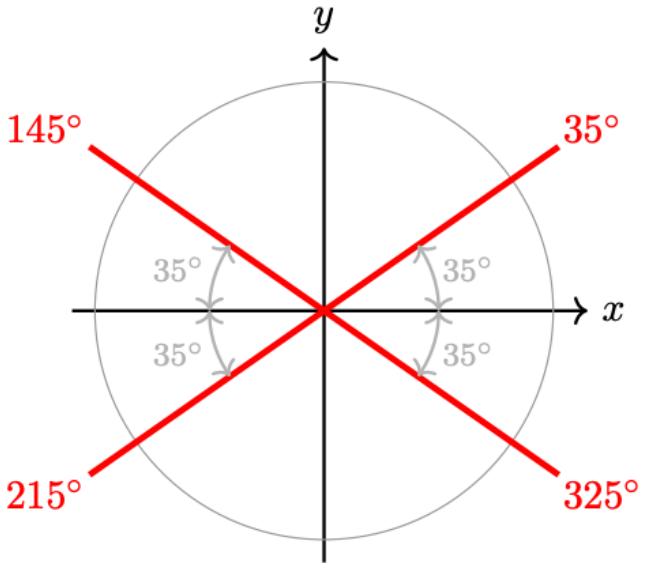


Figure 4.1.2

Interactive Element: One Reference Angle, Four Angles (the "Bow-Tie")

The following Interactive Element might help you visualize why a single acute reference angle belongs to four different angles in standard position, one in each quadrant.

Interact: Drag the first-quadrant point Q to change the shared reference angle.

Observation: All four reference triangles stay congruent — each is a mirror image of the first-quadrant one across the axes — so the four angles share the same reference angle and their trig values agree up to sign.

Example 4.1.1

Find and label the reference angle for the given angle.

a. $A = 125^\circ$
b. $B = 240^\circ$
c. $\alpha = 310^\circ$
d. $\beta = -315^\circ$
e. $t = 820^\circ$

Solutions available at the end of this section.

While you should strive for understanding in mathematics over memorization, some students initially find the following guidelines helpful until they get used to reference angles. Trust me when I say you want to use the following guideline only for a while—it's too bulky.

Let θ be an angle in standard position.

1. If $0^\circ < \theta < 360^\circ$, let $t = \theta$.
2. If θ is not between 0° and 360°, then let t be the angle coterminal to θ that *is* between 0° and 360°.

$$\text{If} \quad t \text{ terminates in QI}, \quad \text{then} \quad \hat{\theta} = t$$

$$\text{If} \quad t \text{ terminates in QII}, \quad \text{then} \quad \hat{\theta} = 180^\circ - t$$

$$\text{If} \quad t \text{ terminates in QIII}, \quad \text{then} \quad \hat{\theta} = t - 180^\circ$$

$$\text{If} \quad t \text{ terminates in QIV}, \quad \text{then} \quad \hat{\theta} = 360^\circ - t$$

Caution: Reference Angles are Acute

By definition, reference angles are acute angles. This means they are *not* 0° or 90°. In fact, quadrantal angles do not have reference angles.

Checkpoint 4.1.1

Draw the angle and state its reference angle.

a. $\alpha = 80^\circ$
b. $\beta = 130^\circ$
c. $\kappa = 225^\circ$
d. $\gamma = 330^\circ$
e. $\lambda = -125^\circ$

Answers available at the end of this section.

Evaluating Trigonometric Functions Using Reference Angles

We can use reference angles to help us evaluate a trigonometric function at a non-acute angle θ. To do so, we construct a **reference triangle** for θ.

Definition: Reference Triangle

A **reference triangle** is a right triangle formed by drawing a perpendicular from a point on the terminal side of an angle (in standard position) to the x-axis.

A reference triangle can easily be drawn by choosing a point P on the terminal side of θ, and then drawing a line from P perpendicular to the x-axis.

The figure below shows angles θ in all four quadrants and the reference angle, $\hat{\theta}$, for each. Study the figures, and make sure you understand how to find the reference angle in each quadrant.

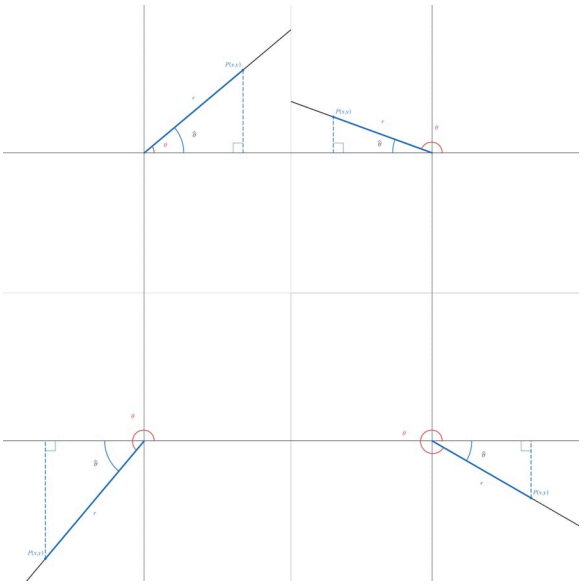


Figure 4.1.10**:** *Reference triangles in all four quadrants*

Summary

These observations may help you remember the formulas:

- The right triangle formed in this way always lies between the terminal side and the x-axis.
- The positive acute angle formed between the terminal side and the x-axis is the reference angle, and the right triangle is the reference triangle.

Interactive Element: Reference Angle and Reference Triangle

The following Interactive Element might help you visualize how the reference triangle and reference angle change as the terminal side sweeps through the four quadrants.

Interact: Drag the point P on the terminal side into each quadrant.

Observation: The shaded right triangle always sits between the terminal side and the x-axis, and the acute angle at the origin is the reference angle $\hat{\theta}$. The legs keep the lengths $|x|$ and $|y|$ in every quadrant, so the trig values of θ match those of $\hat{\theta}$ except for a sign set by the quadrant.

Activity

The following activity reveals the usefulness of reference angles and reference triangles when evaluating trigonometric functions of non-acute angles.

Activity: Reference Angles

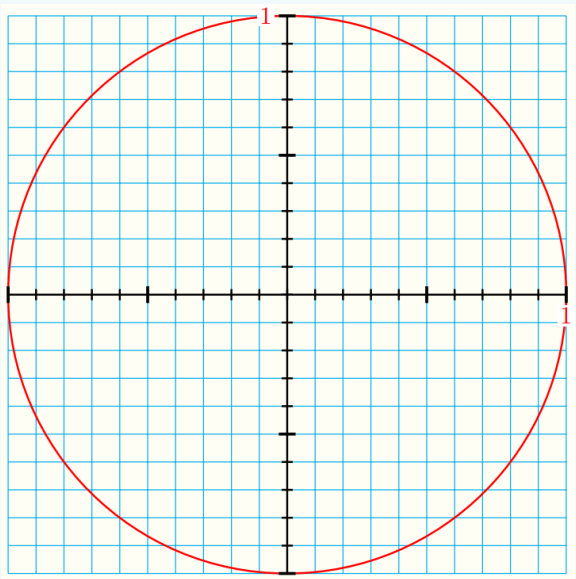


Figure 4.1.11

a. Use a protractor to draw an angle of 56° in standard position. Draw its reference triangle.
b. Use your calculator to find the sine and cosine of 56°, rounded to two decimal places. Label the sides of the reference triangle with their lengths.
c. What are the coordinates of the point P where your angle intersects the unit circle?
d. Draw the reflection of your reference triangle across the y-axis so that you have a congruent triangle in the second quadrant.
e. You now have the reference triangle for a second-quadrant angle in standard position. What is that angle?
f. Use your calculator to find the sine and cosine of your new angle. Label the coordinates of the point Q where the angle intersects the unit circle.
g. Draw the reflection of your triangle from part (d) across the x-axis so that you have a congruent triangle in the third quadrant.
h. You now have the reference triangle for a third-quadrant angle in standard position. What is that angle?
i. Use your calculator to find the sine and cosine of your new angle. Label the coordinates of the point R where the angle intersects the unit circle.
j. Draw the reflection of your triangle from part (g) across the y-axis so that you have a congruent triangle in the fourth quadrant.
k. You now have the reference triangle for a fourth-quadrant angle in standard position. What is that angle?
l. Use your calculator to find the sine and cosine of your new angle. Label the coordinates of the point where the angle intersects the unit circle.
m. **Generalize:** All four of your angles have the same reference angle, 56°. For each quadrant, write a formula for the angle whose reference angle is θ.
    i. Quadrant I:
    ii. Quadrant II:
    iii. Quadrant III:
    iv. Quadrant IV:

Reference Angle Theorem

In all four versions of Figure 4.1.10, coordinate trigonometry states that

$$\begin{aligned} \sin(\theta) &= \frac{y}{r} & \csc(\theta) &= \frac{r}{y} \\ \cos(\theta) &= \frac{x}{r} & \sec(\theta) &= \frac{r}{x} \\ \tan(\theta) &= \frac{y}{x} & \cot(\theta) &= \frac{x}{y} \end{aligned}$$

However, it is also true that in all four versions of Figure 4.1.10, the side of the reference triangle opposite $\hat{\theta}$ has length $|y|$, the side of the reference triangle adjacent to $\hat{\theta}$ has length $|x|$, and the hypotenuse of the reference triangle has a length of r. Therefore, by right triangle trigonometry,

$$\begin{aligned} \sin\left(\hat{\theta}\right) &= \frac{|y|}{r} & \csc\left(\hat{\theta}\right) &= \frac{r}{|y|} \\ \cos\left(\hat{\theta}\right) &= \frac{|x|}{r} & \sec\left(\hat{\theta}\right) &= \frac{r}{|x|} \\ \tan\left(\hat{\theta}\right) &= \frac{|y|}{|x|} & \cot\left(\hat{\theta}\right) &= \frac{|x|}{|y|} \end{aligned}$$

Thus, the values of the trigonometric functions of the angle θ match the values of the trigonometric functions of $\hat{\theta}$, except, possibly, for a difference in sign. The sign of the trigonometric function's value can be determined by the quadrant in which the terminal side of the angle lies. This result is summarized in the following theorem.

Theorem: Reference Angle Theorem

The value of a trigonometric function of any angle is equal to that of the function at its reference angle, except for sign. The quadrant determines the sign of the function.

In the previous chapter, we used a calculator to approximate the values of the trigonometric functions at *acute* angles. Your calculator can easily find the value of a trigonometric function at *any* angle. We showcase this in the following example.

Example 4.1.2

a. Find the reference angle for 200°.
b. Sketch 200° and its reference angle in standard position, along with the reference triangle. Verify that both angles have the same trigonometric values, up to sign.

Solutions available at the end of this section.

Note that the results in Example 4.1.2 align with what we said previously—the sign of the trigonometric function's value can be determined by the quadrant in which the terminal side of the angle lies. Thus, to compute $\sin(200^\circ)$, we could note that $\theta \in \text{QIII}$ and the reference angle is 20°. Since the sine is negative in QIII, we get

$$\begin{aligned} \sin(200^\circ) &= \boxed{?}\sin(20^\circ) && \left(\sin(\theta) \text{ is equal to } \sin\left(\hat{\theta}\right), \text{ except for, possibly, the sign}\right) \\ &= -\sin(20^\circ) && (\text{Since } 200^\circ \in \text{QIII}, \text{ sine is negative}) \end{aligned}$$

From this point forward in Trigonometry, the following Trigonometry Mantra will be helpful.

> Trigonometry is about two things—reference angles and quadrants.

While this mantra oversimplifies the subject, most of Trigonometry is easier once you know the reference angle and quadrant for a given angle.

Checkpoint 4.1.2

a. Find the reference angle for 285°.
b. Sketch 285° and its reference angle in standard position, along with the reference triangle. Verify that both angles have the same trigonometric ratios, up to sign.

Answers available at the end of this section.

When combined with the Reference Angle Theorem, the following corollary improves the speed with which we can evaluate trigonometric functions.

Corollary: Rotational Symmetry

For any integer n,

$$\begin{aligned} \sin(\theta + 360^\circ n) &= \sin(\theta) & \csc(\theta + 360^\circ n) &= \csc(\theta) \\ \cos(\theta + 360^\circ n) &= \cos(\theta) & \sec(\theta + 360^\circ n) &= \sec(\theta) \\ \tan(\theta + 360^\circ n) &= \tan(\theta) & \cot(\theta + 360^\circ n) &= \cot(\theta) \end{aligned}$$

This result is from our previous discussion that the value of a trigonometric function is the same for all angles coterminal with the original given angle.

An Additional Note About the Tangent

As we will investigate later, it turns out that

$$\tan(\theta + 180^\circ n) = \tan(\theta) \quad \text{and} \quad \cot(\theta + 180^\circ n) = \cot(\theta)$$

for any integer n; however, this is not needed for our current discussion.

Let's see how this corollary, along with focusing on reference angles and quadrants, speeds up the evaluation of trigonometric functions whose reference angles are the special angles.

Example 4.1.3

Find the exact value of each trigonometric function.

a. $\cos(225^\circ)$
b. $\cot(120^\circ)$
c. $\tan(-210^\circ)$
d. $\sin(690^\circ)$

Solutions available at the end of this section.

Checkpoint 4.1.3

Find the exact value of each trigonometric function (without touching a calculator).

a. $\sin(210^\circ)$
b. $\sec(-780^\circ)$
c. $\sin(405^\circ)$

Answers available at the end of this section.

Finding Non-Acute Angles

We now turn our attention back to finding angles given values of trigonometric functions. Last chapter, we were able to find θ such that $\tan(\theta) = 0.5662$. At that time, we used the TAN^{-1} button on our calculator to find $\tan^{-1}(0.5662) \approx 29.5185^\circ$. At that time, however, we were only working with right triangle trigonometry, and the angles we were interested in were all acute. When working with non-acute angles, the inverse trigonometric functions can get tricky.

To avoid the trickiness associated with inverse trigonometric functions (and to give us more time with Trigonometry before diving into the more profound theory of the inverse trigonometric functions), it's best to think of them as returning *reference* angles. Since reference angles are always acute, this interpretation works with what we did in the last chapter; however, as we have seen in this section, we are now dealing with angles that are not necessarily acute.

For now, think of the inverse trigonometric functions as returning reference angles.

We must evaluate the inverse trigonometric function at a *positive* value to return a reference angle. The reason for this cannot be fully justified at this time. Still, we *can* say that, for example, $\sin^{-1}(-0.31)$ returns an angle θ so that $\sin(\theta) = -0.31$; however, we know that the values of *all* trigonometric functions are *positive* in QI. Since we want the returned angle to be a reference angle, it must be acute (and, therefore, in the first quadrant). Hence, there is no possibility for a reference angle to be returned if the argument of our inverse trigonometric function is negative.

If we agree that inverse trigonometric functions evaluated at positive numbers return reference angles, we can ask for solutions to equations involving trigonometric functions.

For example, find θ if

$$\sin(\theta) = -0.31,$$

where $\theta \in \text{QIII}$.

First, it should make sense that θ could be in the third quadrant because the sine at θ is negative, and the sine is negative in the third quadrant (as well as in the fourth quadrant). Remember, Trigonometry is about two things—reference angles and quadrants. We have already been given a quadrant. We evaluate the inverse sine function at $|-0.31| = 0.31$ to find the reference angle. Hence, the reference angle is

$$\hat{\theta} = \sin^{-1}(|-0.31|) = \sin^{-1}(0.31) \approx 18.1^\circ.$$

The angle in QIII with reference angle $\hat{\theta} \approx 18.1^\circ$ is $\theta = 180^\circ + \hat{\theta} \approx 180^\circ + 18.1^\circ = 198.1^\circ$.

Summary

To find the angle θ so that the trigonometric function of θ is a given ratio:

1. **Reference Angle.** Use the inverse trigonometric function of the absolute value of the given ratio to determine the reference angle, $\hat{\theta}$.
2. **Quadrant.** Use the given information within the problem to "lock down" the quadrant in which θ terminates.
3. **Actual Angle.** Use the reference angle and quadrant from (1) and (2) to determine θ.

The second step in the summary above assumes you are given extra information to help determine the quadrant in which θ terminates. While this will be true in this section, we will eventually shed that requirement. We will begin to find *all* angles that make a given trigonometric equation valid.

Example 4.1.4

Find θ to the nearest tenth of a degree, if $0^\circ < \theta < 360^\circ$.

a. $\cos(\theta) = -0.8457$, where $\theta \in$ QIII.
b. $\tan(\theta) = -1.9122$, where $\sin(\theta) < 0$.
c. $\csc(\theta) = 2.1174$, where $\cos(\theta) < 0$.

Solutions available at the end of this section.

Checkpoint 4.1.4

a. Solve $\sec(\theta) = -1.8$, where $90^\circ < \theta < 180^\circ$. Round your answer to the nearest tenth of a degree.
b. Find the sine of θ given that $\tan(\theta) = -2$, $\cos(\theta) > 0$, and $0^\circ < \theta < 360^\circ$. Round your answer to four decimal places.

Answers available at the end of this section.

Of course, if given a "special" trigonometric ratio, you should not reach for a calculator.

Example 4.1.5

Find α if $\cos(\alpha) = -\frac{\sqrt{3}}{2}$ and $\alpha \in$ QII, where $0^\circ < \alpha < 360^\circ$.

Solution available at the end of this section.

Checkpoint 4.1.5

Solve $\tan(\beta) = -\sqrt{3}$ given $0^\circ < \beta < 360^\circ$ and $\cos(\beta) > 0$.

Answer available at the end of this section.

Miscellaneous Topics: Thinking About Mathematics (Part 1)

1. What is it about math that you feel is difficult?
2. Would you think it an odd statement if someone told you that they suck at reading and could barely read a comic book? Why do you think people are more apt to proclaim that they suck at math?

Hidden Resources: Microsoft Office 365

Los Rios students have free access to Microsoft Office 365 for as long as they remain enrolled! This is an incredible deal! A warning is necessary - do not use the included OneDrive storage as it gets deleted each semester. Other than that minor issue, this is a fantastic resource. Check out the details on this CRC webpage.

Answers

Section Preview

In this section, we return to considering *any* angle (not just acute angles) — related to the *special angles* or not (although the focus is primarily on angles related to special angles). This is where we tie Coordinate Trigonometry to Right Triangle Trigonometry via the use of reference triangles and reference angles. We increase our calculator usage in two ways:

- We finally have the need to compute trigonometric functions at any angle (e.g., -137°). While we can do so by direct calculator usage, we should not miss the opportunity to show that, for example, $\sin(-137^\circ)$ is nearly the same as the sine of the reference angle for -137°. This foreshadows how we will be working with Trigonometry from this point forward.
- We need to find *non-acute* angles satisfying given trigonometric ratios (e.g., for what θ is $\sin(\theta) = -0.682$?).

Section Preview

- **Solving Equations**

- Solving Linear Equations — One or Two-Steps: This material often requires solving equations of the form $-360^\circ + \hat{\beta} = -315^\circ$.

Section Preview

- Identify the reference angle for an angle given in standard position.
- Evaluate a trigonometric function at a non-acute angle using the reference angle.
- Use technology to approximate the value of a trigonometric function at a non-acute angle.
- Determine an angle given the quadrant in which the angle terminates and the value of a trigonometric function.

Example 4.1.1

a. We begin by sketching the angle and its terminal side.

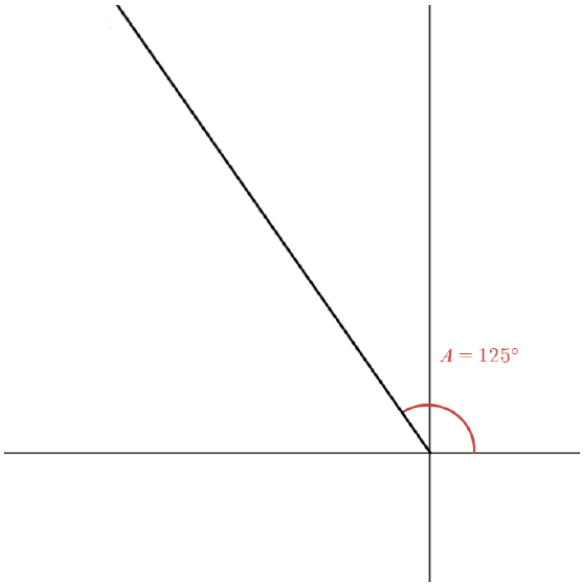


Figure 4.1.3

Since the terminal side of A is in QII, the smallest angle that gets us directly to the x-axis can be found by imagining we rotated to the negative x-axis (180°) and then backed off by $\hat{A}$ to get to $A = 125^\circ$. Mathematically, this is $180^\circ - 125^\circ = 55^\circ$. Hence, the reference angle for A is $\hat{A} = 55^\circ$.

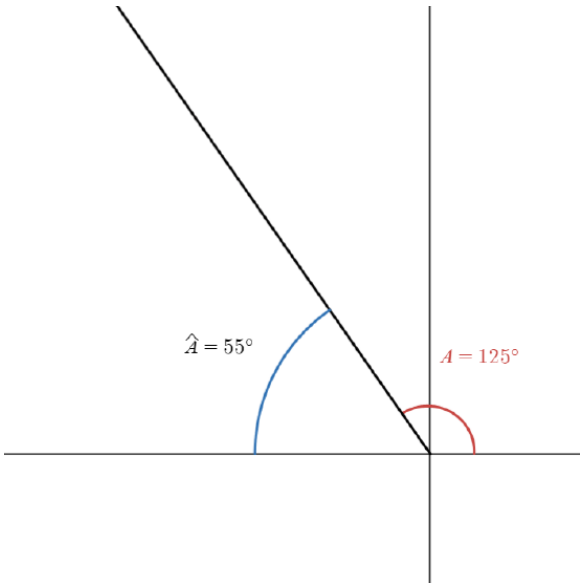


Figure 4.1.4

b. Again, sketch the original angle and its terminal side. $B \in$ QIII, so the smallest angle to the x-axis can be found by imagining we rotated to the negative x-axis (180°) and then added the reference angle $\hat{B}$ to get to $B = 240^\circ$. That is, $180^\circ + \hat{B} = 240^\circ$. Therefore, $\hat{B} = 240^\circ - 180^\circ = 60^\circ$ (see Figure 4.1.5).

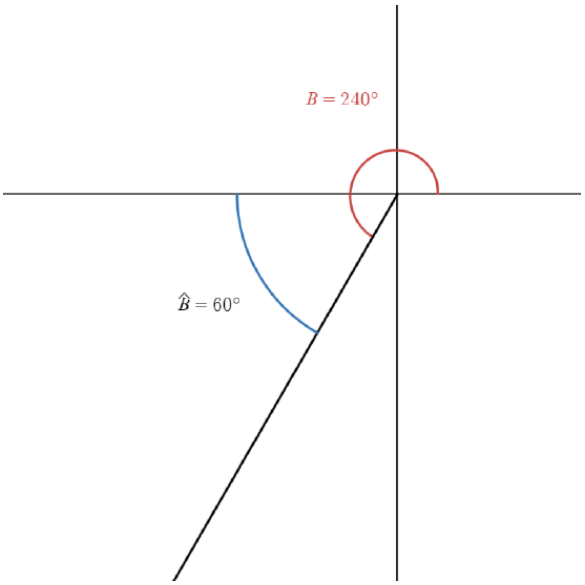


Figure 4.1.5

c. A quick sketch shows that $\alpha \in \text{QIV}$. To find the reference angle, we imagine rotating in a full circle $(360°)$ and backing off by the angle $\hat{\alpha}$ to arrive at $310°$. Therefore, $360° - \hat{\alpha} = 310°$. Hence, $360° - 310° = 50° = \hat{\alpha}$ (see Figure 4.1.6).

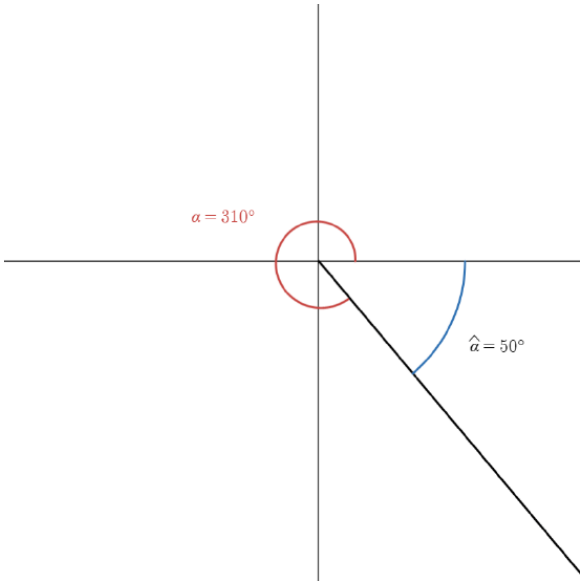


Figure 4.1.6

d. β being a negative angle doesn't change the tactic too much. We sketch $\beta = -315°$ and its terminal side. Since β terminates in QI, we can imagine finding the reference angle by rotating a full revolution in the negative direction $(-360°)$ and backing off by $\hat{\beta}$ to arrive at $\beta = -315°$. "Backing off" in this instance means "adding back" $\hat{\beta}$. Therefore, we want to compute $-360° + \hat{\beta} = -315°$. Solving this equation yields $\hat{\beta} = 45°$ (see Figure 4.1.7).

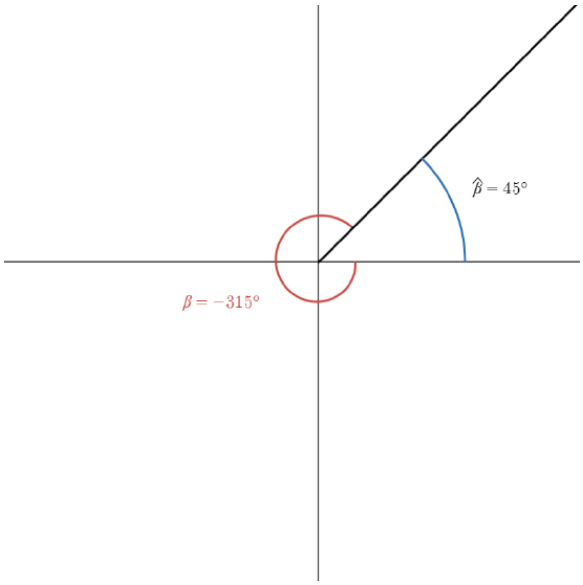


Figure 4.1.7

e. When an angle is outside of the interval $(-360°, 360°)$, a little extra thought is involved (but not much more). Sketching the angle and its terminal side, it's easy to notice that we can find a coterminal angle to $t = 820°$ that is between $0°$ and $360°$. Subtracting $720°$, we arrive at the coterminal angle $100°$ (see Figure 4.1.8).

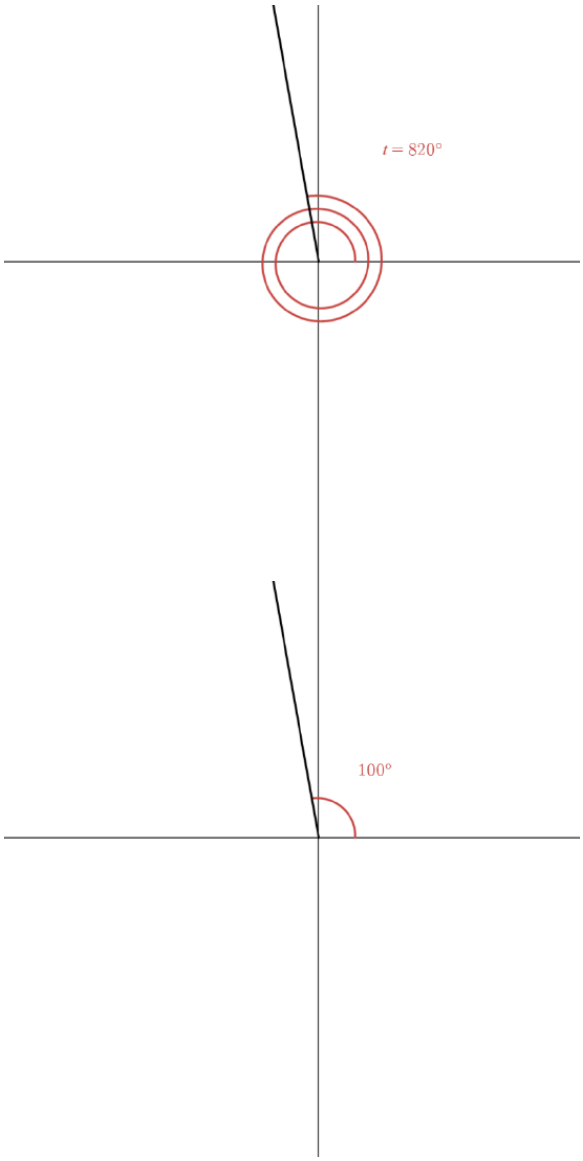


Figure 4.1.8

We are now back to the simpler case of a quadrant II angle. To find the reference angle, we imagine we rotated to the negative x-axis (180°) and then backed off by $\hat{t}$ to arrive at 100°. Mathematically, this means

$$180^\circ - \hat{t} = 100^\circ \implies \hat{t} = 80^\circ.$$

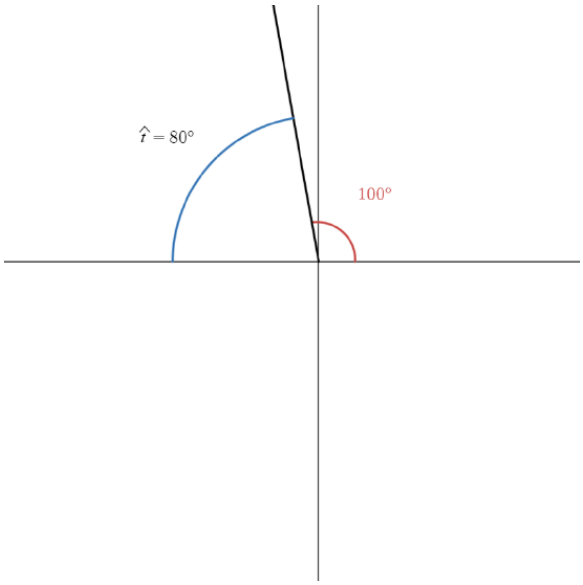


Figure 4.1.9

Checkpoint 4.1.1

a. $\hat{\alpha} = 80^\circ$
b. $\hat{\beta} = 50^\circ$
c. $\hat{\kappa} = 45^\circ$
d. $\hat{\gamma} = 30^\circ$
e. $\hat{\lambda} = 55^\circ$

Example 4.1.2

a. In standard position, an angle of 200° lies in the third quadrant. Therefore, we rotate to 180° and add the reference to get to 200°. Therefore, the reference angle is

$$200^\circ - 180^\circ = 20^\circ = \hat{\theta}.$$

b. Both angles are shown in Figure 4.1.12.

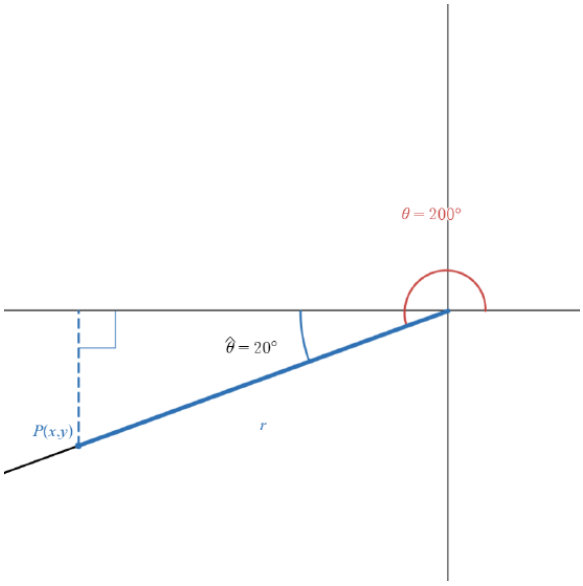


Figure 4.1.12

You can use your calculator to verify the following values.

$$\begin{aligned} \sin(20^\circ) &\approx 0.3420 & \sin(200^\circ) &\approx -0.3420 \\ \cos(20^\circ) &\approx 0.9397 & \cos(200^\circ) &\approx -0.9397 \\ \tan(20^\circ) &\approx 0.3640 & \tan(200^\circ) &\approx 0.3640 \end{aligned}$$

Checkpoint 4.1.2

a. 75°
b. $\sin(285^\circ) = -\sin(75^\circ) \approx -0.9659, \cos(285^\circ) = \cos(75^\circ) \approx 0.2588, \tan(285^\circ) = -\tan(75^\circ) \approx -3.7321$

Example 4.1.3

a. We quickly note that $225^\circ \in$ QIII and the reference angle is $\hat{\theta} = 45^\circ$. Thus,

$$\begin{aligned} \cos(225^\circ) &= \boxed{?}\cos(45^\circ) && \text{(Reference Angle Theorem)} \\ &= -\cos(45^\circ) && \text{(cosine is negative in QIII)} \\ &= -\frac{1}{\sqrt{2}} \end{aligned}$$

b. $120^\circ \in$ QII and the reference angle is $\hat{\theta} = 60^\circ$. Thus,

$$\begin{aligned} \cot(120^\circ) &= \boxed{?}\cot(60^\circ) && \text{(Reference Angle Theorem)} \\ &= -\cot(60^\circ) && \text{(cotangent is negative in QII)} \\ &= -\frac{1}{\sqrt{3}} \end{aligned}$$

c. While we could solve this problem by rotating clockwise by 210°, let's use the corollary.

$$\begin{aligned} \tan(-210^\circ) &= \tan(-210^\circ + 360^\circ) && \text{(corollary)} \\ &= \tan(150^\circ) \\ &= \boxed{?}\tan(30^\circ) && \text{(Reference Angle Theorem)} \\ &= -\tan(30^\circ) && \text{(150° ∈ QII and tangent is negative in QII)} \\ &= -\frac{1}{\sqrt{3}} \end{aligned}$$

d.

$$\begin{aligned} \sin(690^\circ) &= \sin(690^\circ - 360^\circ) && \text{(corollary)} \\ &= \sin(330^\circ) \\ &= \boxed{?}\sin(30^\circ) && \text{(Reference Angle Theorem)} \\ &= -\sin(30^\circ) && \text{(330° ∈ QIV and sine is negative in QIV)} \\ &= -\frac{1}{2} \end{aligned}$$

Checkpoint 4.1.3

a. $-\frac{1}{2}$
b. 2
c. $\frac{1}{\sqrt{2}}$

Example 4.1.4

a. **Reference Angle.** We start by finding the reference angle using the absolute value of the given ratio.

$$\hat{\theta} = \cos^{-1}(|-0.8457|) = \cos^{-1}(0.8457) \approx 32.3^\circ$$

Quadrant. Luckily, we were explicitly given the quadrant in which θ terminates as QIII.
Actual Angle. Thus,

$$\theta = 180^\circ + \hat{\theta} \approx 180^\circ + 32.3^\circ = 212.3^\circ.$$

b. **Reference Angle.**

$$\hat{\theta} = \tan^{-1}(|-1.9122|) = \tan^{-1}(1.9122) \approx 62.4^\circ$$

Quadrant. This example requires a little extra thought than part (a). We are given that both the angle's sine and tangent are negative. The sine is negative in quadrants III and IV, while the tangent is negative in quadrants II and IV. The quadrant where both of these conditions are true is quadrant IV.
Actual Angle. Hence,

$$\theta = 360^\circ - \hat{\theta} \approx 360^\circ - 62.4^\circ = 297.6^\circ.$$

c. **Reference Angle.** Since our calculator (or other technology) does not have an inverse cosecant button, we begin by rewriting the given equation using the Ratio Identities.

$$\csc(\theta) = 2.1174 \implies \frac{1}{\sin(\theta)} = 2.1174 \implies \frac{1}{2.1174} = \sin(\theta)$$

Therefore,

$$\hat{\theta} = \sin^{-1}\left(\frac{1}{2.1174}\right) \approx 28.2^\circ.$$

Quadrant. The cosecant is positive in the same quadrants the sine is positive—namely, QI and QII. The cosine is negative in QII and QIII. Therefore, the quadrant where θ terminates will be the shared quadrant, which is QII.
Actual Angle. Thus,

$$\theta = 180^\circ - \hat{\theta} \approx 180^\circ - 28.2^\circ = 151.8^\circ.$$

Checkpoint 4.1.4

a. $\theta \approx 123.7^\circ$
b. $\sin(\theta) \approx -0.8944$

Example 4.1.5

Reference Angle. It is crucial that you recognize the special ratios. Sketching the 30°-60°-90° triangle, we see that the reference angle for which $\cos(\alpha) = \left|-\frac{\sqrt{3}}{2}\right| = \frac{\sqrt{3}}{2}$ is $\hat{\alpha} = 30^\circ$.
Quadrant. $\alpha \in \text{QII}$
Actual Angle. Thus,

$$\alpha = 180^\circ - \hat{\alpha} = 180^\circ - 30^\circ = 150^\circ.$$

Checkpoint 4.1.5
$\beta = 300^\circ$

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4.1.1: Resources and Key Concepts

Resources

Videos

- Reference Angle
  - Angles - Reference Angles

Key Concepts

Definitions

- **Reference Angle:** For an angle θ in standard position, the reference angle, denoted $\hat{\theta}$, is the positive acute angle formed between the terminal side of θ and the x-axis.
- **Reference Triangle:** A right triangle formed by drawing a perpendicular line from a point on the terminal side of an angle in standard position to the x-axis.

Theorems

- **Reference Angle Theorem:** The value of a trigonometric function of any angle is equal to the value of that same function at its reference angle, except possibly for a difference in sign. The sign is determined by the quadrant in which the angle's terminal side lies.
- **Corollary: Rotational Symmetry:** For any integer n, the value of a trigonometric function is the same for all coterminal angles. For example, $\sin(\theta + 360^\circ n) = \sin(\theta)$.

Common Mistakes

- **Reference Angles are Acute:** By definition, reference angles are always acute (strictly between 0° and 90°). Quadrantal angles do not have reference angles.
- **Measuring from the y-axis:** A reference angle is always measured from the terminal side to the x-axis, never to the y-axis.
- **Negative Reference Angles:** Reference angles are always considered positive, even if the rotation to the x-axis is clockwise.
- **Using Negative Inputs for Inverse Functions:** When using an inverse trigonometric function to find a reference angle, the input must be the positive (absolute) value of the trigonometric ratio.

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4.1.2: Homework

Reading Questions

1. What is a reference angle?
2. How is a reference triangle for an angle θ constructed?
3. Are reference angles ever negative or greater than 90°?
4. What is the reference angle for 150°?
5. What is the reference angle for 300°?
6. Explain the Reference Angle Theorem in your own words.
7. What two pieces of information are needed to evaluate a trigonometric function at a non-acute angle without a calculator?
8. If you are trying to find an angle θ in Quadrant III where $\sin(\theta) = -0.5$, what is the first step in finding the reference angle using a calculator?
9. Once you find the reference angle $\hat{\theta}$, how do you find the actual angle θ if it terminates in Quadrant II?
10. Once you find the reference angle $\hat{\theta}$, how do you find the actual angle θ if it terminates in Quadrant IV?

Homework

Vocabulary Check

1. The ___ for θ is the positive acute angle formed between the terminal side of θ and the x-axis.

Concept Check

2. Camille wants an obtuse angle, θ, that satisfies $\sin(\theta) = 0.3$. Ivan presses some buttons on his calculator and reports that $\theta \approx 17.46^\circ$. Explain Ivan's error and give a correct approximation of θ accurate to two decimal places.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

3. To construct a reference triangle for an angle, we must choose a point P on the terminal side of θ and draw a line from point P perpendicular to the y-axis.

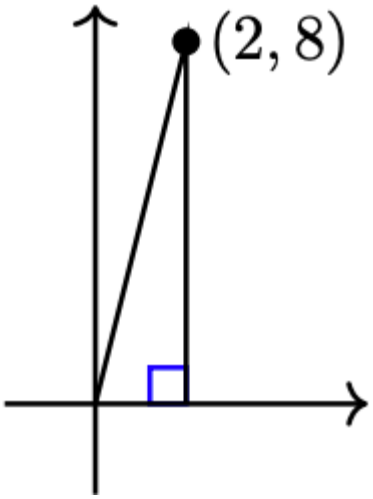
4. The trigonometric ratios of any angle are equal to the ratios of its reference angle, except for sign. The sign of the ratio is determined by the quadrant.

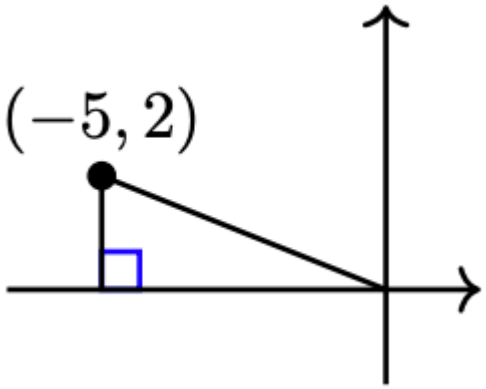
Basic Skills

5. Give the lengths of the legs and the hypotenuse of each right triangle.

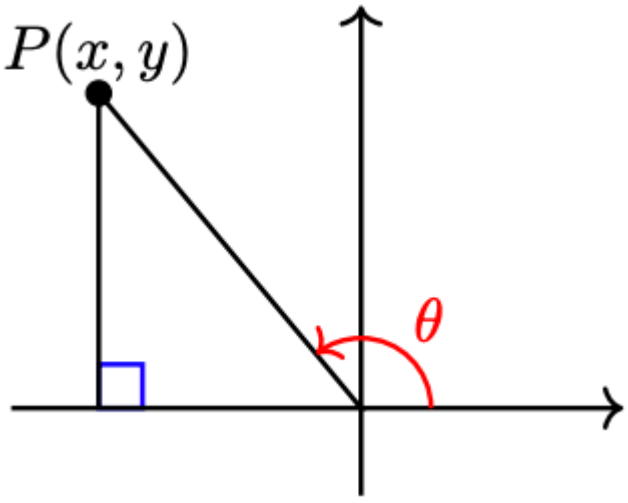
    a.

b.





6. Explain why the length of the horizontal leg of the right triangle is $-x$.



For the following exercises, find the reference angle for the given angle. Make a sketch showing the angle, the reference angle, and the reference triangle.

7. $\theta = 100^\circ$

8. $A = 125^\circ$

9. (MyOpenMath) $\beta = 216^\circ$
10. $\theta = 194^\circ\ 15'$
11. $\alpha = 242^\circ$
12. $\phi = 297^\circ$
13. $\lambda = 137^\circ\ 50'$
14. $\theta = 336^\circ$

For the following exercises, find three angles between 90° and 360° having the given reference angle, and sketch all four angles on the same grid.

15. 15°
16. 26°
17. 40°
18. 50°
19. 68°
20. 75°

For the following exercises, find the exact value of the trigonometric function using reference angles and quadrants (do not use a calculator).

21. $\cos(120^\circ)$
22. $\sin(135^\circ)$
23. $\tan(150^\circ)$
24. $\sec(225^\circ)$
25. $\sin(240^\circ)$
26. $\cot(210^\circ)$
27. $\csc(330^\circ)$
28. $\cos(315^\circ)$
29. $\sin(300^\circ)$
30. $\cos(-150^\circ)$
31. $\sin(-210^\circ)$
32. $\sec(-315^\circ)$
33. $\sin(585^\circ)$
34. $\cos(840^\circ)$
35. $\tan(-585^\circ)$

For the following exercises, use the values below to find the value of the trigonometric function. Do not use a calculator!

$\cos(23^\circ) \approx 0.9205 \quad \sin(46^\circ) \approx 0.7193 \quad \tan(78^\circ) \approx 4.7046$

36. $\cos(157^\circ)$

37. $\sin(226^\circ)$

38. $\csc(314^\circ)$

39. $\sec(203^\circ)$

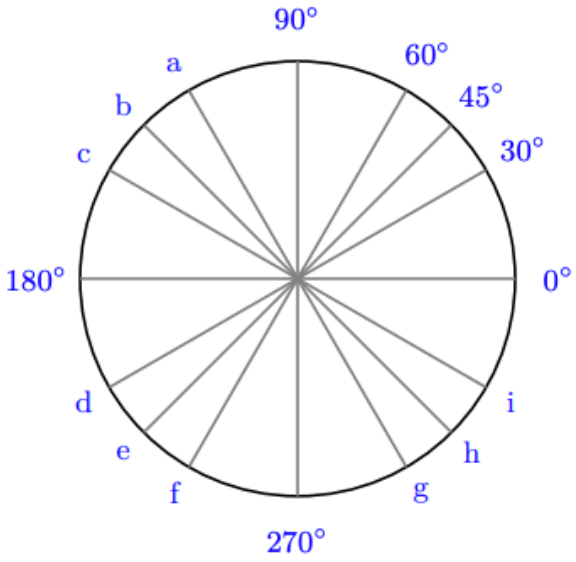
40. $\tan(258^\circ)$

41. $\cot(282^\circ)$

42. $\sin(-134^\circ)$

43. $\cos(-383^\circ)$

44. On the circle in the figure, all angles are shown in standard position. Find the measure in degrees of the angles labeled (a) - (i) and state their reference angles.



45. a. Draw three angles, one in each quadrant except the first, whose reference angle is 60°.

    b. Find exact values for the sine, cosine, and tangent of each angle in part (a).

46. a. Draw three angles, one in each quadrant except the first, whose reference angle is 30°.

    b. Find exact values for the sine, cosine, and tangent of each angle in part (a).

47. a. Draw three angles, one in each quadrant except the first, whose reference angle is 45°.

    b. Find exact values for the sine, cosine, and tangent of each angle in part (a).

48. Complete the table with exact values.

θ	30°	45°	60°	120°	135°	150°	210°	225°	240°	300°	315°	330°
$\sin(\theta)$												
$\cos(\theta)$												

$\tan(\theta)$												
$\csc(\theta)$												
$\sec(\theta)$												
$\cot(\theta)$												

For the following exercises, find a second angle between 0° and 360° with the given trigonometric ratio.

49. $\sin(75^\circ)$

50. $\cos(32^\circ)$

51. $\tan(84^\circ)$

52. $\csc(16^\circ)$

53. $\sec(47^\circ)$

54. $\cot(56^\circ)$

For the following exercises, use the given information to find an approximation of θ to the nearest degree if $0^\circ \leq \theta < 360^\circ$.

55. $\tan(\theta) = 8.1443$ with $\theta \in \text{QIII}$

56. $\sin(\theta) = 0.7880$ with $\theta \in \text{QII}$

57. $\cos(\theta) = 0.9205$ with $\theta \in \text{QIV}$

58. $\tan(\theta) = -3.4874$ with $\theta \in \text{QII}$

59. $\sin(\theta) = -0.9962$ with $\theta \in \text{QIV}$

60. $\cos(\theta) = -0.0349$ with $\theta \in \text{QII}$

61. $\sin(\theta) = 0.8381$ where $\cos(\theta) > 0$

62. $\tan(\theta) = -2.1728$ where $\sin(\theta) > 0$

63. $\cos(\theta) = -0.2155$ where $\tan(\theta) > 0$

64. $\sin(\theta) = -0.1179$ where $\tan(\theta) < 0$

65. $\cos(\theta) = 0.1818$ where $\csc(\theta) < 0$

66. $\tan(\theta) = 1.7531$ where $\sec(\theta) < 0$

67. $\csc(\theta) = 2.8549$ where $\cos(\theta) < 0$

68. $\sec(\theta) = -1.3888$ where $\csc(\theta) > 0$

69. $\cot(\theta) = -0.9411$ where $\cos(\theta) < 0$

For the following exercises, use the given information to find the exact value of θ if $0^\circ \leq \theta < 360^\circ$. Do not use a calculator!

70. $\tan(\theta) = \sqrt{3}$ with $\theta \in \text{QIII}$

71. $\sin(\theta) = \frac{1}{2}$ with $\theta \in \text{QII}$

72. $\cos(\theta) = \frac{1}{\sqrt{2}}$ with $\theta \in \text{QIV}$

73. $\tan(\theta) = -1$ with $\theta \in \text{QII}$

74. $\sin(\theta) = -\frac{\sqrt{3}}{2}$ with $\theta \in \text{QIV}$

75. $\cos(\theta) = -\frac{1}{2}$ with $\theta \in \text{QII}$

76. $\sin(\theta) = \frac{1}{\sqrt{2}}$ where $\cos(\theta) > 0$

77. $\tan(\theta) = -\frac{1}{\sqrt{3}}$ where $\sin(\theta) > 0$

78. $\cos(\theta) = -\frac{\sqrt{3}}{2}$ where $\tan(\theta) > 0$

79. $\sin(\theta) = -\frac{1}{2}$ where $\tan(\theta) < 0$

80. $\cos(\theta) = \frac{1}{2}$ where $\csc(\theta) < 0$

81. $\tan(\theta) = 1$ where $\sec(\theta) < 0$

82. $\csc(\theta) = 2$ where $\cos(\theta) < 0$

83. $\sec(\theta) = -\frac{2}{\sqrt{3}}$ where $\csc(\theta) > 0$

84. $\cot(\theta) = -\sqrt{3}$ where $\cos(\theta) < 0$

Synthesis Questions

For the following exercises, find exact values for all solutions between $0°$ and $360°$.

85. $\cos(\theta) = -\cos(24°)$

86. $\tan(\theta) = -\tan(9°)$

87. $\sin(\theta) = -\sin(66°)$

88. $\cos(\theta) = -\cos(78°)$

89. $\tan(\theta) = -\tan(31°)$

90. $\sin(\theta) = -\sin(42°)$

91. Later we will be able to show that $\sin(18°) = \frac{\sqrt{5}-1}{4}$. What is the exact value of $\sin(162°)$?

92. Later we will be able to show that $\cos(36°) = \frac{\sqrt{5}+1}{4}$. What is the exact value of $\cos(144°)$?

For the following exercises,
(a) Sketch an angle in standard position with the given properties.
(b) Find $\cos(\theta)$, $\sin(\theta)$, and $\tan(\theta)$.
(c) Find the angle θ, rounded to tenths of a degree.

93. The point (–5, 12) is on the terminal side.

94. The point (12, 9) is on the terminal side.

95. $\cos(\theta) = -0.8$

96. $\cos(\theta) = \frac{5}{13}$

97. $\cos(\theta) = \frac{3}{11}$

98. $\cos(\theta) = -\frac{5}{6}$

99. $\tan(\theta) = -\frac{1}{6}$

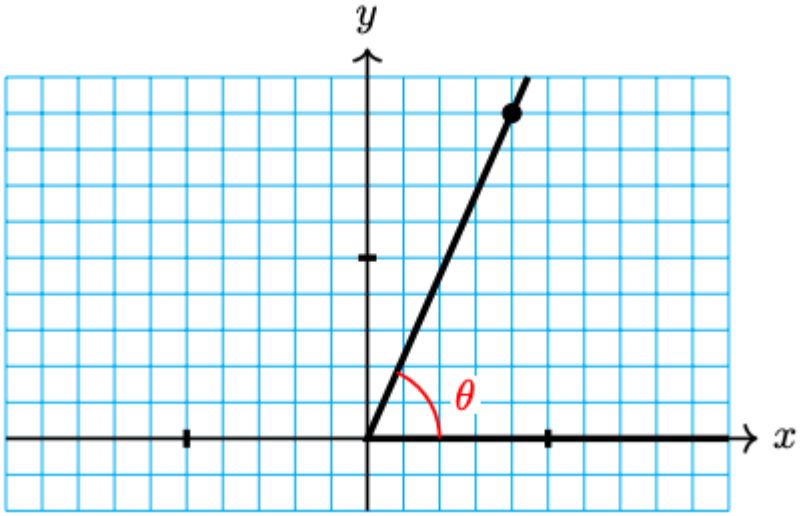
100. $\tan(\theta) = \frac{9}{5}$

101. $\tan(\theta) = 4$

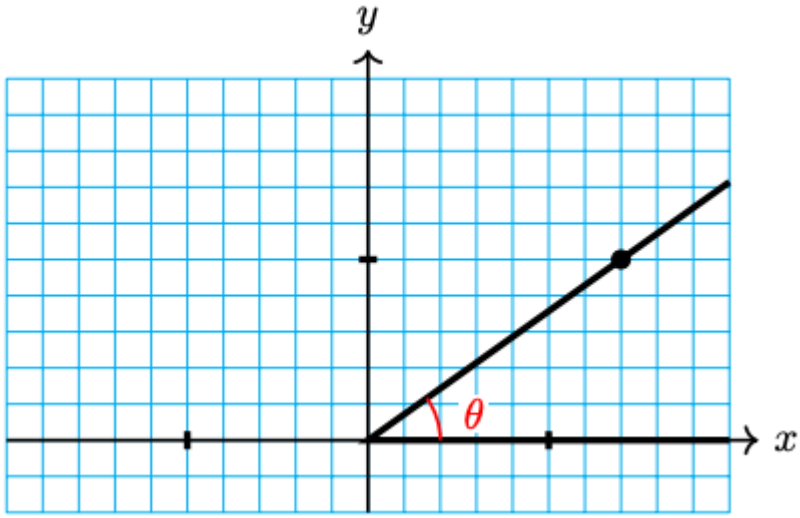
102. $\tan(\theta) = -1$

For the following exercises,
(a) Find the sine and cosine of the angle.
(b) Sketch the supplement of the angle in standard position. (Use congruent triangles.)
(c) Find the sine and cosine of the supplement.
(d) Find the angle and its supplement, rounded to the nearest degree.

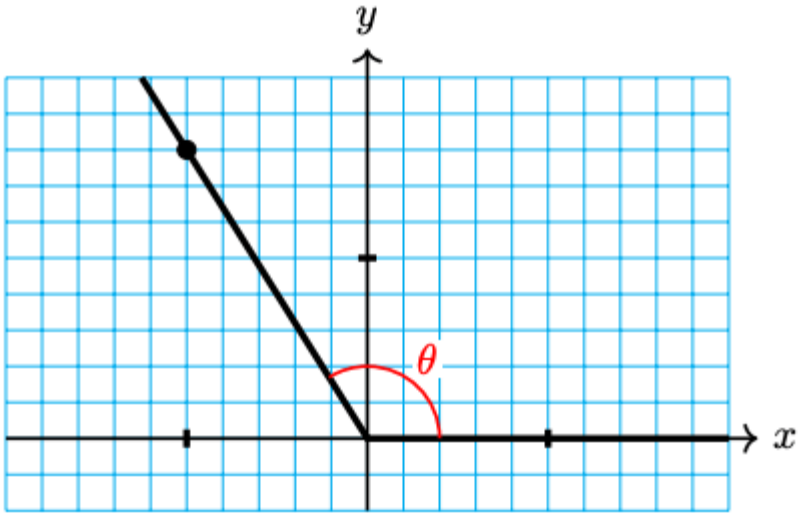
103.



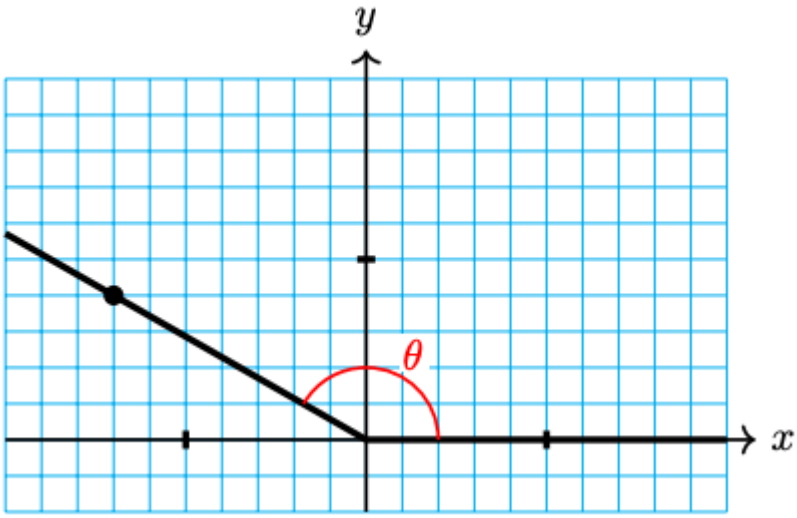
104.



105.



96.



For the following exercises, refer to the following table of values.

θ	15°	25°	65°	75°	105°	115°	155°	165°
$\cos(\theta)$								
$\sin(\theta)$								
$\tan(\theta)$								

97. Use your calculator to fill in the table. Round values to four decimal places.

98. For each angle θ in the table, the angle $180^\circ - \theta$ is also in the table.

a. What is true about $\sin\theta$ and $\sin(180^\circ - \theta)$?

b. What is true about $\cos\theta$ and $\cos(180^\circ - \theta)$?

c. What is true about $\tan\theta$ and $\tan(180^\circ - \theta)$?

The following exercises preview the concept of solving trigonometric equations (a topic reserved for later in the course). For each equation, find two different angles between 0° and 360° that satisfy the equation. Round your answers to the nearest 0.1°.

9. $\sin(\theta) = 0.7$

10. $\sin(\theta) = 0.1$

11. $\dfrac{\sin(\theta)}{6} = 0.14$

12. $\dfrac{5}{\sin(\theta)} = 6$

13. $4.8 = \dfrac{3.2}{\sin(\theta)}$

14. $1.5 = \dfrac{\sin(\theta)}{0.3}$

For the following exercises, fill in the blanks with complements or supplements.

15. If $\sin(57^\circ) = q$, then $\sin(\_\_\_\_) = q$, and $\cos(\_\_\_\_)$ and $\cos(\_\_\_\_)$ both equal q.

16. If $\sin(18^\circ) = w$, then $\sin(\_\_\_\_) = w$, $\cos(\_\_\_\_) = w$, and $\cos(\_\_\_\_) = -w$.

17. If $\cos(74^\circ) = m$, then $\cos(\_\_\_\_) = -m$, and $\sin(\_\_\_\_)$ and $\sin(\_\_\_\_)$ both equal m.

18. If $\cos(36^\circ) = t$, then $\cos(\_\_\_\_) = -t$, and $\sin(\_\_\_\_)$ and $\sin(\_\_\_\_)$ both equal t.

19. a. Sketch the line $y = \dfrac{3}{4}x$.

    b. A point on the line with a positive x-coordinate.

    c. The line $y = \frac{3}{4}x$ makes an angle with the positive x-axis. What is that angle?

    d. Repeat parts (a) through (c) for the line $y = \frac{-3}{4}x$, except find a point with a *negative* x-coordinate.

20. a. Sketch the line $y = \frac{5}{3}x$.

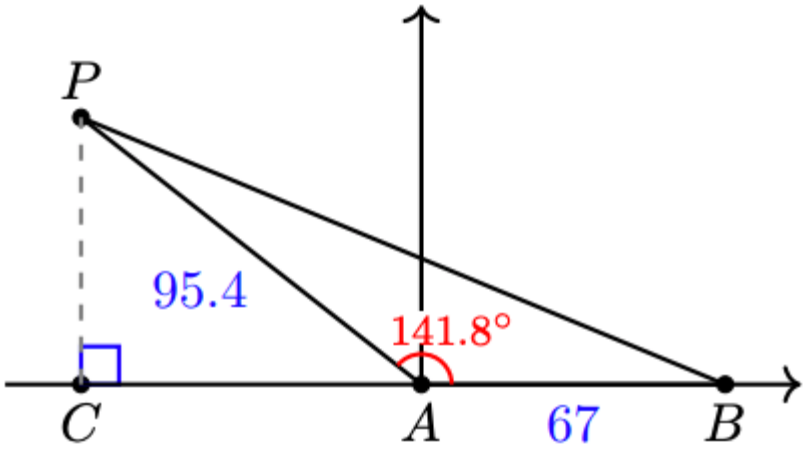
    b. Find a point on the line with a positive x-coordinate.

    c. The line $y = \frac{5}{3}x$ makes an angle with the positive x-axis. What is that angle?

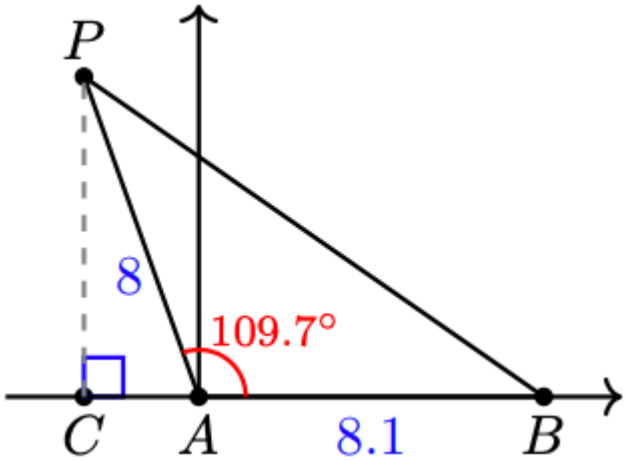
    d. Repeat parts (a) through (c) for the line $y = \frac{-5}{3}x$, except find a point with a *negative* x-coordinate.

For the following exercises,
(a) Find the coordinates of point P. Round to two decimal places.
(b) Find the sides BC and PC of $\triangle PCB$.
(c) Find side PB.

21.



22.



Challenge Problems

23. Why are the sines of supplementary angles equal, but the cosines are not? What about the tangents of supplementary angles?

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4.2: Radian Measure and Arc Length

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Radian Measure as Portion of a Rotation

Imagine riding on a Ferris wheel with a radius of 100 feet, where each rotation takes eight minutes. We can use angles in a standard position to describe your location as you travel around the wheel. Figure $4.2.1$ shows the locations indicated by $\theta = 0°$, $90°$, $180°$, and $270°$.

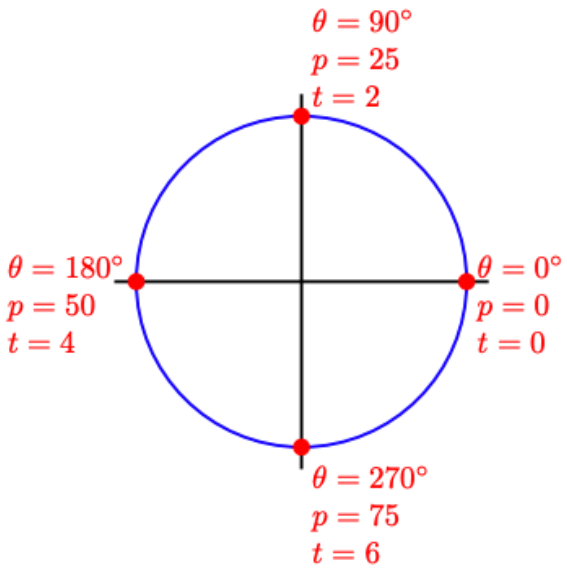


Figure $4.2.1$

However, degrees are not the only way to specify a location on a circle. We could use a percent of one complete revolution and label the same locations by $p = 0$, $p = 25$, $p = 50$, and $p = 75$. Another option is to use the time elapsed so that, for this example, we would have $t = 0$, $t = 2$, $t = 4$, and $t = 6$ minutes (see Figure $4.2.1$).

Note: Revolution and Rotation

The word "revolution" will be used extensively from this point forward. Some texts use "rotation."

Another helpful method uses distance traveled, or arc length, along the circle. In Figure $4.2.1$, how far have you traveled around the Ferris wheel at each of the locations shown, assuming you hopped onto the ride at $\theta = 0°$?

Before we consider that question, let's agree on some vocabulary. To help us understand the vocabulary, let's consider Figure $4.2.2$.

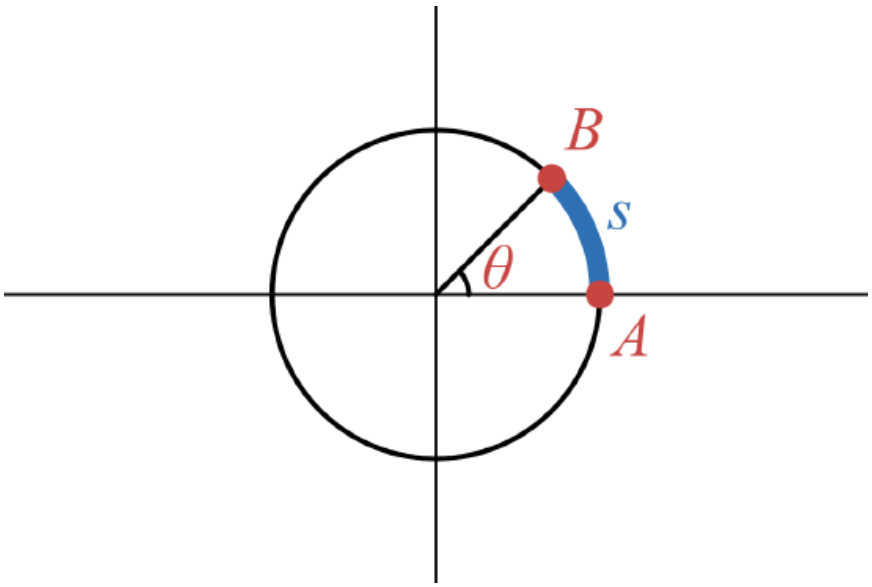


Figure $4.2.2$

We begin by naming a portion of the edge of a circle.

Definition: Arc and Arc Length

An **arc** along a circle is a portion of the circumference of a circle. It's the curved line connecting two points on the circle's edge. The length of this curve is called the **arc length**.

In Figure $4.2.2$, the arc is in blue and its length, which we labeled as s, is called the arc length.

It should be natural to think that any arc along the circumference of a circle is related to an angle "opening" at the center of the circle. This angle (which is labeled as θ in Figure $4.2.2$) gets its own name.

Definition: Central Angle

An angle with vertex at the center of a circle is called a **central angle** of the circle.

Thus, for the given arc in Figure $4.2.2$, the related central angle is θ. If the arc represents a distance traveled, we sometimes refer to such an angle as the **angle of displacement**.

Finally, we need to introduce the conventional language used throughout most of Mathematics when it comes to relating the central angle to the arc.

Definition: Subtend

A central angle whose initial and terminal sides meet the endpoints of an arc is said to **subtend** (or **span**) the arc.

This definition means that the angle θ in Figure $4.2.2$ subtends an arc of length s.

Arc Length

Recall that the circumference of a circle is given by the formula

$$C = 2\pi r. \tag{4.2.1}$$

Another way to say this is that the circumference of a circle is proportional to its radius—increasing the radius of the circle increases the circumference and vice versa.

According to Equation $4.2.1$, if we walk around the entire circumference of a circle, the distance we travel is the product of 2π and the radius, or *approximately* 6.28 times the radius (by the way, just like π, 2π gets its own special name—the Greek letter tau. That is, we define $\tau = 2\pi$). If we walk only part of the way around the circle, then the distance we travel depends also on the angle of displacement.

For example, an angle of 45° is $\frac{1}{8}$ of a complete revolution, so the arc length, s, from point A to point B in Figure $4.2.3$ is $\frac{1}{8}$ of the circumference. Therefore,

$$s = \frac{1}{8}(2\pi r) = \frac{\pi}{4}r.$$

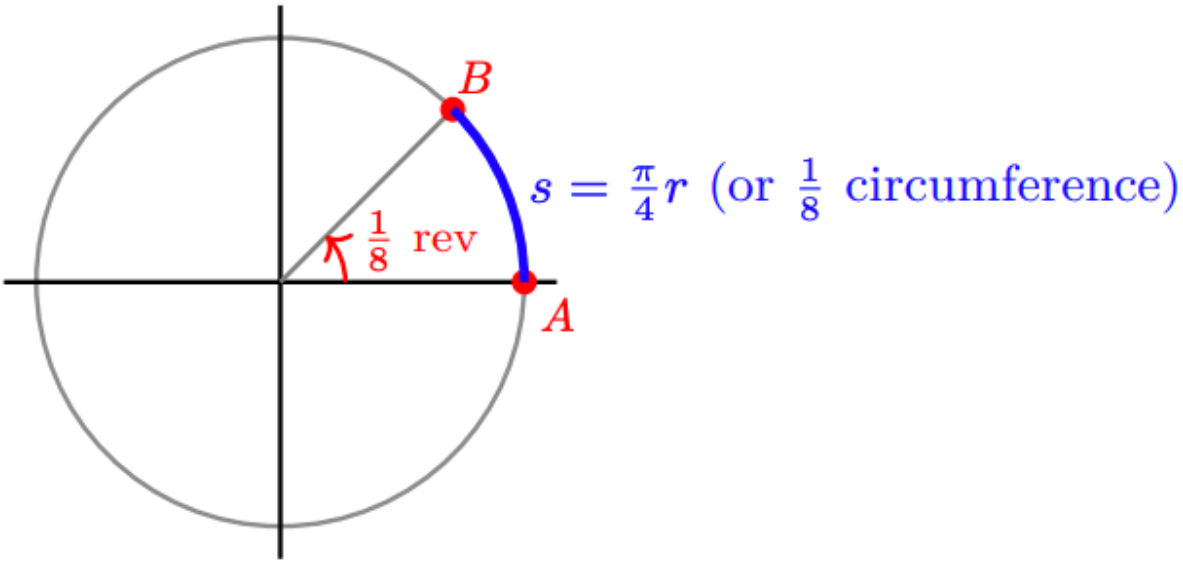


Figure $4.2.3$

Similarly, the angle of displacement from point A to point C in Figure $4.2.4$ is $\frac{3}{4}$ of a complete revolution, so the arc length along the circle from A to C, shown below, is

$$s = \frac{3}{4}(2\pi r) = \frac{3\pi}{2}r.$$

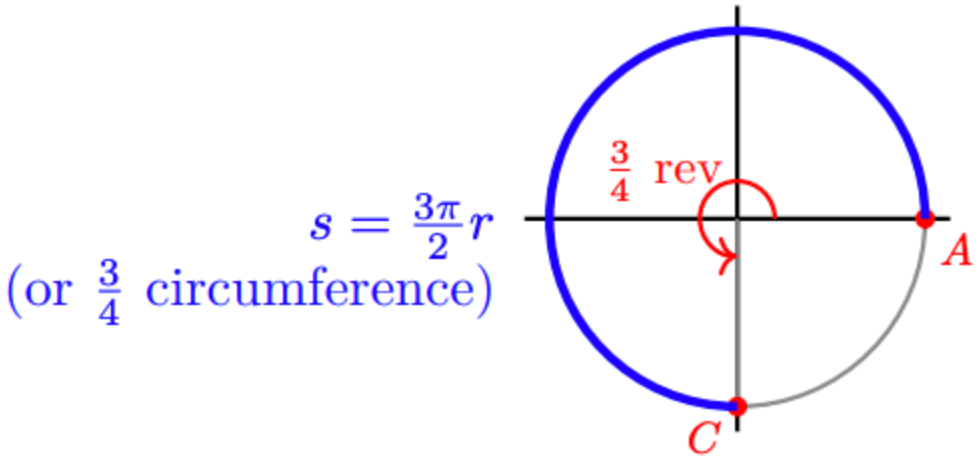


Figure 4.2.4

In general, the length of the arc spanned by an angle is proportional to the size of the angle for a given circle.

Theorem: Arc Length Formula (portion of a circle version)

The length of the arc, s, in a circle of radius r, spanned by a fraction of a revolution, p, is

$$s = p \cdot (2\pi r).$$

The Ferris wheel in the introduction has circumference

$$C = 2\pi(100) \approx 628 \text{ feet},$$

so in half a revolution, you travel approximately 314 feet around the edge. In one-quarter revolution, you travel approximately 157 feet.

To indicate the same four locations from Figure 4.2.1 on the wheel by distance traveled, we would use $s = 0$, $s \approx 157$, $s \approx 314$, and $s \approx 471$, as shown below.

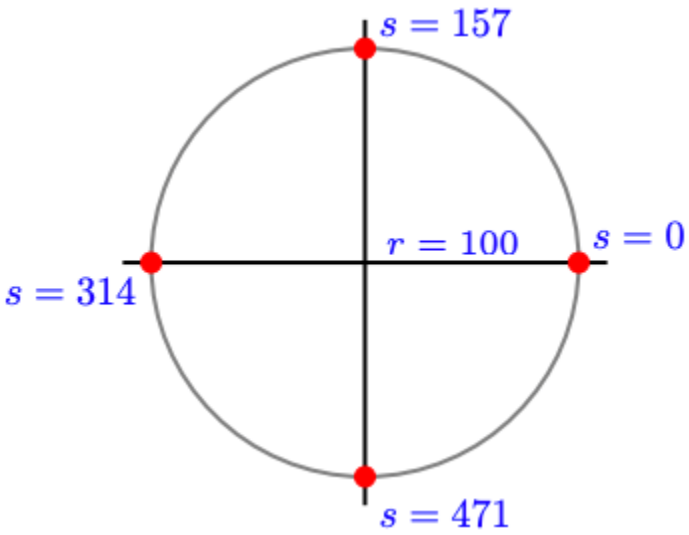


Figure 4.2.5

Example 4.2.1

What length of arc is spanned by an angle of 120° on a circle of radius 12 centimeters?

Solution available at the end of this section.

Checkpoint 4.2.1

How far have you traveled around the edge of a Ferris wheel of radius 100 feet when you have turned through an angle of 150°?

Answer available at the end of this section.

Measuring Angles in Radians

If you think about measuring arc length, you will see that the degree measure of the spanning angle is not as important as the fraction of one revolution it covers. This observation suggests a new unit of measurement for angles, one that is better suited to calculations involving arc length. We'll make one change in our formula for arc length from

$$\textbf{arc length} = (\textbf{fraction of one revolution}) \cdot (2\pi r)$$

to

$$\textbf{arc length} = (\textbf{fraction of one revolution} \cdot 2\pi) \cdot r.$$

You might be asking yourself why we made such a silly change since these two formulas state the same thing (just with the parentheses in different spots).

The subtle difference between these two interpretations of arc length leads to an entirely new angular measurement system. We'll call the quantity in parentheses, (fraction of one revolution $\times 2\pi$), the **radian measure** of the angle that spans the arc.

Definition: Radian Measure (portion of a circle version)

The **radian measure** of an angle is given by

$$\text{fraction of one revolution } \times 2\pi.$$

For example, one complete revolution (360°) is equal to 2π radians, and one-quarter revolution (90°) is equal to $\frac{1}{4}(2\pi) = \frac{\pi}{2}$ radians. Figure 4.2.7 shows the radian measure of the quadrantal angles.

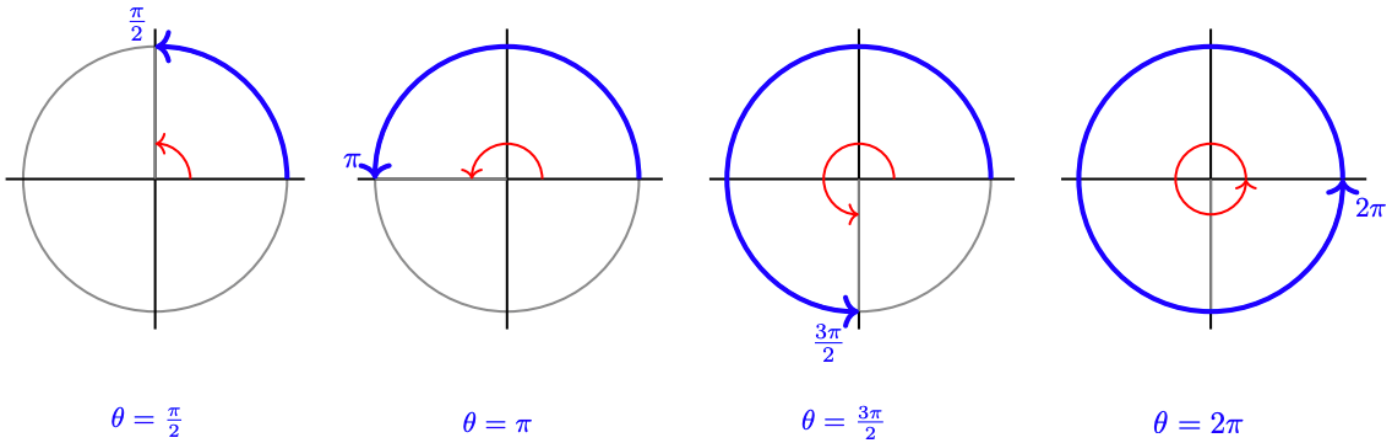


Figure 4.2.7

Note: Radian Measure is Universal

Unlike degree measure, which depends on a human-created construct dependent on either the number of days for the Earth to rotate around the sun or the base-60 numbering system used by the Babylonians (depending on which historical viewpoint of the creation of the degree measurement system you believe), radian measure makes sense on *any* planet and within *any* numbering system. That is, if an alien species were to arrive on Earth, radian measures for angles are likely something we would both have in common.

Example 4.2.2

What is the radian measure of an angle of $120°$?

Solution available at the end of this section.

Checkpoint 4.2.2

What fraction of a revolution is π radians? How many degrees is that?

Answer available at the end of this section.

Radian measure does not have to be expressed in multiples of π. Remember that $\pi \approx 3.14$, so one complete revolution is approximately 6.28 radians, and one-quarter revolution is $\frac{1}{4}(2\pi) = \frac{\pi}{2}$, or about 1.57 radians. Figure 4.2.8 shows decimal approximations for the quadrantal angles.

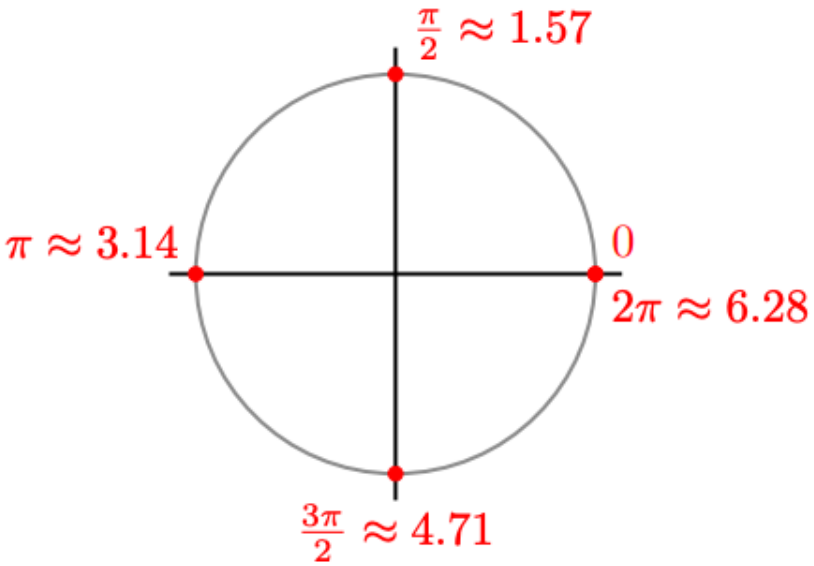


Figure 4.2.8

Table 4.2.1**:** *Quadrantal Angles in Both Degrees and Radians*

Degrees	Radians: Exact Values	Radians: Decimal Approximations
$0°$	0	0
$90°$	$\frac{\pi}{2}$	1.57
$180°$	π	3.14

Degrees	Radians: Exact Values	Radians: Decimal Approximations
270°	$\frac{3\pi}{2}$	4.71
360°	2π	6.28

Memorize the Exact Radian Values of Common Angles

Because they are "benchmarks" for comparing angles, you should memorize the exact values of the angles in the previous table in radians!

Example $4.2.3$

In which quadrant would you find an angle of 2 radians? An angle of 5 radians?

Solution available at the end of this section.

Checkpoint $4.2.3$

Draw a circle centered at the origin and sketch (in standard position) angles of approximately 3 radians, 4 radians, and 6 radians.

Answer available at the end of this section.

Interactive Element: Where Does a Radian Angle Land?

The following Interactive Element might help you visualize where an angle given as a plain number of radians terminates and which quadrant it falls in.

Interact: Drag the orange point to set the terminal side, and read the angle in radians at the point.

Observation: Even without a π, a radian value points to a definite spot — angles between about 1.57 and 3.14 land in Quadrant II, and those between about 4.71 and 6.28 land in Quadrant IV.

It turns out that measuring angles in radians is helpful for many applications besides calculating arc lengths, so we need to start thinking in radians. To help that process, we'll first learn to convert between degrees and radians.

Converting Between Degrees and Radians

It is not difficult to convert the measure of an angle in degrees to its measure in radians or vice versa. One complete revolution is equal to 2π radians or to 360°, so

$$2\pi \text{ radians} = 360^\circ.$$

Dividing both sides of this equation by 2π radians or by 360° gives us an **angle measure conversion theorem**:

Theorem: Angle Measure Conversion Theorem

Assuming the unitless measure of radians,

$$2\pi = 360^\circ$$

$$\Longrightarrow \quad 1 = \frac{180^\circ}{\pi} \quad \text{and} \quad \frac{\pi}{180} = 1^\circ$$

(dividing both sides by 2π radians) (dividing both sides instead by 360)

Example 4.2.4

a. Convert 3 radians to degrees.
b. Convert 3 degrees to radians.

Solutions available at the end of this section.

Technically speaking, a radian is what is known as a **unitless measure**. This is because, from the definition of radian measure, a radian is the product of a fraction of a revolution (no units) and 2π (also, no units). Therefore, it is acceptable to answer Example $4.2.4b$ with "approximately 0.05" instead of "approximately 0.05 radians." Moreover, there is no need to include the word "radian" when the context of the situation implies the constant is an angular measure (and the angular measure is not in degrees). For example, the solution to $4.2.4a$ could have been written as

$$\begin{aligned} 3 &= 3 \times (1) \\ &= 3 \times \left(\frac{180^\circ}{\pi}\right) \\ &= \frac{540^\circ}{\pi} \\ &\approx 171.9^\circ \end{aligned}$$

Checkpoint 4.2.4

a. Convert 60° to radians. Give both an exact answer and an approximation to three decimal places.
b. Convert $\frac{3\pi}{4}$ to degrees.

Answers available at the end of this section.

From our angle conversion theorem, we also learn that

$$1 \text{ radian} = \frac{180^\circ}{\pi} \approx 57.3^\circ.$$

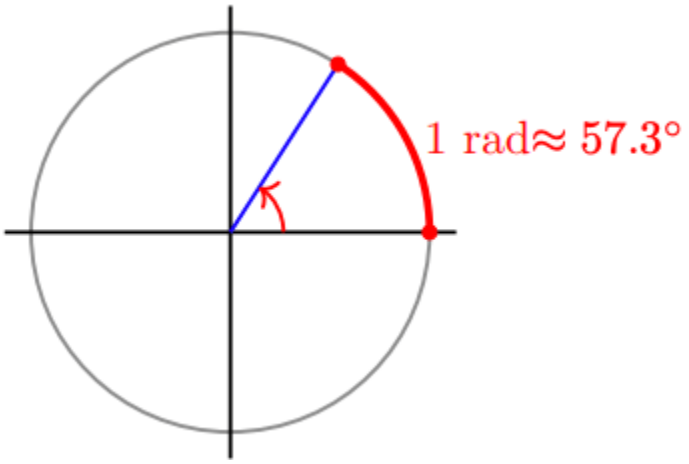


Figure 4.2.10

So while 1° is a relatively small angle, 1 radian is much larger (nearly 60°, in fact). However, this is reasonable because there are only a little more than 6 radians in an entire revolution. An angle of 1 radian is shown above.

We'll soon see that, for many applications, it is easier to work entirely in radians. In fact, in the study of Calculus, you will find that radian measure *must* be used instead of degree measure.

For reference, Figure $4.2.11$ shows a radian protractor.

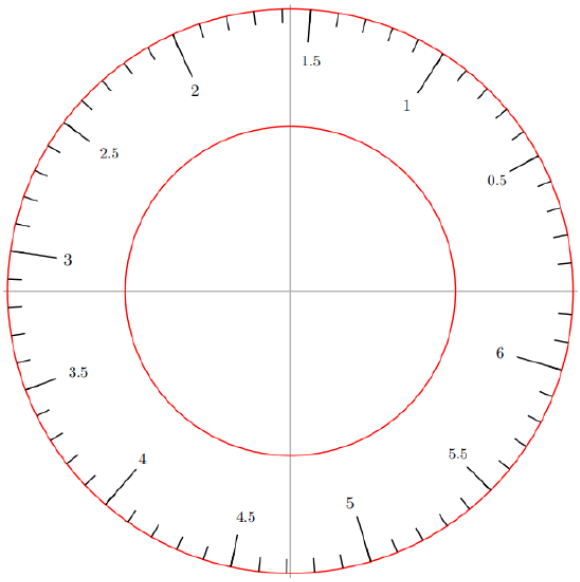


Figure $4.2.11$

Quadrants and Reference Angles in Radians

Radian measure takes a bit to get used to, but you will soon find it easier to work with than degree measure. Understanding the acute special angles and quadrantal angles in radian measure, we now ask how to quickly determine in which quadrant an angle terminates. We also want to know the reference angle for a given angle in radian measure. While angles in radians need not have a π attached to them, it does help when determining the quadrant in which the angle terminates. Therefore, we restrict ourselves to angles of the form $\frac{n\pi}{d}$ for this short discussion.

A beneficial habit to get into when given an angle in radian measure of the form $\frac{n\pi}{d}$ is to think in multiples of the denominator, d. This is because $\frac{d\pi}{d} = \pi$, $\frac{2d\pi}{d} = 2\pi$, $\frac{3d\pi}{d} = 3\pi$, and so on. If you sketch these angles, they terminate on either the positive or negative x-axis—this will be useful.

For example, if asked to find the quadrant in which $\theta = \frac{7\pi}{6}$ terminates, we consider that θ is *slightly* more than $\frac{6\pi}{6} = \pi$ (which is equivalent to 180°). In fact, it is only $\frac{1\pi}{6}$ more than $\frac{6\pi}{6} = \pi$. Thus, to get to θ, we rotate *past* π and into quadrant three by the small amount of $\frac{\pi}{6}$ radians. This small extra angle of rotation is the reference angle for θ. That is, the reference angle for $\theta = \frac{7\pi}{6}$ is $\hat{\theta} = \frac{\pi}{6}$ (see Figure $4.2.12$).

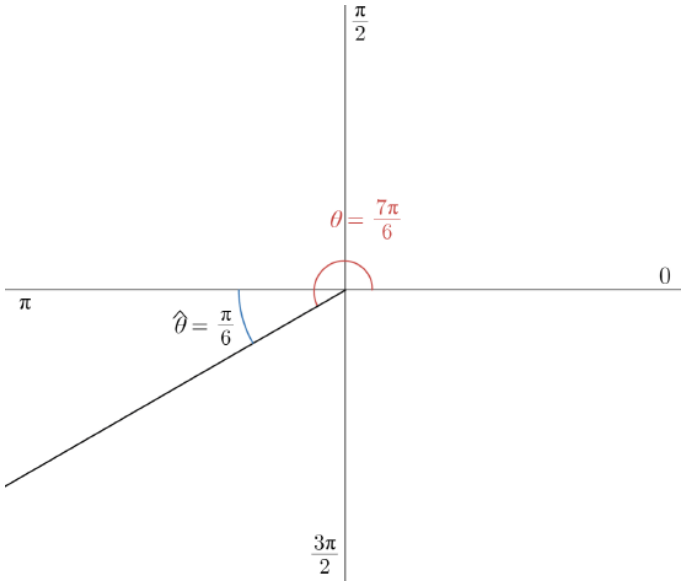


Figure 4.2.12

This concept of using multiples of the denominator need not be limited to positive angles. For example, if $\theta = -\frac{9\pi}{5}$, we imagine rotating in the negative direction to $-\frac{5\pi}{5} = -\pi$; however, this is nowhere near $-\frac{9\pi}{5}$. Therefore, we continue until we get to $-\frac{10\pi}{5} = -2\pi$. At this point, we will have rotated back to the positive x-axis, but we will have rotated *slightly* too far. Therefore, we back off (in the positive direction) by $\frac{1\pi}{5}$ to arrive at $-\frac{9\pi}{5}$, which terminates in the first quadrant. Hence, $\theta = -\frac{9\pi}{5} \in \text{QI}$ with reference angle $\hat{\theta} = \frac{\pi}{5}$ (see Figure 4.2.13).

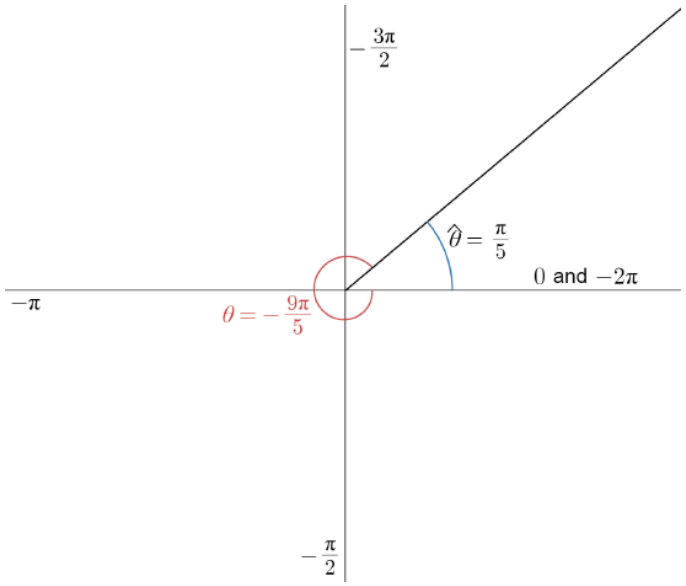


Figure 4.2.13

Example 4.2.5

State the quadrant in which $\beta = \frac{5\pi}{3}$ terminates, state its reference angle and sketch.

Solution available at the end of this section.

Checkpoint 4.2.5

State the quadrant in which $\alpha = -\frac{17\pi}{16}$ terminates, state its reference angle, and sketch.

Answer available at the end of this section.

Arc Length Formula

We now return to our calculation of arc length and see the first instance in which measuring angles in radians is useful. To calculate an arc length, we need only multiply the circle's radius by the radian measure of the spanning angle, θ. Look again at our formula for arc length:

$$\text{arc length} = (\text{fraction of one revolution} \times 2\pi) \cdot r$$

The quantity in parentheses, "fraction of one revolution $\times 2\pi$," is the measure of the spanning angle in radians. This directly leads to a reinvention of the definition of the radian (which is actually the *true* definition of the radian).

Definition: Radian

One **radian** is the measure of a central angle of a circle that is subtended by an arc whose length is equal to the radius of the circle. More generally, the radian measure of a central angle equals the ratio of the length s of the subtended arc to the radius r of the circle; that is, $\theta = \frac{s}{r}$.

Thus, if θ is measured in radians, we have the following formula for arc length, s.

Theorem: Arc Length Formula (radian version)

On a circle of radius r, the length s of an arc spanned by an angle θ in radians is

$$s = r\theta.$$

Caution: The Radian Requirement

It is crucial to remember that the arc length formula, $s = r\theta$, is only valid if θ is in radians.

Thus, there is a special relationship between arc length and radians. An angle of 1 radian spans an arc on a circle equal to the radius of the circle, as shown in Figure 4.2.15. Likewise, the length of any arc is just the product of the measure of its spanning angle in radians and the radius of the circle.

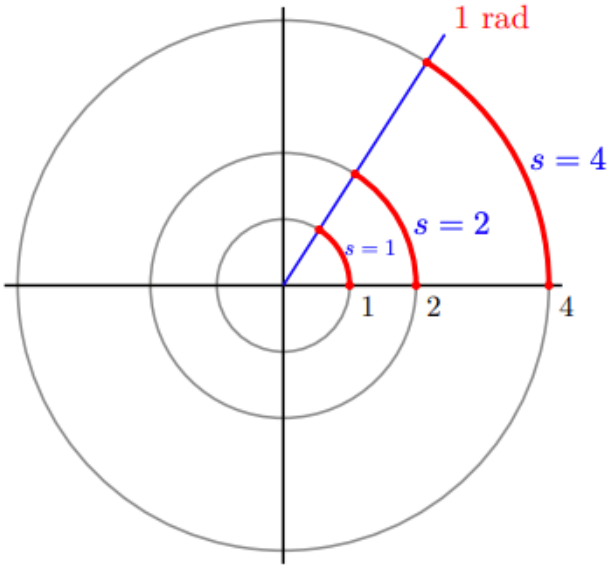


Figure 4.2.15

Interactive Element: What Is a Radian? Measuring an Arc in Radius-Lengths

The following Interactive Element might help you visualize how a radian counts the number of radius-lengths of arc a central angle sweeps out.

Interact: Drag the orange point around the circle to open up the central angle.

Observation: The angle's radian measure equals the arc length measured in radii. At exactly 1 radian the arc is one radius long, and each purple mark adds one more radius of arc.

So, for instance, we can use the formula to find the arc length spanned by an angle of 2.5 radians on a circle of radius six inches:

$$s = r\theta = (6\,\text{inches})\,(2.5) = 15\ \text{inches}.$$

Note that we didn't use the word "radians" when replacing θ with 2.5. This is because, as we mentioned previously, radians is a unitless measure. Therefore, when performing **unit analysis** (which is the act of including units when performing computations to understand the meaning of the result of a series of computations), we leave the word "radians" off the angle.

We can also use the formula in the form $\theta = \frac{s}{r}$ to find an angle that spans a given arc. For example, an arc length equal to one radius determines a central angle of one radian, or approximately $57.3°$.

In the following example, we use the arc length formula to compute a change in latitude on the Earth's surface.

Example 4.2.6

The Earth's radius is roughly 3,960 miles. If you travel 500 miles due north, how many degrees of latitude will you traverse? (Latitude is measured in degrees north or south of the equator.)

Solution available at the end of this section.

Checkpoint 4.2.6

The distance around the face of a large clock from 2 to 3 is five feet. What is the radius of the clock?

Answer available at the end of this section.

Example 4.2.7

The London Eye, located on the South Bank of the River Thames in London, is Europe's tallest Ferris wheel. In fact, when it first opened in 2000, it was the *world*'s tallest Ferris wheel. It has a diameter of 120 meters. Interestingly, the London Eye rotates slowly enough that it never stops—riders easily step on and off at the wheel's bottom platform while the wheel is in motion.



Figure 4.2.17

Loi boards the London Eye to take in an elevated city view. If Loi has moved 80 meters along the circumference of the wheel, how many degrees has the London Eye rotated?

Solution available at the end of this section.

Checkpoint 4.2.7

A central angle in a circle of radius 6 cm cuts off an arc of length 24 cm. What is the radian measure of the angle? What is the degree measure?

Answer available at the end of this section.

Trigonometric Functions of Angles in Radians

Measuring angles in radians has other applications besides calculating arc length. One such use is evaluating trigonometric functions of angles in radians. Many applications in Trigonometry (and beyond) use radians for input values rather than degrees. For this reason, it is crucial to know the trigonometric values for the special angles, which you learned in degrees previously, when the angles are given in radians. Fortunately, the value of a trigonometric function is the same whether the angle is measured in radians or degrees.

For example, $\frac{\pi}{3}$ radians is the same as 60°, because $\frac{\pi}{3} \cdot \frac{180^\circ}{\pi} = 60^\circ$, so

$$\sin\left(\frac{\pi}{3}\right) = \sin\left(60^\circ\right) = \frac{\sqrt{3}}{2}.$$

Table 4.2.2 shows the radian measures for the special angles and the quadrantal angles, as well as the exact values of the sine, cosine, and tangent at these angles.

Table 4.2.2

θ in Degrees	θ in Radians	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	undefined
180°	π	0	-1	0
270°	$\frac{3\pi}{2}$	-1	0	undefined
360°	2π	0	1	0

You should memorize these function values and be able to use them to find trigonometric values for the special angles in all four quadrants, as they will appear frequently throughout the rest of Trigonometry.

Example 4.2.8

Compute the value of $\csc\left(\frac{\pi}{3}\right)$.

Solution available at the end of this section.

Checkpoint 4.2.8

Compute the value of $\cot\left(\frac{\pi}{4}\right)$.

Answer available at the end of this section.

As before, we use reference angles to define the trigonometric functions for angles greater than 90°.

Example 4.2.9

Give exact values for the following.

a. $\tan\left(\frac{2\pi}{3}\right)$
b. $\cos\left(\frac{5\pi}{4}\right)$

Solutions available at the end of this section.

Checkpoint 4.2.9

Give exact values for the following.

a. $\sin\left(\frac{5\pi}{6}\right)$

b. $\tan\left(\frac{7\pi}{4}\right)$

Answers available at the end of this section.

Success in College: Dealing With Failure (Part 2)

A student gets a test back from the professor. The test score is 55%. The professor has written at the top of the exam, "Make an appointment to talk to me about this."

1. Why do you think that the professor wants to talk to the student?
2. Making an appointment to talk to your professor is usually an option in college. You can choose not to do this. Describe the reasons someone may use for making the appointment. Describe the reasons someone may use for not making the appointment.

Success in College: Relying On Others and Fooling Yourself

It is a common mistake that students make:

- You do homework with your friend, and she constantly is the one with the correct answers. Instead of talking about the mistakes, you copy the answer and move on.
- Your classmates have formed a Discord, and you copy the solutions from one of them.
- You only have a couple hours before the homework is due and you use your preferred A.I. model to do most of the work, copying frantically to submit the work before it's too late.

Is this helping you learn? Do you truly think it is helping your grade?

If you answered yes, you're in the majority of students who have been fooled by this method of "get the right answer by any means and move on."

> When I was a student at UC Davis, I concentrated on understanding every concept that came my way. I studied with other students, but more often than not, they relied on my answers and explanations to succeed in the course. The problem is that they were just "parroting" my methods and not understanding them. In the end, most of them did poorly due to their lack of understanding of the material.
>
> - Professor Simpson, Cosumnes River College

This sort of "copying" also leads to another damaging process - a lack of self-confidence. Because they never solve anything by themselves, the students who copy other students' work end up freaking out more before exams and, as a result of their heightened state of panic and their fundamental lack of understanding of the material, they fail the exams.

Think about what kind of student you are, and, without fear of judgement, consider the following questions:

1. A student was having trouble with his homework. For the next few weeks, a friend helped him do his homework. The friend answered his questions and told him what to do when he was stuck on a problem. Why do you think this student may have trouble doing well on a test?
2. A student can do about 75% of the problems on most of her homework assignments without help. She thinks that this is good enough. She thinks she should be able to score about 75% on a test and pass the test. Do you think she is right?

Answers

Section Preview

This section introduces radian measure and, as a natural consequence of the definition, arc length. All of the previous work we did evaluating trigonometric functions (including using reference angles and quadrants) is quickly repeated using radian measure.

Section Preview

- **Geometry**
  - Circles — Properties, Area, and Circumference: The entire section is built on the circle—its circumference, radius, and the proportional relationship between a fraction of a revolution and the arc it subtends. Students must already know the circumference formula $C = 2\pi r$ before this derivation makes sense.

Section Preview

- **Arithmetic**
  - Adding and Subtracting Fractions: Working with reference angles in radian measure of the form $\frac{n\pi}{d}$ requires performing arithmetic on fractions (e.g., $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$).

Section Preview

- Find the radian measure of the central angle given the radius and arc length.
- Convert an angle from degrees to radians or vice versa.
- Evaluate a trigonometric function using radians.
- Identify the reference angle for a given angle measured in radians.
- Solve an application involving arc length.

Example 4.2.1

Because $\frac{120^\circ}{360^\circ} = \frac{1}{3}$, an angle of 120° is $\frac{1}{3}$ of a complete revolution, as shown below.

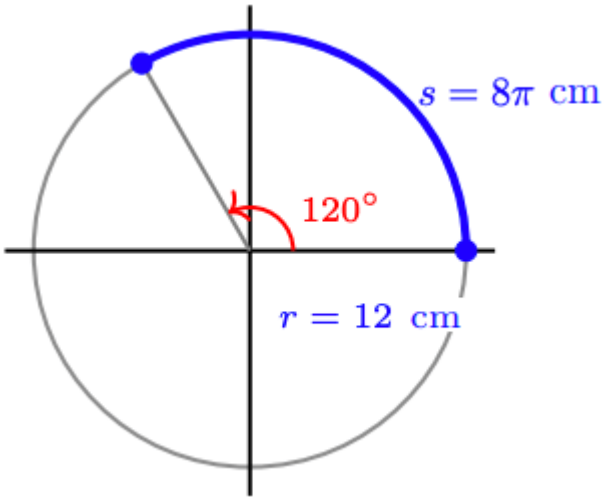


Figure 4.2.6

Using the arc length formula with $r = 12$, we find that

$$s = \frac{1}{3}(2\pi \cdot 12) = \frac{2\pi}{3} \cdot 12 = 8\pi \text{cm}$$

or about 25.1 cm.

Checkpoint 4.2.1

Approximately 261.8 ft

Example 4.2.2

An angle of 120° is $\frac{120^\circ}{360^\circ} = \frac{1}{3}$ of a complete revolution, as we saw in the previous example. Thus, an angle of 120° has a radian measure of $\frac{1}{3}(2\pi)$, or $\frac{2\pi}{3}$.

Checkpoint 4.2.2

Half a revolution, 180°

Example 4.2.3

Look at Figure 4.2.5. The second quadrant includes angles between $\frac{\pi}{2}$ and π, or approximately 1.57 and 3.14 radians, so 2 radians lies in the second quadrant. An angle of 5 radians is between 4.71 and 6.28, or between $\frac{3\pi}{2}$ and 2π radians, so it lies in the fourth quadrant.

Checkpoint 4.2.3

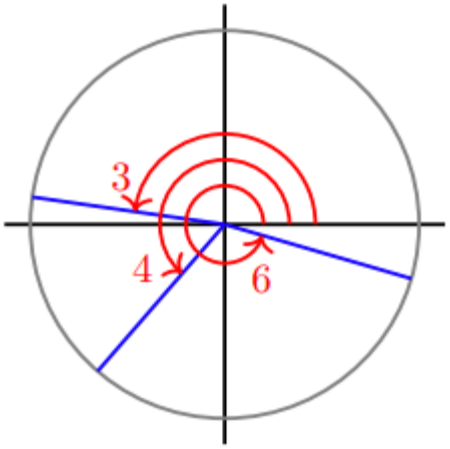


Figure 4.2.9

Example 4.2.4

a.

$$\begin{aligned} 3 \text{ radians} &= 3 \text{ radians} \times (1) \\ &= 3 \text{ radians} \times \left(\frac{180^\circ}{\pi \text{ radians}} \right) \\ &= \frac{540^\circ}{\pi} \\ &\approx 171.9^\circ \end{aligned}$$

b.

$$\begin{aligned} 3^\circ &= 3^\circ \times (1) \\ &= 3^\circ \times \left(\frac{\pi \text{ radians}}{180^\circ} \right) \\ &= \frac{\pi}{60} \text{ radians} \\ &\approx 0.05 \text{ radians} \end{aligned}$$

Checkpoint 4.2.4

a. $\frac{\pi}{3} \approx 1.047$
b. 135°

Example 4.2.5

$\beta = \frac{5\pi}{3}$ is almost all the way to $\frac{6\pi}{3} = 2\pi$, but we have to back off by $\frac{\pi}{3}$. Hence, $\frac{5\pi}{3} \in$ QIV and $\hat{\beta} = \frac{\pi}{3}$ (see Figure 4.2.14).

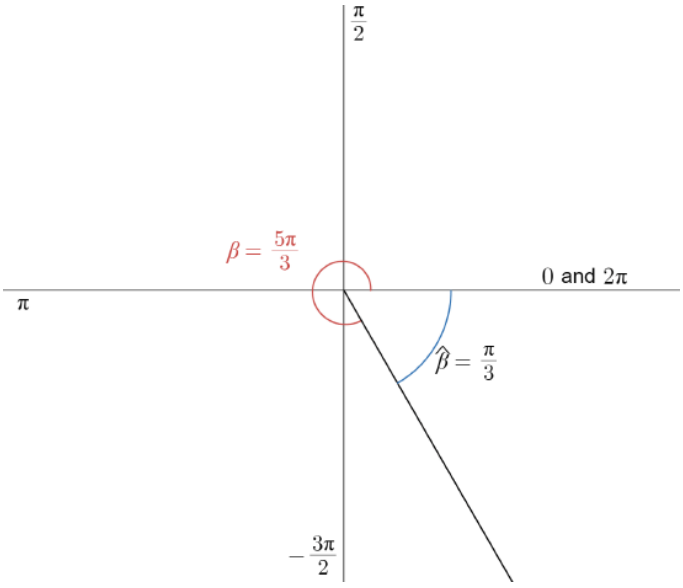


Figure 4.2.14

Checkpoint 4.2.5

$\alpha \in$ QII and $\hat{\alpha} = \frac{\pi}{16}$

Example 4.2.6

We think of the distance of 500 miles as an arc length on the surface of the Earth, as shown below. Substituting $s = 500$ and $r = 3960$ into the arc length formula gives

$$\begin{aligned} 500\,\text{miles} &= 3960\,\text{miles} \times \theta \\ \implies \frac{500\ \cancel{\text{miles}}}{3960\ \cancel{\text{miles}}} &= \theta \\ \implies \theta &\approx 0.1263 \text{ radians} \end{aligned}$$

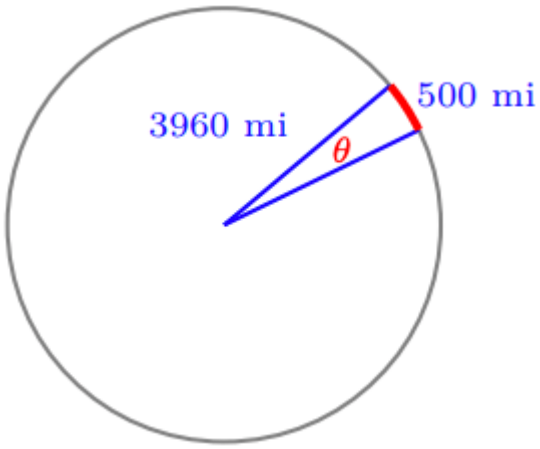


Figure 4.2.16

To convert the angle measure to degrees, we multiply by the conversion factor $\frac{180^\circ}{\pi}$ to get

$$0.1263 \left(\frac{180^\circ}{\pi}\right) \approx 7.23^\circ.$$

Your latitude has changed by about 7.23°.

Checkpoint 4.2.6

9.55 feet

Example 4.2.7

Parsing this question into language associated with our recent material, we have a circle of diameter 120 meters, which means the radius of the circle is 60 meters. Loi is moving through an arc of length 80 meters along the edge of this circle. Our task is to find the central angle that subtends this arc. In the formula

$$s = r\theta,$$

$s = 80$ meters and $r = 60$ meters. Therefore,

$$80\,\text{meters} = (60\,\text{meters})\,\theta \implies \frac{80\ \cancel{\text{meters}}}{60\ \cancel{\text{meters}}} = \frac{4}{3} = \theta.$$

However, this angle is in *radians*. We must convert this to degrees.

$$\frac{4}{3} \times \frac{180^\circ}{\pi} \approx 76.4^\circ.$$

Hence, Loi moves through an approximate angle of 76.4°.

Checkpoint 4.2.7

4 radians $\approx 229^\circ$

Example 4.2.8

Our identities still hold—working in radians does not change that fact. Therefore, by the Reciprocal Identities,

$$\csc\left(\frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}.$$

Checkpoint 4.2.8

1

Example 4.2.9

a. The reference angle for $\theta = \frac{2\pi}{3}$ is $\hat{\theta} = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$, and the tangent is negative in the second quadrant. (See the figure below.) Thus, $\tan\left(\frac{2\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}$.

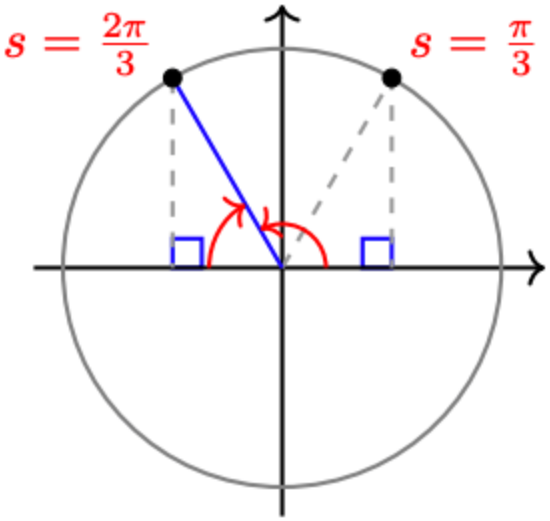


Figure 4.2.18

b. The reference angle for $\theta = \frac{5\pi}{4}$ is $\hat{\theta} = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$, and the cosine is negative in the third quadrant, so $\cos\left(\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

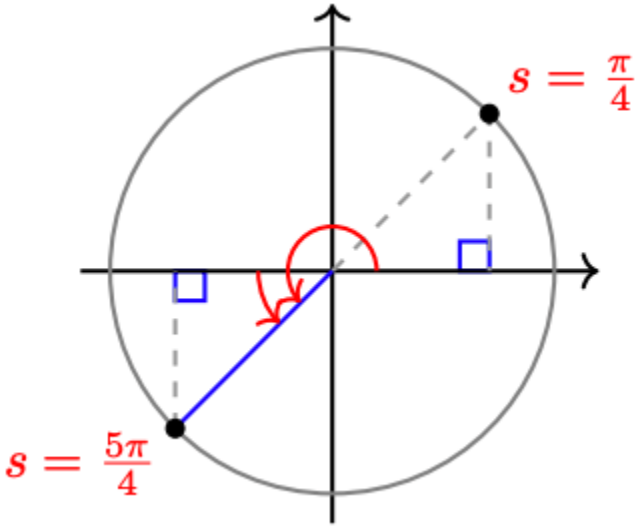


Figure 4.2.19

Checkpoint 4.2.9

a. $\frac{1}{2}$
b. -1

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4.2.1: Resources and Key Concepts

Resources

Videos

- Radian Measure
  - Angles - Angle Measure: Radians
- Arc Length
  - Angles - Arc Length
  - Angle Measure - Arc Length
- Degree to Radian Conversion
  - Angles - Relating Degrees to Radians
- Reference Angles in Radians
  - Angles - Reference Angles

Key Concepts

Definitions

- **Arc and Arc Length:** An arc is the portion of the circumference of a circle between two points on its edge. The arc length is the distance along this curved portion.
- **Central Angle:** An angle whose vertex is at the center of a circle.
- **Subtend:** A central angle is said to subtend (or span) an arc if its sides pass through the endpoints of the arc.
- **Radian Measure:** The radian measure of an angle is defined as the product of its fraction of a complete revolution and 2π.

Theorems

- **Angle Measure Conversion Theorem:** The relationship between radians and degrees can be used to create conversion factors: $1 = \frac{180^\circ}{\pi \text{ radians}}$ and $1 = \frac{\pi \text{ radians}}{180^\circ}$.
- **Arc Length Formula:** On a circle of radius r, the length s of an arc spanned by a central angle θ is given by $s = r\theta$, provided that θ is measured in radians.

Common Mistakes

- **The Radian Requirement:** The arc length formula $s = r\theta$ is only valid if the angle θ is measured in radians. Using an angle in degrees will produce an incorrect result.
- **Confusing Radians and Degrees:** Be mindful of the units for a given angle. An angle of 1 radian is approximately 57.3°, whereas an angle of 1° is very small. Do not use the two interchangeably.
- **Thinking Radians Must Include π:** While many common angles in radians are expressed as multiples of π (e.g., $\frac{\pi}{4}$), an angle can be measured in radians without π (e.g., 2 radians).

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4.2.2: Homework

Reading Questions

1. What is the difference between an arc and an arc length?
2. What defines a central angle?
3. How is the radian measure of an angle defined based on a fraction of a revolution?
4. How many radians are in one full revolution (360°)?
5. What is the formula for arc length, s, if the central angle θ is measured in radians and the radius is r?
6. What is the most important requirement for the angle θ when using the arc length formula $s = r\theta$?
7. What is the conversion factor used to convert an angle from degrees to radians?
8. What is the conversion factor used to convert an angle from radians to degrees?
9. In which quadrant does an angle of 5 radians terminate?
10. What is the reference angle for $\theta = \frac{5\pi}{3}$?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, use the appropriate conversion factor to convert units.

1. $\frac{1 \text{ mile}}{1.609 \text{ kilometers}} = 1$
   a. 10 miles = _____ km
   b. 50 km = _____ miles
2. $\frac{1 \text{ acre}}{0.405 \text{ hectare}} = 1$
   a. 40 acres = _____ hectares
   b. 5 hectares = _____ acres
3. $\frac{1 \text{ horsepower}}{746 \text{ watts}} = 1$
   a. 250 horsepower = _____ watts
   b. 1000 watts = _____ horsepower
4. $\frac{1 \text{ troy ounce}}{480 \text{ grains}} = 1$
   a. 0.5 troy oz = _____ grains
   b. 100 grains = _____ troy oz

For the following exercises, determine how many degrees are in each fraction of one complete revolution.

5. $\frac{1}{4}$

6. $\frac{1}{5}$
7. $\frac{1}{6}$
8. $\frac{1}{8}$

Answers available at the end of this section.

Homework

Vocabulary Check

1. Unlike degree measure, radian measure is a ___ measure (i.e., radian measure does not require a measurement symbol).
2. An angle with vertex at the center of the circle is called a(n) ___.
3. A central angle whose sides meet the endpoints of an arc is said to ___ the arc. Another way to say this is the angle ___ the arc.
4. The ___ length is the distance along the edge of a circle subtended by a central angle θ.
5. The product of 2π and the fraction of one revolution cut out by the angle θ is named the ___ measure of the angle.

Concept Check

6. The length of a circular arc depends on what two variables?
7. Define the radian measure of an angle.
8. What is the conversion factor from radians to degrees?
9. On a unit circle, the length of an arc is equal to what other quantity?

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

10. The distance we travel around a circle of diameter d is proportional to the angle of displacement. In fact,
$$\text{arc length } = (\text{fraction of one revolution}) \cdot (\pi d)$$
11. We measure angles in radians when we work with arc length.
12. An arc length equal to one radius determines a central angle of one degree.
13. An angle given in radian measure must be written as $k\pi$, where k is a real number.
14. $\frac{180^\circ}{\pi \text{ radians}} = 1$
15. On a circle of radius r, the length s of an arc spanned by an angle θ is $s = r\theta$.

Basic Skills

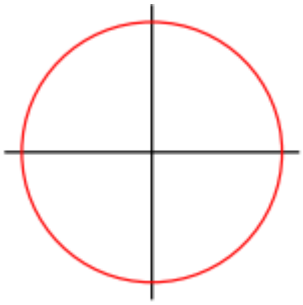
16.

Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

Degrees									

a. Convert each angle to degrees.

b. Sketch each angle on a circle like this, and label in radians.

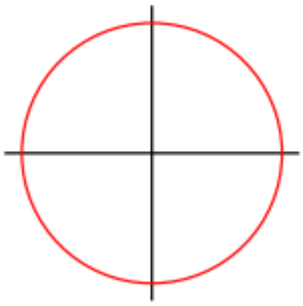


17.

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
Degrees													

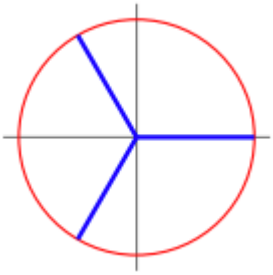
a. Convert each angle to degrees.

b. Sketch each angle on a circle like this, and label in radians.



For the following exercises, express each fraction of one complete rotation in degrees and radians.

18.



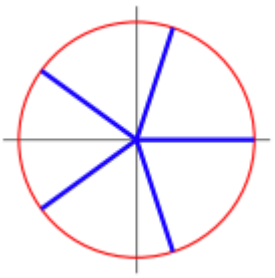
a. $\frac{1}{3}$

b. $\frac{2}{3}$

c. $\frac{4}{3}$

d. $\frac{5}{3}$

19.



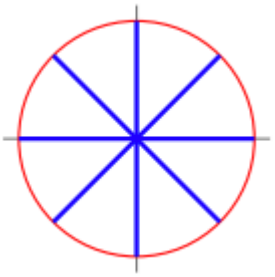
a. $\frac{1}{5}$

b. $\frac{2}{5}$

c. $\frac{3}{5}$

d. $\frac{4}{5}$

20.



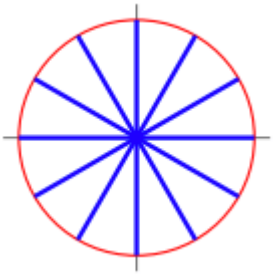
a. $\frac{1}{8}$

b. $\frac{3}{8}$

c. $\frac{5}{8}$

d. $\frac{7}{8}$

21.



a. $\frac{1}{12}$

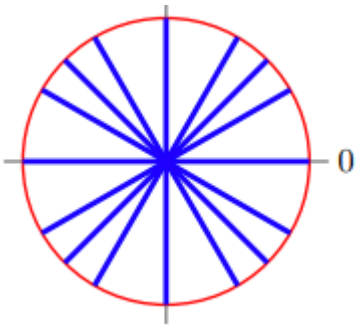
b. $\frac{1}{6}$

c. $\frac{5}{12}$

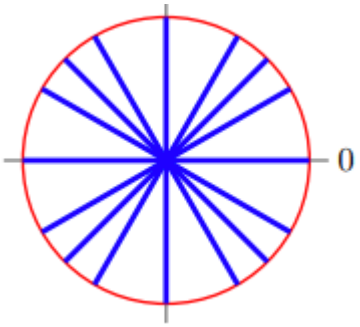
d. $\frac{5}{6}$

For the following exercises, label each angle in standard position with radian measure (assume the angles shown are special angles).

22. Rotate counter-clockwise from 0.



23. Rotate clockwise from 0.



For the following exercises, give a decimal approximation to hundredths for each angle in radians.

24. a. $\frac{\pi}{6}$

b. $\frac{5\pi}{6}$

c. $\frac{7\pi}{6}$

d. $\frac{11\pi}{6}$

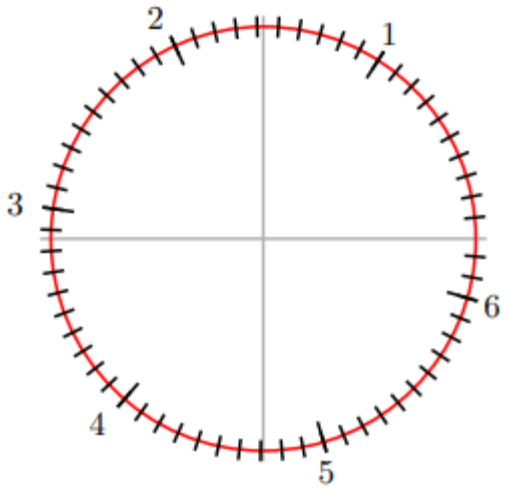
25. a. $\frac{\pi}{4}$

b. $\frac{\pi}{4}$

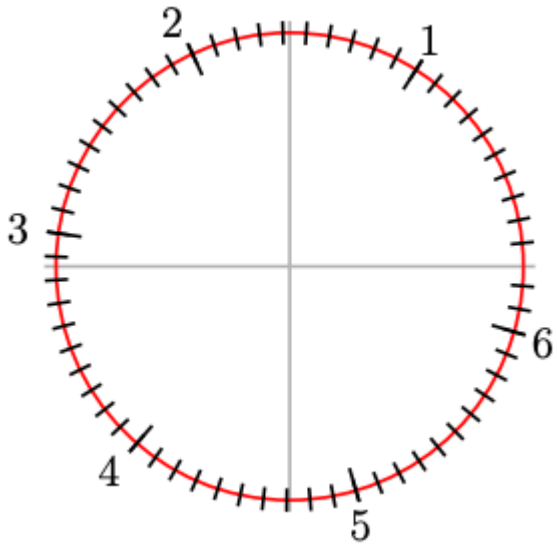
c. $\frac{5\pi}{4}$

d. $\frac{7\pi}{4}$

26. Locate and label each angle from Problem 24 on the unit circle below. (The circle is marked off in tenths of a radian.)



27. Locate and label each angle from Problem 25 on the unit circle below. (The circle is marked off in tenths of a radian.)



For the following exercises, choose the best decimal approximation for each angle in radians from the list below. Do not use a calculator; use the fact that π is a little greater than 3.

$$0.52, 0.79, 2.09, 2.36, 2.62, 3.67, 5.24, 5.50$$

28. $\frac{2\pi}{3}$

29. $\frac{\pi}{4}$

30. $\frac{5\pi}{6}$

31. $\frac{5\pi}{3}$

32. $\frac{\pi}{6}$

33. $\frac{7\pi}{4}$

34. $\frac{3\pi}{4}$

35. $\frac{7\pi}{6}$

For the following exercises, state in which quadrant each angle lies.

36. a. $\frac{\pi}{4}$

    b. $\frac{3\pi}{4}$

    c. $\frac{5\pi}{4}$

    d. $\frac{7\pi}{4}$

37. a. $\frac{\pi}{6}$

    b. $\frac{5\pi}{6}$

    c. $\frac{7\pi}{6}$

    d. $\frac{11\pi}{6}$

38. a. $\frac{\pi}{3}$

    b. $\frac{2\pi}{3}$

    c. $\frac{4\pi}{3}$

    d. $\frac{5\pi}{3}$

39. a. $\frac{\pi}{5}$

    b. $\frac{8\pi}{7}$

    c. $\frac{25\pi}{13}$

    d. $\frac{47\pi}{48}$

For the following exercises, complete the table.

40. (MyOpenMath)

Degrees	18°	60°	90°
Radians			

41. (MyOpenMath)

Radians	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
Degrees			

12. (MyOpenMath)

Radians	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
Degrees			

13.

Radians	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
Degrees			

For the following exercises, convert to radians. Write the answer in exact form and then give approximations rounded to the nearest hundredths.

14. (MyOpenMath)

    a. 120°

    b. 420°

    c. -180°

    d. -225°

15. (MyOpenMath)

    a. 75°

    b. 236°

    c. -327°

    d. $143^\circ\ 40'$

16. a. 138°

    b. 194°

    c. 342°

    d. $281^\circ\ 15'$

For the following exercises, convert to degrees. Round answers to the nearest tenth.

16. (MyOpenMath)

    a. $-\frac{11\pi}{15}$

    b. $\frac{2\pi}{7}$

    c. $-\frac{6\pi}{5}$

17. a. 0.8

    b. 3.5

    c. 5.1

18. a. 1.1

b. 2.6

c. 4.6

For the following exercises, state the quadrant in which the angle terminates and give the reference angle (in radians).

48. $\frac{\pi}{3}$

49. (MyOpenMath) $\frac{3\pi}{4}$

50. $\frac{9\pi}{8}$

51. $\frac{21\pi}{22}$

52. (MyOpenMath) $-\frac{5\pi}{6}$

53. (MyOpenMath) $-\frac{5\pi}{3}$

54. (MyOpenMath) $\frac{11\pi}{4}$

55. $-\frac{15\pi}{7}$

56. 2.4

57. -3.1

For the following exercises, find an angle in each quadrant, rounded to tenths, with the same reference angle as the angle given in radians.

58. 5.8

59. 2.9

60. 3.7

61. 5.1

62. 1.8

63. 4.4

For the following exercises, use the arc length formula to answer the questions. Round answers to hundredths.

64. (MyOpenMath) Find the arc length spanned by an angle of 2 radians on a circle of radius 2 miles.

65. Find the arc length spanned by an angle of 80° on a circle of radius 4 inches.

66. Find the arc length spanned by an angle of 200° on a circle of radius 18 feet.

67. Find the radius of a circle if an angle of 250° spans an arc length of 18 meters.

68. Find the radius of a circle if an angle of 20° spans an arc length of 0.5 kilometers.

69. (MyOpenMath) Find the angle, in radians, subtended by an arc length of 7 feet on a circle of radius 10 feet.

70. (MyOpenMath) Find the angle, in degrees, subtended by an arc length of 7 feet on a circle of radius 6 feet.

71. Find the angle, in radians, subtended by an arc length of 28 centimeters on a circle of diameter 20 centimeters.

72. Find the angle, in radians, subtended by an arc length of 1.6 yards on a circle of diameter 2 yards.

For the following exercises, find the exact value of the trigonometric function. Do not use a calculator!

70. $\sin\left(\frac{\pi}{3}\right)$

71. $\cos\left(\frac{\pi}{6}\right)$

72. $\tan\left(\frac{\pi}{4}\right)$

73. $\csc\left(\frac{\pi}{4}\right)$

74. $2\sec\left(\frac{\pi}{3}\right)$

75. $\cot\left(\frac{\pi}{6}\right)$

76. $-\sin\left(\frac{5\pi}{6}\right)$

77. $\cos\left(\frac{4\pi}{3}\right)$

78. $\tan\left(\frac{11\pi}{6}\right)$

79. $3\csc\left(-\frac{4\pi}{3}\right)$

80. $\sec\left(-\frac{3\pi}{4}\right)$

81. $-\pi\cot\left(-\frac{13\pi}{6}\right)$

82. $\sin\left(\frac{\pi}{2}\right)$

83. $\cos(\pi)$

84. $\tan\left(\frac{3\pi}{2}\right)$

Synthesis Questions

For the following exercises, find an angle between 0 and 2π that is coterminal with the given angle.

85. (MyOpenMath) $\frac{29\pi}{10}$

For the following exercises, find the exact value of the expression for $x = \frac{\pi}{6}$.

85. $\sin(2x)$

86. $\cos(3x)$

87. $\frac{1}{3}\sin(5x)$

88. $6 - \sin(x)$

89. $\frac{2}{3} + \tan\left(\frac{3}{2}x\right)$

90. $\sin(\pi - x)$

91. $\cos(\pi + 2x)$

92. $3\sin\left(2x + \frac{\pi}{3}\right)$

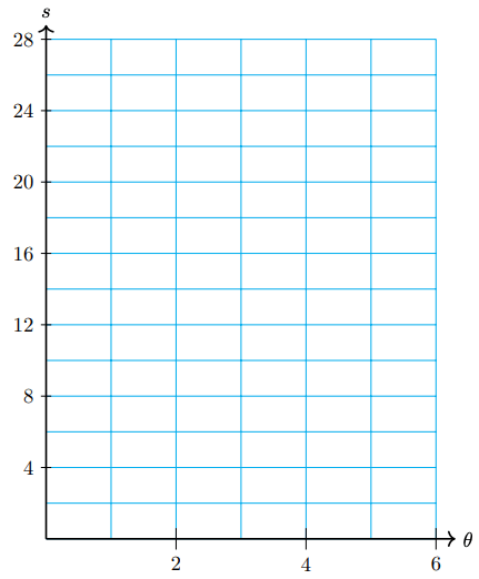
93. $-2 + \frac{3}{4}\cos\left(7x - \frac{\pi}{2}\right)$

94. a. Sketch a circle of radius 4 units, and mark the positions of 1, 2, 3, 4, 5, and 6 radians on the circle.

b. On a circle of radius 4 feet, find the arc length determined by each angle in radians.

θ	1	2	3	4	5	6
s						

c. Graph θ against s. What is the slope of the graph?



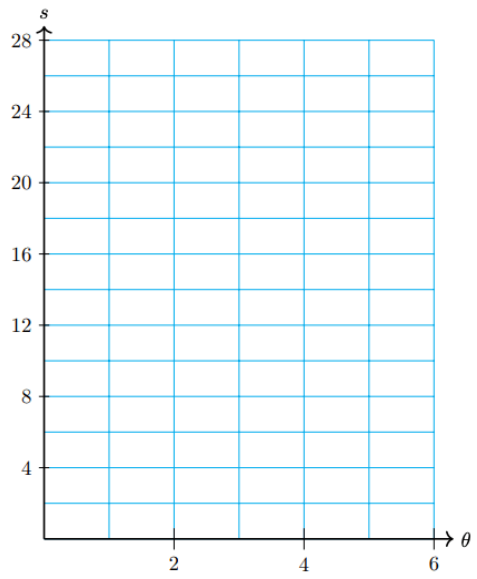
d. If you double the angle, what happens to the arc length? What happens if you triple θ?

95. a. Sketch several concentric circles with increasing radii, and draw an angle of 2 radians through all of them.

b. Find the arc length determined by an angle of 2 radians on circles of given radius.

r	1	2	3	4	5	6
s						

c. Graph s against r. What is the slope of the graph?



d. If you double the radius r, what happens to the arc length? What happens if you triple r?

96. Recall that to calculate a fraction of a revolution in degrees, we divide the angle by $360°$. For example, $90°$ is $\frac{1}{4}$ of a revolution because $\frac{90°}{360°} = \frac{1}{4}$.

a. Write an expression that gives the fraction of a revolution for an angle θ in radians.

b. Use your expression to calculate what fraction of a revolution is represented by each of the following angles: $\theta = \frac{3\pi}{4}$, $\theta = \frac{5\pi}{3}$, $\theta = \frac{7\pi}{6}$.

Applications

For the following exercises, use the arc length formula to answer the questions.

97. **Big Ben.**

a. Through how many radians does the minute hand of a clock sweep between 9:05 pm and 9:30 pm?

b. The dial of Big Ben's clock in London is 23 feet in diameter. How long is the arc traced by the minute hand between $9:05\text{pm}$ and $9:30\text{pm}$?

98. **Floral Clock.** The largest clock ever constructed was the Floral Clock in the garden of the 1904 World's Fair in St. Louis. The hour hand was 50 feet long, the minute hand was 75 feet long, and the radius of the clock's face was 112 feet.

a. If you started at the 12 and walked 500 feet clockwise around the clock face, through how many radians would you walk?

b. If you started your walk at noon, how long would it take the minute hand to reach your position? How far did the tip of the minute hand move in its arc?

99. **The Foucault Pendulum.** In 1851 Jean-Bernard Foucault demonstrated the rotation of the earth with a pendulum installed in the Pantheon in Paris. Foucault's pendulum consisted of a cannonball suspended on a 67 meter wire, and it swept out an arc of 8 meters on each swing. Through what angle did the pendulum swing? Give your answer in radians and then in degrees, rounded to the nearest hundredth.

100. **Linear Distance.** A wheel with radius 40 centimeters is rolled a distance of 1000 centimeters on a flat surface. Through what angle has the wheel rotated? Give your answer in radians and then in degrees, rounded to one decimal place.

101. **Distance Between Cities.** Cosumnes River College (Sacramento, California) and Stony Brook University (Stony Brook, New York) are 2,544 miles apart along the surface of the Earth. Assuming that the radius of the Earth is 3,963 miles, find the

radian measure of the central angle having vertex at the center of the Earth that has Cosumnes River College on one side and Stony Brook University on the other.

92. **Distance on the Surface of the Earth (MyOpenMath).** Find the distance along an arc on the surface of the earth that subtends a central angle of 10 minutes (recall, 1 minute = 1/60 degree). For this exercise, assume the radius of the earth is 3960 miles.

Answers

Skills Refresher

1. a. 16.09 km
   b. 31.08 mi
2. a. 16.2 hectares
   b. 12.35 acres
3. a. 186,500 watts
   b. 1.34 horsepower
4. a. 240 grains
   b. 0.21 troy oz
5. 90°
6. 72°
7. 60°
8. 45°

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4.3: The Trigonometric Functions - Unit Circle Definition

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We now venture into the third and final definition of the trigonometric functions. Many students have already encountered this definition from their high school Pre-Calculus course; however, there are three reasons I have held off on introducing the trigonometric functions via this definition until now:

1. In my experience, learning the unit circle definition of the trigonometric functions before the right triangle definition leads to more memorization and a less accurate understanding of the trigonometric functions.
2. Unit circle trigonometry is theoretically more challenging to grasp. It has two interpretations—one in terms of angles and the other in terms of arc lengths (and, as such, most students who learn it before right triangle trigonometry tend to have a weak grasp of the subject).
3. Unit circle trigonometry relies on radian measure.

Connecting the Sine and Cosine Functions to the Unit Circle

The left image in Figure $4.3.1$ below shows an angle of 30° in standard position in a circle of radius 2. The hypotenuse of its reference triangle is the radius of the circle, so the legs of the triangle have lengths 1 and $\sqrt{3}$. The coordinates of the point P where the terminal side meets the circle are thus $(\sqrt{3}, 1)$.

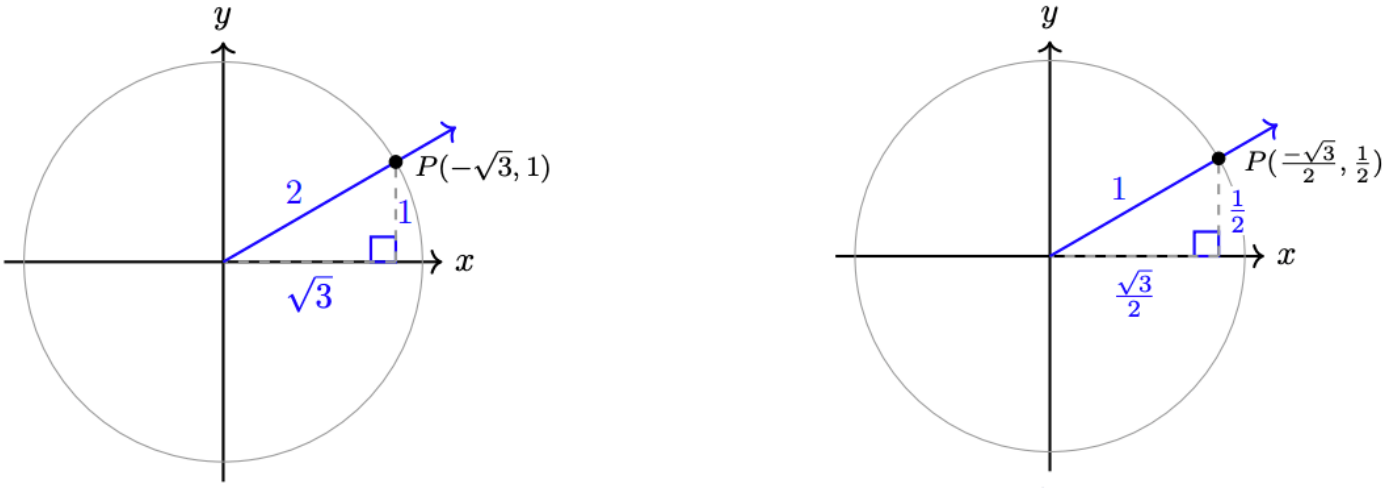


Figure $4.3.1$

The right image in Figure $4.3.1$ is a unit circle. In this figure, the hypotenuse of the reference triangle for 30° has length 1. What are the coordinates of the point P where the terminal side meets the circle? Each side of this triangle is half the length of the sides of the similar triangle in the left image, so the coordinates of P are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

You should recognize the coordinates of the point P as $(\cos(30°), \sin(30°))$. In fact, from the coordinate definition of the trigonometric functions, we should not be surprised to see that, given the point (x, y) on the unit circle on the terminal side of the angle θ,

$$\cos(\theta) = \frac{x}{r} = \frac{x}{1} = x \quad \text{and} \quad \sin(\theta) = \frac{y}{r} = \frac{y}{1} = y$$

We have discovered an important property of unit circles.

Lemma

Let θ be an angle in standard position, and suppose the terminal side of θ intersects the unit circle at the point $P(x, y)$. Then

$$\cos(\theta) = x \quad \text{and} \quad \sin(\theta) = y.$$

While this lemma is not entirely profound, it will lead to a more significant concept. Before we get there, let's use the lemma in an example.

Example 4.3.1

Find the coordinates of point P on the unit circle below.

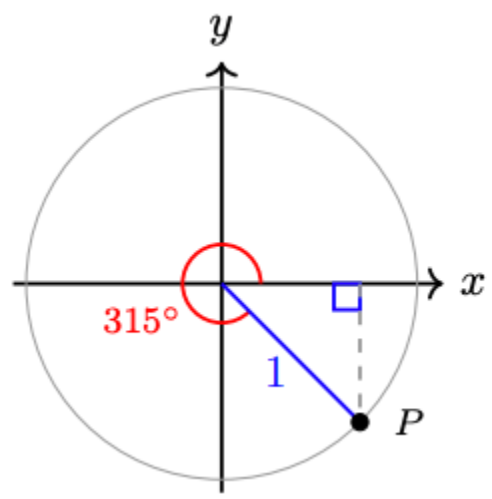


Figure 4.3.2

Solution available at the end of this section.

Checkpoint 4.3.1

Find the sine, cosine, and tangent of the angle φ shown below.

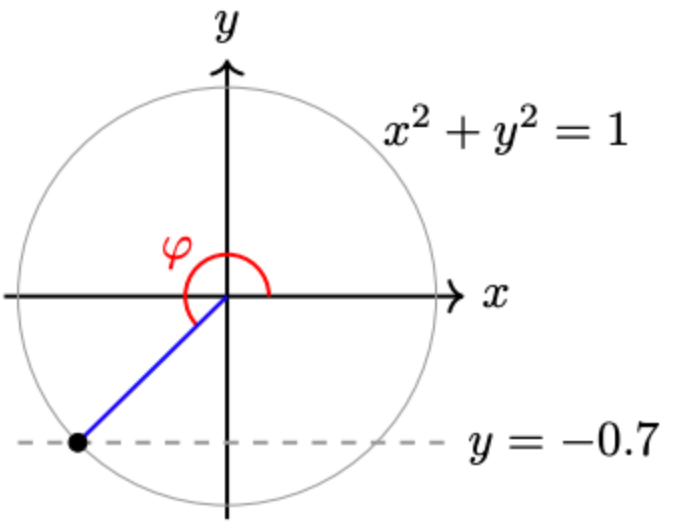


Figure 4.3.3

Answer available at the end of this section.

A Reminder of Radian Measure and Arc Length

For the remainder of this chapter, we will focus on using radians instead of degrees. This change will greatly increase the utility of the trigonometric functions. To illustrate this, and as a refresher of previously learned material, recall that the radius of a unit circle is 1. Therefore, the arc length formula becomes $s = r\theta = (1)\theta = \theta$. Thus, on a *unit circle*, an arc of length 1 determines a central angle of 1 radian. Moreover, the measure of an angle in radians is equal to the length of the arc it spans. In other words, if we walk along the edge of a circle of radius 1, each time we travel through an arc length equivalent to one radius, we have rotated through an angle of one radian.

Example 4.3.2

You are walking on a trail around a circular pond with a radius of one mile. You have hiked for four miles. Sketch your location now.

Solution available at the end of this section.

Checkpoint 4.3.2

An ant walks around the rim of a circular birdbath of a radius of 1 foot. How far has the ant walked when it has turned through an angle of 210°?

Answer available at the end of this section.

Trigonometric Functions of Real Numbers and the Unit Circle Definition

We now venture into a more theoretical area of Trigonometry. For this discussion, we will denote the arc length of a circle subtended by the angle θ with the letter t instead of s (for no better reason than tradition).

Consider the unit circle shown in Figure 4.3.5 below, and the angle θ determined by the arc of length t. The radian measure of θ is the same as the length t of the arc. For example, in this figure $\theta = t = 2$.

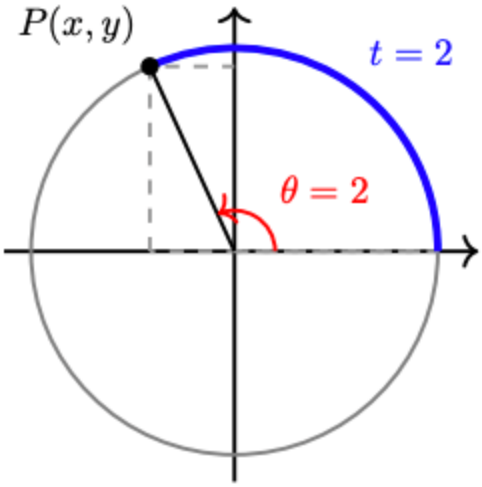


Figure $4.3.5$**:** *Traversing the unit circle*

Therefore, as long as we restrict our conversation to the unit circle, instead of thinking of the sine or cosine as a function of an *angle*, θ, we can change our perspective and think of these as functions of the *length of an arc*, t, along the unit circle. Since the arc length is not an angle but instead a simple real number, we are evaluating the trigonometric function at a real number.

This brings us to our third and final definition of the trigonometric functions.

Definition: Trigonometric Functions (Unit Circle Definition)

Let $P\,(x, y)$ be a point on the unit circle, and let t be the arc length from the point $(1, 0)$ to P along the circumference of the unit circle. The trigonometric functions of the real number t (also called the **circular functions** in this context) are defined as follows:



We call the trigonometric functions of arc length the circular functions because they are defined by the coordinates of points on a unit circle.

We can think of the definition this way: to find the sine or cosine of a real number t, we draw an arc of length t on a unit circle and then find the sine or cosine of the angle θ determined by the arc.

Interactive Element: Tracing an Arc of Length t

The following Interactive Element might help you visualize how the coordinates of the terminal point of an arc of length t are exactly $\cos(t)$ and $\sin(t)$.

Interact: Drag the red point P around the unit circle. The highlighted arc from $(1, 0)$ to P has length t.

Observation: The coordinates of P are always $(\cos(t), \sin(t))$, and the length of the highlighted arc equals the radian value t. Notice that both coordinates stay between -1 and 1, no matter where P is.

Note: The Difference is All in Your Head

In practice, there is no difference between finding the value of a trigonometric function at the number 2 and the value at the angle of 2 radians. In each case, we set the calculator to Radian mode and evaluate, for example,

$$\mathrm{COS}\,2 \approx -0.4161 \quad \text{and} \quad \mathrm{SIN}\,2 \approx 0.9093.$$

So, in some sense, a trigonometric function of a radian is the same as a trigonometric function of a real number. This result is so important that it bears repeating:

> To find the value of a trigonometric function of a real number t, we find the value of the trigonometric function of the angle t radians.

Example 4.3.3

Let $t = \frac{2\pi}{3}$ be the length of the arc along the unit circle from the point $(1, 0)$ to $P(x, y)$. Evaluate each of the following functions.

a. $\sin(t)$
b. $\cot(t)$

Solutions available at the end of this section.

Checkpoint 4.3.3

Evaluate $\sec(t)$, where $t = \frac{5\pi}{4}$ is the arc length along the unit circle from the point $(1, 0)$.

Answer available at the end of this section.

Example 4.3.4

Find the coordinates of the terminal point, P, of an arc of length t starting at $(1, 0)$ on a unit circle.

a. $t = \frac{\pi}{2}$
b. $t = \frac{5\pi}{6}$

Solutions available at the end of this section.

Checkpoint 4.3.4

Find the coordinates of the terminal point, P, of an arc of length t starting at $(1, 0)$ on a unit circle.

a. $t = \pi$
b. $t = \frac{\pi}{3}$

Answers available at the end of this section.

Using a Calculator to Compute the Values of Trigonometric Functions

Before moving forward, let's talk about using your calculator's Radian mode. In the previous section, we evaluated the trigonometric functions at the special radian angles (i.e., multiples of $\pi/6$, $\pi/4$, and $\pi/3$). We now focus on evaluating the trigonometric functions at non-special radian angles (and, therefore, non-special real numbers).

Note: Change to Radian Mode

If you use a scientific or graphing calculator, refer to your instruction manual on setting it to radian mode. The scientific calculator in Desmos is set to degree mode by default, so if you are using that technology, you need to click the "DEG" toggle so that "RAD" is showing; however, the *graphing* calculator in Desmos defaults to radian mode.

Setting the calculator to radian mode allows us to avoid converting from radians to degrees to evaluate trigonometric functions. First, change the calculator setting from Degree mode to Radian mode. Then enter

$$\text{SIN}\,\pi \div 3$$

and the calculator will return 0.8660254038. You can check that this number is a decimal approximation for $\frac{\sqrt{3}}{2}$.

Example 4.3.5

Use your calculator to find the sine and cosine of the following angles in radians. Round your answers to four decimal places.

a. $\theta = \dfrac{7\pi}{4}$
b. $\theta = 3.5$

Solutions available at the end of this section.

Checkpoint 4.3.5

Use your calculator to find the tangents of the following real numbers. Round your answers to four decimal places.

a. $t = \frac{5\pi}{12}$
b. $t = 5.2$

Answers available at the end of this section.

We can now use the trigonometric functions to model periodic behavior as functions of *time* instead of angles. Indeed, we now have the ability to use any variable as the argument of a trigonometric function.

Example 4.3.6

The sunset time in Stockholm, Sweden, on the nth day of the year can be modeled by

$$T = 3.11 \sin(0.017n - 1.38) + 6.03$$

where T is given in hours after noon. Find the sunset time on January 1 (day $n = 1$) and on July 1 (day $n = 182$).

Solution available at the end of this section.

Checkpoint 4.3.6

Variable stars are important in astronomy because they are used to estimate distances. Their magnitude, or brightness, varies periodically and can be modeled by trigonometric functions. The star T Herculis reached its minimum magnitude on December 27, 2004, and t days later its magnitude is approximately

$$M = 10.2 - 2.2 \cos(0.038t)$$

a. What was the magnitude of T Herculis on December 27, 2004?
b. What was the magnitude of T Herculis on December 27, 2006 (730 days later)?

Answers available at the end of this section.

Example 4.3.7

Use the graph of the unit circle shown below to estimate $\cos(2.5)$, $\sin(2.5)$, and $\tan(2.5)$.

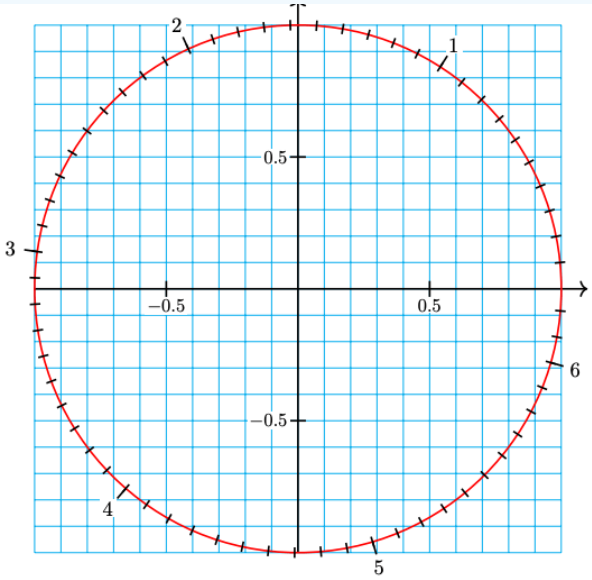


Figure 4.3.8

Solution available at the end of this section.

Checkpoint 4.3.7

Use the graph of the unit circle in the previous example to estimate $\cos(4.2)$, $\sin(4.2)$, and $\tan(4.2)$.

Answer available at the end of this section.

Activities

Activity 4.3.1: Special Angles in Radians

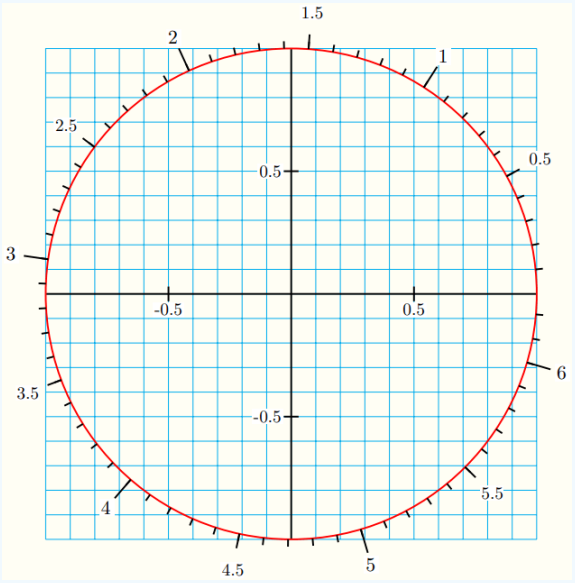


Figure 4.3.9

1. Use the unit circle to estimate the sine, cosine, and tangent of each arc of given length.
   a. 0.6
   b. 2.3
   c. 3.5
   d. 5.3
2. Use the unit circle to estimate two solutions to each equation.
   a. $\cos(t) = 0.3$
   b. $\sin(t) = 0.7$
3. Sketch the angle on the unit circle. Find the reference angle in radians, rounded to two decimal places, and sketch the reference triangle.
   a. 1.8
   b. 5.2
   c. 3.7
4. Give a decimal approximation to two places for each angle, then the degree measure of each.

Radians	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
Decimal Approx.													
Degrees													

Radians	π	$\frac{13\pi}{12}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{17\pi}{12}$	$\frac{3\pi}{2}$	$\frac{19\pi}{12}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	$\frac{23\pi}{12}$	2π
Decimal Approx.													
Degrees													

5. On the unit circle above, plot the endpoint of each arc in standard position.
   a. $\frac{\pi}{3}$
   b. $\frac{7\pi}{6}$
   c. $\frac{7\pi}{4}$

We have already introduced the idea that, given an arc length t along the unit circle, $\cos(t)$ is the x-value of the point where the arc length terminates and $\sin(t)$ is the y-value; however, there are other interesting geometric interpretations of the circular functions of real number. To understand these interpretations better, the following activity should be explored independently or during class.

Activity 4.3.2: Geometric Interpretations of the Circular Functions

Part 1

Figure 4.3.10 shows a unit circle centered at O. The line through AC is tangent to the circle at A. (Recall that a tangent to a circle is perpendicular to the radius that meets it.)

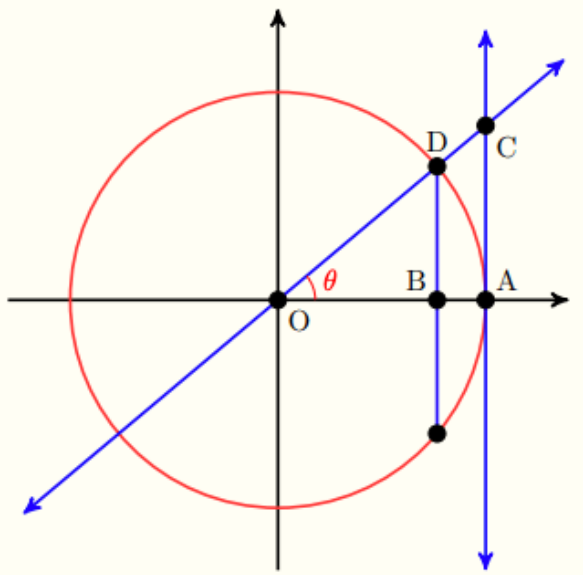


Figure 4.3.10

a. What is the radius of the circle?
b. Find line segments whose lengths are $\cos(\theta)$ and $\sin(\theta)$.
c. Explain why $\triangle OAC$ is similar to $\triangle OBD$.
d. Find a line segment in $\triangle OAC$ whose length is $\tan(\theta)$. Explain why your choice is correct.
e. Find a line segment in $\triangle OAC$ whose length is $\sec(\theta)$. Explain why your choice is correct.

Part 2

Figure 4.3.11 is another unit circle, with tangent BD.

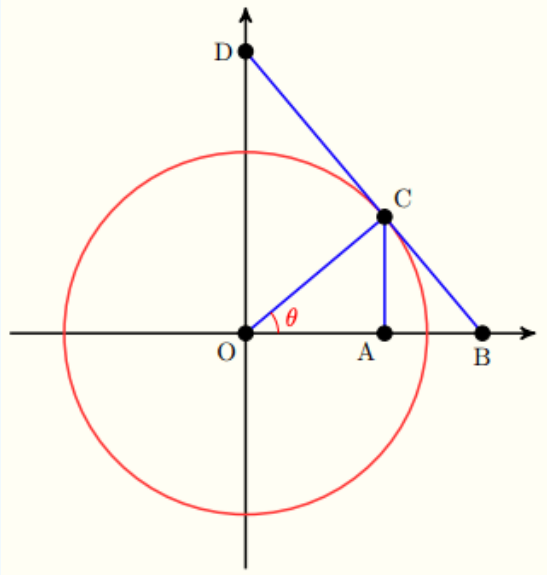


Figure 4.3.11

a. What are the measures of the angles $\angle OCD$ and $\angle OCB$?
b. Explain why $\triangle OCB$ is similar to $\triangle OAC$.
c. Find a line segment in $\triangle OCB$ whose length is $\tan(\theta)$. Explain why your choice is correct.
d. Find a line segment in $\triangle OCB$ whose length is $\sec(\theta)$. Explain why your choice is correct.
e. Explain why $\triangle OCB$ is similar to $\triangle DCO$.
f. Find a line segment in $\triangle DCO$ whose length is $\csc(\theta)$. Explain why your choice is correct.

Domain and Range of the Circular Functions

By interpreting the trigonometric functions in terms of the unit circle definition (as circular functions), we can discuss their domains and ranges.

Interactive Element: Geometric Interpretations of the Circular Functions

Interact: Move the point D to help you investigate the circular functions geometrically.

If t is the length of the arc along the unit circle, then we can see that, no matter how large or small, negative or positive, we make t, the arc length will terminate at a point $P(x, y)$ on the unit circle. Since $x = \cos(t)$ and $y = \sin(t)$, the cosine and sine functions are defined for all real numbers.

Moreover, whenever $x = 0$, we know that $\sec(t) = \frac{1}{x}$ and $\tan(t) = \frac{y}{x}$ are both undefined. We know the values of x on the unit circle become 0 precisely when $t = \pm\frac{\pi}{2}$ or $\pm\frac{3\pi}{2}$ or $\pm\frac{5\pi}{2}$. In general, these two functions are undefined at $t = \frac{\pi}{2} + k\pi$, where k is an integer. A similar argument reveals the cosecant and cotangent functions to be undefined when $t = k\pi$ (again, where k is an integer). The following theorem summarizes these results.

Theorem: Domains of the Circular Functions

Let t be a real number and k be an integer. The domains of the circular functions are

$\sin(t)$ and $\cos(t)$	All real numbers, or $(-\infty, \infty)$
$\tan(t)$ and $\sec(t)$	All real numbers except $t = \frac{\pi}{2} + k\pi$
$\cot(t)$ and $\csc(t)$	All real numbers except $t = k\pi$

While there is a way to write the domains of tangent, cotangent, secant, and cosecant in interval notation, it's a bit complex.

The activity, Geometric Interpretations of the Circular Functions, can be beneficial in visualizing the ranges of the circular functions. For example, Figure $4.3.10$ shows us that the values of the cosine and sine functions (which are the x and y coordinates of points along the unit circle) are always between -1 and 1, inclusive. Thus, the range of these functions is $[-1, 1]$.

Since the values of the tangent function can be interpreted as the slope of the ray emanating from the origin through the point $P(x, y)$ on the unit circle, we can easily see that the tangent can get as large (both positively and negatively) as we want. This is because the slopes of these lines can get as steep as we want (and they can even be zero). Hence, the range of the tangent function is $(-\infty, \infty)$.

The following theorem states the ranges of all of the circular functions. While not incredibly useful now, they will become increasingly important as we move forward, and we should introduce them early so you can get used to them.

Theorem: Ranges of the Circular Functions

Let t be a real number. The ranges of the circular functions are

$\sin(t)$ and $\cos(t)$	$[-1, 1]$
$\tan(t)$ and $\cot(t)$	$(-\infty, \infty)$
$\sec(t)$ and $\csc(t)$	$(-\infty, -1] \cup [1, \infty)$

Example $4.3.8$

Is it possible for $\sec(t) = -\frac{1}{2}$?

Solution available at the end of this section.

Checkpoint $4.3.8$

For each function, state a value the function could return and a value it could not.

a. $\cos(t)$
b. $\csc(t)$

Answers available at the end of this section.

Success in Mathematics: Why Study Mathematics? (Part 2)

Tracy Moody graduated in 2006 with a bachelor's degree in mathematics. She then went to graduate school, studying to become a physical therapist. She writes,

> "Many people cannot seem to fathom how math and physical therapy overlap. Well, one of the obvious things is that physical therapists use statistics to see if new treatments work. However, the main thing I bring as a mathematician to my graduate studies is my ability to problem solve. A degree in mathematics taught me how to look at a problem from several angles. Furthermore, one of the main classes that carried over into my career choice is, believe it or not, real analysis.

I can remember working with my fellow classmates on problems for hours. However, through this process, I learned the importance of perseverance, and I learned how to communicate the problem and solutions in several ways."

Source: www.maa.org, Career Profiles.

1. Do you think studying mathematics has helped you become a better problem solver? Explain.
2. Do you think studying mathematics has helped you learn to "persevere," to keep working on a difficult problem even though you are struggling? Explain.

Success in College: Getting Help

Learning math can be a struggle. Sometimes your professor seems to be talking in a foreign language. Other students look like they get it. The professor keeps going. The stuff you don't understand keeps piling up.

Struggling in a college math class is expected. You must learn to work through your confusion. Your professor expects you to ask questions in class. If needed, you must also get help outside of class. Visit your professor during office hours. Ask if appointments are available outside of office hours. Use the math tutors on campus. It is your responsibility to get the help you need. You must work, ask questions, practice, and think until you get it. You have to take the time to get extra help.

Students who let frustration get to them fail. They put off getting help. They sometimes skip class. They forget that college is different than many high schools. In college, you can fail. If you do not learn enough, you cannot go on.

When successful students are confused, they get help. They work until they understand. You should expect to be confused at times. It would be best if you planned on needing extra time and energy to deal with it.

Here are some questions that may help you succeed in this class (and possibly many other classes as well):

1. Where is your professor's office located?
2. Your professor might keep regular office hours when they are available for help. When are your professor's office hours?

If you are unsure about the answers to either of these questions, feel free to ask your professor - that's what they are there for!

Answers

Section Preview

We wrap up our trio of trig definitions with the Unit Circle definition. I deliberately chose to introduce this definition last because we (human beings) tend to stick with the first or second way we learn something. Since the Unit Circle definition of the trigonometric functions is a completely memorizable definition/method, I consider it the worst to introduce early. Most of your time in this section will be spent on convincing the student that trigonometric functions can also be defined in terms of arc length along the unit circle (hence, the phrase "circular functions"). While I do not spend an insane amount of time on this material, understanding the definition presented in this section (along with its relationship to the area of the sector with arc length t) is important for the development of the hyperbolic functions in Calculus I.

Section Preview

- **Sets and Numbers**
  - Intervals and Interval Notation: The domains of the circular functions are expressed using interval notation (e.g., $(-\infty, \infty)$) and set exclusion (e.g., "all real numbers except $t = \frac{\pi}{2} + k\pi$"). The ranges include unions like $(-\infty, -1] \cup [1, \infty)$. Fluency with interval notation is genuinely required to read and use these theorems.

Section Preview

- Evaluate a trigonometric function using the unit circle.
- Find the value of a trigonometric function given a point on the unit circle.
- Use a calculator to approximate the value of a trigonometric function for an angle in radians.
- Use the unit circle to answer a conceptual question about a trigonometric function.

Example 4.3.1

The coordinates of P are given by

$$x = \cos(315^\circ), \quad y = \sin(315^\circ).$$

The reference angle for 315° is $360^\circ - 315^\circ = 45^\circ$, so the trigonometric functions of 315° are the same as the functions for 45°, up to sign.

Because 315° is in the fourth quadrant, we have $\cos(315^\circ) = \frac{1}{\sqrt{2}}$ and $\sin(315^\circ) = -\frac{1}{\sqrt{2}}$. Thus, the coordinates of P are $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Checkpoint 4.3.1

$\sin(\varphi) = -0.7000, \cos(\varphi) \approx -0.7141, \tan(\varphi) \approx 0.9802$

Example 4.3.2

The pond is a unit circle, so you have traversed an angle in radians equal to the arc length traveled, 4 miles. An angle of 4 radians is in the middle of the third quadrant relative to your starting point, more than halfway but less than three-quarters around the pond.

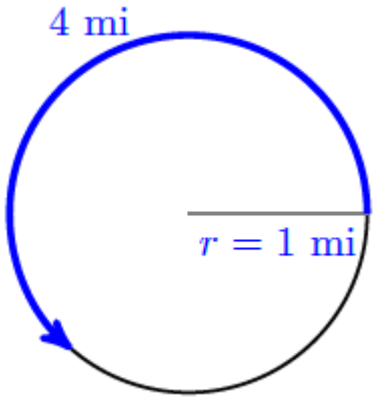


Figure 4.3.4

Checkpoint 4.3.2

3.67 ft

Example 4.3.3

a. Since $t = \frac{2\pi}{3}$, we are being asked to evaluate $\sin\left(\frac{2\pi}{3}\right)$. As stated before, a trigonometric function's value at a real number (an arc length along the unit circle) and the equivalent radian angle are the same. Thus,

$$\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

b. As in part (a), we just evaluate

$$\cot\left(\frac{2\pi}{3}\right) = -\cot\left(\frac{\pi}{3}\right) = -\frac{1}{\sqrt{3}}.$$

Checkpoint 4.3.3

$-\sqrt{2}$

Example 4.3.4

a. On a unit circle, the coordinates of P are $\left(\cos\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right)\right)$, so

$$x = \cos\left(\frac{\pi}{2}\right) = 0 \quad \text{and} \quad y = \sin\left(\frac{\pi}{2}\right) = 1$$

The coordinates of P are $(0, 1)$, as shown below.

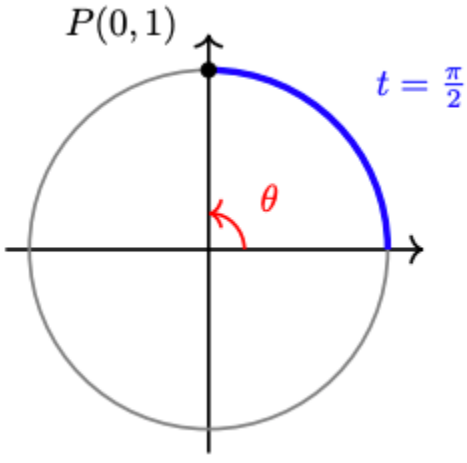


Figure 4.3.6

(We could also observe that an arc of length $\frac{\pi}{2}$ is one-quarter of a unit circle, so the point P sits at the twelve o'clock position on the circle.)

b. The coordinates of P are $\left(\cos\left(\frac{5\pi}{6}\right), \sin\left(\frac{5\pi}{6}\right)\right)$, so

$$x = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$y = \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$$

Thus, the coordinates of P are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, as shown below.

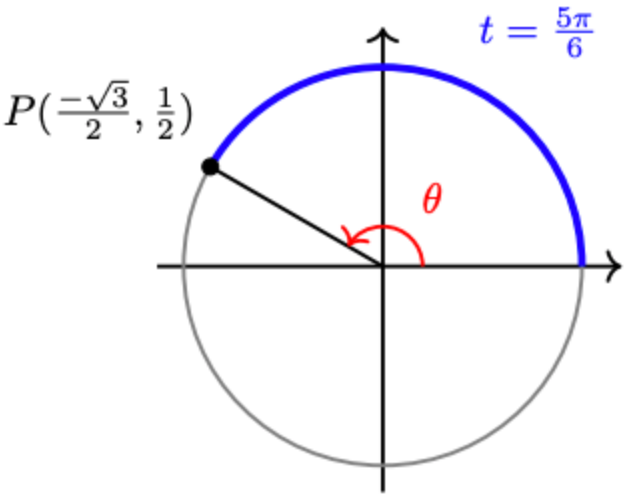


Figure 4.3.7

Checkpoint 4.3.4

a. $(-1, 0)$
b. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Example 4.3.5

a. With your calculator in radian mode, enter

$$\text{COS}\, 7\,\pi \div 4$$

and

$$\text{SIN}\, 7\,\pi \div 4$$

to find

$$\cos\left(\frac{7\pi}{4}\right) \approx 0.7071067812$$

$$\sin\left(\frac{7\pi}{4}\right) \approx -0.7071067812$$

Rounding to four decimal places gives $\cos\left(\frac{7\pi}{4}\right) \approx 0.7071$ and $\sin\left(\frac{7\pi}{4}\right) \approx -0.7071$.

b. Your calculator will give you the values

$$\cos(3.5) \approx -0.9364566873$$

$$\sin(3.5) \approx -0.3507832277$$

Rounding to four places, we have $\cos(3.5) \approx -0.9365$ and $\sin(3.5) \approx -0.3508$. Note that 3.5 radians is a third-quadrant angle, so the signs of the trigonometric values make sense.

Checkpoint 4.3.5

a. 3.7321
b. -1.8856

Example 4.3.6

Evaluate the function for $n = 1$ to find

$$T = 3.11 \sin\left(0.017(1) - 1.38\right) + 6.03 \approx 2.99.$$

On January 1, sunset in Stockholm occurs about 2.99 hours after noon, or at 2:59 pm.

Evaluate the function at $n = 182$ to find

$$T = 3.11 \sin\left(0.017(182) - 1.38\right) + 6.03 \approx 9.11.$$

On July 1, sunset occurs about 9.11 hours after noon, or at 9:07 pm. (Actually, 10:07 pm, because of daylight saving time.)

Checkpoint 4.3.6

a. 8
b. 12.09

Example 4.3.7

The circle is scaled in units of 0.1 radians, and an arc of 2.5 radians in standard position has its terminal point, P, in the second quadrant. The coordinates of P are approximately $(-0.8, 0.6)$, so we have

$$\cos(2.5) \approx -0.8 \quad \text{and} \quad \sin(2.5) \approx 0.6.$$

To find $\tan(2.5)$, we calculate $\frac{y}{x}$.

$$\tan(2.5) = \frac{y}{x} \approx \frac{0.6}{-0.8} = -0.75.$$

Checkpoint 4.3.7

Approximately -0.49, -0.87, and 1.78, respectively.

Example $4.3.8$

Since the range of the secant function is $(-\infty, -1] \cup [1, \infty)$, and $-\frac{1}{2}$ is not in this interval, it is *not* possible for $\sec(t) = -\frac{1}{2}$.

Checkpoint $4.3.8$

a. 2 is not a possible output, but 1/2 is.
b. 1/2 is not a possible output, but 2 is.

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4.3.1: Resources and Key Concepts

Resources

Videos

- The Unit Circle
  - The Unit Circle - The Unit Circle
- Unit Circle Definition of the Trigonometric Functions
  - Unit Circle - Trigonometric Functions: A Unit Circle Approach
  - Unit Circle - Evaluating Trigonometric Functions of Non-Acute, Special Angles
  - Unit Circle - Evaluating Trigonometric Functions at Axial Angles
  - The Unit Circle - Special Triangles
  - The Unit Circle - The Reference Number
  - Trigonometry Mini-Lecture: Graph the Unit Circle with a TI Graphing Calculator & Find Trig Values

Key Concepts

Definitions

- **Trigonometric Functions (Unit Circle Definition):** Let t be a real number representing the arc length from the point (1, 0) to a point $P(x, y)$ on the unit circle. The trigonometric functions of t, also called circular functions, are defined by the coordinates of P. For example, $\cos(t) = x$ and $\sin(t) = y$.

Theorems

- **Lemma:** Let θ be an angle in standard position, and suppose the terminal side of θ intersects the unit circle at the point $P(x, y)$. Then $\cos(\theta) = x$ and $\sin(\theta) = y$.
- **Domains of the Circular Functions:** Let t be a real number and k be an integer. The domains of the circular functions are:
  - $\sin(t)$ and $\cos(t)$: All real numbers, or $(-\infty, \infty)$
  - $\tan(t)$ and $\sec(t)$: All real numbers except $t = \frac{\pi}{2} + k\pi$
  - $\cot(t)$ and $\csc(t)$: All real numbers except $t = k\pi$
- **Ranges of the Circular Functions:** Let t be a real number. The ranges of the circular functions are:
  - $\sin(t)$ and $\cos(t)$: $[-1, 1]$
  - $\tan(t)$ and $\cot(t)$: $(-\infty, \infty)$
  - $\sec(t)$ and $\csc(t)$: $(-\infty, -1] \cup [1, \infty)$

Common Mistakes

- **Using the Wrong Calculator Mode:** When working with the unit circle definition, angles and arc lengths are measured in radians. You must ensure your calculator is set to Radian mode, not Degree mode, to get correct approximations.
- **Confusing Angle Measure with Arc Length:** While the numerical value of an angle in radians is equal to the arc length it subtends on a unit circle, it is important to remember they are conceptually different. An angle is an amount of rotation, while an arc length is a distance.

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4.3.2: Homework

Reading Questions

1. What is the radius of a unit circle?
2. If the terminal side of an angle θ intersects the unit circle at a point $P(x, y)$, what do the x- and y-coordinates represent?
3. On a unit circle, what is the relationship between the radian measure of a central angle and the length of the arc it subtends?
4. When trigonometric functions are defined based on the arc length on a unit circle, what are they sometimes called?
5. To find the value of a trigonometric function for a real number t, what do you do?
6. What calculator mode should be used when evaluating trigonometric functions for angles given in radians or for real numbers?
7. What is the domain of the sine function?
8. What is the domain of the tangent function?
9. What is the range of the cosine function?
10. What is the range of the secant function?
11. Is it possible for $\sin(t)$ to equal 2? Why or why not?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, simplify.

1. a. $\frac{2}{3}+\frac{1}{6}$
   b. $\frac{2x}{3}+\frac{x}{6}$
2. a. $\frac{3}{4}-\frac{5}{8}$
   b. $\frac{3n}{4}-\frac{5n}{8}$
3. a. $2-\frac{3}{4}$
   b. $2b-\frac{3b}{4}$
4. a. $1+\frac{3}{8}$
   b. $m+\frac{3m}{8}$
5. a. $\frac{5}{6}-\frac{7}{6}$
   b. $\frac{5q}{6}-\frac{7q}{6}$
6. a. $\frac{2}{3}-\frac{5}{3}$

b. $\frac{2p}{3} - \frac{5p}{3}$

Answers available at the end of this section.

Homework

Vocabulary Check

1. A circle of radius 1 having center at the origin is called a(n) ___.
2. When considered as functions of arc length, we call the trigonometric functions ___ functions because they are defined by the coordinates of points on a unit circle.

Concept Check

3. Explain why $\cos\theta \leq 1$ for any angle θ.
4. How many solutions are there to the equation $\cos(\theta) = 0.4$ between 0° and 360°?
5. On a unit circle, the ___ measure of the central angle and the ___ length it cuts off are the same value.
6. List the trigonometric functions having a domain of all real numbers.
7. List the trigonometric functions having a range of all real numbers.
8. Sketch a figure on a unit circle to illustrate each equation.
   a. $\sin\frac{3\pi}{4} = 0.7071$
   b. $\cos 2.5 = -0.8011$
9. Write down the multiples of $\frac{\pi}{12}$ from 0 to 2π. Reduce each fraction.
10. Sketch arcs in standard position on a unit circle with the following lengths.

$$s = \frac{\pi}{2}, \quad s = \frac{3\pi}{4}, \quad s = \frac{\pi}{3}, \quad s = \frac{2\pi}{3}$$

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

11. Let $P(x, y)$ be a point on the unit circle determined by an arc of length t from the point $(1, 0)$ to P. Then the coordinates of P are given by

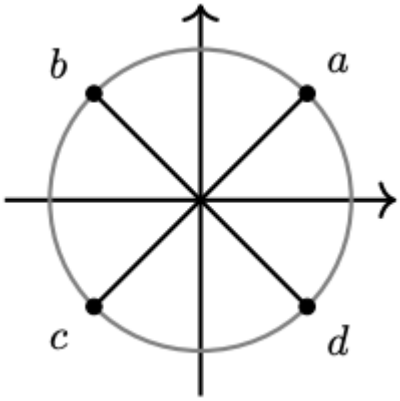
$$x = \sin(t), \quad y = \cos(t).$$

12. If $\beta < \alpha$, then $\sin(\beta) < \sin(\alpha)$.
13. The circular functions are functions of real numbers. As such, we can set our calculator to *either* degree *or* radian mode to get approximations of these functions.
14. The domain of the tangent function includes all real numbers *except* for $\frac{\pi}{2}$.
15. The range of the sine and cosine functions is -1 to 1, inclusive.

Basic Skills

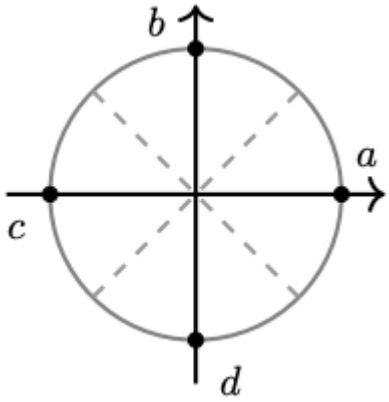
For the following exercises, each point on the unit circle is the terminal point of a special angle in standard position. Give exact values for the radian measure, t, of the angle, and the coordinates (x, y) of the point.

16.



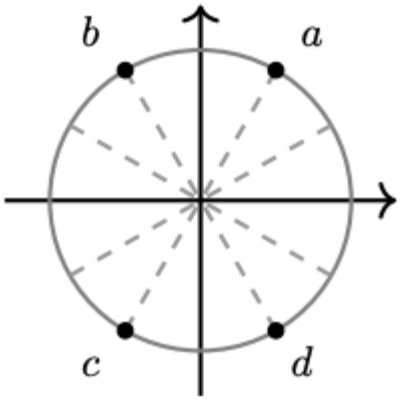
	a	b	c	d
t				
x				
y				

17.



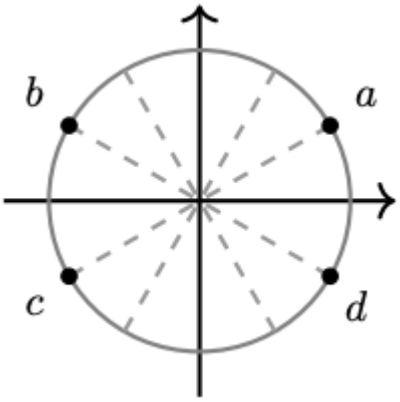
	a	b	c	d
t				
x				
y				

18.



	a	b	c	d
t				
x				
y				

19.



	a	b	c	d
t				
x				
y				

20. Complete the table.

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\frac{2\pi}{3}$			
$\frac{3\pi}{4}$			
$\frac{5\pi}{6}$			

21. Complete the table.

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$

$\frac{7\pi}{6}$			
$\frac{5\pi}{4}$			
$\frac{4\pi}{3}$			

22. Complete the table.

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\frac{5\pi}{3}$			
$\frac{7\pi}{4}$			
$\frac{11\pi}{6}$			

For the following exercises, evaluate each function.

23. $\sin(45^\circ)$

24. $\cos(150^\circ)$

25. $\cot(270^\circ)$

26. $\tan(240^\circ)$

27. $\sec(315^\circ)$

28. $\csc(210^\circ)$

29. $\cos\left(\frac{\pi}{6}\right)$

30. $\sin\left(\frac{3\pi}{2}\right)$

31. $\tan\left(\frac{2\pi}{3}\right)$

32. $\sec\left(\frac{7\pi}{6}\right)$

33. $\csc\left(\frac{5\pi}{4}\right)$

34. $\cot\left(\frac{4\pi}{3}\right)$

For the following exercises, use the unit circle to find all values of θ between 0 and 2π for which the given equation is true.

35. $\sin(\theta) = \frac{\sqrt{3}}{2}$

36. $\cos(\theta) = -\frac{\sqrt{3}}{2}$

37. $\tan(\theta) = -1$

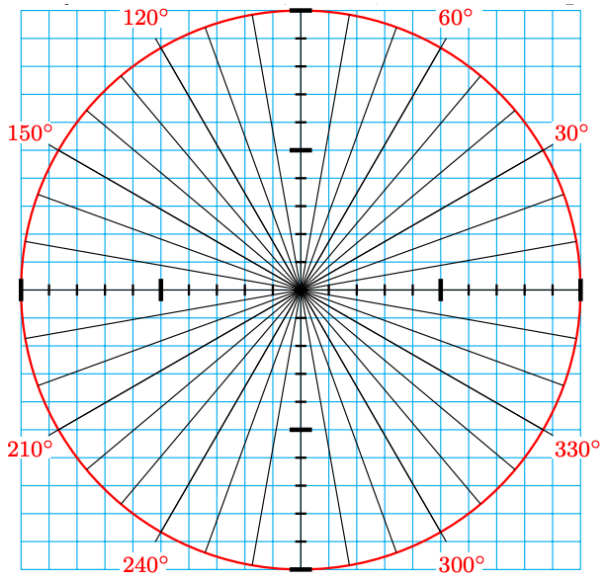
38. $\csc(\theta) = -2$

39. $\sec(\theta) = 1$

40. $\cot(\theta) = \sqrt{3}$

41. Estimate the x-coordinate of the points on the unit circle designated by each angle and complete the table. (Hint: Use

symmetry in the second, third, and fourth quadrants.)



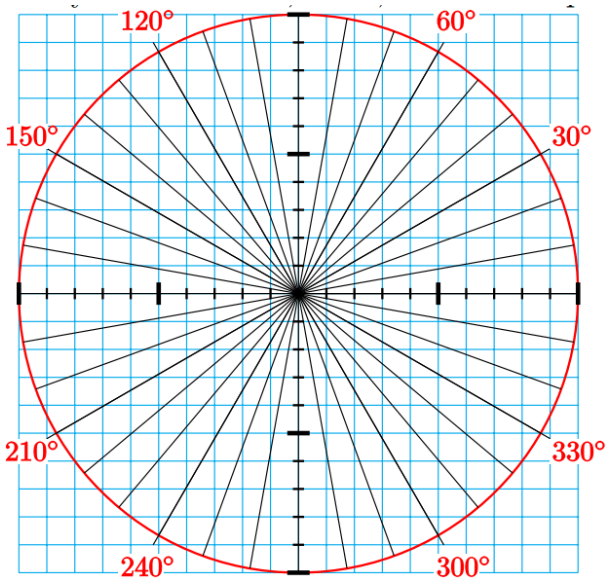
Angle	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
x-coordinate										

Angle	100°	110°	120°	130°	140°	150°	160°	170°	180°
x-coordinate									

Angle	190°	200°	210°	220°	230°	240°	250°	260°	270°
x-coordinate									

Angle	280°	290°	300°	310°	320°	330°	340°	350°	360°
x-coordinate									

42. Estimate the y-coordinate of the points on the unit circle designated by each angle and complete the table. (Hint: Use symmetry in the second, third, and fourth quadrants.)



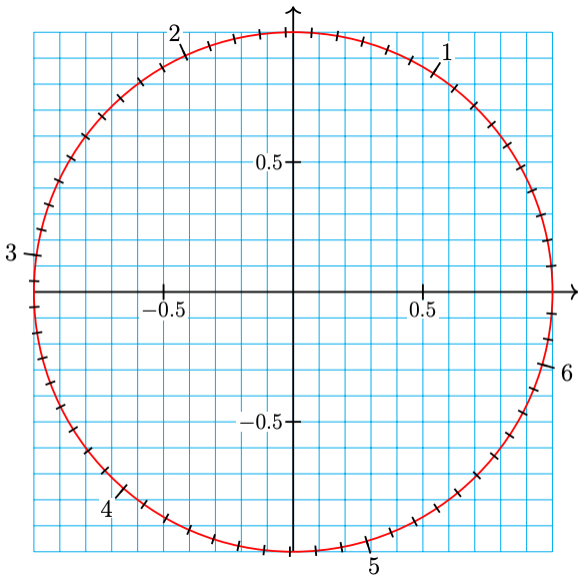
Angle	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
x-coordinate										

Angle	100°	110°	120°	130°	140°	150°	160°	170°	180°
x-coordinate									

Angle	190°	200°	210°	220°	230°	240°	250°	260°	270°
x-coordinate									

Angle	280°	290°	300°	310°	320°	330°	340°	350°	360°
x-coordinate									

For the following exercises, use the unit circle to estimate the sine, cosine, and tangent of each arc of given length.



13. a. 0.4

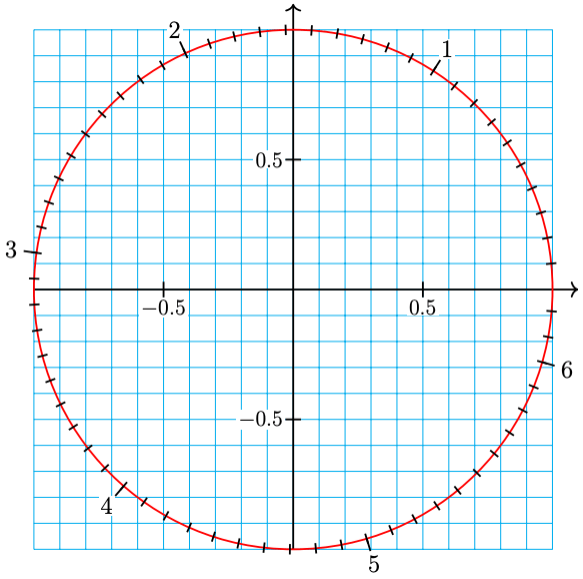
b. 1.2
c. 2

44. a. 0.8
b. 2.6
c. 4

45. a. 2.8
b. 3.5
c. 5

46. a. 3
b. 4.3
c. 5.5

For the following exercises, use the unit circle to estimate two numbers with the given trigonometric value.



47. $\cos t = 0.3$

48. $\sin t = 0.1$

49. $\sin t = -0.7$

50. $\cos t = -0.6$

51. $\tan t = \dfrac{-4}{9}$

52. $\tan t = \dfrac{8}{6}$

Each of the following exercises describes an arc in standard position on the unit circle. In which quadrant does the terminal point of the arc lie?

53. $\sin s > 0, \quad \cos s < 0$

54. $\sin s < 0, \quad \cos s > 0$

55. $\cos t < 0, \quad \tan t < 0$

56. $\cos t > 0, \quad \tan t < 0$

57. $\sin x < 0, \quad \tan x > 0$

58. $\sin x > 0, \quad \tan x < 0$

For the following exercises, without a calculator, decide whether the quantity is positive or negative.

59. $\cos 2.7$

60. $\sin 4.1$

61. $\tan 3.8$

62. $\tan 5.4$

63. $\sin 2.2$

64. $\cos 4.9$

Starting at $(1, 0)$, you walk s units around a unit circle. For the following exercises, sketch a unit circle showing your position. What are your coordinates?

65. $s = 2.5$

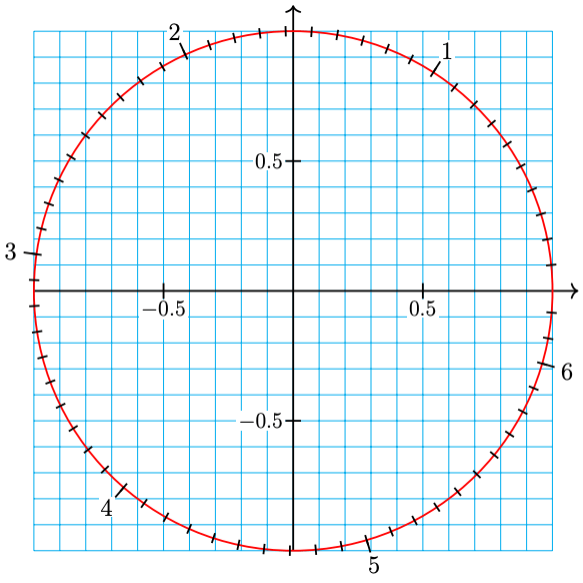
66. $s = 4.3$

67. $s = 8.5$

68. $s = 11$

Synthesis Questions

For the following exercises, place the trigonometric values in order from smallest to largest. Use the figure to help you, but try not to use a calculator!



69. $\sin 0.5, \quad \sin 1.5, \quad \sin 2.5, \quad \sin 3.5$

70. $\cos 1.6, \quad \cos 2.6, \quad \cos 3.6, \quad \cos 5.6$

71. $\cos 2, \quad \cos 3, \quad \cos 4, \quad \cos 5$

72. $\sin 2.8, \quad \sin 3.8, \quad \sin 4.8, \quad \sin 5.8$

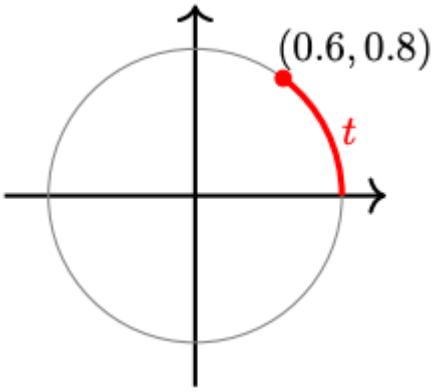
For the following exercises, the figure shows an arc of length t, and the coordinates of its terminal point. Find the terminal point of each related arc given below, and give its sine, cosine, and tangent.

a $2\pi - t$

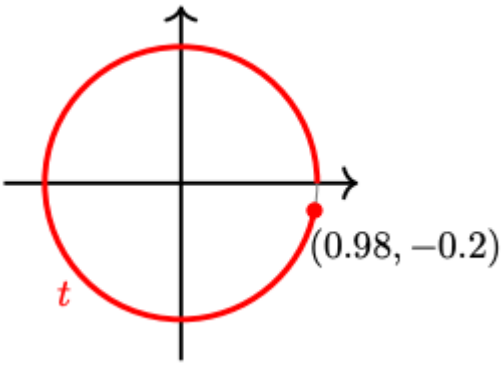
b $\pi - t$

c $\pi + t$

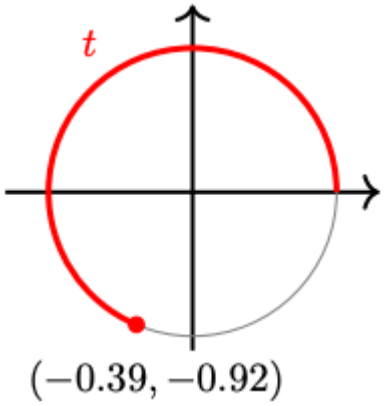
73.



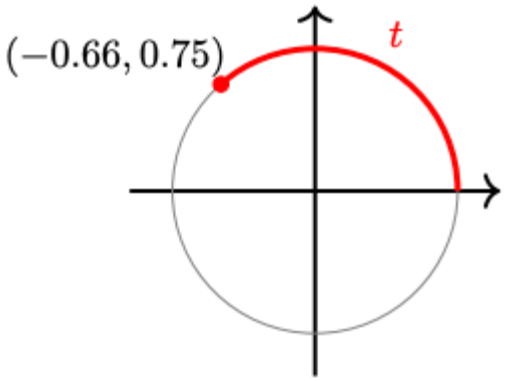
74.



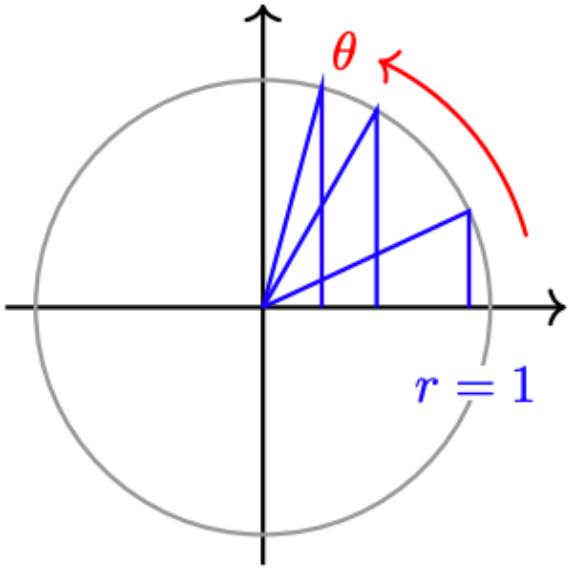
75.



76.



77. Use the figure to help you fill in the blanks.



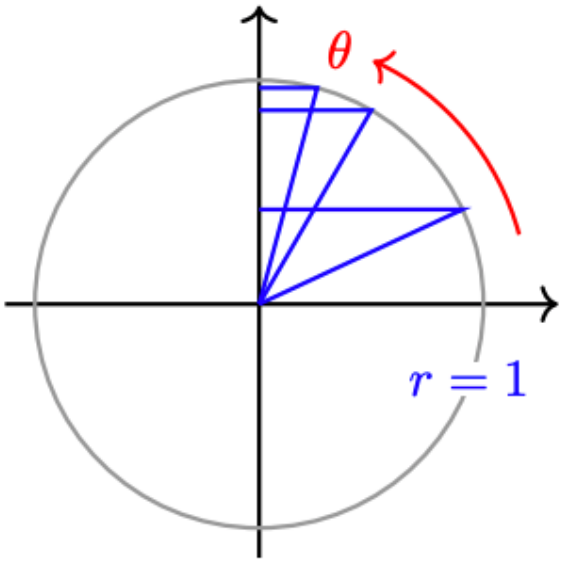
a. As θ increases from 0° to 90°, $f(\theta) = \sin\theta$ _____ from _____ to _____.

b. As θ increases from 90° to 180°, $f(\theta) = \sin\theta$ _____ from _____ to _____.

c. As θ increases from 180° to 270°, $f(\theta) = \sin\theta$ _____ from _____ to _____.

d. As θ increases from 270° to 360°, $f(\theta) = \sin\theta$ _____ from _____ to _____.

78. Use the figure to help you fill in the blanks.



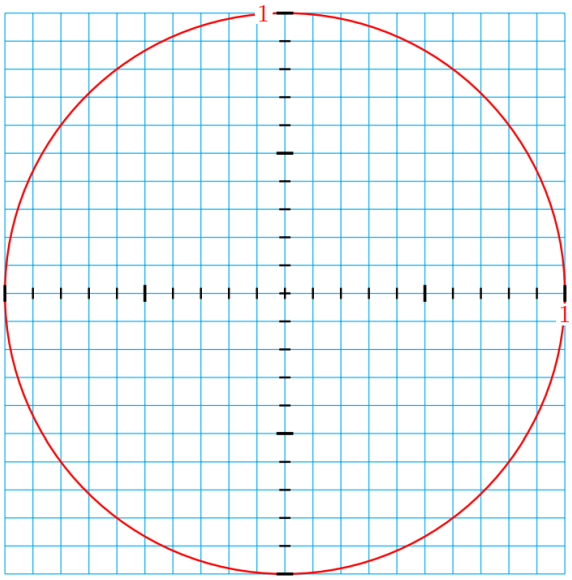
a. As θ increases from 0° to 90°, $f(\theta) = \cos\theta$ _____ from _____ to _____.

b. As θ increases from 90° to 180°, $f(\theta) = \cos\theta$ _____ from _____ to _____.

c. As θ increases from 180° to 270°, $f(\theta) = \cos\theta$ _____ from _____ to _____.

d. As θ increases from 270° to 360°, $f(\theta) = \cos\theta$ _____ from _____ to _____.

For the following exercises, use the grid provided below.



79. Draw two different angles α and β in standard position whose sine is 0.6.

a. Use a protractor to measure α and β.

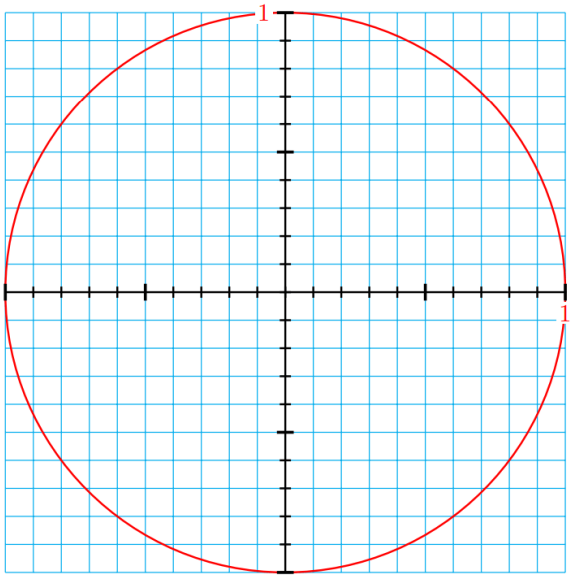
b. Find the reference angles for both α and β. Draw in the reference triangles.

80. Draw two different angles θ and ϕ in standard position whose sine is -0.8.

a. Use a protractor to measure θ and ϕ.

b. Find the reference angles for both θ and ϕ. Draw in the reference triangles.

For the following exercises, use the grid provided below.



81. Draw two different angles α and β in standard position whose cosine is 0.3.

a. Use a protractor to measure α and β.

b. Find the reference angles for both α and β. Draw in the reference triangles.

82. Draw two different angles θ and ϕ in standard position whose cosine is –0.4.

a. Use a protractor to measure θ and ϕ.

b. Find the reference angles for both θ and ϕ. Draw in the reference triangles.

For the following exercises, find all angles between 0° and 360° for which the statement is true.

83. a. $\cos\theta = -1$

b. $\sin\theta = -1$

c. $\tan\theta = -1$

84. a. Find two angles, $0 \le \theta \le 360^\circ$, with $\sin\theta = 0$.

b. Find two angles, $0 \le \theta \le 360^\circ$, with $\cos\theta = 0$.

For the following exercises, find two points on the unit circle with the given coordinate. Sketch the approximate location of the points on the circle. (Hint: what is the equation for the unit circle?)

85. $x = 0.2$

86. $x = -0.6$

87. $y = -0.35$

88. $y = 0.7$

89. $x = \dfrac{-\sqrt{3}}{2}$

90. $y = \frac{1}{\sqrt{2}}$

For the following exercises, find the angle in radians between 0 and 2π determined by the terminal point on the unit circle. Round your answer to hundredths.

91. (−0.1782, 0.9840)

92. (−0.8968, −0.4425)

93. (0.8855, −0.4646)

94. (0.9801, 0.1987)

95. (−0.7659, −0.6430)

96. (0.9602, −0.2794)

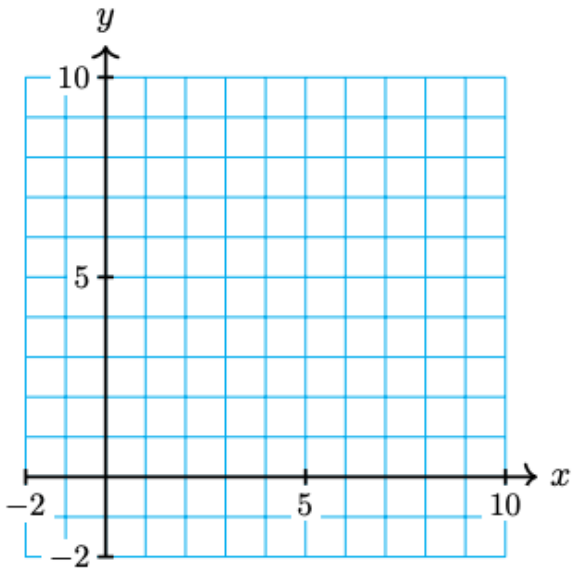
97. a. Sketch a unit circle and the line $y = x$. Find the coordinates of the two points where the line and the circle intersect.

    b. State your answers to part (a) using trigonometric functions.

98. a. Sketch a unit circle and the line $y = -x$. Find the coordinates of the two points where the line and the circle intersect.

    b. State your answers to part (a) using trigonometric functions.

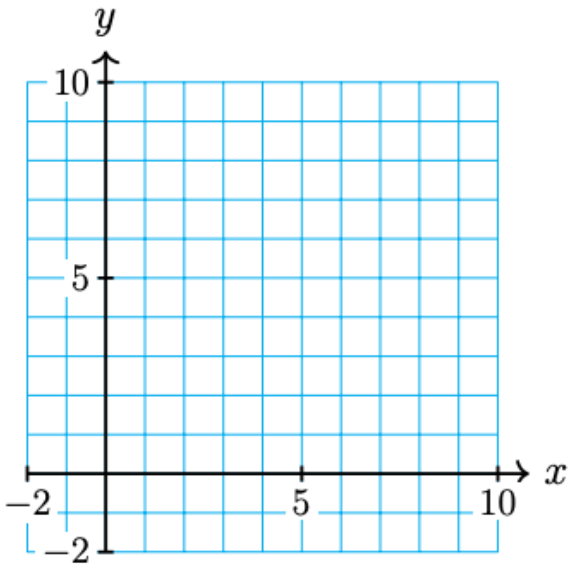
99.



    a. Sketch a line that passes through the origin and the point (8,3). What is the slope of the line?

    b. What is the angle of inclination of the line in radians, measured from the positive x-axis?

100.



a. Sketch a line that passes through the origin and the point (3,8). What is the slope of the line?

b. What is the angle of inclination of the line in radians, measured from the positive x-axis?

For the following exercises, find an equation for the line with the given angle of inclination passing through the given point.

91. $\alpha = \frac{\pi}{3}, (4, 2)$

92. $\alpha = \frac{5\pi}{6}, (-6, 3)$

93. $\alpha = 2.4, (5, -8)$

94. $\alpha = 0.6, (-2, -3)$

95. If $\sec(\theta) = \frac{x}{4}$ for $0 < \theta < \frac{\pi}{2}$, find an expression for $\ln|\sec(\theta) + \tan(\theta)|$ in terms of x.

Applications

96. The sunrise time in Wellington, New Zealand, on the nth day of the year can be modeled by

$$S = 1.93\sin(0.016n - 1.13) + 6.14$$

where S is given in hours after midnight. Find the sunrise time on January 1 (day $n = 1$), April 1 (day $n = 91$), July 1 (day $n = 182$), and on October 1 (day $n = 274$).

97. The variable star RT Cygni reached its maximum magnitude on May 22, 2004, and t days later its magnitude is given by

$$M = 9.55 - 2.25\cos(0.033t)$$

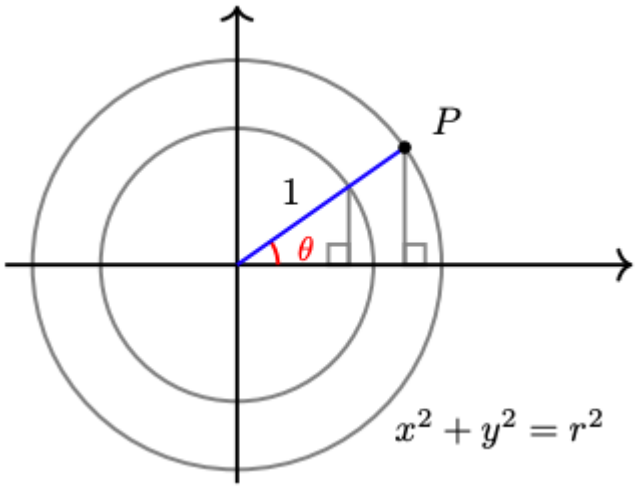
Find the magnitude of RT Cygni on days $t = 0$, $t = 48$, $t = 95$, $t = 142$, and $t = 190$. (Note that smaller values of M denote brighter magnitudes.)

98. City Park features a circular jogging track with a radius of 1 mile centered on the open-air bandstand. You start jogging on the track 1 mile due east of the bandstand and proceed counterclockwise. What are your coordinates relative to the bandstand when you jogged five miles?

99. Silver Reservoir is a circular man-made lake with a radius of 1 kilometer. If you start at the easternmost point on the reservoir and walk counterclockwise for 4 kilometers, how far south of your initial position are you?

Use the results of Problem 110 for Problems 111 - 113.

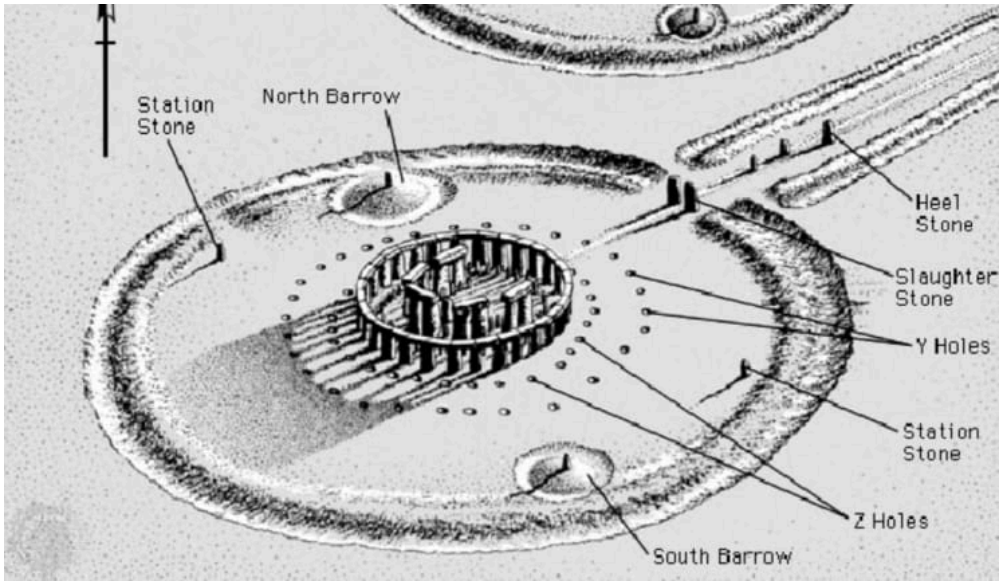
10. Use similar triangles to show that the coordinates of a point P determined by angle θ on a circle of radius r are $x = r\cos\theta, y = r\sin\theta$. (See the figure below.)



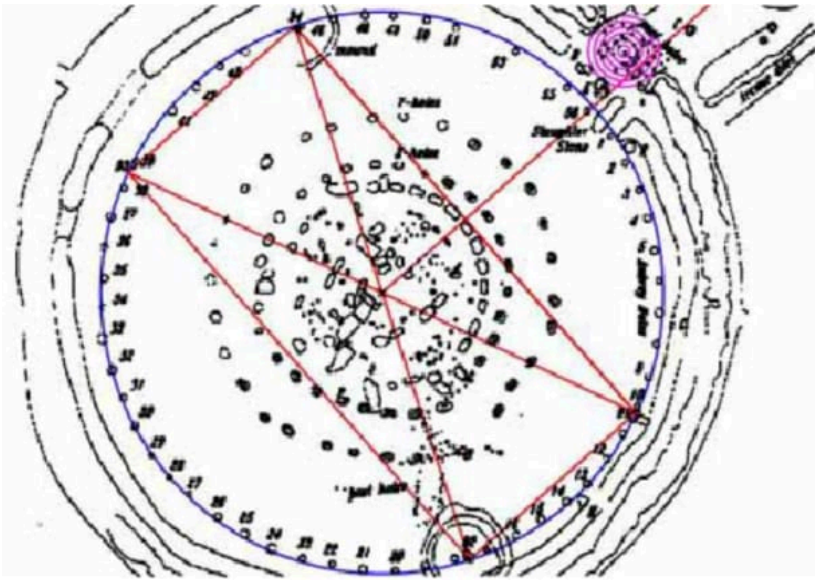
11. The Astrodome in Houston has a diameter of 710 feet. If you start at the easternmost point and walk counterclockwise around its perimeter for 250 feet, how far north of your starting point are you?

12. The Barringer meteor crater near Winslow, Arizona, is 1182 meters in diameter. You start at the easternmost point on the crater's rim and walk counterclockwise around the edge. After walking for 1 kilometer, what is your position relative to your starting point?

13. One of the most intriguing features of Stonehenge is the position of the four Station Stones.



They form the corners of a rectangle inscribed in the Aubrey Circle on the perimeter of the henge, which has a diameter of 288 feet. A line from the center of the circle and perpendicular to the long edge of the rectangle points through the Slaughter Stones at the entrance of the henge and out to the Heel Stone. If you stood in the circle's center on the summer solstice, you would see the sun rise directly over the Heel Stone.

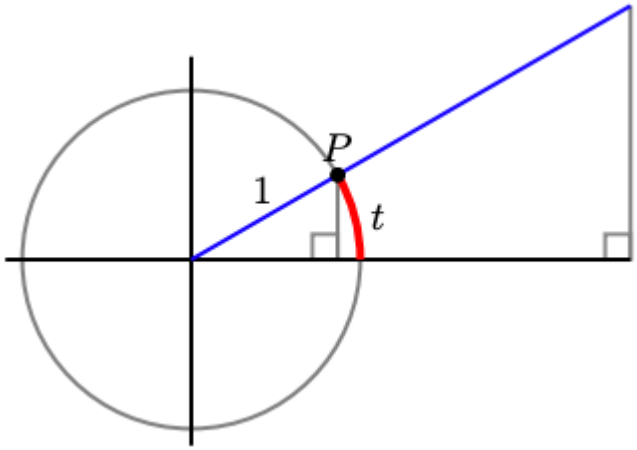


a. The sun rises 48.6° east of north on the summer solstice at Stonehenge. If the positive y-axis points north, find the coordinates of the henge entrance relative to its center.

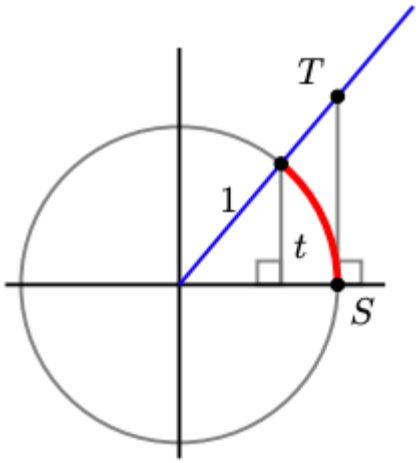
b. The northernmost station stone is located 66.6° of arc counterclockwise from the entrance. Find its coordinates relative to the center of the henge.

Challenge Problems

14. Use similar triangles to show that the coordinates of point P on the unit circle shown below are $(\cos t, \sin t)$.



15. Use similar triangles to show that $ST = \tan t$.



Answers

Skills Refresher

1. a $\frac{5}{6}$, b $\frac{5x}{6}$
2. a $\frac{1}{8}$, b $\frac{n}{8}$
3. a $\frac{5}{4}$, b $\frac{5b}{4}$
4. a $\frac{11}{8}$, b $\frac{11m}{8}$
5. a $\frac{-1}{6}$, b $\frac{-q}{6}$
6. a -1, b $-p$

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4.4: Velocities and Areas of Sectors

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Areas of Sectors

Now that we have paired angles with real numbers via radian measure, a world of applications awaits us.

We recently discovered the relationship between the length, s, of the arc subtended by an angle θ (in radians) in a circle of radius r to be $s = r\theta$. We now investigate the area of the **sector** formed by this central angle θ.

Definition: Sector (of a circle)

The region of a circle bounded by the initial and terminal sides of a central angle θ and the arc subtended by θ is called a **sector of the circle** (or, simply, **sector**).

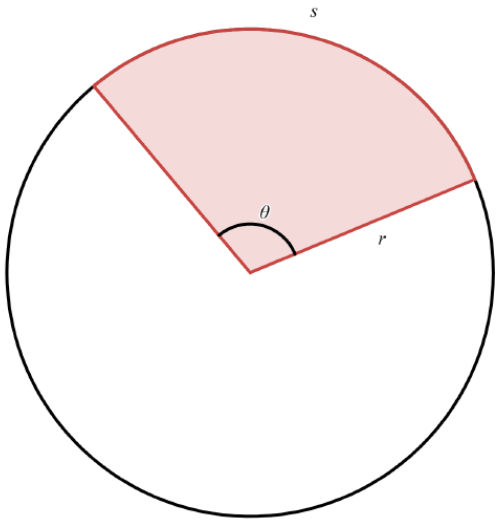


Figure $4.4.1$**:** *The area of a sector of a circle of radius* r *subtended by the angle* θ

To find the area of a sector, recall that the *total area* of a circle of radius r is πr^2; however, the shaded region in Figure $4.4.1$ is not the entire circle. In fact, it is a specific *portion* of this circle. In terms of angular measure, we have only shaded the region covered by an angle rotated through θ out of 2π radians of a full circle. That is, the portion of the circle we have shaded is

$$A = \frac{\theta}{2\pi} \text{ of the area of the full circle} = \frac{\theta}{2\cancel{\pi}} \cdot \cancel{\pi} r^2 = \frac{1}{2} r^2 \theta.$$

This proves the following theorem.

Theorem: Area of a Sector

If θ (in radians) is the central angle in a circle of radius r, then the area of the sector subtended by θ is

$$A = \frac{1}{2}r^2\theta.$$

Finding the areas of sectors becomes increasingly essential as you move into Calculus (specifically, Integral Calculus). Be sure to spend the time now to master this relatively straightforward concept.

Interactive Element: Area of a Sector

The following Interactive Element might help you visualize how a sector's area depends on its central angle and radius.

Interact: Drag the large red point on the rim. Move it around to change the central angle θ, and pull it in or out to change the radius r.

Observation: Sweeping θ fills a fraction $\theta/2\pi$ of the disk — at $\theta = 2\pi$ the sector is the whole circle. The area grows in proportion to θ, but pulling the point outward shows it grows with the square of r: doubling r quadruples the area, matching $A = \frac{1}{2}r^2\theta$.

Example 4.4.1

Find the area of the sector subtended by a central angle of 120° in a circle of radius 4.

Solution available at the end of this section.

Checkpoint 4.4.1

If the area of a sector of a circle having radius 12 meters is 100 square meters, find the measure of the central angle (in radians).

Answer available at the end of this section.

Example 4.4.2: The Midnight Sun Golf Challenge

In certain parts of the world near the Arctic, such as Norway, the sun doesn't set during part of the summer. This phenomenon is known as the Midnight Sun. A golf course in Norway takes advantage of this by hosting a special event: The Midnight Sun Golf Challenge. During this event, golfers can play as many holes as they want throughout the night.

The golf course has a unique challenge on their signature par 3 hole, which features a circular green with a radius of 15 meters. If a golfer lands their ball within a particular marked sector of the green, where the central angle is 30 degrees, they win a prize. To determine the likelihood of winning, calculate the area of the sector where golfers must land their ball.

Solution available at the end of this section.

Checkpoint 4.4.2

A circular park has a sector fenced off for dogs. The sign posted by Sacramento County states that dogs can enjoy 6,000 square feet of play area; however, Paige finds this claim rather dubious. Unafraid about how foolish she might look, Paige takes a very large tape measure to the park and measures the radius of the park to be 120 feet. She then measures the length of fencing along the arc of the dog park area to be roughly 84 feet. Is the sign claiming that dogs have 6,000 square feet of play area correct?

Answer available at the end of this section.

Linear Motion

We have already used the variable s to denote the arc length in a circle of radius r subtended by a central angle θ. We might not have mentioned that this same variable is used throughout Mathematics and Physics to denote spatial position (hence, the letter s). For example, in Algebra and Calculus, you might represent the height of a projectile (relative to the ground) at any time t by the function

$$s(t) = -16t^2 + 20t + 5.$$

Note that we have used the words spatial position and height to represent the same thing. In truth, "height" only applies if an object rises or falls relative to some point in space. "Spatial position," on the other hand, is a bit more flexible and allows us to consider the distance an object is away from a centralized point. Another word commonly used in place of spatial position is **displacement**.

Once we have a function for an object's spatial position (or displacement) at a given time t, we can expand our investigation of the object's motion to include its velocity. In Physics, the **average velocity** (sometimes called the **average linear velocity**) of an object, denoted $\overline{v}$ and read as "v-bar," is defined as the average rate of change of the spatial position of the object with respect to time. As a result, we have

$$\overline{v} = \frac{\text{displacement}}{\text{time}} = \frac{s}{t}.$$

The quantity $\overline{v}$ has units of $\frac{\text{length}}{\text{time}}$ and conveys two ideas: the direction in which the object is moving and how fast the position of the object is changing. For example, if we define movement to the right as positive and movement to the left as negative, an object moving 10 feet to the right in 5 seconds has an average velocity over that period of

$$\overline{v} = \frac{+10\,\text{feet}}{5\,\text{seconds}} = 2\,\text{feet per second}.$$

In this situation, we have written a plus sign on the 10 feet because the object is moving to the right by 10 feet over this period. On the other hand, if the object moving to the left at 10 feet for every 5 seconds, its average velocity over that period would be

$$\overline{v} = \frac{-10\,\text{feet}}{5\,\text{seconds}} = -2\,\text{feet per second}.$$

If we could somehow measure the velocity of an object at a specific moment in time (rather than over an interval of time), we could call that the **instantaneous velocity**, v. The instantaneous velocity of an object is often just called the **velocity** of the object.

It is a common mistake for people to confuse velocity with speed. The **speed** of an object is the magnitude of its instantaneous velocity. That is, $\text{speed} = |v|$. Note that speed is *never* negative, while velocity (and average velocity) can be negative (denoting both speed and direction).

Example 4.4.3

Ants are incredibly fast, especially when you consider their size! The Saharan silver ant (Cataglyphis bombycina) holds the record for the fastest speed ever measured in an ant. It can move at a breathtaking 168.3 inches in five seconds. Find the average velocity of this ant.

Solution available at the end of this section.

Checkpoint 4.4.3

Kenyan athlete Kelvin Kiptum set a men's world record time of 2:00:35 on October 8, 2023, at the 2023 Chicago Marathon (26.2 miles). Find his average velocity for the race in miles per hour. Round your answer to the nearest hundredth.

Answer available at the end of this section.

What if we wanted to know how fast the Kenyan athlete in Checkpoint 4.4.3 was running at exactly 1 hour and 12 minutes into the marathon? That is, what if we wanted to know his instantaneous velocity at precisely 1 hour and 12 minutes into the marathon? We know that he had an *average linear velocity* of approximately 13.04 miles per hour; however, this does not mean he ran at 13.04 miles per hour the entire race. There were many moments when he ran faster than this and others when he ran slower. Since his velocity was not constant throughout the marathon, we would be hard-pressed to compute his instantaneous velocity at 1 hour and 12 minutes into the race—the answer to this type of question is the purview of Calculus.

Since we do not have Calculus at our disposal, if we ever want to answer a question about instantaneous velocity (again, which we call velocity), we must assume that the object's velocity is constant throughout the motion. Then, and only then, it can be shown that the average linear velocity is the same as the instantaneous velocity, v. For the remainder of our discussion, we will make such an assumption. That is, from this sentence forward in this section, we will assume that the object, person, car, particle, or thing we are considering is moving at a constant speed. Hence, its average velocity is equivalent to its instantaneous velocity at any given moment. Again, we will simplify the wording to "velocity."

Circular Motion - Linear Velocity

Our next excursion into the realm of applications comes through circular motion. Suppose an object is moving as pictured below along a circular path of radius r from the point P to the point Q in an amount of time t.

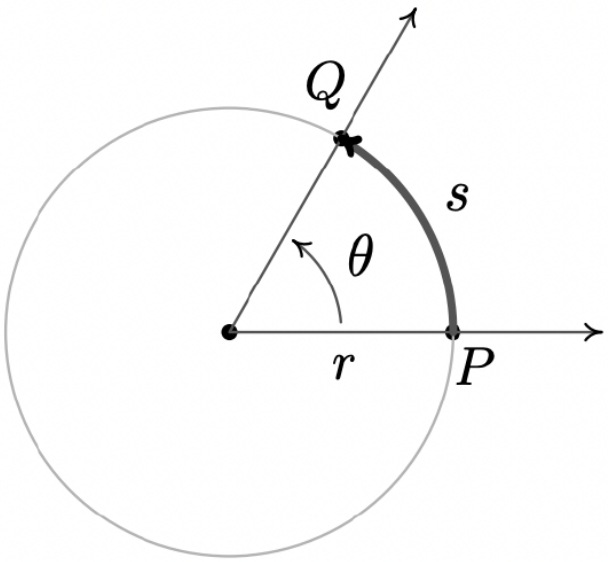


Figure 4.4.2

Like before, s represents a displacement so that $s > 0$ means the object is traveling in a counterclockwise direction, and $s < 0$ indicates movement in a clockwise direction. Note that with this convention, the formula we used to define radian measure, namely $s = r\theta$, still holds since a negative value of s incurred from a clockwise displacement matches the negative we assign to θ for a clockwise rotation.

Again, just as before, the quantity $\frac{s}{t}$ is called the average linear velocity. This is despite the fact that s is a length along the curved edge of a circle. This is because if the path of the object were "uncurled" from a circle to form a line segment, then the average velocity of the object on that line segment would be the same as the average velocity on the circle.

Example 4.4.4

As of 2024, the world's largest Ferris wheel is the Dubai Eye. It opened in 2021 on tiny Bluewaters Island in the Arabian Gulf. The diameter of the wheel is 820 feet, and a single rotation lasts 38 minutes. Assuming the wheel never stops turning, what is a rider's (instantaneous) linear velocity on the Dubai Eye?

Solution available at the end of this section.

Example 4.4.4 showcases that computing the linear velocity of an object along a circular path is very much the same as computing the velocity along a line, we just compute $\frac{\text{distance}}{\text{time}} = \frac{s}{t}$.

Checkpoint 4.4.4

Gum gets stuck to the outer edge of your bike tire. Suppose the tire rotates so that the gum travels along the circular path of its edge 5 inches in 2 seconds. What is the linear velocity of the gum? (By the way, this is also the linear speed of the tire.)

Answer available at the end of this section.

Circular Motion - Angular Velocity

Dividing both sides of $s = r\theta$ by t, we get a very powerful relation:

$$\frac{s}{t} = \frac{r\theta}{t} \implies v = r \cdot \frac{\theta}{t}$$

Definition: Angular Velocity

The quantity $\frac{\theta}{t}$ is called the **average angular velocity** of an object undergoing circular motion. Its units are radians per unit time. If the rate of change of the angle θ with respect to time is constant, then it is simply called the **angular velocity**, and it is denoted by ω (omega). Therefore,

$$\omega = \frac{\theta}{t}.$$

Example 4.4.5

A point on a circle rotates through 38° in 3 hours. What is the angular velocity of the point?

Solution available at the end of this section.

Checkpoint 4.4.5

Suppose a point on a circle rotates through 300° in two hours. Find the *exact* value of the angular velocity of the point.

Answer available at the end of this section.

Circular Motion - Relating Linear and Angular Velocities

We now formalize our recent statement.

Theorem: Linear and Angular Velocity Formula

For an object moving on a circular path of radius r with constant angular velocity ω, the (linear) velocity of the object is given by

$$v = r\omega.$$

We need to talk about units here. The units of v are $\frac{\text{length}}{\text{time}}$, the units of r are length only, and the units of ω are $\frac{\text{radians}}{\text{time}}$. Thus the left hand side of the equation $v = r\omega$ has units $\frac{\text{length}}{\text{time}}$, whereas the right hand side has units $\text{length} \cdot \frac{\text{radians}}{\text{time}} = \frac{\text{length}\cdot\text{radians}}{\text{time}}$. The supposed contradiction in units is resolved by remembering that radians are a dimensionless quantity and angles in radian measure are identified with real numbers so that the units $\frac{\text{length}\cdot\text{radians}}{\text{time}}$ reduce to the units $\frac{\text{length}}{\text{time}}$. We are long overdue for an example.

Example $4.4.6$

Assuming that the surface of the Earth is a sphere, any point on the Earth can be thought of as an object traveling on a circle which completes one revolution in (approximately) 24 hours. The path traced out by the point during these 24 hours is the latitude of that point. Cosumnes River College is at $38^\circ\ 27'\ 14.07''$ north latitude, and it can be shown that the radius of the Earth at this latitude is approximately 3958 miles. Find the linear velocity, in miles per hour, of Cosumnes River College as the world turns.

Solution available at the end of this section.

Checkpoint $4.4.6$

How fast is a bike moving, in miles per hour, if its rear tire has a radius of 15 inches and the angular velocity of the rear tire is 20 radians per second?

Answer available at the end of this section.

It is worth noting that the quantity $\frac{1 \text{ revolution}}{24 \text{ hours}}$ in Example $4.4.6$ is called the **ordinary frequency** of the motion.

Definition: Ordinary Frequency

When an object is undergoing circular motion, the number of cycles (or revolutions) per unit time is called the **ordinary frequency** and is usually denoted by the variable f.

The higher the ordinary frequency, the faster the motion. This is because the circular motion must go through more cycles per unit time. An ordinary frequency that you are familiar with is a radio station's frequency. For example, if you listen to 93.7 on the FM dial, that means you are listening to a signal broadcasting at a frequency of 93.7 MHz. This means 93.7 million Hertz. A Hertz is defined to be 1 cycle per second. Therefore, 93.7 MHz means you are listening to a station broadcasting at 93.7 million cycles per second.

The **angular frequency**, on the other hand, tells us how many radians an object turns through in a given time.

Definition: Angular Frequency

When an object is undergoing circular motion, the number of radians it turns through per unit time is called the **angular frequency** and is usually denoted by the variable ω.

Since there are 2π radians per cycle and f cycles per unit time, the angular frequency is $2\pi f$. However, angular frequency is radians per unit time, and so is angular velocity. Thus, angular frequency and angular velocity mean the same thing! That is, $\omega = 2\pi f$.

One final "measuring stick" for circular motion is very important—the **period** of the circular motion.

Definition: Period (of circular motion)

When an object is undergoing circular motion, we define $T = \frac{1}{f}$ to be the **period** of the motion. It is the amount of time it takes for the object to complete one cycle of the motion.

In the scenario of Example 4.4.6, the period of the motion is 24 hours, or one day.

The concepts of frequency and period help frame the equation $v = r\omega$ in a new light. That is, if ω is fixed, points which are farther from the center of rotation need to travel faster to maintain the same angular frequency since they have farther to travel to make one revolution in one period's time. The distance of the object to the center of rotation is the radius of the circle, r, and is the "magnification factor," which relates ω and v. We will have more to say about frequencies and periods later in this text.

Interactive Element: Linear vs. Angular Velocity

The following Interactive Element might help you visualize why points farther from the center of a spinning object move faster even though every point turns through the same angle in the same time.

Interact: Drag the large red point. Move it around the center to sweep the angle (both points turn together at the same angular velocity), and pull it inward or outward to change its radius.

Observation: Both points sweep the same angle in the same time, so they share one angular velocity ω. Yet the outer (red) arc is always longer than the inner (blue) arc — the farther point covers more distance in the same time, so its linear velocity is larger by exactly the factor r. This is $v = r\omega$.

We close out this section with a beautiful application requiring great attention to detail.

Example 4.4.7

A bicycle with a 27-inch-diameter rear wheel has its gears set so that the chainring has a 4-inch diameter and the rear sprocket has a 3-inch diameter (see image below). The cyclist pedals at 250 rpm. What is the speed of the bicycle in miles per hour?



Figure 4.4.3

Solution available at the end of this section.

Checkpoint 4.4.7

If the chainring of a bicycle is 150 mm in diameter, the sprocket is 80 mm in diameter, and the wheel is 700 mm in diameter, how fast would one need to pedal, in revolutions per minute, in order to maintain a speed of 24 kilometers per hour?

Answer available at the end of this section.

Success in Mathematics: Mathematics in Other Courses

In other courses, we use mathematics to understand and explore the big ideas of the subject. Professors may use different vocabulary, variables, and notations than you learned in your math classes. The mathematics is the same. The way of talking about it may be different.

Also, some courses will use formulas based on mathematics that you have not yet learned. You may not be ready to understand why these formulas work. Sometimes you have to accept that the formula is correct.

Making connections between different courses takes work. It would be best if you were willing to ask questions. You may need to ask your math instructor about topics in your other classes. You may need to ask your other professors why they use different words or notations. It is your responsibility to make these connections. Your ability to do critical thinking in other classes depends on it.

1. There are many formulas that we hear about in daily news or history. Of all these famous formulas, name one.
2. Can you describe the use of the formula you mentioned and what each variable stands for?

Hidden Resources: CRC Hub

As a college student, you have access to so many resources that it can be overwhelming. Luckily, the CRC Hub is your "one-stop shop" for all local resources covering various topics (from healthcare to legal fees). Check out the CRC Hub here.

Answers

Section Preview

There is no subtlety to what is in this section or why I placed it here — you need radian measure to discuss either of the topics in this section, so it could not occur until this chapter. I suppose we could have covered this material directly after introducing radian measure (and prior to the Unit Circle definition of the trigonometric functions); however, I happened to place it here before thinking about that order. Feel free to make the switch if you feel so inclined (it actually might make the connection between the Unit Circle definition of the trigonometric functions and the area of the sector subtended by θ more digestible).

Section Preview

- Calculate the area of a sector formed by a central angle.
- Solve an application involving sector area.
- Calculate the linear velocity of a point moving with uniform circular motion.
- Calculate the angular velocity of a point moving with uniform circular motion.
- Convert from linear velocity to angular velocity and vice versa.
- Solve application problems involving linear and angular velocity.

Example 4.4.1

The most common mistake when using the formula for the area of a sector is not working in radian measure. Since $120^\circ = \frac{2\pi}{3}$, we find the area of the sector to be

$$A = \frac{1}{2} \cdot (4)^2 \left(\frac{2\pi}{3}\right) = \frac{16\pi}{3}.$$

Checkpoint 4.4.1

$\frac{25}{18}$ radians

Example 4.4.2**: The Midnight Sun Golf Challenge**

We are given $\theta = 30^\circ = \frac{\pi}{6}$ radians and $r = 15$ meters. Therefore, the area of the sector is

$$A = \frac{1}{2} \cdot (15)^2 \frac{\pi}{6} = \frac{225\pi}{12} \approx 59 \,\text{square meters}.$$

As an aside, the probability that the golfer's ball would land in this sector (given the ball lands on the green and that all areas are equally likely for the ball to land in) would be the ratio of this sector area to the area of the entire green, which is

$$\frac{A_{\text{sector}}}{A_{\text{green}}} = \frac{225\pi/12}{\pi(15)^2} = \frac{1}{12} \approx 0.083 = 8.3\%.$$

This should make sense because 30° is one-twelfth of a full rotation of a circle.

Checkpoint 4.4.2

No. The given measurements yield a central angle of 0.7 radians. Thus, the area is actually

$$A = \frac{1}{2}(120)^2(0.7) = 5,040 \,\text{square feet}.$$

Example 4.4.3

Let $s = 168.3$ inches and $t = 5$ seconds. Then the average velocity of the Saharan silver ant is

$$\overline{v} = \frac{168.3\,\text{inches}}{5\,\text{seconds}} = 33.66\,\text{inches per second}.$$

Checkpoint 4.4.3

13.04 miles per hour.

Example 4.4.4

We need to compute how far along the edge of the circle formed by the Dubai Eye that a rider moves through in a certain amount of time. Luckily, we were told a rider completes a full circuit in 38 minutes. Therefore, we let $t = 38$ minutes. Moreover, the distance traveled in that time is the same as the circumference of the circle. Here, we can use the faithful formula,

$$C = 2\pi r.$$

The diameter of the Dubai Eye is 820 feet, so the radius is 410 feet. Thus, a rider travels a distance of

$$s = C = 2\pi(410\,\text{feet}) = 820\pi\,\text{feet}.$$

We now use our velocity formula,

$$v = \frac{s}{t} = \frac{820\pi\,\text{feet}}{38\,\text{minutes}} \approx 67.8\,\text{feet per minute}.$$

For reference, this is a glacial 0.8 miles per hour.

Checkpoint 4.4.4

$v = 2.5$ inches per second

Example 4.4.5

Remember that the formula, $s = r\theta$, requires θ to be in radians. This is the same θ as the angle in $\omega = \frac{\theta}{t}$. Therefore, we need to convert the given angle to radians.

$$\omega = \frac{38^\circ}{3\,\text{hours}} = \frac{38\pi\,\text{radians}}{540\,\text{hours}} \approx 0.2211\,\text{radians per hour}.$$

Checkpoint 4.4.5

$\frac{5\pi}{6}$ radians per hour

Example 4.4.6

To use the formula $v = r\omega$, we first need to compute the angular velocity ω. The Earth makes one revolution in 24 hours. One revolution is 2π radians, so

$$\omega = \frac{2\pi\,\text{radians}}{24\,\text{hours}} = \frac{\pi}{12\,\text{hours}},$$

where, once again, we are using the fact that radians are real numbers and are dimensionless. (For simplicity's sake, we are also assuming that we are viewing the rotation of the Earth as counterclockwise, so $\omega > 0$.) Hence, the linear velocity is

$$v = 3958\,\text{miles} \cdot \frac{\pi}{12\,\text{hours}} \approx 1036\,\frac{\text{miles}}{\text{hour}}.$$

Checkpoint 4.4.6

≈ 17 miles per hour

Example 4.4.7

In a complex problem like this, it's best to collect the given data by object. That is, we will have data for the chainring, the sprocket, and the rear wheel. We might not use all of these objects in our computations, but just listing them out is often helpful.

	Chainring	**Sprocket**	**Wheel**
Radius	$r_c = 2\,\text{inches}$	$r_s = 1.5\,\text{inches}$	$r_w = 13.5\,\text{inches}$
LinearVelocity	$v_c = ?$	$v_s = ?$	$v_w = ?$
AngularVelocity	$\omega_c = ?$	$\omega_s = ?$	$\omega_w = ?$

Before we get into computations, a little bit of logic is helpful. First, how quickly the sprocket is spinning should match how quickly the rear wheel is spinning (in angles per unit time). Therefore, $\omega_s = \omega_w$. Additionally, we were told the cyclist is pedaling at 250 rpm (revolutions per minute). Thus, the chainring (which the pedal is attached to) is spinning at 250 rpm. We need to convert this to "radians per minute" using the fact that there are 2π radians per revolution. Therefore,

$$250\,\text{rpm} = \frac{250\,\text{revolutions}}{\text{minute}} = \frac{250\ \cancel{\text{revolutions}}}{\text{minute}} \cdot \frac{2\pi\,\text{radians}}{\cancel{\text{revolution}}} = \frac{500\pi\,\text{radians}}{\text{minute}}.$$

Notice that we kept the word "radians" in this computation. It's not necessary to do this, but it allows us to easily interpret the meaning of the quantity we are working with. At this point, let's update our table.

	Chainring	Sprocket	Wheel
Radius	$r_c = 2\,\text{inches}$	$r_s = 1.5\,\text{inches}$	$r_w = 13.5\,\text{inches}$
LinearVelocity	$v_c = ?$	$v_s = ?$	$v_w = ?$
AngularVelocity	$\omega_c = 500\pi\,\dfrac{\text{radians}}{\text{minute}}$	$\omega_s = \omega_w$	$\omega_w = \omega_s$

We now use the completed information about the chainring (the radius and angular velocity) to find the linear velocity of a point on the edge of the chainring.

$$v_c = r_c\omega_c = (2\,\text{inches})(500\pi\,\text{radians/minute}) = 1000\pi\,\frac{\text{inches}}{\text{minute}}$$

Remember, radians is a unitless measure. Since a point along the edge of the chainring is just a link in the chain connecting the chainring to the sprocket, this will also be the linear velocity of a link moving along the edge of the sprocket. Thus, our table gets updated to the following.

	Chainring	Sprocket	Wheel
Radius	$r_c = 2\,\text{inches}$	$r_s = 1.5\,\text{inches}$	$r_w = 13.5\,\text{inches}$
LinearVelocity	$v_c = 1000\pi\,\dfrac{\text{inches}}{\text{minute}}$	$v_s = 1000\pi\,\dfrac{\text{inches}}{\text{minute}}$	$v_w = ?$
AngularVelocity	$\omega_c = 500\pi\,\dfrac{\text{radians}}{\text{minute}}$	$\omega_s = \omega_w$	$\omega_w = \omega_s$

We now can find the angular velocity of the sprocket.

$$v_s = r_s\omega_s \implies \omega_s = \frac{v_s}{r_s} = \frac{1000\pi\ \cancel{\text{inches}}/\text{minute}}{1.5\ \cancel{\text{inches}}} = \frac{2000\pi}{3}\,\frac{\text{radians}}{\text{minute}}$$

Again, this angular velocity is exactly the same as the angular velocity of the wheel. Thus, our table becomes

	Chainring	Sprocket	Wheel
Radius	$r_c = 2\,\text{inches}$	$r_s = 1.5\,\text{inches}$	$r_w = 13.5\,\text{inches}$
LinearVelocity	$v_c = 1000\pi\,\dfrac{\text{inches}}{\text{minute}}$	$v_s = 1000\pi\,\dfrac{\text{inches}}{\text{minute}}$	$v_w = ?$
AngularVelocity	$\omega_c = 500\pi\,\dfrac{\text{radians}}{\text{minute}}$	$\omega_s = \dfrac{2000\pi}{3}\,\dfrac{\text{radians}}{\text{minute}}$	$\omega_w = \dfrac{2000\pi}{3}\,\dfrac{\text{radians}}{\text{minute}}$

We finally arrive at the penultimate computation—the linear velocity of the wheel.

$$v_w = r_w\omega_w = (13.5\,\text{inches})\left(\frac{2000\pi}{3}\,\frac{\text{radians}}{\text{minute}}\right) = 9000\pi\,\frac{\text{inches}}{\text{minute}}$$

Did you know the word "penultimate" means "second-to-last?" This means we have one more step (the ultimate step?). First, the linear velocity of a point on the wheel gives the velocity of the bicycle; however, this is in inches per minute and we were asked to find the answer in miles per hour. This is a nice **unit analysis** problem. Let's convert the distance units first.

$$9000\pi\,\frac{\cancel{\text{inches}}}{\text{minute}}\cdot\frac{\cancel{\text{foot}}}{12\ \cancel{\text{inches}}}\cdot\frac{\text{mile}}{5280\ \cancel{\text{feet}}} = \frac{9000\pi}{(12)(5280)}\,\frac{\text{miles}}{\text{minute}}$$

Now let's convert the time units.

$$\frac{9000\pi}{(12)(5280)}\,\frac{\text{miles}}{\cancel{\text{minute}}}\cdot\frac{60\ \cancel{\text{minutes}}}{\text{hour}} = \frac{(9000\pi)(60)}{(12)(5280)}\,\text{mph} \approx 26.8\,\text{mph}$$

Checkpoint 4.4.7

Approximately 97 rpm.

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4.4.1: Resources and Key Concepts

Resources

Videos

- Area of a Sector
  - Angle Measure - Area of a Sector
- Circular Motion
  - Angle Measure - Circular Motion

Key Concepts

Definitions

- **Sector (of a circle):** The region of a circle that is bounded by a central angle's initial and terminal sides and the arc it subtends.
- **Average Velocity:** The average rate of change of an object's spatial position with respect to time.
- **Instantaneous Velocity:** The velocity of an object at a specific moment in time. Often simply called "velocity."
- **Speed:** The magnitude of an object's instantaneous velocity, given by the formula $\text{speed} = |v|$.
- **Angular Velocity:** The rate of change of an angle with respect to time, denoted by ω. It is calculated as $\omega = \frac{\theta}{t}$, where θ is in radians.
- **Ordinary Frequency:** The number of cycles or revolutions an object completes per unit of time, denoted by f.
- **Angular Frequency:** The number of radians an object turns through per unit of time. It is equivalent to angular velocity, ω.
- **Period (of circular motion):** The amount of time it takes for an object in circular motion to complete one full cycle. It is the reciprocal of the ordinary frequency, $T = \frac{1}{f}$.

Theorems

- **Area of a Sector:** If θ (in radians) is the central angle in a circle of radius r, then the area of the sector subtended by θ is $A = \frac{1}{2}r^2\theta$.
- **Linear and Angular Velocity Formula:** For an object moving on a circular path of radius r with constant angular velocity ω, the linear velocity of the object is given by $v = r\omega$.

Common Mistakes

- **Using Degrees in Formulas:** The formulas for the area of a sector ($A = \frac{1}{2}r^2\theta$) and the relationship between linear and angular velocity ($v = r\omega$) both require the angle θ and the angular velocity ω to be in radian-based units. Using degrees will lead to incorrect answers.
- **Confusing Velocity and Speed:** Velocity can be negative, indicating direction, while speed is the magnitude of velocity and is never negative.

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4.4.2: Homework

Reading Questions

1. What is a sector of a circle?
2. What is the formula for the area of a sector, and what unit must the central angle be in for the formula to be valid?
3. What is the difference between linear velocity and angular velocity?
4. What is the formula that relates linear velocity (v) and angular velocity (ω) for an object moving in a circle of radius r?
5. What are the typical units for angular velocity?
6. What is the difference between ordinary frequency and angular frequency?
7. What is the period of circular motion, and how does it relate to ordinary frequency?
8. If a point on a circle rotates through an angle of 38° in 3 hours, what must you do first before calculating its angular velocity?
9. Why is it acceptable to say that the units of $r\omega$ are $\frac{\text{length}}{\text{time}}$ even though the units of ω are $\frac{\text{radians}}{\text{time}}$?

Homework

Vocabulary Check

1. The region bounded by the initial and terminal sides of a central angle θ and the edge of the circle is called a ___ of a circle.

2. A word commonly used in place of spatial position is ___.

3. The ___ of a particle is the average rate of change of its position with respect to a change in time. If we could take an instantaneous "snapshot" of this form of motion, we would call it the ___ of the particle.

4. The ___ of a particle is the magnitude of its velocity.

Concept Check

5. Write each statement as a mathematical formula.

   a. The length of the arc, s, is equal to:

   $$(\text{the fraction of a revolution}) \times (\text{circumference of the circle})$$

   b. The area of the sector, A, is equal to:

   $$(\text{the fraction of a revolution}) \times (\text{area of the circle})$$

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

6. To compute the area of a sector, we multiply the given angle by half of the square of the circle's radius.

7. Given two circles, one of radius r_1 and the other of radius r_2, particles on each circle moving at a constant angular velocity, ω, move at the same linear velocity.

Basic Skills

For the following exercises, θ is the central angle in a circle of radius r. In each case, find the area of the sector.

8. $\theta = \frac{2\pi}{3}, \qquad r = 7 \text{ yards}$

9. $\theta = \frac{7\pi}{6}, \qquad r = \sqrt{2} \text{ m}$

10. $\theta = \frac{6\pi}{7}, \quad r = \sqrt{3}$ miles

11. $\theta = 3, \quad r = \frac{1}{2}$

12. $\theta = 5, \quad r = 4.1$ cm

13. $\theta = 60^\circ, \quad r = 2$ feet

14. $\theta = 330^\circ, \quad r = \pi$ inches

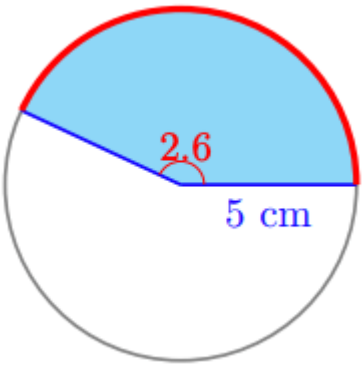
15. $\theta = 32^\circ, \quad r = 8$ parsecs

16. (MyOpenMath) $\theta = 212^\circ, \quad r = 0.88$ leagues

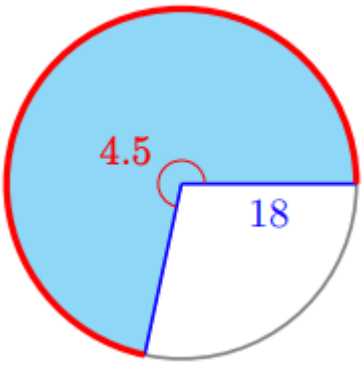
For the following exercises, find the area of the shaded sector.

17.

a.



b.



Determining Areas of Sectors

18. An arc of length 8 feet is cut off by a central angle of $2\pi/3$ radians. Find the area of the resulting sector.

19. A central angle of 4 radians cuts off an arc of length 16 meters. Find the area of the resulting sector.

20. (MyOpenMath) Calculate the area of a sector with a radius of 6 cm and a central angle of 30°.

21. A sector has a radius of 4 meters and a central angle of 45°. Find its area.

22. (MyOpenMath) If the radius of a sector is 7.4 inches and the central angle is 225°, what is the area of the sector?

23. Determine the area of a sector with a radius of 8 feet and central angle $2\pi/3$.

24. Calculate the area of a sector with a 3-meter radius and central angle $5\pi/6$.

Determining the Radius of a Sector

25. If the area of a sector is 25 cm^2 and the central angle is $45°$, find the radius.

26. Determine the radius of a sector with an area of 30 m^2 and a central angle of 60 degrees.

27. A sector has an area of 14 in^2 and central angle $\pi/6$. What is its radius?

28. What is the radius of a sector with an area of 78 cm^2 and a central angle of 120 degrees?

29. What is the radius of a sector if the area is 100 cm^2 and the central angle is 250 degrees?

Determining the Central Angle of a Sector

30. Find the radian measure of the central angle of a sector with a radius of 5 cm and an area of 20 cm^2.

31. Determine the radian measure of the central angle if a sector has a radius of 3 meters and an area of 15 m^2.

32. What is the degree measure of the central angle of a sector with a radius of 7 inches and an area of 24 in^2?

33. Calculate the degree measure of the central angle for a sector with a radius of 10 feet and an area of 50 ft^2.

34. Calculate the degree measure of the central angle for a sector with a radius of 11 cm and an area of 30.25 cm^2.

Determining Angular Velocity

35. **Wheel Rotation.** If a wheel rotates 300 times in one minute, find its angular velocity in radians per second.

36. **Merry-go-Round.** How many complete revolutions does a merry-go-round make in 10 minutes if it has an angular velocity of 0.5 radians per second?

37. **Mixing Machine.** A mixing blade in a blender rotates at 1200 revolutions per minute. Calculate its angular velocity in radians per second.

38. **Computer Hard Drive.** The spindle of a computer hard drive spins at 7200 RPM. What is the angular velocity of the spindle in radians per second?

39. **Electric Generator.** An electric generator's rotor spins at 3600 revolutions per minute. Determine the rotor's angular velocity in radians per second.

40. **Laundry Machine.** During the spin cycle, a washing machine drum rotates at 800 revolutions per minute. Calculate the angular velocity in radians per second.

41. **Vehicle's Alternator.** The alternator in a car operates at 6000 RPM when the engine is at full speed. Find the angular velocity of the alternator in radians per second.

Converting Between Linear and Angular Velocities

42. **Bike Tire.** A bicycle tire has a radius of 35 cm. If the tire rotates at an angular velocity of 2 radians per second, calculate the linear velocity of a point on the tire's rim.

43. **Circular Machinery.** A part of a machine moves in a circle of radius 15 cm at a constant speed. If the linear velocity of the part is 90 cm/s, calculate its angular velocity in radians per second.

44. **Ceiling Fan Speed.** The blade of a ceiling fan is 0.7 meters long from the center to the tip. If the tip of the blade moves at a

linear velocity of 4.5 meters per second, calculate the angular velocity of the blade in radians per second.

45. **Amusement Park Ride.** A merry-go-round has a radius of 5 meters. If a horse on the edge of the ride moves at a linear velocity of 3 meters per second, find the angular velocity of the merry-go-round in radians per second.

46. **Industrial Flywheel.** An industrial flywheel has a radius of 2 meters. If it is required to achieve a linear speed of 10 meters per second at its rim, determine the necessary angular velocity in radians per second.

47. **Athletics Track Runner.** A runner is jogging along a circular track with a radius of 25 meters, maintaining a constant linear speed of 2.5 meters per second. Calculate the runner's angular velocity in radians per second.

48. **Spinning Top.** A spinning top has a radius of 8 cm. If the linear velocity of a point on its edge is 1.6 meters per second during a spin, what is the angular velocity of the top in radians per second?

49. **Windmill Blades.** A windmill's blade has an angular velocity of 0.8 radians per second and each blade is 3 meters long. Calculate the linear velocity at the tip of a blade.

50. **Rotating Sign.** A circular sign rotates at an angular velocity of 2 radians per second. If the radius of the sign is 1.5 meters, what is the linear velocity of the edge of the sign?

51. **Gear in a Machine.** A gear in a factory machine has a radius of 20 cm and rotates with an angular velocity of 5 radians per second. Determine the linear velocity of a point on the edge of the gear.

52. **Carousel Horse.** A carousel horse moves in a circular path with a radius of 4 meters. If the angular velocity of the carousel is 1 radian per second, find the linear velocity of the horse.

53. **Bicycle Pedaling Speed.** A bicycle's pedal arm is 17.5 cm long, and the cyclist is pedaling at an angular velocity of 4 radians per second. Calculate the linear velocity at the end of the pedal arm.

Synthesis Questions

54. Calculate the radius of a sector whose area is one-fifth of the area of a circle with a radius of 10 cm.

55. Determine the radius of a sector if its area is equivalent to half the area of a circle with a radius of 14 meters.

56. Two circular saw blades are spinning, one at 120 RPM and the other at 180 RPM. If both blades have a diameter of 30 cm, which blade has a higher linear velocity at the rim, and by how much?

Applications

Sector Area Applications

57. **Land Allocation.** A farmer plans to allocate a portion of his circular field for a special crop. The field has a radius of 100 meters. If he uses a 60-degree sector for this crop, calculate the area that will be cultivated.

58. **Clock Face Design.** A designer is creating a new clock face where each hour marker is represented by a different colored sector. The clock has a radius of 12 cm. Calculate the area of one hour sector assuming a simple 12-hour clock.

59. **Architectural Design.** An architect designs a circular window with decorative sectors. Each sector has a central angle of 45 degrees and the window has a diameter of 2 meters. Calculate the area of one decorative sector.

60. **City Planning.** A city planner needs to design a roundabout garden that occupies a 38-degree sector of a circle with a radius of 20 meters. Calculate the area available for planting.

61. **Pizza Slicing.** A pizza place offers a unique pizza slice shaped as a sector. The pizza has a diameter of 40 cm. If the central angle for each slice is 20 degrees, find the area of one slice of pizza.

62. **Sports Fields.** A sports complex includes a sector-shaped running track portion used for sprinting drills. If the track follows

a 40-degree sector of a circle with a radius of 100 meters, calculate the area of this track section.

53. **Public Park Design.** A public park is designed with a circular area for a water fountain, surrounded by flower beds laid out in sectors. If the circular area has a radius of 15 meters and each sector for the flower beds is 30 degrees, find the area of one flower bed.

54. **Non-Uniform Rainfall.** A circular region with a radius of 20 km receives rainfall. However, the rain is not uniformly distributed. If a 120-degree sector receives twice the amount of rain per square kilometer as the rest of the region, calculate the total rainfall if the sector received 600 mm.

55. **Engineering Design.** An engineer is designing a gear in the shape of a sector. The gear must mesh with another gear with an area three times larger but with the same arc length. If the radius of the smaller gear is 15 cm, determine the central angle of the sector for the smaller gear.

56. **Advanced Pizza Slicing.** A circular pizza is sliced such that each consecutive slice has a central angle 10 degrees greater than the previous one, starting from 10 degrees. Calculate the area of the 5th slice if the pizza has a radius of 30 cm.

57. **Sector-based Cooling Zones.** A circular mechanical part is designed with cooling zones in the form of sectors. Each zone dissipates heat differently. If one zone with a 60-degree angle dissipates heat at 100 watts per square meter and the radius of the part is 30 cm, find the total heat dissipation for this zone.
(*Hint:* Heat dissipation is the total number of watts of heat released in a given area.)

Velocity Applications

58. Find the angular velocity of the following objects.

    a. The London Eye Ferris wheel, which makes one revolution every 20 minutes.

    b. An old-fashioned long-playing record, which revolved $33\frac{1}{3}$ times in 60 seconds.

59. **Conveyor Belt.** A conveyor belt moves at a linear velocity of 1.5 meters per second. If the driving drum has a radius of 0.5 meters, calculate the angular velocity of the drum in radians per second.

70. **Bike Wheel Spin.** A bicycle wheel has a radius of 0.4 meters. If the wheel is spinning with an angular velocity of 5 radians per second, what is the linear velocity of a point on the rim?

71. **Rollerblade Wheel.** A rollerblade wheel has a radius of 40 mm. If a rollerblader is moving at a linear speed of 10 km/h, find the angular velocity of the wheels in radians per second.

72. **Ferris Wheel.** A Ferris wheel has a radius of 25 meters and rotates at a linear velocity of 2 meters per second at its rim. Calculate its angular velocity in radians per second.

73. **Tire Rotation (MyOpenMath).** A truck with 20-inch-diameter tires is traveling at 50 miles per hour.

    a. Find the angular velocity of the tire in radians per minute.

    b. How many revolutions per minute do the tires make?

74. **Tire Rotation.** A car tire has a radius of 35 cm. If the car is traveling at 80 km/h, find the angular velocity of the tire in radians per second.

75. **Potter's Wheel Speed.** If the edge of a potter's wheel moves at a linear velocity of 15 m/s and the wheel has a radius of 1.2 meters, determine the angular velocity in radians per second.

76. **Drill Bit Rotation.** A drill bit has a radius of 1 cm. If it needs to achieve a linear velocity of 20 m/s at its tip, calculate the required angular velocity in radians per second.

77. **Record Player.** If a point on a record player is moving with a linear velocity of 0.5 m/s and the record has a radius of 15 cm, what is the angular velocity in radians per second?

78. **Electric Fan.** An electric fan has blades that are each 0.25 meters long. If a point on the tip of a blade moves with a linear velocity of 3 m/s, what is the angular velocity of the blades in radians per second?

79. **Satellite Dish Rotation.** A satellite dish has a radius of 2 meters. If it rotates such that a point on its edge moves at a linear velocity of 0.1 m/s, calculate the angular velocity in radians per second.

80. **Fan Speed.** A ceiling fan has blades that are each 0.75 meters long. If the fan spins at 200 revolutions per minute (RPM), calculate the angular velocity in radians per second.

81. **Merry-Go-Round.** Children are riding on a merry-go-round that has a radius of 2 meters and makes one complete rotation every 4 seconds. Find the angular velocity in radians per second.

82. **Driving Speed and Wheel Rotation.** A car with wheels of radius 30 cm is traveling at a speed of 60 km/h. Calculate the angular velocity of the wheels in radians per second.

83. **Bicycle Dynamics.** A bicycle's front wheel has a diameter of 70 cm and rotates at 150 RPM while cruising. Determine the linear velocity of the bicycle.

84. **Angular Velocity of a Clock's Second Hand.** Calculate the angular velocity in radians per hour of the second hand of a clock.

85. **Disk Rotation.** A disk in a laboratory experiment rotates at a speed of 600 RPM. Calculate its angular velocity in radians per second.

86. **Jogging on a Track.** A jogger runs around a circular track with a diameter of 200 meters at a speed of 8 km/h. Compute the angular velocity of the jogger in radians per hour.

87. **Wheelchair Wheel Rotations.** A wheelchair has wheels of diameter 60 cm. If a person moves at a speed of 4 km/h, determine the angular velocity of the wheels in radians per second.

88. **Spinner Toy.** A spinner toy rotates 4000 times in one minute. Calculate its angular velocity in radians per second.

89. **Record Player.** A record player spins a record at 33 revolutions per minute. Find the angular velocity in radians per second and the linear velocity at a point 10 cm from the center.

90. **Dryer.** Clothes dryers draw 3.5 times as much power as washing machines, so newer machines have been engineered for greater efficiency. A vigorous spin cycle reduces the time needed for drying, and some front-loading models spin at a rate of 1500 rotations per minute.

    a. If the radius of the drum is 11 inches, how far do your socks travel in one minute?

    b. How fast are your socks traveling during the spin cycle?

91. **Hubble Telescope.** The Hubble telescope is in orbit around the earth at an altitude of 600 kilometers, and completes one orbit in 97 minutes.

    a. How far does the telescope travel in one hour? (The radius of the earth is 6400 kilometers.)

    b. What is the speed of the Hubble telescope?

92. **Energy-Producing Windmill.** The first large windmill used to generate electricity was built in Cleveland, Ohio in 1888. Its sails were 17 meters in diameter, and moved at 10 rotations per minute. How fast did the ends of the sails travel?

93. **Largest Windmill.** The largest windmill operating today has wings 54 meters in length. To be most efficient, the tips of the wings must travel at 50 meters per second. How fast must the wings rotate?

94. **Planetary Motion.** Assume a simplified model where a planet moves in a circular orbit around its star with a constant angular velocity. If the radius of the orbit is 1.5×10^6 kilometers and it completes one orbit in 215 days, find the linear velocity of the planet.

95. **Wheel Rotation.** A car's tire has a diameter of 60 cm. If the car is traveling at a speed of 72 km/h, calculate the angular velocity of the tire in radians per second.

96. **CD Player Speed.** A CD rotates at 500 revolutions per minute (RPM) when playing. What is the linear velocity at a point 3 cm from the center of the CD?

97. **Angular Speed of a Wind Turbine.** A wind turbine blade is 45 meters long, and it completes one rotation every three seconds. Calculate the angular velocity in radians per second and the linear velocity at the tip of the blade.

98. **Bicycle Wheel Dynamics.** A bicycle wheel has a radius of 35 cm. If the bicycle is moving at a speed of 15 km/h, find the wheel's angular velocity in radians per second.

99. **Roller Coaster Speed.** A roller coaster moves in a vertical loop of radius 12 meters at a speed of 36 km/h at the highest point of the loop. Calculate the angular velocity at this point.

100. **Flywheel Operation.** A flywheel in a machine spins at 1800 RPM. If the radius of the flywheel is 20 cm, calculate the linear velocity at the edge of the flywheel.

101. **Satellite Orbiting.** A satellite orbits the Earth in a circular path with a radius of 10,000 km at a constant speed of 7 km/s. Determine the angular velocity of the satellite in radians per hour.

102. **Speed of a Ferris Wheel.** A Ferris wheel with a diameter of 100 meters takes 4 minutes to make one complete revolution. Calculate the linear velocity and angular velocity of a cabin at the edge of the Ferris wheel.

103. **Potter's Wheel.** A potter's wheel moves from rest to an angular speed of 120 RPM in 5 seconds. Calculate the angular acceleration and the linear velocity at a point 15 cm from the center at the end of 5 seconds.

104. **Circular Saw.** A circular saw blade has a radius of 15 cm and rotates at a speed of 3000 RPM. Find the angular velocity in radians per second and the linear velocity at the perimeter of the blade.

Challenge Problems

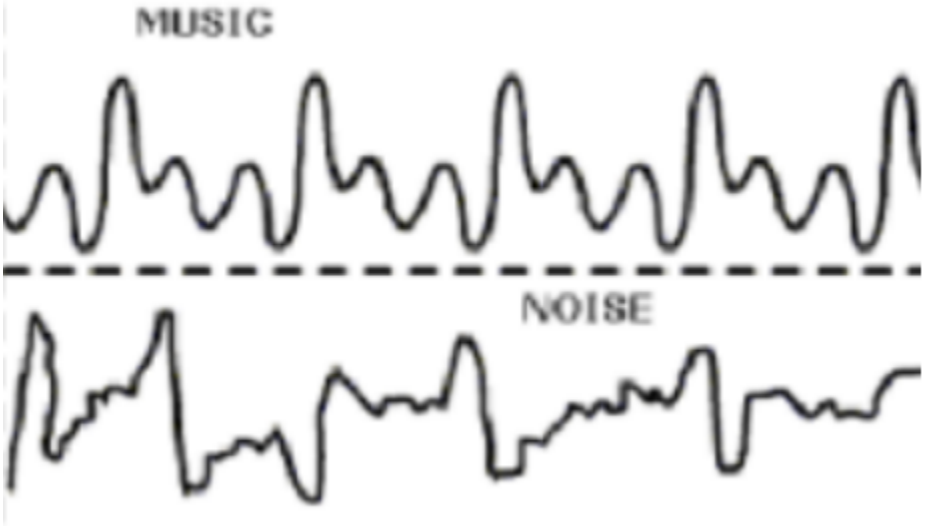
94. If the radius of a sector is doubled while the central angle remains the same, by what factor does the area increase?

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5: GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

Musical instruments produce sound by the vibration of a string on a violin, guitar, or piano, or a column of air in a brass or woodwind instrument. The vibration causes a periodic variation in air pressure that is heard as sound. Periodic vibrations create a pleasing, musical sound, while random vibrations sound like noise.

Figure 1

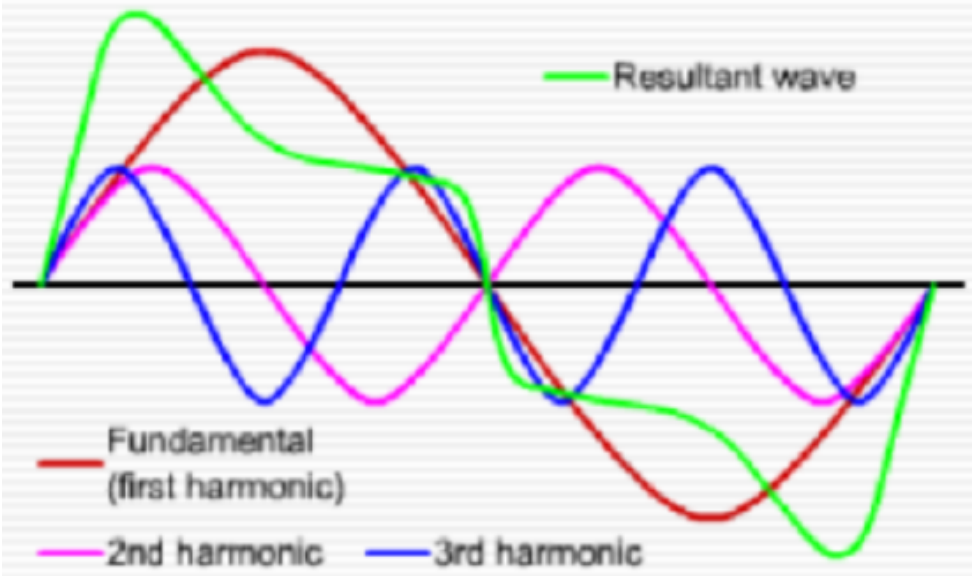


There are three distinguishing characteristics of musical notes: loudness, pitch, and timbre.

- The **loudness** of a note is measured by the magnitude, or amplitude, of the changes in air pressure.
- The **pitch** of the note is determined by its frequency, the number of times its basic pattern is repeated each second. For example, a note with pitch 440 hertz repeats its pattern 440 times per second. Humans can hear frequencies that range roughly from 20 to 18,000 hertz.
- The **timbre** of the note allows us to tell a piano note from a guitar note with the same loudness and pitch.

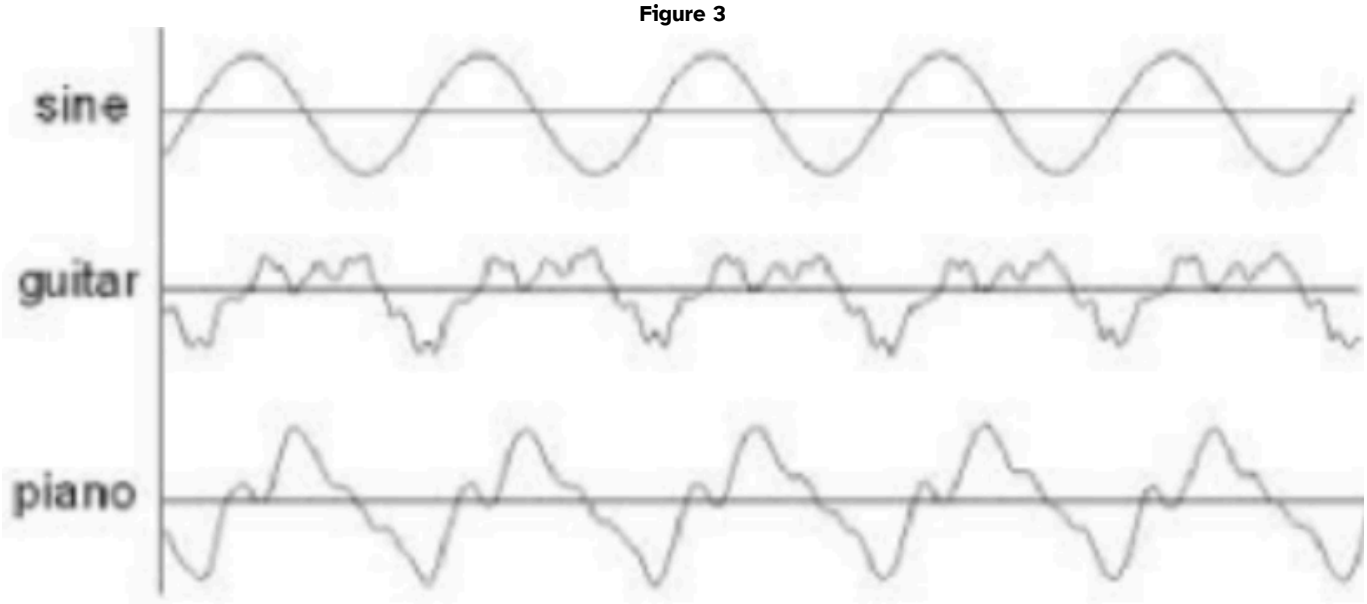
The simplest pressure vibration is produced by a tuning fork. The graph of its pressure function resembles a sine curve, and the corresponding sound is called a **pure tone**. When a musical instrument produces a pure tone, not only is its frequency produced, but integer multiples of the **fundamental frequency**, called **harmonics**, are produced as well. Sine curves of different frequencies combine to create the timbre of the note.

Figure 2



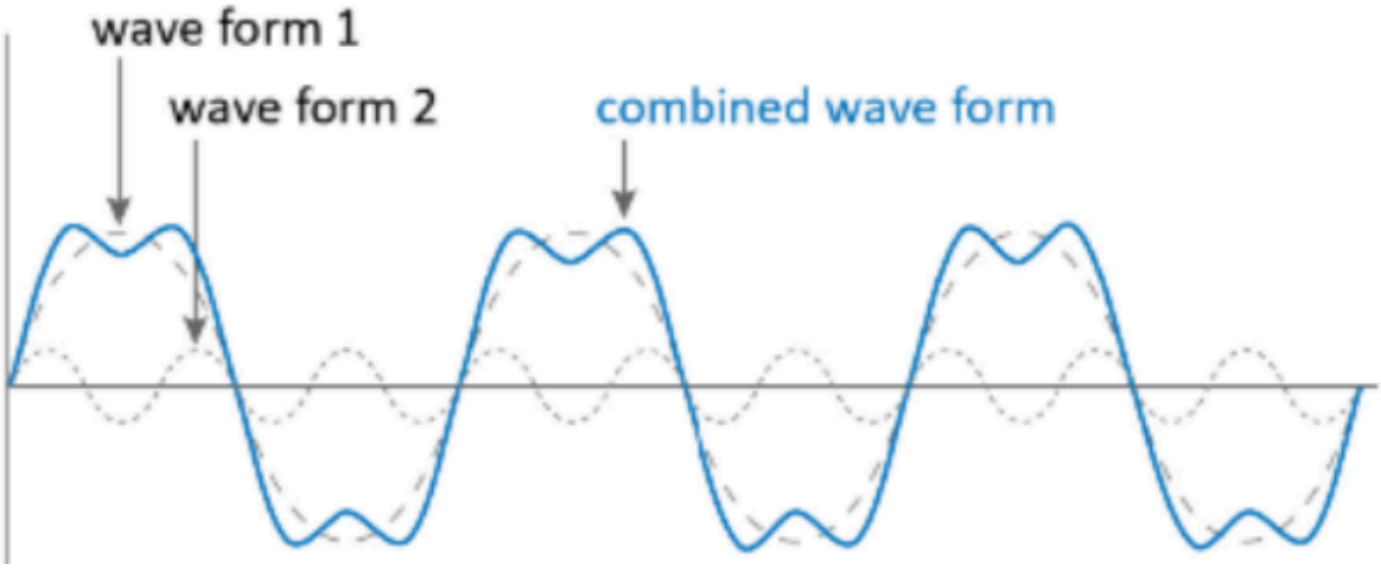
If two musical instruments play the same note, the notes have the same pitch, but they sound different because the amplitudes of each of the harmonics is different for the two instruments.

Figure 3



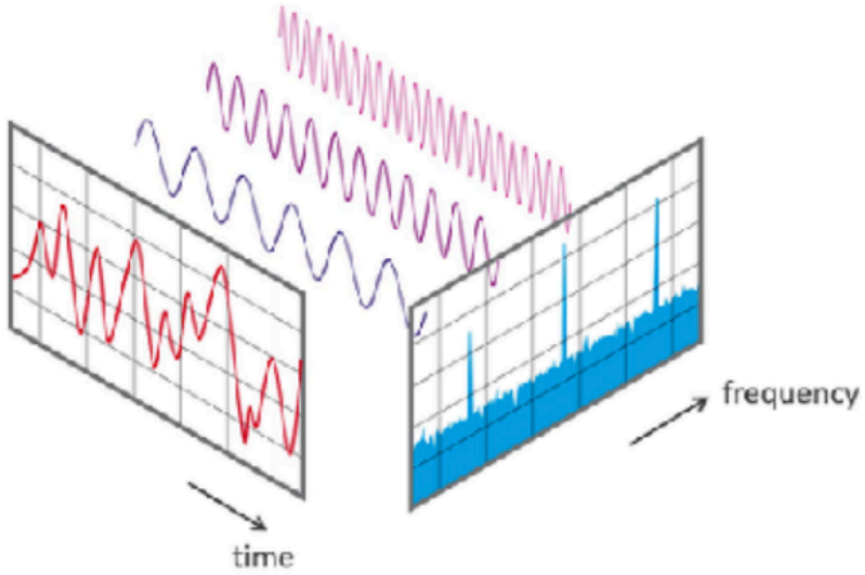
The French mathematician Joseph Fourier, who lived from 1768 to 1830, discovered that any periodic wave can be written as the sum of a number of sines and cosines with differing amplitudes.

Figure 4



Fourier created a method for determining the frequencies and amplitudes of the simpler waves that make up a more complicated periodic function. Applying the Fourier transform to a sampled musical note reveals which component frequencies are present in the note. The same sound can then be re-synthesized by including those frequency components.

Figure 5



Today, Fourier analysis is the foundation of signal processing, not only for audio waves, but for radio waves, light waves, seismic waves, and even images. By analyzing a compound waveform, the "noise" components can be isolated and removed in order to smooth out the signal.

Fourier analysis is also used in X-ray crystallography to reconstruct a crystal structure from its diffraction pattern, and in nuclear magnetic resonance spectroscopy to determine the mass of ions from the frequency of their motion in a magnetic field.

5.1: Base Graphs of the Fundamental Trigonometric Functions

5.1.1: Resources and Key Concepts

5.1.2: Homework

5.2: Non-Rigid Transformations

5.2.1: Resources and Key Concepts

5.2.2: Homework

5.3: Rigid Transformations

5.3.1: Resources and Key Concepts

5.3.2: Homework

5.4: Sinusoidal Models

5.4.1: Resources and Key Concepts

5.4.2: Homework

5.5: Harmonic Motion

5.5.1: Resources and Key Concepts

5.5.2: Homework

5.6: Graphs of the Remaining Trigonometric Functions

5.6.1: Resources and Key Concepts

5.6.2: Homework

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5.1: Base Graphs of the Fundamental Trigonometric Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

When you were learning Algebra, you were introduced to variables, expressions involving variables, equations involving variables, and graphs of those equations (in that order). In Trigonometry, this order is slightly modified. We have learned about trigonometric functions and manipulated expressions involving those functions; however, before solving equations involving those functions, it's best to be able to visualize their graphs.

This section focuses on the base graphs of the **fundamental trigonometric functions**.

Definition: Fundamental Trigonometric Functions

The sine, cosine, and tangent are called the **fundamental trigonometric functions**.

The graphs of the remaining trigonometric functions will be introduced after we have a few sections to get used to the graphs of the fundamental trigonometric functions.

Motivating Periodic Functions of Angles

Imagine that you are riding on a Ferris wheel. As the wheel turns, your height above the ground increases and decreases. This pattern repeats each time the Ferris wheel makes a complete rotation. This type of motion is called **periodic motion**.

Definition: Periodic Motion

When a body repeats its movement along a path in a regular and predictable manner over a specific time interval, its movement is called **periodic motion**.

In Mathematics, we seek to model periodic motion with a function.

Definition: Periodic Function

A mathematical function modeling periodic motion is called a **periodic function**. A periodic function repeats its values at regular intervals, meaning its graph has a pattern that repeats itself over a specific horizontal distance or time interval.

We use periodic functions to model phenomena that exhibit cyclical behavior, such as the height of tides, seasonal growth

patterns in plants and animals, radio waves, and planetary motion.

Let's create a mathematical model for a ride on a Ferris wheel. Suppose the Ferris wheel has a radius of 100 feet and rotates counterclockwise. Our model will be a function that describes the rider's height above the ground as they ride the wheel. To graph the Ferris wheel function, we must first specify the input and output variables and then choose a coordinate system to display their values.

While developing the model, let's place the origin at the center of the Ferris wheel (see Figure 5.1.1 below).

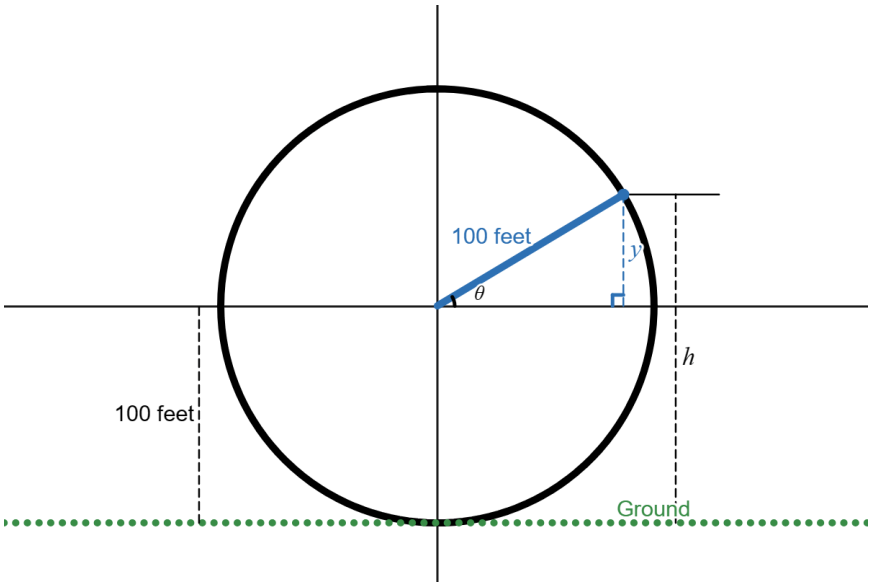


Figure 5.1.1**:** *A graphical representation of a 100-foot-radius Ferris wheel centered at the origin.*

While this is not the most realistic placement for us to imagine (it makes us think that the bottom of the Ferris wheel is below ground level), this temporary placement will reveal an important characteristic of the rider's motion. The line from the origin to the rider's position on the wheel makes an angle, θ, with the horizontal. This angle will be the independent variable for the function. The height of the rider, h, is also a variable and is related to the y-coordinate of the rider's position. Figure 5.1.1 shows that $h = y + 100$ because the center of the wheel is 100 feet above the ground.

Let's focus on the rider's height as a function of angle. Therefore, we'll first graph y as the dependent variable instead of h. As the angle θ increases from 0° to 90°, the rider's y-coordinate increases from 0 to 100. At that point, they are at the top of the wheel. Then, as θ increases from 90° to 180°, the rider's y-coordinate decreases from 100 back to 0 (see Figure 5.1.2).

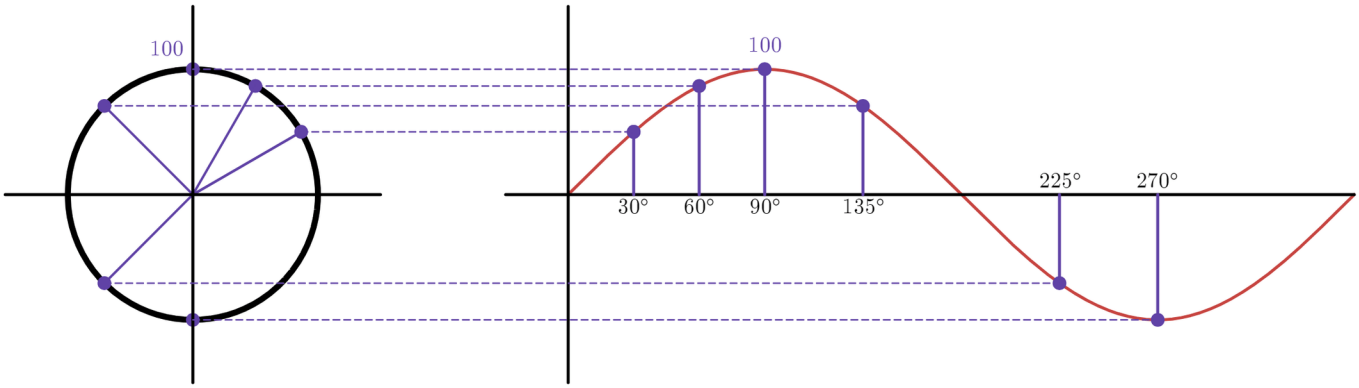


Figure 5.1.2**:** *"Unwinding" the circle to get a visual model of the height function.*

Finally, as θ increases from 180° to 360°, the rider's y-coordinate decreases from 0 to -100 and then increases from -100 back to 0. At that point, the rider has made one complete rotation on the Ferris wheel. If she goes around again, θ increases from 360° to 720°, and the graph of her y-coordinate will repeat the pattern of the first rotation. Again, Figure 5.1.2 shows how the rider's y

-coordinate is plotted as a function of the angle θ.

Look back at Figure $5.1.1$ and notice from the reference triangle that $\sin(\theta) = \frac{y}{100}$. Therefore, $y = 100\sin(\theta)$. For example, when $\theta = 30^\circ$, the y-coordinate of the rider is

$$y = 100\sin(30^\circ) = 100\left(\frac{1}{2}\right) = 50$$

and the rider's height above the ground is $h = y + 100 = 150$ feet. In general, then, h is given as a function of θ by

$$h = y + 100 = 100\sin(\theta) + 100.$$

This is our model for a rider's height on a Ferris wheel as a function of angle.

The graph on the right side of Figure $5.1.2$ is called a **sine curve**.

Definition: Sine Curve

A smooth, wave-like curve that oscillates (moves up and down) periodically and which can be derived from either the sine or cosine function is called a **sine curve**. A curve of this nature is often called **sinusoidal**,

While it might seem odd that we use the blanket phrase "sine curve" when describing a graph derived from the sine *or* the cosine, this consolidation will make sense shortly.

An Aside: Degrees or Radians?

Before we go any further, we need to make one thing clear:

> Figure $5.1.2$ is the first and the last time you will see a graph of the sine (or any other trigonometric) function in terms of degrees.

While there are several reasons that we endeavor only to graph trigonometric functions using radians, the two most impactful are the relation to the unit circle and scale. When we derived the unit circle definition of the trigonometric functions, we did so using radian measure. If we redraw Figure $5.1.2$ using radians as the angular measure (and the unit circle instead of a circle of radius 100), we get Figure $5.1.3$.

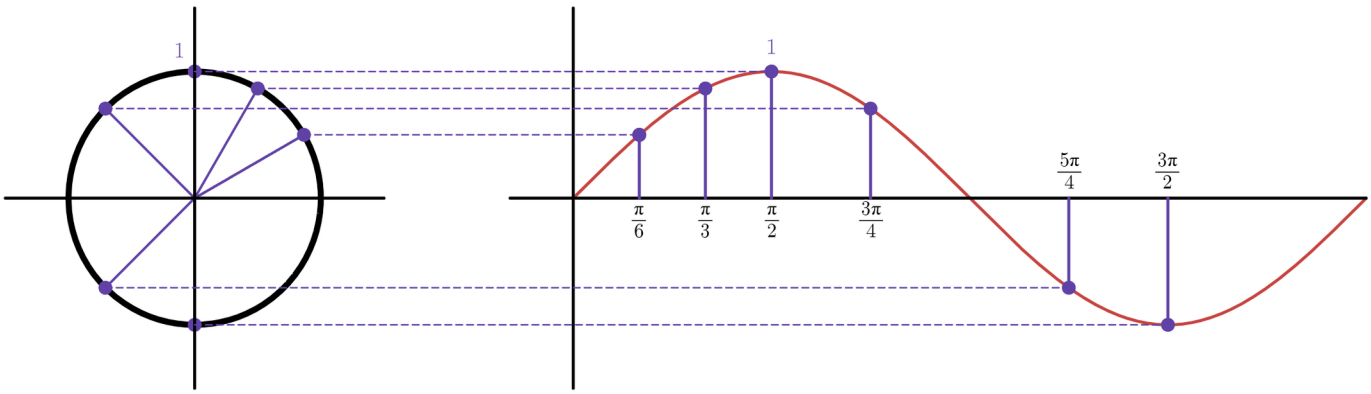


Figure $5.1.3$**:** *Drawing the sine curve using the unit circle.*

Note that the scale on the vertical axis (namely, that the largest and smallest y-values are 1 and -1, respectively) and the scale on the horizontal axis (from $x = 0$ to $x = 2\pi$) are more closely aligned than had we kept the angular measure in degrees.

Graphing the Sine Function

There are two main ways students initially learn to graph the sine function—by plotting points or by tracing the y-values along the

unit circle. The latter is what we did in Figure 5.1.3. The former is not the ideal method, but it's a good idea to showcase it right now so we can link the computed values of the sine function to points on a graph.

Point-Plotting Approach

Up to this point, we have mostly let sine be a function of the angle θ; however, we recently discovered (via our discussion of circular functions) that the trigonometric functions are also functions of real numbers. Therefore, we should feel fine working with $f(x) = \sin(x)$. The independent variable, x, could represent an angle, the arc length along a circle, or time—our application determines the context.

Example 5.1.1

Sketch a portion of the graph of $y = \sin(x)$ on the interval $[0, 2\pi]$ by making a table of values.

Solution available at the end of this section.

Checkpoint 5.1.1

Sketch a portion of the graph of $y = \sin(x)$ on the interval $\left[-\frac{3\pi}{2}, \frac{\pi}{2}\right]$ by making a table of values.

Answer available at the end of this section.

There are several issues with plotting points to obtain the sine function graph. The first is the amount of time spent computing values. We will be graphing the sine curve throughout the rest of our mathematical journey, and if we need to plot points every time we graph, we will be wasting time and effort. The second issue has to do with "connecting the dots." Those who have no idea of the curvature of the sine graph end up connecting the points with lines and arrive at something like Figure 5.1.6.

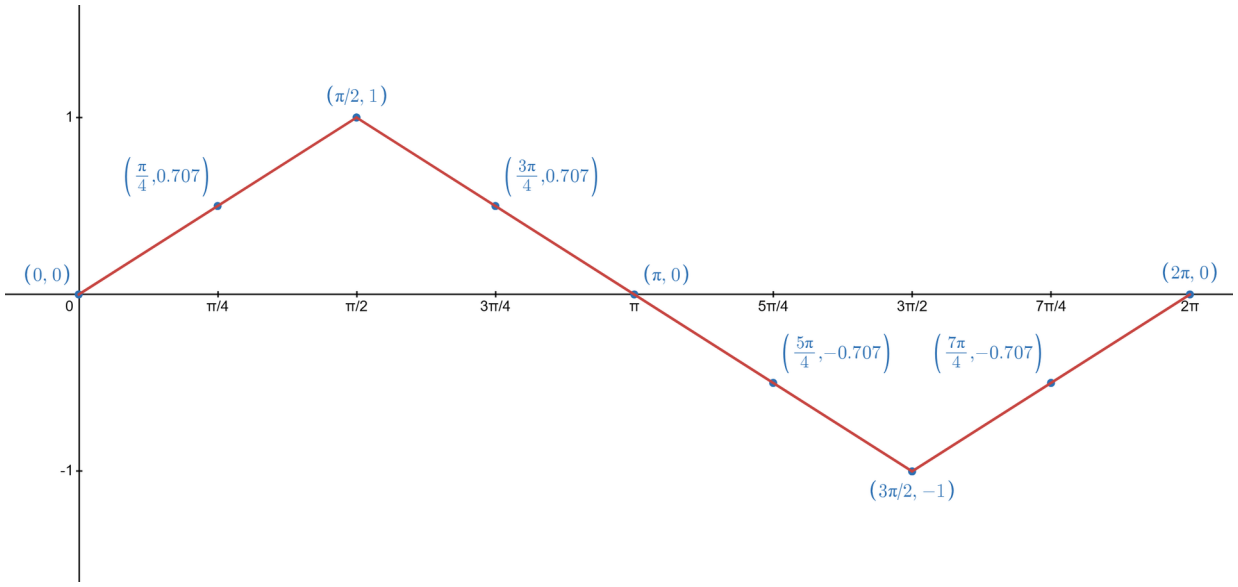


Figure 5.1.6: *NEVER GRAPH A TRIGONOMETRIC FUNCTION USING LINES*

I hope you can see the difference between the correct graph of the sine function from Example 5.1.1 and the incorrect graph represented in Figure 5.1.6.

The third issue with graphing using point-plotting involves building a natural understanding of the sine curve. Many features, like domain and range, slip out of the student's grasp, who consistently requires the "connect the dots" method when graphing trigonometric functions.

Finally, the fourth issue occurs when needing to extend the graph of the sine function beyond the interval $[0, 2\pi]$. As Checkpoint $5.1.1$ shows, and as we shall see in more concrete form shortly, the sine curve oscillates in both directions (as $x \to \infty$ and as $x \to -\infty$). If we needed to plot points to extend the graph, we would be plotting and computing for a long time.

With all these considerations in mind, we will avoid using point-plotting when graphing trigonometric functions (other than introducing the base graphs of the trigonometric functions).

Unit Circle Approach

We have already motivated the sine curve using the unit circle in Figure $5.1.3$; however, let's spend a few more moments solidifying our understanding of the relationship between the unit circle and the sine curve.

Recall, $\sin(t)$ is the value of the y-coordinate for the point P on the unit circle, where t is simultaneously the arc length from $(1, 0)$ to the point P and the angle subtended by this arc. The following Interactive Element shows how the sine curve is created by tracing these y-coordinates as the angle (or arc length) t increases from 0 to 2π (if you only have access to the PDF of this textbook, then please go to the following link: https://www.desmos.com/calculator/aq5r4ka4dd)

Interactive Element: Developing the Graph of the Sine Function via the Unit Circle

Observation: As the angle (or arc length) t increases, the y-coordinate increases to 1, then decreases to -1, then increases back to $y = 0$.

We now can easily "extend" the graph of the sine function by allowing t to go beyond 2π (https://www.desmos.com/calculator/imbgdopjtz).

Interactive Element: Extending the Graph of the Sine Function

Basic Features of the Sine Curve

Now that we know how to graph the base sine curve, we can dive into the details of this function.

Domain and Range

We now have enough knowledge to graph the *entire* sine curve. Figure $5.1.7$ shows the general sine curve along with a single cycle (in blue) on the interval $[0, 2\pi]$.

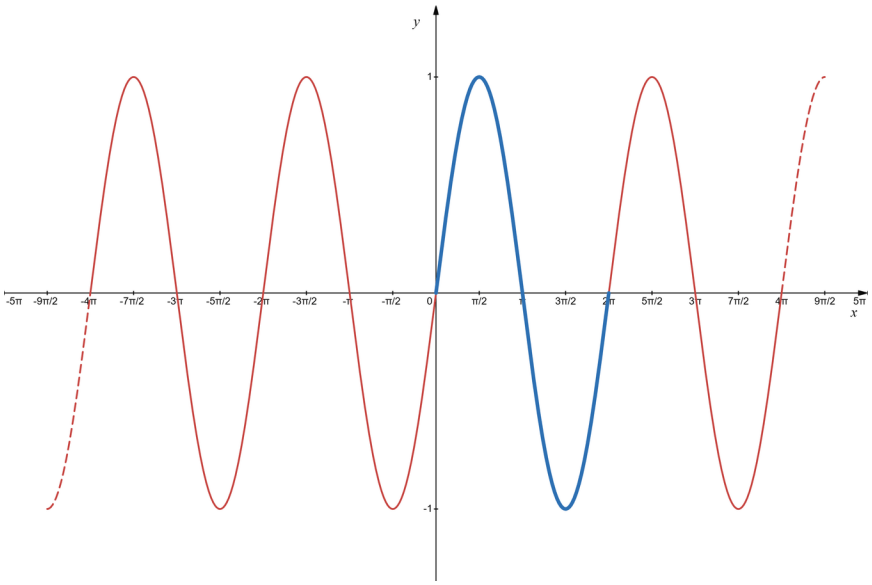


Figure 5.1.7**:** $y = \sin(x)$

Since the sine curve oscillates forever in both directions, it should be evident that the sine function has an output for every input. Therefore, the domain of the sine function is all real numbers. Moreover, we can see from the graph that the values of the sine function oscillate between -1 and 1 (inclusive). Hence, the range must be $[-1, 1]$. We summarize these results in the following theorem.

Theorem: Domain and Range of the Sine Function

The domain of $f(x) = \sin(x)$ is $(-\infty, \infty)$ and the range is $[-1, 1]$.

Caution: Is it $\sin(x)$ or $\sin(t)$?

It is common for those learning Trigonometry to need clarification about the meaning of the argument of the sine function at this point. When we speak of the unit circle, we are speaking of the graph of all solutions, (x, y), to the equation

$$x^2 + y^2 = 1.$$

As was mentioned in a previous section, the x-coordinate of a point on the unit circle is defined to be $\cos(t)$ and the y-coordinate is defined to be $\sin(t)$, where t is simultaneously the arc length along the unit circle to the point in question and the angle subtended by that arc length.

Unfortunately, we will now be graphing $f(x) = \sin(x)$. This value of x differs from the related x-coordinate on the unit circle! You can (and should) think of the x in $f(x) = \sin(x)$ as a "dummy variable." If you prefer to replace it with θ or t, please feel free to do so; however, we will continue to write sine as a function of x to get you used to this often confusing convention.

Period

Another observation is that the sine function repeats its pattern every 2π units along the x-axis.

Definition: Period

For any function $f(x)$, the smallest positive number k for which

$$f(x + k) = f(x)$$

for all x in the domain of f is called the **period** of $f(x)$.

If we take any interval of length 2π along the x-axis, we will see that the corresponding section of the sine curve over that interval can be "copied-and-pasted" an infinite number of times to create the entire sine curve. For example, while not convenient, we could consider the interval $[-5, -5+2\pi]$ along the x-axis, as shown in Figure $5.1.8$ below.

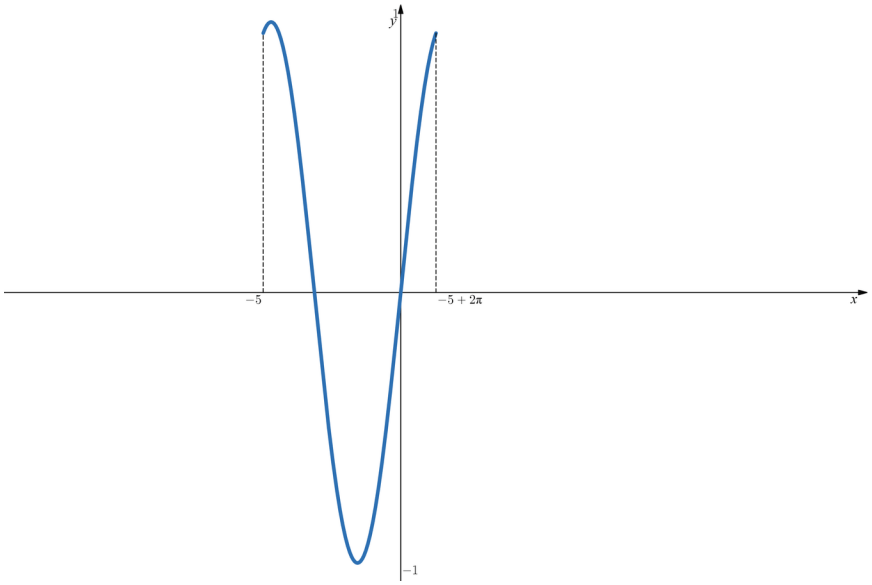


Figure $5.1.8$**:** *The sine curve on the interval* $[-5, -5+2\pi]$

"Copying-and-pasting" that segment an infinite number of times in both directions, we rebuild the entire sine curve.

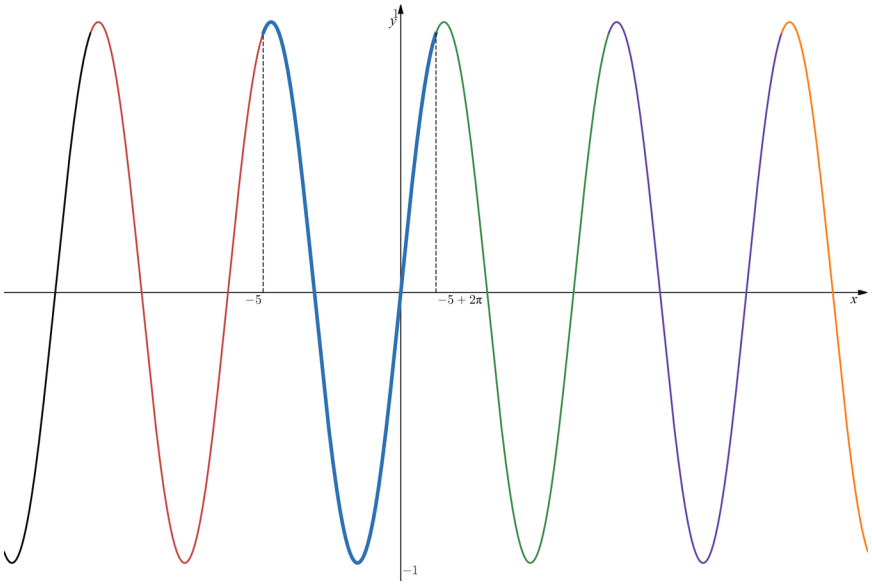


Figure $5.1.9$

Moreover, if we had selected a smaller interval along the x-axis, say $\left[-5, -5+\frac{3\pi}{2}\right]$, we would not have captured enough of the behavior of the sine curve to appropriately recreate the entire curve by "copying-and-pasting" the smaller section.

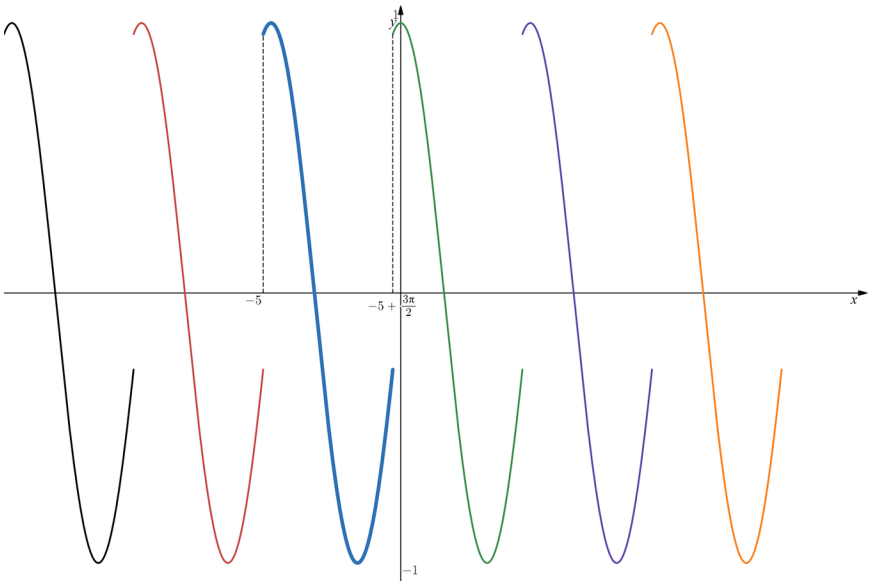


Figure 5.1.10

Hence, by the definition of the period of a function, the following theorem should make sense.

Theorem: Period of the Sine Function

The period of $f(x) = \sin(x)$ is 2π. This is also known as the **natural period** of the sine function.

While the graph of the sine function (and all of the other trigonometric functions) extends in both directions forever (see Figure 5.1.7), the section of the graph starting at $x = 0$ and ending at $x = 2\pi$ is going to be very important as we move forward.

Definition: Principal Cycle (of a trigonometric function)

The **principal cycle** of a trigonometric function is the interval starting at $x = 0$ and terminating at $x = p$, where p is the period of the trigonometric function.

Thus, by definition, we get the following theorem.

Theorem: Principal Cycle of the Sine Function

The principal cycle of the sine function is $[0, 2\pi]$.

It is important to note that there are other periodic functions besides the sine function.

Example 5.1.2

Which of the functions shown below are periodic? If the function is periodic, give its period.

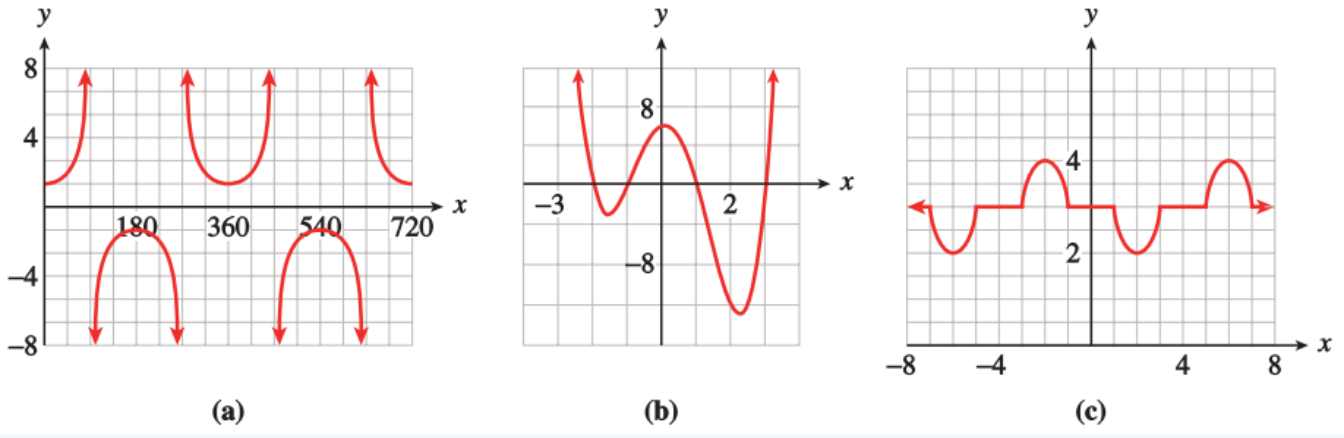


Figure 5.1.11

Solutions available at the end of this section.

Checkpoint 5.1.2

Which of the functions shown below are periodic? If the function is periodic, give its period.

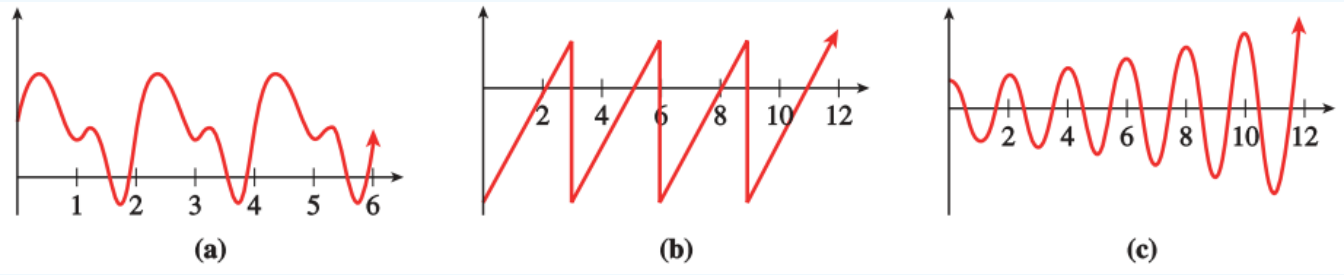


Figure 5.1.12

Answers available at the end of this section.

Example 5.1.3

A patient receives regular doses of medication to maintain a certain level of the drug in his body. After each dose, the patient's body eliminates a certain percentage of the medication before the next dose. The graph shows the amount of the drug, in milliliters, in the patient's body as a function of time in hours.

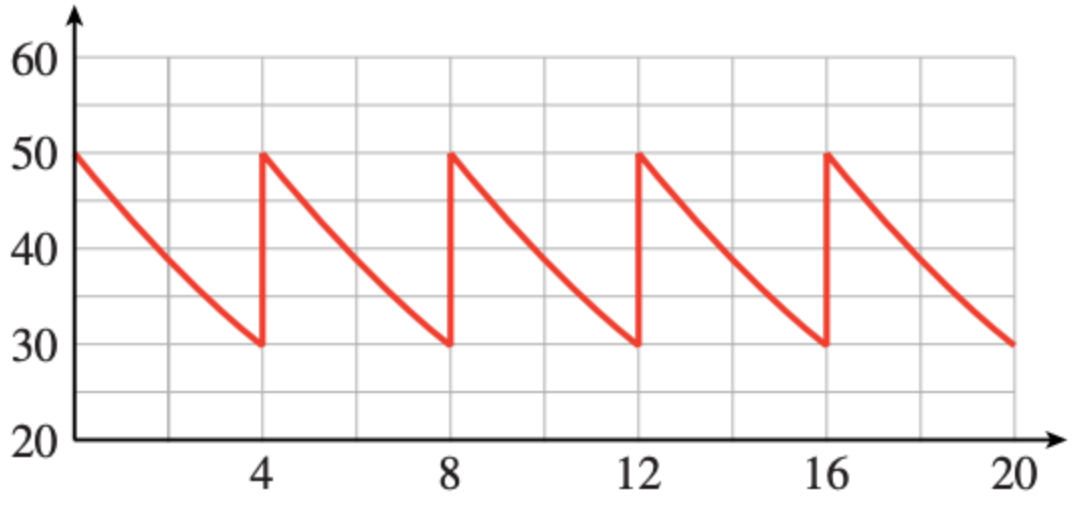


Figure 5.1.13

a. How much of the medication is administered with each dose?
b. How often is the medication administered?
c. What percent of the drug is eliminated from the body between doses?

Solutions available at the end of this section.

Checkpoint 5.1.3

You are sitting on your front porch late one evening, and you see a light coming down the road tracing out the path shown below, with distances in inches. You realize you are seeing a bicycle light fixed on a bike's front wheel.

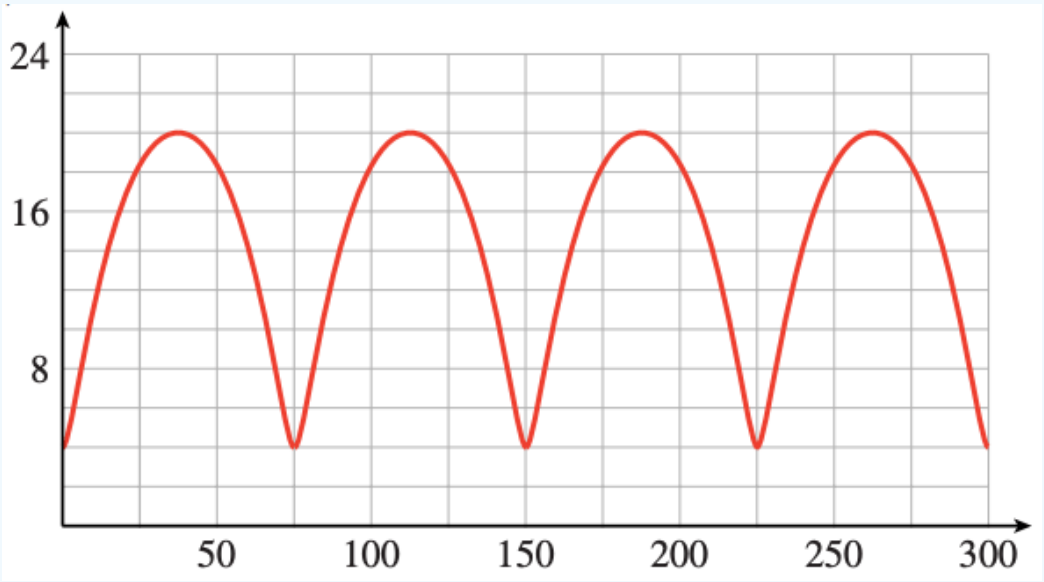


Figure 5.1.14

a. Approximately what is the period of the graph?
b. How far above the ground is the light?
c. What is the diameter of the bicycle wheel?

Answers available at the end of this section.

Amplitude

The sine curve (and, as we shall soon see, the cosine curve) has a feature almost exclusively unique among the graphs of all other functions—**amplitude**.

Definition: Amplitude (of a function)

Let M and m be a function's greatest and least values, respectively. The **amplitude** of the graph of the function is defined to be

$$\text{Amplitude} = \frac{1}{2}|M - m|.$$

That is, the amplitude is half the distance between the highest and lowest values of the function.

Example 5.1.4

Determine the amplitude of each function.

a.

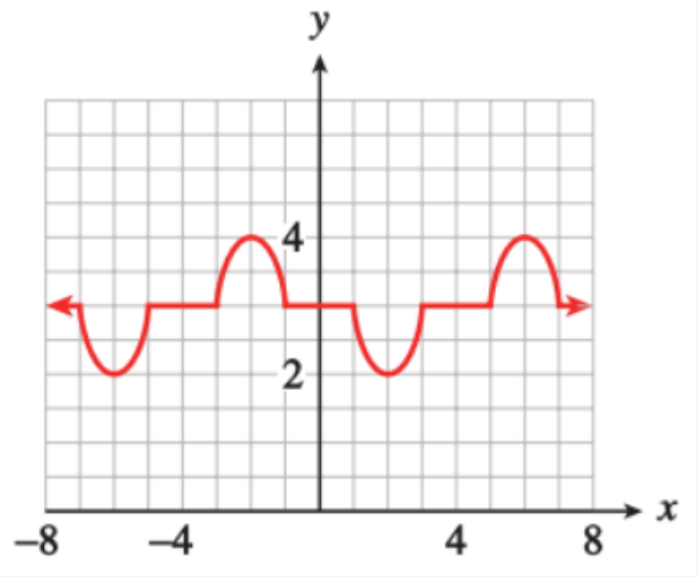


Figure 5.1.15

b.

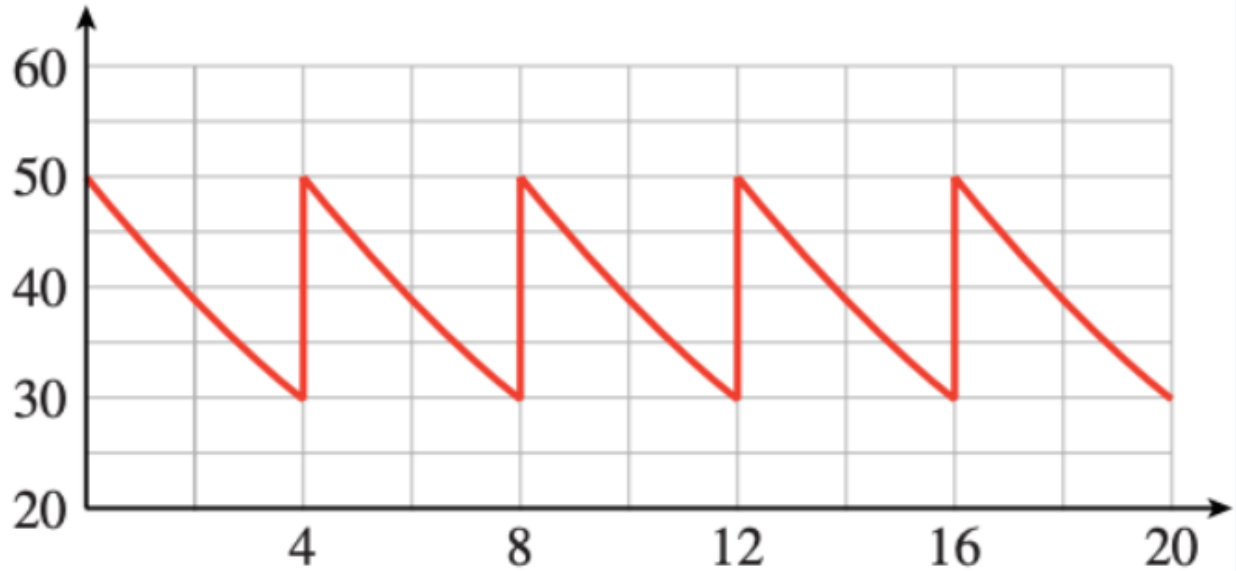


Figure 5.1.16

Solutions available at the end of this section.

Checkpoint 5.1.4

Determine the amplitude and period of the function.

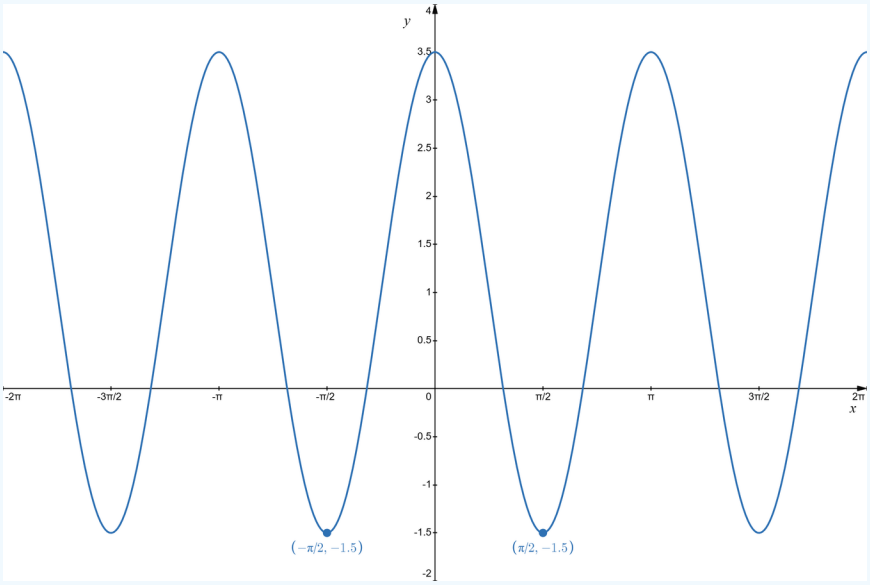


Figure 5.1.17

Answer available at the end of this section.

The concept of amplitude is generally reserved for the sine and cosine functions; however, it occasionally gets referenced when working with other periodic functions.

Theorem: Amplitude of the Sine Function

The amplitude of $f(x) = \sin(x)$ is 1.

Proof available at the end of this section.

Zeros

We know that the graph of $f(x) = \sin(x)$ crosses the x-axis when $\sin(x) = 0$; however, using the intuitive visualization of the sine function being the y-coordinates of points along the edge of the unit circle, we can see (from Figure $5.1.3$, Example $5.1.1$, or either of the Interactive Elements above) that this occurs when $x = 0, \pi, 2\pi, -\pi, -2\pi, \ldots$. The following theorem provides a much better way to write this.

Theorem: Zeros of the Sine Function

The zeros of $f(x) = \sin(x)$ occur at $x = \pi k$, where $k \in \mathbb{Z}$.

As a reminder, the notation, $\in$, means "belongs to" and the notation, $\mathbb{Z}$, means "the set of all integers." Therefore, as long as $k \in \mathbb{Z}$, $x = \pi k$ is shorthand for

$$x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, \cdots.$$

We will be using this notation extensively from this point forward in Mathematics.

Graphing the Cosine Function

Now that we have set the foundation for all the concepts we need, our discussion of the cosine function graph will be shorter than that of the sine function. We begin using the worst method for graphing a function—point-plotting.

Point-Plotting Approach

Example $5.1.5$

Sketch a portion of the graph of $y = \cos(x)$ on the interval $[0, 2\pi]$ by making a table of values.

Solution available at the end of this section.

Checkpoint $5.1.5$

Sketch a portion of the graph of $y = \cos(x)$ on the interval $\left[-\frac{3\pi}{2}, \frac{\pi}{2}\right]$ by making a table of values.

Answer available at the end of this section.

Unit Circle Approach

Relating the cosine curve to the unit circle is similar to what we did with the sine curve; however, we must remember that $\cos(t)$ is the x-coordinate of the points on the unit circle. This makes the visual a little more challenging, but you can play with the following Interactive Element (https://www.desmos.com/calculator/rr51fjmchw) to help cement the concept.

Interactive Element: Developing the Graph of the Cosine Function via the Unit Circle

Observation: As the angle (or arc length) t increases, the x-coordinate decreases from 1 to -1, then increases from -1 back to 1.

As before, we show the general cosine curve along with a single cycle (in blue) on the interval $[0, 2\pi]$ in Figure $5.1.20$ below.

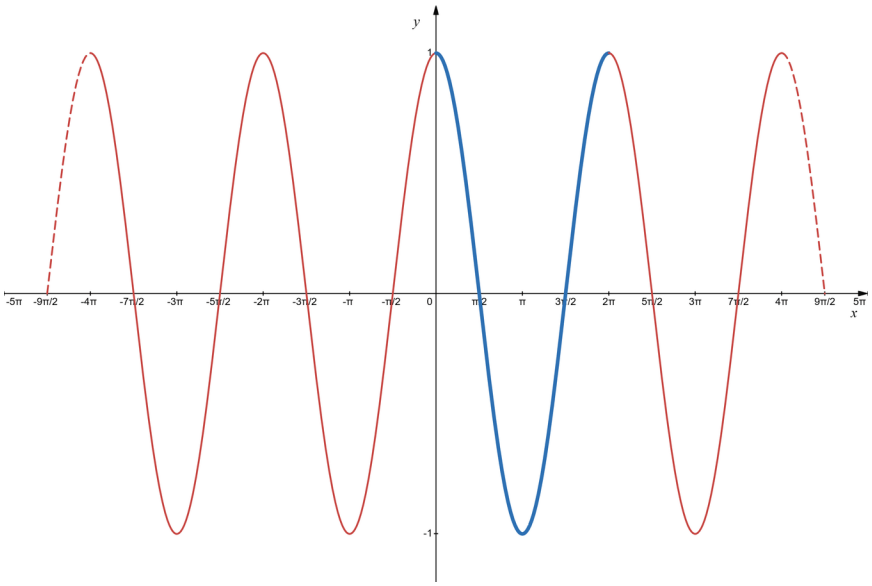


Figure $5.1.20$: $y = \cos(x)$

It is no coincidence that the graph in Figure $5.1.20$ has many characteristics resembling the graph of the sine function (Figure $5.1.7$). The sine and cosine functions are highly related, so their graphs are similar (but not the same). We will explore this similarity later.

Domain, Range, Period, Amplitude, and Zeros

As with the graph of the sine function, the graph of the cosine function reveals the domain and range of the cosine, its period, and the amplitude.

Theorem: Domain and Range of the Cosine Function

The domain of $f(x) = \cos(x)$ is $(-\infty, \infty)$ and the range is $[-1, 1]$.

Theorem: Period of the Cosine Function

The period of $f(x) = \cos(x)$ is 2π. Thus, the natural period of the cosine function is 2π.

Theorem: Principal Cycle of the Cosine Function

The principal cycle of the cosine function is $[0, 2\pi]$.

Theorem: Amplitude of the Cosine Function

The amplitude of $f(x) = \cos(x)$ is 1.

Finally, we know that the graph of $f(x) = \cos(x)$ crosses the x-axis when $\cos(x) = 0$. This happens when the x-coordinates of points along the edge of the unit circle become 0. This occurs when the angular argument of the cosine function is $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$, $\pm\frac{7\pi}{2}$, ... The following theorem summarizes this result.

Theorem: Zeros of the Cosine Function

The zeros of $f(x) = \cos(x)$ occur at $x = \frac{\pi}{2} + \pi k$, where $k \in \mathbb{Z}$.

There are other ways you could write this set of zeros. For example,

$$x = \frac{\pi}{2} + \pi k = \frac{\pi}{2} + \frac{2\pi k}{2} = \frac{\pi + 2\pi k}{2} = \frac{\pi}{2}(1 + 2k) = (2k+1)\frac{\pi}{2}.$$

I find that students remember the form given in the theorem the best.

Graphing the Tangent Function

The tangent function is the more abstract graph of the fundamental trigonometric functions. Therefore, we will spend much time deriving the graph and discussing how it is generated. In the end, students will gravitate to either the conceptual approach (which uses the graphs of the sine and cosine to help develop the tangent curve) or the unit circle approach (which uses the slope of the terminal side of the angle). Understanding both approaches to graphing the tangent curve is best because both will be needed in future mathematics courses.

Before we get to those "headier" approaches to graphing the tangent function, we start, as we did with the sine curve, by plotting points.

Point-Plotting Approach

As with the sine and cosine functions, we begin by making a table of values and plotting points. If we decide to use the same "test points" to build our table of values, something odd happens.



The fact that the tangent function is undefined at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ should not be surprising. By the Ratio Identity, $\tan(x) = \frac{\sin(x)}{\cos(x)}$, and we have just mentioned that the cosine has zeros at $x = \frac{\pi}{2} + \pi k$. That is, the tangent function encounters division by zero at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.

What does this mean about the graph of the tangent function? If we decide to plot the points in our list where the tangent *is* defined, all we would get is the following graph.

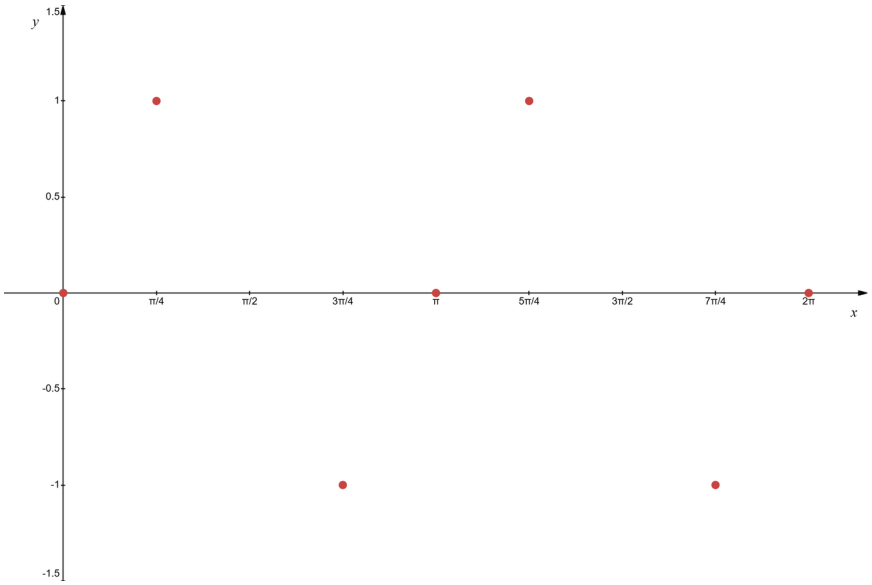


Figure 5.1.21

We are now faced with the question, "How do we connect these points?"

This is one of the many weaknesses of graphing via point-plotting—we are often left guessing how the graph behaves between the points.

In these situations, most people plot more points to see if the true behavior of the graph reveals itself. If we make our table more robust by including every special angle from 0 to 2π, we get the following:



However, as detailed and exhaustive as this might seem, the graphical result is only slightly better.

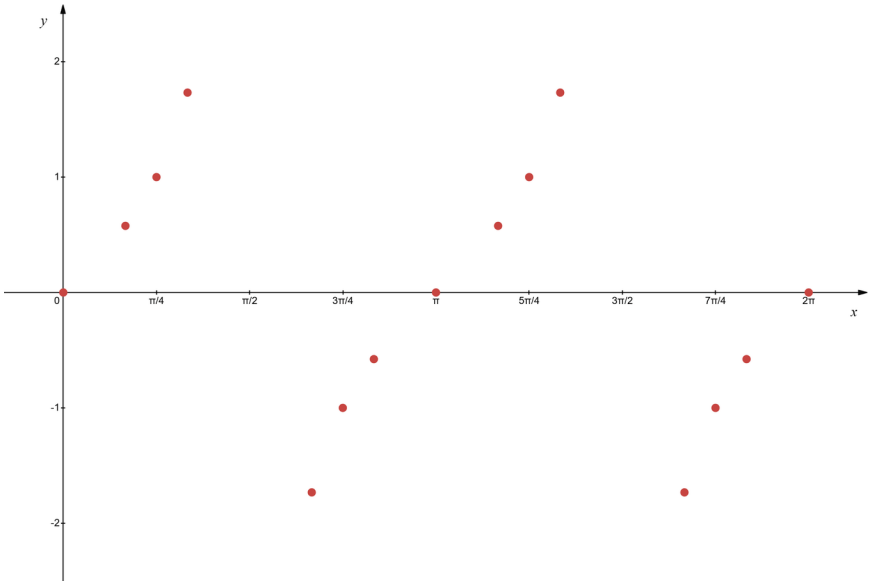


Figure 5.1.22

While Figure 5.1.22 begins to reveal some of the patterns of the tangent graph, it's not quite revealing everything we need.

Therefore, we must throw in the proverbial towel and try a different method of graphing the tangent function.

Unit Circle Approach

From the unit circle definition of the trigonometric functions, if $P(x, y)$ is a point on the unit circle, then $\tan(\theta) = \frac{y}{x}$, where θ is the angle from the positive x-axis to the ray connecting the origin to P (see Figure 5.1.23).

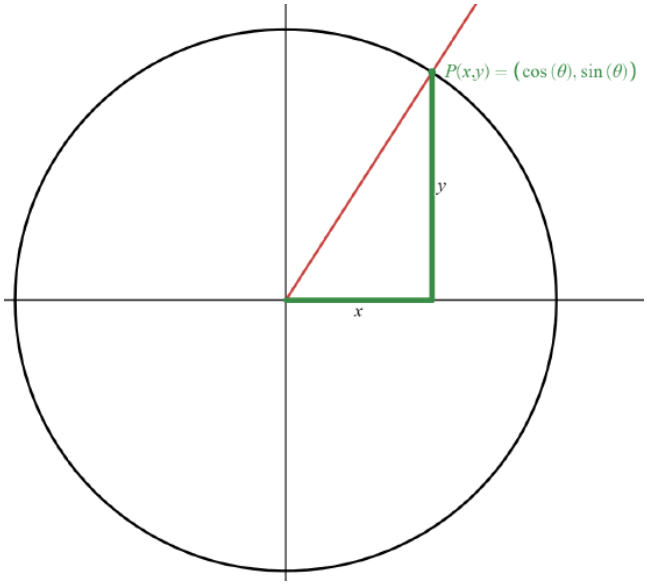


Figure 5.1.23

Therefore, $\tan(\theta)$ is the slope of the ray connecting the origin to P. When $\theta = 0$, this slope is 0. As θ increases from 0, the slope of this ray also increases; however, as θ approaches $\frac{\pi}{2}$, something interesting happens—the slope of the ray increases without bound. Eventually, at $\theta = \frac{\pi}{2}$, the ray becomes vertical; however, we know from Algebra that vertical lines have undefined slopes. As such, this results in a vertical asymptote in our tangent function graph. The following Interactive Element (https://www.desmos.com/calculator/tv71jqbwil) displays the resulting curve. I recommend you spend a while truly trying to understand what is going on here.

Interactive Element

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Desmos Interactive online

Conceptual Approach

To truly understand what is going on with the tangent function graph, we need to understand the nature of a ratio when its denominator is approaching zero. Consider the values of the three functions, $f(x) = \frac{1}{x}$, $g(x) = \frac{13}{x}$, and $h(x) = -\frac{2}{x^2}$, as the values of x approach 0.

$f(x) = \frac{1}{x}$ **as** $x \to 0^+$	$g(x) = \frac{13}{x}$ **as** $x \to 0^-$	$h(x) = -\frac{2}{x^2}$ **as** $x \to 0^-$
$\frac{1}{0.1} = 10$	$\frac{13}{-0.1} = -130$	$-\frac{2}{(-0.1)^2} = -200$
$\frac{1}{0.01} = 100$	$\frac{13}{-0.01} = -1300$	$-\frac{2}{(-0.01)^2} = -20000$

$f(x)=\frac{1}{x}$ as $x\to 0^+$	$g(x)=\frac{13}{x}$ as $x\to 0^-$	$h(x)=-\frac{2}{x^2}$ as $x\to 0^-$
$\frac{1}{0.001}=1000$	$\frac{13}{-0.001}=-13000$	$-\frac{2}{(-0.001)^2}=-2000000$
$\frac{1}{0.0001}=10000$	$\frac{13}{-0.0001}=-130000$	$-\frac{2}{(-0.0001)^2}=-200000000$

Before we continue, it's important to clarify the notations 0^+ and 0^-.

Definition

The notation $x\to a^+$ is read as "x is approaching a from the right side of a," and it means that x is strictly greater than a and is getting closer and closer to a.

The notation $x\to a^-$ is read as "x is approaching a from the left side of a," and it means that x is strictly less than a and is getting closer and closer to a.

In both cases, x will never take on the value of a.

Hence, $x\to 0^+$ means that x is getting close to 0, but is still greater than 0, and $x\to 0^-$ means x is getting close to 0, but is still less than 0.

In all three functions from the table, the denominators are approaching zero. In the first and third columns, the denominators approach 0^+ (zero from the right). In the second column, the denominators approach 0^- (zero from the left). Notice that, in all three cases, the smaller the denominator gets, the larger the value of the fraction becomes. Understanding this property is critical to conceptually understanding vertical asymptotes. I have summarized this result below.

If the numerator is approaching a **nonzero** constant and the denominator is approaching (but never taking on the value of) 0, the ratio $\frac{N(x)}{D(x)}$ approaches either $+\infty$ or $-\infty$. The sign of the result is determined by comparing the signs of the numerator and denominator of the ratio of functions as x is approaching a.

Thus, a vertical asymptote for a function results from what happens as its numerator approaches some nonzero constant while its denominator approaches (but does not become) 0.

We have already mentioned that the tangent function is undefined when cosine is 0 and that this occurs precisely when $x=\frac{\pi}{2}+k\pi$, where $k\in\mathbb{Z}$. If we use the knowledge from the conversation above, we will be able to determine if the tangent has vertical asymptotes at these values (and the behavior of the tangent curve near these vertical asymptotes).

Let's focus our efforts on what happens as the x-values approach $\frac{\pi}{2}$. Since $\tan(x)=\frac{\sin(x)}{\cos(x)}$, we start by sketching both the sine and cosine functions near $x=\frac{\pi}{2}$.

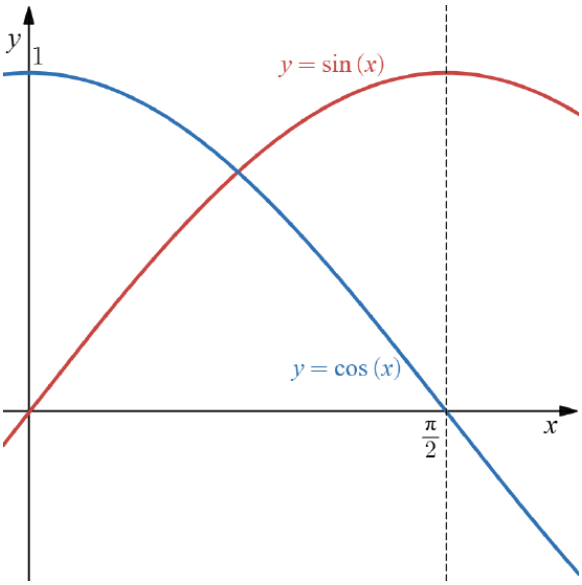


Figure 5.1.24

Figure 5.1.25 shows the values of the sine as $x \to \frac{\pi}{2}^-$.

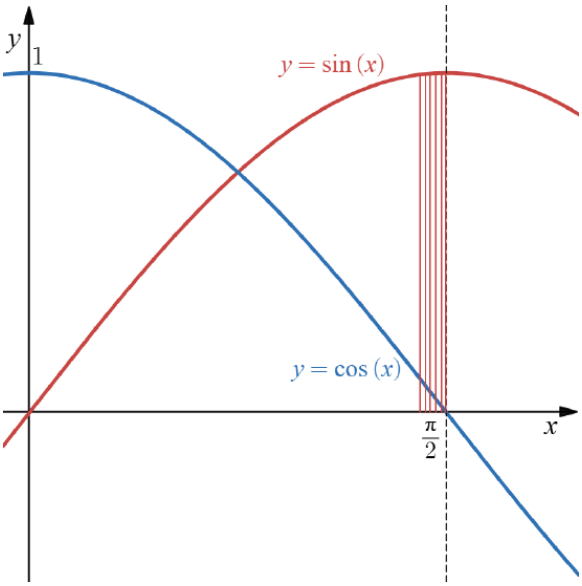


Figure 5.1.25**:** *The values of the sine function as* $x \to \frac{\pi}{2}^-$

Likewise, Figure 5.1.26 shows the values of cosine as $x \to \frac{\pi}{2}^-$.

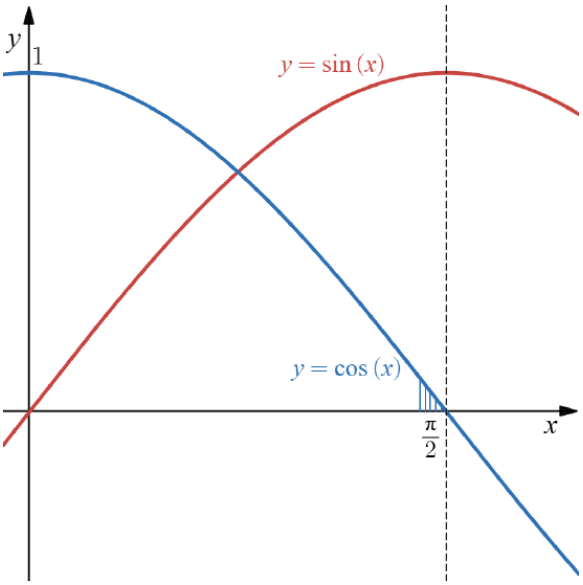


Figure 5.1.26**:** *The values of the cosine function as* $x \to \frac{\pi}{2}^-$

From a graphical investigation, as x approaches $\frac{\pi}{2}$ from the left,

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \text{ approaches } \frac{1}{0^+} \text{ which, in turn, approaches } +\infty. \tag{5.1.1}$$

We now consider what happens as $x \to \frac{\pi}{2}^+$. Consider Figures 5.1.27 and 5.1.28.

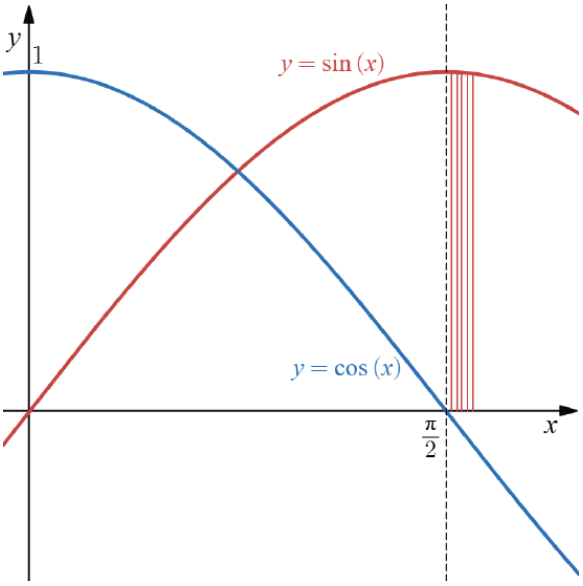


Figure 5.1.27**:** *The values of the sine function as* $x \to \frac{\pi}{2}^+$

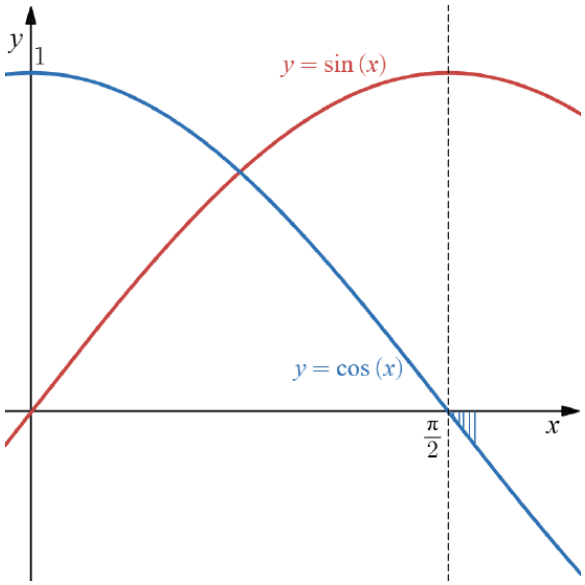


Figure 5.1.28**:** *The values of the cosine function as* $x \to \frac{\pi}{2}^+$

In this case, we can see the sine function is approaching 1, but the cosine is approaching 0 from the negative side of 0. Hence,

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \text{ approaches } \frac{1}{0^-} \text{ which, in turn, approaches } -\infty. \tag{5.1.2}$$

Combining our point-plotting in Figure 5.1.22 with the results from 5.1.1 and 5.1.2, we get the following graph.

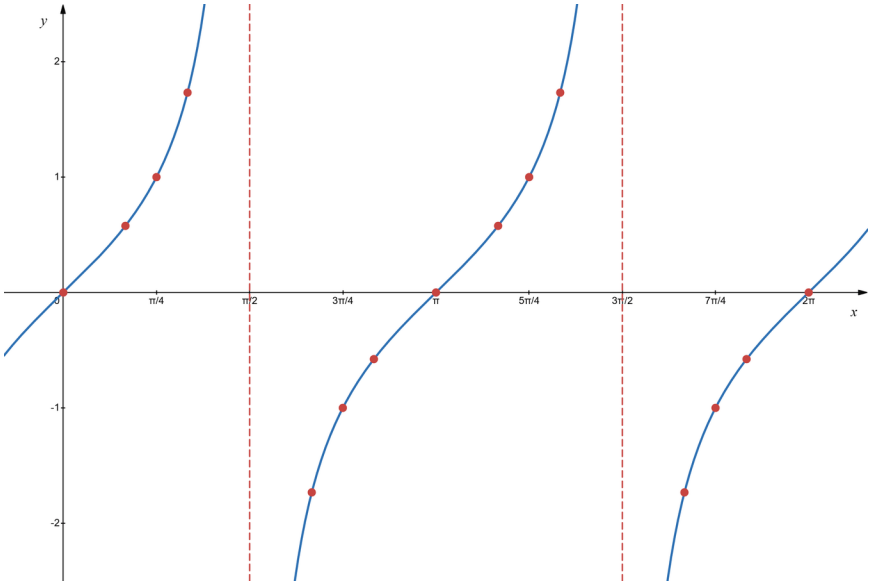


Figure 5.1.29

Notice that we took the liberty to draw in the vertical asymptote at $x = \frac{3\pi}{2}$.

We now have enough knowledge to graph the *entire* tangent curve. Figure 5.1.30 shows the general tangent curve along with a single cycle (in blue) on the interval $[0, \pi]$.

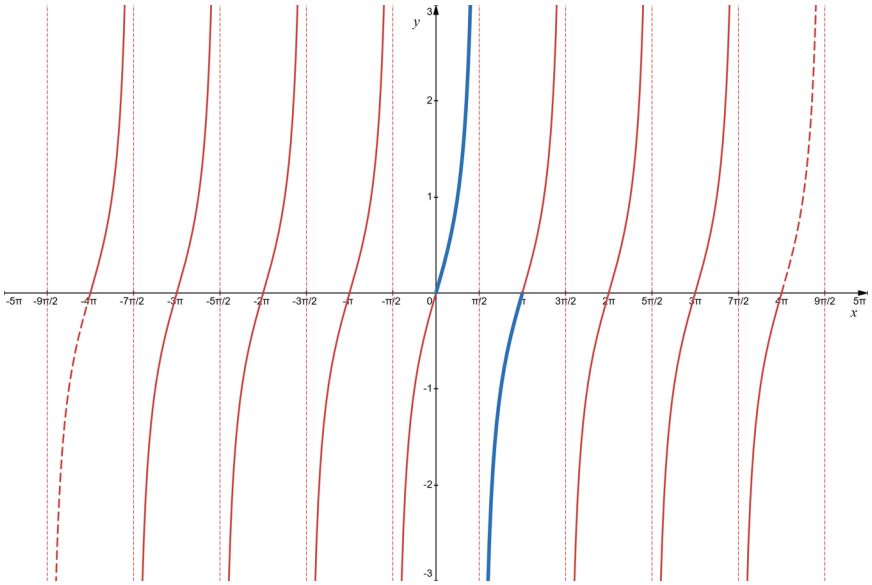


Figure 5.1.30**:** $y = \tan(x)$

Domain, Range, Period, and Zeros

There are a few essential features to mention about the tangent curve. First, the pattern repeats every π units. Therefore, the period of the tangent function is π. The domain is missing the values of x for which the cosine is zero, and the range is all real numbers. Finally, the zeros of the tangent function occur when the sine function has zeros. All of these facts are summarized below.

Theorem: Domain and Range of the Tangent Function

The domain of the tangent function is $\{x \mid x \neq \frac{\pi}{2} + \pi k\}$, where $k \in \mathbb{Z}$. The range is $(-\infty, \infty)$.

Theorem: Period of the Tangent Function

The period of $f(x) = \tan(x)$ is π. Thus, the natural period of the tangent function is π.

Theorem: Principal Cycle of the Tangent Function

The principal cycle of the tangent function is $[0, \pi]$.

Theorem: Zeros of the Tangent Function

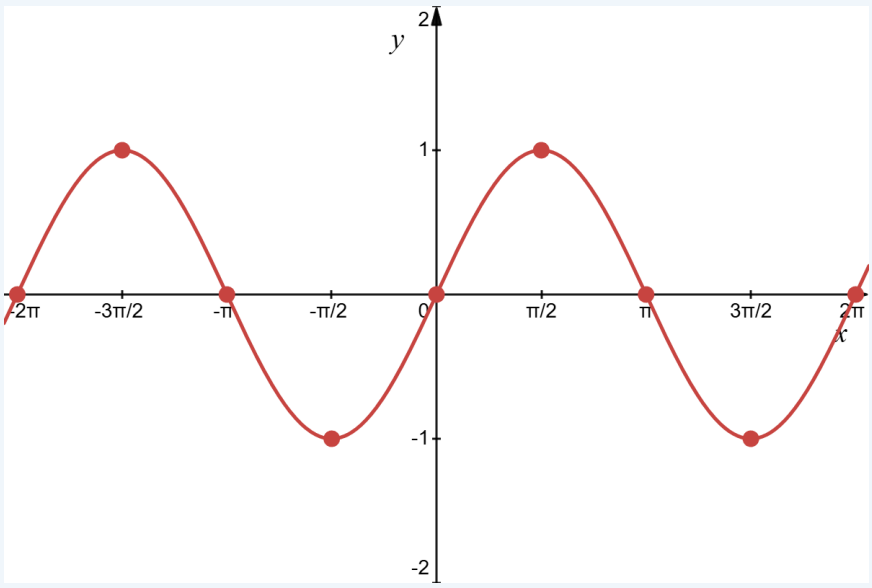
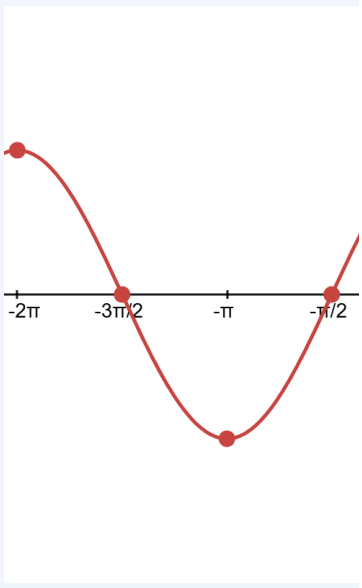
The zeros of the tangent function occur at $x = \pi k$, where $k \in \mathbb{Z}$.

In Figure 5.1.30, we highlighted the single period on the interval $[0, \pi]$. Most textbooks use the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$; however, our choice of highlighted interval will serve an important purpose in the next section.

Also, note that we didn't state the amplitude of the tangent function. This is because the tangent function does not have an amplitude. To have an amplitude, a function must have a greatest and least value; however, the range of the tangent is $(-\infty, \infty)$. Therefore, it is **unbounded** (i.e., it does not have a greatest or least value). As such, it cannot have an amplitude.

Summary of the Graphs of the Fundamental Trigonometric Functions

Before we continue, let's summarize what we have discovered about the graphs of the fundamental trigonometric functions.

	Sine Function	C
Graph		
Domain	$(-\infty, \infty)$	
Range	$[-1, 1]$	
Amplitude	1	
Natural Period	2π	
Principal Cycle	$[0, 2\pi]$	
Zeros	$x = \pi k,\ k \in \mathbb{Z}$	x

Finally, when asked to graph the sine or cosine functions, we only need to know their values at the quadrantal angles (these correspond to the "dots" on the graphs of the sine and cosine curves above). This concept will lead to a graphing method for transformations of trigonometric functions in our next section.

Symmetry Identities

Now that we have the base graphs of the fundamental trigonometric functions, it's natural to discuss their graphs' symmetric behavior.

Recall from Algebra the following definition.

Definition: Even and Odd Functions

The function f is called an **even function** if for every input x in the domain of f

$$f(-x) = f(x).$$

The function f is called an **odd function** if for every input x in the domain of f

$$f(-x) = -f(x).$$

Figure 5.1.31 is the key to proving the symmetric behavior of the trigonometric functions.

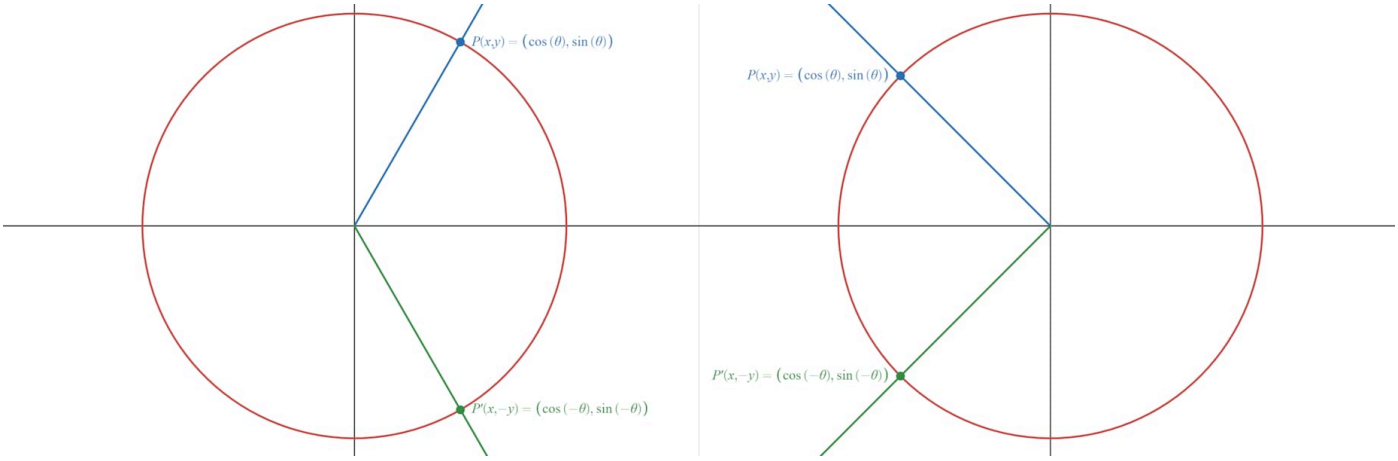


Figure 5.1.31

Note that, in both graphs in Figure 5.1.31, the point P on the terminal side of θ is a reflection across the x-axis of the point P' on the terminal side of $-\theta$. Hence, each point's x-coordinate remains the same, but the y-coordinates are opposite in sign. Therefore, no matter what quadrant in which θ terminates,

$$x = \cos(\theta) \qquad \text{and} \qquad x = \cos(-\theta).$$

Thus,

$$\cos(\theta) = \cos(-\theta),$$

and so cosine is an even function.

Moreover,

$$y = \sin(\theta) \qquad \text{and} \qquad -y = \sin(-\theta).$$

The equation on the right side can be solved for y to become $y = -\sin(-\theta)$. Hence, $\sin(\theta) = -\sin(-\theta)$. Stated more clearly,

$$\sin(-\theta) = -\sin(\theta).$$

Thus, sine is an odd function.

From these two results, the proofs of the symmetric behaviors of the remaining trigonometric functions are much easier. For example,

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\frac{\sin(\theta)}{\cos(\theta)} = -\tan(\theta).$$

Thus, tangent is an odd function. The rest of the proofs are left as homework exercises.

These results are summarized in the following theorem.

Theorem: Symmetry Identities

The sine, cosecant, tangent, and cotangent are odd functions. The cosine and secant are even functions.

Of course, we could have guessed that the sine and tangent are odd functions because their graphs are symmetric about the origin. Similarly, we could have guessed the cosine is an even function because its graph is symmetric about the y-axis.

Advice: Memorize These Identities

The Symmetry Identities are needed throughout the rest of Trigonometry and Calculus. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated.

Example 5.1.6

Is the function even, odd, or neither?

$$f(x) = x \sin\left(x^3 - x\right) + x^2$$

Solution available at the end of this section.

Checkpoint 5.1.6

Determine if the function is even, odd, or neither.

$$g(x) = x^3 \cos(x) - \tan(x)$$

Answer available at the end of this section.

Finally, we can take advantage of the symmetric behavior of a trigonometric function to speed up evaluations.

Example 5.1.7

Evaluate each function using symmetry.

a. $\sec\left(-\frac{5\pi}{6}\right)$
b. $\sin\left(-\frac{5\pi}{4}\right)$

Solutions available at the end of this section.

Checkpoint 5.1.7

Evaluate $\cot\left(-\frac{2\pi}{3}\right)$ using symmetry.

Answer available at the end of this section.

Evaluating trigonometric functions using symmetry can be thought of as an alternative, not a replacement, to how we have been evaluating them up to this point. I recommend practicing using symmetry for evaluation because it can speed up computations.

Activity

Activity: Solving Basic Trigonometric Equations

1.

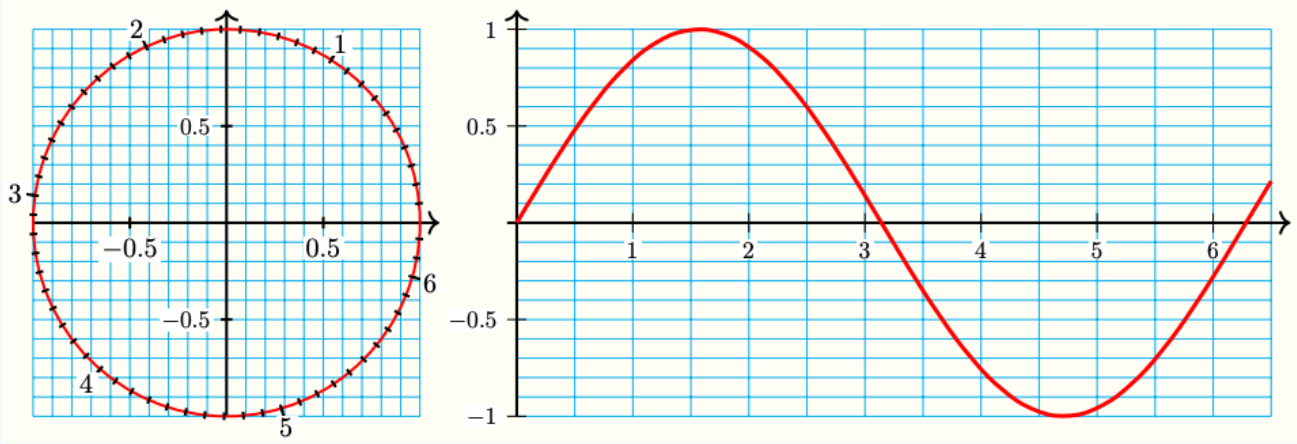


Figure 5.1.33

a. Use the graph of $y = \sin x$ to estimate two solutions of the equation $\sin(x) = 0.65$. Show your solutions on the graph.
b. Use the unit circle to estimate two solutions of the equation $\sin(x) = 0.65$. Show your solutions on the circle.
c. Use the graph of $y = \sin x$ to estimate two solutions of the equation $\sin(x) = -0.2$. Show your solutions on the graph.
d. Use the unit circle to estimate two solutions of the equation $\sin(x) = -0.2$. Show your solutions on the circle.

2.

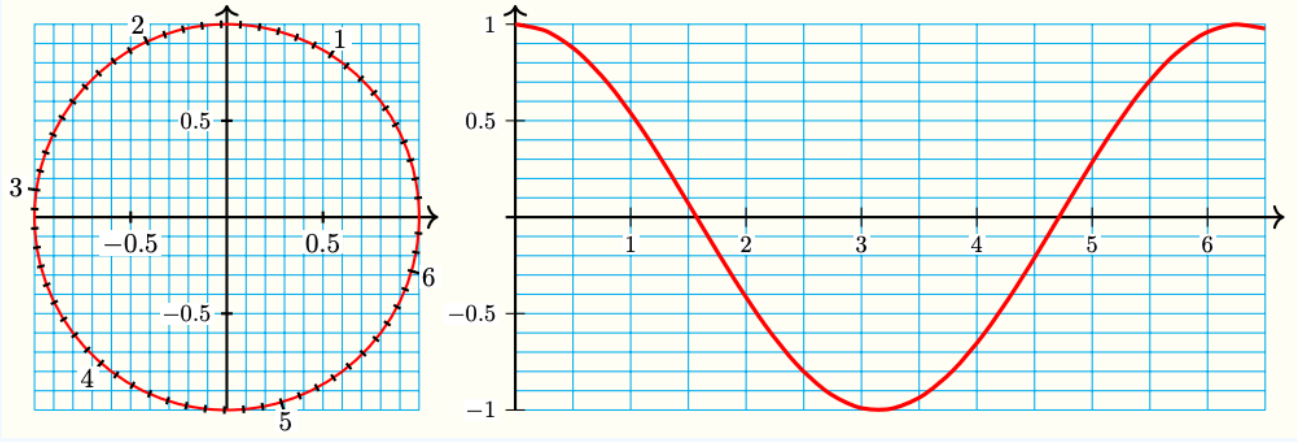


Figure 5.1.34

a. Use the graph of $y = \cos x$ to estimate two solutions of the equation $\cos(x) = 0.15$. Show your solutions on the graph.
b. Use the unit circle to estimate two solutions of the equation $\cos(x) = 0.15$. Show your solutions on the circle.
c. Use the graph of $y = \cos x$ to estimate two solutions of the equation $\cos(x) = -0.4$. Show your solutions on the graph.
d. Use the unit circle to estimate two solutions of the equation $\cos(x) = -0.4$. Show your solutions on the circle.

Solving Trigonometric Equations Using Graphs

Now that we have a basic understanding of the graphs of the fundamental trigonometric functions, we can begin to visualize the solutions to basic trigonometric equations.

Example 5.1.8

Use a graph to find all solutions of $\cos(t) = -0.62$ between 0 and 2π.

Answer available at the end of this section.

Caution: Graphs Are a Terrible Method for Solving Equations

Example $5.1.8$ showcases the issue with using a graph to solve *any* equation—you get very rough approximations. In truth, the solutions to $\cos(t) = -0.62$ on the interval $[0, 2\pi]$ are approximately 2.2395 and 4.0436 (rounded to four decimal places). There is virtually no way to gain this level of accuracy using a hand-drawn graph.

An interesting observation comes from the fact that the sine and cosine curves oscillate forever in both directions. Had we not restricted our answers in Example $5.1.8$ to lie between 0 and 2π, we would have encountered an infinite number of times where the cosine curve intersects the line $y = -0.62$. Thus, the equation $\cos(t) = -0.62$ has an infinite number of solutions when considering all real numbers instead of restricting ourselves to the interval $[0, 2\pi]$.

Checkpoint $5.1.8$

Use a graph to find the two angles between 0 and 2π that satisfy $\sin(t) = -0.85$.

Answer available at the end of this section.

Time-Sensitive Advice: Drop Deadline

Within the Los Rios Community College District, the last day you can drop a class is the Monday (or, sometimes, the Tuesday) of the thirteenth week of the semester. After this date, you will no longer be able to drop a class for that semester and the only grades you can receive in the classes you remain in are A, B, C, D, or F.

Will you still get a "W?"

Dropping any class between the third week and this final drop deadline will result in a "W" notation on your record for that class.

What does this mean?

Once you withdraw (W) or fail (D or F) a course three times within the Los Rios Community College District (CRC, SCC, FLC, or ARC), you are never allowed to take that course again at any of the colleges within the district. Thus, if you wanted to continue through the math sequence, you would need to take this course outside the Los Rios Community College District.

Success in College: Memorization Learning Preferences

> Students may be inclined to approach their courses in one of three ways. Those with a reproducing orientation tend to take a surface approach to learning, relying on rote memorization and mechanical formula substitution and making little or no effort to understand the material being taught. Those with a meaning orientation tend to adopt a deep approach, probing and questioning and exploring the limits of applicability of new material. Those with an achieving orientation tend to use a strategic approach, doing whatever is necessary to get the highest grade they can, taking a surface approach if that suffices and a deep approach when necessary.
>
> Source: R. Felder, R. Brent , Journal of Engineering Education, 94 (1), 57-72 (2005).

1. Students have different motivations for going to math class. What do you think the main motivation is for students who have a reproducing orientation and who take a surface approach?
2. What do you think is the main motivation for students who have a meaning orientation and who take a deep approach?

Answers

Section Preview

I chose to only introduce the graphs of the sine, cosine, and tangent because these are the foundational graphs for the remainder of trigonometry. The graphs of the remaining trigonometric functions are held off until the very end of the chapter. Additionally, this section introduces the symmetry of these functions. Therefore, we can spend some time doing proofs involving symmetry and these functions.

Section Preview

- **Sets and Numbers**
  - Set-Builder Notation: Needed for the domain of tangent.
- **Rational Functions**
  - Asymptotes: Needed to understand what a vertical asymptote is before encountering it in the tangent graph.

Section Preview

- Sketch the base graphs of the fundamental trigonometric functions.
- Determine whether the graph of an equation involving trigonometric functions is even, odd, or neither.
- Evaluate a trigonometric function using the Symmetry Identities.

Example 5.1.1

To graph $f(x) = \sin(x)$ without relating to the unit circle, we need to list out several values of the sine function.

[redacted]

Plotting these values and connecting them with a smooth curve, we get the following:

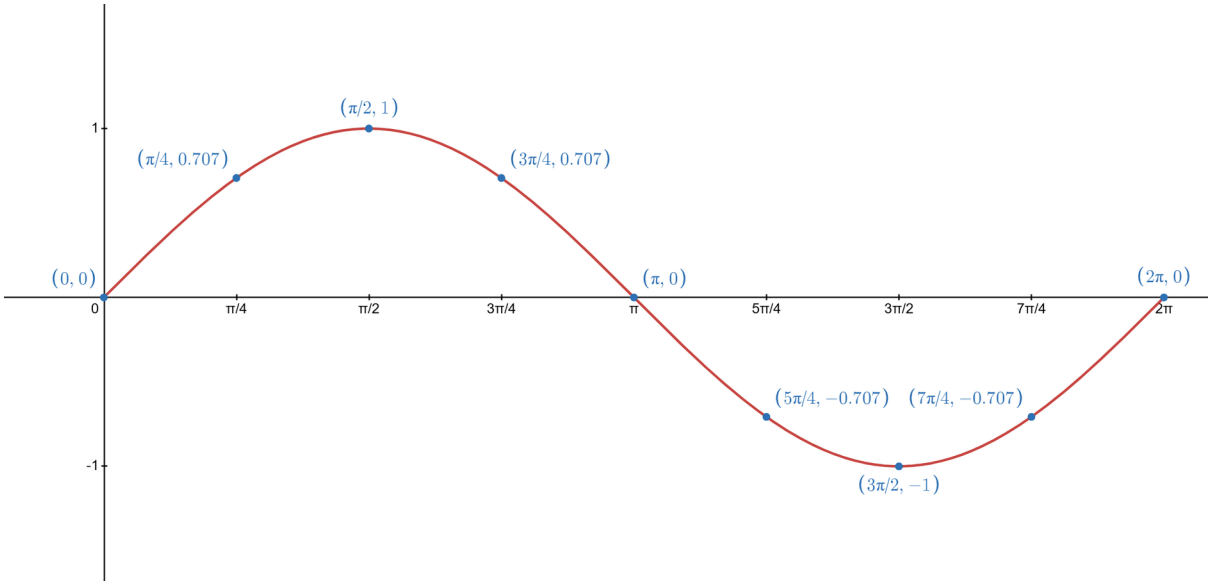


Figure 5.1.4

Checkpoint 5.1.1

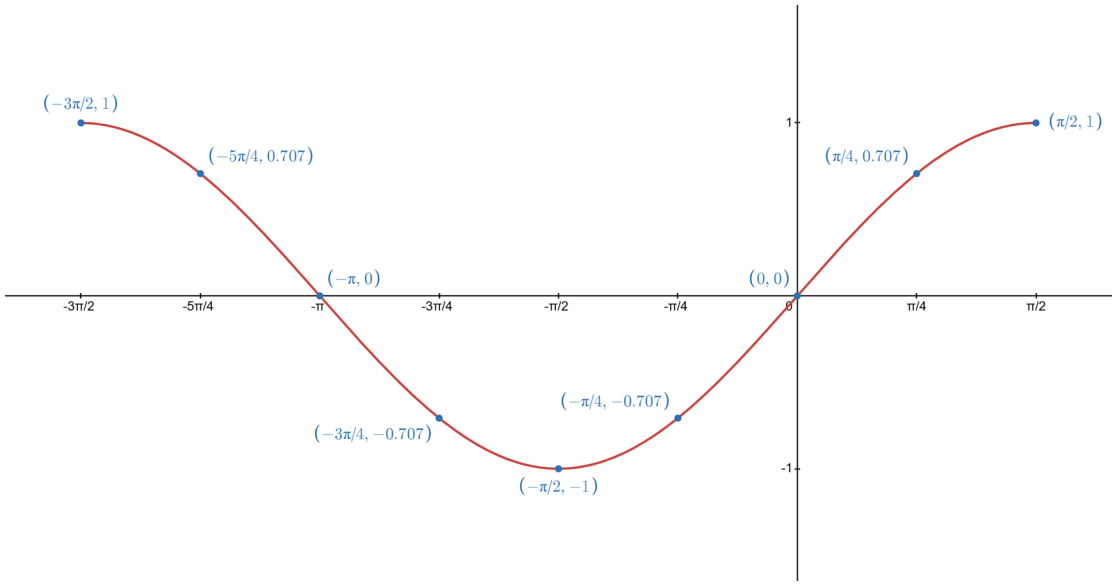


Figure 5.1.5

Example 5.1.2

a. This graph is periodic with period 360.
b. This graph is not periodic.
c. This graph is periodic with period 8.

Checkpoint 5.1.2

a. Period: 2
b. Period: 3
c. Not periodic

Example 5.1.3

a. The medication level increases from 30 ml to 50 ml at each cycle of the graph, so 20 ml of medication is administered at each dose.
b. The medication level peaks sharply every four hours after administering each new dose.
c. The medication level declines by 20 ml between doses, or $\frac{20}{50} = 0.6$, or 60%.

Checkpoint 5.1.3

a. 75 in
b. 4 in
c. 24 in

Example 5.1.4

a. The maximum value for the graph of this function is $M = 4$ and the minimum is $m = 2$. Therefore, the amplitude of the graph is

$$\text{Amplitude} = \frac{1}{2}|4 - 2| = 1.$$

b. The maximum value is $M = 50$ and the minimum is $m = 30$. Thus,

$$\text{Amplitude} = \frac{1}{2}|50 - 30| = 10.$$

Checkpoint 5.1.4

Period: π
Amplitude: 2.5

Theorem: Amplitude of the Sine Function

The maximum value of $f(x) = \sin(x)$ is $M = 1$ and the minimum value is $m = -1$. Hence,

$$\text{Amplitude} = \frac{1}{2}|1 - (-1)| = 1.$$

Example 5.1.5

[redacted]

Plotting these values and connecting them with a smooth curve, we get the following:

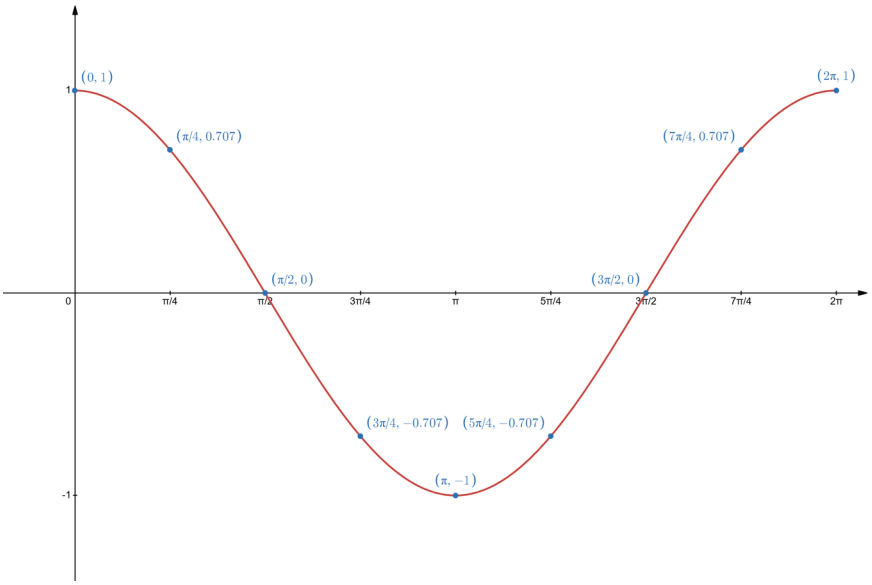


Figure 5.1.18

Checkpoint 5.1.5

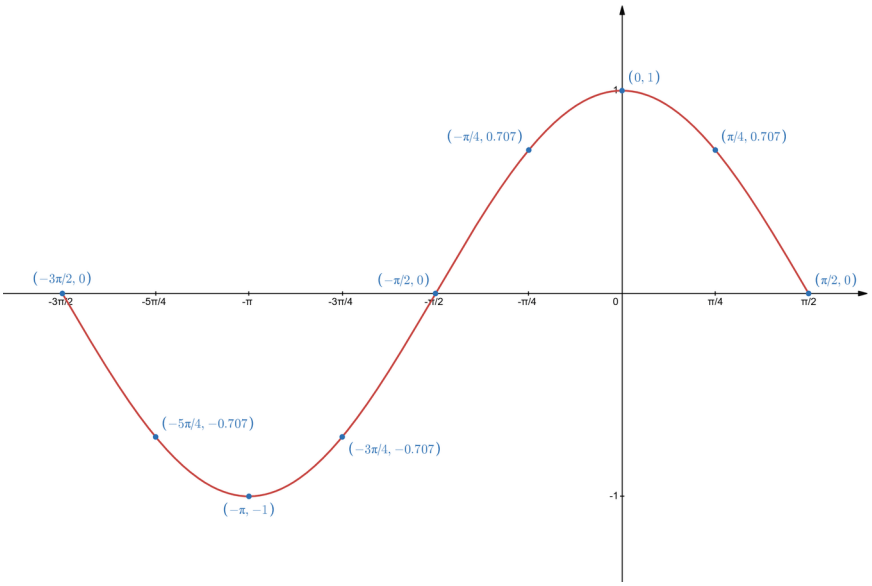


Figure 5.1.19

Example 5.1.6

$$\begin{aligned} f(-x) &= (-x)\sin\left((-x)^3-(-x)\right)+(-x)^2 \\ &= -x\sin\left(-x^3+x\right)+x^2 \\ &= -x\sin\left(-(x^3-x)\right)+x^2 \\ &= x\sin\left(x^3-x\right)+x^2 \qquad \text{(sine is an odd function)} \\ &= f(x) \end{aligned}$$

Thus, f is an even function.

Checkpoint 5.1.6

Odd

Example 5.1.7

a.

$$\begin{aligned}
\sec\left(-\frac{5\pi}{6}\right) &= \sec\left(\frac{5\pi}{6}\right) && \text{(secant is even)}\\
&= \frac{1}{\cos\left(\frac{5\pi}{6}\right)} && \text{(Reciprocal Identity)}\\
&= \frac{1}{-\cos\left(\frac{\pi}{6}\right)} && \text{(reference angle and quadrant)}\\
&= -\frac{1}{\sqrt{3}/2}\\
&= -\frac{2}{\sqrt{3}}
\end{aligned}$$

b.

$$\begin{aligned}
\sin\left(-\frac{5\pi}{4}\right) &= -\sin\left(\frac{5\pi}{4}\right) && \text{(sine is odd)}\\
&= \sin\left(\frac{\pi}{4}\right) && \text{(reference angle and quadrant)}\\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

Checkpoint 5.1.7

$\frac{1}{\sqrt{3}}$

Example 5.1.8

Figure 5.1.32 shows a graph of $y = \cos(t)$, with t in radians, and the horizontal line $y = -0.62$. The solutions of the equation are the t-coordinates of the intersection points, and these *appear* to be about $t = 2.25$ and slightly over 4.0, perhaps 4.05. You can check that these values roughly satisfy the equation; it is difficult to read the graph with any greater accuracy.

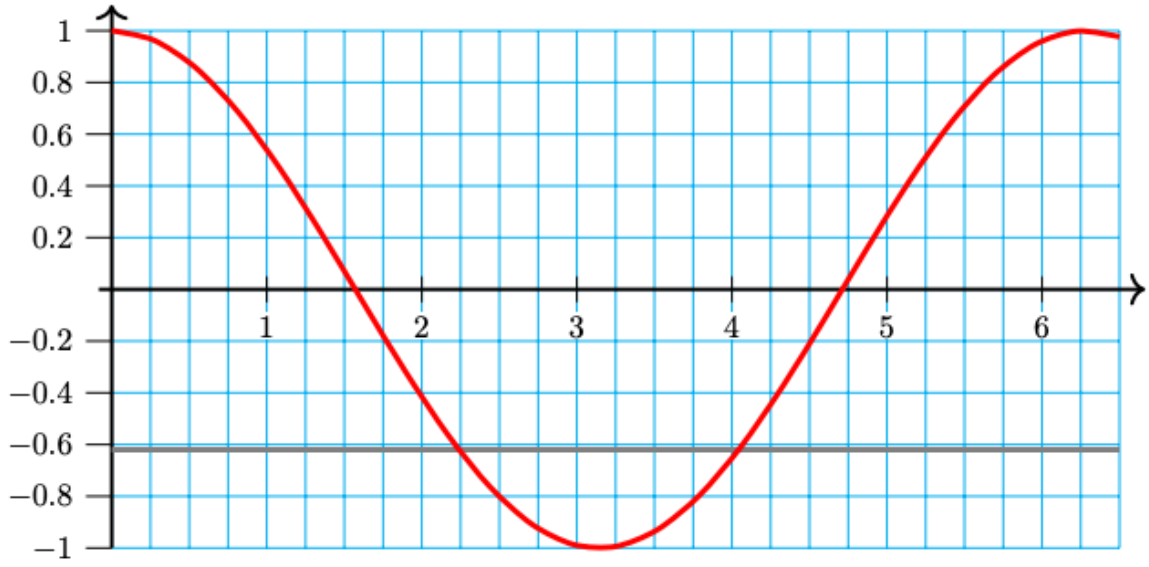


Figure 5.1.32

Checkpoint 5.1.8

Approximately 4.14 and 5.25

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5.1.1: Resources and Key Concepts

Resources

Videos

- Graphs of the Sine and Cosine Functions
  - Trigonometric Graphs - Review of Graphs of Sine and Cosine Functions
- Graphs of the Tangent Function
  - Graphs of the Other Trigonometric Functions - Deriving the Base Graphs for Tangent and Cotangent

Key Concepts

Definitions

- **Fundamental Trigonometric Functions:** The sine, cosine, and tangent functions.
- **Periodic Motion:** Movement that repeats along a path in a regular and predictable manner over a specific time interval.
- **Periodic Function:** A function that models periodic motion by repeating its values at regular intervals.
- **Sine Curve (Sinusoidal):** A smooth, wave-like curve that oscillates periodically and can be derived from either the sine or cosine function.
- **Period:** For a function $f(x)$, the period is the smallest positive number k such that $f(x+k)=f(x)$ for all x in the domain of f.
- **Principal Cycle (of a trigonometric function):** The portion of the graph of a trigonometric function on the interval starting at $x=0$ and ending at $x=p$, where p is the period of the function.
- **Amplitude (of a function):** Half the distance between the function's maximum (M) and minimum (m) values, calculated as $\frac{1}{2}|M-m|$.
- **Even Function:** A function f is even if $f(-x)=f(x)$ for every x in its domain. Its graph is symmetric with respect to the y-axis.
- **Odd Function:** A function f is odd if $f(-x)=-f(x)$ for every x in its domain. Its graph is symmetric with respect to the origin.

Theorems

- **Domain and Range of the Sine Function:** The domain of $f(x)=\sin(x)$ is $(-\infty,\infty)$ and the range is $[-1,1]$.
- **Period of the Sine Function:** The period of $f(x)=\sin(x)$ is 2π.
- **Principal Cycle of the Sine Function:** The principal cycle of the sine function is $[0,2\pi]$.
- **Amplitude of the Sine Function:** The amplitude of $f(x)=\sin(x)$ is 1.
- **Zeros of the Sine Function:** The zeros of $f(x)=\sin(x)$ occur at $x=\pi k$, where $k\in\mathbb{Z}$.
- **Domain and Range of the Cosine Function:** The domain of $f(x)=\cos(x)$ is $(-\infty,\infty)$ and the range is $[-1,1]$.
- **Period of the Cosine Function:** The period of $f(x)=\cos(x)$ is 2π.
- **Principal Cycle of the Cosine Function:** The principal cycle of the cosine function is $[0,2\pi]$.
- **Amplitude of the Cosine Function:** The amplitude of $f(x)=\cos(x)$ is 1.
- **Zeros of the Cosine Function:** The zeros of $f(x)=\cos(x)$ occur at $x=\frac{\pi}{2}+\pi k$, where $k\in\mathbb{Z}$.
- **Domain and Range of the Tangent Function:** The domain of the tangent function is $\{x|x\neq\frac{\pi}{2}+\pi k\}$, where $k\in\mathbb{Z}$. The range is $(-\infty,\infty)$.
- **Period of the Tangent Function:** The period of $f(x)=\tan(x)$ is π.
- **Principal Cycle of the Tangent Function:** The principal cycle of the tangent function is $[0,\pi]$.
- **Zeros of the Tangent Function:** The zeros of the tangent function occur at $x=\pi k$, where $k\in\mathbb{Z}$.
- **Symmetry Identities:** The sine, cosecant, tangent, and cotangent functions are odd. The cosine and secant functions are even.

Common Mistakes

- **Graphing in Degrees:** From this point forward, graphs of trigonometric functions will use radians for the horizontal axis, not degrees. This is critical for maintaining proper scale and for applications in calculus.
- **Connecting Plotted Points with Straight Lines:** The graphs of trigonometric functions are smooth curves. Never connect plotted points with straight line segments.
- **Confusing the Argument of a Function with a Coordinate:** The input variable for a function like $y = \sin(x)$ is a dummy variable representing an angle or real number. It should not be confused with the x-coordinate of a point on the unit circle.
- **Using Graphs to Solve Equations:** Using a graph to find the solution to an equation provides only a rough approximation and is not a valid method for finding exact solutions.
- **Using Graphs as Proofs:** A graph cannot be used to prove an equation is an identity, as it may be deceiving. However, a graph can be used to show an equation is *not* an identity if the graphs of the two sides are different.

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5.1.2: Homework

Reading Questions

1. What are the three fundamental trigonometric functions?
2. What is a periodic function, and what does its "period" represent?
3. What is the period of the sine function? What is the period of the tangent function?
4. What is the "amplitude" of a function?
5. What is the amplitude of the cosine function? Why does the tangent function not have an amplitude?
6. What is the domain and range of the sine function?
7. What is the domain and range of the tangent function?
8. What is the "principal cycle" of the sine function?
9. At what values of x does the graph of $y = \cos(x)$ have x-intercepts?
10. At what values of x does the graph of $y = \tan(x)$ have vertical asymptotes?
11. What does it mean for a function to be "even"? Which of the fundamental trigonometric functions are even?
12. What does it mean for a function to be "odd"? Which of the fundamental trigonometric functions are odd?

Homework

Vocabulary Check

1. Which trigonometric functions are called the *fundamental* trigonometric functions?
2. The smallest distance (along the x-axis) for which a function starts a repeating pattern (that continues forever) is called the ___ of the function.
3. The ___ for a function is found by taking one-half the sum of the function's maximum and minimum values.
4. If the graph of a function is symmetric about the origin, we say that the function is ___.

Concept Check

5. List the values from 0 to 2π by multiples of $\frac{\pi}{4}$.
6. List the values from 0 to 2π by multiples of $\frac{\pi}{6}$.
7. List several ways in which the graph of $y = \tan(\theta)$ is different from the graphs of $y = \sin(\theta)$ and $y = \cos(\theta)$.
8. Sketch two examples of a function with period 8: one that is sinusoidal and one that is not.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

9. The amplitude of a function is the vertical distance between the maximum and minimum values (if these exist).
10. $\tan(\theta + 180^\circ) = \tan\theta$ for any value of θ.
11. If $f(x + p) = f(x)$ for all x, then the constant p is called the period of the function.
12. The tangent function has period π. It is undefined at odd multiples of $\frac{\pi}{2}$ and is increasing on each interval of its domain.
13. Functions that have graphs shaped like sines or cosines are called sinusoidal.
14. The cosine graph looks just like the sine graph except flipped upside down.

15. The period of a sine or cosine function is the distance between horizontal intercepts.

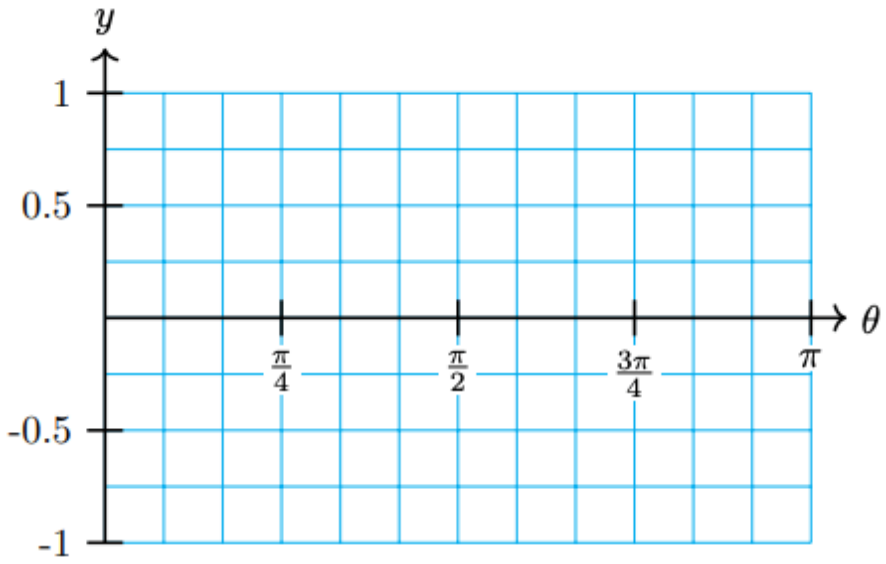
16. Periodic functions are used to model phenomena that exhibit cyclical behavior.

Basic Skills

17. a. Use your calculator to complete the table of values. Round values to hundredths.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$\cos\theta$													

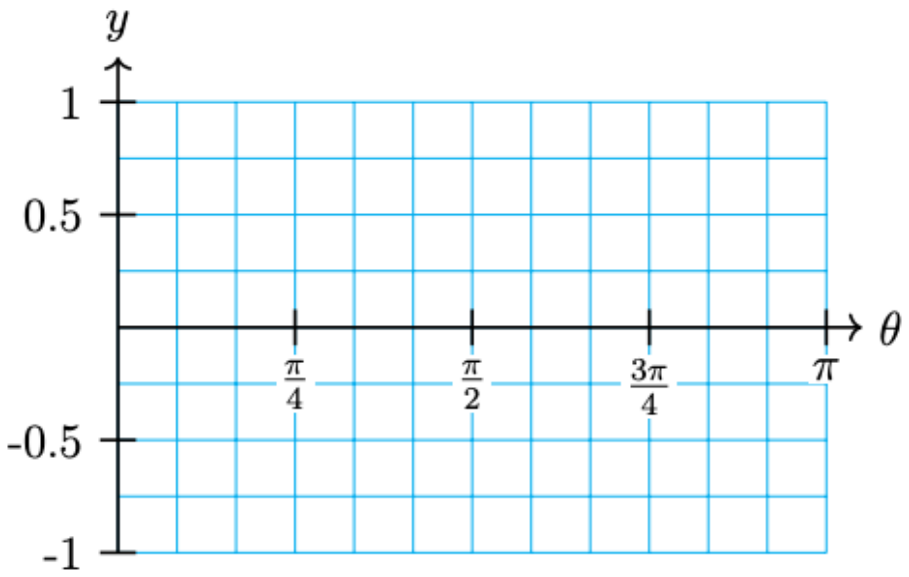
b. Sketch a graph of $y = \cos\theta$ on the grid.



18. a. Use your calculator to complete the table of values. Round values to hundredths.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$\sin\theta$												

b. Sketch a graph of $y = \sin\theta$ on the grid.



19. a. Sketch a graph of $y = \cos x$ using the guide points in the table below.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos x$					

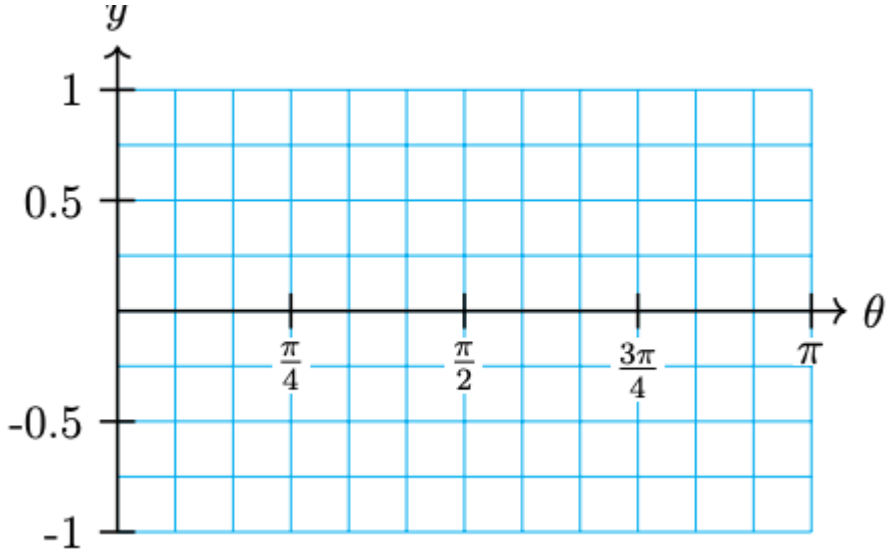
b. State the domain and range of the function $y = \cos x$.

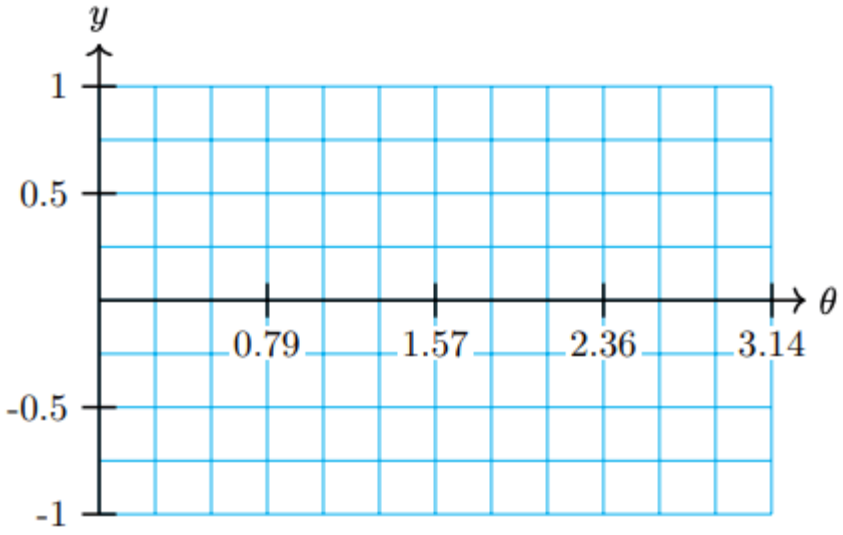
20. a. Sketch a graph of $y = \tan x$ using the guide points in the table below.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$\cos x$					

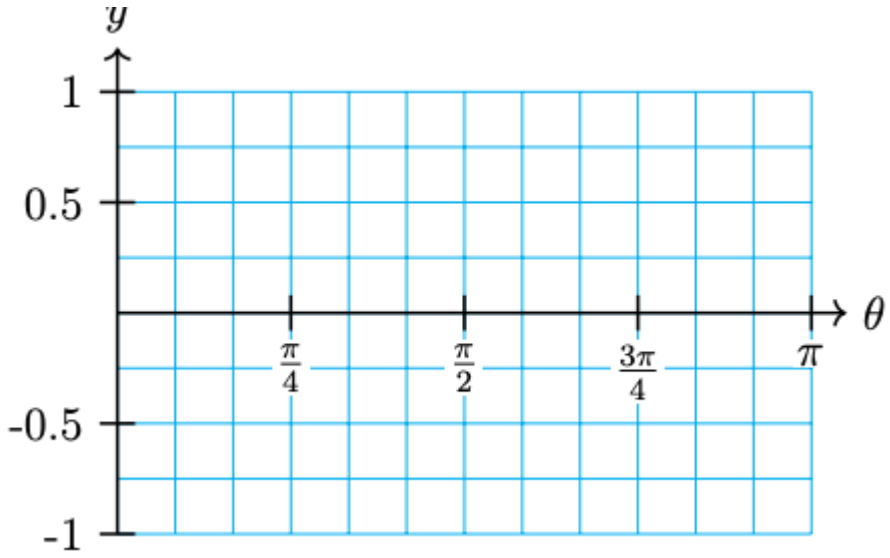
b. State the domain and range of the function $y = \tan x$.

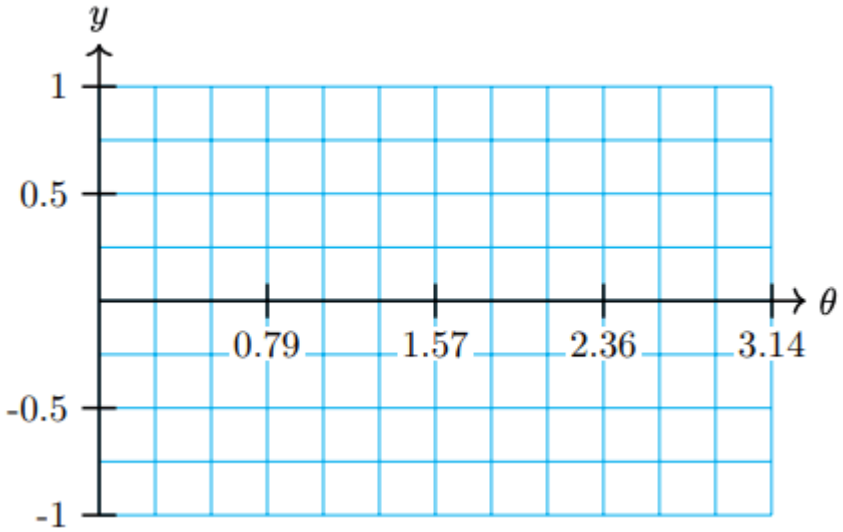
21. Sketch a graph of $y = \sin x$ on each grid.





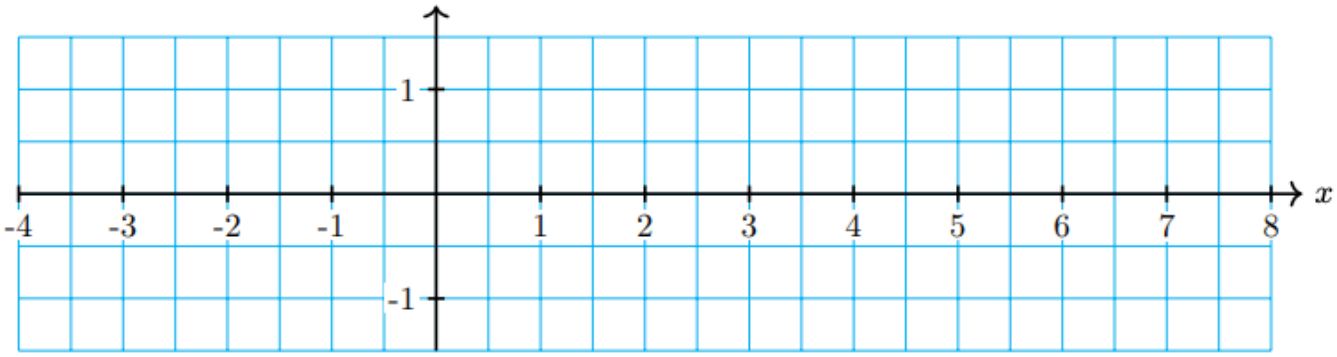
22. Sketch a graph of $y = \cos x$ on each grid.





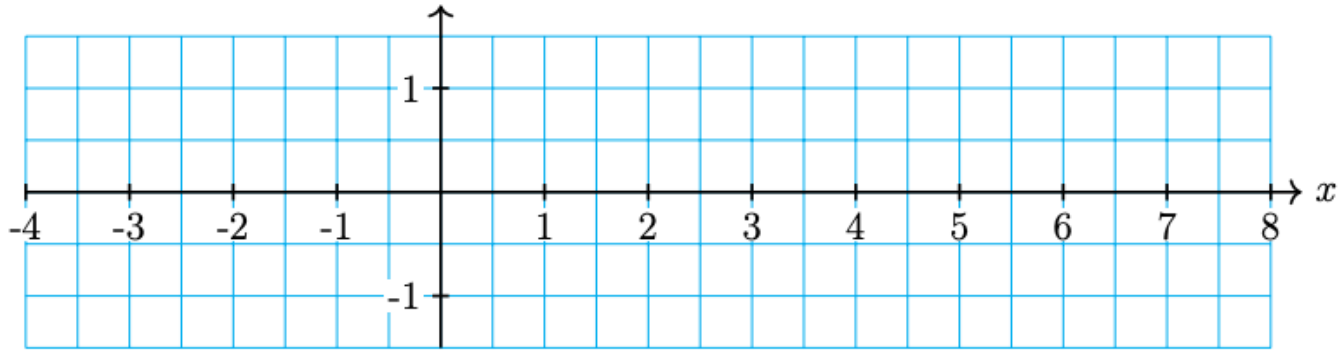
23. a. Sketch a graph of $f(x) = \sin x$ where x is a real number.

b. State the domain and range of $f(x) = \sin x$.



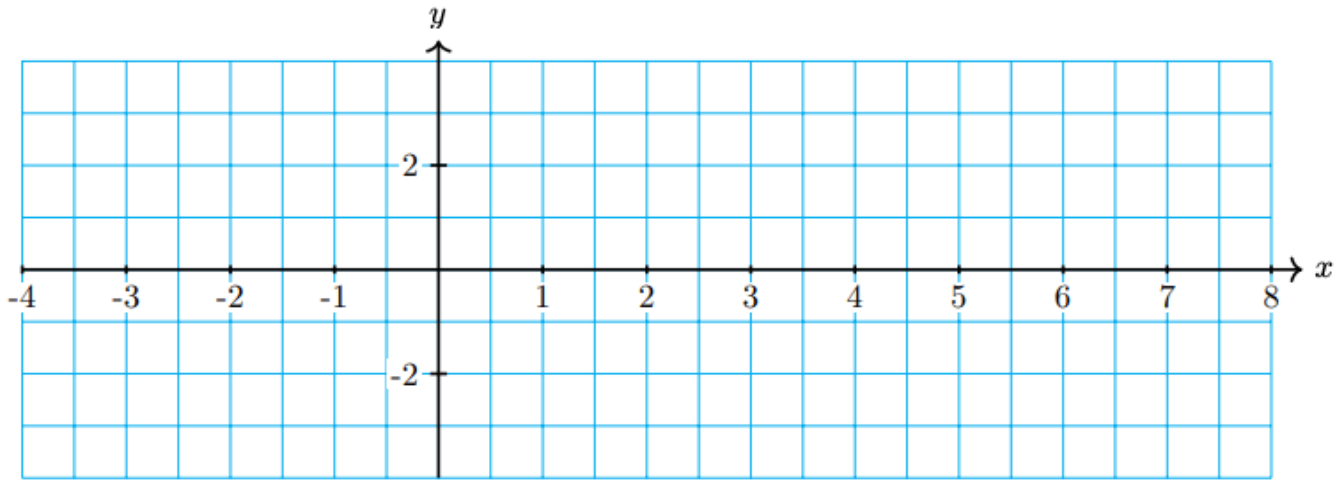
24. a. Sketch a graph of $g(x) = \cos x$ where x is a real number.

b. State the domain and range of $g(x) = \cos x$.

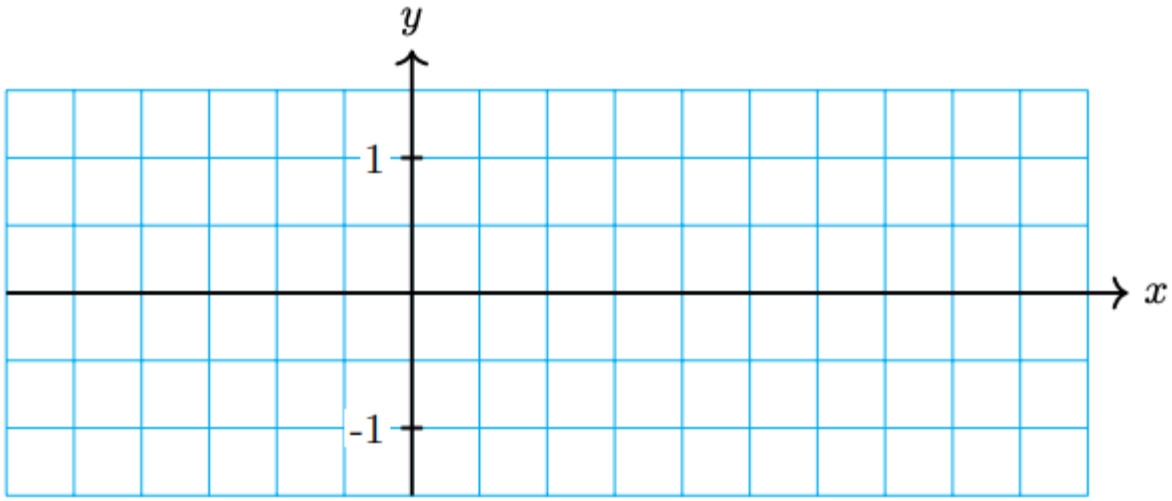


25. a. Sketch a graph of $h(x) = \tan x$ where x is a real number.

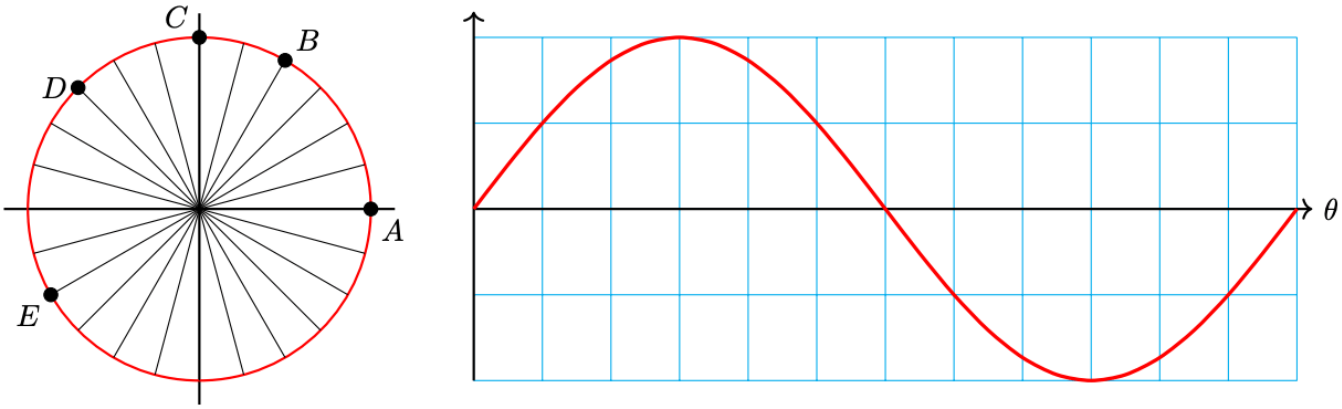
b. State the domain and range of $h(x) = \tan x$.



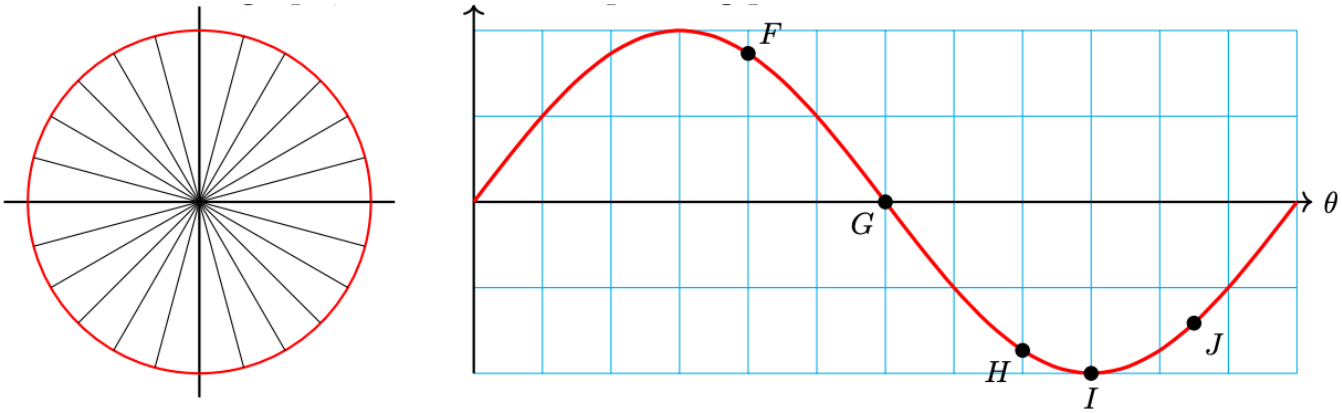
26. Sketch $f(x) = \cos x$ and $g(x) = \sin x$ on the same grid.



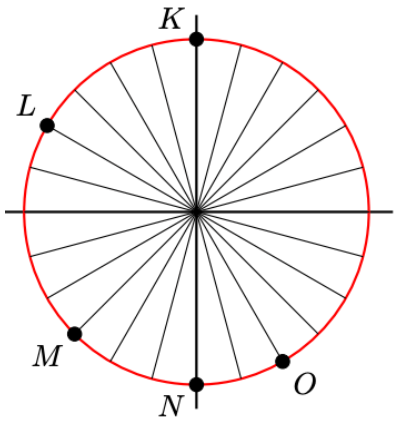
27. The graph shows your height as a function of angle as you ride the Ferris wheel. For each location A - E on the Ferris wheel, mark the corresponding point on the graph.

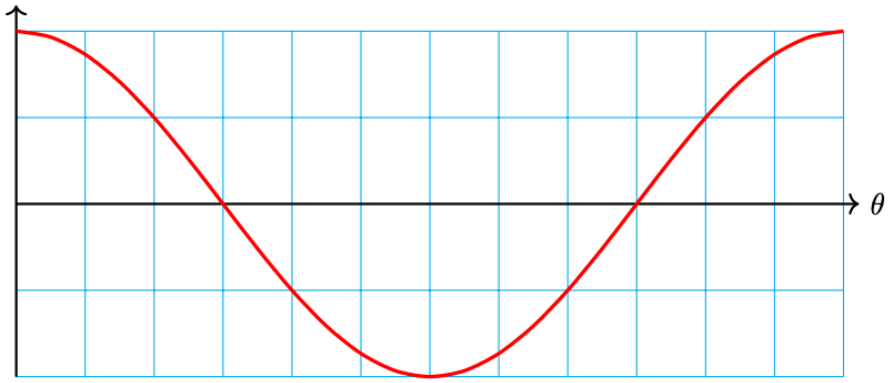


28. The graph shows your height as a function of angle as you ride the Ferris wheel. For each location F - J on the graph, mark the corresponding point on the Ferris wheel.

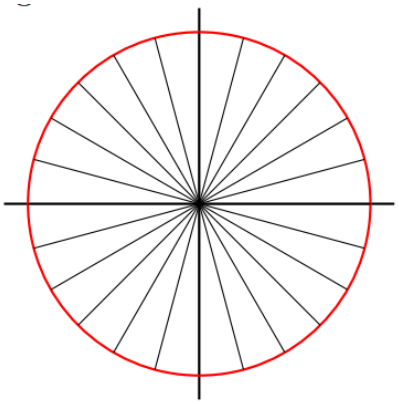


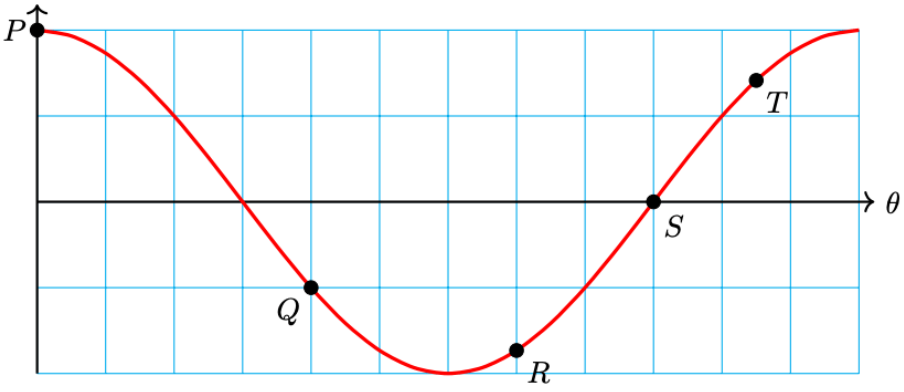
29. The graph shows the horizontal displacement of your foot from the center of the chain gear as you pedal a bicycle. For each location K - O on the chain gear, mark the corresponding point on the graph.



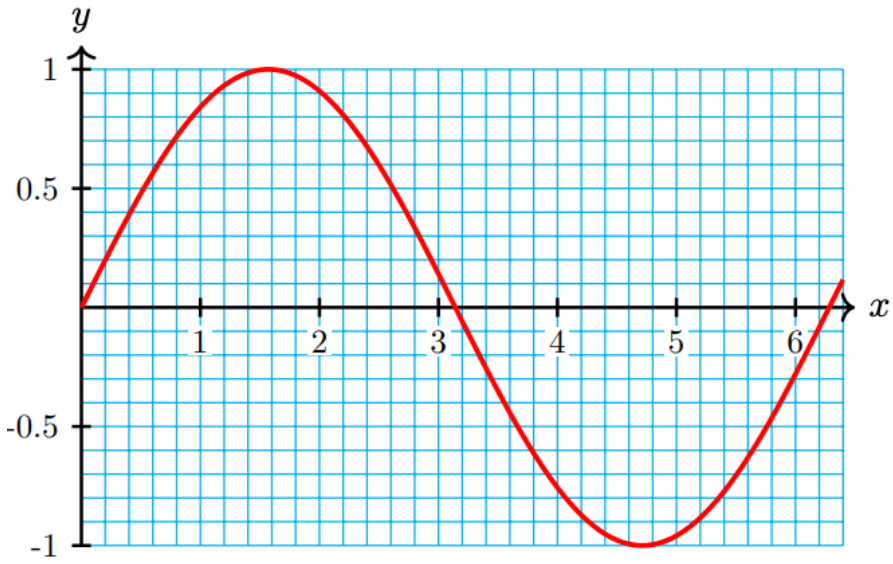


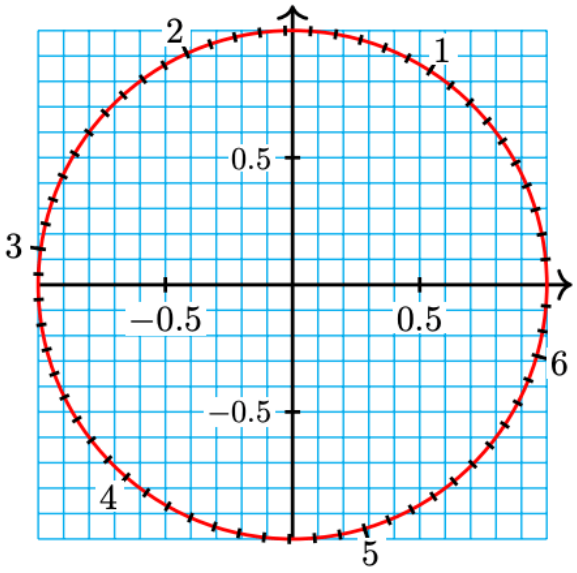
30. The graph shows the horizontal displacement of your foot from the center of the chain gear as you pedal a bicycle. For each location P - T on the graph, mark the corresponding point on the chain gear.





For the following exercises, use the figures below. Show your solutions on the graphs.





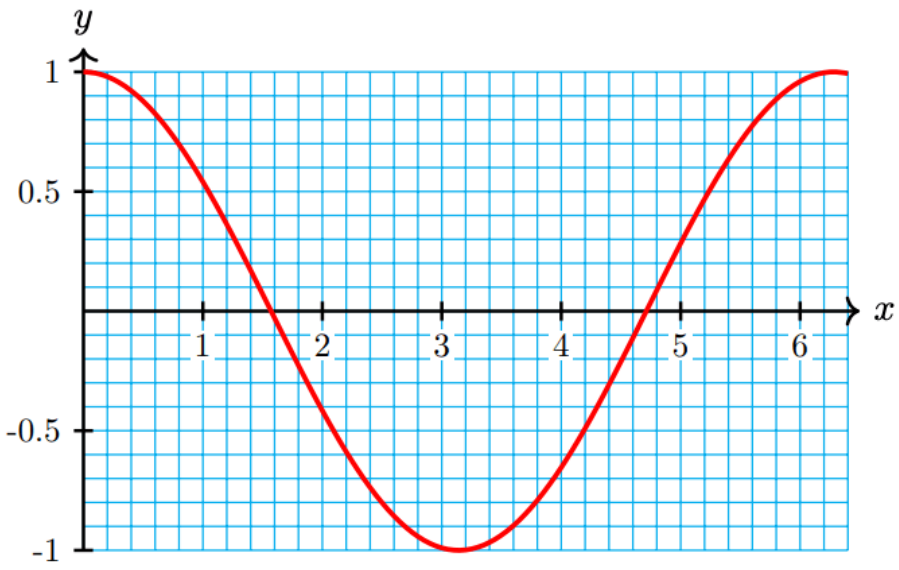
31. a. Use the graph of $y = \sin x$ to estimate two solutions of the equation $\sin x = 0.65$.

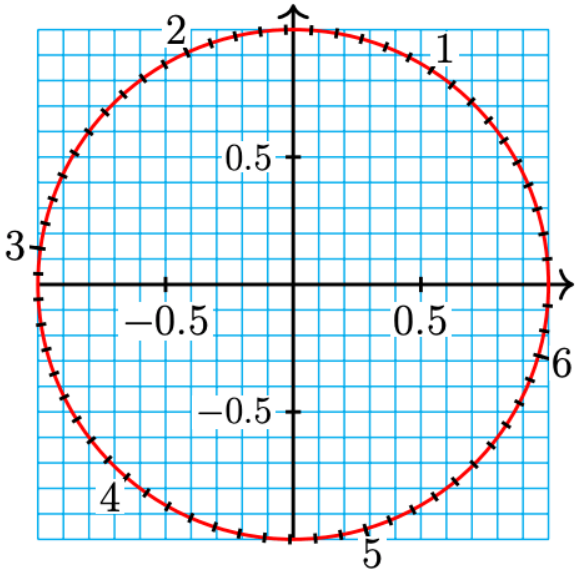
    b. Use the unit circle to estimate two solutions of the equation $\sin x = 0.35$.

32. a. Use the graph of $y = \sin x$ to estimate two solutions of the equation $\sin x = -0.2$.

    b. Use the unit circle to estimate two solutions of the equation $\sin x = -0.6$.

For the following exercises, use the figures below. Show your solutions on the graphs.





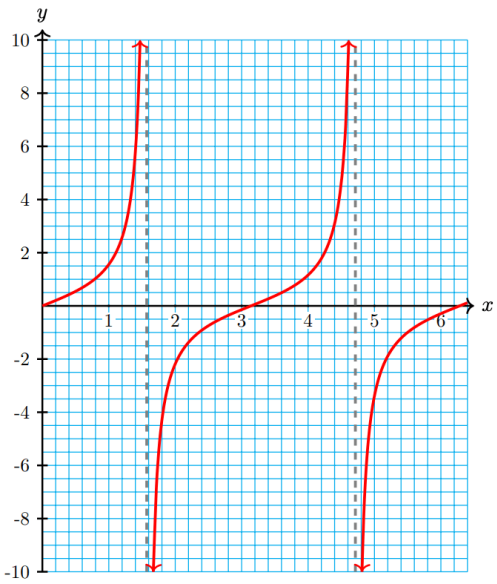
33. a. Use the graph of $y = \cos x$ to estimate two solutions of the equation $\cos x = -0.4$.

    b. Use the unit circle to estimate two solutions of the equation $\cos x = -0.8$.

34. a. Use the graph of $y = \cos x$ to estimate two solutions of the equation $\cos x = 0.15$.

    b. Use the unit circle to estimate two solutions of the equation $\cos x = 0.55$.

For the following exercises, use the graph of $y = \tan x$ to estimate two solutions to the equation.



35. $\tan x = 4$

36. $\tan x = 7$

37. $\tan x = -0.5$

38. $\tan x = -2.5$

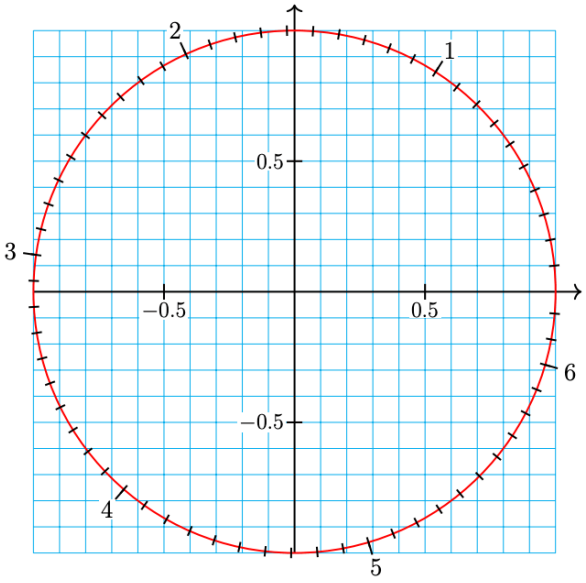
39. $\sin x = 5.5 \cos x$

40. $\sin x = -4.5 \cos x$

41. $-2 \sin x = 3 \cos x$

42. $2 \sin x = \cos x$

For the following exercises, find all solutions between 0 and 2π. Round to two decimal places. Sketch your solutions on a unit circle.



43. $\cos t = 0.74$

44. $\sin t = 0.58$

45. $\tan t = 1.6$

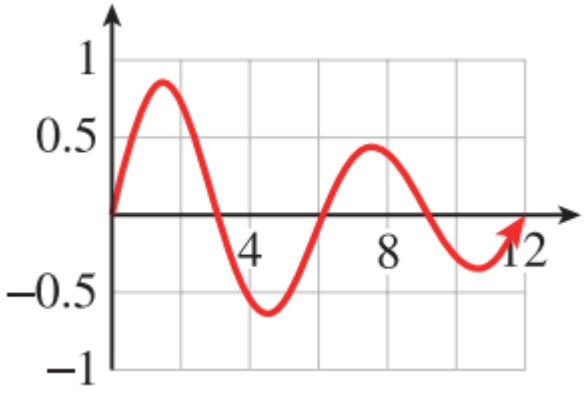
46. $\tan x = -0.6$

47. $\sin x = -0.72$

48. $\cos x = -0.48$

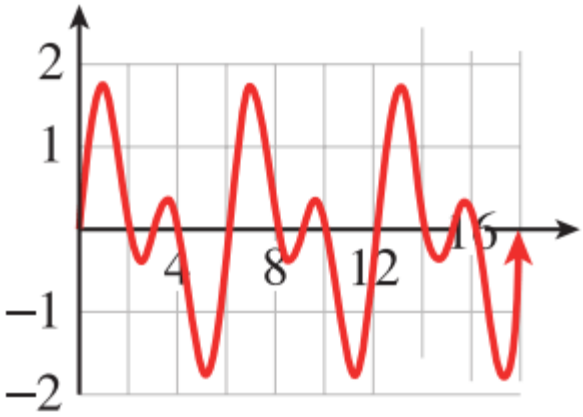
Which of the graphs in the following exercises are periodic? If the graph is periodic, give its period.

49.

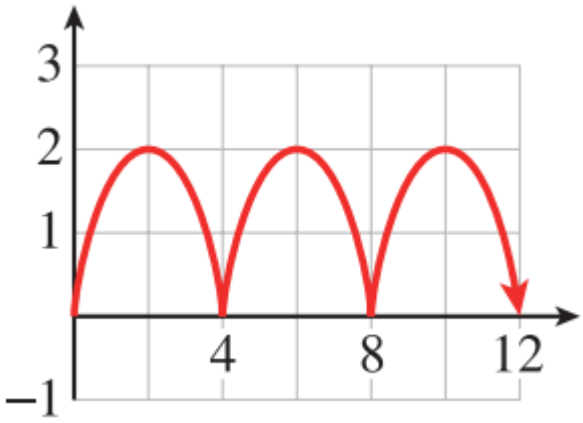


50.

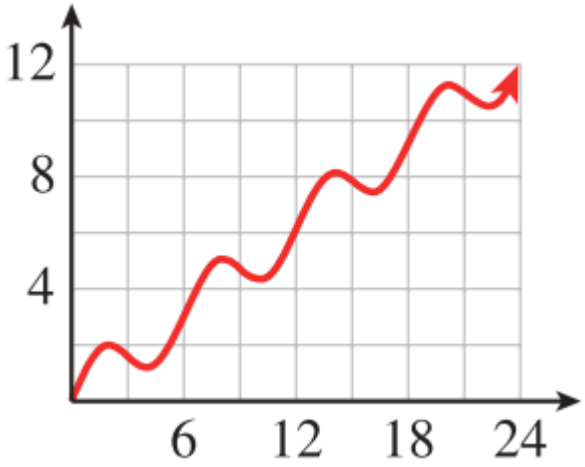
51.

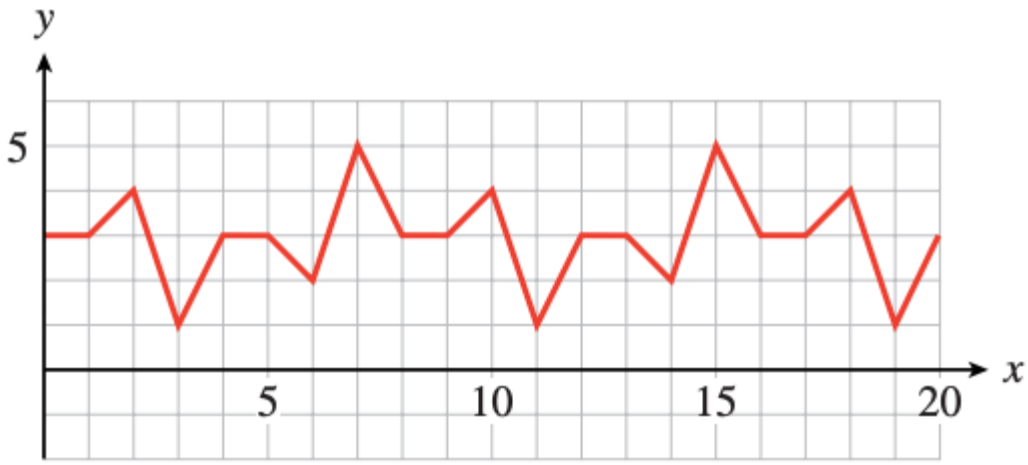


52.



53. a. Is the function shown periodic? If so, what is its period? If not, explain why not.





b. Compute the difference between the maximum and minimum function values. Sketch in the midline of the graph.

c. Find the smallest positive value of k for which $f(x) = f(x + k)$ for all x.

d. Find the smallest positive values of a and b for which $f(b) - f(a)$ is a maximum.

For the following exercises, use symmetry to help in evaluating the given trigonometric function.

54. $\cos(-30^\circ)$

55. $\sin\left(-\frac{5\pi}{6}\right)$

56. $\tan(-135^\circ)$

57. $\sec\left(-\frac{5\pi}{3}\right)$

58. $\csc(-315^\circ)$

59. $\cot\left(-\frac{7\pi}{6}\right)$

Synthesis Questions

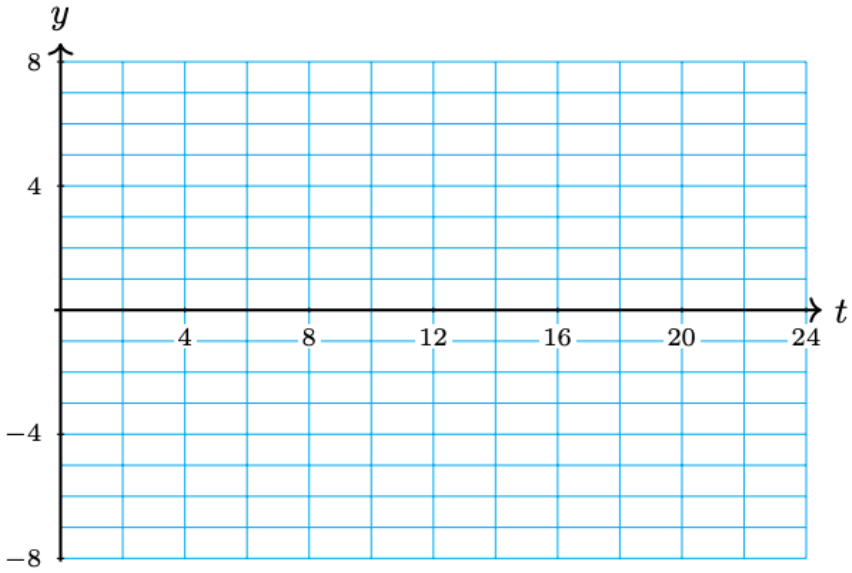
60. An ant is walking clockwise around the face of a sundial in the garden. The sundial is a circle with a 12-inch diameter, and the ant makes one circuit of the sundial in 24 seconds.

a. Sketch the sundial and a coordinate system with its origin at the center of the dial. Draw the y-axis to align with the gnomon (pointer) of the sundial.

b. Suppose we start timing the ant when it is at the tip of the gnomon. Complete the table showing the ant's location as an angle in standard position and its y-coordinate at that time.

t	0	2	4	6	8	10	12	14	16	18	20	22	24
θ	90°	60°											
$y = f(t)$	6	$3\sqrt{3}$											

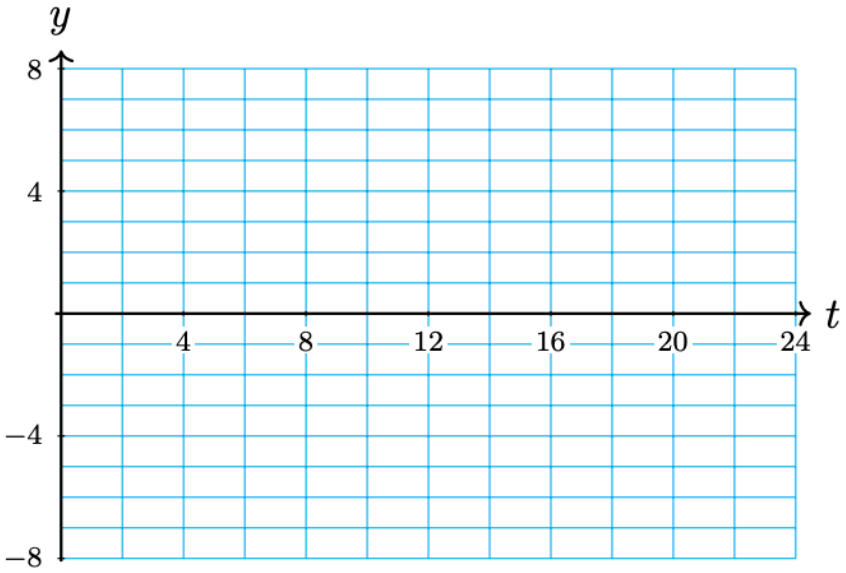
c. Sketch a graph of the ant's y-coordinate as a function of time.



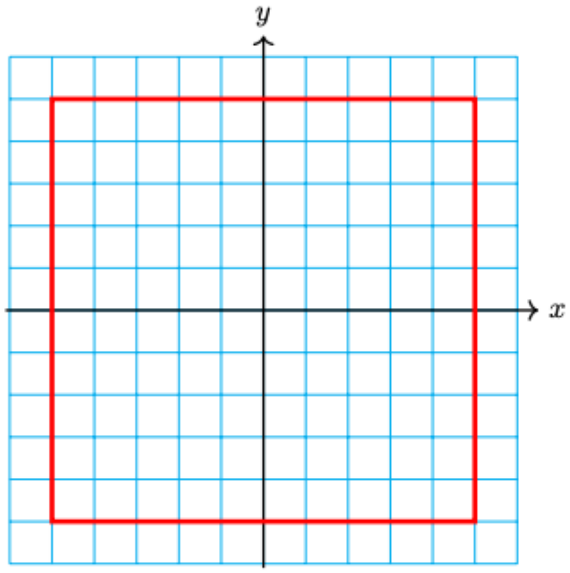
d. If the ant makes a second trip around the sundial, what will the graph look like from $t = 24$ to $t = 48$? Explain the statement $f(t + 24) = f(t)$.

61. Repeat the previous exercise, but make a table and graph of the ant's x-coordinate as a function of time.

t	0	2	4	6	8	10	12	14	16	18	20	22	24
θ	90°	60°											
$x = g(t)$	0	3											



62. Jorge is standing at the point $(5, 0)$ on the square shown. If he walks around the square in the counter-clockwise direction, his y-coordinate is a function of the distance, d, he has walked.



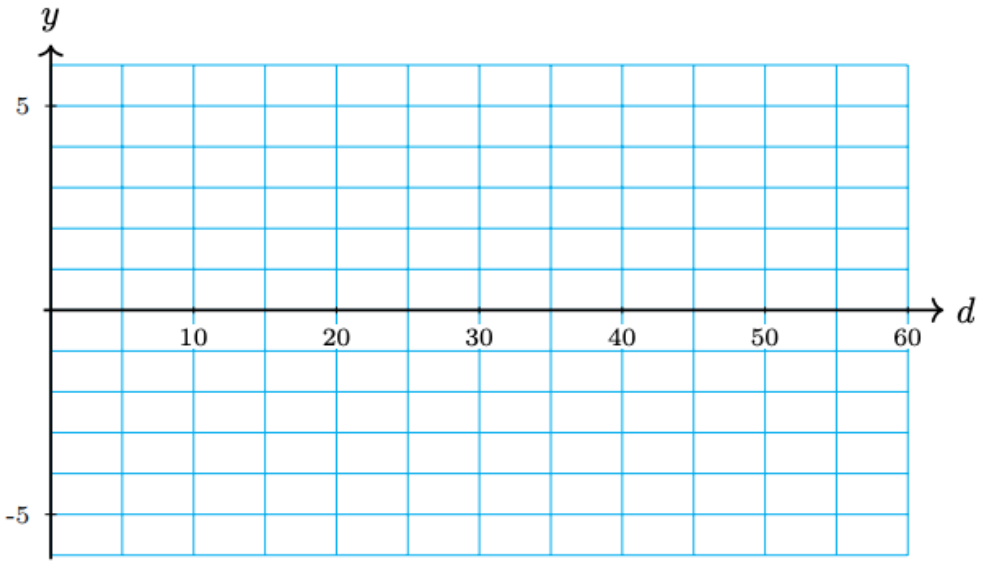
a. What is the y-coordinate of Jorge's position when he has walked 2 units? 5 units? 8 units? (Start filling in the table in part (c).)

b. After Jorge reaches the upper right corner of the square, he will turn left and start walking along the top of the square. What is his y-coordinate from $d = 5$ to $d = 15$?

c. Fill in the rest of the table with the y-coordinates of Jorge's position when he has walked a distance d.

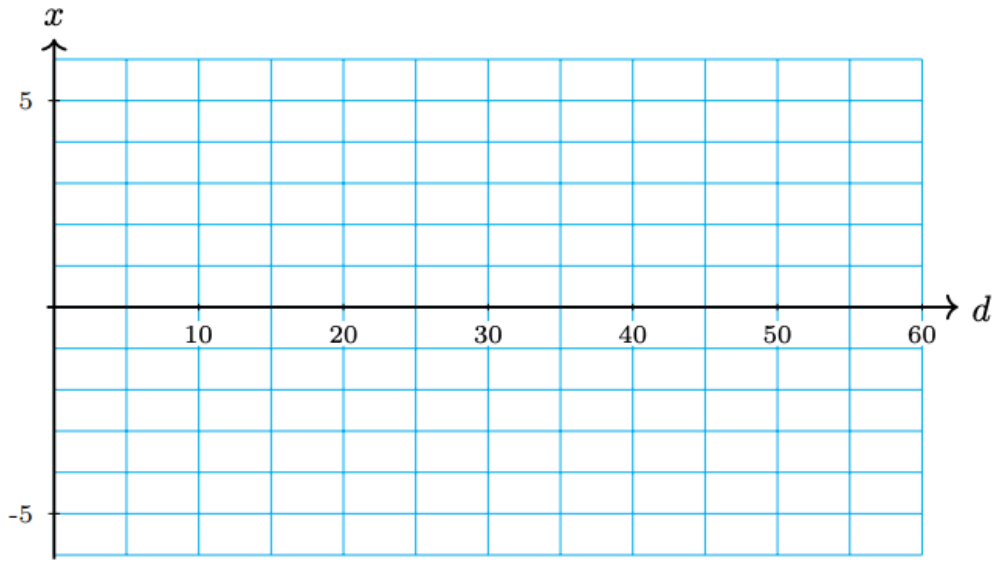
d	0	2	5	8	10	12	15	18	20	22	25	28	30	32	35	38	40
y																	

d. Plot a graph of Jorge's coordinate as a function of the distance he has walked.



53. Repeat the previous exercise, except consider Jorge's x-coordinate as a function of the distance he has walked.

d	0	2	5	8	10	12	15	18	20	22	25	28	30	32	35	38	40
x																	



a. Refer to the previous exercise. Suppose you know where Jorge is on the square at some given time. Can you predict where he will be after he has walked another 40 units?

b. Let $y = f(d)$. What does your answer say about $f(d)$ and $f(d + 40)$ for any positive value of d?

c. How does the graph of f for $0 \leq d \leq 40$ compare with the graph for $40 \leq d \leq 80$?

d. Describe how the graph of f would continue for $0 \leq d \leq 400$.

64. Refer to the previous exercise. Sketch a graph of Jorge's x-coordinate, $x = g(d)$, for $0 \leq d \leq 120$.

Proofs

For the following exercises, you may assume that sine is an odd function and cosine is an even function.

65. Prove that tangent is an odd function.

66. Prove that secant is an even function.

67. Prove that cosecant is an odd function.

68. Prove that cotangent is an odd function.

For the following exercises, prove the identity.

69. $\cot(-\theta)\sin(-\theta) = \cos(\theta)$

70. $\dfrac{\cos(-x)\sec(x)}{\tan(-x)} = -\cot(x)$

71. $\sin(-y)\sec(-y) = \tan(-y)$

72. $\sin(-\theta)\sec(-\theta)\cot(-\theta) = 1$

73. $1 = 2\cos(x)\sec(-x) + \tan(-y)\cot(y)$

74. $\dfrac{\tan(-\gamma)}{\sin(\gamma)} = -\sec(\gamma)$

75. $\tan(A) - \dfrac{1}{\tan(-A)} = \dfrac{\sec(-A)}{\sin(A)}$

76. $\sec(z) + \tan(z)\sin(-z) = \cos(-z)$

77. $\sin(-\phi)[\csc(-\phi) + \sec(-\phi)] = 1 + \tan(-\phi)$

Applications

For the following exercises, sketch a periodic function that models the situation.

78. At a ski slope, the lift chairs take 5 minutes to travel from the bottom, at an elevation of 3000 feet, to the top, at elevation 4000 feet. The cable supporting the ski lift chairs is a loop turning on pulleys at a constant speed. At the top and bottom, the chairs are at a constant elevation for a few seconds to allow skiers to get on and off.

    a. Sketch a graph of $h(t)$, the height of one chair at time t. Show at least two complete trips.

    b. What is the period of $h(t)$?

79. The heater in Wyatt's house doesn't have a thermostat; it runs on a timer. It uses 300 watts when it is running. Wyatt sets the heater to run from 6 am to noon, and again from 4 pm to 10 pm in the evening.

    a. Sketch a graph of $P(t)$, the power drawn by the heater as a function of time. Show at least two days of heater use.

    b. What is the period of $P(t)$?

80. Piyali adds water to her fish pond once a week to keep the depth at 30 centimeters. During the week, the water evaporates at a constant rate of 0.5 centimeters per day.

    a. Sketch a graph of $D(t)$, the depth of the water as a function of time. Show at least two weeks.

    b. What is the period of $D(t)$?

81. Roy's miniature schnauzer, Sophie, is very energetic and bounces excitedly at dinner time. Sophie can jump 30 inches high, and each jump takes him 0.8 seconds.

    a. Sketch a graph of Sophie's height, $D(t)$, as a function of time. Show at least two jumps.

    b. What is the period of $h(t)$?

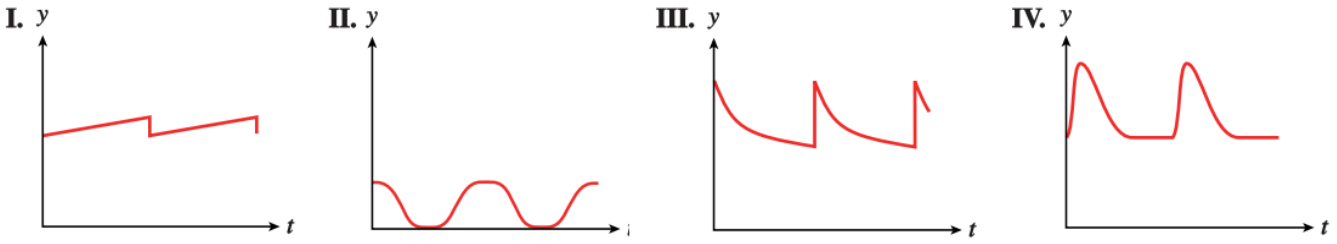
82. Each situation describes a periodic function. Match each situation with the appropriate graph.

    a. When the heart contracts, blood pressure in the arteries rises rapidly to a peak (systolic blood pressure) and then falls off quickly to a minimum (diastolic blood pressure). Blood pressure is a function of time.

    b. After an injection is given to a patient, the amount of the drug present in his bloodstream decreases over time. The patient receives injections at regular intervals to restore the drug level to the prescribed level. The amount of the drug present is a function of time.

    c. The monorail shuttle train between the north and south terminals at Gatwick Airport departs from the south terminal every 12 minutes. The distance from the train to the south terminal is a function of time.

    d. Ron gets a haircut every two weeks. The length of his hair is a function of time.



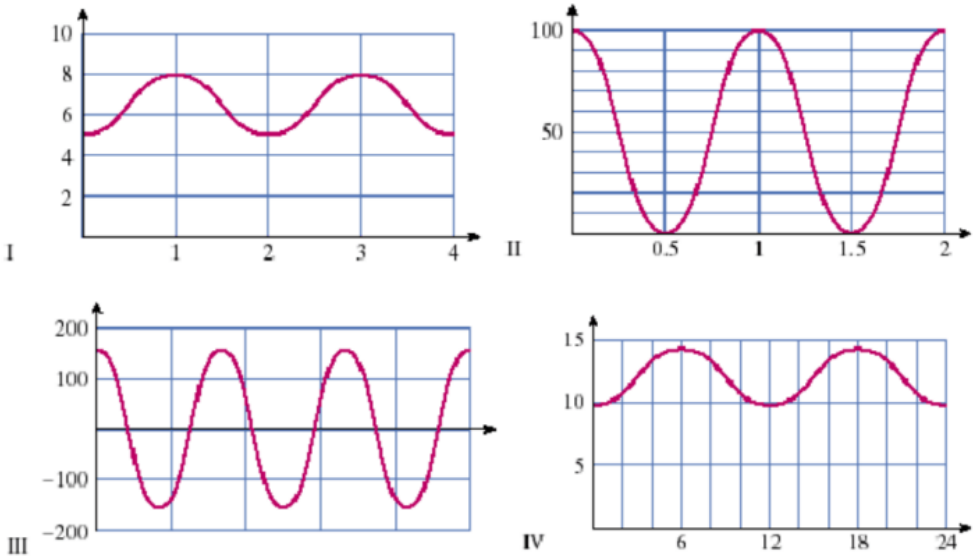
33. Match each of the following situations with an appropriate graph below.

a. The number of hours of daylight in Salt Lake City varies from a minimum of 9.6 hours on the winter solstice to a maximum of 14.4 hours on the summer solstice.

b. A weight is 6.5 feet above the floor, suspended from the ceiling by a spring. The weight is pulled down to 5 feet above the floor and released, rising past 6.5 feet in 0.5 seconds before attaining its maximum height of feet. Neglecting the effects of friction, the height of the weight will continue to oscillate between its minimum and maximum height.

c. The voltage used in U.S. electrical current changes from 155V to –155V and back 60 times each second.

d. Although the moon is spherical, what we see from earth looks like a disk, sometimes only partly visible. The percentage of the moon's disk that is visible varies between 0 (at new moon) to 100 (at full moon).



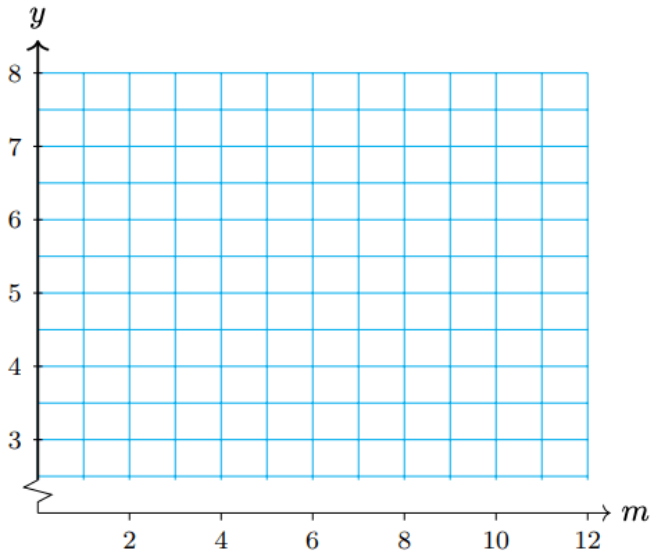
34. The table shows sunrise and sunset times in Los Angeles on the fifteenth of each month.

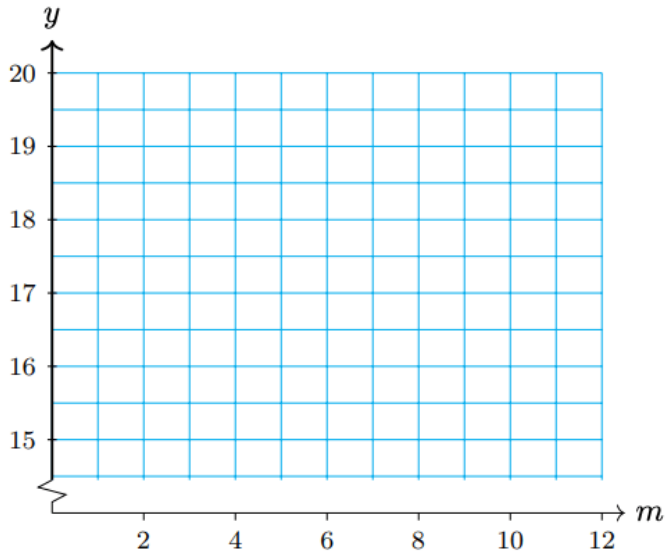
Month	Oct	Nov	Dec	Jan	Feb	Mar
Sunrise	5:58	6:26	6:51	6:59	6:39	6:04
Sunset	17:20	16:50	16:45	17:07	17:37	18:01

Month	Apr	May	Jun	Jul	Aug	Sep
Sunrise	5:22	4:52	4:42	4:43	5:15	5:37
Sunset	18:25	18:48	19:07	19:05	18:40	18:00

a. Use the left-hand grid to plot the sunrise times and sketch a sinusoidal graph through the points.

b. Use the right-hand grid to plot the sunset times and sketch a sinusoidal graph through the points.



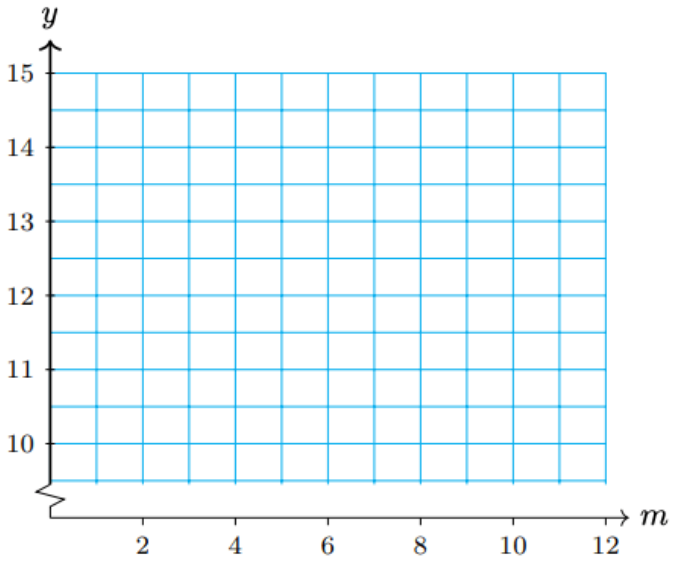


35. a. Use the data from Problem 65 to complete the table with the hours of sunlight in Los Angeles on the fifteenth of each month.

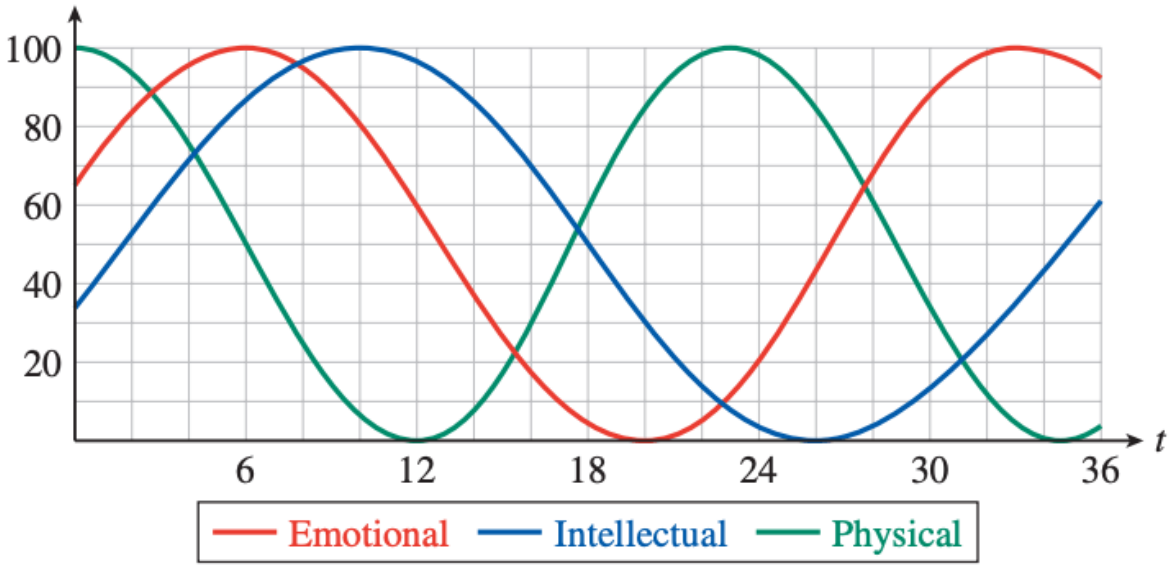
Month	Oct	Nov	Dec	Jan	Feb	Mar
Hours of Daylight						

Month	Apr	May	Jun	Jul	Aug	Sep
Hours of Daylight						

b. Plot the daylight hours and sketch a sinusoidal graph through the points.



36. Many people who believe in astrology also believe in biorhythms. The graph shows an individual's three biorhythms, physical, emotional, and intellectual, for 36 days, from $t = 0$ on September 30 to November 5.

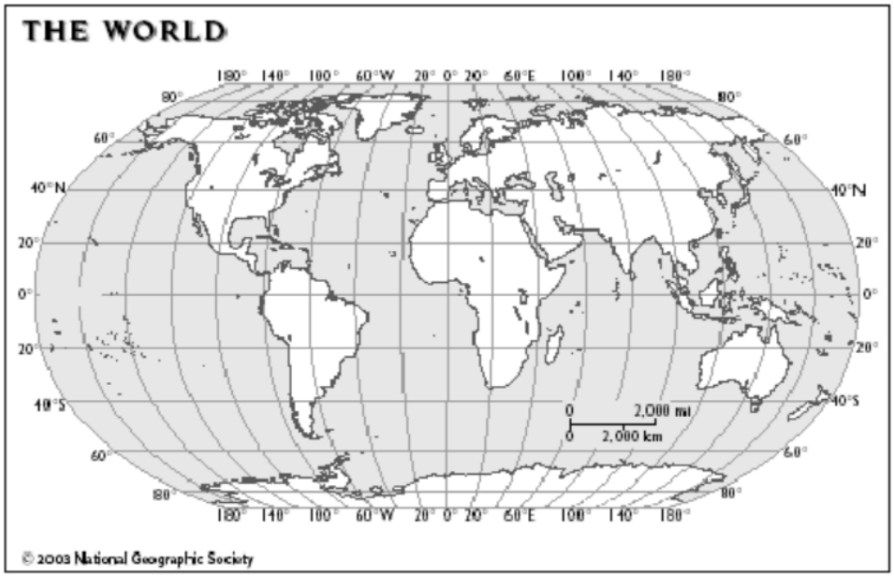


a. Find the dates of highest and lowest activity for each biorhythm during the month of October.

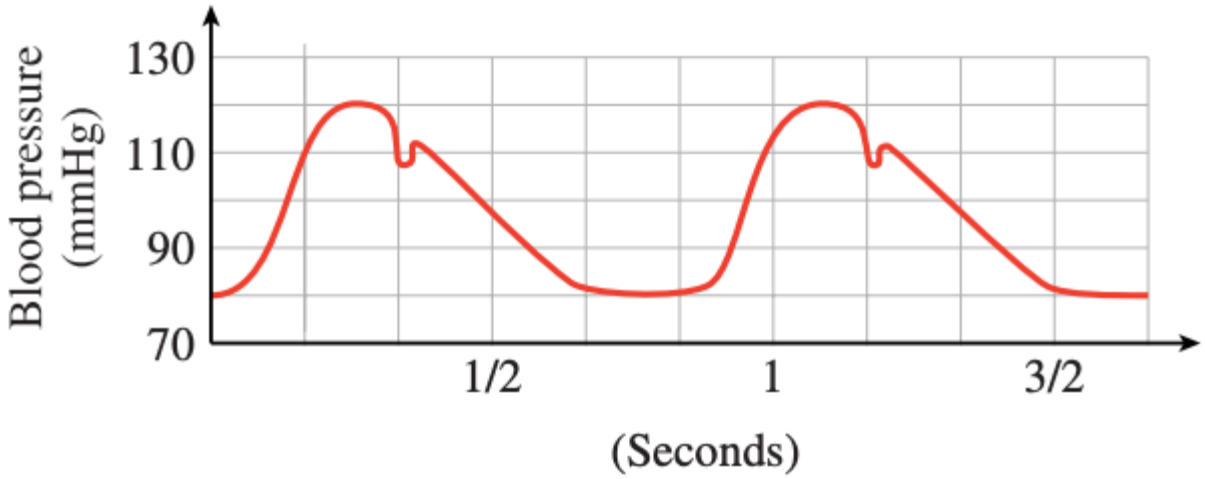
b. Find the period of each biorhythm in days.

c. On the day of your birth, all three biorhythms are at their maximum. How old will you be before all three are again at the maximum level?

37. The path of a satellite orbiting above the earth makes a sinusoidal graph on a map of the earth, with its midline at the equator. On the map on the next page, sketch a graph for a satellite that orbits the earth every 90 minutes, and strays no farther than 4000 km from the equator. (One degree of latitude is equal to 111 kilometers.) The satellite passes over the spot 0° latitude and 0° longitude at time $t = 0$. Label a scale on the equator to serve as a time axis for your graph.



38. The graph shows arterial blood pressure, measured in millimeters of mercury (mmHg), as a function of time.

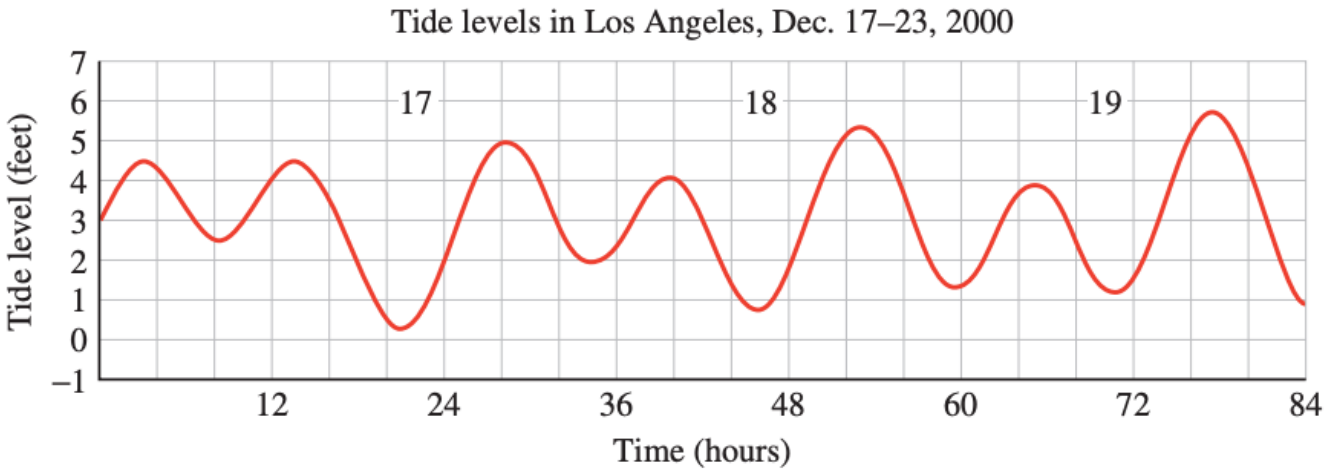


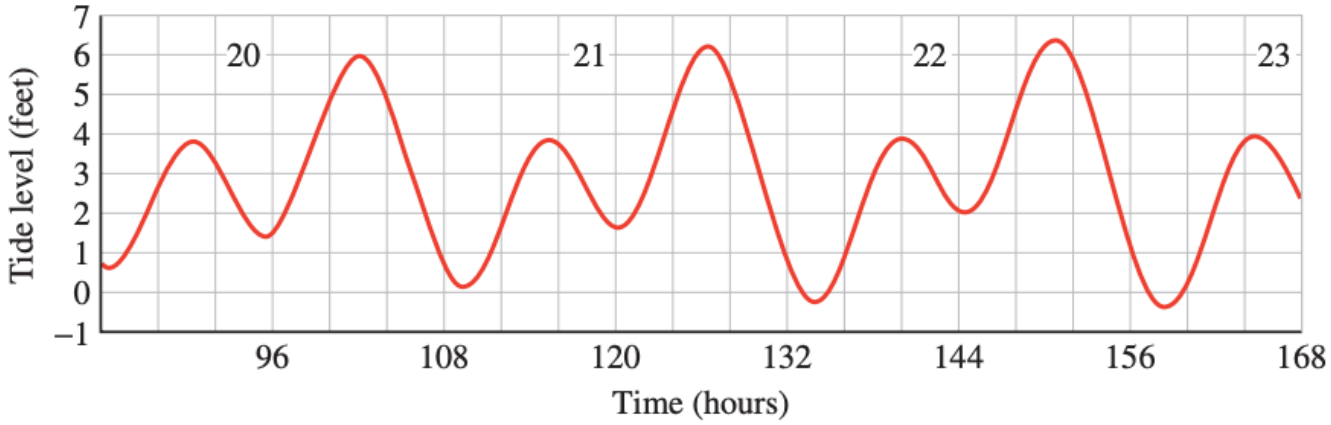
a. What are the maximum (systolic) and minimum (diastolic) pressures? The pulse pressure is the difference of systolic and diastolic pressures. What is the pulse pressure?

b. The mean arterial pressure is the diastolic pressure plus one-third of the pulse pressure. Calculate the mean arterial pressure, and draw a horizontal line on the graph at that pressure.

c. The blood pressure graph repeats its cycle with each heartbeat. What is the heart rate, in beats per minute, of the person whose blood pressure is shown in the graph?

39. Here is a tide chart for Los Angeles for the week of December 17–23, 2000. The horizontal axis shows time in hours, with $t = 12$ corresponding to noon on December 17. The vertical axis shows the height of the tide in feet above mean sea level.





a. High tides occurred at 3:07 am and 2:08 pm on December 17, and low tides at 8:41 am and 9:02 pm. Estimate the heights of the high and low tides on that day.

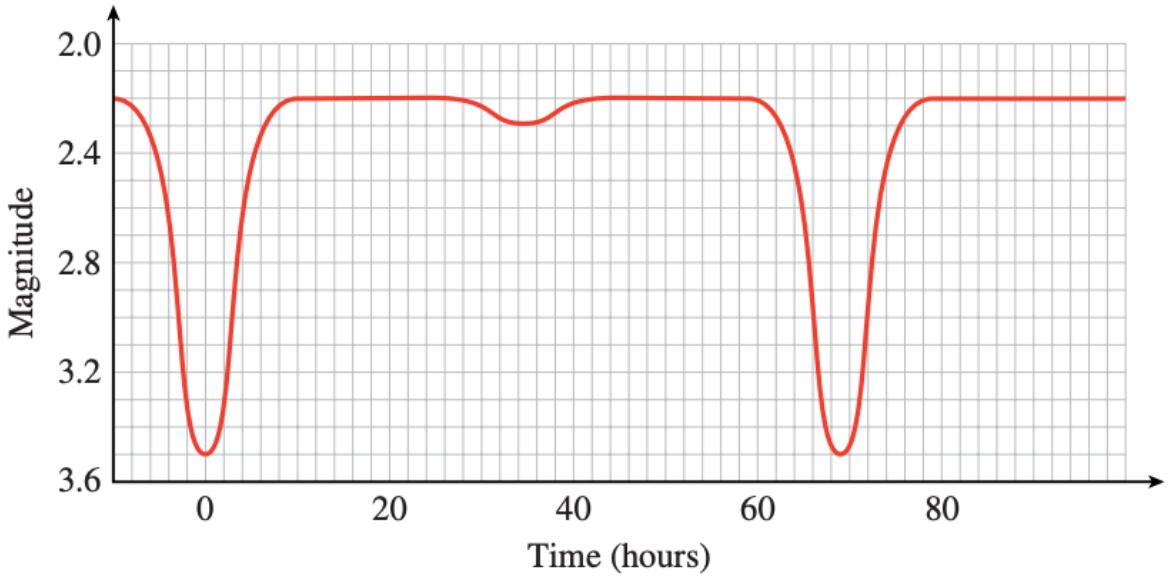
b. Is tide height a periodic function of time? Use the information from part (a) to justify your answer.

c. Make a table showing approximate times and heights for the high tides throughout the week. Make a similar table for the low tides.

d. Describe the trend in the heights of the high tides over the week. Describe the trend in the heights of the low tides.

e. What is the largest height difference between consecutive high and low tides during the week shown? When does it occur?

20. The apparent magnitude of a star is a measure of its brightness as seen from earth. Smaller values of apparent magnitude correspond to brighter stars. The graph below, called a light curve, shows the apparent magnitude of the star Algol as a function of time. Algol is actually a system of two stars, a bright principal star and its dimmer companion, in orbit around each other. As each star passes in front of the other it eclipses some of the light that reaches earth.

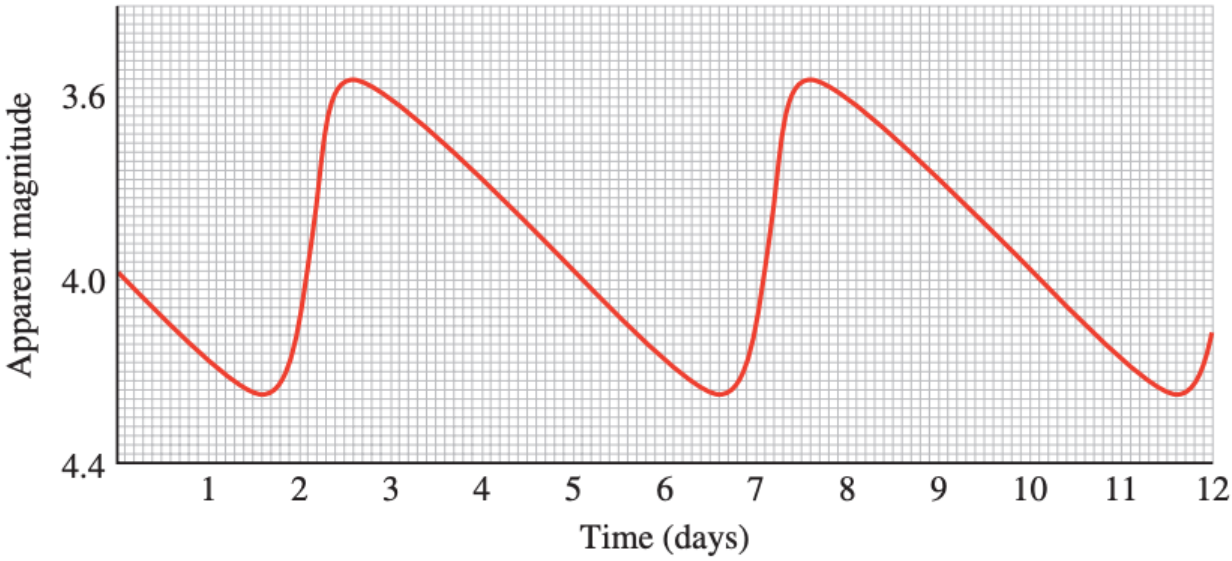


a. The light curve is periodic. What is its period?

b. What is the range of apparent magnitudes of the Algol system?

c. Explain the large and small dips in the light curve. What is happening to cause the dips?

91. Some stars, called Cepheid variable stars, appear to pulse, getting brighter and dimmer periodically. The graph shows the light curve for the star Delta Cephei.



a. What is the period of the graph?

b. What is the range of apparent magnitudes for Delta Cephei?

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5.2: Non-Rigid Transformations

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The material in the next few sections relies heavily on your prerequisite graphing skills from Algebra. As such, we will try our best to "backfill" some of that information here. We begin by reintroducing the definition of a **transformation** of the graph of a function.

Definition: Transformation (of the graph of a function)

A **transformation** of the graph of a function is an alteration of the graph's orientation (by reflections across axes), scale, and position through various operations.

What you may *not* have learned in your Algebra course is that these transformations can be categorized into two distinct groups—**rigid** and **non-rigid transformations**.

Definition: Rigid Transformation

A **rigid transformation** of a graph is a transformation that changes the position of the graph without changing its shape or size.

You likely remember rigid transformations as **shifts**. As a reminder of the basic idea of a rigid transformation, let's consider the adorable "base graph" of my dog, $y = \text{Sophie}\,(x)$ in Figure $5.2.1$.

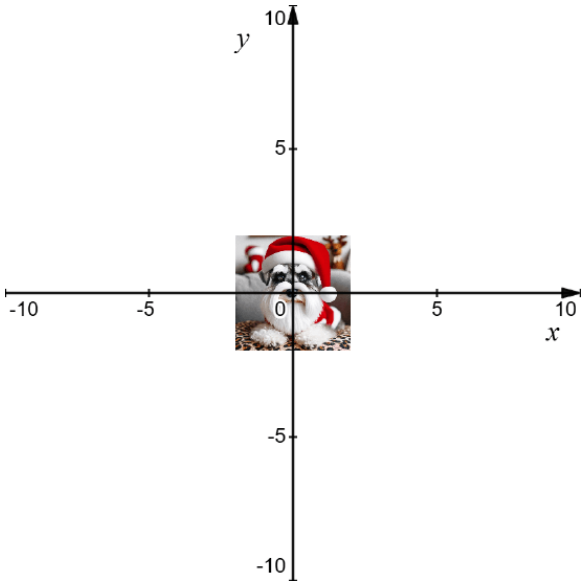


Figure 5.2.1**:** *The base graph of* $y = \text{Sophie}\,(x)$

Note: This is *not* a function

Technically, this is not a function (Sophie doesn't pass the Vertical Line Test); however, using an image usually conveys the point of transformations to students more efficiently than traditional functions.

From Algebra, we know that if we wanted to shift the graph of Sophie 2 units to the right and 3 units down, the resulting function would be

$$g(x) = \text{Sophie}\,(x-2) - 3.$$

The orientation and scale of the graph of this new "function" (seen in Figure 5.2.2) would be the same as the base graph of the "Sophie" function, but the position is different.

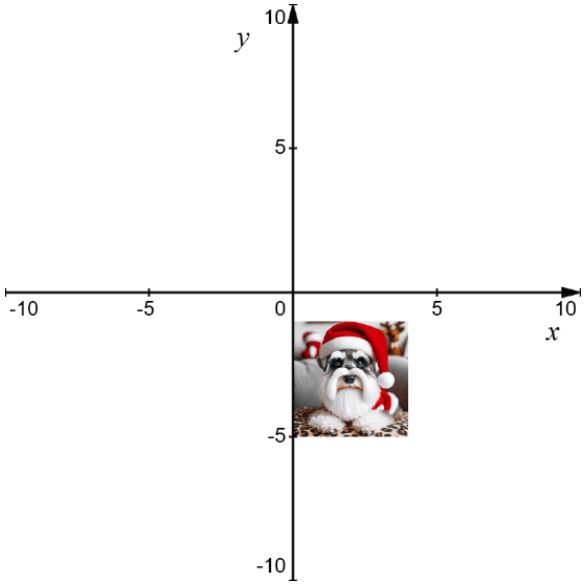


Figure 5.2.2**:** $g(x) = \text{Sophie}\,(x-2) - 3$

In general, given a function $f(x)$, we can obtain the graph of $g(x) = f(x + C) + D$ by *subtracting* C from all of the x-coordinates of the graph of $f(x)$ and *adding* D to all the y-coordinates.

Definition: Non-rigid Transformation

A **non-rigid transformation** of a graph if a transformation that retains the graph's original center position but scales the graph vertically or horizontally, or changes the graph's orientation by reflecting the graph about a line.

For example, if we took the base graph of our "Sophie function" from Figure $5.2.1$ and decided to scale it vertically (also known as stretching) by a factor of 2, we would need to multiply all the y-coordinates of the points in the base graph by 2. From Algebra, we know this is achieved by multiplying the function by 2 to get $h(x) = 2\ \text{Sophie}\,(x)$. The graph of this new image would look like the following:

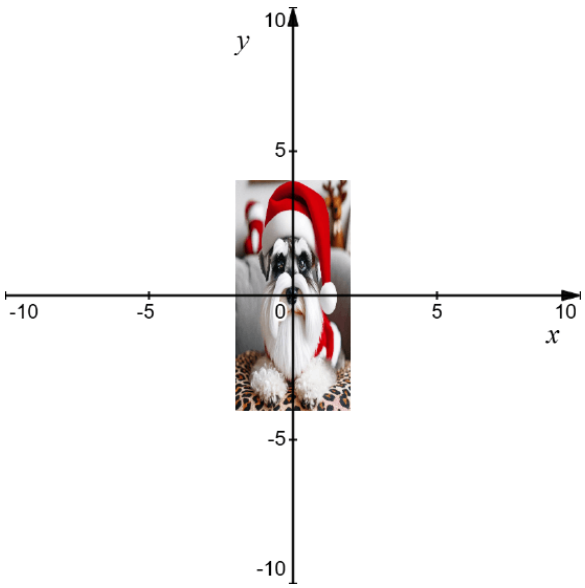


Figure $5.2.3$**:** $h(x) = 2\ \text{Sophie}\,(x)$

We also learned (again, from Algebra) to combine transformations to get a new function from an old one. For example, we could take $\text{Sophie}\,(x)$, scale it horizontally by a factor of $1/2$ (also known as compressing), reflect it about the y-axis, shift it 3 units left, and shift it 4 units up. The result of all of these manipulations is the function $k(x) = \text{Sophie}\,(-2\,(x+3)) + 4$, and it looks like the following image:

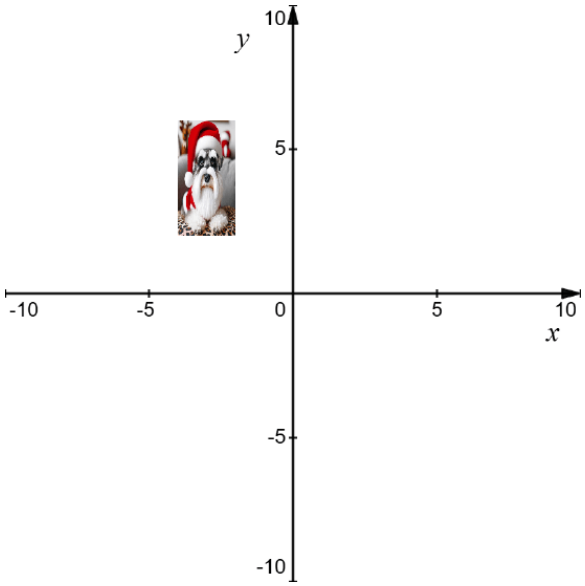


Figure $5.2.4$**:** $k(x) = \text{Sophie}\,(-2\,(x+3)) + 4$

In this section, we will perform these same transformations on trigonometric functions; however, because these functions are periodic, we can only capture a small section of their graphs. The process we introduce to apply these transformations and the names we define during this development will be used throughout the rest of Trigonometry and Calculus.

Our ultimate goal is to graph a trigonometric function of the form

$$f(x) = A\,\mathrm{trig}\,(Bx + C) + D, \tag{5.2.1}$$

where "trig" could be any of the trigonometric functions. The unknown values, A, B, C, and D, are called **parameters**.

Definition: Parameter

A **parameter** is a variable that influences the behavior or output of a mathematical object (like a function or equation) but is held constant within a specific context or scenario. It's a quantity that helps define or constrain a system, function, or dataset.

We will review what happens to the graph of the base trigonometric function when we adjust these parameters.

Amplitude and the Graph of $y = A\ \mathrm{trig}(x)$

We start our investigation of the effect of the parameters in Equation 5.2.1 by setting $B = 1$, $C = 0$, and $D = 0$. Therefore, we are interested in the effect of A on the graph of

$$f(x) = A\,\mathrm{trig}\,(x)\,.$$

If $A > 0$, then from Algebra, we know that the graph of $y = Ag(x)$ is a vertical stretch (if $A > 1$) or compression (if $0 < A < 1$) of the graph of g; however, does this still hold in Trigonometry? The following example might lead us to the answer.

Example 5.2.1

Make a table of values and sketch the graph of each function. Compare your graph to the function's base graph.

a. $f(x) = 3\sin(x)$
b. $g(x) = \frac{1}{5}\cos(x)$
c. $h(x) = \pi\tan(x)$

Solutions available at the end of this section.

Note: Abusive Language

In Example 5.2.1b, I state that " the graph of $g(x) = \frac{1}{5}\cos(x)$ is a vertical compression of the base graph by a factor of $1/5$." Some authors (and instructors) will say, "$g(x)$ is a compression of the graph of $y = \cos(x)$ by a factor of 5." I have always found this language a bit confusing. Therefore, I will adhere to the convention in this textbook of stating "a vertical compression by a factor of $\frac{1}{5}$." The fact that the word "compression" is involved gives the context that the y-values of the original function are being "squished" to make them smaller than before.

Example 5.2.1 verifies that our experience in Algebra still holds in Trigonometry; however, we artificially restricted the values of A to be positive. What happens if $A < 0$? Again, from Algebra, we know the graph of $y = -g(x)$ would be a reflection of the graph of the function g about the x-axis. Does this still hold in Trigonometry?

Checkpoint 5.2.1

Make a table of values and sketch the graph of each function. Compare your graph to the function's base graph.

a. $f(x) = -2\cos(x)$
b. $g(x) = -\frac{1}{2}\sin(x)$

Answers available at the end of this section.

The following theorem summarizes these results.

Theorem: Vertical Scaling

The graph of

$$f(x) = A\,\text{trig}(x)$$

vertically scales the graph of $y = \text{trig}(x)$ by a factor of $|A|$.

- If $|A| > 1$, it is a vertical stretch by a factor of $|A|$.
- If $0 < |A| < 1$, it is a vertical compression by a factor of $|A|$.

Additionally, if $A < 0$, the graph is reflected about the horizontal axis.

In either case, the y-coordinates of points on the graph of $y = \text{trig}(x)$ are multiplied by A to become the new points on the graph of $f(x) = A\ \text{trig}(x)$.

Note: Vertical Scaling Changes the Range

Except for the tangent function (whose range is all real numbers), the range of a trigonometric function will change under vertical scaling. For example, in Example $5.2.1b$ the range of $g(x) = \frac{1}{5}\cos(x)$ is $\left[-\frac{1}{5}, \frac{1}{5}\right]$.

When we introduced the definition of amplitude, we also mentioned that while the *base* sine and cosine functions have amplitudes of 1, the tangent does not have an amplitude because its range is unbounded. We now introduce the amplitude of *any* sinusoidal function.

Example $5.2.2$

Compute the amplitude of each function.

a. $f(x) = 3\sin(x)$
b. $g(x) = \frac{1}{5}\cos(x)$

Solutions available at the end of this section.

Example $5.2.2$ provides an observation that will allow us to state the amplitude of a sinusoidal function quickly.

Theorem: Amplitude of a Sinusoidal Function

The amplitude of $f(x) = A\cos(x)$ or $g(x) = A\sin(x)$ is $|A|$.

Checkpoint $5.2.2$

State the amplitude of each function.

a. $f(x) = -2\cos(x)$

b. $g(x) = -\frac{1}{2}\sin(x)$
c. $h(x) = 12\tan(x)$

Answers available at the end of this section.

If you are using the digital version of this textbook, feel free to use the following Interactive Element to help solidify the concept of amplitude and the effect of A on the graph of $f(x) = A\sin(x)$.

Interactive Element: Investigating the Effects of A on the Graph of $y = A\sin(x)$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Desmos Interactive online

Interaction: Adjust the slider for A to see the effect on the graph of $y = A\sin(x)$.

Period

The next parameter in Equation 5.2.1 is B. Very technically, you should have already encountered graphs of functions of the form $y = g(Bx)$ in Algebra, but I find that many people are confounded by what is truly happening and, very honestly, you don't see the true effect of that transformation until Trigonometry. Therefore, we are going to cover it in detail here. To determine this parameter's effects on the trigonometric function's graph, we take an example and build a table of values.

Suppose we are asked to graph $y = \cos(2x)$. By building a table of values using the quadrantal angles, we get the following:

[redacted]

Graphing the base cosine curve along with these points yields Figure 5.2.10.

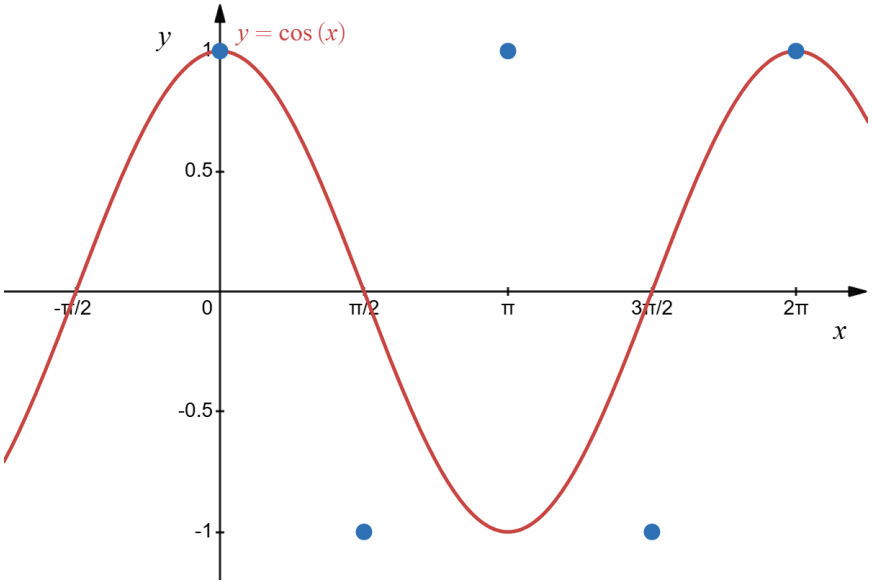


Figure 5.2.10

The issue should be readily apparent—we need more points to plot. The quadrantal angles provide too coarse of a picture. Let's

add some values to our table.

[redacted]

Plotting these additional points (and connecting them with a nice sinusoidal curve), we get the following figure.

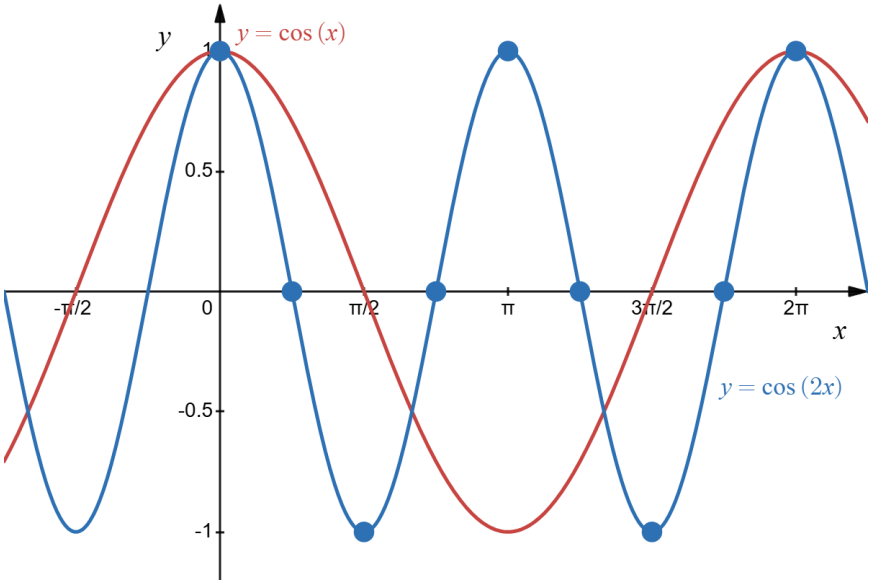


Figure 5.2.11

Before moving on, we need to mention a few things about Figure 5.2.11. First, recall that the principal cycle of the cosine function is $[0, 2\pi]$. This means that standard cosine function goes through a single cycle on that interval; however, $y = \cos(2x)$ goes through *two* complete cycles on the same interval. This is a beautiful way of thinking about the effect of B on the graph of $y = A\ \text{trig}\,(Bx)$. B tells us how many "cycles" of the graph get squeezed into the principal cycle of the base function. Another interpretation is that B tells us how many times faster the curve is "oscillating" than the base curve. In this case, $y = \cos(2x)$ oscillates twice as fast as $y = \cos(x)$. This being said, we can safely conjecture that B modifies the function's period (how long it takes to complete one cycle).

Note: "Oscillating" doesn't work for the tangent

I placed "oscillating" in quotes because this interpretation doesn't work for the tangent function. That is, $y = \tan(2x)$ doesn't oscillate twice as fast as the base tangent function because the tangent doesn't oscillate.

The second item worth mentioning is that you will usually only be asked to graph a single cycle of a trigonometric function, and this single cycle will be based on the principal cycle. Hand-drawn (and computer-generated) graphs look crowded when more than one cycle is sketched. In general, your instructor will know you just copied a computer-generated graph of a given trigonometric function if you include several cycles.

This brings us to our third and most important observation. For any trigonometric function, we only need five special x-values to obtain an accurate sketch of a single cycle (again, based on the principal cycle). For simplicity, we label these x_0, x_1, x_2, x_3, and x_4. These are

- x_0**:** the beginning x-value
- x_4**:** the ending x-value
- x_2**:** the value of x directly between the beginning and ending x-values
- x_1 **and** x_3**:** the two values of x between the beginning and middle, and middle and ending x-values

As they say, a picture is worth a thousand words, so let's visualize this using $y = \cos(2x)$.

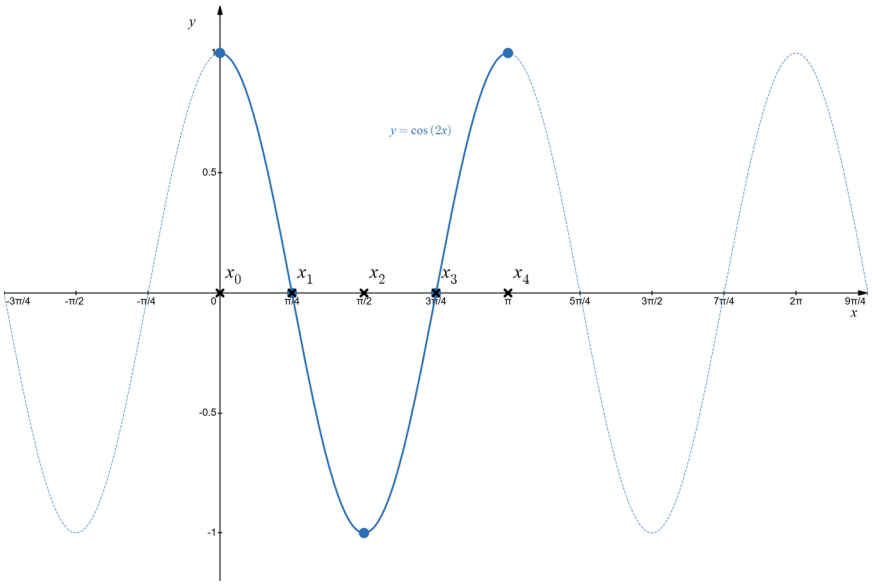


Figure 5.2.12

Each of the x-values in Figure 5.2.12 is called a **key number** of a single graph cycle.

Definition: Key Numbers

The **key numbers** of a cycle for a trigonometric function are the x-values of the of the initial, final, middle, and quarter-points in the principal cycle. These can be found by partitioning the cycle into four equally-spaced subintervals.

To showcase how the key numbers help graph trigonometric functions, suppose we are asked to graph the principal cycle of the base cosine function, $y = \cos(\theta)$. Again, the principal cycle for the cosine function is $0 \leq \theta \leq 2\pi$. Partitioning this interval into four equally-spaced subintervals requires taking the length of this interval (2π) and dividing by four to get

$$\frac{2\pi}{4} = \frac{\pi}{2}.$$

We will call the length of this subinterval the **step size** for this function.

Definition: Step Size

The distance between any two consecutive key numbers is called the **step size**.

To graph, we mark the first x-value in the *principal cycle*, which is $x = 0$. Hence, $x_0 = 0$. From there, we "mark by steps" until we reach the end of the single cycle (see the figure below).

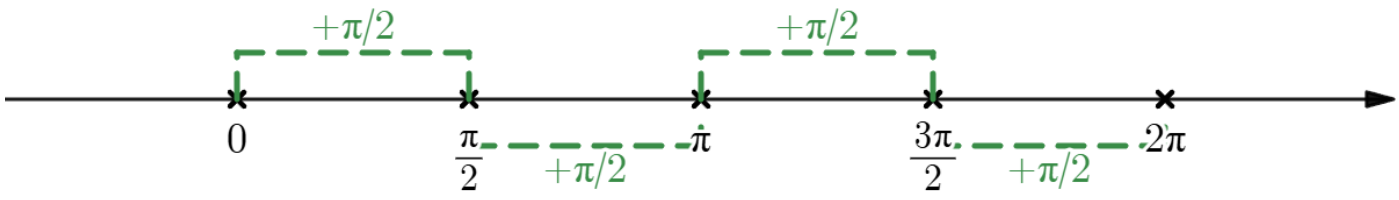


Figure 5.2.13

Since we know the basic shape of the cosine graph, we can complete the graph using these key numbers.

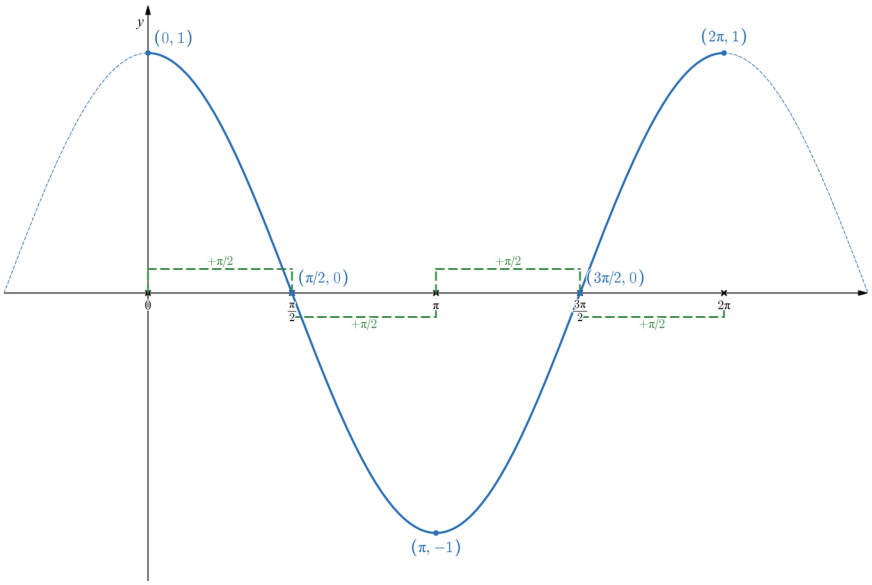


Figure 5.2.14

Before going into an example, let's amend a definition we created in the previous section. We know that the *natural period* of the cosine function (and sine function) is 2π; however, what is the period of the modified function $y = \cos(2x)$?

Recall that a single cycle of $y = \cos(\theta)$ occurs on $0 \leq \theta \leq 2\pi$. Replacing θ with $2x$, we have $0 \leq 2x \leq 2\pi$. Dividing all three sides of this inequality by 2, we get $0 \leq x \leq \pi$. That is, $y = \cos(2x)$ goes through a single cycle on the interval $[0, \pi]$. Hence, the period of $y = \cos(2x)$ is π. Generalizing this observation leads us to the following theorem.

Theorem: Period of a Trigonometric Function

Let p be the natural period of a trigonometric function. Then the period of $y = A\ \text{trig}(Bx)$ is $\frac{p}{B}$.

Thus, the period of $y = A\cos(Bx)$ is $2\pi/B$, and the period of $y = A\tan(Bx)$ is π/B (remember, the natural period of the tangent is π).

While we are at it, let's create a formula to help us compute the step size.

Theorem: Step Size Formula

The step size for a trigonometric function is $\frac{\text{Period}}{4}$.

The step size simplifies the process of finding the key numbers. By starting with the first key number, x_0, we add the step size to arrive at the second key number, x_1. To this, we add the step size to get the third key number, x_2. And so on.

We are overdue for some examples.

Example 5.2.3

Sketch a single cycle of each function, stating its amplitude (if applicable), period, and step size.

a. $f(x) = \cos(2x)$
b. $g(x) = -3\sin\left(\frac{x}{3}\right)$
c. $y = \frac{1}{5}\tan(-4x)$

Solutions available at the end of this section.

Did you notice I didn't bother simplifying fractions when I labeled the key numbers in Examples $5.2.3b$ and $5.2.3c$? This was purposeful and is a technique we will use moving forward.

Note: Changes in the Period Affect the Domain of the Tangent

While the domain of the sine and cosine functions remain all real numbers no matter the period, the tangent's domain changes when we change the period. For example, in Example $5.2.3c$ the domain of the tangent does not include $\frac{\pi}{8}$ (whereas, in the base tangent function, $\frac{\pi}{8}$ is in the domain). This is because a change in the period of a function causes **horizontal scaling** (stretching or compressing along the horizontal axis). When this scaling occurs, any vertical asymptotes the function might have also move.

Checkpoint $5.2.3$

Sketch a single cycle of $y = -\frac{1}{2}\cos(7x)$, stating its amplitude, period, and step size.

Answer available at the end of this section.

As before, the following Interactive Element (accessible through the digital version of the textbook) can help to solidify the concept of period and the effect of B on the graph of $f(x) = A\sin(Bx)$.

Interactive Element: Investigating the Effects of B on the Graph of $y = A\sin(Bx)$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Desmos Interactive online

Interaction: Adjust the slider for A and B to see the effect on the graph of $y = A\sin(Bx)$.

Summary of Graphing Trigonometric Functions Using Non-Rigid Transformations

Summary of Graphing $f(x) = \overline{A}\ \text{trig}\left(\overline{B}x\right)$

To graph the function

$$f(x) = \overline{A}\,\text{trig}\left(\overline{B}x\right):$$

Precondition Stage

We begin by cleaning up the trigonometric function and ensuring that the coefficient of x is positive. This is done using the Symmetry Identities. Once the Symmetry Identities have been used, your new function will be of the form

$$f(x) = A\text{trig}(Bx).$$

Information Stage

We now identify the following information (in the order listed):

- $\text{Amplitude} = |A|$
- $\text{Period} = \dfrac{\text{Natural Period}}{B}$
- $\text{Step Size} = \dfrac{\text{Period}}{4}$

where the natural period is 2π if given the sine or cosine functions, and π if given the tangent function.

Graphing Stage

We then use each of these *in reverse order* to obtain a sketch of the trigonometric function.

- Draw a horizontal line, mark the initial key number as 0, and mark the remaining key numbers by successively adding the **step size** to the previous key number.
- Double-check that the distance from the initial key number to the final key number matches the **period**, then sketch one cycle of the base trigonometric function, making sure to account for the sign of A and drawing any vertical asymptotes (if the given function is the tangent).
- Draw the vertical axis at $x = 0$ and use the **amplitude** to either
  - label the maximum and minimum values of the trigonometric function on the vertical axis (if the given function is sinusoidal), or
  - label the remaining key points on the graph (if the given function is the tangent).

To be clear, let's do a final example. This time, however, let's practice extending the graph as well.

Example 5.2.4

Sketch the graph of each function over the given interval.

a. $g(x) = -\dfrac{1}{2}\cos\left(-\dfrac{2}{5}x\right)$ for $-10\pi \le x \le 10\pi$
b. $h(x) = 3\tan(6x)$ for $-\frac{\pi}{12} \le x \le \frac{\pi}{2}$

Solutions available at the end of this section.

Finding Equations Given Graphs

We wrap up this section by inverting our process. Instead of graphing a function given its equation, we find the equation for a function given its graph. This process is similar to what we introduced in the summary above.

Example 5.2.5

Find the equation of the graphed function.

a.

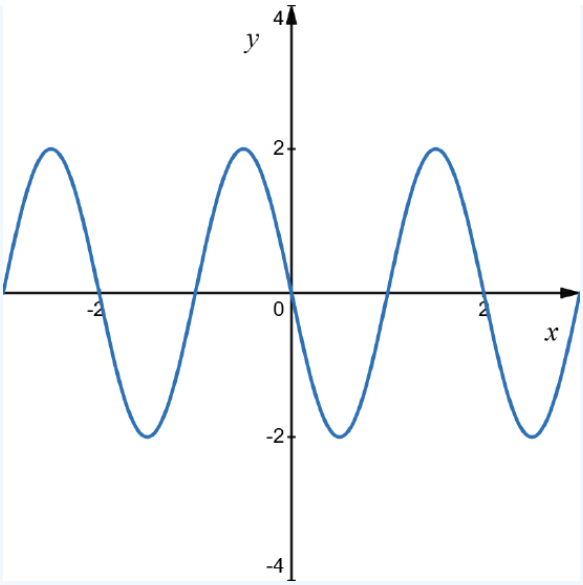


Figure 5.2.31

b.

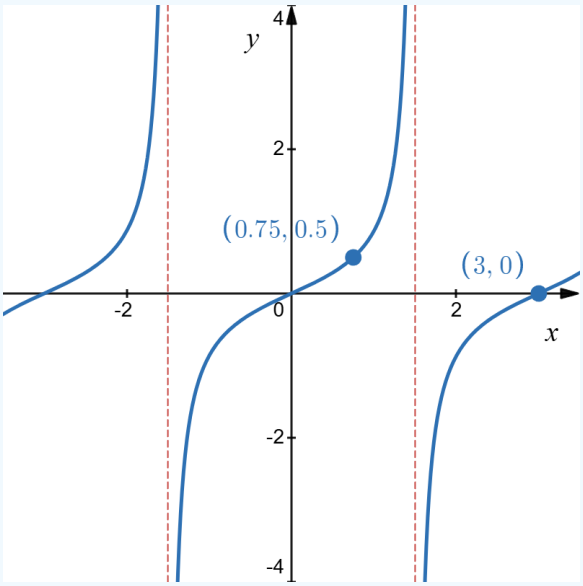


Figure 5.2.32

Solutions available at the end of this section.

Checkpoint 5.2.5

Find the equation of the function graphed below.

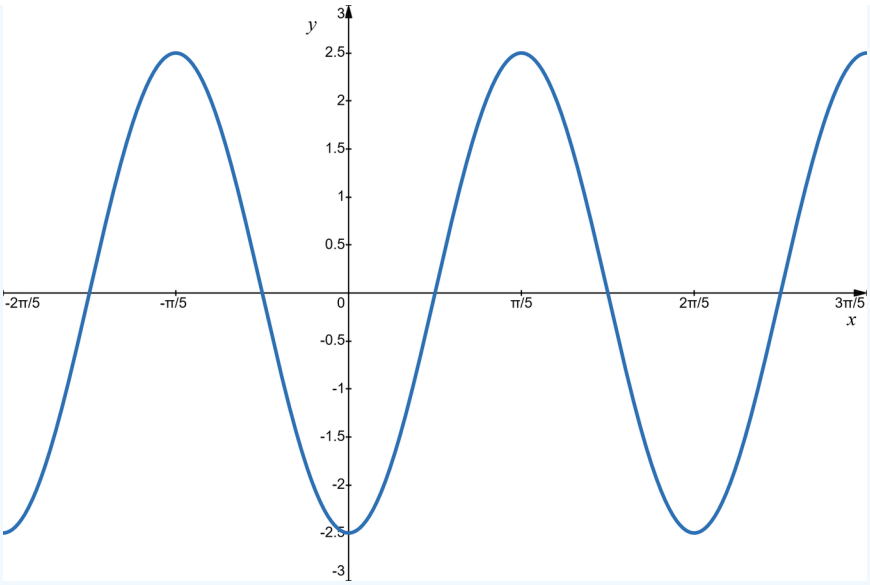


Figure 5.2.33

Answer available at the end of this section.

Success in College: Dealing With Failure (Part 3)

Sometimes unexpected events happen during a term. You get sick and are in bed for a week. Someone that you need to help care for gets sick. You are in a car accident and lose your transportation. A daycare provider quits. A relationship falls apart. Your boss at work needs you to increase your hours. Because this happens, you don't have time to keep up with your courses. You miss class. You fail one or more tests.

1. A student in this situation decides to withdraw from the math class. She reasons that it is better to concentrate on passing her other classes. She will re-enroll in math the next term. Are there any adverse consequences of making this choice? Explain.
2. A student in this situation decides to keep coming to class and trying to do the homework. He plans to take the course over the next term but wants to learn as much as he can now. This should make the course easier next term. Are there any adverse consequences of making this choice? Explain.

Success in Mathematics: Math Anxiety (Part 2)

I'm going to keep this short (for a change). Please take this math anxiety test to help you better understand your math anxiety. While at that website, check out the hints on overcoming math anxiety.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. A suggestion for reducing math anxiety is to come to class prepared. If you are prepared, then you are ready to be successful. When you are successful, you start learning that you do not have to be anxious. What do you think it means to be prepared for your math class?
2. A suggestion for reducing math anxiety is to practice math every day. How many days a week do you do math? How long, on average, do you spend doing math on those days?

Answers

Section Preview

The choice to cover non-rigid transformations (amplitude and period) before the rigid transformations is purposeful. Traditionally, when performing transformations,

it is best to save the rigid transformations for the end.

Section Preview

- **Graphing Functions**
  - Transformations — Non-Rigid Transformations: Students need to already know what a vertical stretch/compression of a general function looks like before the section can apply those ideas to trig functions.

Section Preview

- Determine the amplitude of a sinusoidal function.
- Determine the period of a sine, cosine, or tangent function.
- Graph a sine, cosine, or tangent function having a non-standard amplitude or period.
- Determine the equation of the graph of a trigonometric function.

Example 5.2.1

a. Now that we know the basic shape of the sine function, we only need to make a table of the quadrantal angles.

[redacted]

Plotting these points and graphing the base sine function, we get the following:

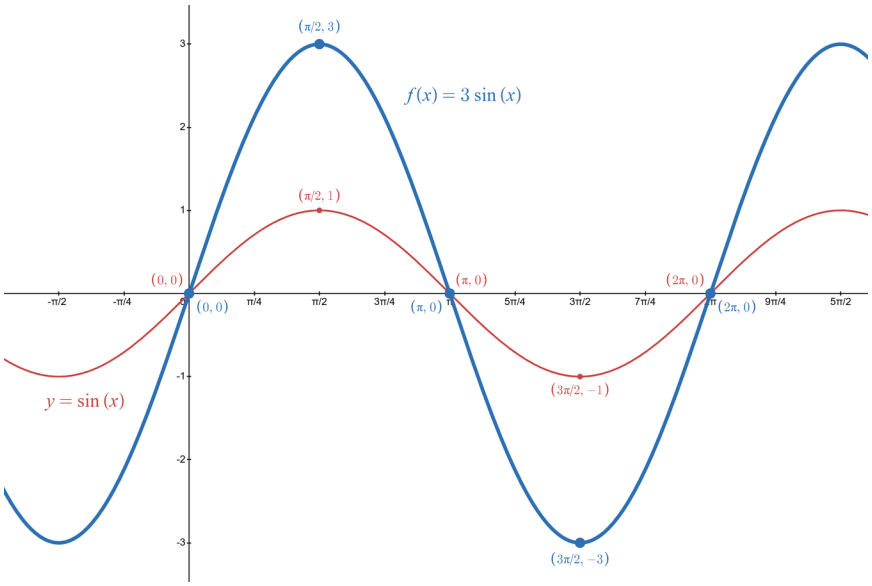


Figure 5.2.5

We can immediately see that the graph of $f(x) = 3\sin(x)$ is a vertical stretch of the base graph by a factor of 3. That is, all y-values of $f(x) = 3\sin(x)$ are three times that of the y-values for $y = \sin(x)$. Judging from this single example, it *looks like* the role of A in $f(x) = A\ \text{trig}(x)$ is to multiply the y-values of the base trigonometric function by A. However, looks can be deceiving, and this is *not* a proof. Let's do the second example and see if our insight holds.

b. Again, we use our knowledge of the basic shape of the cosine graph, along with a table of quadrantal angles.

[redacted]

Plotting these points and graphing the base cosine function, we get the following:

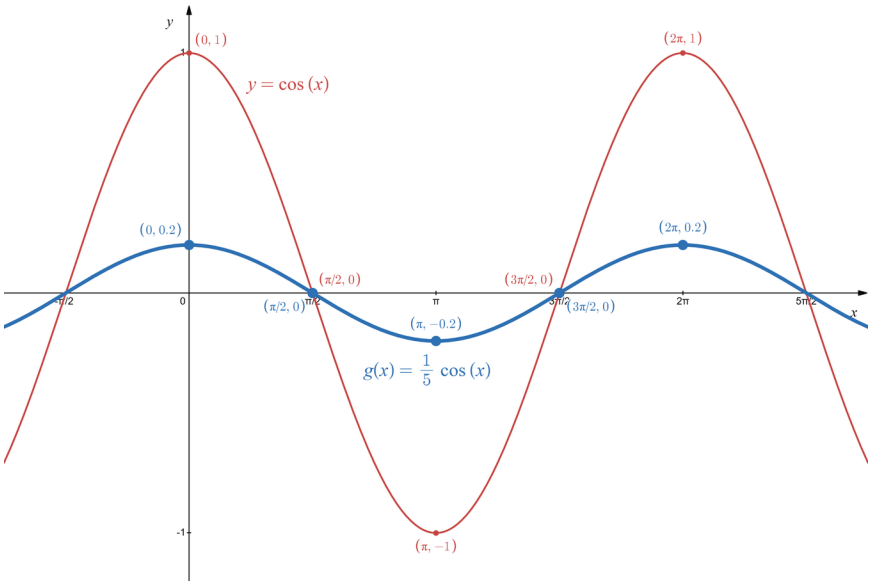


Figure 5.2.6

As before, we see that the graph of $g(x) = \frac{1}{5}\cos(x)$ is a vertical compression of the base graph by a factor of $1/5$. That is, all y-values of $g(x) = \frac{1}{5}\cos(x)$ are one-fifth of the y-values for $y = \cos(x)$.

c. The tangent function always requires special consideration because it is undefined at certain quadrantal angles. Therefore, our approach from parts (a) and (b) needs to be modified. Again, this is where having a strong grasp of the base graph for the tangent function comes into play. Since the natural period of the tangent is π, we will plot coordinates for $x = 0, \pi/4, 3\pi/4$, and π (completely avoiding the domain issue by skipping over $x = \pi/2$).

[redacted]

Plotting these points and graphing the base tangent function, we get the following:

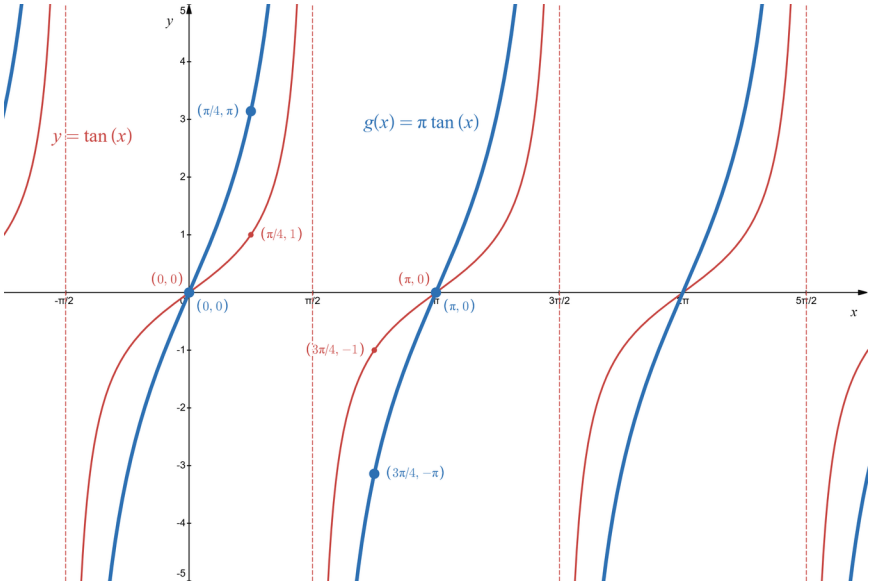


Figure 5.2.7

Again, $h(x) = \pi\tan(x)$ is a vertical stretch of the base graph by a factor of π.

Checkpoint 5.2.1

a.

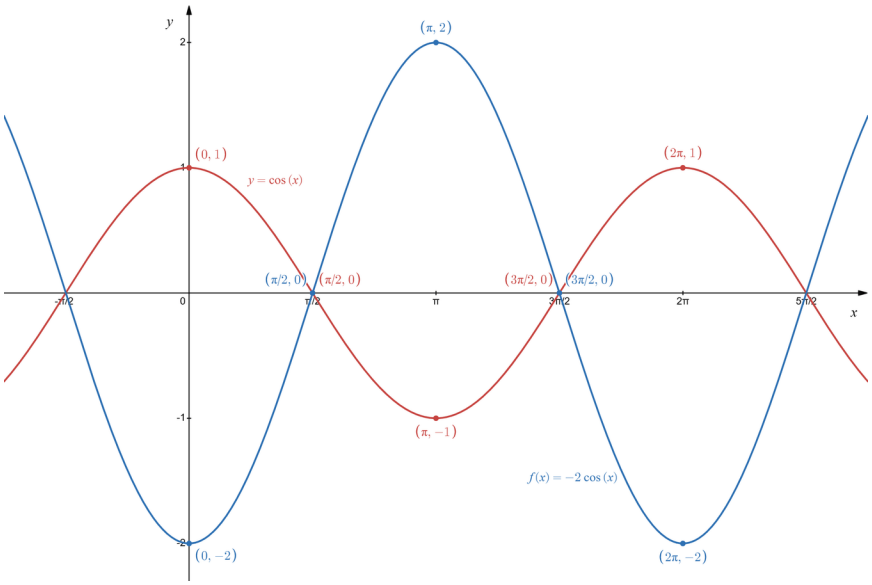


Figure 5.2.8

The coefficient of -2 multiplies all y-values of the base graph by -2. This effectively causes a vertical stretch (by a factor of 2) and a reflection about the x -axis.

b.

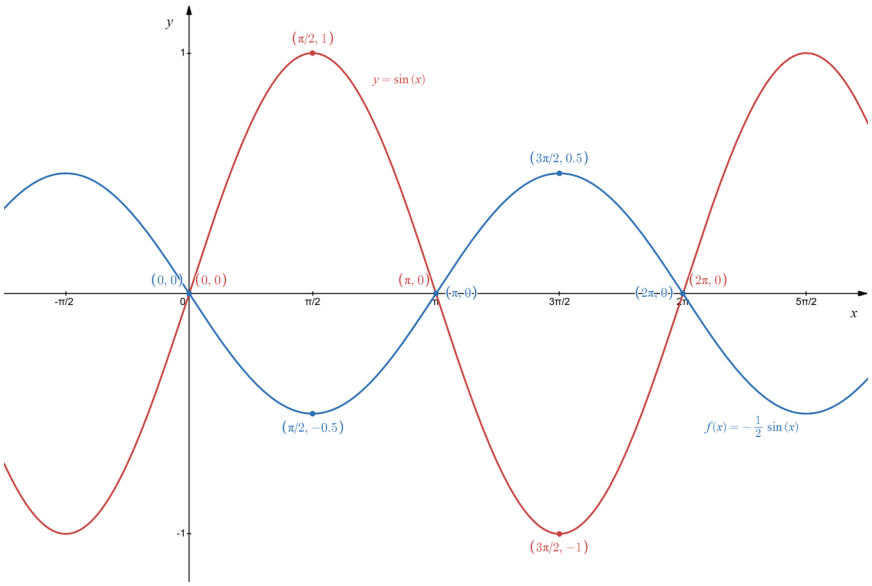


Figure 5.2.9

The coefficient of $-1/2$ multiplies all y-values of the base graph by $-1/2$, effectively compressing the graph vertically by a factor of $1/2$ and reflecting it about the x-axis.

Example 5.2.2

a. From Example $5.2.1a$, we see the greatest and least values of the graph are 3 and -3, respectively. Thus, the amplitude of $f(x) = 3\sin(x)$ is

$$\text{Amplitude} = \frac{1}{2}|3-(-3)| = \frac{1}{2}|6| = 3.$$

b. From Example $5.2.1b$, we see the greatest and least values of the graph are $1/5$ and $-1/5$, respectively. Thus, the amplitude of $g(x) = \frac{1}{5}\cos(x)$ is

$$\text{Amplitude} = \frac{1}{2}\left|\frac{1}{5}-\left(-\frac{1}{5}\right)\right| = \frac{1}{2}\left|\frac{2}{5}\right| = \frac{1}{5}.$$

Checkpoint 5.2.2

a. 2
b. $\frac{1}{2}$
c. Trick question... the tangent function does not have an amplitude.

Example 5.2.3

a. The process we illustrate here (and that we build on moving forward) is a bit algorithmic but very helpful in graphing trigonometric functions. It's based on the "template" trigonometric function

$$y = A\,\text{trig}\,(Bx + C) + D.$$

In this section, $C = D = 0$, so we are only dealing with functions of the form

$$y = A\,\mathrm{trig}\,(Bx)\,.$$

Specifically, for this problem, we have $f(x) = 1\cos(2x)$. We read this from left to right.

A	$=$	1	$\Longrightarrow$	Amplitude	$=$	1
B	$=$	2	$\Longrightarrow$	Period	$=$	$\dfrac{\text{natural period}}{B} = \dfrac{2\pi}{2} = \pi$
				Step Size	$=$	$\dfrac{\text{Period}}{4} = \dfrac{\pi}{4}$

We can now sketch an accurate graph of a single cycle. The initial x-value in the principal cycle of all trigonometric functions is $x = 0$. To find the next key number, we add the step size. See the figure below.

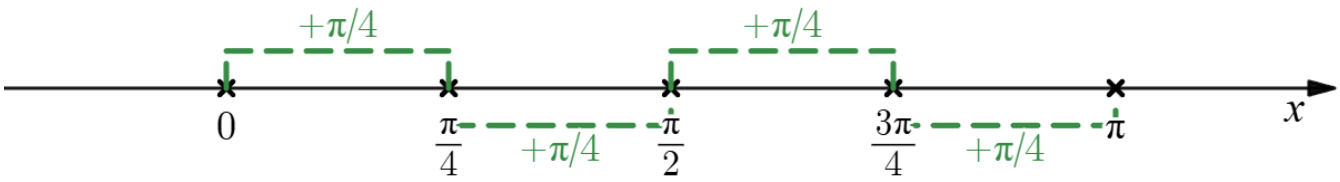


Figure 5.2.15

The length of this interval is π, which matches the period, so we are on track for success. Since the initial key number is at $x = 0$, we now know where the y-axis is, so let's draw that in.

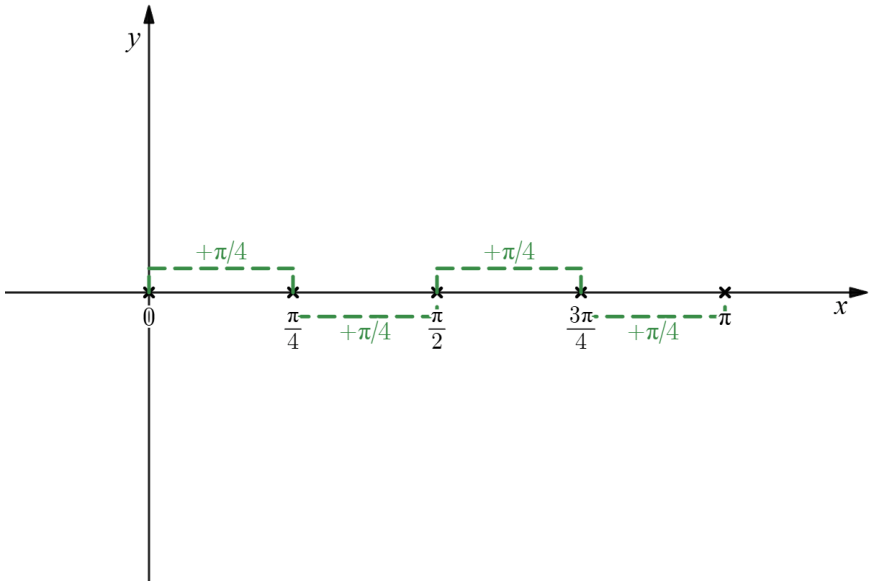


Figure 5.2.16

The function we are graphing is the *positive* cosine, so we can fill in the basic shape of a single cycle.

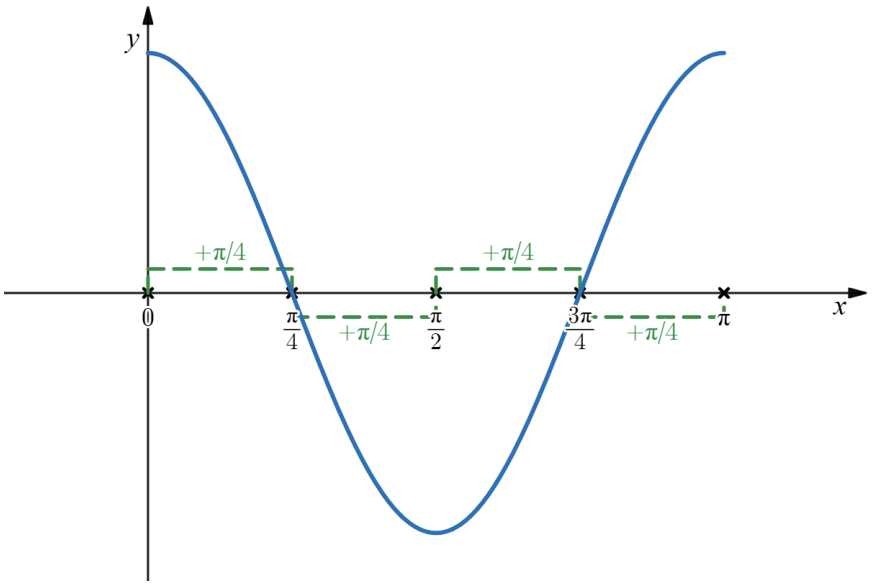


Figure 5.2.17

Finally, we use the amplitude to label the y-axis. Since the amplitude is 1, the maximum value is 1 and the minimum value of the function is -1.

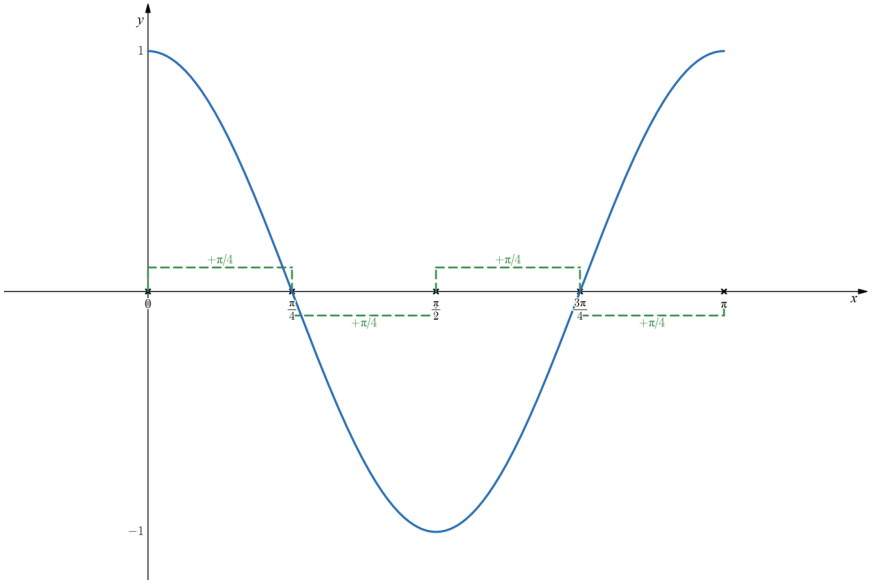


Figure 5.2.18

I know I said that, unless asked, you should refrain from graphing more than a single cycle, however, it is nice to let your reader know that this pattern is cyclic. We showcase this by extending the graph for a short distance in both directions.

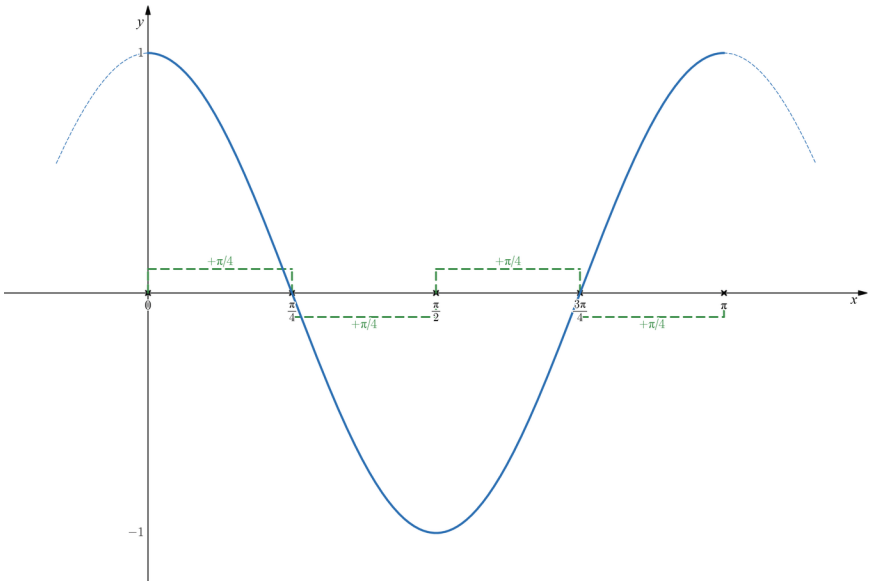


Figure 5.2.19

b. We have $g(x) = -3\sin\left(\frac{x}{3}\right) = -3\sin\left(\frac{1}{3}x\right)$. Reading this from left-to-right, we get the following.

A	$=$	-3	$\Longrightarrow$	Amplitude	$=$	3
B	$=$	$\frac{1}{3}$	$\Longrightarrow$	Period	$=$	$\frac{\text{natural period}}{B} = \frac{2\pi}{1/3} = 6\pi$
				Step Size	$=$	$\frac{\text{Period}}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$

Remembering that the initial x-value for the principal cycle is $x = 0$, we find all of the key numbers and label the x-axis.

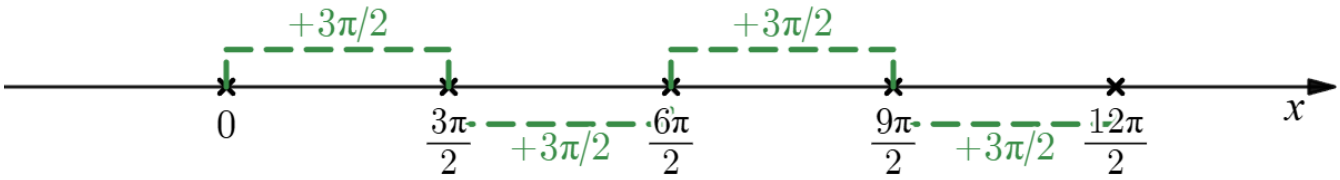


Figure 5.2.20

Again, we see the length of this interval is $\frac{12\pi}{2} = 6\pi$, which is the period. Therefore, we must be doing something right! Placing the y-axis appropriately, we get the following.

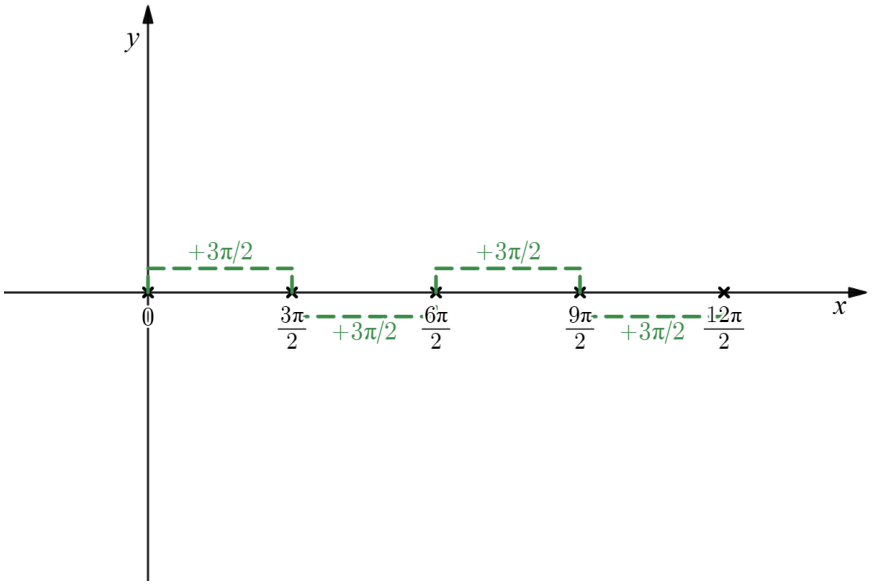


Figure 5.2.21

Here is where paying attention to the original function is essential (technically, paying attention is always important)—the amplitude of the function is 3, but

the sine function has been reflected about the x-axis because $A < 0$. Therefore, we label the y-axis with the maximum and minimum values of 3 and -3 but draw our sine function reflected about the x-axis.

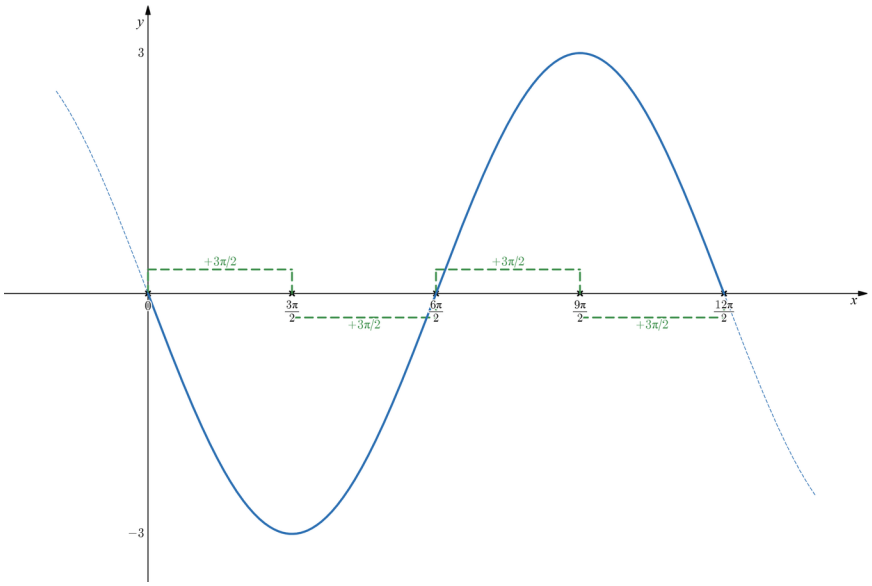


Figure 5.2.22

c. The tangent function might seem like the oddball of the trigonometric functions. Still, the process of sketching a single cycle remains the same, except that the tangent does not have an amplitude. One difference you should notice with this specific example is that the coefficient on x is *negative*. This is where remembering the Symmetry Identities pays off. Since tangent is an odd function,

$$y = \frac{1}{5}\tan(-4x) = -\frac{1}{5}\tan(4x).$$

Now that we have taken care of the negative argument in the function, we get to work. It is essential to remember that the natural period of the tangent function is π.

$$\begin{array}{rcl} B = 4 \implies \text{Period} & = & \dfrac{\text{natural period}}{B} = \dfrac{\pi}{4} \\ \text{Step Size} & = & \dfrac{\text{Period}}{4} = \dfrac{\pi/4}{4} = \dfrac{\pi}{16} \end{array}$$

Again, the initial x-value in the principal cycle is $x = 0$. To find the next key number, we add the step size. See the figure below.

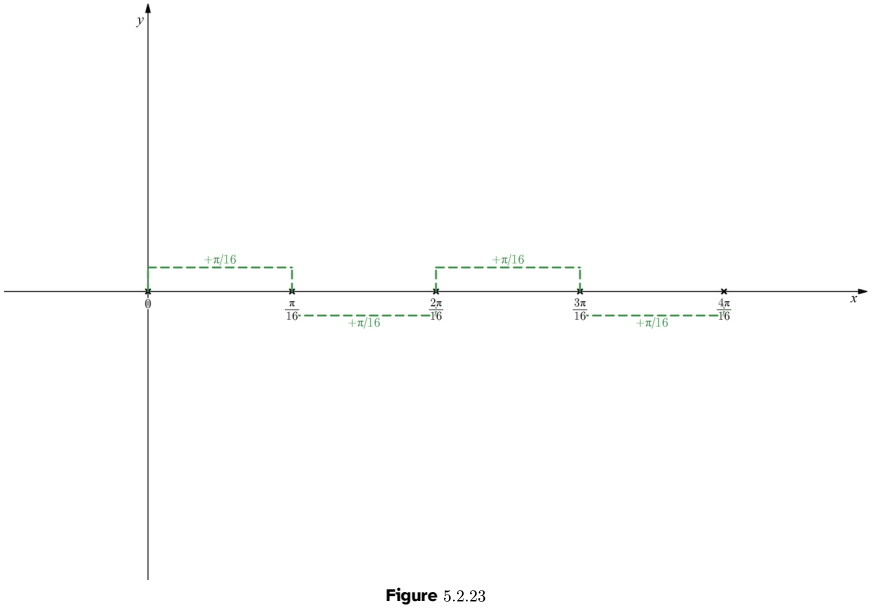


Figure 5.2.23

We know what the base graph of the tangent looks like, so we know the beginning and ending key numbers (0 and $4\pi/16$) are going to be the zeros of the function and the middle key number ($2\pi/16$) is the vertical asymptote; however, what do we do with $\pi/16$ and $3\pi/16$? Up to this point, we have not bothered with the coefficient of the function, $-1/5$. Here is where it comes into play. If we were to evaluate the given function at $\pi/16$, we would get

$$-\frac{1}{5}\tan\left(4\left(\frac{\pi}{16}\right)\right) = -\frac{1}{5}\tan\left(\frac{\pi}{4}\right) = -\frac{1}{5}(1) = -\frac{1}{5}.$$

That is, the excess key numbers, $\pi/16$ and $3\pi/16$, give us two convenient points to plot on the graph, and the value at these points is $\pm A$. Sketching a tangent function that has been reflected about the x-axis and plotting those two points, we get the following graph.

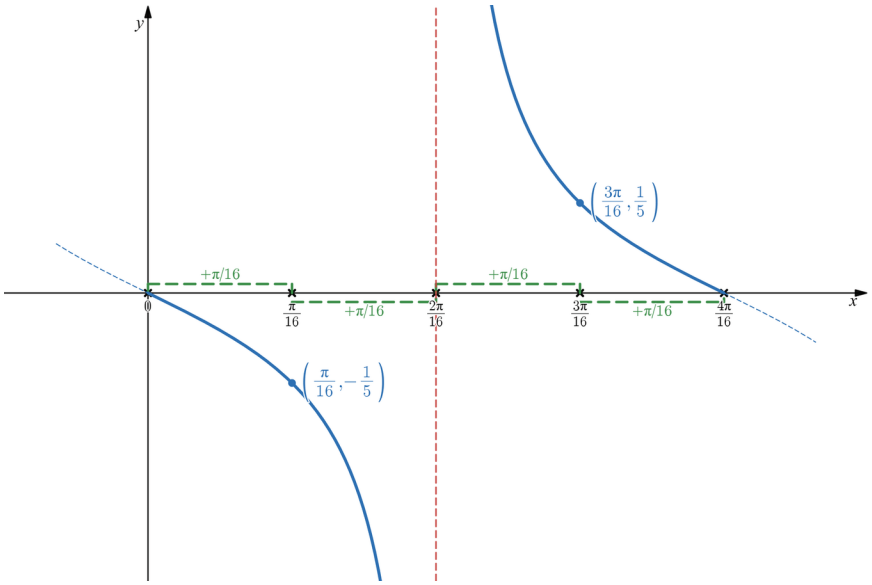
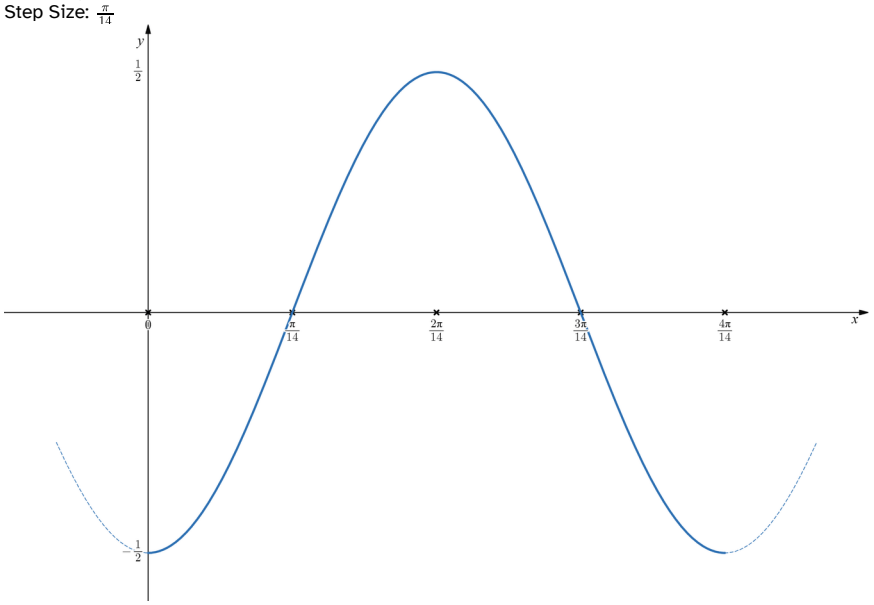


Figure 5.2.24

Checkpoint 5.2.3

Amplitude: $\frac{1}{2}$
Period: $\frac{2\pi}{7}$
Step Size: $\frac{\pi}{14}$



Example 5.2.4

a. **Precondition Stage:** Since the coefficient of x is negative, we use the Symmetry Identities to rewrite the given function. Cosine is an even function. Therefore, $\cos(-\theta) = \cos(\theta)$. Hence, $g(x)$ can be rewritten as

$$g(x) = -\frac{1}{2}\cos\left(\frac{2}{5}x\right).$$

Information Stage: Since the given function is sinusoidal, it has an amplitude, and its natural period is 2π.

$$\begin{array}{rcl} A = -\frac{1}{2} & \Longrightarrow & \text{Amplitude} = |A| = \frac{1}{2} \\ B = \frac{2}{5} & \Longrightarrow & \text{Period} = \frac{\text{Natural Period}}{B} = \frac{2\pi}{2/5} = 10\pi \\ & \Longrightarrow & \text{Step Size} = \frac{\text{Period}}{4} = \frac{10\pi}{4} = \frac{5\pi}{2} \end{array}$$

Graphing Stage: Drawing a horizontal line (the x-axis), marking the initial key number ($x = 0$), and marking the remaining key numbers using the step size, we get the following.

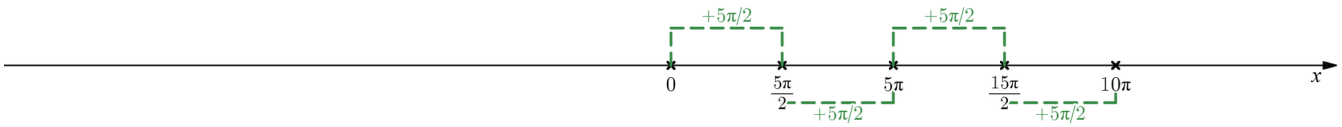


Figure 5.2.25

Notice that I left quite a bit of open space on the left half of the number line. This is because we will eventually extend the single cycle of our cosine function so that our graph covers the interval $[-10\pi, 10\pi]$. We take a moment to make sure the distance from the initial key number ($x = 0$) to the final key number ($x = 10\pi$) is the same as the period. In this case, the distance is

$$|10\pi - 0| = 10\pi = \text{Period},$$

so we are on the right track. We now sketch a single cycle of the *reflected* cosine (because $A < 0$).

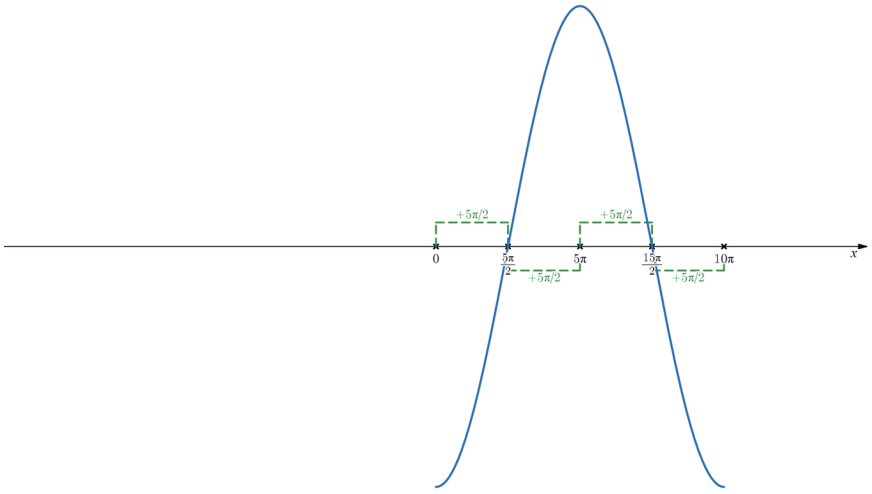


Figure 5.2.26

Finally, we draw in the y-axis at $x = 0$, label the maximum and minimum values using the amplitude, and extend the graph to cover the interval $[-10\pi, 10\pi]$.

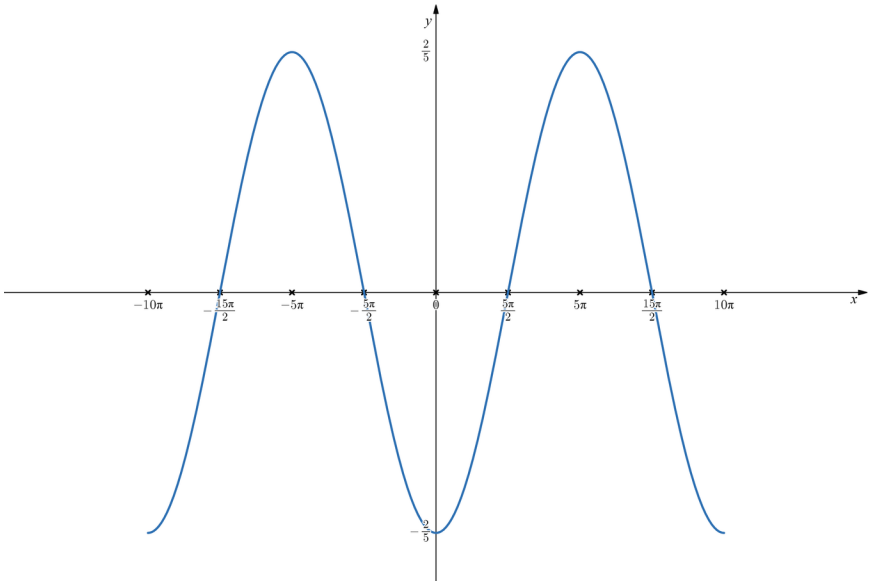


Figure 5.2.27

b. **Precondition Stage:** Since the coefficient of x is already positive, we do not need to clean up the trigonometric function.
Information Stage: Since h is a tangent function, it does not have an amplitude, and its natural period is π.

$$\begin{array}{rcl} B = 6 \implies \text{Period} = \dfrac{\text{Natural Period}}{B} = \dfrac{\pi}{6} \\ \implies \text{Step Size} = \dfrac{\text{Period}}{4} = \dfrac{\pi/6}{4} = \dfrac{\pi}{24} \end{array}$$

Graphing Stage: Starting at $x = 0$ and marking the key points, we get the following number line (which has been extended because of the interval over which we are asked to graph).



Figure 5.2.28

Note that the distance between the beginning key number and the ending key number is

$$\left|\frac{4\pi}{24} - 0\right| = \frac{4\pi}{24} = \frac{\pi}{6} = \text{Period}.$$

Thus, we are good to go! We now graph a single cycle of the tangent function. Remember that there is a vertical asymptote in the middle of the cycle (at $x = 2\pi/24$).

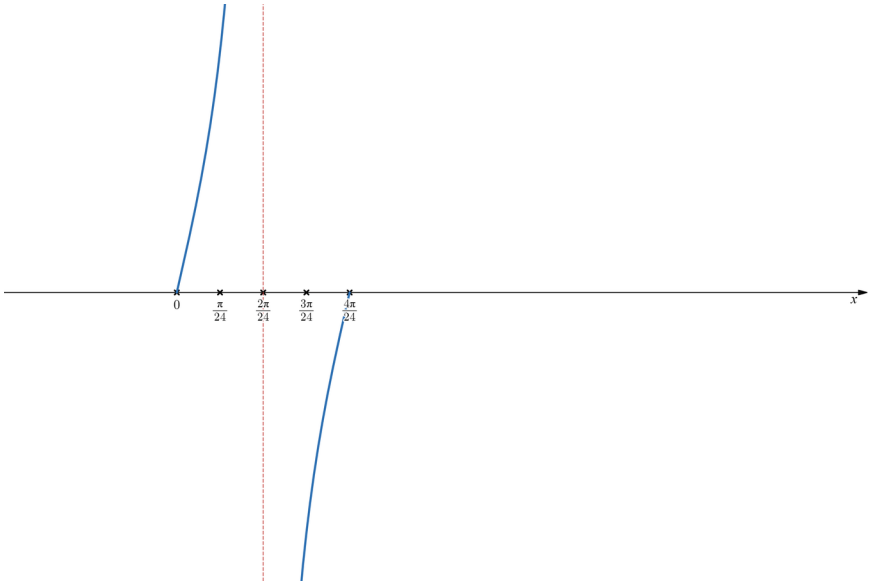


Figure 5.2.29

From here, we draw the y-axis at $x = 0$, label the points on the curve at $x = \pi/24$ and $x = 3\pi/24$, and extend the graph to cover the requested interval.

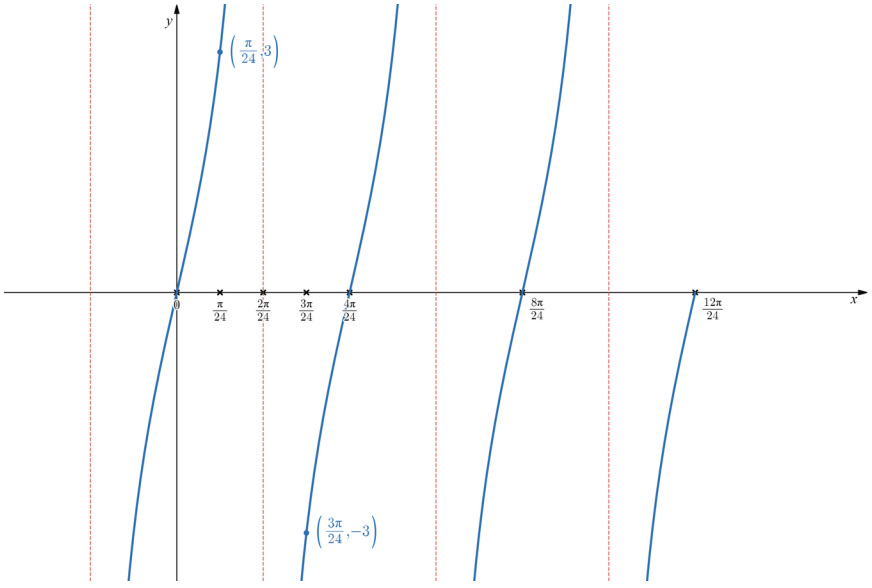


Figure 5.2.30

Example 5.2.5

a. Our first task is to identify the base function. The shape of this function is sinusoidal. Moreover, because it goes through the origin, it is a sine function (although, technically, it can be written as a cosine function as well). However, it has been reflected about the x-axis, so we know that $A < 0$. We can also see that the amplitude is 2. Thus, $A = -2$ so that

$$f(x) = -2\sin(Bx).$$

All we need to do is identify B. We know that B is *associated* with the period of the function, but it is not *equal* to the period. The relationship between B and the period is

$$\text{Period} = \frac{\text{Natural Period}}{B}.$$

However, the natural period of the sine function is 2π, so

$$\text{Period} = \frac{2\pi}{B} \implies B = \frac{2\pi}{\text{Period}}.$$

The graph shows that the function completes one full cycle every 2 units. Hence, the period is 2. Substituting that into our formula, we get

$$B = \frac{2\pi}{2} = \pi.$$

Thus, our function is

$$f(x) = -2\sin(\pi x).$$

b. The base function here is the tangent. So we know we are dealing with a function of the form

$$g(x) = A\tan(Bx).$$

The value of A will be a bit trickier to get here because the tangent does not have an amplitude; however, all is not lost. We instead start by finding B. As in part (a),

$$\text{Period} = \frac{\text{Natural Period}}{B}$$

and we know the natural period of the tangent function is π. Therefore,

$$\text{Period} = \frac{\pi}{B} \implies B = \frac{\pi}{\text{Period}}.$$

We can see that a single cycle gets completed in 3 units; thus, the period of this function is 3. Hence,

$$B = \frac{\pi}{3}.$$

Now we have

$$g(x) = A\tan\left(\frac{\pi}{3}x\right).$$

To find the value of A, we recall that any point on the graph of an equation must also satisfy the equation. In particular, because the point $\left(\frac{3}{4}, \frac{1}{2}\right)$ lies on this tangent curve, when $x = \frac{3}{4}$, it must be the case that $y = \frac{1}{2}$. That is,

$$\frac{1}{2} = A\tan\left(\frac{\pi}{3}\cdot\frac{3}{4}\right) \implies \frac{1}{2} = A\tan\left(\frac{\pi}{4}\right) \implies \frac{1}{2} = A\cdot 1 \implies A = \frac{1}{2}.$$

Thus, the equation of this tangent function is

$$g(x) = \frac{1}{2}\tan\left(\frac{\pi}{3}x\right).$$

Checkpoint 5.2.5

$f(x) = -\frac{5}{2}\cos(5x)$

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5.2.1: Resources and Key Concepts

Resources

Prerequisite Topics

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Graphing Functions**
  - Transformations: This section is entirely about applying non-rigid transformations (stretches, compressions, and reflections) to the base graphs of trigonometric functions. A prior understanding of how parameters alter a base graph is essential.

Videos

- Transformations
  - Trigonometric Graphs - Review of Graphs of Sine and Cosine Functions
  - Trigonometric Graphs - Review of Identifying the Equation for a Sinusoidal Function from a Graph
  - Graphs of the Other Trigonometric Functions - Graphing Tangent and Cotangent Functions
  - Graphs of the Other Trigonometric Functions - Identifying Equations of Trig Functions from Graphs

Key Concepts

Definitions

- **Transformation (of the graph of a function):** An alteration of a graph's orientation, scale, and/or position through various mathematical operations.
- **Rigid Transformation:** A transformation that changes a graph's position (e.g., shifts) but keeps its original orientation and scale.
- **Non-rigid Transformation:** A transformation that keeps the graph's original center position but changes its scale (stretching or compressing) or orientation (reflecting).
- **Parameter:** A variable that influences the output or behavior of a function but is held constant within a specific context. In this section, A and B in $y = A\text{trig}(Bx)$ are parameters.
- **Key Numbers:** The x-values of the initial, final, middle, and quarter-points of a single cycle of a trigonometric function's graph.
- **Step Size:** The horizontal distance between any two consecutive key numbers.

Theorems

- **Vertical Scaling:** The graph of $f(x) = A\text{trig}(x)$ vertically scales the graph of $y = \text{trig}(x)$ by a factor of $|A|$. If $|A| > 1$, it is a vertical stretch. If $0 < |A| < 1$, it is a vertical compression. If $A < 0$, the graph is also reflected about the horizontal axis.
- **Amplitude of a Sinusoidal Function:** The amplitude of $f(x) = A\cos(x)$ or $g(x) = A\sin(x)$ is $|A|$.
- **Period of a Trigonometric Function:** If p is the natural period of a trigonometric function, then the period of $y = A\text{trig}(Bx)$ is $\frac{p}{|B|}$. For sine and cosine, the period is $\frac{2\pi}{|B|}$. For tangent, the period is $\frac{\pi}{|B|}$.
- **Step Size Formula:** The step size for graphing a trigonometric function is given by the formula $\frac{\text{Period}}{4}$.

Common Mistakes

- **Confusing Period and the Parameter B:** The period of a function like $y = \sin(Bx)$ is not B. The period is calculated by

dividing the natural period of the base function by $|B|$.

- **Using the Wrong Natural Period:** The natural period of sine and cosine is 2π, while the natural period of tangent is π. Using the wrong natural period when calculating the new period is a common error.
- **Ignoring a Negative Coefficient on the Angle:** When graphing a function like $y = \tan(-4x)$, you must first use the Symmetry Identities to handle the negative. Since tangent is an odd function, $y = \tan(-4x) = -\tan(4x)$. Graphing without this first step will lead to an incorrect orientation.
- **Forgetting Tangent Has No Amplitude:** The concept of amplitude applies to sinusoidal functions (sine and cosine) because they have maximum and minimum values. The tangent function is unbounded, so it does not have an amplitude.

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5.2.2: Homework

Reading Questions

1. What is the difference between a rigid and a non-rigid transformation?
2. In the function $y = A\sin(Bx)$, what effect does the parameter A have on the graph? What happens if A is negative?
3. In the function $y = A\cos(Bx)$, what effect does the parameter B have on the graph?
4. What is the amplitude of the function $y = -5\cos(x)$?
5. What is the period of the function $y = \sin(3x)$?
6. What is the period of the function $y = \tan(2x)$?
7. What are the five "key numbers" used to sketch one cycle of a trigonometric graph?
8. How do you calculate the "step size" for a graph?
9. When graphing $y = \cos(-2x)$, what property of the cosine function allows you to rewrite it first?
10. Why does the tangent function not have an amplitude?

Homework

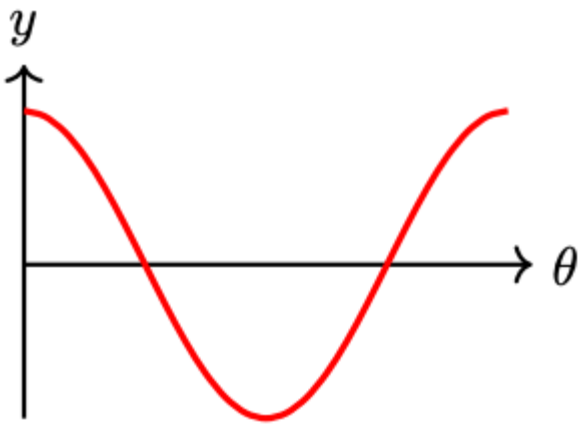
Vocabulary Check

1. To ___ the graph of a function, we mean to stretch or compress the graph horizontally or vertically.
2. A ___ transformation keeps the function's original scale and orientation but changes its position.
3. A ___ transformation keeps the function's original central position but changes its scale or orientation.
4. If a graph is symmetric about the y-axis, we say it is an ___ function.
5. An___ function is one whose graph is symmetric about the origin.
6. If $A > 1$, then the graph of $y = A\sin(Bx)$ is ___ vertically by a factor of ___.
7. If $0 < A < 1$, then the graph of $y = A\tan(Bx)$ is ___ vertically by a factor of ___.

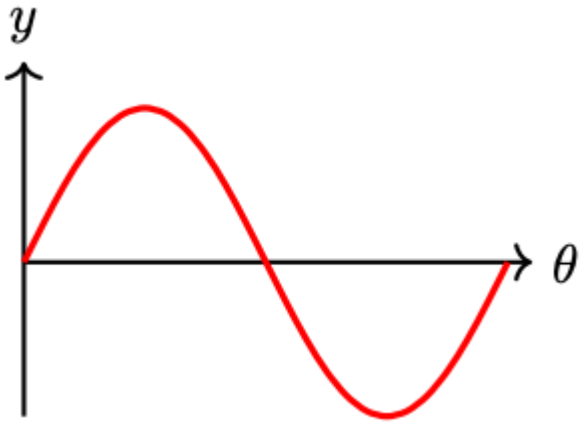
Concept Check

8. What is the range of $f(x) = A\sin(Bx)$?
9. What is the effect of $A < 0$ on the graph of the function $g(x) = A\tan(Bx)$?
10. What is the period of $h(x) = A\cos(Bx)$?
11. What is the period of $k(x) = A\tan(Bx)$?
12. When graphing $y = \cos(-Bx)$, we first rewrite the function as $y =$ ___ because cosine is an ___ function.
13. When graphing $y = \tan(-Bx)$, we first rewrite the function as $y =$ ___ because tangent is an ___ function.
14. One of these graphs is $y = A\sin k\theta$, and the other is $y = A\cos k\theta$. Explain how you know which is which.

    a.



b.



True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

15. The natural period of the cosine function is 2π.

16. The natural period of the tangent function is 2π.

17. Dividing the period by 4 gives us the step size.

18. Dividing the period of the function by 4 partitions a single cycle of the trigonometric function into four subintervals.

19. There are four key numbers.

Basic Skills

20. Write an equation for a sine function with amplitude 6.

21. Write an equation for a cosine function with amplitude $\frac{1}{2}$.

22. Write an equation for a sine function with period $\frac{\pi}{2}$.

23. Write an equation for a cosine function with period 4π.

For the following exercises, state the amplitude (if applicable) and period of the graph.

24. $y = -\cos(4x)$

25. $y = -\sin(3x)$

26. $y = -5\sin\left(\frac{x}{3}\right)$

27. $y = 6\cos\left(\frac{x}{2}\right)$

28. $y = \tan(2x)$

29. $y = \tan(4x)$

For the following exercises,
(i) Graph the function.
(ii) State the amplitude and period of the function.

30. $y = 3\cos(\theta)$

31. $y = 4\sin(\theta)$

32. $y = \cos(3\theta)$

33. $y = \sin(2\theta)$

For the following exercises, sketch one cycle of each graph by hand and label scales on the axes.

34. a. $y = \cos x$
    b. $y = \cos 4x$
    c. $y = -\cos x$

35. a. $y = \sin x$
    b. $y = \sin 3x$
    c. $y = -\sin 3x$

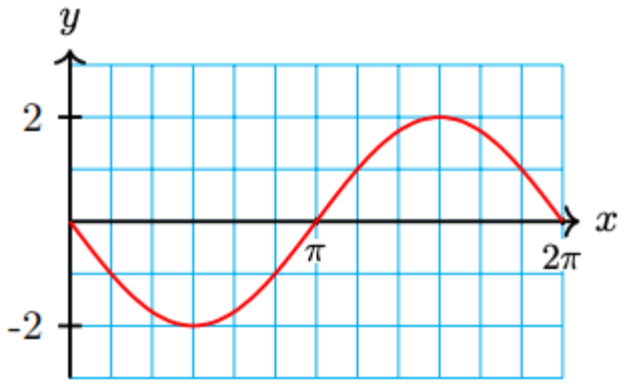
36. a. $y = \sin x$
    b. $y = \sin\frac{x}{3}$
    c. $y = -5\sin\frac{x}{3}$

37. a. $y = \cos x$
    b. $y = \cos\frac{x}{2}$
    c. $y = 6\cos\frac{x}{2}$

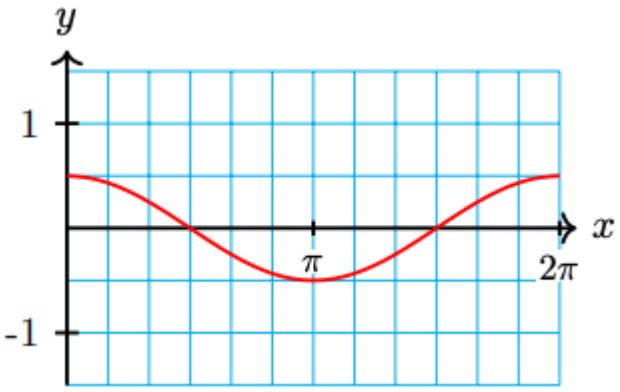
For the following exercises, write an equation for the graph using sine or cosine.

38.

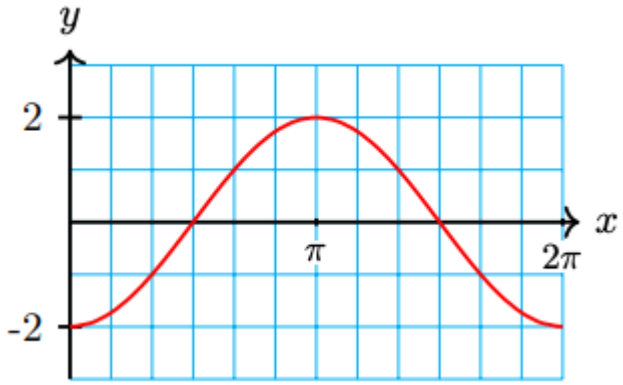
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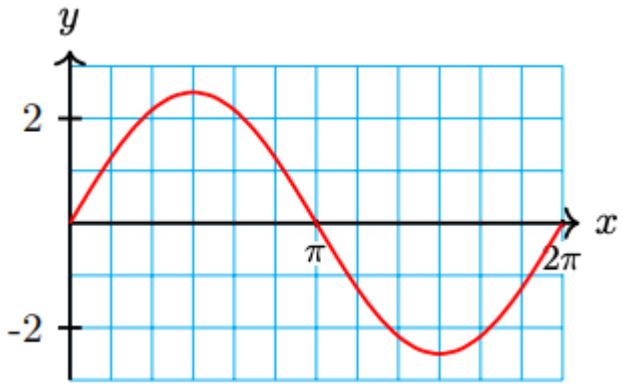
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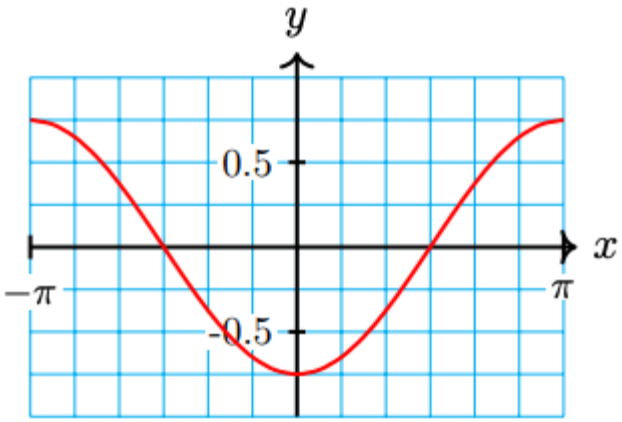


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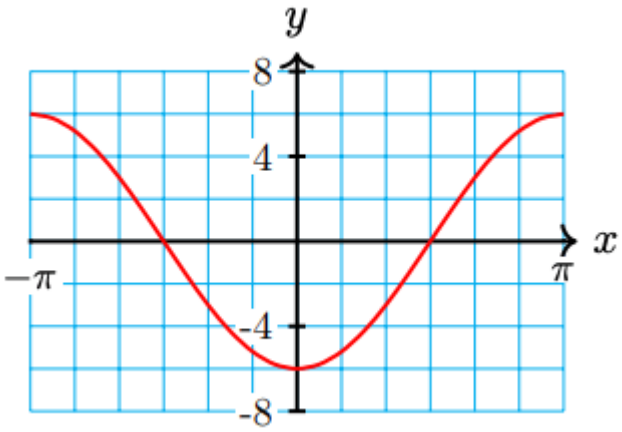


42.



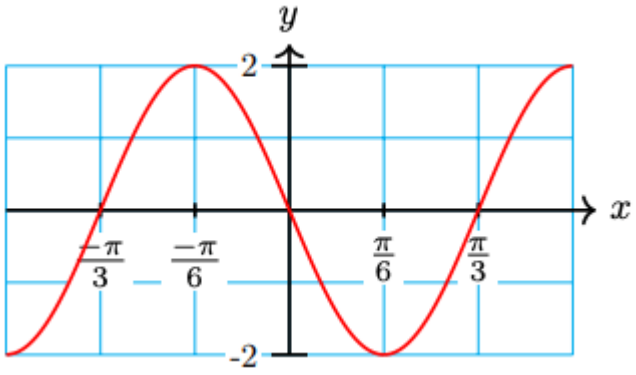


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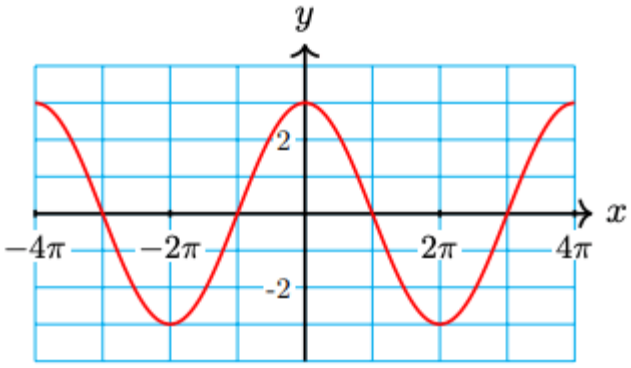


For the following exercises,
(i) State the graph's amplitude and period.
(ii) Write an equation for the graph using sine or cosine.

14.



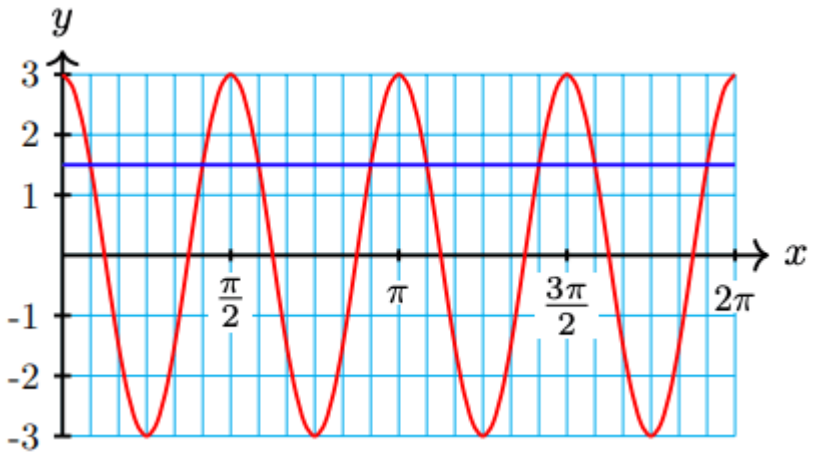
15.



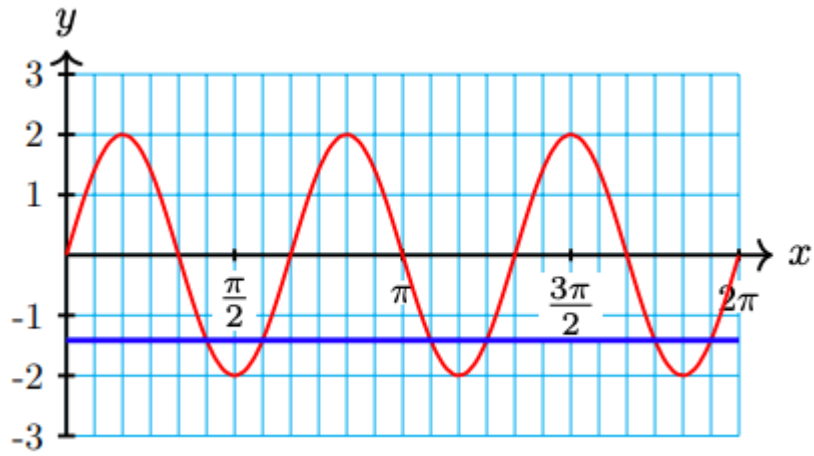
Synthesis Questions

For the following exercises, use the graph to find all solutions between 0 and 2π.

16. $3\cos(4x) = 1.5$



17. $2\sin(3x) = -\sqrt{2}$



For the following exercises,
(i) Use technology to graph the function for $0 \le x \le 2\pi$.
(ii) Use the graph from part (i) to approximate all solutions between 0 and 2π. Round your answers to hundredths.

18. a. $f(x) = 3\sin(2x)$

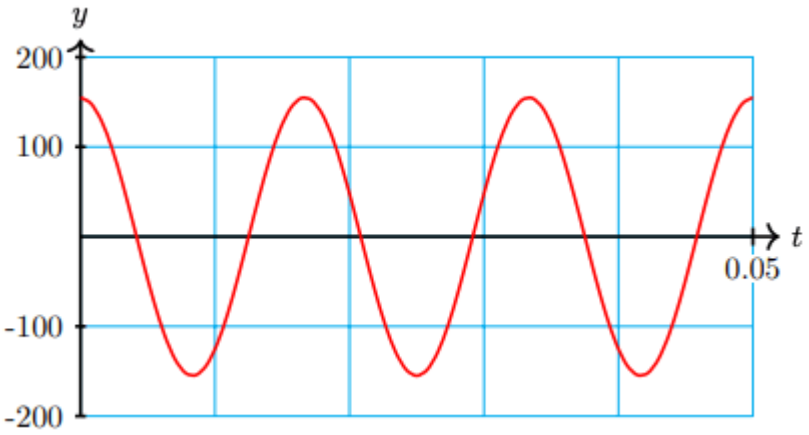
b. $3\sin(2x) = -1.5$

49. a. $g(x) = -2\cos(3x)$

b. $-2\cos(3x) = 1$

Applications

50. **Electric Current.** The following graph illustrates that the voltage used in U.S. electrical current changes from 155V to -155V and back 60 times each second. Write an equation for this graph.



51. **Electric Current.** If I represents the intensity of the current in an alternating circuit and t represents time, then

$$I(t) = 50\sin(120\pi t),$$

where I is measured in amperes and t is measured in seconds. Find the maximum value of I and the time it takes for I to go through one complete cycle.

52. **Sound Waves.** The oscillations in air pressure representing the sound wave for a tone at the standard pitch of C can be modeled by the equation

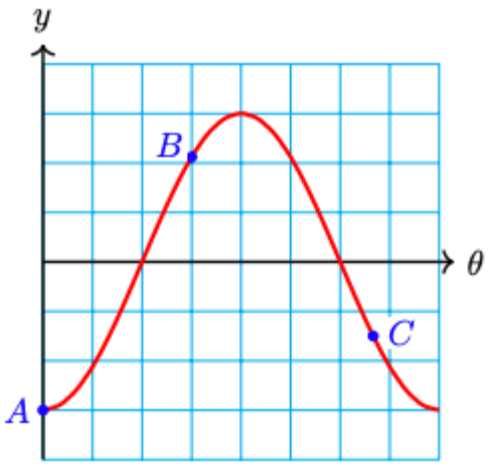
$$S(t) = 0.05\sin(524\pi),$$

where S is the sound pressure in pascals after t seconds. Sketch a single cycle of the graph of this function.

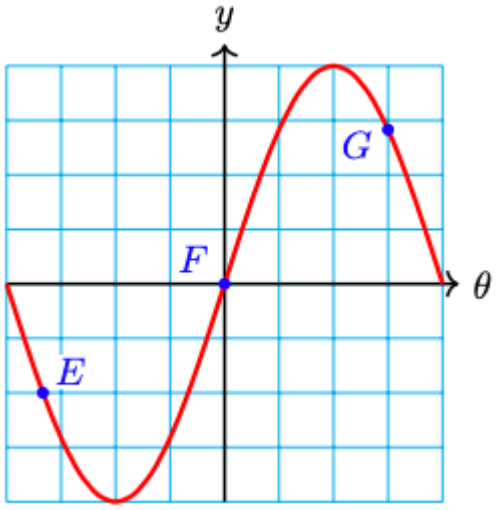
Challenge Problems

For the following exercises, give the coordinates of the points on the graph.

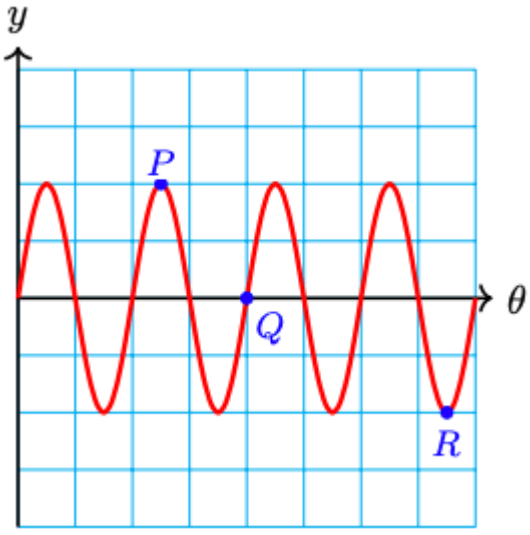
53. $f(\theta) = -3\cos(\theta)$



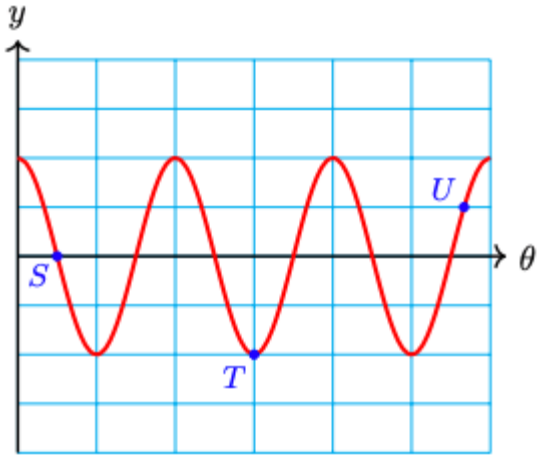
54. $f(\theta) = 4\sin(\theta)$



55. $f(\theta) = \sin(4\theta)$



56. $f(\theta) = \cos(3\theta)$



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5.3: Rigid Transformations

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

In the previous section, we stated that our ultimate goal (when it comes to graphing the trigonometric functions) is to graph any trigonometric function of the form

$$f(x) = A\,\mathrm{trig}\,(Bx + C) + D. \tag{5.3.1}$$

Now that we know the effect of the parameters A and B on the graph of the fundamental trigonometric functions, it's time to turn our attention to the parameters C and D.

Phase Shifts (also known as Horizontal Shifts)

Continuing with alphabetical order, let's focus on the effect of the parameter C; however, let's make things simple (for now). Recall (from Algebra), the graph of the function $y = g(x + C)$ is a horizontal shift of the graph of g. It moves the graph to the *left* C units if $C > 0$, and *right* C units if $C < 0$. Most students in Algebra accept this as truth and memorize the result; however, as you are taking a college-level Trigonometry course, it might serve you well to understand *why* this is true.

Click here to delve into why the graph of $y = g(x + C)$ is a horizontal shift of the graph of $g(x)$

Let $g(x)$ be a function. Then each input leads to a single output. Let's grab an *arbitrary* value from the domain of g and call it x_1 (you can call it "happy face" or "star," but x_1 is easier to write). Again, since g is a function, $g(x_1)$ must return a single value. Let's call this output y_1.

So far, all we have said is that g is a function and that $g(x_1) = y_1$; however, this last statement means the point $(x_1, g(x_1))$ is on the graph of g.

Now, your friend comes along and says, "Hey, I need to graph $g(x + C)$. All I have been given is that C is positive."

You pause and think for a moment. Turning to your friend, you say, "I don't know what that graph looks like, but I know that when I plug in x_1 to g, I get out y_1."

Your friend smirks and says, "Okay, let's plug x_1 into $g(x + C)$ to find its output!"

You watch him work for a few moments until he gets stuck. He ends up with $g(x_1 + C)$, and you look at him quizzically, "I don't know what the value of g is at $x_1 + C$. I only know its value at x_1."

Then it hits you!

"You did it wrong! You want to evaluate $g(x_1)$. You need to let the argument of your function, which is $x + C$, be x_1."

You start doing the math,

$$x + C = x_1 \implies x = x_1 - C.$$

You pause... thinking.

"So, to get the argument of $g(x+C)$ to be x_1, we need to plug in $x_1 - C$ for x."

Thinking even harder, you have the epiphany.

"With the original function, we plugged in x_1 and got out y_1. With $g(x+C)$ we needed to plug in $x_1 - C$ to get $g(x_1 - C + C) = g(x_1) = y_1$. That is, the original point, x_1, gets shifted to the left C units!"

Since our point, $(x_1, g(x_1))$ was arbitrary, all points on the graph of g experience the same shift. Hence, the graph of $g(x+C)$, where $C > 0$, is the graph of g shifted C units to the left.

Example 5.3.1

Make a table of values and sketch a graph of $y = -2\cos\left(x + \frac{\pi}{3}\right)$.

Solution available at the end of this section.

Checkpoint 5.3.1

Complete the table and sketch a graph of $y = -3\sin\left(x - \frac{\pi}{6}\right)$. (Do you expect the graph to be shifted to the left or the right, compared to the graph of $y = -3\sin(x)$?)

x	$x - \frac{\pi}{6}$	$\sin\left(x - \frac{\pi}{6}\right)$	$y = -3\sin\left(x - \frac{\pi}{6}\right)$
	0		
	$\frac{\pi}{2}$		
	π		
	$\frac{3\pi}{2}$		
	2π		

Answer available at the end of this section.

Example 5.3.1 and Checkpoint 5.3.1 should convince you that our previous experience from Algebra with horizontal shifts of functions still works with trigonometric functions; however, let's solidify that concept to drive the point home.

We know that all the fundamental trigonometric functions have principal cycles occurring on the interval $0 \le \theta \le p$, where θ is the argument and p is the function's natural period. This means that the modified function, $f(x) = \text{trig}(x + C)$, goes through that same cycle when $0 \le x + C \le p$. Subtracting C from all three sides, we discover that this modified trigonometric function goes through the same single cycle on the interval $-C \le x \le p - C$. That is, the starting point for the principal cycle has been shifted from 0 to $-C$. This shift in the starting point from the principal cycle is known as the **phase shift** of the trigonometric function. This definition is so important that we should codify it.

Definition: Phase Shift

The **phase shift** of a trigonometric function is where the original x-value from the principal cycle (which is $x = 0$) has been shifted.

Moreover, we just proved the following result.

Lemma: Phase Shift Formula for $y = \text{trig}(x + C)$

The phase shift for a trigonometric function of the form $y = \text{trig}(x + C)$ is $-C$.

Thus, the phase shift for the function in Example $5.3.1$ is $-C = -\frac{\pi}{3}$ and the phase shift for the function in Checkpoint $5.3.1$ is $-C = -\left(-\frac{\pi}{6}\right) = \frac{\pi}{6}$.

If you're wondering why we called that previous statement a lemma instead of a theorem, it is because a lemma is a stepping stone to a bigger theorem, and the bigger theorem (which we will get to shortly) is going to supercede that lemma. Before we get there, however, we still have some work to do.

We now can expertly graph a trigonometric function of the form

$$f(x) = A\,\text{trig}\,(x + C)\,,$$

but what happens when the coefficient of x is not 1? That is, if we are given

$$f(x) = A\,\text{trig}\,(Bx + C)\,,$$

is the phase shift still $-C$?

The answer to this question is an adamant **no** (and this is why we only wrote that last conjecture as a lemma).

Using our previous derivation, the modified function, $f(x) = A\ \text{trig}\,(Bx + C)$, goes through its principal cycle when $0 \leq Bx + C \leq p$, where p is the natural period of the trigonometric function. Subtracting C from all three sides, we get $-C \leq Bx \leq p - C$. *Assuming* $B > 0$, we can divide all three sides by B to find that this modified trigonometric function goes through the same single cycle on the interval $-\frac{C}{B} \leq x \leq \frac{p-C}{B}$. That is, the starting point for the principal cycle has been shifted from 0 to $-\frac{C}{B}$. Therefore, the phase shift is $-\frac{C}{B}$. The following theorem summarizes all of this into a nice, clean statement.

Theorem: Phase Shift Formula

The phase shift of the function $y = A\ \text{trig}\,(Bx + C)$ is $-\frac{C}{B}$.

If $B < 0$, we use the Symmetry Identities to force it to become positive before attempting to find the phase shift.

Caution: C is Not Always the Phase Shift

When considering the function

$$f(x) = A\,\text{trig}\,(Bx + C)\,,$$

it is crucial to remember that B is ***not*** the period and C is ***not necessarily*** the phase shift (although, it does adopt the name *phase angle* in Physics).

- $\text{Period} = \dfrac{\text{natural period}}{B}$
- $\text{Phase Shift} = -\dfrac{C}{B}$

Example $5.3.2$

State the phase shift of each function.

a. $y = 2\tan\,(6x - \pi)$
b. $y = \sqrt{5}\sin\left(-\frac{\pi}{2}x + \frac{\pi}{4}\right)$

Solutions available at the end of this section.

Checkpoint 5.3.2

State the phase shift of $y = -2\cos\left(2x - \frac{\pi}{3}\right)$

Answer available at the end of this section.

Some instructors (including myself) introduce an alternative way of manipulating the argument of the trigonometric function $f(x) = A\ \text{trig}\,(Bx + C)$. If you factor B from all terms in the argument, your function turns into

$$f(x) = A\,\text{trig}\left(B\left(x + \frac{C}{B}\right)\right).$$

In this form, the phase shift reveals itself because $\frac{C}{B} = -\left(-\frac{C}{B}\right)$ and so

$$f(x) = A\,\text{trig}\left(B\left(x - \left(-\frac{C}{B}\right)\right)\right).$$

We will include this in an example or two, just in case your instructor likes this form.

We now modify our graphing tactic from the previous section to include this new transformation.

Summary of Graphing $f(x) = \overline{A}\ \text{trig}\left(\overline{B}x + \overline{C}\right)$

To graph the function

$$f(x) = \overline{A}\,\text{trig}\left(\overline{B}x + \overline{C}\right):$$

Precondition Stage

We begin by cleaning up the trigonometric function and ensuring that the coefficient of x is positive. This is done using the Symmetry Identities. Again, as a warning, if $\overline{B} < 0$, you will factor a negative off both terms in the argument of the trigonometric function. Once the Symmetry Identities have been used, your new function will be of the form

$$f(x) = A\,\text{trig}\,(Bx + C).$$

Information Stage

We now identify the following information (in the order listed):

- $\text{Amplitude} = |A|$
- $\text{Period} = \dfrac{\text{Natural Period}}{B}$
- $\text{Step Size} = \dfrac{\text{Period}}{4}$
- $\text{Phase Shift} = -\dfrac{C}{B}$

where the natural period is 2π if given the sine or cosine functions, and π if given the tangent function.

Graphing Stage

We then use each of these *in reverse order* to obtain a sketch of the trigonometric function.

- Draw a horizontal line and mark the initial key number as the **phase shift**.
- Mark the remaining key numbers by successively adding the **step size** to the previous key number.
- Double-check that the distance from the initial key number to the final key number matches the **period**, then sketch one cycle of the base trigonometric function, making sure to account for the sign of A and drawing any vertical asymptotes (if the given function is the tangent).
- Draw the vertical axis at $x = 0$ and use the **amplitude** to either
  - label the maximum and minimum values of the trigonometric function on the vertical axis (if the given function is sinusoidal), or

- label the remaining key points on the graph (if the given function is the tangent).

Example 5.3.3

Graph a single cycle of the given trigonometric function, labeling key numbers, the y-axis, stating the period, step size, amplitude, and phase shift.

a. $y = 2\tan(6x - \pi)$
b. $y = \sqrt{5}\sin\left(-\frac{\pi}{2}x + \frac{\pi}{4}\right)$

Solutions available at the end of this section.

Checkpoint 5.3.3

Graph $y = -2\cos\left(2x + \frac{\pi}{3}\right)$ and state the period, amplitude, step size, and phase shift.

Answer available at the end of this section.

As we have done previously, you can use the following Interactive Element to help visualize the effect of C on the graph of the sine function.

Interactive Element: Investigating the Effects of C on the Graph of $y = A\ \sin(Bx + C)$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Desmos Interactive online

Interaction: Adjust the slider for A, B, and C to see the effect on the graph of $y = A\ \sin(Bx + C)$.

Vertical Shifts and the Midline

We finish our discussion of transformations with the parameter D in the function

$$f(x) = A\,\mathrm{trig}\,(Bx + C) + D.$$

Of all the transformations from your Algebra class, this is the one most students are pretty good with. Recall, the graph of $y = g(x) + D$ is the graph of g shifted up (if $D > 0$) or down (if $D < 0$) $|D|$ units. That is, the "+ D" is just adding D to all the y-values of the original function.

The only new concept here is the idea of a **midline** for a trigonometric function.

Definition: Midline

The **midline** of a trigonometric function is a horizontal line that represents the average value of the function. For sinusoidal functions, it is situated halfway between the maximum and minimum points. The midline for all base graphs of the trigonometric functions is $y = 0$.

The midline is essentially the center line around which the graph of the trigonometric function oscillates (for the case of the

sinusoidal curves). Once a vertical shift occurs, this "hidden" axis appears, and it is handy when graphing trigonometric functions.

For example, consider the function

$$y = -2\sin(\pi x) + 3.$$

From the previous section, we should be able to quickly sketch *most* of this function, but the "$+3$" at the end is new. We know the base function is the sine, the fact that the coefficient of the sine is negative means the base graph has been reflected about the x-axis, the amplitude (because the graph is sinusoidal) is $|-2| = 2$, the period is $\frac{2\pi}{B} = \frac{2\pi}{\pi} = 2$, and the step size is $\frac{\text{Period}}{4} = \frac{2}{4} = \frac{1}{2}$. If this is all we had, then the graph would look like the following:

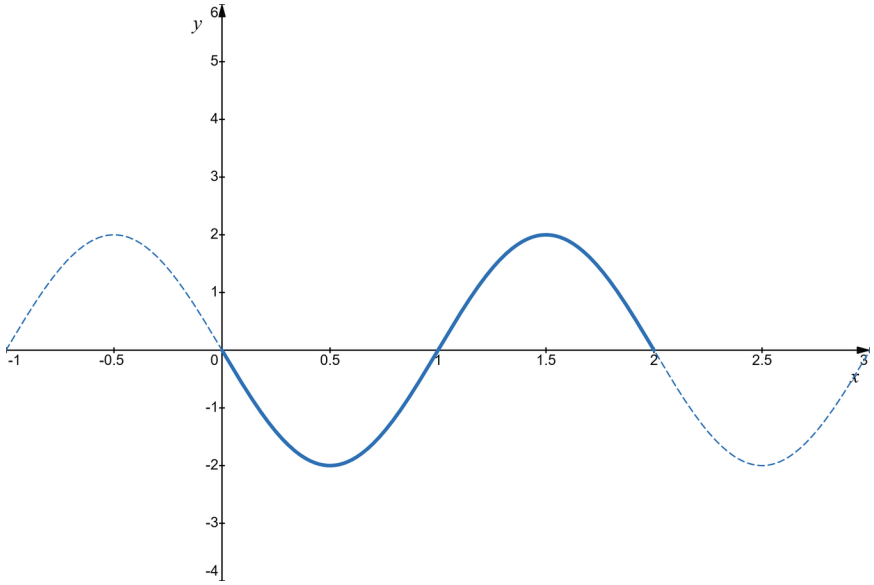


Figure $5.3.10$**:** $y = -2\sin(\pi x)$

Since our given function is $y = -2\sin(\pi x) + 3$, we need to add three to all of the outputs in Figure $5.3.10$. As we redraw our graph and shift up three units, it is helpful to move up the hidden midline.

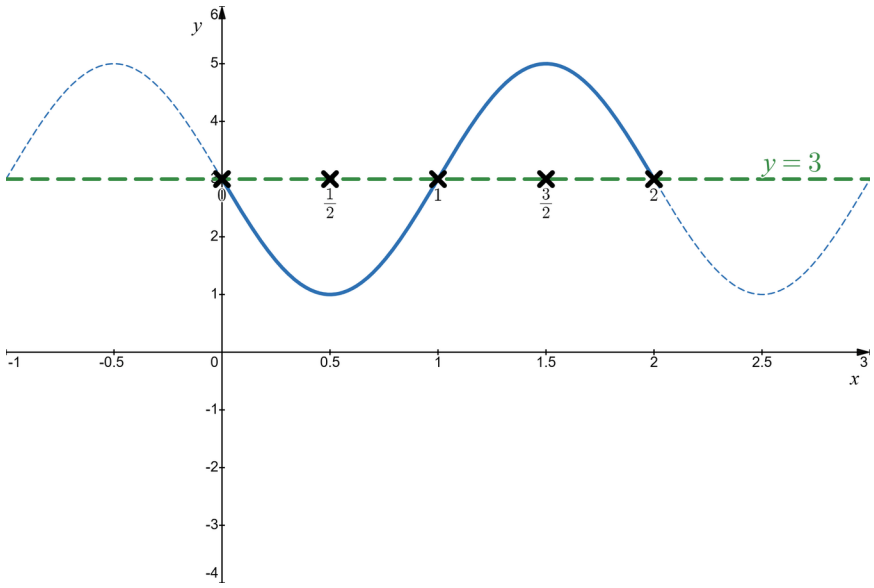


Figure $5.3.11$**:** $y = -2\sin(\pi x) + 3$

Let's update our graphing tactic a final time.

Summary of Graphing $f(x) = \overline{A}\ \text{trig}\left(\overline{B}x + \overline{C}\right) + D$

To graph the function

$$f(x) = \overline{A}\,\text{trig}\left(\overline{B}x + \overline{C}\right) + D:$$

Precondition Stage

- If the trigonometric function is not written in the order listed above (e.g. if you are given $y = 3 - 2\sin(5 - x)$), rewrite it so it matches the form above—this is critical!
- Clean up the trigonometric function using the Symmetric Identities to ensure the coefficient of x is positive. Again, as a warning, if $\overline{B} < 0$, you will factor a negative off both terms in the argument of the trigonometric function. Once the Symmetry Identities have been used, your new function will be of the form

$$f(x) = A\,\text{trig}\,(Bx + C) + D.$$

Information Stage

We now identify the following information (in the order listed):

- $\text{Amplitude} = |A|$
- $\text{Period} = \dfrac{\text{Natural Period}}{B}$
- $\text{Step Size} = \dfrac{\text{Period}}{4}$
- $\text{Phase Shift} = -\dfrac{C}{B}$
- $\text{Vertical Shift} = D$

where the natural period is 2π if given the sine or cosine functions, and π if given the tangent function.

Graphing Stage

We then use each of these *in reverse order* to obtain a sketch of the trigonometric function.

- Draw a horizontal line and label it $y = D$. This is your **vertical shift**.
- Mark the initial key number as the **phase shift**.
- Mark the remaining key numbers by successively adding the **step size** to the previous key number.
- Double-check that the distance from the initial key number to the final key number matches the **period**, then sketch one cycle of the base trigonometric function, making sure to account for the sign of A and drawing any vertical asymptotes (if the given function is the tangent).
- Draw the vertical axis at $x = 0$ and use the **amplitude** to either
  - label the maximum and minimum values of the trigonometric function on the vertical axis (if the given function is sinusoidal), or
  - label the remaining key points on the graph (if the given function is the tangent).

Example 5.3.4

Graph each function. State the amplitude (if applicable), period, step size, phase shift, and vertical shift. Your graph should have five key points (or four key points and an asymptote) labeled, as well as the midline.

a. $y = -5 + 5\sin\left(3\pi x + \frac{\pi}{2}\right)$
b. $y = \frac{1}{3}\tan(2x - 1) + 1$

Solutions available at the end of this section.

Checkpoint 5.3.4

Graph the function. State the amplitude (if applicable), period, step size, phase shift, and vertical shift. Your graph should have five key points (or four key points and an asymptote) labeled, as well as the midline.

$$f(x) = 3 - \frac{7}{5}\tan\left(2x - \frac{\pi}{2}\right)$$

Answer available at the end of this section.

Interactive Element: Investigating the Effects of D on the Graph of $y = A \sin(Bx + C) + D$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Desmos Interactive online

Interaction: Adjust the sliders for each parameter to see the effects on the graph of $y = A\sin(Bx + C) + D$.

Finding Equations Given Graphs

Finding the equation of a trigonometric function given its graph is only slightly more complex when there has been a phase shift or vertical shift. The key to success with these problems is identifying two points on the curve. This is *usually* enough to find the equation.

Example 5.3.5

Find the equation of the graphed function.

a.

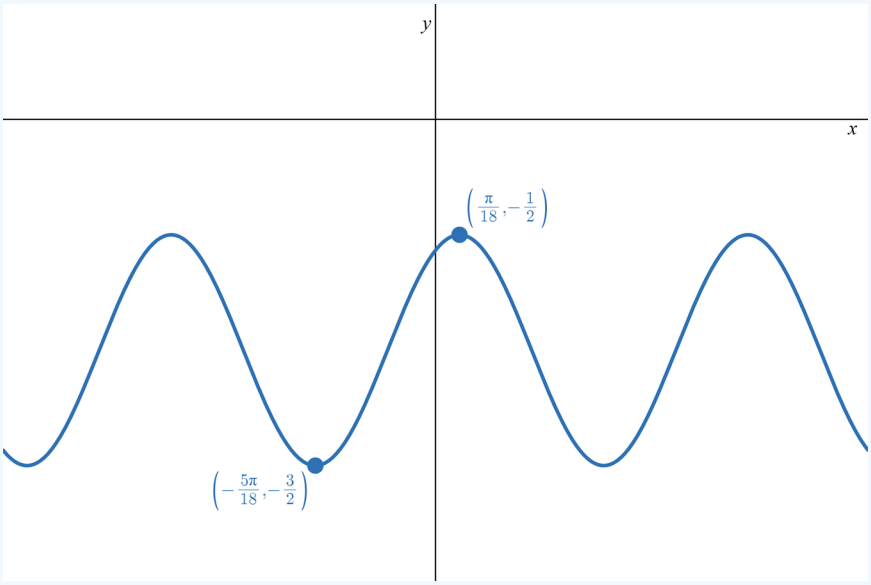


Figure 5.3.22

b.

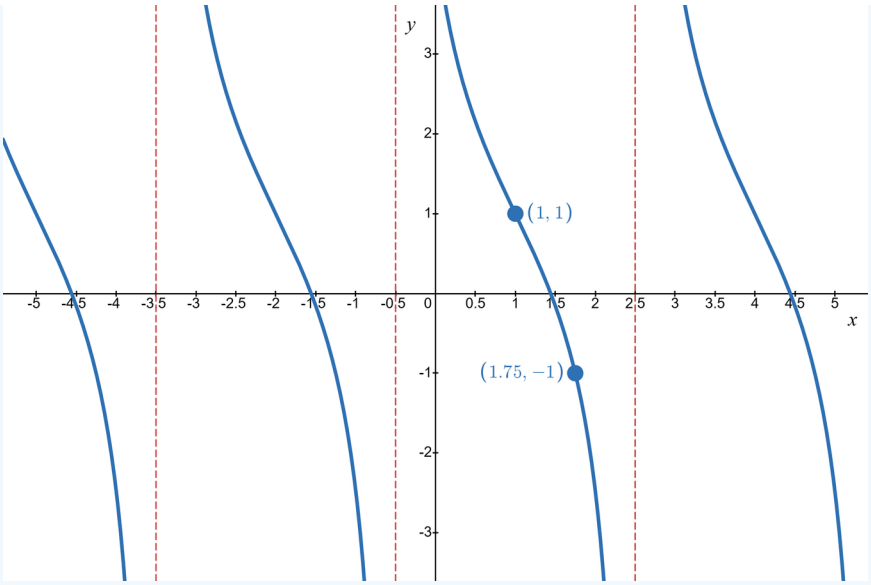


Figure 5.3.23

Solutions available at the end of this section.

Checkpoint 5.3.5

Find the equation of the graphed function.

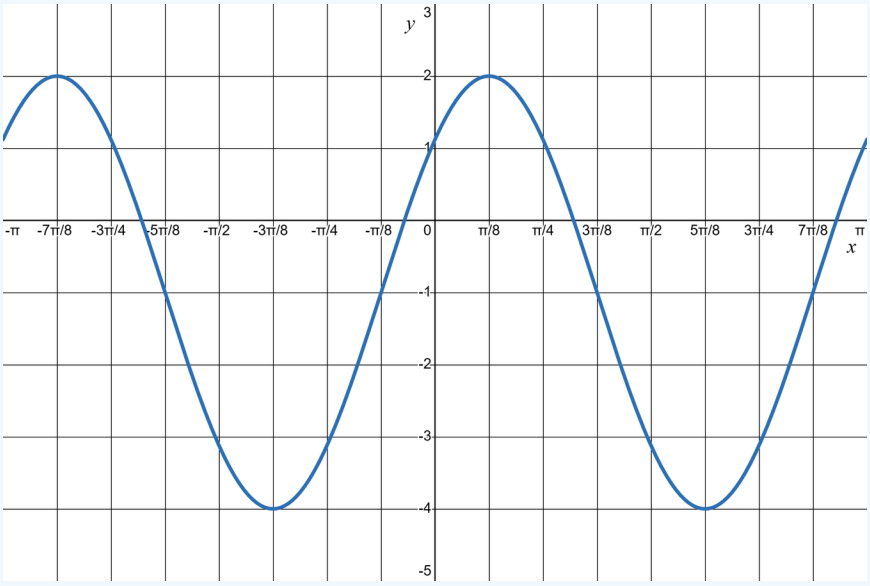


Figure 5.3.24

Answer available at the end of this section.

Revisiting the Cofunction Identities

Recall the Cofunction Identities (which we rewrite here in radian measure).

Theorem: Cofunction Identities (radians)

The trigonometric function of an angle is equal to the cofunction evaluated at the complement of the angle. That is,

$$\sin(\theta) = \cos\left(\frac{\pi}{2}-\theta\right) \quad \text{and} \quad \cos(\theta) = \sin\left(\frac{\pi}{2}-\theta\right)$$
$$\sec(\theta) = \csc\left(\frac{\pi}{2}-\theta\right) \quad \text{and} \quad \csc(\theta) = \sec\left(\frac{\pi}{2}-\theta\right)$$
$$\tan(\theta) = \cot\left(\frac{\pi}{2}-\theta\right) \quad \text{and} \quad \cot(\theta) = \tan\left(\frac{\pi}{2}-\theta\right)$$

At the time, we could only prove these identities for acute angles. We are still not at a point where we can prove these for all angles, but we can now *visualize* these identities. For example, let's consider the function $y = \cos\left(\frac{\pi}{2}-x\right)$. The Cofunction Identities tell us that this should be the same as $y = \sin(x)$. Let's manipulate that cosine function a little.

$$\begin{aligned} y &= \cos\left(\tfrac{\pi}{2}-x\right) \\ &= \cos\left(-\left(-\tfrac{\pi}{2}+x\right)\right) \\ &= \cos\left(-\left(x-\tfrac{\pi}{2}\right)\right) \\ &= \cos\left(x-\tfrac{\pi}{2}\right) \qquad \text{(cosine is an even function)} \end{aligned}$$

So, that cosine function is just like a regular cosine function, but with a phase shift of $\frac{\pi}{2}$. Graphing this, we get the following:

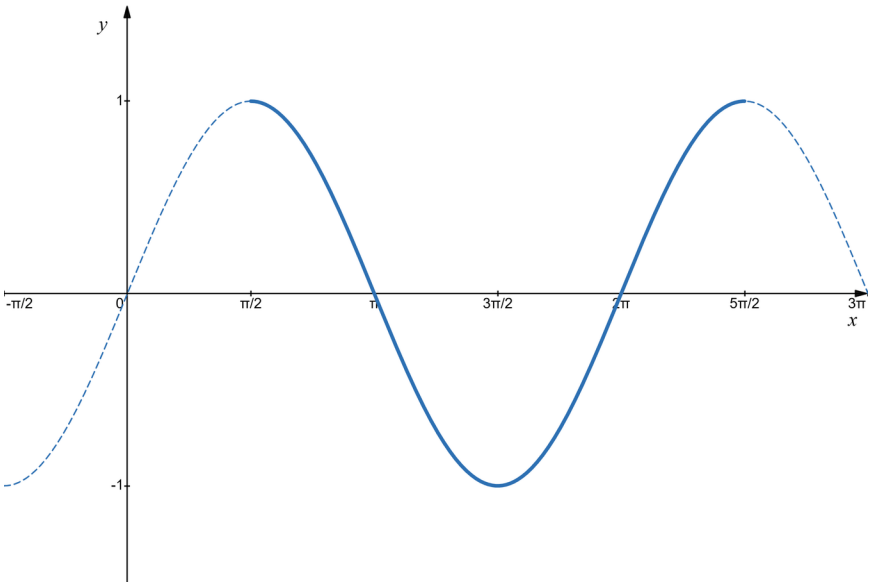


Figure 5.3.25

However, this is *exactly* the graph of $y = \sin(x)$. Therefore, we feel confident in stating that $\cos\left(\frac{\pi}{2}-x\right) = \sin(x)$. You can use the same technique to convince yourself that the other Cofunction Identities are true (go ahead, it's a great exercise!).

Introduction to Sinusoidal Models

Many natural phenomena can be modeled with sinusoidal functions. In this subsection, we will work with applications where we have either been given the explicit sinusoidal model or the graph that models a situation. In the next section, we will learn to build sinusoidal models from language.

Example 5.3.6

The formula

$$D(t) = 12.25 - 5.25\cos\left(\frac{\pi}{6}t\right)$$

gives the number of hours of daylight in Glasgow, Scotland, as a function of time, in months after January 1.

a. Graph the function by hand.
b. State the midline, amplitude, and period of the graph.
c. How many hours of daylight does Glasgow enjoy on the longest day of the year?
d. In which months does Glasgow experience less than 8 hours of light daily?

Solutions available at the end of this section.

Checkpoint 5.3.6

The pistons in an automobile engine move up and down in the cylinders. If t is in milliseconds, the distance from the top of the piston to the top of the cylinder is given in centimeters by

$$D = 7 + 6\sin(4\pi t)$$

a. Graph the function by hand.
b. State the midline, amplitude, and period of the graph.
c. Find the largest and the smallest clearance between the piston and the top of the cylinder.

Answer available at the end of this section.

Example 5.3.7

Sunspots are dark regions on the Sun first observed by Galileo in 1610. The number of sunspots is not constant but varies with a period of approximately 11 years, called the solar cycle. Solar activity is directly related to this cycle and can cause disturbances in the Earth's upper atmosphere. Planning for satellite orbits and space missions requires knowledge of solar activity levels years in advance.

The figure below shows sunspot data for the last solar cycle, from July 1985 through June 1996, and a sinusoidal function that models the data.

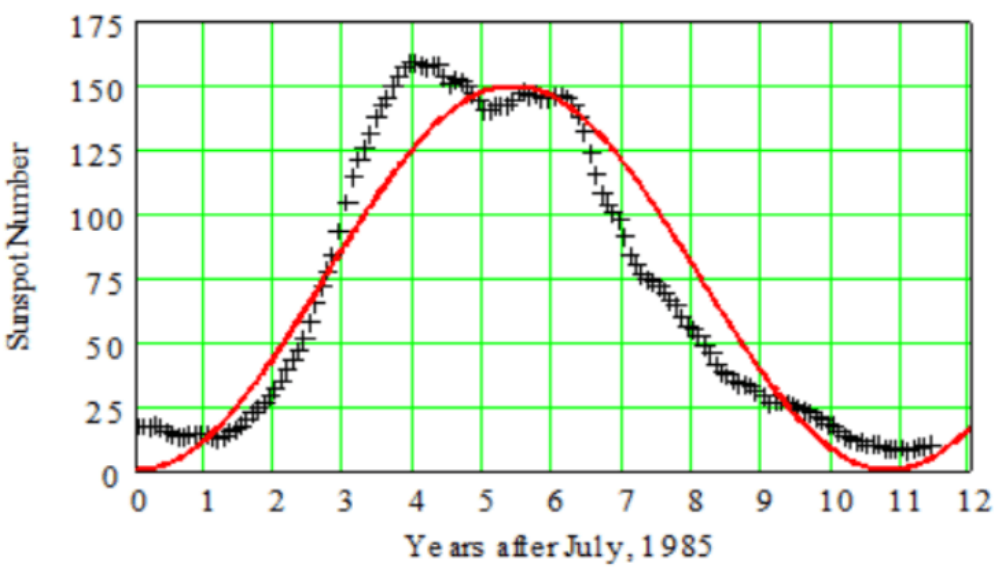


Figure 5.3.28

The period of the solar cycle shown is 10.8 years. Use this information and the graph to write a formula for the model.

Solution available at the end of this section.

Checkpoint 5.3.7

The figure below shows the sunspot data for the solar cycle that began in July 1996, and a curve of best fit calculated by NASA. This curve is not sinusoidal, but has a similar shape.

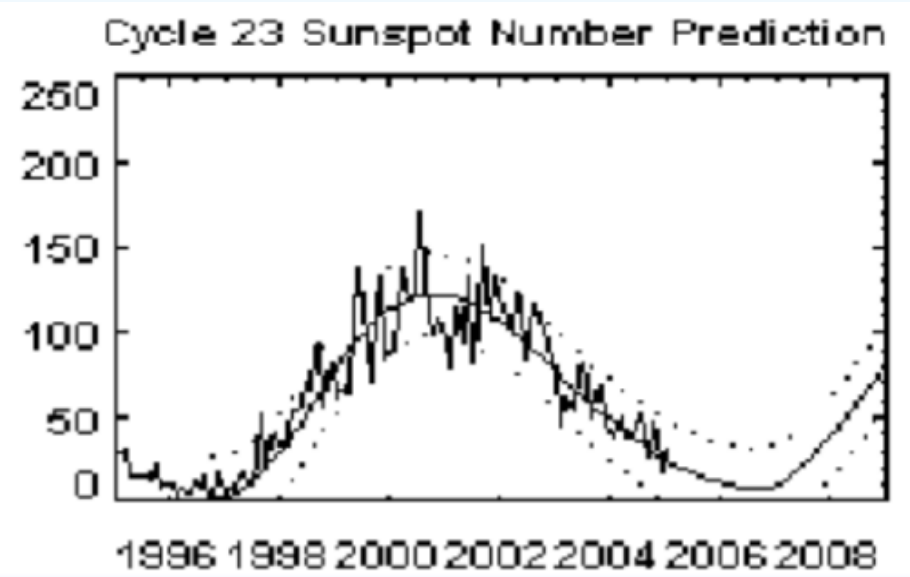


Figure 5.3.29

a. The minimum sunspot number occurred in October 1996. Use the graph to estimate the period, midline, and amplitude of a sinusoidal function that approximates the data.
b. Write a function, $s(t)$, that approximates the data, where t is the number of years since July 1996.
c. Use your function to predict the sunspot number in January 2025.

Answer available at the end of this section.

Miscellaneous Topics: Thinking About Mathematics (Part 2)

A significant concern in mathematics education is the trend that students rely on calculators more. An argument for calculator usage is that, in "real life," a person will always have access to a computer or a calculator. Therefore, forcing students to do problems "by hand" is useless.

1. Give an argument against high calculator usage in math classes.
2. Do you think that your professor relies too much on calculator usage in class? What about their use of technology? Would you like more or less?

Hidden Resources: Adobe Creative Cloud

The Adobe Creative Cloud is a collection of Adobe software products. In addition to offering the products, the Creative Cloud also offers cloud storage.

Apps Included in the Adobe Creative Suite are:

- Design Apps: Photoshop, InCopy, Spark, Illustrator, Adobe XD, Dimension, InDesign
- Web & UX Apps: Adobe XD, Spark, Dreamweaver, Animate, InCopy, InDesign, Photoshop
- Photography Apps: Photoshop, Lightroom
- Video & Audio Apps: Premiere Pro, Premiere Rush, After Effects, Spark, Audition

Typically, the Adobe Creative Suite costs $55/month, but Los Rios students have access to Adobe Creative Cloud for $39.99 per semester or $79.99 per year. Check out this page for more details.

Answers

Section Preview

While the rigid transformations are introduced in this section, the majority of the instruction will likely be focused on graphing trigonometric functions requiring *all* transformations. There isn't another section on graphing transformations of the fundamental trigonometric functions, so make sure students really have practice graphing lots of transformations here.

Section Preview

- **Graphing Functions**
  - Transformations — Rigid Transformations: The opening explicitly says "Recall, the graph of $y = g(x + C)$ is a horizontal shift of the graph of g." The rigid transformation framework — horizontal and vertical shifts of a general function — is the direct prior-course prerequisite.

Section Preview

- Graph fundamental trigonometric functions having vertical transformations.
- Graph fundamental trigonometric functions having horizontal transformations.
- Graph fundamental trigonometric functions having combinations of all transformations.
- Given the graph, write the corresponding trigonometric function.
- Evaluate given sinusoidal models for applications.
- Build a model for an application based on the graph.

Example 5.3.1

From Algebra, we expect the graph to be shifted $\frac{\pi}{3}$ units to the left compared to the graph of $y = \cos(x)$.

As we commonly try to do, we want the argument of $\cos\left(x + \frac{\pi}{3}\right)$ (which is the expression $x + \frac{\pi}{3}$) to be quadrantal angles.

x	$x + \frac{\pi}{3}$	$\cos\left(x + \frac{\pi}{3}\right)$	$y = 2\cos\left(x + \frac{\pi}{3}\right)$
	0		
	$\frac{\pi}{2}$		
	π		
	$\frac{3\pi}{2}$		
	2π		

The second column of this table is what we desire $x+\frac{\pi}{3}$ to be. For $x+\frac{\pi}{3}$ to become 0, we solve

$$x+\frac{\pi}{3}=0 \implies x=-\frac{\pi}{3}.$$

Likewise, for $x+\frac{\pi}{3}$ to become $\frac{\pi}{2}$, we solve

$$x+\frac{\pi}{3}=\frac{\pi}{2} \implies x=\frac{\pi}{6}.$$

We continue this process to fill out the first column of the table.

x	$x+\frac{\pi}{3}$	$\cos\left(x+\frac{\pi}{3}\right)$	$y=2\cos\left(x+\frac{\pi}{3}\right)$
$-\frac{\pi}{3}$	0		
$\frac{\pi}{6}$	$\frac{\pi}{2}$		
$\frac{2\pi}{3}$	π		
$\frac{7\pi}{6}$	$\frac{3\pi}{2}$		
$\frac{5\pi}{3}$	2π		

Finally, we fill out the third and fourth columns using the second column as the arguments.

x	$x+\frac{\pi}{3}$	$\cos\left(x+\frac{\pi}{3}\right)$	$y=2\cos\left(x+\frac{\pi}{3}\right)$
$-\frac{\pi}{3}$	0	1	-2
$\frac{\pi}{6}$	$\frac{\pi}{2}$	0	0
$\frac{2\pi}{3}$	π	-1	2
$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	0	0
$\frac{5\pi}{3}$	2π	1	-2

We now plot the x-values (in the first column) and their corresponding function values (in the fourth column) to make the graph. One cycle of the graph is shown below. As we expected, the graph is shifted $\frac{\pi}{3}$ units to the left compared to the graph of $y=-2\cos(x)$.

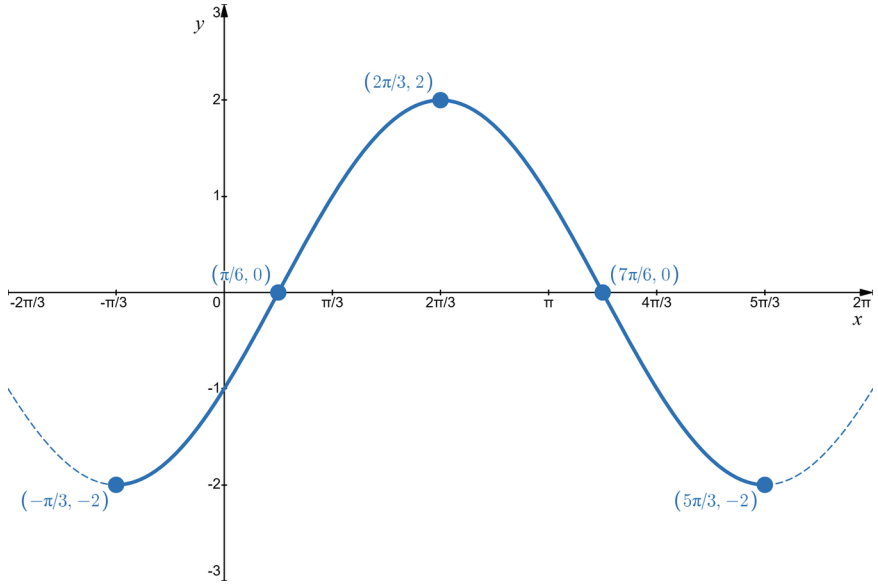


Figure 5.3.1

Checkpoint 5.3.1

x	$x-\frac{\pi}{6}$	$\sin\left(x-\frac{\pi}{6}\right)$	$y=-3\sin\left(x-\frac{\pi}{6}\right)$
$\frac{\pi}{6}$	0	0	0
$\frac{2\pi}{3}$	$\frac{\pi}{2}$	1	-3
$\frac{7\pi}{6}$	π	0	0
$\frac{5\pi}{3}$	$\frac{3\pi}{2}$	-1	3
$\frac{13\pi}{6}$	2π	0	0

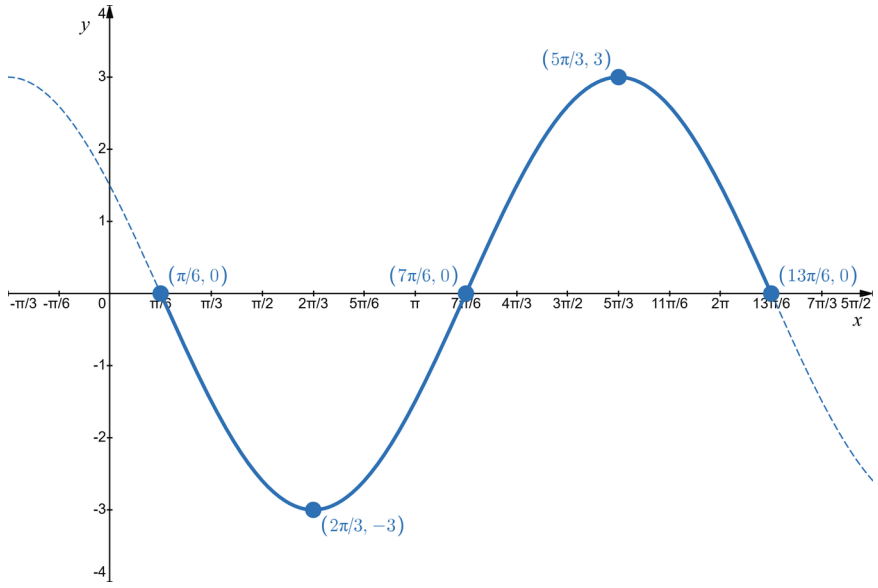


Figure 5.3.2

Example 5.3.2

a. In this case, $B=6$ and $C=-\pi$. Hence, the phase shift for this function is

$$-\frac{C}{B}=-\frac{-\pi}{6}=\frac{\pi}{6}.$$

b. As in the previous section, we must use the Symmetry Identities when $B<0$. When using these identities, be sure to factor the negative off *both* B and C.

$$\begin{aligned} y &= \sqrt{5}\sin\left(-\frac{\pi}{2}x+\frac{\pi}{4}\right) \\ &= \sqrt{5}\sin\left(-\left(\frac{\pi}{2}x-\frac{\pi}{4}\right)\right) \\ &= -\sqrt{5}\sin\left(\frac{\pi}{2}x-\frac{\pi}{4}\right) \qquad \text{(sine is an odd function)} \end{aligned}$$

Hence, the phase shift is

$$-\frac{C}{B}=-\frac{-\pi/4}{\pi/2}=\frac{1}{2}.$$

Checkpoint 5.3.2

$\frac{\pi}{6}$

Example 5.3.3

a. **Precondition Stage:** Since $B=6>0$, there is nothing to precondition; however, if we wanted to factor the 6 from both terms in the argument, we would get

$$y=2\tan\left(6\left(x-\frac{\pi}{6}\right)\right),$$

which reveals the phase shift to be $\frac{\pi}{6}$.
Information Stage: This is not a sinusoidal function, so we will not have an amplitude, and the natural period is π.

$$\begin{array}{rcl} B = 6 & \Longrightarrow & \text{Period} = \dfrac{\text{Natural Period}}{B} = \dfrac{\pi}{6} \\ & \Longrightarrow & \text{Step Size} = \dfrac{\text{Period}}{4} = \dfrac{\pi/6}{4} = \dfrac{\pi}{24} \\ C = -\pi & \Longrightarrow & \text{Phase Shift} = -\dfrac{C}{B} = -\dfrac{-\pi}{6} = \dfrac{\pi}{6} = \dfrac{4\pi}{24} \end{array}$$

It's a good idea to get the phase shift and the step size to have the same denominator, as you will add the step size to the phase shift in the graphing stage.
Graphing Stage: Graphing a number line (the x-axis), we plot the phase shift and, from there, mark the remaining key numbers using the step size.

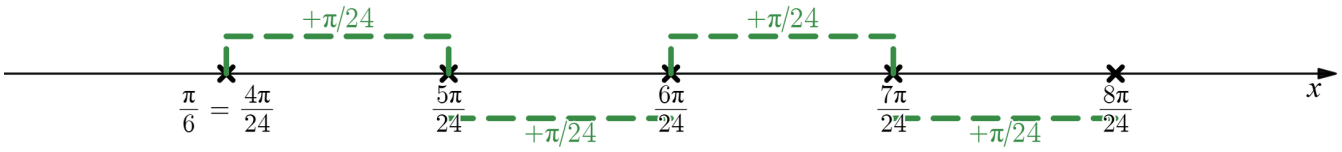


Figure 5.3.3

The distance between the initial key number and the final key number is

$$\left|\frac{8\pi}{24} - \frac{4\pi}{24}\right| = \frac{4\pi}{24} = \frac{\pi}{6} = \text{Period}$$

so we have done well so far. We now sketch a single cycle of the unreflected tangent function, drawing in the vertical asymptote halfway through the cycle.

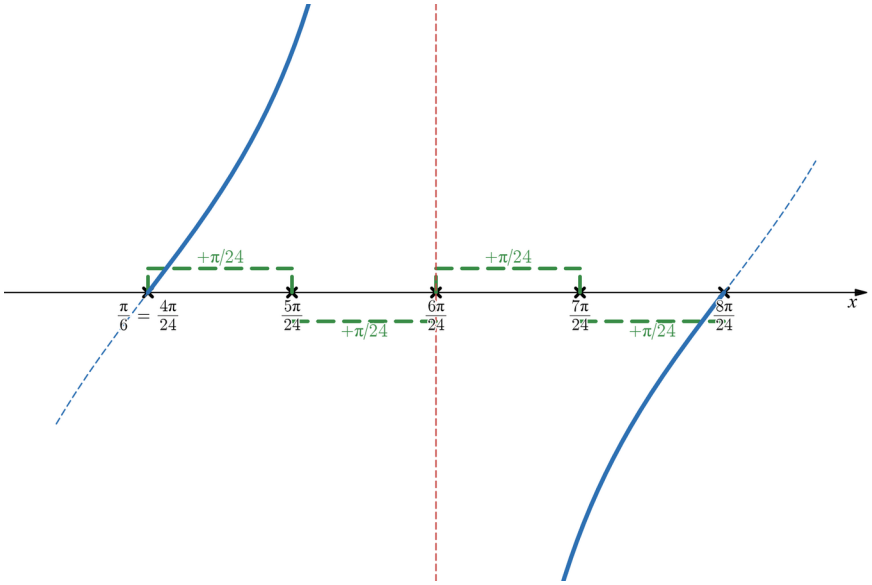


Figure 5.3.4

We can see that the initial key number is positive and, therefore, to the right of the y-axis. Consequently, we draw in the y-axis where we think it should be. Simultaneously, let's plot the points corresponding to the key numbers $\frac{5\pi}{24}$ and $\frac{7\pi}{24}$.

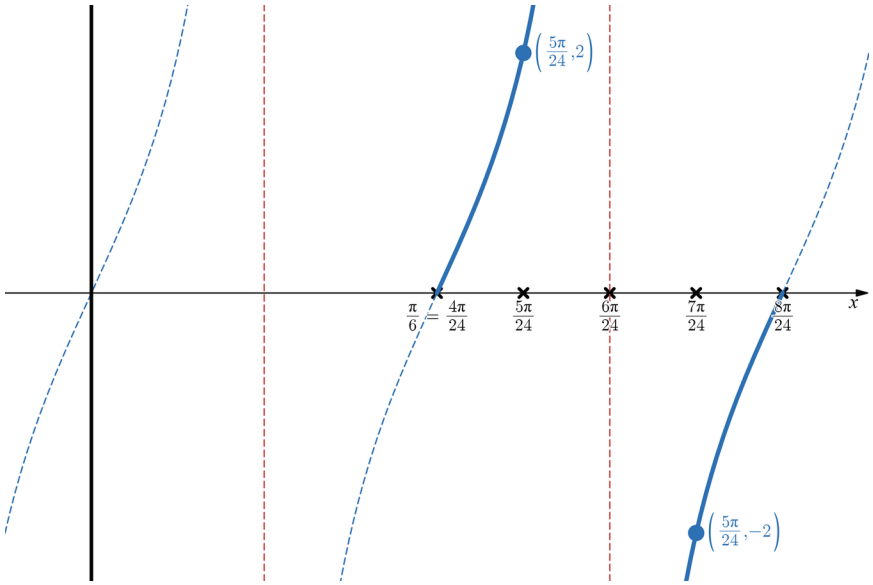


Figure 5.3.5

This completes one full cycle of the graph of the given function (although we decided to show some extra cycles, it's not a requirement).

b. **Precondition Stage:** Since $B = -\frac{\pi}{2} < 0$, we need to factor a negative from both terms in the argument of the function.

$$\begin{aligned} y &= \sqrt{5}\sin\left(-\frac{\pi}{2}x + \frac{\pi}{4}\right) \\ &= \sqrt{5}\sin\left(-\left(\frac{\pi}{2}x - \frac{\pi}{4}\right)\right) \\ &= -\sqrt{5}\sin\left(\frac{\pi}{2}x - \frac{\pi}{4}\right) \qquad \text{(sine is an odd function)} \end{aligned}$$

Moreover, if your instructor prefers the factored form, we could factor the $\frac{\pi}{2}$ from both terms in the argument to get

$$y = -\sqrt{5}\sin\left(\frac{\pi}{2}\left(x - \frac{1}{2}\right)\right).$$

Doing this reveals the phase shift to be $\frac{1}{2}$.

Information Stage: This is a sinusoidal function, so we will have an amplitude, and the natural period is 2π.

$$\begin{array}{rcl|rcl} A = -\sqrt{5} & \Longrightarrow & \text{Amplitude} & = & |A| = \left|-\sqrt{5}\right| = \sqrt{5} \\ B = \frac{\pi}{2} & \Longrightarrow & \text{Period} & = & \dfrac{\text{Natural Period}}{B} = \dfrac{2\pi}{\pi/2} = 4 \\ & \Longrightarrow & \text{Step Size} & = & \dfrac{\text{Period}}{4} = \dfrac{4}{4} = 1 \\ C = -\frac{\pi}{4} & \Longrightarrow & \text{Phase Shift} & = & -\dfrac{C}{B} = -\dfrac{-\pi/4}{\pi/2} = \dfrac{1}{2} \end{array}$$

Graphing Stage: It's important to remember that we are graphing the "cleaned" function

$$y = -\sqrt{5}\sin\left(\frac{\pi}{2}x - \frac{\pi}{4}\right).$$

Graphing the x-axis, plotting the phase shift, and marking the subsequent key numbers by increasing by the step size, we get the following:

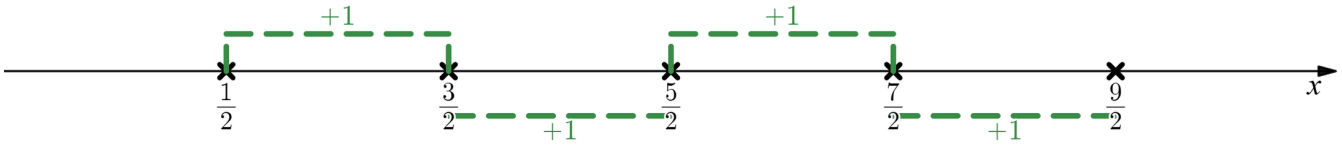


Figure 5.3.6

We then graph the *reflected* sine function.

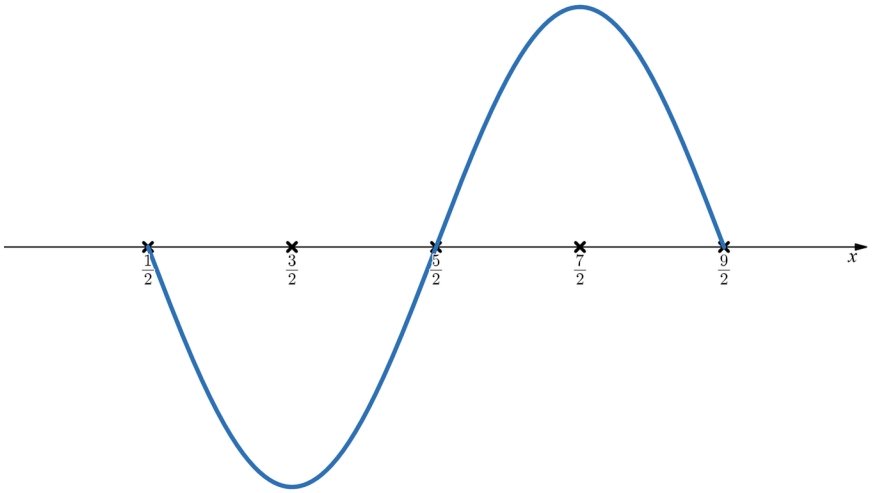


Figure 5.3.7

Finally, we place the y-axis appropriately, label it using the amplitude, and label the maximum and minimum points on the sine curve (this is something we didn't do before, but some instructors expect it, so we want to showcase good behavior).

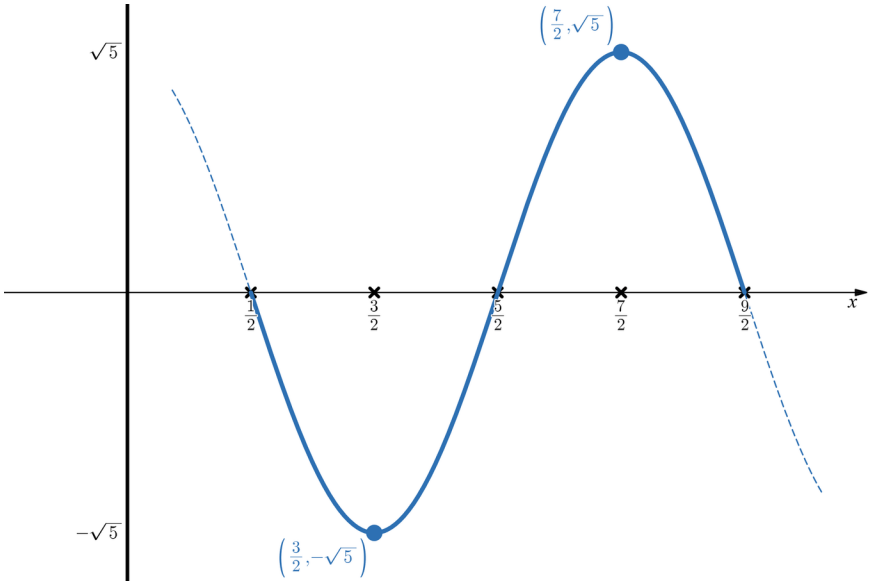


Figure 5.3.8

Checkpoint 5.3.3

Amplitude: 2
Period: π
Step Size: $\frac{\pi}{4}$
Phase Shift: $-\frac{\pi}{6}$

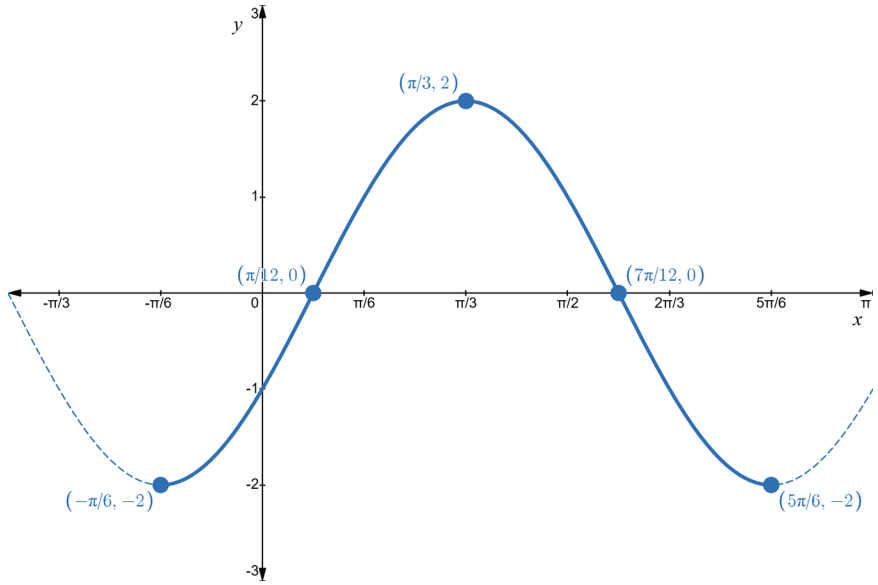


Figure 5.3.9

Example 5.3.4

a. **Precondition Stage:** Rewriting the given function into proper form, we get

$$y = 5\sin\left(3\pi x + \frac{\pi}{2}\right) - 5.$$

Information Stage:

A	$=$	5	$\Longrightarrow$	Amplitude	$=$	5	
B	$=$	3π	$\Longrightarrow$	Period	$=$	$\frac{\text{Natural Period}}{B} = \frac{2\pi}{3\pi} = \frac{2}{3}$	
			$\Longrightarrow$	Step Size	$=$	$\frac{\text{Period}}{4} = \frac{2/3}{4} = \frac{1}{6}$	
C	$=$	$\frac{\pi}{2}$	$\Longrightarrow$	Phase Shift	$=$	$-\frac{C}{B} = -\frac{\pi/2}{3\pi} = -\frac{1}{6}$	
D	$=$	-5	$\Longrightarrow$	Vertical Shift	$=$	-5	

That last line tells us the midline is $y = -5$.
Graphing Stage: We start by graphing a horizontal line and labeling it $y = -5$.



Figure 5.3.12

We then plot five equally-spaced points, labeling the first using the phase shift, and adding the step size to reach each subsequent point.

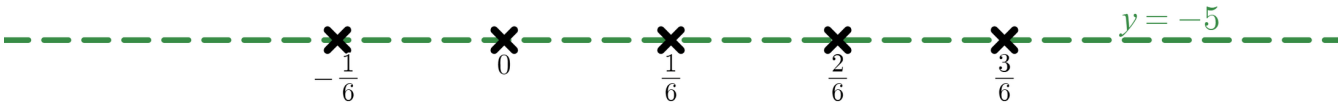


Figure 5.3.13

Double-checking the length of the interval, we see that

$$\left|\frac{3}{6} - \left(-\frac{1}{6}\right)\right| = \frac{4}{6} = \frac{2}{3} = \text{Period}.$$

So far, so good! The function we want to graph is an *unreflected* sine function, so let's use these key numbers to sketch.

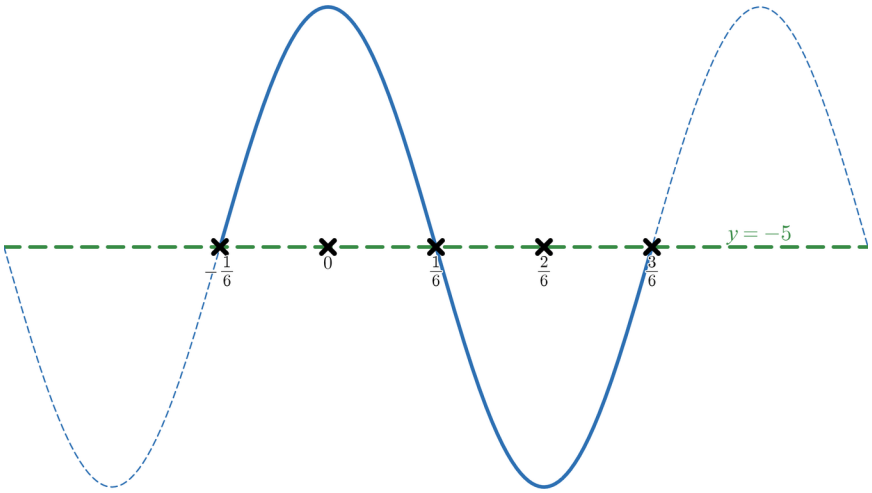


Figure 5.3.14

From here, we can see where $x = 0$ is, so we know where the y-axis goes.

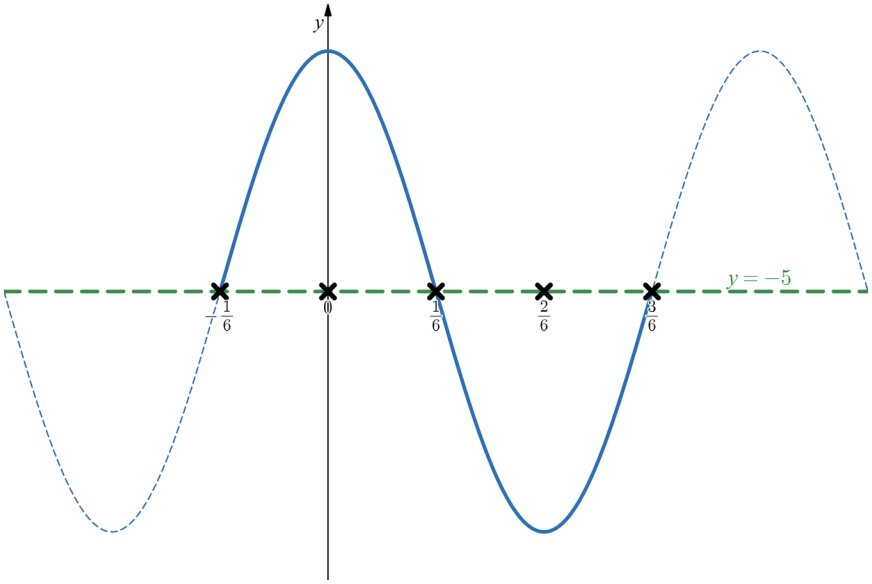


Figure 5.3.15

Since the midline is at $y = -5$ and the amplitude is 5, we can label the y-axis by adding and subtracting the amplitude from the midline.

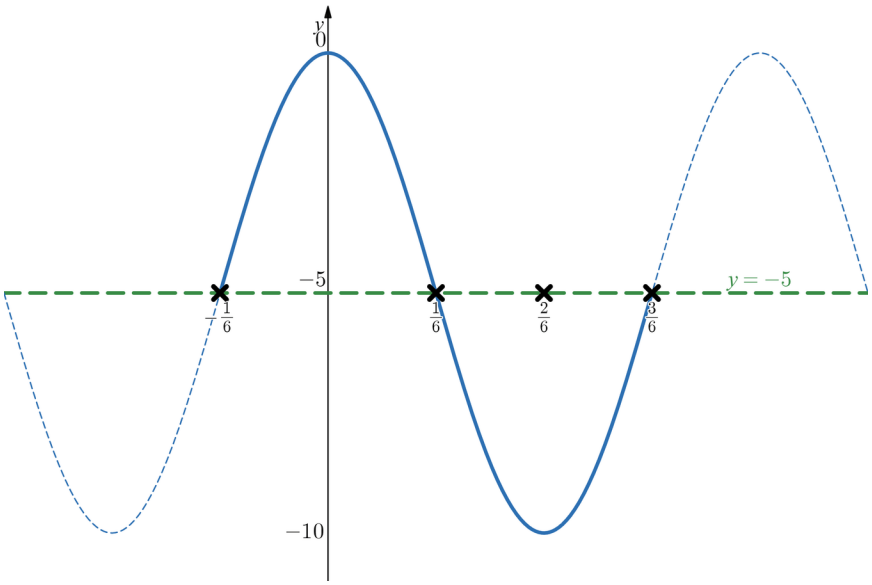


Figure 5.3.16

Finally, now that we see where $y = 0$, we can draw the x-axis.

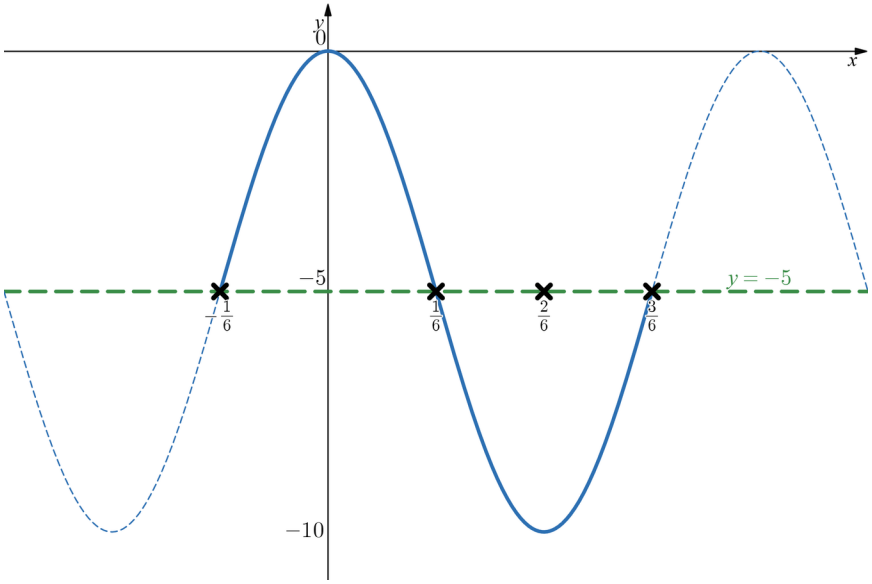


Figure 5.3.17

That is a complete cycle of the graph of $y = -5 + 5\sin\left(3\pi x + \frac{\pi}{2}\right)$.

b. **Precondition Stage:** There is nothing to precondition here.
Information Stage:

B	$=$	2	$\Longrightarrow$	Period	$=$	$\dfrac{\text{Natural Period}}{B} = \dfrac{\pi}{2}$
			$\Longrightarrow$	Step Size	$=$	$\dfrac{\text{Period}}{4} = \dfrac{\pi/2}{4} = \dfrac{\pi}{8}$
C	$=$	-1	$\Longrightarrow$	Phase Shift	$=$	$-\dfrac{C}{B} = -\dfrac{-1}{2} = \dfrac{1}{2}$
D	$=$	1	$\Longrightarrow$	Vertical Shift	$=$	1

This is a case where the phase shift ($\frac{1}{2}$) and the step size ($\frac{\pi}{8}$) are not easily added together. Please pay close attention to how we handle this during the graphing stage.
Graphing Stage: Graph the midline, plot five equally-spaced points, call the first point the phase shift, and label the remaining using the step size.

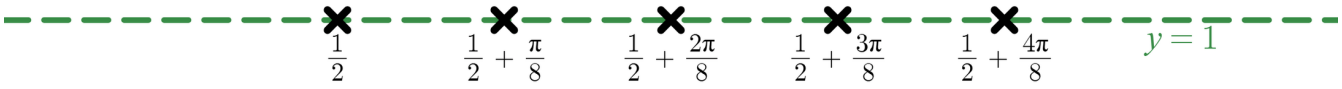


Figure 5.3.18

We now graph the *unreflected* tangent function and graph the vertical asymptote in the middle of the cycle. Moreover, we place the y-axis.

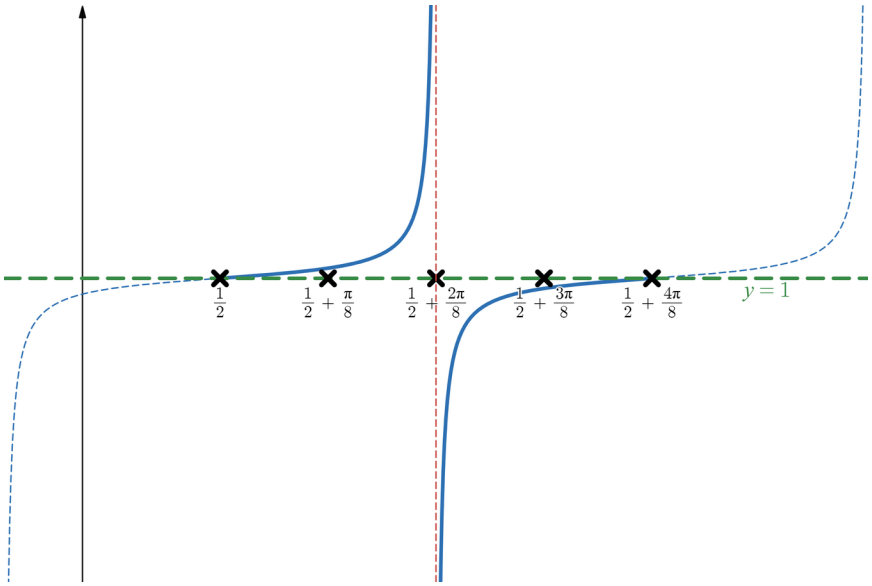


Figure 5.3.19

We can see that two of our key points are "labeled" ($(\frac{1}{2},1)$ and $(\frac{1}{2}+\frac{4\pi}{8},1)$) and we have the asymptote labeled ($x=\frac{1}{2}+\frac{2\pi}{8}$); however, we have two extra key numbers that need to be used. Despite the tangent function not having an amplitude, $|A|=\frac{1}{3}$ still serves a purpose. It tells us how far away from the midline the points corresponding to those unlabeled key numbers should be. That is, the y-value associated with $x=\frac{1}{2}+\frac{\pi}{8}$ is $1+\frac{1}{3}=\frac{4}{3}$. Likewise, the y-value for $x=\frac{1}{2}+\frac{3\pi}{8}$ is $1-\frac{1}{3}=\frac{2}{3}$. We label these points and draw in the x-axis (appropriately).

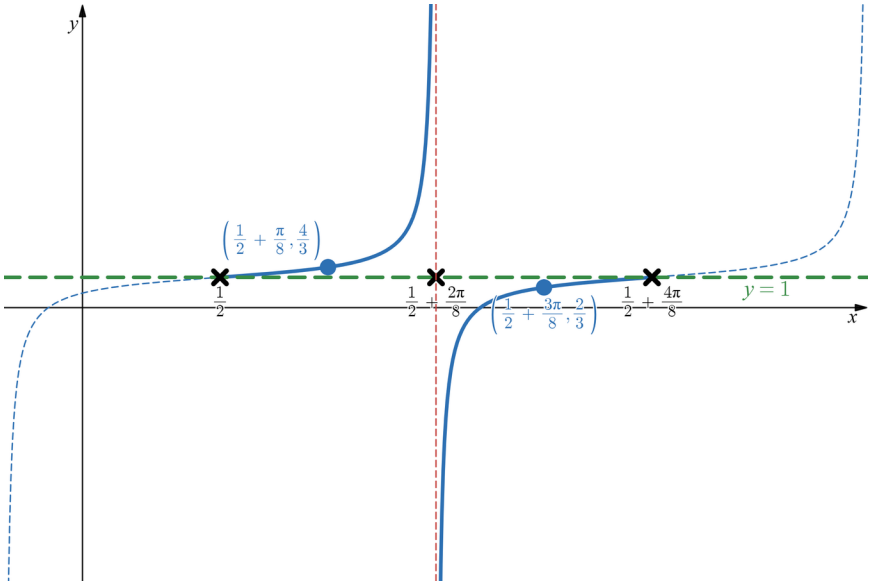


Figure 5.3.20

Checkpoint 5.3.4

Period: $\frac{\pi}{2}$
Step Size: $\frac{\pi}{8}$
Phase Shift: $\frac{\pi}{4}$
Vertical Shift: 3

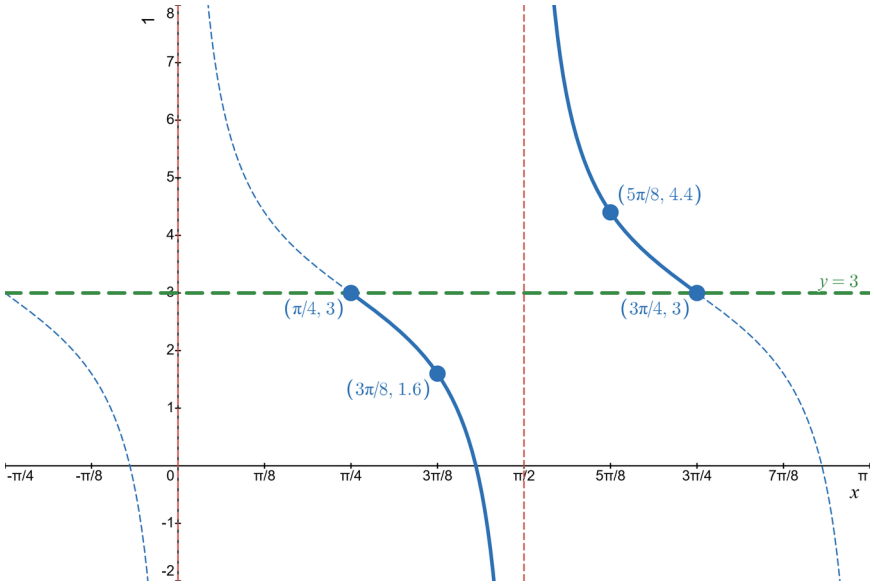


Figure 5.3.21

Example 5.3.5

a. When rebuilding an equation from a graph, it's best to figure out the information using the order with which we graph. That is, try to identify the vertical shift, phase shift, period (which might require finding the step size), and amplitude (if applicable). This is more of a "rule of thumb," so there might be times you do things out of order.

Vertical Shift: Since the maximum and minimum of this sinusoidal function are $-\frac{1}{2}$ and $-\frac{3}{2}$, respectively, the midline must sit directly between these two values. That is, the vertical shift is the *average* of these.

$$\text{Vertical Shift} = D = \frac{-\frac{1}{2} + \left(-\frac{3}{2}\right)}{2} = -1.$$

Phase Shift: Given the graph of a sinusoidal function and asked to find the equation, you have *a lot* of options. For example, the given graph could be interpreted as a sine function shifted slightly to the left (or slightly to the right and reflected about $y = -1$); however, try to use the given information to make the process easier. We will assume the leftmost point we were given, $\left(-\frac{5\pi}{18}, -\frac{3}{2}\right)$, is the phase shift. Thus,

$$\text{Phase Shift} = -\frac{C}{B} = -\frac{5\pi}{18}. \tag{5.3.2}$$

While this does not immediately give us B or C, it does tell us that we are dealing with a reflected cosine function. We will use Equation 5.3.2 in a moment.

Period or Step Size: Assuming the cycle begins at $x = -\frac{5\pi}{18}$, we can see that the curve is *halfway* through a full cycle by $x = \frac{\pi}{18}$. That is, half the period is $\left|-\frac{5\pi}{18} - \frac{\pi}{18}\right| = \frac{6\pi}{18} = \frac{\pi}{3}$. Therefore, the full period is $\frac{2\pi}{3}$, and so

$$\text{Period} = \frac{2\pi}{B} = \frac{2\pi}{3} \implies B = 3.$$

We can now go back to Equation 5.3.2 and let $B = 3$ to get

$$-\frac{C}{3} = -\frac{5\pi}{18} \implies C = \frac{5\pi}{6}.$$

Amplitude: The amplitude is the distance from the midline to the maximum (or minimum). In this case, the amplitude is

$$\text{Amplitude} = |A| = \frac{1}{2}.$$

Putting this altogether, and remembering that the function is a *reflected* cosine function, we get

$$f(x) = -\frac{1}{2}\cos\left(3x + \frac{5\pi}{6}\right) - 1.$$

It's *crucial* for me to state that there are an infinite number of correct answers when it comes to stating the equation of a trigonometric function. If checking your work against a friend's work and if your friend has a different function, you should graph both equations to see if they are equivalent.

b. **Vertical Shift:** If you imagine a horizontal midline for this tangent-like curve, it would be at $y = 1$. Thus,

$$D = 1.$$

Phase Shift: Remember that the principal cycle of the tangent function starts at the midline (x-axis). Thus, we will assume the first point, $(1, 1)$, is where the cycle of this shifted function starts. This means the function is a reflected tangent function and

$$\text{Phase Shift} = -\frac{C}{B} = 1. \tag{5.3.3}$$

Period or Step Size: The tangent is a nice function to deal with sometimes because we can see the period from the vertical asymptotes. The distance between these matches the period of the given tangent function. Therefore,

$$\text{Period} = \frac{\pi}{B} = |2.5 - (-0.5)| = 3 \implies B = \frac{\pi}{3}.$$

Plugging this back into Equation 5.3.3, we get

$$-\frac{C}{\pi/3} = 1 \implies C = -\frac{\pi}{3}.$$

Amplitude: Since we are dealing with a tangent function, there is no amplitude, but we still need to determine the coefficient of the tangent function. Luckily, we were given a second point, $(\frac{7}{4}, -1)$. So far we have

$$g(x) = A\tan(Bx + C) + D = A\tan\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) + 1.$$

Letting the output be -1 and $x = \frac{7}{4}$, we get

$$\begin{aligned} -1 &= A\tan\left(\frac{7\pi}{12} - \frac{\pi}{3}\right) + 1 \\ \implies -2 &= A\tan\left(\frac{7\pi}{12} - \frac{4\pi}{12}\right) \\ \implies -2 &= A\tan\left(\frac{3\pi}{12}\right) \\ \implies -2 &= A\tan\left(\frac{\pi}{4}\right) \\ \implies -2 &= A(1) \\ \implies -2 &= A \end{aligned}$$

Hence,

$$g(x) = -2\tan\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) + 1.$$

Checkpoint 5.3.5

$y = 3\sin(2x + \frac{\pi}{4}) - 1$

Example 5.3.6

a. We begin by rewriting the function as

$$D(t) = -5.25\cos\left(\frac{\pi}{6}t\right) + 12.25.$$

The amplitude is $A = 5.25$, the period is $\frac{2\pi}{\pi/6} = 12$. Therefore, the step size is 3. Since $C = 0$, there is no phase shift. Finally, the vertical shift is 12.25. Graphing the reflected cosine starting at $t = 0$, stepping by 3, with an amplitude of 5.25, we get the following graph.

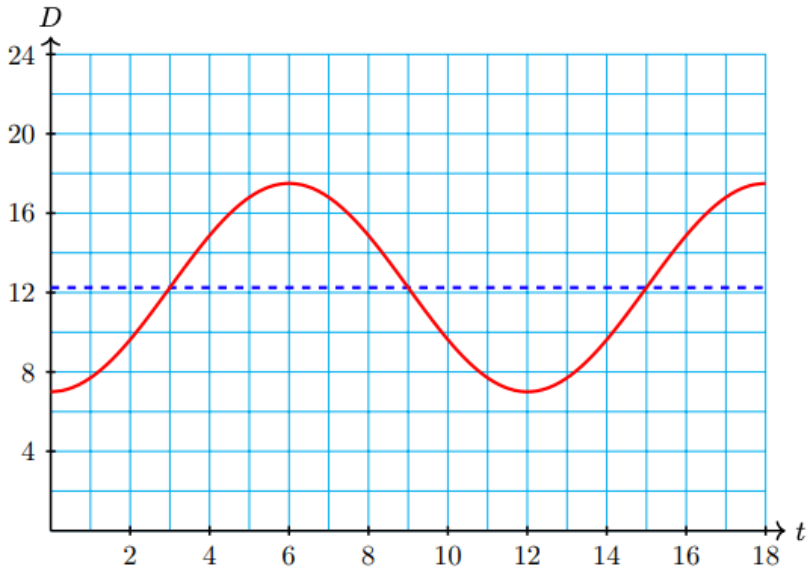


Figure 5.3.26

b. The midline of the graph is $D = 12.25$ hours, the amplitude is 5.25 hours, and the period is 12 months.
c. On the longest day of the year, there are $12.25 + 5.25 = 17.5$ hours of daylight in Glasgow.
d. From the graph, we see that $D(t) < 8$ for t between 0 and 2, or in January and February. $D(t)$ is also less than 8 for a short time between $t = 11$ and $t = 12$ at the end of December.

Checkpoint 5.3.6

a.

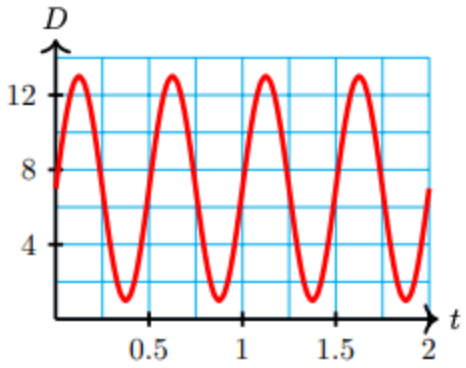


Figure 5.3.27

b. Midline: $D = 7$, amplitude: 6, period: 0.5 millisec
c. Largest: 13 cm, smallest: 1 cm

Example 5.3.7

The graph has the shape of $y = -\cos(x)$. It appears to have midline $y = 75$ and amplitude 75, so $|A| = 75$ and $A = -75$. To find B, we compute

$$\text{Period} = \frac{2\pi}{B} \implies B = \frac{2\pi}{\text{Period}} = \frac{2\pi}{10.8} \approx 0.58.$$

Thus, a formula for the model is

$$y = -75\cos(0.58x + C) + 75.$$

If we are trying to fit the red curve, the phase shift would be 0 and so $C = 0$. Hence,

$$y = -75\cos(0.58x) + 75.$$

Checkpoint 5.3.7

a. Period: 10 years, midline: $y = 61.5$, amplitude: 53
b. $s(t) = 61.5 - 53\cos\left(\frac{\pi}{5}t - \frac{\pi}{20}\right)$
c. ≈ 18.6

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5.3.1: Resources and Key Concepts

Resources

Videos

- Graphing Trigonometric Functions
  - Graphs of the Other Trigonometric Functions - Graphing Tangent and Cotangent Functions
- Sinusoidal Models
  - Trigonometric Graphs - Technology and Graphs of Trigonometric Functions (Part 1)
  - Graphs of the Other Trigonometric Functions - An Application Involving the Tangent Function
- Finding Equations from Graphs
  - Trigonometric Graphs - Review of Identifying the Equation for a Sinusoidal Function from a Graph

Key Concepts

Definitions

- **Phase Shift:** The horizontal shift of a trigonometric function. It represents the x-value to which the starting point of the principal cycle (originally at $x = 0$) has been moved.
- **Midline:** A horizontal line that represents the average value of a trigonometric function. For sinusoidal functions, it is the line situated halfway between the maximum and minimum points.

Theorems

- **Lemma: Phase Shift Formula for y = trig(x + C):** The phase shift for a trigonometric function of the form $y = \text{trig}(x + C)$ is $-C$.
- **Phase Shift Formula:** The phase shift of the function $y = A\text{trig}(Bx + C)$ is $-\frac{C}{B}$.
- **Cofunction Identities (radians):** The trigonometric function of an angle is equal to the cofunction evaluated at the complement of the angle. For example, $\sin(\theta) = \cos(\frac{\pi}{2} - \theta)$.

Common Mistakes

- **Confusing C with the Phase Shift:** In the function $y = A\text{trig}(Bx + C)$, the phase shift is $-\frac{C}{B}$, not simply $-C$. The value of B affects the horizontal shift.
- **Ignoring Negative B-values:** Before determining the period or phase shift, if the parameter B is negative, you must first use the Symmetry Identities to make the coefficient of x positive. Forgetting this step will result in an incorrect phase shift.
- **Incorrectly Identifying the Midline:** The midline is the average of the maximum and minimum values, not simply the lowest point of the graph. It is the horizontal line the graph oscillates around.

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5.3.2: Homework

Reading Questions

1. What is a phase shift and what does it represent on a graph?
2. What is the midline of a trigonometric function?
3. For the general function $f(x) = A\text{trig}(Bx + C) + D$, which parameter determines the midline?
4. What is the formula for the phase shift of the function $y = A\sin(Bx + C)$?
5. How can you find the midline of a sinusoidal graph if you know its maximum and minimum values?
6. When graphing a function like $y = \sin(x + \frac{\pi}{3})$, does the graph shift to the left or to the right?
7. Before finding the phase shift for a function like $y = \cos(-2x + \pi)$, what is the necessary first step?
8. The text shows that the graph of $y = \cos(\frac{\pi}{2} - x)$ is identical to the graph of $y = \sin(x)$. What transformation of the cosine graph produces the sine graph?
9. When finding the equation of a sinusoidal function from its graph, why is it helpful to identify the vertical shift first?
10. When finding the period of a tangent function from its graph, what graphical feature can you measure the distance between?

Homework

Vocabulary Check

1. The ___ tells us the x-value of where the original initial number in the principal cycle has moved.
2. Given the function $f(x) = A\ \text{trig}\,(Bx + C) + D$, the graph of the equation $y = D$ is called the ___ of the trigonometric function.
3. Given the function $f(x) = A\ \text{trig}\,(Bx + C) + D$, D represents the ___.

Concept Check

4. Write an equation for a cosine function with midline -5.
5. Write an equation for a sine function with midline 2.
6. Which of the following functions are the same? Explain your answer.
$$f(x) = \cos\left(2x - \frac{\pi}{3}\right), \quad g(x) = (\cos 2x) - \frac{\pi}{3}, \quad h(x) = \cos 2x - \cos\frac{\pi}{3}$$
7. Which of the following functions are the same? Explain your answer.
$$f(x) = \sin\left(2x - \frac{\pi}{3}\right), \quad g(x) = \sin\left[2\left(x - \frac{\pi}{6}\right)\right], \quad h(x) = 2\sin\left(x - \frac{\pi}{6}\right)$$

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

8. The phase shift is also known as the horizontal shift.
9. The amplitude of $f(x) = -3\tan(2x - 1)$ is 3.
10. The phase shift of the function $f(x) = -3\tan(2x - 1)$ is 1.
11. The phase shift of the function $f(x) = -3\tan(2x - 1)$ is -1.
12. The functions $y = 2\cos(\pi x - 4)$ and $y = -3\sin\left(\pi\left(x - \frac{4}{\pi}\right)\right)$ have the same phase shift.

Basic Skills

For the following exercises, calculate the phase shift.

13. $y = -2\sin\left(3x - \frac{\pi}{4}\right)$

14. $y = 6\cos\left(2x + \frac{3\pi}{4}\right)$

15. $y = 12\cos\left(\frac{t}{3} + 0.8\right)$

16. $y = -5\sin\left(\frac{t}{4} - 0.6\right)$

For the following exercises, state the amplitude, period, and midline of the graph.

17. $y = -3 + 2\sin(x)$

18. $y = 4 - 3\cos(x)$

19. $y = 1 - \cos(\pi x)$

20. $y = 2 + \sin(2\pi x)$

For the following exercises,
(a) Sketch the graphs of f and g on the same axes.
(b) State the phase shift from f to g.
(c) Find all values of x for which $g(x) = 1$ for $-\pi \le x \le \pi$.
(d) Find all values of x for which $g(x) = 0$ for $-\pi \le x \le \pi$.

21. $f(x) = \sin(x)$ and $g(x) = \sin\left(x - \frac{\pi}{3}\right)$.

22. $f(x) = \cos(x)$ and $g(x) = \cos\left(x + \frac{\pi}{3}\right)$.

23. $f(x) = \tan(x)$ and $g(x) = \tan\left(x + \frac{\pi}{4}\right)$.

24. $f(x) = \tan(x)$ and $g(x) = \tan\left(x - \frac{\pi}{2}\right)$.

For the following exercises,
(a) What are the amplitude and phase shift of the given function?
(b) Sketch the graph.
(c) Find all values of x for which $f(x) = 0$ for $0 \le x \le 2\pi$.

25. $f(x) = -2\cos\left(x + \frac{\pi}{6}\right)$

26. $f(x) = -6\sin\left(x - \frac{2\pi}{3}\right)$

27. $f(x) = \frac{1}{2}\tan\left(x + \frac{\pi}{5}\right)$

28. $f(x) = -\frac{3}{7}\cos\left(x - \frac{\pi}{4}\right)$

For the following exercises,
(a) What are the period and phase shift of the given function?
(b) Sketch the graph.
(c) Find all values of x for which $f(x) = 1$ for $0 \le x \le 2\pi$.
(d) Find all values of x for which $f(x) = 0$ for $0 \le x \le 2\pi$.

29. $f(x) = \cos\left(2x - \frac{\pi}{3}\right)$

30. $f(x) = \sin\left(3x + \frac{\pi}{2}\right)$

31. $f(x) = \sin\left(\pi x + \frac{\pi}{3}\right)$

32. $f(x) = \cos\left(\pi x - \frac{\pi}{3}\right)$

For the following exercises,
(a) What are the midline, period, phase shift, and amplitude of the given function?
(b) Sketch the graph.
(c) Find all values of x for which $f(x) = 1$ for $0 \le x \le 2\pi$.
(d) Find all values of x for which $f(x) = k$ for $0 \le x \le 2\pi$, where k is stated within the problem.

33. $y = 3\sin\left(\frac{x}{2} - \frac{\pi}{6}\right) + 4,\ k = 4$

34. $y = 2\cos\left(\frac{x}{3} - \frac{\pi}{4}\right) - 1,\ k = -1$

For the following exercises,
(a) State the amplitude.
(b) State the period.
(c) State the step size.
(d) State the phase shift.
(e) State the vertical shift.
(f) Use this information to create an accurate graph of a single cycle of the trigonometric function. Be sure to label any asymptotes and all key points.

35. $y = 2 - 5\cos(2t)$

36. $y = -2 + 4\sin(3t)$

37. $y = 1 + 3\cos\left(\frac{t}{2}\right)$

38. $y = -2 - 3\sin\left(\frac{t}{4}\right)$

39. $y = 4 + 2\tan(3x)$

40. $y = 3 - \tan\left(\frac{x}{4}\right)$

41. $y = 1 - 2\tan\left(\frac{x}{3}\right)$

42. $y = 2\sin(2x - \pi) + 1$

43. $y = -1 + 3\cos(2x + \pi)$

44. $y = -2 - \tan\left(3x - \frac{\pi}{2}\right)$

45. $y = \frac{3}{2} - \frac{1}{2}\cos(3x + \pi)$

46. $y = -\frac{1}{3}\sin\left(\pi x + \frac{3\pi}{2}\right) + 2$

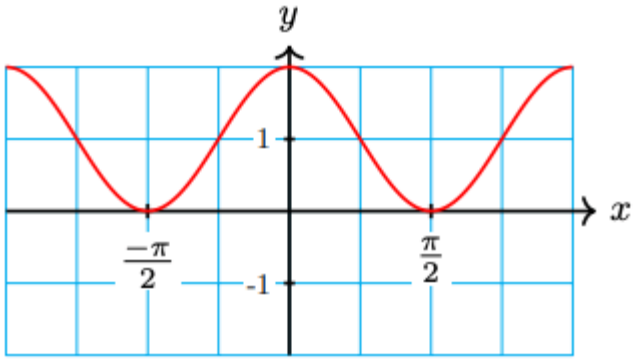
47. $y = -3\tan(2x - 1) + 1$

48. $y = -4\cos(\pi x - 3) - 4$

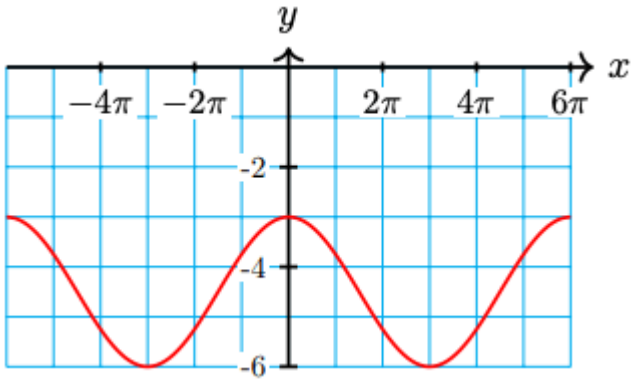
For the following exercises, state the equation of the function.

49.

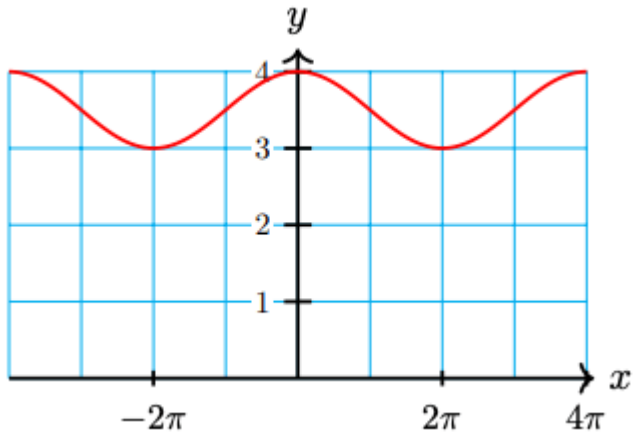
50.


y
1
x
$\frac{-\pi}{2}$
$\frac{\pi}{2}$
-1

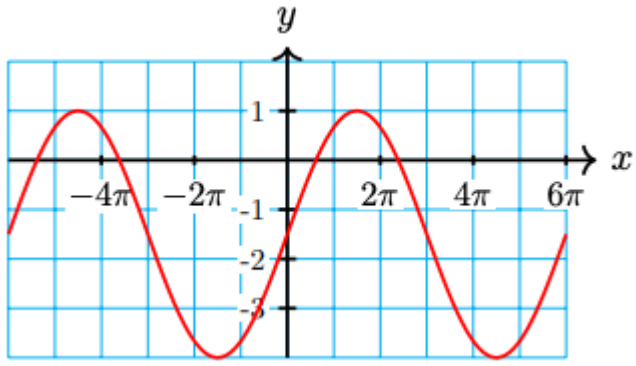
51.


y
x
-4π
-2π
2π
4π
6π
-2
-4
-6

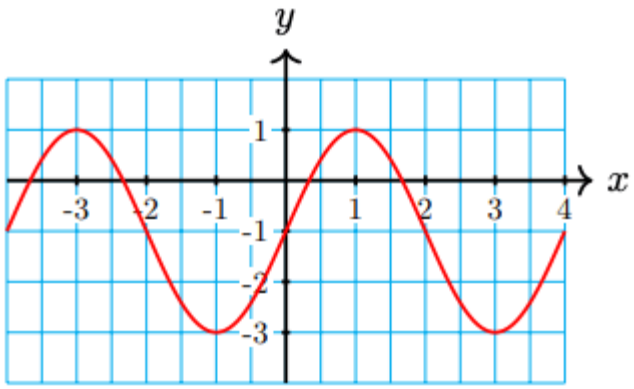
52.


y
4
3
2
1
x
-2π
2π
4π

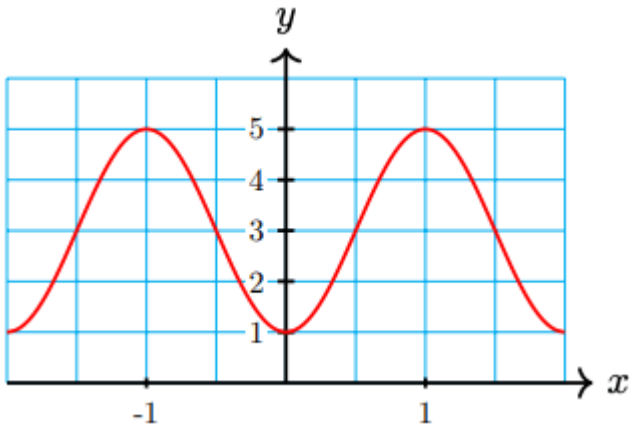
53.


y
1
x
-4π
-2π
-1
2π
4π
6π
-2
-3

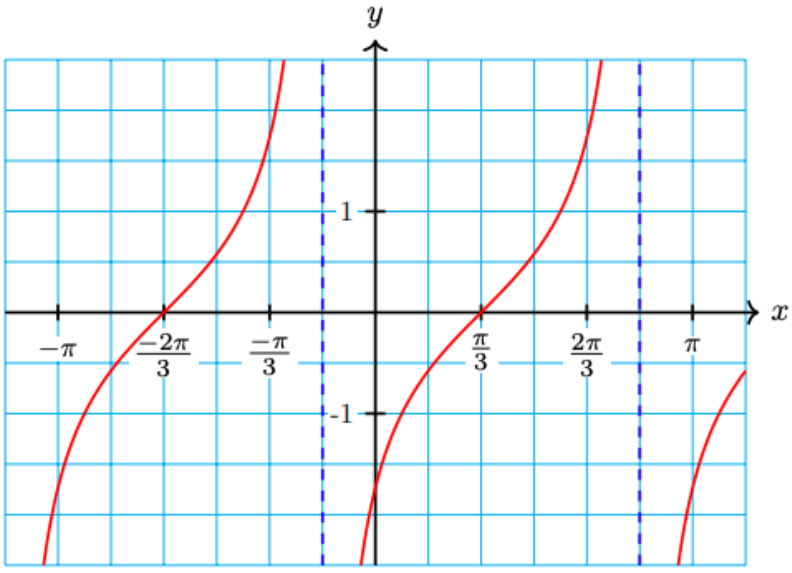
54.

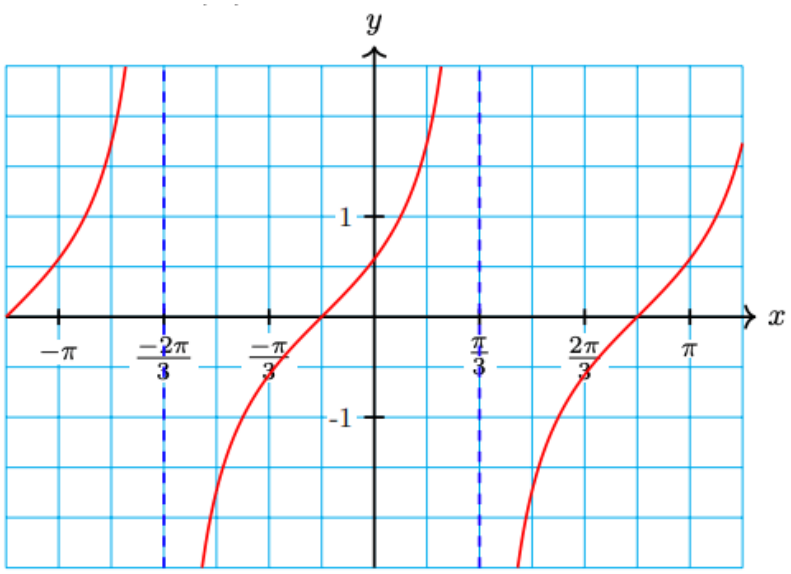

y
x
1
-1
-2
-3
-3
-2
-1
1
2
3
4

55.


y
x
5
4
3
2
1
-1
1

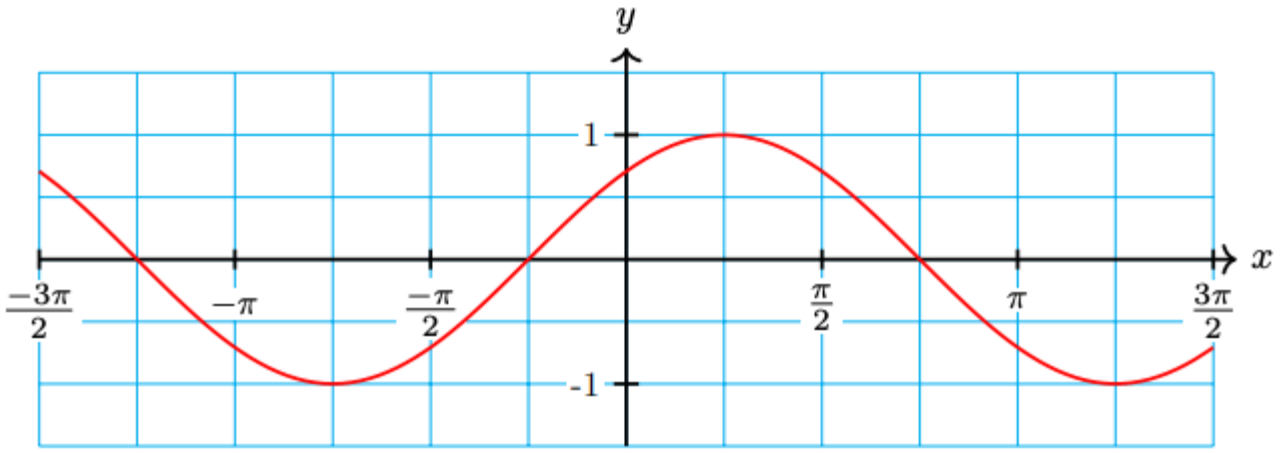
56.


y
x
1
-1
$-\pi$
$\frac{-2\pi}{3}$
$\frac{-\pi}{3}$
$\frac{\pi}{3}$
$\frac{2\pi}{3}$
π

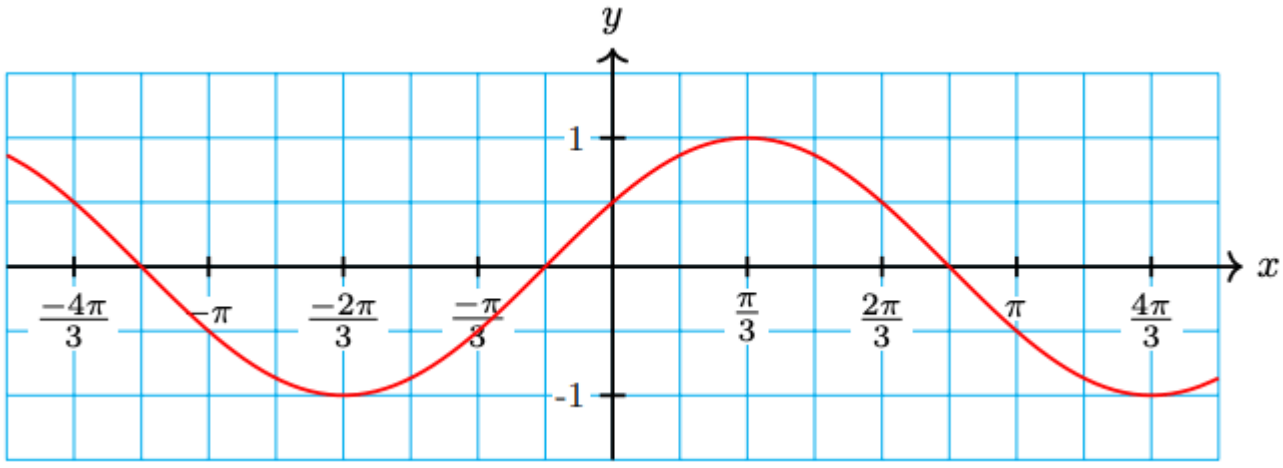


For the following exercises, find a formula for the graph of the given function in terms of
(a) sine
(b) cosine

57.

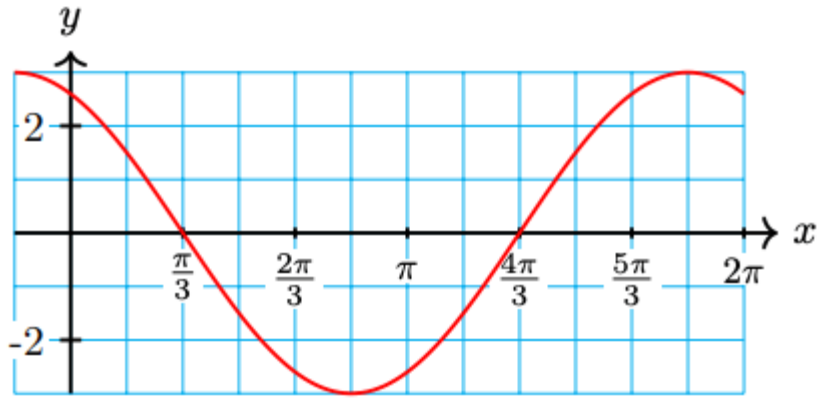


58.

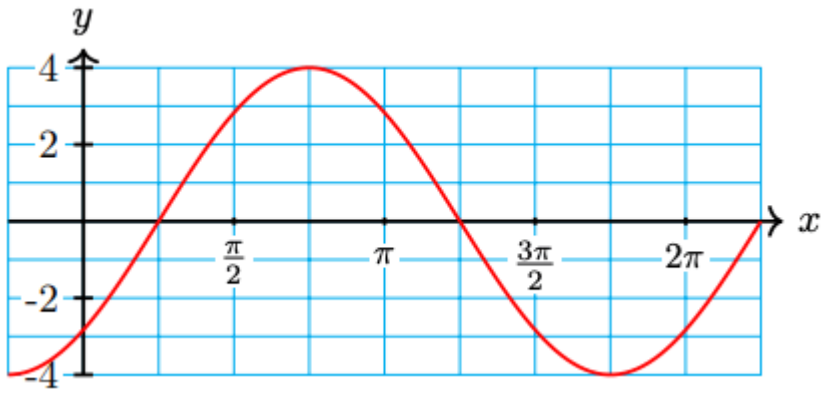


59.

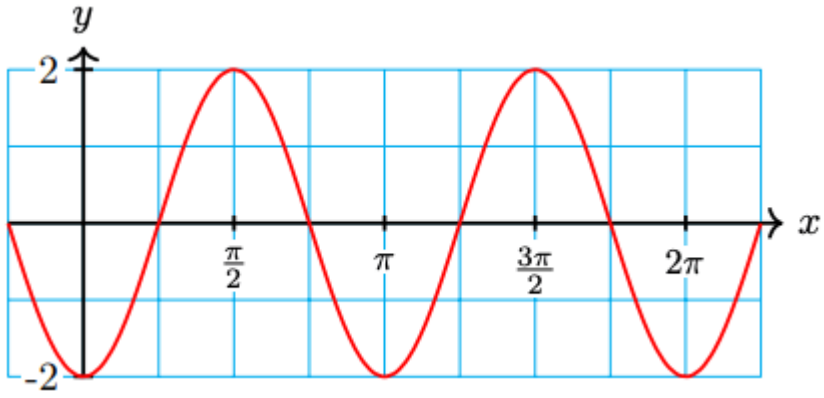
60.

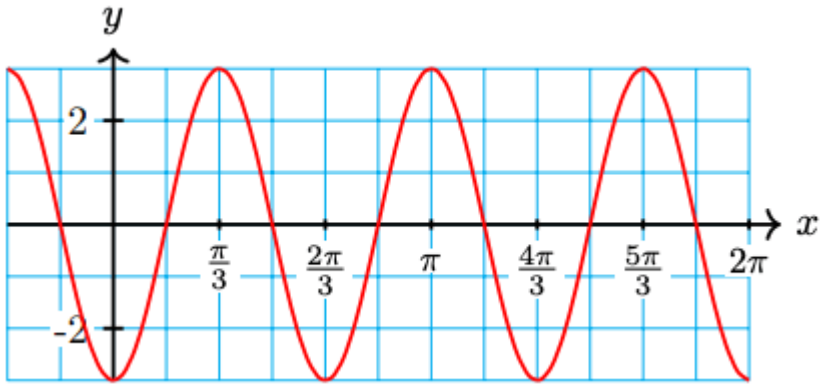

y
2
-2
x
$\frac{\pi}{3}$
$\frac{2\pi}{3}$
π
$\frac{4\pi}{3}$
$\frac{5\pi}{3}$
2π

61.

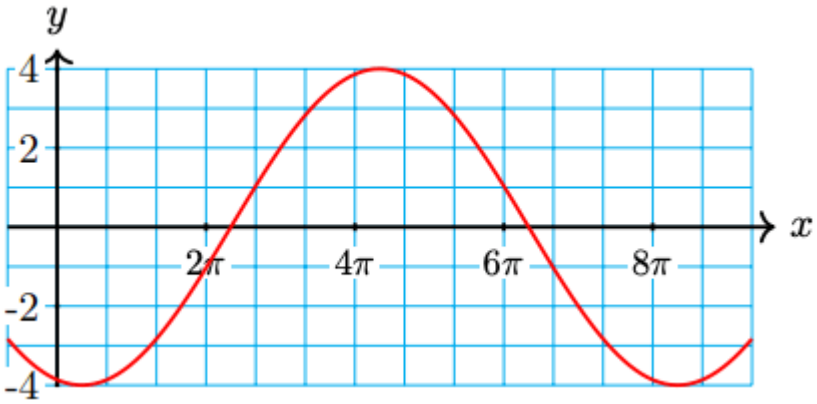

y
4
2
-2
-4
x
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

62.

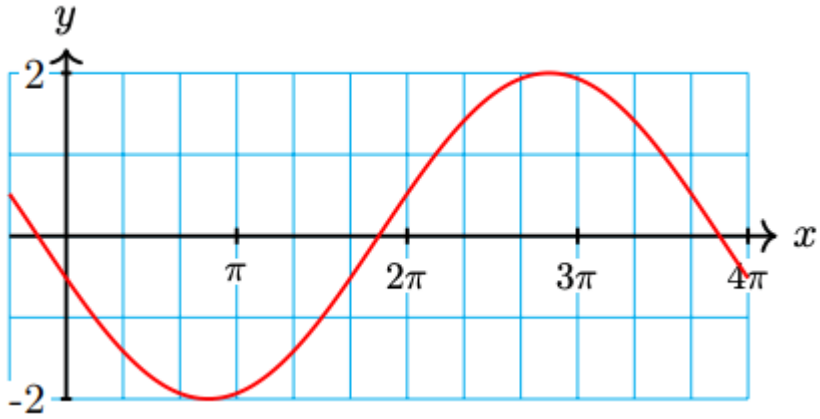

y
2
-2
x
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π


y
2
-2
x
$\frac{\pi}{3}$
$\frac{2\pi}{3}$
π
$\frac{4\pi}{3}$
$\frac{5\pi}{3}$
2π

53.



54.



For the following exercises, find and graph a single cycle of a sinusoidal function with the given properties.

55. Amplitude of 2, a period of 3, and is shifted 4 units to the left and 5 units upwards compared with the sine function.

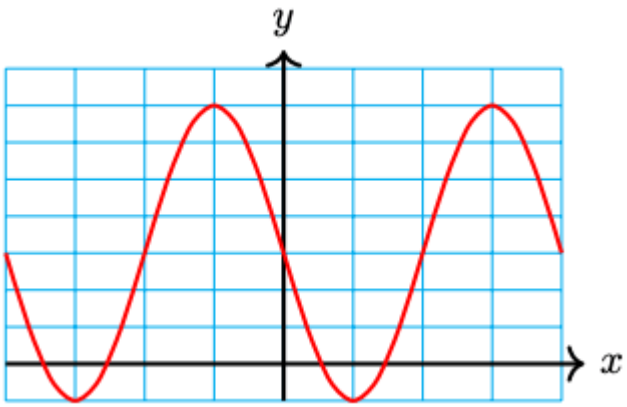
56. Amplitude of 3, a period of 24, and is shifted 2 units to the right and 4 units upwards compared with the cosine function.

57. Amplitude of 5, a period of 360, its midline at $y = 12$, and passes through (0,7).

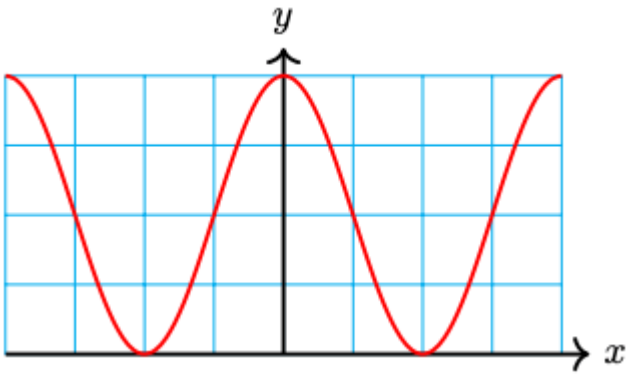
58. Amplitude of 50, a period of 30, its midline at $y = 50$, and passes through (0,100).

For the following exercises, label the scales on the axes for the graph.

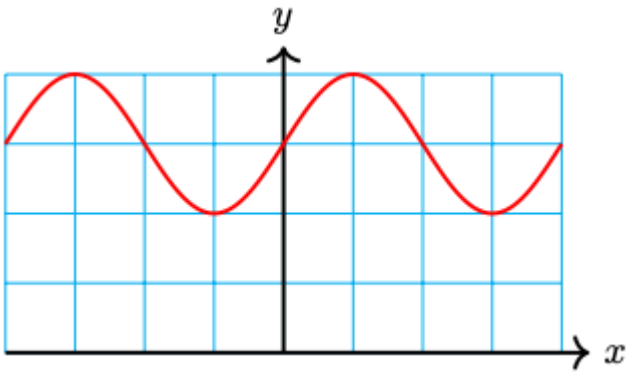
59. $y = 3 - 4\sin(2x)$



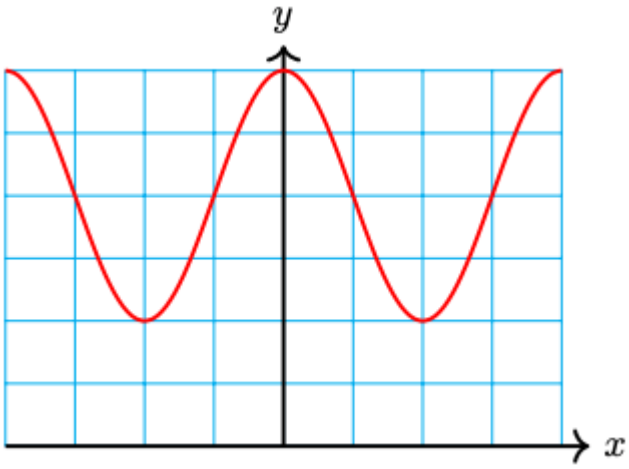
70. $y = 2\cos(5x) + 2$



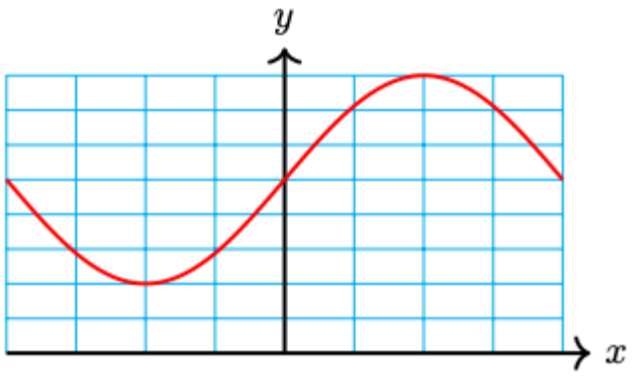
71. $y = \frac{1}{2}\sin(3x) + \frac{3}{2}$



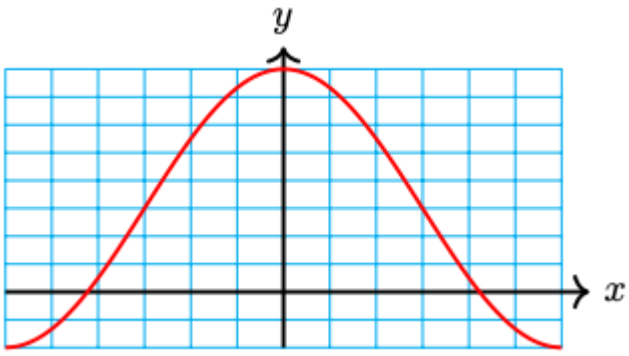
72. $y = \frac{2}{5}\cos(6x) + \frac{4}{5}$



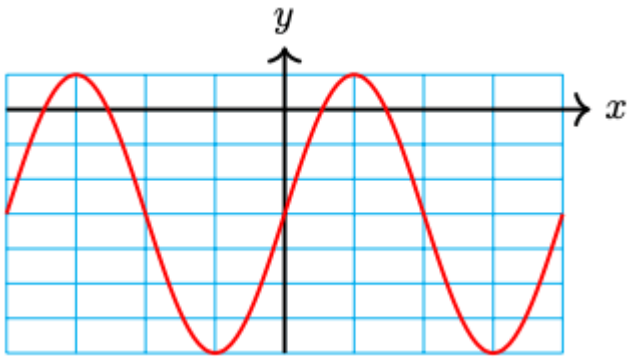
73. $50 - 30\sin\left(\frac{x}{4}\right)$



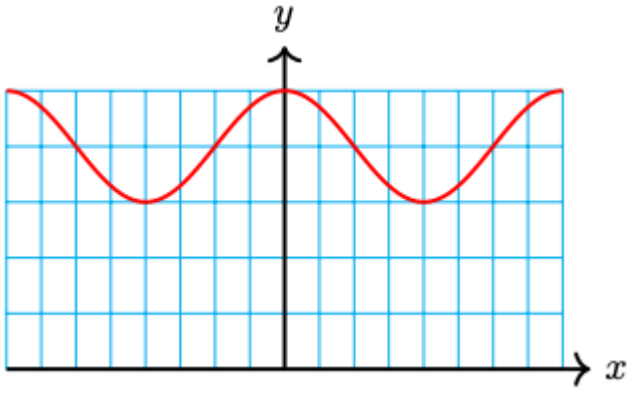
74. $25\cos\left(\frac{x}{3}\right)+15$



75. $y=4\sin(\pi x)-3$



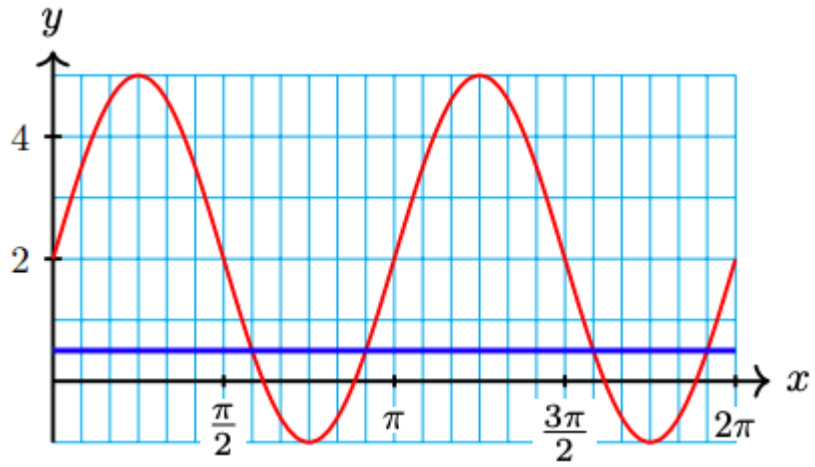
76. $y=\frac{1}{2}\cos\left(\frac{\pi x}{2}\right)+2$



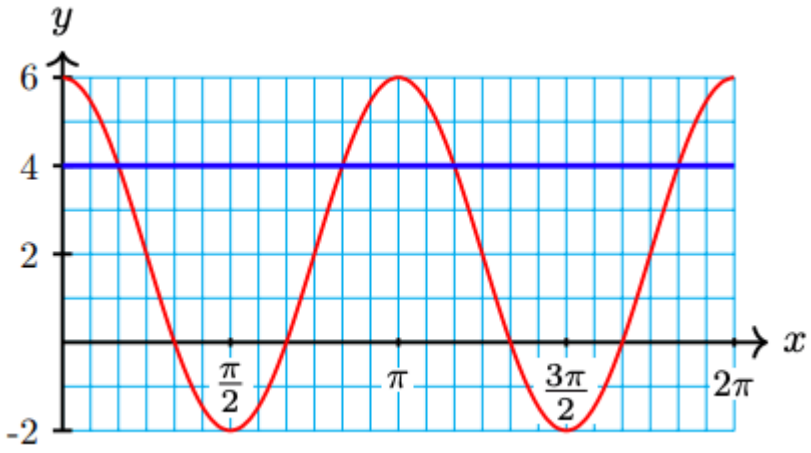
Synthesis Questions

For the following exercises, use the graph to find all solutions between 0 and 2π.

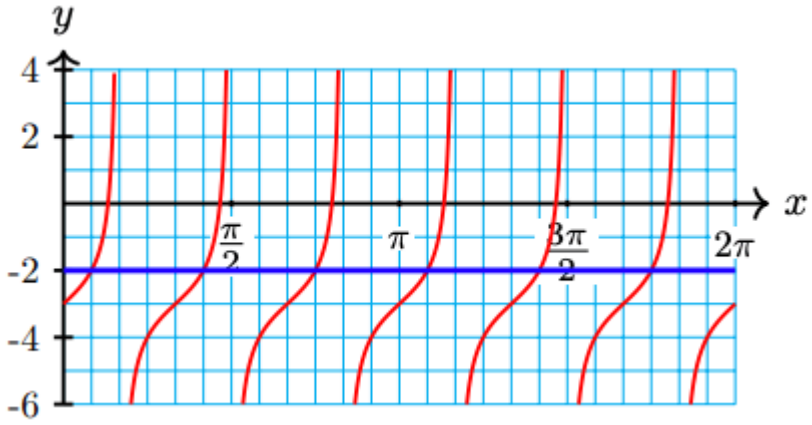
77. $2 + 3\sin(2x) = 0.5$



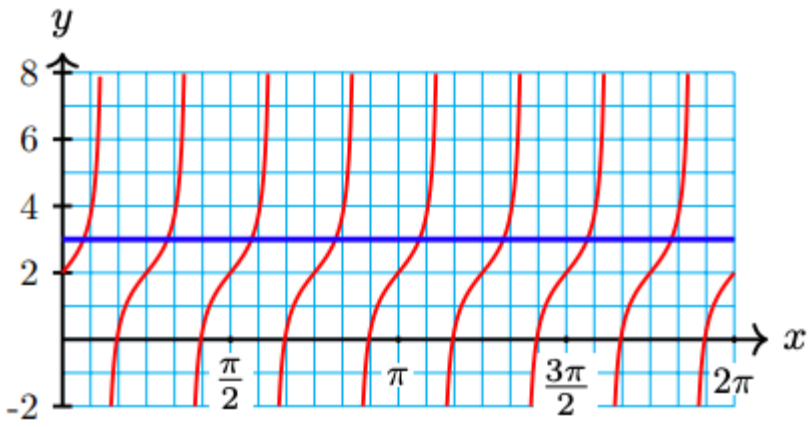
78. $2 + 4\cos(2x) = 4$



79. $-3 + \tan(3x) = -2$



80. $2 + \tan(4x) = 3$



For the following exercises,
(i) Use a calculator to graph the function for $0 \le x \le 2\pi$.
(ii) Use the intersect feature to find all solutions between 0 and 2π. Round your answers to hundredths.

81. a. $h(x) = 2 - 4\cos\left(\frac{x}{4}\right)$

    b. $2 - 4\cos\left(\frac{x}{4}\right) = 0$

82. a. $H(x) = 3 + 2\sin\left(\frac{x}{2}\right)$

    b. $3 + 2\sin\left(\frac{x}{2}\right) = 5$

83. a. $G(x) = -1 + 3\cos(3x)$

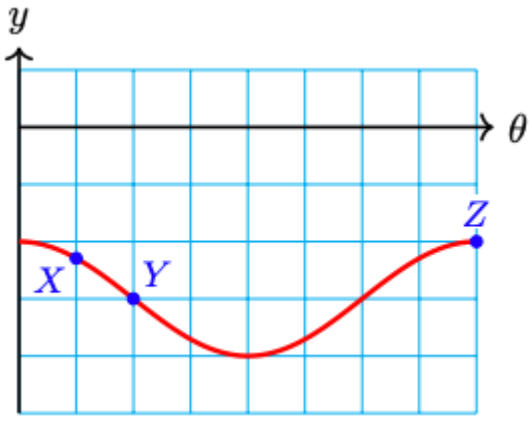
    b. $-1 + 3\cos(3x) = 1$

84. a. $F(x) = 4 - 3\sin(2x)$

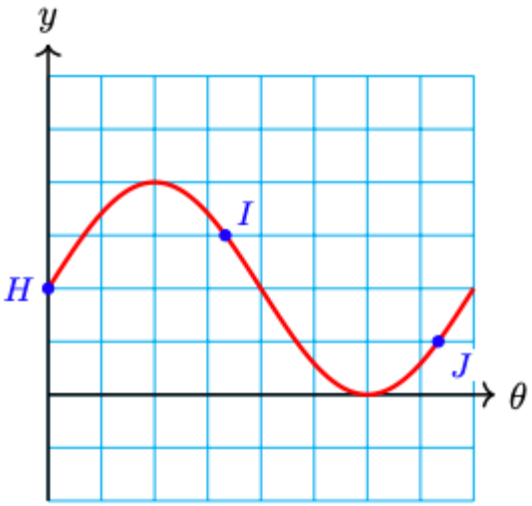
    b. $4 - 3\sin(2x) = 2.5$

For the following exercises, give the coordinates of the points on the graph.

85. $f(\theta) = -3 + \cos\theta$



86. $f(\theta) = 1 + \sin\theta$



Applications

37. The tide in Yorktown is approximated by the function

$$h(t) = 1.4 - 1.4\cos(0.51t)$$

measured in feet above low tide, where t is the number of hours since the last low tide.

a. What are the midline, period, and amplitude?

b. Graph $h(t)$ for two periods, labeling the points that correspond to high tide and low tide.

c. If the last low tide occurred at 5:00 am, predict when the next high and low tides will occur.

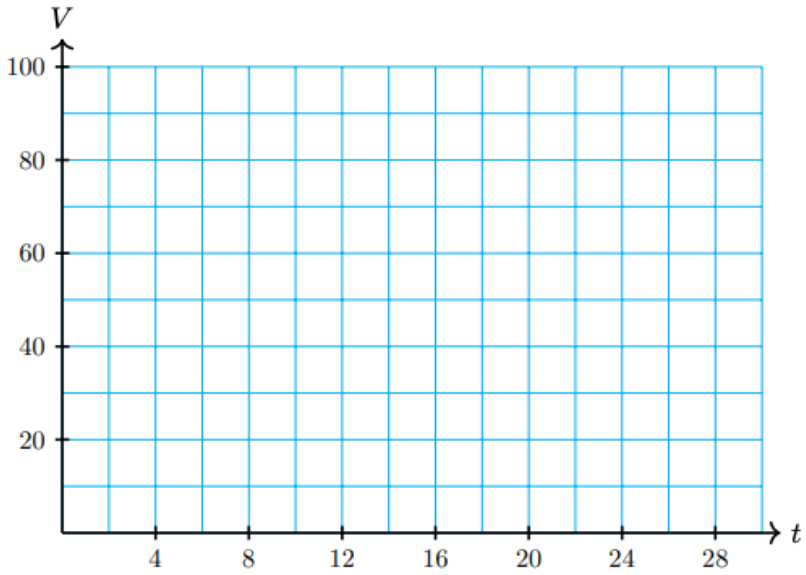
38. When observed from Earth, the moon looks like a disk that is partially visible and partially in shadow. The percentage of the disk that is visible can be approximated by

$$V(t) = 50\cos(0.21t) + 50$$

where t is the number of days since the last full moon.

a. Graph $V(t)$ in the window

$$X_{\min} = 0, X_{\max} = 30$$
$$Y_{\min} = 0, Y_{\max} = 120$$



b. Sketch the graph on the grid.

c. Label on your graph the points that correspond to full moon, half moon, and new moon. (New moon occurs when the part of the moon receiving sunlight is facing directly away from the Earth.)

d. At what times during the lunar month is 25% of the moon visible? Mark those points on your graph.

e. During which days is less than 50% of the moon visible? Mark the corresponding points on your graph.

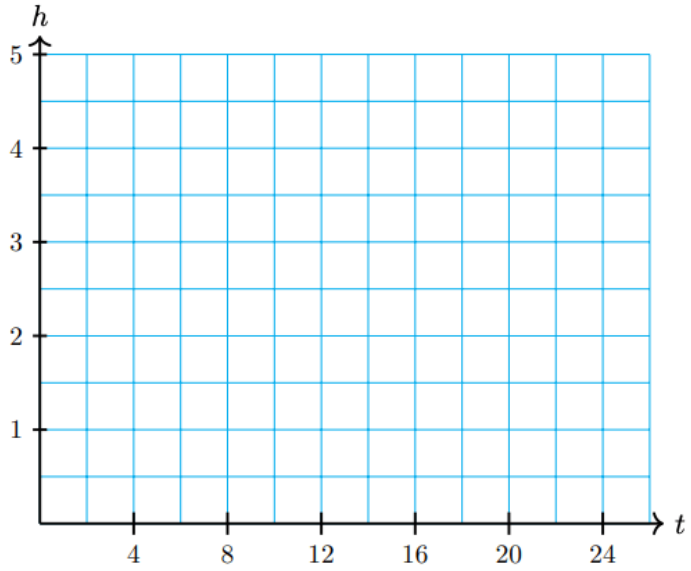
39. The tide in Malibu is approximated by the function

$$h(t) = 2.5 - 2.5\cos(0.5t)$$

measured in feet above low tide, where t is the number of hours since the last low tide.

a. Graph $h(t)$ in the window

$$X_{\min} = 0, X_{\max} = 25$$
$$Y_{\min} = 0, Y_{\max} = 6$$



b. Sketch the graph on the grid.

c. Label on your graph the points that correspond to high tide and low tide.

d. How high is high tide, and at what times does it occur?

e. At what times during the 25-hour period is the tide 4 feet above low tide? Mark those points on your graph.

f. Kathie walks along the beach only when the tide is below 1 foot. Find the intervals on your graph when the tide is below 1 foot.

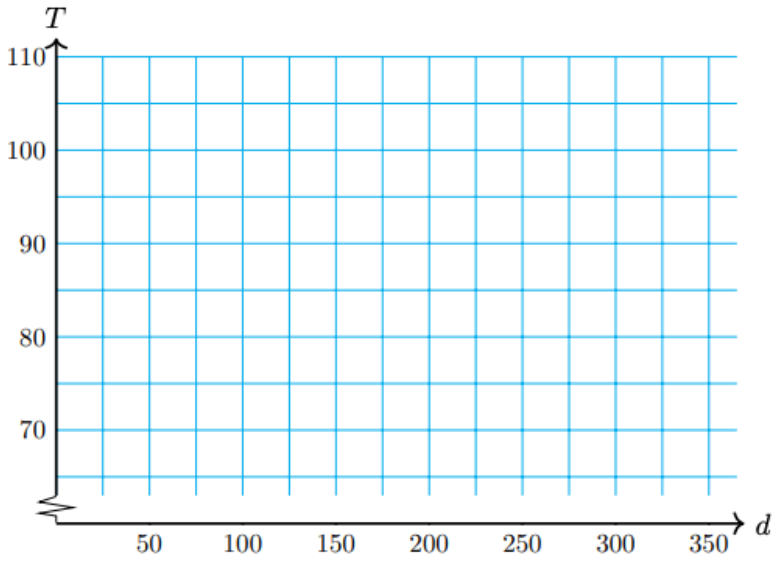
90. The average daily high temperature in the town of Beardsley, Arizona is approximated by the function

$$T(d) = 85.5 - 19.5\cos(0.0175d - 0.436)$$

where the temperature is measured in degrees Fahrenheit, and d is the number of days since January 1.

a. Graph $T(d)$ in the window

$$X_{\min} = 0, X_{\max} = 365$$
$$Y_{\min} = 0, Y_{\max} = 110$$



b. Sketch the graph on the grid.

c. Label on your graph the points that correspond to highest and lowest average temperature.

d. What is the hottest day and what is its average temperature? What is the day with the lowest average temperature, and what is that temperature?

e. At what times during the year are average high temperatures above $90°$? Mark those points on your graph.

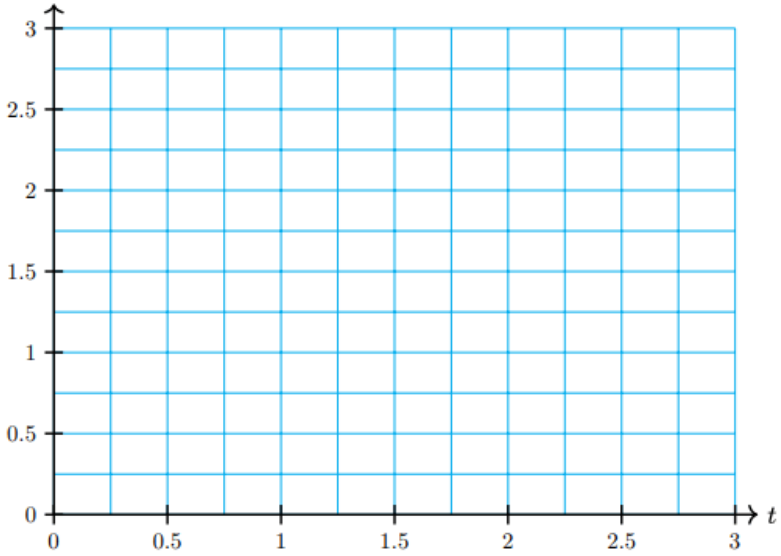
91. A weight is suspended from the ceiling on a spring. The weight is pushed straight up, compressing the spring, then released. The height of the weight above the ground is given by the function

$$H(t) = 2 + 0.5\cos(6t)$$

where the height is measured in meters, and t is the number of seconds since the mass was released.

a. Graph $H(t)$ in the window

$$X_{\min} = 0, X_{\max} = 3$$
$$Y_{\min} = 0, Y_{\max} = 3$$



b. Sketch the graph on the grid.

c. Label on your graph the points that correspond to highest and lowest positions of the mass.

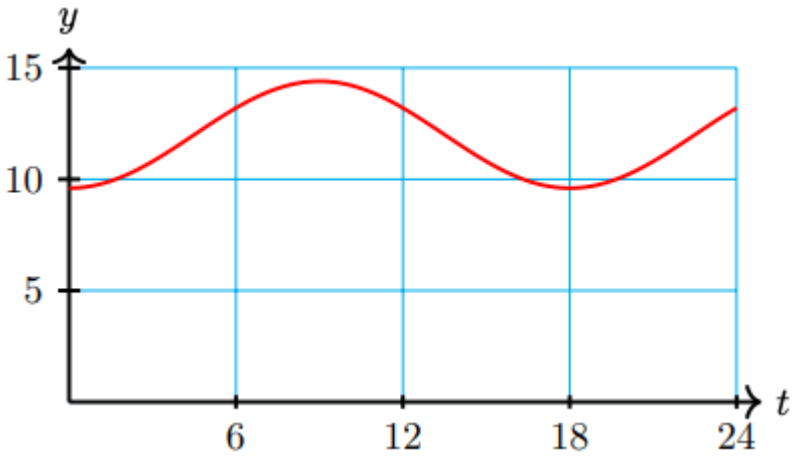
d. How high is highest point, and when is that height attained during the first 3 seconds?

e. How high is lowest point, and when is that height attained during the first 3 seconds?

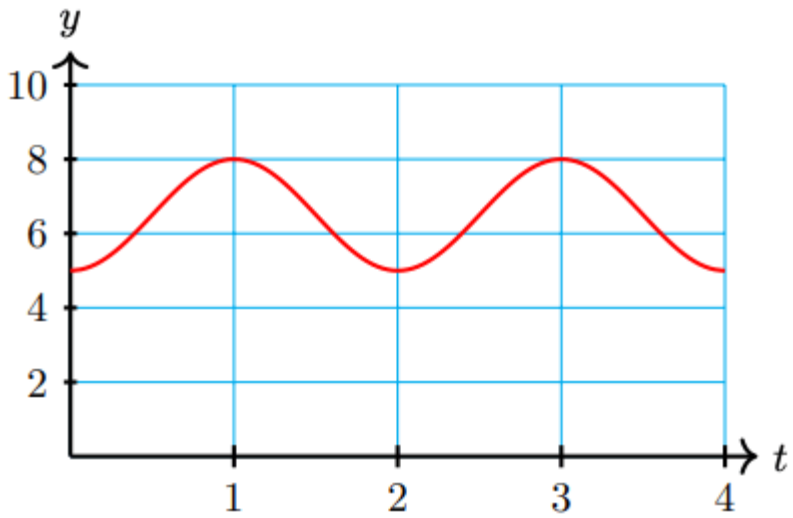
f. Find the intervals during the first 3 seconds when the mass is less than 2 meters above the ground.

For the following exercises, write an equation for the sinusoidal function whose graph is shown.

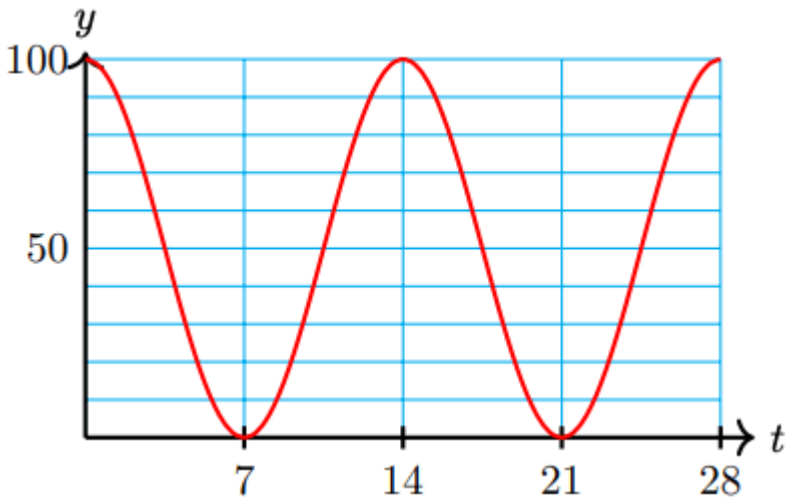
92. The number of hours of daylight in Salt Lake City varies from a minimum of 9.6 hours on the winter solstice to a maximum of 14.4 hours on the summer solstice. Time is measured in months, starting at the winter solstice.



93. A weight is 6.5 feet above the floor, suspended from the ceiling by a spring. The weight is pulled down to 5 feet above the floor and released, rising past 6.5 feet in 0.5 seconds before attaining its maximum height of feet. The weight oscillates between its minimum and maximum height.



94. Although the moon is spherical, what we see from Earth looks like a disk, sometimes only partly visible. The percentage of the moon's disk that is visible varies between 0 (at new moon) to 100 (at full moon), over a 28-day cycle.



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5.4: Sinusoidal Models

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Rewriting the Coordinate Definition

We begin this section with a brief interlude. Recall the coordinate definition of the trigonometric functions, which is presented below for our discussion.

Definition: Trigonometric Functions (Coordinate Definition)

Let θ be an angle in standard position with the point $P(x, y)$ on the terminal side of θ, where P is not the origin. Define $r = \sqrt{x^2 + y^2}$. The trigonometric functions of the angle θ are defined as follows:



If we hold r to be constant and plot all points a distance of r units from the origin, then we will have graphed a circle of radius r centered at the origin. By the coordinate definition of the trigonometric functions, if the point (x, y) is on this circle and is on the terminal side of θ, then

$$\cos(\theta) = \frac{x}{r} \quad \text{and} \quad \sin(\theta) = \frac{y}{r}.$$

We have reached a point in this text where writing these formulas in an alternate form is more useful. Solving the formulas for x and y, respectively, we get the following:

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta). \tag{5.4.1}$$

The formulas in 5.4.1 apply to any point on the terminal side of θ on a circle of radius r. Understanding these formulas is essential for our success moving forward in Trigonometry.

Sinusoidal Models

Now that we have a solid understanding of the graphs for the fundamental trigonometric functions, it's time we focused on their applications. In the same way exponential functions can be used to model a wide variety of phenomena in nature, the cosine and sine functions can be used to model their fair share of natural behaviors. In particular, sinusoidal waves are common in physics, engineering, computer science, signal processing, and many other fields. For consistency, we will focus on building applications using the sine function (although, you are welcome to use the Cofunction Identities to transform our sine models into cosine models). Moreover, we switch to the independent variable t (instead of x or θ) since the applications in this section are time-dependent.

Before diving into the applications, let's summarize some important facts and definitions about the sine function. Let

$$f(t) = A\sin(Bt + C) + D. \tag{5.4.2}$$

From the material so far in this chapter, we know that $|A|$ is the amplitude, the period is $\frac{2\pi}{B}$, the phase shift is $-\frac{C}{B}$, and the vertical shift (or midline) is D.

Along with knowing these formulas, it is helpful to remember what these quantities mean in context. As we discuss these contextual meanings, we will inevitably introduce more definitions (mostly from physics) that we will occasionally be using throughout the remainder of this text.

- The amplitude measures the sine wave's maximum **displacement** (change in position) from its midline. Therefore, if the midline of a sinusoidal function is $y = 120$ and the amplitude is 30, the maximum displacement of the sinusoidal wave is 30.
- The period is the length of time it takes to complete one cycle of the sinusoid. For a slightly oversimplified example, the signal your antenna is receiving from a broadcasting radio station is a sinusoidal wave. If this wave takes 1.0152×10^{-8} seconds to complete a single cycle, then its period is 1.0152×10^{-8}.
  - A more physically interesting, and important, interpretation related to the period is the ordinary frequency of the sinusoid (often just called the frequency). This was introduced previously but not really used, so let's recall some information about the frequency. This is the number of cycles the sinusoid goes through in a given unit of time (generally, per second). Since the period is in seconds per cycle and frequency is cycles per second, it should make sense that period and frequency are reciprocals. That is, if f is the frequency of a sinusoid and T is its period,

    $$f = \frac{1}{T} \quad \text{and} \quad T = \frac{1}{f}.$$

    Thus, the frequency of the radio station mentioned above is

    $$f = \frac{1}{T} = \frac{1}{1.0152 \times 10^{-8}} \approx 98{,}502{,}758 \text{ cycles per second.}$$

    The unit for "cycles per second" is called the **Hertz** (**Hz**). Therefore, the radio station is operating at roughly 98.5 million Hertz, or 98.5 MHz (megahertz).
  - Another concept we introduced previously is the angular frequency. In terms of Equation 5.4.2, the angular frequency is defined as the value of B. This tells how many cycles are completed over an interval of length 2π.
- The phase shift represents how much of a "head start" the sinusoid has over the unshifted sine function.
  - The **phase angle** (also called the **phase**), which was only mentioned in a footnote in the previous section, is the value of C. It is the angle corresponding to $t = 0$. Since we have not officially written the definition for this, it is provided below.

Definition: Phase Angle

The **phase angle** (also called the **phase**) is defined to be the value of C in the sinusoidal functions $A\ \sin(Bx + C) + D$ and $A\ \cos(Bx + C) + D$.

All of this information is summarized below.

Summary: Sine Curve Properties

- The **amplitude** is $|A|$
- The **angular frequency** is B and the **ordinary frequency** is $f = \frac{B}{2\pi}$

- The **period** is $T = \frac{2\pi}{B} = \frac{1}{f}$
- The **phase angle** is C and the **phase shift** is $-\frac{C}{B}$
- The vertical shift or **midline** is D

From this point forward, when dealing with applications, we will use the terms *angular frequency*, *frequency*, and *phase* when possible. Therefore, it's best to make a note of these definitions so you don't forget them.

Our first foray into sinusoidal motion puts these definitions and the equations in 5.4.1 to good use.

Example 5.4.1

The Giant Wheel at Cedar Point is a circle with a diameter of 128 feet, which sits on an 8-foot-tall platform, making its overall height 136 feet. It completes two revolutions in 2 minutes and 7 seconds. Assuming that the riders are at the edge of the circle, find a sinusoid that describes the height of the passengers above the ground t seconds after they pass the point on the wheel closest to the ground.

Solution available at the end of this section.

A few remarks about Example 5.4.1 are in order. First, note that the amplitude of 64 in our answer corresponds to the radius of the Giant Wheel. This means that passengers on the Giant Wheel never stray more than 64 feet vertically from the center of the Wheel, which makes sense. Another way to say this (using our new definitions) is that the maximum displacement of a rider on the Great Wheel is 64 feet.

The second thing to note is our process of finding the value of the phase angle. Trigonometry is a fun topic because there are usually several ways to do something, and not one is better than another. In fact, if I were investigating this problem and were not concerned about instructing you on new language, I would have found the value of C using a different approach. To figure out the phase angle, we could have sketched the path of a rider on the Great Wheel from the time they enter $(t = 0)$.

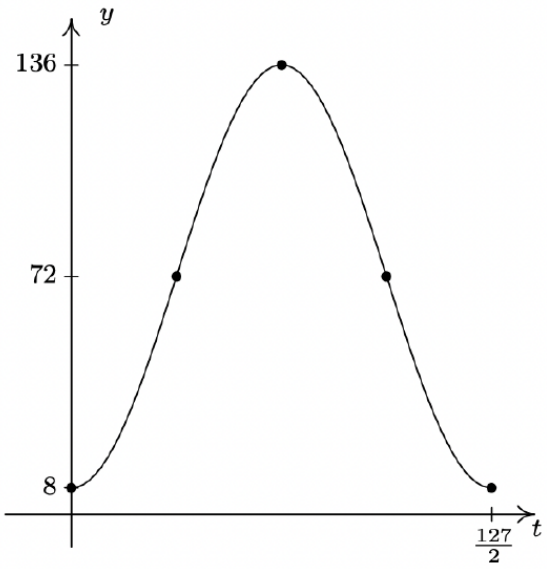


Figure 5.4.4

Notice that this sketch looks more like a reflected cosine than a sine; however, we made our choice to work with sine, and, darn it, we are going to stick with it. The actual sine function starts one-fourth of the way through this cycle. That is, at time $\frac{1}{4}\left(\frac{127}{2}\right) = \frac{127}{8}$. Hence, the sinusoidal model has a phase shift of $\frac{127}{8}$. Using the relation $\text{Phase Shift} = -\frac{C}{B}$, we get

$$\begin{aligned} \text{Phase Shift} &= -\frac{C}{B} \\ \Longrightarrow \quad \frac{127}{8} &= -\frac{C}{4\pi/127} \\ \Longrightarrow \quad \frac{127}{8} &= -\frac{127C}{4\pi} \\ \Longrightarrow \quad -\frac{4\pi}{8} &= C \\ \Longrightarrow \quad -\frac{\pi}{2} &= C \end{aligned}$$

Thus, our sinusoidal model is

$$h(t) = 64\sin\left(\frac{4\pi}{127}t - \frac{\pi}{2}\right) + 72.$$

Now, you might say, "Hey, that's not what we got in Example 5.4.1!" I beg to differ. Remember, the value of $\sin(\theta)$ is the same as the value of $\sin(\theta - 2\pi)$. If you subtract 2π from the argument in the answer to Example 5.4.1, you would get $h(t) = 64\sin\left(\frac{4\pi}{127}t - \frac{\pi}{2}\right) + 72$.

The third and final thing to mention is the phase shift of our answer, which is $\frac{127}{8} = 15.875$. This represents the "time delay" (in seconds) we introduce by starting the motion at the point P as opposed to the point Q. Said differently, passengers who "start" at P take 15.875 seconds to "catch up" to the point Q.

Checkpoint 5.4.1

Outside temperature over a day can be modeled as a sinusoidal function. Suppose you know the high temperature for the day is 78 degrees and the low temperature of 42 degrees occurs at 6 AM. Assuming t is the number of hours since midnight, find an equation for the temperature, T, in terms of t.

Answer available at the end of this section.

Our next example shows how to deal with data that is sinusoidal in nature.

Example 5.4.2

According to the U.S. Naval Observatory website, the number of hours, H, of daylight that Fairbanks, Alaska received on the 21st day of the nth month of 2009 is given below. Here, $t = 1$ represents January 21, 2009, $t = 2$ represents February 21, 2009, and so on.



a. Find a sinusoid that models these data and use graphing technology to graph your answer along with the data.
b. Compare your answer to part (a) to one obtained using the regression feature of your graphing technology.

Solutions available at the end of this section.

Checkpoint 5.4.2

The table shows the number of hours of daylight in Glasgow, Scotland on the first of each month.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Daylight Hours	7.1	8.7	10.7	13.1	15.3	17.2	17.5	16.7	13.8	11.5	9.2	7.5

a. Sketch a sinusoidal graph of daylight hours as a function of time, with $t = 1$ in January.
b. Estimate the period, amplitude, and midline of the graph.

Answers available at the end of this section.

Success in College: Negative Talk

Sometimes math students use “negative self-talk” about math. They say, “I can’t do math” instead of, “I can do some math and, with hard work, I’ll be able to do more.” They say, “I should be able to work faster” instead of “I can work fast enough to pass the course.”

It has been proven that positive thinking is part of success. Being an optimist about math, but more importantly, about life, is a crucial building block of success.

A favorite quote of mine comes from Thomas Edison. When asked how he felt about so many failed tests before finally inventing the lightbulb, he responded, “I didn’t fail the tests; I just found 100 ways to do it wrong.”

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. What effect(s) do you think negative self-talk can have on a student’s attitude about math? Why?
2. What effect do you think negative self-talk can have on a student’s performance in math? Why?

Success in Mathematics: Why Study Mathematics? (Part 3)

Matthew Jannsen graduated with a degree in mathematics in 2000. He went to law school and is now an attorney. He writes,

> "At this stage of my career I spend a majority of my time conducting legal research and drafting legal memoranda. Most people assume that mathematics would not be the most appropriate preparation for a person who makes their living engaging in these activities. I would disagree. In fact, I would suggest that my mathematics background has been one of my greatest assets throughout my law school and legal careers and will continue to help me develop as a lawyer throughout my career. In order to succeed as a math major, I always had to engage in logical thought... I constantly rely on the logical thinking skills that I developed as a math major in executing my tasks as an attorney and I am grateful for my mathematics education."
>
> *Source: http://www.maa.org/careers/janssen.html*

1. Describe, in detail, what you think it means to "think logically."
2. Is logical thought a vital skill in the career that you hope to have? Explain.

Answers

Section Preview

This section covers an introduction to applications (or data models) that can be described using the sine or cosine functions. The exception is simple and damped harmonic motion—that material is reserved for the next section. For now, it's important that students understand the concepts of angular and ordinary frequency. Playing with data in Desmos can be very useful when it comes to instruction for this section.

Section Preview

- **Linear Functions**
  - Linear Models: The section introduces displacement and describes sinusoidal waves as phenomena modeled by functions.

Section Preview

- **Sets and Numbers**
  - Scientific Notation: Scientific notation appears explicitly in the radio frequency example (1.0152×10^{-8} seconds), making this a genuine operational

prerequisite for reading and interpreting the context.

- **Arithmetic**
  - Mathematical Operations — Division: Computing D from the average of max and min values requires adding and dividing fractions/decimals: $\frac{3.3+21.8}{2} = 12.55$.

Section Preview

- Build a sinusoidal model from an application.
- Build a sinusoidal model from data.

Example 5.4.1

As is most often the case, we begin by sketching the situation below. We will assume a counterclockwise rotation.

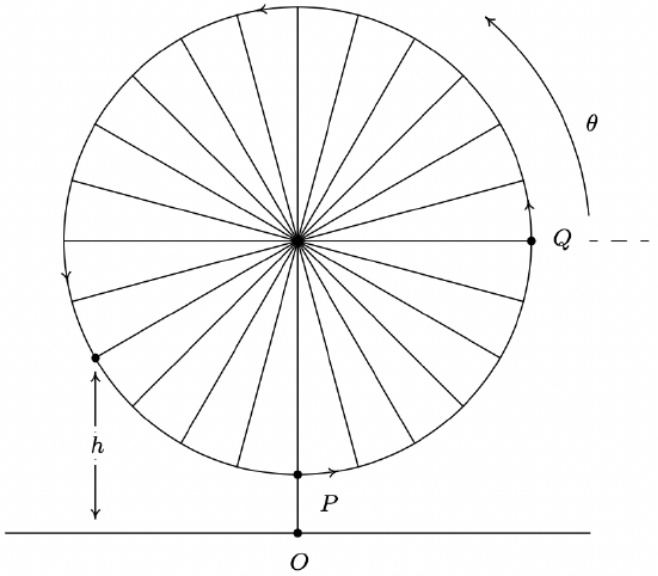


Figure 5.4.1

We were told the diameter of the circle is 128 feet. Therefore, its radius is 64 feet. For simplicity, and to introduce a coordinate system to the situation, let's graph this circle centered at *the origin*. We will raise it up eventually, but this initial sketch can be very helpful.

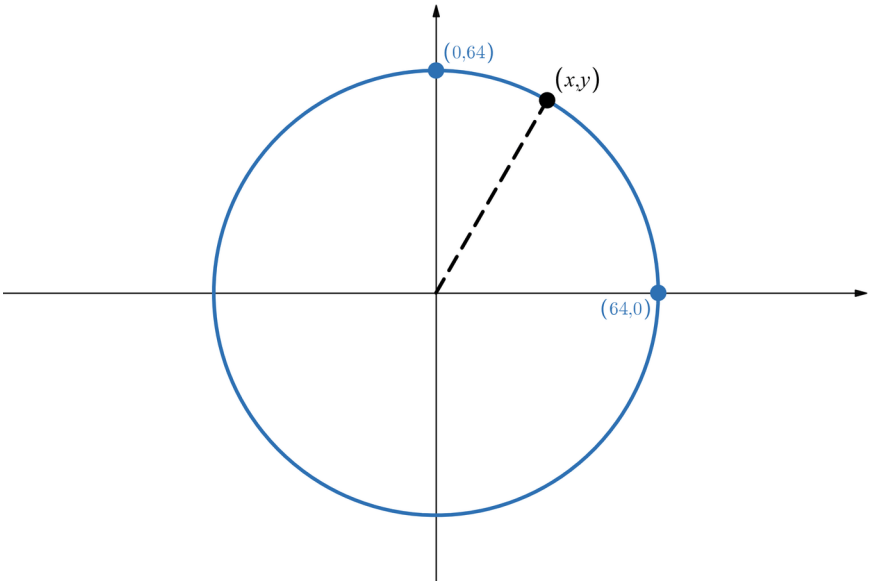


Figure 5.4.2

Our eventual goal is to find the height of a rider, which is related to the y-coordinates of this circle. However, we know from 5.4.1 that

$$y = r\sin(\theta) = 64\sin(\theta).$$

One thing you must get used to is that our sinusoidal models are functions of time—not angles. Therefore, we will trade θ out for $Bt + C$—this is the most general argument of the sine function and something we have been working with recently. Hence, the height of a rider on the Great Wheel will be related to the function

$$64\sin(\theta) = 64\sin(Bt + C).$$

Since the Great Wheel is not partially underground, we need to lift this function up. Specifically, we need to shift the center of our circle so that the bottom of the Great Wheel is 8 feet above the ground (which is the x-axis in our figure above). Thus, we need to move this circle up 64 feet plus an additional 8 feet, for a total of 72 feet.

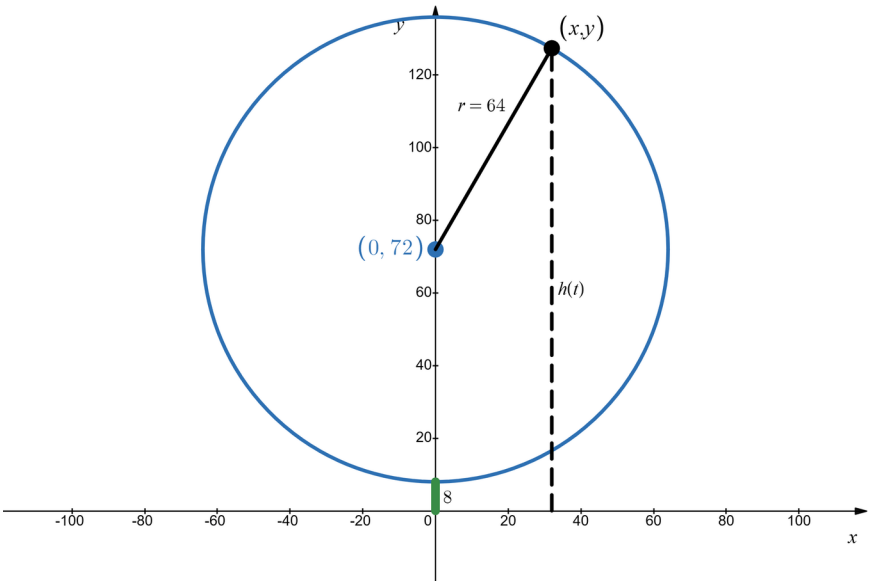


Figure 5.4.3

Therefore, we are considering a sinusoidal model of the form

$$h(t) = 64\sin(Bt + C) + 72.$$

There are only two things left to consider—the angular frequency and the phase angle of the sinusoidal model.

The angular frequency is B, which is known to be related to the period, T, by the formula

$$T = \frac{2\pi}{B}$$
$$\implies B = \frac{2\pi}{T}. \tag{5.4.3}$$

The period is the time it takes for the sinusoidal function to complete one cycle. For this application, the period is the time it takes the Giant Wheel to perform *one* full revolution. We are told the Giant Wheel goes through *two* full revolutions in 2 minutes and 7 seconds. Therefore, it takes 127 seconds to go through *two* revolutions and half that to go through *one*. Hence, the period is

$$\text{Period} = T = \frac{\text{time}}{\text{one revolution}} = \frac{127}{2} \text{ seconds per revolution.}$$

Substituting this into 5.4.3, we get

$$B = \frac{2\pi}{T} = \frac{2\pi}{127/2} = \frac{4\pi}{127}.$$

The sinusoidal model is really taking shape now.

$$h(t) = 64\sin\left(\frac{4\pi}{127}t + C\right) + 72.$$

The height of a rider at $t = 0$ is 8. Therefore, letting $t = 0$ and $h = 8$, we get

$$8 = 64\sin(C) + 72 \implies -1 = \sin(C) \implies C = \frac{3\pi}{2}.$$

With this, we have found the equation of motion for the rider,

$$h(t) = 64\sin\left(\frac{4\pi}{127}t + \frac{3\pi}{2}\right) + 72.$$

We can check the reasonableness of our answer by graphing $y = h(t)$ over the interval $\left[0, \frac{127}{2}\right]$.

Checkpoint 5.4.1

$T(t) = 60 + 18\sin\left(\frac{\pi}{12}t - \pi\right)$

Example 5.4.2

a. We plot the data below (using Desmos' table feature) to get a feel for it.

x_1	y_1
1	5.8
2	9.3
3	12.4
4	15.9
5	19.4
6	21.8
7	19.4
8	15.6
9	12.4
10	9.1
11	5.6
12	3.3

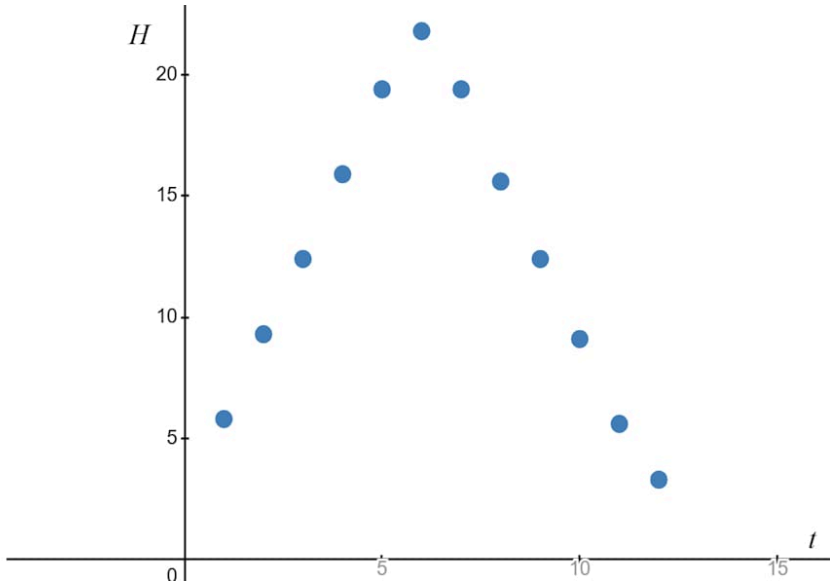


Figure 5.4.5

The data certainly appear sinusoidal, but when it comes down to it, manually fitting a sinusoid to data is not an exact science. We do our best to find the constants A, B, C and D so that the function

$$H(t) = A\sin(Bt + C) + D$$

closely matches the data. We first go after the vertical shift D whose value determines the midline. In a typical sinusoid, the value of D is the average of the maximum and minimum values. So here we take

$$D = \frac{3.3 + 21.8}{2} = 12.55.$$

Next is the amplitude, $|A|$, which is the displacement from the midline to the maximum (and minimum) values. We find

$$|A| = |21.8 - 12.55| = 9.25.$$

As long as we know we are going to choose our sine function to be positive, we have

$$H(t) = 9.25\sin(Bt + C) + 12.55.$$

Next, we go after the angular frequency, B. Since the data collected is over the span of a year (12 months), we take the period to be $T = 12$ months. This means

$$B = \frac{2\pi}{T} = \frac{2\pi}{12} = \frac{\pi}{6}.$$

The last quantity to find is the phase, C. It is easier in this case to find the phase *shift*, $-\frac{C}{B}$. Since we picked $A > 0$, the phase shift corresponds to the first value of t with $H(t) = 12.55$ (the midline value). Here, we choose $t = 3$, since the output of 12.4 is closer to 12.55 than the next value, 15.9, which corresponds to $t = 4$. Hence, $-\frac{C}{B} = 3$, so $C = -3B = -3\left(\frac{\pi}{6}\right) = -\frac{\pi}{2}$. Thus, we have

$$H(t) = 9.25\sin\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) + 12.55.$$

Below is a graph of our data with the curve $y = H(t)$.

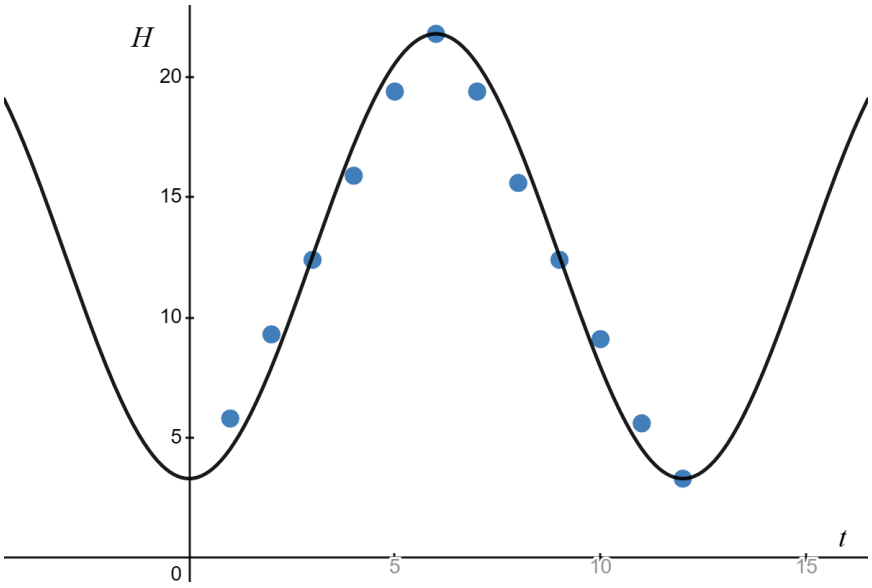
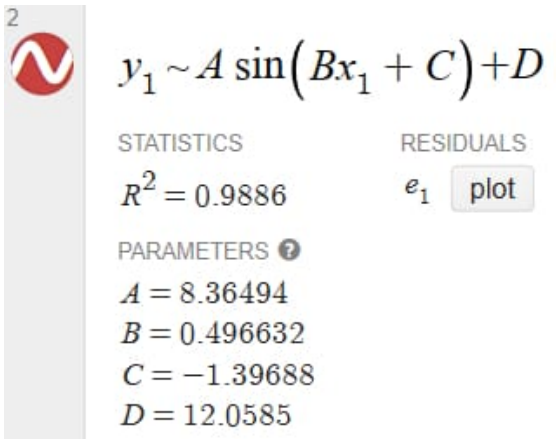


Figure 5.4.6

b. Using the following command in Desmos, we get



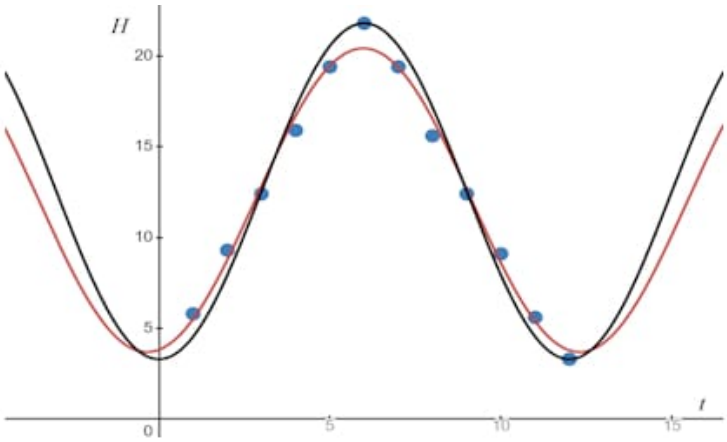


Figure 5.4.7

While both models seem to fit the data reasonably, Desmos' model is the better fit.

Checkpoint 5.4.2

a. Plot the data points and fit a sinusoidal curve by eye, as shown below.

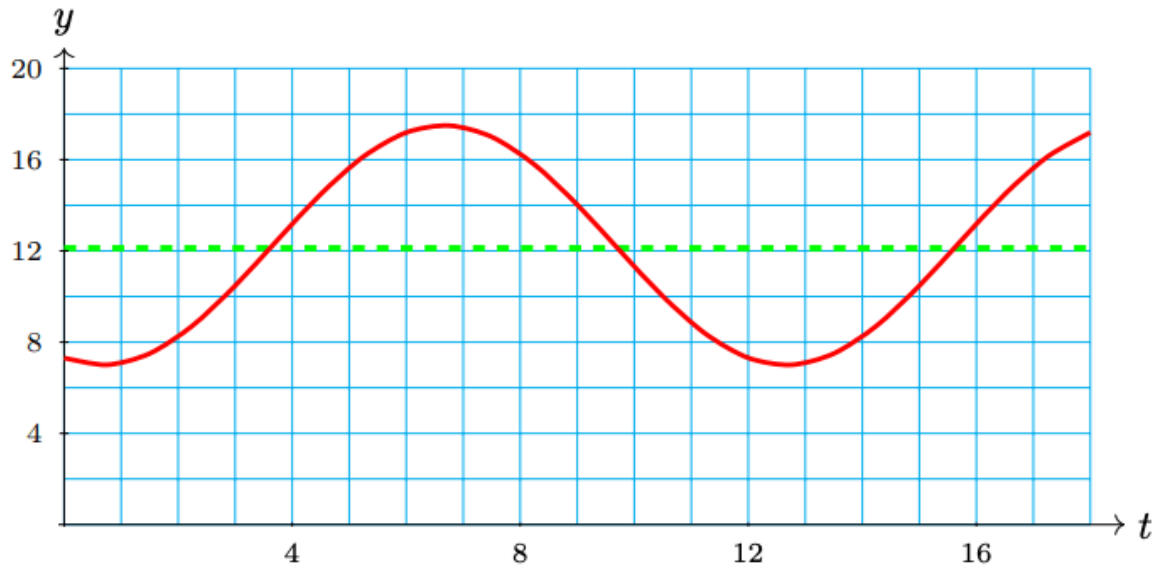


Figure 5.4.8

b. The period of the graph is 12 months. The midline is approximately $y = 12.25$, and the amplitude is approximately 5.25.

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5.4.1: Resources and Key Concepts

Resources

Videos

- Sinusoidal Models
  - Graphs of the Sine and Cosine Functions - Applications Involving Sinusoidal Functions
  - Trigonometric Graphs - Review of Applications Involving Sinusoidal Functions

Key Concepts

Definitions

- **Trigonometric Functions (Coordinate Definition):** For an angle θ in standard position, a point $P(x, y)$ on its terminal side at a distance r from the origin has coordinates given by $x = r\cos(\theta)$ and $y = r\sin(\theta)$.
- **Maximum Displacement:** In the context of a sinusoidal function, the maximum displacement is the amplitude, which measures the wave's maximum change in position from its midline.
- **Ordinary Frequency (f):** The number of cycles a sinusoid completes in a given unit of time. It is the reciprocal of the period ($f = \frac{1}{T}$).
- **Hertz (Hz):** The unit for ordinary frequency, equivalent to one cycle per second.
- **Angular Frequency (ω or B):** The value of the parameter B in a sinusoidal function. It is related to the ordinary frequency by the formula $\omega = 2\pi f$.
- **Phase Angle:** The value of the parameter C in the sinusoidal function $f(t) = A\sin(Bt + C) + D$. It corresponds to the angle at $t = 0$.

Theorems

- **Sine Curve Properties:** For a function of the form $f(t) = A\sin(Bt + C) + D$:
  - The amplitude is $|A|$.
  - The angular frequency is B.
  - The ordinary frequency is $f = \frac{B}{2\pi}$.
  - The period is $T = \frac{2\pi}{B} = \frac{1}{f}$.
  - The phase angle is C.
  - The phase shift is $-\frac{C}{B}$.
  - The vertical shift, or midline, is D.

Common Mistakes

- **Confusing Period and Frequency:** The period is the time it takes to complete one cycle (e.g., seconds per cycle), while frequency is the number of cycles completed in a unit of time (e.g., cycles per second). They are reciprocals of each other.
- **Confusing Angular Frequency and Ordinary Frequency:** The angular frequency is the parameter B in the sinusoidal model, while the ordinary frequency f must be calculated using $f = B/2\pi$.
- **Confusing Phase Angle and Phase Shift:** The phase angle is the parameter C, while the phase shift is a horizontal translation of the graph calculated as $-C/B$.
- **Assuming Only One Correct Model:** For a given sinusoidal graph, there are infinitely many correct equations that can model it. A cosine function can be used instead of a sine function, and different phase shifts can produce the same graph.

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5.4.2: Homework

Reading Questions

1. If a point (x, y) is on the terminal side of an angle θ on a circle of radius r, what are the formulas for x and y in terms of r and θ?
2. In the sinusoidal model $f(t) = A\sin(Bt + C) + D$, what does each parameter (A, B, C, D) represent in terms of the graph's properties?
3. What is the difference between ordinary frequency and angular frequency?
4. How is the period of a sinusoidal function related to its ordinary frequency?
5. What is the difference between a phase angle and a phase shift?
6. If a sinusoidal model has a maximum value of 50 and a minimum value of 10, what are its amplitude and midline?
7. If a Ferris wheel completes 2 revolutions in 120 seconds, what is its period in seconds?
8. Using the period from the previous question, what would be the angular frequency, B, for a sinusoidal model of the Ferris wheel's motion?
9. When building a sinusoidal model from data, what feature of the data helps you determine the vertical shift (midline)?
10. When building a sine model $(A > 0)$ from data, what feature of the data helps you determine the phase shift?

Homework

Vocabulary Check

1. Change in position is also known as ___.
2. The period of a function measures the number of ___ per ___.
3. The ordinary frequency (or, simply, frequency) of a sinusoid measures the number of ___ per ___.
4. The unit for frequency is the ___.
5. In the function
$$f(t) = A\,\sin\,(Bt + C) + D,$$
B is called the ___, C is called the ___, and $-\frac{C}{B}$ is called the ___.

Concept Check

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

The following questions concern the sinusoidal function
$$f(t) = A\,\sin\,(Bt + C) + D.$$

6. B is the period
7. C is the phase shift
8. $y = D$ is the midline
9. $-\frac{C}{B}$ is the phase shift
10. If T is the period of the sinusoid and f is the frequency, $f = \frac{1}{T}$.

Basic Skills

For the following exercises,
(a) Estimate the amplitude, period, and midline of a sinusoidal function that fits the data.
(b)Write a formula for the function.

11.

t	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2	2.25	2.5
$f(t)$	5.2	4.26	2	-0.26	-1.2	-0.26	2	4.26	5.2	4.26	2

12.

s	0	1	2	3	4	5	6	7	8	9	10
$g(s)$	3	2.58	2.4	2.58	3	3.42	3.6	3.42	3	2.58	2.4

13.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$H(x)$	5	7.9	9.8	9.8	7.9	5	2.1	0.2	0.2

14.

t	0	0.79	1.57	2.36	3.14	3.93	4.71	5.50	6.28
$V(t)$	1	-0.17	-3	-5.8	-7	-5.3	-3	-0.17	1

Applications

15. The sounds we hear are made up of mechanical waves. The note "A" above the note "middle C" is a sound wave with ordinary frequency

$$f = 440 \text{ Hertz } = 440\frac{\text{cycles}}{\text{second}}.$$

Find a sinusoid which models this note, assuming that the amplitude is 1 and the phase shift is 0.

16. The average daily high temperature in Fairbanks, Alaska can be approximated by a sinusoidal function with a period of 12 months. The low temperature of -1.6° occurs in January, and the high temperature of 72.3° in July.

    a. What are the midline, period, and amplitude?

    b. Write a formula for the average daily high temperature $T(m)$, where m is the number of months since January.

    c. What is the displacement of the temperature?

    d. What is the ordinary frequency?

    e. What is the angular frequency?

    f. Graph $T(m)$ for two periods, labeling the points that correspond to highest and lowest average temperature.

17. Depending on its phase, the moon looks like a disk that is partially visible and partially in shadow. The visible fraction ranges from 0% to 100% and can be approximated by a sinusoidal function $V(t)$, where t is the number of days since the last full moon. The time between successive full moons (a lunar month) is 29.5 days.

    a. What are the period, midline, and amplitude of $V(t)$?

    b. Write a formula for $V(t)$.

    c. What is the ordinary frequency?

    d. What is the angular frequency?

    e. Graph your function over two periods, labeling the points that correspond to full moon, half moon, and new moon.

18. The height of a child's toy suspended at the end of a spring is approximated by a sinusoidal function. The toy's height ranges between 200 centimeters and 260 centimeters above the ground, and it completes one up-and-down cycle every 0.8 seconds.

    a. What are the midline, period, and amplitude?

    b. Let $h(t)$ be the height of the toy in centimeters, where $t = 0$ seconds corresponds to a time when the object was at the midline and moving upwards. Graph $h(t)$ for two periods, labeling the points that correspond to the high and low positions of the toy.

    c. What is the ordinary frequency of $h(t)$?

    d. What is the angular frequency of $h(t)$?

    e. When does the toy reach its maximum height the second time?

19. The voltage, V, in an alternating current source has amplitude $220\sqrt{2}$ and ordinary frequency $f = 60$ Hertz. Find a sinusoid which models this voltage. Assume that the phase is 0.

20. The London Eye is a popular tourist attraction in London, England and is one of the largest Ferris Wheels in the world. It has a diameter of 135 meters and makes one revolution (counterclockwise) every 30 minutes. It is constructed so that the lowest part of the Eye reaches ground level, enabling passengers to simply walk on and off the ride. Find a sinusoid which models the height h of the passenger above the ground in meters t minutes after they board the Eye at ground level.

21. Repeat the previous problem, but find a sinusoid which models the horizontal displacement, x, of the passenger from the center of the Eye in meters t minutes after they board the Eye. Here we take $x(t) > 0$ to mean the passenger is to the right of the center, while $x(t) < 0$ means the passenger is to the left of the center.

22. The average daily maximum temperature in Stockholm, Sweden is 30°F in January and 72°F in July.

    a. Sketch a sinusoidal graph of $S(t)$, the average maximum temperature in Stockholm as a function of time, for one year.

    b. Give the period, midline, and amplitude of your graph.

    c. Write an equation for the function.

23. The average daily maximum temperature in Riyadh, Saudi Arabia is 86°F in January and 113°F in July.

    a. Sketch a sinusoidal graph of $R(t)$, the average maximum temperature in Riyadh as a function of time, for one year.

    b. Give the period, midline, and amplitude of your graph.

    c. Write an equation for the function.

24. The height of the tide in Cabot Cove can be approximated by a sinusoidal function. At 5 am on July 23, the water level reached its high mark at the 20-foot line on the pier, and at 11 am, the water level was at its lowest at the 4-foot line.

    a. Sketch a graph of $W(t)$, the water level as a function of time, from 5 am on July 23 to 5 am on July 24.

    b. Give the period, midline, amplitude of your graph.

    c. State the phase angle and phase shift.

    d. State the ordinary frequency and angular frequency.

    e. Write an equation for the function.

25. The population of mosquitoes at Marsh Lake is a sinusoidal function of time. The population peaks around June 1 at about 6000 mosquitoes per square kilometer, and is smallest on December 1, at 1000 mosquitoes per square kilometer.

    a. Sketch a graph of $M(t)$, the number of mosquitoes as a function of the month, where $t = 0$ on June 1.

b. Give the period, midline, amplitude of your graph.

c. State the phase angle and phase shift.

d. State the ordinary frequency and angular frequency.

e. Write an equation for the function.

26. The paddlewheel on the Delta Queen steamboat is 28 feet in diameter, and is rotating once every ten seconds. The bottom of the paddlewheel is 4 feet below the surface of the water.

a. The ship's logo is painted on one of the paddlewheel blades. At $t = 0$, the blade with the logo is at the top of the wheel. Sketch a graph of the logo's height above the water as a function of t.

b. Give the period, midline, amplitude of your graph.

c. State the phase angle and phase shift.

d. State the ordinary frequency and angular frequency.

e. Write an equation for the function.

27. Min's bicycle wheel is 24 inches in diameter, and she has a light attached to the spokes 10 inches from the center of the wheel. It is dark, and she is cycling home slowly from work. The bicycle wheel makes one revolution every second.

a. At $t = 0$, the light is at its highest point the bicycle wheel. Sketch a graph of the light's height as a function of t.

b. Give the period, midline, amplitude of your graph.

c. State the phase angle and phase shift.

d. State the ordinary frequency and angular frequency.

e. Write an equation for the function.

28. You are performing the yo-yo trick, "Around the World," in which the yo-yo is thrown so it sweeps out a vertical circle. Suppose the yo-yo string is 28 inches and it completes one revolution in 3 seconds. If the closest the yo-yo ever gets to the ground is 2 inches, find a sinusoid which models the height, h, of the yo-yo above the ground in inches t seconds after it leaves its lowest point.

29. The table below lists the average temperature of Lake Erie as measured in Cleveland, Ohio on the first of the month for each month during the years 1971 – 2000. For example, $t = 3$ represents the average of the temperatures recorded for Lake Erie on every March 1 for the years 1971 through 2000.



a. Fit a sinusoid to these data.

b. Using graphing technology, graph your model along with the data set to judge the reasonableness of the fit.

c. Use the model you found in part (a) to predict the average temperature recorded for Lake Erie on April 15th and September 15th during the years 1971–2000.

d. Compare your results to those obtained using graphing technology.

30. The fraction of the moon illuminated at midnight Eastern Standard Time on the tth day of June, 2009 is given in the table

below.



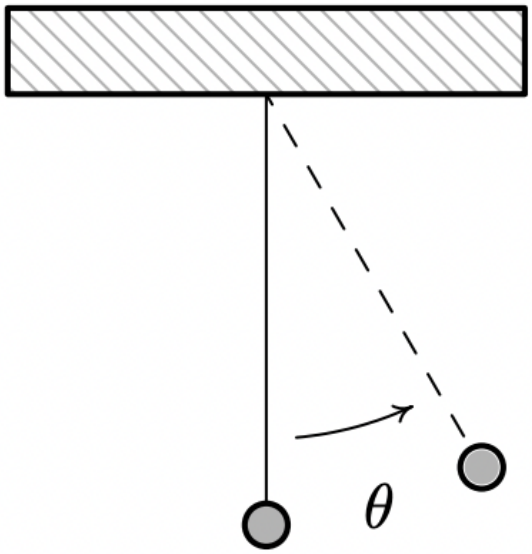
a. Fit a sinusoid to these data.

b. Using graphing technology, graph your model along with the data set to judge the reasonableness of the fit.

c. Use the model you found in part (a) to predict the fraction of the moon illuminated on June 1, 2009.

d. Compare your results to those obtained using graphing technology.

31. Consider the pendulum below. Ignoring air resistance, the angular displacement of the pendulum from the vertical position, θ, can be modeled as a sinusoid.



The amplitude of the sinusoid is the same as the initial angular displacement, θ_0, of the pendulum and the period of the motion is given by

$$T = 2\pi\sqrt{\frac{l}{g}},$$

where l is the length of the pendulum and $g \approx 9.8\frac{\text{m}}{\text{s}^2}$ is the acceleration due to gravity.

a. Find a sinusoid which gives the angular displacement θ as a function of time, t. Arrange things so $\theta(0) = \theta_0$.

b. Suppose Sang's Seth-Thomas antique schoolhouse clock needs a period of $T = \frac{1}{2}$ second and she can adjust the length of the pendulum via a small dial on the bottom of the bob. At what length should she set the pendulum?

c. Continuing with part (b) and assuming the initial displacement of the pendulum is 15°, find a sinusoid which models the displacement of the pendulum, θ, as a function of time, t, in seconds.

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5.5: Harmonic Motion

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Harmonic motion is a form of periodic motion, but there are factors to consider that differentiate the two types. Applications involving general periodic motion cycle through their periods with no outside interference. This type of motion is what we have dealt with up to this point in Trigonometry. In terms of sinusoidal functions, we think of this as **oscillatory motion**.

Definition: Oscillatory Motion

Oscillatory motion is a repetitive back-and-forth or to-and-fro movement of an object around a central point or equilibrium position. This movement is characterized by a regular pattern of variation in a quantity, like displacement, velocity, or acceleration, over time.

Examples of oscillatory motion include temperature measurements throughout the day, the elevation of a rider on a Ferris wheel, and sound waves. In all of these examples, the motion is periodic. Still, there is no physical force requiring the measured quantity to tend back to the midline of the periodic motion.

Harmonic motion, on the other hand, requires a restorative force.

Definition: Harmonic Motion

Harmonic motion is a repetitive movement back and forth through an equilibrium point. It is characterized by a restorative force that's proportional to the displacement from equilibrium and acts in the opposite direction.

As a reminder from Physics, a force can be considered a push or pull. Thus, an object goes through harmonic motion if it oscillates (much like periodic motion) but is constantly being pulled by a force back to the midline of the oscillatory motion. Harmonic motion is crucial in Differential Equations, Engineering, and Physics. Harmonic motion includes the motion of springs, gravitational force, and magnetic force. It can further be broken into two major types—**simple harmonic motion** (also called **undamped harmonic motion**) and **damped harmonic motion**.

Note: Gazing into the future

Within both simple and damped harmonic motion, you also encounter free and driven (or forced) motion. These topics are reserved for Differential Equations; however, you can think of **free motion** as motion without extra external forces being applied. **Driven** (or **forced**) **motion** occurs when the object gains or loses kinetic energy due to an extra external force being applied.

For example, if we attach a mass to a spring, allow it to stretch to equilibrium, pull down slightly once it has reached equilibrium, and then let go, the motion of the mass will be free, damped motion. It starts oscillating, and we do not add any other forces to its movement; however, the spring constantly tries to pull (or push) the mass back to equilibrium. As time goes on, the amplitude of the oscillations decreases as the mass settles to the equilibrium position.

On the other hand, if we attach a mass to a spring, allow it to stretch to equilibrium, pull down slightly once it has reached equilibrium, let go, and then give it an extra push each time it reaches its lowest point (kind of like pushing a child on a swing), the motion is driven, damped motion. It's still damped because we are trying to be realistic by assuming that, when left alone, the amplitude of the oscillations should get smaller; however, it is driven because we add energy to the mass by pushing it each time it reaches its lowest point.

Solving Exponential Equations

The material in this section relies heavily on your knowledge of exponential and logarithmic functions, the Laws of Logarithms, and your ability to solve exponential equations. Since it might have been a while since you have seen such topics, let's take a moment to review the "nuts and bolts" needed for the material at hand.

Definition: Exponential Function

An **exponential function** is a function with the form

$$f(x) = ab^x,$$

where a is a non-zero real number called the **initial value** and b is any positive real number, called the **base**, such that $b \neq 1$. This function is also referred to as the **exponential function base** b.

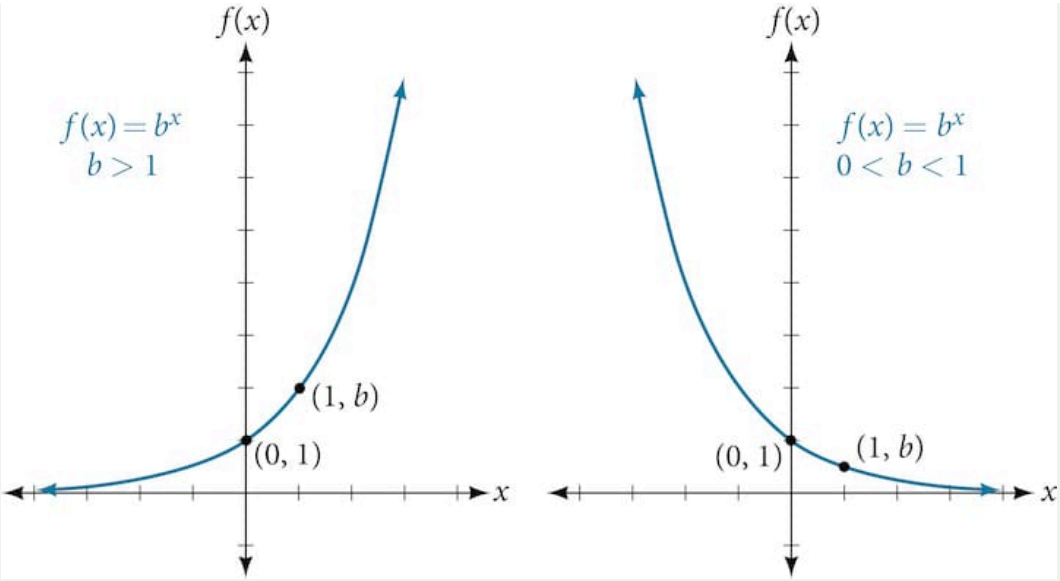
Many exponential functions appear in population models, where the function values rely on time. Therefore, you will commonly encounter exponential functions in the form $P(t) = P_0 \cdot b^t$, where $P(0) = P_0 \cdot b^0 = P_0$ and, as such, P_0 is called the **initial population**.

While there is a chapter's-worth of theory and material we could cover when talking about exponential functions (and the interested student *should* take a College Algebra course to explore such theory), for this section we just need to recall two pieces of theory. First up is the graph of the exponential function.

Theorem: Properties of the Graphs of Exponential Functions

An exponential function with the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$, has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x-intercept: none
- y-intercept: $(0, 1)$
- Secondary key point: $(1, b)$
- increasing if $b > 1$
- decreasing if $0 < b < 1$



The leftmost graph in the theorem above is of significant value when we discuss damped harmonic motion near the end of this section.

The second piece of theory we must recall from Algebra is how to solve a simple exponential equation of the form $k = e^x$, where $k > 0$.

To solve such an equation, we need to remember that logarithms and exponentials are inverses to one another. More specifically,

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln(x)} = x.$$

Therefore, to solve $k = e^x$, we apply the natural logarithmic function to both sides of the equation to get the following:

$$\begin{aligned} k &= e^x \\ \implies \ln(k) &= \ln(e^x) \\ \implies \ln(k) &= x. \end{aligned}$$

Simple Harmonic Motion

The "prototype" motion, called **simple harmonic motion**, involves a restorative force but assumes that the motion will continue forever.

Definition: Simple Harmonic Motion

Simple harmonic motion is a special type of harmonic motion resulting from a sinusoidal oscillation which continues indefinitely. We call this type of harmonic motion a **non-driven**, **undamped** motion.

There are two words in this definition that you might be unfamiliar with.

A **driven** motion is one that experiences an additional pulse of energy. Think of this as the energy a parent gives to a child when they are on a swing. The child's motion on the swing as they go back and forth is harmonic motion. Every once in a while, the parent gives an extra push to the child so that their harmonic motion speeds back up—they are *driving* the harmonic motion with extra energy (in fact, the child swinging her legs also adds a bit of extra energy to keep her in motion).

In **non-driven** harmonic motion, the parent would not be pushing the child (and the child would not be swinging her legs to gain more "swing energy"). You can imagine what would happen in this scenario—the swing would oscillate back and forth, but with less magnitude each time. Eventually, the child would stop swinging.

But *why* would the child stop swinging?

In reality, there is almost always a **damping** force—a force that slows the oscillatory motion. In the case of the child on the swing, the damping force is due to friction (specifically, the resistive force of friction due to air and likely the friction on the chains connected to the swing set.

Undamped motion, on the other hand, never experiences loss of energy due to friction or other external factors. As such, simple harmonic motion (being both undamped and non-driven) is not the most realistic type of harmonic motion; however, it's a necessary starting point to understand the robust forms.

Imagine a weighted object hanging on a spring. When that object is not disturbed, we say that the object is at rest or in **equilibrium**. If the object is pulled down and then released, the spring's force (called the spring's **restorative force**) pulls the object back toward equilibrium, and harmonic motion begins. The restorative force of the spring is directly proportional to the object's displacement from its equilibrium point.

Note: Recalling some Algebra

From Algebra, F is **directly proportional** to x if there exists a constant, k, such that $F = kx$. k is called the **constant of proportionality**.

Interactive Element: Simple Harmonic Motion

Observation: The simple harmonic motion of the object is sinusoidal.

Theorem: Simple Harmonic Motion Model

The motion of a particle undergoing simple harmonic motion is modeled using a sinusoidal function. It can be written as either

$$s(t) = A\sin(\omega t + \phi) \quad \text{or} \quad s(t) = A\cos(\omega t + \phi),$$

where the frequency of the motion is $f = \frac{1}{\text{Period}} = \frac{\omega}{2\pi}$ cycles per unit time.

The position of a particle relative to the ground is often unimportant when building a model for simple harmonic motion. This is because we are only concerned with modeling the particle's motion—not the height. Moreover, all our previous training with graphs of sinusoidal functions still holds here. The only difference is that we are replacing B with ω, and C with ϕ. This is mainly because ω and ϕ are the typical notations used in Physics.

Finally, the sinusoidal nature of these phenomena observed in physics is just that—an observation. We have not, and cannot, prove this result in Trigonometry. You do, however, prove this result in Differential Equations.

Example 5.5.1

The displacement of a mass suspended by a spring is modeled by the function

$$s(t) = 1.5\sin(6\pi t),$$

where $s(t)$ is measured in cm and t in seconds.

a. Find the amplitude, period, and frequency of the motion of the mass.
b. Sketch a graph of the displacement of the mass.

Solutions available at the end of this section.

Checkpoint 5.5.1

A mass suspended from a spring is pulled 3 cm below its resting position. The mass is released at time $t = 0$ and allowed to oscillate. After $\frac{1}{3}$ second, it is observed that the mass returns to its lowest position. Find a function to model this motion relative to its initial resting position.

Answer available at the end of this section.

Damped Harmonic Motion

In reality, a pendulum does not swing back and forth forever, and an object on a spring does not bounce up and down forever. Eventually, the pendulum stops swinging, the object stops bouncing, and both return to equilibrium. As mentioned previously, periodic motion in which a damping force (typically friction) acts is known as **damped harmonic motion**.

Definition: Damped Harmonic Motion

Damped harmonic motion results from a sinusoidal oscillation that decreases in amplitude over time due to energy dissipation. This is typically caused by frictional forces or other dissipative forces that remove energy from the system.

Interactive Element: Damped Harmonic Motion

Observation: The quasi-sinusoidal harmonic motion of the object is being damped.

In Physics, various formulas account for the damping factor on the moving object. Some of these are Calculus-based formulas that involve derivatives. We will use damping functions based on decreasing exponential functions (as these are the most common damping functions).

Theorem: Damped Harmonic Motion

The motion of a particle undergoing damped harmonic motion is modeled using a **quasi-sinusoidal function**. The displacement of the particle can be written as either

$$s(t) = A\,e^{-ct}\sin(\omega t + \phi) \quad \text{or} \quad s(t) = A\,e^{-ct}\cos(\omega t + \phi),$$

where c is called the **damping factor**, $\frac{2\pi}{\omega}$ is called the **quasi-period**, and $\frac{1}{\text{Quasi-Period}} = \frac{\omega}{2\pi}$ is called the **quasi-frequency**.

The words quasi-period and quasi-frequency reflect the fact that these models are not true sinusoidal models.

Conceptually (and graphically), the decreasing exponential function forces a boundary on the model's position values (outputs). The following Interactive Element should provide more understanding than I could write in a few paragraphs.

Interactive Element: Damped Harmonic Model (and the exponential boundaries)

Interact: Move the slider to adjust the damping factor.

Observation: The boundary of the damped harmonic motion is always $y = \pm A\ e^{-ct}$.

Example 5.5.2

Model the equations that fit the two scenarios and use graphing technology to graph the functions: Two mass-spring systems exhibit damped harmonic motion at a quasi-frequency of 0.5 cycles per second. Both have an initial displacement of 10 cm. The first has a damping factor of 0.5, and the second has a damping factor of 0.1.

Solutions available at the end of this section.

Checkpoint 5.5.2

Find and graph a function of the form $s(t) = A\ e^{-ct}\cos(\omega t)$ that models the information given.

a. $A = 20,\ c = 0.05$, and the quasi-period is 4
b. $A = 2,\ c = 1.5$, and the quasi-frequency is 3

Answers available at the end of this section.

Example 5.5.3

Find and graph a function of the form $s(t) = A\ e^{-ct}\sin(\omega t)$ that models the information given.

a. $A = 7$, $c = 10$, and the quasi-period is $p = \frac{\pi}{6}$
b. $A = 0.3$, $c = 0.2$, and the quasi-frequency is $f = 20$

Solutions available at the end of this section.

A comparison of Examples $5.5.3a$ and $5.5.3b$ illustrates how we choose between the sine or cosine functions to model sinusoidal criteria. We see that the cosine function is at the maximum displacement when $t = 0$, and the sine function is at the equilibrium point when $t = 0$. For example, consider the equation

$$y = 20e^{-0.05t}\cos\left(\frac{\pi}{2}t\right)$$

from Checkpoint $5.5.2$. We can see from the graph that when $t = 0$, $y = 20$, which is the initial amplitude. Check this by setting $t = 0$ in the cosine equation:

$$\begin{aligned} y &= 20e^{-0.05(0)}\cos\left(\frac{\pi}{2}\cdot 0\right) \\ &= 20(1)(1) \\ &= 20 \end{aligned}$$

Using the sine function yields

$$\begin{aligned} y &= 20e^{-0.05(0)}\sin\left(\frac{\pi}{2}\cdot 0\right) \\ &= 20(1)(0) \\ &= 0 \end{aligned}$$

Thus, cosine is the correct function.

Checkpoint 5.5.3

Write the equation for damped harmonic motion given $A = 10$, $c = 0.5$, and the quasi-period is $p = 2$, where the initial value of the function is $s(0) = -10$.

Answer available at the end of this section.

Example 5.5.4

A spring measuring 10 inches in natural length is compressed by 5 inches and released. It oscillates once every 3 seconds, and its amplitude decreases by 30% every second. Find an equation that models the position of the spring t seconds after being released.

Solution available at the end of this section.

Example 5.5.5

A guitar string is plucked and vibrates in a damped harmonic motion. The string is pulled and displaced 2 cm from its resting position. After 3 seconds, the displacement measures 1 cm. Find the damping constant.

Solution available at the end of this section.

Miscellaneous Topics: Thinking About Mathematics (Part 3)

> When I would get frustrated in school, I often found it good to reflect upon some of my favorite teachers and think about how they would approach the subject.
>
> - Professor Simpson, Cosumnes River College

1. Who was your favorite teacher, and what subject did they teach?
2. Can you make any suggestions I can incorporate into the classroom that worked for that teacher?

Success in College: Self-Motivation Learning Preferences

Choose the statements that best describe you.

I come to class because I don't want to miss anything.	or	I come to class because it is required.
I would probably do less work in this class if I were graded pass/fail instead of a letter grade.	or	I would probably do the same amount of work in this class if I was graded pass/fail instead of with a letter grade.
I need frequent homework assignments to keep me working.	or	I do homework frequently to know what is going on in class.

1. Do you consider the choices above to be good or bad for your education?
2. Some professors provide detailed schedules with assignments and test dates at the beginning of the term. Others announce the homework assignments at each class meeting and test days during the term. Which do you prefer?

Answers

Section Preview

The material in this section could easily be taught when covering the material in the previous section. I broke it into its own section because I didn't want to have a massive sinusoidal model section. Also, harmonic motion is a big topic in Differential Equations (nearly a full chapter's worth) and having a single section for it here might make it stand out more for the student.

Section Preview

- **Exponential Functions**
  - The Basics of Exponential Functions: Exponential functions are needed to understand the decaying envelope Ae^{-ct}.
- **Logarithmic Functions**

- The Basics of Logarithmic Functions: The basics of logarithmic functions are needed to understand the inverse relationship $\ln(e^x) = x$ used to solve for the damping constant.

Section Preview

- **Logarithmic Functions**
  - Laws of Logarithms — Power Law: The Power Law is used explicitly in Example 5 to rewrite $\ln(1/2)^{-1}$.
- **Solving Equations**
  - Solving Exponential Equations: Solving exponential equations is the direct skill used throughout Examples 4 and 5 when solving $0.7 = e^{-c}$ and $\frac{1}{2} = e^{-3c}$ for the damping constant c.

Section Preview

- Build a model for simple harmonic motion.
- Build a model for damped harmonic motion.

Example 5.5.1

a. From our previous experience, the amplitude is $|1.5| = 1.5$, the period is $\frac{2\pi}{\omega} = \frac{2\pi}{6\pi} = \frac{1}{3}$ seconds per cycle, and the frequency is $f = \frac{1}{\text{Period}} = 3$ cycles per second.

b.

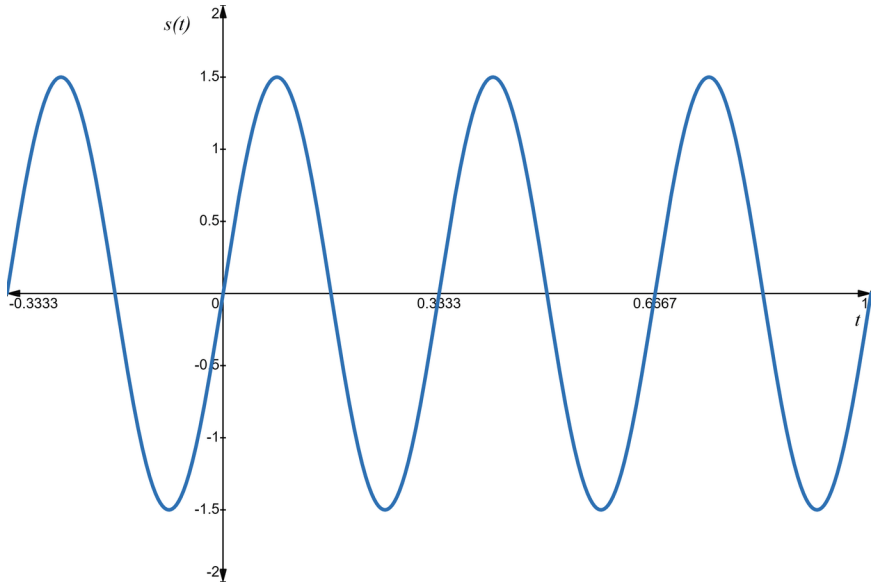


Figure 5.5.1

Checkpoint 5.5.1

$s(t) = -3\cos(6\pi t)$

Example 5.5.2

At time $t = 0$, the displacement is the maximum of 10 cm, which calls for the cosine function. The cosine function will apply to both models. We are given the quasi-frequency, $f = \frac{\omega}{2\pi}$, of 0.5 cycles per second. Thus,

$$\begin{aligned} \frac{\omega}{2\pi} &= 0.5 \\ \Longrightarrow \quad \omega &= \pi \end{aligned}$$

The first spring system has a damping factor of $c = 0.5$. Following the general model for damped harmonic motion, we have

$$s_1(t) = 10e^{-0.5t}\cos(\pi t).$$

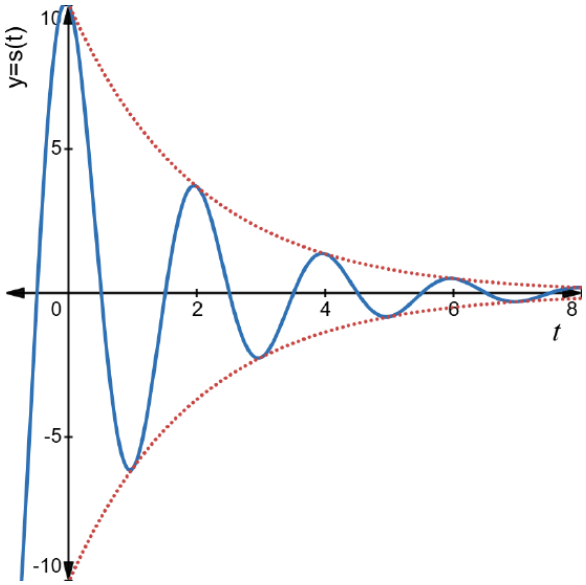


Figure 5.5.2

The second spring system has a damping factor of $c = 0.1$ and can be modeled as

$$s_2(t) = 10e^{-0.1t}\cos(\pi t).$$

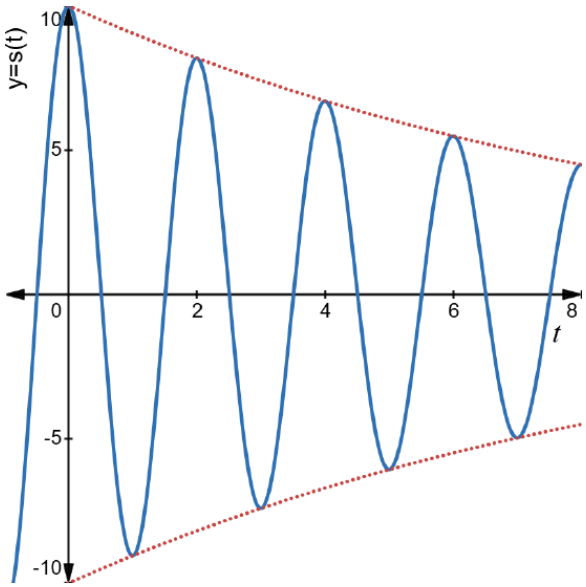


Figure 5.5.3

Notice the differing effects of the damping constant. The function's local maximum and minimum values with the damping factor $c = 0.5$ decrease much more rapidly than the function with $c = 0.1$.

Checkpoint 5.5.2

a. $s(t) = 20e^{-0.05t}\cos\left(\frac{\pi}{2}t\right)$.

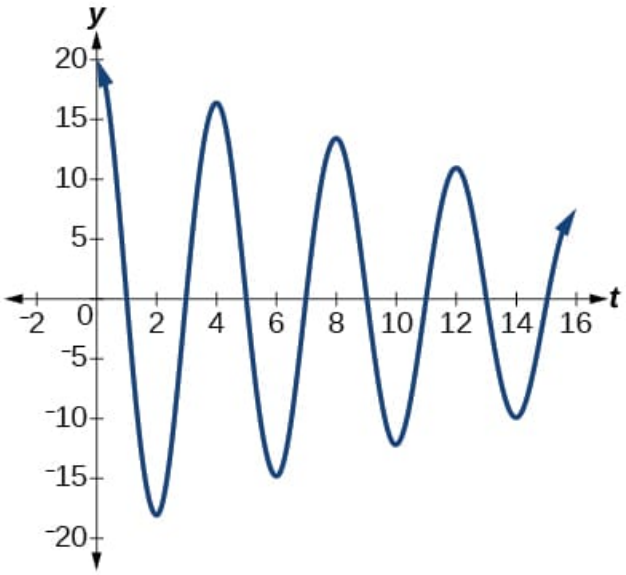


Figure 5.5.4

b. $s(t) = 2e^{-1.5t}\cos(6\pi t)$.

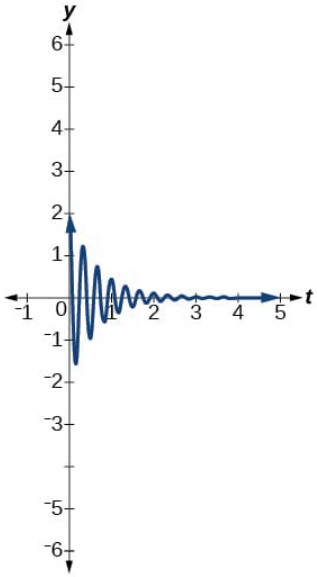


Figure 5.5.5

Example 5.5.3

Calculate the value of ω and substitute the known values into the model.

a. As the quasi-period is $\frac{2\pi}{\omega}$, we have

$$\begin{aligned} \frac{\pi}{6} &= \frac{2\pi}{\omega} \\ \Longrightarrow \quad \omega\pi &= 6(2\pi) \\ \Longrightarrow \quad \omega &= 12 \end{aligned}$$

The damping factor is given as 10 and A is 7. Thus, the model is $s(t) = 7e^{-10t}\sin(12t)$.

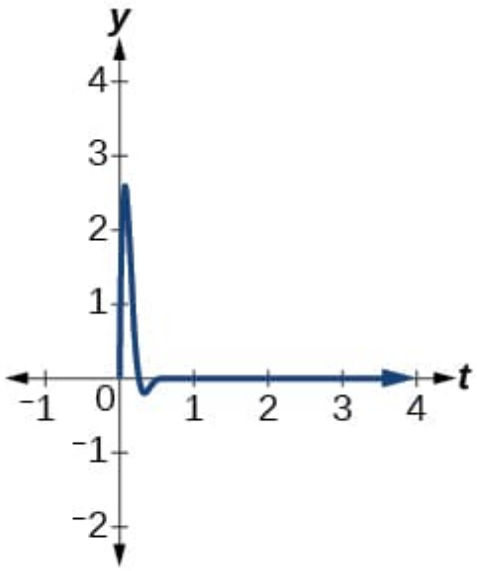


Figure 5.5.6

b. As the quasi-frequency is $\frac{\omega}{2\pi}$, we have

$$\begin{aligned} 20 &= \frac{\omega}{2\pi} \\ \implies \quad 40\pi &= \omega \end{aligned}$$

The damping factor is given as 0.2 and $A = 0.3$. The model is $s(t) = 0.3e^{-0.2t}\sin(40\pi t)$.

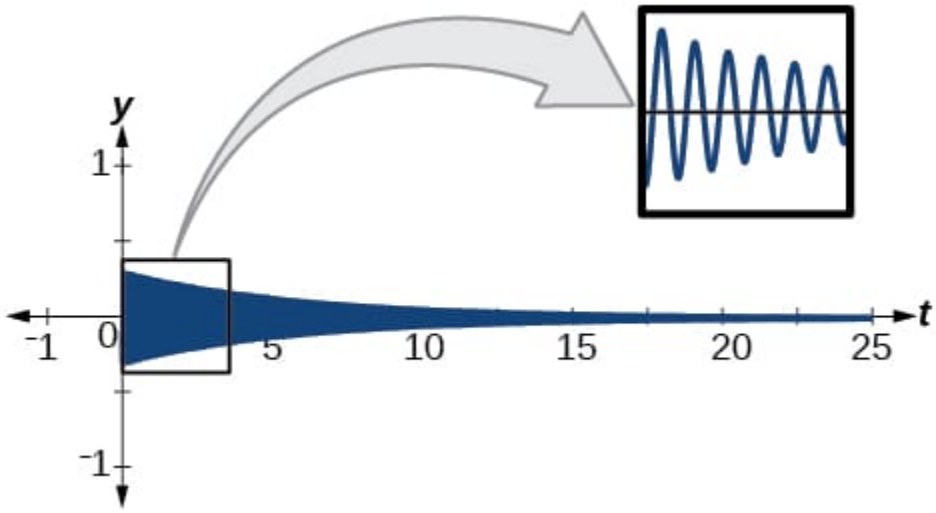


Figure 5.5.7

Checkpoint 5.5.3

$s(t) = -10e^{-0.5t}\cos(\pi t)$

Example 5.5.4

The amplitude begins at 5 inches and decreases by 30% each second. We will make an "amplitude function" as an intermediate step. We can write the amplitude portion of the function as

$$\text{Amp}(t) = Ae^{-ct}.$$

We know that the amplitude is 5 when $t = 0$. Hence,

$$5 = Ae^{-c(0)} \implies 5 = Ae^{0} \implies 5 = A(1) \implies 5 = A.$$

Therefore, our amplitude function becomes

$$\text{Amp}(t) = 5e^{-ct}.$$

We also know the amplitude drops by 30% when $t = 1$. "Dropping by 30%" is the same as saying the amplitude at $t = 1$ is 70% (100% minus 30%) of its original value at $t = 1$. Thus, we want to solve $0.7(5) = 5e^{-c(1)}$.

$$\begin{aligned} 0.7(5) &= 5e^{-c(1)} \\ \Longrightarrow \quad 0.7 &= e^{-c} && \text{(dividing)} \\ \Longrightarrow \quad \ln(0.7) &= -c && \text{(taking the log of both sides)} \\ \Longrightarrow \quad -\ln(0.7) &= c && \text{(solving for } c\text{)} \\ \Longrightarrow \quad c &\approx -0.357 \end{aligned}$$

Let's address the quasi-period. The spring completes a cycle every 3 seconds. This is the quasi-period, and we can use the formula to find ω.

$$\begin{aligned} 3 &= \frac{2\pi}{\omega} \\ \Longrightarrow \quad \omega &= \frac{2\pi}{3} \end{aligned}$$

The natural length of 10 inches is the midline. We will use the cosine function since the spring starts at maximum displacement. This portion of the equation is represented as

$$y = \cos\left(\frac{2\pi}{3}t\right) + 10.$$

Finally, we put both functions together. Our model for the position of the spring at t seconds is given as

$$y \approx 5e^{-0.357t}\cos\left(\frac{2\pi}{3}t\right) + 10.$$

The approximation sign is needed because the coefficient of t is an approximation.

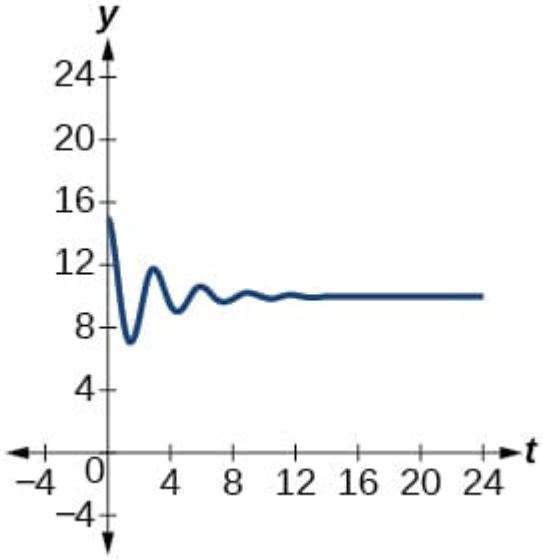


Figure 5.5.8

Example 5.5.5

Following our tactic from Example 5.5.4, we recognize that the initial displacement, 2 cm, represents A. Therefore, the amplitude function for our model is

$$\text{Amp}(t) = 2e^{-ct}.$$

It is stated that after 3 seconds, maximum displacement measures one-half of its original value. Therefore, we have the equation

$$\frac{2}{2} = 2\,e^{-c(3)}.$$

Solving for c, we get the following:

$$\begin{aligned} \frac{2}{2} &= 2\,e^{-c(3)} \\ \Longrightarrow \quad 1 &= 2e^{-3c} && \text{(simplifying)} \\ \Longrightarrow \quad \frac{1}{2} &= e^{-3c} && \text{(dividing both sides by 2)} \\ \Longrightarrow \quad \ln\left(\frac{1}{2}\right) &= -3c && \text{(taking the ln of both sides)} \\ \Longrightarrow \quad -\frac{\ln\left(\frac{1}{2}\right)}{3} &= c && \text{(dividing both sides by } -3\text{)} \\ \Longrightarrow \quad \frac{\ln\left(\frac{1}{2}\right)^{-1}}{3} &= c && \text{(Laws of Logarithms)} \\ \Longrightarrow \quad \frac{\ln(2)}{3} &= c && \text{(Laws of Exponents)} \end{aligned}$$

Thus, the damping constant is $c = \frac{\ln(2)}{3}$.

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5.5.1: Resources and Key Concepts

Resources

Prerequisite Topics

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Exponential Functions**
  - The Basics of Exponential Functions: This section introduces damped harmonic motion, which is modeled using an exponential decay factor. Understanding the behavior of exponential functions is crucial.
  - Graphing Exponential Functions: Visualizing the exponential decay that bounds the graph of a damped harmonic motion function is a key concept in this section.
- **Logarithmic Functions**
  - The Basics of Logarithmic Functions: Since the Laws of Logarithms are needed, it is necessary to understand the fundamentals of logarithms.
  - Laws of Logarithms: These laws are used when solving for the damping constant in application problems, as seen in Example 5.
- **Solving Equations**
  - Solving Exponential Equations: This skill is required to find the damping constant, c, from an amplitude function of the form $A(t) = A_0 e^{-ct}$, as demonstrated in Example 4.

Videos

- Simple Harmonic Motion
  - Modeling Harmonic Motion - Simple Harmonic Motion
- Damped Harmonic Motion
  - Modeling Harmonic Motion - Damped Harmonic Motion

Key Concepts

Definitions

- **Oscillatory Motion:** A repetitive back-and-forth or to-and-fro movement of an object around a central point or equilibrium position.
- **Harmonic Motion:** A repetitive movement back and forth through an equilibrium point, characterized by a restorative force that is proportional to the displacement from equilibrium and acts in the opposite direction.
- **Exponential Function:** A function of the form $f(x) = ab^x$, where a is the non-zero initial value and b is a positive real number (not equal to 1) called the base.
- **Simple Harmonic Motion:** A special type of harmonic motion resulting from a sinusoidal oscillation that continues indefinitely without any loss of energy. It is also known as non-driven, undamped motion.
- **Damped Harmonic Motion:** A sinusoidal oscillation that decreases in amplitude over time due to energy dissipation, typically caused by frictional or other dissipative forces.

Theorems

- **Properties of the Graphs of Exponential Functions:** An exponential function of the form $f(x) = b^x$ (where $b > 0, b \neq 1$) is one-to-one, has a horizontal asymptote at $y = 0$, a domain of $(-\infty, \infty)$, a range of $(0, \infty)$, and a y-intercept at $(0, 1)$.
- **Simple Harmonic Motion Model:** The motion of a particle undergoing simple harmonic motion is modeled by a sinusoidal

function, which can be written as either $s(t) = A\sin(\omega t + \phi)$ or $s(t) = A\cos(\omega t + \phi)$, where $f = \frac{\omega}{2\pi}$ is the frequency of the motion.
- **Damped Harmonic Motion:** The displacement of a particle undergoing damped harmonic motion is modeled using a quasi-sinusoidal function, which can be written as either $s(t) = Ae^{-ct}\sin(\omega t + \phi)$ or $s(t) = Ae^{-ct}\cos(\omega t + \phi)$. In these models, c is the damping factor, $\frac{2\pi}{\omega}$ is the quasi-period, and $\frac{\omega}{2\pi}$ is the quasi-frequency.

Common Mistakes

- **Confusing Harmonic and Oscillatory Motion:** While all harmonic motion is oscillatory, not all oscillatory motion is harmonic. Harmonic motion specifically requires a restorative force that pulls an object back toward an equilibrium position.
- **Confusing Angular Frequency with Ordinary Frequency:** In the context of harmonic motion, ω represents the angular frequency. The ordinary frequency, f, which measures cycles per unit time (Hertz), must be calculated using the formula $f = \frac{\omega}{2\pi}$.
- **Choosing the Incorrect Model Function:** When modeling harmonic motion, a cosine function is typically used if the motion starts at maximum displacement ($t = 0$). A sine function is used if the motion starts at the equilibrium point ($t = 0$).
- **Ignoring the Damping Factor:** Forgetting that the amplitude in damped harmonic motion is not constant but is a decreasing exponential function, $A(t) = A_0e^{-ct}$, will lead to an incorrect model.

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5.5.2: Homework

Reading Questions

1. What is the key feature that distinguishes harmonic motion from general oscillatory motion?
2. What is the difference between simple harmonic motion and damped harmonic motion?
3. In the model for simple harmonic motion, $s(t) = A\sin(\omega t + \phi)$, what do A and ω represent?
4. How is the ordinary frequency, f, related to the angular frequency, ω?
5. What is the general form of the function that models damped harmonic motion?
6. In the damped harmonic motion model, what is the term for the parameter c?
7. What is a "quasi-period"?
8. When modeling the motion of a spring, what initial condition suggests using a cosine function?
9. When modeling the motion of a spring, what initial condition suggests using a sine function?
10. To solve for the damping constant c in an amplitude function like $A(t) = A_0 e^{-ct}$, what mathematical function is typically used?

Homework

Vocabulary Check

1. The force associated with an object's tendency to return to equilibrium is called the ___ force.
2. Of the two types of harmonic motion discussed in this section, ___ harmonic motion is more realistic as it considers friction.
3. While an object undergoing simple harmonic motion has a period and frequency, an object undergoing damped harmonic motion has a ___ and ___.
4. Another name for simple harmonic motion is ___ harmonic motion.

Concept Check

5. Explain the effect of a damping factor on the graphs of harmonic motion functions.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

6. A particle undergoing harmonic motion will theoretically oscillate forever.
7. A particle undergoing damped harmonic motion will theoretically oscillate forever.
8. The frequency of the particle undergoing damped harmonic motion described by
$$s(t) = 2e^{-0.01t}\sin\left(\frac{2\pi}{3}t\right)$$
is $f = \frac{\omega}{2\pi} = \frac{2\pi/3}{2\pi} = \frac{1}{3}$.

Basic Skills

For the following exercises, find the amplitude, period, and frequency of the given function.

9. The displacement, $h(t)$, in centimeters of a mass suspended by a spring is modeled by the function
$$h(t) = 14\sin(120\pi t),$$
where t is measured in seconds.

10. The displacement, $h(t)$, in centimeters of a mass suspended by a spring is modeled by the function
$$h(t) = 8\sin(6\pi t),$$
where t is measured in seconds.

11. The displacement, $h(t)$, in centimeters of a mass suspended by a spring is modeled by the function
$$h(t) = 11\sin(12\pi t),$$
where t is measured in seconds.

12. The displacement, $h(t)$, in centimeters of a mass suspended by a spring is modeled by the function
$$h(t) = 4\cos(2\pi t),$$
where t is measured in seconds.

13. The displacement, $h(t)$, in centimeters, of a mass suspended by a spring is modeled by the function
$$h(t) = -5\cos(60\pi t),$$
where t is measured in seconds.

Applications

For the following exercises, construct an equation that models the described behavior.

14. A spring attached to the ceiling is pulled 10 cm down from equilibrium and released. The amplitude decreases by 15% each second. The spring oscillates 18 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

15. A spring attached to the ceiling is pulled 7 cm down from equilibrium and released. The amplitude decreases by 11% each second. The spring oscillates 20 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

16. A spring attached to the ceiling is pulled 17 cm down from equilibrium and released. After 3 seconds, the amplitude has decreased to 13 cm. The spring oscillates 14 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

17. A spring attached to the ceiling is pulled 19 cm down from equilibrium and released. After 4 seconds, the amplitude has decreased to 14 cm. The spring oscillates 13 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

For the following exercises, create a function modeling the described behavior. Then, calculate the desired result using a calculator.

18. A spring attached to a ceiling is pulled down 11 cm from equilibrium and released. After 2 seconds, the amplitude has decreased to 6 cm. The spring oscillates 8 times each second. Find when the spring first comes between -0.1 and 0.1 cm, effectively at rest.

19. A spring attached to a ceiling is pulled down 21 cm from equilibrium and released. After 6 seconds, the amplitude has decreased to 4 cm. The spring oscillates 20 times each second. Find when the spring first comes between -0.1 and 0.1 cm, effectively at rest.

20. Two springs are pulled down from the ceiling and released at the same time. The first spring, which oscillates 8 times per second, was initially pulled down 32 cm from equilibrium, and the amplitude decreases by 50% each second. The second spring, oscillating 18 times per second, was initially pulled down 15 cm from equilibrium and after 4 seconds has an amplitude of 2 cm. Which spring comes to rest first, and at what time? Consider “rest” as an amplitude less than 0.1 cm.

21. Two springs are pulled down from the ceiling and released at the same time. The first spring, which oscillates 14 times per second, was initially pulled down 2 cm from equilibrium, and the amplitude decreases by 8% each second. The second spring, oscillating 22 times per second, was initially pulled down 10 cm from equilibrium and after 3 seconds has an amplitude of 2 cm. Which spring comes to rest first, and at what time? Consider “rest” as an amplitude less than 0.1 cm.

22. A spring attached to a ceiling is pulled down 20 cm. After 3 seconds, wherein it completes 6 full periods, the amplitude is only 15 cm. Find the function modeling the position of the spring t seconds after being released. At what time will the spring come to rest? In this case, use 1 cm amplitude as rest.

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5.6: Graphs of the Remaining Trigonometric Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

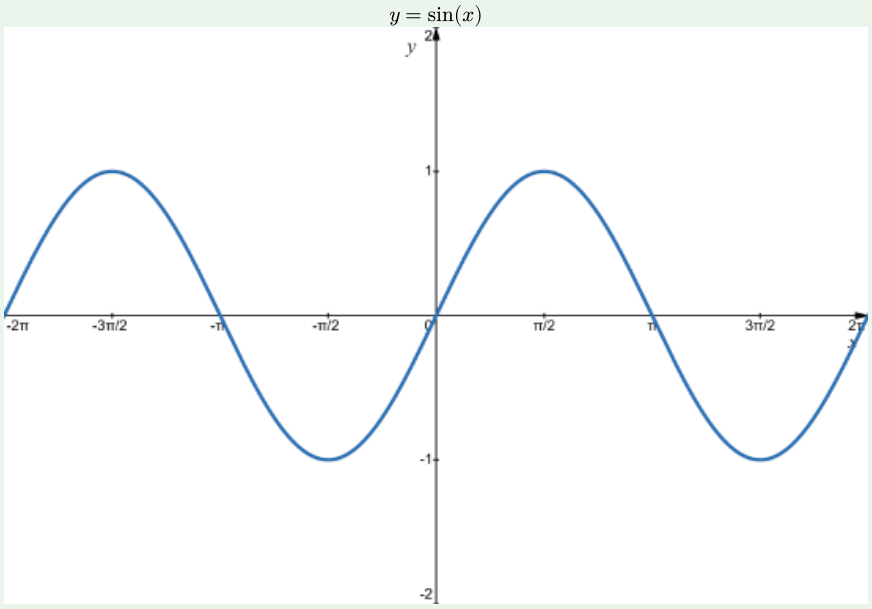
The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

A Summary of the Graphs of the Fundamental Trigonometric Functions

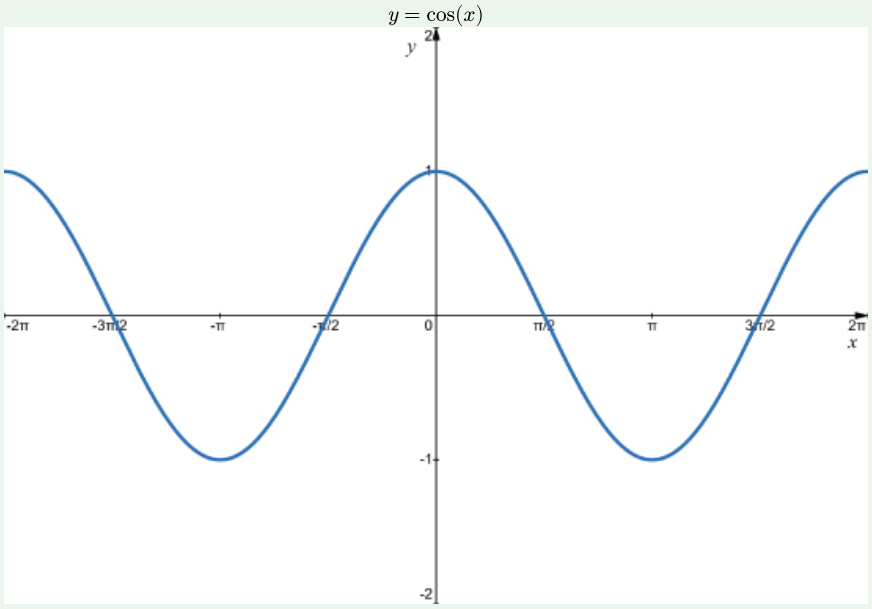
Before we go too far, let's summarize what we know about the fundamental trigonometric functions.

Theorem: Properties of the Graph of the Sine Function



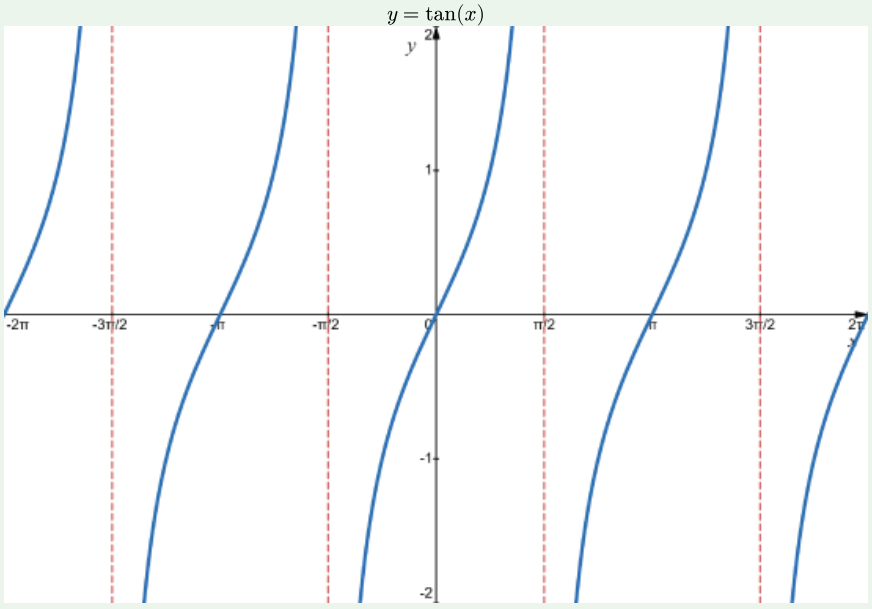
- **Domain:** $(-\infty, \infty)$
- **Range:** $[-1, 1]$
- **Symmetry:** Odd (symmetric about the origin)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** 1
- **Zeros:** $x = k\pi$, where $k \in \mathbb{Z}$

Theorem: Properties of the Graph of the Cosine Function



- **Domain:** $(-\infty, \infty)$
- **Range:** $[-1, 1]$
- **Symmetry:** Even (symmetric about the y-axis)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** 1
- **Zeros:** $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$

Theorem: Properties of the Graph of the Tangent Function



- **Domain:** All $x \neq \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$
- **Range:** $(-\infty, \infty)$
- **Symmetry:** Odd (symmetric about the origin)

- **Principal Cycle:** $[0, \pi]$
- **Natural Period:** π
- **Amplitude:** N/A
- **Zeros:** $x = k\pi$, where $k \in \mathbb{Z}$

The principal cycle of the tangent is $[0, \pi]$, even though the tangent is undefined halfway through the cycle. It's just the cycle we use as the "standard" for our base graph, and it is okay that the tangent is undefined at some point within the cycle. The principal cycle is *not* the domain.

An Interlude—Vertical Asymptotes

We now turn our attention to the graphs of the remaining trigonometric functions—$\csc(x)$, $\sec(x)$, and $\cot(x)$. In each of these cases, we must understand the conceptual approach to graphing we developed in when we introduced the graphs of the fundamental trigonometric functions.

Recall, if given a fraction where the numerator is held constant and the denominator is *approaching* zero (but not equal to zero), then the value of the fraction is approaching either $+\infty$ or $-\infty$. To drive the point home, let's demonstrate this behavior both numerically and graphically.

For the numerical demonstration, you can create any rational function you want where the numerator is a constant and the denominator is an algebraic expression. All we require is that the denominator approaches zero as the values of x approach some value. For example, suppose we choose the numerator to be 7 and the denominator to be $6x^2 + 11x + 3$. Then the function we are working with is

$$R(x) = \frac{7}{6x^2 + 11x + 3}.$$

We now want the values of x to approach some number that causes division by zero. To reveal what causes division by zero for this function, we factor the denominator as follows:

$$R(x) = \frac{7}{(3x + 1)(2x + 3)}.$$

We can see this function would have division by zero at $x = -\frac{1}{3}$ and $x = -\frac{3}{2}$.

Let's build two tables of values—the first to see what happens to the function as $x \to -\frac{3}{2}^-$, and the second to see what happens as $x \to -\frac{3}{2}^+$. For convenience, we will rewrite $-\frac{3}{2}$ in its decimal form, -1.5.





In the first table, as x is approaching -1.5 from the left, the denominator of $R(x)$ is approaching zero from the right. Simultaneously, the function values are becoming larger and larger positive numbers. That is,

$$\text{As } x \to -\frac{3}{2}^-,\ R(x) \to \infty.$$

Similarly, the second table demonstrates that, as x is approaching -1.5 from the right, the denominator of $R(x)$ approaches zero from the left. This causes the function values to become larger and larger negative numbers. We would say,

$$\text{As } x \to -\frac{3}{2}^+,\ R(x) \to -\infty.$$

For the graphical demonstration, we quickly sketch the graph of $R(x)$ around $x = -\frac{3}{2}$.

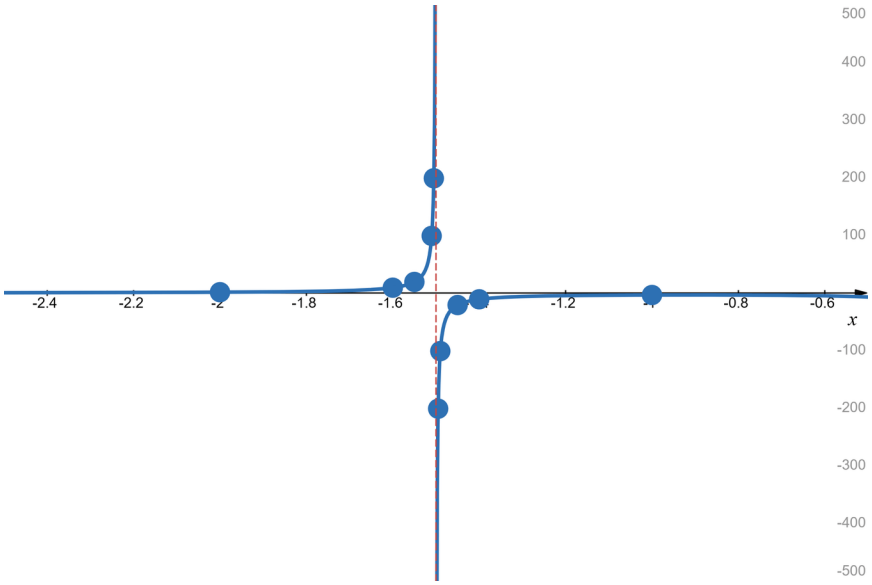


Figure 5.6.1

As you can see from the graph, the y-values of the function (also called the function values) become more and more positive as the x-values get closer and closer to -1.5 from the left. Likewise, the function values become increasingly more negative as the x -values approach -1.5 from the right.

The moral of the story can be stated as follows:

> If the numerator is a nonzero constant and the denominator is tending to (but not becoming) zero, the ratio
>
> $$\frac{\text{numerator}}{\text{denominator}}$$
>
> approaches either $+\infty$ or $-\infty$. The sign can be determined by comparing the signs of the numerator and denominator as the denominator approaches zero. Graphically, this means there is a vertical asymptote at this value of x.

Graphing the Cosecant Function

We know that $\csc(x) = \frac{1}{\sin(x)}$. Therefore, by the preceding discussion, as the denominator tends to zero, the overall ratio (which is the cosecant) will tend to $+\infty$ or $-\infty$. The zeros of the sine function occur at $x = k\pi$, where $k \in \mathbb{Z}$ (see Figure 5.6.2). Let's start by focusing on the zero at $x = 0$.

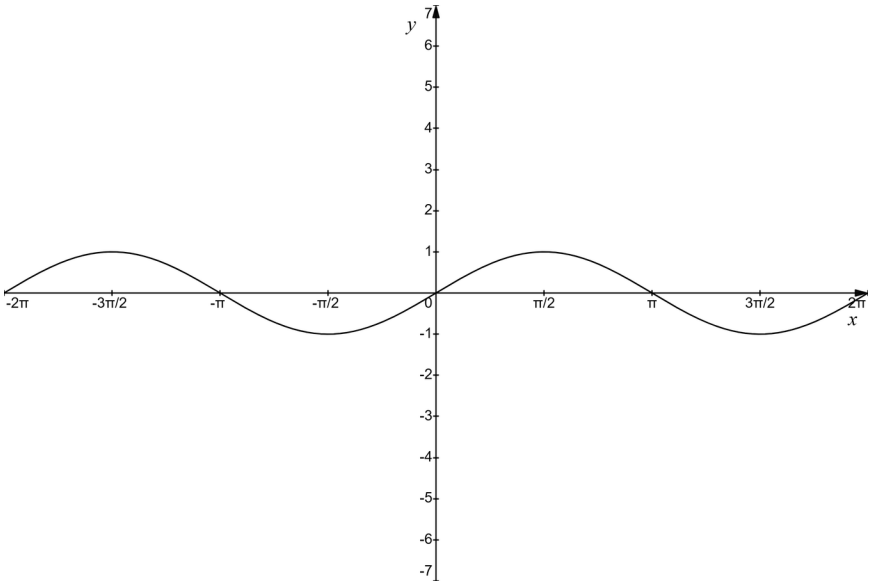


Figure 5.6.2**:** $y = \sin(x)$

From Figure 5.6.2, we can see as $x \to 0^-$, the y-values of $\sin(x)$ are approaching 0^-. Therefore,

$$\text{as } x \to 0^-,\ \frac{1}{\sin(x)} \to \frac{1}{0^-} \to -\infty.$$

It is *negative* infinity because the numerator is *positive*, but the denominator is negative (albeit, very close to zero). Similarly, as $x \to 0^+$, the y-values of $\sin(x)$ are approaching 0^+. Therefore,

$$\text{as } x \to 0^+,\ \frac{1}{\sin(x)} \to \frac{1}{0^+} \to +\infty = \infty.$$

Again, we know the result is *positive* infinity because the numerator is positive and the denominator is positive. Sketching this behavior, we get the following:

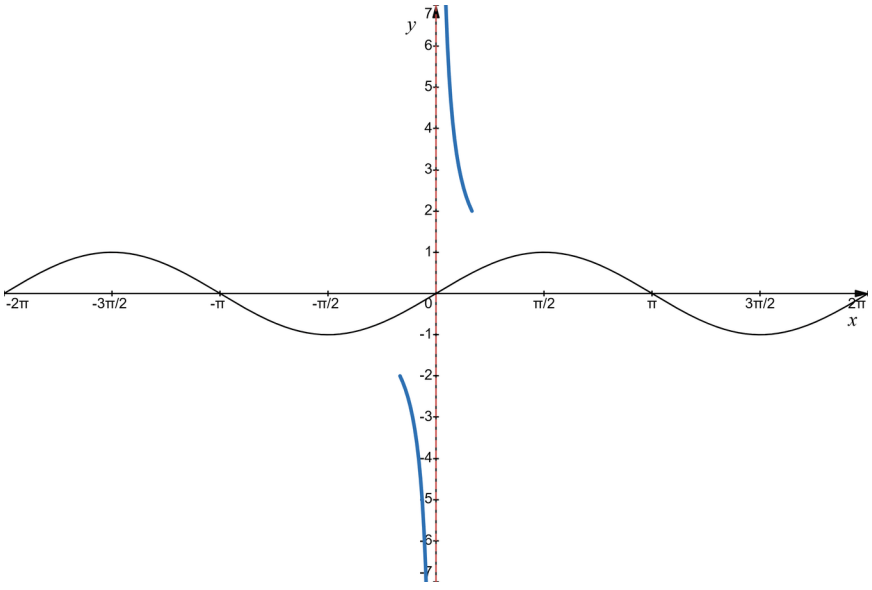


Figure 5.6.3**:** *Beginning the sketch of* $y = \csc(x)$

It's hard to see in Figure 5.6.3, but we drew a vertical asymptote at $x = 0$ (the y-axis). A similar investigation shows that

$$\text{as } x \to \pi^-,\ \frac{1}{\sin(x)} \to \frac{1}{0^+} \to \infty$$

and

$$\text{as } x \to \pi^+, \ \frac{1}{\sin(x)} \to \frac{1}{0^-} \to -\infty.$$

Using this to fill in a little more of the graph, we get the following:

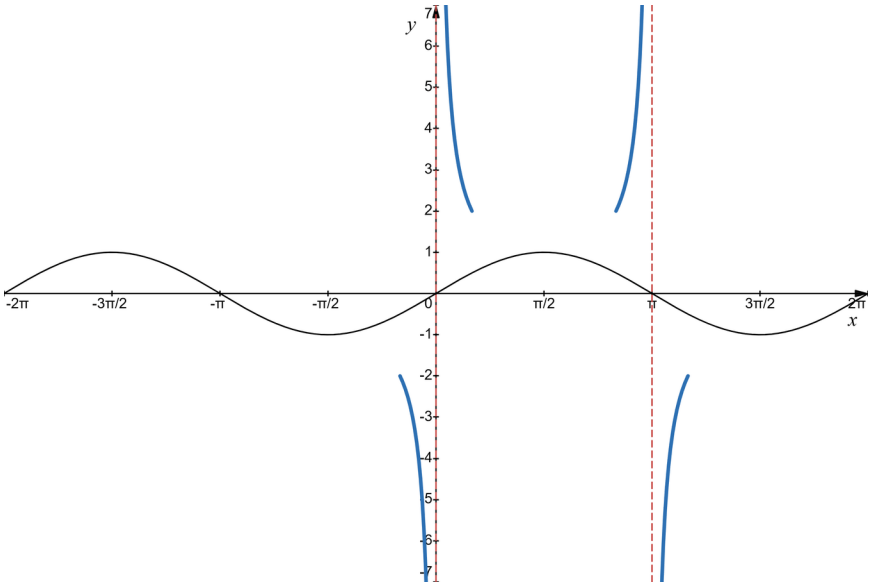


Figure 5.6.4**:** *Getting closer to the graph of* $y = \csc(x)$

From this, we expect the cosecant function to have vertical asymptotes at these zeros of $\sin(x)$ (due to the division by zero). Furthermore, we expect the cosecant to be positive when its reciprocal (the sine) is positive, and negative when the reciprocal is negative. This gets us very close to the base graph of the cosecant.

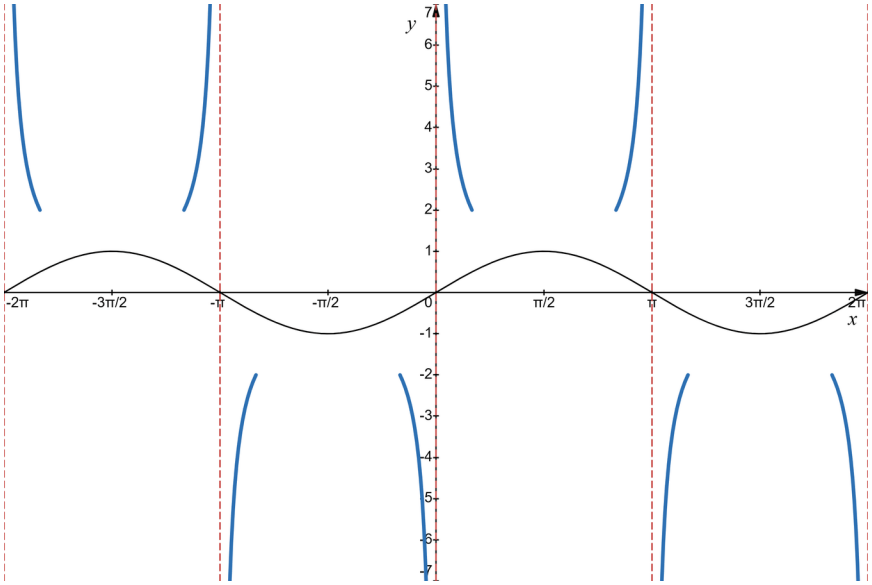


Figure 5.6.5**:** *Almost there*

To finish out the graph of the base cosecant function, we will build a table of values for the interval $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ and use this to extrapolate the features for the rest of the graph.

[redacted]

Plotting these values, we get the following:

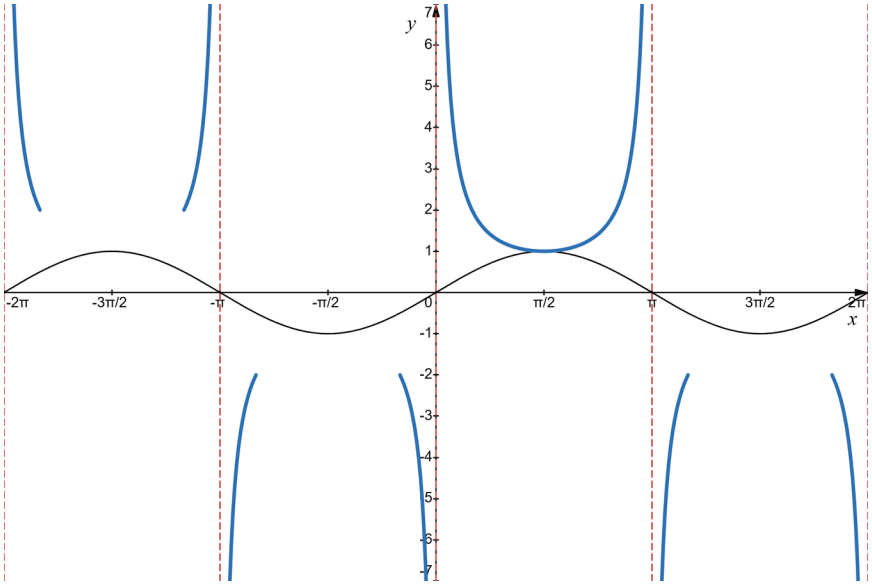


Figure 5.6.6**:** *So close!*

We can now see the pattern for the entire graph of the cosecant. Filling in the blanks, we get our final graph.

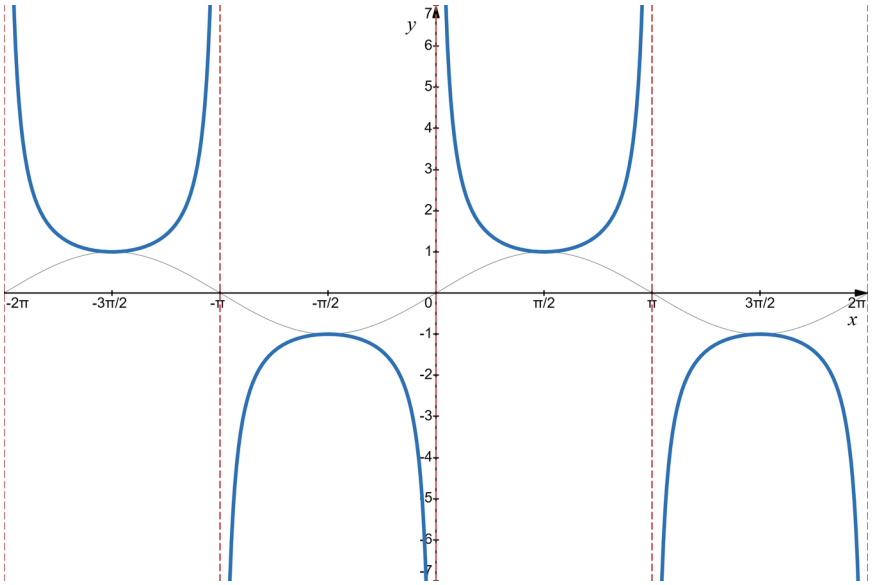
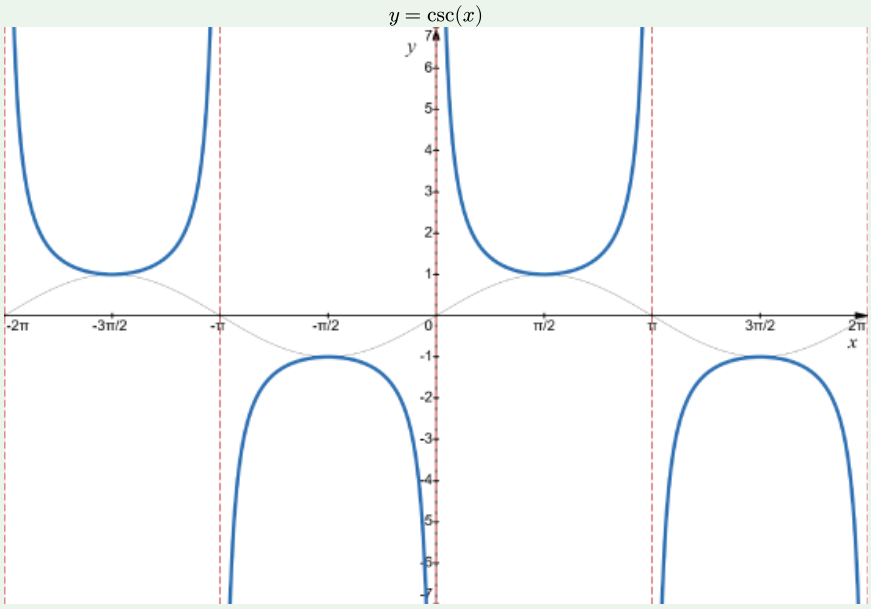


Figure 5.6.7**:** *The base graph of the cosecant,* $y = \csc(x)$

I have left a very lightly sketched sine function in this graph. This is because when you are graphing the cosecant, you will inevitably need to start by graphing its reciprocal.

Just as we did with the fundamental trigonometric functions, we now summarize the properties of the cosecant function (with the help of its graph as a visual aid).

Theorem: Properties of the Graph of the Cosecant Function

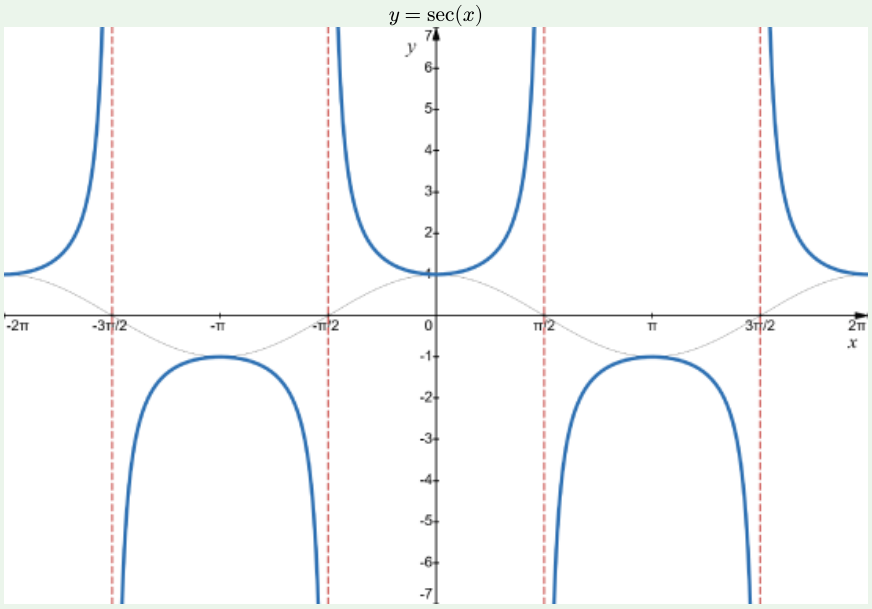


- **Domain:** All $x \neq k\pi$, where $k \in \mathbb{Z}$
- **Range:** $(-\infty, -1] \cup [1, \infty)$
- **Symmetry:** Odd (symmetric about the origin)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** N/A
- **Zeros:** None

Graphing the Secant Function

The derivation of the graph of the secant function is similar to that of the cosecant function. It is left for you as an exercise.

Theorem: Properties of the Graph of the Secant Function



- **Domain:** All $x \neq \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$
- **Range:** $(-\infty, -1] \cup [1, \infty)$
- **Symmetry:** Even (symmetric about the y-axis)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** N/A
- **Zeros:** None

As with the graph of the cosecant, we usually lightly graph the reciprocal of the secant (which is the cosine) to help us in graphing the secant.

Graphing the Cotangent Function

As with the cosecant and the secant, to graph the cotangent function, we will use its reciprocal. That is, because $\cot(x) = \frac{1}{\tan(x)}$, we will first graph the tangent and discuss what happens as the tangent function tends to zero.

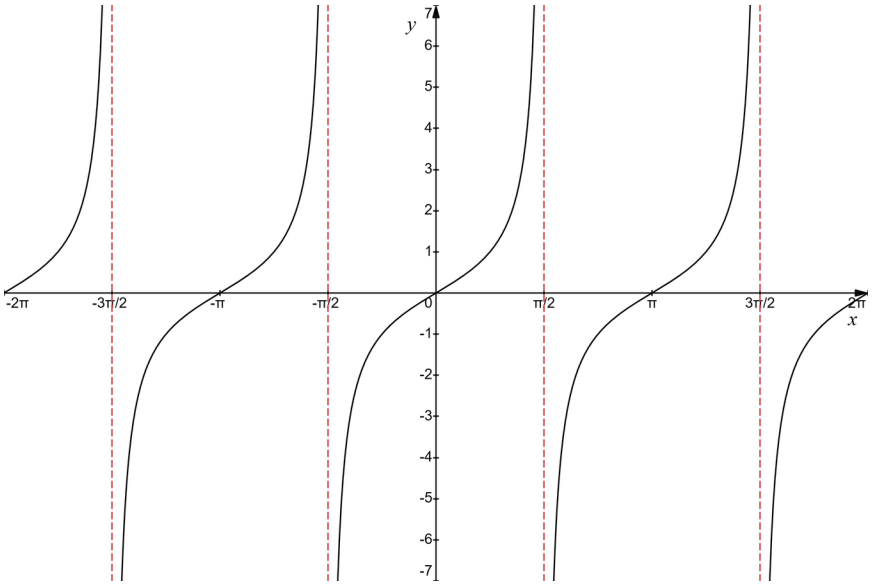


Figure 5.6.8**:** $y = \tan(x)$

Figure 5.6.8 shows the base graph of the tangent function. Using the same arguments as before, we can get the behavior of the reciprocal function, $\frac{1}{\tan(x)} = \cot(x)$, near the zeros of the tangent.

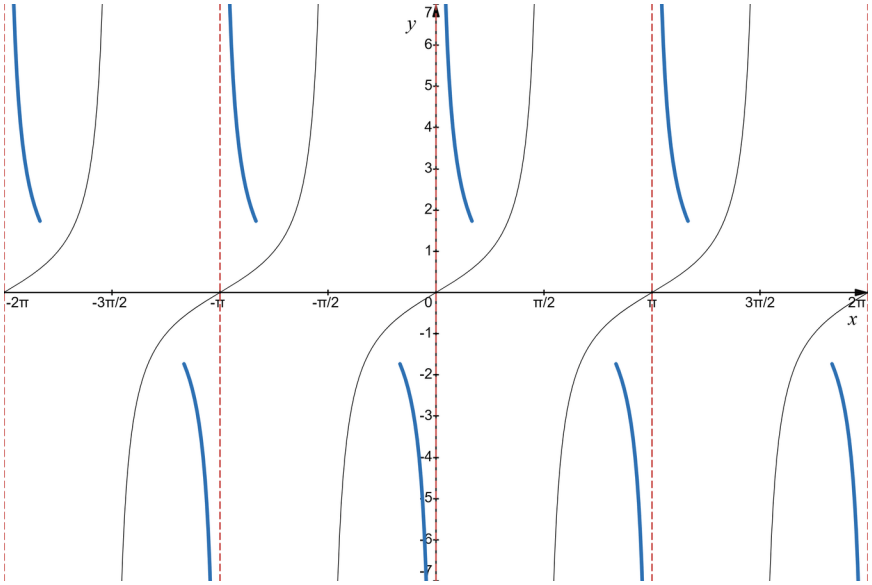


Figure 5.6.9**:** *Almost done graphing the base graph of the cotangent*

Notice in Figure 5.6.9 that the vertical asymptotes are not in the same places as the vertical asymptotes for the tangent. This is because the tangent is undefined when the cosine function is zero (at $x = \frac{\pi}{2} + k\pi$) and the cotangent is undefined when the sine is zero (at $x = k\pi$). Evaluating the cotangent function for a few special values between $x = 0$ and $x = \pi$ should give us the pattern of those missing middle regions.



Plotting these points, filling out the rest of the curve with the same pattern, and erasing the underlying tangent graph, we get the following:

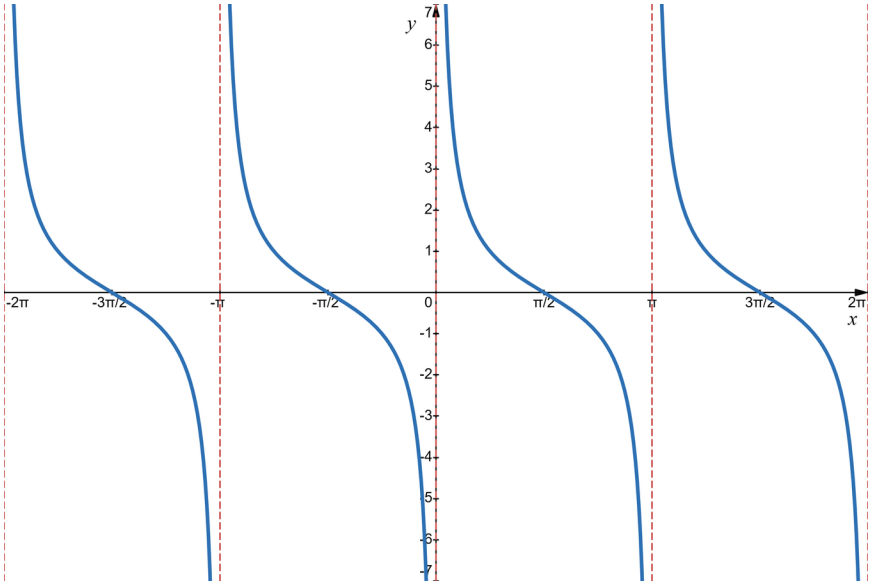
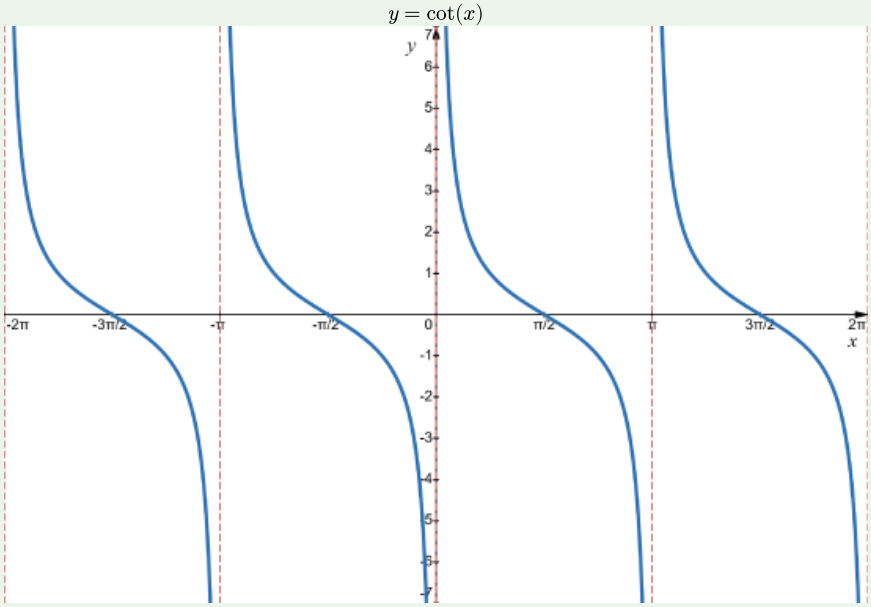


Figure 5.6.10**:** *The base graph of the cotangent,* $y = \cot(x)$

As before, we list all of the properties in a convenient theorem.

Theorem: Properties of the Graph of the Cotangent Function

$y = \cot(x)$



- **Domain:** All $x \neq k\pi$, where $k \in \mathbb{Z}$
- **Range:** $(-\infty, \infty)$
- **Symmetry:** Odd (symmetric about the origin)
- **Principal Cycle:** $[0, \pi]$
- **Natural Period:** π
- **Amplitude:** N/A
- **Zeros:** $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$

Transformations of the Reciprocal Functions

Graphing transformations of the reciprocal functions is the same as graphing the transformations of the fundamental trigonometric functions. In fact, to graph transformations of the cosecant and secant functions, we first fully graph the transformations of their related fundamental functions. After that, graphing the transformed cosecant and secant functions is a somewhat trivial move. Let's see this in action.

Example 5.6.1

Graph a single cycle of each function. State all key characteristics of the graph (including domain, range, period, phase shift, etc.).

a. $f(x) = -1.2 + 2.8 \sec\left(-3x - \frac{\pi}{2}\right)$
b. $g(x) = 2 \cot(2x - 2) + 1$

Solutions available at the end of this section.

Checkpoint 5.6.1

Graph the function. Clearly state all key information (including domain and range).

$$y = 3 - 4 \csc\left(-\pi x + \frac{2}{3}\pi\right)$$

Answer available at the end of this section.

Finding Equations Given Graphs

Finding the equation of a reciprocal function given its graph requires a little bit more work than with the fundamental trigonometric functions.

Example 5.6.2

Find the equation of the graphed function.

a.

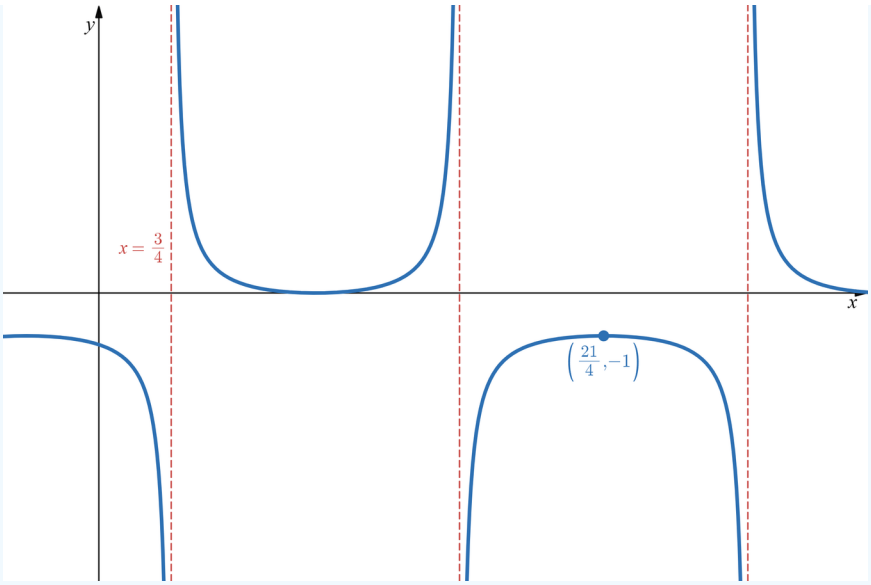


Figure 5.6.19

b.

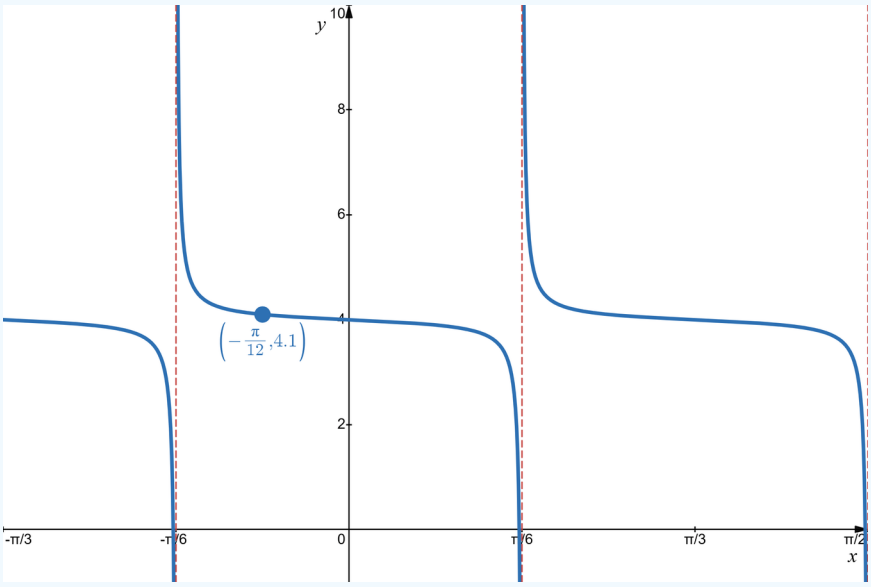


Figure 5.6.20

Solutions available at the end of this section.

Checkpoint 5.6.2

Find the equation of the graphed function.

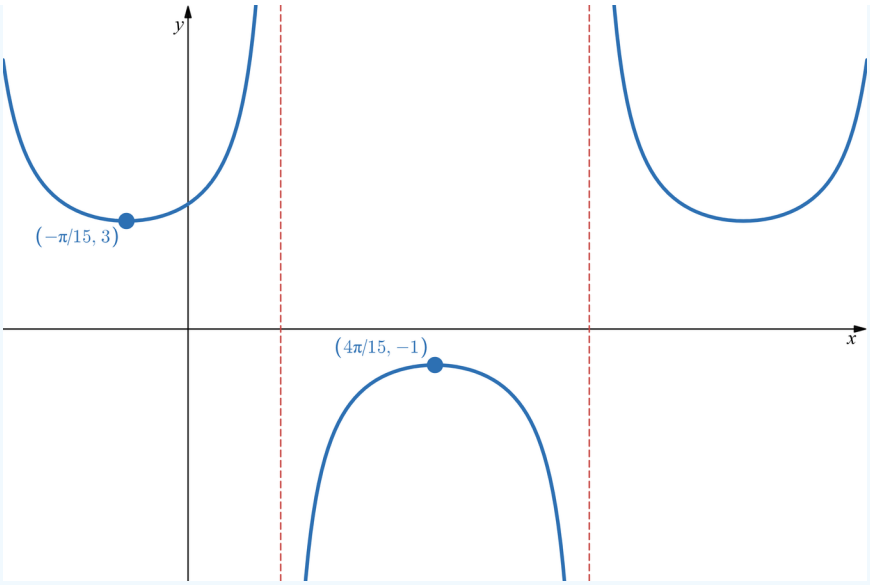


Figure 5.6.22

Answer available at the end of this section.

Success in Mathematics: Math Anxiety (Part 3)

For some math students, anxiety changes their ability to show what they know. They have physical and emotional symptoms. You are responsible for recognizing if math anxiety is interfering with your success. Do you need to make some changes to reduce your math anxiety? Do you need help to do so?

Researchers have identified various events that seem to result in math anxiety. Perhaps you had negative experiences learning math. For you, learning math meant feeling dumb or embarrassed. Perhaps a teacher or parent who struggled with math has passed on their fears to you. Or you believe that your gender or ethnic heritage means you cannot do the math.

Some students learn to cope on their own with their math anxiety. They use relaxation techniques. They practice problems over and over. At test time, their anxiety does not block them from doing the work. Others find help from counselors at their college.

Math anxiety is common among college students. It is your responsibility to think about it and, if needed, get help so that you can cope with your fears.

Math anxiety affects students in physical ways such as increasing blood pressure and heart rate, breathing faster, excess sweating, dry mouth, restlessness, or nausea. Emotionally, it can cause fear, anger, nervousness, worry, and feelings of hopelessness. Students may have trouble concentrating, sitting still, reading, or keeping their legs from bouncing. They may feel desperate - like they need to run away or cry.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Using a scale of 1 to 10, with 10 describing overwhelming anxiety that makes it extremely difficult to do math, rate your math anxiety. How does it affect you?
2. Most students have some anxiety about their classes. They worry about giving speeches, taking tests, writing a good paper, or completing all their work. Worrying can be motivating. It can help us get going and to get things done. Or, worrying can be a bad thing. It paralyzes us. In general, how does worrying affect you?

Applications of Mathematics: Poker

So, where does mathematics apply in poker? Poker players can use mathematics in many different situations; however, it is most commonly used when a player is on a draw, such as a flush or straight draw. If an opponent makes a bet and you are on a draw, you are faced with the decision of whether or not you should call to try and complete the draw by the next card or fold and let your opponent take the pot. In this situation, a player with good knowledge of poker mathematics will always know whether or not to call, while a player who does not know mathematics will be unsure.

Players unfamiliar with poker odds will make an educated guess on whether or not they should call. If the bet is large, they may feel it is too expensive to try and catch the right card, but if the bet is small, they will be more inclined to call. On the other hand, a player that can use mathematics correctly will be able to work out the pot odds they are getting on the hand and act accordingly. Pot odds consider the amount your opponent has bet in relation to the pot size and the likelihood of completing your draw to inform you about whether or not you should call or fold.

1. Can you think of three other games that rely explicitly or implicitly on mathematics? Explain.
2. Oddly enough, most of us are superstitious about numbers. Some people even play their "favorite" numbers when they play the lottery. Suppose you pick your favorite numbers, and I pick the numbers 1, 2, 3, 4, 5, and 6. Who has better odds of winning? Why are you still likely to win more if the odds are the same?

Answers

Section Preview

The graphs of the rest of the trigonometric functions are covered in this section. Try your best to avoid treating this material like an afterthought. The graphs of the secant and cosecant spring up in odd places in Calculus, so the students need to be aware that this material, while held off for the end of the chapter, is critical to understand.

Section Preview

- Graph the cosecant, secant, and cotangent functions, along with transformations of these functions.
- Determine the equation of a trigonometric function given its graph.

Example 5.6.1

a. Just as we have done previously, we rewrite it in the better form

$$f(x) = 2.8 \sec\left(-3x - \frac{\pi}{2}\right) - 1.2$$

Using the fact that secant is an even function, we clean this up a little more to the following:

$$f(x) = 2.8 \sec\left(-\left(3x + \frac{\pi}{2}\right)\right) - 1.2 = 2.8 \sec\left(3x + \frac{\pi}{2}\right) - 1.2$$

Now that we are finished "preconditioning" the function, it's time we discussed how we are going to graph it. With the secant and cosecant, we perform the intermediate step of graphing their reciprocal functions, the cosine and sine, respectively. So, let's graph

$$y = 2.8 \cos\left(3x + \frac{\pi}{2}\right) - 1.2.$$

We list out all the essential information.

A	$=$	2.8	$\Longrightarrow$	Amplitude	$=$	2.8
B	$=$	3	$\Longrightarrow$	Period	$=$	$\dfrac{\text{Natural Period}}{B} = \dfrac{2\pi}{3}$
			$\Longrightarrow$	Step Size	$=$	$\dfrac{\text{Period}}{4} = \dfrac{2\pi/3}{4} = \dfrac{\pi}{6}$
C	$=$	$\dfrac{\pi}{2}$	$\Longrightarrow$	Phase Shift	$=$	$-\dfrac{C}{B} = -\dfrac{\pi/2}{3} = -\dfrac{\pi}{6}$
D	$=$	-1.2	$\Longrightarrow$	Vertical Shift	$=$	-1.2

As usual, we graph and label the midline, label the initial number as the phase shift, and step to the other key numbers using the step size.

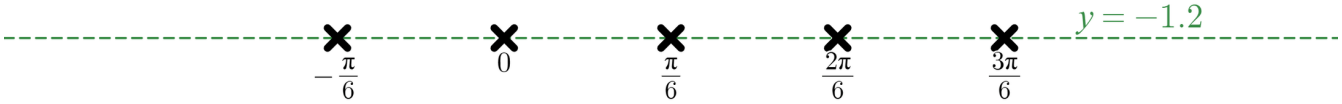


Figure 5.6.11

Note that the distance between the first and last key number is

$$\left|\frac{3\pi}{6} - \left(-\frac{\pi}{6}\right)\right| = \frac{2\pi}{3} = \text{Period},$$

so we have done something right! We then graph the non-reflected cosine, draw in and label the y-axis, and use that to place the x-axis.

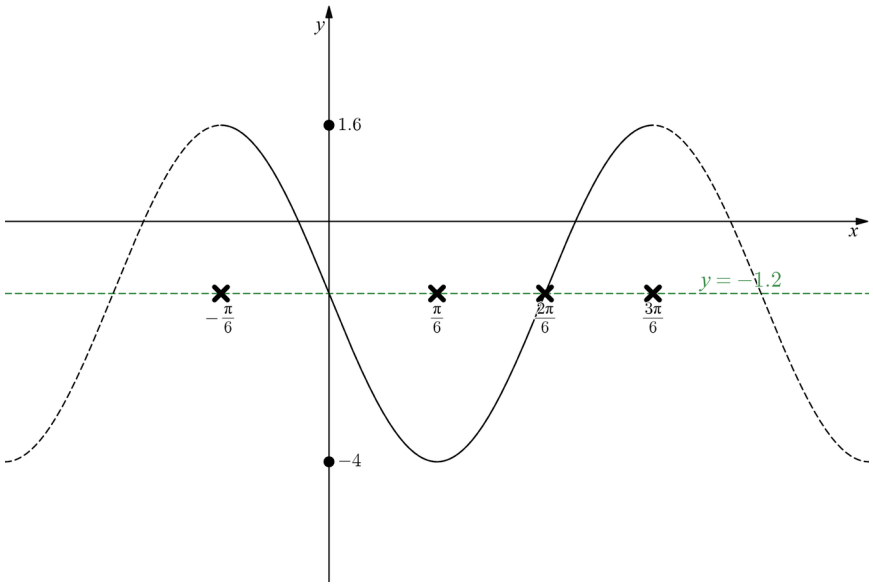


Figure 5.6.12

Now comes the new part. Wherever the cosine function crosses the midline, the original function is undefined. This gives us our vertical asymptotes at $x = 0$, $x = \pm\frac{2\pi}{6} = \frac{\pi}{3}$, $x = \pm\frac{4\pi}{6} = \frac{2\pi}{3}$, and so on. Graphing these vertical asymptotes allows us to graph the secant.

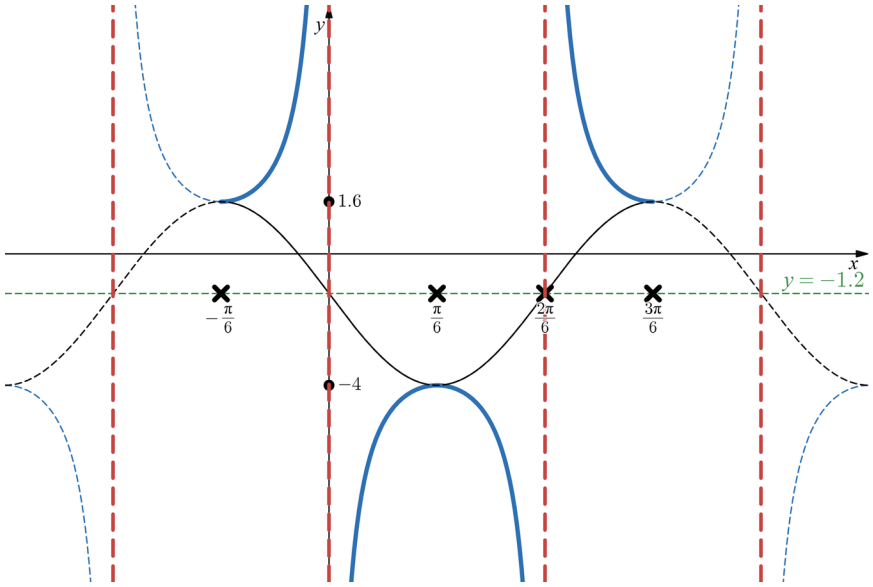


Figure 5.6.13

Notice that I didn't remove the sketch of the cosine. This is perfectly fine (and some instructors expect you to leave it). Finally, we should state as much about this function as we can. The domain (by looking at our spectacular graph) is all $x \neq \frac{\pi}{3}k$, where $k \in \mathbb{Z}$. The range (again, from the graph) is $(-\infty, -4] \cup [1.6, \infty)$. The only thing that we cannot state at this point are the zeros of this function (luckily, I didn't ask for them).

b. Before we start, I need to be clear and state that the secant and cosecant functions are the only ones where we must first graph their reciprocal functions. The base graph for the cotangent is simple enough to handle without bothering with the tangent.

Now, there isn't any preconditioning necessary for this function, so we go ahead and jump right into the other stages. Just like the tangent, the cotangent does not have an amplitude and its natural period is π.

$$
\begin{array}{rcl}
B = 2 \implies \text{Period} & = & \dfrac{\text{Natural Period}}{B} = \dfrac{\pi}{2} \\
\implies \text{Step Size} & = & \dfrac{\text{Period}}{4} = \dfrac{\pi/2}{4} = \dfrac{\pi}{8} \\
C = -2 \implies \text{Phase Shift} & = & -\dfrac{C}{B} = -\dfrac{-2}{2} = 1 \\
D = 1 \implies \text{Vertical Shift} & = & 1
\end{array}
$$

Graphing and labeling the midline, plotting the phase shift as the first key number, and stepping to each subsequent key number using the step size, we get the following:

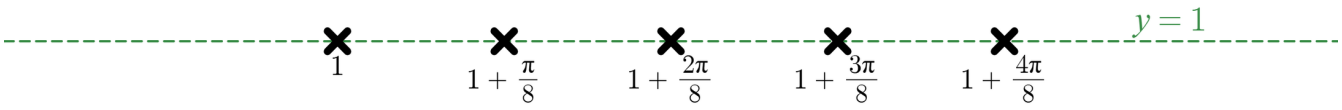


Figure 5.6.14

Because the phase shift was a rational number and the step size was an irrational number, we could not easily add them. Leaving them in the form above is preferred. If you look at the base graph for the cotangent, its principal cycle occurs from $x = 0$ to $x = \pi$. This cycle starts and ends with a vertical asymptote, decreases from left to right, and crosses the midline halfway through. That's precisely what we are going to do here.

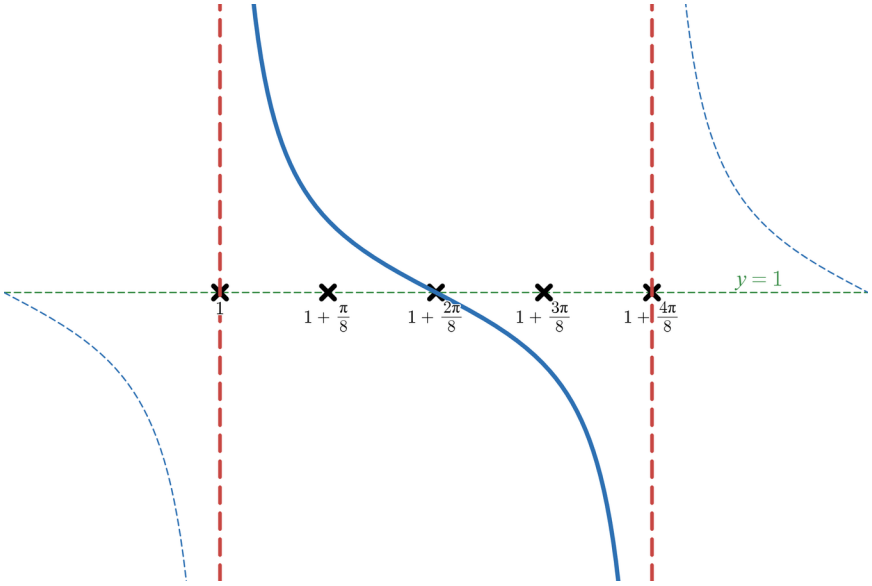


Figure 5.6.15

Now we place the y-axis one unit to the left of the phase shift and the x-axis one unit down from the midline. How do I know how far one unit is? It's more of an art of approximation.

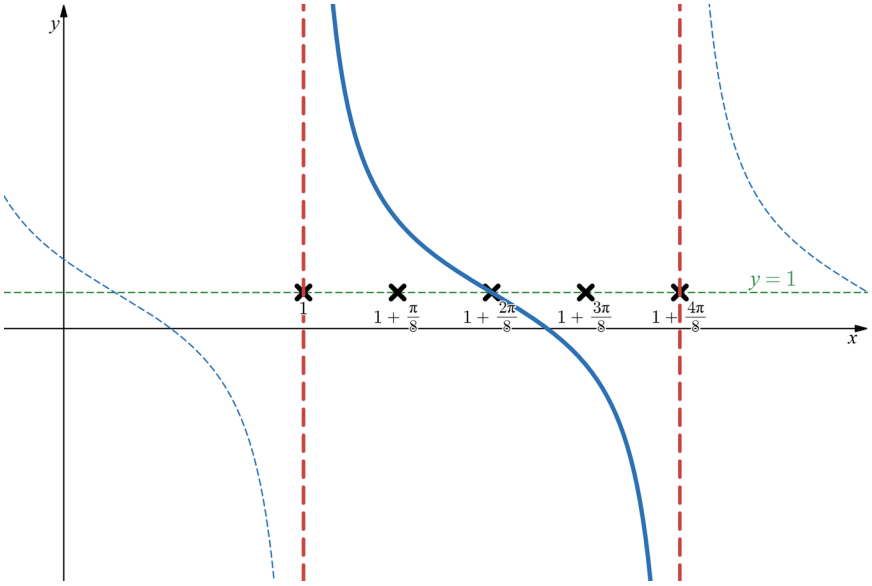


Figure 5.6.16

Finally, we *should* label the two points on the curve at $x = 1 + \frac{\pi}{8}$ and $x = 1 + \frac{3\pi}{8}$. The good news is that the y-values of these points vary from the midline by $|A|$. Thus, the corresponding y-values are $y = 1 + 2 = 3$ and $y = 1 - 2 = -1$.

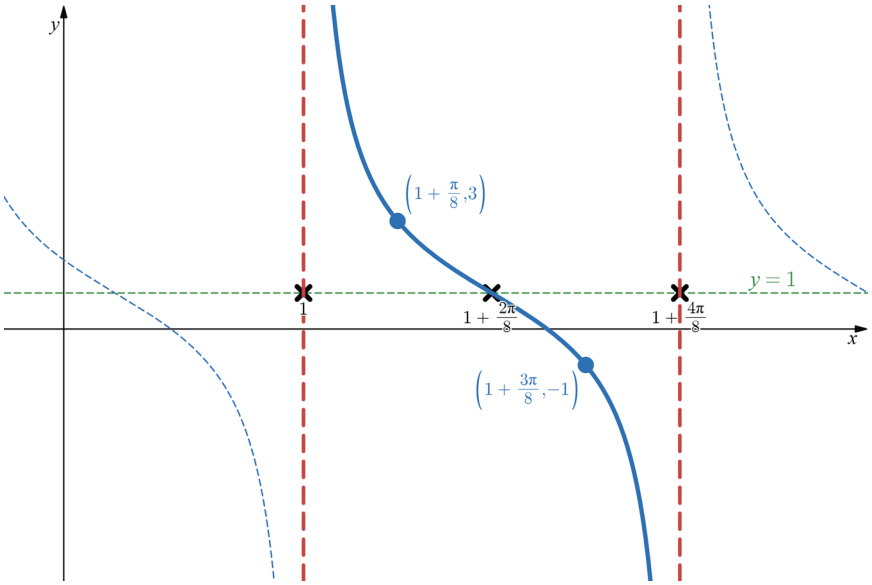


Figure 5.6.17

The domain of the function is all $x \neq 1+\frac{\pi}{2}k$, where $k \in \mathbb{Z}$. The range is all real numbers, $\mathbb{R}$.

Checkpoint 5.6.1

Amplitude of the related sine function: 4
Period: 2
Step Size: $\frac{1}{2}$
Phase Shift: $\frac{2}{3}$
Vertical Shift: 3
Domain: All $x \neq \frac{2}{3}+k$, where $k \in \mathbb{Z}$
Range: $(-\infty,1] \cup [7,\infty)$

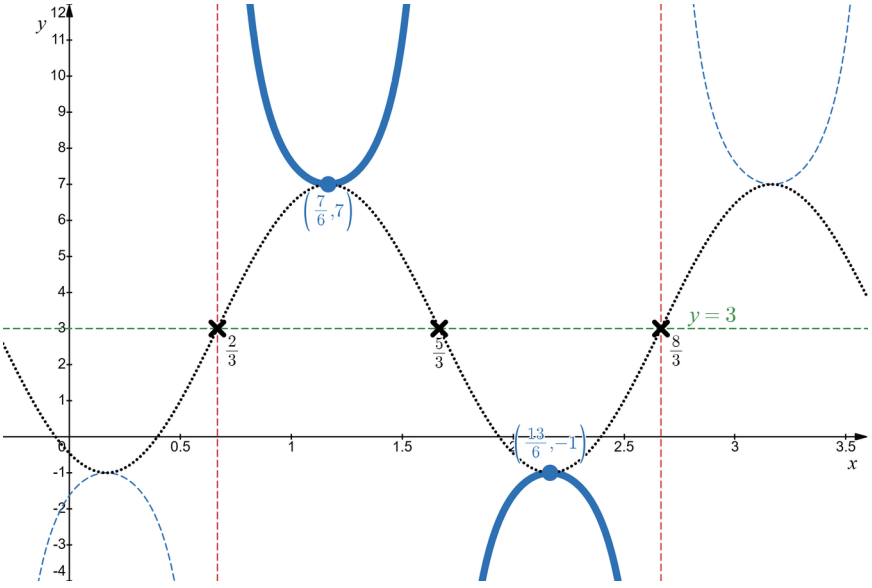


Figure 5.6.18

Example 5.6.2

a. As with the sine and cosine functions, graphs of the cosecant and secant are so similar that you could use either as an equation for the given graph (they would just be shifts of one another). A good rule of thumb when building trigonometric equations from graphs is to read the graph from left to right. The piece of information (whether a point or an equation of an asymptote) you have been given will set the beginning of the cycle for the trigonometric function. Therefore, in this case, we will assume the cycle of the trigonometric function starts at $x=\frac{3}{4}$. With the cosecant and secant, it's beneficial to draw a sinusoid "helper" function. Looking at the given graph, we can see that the midline should be at $y=-\frac{1}{2}$. Drawing this in allows us to sketch our "helper" function.

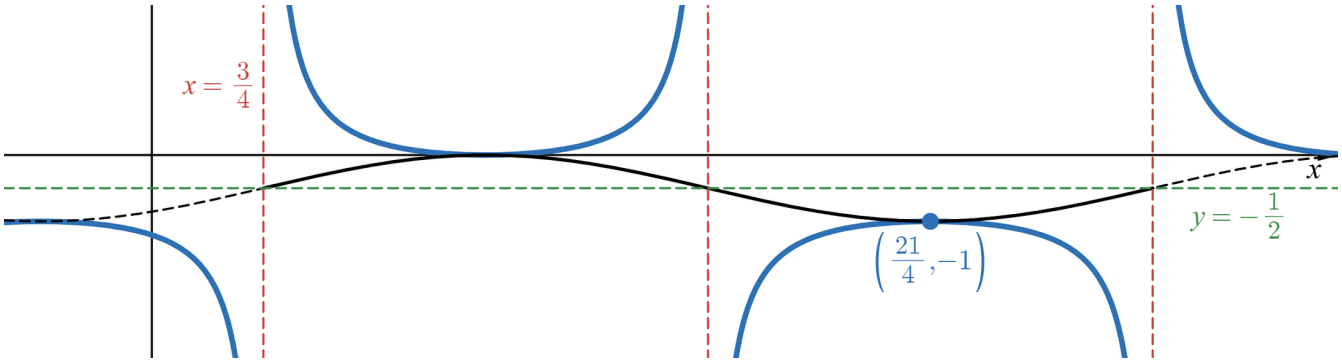


Figure 5.6.21

We now find the equation of the "helper" function. It is definitely a sine function starting at $x = \frac{3}{4}$, so the phase shift is

$$-\frac{C}{B} = \frac{3}{4}.$$

The midline is $y = -\frac{1}{2}$, so the vertical shift is

$$D = -\frac{1}{2}.$$

Additionally, the amplitude is the distance between the midline and the maximum or minimum values of the sinusoid. Since the maximum is $y = 0$ and the midline is $y = -\frac{1}{2}$, the amplitude is $\frac{1}{2}$. Moreover, since this sine function is not reflected,

$$A = \frac{1}{2}.$$

The only thing left to find is the period. This would give us an equation to find B, which in turn would allow us to find C. The cycle begins at $x = \frac{3}{4}$ and three-quarters of the way through the cycle, we are given the point $\left(\frac{21}{4}, -1\right)$. That is, the distance between $\frac{3}{4}$ and $\frac{21}{4}$ is three-quarters of the period.

$$\begin{aligned} \frac{3}{4} \text{ of the Period} &= \left|\frac{3}{4} - \frac{21}{4}\right| \\ &= \left|-\frac{18}{4}\right| \\ &= \frac{9}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Period} &= \left(\frac{9}{2}\right)\left(\frac{4}{3}\right) \\ &= 6 \end{aligned}$$

Since the period is $\frac{2\pi}{B}$,

$$\frac{2\pi}{B} = 6 \implies \frac{2\pi}{6} = B \implies \frac{\pi}{3} = B.$$

Earlier we stated that $-\frac{C}{B} = \frac{3}{4}$, so

$$-\frac{C}{\pi/3} = \frac{3}{4} \implies C = -\frac{\pi}{4}.$$

Thus, the "helper" function is

$$y = A\sin(Bx + C) + D = \frac{1}{2}\sin\left(\frac{\pi}{3}x - \frac{\pi}{4}\right) - \frac{1}{2}.$$

Finally, the *actual* function is the reciprocal of the sine. That is, the function in the initial graph is

$$y = \frac{1}{2}\csc\left(\frac{\pi}{3}x - \frac{\pi}{4}\right) - \frac{1}{2}.$$

b. Again, you *could* write the equation for this in terms of the (reflected) tangent function; however, we will write our equation in terms of the cotangent. Sometimes, you have to "eyeball" the features of a graph. In this example, the midline *looks* to be $y = 4$, so we will say $D = 4$. The phase shift could be $-\frac{\pi}{6}$ or $+\frac{\pi}{6}$, but the extra point they gave us is in the interval $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$, so we will choose our phase shift to be $-\frac{\pi}{6}$. This means

$$-\frac{C}{B} = -\frac{\pi}{6}.$$

Furthermore, the period is easy to spot on this graph. From vertical asymptote to vertical asymptote, the distance is $\frac{2\pi}{6} = \frac{\pi}{3}$. Therefore, the period is $\frac{\pi}{3}$ and, as such,

$$\frac{\pi}{B} = \frac{\pi}{3} \implies B = 3.$$

With this, we can find C.

$$-\frac{C}{B} = -\frac{\pi}{6} \implies \frac{C}{3} = \frac{\pi}{6} \implies C = \frac{\pi}{2}.$$

We are close to the finish line. Our function is

$$y = A\cot\left(3x + \frac{\pi}{2}\right) + 4.$$

Now we use the given data point to help us find the value of A.

$$\begin{aligned} 4.1 &= A\cot\left(3\left(-\frac{\pi}{12}\right)+\frac{\pi}{2}\right)+4 \\ \Longrightarrow \quad 0.1 &= A\cot\left(-\frac{\pi}{4}+\frac{\pi}{2}\right) \\ \Longrightarrow \quad 0.1 &= A\cot\left(\frac{\pi}{4}\right) \\ \Longrightarrow \quad 0.1 &= A\,(1) \\ \Longrightarrow \quad 0.1 &= A \end{aligned}$$

Thus, the function is

$$y = 0.1\cot\left(3x+\frac{\pi}{2}\right)+4.$$

Checkpoint 5.6.2

$y = 2\sec\left(3x+\frac{\pi}{5}\right)+1$

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5.6.1: Resources and Key Concepts

Resources

Videos

- Graphs of Cosecant and Secant
  - Graphs of the Other Trigonometric Functions - Deriving the Base Graphs for Secant and Cosecant
  - Graphs of the Other Trigonometric Functions - Graphing Secant and Cosecant Functions
- Graph of Cotangent
  - Graphs of the Other Trigonometric Functions - Deriving the Base Graphs for Tangent and Cotangent
  - Graphs of the Other Trigonometric Functions - Graphing Tangent and Cotangent Functions

Key Concepts

Theorems

- **Properties of the Graph of the Cosecant Function:**
  - Domain: All $x \neq k\pi$, where $k \in \mathbb{Z}$
  - Range: $(-\infty, -1] \cup [1, \infty)$
  - Symmetry: Odd (symmetric about the origin)
  - Principal Cycle: $[0, 2\pi]$
  - Natural Period: 2π
  - Amplitude: N/A
  - Zeros: None
- **Properties of the Graph of the Secant Function:**
  - Domain: All $x \neq \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$
  - Range: $(-\infty, -1] \cup [1, \infty)$
  - Symmetry: Even (symmetric about the y-axis)
  - Principal Cycle: $[0, 2\pi]$
  - Natural Period: 2π
  - Amplitude: N/A
  - Zeros: None
- **Properties of the Graph of the Cotangent Function:**
  - Domain: All $x \neq k\pi$, where $k \in \mathbb{Z}$
  - Range: $(-\infty, \infty)$
  - Symmetry: Odd (symmetric about the origin)
  - Principal Cycle: $[0, \pi]$
  - Natural Period: π
  - Amplitude: N/A
  - Zeros: $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$

Common Mistakes

- **Forgetting the Reciprocal Relationship:** The easiest way to graph the cosecant and secant functions is to first lightly sketch the graph of their reciprocal functions (sine and cosine, respectively). The key features of the reciprocal graph (zeros, maximums, minimums) determine the key features of the cosecant/secant graph (asymptotes, minimums, maximums).
- **Misplacing Asymptotes:** The vertical asymptotes for $y = \csc(x)$ and $y = \cot(x)$ occur where $\sin(x) = 0$. The vertical asymptotes for $y = \sec(x)$ occur where $\cos(x) = 0$. Confusing these will lead to incorrect graphs.
- **Incorrectly Transforming Asymptotes:** When graphing a transformed function like $y = A\sec(Bx + C) + D$, the vertical asymptotes are located where the "helper" function, $y = A\cos(Bx + C) + D$, crosses its midline, $y = D$.

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5.6.2: Homework

Reading Questions

1. How is the graph of the cosecant function related to the graph of the sine function?
2. Where do the vertical asymptotes of the secant function occur?
3. What is the natural period of the cotangent function?
4. What is the domain of the cosecant function?
5. What is the range of the secant function?
6. Which of the reciprocal trigonometric functions are odd? Which are even?
7. What is the recommended first step when graphing a transformed cosecant or secant function?
8. Why do the cosecant, secant, and cotangent functions not have an amplitude?
9. The zeros of the sine function correspond to what features on the graph of the cosecant function?
10. The local maximums on the graph of the sine function correspond to what features on the graph of the cosecant function?

Homework

Concept Check

1. Describe the general shape of the graph of the cosecant function.
2. What is the relationship between the sine function and the cosecant function? How does this relationship affect their graphs?
3. Describe the general shape of the graph of the secant function.
4. What is the relationship between the cosine function and the secant function? How does this relationship affect their graphs?
5. Describe the general shape of the graph of the cotangent function.
6. What is the relationship between the tangent function and the cotangent function? How does this relationship affect their graphs?

Basic Skills

7. Determine the period, phase shift, and midline of the function $y = 3\sec(2x - \pi) + 1$.
8. Determine the period, phase shift, and midline of the function $y = 2\csc(x + \frac{\pi}{4}) - 3$.
9. Determine the period, phase shift, and midline of the function $y = -\cot\left(\frac{1}{2}x - \frac{\pi}{3}\right) + 4$.
10. Determine the period, phase shift, and midline of the function $y = 4\sec\left(\frac{1}{3}x + \frac{\pi}{6}\right) - 2$.
11. Determine the period, phase shift, and midline of the function $y = -2\csc(3x - \pi) + 1$.
12. Determine the period, phase shift, and midline of the function $y = 3\cot\left(4x + \frac{\pi}{2}\right) - 5$.
13. Graph $y = \sec(x - \frac{\pi}{2}) + 2$.
14. Graph $y = 2\csc\left(\frac{\pi}{2}x\right) - 1$.
15. Graph $y = -\cot\left(x + \frac{\pi}{4}\right) + 3$.
16. Graph $y = 3\sec(2x) - 4$.

17. Graph $y = -\csc\left(\frac{1}{2}x - \frac{\pi}{3}\right) + 2$.

18. Graph $y = \cot\left(\frac{3}{2}x + \frac{\pi}{6}\right) - 1$.

19. Graph $y = -2\sec\left(\frac{1}{3}x - \frac{\pi}{4}\right) + 5$.

20. Graph $y = 4\csc(3x) - 3$.

21. Graph $y = \cot(2x) + 4$.

22. Graph $y = -\sec\left(\frac{2}{3}x + \frac{\pi}{2}\right) - 2$.

23. Graph $y = \csc\left(\frac{\pi}{3}x - \frac{\pi}{6}\right) + 1$.

24. Graph $y = -3\cot\left(\frac{1}{2}x + \frac{\pi}{3}\right) + 2$.

For the following exercises, assume that the related sinusoidal function has an amplitude of 1.

25. Given the graph of a secant function that has a period of π, a phase shift of $\frac{\pi}{4}$ to the right, and a vertical shift up by 2 units, determine the equation of the graph.

26. Given the graph of a cosecant function that has a period of 2π, a phase shift of $\frac{\pi}{6}$ to the left, and a vertical shift down by 3 units, determine the equation of the graph.

27. Given the graph of a reflected secant function that has a period of $\frac{\pi}{2}$, a phase shift of $\frac{\pi}{2}$ to the left, and a vertical shift down by 4 units, determine the equation of the graph.

28. Given the graph of a reflected cosecant function that has a period of 4π, a phase shift of $\frac{\pi}{4}$ to the right, and a vertical shift up by 5 units, determine the equation of the graph.

29. Given the graph of a secant function with a period of 2π, a vertical shift up by 1 unit, and where the related sinusoidal function has an amplitude of 3, determine the equation of the graph.

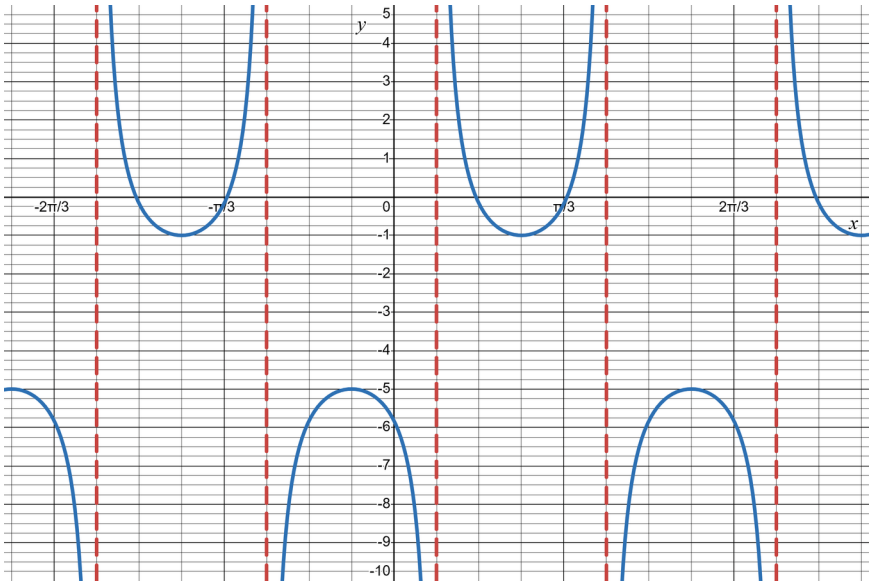
30. Given the graph of a cosecant function with a period of π, a vertical shift down by 4 units, and where the related sinusoidal function has an amplitude of 2, determine the equation of the graph.

31. Given the graph of a reflected secant function with a period of $\frac{\pi}{3}$, a phase shift of $\frac{\pi}{2}$ to the left, a vertical shift down by 3 units, and where the related sinusoidal function has an amplitude of 1.2, determine the equation of the graph.

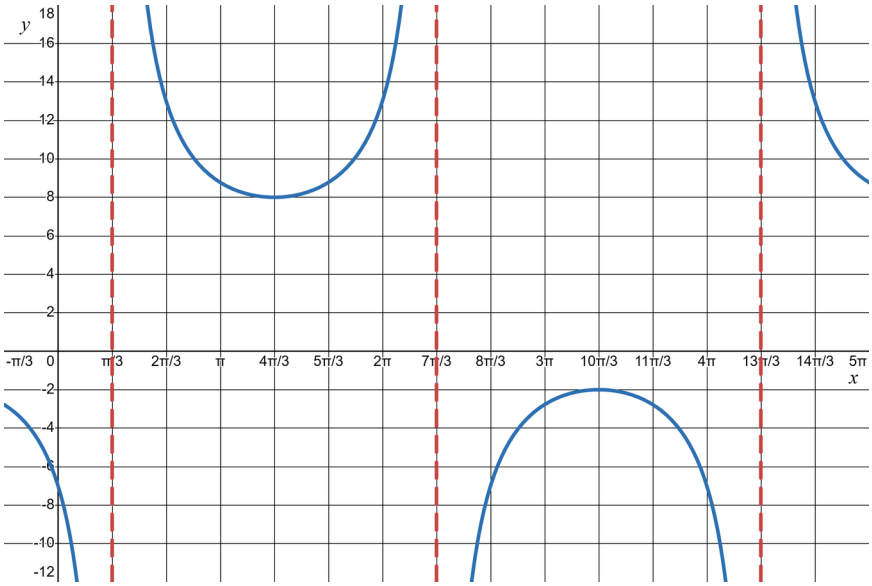
32. Given the graph of a reflected cosecant function with a period of $\frac{\pi}{4}$, a phase shift of $\frac{\pi}{6}$ to the right, a vertical shift up by 2 units, and where the related sinusoidal function has an amplitude of 2, determine the equation of the graph.

For the following exercises, find the equation of the function in terms of cosecant, secant, or cotangent.

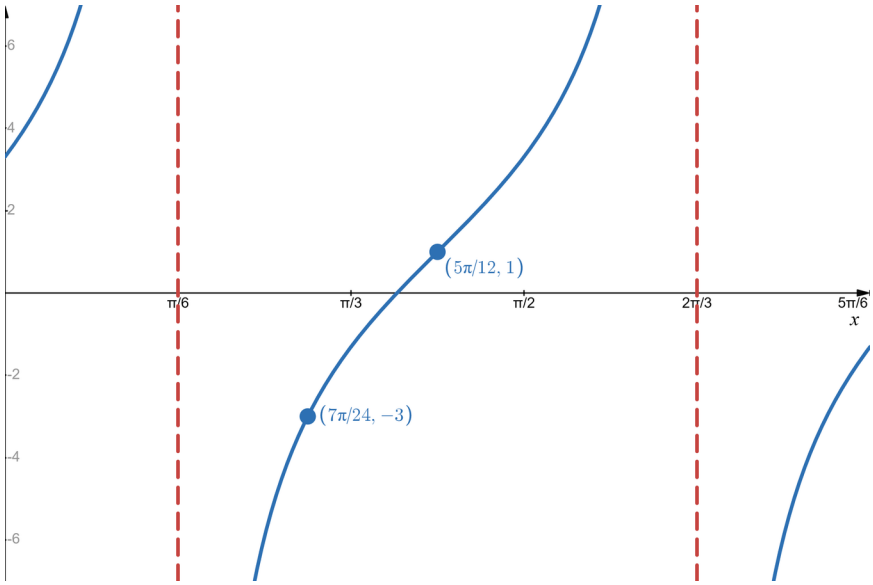
33.


y
x
-2π/3
-π/3
0
π/3
2π/3

34.


y
x
-π/3
0
π/3
2π/3
π
4π/3
5π/3
2π
7π/3
8π/3
3π
10π/3
11π/3
4π
13π/3
14π/3
5π

35.



Synthesis Questions

36. Given the function $y = 3\csc(2x - \frac{\pi}{2}) + 4$, find all asymptotes within the interval $[0, 2\pi]$. Explain the process of identifying these asymptotes.

37. Given the function $y = 2\sec\left(\frac{1}{2}x - \frac{\pi}{3}\right) - 1$, find all asymptotes within the interval $[-2\pi, 2\pi]$. Describe the method used to determine these asymptotes.

38. For the function $y = -3\csc(4x + \frac{\pi}{4}) + 2$, determine all asymptotes within the interval $[0, \pi]$. Explain your reasoning and the steps taken to find these asymptotes.

39. Given the function $y = \sec(3x - \frac{\pi}{6}) - 2$, find the equations of all vertical asymptotes within the interval $[-\pi, \pi]$. Describe the procedure used to identify these asymptotes.

40. For the function $y = -\csc\left(\frac{3}{2}x + \frac{\pi}{3}\right) + 5$, find the equations of all vertical asymptotes within the interval $[-\pi, \pi]$. Detail the steps and logic used to locate these asymptotes.

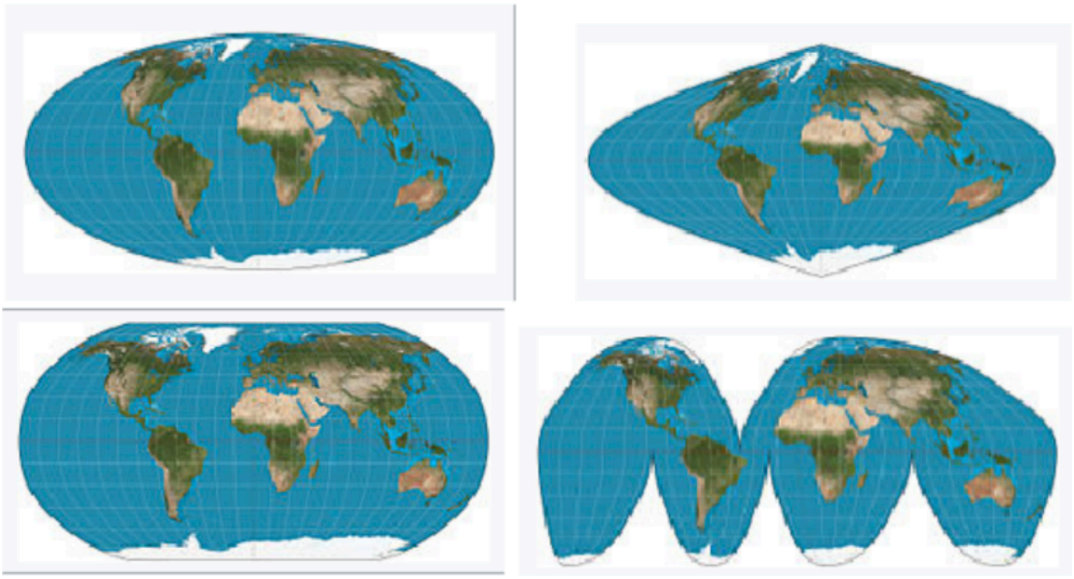
41. Consider the function $y = 4\sec\left(2x - \frac{\pi}{2}\right) + 1$. Identify all vertical asymptotes within the interval $[0, 2\pi]$ and explain how you determined their locations.

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6: ANALYTIC TRIGONOMETRY

Mapmakers have always faced an unavoidable challenge: It is impossible to translate the surface of a sphere onto a flat map without some form of distortion. Over the years, a variety of map projections have been developed to suit different uses.

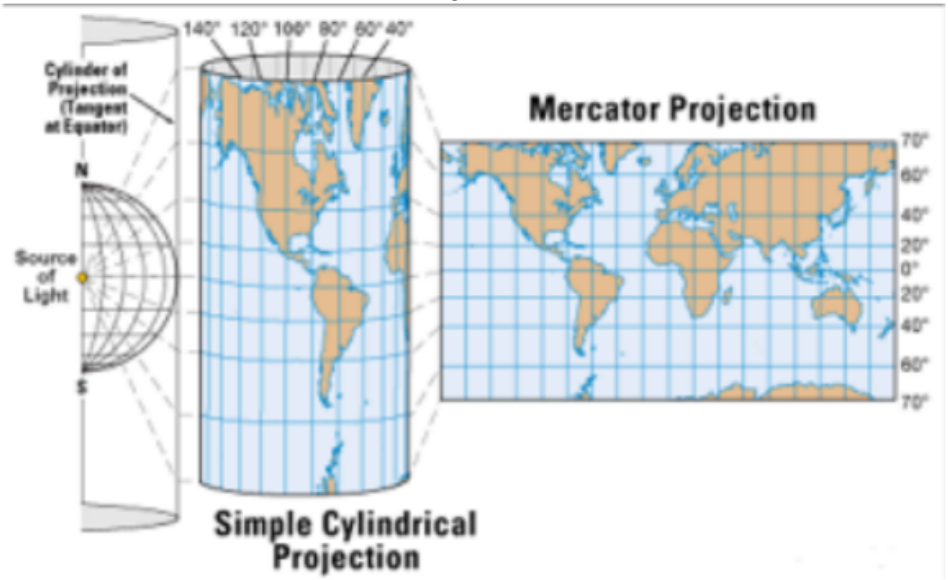
Figure 1



The sixteenth century was an age of discovery, when explorers and merchants began sailing to distant and previously unknown lands. But at that time there was no reliable technology for navigation. Although more regions of the world were being mapped more accurately, a flat map by itself was not enough to help a sailor in the middle of the ocean.

In 1569, the Flemish cartographer Gerardus Mercator published a new map using what is known as a cylindrical projection. To imagine how a Mercator projection works, picture shining a light through a glass globe onto a piece of paper rolled into a cylinder and wrapped around the globe. The cylinder is tangent to the globe at its equator.

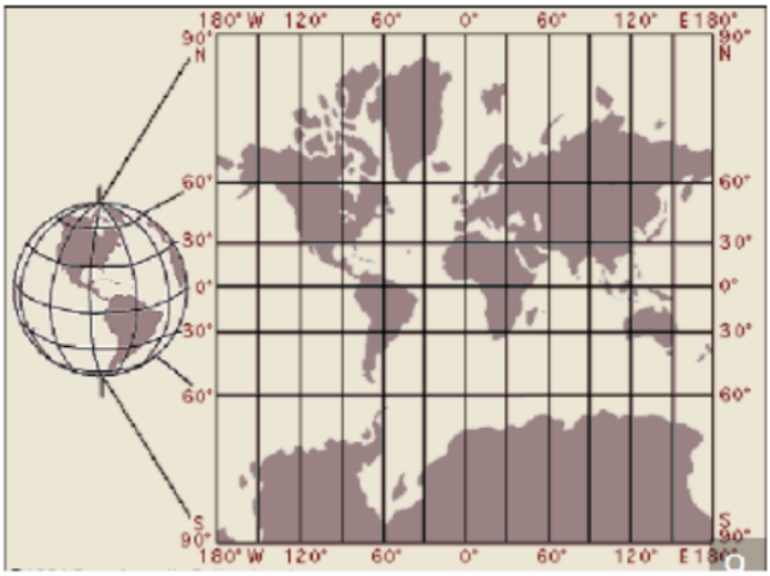
Figure 2



Notice how the latitude lines are farther apart the farther you get from the Equator. This projection distorts the size of objects as

the latitude increases, so that Greenland and Antarctica appear much larger than they actually are.

Figure 3



But the Mercator projection map is ideally suited for navigation, because any straight line on the map is a line of constant true bearing. If a navigator measures the bearing on the map from his location to his destination, he can set his ship's compass for the same bearing and maintain that course.

However, the Mercator projection does not preserve distances. On a globe, circles of latitude (also known as parallels) get smaller as they move away from the Equator towards the poles. Thus, in the Mercator projection, when a globe is "unwrapped" on to a rectangular map, the parallels need to be stretched to the length of the Equator. Mercator had to increase the scale of his map gradually as it moved away from the equator, so that the latitude lines appear equal in length to the equator.

The horizontal scale factor at any latitude must be inversely proportional to lengths on that latitude. Because the radius of the circle of latitude θ is $R\cos\theta$, the corresponding parallel on the map must be stretched by a factor of $\frac{1}{\cos\theta}$. And because the secant is the reciprocal of the cosine, the scale factor is proportional to the secant of the latitude.

A variety of "equal area" projections have been developed in modern times, but the Mercator projection is still widely used, in classrooms and atlases. For many people, it represents our image of the world.

Figure 4



6.1: Basic Trigonometric Identities and Proof Techniques

6.1.1: Resources and Key Concepts

6.1.2: Homework

6.2: Sum and Difference Identities

6.2.1: Resources and Key Concepts

6.2.2: Homework

6.3: Double-Angle Identities

6.3.1: Resources and Key Concepts

6.3.2: Homework

6.4: Half-Angle and Power Reduction Identities

6.4.1: Resources and Key Concepts

6.4.2: Homework

6.5: Sum-to-Product and Product-to-Sum Identities

6.5.1: Resources and Key Concepts

6.5.2: Homework

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6.1: Basic Trigonometric Identities and Proof Techniques

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We already introduced the concept of an identity early on in this textbook. At the time, we had only recently been introduced to the trigonometric functions; however, we had enough knowledge to understand and use the Reciprocal, Ratio, and Pythagorean Identities. Since then, we have encountered two other sets of identities—the Symmetry Identities and the Cofunction Identities. For convenience, the following tables summarize all of these basic identities and some of their more common equivalent forms.

Reciprocal Identities	
Basic Identity	***Alternate Form***
$\sin(\theta) = \dfrac{1}{\csc(\theta)}$	$\csc(\theta) = \dfrac{1}{\sin(\theta)}$
$\cos(\theta) = \dfrac{1}{\sec(\theta)}$	$\sec(\theta) = \dfrac{1}{\cos(\theta)}$
$\tan(\theta) = \dfrac{1}{\cot(\theta)}$	$\cot(\theta) = \dfrac{1}{\tan(\theta)}$

Ratio Identities	
Basic Identity	***Alternate Form***
$\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$

Pythagorean Identities	
Basic Identity	***Alternate Forms***
$\cos^2(\theta) + \sin^2(\theta) = 1$	$\cos^2(\theta) = 1 - \sin^2(\theta)$ and $\cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$
	$\sin^2(\theta) = 1 - \cos^2(\theta)$ and $\sin(\theta) = \pm\sqrt{1 - \cos^2(\theta)}$
$1 + \tan^2(\theta) = \sec^2(\theta)$	$\tan^2(\theta) = \sec^2(\theta) - 1$
$\cot^2(\theta) + 1 = \csc^2(\theta)$	$\cot^2(\theta) = \csc^2(\theta) - 1$

Symmetry Identities	
Basic Identity	***Alternate Form***
$\sin(-\theta) = -\sin(\theta)$	$\csc(-\theta) = -\csc(\theta)$
$\cos(-\theta) = \cos(\theta)$	$\sec(-\theta) = \sec(\theta)$
$\tan(-\theta) = -\tan(\theta)$	$\cot(-\theta) = -\cot(\theta)$

Cofunction Identities	
Basic Identity	***Alternate Form***

Cofunction Identities	
$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$
$\sec(\theta) = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc(\theta) = \sec\left(\frac{\pi}{2} - \theta\right)$
$\tan(\theta) = \cot\left(\frac{\pi}{2} - \theta\right)$	$\cot(\theta) = \tan\left(\frac{\pi}{2} - \theta\right)$

Students who focus on memorization over understanding tend to panic at this point. They look at the preceding tables as huge lists of things to memorize. While it is true that you will have to recall these identities without looking them up or having them on a note card, it is crucial you understand most of these identities can be derived from the basic definitions of Trigonometry—this will not only shorten the list of what you feel you need to memorize, it will also give you a stronger foundation for success in Mathematics.

For example, the Pythagorean Identities are built from the unit circle definition of the trigonometric functions, the Ratio Identities, and the Reciprocal Identities. The unit circle definition tells us that any point, (x, y), on the unit circle can be represented as $(\cos(\theta), \sin(\theta))$, where θ is the angle in standard position whose terminal side contains the point (x, y). Since the equation of the unit circle is $x^2 + y^2 = 1$, this gives us the main Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$. The alternate forms come from algebraic manipulations.

If we divide both sides of the main Pythagorean Identity by $\cos^2(\theta)$, we get

$$\frac{\cos^2(\theta) + \sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} \quad \Longrightarrow \quad \frac{\cos^2(\theta)}{\cos^2(\theta)} + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \sec^2(\theta) \quad \Longrightarrow \quad 1 + \tan^2(\theta) = \sec^2(\theta).$$

A similar approach proves $\cot^2(\theta) + 1 = \csc^2(\theta)$.

The Symmetry Identities can be recalled very quickly from our graphical understanding of the trigonometric functions—cosine and its reciprocal are the only two trigonometric functions whose graphs are symmetric about the y-axis, so they are even functions. The rest of the trigonometric functions have graphs symmetric about the origin (so they are odd functions).

The Cofunction Identities are the only set of identities that we have not proved so far; however, we will remedy that in the next section.

Proving Identities

We have used identities to manipulate trigonometric expressions into different forms and to prove other identities. The remainder of this section reinforces the latter skill (which, in turn, will strengthen the former).

We start with a tactic to help us prove trigonometric identities.

How To... Prove a Trigonometric Identity

1. **Complex to Simple:** It is usually better to start with the more complex side, as it is easier to simplify than to build.
2. **Algebra Before Trigonometry:** Follow the Mathematical Mantra. Look for opportunities to factor expressions, square a binomial, or add fractions.
3. **Trigonometric Identities:** Always look for fundamental trigonometric identities - these can often be used as a key to proving a given statement. As we move forward, we will learn new identities to add to our list of fundamental identities.
4. **Keep an "Eye on the Prize:"** Always take note of the side you are not currently working on. It might contain clues as to what you should do next.
5. **Switch to Sines and Cosines:** If these steps do not yield the desired result, try converting all terms to sines and cosines.
6. **Switch Sides:** If you get stuck manipulating one side of the statement, move to the other side and work on that for a while.

It's important to note that the preceding "How To" is a "rule of thumb." As with most rules of thumb, there are exceptions to what to do and when to do it. For example, you might be given a statement to prove where *both* sides appear to be equal in complexity. In this case, you just choose a side to start with based on other factors. Likewise, you might find that switching

trigonometric functions to sines and cosines makes things too messy. You might look for fundamental identities involving the reciprocal functions in these cases.

In the end, becoming proficient in proving trigonometric identities is all about practice, and the method is entirely heuristic. Despite having a tactic for tackling proofs of trigonometric identities, most of your understanding and mastery will come from proving them on your own.

Example 6.1.1

Verify $\tan(\theta)\cos(\theta) = \sin(\theta)$.

Solution available at the end of this section.

In the solution to Example 6.1.1, we created a table where the first column contained the left-hand side (and its manipulations) and the second column contained the right-hand side (and, if we needed, it would have had manipulations of the right-hand side as well). The vertical bar between these two sides is purposeful. When trying to prove an identity, you are trying to show, through logical, deductive steps, that one side of the identity is equivalent to the other side. Until, or unless, you have proven these two sides are equivalent, you cannot assume they are the same. Therefore, you cannot treat it as an equation. This means you cannot add, subtract, multiply, divide, or square both sides because you assume the equation is true once you do so.

To be clear, when proving the identity in Example 6.1.1, it would be illegal for us to do the following

$$\begin{aligned} & & \mathbf{LHS} &= \mathbf{RHS} \\ \Longrightarrow & & \tan(\theta)\cos(\theta) &= \sin(\theta) \\ \Longrightarrow & & \left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos(\theta) &= \sin(\theta) \\ \Longrightarrow & \cancel{\cos(\theta)} \cdot \left(\frac{\sin(\theta)}{\cancel{\cos(\theta)}}\right)\cos(\theta) &= \cos(\theta)\cdot\sin(\theta) \\ \Longrightarrow & & \sin(\theta)\cos(\theta) &= \sin(\theta)\cos(\theta) \end{aligned}$$

Doing this assumes that the two statements (the left side and the right side) are equal from the beginning.

Caution: Do not cross the line

When proving any identity (whether in Trigonometry or other Mathematics courses), treat each side separately and do not perform operations *between* sides. Do not add, subtract, multiply, or divide both sides by the same expression. Moreover, do not square both sides or perform any other operations between sides simultaneously.

Checkpoint 6.1.1

Verify the identity $\csc(\theta)\cos(\theta)\tan(\theta) = 1$.

Example 6.1.2

Prove the identity.

a. $[1+\sin(x)][1+\sin(-x)] = \cos^2(x)$
b. $\sin^2(\theta) = \frac{\sec^2(\theta)-1}{\sec^2(\theta)}$
c. $\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} = \cos(\theta)+\sin(-\theta)$
d. $[1-\cos^2(x)][1+\cot^2(x)] = 1$

e. $\frac{\cos(\theta)}{1+\sin(\theta)} = \frac{1-\sin(\theta)}{\cos(\theta)}$

Solutions available at the end of this section.

Example $6.1.2d$ is a great example of a proof where making a different choice from the start can change the proof dramatically. For example, most people new to proofs would likely have distributed the expressions on the left side to get

	LHS	**RHS**
	$\left[1-\cos^2(x)\right]\left[1+\cot^2(x)\right]$	1
$=$	$1+\cot^2(x)-\cos^2(x)-\cos^2(x)\cot^2(x)$	(distribution)
$=$	$1-\cos^2(x)+\cot^2(x)-\cos^2(x)\cot^2(x)$	(Commutative Property of Addition)
$=$	$\sin^2(x)+\cot^2(x)-\cos^2(x)\cot^2(x)$	(Pythagorean Identity)
$=$	$\sin^2(x)+\cot^2(x)\left[1-\cos^2(x)\right]$	(factor)
$=$	$\sin^2(x)+\cot^2(x)\sin^2(x)$	(Pythagorean Identity)
$=$	$\sin^2(x)+\frac{\cos^2(x)}{\sin^2(x)}\sin^2(x)$	(Ratio Identity)
$=$	$\sin^2(x)+\frac{\cos^2(x)}{\cancel{\sin^2(x)}}\cancel{\sin^2(x)}$	(cancel like factors)
$=$	$\sin^2(x)+\cos^2(x)$	
$=$	1	(Pythagorean Identity)
$=$	**RHS**	

The moral of the story here is that there are often many ways to prove an identity. With practice, you begin to see the fastest routes through the proof.

Checkpoint 6.1.2

Prove the identity.

a. $\frac{\cot(\theta)}{\csc(\theta)} = \cos(\theta)$
b. $\frac{\sin^2(\theta)-1}{\tan(\theta)\sin(\theta)-\tan(\theta)} = \frac{\sin(\theta)+1}{\tan(\theta)}$

Hidden Resources: Kanopy

Did you know that as a Los Rios Community College student, you have free access to Kanopy?

What is Kanopy?

According to Wikipedia, Kanopy "is an on-demand streaming video platform for public libraries and universities that offers films and documentaries."

Why would you be interested?

There are a lot of great movies, documentaries, and educational videos on this service - and it's free for you because you are a student of the Los Rios Community College District.

Is it as good as Hulu or Netflix?

No, but it does have a lot of the same titles.

Why aren't any of the Great Courses available to us?

Not being able to access the Great Courses is a disappointment. That was one of the options Los Rios chose not to purchase; however, drown your sorrows in What We Do in the Shadows instead!

How do I sign up?

All you need to do is go to https://losrios.kanopy.com and start typing Los Rios when asked for your institution. That should be it.

Success in College: Environment and Learning Preferences

Throughout your academic career, you have completed many exams. While you might not have thought about it, you should be able to identify a strong correlation between the environment you study in and your performance on exams.

Learning preferences describe the situations in which you best learn math. Are you a visual, auditory, or kinesthetic learner? Should you take notes as you listen? Should you record your class? Should you trace the graphs in your book as you study?

Think about whether you learn better by working by yourself or with others. Do you prefer to work in quiet or with music in the background? Do you like to sit at a desk or recline in an easy chair? Do you learn best at a particular time of the day? When you study, does it help if you can move around? All of these are part of your learning preferences.

If you know your learning preferences, you can try to adjust your studying and class behavior to improve your work. It is your responsibility to think about and, if needed, make changes in how you are learning and studying.

Think about the following list. When you study, do you prefer...

- a quiet room, or background noise (music or TV)
- bright light shining on your work, or dim lighting
- sitting at a desk or table, or being relaxed on a couch or a bed
- eating or drinking while you learn, or only eating or drinking during breaks
- working hard for a reasonably long time or taking frequent breaks

1. Do you work in conditions that match these preferences when you do your math homework?
2. Do you work in conditions that match these preferences when you study for math tests?

Answers

Section Preview

This section might seem awkward to you and your students if you don't buy into the purpose of its existence. The point of this material is to clarify what identities and contradictions are. Use this time to summarize the identities the students already know and focus on proof tactics. **Be very clear that performing mathematical operations "between sides of the equality" is illegal!** Doing such tom-foolery is assuming that the identity is true to begin with.

Section Preview

- Prove a trigonometric identity.

Example 6.1.1

We will start on the left side, as it is the more complicated side:

	LHS		RHS
	$\tan(\theta)\cos(\theta)$		$\sin(\theta)$
$=$	$\left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos(\theta)$	(Ratio Identity)	
$=$	$\sin(\theta)$		
$=$	**RHS**		

Example 6.1.2

a. Working on the left side of the equation, we have

	LHS		RHS
	$[1+\sin(x)][1+\sin(-x)]$		$\cos^2(x)$
$=$	$[1+\sin(x)][1-\sin(x)]$	(Symmetry Identity)	
$=$	$1-\sin^2(x)$	(distribution)	
$=$	$\cos^2(x)$	(Pythagorean Identity)	
$=$	**RHS**		

b. This is a case where the right side is more complex. Let's start there.

	LHS		RHS
	$\sin^2(\theta)$		$\frac{\sec^2(\theta)-1}{\sec^2(\theta)}$
$=$	$\frac{\sec^2(\theta)}{\sec^2(\theta)}-\frac{1}{\sec^2(\theta)}$	(arithmetic)	
$=$	$1-\frac{1}{\sec^2(\theta)}$	(arithmetic)	
$=$	$1-\cos^2(\theta)$	(Reciprocal Identity)	
$=$	$\sin^2(\theta)$	(Pythagorean Identity)	
$=$	**LHS**		

c. It should be obvious that the left side is the more complicated side in this case; however, notice that both sides have negative angular arguments for some of their trigonometric functions. Let's start by cleaning those up on both sides.

	LHS		RHS
	$\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)}$		$\cos(\theta)+\sin(-\theta)$
		$=$	$\cos(\theta)-\sin(\theta)$ (Symmetry Identity)
$=$	$\frac{[\sin(-\theta)]^2-[\cos(-\theta)]^2}{\sin(-\theta)-\cos(-\theta)}$	(it's best to rewrite powers first)	
$=$	$\frac{[-\sin(\theta)]^2-[\cos(\theta)]^2}{-\sin(\theta)-\cos(\theta)}$	(Symmetry Identities)	
$=$	$\frac{[\sin(\theta)]^2-[\cos(\theta)]^2}{-\sin(\theta)-\cos(\theta)}$	(squaring the negative)	
$=$	$-\frac{[\sin(\theta)]^2-[\cos(\theta)]^2}{\sin(\theta)+\cos(\theta)}$	(factoring the negative from the denominator)	
$=$	$-\frac{[\sin(\theta)-\cos(\theta)][\sin(\theta)+\cos(\theta)]}{\sin(\theta)+\cos(\theta)}$	(Difference of Squares)	
$=$	$-\frac{[\sin(\theta)-\cos(\theta)]\ \cancel{[\sin(\theta)+\cos(\theta)]}}{\cancel{\sin(\theta)+\cos(\theta)}}$	(canceling like factors)	
$=$	$-[\sin(\theta)-\cos(\theta)]$		
$=$	$-\sin(\theta)+\cos(\theta)$	(distribution)	
$=$	$\cos(\theta)-\sin(\theta)$	(Commutative Property of Addition)	
$=$	**RHS**		

d. Starting with the left side, we get the following:

	LHS		RHS
	$[1-\cos^2(x)]\,[1+\cot^2(x)]$		1
$=$	$[1-\cos^2(x)]\left[1+\dfrac{\cos^2(x)}{\sin^2(x)}\right]$	(rewriting in terms of sines and cosines)	
$=$	$[1-\cos^2(x)]\left[\dfrac{\sin^2(x)}{\sin^2(x)}+\dfrac{\cos^2(x)}{\sin^2(x)}\right]$	(getting a common denominator)	
$=$	$[1-\cos^2(x)]\left[\dfrac{\sin^2(x)+\cos^2(x)}{\sin^2(x)}\right]$	(adding fractions)	
$=$	$[1-\cos^2(x)]\left[\dfrac{1}{\sin^2(x)}\right]$	(Pythagorean Identity)	
$=$	$[\sin^2(x)]\left[\dfrac{1}{\sin^2(x)}\right]$	(Pythagorean Identity)	
$=$	$\dfrac{\sin^2(x)}{\sin^2(x)}$		
$=$	1		
$=$	**RHS**		

e. Both sides of this identity are equally complex. Let's start with the left side and see how far we get. This example illustrates our need to "keep our eyes on the prize." If you want to somehow transform the left side into the right, we will eventually need to get a $1-\sin(\theta)$ in the numerator of the left side. That tells us where to start with this proof.

	LHS		RHS
	$\dfrac{\cos(\theta)}{1+\sin(\theta)}$		$\dfrac{1-\sin(\theta)}{\cos(\theta)}$
$=$	$\dfrac{\cos(\theta)\,[1-\sin(\theta)]}{1-\sin^2(\theta)}$	(distribution)	
$=$	$\dfrac{\cos(\theta)\,[1-\sin(\theta)]}{\cos^2(\theta)}$	(Pythagorean Identity)	
$=$	$\dfrac{\cancel{\cos(\theta)}\,[1-\sin(\theta)]}{\cos\cancel{^2}(\theta)}$	(cancel like factors)	
$=$	$\dfrac{1-\sin(\theta)}{\cos(\theta)}$		
$=$	**RHS**		

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6.1.1: Resources and Key Concepts

Resources

Videos

- Proof Techniques
  - Fundamental Identities - Tactics for Proving Trigonometric Identities
  - A Review of the Basic Trigonometric Identities - Proving Trigonometric Identities (Part 2)

Key Concepts

Common Mistakes

- **Do Not Cross the Line:** When proving an identity, you must treat each side of the equation separately. Do not add, subtract, multiply, divide, or perform any other operation on both sides simultaneously, as this assumes the identity is already true.
- **Giving Up on One Side:** If you get stuck while manipulating one side of an identity, you should try working with the other side to see if you can meet in the middle or gain insight into the necessary steps.
- **Not Using Sines and Cosines:** When other algebraic or identity-based strategies fail, a reliable technique is to convert all trigonometric functions in the expression into their sine and cosine equivalents.
- **Building Instead of Simplifying:** It is almost always easier to start with the more complex side of an identity and simplify it to match the other side, rather than trying to build up the simpler side.

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6.1.2: Homework

Reading Questions

1. What are the five main families of basic trigonometric identities summarized in this section?
2. What is the first step recommended in the "How To" guide for proving a trigonometric identity?
3. When proving an identity, why is it considered "illegal" to perform an operation on both sides of the equals sign?
4. If you are stuck while trying to prove an identity, what are two strategies the text suggests trying?
5. What does it mean to "keep an eye on the prize" when proving an identity?
6. In the proof of $[1+\sin(x)][1+\sin(-x)]=\cos^2(x)$, which two fundamental identities were used?
7. What is the general strategy for proving an identity if you are unsure where to begin?
8. Explain why it is generally easier to start with the more complex side of an identity.

Homework

Concept Check

1. We showed that $\sin(\theta)=\pm\sqrt{1-\cos^2(\theta)}$ for all θ. Give an example of an angle θ such that $\sin(\theta)=-\sqrt{1-\cos^2(\theta)}$.
2. We showed that $\cos(\theta)=\pm\sqrt{1-\sin^2(\theta)}$ for all θ. Give an example of an angle θ such that $\cos(\theta)=-\sqrt{1-\sin^2(\theta)}$.
3. A student claims that $\cos(\theta)+\sin(\theta)=1$ is an identity because $\cos(0)+\sin(0)=1+0=0$. How would you respond to this student?

Proofs

For the following exercises, prove the identity. Assume all quantities are defined.

4. $\tan(\theta)+\tan(-\theta)=0$
5. $\cos^2(\theta)-\sin^2(\theta)=2\cos^2(\theta)-1$
6. $\cos^2(\theta)-\sin^2(\theta)=1-2\sin^2(\theta)$
7. $\dfrac{\sin^2(\theta)}{1-\sin^2(\theta)}=\tan^2(\theta)$
8. $\sin^4(\theta)-\cos^4(\theta)=\sin^2(\theta)-\cos^2(\theta)$
9. $\sin^2(x)+\tan^2(x)+\cos^2(x)=\sec^2(x)$
10. $\cos^4(\theta)-\sin^4(\theta)=1-2\sin^2(\theta)$
11. $\dfrac{\sin(x)+\cos(x)}{\cos(x)}=\tan(x)+1$
12. $\dfrac{\cos^2(\theta)}{1+\sin(\theta)}=1-\sin(\theta)$
13. $\dfrac{\sin^2(x)}{1+\cos(x)}=1-\cos(x)$
14. $\dfrac{\cos(x)}{1+\sin(x)}=\sec(x)-\tan(x)$

15. $\dfrac{\sin(x)-\cos(x)}{\sin(x)+\cos(x)}=\dfrac{\tan(x)-1}{1+\tan(x)}$

16. $\dfrac{1-\tan(\theta)}{1+\tan(\theta)}=\dfrac{\cot(\theta)-1}{\cot(\theta)+1}$

17. $\dfrac{1}{\tan(x)+\cot(x)}=\sin(x)\cdot\cos(x)$

18. $\dfrac{\tan(x)+\sec(x)-1}{\tan(x)-\sec(x)+1}=\dfrac{1+\sin(x)}{\cos(x)}$

19. $\dfrac{1+\sin(x)}{\cos(x)}=\sec(x)+\tan(x)$

20. $\dfrac{\sin(x)-\cos(x)}{\cos(-x)-\sin(-x)}=\dfrac{-1-\tan(-x)}{1+\tan(x)}$

21. $\dfrac{\sin(-x)}{1+\cos(x)}=\dfrac{\cos(-x)-1}{\sin(x)}$

22. $\dfrac{\sec(x)-1}{\sec(x)+1}=\dfrac{\sin^2(x)}{(\cos(x)+1)^2}$

23. $\dfrac{\sec(x)-1-\tan(-x)}{\tan(x)-\sec(-x)+1}=\dfrac{1-\sin(-x)}{\cos(x)}$

24. $\dfrac{\cos(-x)}{1-\sin(-x)}=\sec(-x)+\tan(-x)$

25. $\dfrac{1-\sin(-x)}{\cos(x)}=\sec(-x)-\tan(-x)$

26. $\sin(3\pi-2\theta)=-\sin(2\theta-3\pi)$

27. $\dfrac{\cos(x)}{1-\sin(x)}=\sec(x)+\tan(x)$

28. $\dfrac{1+\cos\left(\frac{\pi}{2}-x\right)}{\cos(x)}=\csc\left(\frac{\pi}{2}-x\right)+\tan(x)$

29. $\dfrac{1-\cos(x)}{\cos(x)}=\dfrac{\sin^2(x)}{\cos(x)(1+\cos(x))}$

30. $\dfrac{\sin^2(x)}{\cos(x)}=\tan(x)\cdot\sin(x)$

31. $\dfrac{\sin(-x)-\cos(x)}{\cos(x)}=\tan(-x)-1$

32. $\dfrac{1-2\cos^2(\theta)}{\sin(\theta)\cos(\theta)}=\tan(\theta)-\cot(\theta)$

33. $\cos\left(-\dfrac{\pi}{4}-5t\right)=\cos\left(5t+\dfrac{\pi}{4}\right)$

34. $\dfrac{\cos\left(\frac{\pi}{2}-x\right)}{\cos(x)+1}=\dfrac{1-\sin\left(\frac{\pi}{2}-x\right)}{\sin(x)}$

35. $\dfrac{1-\tan^2(\theta)}{1-\cot^2(\theta)}=1-\sec^2(\theta)$

36. $\dfrac{\cos(-x)}{1+\sin(-x)} = \sec(x) - \tan(-x)$

37. $\dfrac{1+\cos(x)}{\sin(x)} = \csc(x) + \cot(x)$

38. $\dfrac{1-\sin^2(x)}{\cos(x)} = \cos(x)$

39. $\dfrac{1-\sin(x)}{1+\sin(x)} = \dfrac{\sec(x)-\tan(x)}{\sec(x)+\tan(x)}$

40. $\dfrac{\sec(x)-1}{\sec(x)+1} = \dfrac{1-\cos(x)}{1+\cos(x)}$

41. $\dfrac{1+\sin(x)}{\cos(x)} = \dfrac{\cos(x)}{1-\sin(x)}$

42. $\tan(-t^2+1) = -\tan(t^2-1)$

43. $\dfrac{1-\cos\left(\frac{\pi}{2}-x\right)}{\cos(x)} = \csc\left(\frac{\pi}{2}-x\right) - \tan(x)$

44. $\dfrac{\sin(x)-\sin\left(\dfrac{\pi}{2}-x\right)}{\sin(x)+\cos(x)} = \dfrac{\cot\left(\dfrac{\pi}{2}-x\right)-1}{1+\tan(x)}$

45. $\dfrac{\cot\left(\frac{\pi}{2}-B\right)+\sec(B)-1}{\tan(B)-\csc\left(\frac{\pi}{2}-B\right)+1} = \dfrac{1-\sin(-B)}{\cos(B)}$

46. $\dfrac{\sin(\phi)}{1+\cos(\phi)} = \dfrac{1-\cos(\phi)}{\sin(\phi)}$

47. $\dfrac{\cos^2(\theta)}{1-\sin(\theta)} = 1+\sin(\theta)$

48. $\csc(-\theta-5) = -\csc(\theta+5)$

49. $\dfrac{\sin\left(\dfrac{\pi}{2}-\alpha\right)}{1+\sin(\alpha)} = \csc\left(\dfrac{\pi}{2}-\alpha\right) - \tan(\alpha)$

50. $\dfrac{1+\sin(-\theta)}{1+\sin(\theta)} = \dfrac{\sec(-\theta)+\tan(-\theta)}{\sec(-\theta)+\tan(\theta)}$

51. $\dfrac{1+\cos(-A)}{\sin(-A)} = -\csc(A) + \cot(-A)$

52. $\dfrac{\cos(-\phi)-1}{\cos(\phi)} = \dfrac{\sin^2(-\phi)}{\cos(\phi)(1+\cos(\phi))}$

53. $\dfrac{\sin\left(\frac{\pi}{2}-\theta\right)}{1-\sin(\theta)} = \csc\left(\frac{\pi}{2}-\theta\right) + \cot\left(\frac{\pi}{2}-\theta\right)$

54. $\dfrac{\sec(-\alpha)-1}{\sec(\alpha)+1} = \dfrac{\sin^2(-\alpha)}{(\cos(-\alpha)+1)^2}$

55. $\sec(-6t) = \sec(6t)$

56. $-\dfrac{\cos^2(-A)}{1+\sin(-A)} = -1+\sin(-A)$

57. $\dfrac{\tan(\theta)+\tan(\phi)}{\cot(\theta)+\cot(\phi)} = \tan(\theta)\tan(\phi)$

58. $\dfrac{\sin^2(-\alpha)}{\cos(-\alpha)} = -\tan(\alpha)\cdot\sin(-\alpha)$

59. $\dfrac{\sec(y)-1}{\sec(y)+1} = \dfrac{1-\cos(-y)}{1+\cos(-y)}$

60. $\dfrac{\sin^2\left(\frac{\pi}{2}-A\right)}{1-\sin(A)} = \dfrac{\cos^2(A)}{1-\cos\left(\frac{\pi}{2}-A\right)}$

61. $\dfrac{1+\sin\left(\frac{\pi}{2}-B\right)}{\sin(B)} = \sec\left(\frac{\pi}{2}-B\right)+\cot(B)$

62. $\dfrac{\sin\left(\frac{\pi}{2}-\phi\right)-1}{\sin\left(\frac{\pi}{2}-\phi\right)} = \dfrac{\sin^2(\phi)}{\sin\left(\frac{\pi}{2}-\phi\right)\left(1+\sin\left(\frac{\pi}{2}-\phi\right)\right)}$

63. $\cot(9-7\theta) = -\cot(7\theta-9)$

64. $\dfrac{\csc\left(\frac{\pi}{2}-B\right)-1}{\sec(B)+1} = \dfrac{\cos^2\left(\frac{\pi}{2}-B\right)}{\left(\sin\left(\frac{\pi}{2}-B\right)+1\right)^2}$

65. $\dfrac{1-\cos(\alpha)}{\sin^2(\alpha)} = \dfrac{1}{1+\cos(\alpha)}$

Challenge Problems

66. Suppose that you are given a system of two equations of the following form:

$$\begin{aligned} A\cos(\phi) &= B\nu_1 - B\nu_2\cos(\theta) \\ A\sin(\phi) &= B\nu_2\sin(\theta) \end{aligned}$$

Show that

$$A^2 = B^2\left(\nu_1^2+\nu_2^2-2\nu_1\nu_2\cos(\theta)\right).$$

67. Suppose that a point with coordinates

$$(x,y) = \left(a\left(\cos(\psi)-\epsilon\right), a\sqrt{1-\epsilon^2}\sin(\psi)\right)$$

is a distance $r>0$ from the origin, where $a>0$ and $0<\epsilon<1$. Use $r^2=x^2+y^2$ to show that $r=a\left(1-\epsilon\cos(\psi)\right)$. (Note: These coordinates arise in the study of elliptical orbits of planets.)

68. Show that each trigonometric function can be written solely in terms of the sine function.

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6.2: Sum and Difference Identities

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We have come a long way in Trigonometry. We have defined the trigonometric functions (in terms of angles, ratios of sides of right triangles, and real numbers), proved trigonometric identities with them, used them to solve applications, graphed them, and evaluated them by hand at the special angles. If we needed to evaluate a trigonometric function at a non-special angle, however, we must use technology. But, is this always the case?

For example, if we know the sine of 27°, can we find the sine of $2\,(27^\circ) = 54^\circ$ without using a calculator? Or, if we know $\cos(\alpha)$ and $\cos(\beta)$, can we calculate $\cos(\alpha+\beta)$?

The Sum and Difference Identities for the Cosine

All the identities that relate the trigonometric ratios of different angles are derived from the Sum and Difference Identities (which really should be named the "Sum and Difference of Angles Identities," but that's too long to write). Let's see why we need these formulas.

Is it true that

$$\cos(\alpha+\beta) = \cos(\alpha) + \cos(\beta)?$$

We can test this hypothesis by evaluating both expressions for some specific values of α and β, say $\alpha = 45^\circ$ and $\beta = 30^\circ$, as shown below.

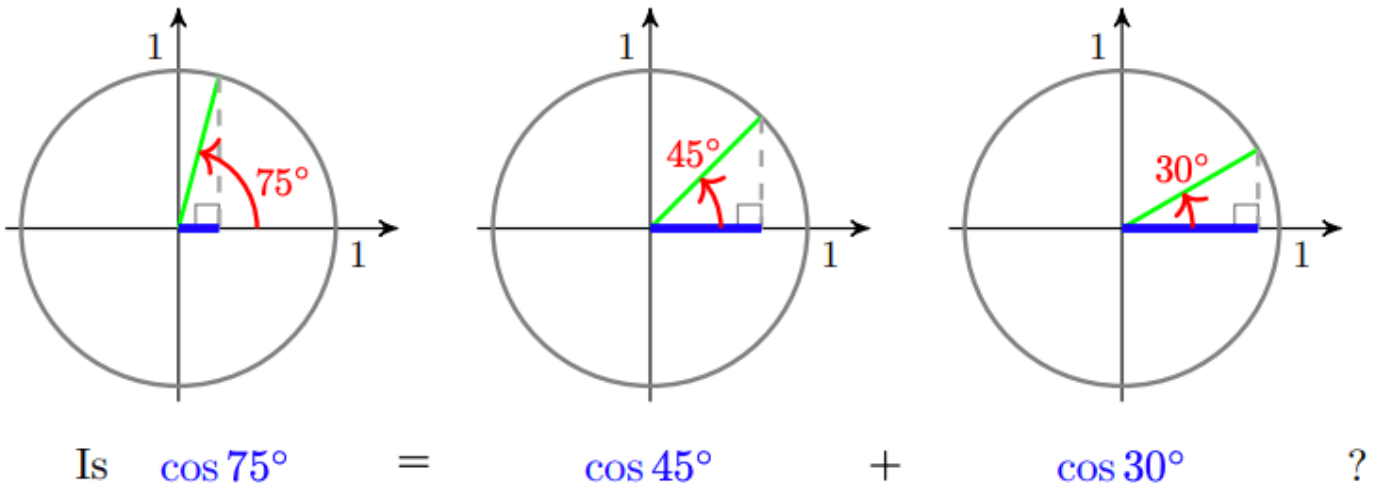


Figure 6.2.1

Taking a unit circle approach, you should be able to see that $\cos(75^\circ)$ (which is the x-value of the point on the terminal side of 75°) is in fact smaller than either $\cos(45^\circ)$ or $\cos(30^\circ)$. Therefore, it cannot be true that $\cos(75^\circ)$ is equal to $\cos(45^\circ) + \cos(30^\circ)$.

Example 6.2.1

Verify that $\cos(45^\circ + 30^\circ)$ is not equal to $\cos(45^\circ) + \cos(30^\circ)$.

Solution available at the end of this section.

Checkpoint 6.2.1

Show that $\sin(60^\circ + 30^\circ)$ is not equal to $\sin(60^\circ) + \sin(30^\circ)$.

Answer available at the end of this section.

Caution: You Cannot "Split" a Trigonometric Function Across its Arguments

Because the values of the expressions in the Example 6.2.1 and Checkpoint 6.2.1 are different, it is *not* true that $\cos(\alpha + \beta)$ is equal to $\cos(\alpha) + \cos(\beta)$ for all angles α and β, or that $\sin(\alpha + \beta)$ is equal to $\sin(\alpha) + \sin(\beta)$.

This is a general property of almost all functions. That is, if f is a function, it is generally the case that

$$f(a \pm b) \neq f(a) \pm f(b).$$

It turns out that there is a relationship between the trigonometric ratios for $\alpha \pm \beta$ and the trigonometric ratios of α and β.

Theorem: Sum and Difference Identities for the Cosine

For angles α and β,

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta).$$

Proof available at the end of this section.

The $\pm$ and $\mp$ signs on the left and right side of this identity are always opposite.

Advice: Memorize These Identities

The Sum and Difference Identities for the cosine are needed throughout the rest of Trigonometry, Calculus, and Differential Equations. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated.

Notice that to find the cosine of $\alpha + \beta$ we must know (or be able to find) both trigonometric ratios for both and α and β.

The Sum and Difference Identities can be used to find exact values for trigonometric ratios of various angles.

Example 6.2.2

Find an exact value for $\cos(105^\circ)$.

Solution available at the end of this section.

Checkpoint 6.2.2

Suppose that $\sin(\theta) = \frac{-5}{13}$ and $\cos(\theta) = \frac{12}{13}$. Find an exact value for $\cos\left(\frac{\pi}{4} + \theta\right)$.

Answer available at the end of this section.

Proving the Cofunction Identities

We have been using the Cofunction Identities for quite a while, but have yet to see a proof. This is usually unforgivable in Mathematics; however, I wanted you to have some experience with these identities and, at the time, we didn't have the "mathematical technology" (read as, the knowledge) to prove them—now, we do. Let's prove $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$.

$$\begin{aligned} \cos\left(\frac{\pi}{2} - \theta\right) &= \cos\left(\frac{\pi}{2}\right)\cos(\theta) + \sin\left(\frac{\pi}{2}\right)\sin(\theta) && \text{(Difference Identity for the Cosine)} \\ &= (0)\cos(\theta) + (1)\sin(\theta) && \text{(evaluating trig functions)} \\ &= \sin(\theta) \end{aligned}$$

The remaining Cofunction Identities are left for you to prove as exercises.

The Sum and Difference Identities for the Sine

Theorem: Sum and Difference Identities for the Sine

For angles α and β,

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta).$$

Proof available at the end of this section.

Notice that the sine function "keeps its sign." That is, if we are given $\alpha + \beta$, the result will be a sum of functions. If given $\alpha - \beta$, we will have a difference of functions.

Advice: Memorize These Identities

The Sum and Difference Identities for the sine are needed throughout the rest of Trigonometry, Calculus, and Differential Equations. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated.

Example 6.2.3

Suppose that $\sin(\theta) = 0.6$ and $\cos(\theta) = -0.8$. Find an exact value for $\sin\left(\theta + \frac{2\pi}{3}\right)$.

Solution available at the end of this section.

Example 6.2.4

Find the exact value of $\sin\left(\frac{19\pi}{12}\right)$.

Solution available at the end of this section.

Checkpoint 6.2.4

Evaluate $\sin\left(\frac{7\pi}{12}\right)$ exactly.

Answer available at the end of this section.

Example 6.2.5

If α is a Quadrant II angle with $\sin(\alpha) = \frac{5}{13}$, and β is a Quadrant III angle with $\tan(\beta) = 2$, find $\sin(\alpha - \beta)$.

Solution available at the end of this section.

Checkpoint 6.2.5

If α is a Quadrant II angle with $\sin(\alpha) = \frac{3}{5}$, and β is a Quadrant III angle with $\cos(\beta) = -\frac{5}{13}$, find $\cos(\alpha - \beta)$.

Answer available at the end of this section.

The Sum and Difference Identities for the Tangent

There are also sum and difference formulas for the tangent. The proof for the tangent versions involves using the Quotient Identities to rewrite the tangent in terms of the sine and cosine, which we leave for you in the homework area.

Theorem: Sum and Difference Identities for the Tangent

For angles α and β where the following formula is defined,

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

There is a subtle restriction in the last theorem that the two before it didn't have. Since the tangent function does not have a domain consisting of all real numbers, the Sum and Difference Identities for the tangent have exceptions whenever α or β are odd multiples of $\frac{\pi}{2}$; however, this theorem is rarely used and, as such, this exception is not going to be encountered (at least, not at this level).

Example 6.2.6

Find an exact value for $\tan(75^\circ)$.

Solution available at the end of this section.

Checkpoint 6.2.6

Evaluate $\tan\left(\frac{\pi}{12}\right)$ exactly.

Answer available at the end of this section.

Revisiting Proofs of Identities

Every time we introduce a new set of major identities (like the Sum and Difference Identities), we get to practice proofs again. In some sense, proofs begin to get much easier once we learn identities like the Sum and Difference Identities. This is because one of the first things we should look for when trying to prove an identity is the existence of other identities within the claim. That is, we look for Pythagorean Identities, Symmetry Identities, and so on. Now we have a new set of identities to look for.

Example 6.2.7

Prove the following identity.

$$\frac{\sin(\alpha-\beta)}{\cos(\alpha)\cos(\beta)} = \tan(\alpha) - \tan(\beta)$$

Solution available at the end of this section.

Checkpoint 6.2.7

Prove.

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta)$$

Revisiting Graphs of Trigonometric Functions

The Sum and Difference Identities allow us to expand the set of trigonometric functions we can graph. Up to this point, we have only been able to graph functions of the form

$$f(x) = A\,\text{trig}(Bx+C) + D.$$

However, now we can graph certain *combinations* of trigonometric functions. Specifically, those having the form of the Sum and Difference Identities.

Example 6.2.8

Graph the function.

$$f(x) = \frac{1}{2}\cos(2x)\cos(3x) - \frac{1}{2}\sin(2x)\sin(3x)$$

Solution available at the end of this section.

Checkpoint 6.2.8

Graph the function.

$$g(x) = -2\sin(\pi x)\cos\left(\frac{\pi}{2}x\right) + 2\sin\left(\frac{\pi}{2}x\right)\cos(\pi x)$$

Answer available at the end of this section.

Success in College: Learning From Tests (Part 3)

1. Look at your last test from this class. How many points did you miss because you could not remember something or did not know how to do the problem?

2. Knowing what you know now, do you think you could do better on this exam?

Success in Mathematics: Sequential Versus Global Learners

Sequential learners tend to think in a linear manner and are able to function with only partial understanding of material they have been taught. Global learners think in a systems-oriented manner, and may have trouble applying new material until they fully understand it and see how it relates to material they already know about and understand. Once they grasp the big picture, however, their holistic perspective enables them to see innovative solutions to problems that sequential learners might take much longer to reach.

Source: R. Felder, R. Brent , Journal of Engineering Education, 94 (1), 57-72 (2005).

1. When learning math, do you think you are more of a sequential learner or a global learner?
2. Students often have different learning preferences for different subjects. When learning history or another subject in the humanities, do you think you are more a sequential learner or a global learner?

Answers

Section Preview

Do not succumb to laziness! Prove one of the Sum and Difference Identities. It's important that students see the logic necessary to derive such foundational theorems.

Section Preview

- Use the Sum and Difference Identities to find the exact value of a trigonometric function.
- Prove an equation is an identity using the Sum and Difference Identities.
- Simplify an expression using the Sum and Difference Identities.
- Use the Sum and Difference Identities to graph an equation.

Example 6.2.1

We know that $\cos(45^\circ) = \frac{1}{\sqrt{2}} \approx 0.7071$ and $\cos(30^\circ) = \frac{\sqrt{3}}{2} \approx 0.8660$. Thus, $\cos(45^\circ) + \cos(30^\circ) \approx 1.5731$. For $\cos(45^\circ + 30^\circ) = \cos(75^\circ)$, we use a calculator to get the approximation 0.2588. Therefore, the two expressions are not equal.

Checkpoint 6.2.1

$\sin(60^\circ + 30^\circ) = 1$, but $\sin(60^\circ) + \sin(30^\circ) = \frac{\sqrt{3}}{2} + \frac{1}{2}$

Theorem: Sum and Difference Identities for the Cosine

Let α and β be angles in standard position and let α_0 and β_0 be the angles coterminal to α and β, respectively, each of which measure between 0 and 2π radians. Moreover, without loss of generality, assume that $\alpha_0 > \beta_0$ (if this assumption were not true, we would just relabel the variables accordingly).

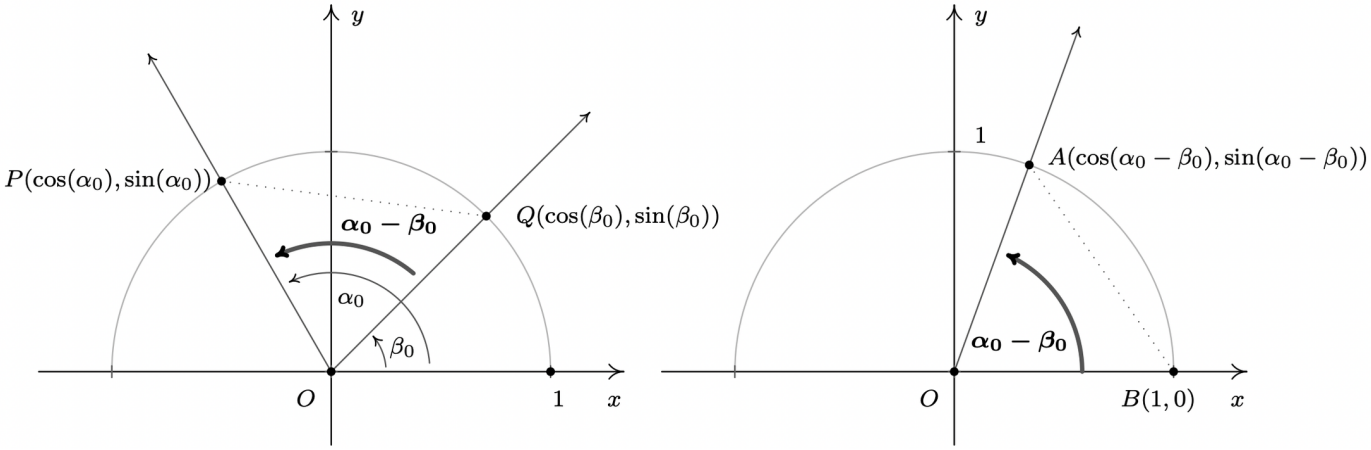
Since α_0 is coterminal with α, $\alpha = \alpha_0 + 2\pi m$ for some $m \in \mathbb{Z}$. Likewise, $\beta = \beta_0 + 2\pi n$ for some $n \in \mathbb{Z}$. Therefore,

$$\cos(\alpha - \beta) = \cos(\alpha_0 + 2\pi m - (\beta_0 + 2\pi n)) = \cos(\alpha_0 - \beta_0 + 2\pi(m - n)).$$

Since m and n are both integers, $m - n$ is also an integer. Moreover, cosine has a period of 2π means that $\cos(\alpha_0 - \beta_0 + 2\pi(m - n)) = \cos(\alpha_0 - \beta_0)$. Thus,

$$\cos(\alpha - \beta) = \cos(\alpha_0 - \beta_0).$$

All of this is to reduce our work to the angles α_0 and β_0. We now draw and label the points P and Q on the unit circle which are on the terminal sides of α_0 and β_0, respectively.



The image on the right side of this figure is a rotation of the image on the left clockwise by β_0 radians. Since the angles $\angle POQ$ and $\angle AOB$ are congruent, the distance between P and Q is equal to the distance between A and B. The Distance Formula yields

$$\begin{aligned}
&& \text{Disatnce between } P \text{ and } Q &= \text{Distance between } A \text{ and } B && \\
\Longrightarrow\quad & \sqrt{(\cos(\alpha_0)-\cos(\beta_0))^2+(\sin(\alpha_0)-\sin(\beta_0))^2} &= \sqrt{(\cos(\alpha_0-\beta_0)-1)^2+(\sin(\alpha_0-\beta_0)-0)^2} && \text{(Distance Formula)} \\
\Longrightarrow\quad & (\cos(\alpha_0)-\cos(\beta_0))^2+(\sin(\alpha_0)-\sin(\beta_0))^2 &= (\cos(\alpha_0-\beta_0)-1)^2+(\sin(\alpha_0-\beta_0)-0)^2 && \text{(squaring both sides)} \\
\Longrightarrow\quad & \begin{array}{c}\cos^2(\alpha_0)-2\cos(\alpha_0)\cos(\beta_0)+\cos^2(\beta_0)\\ +\sin^2(\alpha_0)-2\sin(\alpha_0)\sin(\beta_0)+\sin^2(\beta_0)\end{array} &= \begin{array}{c}\cos^2(\alpha_0-\beta_0)-2\cos(\alpha_0-\beta_0)+1\\ +\sin^2(\alpha_0-\beta_0)\end{array} && \text{(distributing)} \\
\Longrightarrow\quad & \begin{array}{c}\cos^2(\alpha_0)+\sin^2(\alpha_0)\\ -2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0)\\ +\cos^2(\beta_0)+\sin^2(\beta_0)\end{array} &= \begin{array}{c}\cos^2(\alpha_0-\beta_0)+\sin^2(\alpha_0-\beta_0)\\ -2\cos(\alpha_0-\beta_0)\\ +1\end{array} && \text{(Commutative Property of Addition)} \\
\Longrightarrow\quad & \begin{array}{c}1\\ -2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0)\\ +1\end{array} &= \begin{array}{c}1\\ -2\cos(\alpha_0-\beta_0)\\ +1\end{array} && \text{(Pythagorean Identity)} \\
\Longrightarrow\quad & 2-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0) &= 2-2\cos(\alpha_0-\beta_0) && \text{(simplifying)} \\
\Longrightarrow\quad & -2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0) &= -2\cos(\alpha_0-\beta_0) && \text{(subtracting 2 from both sides)} \\
\Longrightarrow\quad & \cos(\alpha_0)\cos(\beta_0)+\sin(\alpha_0)\sin(\beta_0) &= \cos(\alpha_0-\beta_0) && \text{(dividing both sides by } -2)
\end{aligned}$$

Combining this result with the fact that $\cos(\alpha-\beta)=\cos(\alpha_0-\beta_0)$, we have the identity

$$\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta).$$

To get the sum identity for cosine, we use this result along with the Symmetry Identities.

$$\cos(\alpha+\beta)=\cos(\alpha-(-\beta))=\cos(\alpha)\cos(-\beta)+\sin(\alpha)\sin(-\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$$

Hence,

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta).$$

Example 6.2.2

We can write 105° as the sum of two special angles: $105^\circ=60^\circ+45^\circ$. Now apply the sum of angles identity for the cosine.

$$\begin{aligned}
\cos(60^\circ+45^\circ) &= \cos(60^\circ)\cos(45^\circ)-\sin(60^\circ)\sin(45^\circ)\\
&= \frac{1}{2}\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}\\
&= \frac{\sqrt{2}-\sqrt{6}}{4}
\end{aligned}$$

Thus, $\cos 105^\circ=\frac{\sqrt{2}-\sqrt{6}}{4}$. You can check that your calculator gives the same decimal approximation of about -0.2588 for both $\cos 105^\circ$ and $\frac{\sqrt{2}-\sqrt{6}}{4}$.

Checkpoint 6.2.2

$$\frac{17}{13\sqrt{2}}=\frac{17\sqrt{2}}{26}$$

Theorem: Sum and Difference Identities for the Sine

To prove the Sum and Difference Identities for the sine, we make use of the Cofunction Identities.

$$\begin{aligned}
\sin(\alpha+\beta) &= \cos\left(\tfrac{\pi}{2}-(\alpha+\beta)\right) && \text{(Cofunction Identities)}\\
&= \cos\left(\left(\tfrac{\pi}{2}-\alpha\right)-\beta\right) && \text{(Associative Property of Addition)}\\
&= \cos\left(\tfrac{\pi}{2}-\alpha\right)\cos(\beta)+\sin\left(\tfrac{\pi}{2}-\alpha\right)\sin(\beta) && \text{(Difference Identity for Cosine)}\\
&= \sin(\alpha)\cos(\beta)+\sin\left(\tfrac{\pi}{2}-\alpha\right)\sin(\beta) && \text{(Cofunction Identities)}\\
&= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) && \text{(Cofunction Identities)}
\end{aligned}$$

Thus, we get the identity.

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta).$$

A similar proof shows that

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta).$$

Example 6.2.3

Recall that $\sin\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2}$ and $\cos\left(\frac{2\pi}{3}\right)=\frac{-1}{2}$. Substituting these values into the sum formula for sine, we find

$$\begin{aligned}\sin\left(\theta+\frac{2\pi}{3}\right) &= \sin(\theta)\cos\left(\frac{2\pi}{3}\right)+\cos(\theta)\sin\left(\frac{2\pi}{3}\right)\\ &= 0.6\left(\frac{-1}{2}\right)+(-0.8)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{3}{5}\cdot\frac{-1}{2}+\frac{-4}{5}\cdot\frac{\sqrt{3}}{2}\\ &= \frac{-3-4\sqrt{3}}{10}\end{aligned}$$

Example 6.2.4

We need to write the angle $\frac{19\pi}{12}$ as a sum or difference of common angles. The denominator of 12 suggests a combination of angles with denominators 3 and 4. One such combination is $\frac{19\pi}{12}=\frac{16\pi}{12}+\frac{3\pi}{12}=\frac{4\pi}{3}+\frac{\pi}{4}$. We get

$$\begin{aligned}\sin\left(\frac{19\pi}{12}\right) &= \sin\left(\frac{4\pi}{3}+\frac{\pi}{4}\right)\\ &= \sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)\\ &= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{-\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Checkpoint 6.2.4

$$\frac{\sqrt{3}+1}{2\sqrt{2}}$$

Example 6.2.5

This is one of my favorite styles of problems. There are two main ways to find the solution—using identities or using right triangle trigonometry. I will cover both methods here, but I must confess—I highly prefer the right triangle approach.

Using Identities

In order to find $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$, we need to find $\cos(\alpha)$ and both $\cos(\beta)$ and $\sin(\beta)$. To find $\cos(\alpha)$, we use the Pythagorean Identity $\cos^2(\alpha)+\sin^2(\alpha)=1$. Since $\sin(\alpha)=\frac{5}{13}$, we have $\cos^2(\alpha)+\left(\frac{5}{13}\right)^2=1$, or $\cos(\alpha)=\pm\frac{12}{13}$. Since α is a Quadrant II angle, $\cos(\alpha)=-\frac{12}{13}$. We now set about finding $\cos(\beta)$ and $\sin(\beta)$. We have several ways to proceed, but the Pythagorean Identity $1+\tan^2(\beta)=\sec^2(\beta)$ is a quick way to get $\sec(\beta)$, and hence, $\cos(\beta)$. With $\tan(\beta)=2$, we get $1+2^2=\sec^2(\beta)$ so that $\sec(\beta)=\pm\sqrt{5}$. Since β is a Quadrant III angle, we choose $\sec(\beta)=-\sqrt{5}$ so $\cos(\beta)=\frac{1}{\sec(\beta)}=\frac{1}{-\sqrt{5}}=-\frac{\sqrt{5}}{5}$. We now need to determine $\sin(\beta)$. We could use The Pythagorean Identity $\cos^2(\beta)+\sin^2(\beta)=1$, but we opt instead to use a quotient identity. From $\tan(\beta)=\frac{\sin(\beta)}{\cos(\beta)}$, we have $\sin(\beta)=\tan(\beta)\cos(\beta)$ so we get $\sin(\beta)=(2)\left(-\frac{\sqrt{5}}{5}\right)=-\frac{2\sqrt{5}}{5}$. We now have all the pieces needed to find $\sin(\alpha-\beta)$:

$$\begin{aligned}\sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\\ &= \left(\frac{5}{13}\right)\left(-\frac{\sqrt{5}}{5}\right)-\left(-\frac{12}{13}\right)\left(-\frac{2\sqrt{5}}{5}\right)\\ &= -\frac{29\sqrt{5}}{65}\end{aligned}$$

Using Right Triangle Trigonometry

For this method, we remind ourselves that Trigonometry is mostly about quadrants and reference angles. We draw $\alpha\in$ QII and $\beta\in$ QIII and their corresponding reference angles.

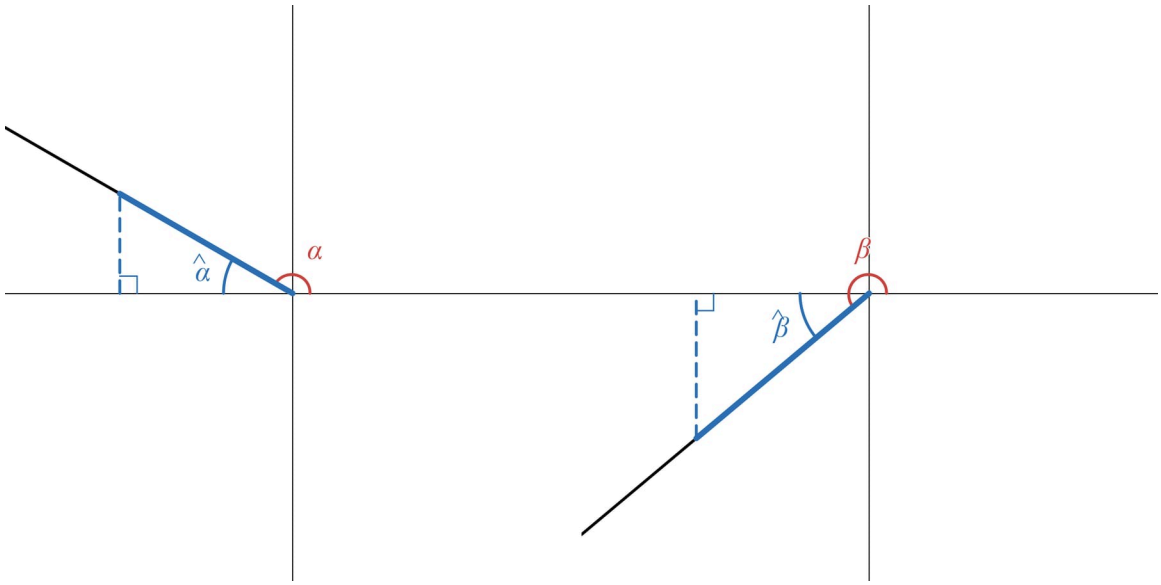


Figure 6.2.2

Neither of these graphs are drawn to scale, nor do they need to be. Since $\sin(\alpha) = \frac{5}{13}$, we wisely choose the terminal point so that its y-value is 5 and its distance to the origin is 13. Looking at the reference triangle, this should remind you that sine is opposite over hypotenuse. Likewise, since $\tan(\beta) = 2 = \frac{2}{1}$, we choose the terminal point of β to have a y-value of -2 (remember, $\beta \in$ QIII) and an x-value of -1. Again, the tangent is opposite over adjacent, so the tangent of $\hat{\beta}$ is $\frac{-2}{-1} = 2$.

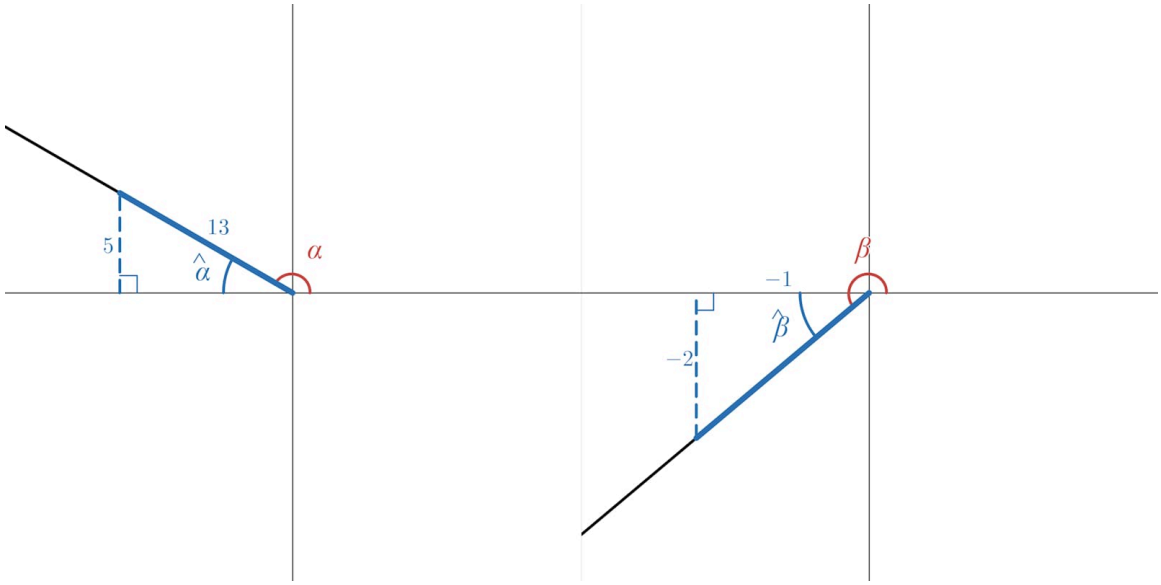


Figure 6.2.3

Finally, we use the Pythagorean Theorem to help fill in the missing sides (taking into account the quadrants).

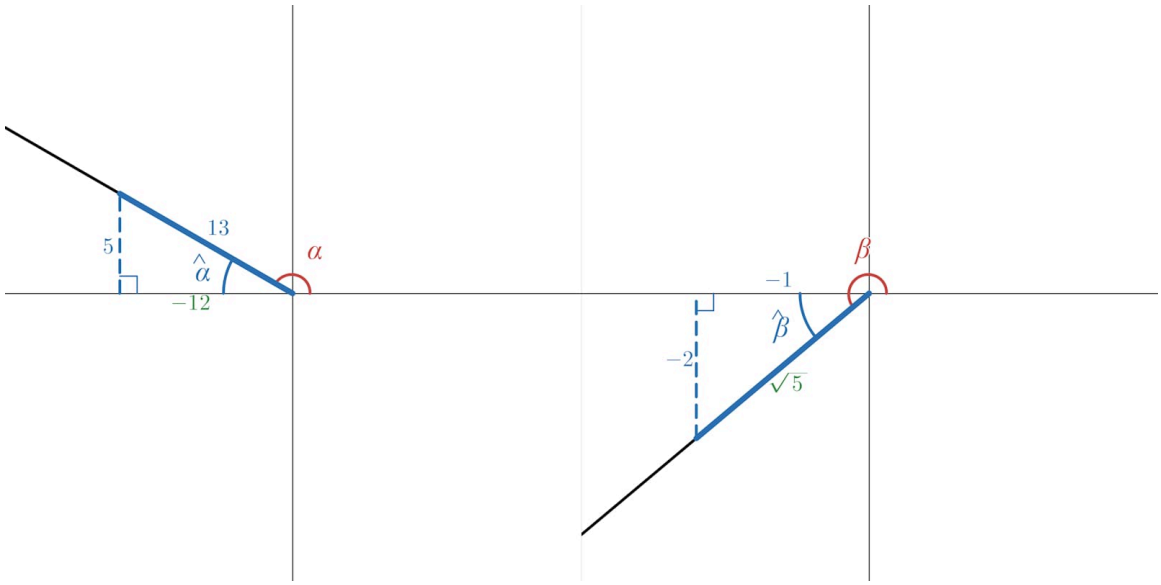


Figure 6.2.4

From these figures, we read off the information we need.

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)=\frac{5}{13}\cdot\frac{-1}{\sqrt{5}}-\frac{-12}{13}\cdot\frac{-2}{\sqrt{5}}=\frac{-5+(-24)}{13\sqrt{5}}=-\frac{29}{13\sqrt{5}}.$$

Checkpoint 6.2.5

$-\frac{56}{65}$

Example 6.2.6

We observe that $75^\circ=45^\circ+30^\circ$, so $\tan(75^\circ)=\tan(45^\circ+30^\circ)$. We can apply the sum formula for tangent.

$$\begin{aligned}\tan(45^\circ+30^\circ) &= \frac{\tan(45^\circ)+\tan(30^\circ)}{1-\tan(45^\circ)\tan(30^\circ)}\\ &= \frac{1+\frac{1}{\sqrt{3}}}{1-1\left(\frac{1}{\sqrt{3}}\right)}\cdot\frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1}\end{aligned}$$

Checkpoint 6.2.6

$\frac{\sqrt{3}-1}{\sqrt{3}+1}$

Example 6.2.7

The left side is the more complex side, so we start there. Also, we immediately spot our newly-discovered Sum and Difference Identities, so we know where to begin.

$$\begin{aligned}\frac{\sin(\alpha-\beta)}{\cos(\alpha)\cos(\beta)} &= \frac{\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)} && \text{(Sum and Difference Identities)}\\ &= \frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}-\frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)} && \text{(property of fractions)}\\ &= \frac{\sin(\alpha)\,\cancel{\cos(\beta)}}{\cos(\alpha)\,\cancel{\cos(\beta)}}-\frac{\cancel{\cos(\alpha)}\,\sin(\beta)}{\cancel{\cos(\alpha)}\,\cos(\beta)} && \text{(cancel like factors)}\\ &= \frac{\sin(\alpha)}{\cos(\alpha)}-\frac{\sin(\beta)}{\cos(\beta)}\\ &= \tan(\alpha)-\tan(\beta) && \text{(Ratio Identities)}\end{aligned}$$

Example 6.2.8

Prior to this moment, there would have been no possibility for us to graph this function by hand; however, we now know, from the Sum and Difference Identities, that

$$f(x)=\frac{1}{2}\cos(2x)\cos(3x)-\frac{1}{2}\sin(2x)\sin(3x)=\frac{1}{2}\left[\cos(2x)\cos(3x)-\sin(2x)\sin(3x)\right]=\frac{1}{2}\cos(2x+3x)=\frac{1}{2}\cos(5x).$$

The amplitude of this cosine function is $\frac{1}{2}$, the period is $\frac{2\pi}{5}$, the step size is $\frac{\pi}{10}$, the phase shift is 0, and the vertical shift is 0. Putting this together, we get the following graph.

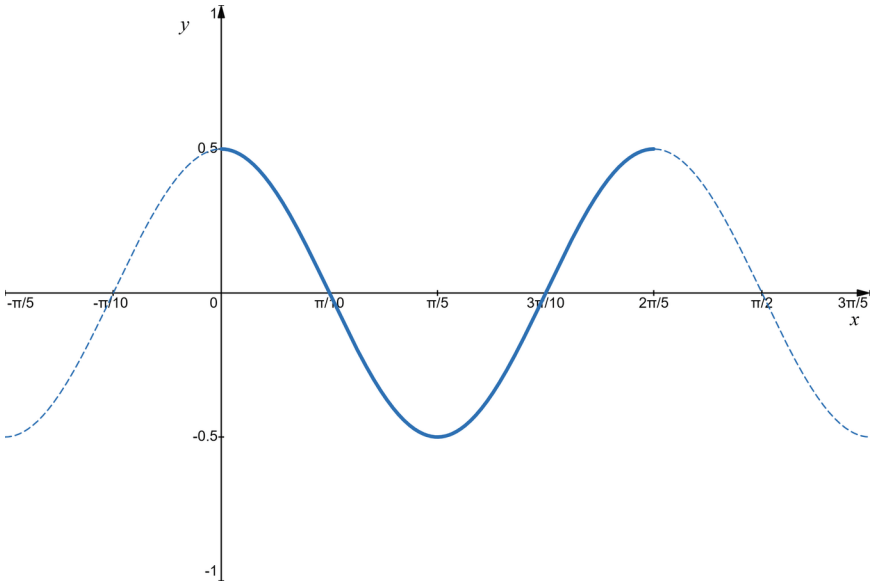


Figure 6.2.5

Checkpoint 6.2.8

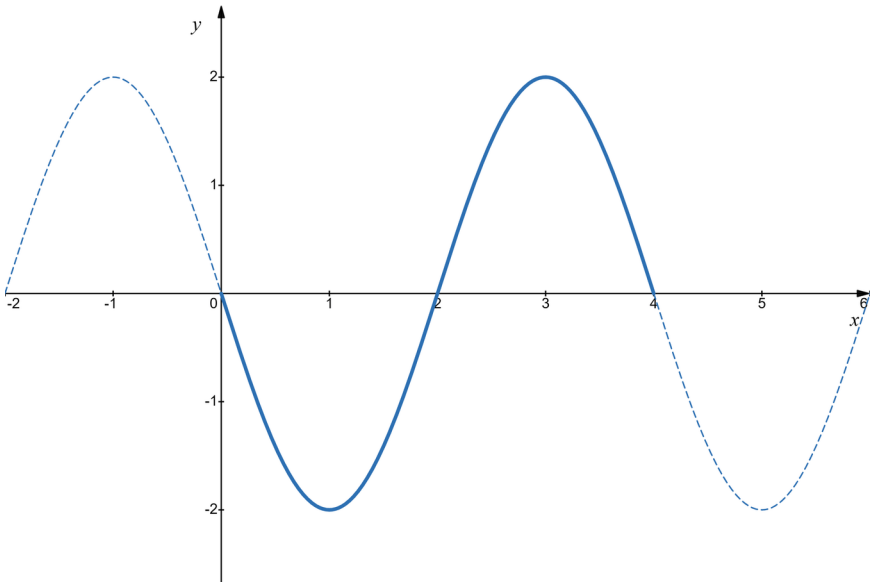


Figure 6.2.6

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6.2.1: Resources and Key Concepts

Resources

Videos

- Sum and Difference Identities
  - Sum and Difference Identities - The Sum and Difference Identities: Theorem and Proof
  - Sum and Difference Identities - Evaluating Trigonometric Functions of Non-Special Angles
  - Sum and Difference Identities - Using the Sum and Difference Identities
  - Sum and Difference Identities - Proving Trigonometric Identities (Part 3)

Key Concepts

Theorems

- **Sum and Difference Identities for the Cosine:** For any angles α and β:
  - $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$
  - $\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$
- **Sum and Difference Identities for the Sine:** For any angles α and β:
  - $\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$
  - $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$
- **Sum and Difference Identities for the Tangent:** For any angles α and β where the functions are defined:
  - $\tan(\alpha+\beta)=\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$
  - $\tan(\alpha-\beta)=\frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$

Common Mistakes

- **You Cannot "Split" a Trigonometric Function Across its Arguments:** A very common mistake is to assume that a trigonometric function distributes over a sum or difference. For example, $\cos(\alpha+\beta)$ is not equal to $\cos(\alpha)+\cos(\beta)$. The Sum and Difference Identities must be used instead.
- **Confusing the Signs in the Identities:** The signs in the Sum and Difference Identities can be tricky. For the cosine identities, the sign in the middle of the expression is the opposite of the operation in the argument (e.g., $\cos(\alpha+\beta)$ has a minus sign). For the sine identities, the sign in the middle matches the operation in the argument (e.g., $\sin(\alpha+\beta)$ has a plus sign).

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6.2.2: Homework

Reading Questions

1. Is $\cos(A+B)$ equivalent to $\cos(A)+\cos(B)$? Explain why or why not.
2. What is the Sum Identity for the cosine function?
3. What is the Difference Identity for the sine function?
4. How can the Sum and Difference Identities be used to find an exact value for $\cos(105^\circ)$?
5. This section proves the Cofunction Identity $\cos(\frac{\pi}{2}-\theta)=\sin(\theta)$. What primary identity was used to begin this proof?
6. What is the Sum Identity for the tangent function?
7. If you need to find $\sin(\alpha-\beta)$ and you are only given $\sin(\alpha)$ and $\tan(\beta)$ (along with their quadrants), what other trigonometric values must you find first?
8. How can the Sum and Difference Identities be used to simplify an expression like $\cos(2x)\cos(3x)-\sin(2x)\sin(3x)$ into a single trigonometric function?

Skills Refresher

Skills Refresher

For the following exercises,
(a) Compute $f(2), f(3)$, and $f(2+3)$.
(b) Determine if it is true that $f(a+b)=f(a)+f(b)$ whenever the function values are defined.

1. $f(x)=3x+2$
2. $f(x)=x^2$
3. $f(x)=\sqrt{x}$
4. $f(x)=\dfrac{1}{x}$
5. $f(x)=|x|$
6. $f(x)=2^x$

Answers available at the end of this section.

Homework

Concept Check

1. Explain why $f(a+b)=f(a)+f(b)$ is not a valid application of the Distributive Law.
2. Nhat says that $\sin\left(\theta+\frac{\pi}{6}\right)=\frac{1}{2}+\sin(\theta)$. Is she correct? Explain.
3. Piyali says that $\tan\left(\theta+\frac{\pi}{4}\right)=\frac{1+\tan(\theta)}{1-\tan(\theta)}$. Is she correct? Explain.

Basic Skills

4. Show that $\cos(45^\circ + 45^\circ)$ is not equal to $\cos(45^\circ) + \cos(45^\circ)$.
5. Show that $\tan(60^\circ - 30^\circ)$ is not equal to $\tan(60^\circ) - \tan(30^\circ)$.
6. Use your calculator to verify that $\tan(87^\circ - 29^\circ)$ is not equal to $\tan(87^\circ) - \tan(29^\circ)$.
7. Use your calculator to verify that $\cos(52^\circ + 64^\circ)$ is not equal to $\cos(52^\circ) + \cos(64^\circ)$.
8. Use graphs to show that $\sin\left(x - \frac{\pi}{6}\right)$ is not equivalent to $\sin(x) - \sin\left(\frac{\pi}{6}\right)$.
9. Use graphs to show that $\tan\left(x + \frac{\pi}{4}\right)$ is not equivalent to $\tan(x) + \tan\left(\frac{\pi}{4}\right)$.

For the following exercises, use the Sum and Difference Identities to find the exact value. You may have need of the Ratio, Reciprocal, or Symmetry Identities as well.

10. $\cos(15^\circ)$
11. $\sin(15^\circ)$
12. $\tan(15^\circ)$
13. $\cos(75^\circ)$
14. $\cos(165^\circ)$
15. $\sin(165^\circ)$
16. $\sec(165^\circ)$
17. $\sin(105^\circ)$
18. $\csc(195^\circ)$
19. $\cot(255^\circ)$
20. $\tan(375^\circ)$
21. $\cos\left(\frac{13\pi}{12}\right)$
22. $\sin\left(\frac{11\pi}{12}\right)$
23. $\tan\left(\frac{13\pi}{12}\right)$
24. $\cos\left(\frac{7\pi}{12}\right)$
25. $\tan\left(\frac{17\pi}{12}\right)$
26. $\sin\left(\frac{\pi}{12}\right)$
27. $\cot\left(\frac{11\pi}{12}\right)$
28. $\csc\left(\frac{5\pi}{12}\right)$

29. $\sec\left(-\frac{\pi}{12}\right)$

30. $\sin\left(\frac{29\pi}{12}\right)$

For the following exercises, find the exact value of the expression without using a calculator.

31. $\cos(-10^\circ)\cos(35^\circ)+\sin(-10^\circ)\sin(35^\circ)$

32. $\cos\left(\frac{7\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)-\sin\left(\frac{7\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)$

33. $\sin\left(\frac{7\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)+\cos\left(\frac{7\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)$

34. $\sin(80^\circ)\cos(55^\circ)+\cos(80^\circ)\sin(55^\circ)$

For the following exercises, find exact values for the trigonometric ratios. (Do not use a calculator!)

35. Suppose $\cos(\alpha)=\frac{3}{5}, \sin(\alpha)=\frac{4}{5}, \cos(\beta)=\frac{5}{13}$, and $\sin(\beta)=\frac{-12}{13}$. Evaluate the following.

    a. $\cos(\alpha+\beta)$

    b. $\sin(\alpha+\beta)$

    c. $\tan(\alpha+\beta)$

36. Suppose $\cos(\alpha)=\frac{-2}{3}, \sin(\alpha)=\frac{\sqrt{5}}{3}, \cos(\beta)=\frac{\sqrt{3}}{2}$, and $\sin(\beta)=\frac{-1}{2}$. Evaluate the following.

    a. $\cos(\alpha-\beta)$

    b. $\sin(\alpha-\beta)$

    c. $\tan(\alpha-\beta)$

37. If $\tan(t)=\frac{3}{4}$ and $\tan(s)=\frac{-7}{24}$, find exact values for the following.

    a. $\tan(s+t)$

    b. $\tan(s-t)$

38. If $\tan(x)=-3$ and $\tan(y)=-5$, find exact values for the following.

    a. $\tan(x+y)$

    b. $\tan(x-y)$

39. Suppose $\cos(\theta)=\frac{15}{17}$ and $\sin(\phi)=\frac{3}{5}$, where θ and ϕ are in quadrant I. Evaluate the following.

    a. $\cos(\theta+\phi)$

    b. $\tan(\theta-\phi)$

40. Suppose $\cos(\theta)=\frac{15}{17}$, where θ is in quadrant IV, and $\sin(\phi)=\frac{3}{5}$, where ϕ is in quadrant II. Evaluate the following.

    a. $\sin(\theta-\phi)$

    b. $\tan(\theta+\phi)$

41. If $\sin(\alpha) = \frac{12}{13}, \frac{\pi}{2} < \alpha < \pi$, and $\cos(\beta) = \frac{-3}{5}, \pi < \beta < \frac{3\pi}{2}$, find exact values for the following.
    a. $\sin(\alpha + \beta)$
    b. $\cos(\alpha + \beta)$
    c. $\tan(\alpha + \beta)$
    d. Sketch the angles α, β and $\alpha + \beta$.

42. If $\cos(\alpha) = \frac{3}{8}, \frac{3\pi}{2} < \alpha < 2\pi$, and $\sin(\beta) = \frac{-1}{4}, \pi < \beta < \frac{3\pi}{2}$, find exact values for the following.
    a. $\sin(\alpha - \beta)$
    b. $\cos(\alpha - \beta)$
    c. $\tan(\alpha - \beta)$
    d. Sketch the angles α, β and $\alpha - \beta$.

43. If α is a Quadrant IV angle with $\cos(\alpha) = \frac{\sqrt{5}}{5}$, and $\sin(\beta) = \frac{\sqrt{10}}{10}$, where $\frac{\pi}{2} < \beta < \pi$, find
    a. $\cos(\alpha + \beta)$
    b. $\sin(\alpha + \beta)$
    c. $\tan(\alpha + \beta)$
    d. $\cos(\alpha - \beta)$
    e. $\sin(\alpha - \beta)$
    f. $\tan(\alpha - \beta)$

44. If $\csc(\alpha) = 3$, where $0 < \alpha < \frac{\pi}{2}$, and β is a Quadrant II angle with $\tan(\beta) = -7$, find
    a. $\cos(\alpha + \beta)$
    b. $\sin(\alpha + \beta)$
    c. $\tan(\alpha + \beta)$
    d. $\cos(\alpha - \beta)$
    e. $\sin(\alpha - \beta)$
    f. $\tan(\alpha - \beta)$

45. If $\sin(\alpha) = \frac{3}{5}$, where $0 < \alpha < \frac{\pi}{2}$, and $\cos(\beta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \beta < 2\pi$, find
    a. $\sin(\alpha + \beta)$
    b. $\cos(\alpha - \beta)$
    c. $\tan(\alpha - \beta)$

46. If $\sec(\alpha) = -\frac{5}{3}$, where $\frac{\pi}{2} < \alpha < \pi$, and $\tan(\beta) = \frac{24}{7}$, where $\pi < \beta < \frac{3\pi}{2}$, find

a. $\csc(\alpha - \beta)$

b. $\sec(\alpha + \beta)$

c. $\cot(\alpha + \beta)$

47. If $\sin(\theta) = 0.2$ and $\cos(\theta) > 0$, find $\sin\left(\theta + \frac{\pi}{3}\right)$.

48. If $\cos(\theta) = 0.6$ and $\sin(\theta) < 0$, find $\cos\left(\theta + \frac{3\pi}{4}\right)$.

For the following exercises, use the Sum and Difference Identities to expand each expression. Simplify your answers completely.

49. $\sin(\theta - 270^\circ)$

50. $\cos(270^\circ + \theta)$

51. $\cos\left(t + \frac{\pi}{6}\right)$

52. $\sin\left(t - \frac{2\pi}{3}\right)$

53. $\tan\left(\beta - \frac{\pi}{6}\right)$

54. $\tan\left(\phi + \frac{\pi}{4}\right)$

55. Use the Sum Identities for sine and cosine to derive a formula for each expression. Then use graphs to verify your formula.

a. $\cos(\theta + 90^\circ)$

b. $\sin(\theta + 90^\circ)$

56. Use the Sum Identities for sine and cosine to derive a formula for each expression. Then use graphs to verify your formula.

a. $\cos(\theta + \pi)$

b. $\sin(\theta + \pi)$

57. Use the Difference Identities for sine and cosine to derive formulas for the following.

a. $\cos\left(\theta - \frac{\pi}{2}\right)$

b. $\sin\left(\theta - \frac{\pi}{2}\right)$

For the following exercises, graph one cycle of the function by hand.

58. $f(x) = \sin(4x)\cos(3x) - \cos(4x)\sin(3x)$

59. $f(x) = \sin(x)\cos(2x) + \cos(2x)\sin(x)$

60. $f(x) = \frac{1}{5}\cos(5x)\cos(2x) + \frac{1}{5}\sin(5x)\sin(2x)$

61. $f(x) = 2\cos(x)\cos(2x) - 2\sin(x)\sin(2x)$

Synthesis Questions

For the following exercises, simplify the function using the Sum and Difference Identities.

62. $f(x) = \sin\left(x + \frac{\pi}{2}\right)$

53. $f(x) = \cos\left(x - \frac{\pi}{4}\right)$

54. $f(x) = \tan(\pi - x)$

55. $f(x) = \sin\left(\frac{\pi}{6} - x\right)$

56. $f(x) = \cos\left(x + \frac{11\pi}{12}\right)$

57. $f(x) = \cos\left(\frac{2\pi}{3} - x\right)$

For the following exercises,
(a) Use graphs to decide if the equation is an identity.
(b) If the equation is not an identity, find a value of the variable that makes the equation false.

58. $\sin\left(\frac{\pi}{2} + \beta\right) = 1 + \sin(\beta)$

59. $\cos\left(\frac{\pi}{3} - \beta\right) = \cos\left(\beta - \frac{\pi}{3}\right)$

70. $\sin(A + 180^\circ) = -\sin(A)$

71. $\cos\left(\phi + \frac{\pi}{3}\right) = \frac{1}{2} + \cos(\phi)$

72. $\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\sin(x) + \cos(x))$

73. $2\cos\left(x - \frac{\pi}{6}\right) = \sin(x) + \sqrt{3}\cos(x)$

74. $\sin\left(x - \frac{\pi}{3}\right) + \cos\left(x + \frac{\pi}{6}\right) = 0$

75. a. Use a graphing device to draw the graph of $f(x) = \sin(x) + \cos(x)$ using $-\pi \le x \le 2\pi$ and $-2 \le y \le 2$. Does the graph of this function appear to be a sinusoid? If so, approximate the amplitude and phase shift of the sinusoid. What is the period of this sinusoid.

    b. Use one of the sum identities to rewrite the expression $\sin\left(x + \frac{\pi}{4}\right)$. Then use the values of $\sin\left(\frac{\pi}{4}\right)$ and $\cos\left(\frac{\pi}{4}\right)$ to further rewrite the expression.

    c. Use the result from part (b) to show that the function $f(x) = \sin(x) + \cos(x)$ is indeed a sinusoidal function. What is its amplitude, phase shift, and period?

76. a. Use a graphing device to draw the graph of $g(x) = \sin(x) + \sqrt{3}\cos(x)$ using $-\pi \le x \le 2\pi$ and $-2.5 \le y \le 2.5$. Does the graph of this function appear to be a sinusoid? If so, approximate the amplitude and phase shift of the sinusoid. What is the period of this sinusoid.

    b. Use one of the sum identities to rewrite the expression $\sin\left(x + \frac{\pi}{3}\right)$. Then use the values of $\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{3}\right)$ to further rewrite the expression.

    c. Use the result from part (b) to show that the function $g(x) = \sin(x) + \sqrt{3}\cos(x)$ is indeed a sinusoidal function. What is its amplitude, phase shift, and period?

Proofs

77. Use the Difference Identities for sine and cosine to prove the statement.

    a. $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$

b. $\sin\left(\frac{\pi}{2}-\theta\right)=\cos(\theta)$

For the following exercises, prove the identity.

78. $\cos(\theta-\pi)=-\cos(\theta)$

79. $\sin(\pi-\theta)=\sin(\theta)$

80. $\tan\left(\theta+\frac{\pi}{2}\right)=-\cot(\theta)$

81. $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin(\alpha)\cos(\beta)$

82. $\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos(\alpha)\sin(\beta)$

83. $\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos(\alpha)\cos(\beta)$

84. $\cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin(\alpha)\sin(\beta)$

85. $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{1+\cot(\alpha)\tan(\beta)}{1-\cot(\alpha)\tan(\beta)}$

86. $\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}=\frac{1-\tan(\alpha)\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$

87. $\frac{\tan(\alpha+\beta)}{\tan(\alpha-\beta)}=\frac{\sin(\alpha)\cos(\alpha)+\sin(\beta)\cos(\beta)}{\sin(\alpha)\cos(\alpha)-\sin(\beta)\cos(\beta)}$

88. $\frac{\sin(t+h)-\sin(t)}{h}=\cos(t)\left(\frac{\sin(h)}{h}\right)+\sin(t)\left(\frac{\cos(h)-1}{h}\right)$

89. $\frac{\cos(t+h)-\cos(t)}{h}=\cos(t)\left(\frac{\cos(h)-1}{h}\right)-\sin(t)\left(\frac{\sin(h)}{h}\right)$

90. $\frac{\tan(t+h)-\tan(t)}{h}=\left(\frac{\tan(h)}{h}\right)\left(\frac{\sec^2(t)}{1-\tan(t)\tan(h)}\right)$

91. $\cot(A+B)=\frac{\cot(A)\cot(B)-1}{\cot(A)+\cot(B)}$

92. $\cot(A-B)=\frac{\cot(A)\cot(B)+1}{\cot(B)-\cot(A)}$

93. $\tan(\theta+45^\circ)=\frac{1+\tan(\theta)}{1-\tan(\theta)}$

94. $\frac{\cos(A+B)}{\sin(A)\cos(B)}=\cot(A)-\tan(B)$

95. $\cot(A)+\cot(B)=\frac{\sin(A+B)}{\sin(A)\sin(B)}$

96. $\frac{\sin(A-B)}{\sin(A+B)}=\frac{\cot(B)-\cot(A)}{\cot(B)+\cot(A)}$

97. $\cos(A+B+C)=\cos(A)\cos(B)\cos(C)-\cos(A)\sin(B)\sin(C)-\sin(A)\cos(B)\sin(C)-\sin(A)\sin(B)\cos(C)$

98. $\tan(A+B+C)=\frac{\tan(A)+\tan(B)+\tan(C)-\tan(A)\tan(B)\tan(C)}{1-\tan(B)\tan(C)-\tan(A)\tan(C)-\tan(A)\tan(B)}$

99. Verify the Cofunction Identities for tangent, secant, cosecant and cotangent.

90. Verify the Difference Identities for sine and tangent.

Challenge Problems

91. Prove the identity

$$\sin(\theta) + \cos(\theta) = \sqrt{2}\sin(\theta + 45^\circ).$$

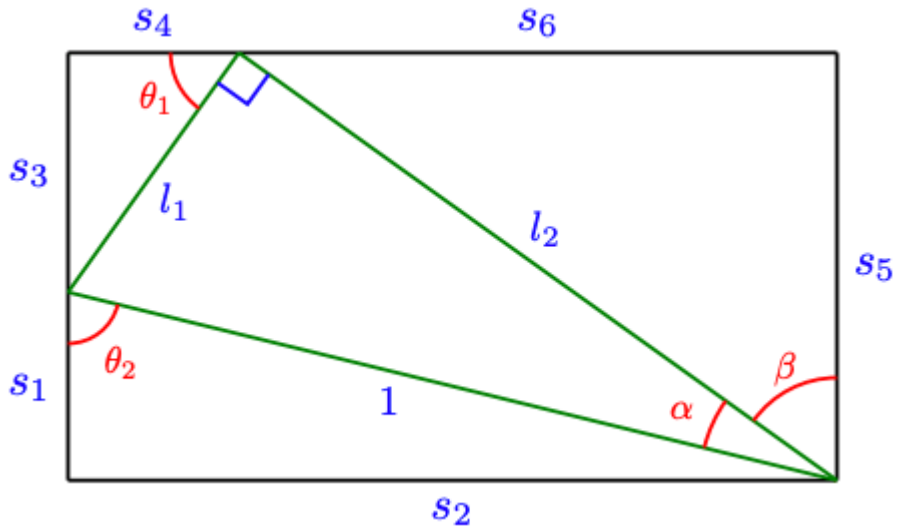
Explain why this shows that

$$-\sqrt{2} \leq \sin(\theta) + \cos(\theta) \leq \sqrt{2}$$

for all angles θ. For which θ between 0° and 360° would $\sin(\theta) + \cos(\theta)$ be the largest?

For the following exercises, verify the Addition and Subtraction Identities for acute angles.

92. The figure below shows a right triangle inscribed in a rectangle.



a. Label the legs l_1 and l_2 of the right triangle with their lengths.

b. Explain why $\theta_1 = \beta$ and $\theta_2 = \alpha + \beta$. Label the diagram with these angles.

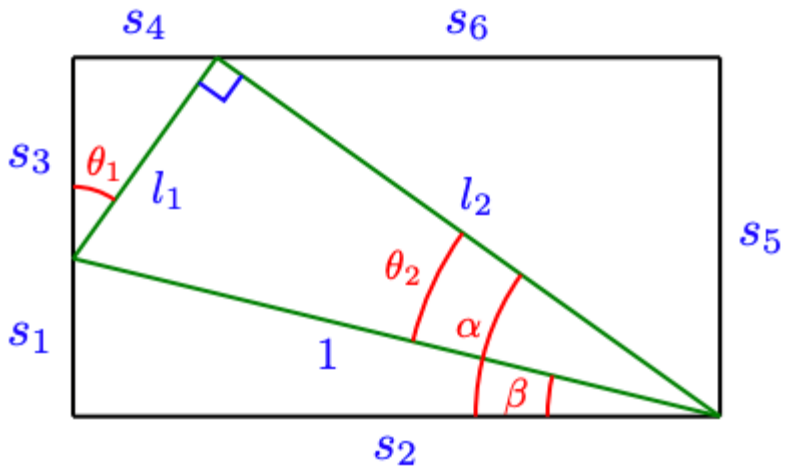
c. Label the legs s_1 and s_2 of the bottom triangle with their lengths.

d. Label the legs s_3 and s_4 of the top left triangle with their lengths.

e. Label the legs s_5 and s_6 of the top right triangle with their lengths.

f. Use the fact that the opposite sides of a rectangle are equal to state the addition formulas for sine and cosine.

93. The figure below shows a right triangle inscribed in a rectangle.



a. Label the legs l_1 and l_2 of the right triangle with their lengths.

b. Explain why $\theta_1 = \beta$ and $\theta_2 = \alpha - \beta$. Label the diagram with these angles.

c. Label the legs s_1 and s_2 of the bottom triangle with their lengths.

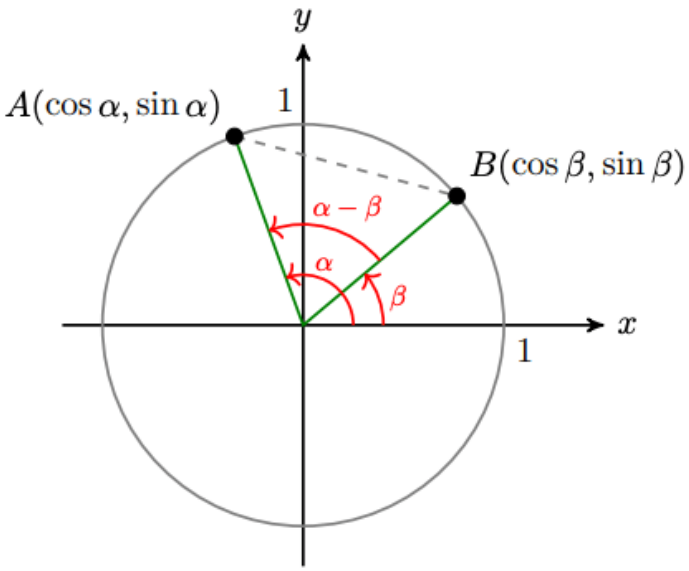
d. Label the legs s_3 and s_4 of the top left triangle with their lengths.

e. Label the legs s_5 and s_6 of the top right triangle with their lengths.

f. Use the fact that the opposite sides of a rectangle are equal to state the subtraction formulas for sine and cosine.

94. Follow the steps to prove the difference of angles formula for cosine,

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$



a. Write an expression for $(AB)^2$, the square of the distance between the points A and B, using the Law of Cosines for $\triangle AOB$.

b. Write another expression for $(AB)^2$ using the Distance Formula and the coordinates of A and B.

c. Equate the two expressions for $(AB)^2$ you obtained in (a) and (b). Simplify the equation to obtain $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

95. For any positive angles A, B, and C such that $A+B+C=90^{\circ}$, show that

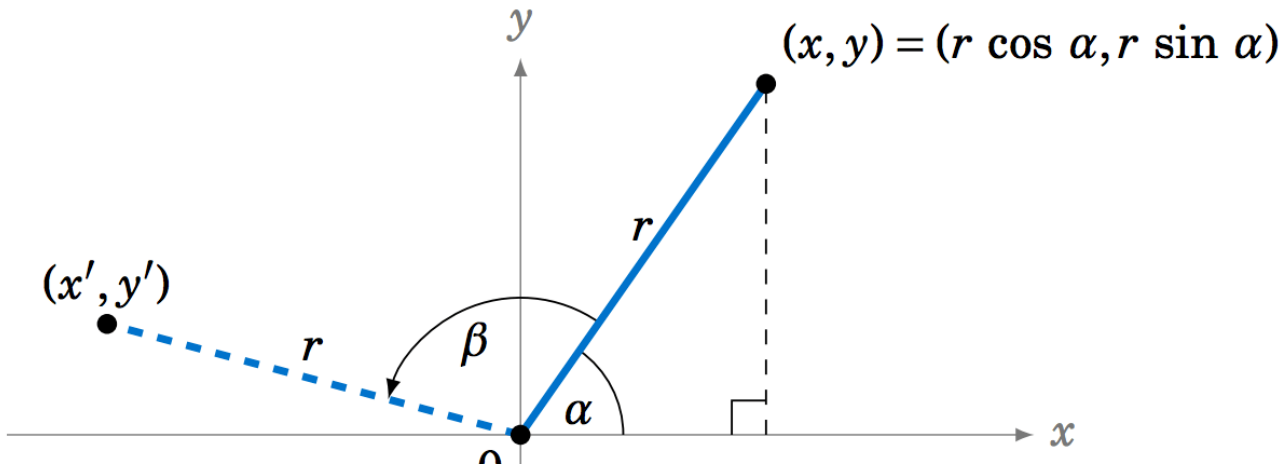
$$\tan(A)\tan(B)+\tan(B)\tan(C)+\tan(C)\tan(A)=1.$$

96. Prove the identity

$$\sin(A+B)\cos(B)-\cos(A+B)\sin(B)=\sin(A).$$

Note that the right side depends only on A, while the left side depends on both A and B.

97. A line segment of length $r>0$ from the origin to the point (x,y) makes an angle α with the positive x-axis, so that $(x,y)=(r\cos(\alpha), r\sin(\alpha))$, as in the figure below. What are the endpoint's new coordinates (x',y') after a counterclockwise rotation by an angle β? Your answer should be in terms of r, α, and β.



Answers

Skills Refresher

1. a. 8, 11, 17
   b. Not true
2. a. 4, 9, 25
   b. Not true
3. a. $\sqrt{2}, \sqrt{3}, \sqrt{5}$
   b. Not true
4. a. $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$
   b. Not true
5. a. 2, 3, 5
   b. Not true
6. a. 4, 8, 32
   b. Not true

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6.3: Double-Angle Identities

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The Double Angle Identities

Several other useful identities can be derived from the Sum and Difference Identities. The Double-Angle Identities are a special case of the Sum Identities, where $\alpha = \beta$.

Theorem: Double-Angle Identities

$$\begin{aligned}
\sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\
&= 2\cos^2(\theta) - 1 \\
&= 1 - 2\sin^2(\theta) \\
\tan(2\theta) &= \frac{2\tan(\theta)}{1 - \tan^2(\theta)}
\end{aligned}$$

Proof available at the end of this section.

Advice: Memorize The Sine and Cosine Versions of These Identities

The Double-Angle Identities for the sine and cosine are needed throughout the rest of Trigonometry, Calculus, and Differential Equations. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated. The tangent version is less common and can be derived from the sine and cosine versions if needed.

You can also justify the identities to yourself by graphing both sides of the formula to see that the graphs are identical.

Caution: Don't Factor Out of Functions!

The first thing we can learn from the Double-Angle Identities is that $\sin(2\theta)$ is not equal to $2\sin(\theta)$! You can check this very easily by choosing a value for θ, say 45°. Then

$$\sin(2\theta) = \sin(90^\circ) = 1$$

but

$$2\sin(\theta) = 2\sin(45^\circ) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

and $\sqrt{2} \neq 1$. Similarly, $\cos(2\theta) \neq 2\cos(\theta)$ and $\tan(2\theta) \neq 2\tan(\theta)$.

Finding Exact Values of Trigonometric Functions Involving Double Angles

We now reach a point in Trigonometry where we are told the value of a trigonometric function of θ, but we are interested in the value of a trigonometric function of 2θ. In these instances, the Double-Angle Identities (which, again, are special sub-cases of the Sum and Difference Identities) are what we need.

Example 6.3.1

Find $\sin(2\theta)$ for the angle θ shown in Figure 6.3.1.

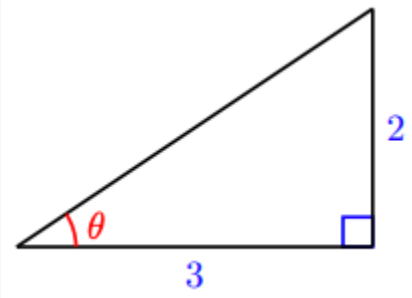


Figure 6.3.1

Solution available at the end of this section.

Checkpoint 6.3.1

Find $\cos(2\theta)$ and $\tan(2\theta)$ for the angle θ shown in Example 6.3.1.

Answer available at the end of this section.

Example 6.3.2

Given that $\cot(\theta) = -\frac{3}{4}$ and θ is in quadrant II, find the following:

a. $\sin(2\theta)$
b. $\cos(2\theta)$
c. $\tan(2\theta)$

Solutions available at the end of this section.

A note about Example $6.3.2c$ is in order. As we have seen throughout Trigonometry, several ways to approach a problem usually exist. It's easy to get so intent on using one method with a problem that we blind ourselves to simpler approaches. For example, we could have used the Ratio Identities to get

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{-24/25}{-7/25} = \frac{24}{7}.$$

Checkpoint 6.3.2

Given $\sin(\alpha) = \frac{5}{8}$, with α in quadrant I, find $\cos(2\alpha)$.

Answer available at the end of this section.

We can work with algebraic expressions instead of numerical values for the trigonometric ratios.

Example 6.3.3

Use Figure $6.3.3$ to express $\cos(2\phi)$ in terms of a.

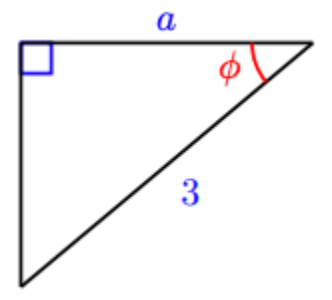


Figure 6.3.3

Solution available at the end of this section.

Checkpoint 6.3.3

For the triangle in Example $6.3.3$, find expressions for $\sin(2\phi)$ and $\tan(2\phi)$.

Answer available at the end of this section.

Example 6.3.4

Find an expression for $\cos(2\beta)$ if you know that $\cos(\beta) = \frac{x}{4}$.

Solution available at the end of this section.

Checkpoint 6.3.4

Find an expression for $\cos(2\alpha)$ if you know that $\sin(\alpha) = \frac{6}{w}$.

Answer available at the end of this section.

Revisiting Proofs of Identities

Establishing identities using the Double-Angle Identities is performed using the same steps we used previously.

Example 6.3.5

Verify the following identity.

$$1+\sin(2\theta)=(\sin(\theta)+\cos(\theta))^2$$

Solution available at the end of this section.

The process in Example 6.3.5 is not complicated, as long as we recall the perfect square formula from algebra:

$$(a\pm b)^2=a^2\pm 2ab+b^2,$$

where $a=\sin(\theta)$ and $b=\cos(\theta)$. Part of being successful in Mathematics is the ability to recognize patterns. While the terms or symbols may change, the Algebra remains consistent.

Checkpoint 6.3.5

Verify the identity.

$$\cos^4(\theta)-\sin^4(\theta)=\cos(2\theta).$$

Answer available at the end of this section.

Example 6.3.6

Prove the identity.

$$\tan(2\theta)=\frac{2}{\cot(\theta)-\tan(\theta)}$$

Solution available at the end of this section.

In Example 6.3.6, if we had chosen to work on the right side, we would have worked backward to arrive at the equivalency. While this is possible (and not too tricky), I like taking advantage of obvious identities. There are usually several paths to a desired result when using the identities to simplify a trigonometric expression or solve a trigonometric equation.

Checkpoint 6.3.6

Prove the identity.

$$\cos(2\theta)\cos\theta=\cos^3(\theta)-\cos(\theta)\sin^2(\theta)$$

Answer available at the end of this section.

Revisiting Graphs of Trigonometric Functions

As with the previous section, we use this opportunity to expand the types of trigonometric functions we can graph.

Example 6.3.7

Graph the function.

$$f(x) = -8\cos^2(3x) + 4$$

Solution available at the end of this section.

Checkpoint 6.3.7

Graph the function.

$$g(x) = \frac{6\tan\left(\frac{\pi}{4}x\right)}{1 - \tan^2\left(\frac{\pi}{4}x\right)}$$

Answer available at the end of this section.

Miscellaneous Topics: What Are Your Top Degree Choices?

I have seen all the statistics regarding the correlation between income and education. In general, the higher your education, the higher your income.

1. Name three benefits, besides money, that a higher degree can give you.
2. What are your top three choices for degrees (e.g., English, Mathematics, Psychology)?

Success in College: Affirmations

Do you ever tell yourself (or others) that you're not good at math? This is called negative self-talk. The more you talk that way, the more likely your subconscious will believe it - and you will do poorly in math.

You can counteract years of negative self-talk by telling yourself with conviction that you're good at math. It might seem strange to state that something that hasn't happened yet is true, but it works! Such statements are called affirmations.

There are four guiding principles for getting the most out of affirmations:

I. Say affirmations that imply that the desired event is currently happening. For example, say, "I am good at algebra," not "I will be good at algebra."
II. Say affirmations that you are making progress toward the desired result. For example, say, "I am good at trigonometry, and I continue to improve at it."
II. Say affirmations in the positive rather than the negative. For example, say "I attend each class," not "I don't cut classes."
IV. Say affirmations with conviction.

1. Do you practice affirmations? If so, how?
2. After reading this, will you try to practice affirmations? Where else in life could affirmations be put to work?

Answers

Section Preview

There is not much I can clarify here that you don't already know. I suppose the only thing that I should repeat (from the text) is that the Sum and Difference Identities, and the Double-Angle Identities must be memorized (at least for the sine and cosine). Do *not* fall into the trap of allowing students to have these on a note card during exams — they will not have such luxury in Calculus.

Section Preview

- **Arithmetic**
  - Simplifying Fractions and Mixed Numbers: The three forms of $\cos(2\theta)$ allow selection based on what information is given, with the Pythagorean Identity substitutions requiring basic arithmetic and fraction simplification.
- **Solving Equations**
  - Solving Quadratic Equations by Factoring: Solving for a trig function value when another is given (Examples 1–4) requires solving a linear or quadratic equation in the trig function — particularly solving quadratic equations by factoring when the double-angle substitution produces a quadratic.

Section Preview

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Section Preview

- Use a Double-Angle Identity to find the exact value of a trigonometric function.
- Prove an identity using a Double-Angle Identity.
- Use a Double-Angle Identity to graph an equation.

Theorem: Double-Angle Identities

Deriving the Double-Angle Identity for sine begins with the Sum Identity,

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta).$$

If we let $\alpha=\beta=\theta$, then we have

$$\begin{aligned}\sin(\theta+\theta) &= \sin(\theta)\cos(\theta)+\cos(\theta)\sin(\theta)\\ \Longrightarrow\quad \sin(2\theta) &= 2\sin(\theta)\cos(\theta)\end{aligned}$$

Deriving the Double-Angle Identity for cosine gives us three options. First, starting from the Sum Identity,

$$\cos(\alpha+\beta)=\cos\ (\alpha)\ \cos\ (\beta)-\sin\ (\alpha)\ \sin\ (\beta),$$

and letting $\alpha=\beta=\theta$, we have

$$\begin{aligned}\cos(\theta+\theta) &= \cos(\theta)\cos(\theta)-\sin(\theta)\sin(\theta)\\ \Longrightarrow\quad \cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta)\end{aligned}$$

Using the Pythagorean Identities, we can expand this Double-Angle Identity for cosine and get two more variations. The first variation is

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta)\\ &= (1-\sin^2(\theta))-\sin^2(\theta)\\ &= 1-2\sin^2(\theta)\end{aligned}$$

The second variation is

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta)\\ &= \cos^2(\theta)-(1-\cos^2(\theta))\\ &= 2\cos^2(\theta)-1\end{aligned}$$

Similarly, to derive the Double-Angle Identity for the tangent, replacing $\alpha=\beta=\theta$ in the sum formula gives

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\\ \Longrightarrow\quad \tan(\theta+\theta) &= \frac{\tan(\theta)+\tan(\theta)}{1-\tan(\theta)\tan(\theta)}\\ \Longrightarrow\quad \tan(2\theta) &= \frac{2\tan(\theta)}{1-\tan^2(\theta)}\end{aligned}$$

Example 6.3.1

We start by using the Pythagorean Theorem to find the hypotenuse of the triangle.

$$c^2=2^2+3^2=13.$$

So $c=\sqrt{13}$. Thus, $\cos(\theta)=\frac{3}{\sqrt{13}}$ and $\sin(\theta)=\frac{2}{\sqrt{13}}$. Now we can use these values in the Double-Angle Identity to find $\sin(2\theta)$.

$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta)\\ &= 2\left(\frac{2}{\sqrt{13}}\right)\left(\frac{3}{\sqrt{13}}\right)\\ &= \frac{12}{13}\end{aligned}$$

Checkpoint 6.3.1

$\cos(2\theta)=\dfrac{5}{13}, \quad \tan(2\theta)=\dfrac{12}{5}$

Example 6.3.2

If we draw a triangle to reflect the information given, we can find the values needed to help us. We are given $\cot(\theta)=-\frac{3}{4}$, such that θ is in quadrant II. The

cotangent of an angle is equal to the adjacent side over the opposite side, and because θ is in the second quadrant, the adjacent side is on the x-axis and is negative. We use the Pythagorean Theorem to find the length of the hypotenuse:

$$\begin{aligned} (-4)^2 + (3)^2 &= c^2 \\ \Longrightarrow \quad 16 + 9 &= c^2 \\ \Longrightarrow \quad 25 &= c^2 \\ \Longrightarrow \quad c &= 5 \end{aligned}$$

Now we can draw the situation appropriately.

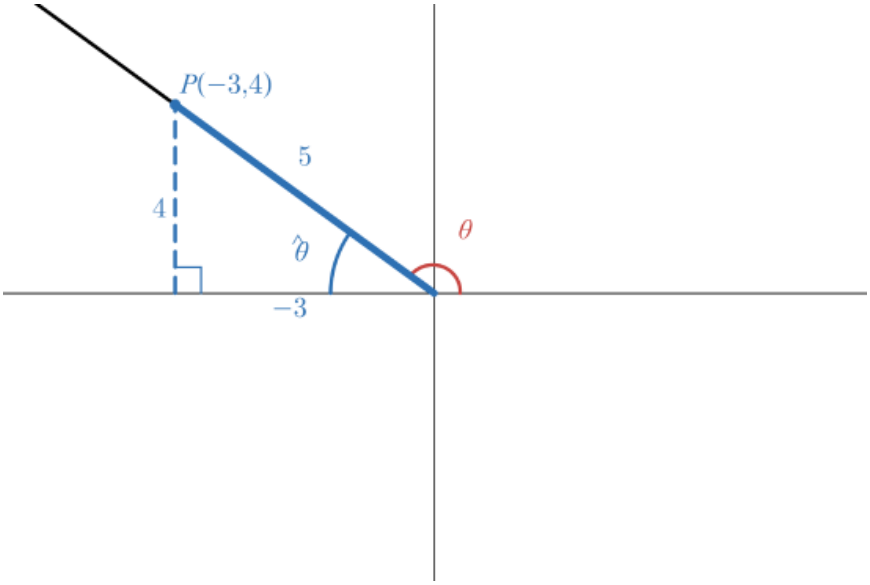


Figure 6.3.2**:** *Not drawn to scale!*

When drawn like this, you can use the coordinate definition of the trigonometric functions or the right triangle definition using $\hat{\theta}$ and the fact that θ terminates in QII — it's your choice.

a. Let's begin by writing the Double-Angle Identity for sine.

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

We see that we need to find $\sin(\theta)$ and $\cos(\theta)$. Based on our drawing, we see that the hypotenuse equals 5, so $\sin(\theta) = \frac{4}{5}$, and $\cos(\theta) = -\frac{3}{5}$. Substitute these values into the equation and simplify. Thus,

$$\begin{aligned} \sin(2\theta) &= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) \\ &= -\frac{24}{25} \end{aligned}$$

b. Write the Double-Angle Identity for cosine.

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

Again, substitute the values of the sine and cosine into the equation and simplify.

$$\begin{aligned} \cos(2\theta) &= \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 \\ &= \frac{9}{25} - \frac{16}{25} \\ &= -\frac{7}{25} \end{aligned}$$

c. Write the Double-Angle Identity for tangent.

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

In this formula, we need the tangent. Using the Reciprocal Identity, we have $\tan(\theta) = -\frac{4}{3}$. Substitute this value into the equation, and simplify.

$$\begin{aligned}\tan(2\theta) &= \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2}\\ &= \frac{-\frac{8}{3}}{1-\frac{16}{9}}\\ &= -\frac{8}{3}\left(-\frac{9}{7}\right)\\ &= \frac{24}{7}\end{aligned}$$

Checkpoint 6.3.2

$\cos(2\alpha) = \frac{7}{32}$

Example 6.3.3

We use the Pythagorean Theorem to find an expression for the third side of the triangle.

$$\begin{aligned}a^2+b^2 &= 3^2\\ \Longrightarrow \qquad b &= \sqrt{9-a^2}\end{aligned}$$

Now we can write expressions for the sine and cosine of ϕ.

$$\cos(\phi) = \frac{a}{3} \quad \text{and} \quad \sin(\phi) = \frac{\sqrt{9-a^2}}{3}$$

Finally, we substitute these expressions into the Double-Angle Identity.

$$\begin{aligned}\cos(2\phi) &= \cos^2(\phi) - \sin^2(\phi)\\ &= \left(\frac{a}{3}\right)^2 - \left(\frac{\sqrt{9-a^2}}{3}\right)^2\\ &= \frac{a^2}{9} - \frac{9-a^2}{9}\\ &= \frac{2a^2-9}{9}\end{aligned}$$

Checkpoint 6.3.3

$\sin(2\phi) = \frac{2a\sqrt{9-a^2}}{9}, \quad \tan(2\phi) = \frac{2a\sqrt{9-a^2}}{2a^2-9}$

Example 6.3.4

We use the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$.

$$\begin{aligned}\cos(2\beta) &= 2\cos^2(\beta) - 1\\ &= 2\left(\frac{x}{4}\right)^2 - 1\\ &= \frac{2x^2}{16} - 1\\ &= \frac{2x^2-16}{16}\\ &= \frac{x^2-8}{8}\end{aligned}$$

Checkpoint 6.3.4

$\frac{w^2-76}{w^2}$

Example 6.3.5

We will work on the right side of the equal sign and rewrite the expression until it matches the left side.

$$\begin{aligned}\text{RHS} &= (\sin(\theta)+\cos(\theta))^2\\ &= \sin^2(\theta) + 2\sin(\theta)\cos(\theta) + \cos^2(\theta) && \text{(distribution)}\\ &= (\sin^2(\theta)+\cos^2(\theta)) + 2\sin(\theta)\cos(\theta) && \text{(Commutative and Associative Properties of Addition)}\\ &= 1 + 2\sin(\theta)\cos(\theta) && \text{(Pythagorean Identities)}\\ &= 1 + \sin(2\theta) && \text{(Double-Angle Identities)}\\ &= \text{LHS}\end{aligned}$$

Checkpoint 6.3.5

$\cos^4(\theta) - \sin^4(\theta) = (\cos^2(\theta) - \sin^2(\theta))(\cos^2(\theta) + \sin^2(\theta)) = \cos(2\theta)\cdot 1 = \cos(2\theta)$

Example 6.3.6

Despite the right side looking more complicated, the obvious Double-Angle Identity on the left side tells us that we should at least start on that side and simplify or rewrite until it equals the right side of the equation.

$$\begin{aligned}
\text{LHS} &= \tan(2\theta) \\
&= \frac{2\tan(\theta)}{1-\tan^2(\theta)} && \text{(Double-Angle Identity)} \\
&= \frac{2\tan(\theta)\left(\frac{1}{\tan(\theta)}\right)}{(1-\tan^2(\theta))\left(\frac{1}{\tan(\theta)}\right)} && \text{(multiply by a term that results in desired numerator)} \\
&= \frac{2}{\frac{1}{\tan(\theta)}-\frac{\tan^2(\theta)}{\tan(\theta)}} \\
&= \frac{2}{\cot(\theta)-\tan(\theta)} && \text{(Reciprocal Identity)} \\
&= \text{RHS}
\end{aligned}$$

Checkpoint 6.3.6

$\cos(2\theta)\cos(\theta) = (\cos^2(\theta)-\sin^2(\theta))\cos(\theta) = \cos^3(\theta)-\cos(\theta)\sin^2(\theta)$

Example 6.3.7

Careful observation and a little pre-factoring will reveal that this function contains a Double-Angle Identity.

$$\begin{aligned}
f(x) &= -8\cos^2(3x)+4 \\
&= -4(2\cos^2(3x)-1) && \text{(factoring out the GCF)} \\
&= -4\cos(6x) && \text{(Double-Angle Identity)}
\end{aligned}$$

This is a reflected cosine function with amplitude 4, period $\frac{\pi}{3}$, step size $\frac{\pi}{12}$, and no phase shift or vertical shift.

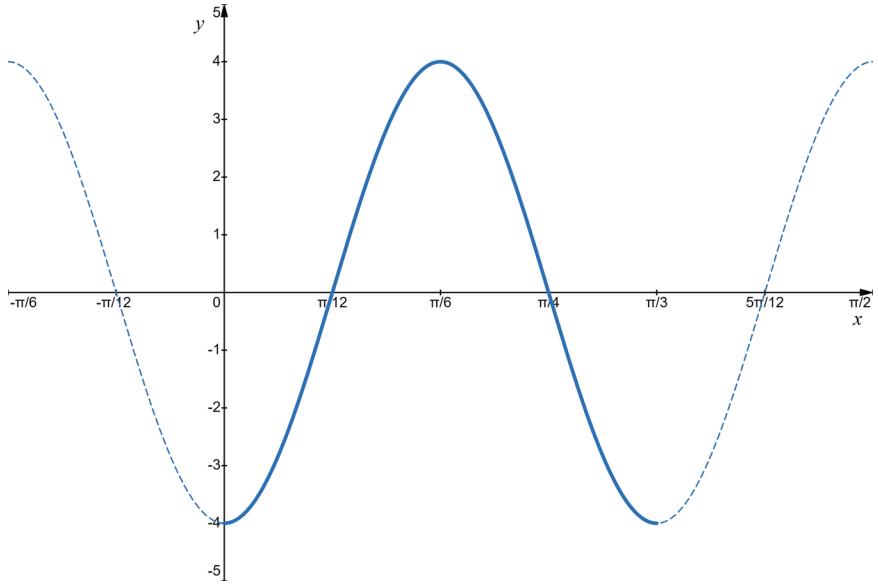


Figure 6.3.4

Checkpoint 6.3.7

$g(x) = 3\tan\left(\frac{\pi}{2}x\right)$.

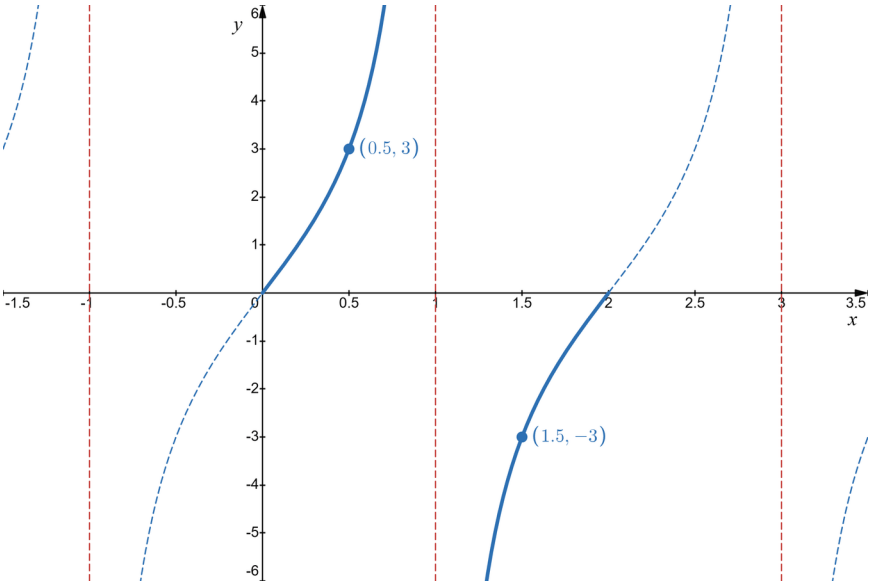


Figure 6.3.5

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6.3.1: Resources and Key Concepts

Resources

Videos

- Double-Angle Identities
  - Double-Angle Identities - Theorem and Proof
  - Double-Angle Identities - Using the Double-Angle Identities
  - Double-Angle Identities - Proving Trigonometric Identities (Part 4)

Key Concepts

Theorems

- **Double-Angle Identities:**
  - $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
  - $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
  - $\cos(2\theta) = 2\cos^2(\theta) - 1$
  - $\cos(2\theta) = 1 - 2\sin^2(\theta)$
  - $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$

Common Mistakes

- **Don't Factor Out of Functions:** A common mistake is to treat a trigonometric function as if it distributes over its argument. For example, $\sin(2\theta)$ is not equal to $2\sin(\theta)$. The Double-Angle Identities must be used to correctly evaluate or expand such expressions.
- **Ignoring Simpler Approaches:** It is easy to get so focused on using one specific method that you miss a simpler path. For example, after finding $\sin(2\theta)$ and $\cos(2\theta)$, the easiest way to find $\tan(2\theta)$ is often to use the Ratio Identity rather than the Double-Angle Identity for tangent.

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6.3.2: Homework

Reading Questions

1. How are the Double-Angle Identities derived from the Sum and Difference Identities?
2. What is the Double-Angle Identity for $\sin(2\theta)$?
3. List the three different forms of the Double-Angle Identity for $\cos(2\theta)$.
4. If you are given the value of $\cos(\theta)$ and need to find $\cos(2\theta)$, which of the three forms of the cosine double-angle identity would be the most direct to use?
5. Provide a numerical example that demonstrates that $\sin(2\theta) \neq 2\sin(\theta)$.
6. If you need to find the value of $\sin(2\theta)$, what two pieces of information about θ must you know?
7. How can a Double-Angle Identity be used to help graph a function like $f(x) = -8\cos^2(3x) + 4$?
8. When proving an identity, what should you look for if you see an expression like $\cos(2x)$?

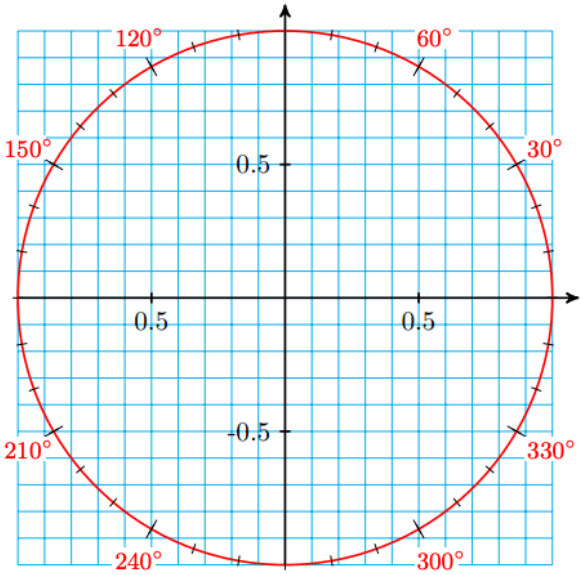
Homework

Concept Check

1. Provide an example to show that doubling an angle does not double its sine or cosine.

Basic Skills

For the following exercises, use the unit circle to estimate trigonometric values. Then verify with your calculator.



2. Does $\sin(2 \cdot 90^\circ) = 2\sin(90^\circ)$?
3. Does $\cos(2 \cdot 25^\circ) = 2\cos(25^\circ)$?
4. Does $\tan(2 \cdot 70^\circ) = 2\tan(70^\circ)$?
5. Does $\tan(2 \cdot 100^\circ) = 2\tan(100^\circ)$?

For the following exercises, verify that each statement is true

6. $\sin(90^\circ) = 2\sin(45^\circ)\cos(45^\circ)$

7. $\sin(60^\circ) = 2\sin(30^\circ)\cos(30^\circ)$

8. $\cos(60^\circ) = \cos^2(30^\circ) - \sin^2(30^\circ)$

9. $\tan(60^\circ) = \dfrac{2\tan(30^\circ)}{1-\tan^2(30^\circ)}$

For the following exercises, determine if the statement is true or false. Explain your answer.

10. If $\cos(\alpha) = 0.32$, then $\cos(2\alpha) = 0.64$.

11. If $\sin(\phi) = r$, then $\sin(2\phi) = 2r$.

For the following exercises, use the Double Angle Identities to simplify the expression. Leave your answer in exact form.

12. $2\sin(34^\circ)\cos(34^\circ)$

13. $\cos^2\left(\dfrac{\pi}{10}\right) - \sin^2\left(\dfrac{\pi}{10}\right)$

14. $1 - 2\sin^2\left(\dfrac{\pi}{16}\right)$

15. $2\cos^2(18^\circ) - 1$

16. $\cos^2(3\theta) - \sin^2(3\theta)$

17. $2\sin(2\alpha)\cos(2\alpha)$

18. $2\sin(5t)\cos(5t)$

19. $\cos^2(4w) - \sin^2(4w)$

20. $\dfrac{2\tan(64^\circ)}{1-\tan^2(64^\circ)}$

21. $\dfrac{2\tan\left(\frac{\pi}{5}\right)}{1-\tan^2\left(\frac{\pi}{5}\right)}$

22. $2\cos^2(2\beta) - 1$

23. $1 - 2\sin^2(6s)$

24. $\cos^2(28^\circ) - \sin^2(28^\circ)$

25. $2\cos^2(37^\circ) - 1$

26. $1 - 2\sin^2(17^\circ)$

27. $\cos^2(37^\circ) - \sin^2(37^\circ)$

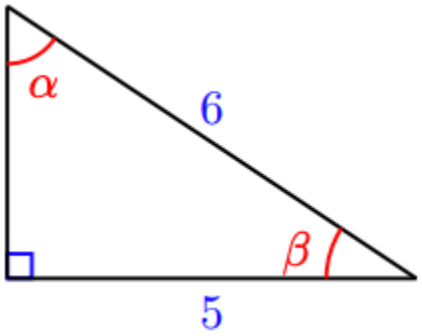
28. $\cos^2(9x) - \sin^2(9x)$

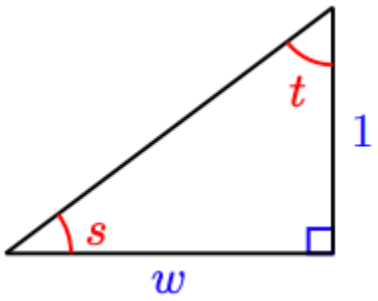
29. $\cos^2(6x) - \sin^2(6x)$

30. $4\sin(8x)\cos(8x)$

31. $6\sin(5x)\cos(5x)$

For the following exercises, use the figures to find the trigonometric ratios.





32. a. $\sin(\alpha)$
    b. $\cos(\alpha)$
    c. $\tan(\alpha)$
    d. $\sin(2\alpha)$
    e. $\cos(2\alpha)$
    f. $\tan(2\alpha)$

33. a. $\sin(\beta)$
    b. $\cos(\beta)$
    c. $\tan(\beta)$
    d. $\sin(2\beta)$
    e. $\cos(2\beta)$
    f. $\tan(2\beta)$

34. a. $\sin(s)$
    b. $\cos(s)$
    c. $\tan(s)$
    d. $\sin(2s)$
    e. $\cos(2s)$
    f. $\tan(2s)$

35. a. $\sin(t)$
    b. $\cos(t)$
    c. $\tan(t)$
    d. $\sin(2t)$
    e. $\cos(2t)$

f. $\tan(2t)$

36. Suppose $\cos(\theta) = \frac{12}{13}$ and $\frac{3\pi}{2} < \theta < 2\pi$. Compute exact values for the following.

a. $\sin(\theta)$

b. $\sin(2\theta)$

c. $\cos(2\theta)$

d. $\tan(2\theta)$

e. Sketch the angles θ and 2θ.

37. Suppose $\sin(\phi) = \frac{5}{6}$ and $\frac{\pi}{2} < \phi < \pi$. Compute exact values for the following.

a. $\cos(\phi)$

b. $\sin(2\phi)$

c. $\cos(2\phi)$

d. $\tan(2\phi)$

e. Sketch the angles ϕ and 2ϕ.

38. If $\tan(u) = -4$ and \(270^{\circ}

a. $\tan(2u)$

b. $\cos(2u)$

c. $\sin(2u)$

39. If $\tan(v) = \frac{2}{3}$ and \(180^{\circ}

a. $\tan(2v)$

b. $\cos(2v)$

c. $\sin(2v)$

For the following exercises, use the given information about θ to find the exact values of $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.

40. $\sin(\theta) = -\frac{7}{25}$ where $\frac{3\pi}{2} < \theta < 2\pi$

41. $\cos(\theta) = \frac{28}{53}$ where $0 < \theta < \frac{\pi}{2}$

42. $\tan(\theta) = \frac{12}{5}$ where $\pi < \theta < \frac{3\pi}{2}$

43. $\csc(\theta) = 4$ where $\frac{\pi}{2} < \theta < \pi$

44. $\cos(\theta) = \frac{3}{5}$ where $0 < \theta < \frac{\pi}{2}$

45. $\sin(\theta) = -\frac{4}{5}$ where $\pi < \theta < \frac{3\pi}{2}$

46. $\cos(\theta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \theta < 2\pi$

47. $\sin(\theta) = \frac{5}{13}$ where $\frac{\pi}{2} < \theta < \pi$

48. $\sec(\theta) = \sqrt{5}$ where $\frac{3\pi}{2} < \theta < 2\pi$

49. $\tan(\theta) = -2$ where $\frac{\pi}{2} < \theta < \pi$

50. Suppose θ is a Quadrant I angle with $\sin(\theta) = x$. Verify the following formulas

    a. $\cos(\theta) = \sqrt{1 - x^2}$

    b. $\sin(2\theta) = 2x\sqrt{1 - x^2}$

    c. $\cos(2\theta) = 1 - 2x^2$

51. Suppose θ is a Quadrant I angle with $\tan(\theta) = x$. Verify the following formulas

    a. $\cos(\theta) = \frac{1}{\sqrt{x^2 + 1}}$

    b. $\sin(\theta) = \frac{x}{\sqrt{x^2 + 1}}$

    c. $\sin(2\theta) = \frac{2x}{x^2 + 1}$

    d. $\cos(2\theta) = \frac{1 - x^2}{x^2 + 1}$

52. If $\sin(\theta) = \frac{x}{2}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\cos(2\theta)$ in terms of x.

53. If $\tan(\theta) = \frac{x}{7}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

For the following exercises, graph one cycle of the given function by hand.

54. $f(x) = 4\sin(3x)\cos(3x)$

55. $g(x) = \frac{2}{3}\cos^2\left(\frac{\pi}{3}x\right) - \frac{1}{2}$

56. $h(x) = -\frac{5}{7}\sin\left(\frac{1}{3}x\right)\cos\left(\frac{1}{3}x\right)$

57. $k(x) = \frac{3\tan\left(\frac{1}{2}x\right)}{2 - 2\tan^2\left(\frac{1}{2}x\right)}$

58. $m(x) = 2\cos^2(5x) - 2\sin^2(5x)$

Proofs

For the following exercises, prove the given identity for any right triangle $\triangle ABC$, where C is a right angle.

59. $\sin(A - B) = \cos(2B)$

60. $\cos(A - B) = \sin(2A)$

61. $\sin(2A) = \frac{2ab}{c^2}$

62. $\cos(2A) = \frac{b^2 - a^2}{c^2}$

53. $\tan(2A) = \dfrac{2ab}{b^2 - a^2}$

54. Prove the Double-Angle Identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.

55. Prove the Double-Angle Identity $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.

For the following exercises,
(a) Use graphs to decide if the equation is an identity.
(b) If the equation is not an identity, find a value of the variable that makes the equation false.

56. $\cos(4\theta) = 4\cos(\theta)$

57. $\cos(2x) = (\cos(x) + \sin(x))(\cos(x) - \sin(x))$

For the following exercises, prove the identity. Assume all quantities are defined.

58. $(\cos(\theta) + \sin(\theta))^2 = 1 + \sin(2\theta)$

59. $(\cos(\theta) - \sin(\theta))^2 = 1 - \sin(2\theta)$

60. $\dfrac{\sin(2\theta)}{\sin(\theta)} - \dfrac{\cos(2\theta)}{\cos(\theta)} = \sec(\theta)$

61. $\dfrac{\sin(3\theta)}{\sin(\theta)} - \dfrac{\cos(3\theta)}{\cos(\theta)} = 2$

62. $\tan(2\theta) = \dfrac{1}{1 - \tan(\theta)} - \dfrac{1}{1 + \tan(\theta)}$

63. $\csc(2\theta) = \dfrac{\cot(\theta) + \tan(\theta)}{2}$

64. $(\sin^2 x - 1)^2 = \cos(2x) + \sin^4 x$

65. $\sin(2x) = \dfrac{2\tan(x)}{1 + \tan^2(x)}$

66. $\tan(2x) = \dfrac{2\sin(x)\cos(x)}{2\cos^2(x) - 1}$

67. $\cot(x) - \tan(x) = 2\cot(2x)$

68. $\dfrac{\sin(2\theta)}{1 + \cos(2\theta)} = \tan(\theta)$

69. $\cos(2\alpha) = \dfrac{1 - \tan^2(\alpha)}{1 + \tan^2(\alpha)}$

70. $\dfrac{1 + \cos(2t)}{\sin(2t) - \cos(t)} = \dfrac{2\cos(t)}{2\sin(t) - 1}$

71. $\sin(3x) = 3\sin(x)\cos^2(x) - \sin^3(x)$

72. $\cos(3x) = \cos^3(x) - 3\sin^2(x)\cos(x)$

73. $8\sin^4(\theta) = \cos(4\theta) - 4\cos(2\theta) + 3$

74. $8\cos^4(\theta) = \cos(4\theta) + 4\cos(2\theta) + 3$

75. $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$

86. $\sin(4\theta) = 4\sin(\theta)\cos^3(\theta) - 4\sin^3(\theta)\cos(\theta)$

87. $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$

88. $\cos(8\theta) = 128\cos^8(\theta) - 256\cos^6(\theta) + 160\cos^4(\theta) - 32\cos^2(\theta) + 1$

89. $\sec(2\theta) = \dfrac{\cos(\theta)}{\cos(\theta) + \sin(\theta)} + \dfrac{\sin(\theta)}{\cos(\theta) - \sin(\theta)}$

90. $\dfrac{1}{\cos(\theta) - \sin(\theta)} + \dfrac{1}{\cos(\theta) + \sin(\theta)} = \dfrac{2\cos(\theta)}{\cos(2\theta)}$

91. $\dfrac{1}{\cos(\theta) - \sin(\theta)} - \dfrac{1}{\cos(\theta) + \sin(\theta)} = \dfrac{2\sin(\theta)}{\cos(2\theta)}$

92. $\dfrac{\cos^2(\psi)}{\cos^2(\theta)} = \dfrac{1 + \cos(2\psi)}{1 + \cos(2\theta)}$

Challenge Problems

93. Some trigonometry textbooks used to claim (incorrectly) that $\sin(\theta) + \cos(\theta) = \sqrt{1 + \sin(2\theta)}$ was an identity. Give an example of a specific angle θ that would make that equation false. Is $\sin(\theta) + \cos(\theta) = \pm\sqrt{1 + \sin(2\theta)}$ an identity? Justify your answer.

94. In general, what is the largest value that $\sin(\theta)\cos(\theta)$ can take? Justify your answer.

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6.4: Half-Angle and Power Reduction Identities

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The Power Reduction Identities

The Double-Angle Identities can be used to derive the Power Reduction Identities, which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine. They allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine. These formulas are essential in higher-level math courses, Calculus in particular. Also called the Power Reduction Formulas (or, simply, the Reduction Formulas), these form three identities and are easily derived from the Double-Angle Identities.

Theorem: Power Reduction Identities

$$\begin{aligned} \sin^2(\theta) &= \frac{1-\cos(2\theta)}{2} \\ \cos^2(\theta) &= \frac{1+\cos(2\theta)}{2} \\ \tan^2(\theta) &= \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \end{aligned}$$

Proof available at the end of this section.

Advice: Memorize the Sine and Cosine Versions of These Identities

The Power Reduction Identities for the sine and cosine are needed throughout the rest of Trigonometry, Calculus, and Differential Equations. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated. The tangent version is less common and can be derived from the sine and cosine versions if needed.

Using the Reduction Identities to Rewrite an Expression Without Powers Greater Than 1

The Power Reduction Identities are frequently used in Calculus to make an otherwise impossible problem possible. This is because working with higher powers of trigonometric functions can be problematic in Integral Calculus.

Example 6.4.1

Write an equivalent expression for $\cos^4(x)$ that does not involve any powers of sine or cosine greater than 1.

Solution available at the end of this section.

Checkpoint 6.4.1

Write an equivalent expression for $\sin^4(3x)$ that does not involve any powers of sine or cosine greater than 1.

Answer available at the end of this section.

Revisiting Proofs of Identities

The Power Reduction Identities can come in handy when trying to prove identities.

Example 6.4.2

Prove

$$\sin^3(2x) = \frac{1}{2}\sin(2x)\left[1 - \cos(4x)\right].$$

Solution available at the end of this section.

In Example 6.4.2, we substituted $\frac{1-\cos(4x)}{2}$ for $\sin^2(2x)$. This is because the Power Reduction Identity states

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}.$$

We let $\theta = 2x$, so $2\theta = 4x$.

Checkpoint 6.4.2

Prove that

$$10\cos^4(x) = \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x).$$

Answer available at the end of this section.

The Half-Angle Identities

The next set of identities can be quickly derived from the Power Reduction Identities by replacing θ with $\frac{\alpha}{2}$ and taking the square root of both sides. These are known as the Half-Angle Identities.

Theorem: Half-Angle Identities

$$\begin{aligned}
\sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos(\theta)}{2}} \\
\cos\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1+\cos(\theta)}{2}} \\
\tan\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} = \frac{\sin(\theta)}{1+\cos(\theta)} = \frac{1-\cos(\theta)}{\sin(\theta)}
\end{aligned}$$

Proof available at the end of this section.

Note that the Half-Angle Identities are preceded by a $\pm$ sign. This does not mean that both the positive and negative expressions are valid. Instead, it depends on the quadrant in which $\frac{\theta}{2}$ terminates.

Note: Naming convention

I have been using the convention in this textbook to call all our theorems based on the trigonometric functions by the term "identities." That is, we have learned the Pythagorean Identities, the Ratio Identities, the Symmetry Identities, the Double-Angle Identities, and so on. You will also hear of these referred to as "formulas." That is, a student from another class might say, "Double-Angle Formulas." This is fine. The only reason for my convention is that the word "identity" has more meaning. It implies that the statement is always true. On the other hand, the word "formula" is a bit more circumstantial (i.e., not always true). In the context of the theorems we have presented so far, you could use either terminology and you would still be correct.

Using the Half-Angle Identities to Evaluate Trigonometric Functions

Example 6.4.3

Find the exact value of $\sin(15^\circ)$.

Solution available at the end of this section.

Checkpoint 6.4.3

Find the exact value of $\cos\left(\frac{\pi}{8}\right)$.

Answer available at the end of this section.

When using a Half-Angle Identity, you must address the $\pm$ sign. To do this, you consider the quadrant within which the *half-angle* terminates. For example, suppose you are given $\cos(\theta) = \frac{5}{7}$, where $\theta \in \text{QIV}$. If you are then asked to find $\cos\left(\frac{\theta}{2}\right)$, you should start by determining the quadrant within which $\frac{\theta}{2}$ lies because that will determine the sign of $\cos\left(\frac{\theta}{2}\right)$. We use the fact that $\theta \in \text{QIV}$ to do this. This implies

$$\begin{aligned} \frac{3\pi}{2} &< \theta < 2\pi \\ \Rightarrow \quad \frac{3\pi}{4} &< \frac{\theta}{2} < \pi \end{aligned}$$

Hence, $\frac{\theta}{2} \in \text{QII}$. We then use the Half-Angle Identity and the given information to finish the problem.

$$\begin{aligned} \cos\left(\frac{\theta}{2}\right) &= -\sqrt{\frac{1+\cos(\theta)}{2}} \\ &= -\sqrt{\frac{1+\frac{5}{7}}{2}} \\ &= -\sqrt{\frac{7+5}{14}} \\ &= -\sqrt{\frac{12}{14}} \\ &= -\sqrt{\frac{6}{7}} \end{aligned}$$

Example 6.4.4

Given that $\tan(\alpha) = \frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

a. $\sin\left(\frac{\alpha}{2}\right)$
b. $\cos\left(\frac{\alpha}{2}\right)$
c. $\tan\left(\frac{\alpha}{2}\right)$

Solutions available at the end of this section.

Checkpoint 6.4.4

Given that $\sin(\alpha) = -\frac{4}{5}$ and α lies in quadrant IV, find the exact value of $\cos\left(\frac{\alpha}{2}\right)$.

Answer available at the end of this section.

The following example requires us to recall how to find an acute angle using the inverse trigonometric function. For a review of this skill, please see section 3.2.

Example 6.4.5

A bicycle ramp is constructed for high-level competition with an angle of θ formed by the ramp and the ground. Another ramp is to be constructed half as steep for novice competition. If $\tan(\theta) = \frac{5}{3}$ for higher-level competition, what is the measurement of the angle for novice competition?

Solution available at the end of this section.

Revisiting Graphs of Trigonometric Functions

Example 6.4.6

Graph the function.

$$y = 3\sin^2\left(\frac{x}{2}\right)$$

Solution available at the end of this section.

Checkpoint 6.4.6

Graph the function.

$$y = -2\cos^2(3x) - 1$$

Answer available at the end of this section.

Success in College: Learning From Tests (Part 4)

Tests and quizzes have two purposes. First, they assess your performance. Your test scores are part of a professor's information to assign you a grade for the course. Tests also allow you to see what you do not know. It is your responsibility to use tests to improve your skills and knowledge.

Often, a passing score on a test is 70%. However, the missing 30% is still critically important. When preparing for the next test, you must study your mistakes from the last test. Perhaps you will realize that you still do not understand an important concept. This concept will not disappear. Now is the time to get extra help. When it reappears, you will be ready. Perhaps you will realize that you are making many careless mistakes. Now is the time to plan strategies for checking your work on the next test. Perhaps you will realize that you made most of your mistakes on the last few questions of the exam. Now is the time to ponder how to pace yourself as you work on a test.

Learning from tests takes self-discipline. It is tempting to throw the test away or hide it in a folder. It is your responsibility to review your tests critically. Your ability to do well in the future depends on it.

1. Describe what you usually do with a returned math test.
2. Does what you do with a returned math test depend on whether you earned a good or poor grade on the test? Explain.

Success in Mathematics: Process-Oriented Learning Preferences

Choose the statements that best describe you.

I learn better if I know why I'm learning something.	or	I don't need to know why I'm doing something before learning how to do it.
I find it annoying to learn something I don't think I will use anywhere else.	or	I believe that learning new things is always worth it because it exercises my brain.
I naturally think in steps.	or	It is hard for me to learn something step by step.

1. Do you consider your choices above good or bad for your education?
2. In specific textbooks, many procedures are summarized as a list of steps. This list is shown in a separate box. Do you refer to these boxes when you do your homework? Why or why not?

Answers

Section Preview

The Sum and Difference, Double-Angle, and Half-Angle Identities must be memorized (at least for the sine and cosine). Do *not* fall into the trap of allowing students to have these on a note card during exams — they will not have such luxury in Calculus.

Section Preview

- Use the Power Reduction Identities to rewrite the power of a trigonometric function in terms of single powers.
- Use a Half-Angle Identity to find the exact value of a trigonometric function.
- Prove an identity using a Half-Angle Identity.
- Use the Half-Angle Identities to graph a function.

Theorem: Power Reduction Identities

We can use two of the three Double-Angle Identities for cosine to derive the Power Reduction Identities for sine and cosine. Let's begin with $\cos(2\theta) = 1 - 2\ \sin^2(\theta)$. Solving for $\sin^2(\theta)$, we get:

$$\begin{aligned} & \cos(2\theta) &=& \ 1 - 2\sin^2(\theta) \\ \Longrightarrow\ & 2\sin^2(\theta) &=& \ 1 - \cos(2\theta) \\ \Longrightarrow\ & \sin^2(\theta) &=& \ \frac{1-\cos(2\theta)}{2} \qquad \text{Power Reduction Identity for Sine} \end{aligned}$$

Next, we use the formula $\cos(2\theta) = 2\ \cos^2(\theta) - 1$. Solving for $\cos^2(\theta)$, we get:

$$\begin{aligned} \cos(2\theta) &= 2\cos^2(\theta) - 1 \\ \implies 1 + \cos(2\theta) &= 2\cos^2(\theta) \\ \implies \cos^2(\theta) &= \frac{1+\cos(2\theta)}{2} \qquad \text{Power Reduction Identity for Cosine} \end{aligned}$$

The last Reduction Identity is derived by writing tangent in terms of sine and cosine:

$$\begin{aligned} \tan^2(\theta) &= \frac{\sin^2(\theta)}{\cos^2(\theta)} \\ &= \frac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}} \qquad \text{(substitute the Reduction Identities)} \\ &= \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \qquad \text{Power Reduction Identity for Tangent} \end{aligned}$$

Example 6.4.1

We will apply the Reduction Identity for cosine twice.

$$\begin{aligned} \cos^4(x) &= \left(\cos^2(x)\right)^2 \\ &= \left(\frac{1+\cos(2x)}{2}\right)^2 && \text{(Reduction Identity)} \\ &= \frac{1}{4}\left(1 + 2\cos(2x) + \cos^2(2x)\right) && \text{(distribution)} \\ &= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4}\left(\frac{1+\cos(4x)}{2}\right) && \text{(distribution and Reduction Identity)} \\ &= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{8} + \frac{1}{8}\cos(4x) && \text{(distribution)} \\ &= \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x) && \text{(combining like terms)} \end{aligned}$$

Checkpoint 6.4.1

$\frac{3}{8} - \frac{1}{2}\cos(6x) + \frac{1}{8}\cos(12x)$

Example 6.4.2

We will work on simplifying the left side of the equation.

$$\begin{aligned} \text{LHS} &= \sin^3(2x) \\ &= \sin(2x)\cdot\sin^2(2x) \\ &= \sin(2x)\left[\frac{1-\cos(4x)}{2}\right] && \text{(Power Reduction Identity)} \\ &= \sin(2x)\left(\frac{1}{2}\right)[1-\cos(4x)] \\ &= \frac{1}{2}\sin(2x)[1-\cos(4x)] \\ &= \text{RHS} \end{aligned}$$

Checkpoint 6.4.2

$$\begin{aligned} 10\cos^4(x) &= 10\left(\cos^2(x)\right)^2 \\ &= 10\left[\frac{1+\cos(2x)}{2}\right]^2 && \text{(Power Reduction Identity)} \\ &= \frac{10}{4}\left[1 + 2\cos(2x) + \cos^2(2x)\right] && \text{(distribution)} \\ &= \frac{10}{4} + \frac{10}{2}\cos(2x) + \frac{10}{4}\left(\frac{1+\cos(4x)}{2}\right) && \text{(distribution and Power Reduction Identity)} \\ &= \frac{10}{4} + \frac{10}{2}\cos(2x) + \frac{10}{8} + \frac{10}{8}\cos(4x) \\ &= \frac{30}{8} + 5\cos(2x) + \frac{10}{8}\cos(4x) \\ &= \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x) \end{aligned}$$

Theorem: Half-Angle Identities

The Half-Angle Identity for sine is derived as follows:

$$\begin{aligned} \sin^2(\theta) &= \frac{1-\cos(2\theta)}{2} && \text{(Power Reduction Identity)} \\ \Longrightarrow \quad \sin^2\left(\frac{\alpha}{2}\right) &= \frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{2} && \left(\text{substituting } \frac{\alpha}{2} \text{ for } \theta\right) \\ &= \frac{1-\cos(\alpha)}{2} \\ \Longrightarrow \quad \sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos(\alpha)}{2}} && \text{(taking the square root of both sides)} \end{aligned}$$

The derivations of the Half-Angle Identities for cosine and tangent are left for the reader.

Example 6.4.3

Since $15^\circ = \frac{30^\circ}{2}$, we use the Half-Angle Identity for sine:

$$\begin{aligned} \sin(15^\circ) &= \sin\left(\frac{30^\circ}{2}\right) \\ &= +\sqrt{\frac{1-\cos(30^\circ)}{2}} && \text{(since } 15^\circ \in \text{QI, } \sin(15^\circ) \text{ is positive)} \\ &= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} \\ &= \sqrt{\frac{2-\sqrt{3}}{4}} \\ &= \frac{\sqrt{2-\sqrt{3}}}{2} \end{aligned}$$

Remember that we can check the answer with a graphing calculator.

Checkpoint 6.4.3

$$\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

Example 6.4.4

Using the given information, we draw the triangle shown below.

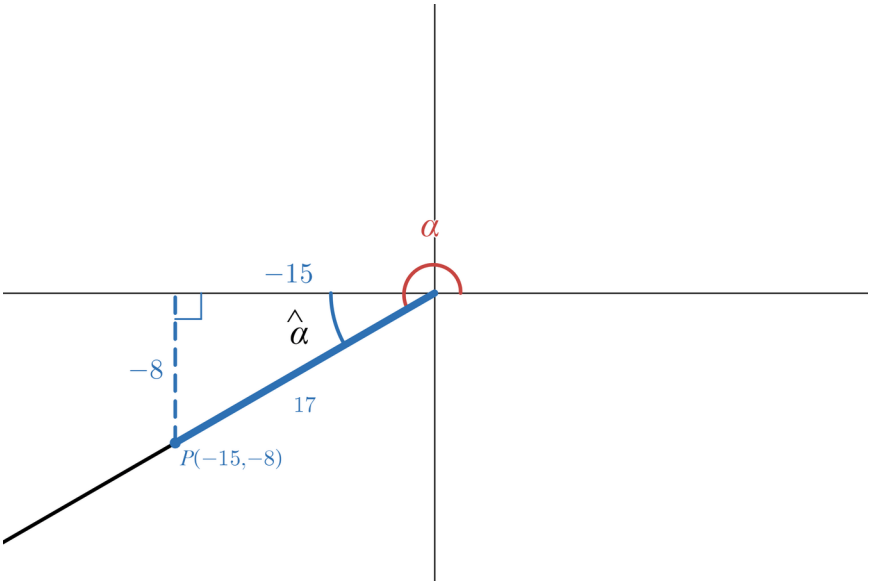


Figure 6.4.1

Using the Pythagorean Theorem, we find the hypotenuse to be 17. Therefore, we can calculate

$$\sin(\alpha) = -\frac{8}{17}$$

and

$$\cos(\alpha) = -\frac{15}{17}.$$

Before we start, we must remember that if α is in quadrant III, then $\pi < \alpha < \frac{3\pi}{2}$, so $\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}$. This means that the terminal side of $\frac{\alpha}{2}$ is in quadrant II.

a. To find $\sin\left(\frac{\alpha}{2}\right)$, we begin by writing the Half-Angle Identity for sine. Then, we substitute the value of the cosine we found from the triangle above and simplify.

$$\begin{aligned}
\sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos(\alpha)}{2}} && \text{(Half-Angle Identity)}\\
&= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{2}}\\
&= \pm\sqrt{\frac{\frac{32}{17}}{2}}\\
&= \pm\sqrt{\frac{32}{34}}\\
&= \pm\sqrt{\frac{16}{17}}\\
&= \pm\frac{4}{\sqrt{17}}\\
&= \frac{4}{\sqrt{17}} && \left(\frac{\alpha}{2}\in \text{QII}\right)
\end{aligned}$$

We choose the positive value of $\sin\left(\frac{\alpha}{2}\right)$ because the angle terminates in quadrant II and sine is positive in quadrant II.

b. To find $\cos\left(\frac{\alpha}{2}\right)$, we will write the Half-Angle Identity for cosine, substitute the value of the cosine we found from the triangle, and simplify.

$$\begin{aligned}
\cos\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1+\cos(\alpha)}{2}} && \text{(Half-Angle Identity)}\\
&= \pm\sqrt{\frac{1+\left(-\frac{15}{17}\right)}{2}}\\
&= \pm\sqrt{\frac{\frac{2}{17}}{2}}\\
&= \pm\sqrt{\frac{2}{34}}\\
&= \pm\sqrt{\frac{1}{17}}\\
&= -\frac{1}{\sqrt{17}} && \left(\frac{\alpha}{2}\in \text{QII}\right)
\end{aligned}$$

We choose the negative value of $\cos\left(\frac{\alpha}{2}\right)$ because the angle is in quadrant II and the cosine is negative in quadrant II.

c. To find $\tan\left(\frac{\alpha}{2}\right)$, we write the Half-Angle Identity for the tangent. Again, we substitute the value of the cosine we found from the triangle and simplify.

$$\begin{aligned}
\tan\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}} && \text{(Half-Angle Identity)}\\
&= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{1+\left(-\frac{15}{17}\right)}}\\
&= \pm\sqrt{\frac{\frac{32}{17}}{\frac{2}{17}}}\\
&= \pm\sqrt{\frac{32}{2}}\\
&= -\sqrt{16} && \left(\frac{\alpha}{2}\in \text{QII}\right)\\
&= -4
\end{aligned}$$

We choose the negative value of $\tan\left(\frac{\alpha}{2}\right)$ because $\frac{\alpha}{2}$ lies in quadrant II, and tangent is negative in quadrant II.

Checkpoint 6.4.4

$-\frac{2}{\sqrt{5}}$

Example 6.4.5

Since the angle for novice competition measures half the steepness of the angle for the high-level competition, and $\tan(\theta)=\frac{5}{3}$ for high competition, we can find $\cos(\theta)$ from the right triangle and the Pythagorean Theorem so that we can use the Half-Angle Identities (see the figure below).

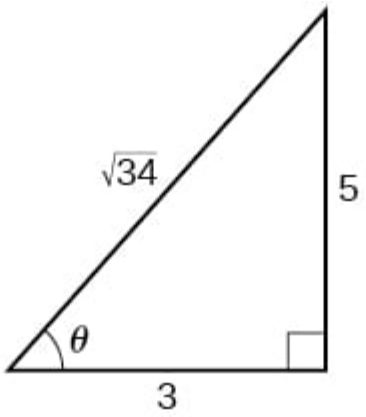


Figure 6.4.2

We see that $\cos(\theta) = \frac{3}{\sqrt{34}}$. We can use the Half-Angle Identity for tangent:

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}.$$

Since $\tan(\theta)$ is in the first quadrant, so is $\tan\left(\frac{\theta}{2}\right)$.

$$\begin{aligned}\tan\left(\frac{\theta}{2}\right) &= \sqrt{\frac{1-\frac{3}{\sqrt{34}}}{1+\frac{3}{\sqrt{34}}}} \\ &= \sqrt{\frac{\sqrt{34}-3}{\sqrt{34}+3}} \\ &\approx 0.57\end{aligned}$$

We can take the inverse tangent to find the angle: $\tan^{-1}(0.57) \approx 29.7^\circ$. So, the angle of the ramp for novice competition is $\approx 29.7^\circ$.

Example 6.4.6

We begin by recognizing this as the square of a Half-Angle Identity.

$$\begin{aligned}y &= 3\sin^2\left(\frac{x}{2}\right) \\ &= 3\left[\pm\sqrt{\frac{1-\cos(x)}{2}}\right]^2 \\ &= 3\left(\frac{1-\cos(x)}{2}\right) \\ &= -\frac{3}{2}\cos(x)+\frac{3}{2}\end{aligned}$$

We see this is a reflected cosine function with amplitude $\frac{3}{2}$, period 2π, step size $\frac{\pi}{2}$, phase shift 0, and vertical shift $\frac{3}{2}$. Using all this information, we obtain the following graph.

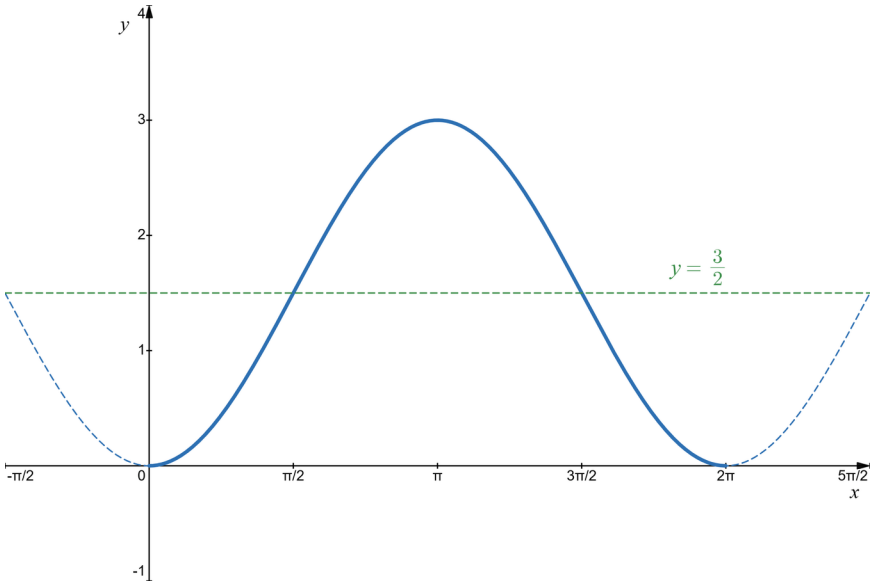


Figure 6.4.3

Checkpoint 6.4.6

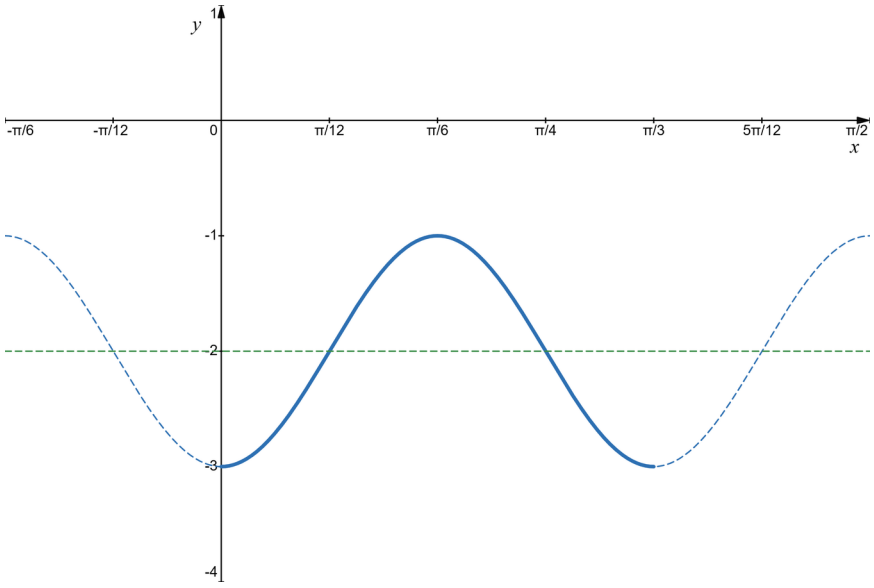


Figure 6.4.4

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6.4.1: Resources and Key Concepts

Resources

Keywords

- Power Reduction Identities
  - Double-Angle Identities - Power-Reduction Formulas and Their Use
- Half-Angle Identities
  - Half-Angle Identities - Theorem and Proof
  - Half-Angle Identities - Evaluating Trig Functions of Non-Special Angles Using Half-Angle Identities
  - Half-Angle Identities - Using the Half-Angle Identities
  - Half-Angle Identities - Proving Trigonometric Identities (Part 5)

Key Concepts

Theorems

- **Power Reduction Identities:**
  - $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$
  - $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$
  - $\tan^2(\theta) = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$
- **Half-Angle Identities:**
  - $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
  - $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$
  - $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} = \frac{\sin(\theta)}{1+\cos(\theta)} = \frac{1-\cos(\theta)}{\sin(\theta)}$

Common Mistakes

- **Forgetting the $\pm$ Sign:** The Half-Angle Identities for sine and cosine include a $\pm$ symbol. This does not mean both signs are correct. You must determine the correct sign (positive or negative) based on the quadrant in which the half-angle $\frac{\theta}{2}$ terminates.
- **Determining the Sign from the Wrong Angle:** The sign for a half-angle identity, like $\sin(\frac{\theta}{2})$, depends on the quadrant of $\frac{\theta}{2}$, not the quadrant of the original angle θ.
- **Incorrectly Applying Power Reduction:** The Power Reduction Identities are used to rewrite even powers of sine and cosine in terms of the first power of cosine. They cannot be used directly on odd powers.

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6.4.2: Homework

Reading Questions

1. What is the primary purpose of the Power Reduction Identities?
2. From which identities are the Power Reduction Identities derived?
3. What is the Power Reduction Identity for $\sin^2(\theta)$?
4. How are the Half-Angle Identities derived from the Power Reduction Identities?
5. What is the Half-Angle Identity for $\cos(\frac{\theta}{2})$?
6. What does the $\pm$ symbol in the Half-Angle Identities signify?
7. How do you determine whether to use the positive or negative root when applying a Half-Angle Identity?
8. If θ is in Quadrant IV, in which quadrant must $\frac{\theta}{2}$ lie?
9. How can you use the Half-Angle Identities to help graph a function like $y = 3\sin^2(\frac{x}{2})$?
10. To find the exact value of $\sin(15^\circ)$ using a Half-Angle Identity, what angle would you use for θ?

Homework

Basic Skills

For the following exercises, rewrite the expression without exponents.

1. $\cos^2(5x)$
2. $\cos^2(6x)$
3. $\sin^4(8x)$
4. $\sin^4(3x)$
5. $\cos^2(x)\sin^4(x)$
6. $\cos^4(x)\sin^2(x)$

For the following exercises, use the Half-Angle Identities to find the exact value.

7. $\cos(75^\circ)$
8. $\sin(105^\circ)$
9. $\cos(67.5^\circ)$
10. $\sin(157.5^\circ)$
11. $\tan(112.5^\circ)$
12. $\cos\left(\frac{7\pi}{12}\right)$
13. $\sin\left(\frac{\pi}{12}\right)$
14. $\sin\left(\frac{\pi}{24}\right)$
15. $\cos\left(\frac{\pi}{24}\right)$

16. $\sin\left(\frac{9\pi}{8}\right)$

17. $\tan\left(\frac{3\pi}{8}\right)$

18. $\cos\left(\frac{\pi}{8}\right)$

19. $\sin\left(\frac{5\pi}{8}\right)$

20. $\tan\left(\frac{7\pi}{8}\right)$

21. $\sin\left(-\frac{\pi}{8}\right)$

22. $\sin\left(\frac{13\pi}{24}\right)$

For the following exercises, use the given information about θ to find the exact values of

- $\sin\left(\frac{\theta}{2}\right)$
- $\cos\left(\frac{\theta}{2}\right)$
- $\tan\left(\frac{\theta}{2}\right)$

23. $\sin(\theta) = -\frac{7}{25}$ where $\frac{3\pi}{2} < \theta < 2\pi$

24. $\cos(\theta) = \frac{28}{53}$ where $0 < \theta < \frac{\pi}{2}$

25. $\tan(\theta) = \frac{12}{5}$ where $\pi < \theta < \frac{3\pi}{2}$

26. $\csc(\theta) = 4$ where $\frac{\pi}{2} < \theta < \pi$

27. $\cos(\theta) = \frac{3}{5}$ where $0 < \theta < \frac{\pi}{2}$

28. $\sin(\theta) = -\frac{4}{5}$ where $\pi < \theta < \frac{3\pi}{2}$

29. $\cos(\theta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \theta < 2\pi$

30. $\sin(\theta) = \frac{5}{13}$ where $\frac{\pi}{2} < \theta < \pi$

31. $\sec(\theta) = \sqrt{5}$ where $\frac{3\pi}{2} < \theta < 2\pi$

32. $\tan(\theta) = -2$ where $\frac{\pi}{2} < \theta < \pi$

Synthesis Problems

33. Let θ be a Quadrant III angle with $\cos(\theta) = -\frac{1}{5}$. Show that this is not enough information to determine the sign of $\sin\left(\frac{\theta}{2}\right)$ by first assuming $3\pi < \theta < \frac{7\pi}{2}$ and then assuming $\pi < \theta < \frac{3\pi}{2}$ and computing $\sin\left(\frac{\theta}{2}\right)$ in both cases.

For the following exercises, graph *two* cycles of the function.

34. $y = 8\sin^2\left(\frac{x}{2}\right)$

35. $y = 2\cos^2\left(\frac{x}{2}\right)$

36. $y = -2\cos^2\left(\frac{x}{2}\right) + 4$

37. $y = \frac{1}{3}\cos^2\left(\frac{x}{2}\right) - \frac{1}{6}$

Proofs

For the following exercises, prove the identity.

38. $\tan\left(\frac{1}{2}\theta\right) = \csc(\theta) - \cot(\theta)$

39. $\tan^2\left(\frac{1}{2}\theta\right) = \frac{\tan(\theta) - \sin(\theta)}{\tan(\theta) + \sin(\theta)}$

40. $\sin^2\left(\frac{\theta}{2}\right) = \frac{\csc(\theta) - \cot(\theta)}{2\csc(\theta)}$

41. $2\sin^2\left(\frac{\theta}{2}\right) = \frac{\sin^2(\theta)}{1 + \cos(\theta)}$

42. $2\cos^2\left(\frac{\theta}{2}\right) = \frac{\sin^2(\theta)}{1 - \cos(\theta)}$

43. $\cos^2\left(\frac{\theta}{2}\right) = \frac{\tan(\theta) + \sin(\theta)}{2\tan(\theta)}$

44. $\sec^2\left(\frac{\theta}{2}\right) = \frac{2\sec(\theta)}{\sec(\theta) + 1}$

45. $\csc^2\left(\frac{\theta}{2}\right) = \frac{2\sec(\theta)}{\sec(\theta) - 1}$

46. $4\sin^4(\theta) = 1 - 2\cos(2\theta) + \cos^2(2\theta)$

47. $\cos^4(\theta) = \frac{1}{4} + \frac{\cos(2\theta)}{2} + \frac{\cos^2(2\theta)}{4}$

Challenge Questions

48. Without using your calculator, show that $\frac{\sqrt{2+\sqrt{3}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$

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6.5: Sum-to-Product and Product-to-Sum Identities

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

You should commit to memory the identities we have covered up to this point. Specifically, you need to be able to recall the following identities without access to a note card.

Reciprocal Identities	
Basic Identity	***Alternate Form***
$\sin(\theta) = \frac{1}{\csc(\theta)}$	$\csc(\theta) = \frac{1}{\sin(\theta)}$
$\cos(\theta) = \frac{1}{\sec(\theta)}$	$\sec(\theta) = \frac{1}{\cos(\theta)}$
$\tan(\theta) = \frac{1}{\cot(\theta)}$	$\cot(\theta) = \frac{1}{\tan(\theta)}$

Ratio Identities	
Basic Identity	***Alternate Form***
$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$

Pythagorean Identities	
Basic Identity	***Alternate Forms***
$\cos^2(\theta) + \sin^2(\theta) = 1$	$\cos^2(\theta) = 1 - \sin^2(\theta)$ and $\cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$
	$\sin^2(\theta) = 1 - \cos^2(\theta)$ and $\sin(\theta) = \pm\sqrt{1 - \cos^2(\theta)}$
$1 + \tan^2(\theta) = \sec^2(\theta)$	$\tan^2(\theta) = \sec^2(\theta) - 1$
$\cot^2(\theta) + 1 = \csc^2(\theta)$	$\cot^2(\theta) = \csc^2(\theta) - 1$

Symmetry Identities	
Basic Identity	***Alternate Form***
$\sin(-\theta) = -\sin(\theta)$	$\csc(-\theta) = -\csc(\theta)$
$\cos(-\theta) = \cos(\theta)$	$\sec(-\theta) = \sec(\theta)$
$\tan(-\theta) = -\tan(\theta)$	$\cot(-\theta) = -\cot(\theta)$

Cofunction Identities	
Basic Identity	***Alternate Form***
$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$
$\sec(\theta) = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc(\theta) = \sec\left(\frac{\pi}{2} - \theta\right)$
$\tan(\theta) = \cot\left(\frac{\pi}{2} - \theta\right)$	$\cot(\theta) = \tan\left(\frac{\pi}{2} - \theta\right)$

Sum and Difference of Angles Identities
$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$

Double-Angle Identities
$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\ &= 2\cos^2(\theta) - 1\\ &= 1 - 2\sin^2(\theta)\end{aligned}$

Power Reduction Identities
$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$
$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$

Half-Angle Identities
$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$

You should note that the tangent identities are not listed for our recent formulas. This is because they are rarely used and, if needed, can be derived from the identities for sine and cosine.

The Sum-to-Product and Product-to-Sum Identities

Our next batch of identities, the Product-to-Sum Identities (also known as the **Prosthaphaeresis Formulas**), can be easily verified by expanding each of the right-hand sides per the Sum and Difference Identities (we leave the details as exercises). They are of particular use in Calculus, and we list them here for reference.

Theorem: Product-to-Sum Identities

For angles α and β,

$$\begin{aligned}\sin(\alpha)\sin(\beta) &= \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]\\ \cos(\alpha)\cos(\beta) &= \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]\\ \sin(\alpha)\cos(\beta) &= \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]\end{aligned}$$

The final set of identities in this chapter, the Sum-to-Product Identities, are related to the Product-to-Sum Identities. These are easily verified using the Product-to-Sum Identities, and as such, their proofs are left as exercises.

Theorem: Sum-to-Product Identities

For angles α and β,

$$\begin{aligned}
\sin(\alpha) \pm \sin(\beta) &= 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \\
\cos(\alpha) + \cos(\beta) &= 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \\
\cos(\alpha) - \cos(\beta) &= -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)
\end{aligned}$$

Do Not Memorize These Identities!

I cannot think of any situation where committing these identities to memory is beneficial. It's enough to know they exist and when they are needed.

Example 6.5.1

a. Write $\cos(2\theta)\cos(6\theta)$ as a sum.
b. Write $\sin(4\theta)\cos(2\theta)$ as a sum.
c. Write $\sin(\theta) - \sin(3\theta)$ as a product.

Solutions available at the end of this section.

The reader is reminded that all of the identities presented in this section, which regard the circular functions as functions of angles (in radian measure), apply equally well to the circular (trigonometric) functions considered as functions of real numbers.

Checkpoint 6.5.1

a. Write the product as a sum or difference.

$$\sin(x + y)\cos(x - y)$$

b. Write the sum as a product.

$$\sin(3\theta) + \sin(\theta)$$

Answer available at the end of this section.

Using the Sum-to-Product and Product-to-Sum Identities to Evaluate Trigonometric Functions

There are times when obtaining exact values of trigonometric functions is easier when using the Sum-to-Product and Product-to-Sum Identities. Again, you are not expected to memorize these identities, but you *are* expected to be able to recognize when they might be of use.

Example 6.5.2

Find the exact value of $\cos(15^\circ) - \cos(75^\circ)$.

Solution available at the end of this section.

Checkpoint 6.5.2

Find the exact value of $\cos\left(\frac{11\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$.

Answer available at the end of this section.

Revisiting Proofs of Identities

While proofs of identities that require Sum-to-Product or Product-to-Sum Identities are no more challenging than proofs involving the other trigonometric identities, using these newer identities less frequently makes recognizing the need for them more difficult. Try your best to remember that when all else fails, you might need to use a Sum-to-Product or Product-to-Sum Identity.

Example 6.5.3

Prove the identity.

$$\frac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)} = -\tan(t)$$

Solution available at the end of this section.

Miscellaneous Topics: How Does Math Relate to Your Field?

1. Please tell me what degree you are currently pursuing.
2. Name at least five things in your chosen field with which mathematics can help.

Success in College: To Be Smart or Not To Be Smart... Is This Your Question?

> When I attended CRC, I had a computer science professor who had some great sayings. One of my favorites occurred when a student raised her hand and said, in a sheepish voice, "This may be a stupid question, but..."
>
> The brilliantly opinionated professor immediately interrupted the student and stated, in a voice beaming with inspiring confidence, "There are no stupid questions... only stupid people who don't ask them."
>
> "That is so true," I thought to myself. Here we are, learning, experiencing, and becoming better citizens of this great wide world. Still, most students refuse to involve themselves in their education by remaining silent when they should be speaking up. This single abrasive comment by the professor summed up the majority of my fellow students!
>
> I immediately swore to myself not to be shy anymore about asking questions. If I annoy my neighbor by slowing the lecture, so be it! If I look like I am lost in front of others, that's a small price to pay to gain understanding.
>
> And so it went... I carried this habit throughout the universities I attended, in every course, with every professor. As time went on, and I was in classes with more capable students because those earlier students fell by the wayside, I found I was surrounded by those with the same habit. It turned out that one of the (many) keys to success on the road to becoming a critical thinker was being engaged and asking questions.
>
> - Professor Simpson, Cosumnes River College

So what about you? Are you willing to let go of that social stigma that asking questions makes you look "dumb?" Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. In some high schools, it is not "socially acceptable" to act as if you care about classwork. Students do not ask for help in front of others. This attitude may or may not continue in beginning college classes, even for older students. Is this a typical attitude at your college?
2. Some professors check for understanding by calling on students in class. This way, the professor knows if more examples or discussions are needed. Some students have no problem answering questions in front of the class. Other students struggle to answer in front of the class even if they think they know the correct answer. How do you react when a professor asks you a question in front of the class? Why?

Answers

Section Preview

Students need to know these identities exist — they are used infrequently in Calculus, and having knowledge of their existence is assumed; however, do not force students to memorize these.

Section Preview

- Rewrite a product of sines and cosines as a sum or a difference.
- Express a sum or difference of sines and cosines as a product.
- Prove an identity using the Sum-to-Product and Product-to-Sum Identities.

Example 6.5.1

a. Identifying $\alpha = 2\theta$ and $\beta = 6\theta$, we find

$$\begin{aligned} \cos(2\theta)\cos(6\theta) &= \frac{1}{2}\left[\cos(2\theta - 6\theta) + \cos(2\theta + 6\theta)\right] \\ &= \frac{1}{2}\cos(-4\theta) + \frac{1}{2}\cos(8\theta) \\ &= \frac{1}{2}\cos(4\theta) + \frac{1}{2}\cos(8\theta), \end{aligned}$$

where the last equality is courtesy of the even identity for cosine, $\cos(-4\theta) = \cos(4\theta)$.

b. Write the formula for the product of sine and cosine. Then, substitute the given values into the formula and simplify.

$$\begin{aligned} \sin(\alpha)\cos(\beta) &= \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)] \\ \implies \sin(4\theta)\cos(2\theta) &= \frac{1}{2}[\sin(4\theta + 2\theta) + \sin(4\theta - 2\theta)] \\ &= \frac{1}{2}[\sin(6\theta) + \sin(2\theta)] \end{aligned}$$

c. Identifying $\alpha = \theta$ and $\beta = 3\theta$ yields

$$\begin{aligned} \sin(\theta) - \sin(3\theta) &= 2\sin\left(\frac{\theta - 3\theta}{2}\right)\cos\left(\frac{\theta + 3\theta}{2}\right) \\ &= 2\sin(-\theta)\cos(2\theta) \\ &= -2\sin(\theta)\cos(2\theta), \end{aligned}$$

where the last equality is courtesy of the odd identity for sine, $\sin(-\theta) = -\sin(\theta)$.

Checkpoint 6.5.1

a. $\frac{1}{2}(\sin(2x) + \sin(2y))$

b. $2\sin(2\theta)\cos(\theta)$

Example 6.5.2

We could tackle this problem by using a Half-Angle Identity for the first term and a Sum of Angles Identity for the second; however, recognizing that this is a difference of cosines also gives us the clue that we could use a Sum-to-Product Identity.

$$\begin{aligned} \cos(15^\circ) - \cos(75^\circ) &= -2\sin\left(\frac{15^\circ + 75^\circ}{2}\right)\sin\left(\frac{15^\circ - 75^\circ}{2}\right) \qquad \text{(Sum-to-Product Identity)} \\ &= -2\sin(45^\circ)\sin(-30^\circ) \\ &= -2\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2}\right) \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

Checkpoint 6.5.2

$\frac{-2 - \sqrt{3}}{4}$

Example 6.5.3

We will start with the left side, the more complicated side of the equation, and rewrite the expression until it matches the right side.

$$
\begin{aligned}
\text{LHS} &= \frac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)} \\
&= \frac{-2\sin\left(\frac{4t+2t}{2}\right)\sin\left(\frac{4t-2t}{2}\right)}{2\sin\left(\frac{4t+2t}{2}\right)\cos\left(\frac{4t-2t}{2}\right)} && \text{(Sum-to-Product Identities)} \\
&= \frac{-2\sin(3t)\sin(t)}{2\sin(3t)\cos(t)} \\
&= -\frac{\sin(t)}{\cos(t)} \\
&= -\tan(t) \\
&= \text{RHS}
\end{aligned}
$$

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6.5.1: Resources and Key Concepts

Resources

Videos

- Sum-to-Product Identities and Product-to-Sum Identities
  - The Product to Sum and Sum to Product Formulas - Theorem, Proof, and Examples

Key Concepts

Theorems

- **Product-to-Sum Identities:** For any angles α and β:
  - $\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$
  - $\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$
  - $\sin(\alpha)\cos(\beta) = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$
- **Sum-to-Product Identities:** For any angles α and β:
  - $\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$
  - $\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
  - $\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

Common Mistakes

- **Unnecessary Memorization:** It is not beneficial to commit the Sum-to-Product and Product-to-Sum identities to memory. It is sufficient to know that they exist and to be able to recognize when they might be useful.
- **Forgetting Negative Signs or Coefficients:** It is easy to forget the negative sign in the formula for $\cos(\alpha) - \cos(\beta)$ or the coefficient of $\frac{1}{2}$ in the Product-to-Sum identities. Always double-check the formula when applying it.

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6.5.2: Homework

Reading Questions

1. How can the Product-to-Sum identities be verified?
2. What is the Product-to-Sum identity used to rewrite a product of two cosines, $\cos(\alpha)\cos(\beta)$, as a sum?
3. What is the Sum-to-Product identity used to rewrite a difference of two sines, $\sin(\alpha)-\sin(\beta)$, as a product?
4. According to the text, is it beneficial to memorize the Sum-to-Product and Product-to-Sum identities?
5. What two different methods are mentioned in Example 2 for finding the exact value of $\cos(15^\circ)-\cos(75^\circ)$?
6. In Example 1, what previously learned identity is used to rewrite $\cos(-4\theta)$ as $\cos(4\theta)$?
7. When might you need to use a Sum-to-Product or Product-to-Sum identity?

Homework

Basic Skills

For the following exercises, write the given product as a sum.

1. $\cos(3\theta)\cos(5\theta)$
2. $\sin(2\theta)\sin(7\theta)$
3. $\sin(9\theta)\cos(\theta)$
4. $\cos(2\theta)\cos(6\theta)$
5. $\sin(3\theta)\sin(2\theta)$
6. $\cos(\theta)\sin(3\theta)$

For the following exercises, write the given sum as a product.

7. $\cos(3\theta)+\cos(5\theta)$
8. $\sin(2\theta)-\sin(7\theta)$
9. $\cos(5\theta)-\cos(6\theta)$
10. $\sin(9\theta)-\sin(-\theta)$
11. $\sin(\theta)+\cos(\theta)$
12. $\cos(\theta)-\sin(\theta)$

For the following exercises, find the exact values of the trigonometric expressions.

13. $2\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{3}\right)$
14. $\sin\left(\frac{\pi}{12}\right)-\sin\left(\frac{7\pi}{12}\right)$
15. $\cos\left(\frac{5\pi}{12}\right)+\cos\left(\frac{7\pi}{12}\right)$

Proofs

16. Verify the Product to Sum Identities.

17. Verify the Sum to Product Identities.

For the following exercises, prove the identity.

18. $\dfrac{\sin(x)+\sin(y)}{\cos(x)+\cos(y)} = \tan\left(\dfrac{x+y}{2}\right)$

19. $\dfrac{\sin(x)-\sin(y)}{\cos(x)-\cos(y)} = -\cot\left(\dfrac{x+y}{2}\right)$

20. $\sin(x)+\sin(2x)+\sin(3x) = 4\cos\left(\dfrac{x}{2}\right)\cos(x)\sin\left(\dfrac{3x}{2}\right)$

21. $\cos(x)+\cos(2x)+\cos(3x) = 4\cos\left(\dfrac{x}{2}\right)\cos(x)\cos\left(\dfrac{3x}{2}\right)-1$

22. $\dfrac{\cos(x)-\cos(3x)}{\sin(3x)-\sin(x)} = \tan(2x)$

23. $-\tan(4x) = \dfrac{\cos(3x)-\cos(5x)}{\sin(3x)-\sin(5x)}$

24. $32\sin^2(\theta)\cos^4(\theta) = 2+\cos(2\theta)-2\cos(4\theta)-\cos(6\theta)$

25. $32\sin^4(\theta)\cos^2(\theta) = 2-\cos(2\theta)-2\cos(4\theta)+\cos(6\theta)$

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7: TRIGONOMETRIC EQUATIONS

7.1: The Fundamental Inverse Trigonometric Functions

7.1.1: Resources and Key Concepts

7.1.2: Homework

7.2: The Remaining Inverse Trigonometric Functions

7.2.1: Resources and Key Concepts

7.2.2: Homework

7.3: Solving Trigonometric Equations - Algebraic Techniques

7.3.1: Resources and Key Concepts

7.3.2: Homework

7.4: Solving Trigonometric Equations Using Identities

7.4.1: Resources and Key Concepts

7.4.2: Homework

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7.1: The Fundamental Inverse Trigonometric Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We previously introduced the idea of finding an *acute* angle given the value of a trigonometric function. We learned to use the calculator keys $\mathrm{SIN}^{-1}, \mathrm{COS}^{-1}$, and TAN^{-1} to find approximate values of θ when we know either $\sin(\theta), \cos(\theta)$, or $\tan(\theta)$. For example, if we know that $\cos(\theta) = 0.3$, then the *acute* angle θ is approximated by

$$\theta = \cos^{-1}(0.3) \approx 1.2661 \text{ radians}.$$

If you consider the initial piece of information, namely that $\cos(\theta) = 0.3$, this implies that we can use the $\mathrm{SIN}^{-1}, \mathrm{COS}^{-1}$, and TAN^{-1} keys to solve trigonometric equations (as long as the angle we are interested in is acute), just as we use square roots to solve quadratic equations.

Up to this point, that's all we needed to know about the inverse trigonometric functions; however, we will soon be solving more complex trigonometric equations, requiring a deeper discussion of them.

Does Every Function Have an Inverse?

Consider the function $F(x) = x^2 - 4$. First, we'll find a formula for the inverse. We let $y = F(x)$ and solve for x

$$\begin{aligned} y &= x^2 - 4 \\ \Longrightarrow \quad y + 4 &= x^2 \\ \Longrightarrow \quad \pm\sqrt{y+4} &= x \end{aligned}$$

We then interchange x and y to get $y = \pm\sqrt{x+4}$.

The graphs of F and its inverse, $F^{-1}(x) = \pm\sqrt{x+4}$, are shown in Figure 7.1.1. You can see that although F is a function, its inverse is not because the graph of the inverse does not pass the Vertical Line Test.

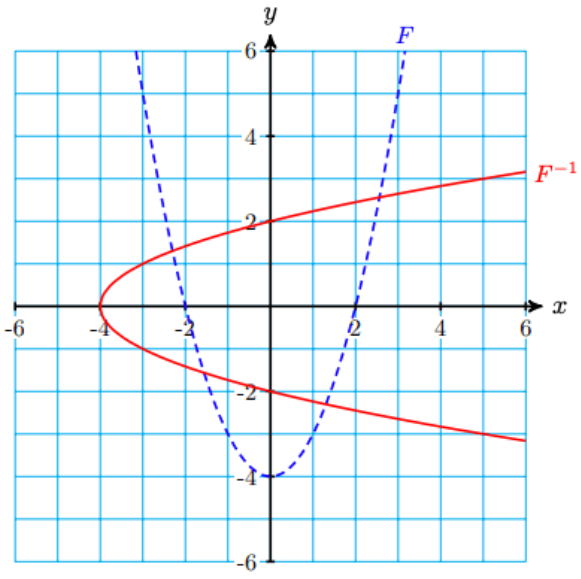


Figure 7.1.1

From Algebra, for a function, f, to have an inverse that is also a function, we require f to pass the Horizontal Line Test. Recall that a function that passes the Horizontal Line Test is called one-to-one.

At this point, I want to be *very* clear about a few points:

1. You can always find the inverse of a function graphically. All you need to do is go through each point on the graph of the function and swap the x- and y-coordinates. In doing so, the domain of your new graph is the old function's range, and the new graph's range is the old function's domain.
2. It might be impossible to find the equation representing the inverse. For example, $f(x) = e^x - 3x^2 + \sin(x)$ is definitely a function, but solving $y = e^x - 3x^2 + \sin(x)$ for x is not going to happen.
3. The inverse of a function need not be a function itself. We have just seen that $F(x) = x^2 - 4$ is an example of a function with an inverse that is *not* a function.
4. If a function f passes the Horizontal Line Test, then the function is called one-to-one and its inverse is also a function that we call the inverse function of f.

Example 7.1.1

a. Which of the graphs below represent functions?
b. Which of the functions are one-to-one?
c. Which of these functions have inverse functions?

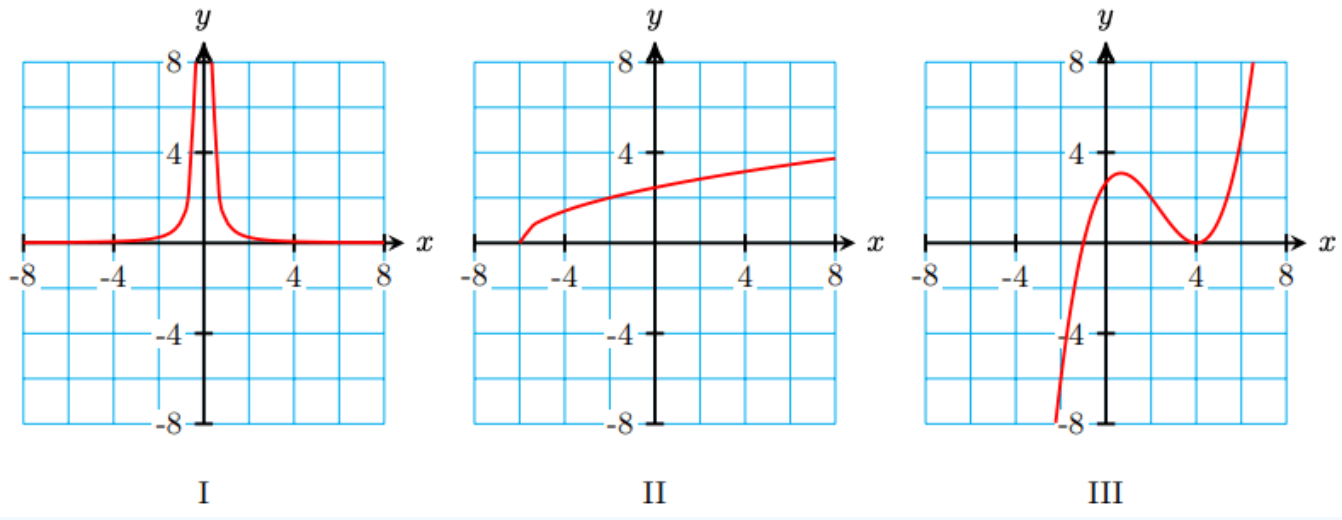


Figure 7.1.2

Solutions available at the end of this section.

Checkpoint 7.1.1

Which of the functions below are one-to-one?

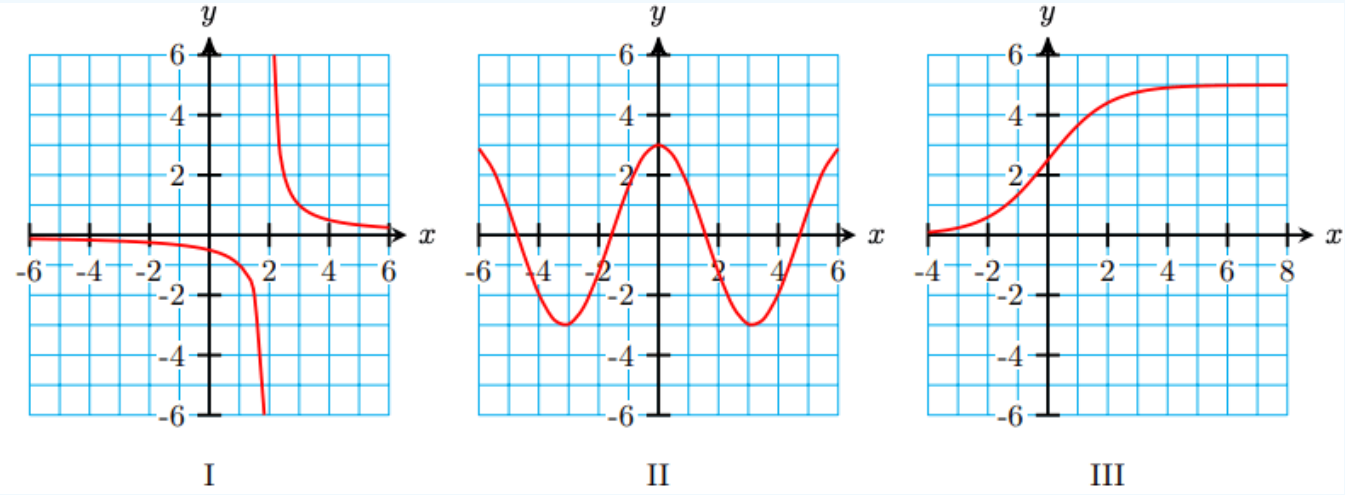


Figure 7.1.3

Answer available at the end of this section.

Restricting the Domain: The Inverse Sine Function

Sometimes, it is so crucial that the inverse be a function that we are willing to sacrifice part of the original function to achieve this result.

Look again at the graph of $F(x) = x^2 - 4$. If we use only nonnegative x-values for the domain, we create a new function,

$$f(x) = x^2 - 4, \quad x \geq 0.$$

The graph of this new function is shown as a dashed curve in Figure 7.1.4.

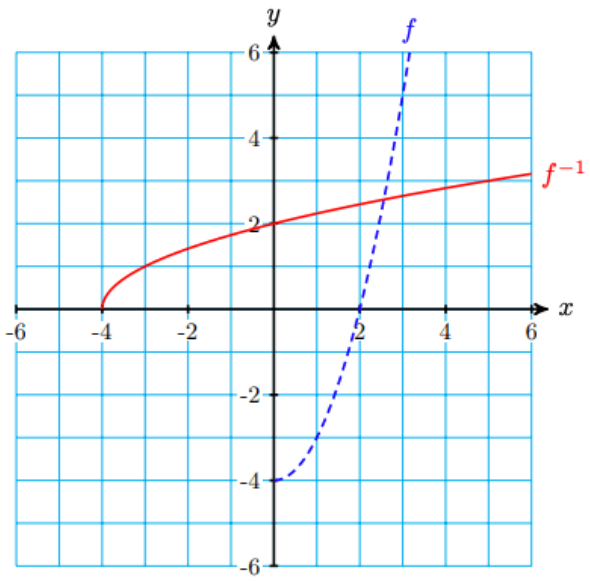


Figure 7.1.4

The new function is one-to-one, and its inverse, $f^{-1}(x) = \sqrt{x+4}$, is also a function.

We say that we have **restricted the domain** of the original function, and we will use this technique to define inverse functions for the trigonometric functions.

Theorem: Forcing Invertibility by Restricting Domain

If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.

The sine function is *not* one-to-one; there are many angles that have the same sine value. In order to define its inverse function, we must restrict the domain of the sine to an interval on which the y-values do not repeat. But there are many candidates for such an interval; which one do we choose?

The most useful interval starts at $x = 0$ and moves as far as we can in either direction until the y-values begin to repeat. By doing so, we obtain the restricted domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, as shown in Figure 7.1.5. This piece of the function includes all of the original range values, from -1 to 1.

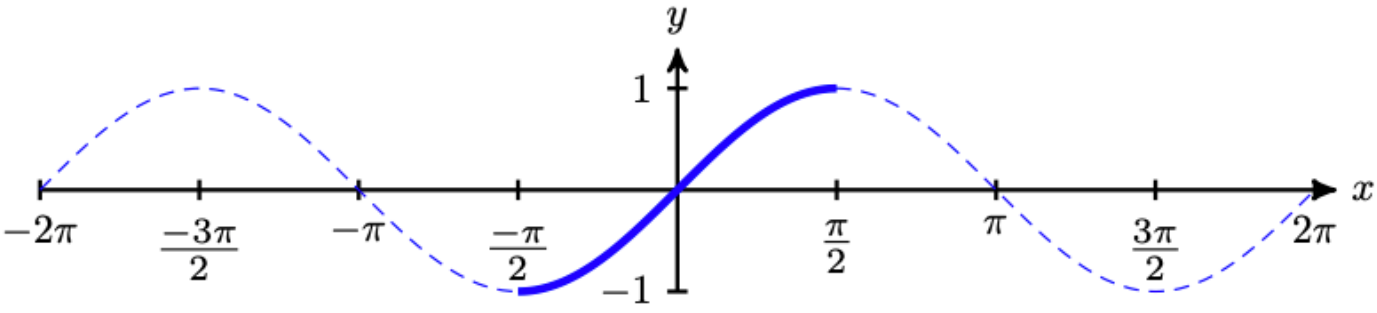


Figure 7.1.5

Because the sine function is one-to-one in this domain, its inverse is a function.

The graph of the inverse sine function, $y = \sin^{-1}(x)$, is shown in Figure 7.1.6. Its domain is the same as the range of the sine function, namely $-1 \leq x \leq 1$, and its range is our restricted domain for the sine, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

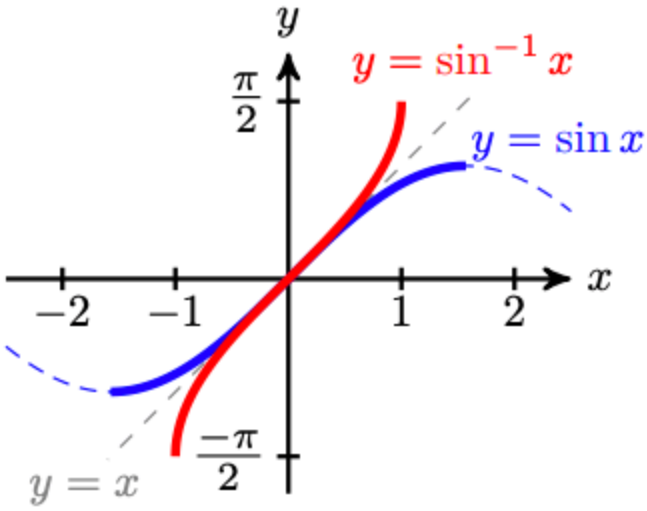


Figure 7.1.6

Definition: The Inverse Sine Function

The function $f(x) = \sin^{-1}(x)$ is defined as follows:

$$\sin^{-1}(x) = \theta \quad \text{if and only if} \quad \sin(\theta) = x \quad \text{and} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

In other words, $\sin^{-1}(x)$ is the angle in radians between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, whose sine is x. There are many angles with a given sine value x, but only one of these angles can be $\sin^{-1}(x)$. This is why the calculator's SIN^{-1} key only returns angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Prior to this moment, we have only considered inverse trigonometric functions returning acute (and, therefore, positive) angles; however, with this new definition (based upon domain-restriction of the sine function), we have something more useful:

- If $x = \sin(\theta)$ is positive, the inverse sine function delivers a first quadrant angle, $0 \leq \theta \leq \frac{\pi}{2}$. This should make sense because all trigonometric functions are positive in the first quadrant.
- If $x = \sin(\theta)$ is negative, the inverse sine function delivers a fourth quadrant angle, $-\frac{\pi}{2} \leq \theta \leq 0$.

As the next example illustrates, when dealing with inverse trigonometric functions, we focus on finding the *quadrant* and the *reference angle* for the function.

Example 7.1.2

Simplify each expression without using a calculator.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
b. $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$
c. $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$
d. $\sin^{-1}(\sin(\pi))$
e. $\sin\left(\sin^{-1}(0.2)\right)$

Solutions available at the end of this section.

The process we performed in Example 7.1.2 is crucial to understand. It reinforces the statement, "Trigonometry is mostly about quadrants and reference angles."

When evaluating $\sin^{-1}(r)$, where r is some number (often a ratio), we let the inverse equal θ (this is to remind us that we are looking for an angle). That is, we begin by writing $\sin^{-1}(r) = \theta$. In *most* cases, we are looking for a *reference angle*, $\hat{\theta}$, such that

$\sin\left(\hat{\theta}\right) = |r|$. We use the sign of r to inform us as to which *quadrant* θ is in. Once we have figured out these two details, we easily state the value of θ (remembering the requirement that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$).

Example $7.1.2d$ showcases the exception to the "find the reference angle and quadrant" rule. If r is 0, 1, or -1, then we want an angle such that $\sin(\theta)$ is equal to that value; however, these values are from quadrantal angles (0, $\frac{\pi}{2}$, and $-\frac{\pi}{2}$, respectively). As such, we will not have a reference angle to find.

Finally, while Example $7.1.2e$ illustrates that $\sin\left(\sin^{-1}(r)\right) = r$ as long as $r \in [-1, 1]$, Examples $7.1.2c$ and $7.1.2d$ give us the following caution.

Caution: $\sin^{-1}(\sin(\theta))$

$\sin^{-1}(\sin(\theta))$ is not necessarily θ.

The following theorem codifies the properties of the inverse sine function.

Theorem: Properties of the Inverse Sine Function

Properties of $f(x) = \sin^{-1}(x)$:

- $\theta = \sin^{-1}(x)$ if and only if $\sin(\theta) = x$ where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- Domain: $[-1, 1]$
- Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\sin\left(\sin^{-1}(x)\right) = x$ for all $x \in [-1, 1]$
- $\sin^{-1}(\sin(x)) = x$ provided $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Checkpoint 7.1.2

Simplify each expression without using a calculator.

a. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$
b. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$
c. $\sin\left(\sin^{-1}(-0.389)\right)$

Answers available at the end of this section.

The Inverse Cosine and Inverse Tangent Functions

The cosine and tangent functions are also periodic, so, just as with the sine function, to define their inverse functions we must restrict their domains to intervals where they are one-to-one. The graph of cosine is shown in Figure $7.1.7$.

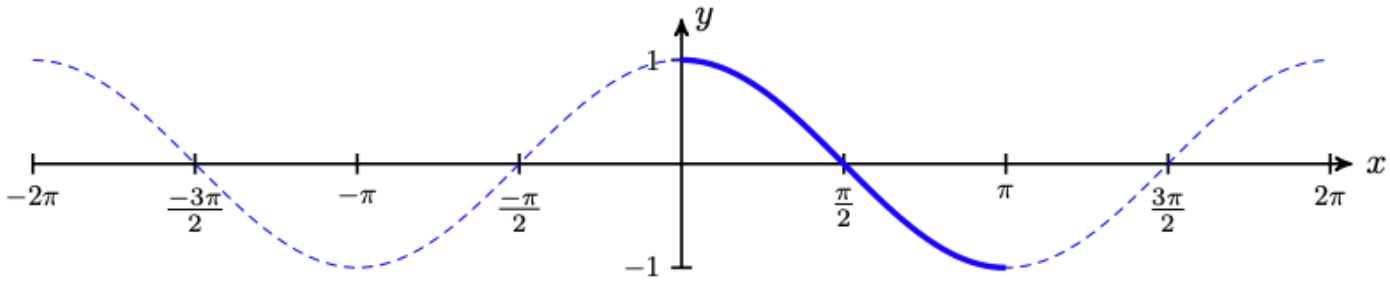


Figure $7.1.7$

Once again, the choice of these intervals is arbitrary. Suppose we start at $\theta = 0$ on the cosine graph. In that case, we can move in only one direction, either right or left, without encountering repeated y-values. As shown in Figure 7.1.7, we choose to move in the positive direction to obtain the interval $0 \leq \theta \leq \pi$. In this domain, the inverse of cosine is a function (see Figure 7.1.8).

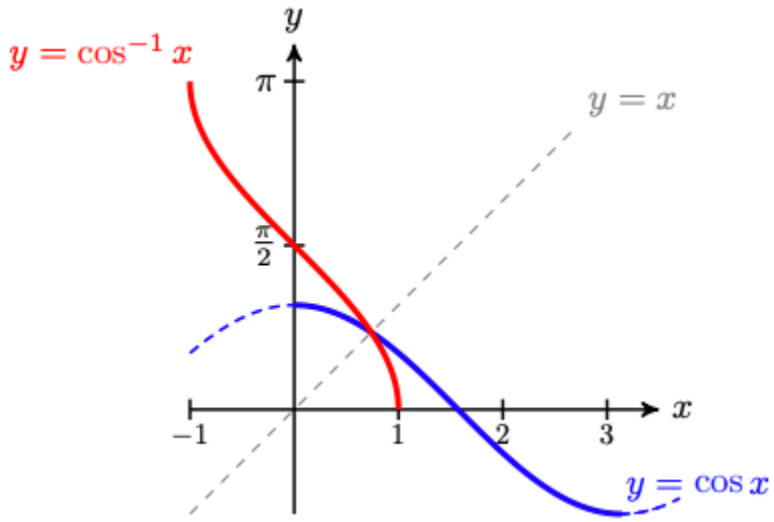


Figure 7.1.8

Definition: Inverse Cosine Function

The function $f(x) = \cos^{-1}(x)$ is defined as follows:

$$\cos^{-1}(x) = \theta \quad \text{if and only if} \quad \cos(\theta) = x \quad \text{and} \quad 0 \leq \theta \leq \pi.$$

The range of the inverse cosine function is $0 \leq y \leq \pi$, so it delivers angles in the first and second quadrants. (Compare to the inverse sine, whose outputs are angles in the first or fourth quadrants.)

Theorem: Properties of the Inverse Cosine Function

Properties of $f(x) = \cos^{-1}(x)$:

- $\theta = \cos^{-1}(x)$ if and only if $\cos(\theta) = x$ where $0 \leq \theta \leq \pi$
- Domain: $[-1, 1]$
- Range: $[0, \pi]$
- $\cos(\cos^{-1}(x)) = x$ for all $x \in [-1, 1]$
- $\cos^{-1}(\cos(x)) = x$ provided $x \in [0, \pi]$

Example 7.1.3

Simplify each expression without using a calculator.

a. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
b. $\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right)$

Solutions available at the end of this section.

Let's take a moment to highlight an important takeaway from Example 7.1.3b. First, we know that the inverse trigonometric functions *return* angles. What part (b) of 7.1.3 shows is that, given $\cos^{-1}(\cos(\alpha)) = \theta$, the reference angle for both θ and α are the same. That is, we had $\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right) = \theta$. The reference angle for $\frac{7\pi}{4}$ is $\frac{\pi}{4}$, which is exactly the reference angle $\hat{\theta}$.

This same principle holds for $\sin^{-1}(\sin(\alpha)) = \theta$. Therefore, given $\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$, we know this will return an angle θ such that $\hat{\theta} = \frac{\pi}{3}$.

The reason for this common reference angle stems from the definition of the inverse trigonometric function,

$$\sin^{-1}(\sin(\alpha)) = \theta \quad \text{if and only if} \quad \sin(\theta) = \sin(\alpha).$$

Therefore, $\hat{\alpha}$ must equal $\hat{\theta}$.

Checkpoint 7.1.3

Simplify $\cos^{-1}(-1)$ without using a calculator.

Answer available at the end of this section.

Finally, consider the graph of the tangent function. To choose a convenient interval on which the tangent is one-to-one, we start at $x = 0$ and move as far as we can in either direction along the x-axis. In this way, we obtain one cycle of the graph on the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, as shown in Figure 7.1.9.

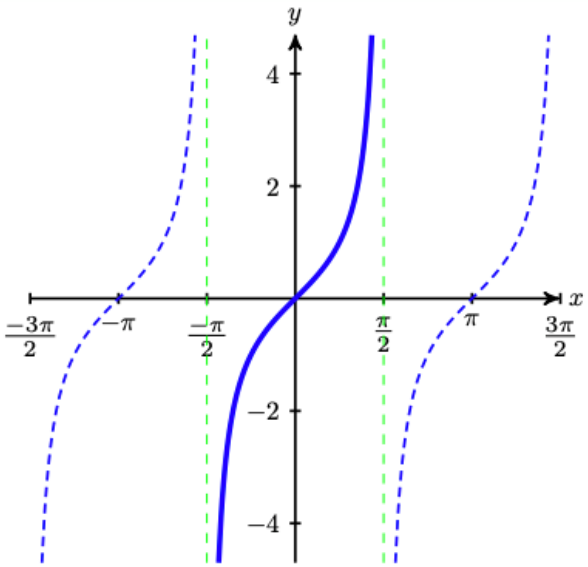


Figure 7.1.9

The range of the tangent on that interval includes all real numbers. Consequently, the domain of the inverse tangent function consists of all real numbers, and its range is the interval \(-\frac{\pi}{2}

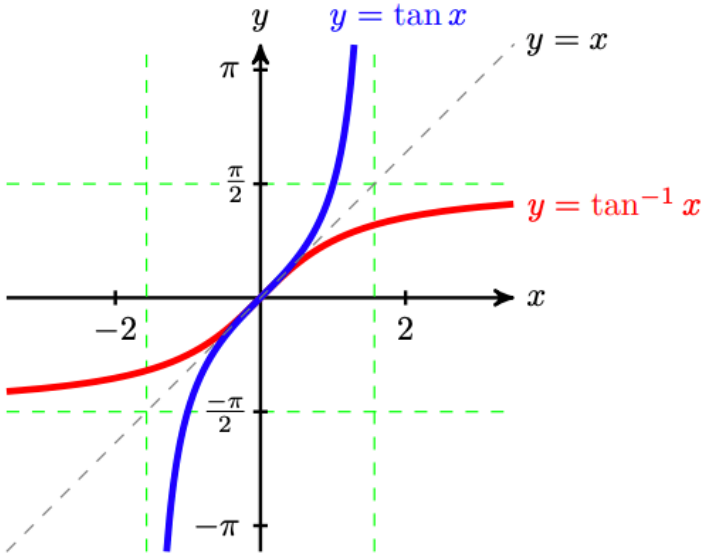


Figure 7.1.10

Note: The Inverse Tangent Graph

While you don't see the inverse sine and inverse cosine graphs in Mathematics beyond this point, the graph of the inverse tangent function is vital for Calculus concepts. This is a good one to be able to derive quickly.

Definition: The Inverse Tangent Function

The function $f(x) = \tan^{-1}(x)$ is defined as follows:

$$\tan^{-1}(x) = \theta \quad \text{if and only if} \quad \tan(\theta) = x \qquad \text{and} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Theorem: Properties of the Inverse Tangent Function

Properties of $f(x) = \tan^{-1}(x)$:

- $\theta = \tan^{-1}(x)$ if and only if $\tan(\theta) = x$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- Domain: $(-\infty, \infty)$
- Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$
- As $x \to -\infty$, $\tan^{-1}(x) \to -\frac{\pi}{2}$, and as $x \to \infty$, $\tan^{-1}(x) \to \frac{\pi}{2}$
- $\tan\left(\tan^{-1}(x)\right) = x$ for all $x \in \mathbb{R}$
- $\tan^{-1}\left(\tan(x)\right) = x$ provided $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Example 7.1.4

Simplify each expression without using a calculator.

a. $\tan^{-1}(-\sqrt{3})$
b. $\tan^{-1}\left(\tan\left(\frac{7\pi}{4}\right)\right)$

Solutions available at the end of this section.

Checkpoint 7.1.4

Simplify $\tan^{-1}(-1)$ without using a calculator.

Answer available at the end of this section.

Summary of Inverse Trigonometric Functions

Summarizing the inverse trigonometric functions and their properties before moving forward is helpful.

Summary: Inverse Trigonometric Functions

Key Concepts

Trigonometric functions take angles as arguments and return ratios. Inverse trigonometric functions take ratios as arguments and return (restricted) angles. That is,

$$\text{trig}\,(\text{angle}) = \text{ratio} \quad \text{and} \quad \text{trig}^{-1}\,(\text{ratio}) = \text{angle}.$$

To drive this point home, the rest of this summary uses r for "ratio" and θ for the angle.

Basic Definitions

$$\begin{aligned}
\sin^{-1}(r) = \theta \quad &\text{if and only if} \quad \sin(\theta) = r, \quad \text{where} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\
\cos^{-1}(r) = \theta \quad &\text{if and only if} \quad \cos(\theta) = r, \quad \text{where} \quad 0 \le \theta \le \pi \\
\tan^{-1}(r) = \theta \quad &\text{if and only if} \quad \tan(\theta) = r, \quad \text{where} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}
\end{aligned}$$

Domains and Ranges

Function	Domain	Range	Function	Domain	Range
$r = \sin(\theta)$	$\theta \in (-\infty, \infty)$	$r \in [-1, 1]$	$\theta = \sin^{-1}(r)$	$r \in [-1, 1]$	$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$r = \cos(\theta)$	$\theta \in (-\infty, \infty)$	$r \in [-1, 1]$	$\theta = \cos^{-1}(r)$	$r \in [-1, 1]$	$\theta \in [0, \pi]$
$r = \tan(\theta)$	$\theta \neq \frac{\pi}{2} + \pi k$	$r \in (-\infty, \infty)$	$\theta = \tan^{-1}(r)$	$r \in (-\infty, \infty)$	$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Invertibility Properties

$$\begin{aligned}
\sin\left(\sin^{-1}(x)\right) &= x \quad \text{for } -1 \le x \le 1 \\
\cos\left(\cos^{-1}(x)\right) &= x \quad \text{for } -1 \le x \le 1 \\
\tan\left(\tan^{-1}(x)\right) &= x \quad \text{for } -\infty < x < \infty \\
\sin^{-1}\left(\sin(x)\right) &= x \quad \text{only for } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\
\cos^{-1}\left(\cos(x)\right) &= x \quad \text{only for } 0 \le x \le \pi \\
\tan^{-1}\left(\tan(x)\right) &= x \quad \text{only for } -\frac{\pi}{2} < x < \frac{\pi}{2}
\end{aligned}$$

Advice for Use

When asked to find the value of an inverse trigonometric function, $\text{trig}^{-1}(r)$, we start by recalling that the inverse trigonometric functions return angles. Therefore, we write

$$\text{trig}^{-1}(r) = \theta.$$

Our goal is to find *an appropriately restricted angle*, θ, such that

$$\text{trig}(\theta) = r.$$

A good piece of advice is to find the *reference angle*, $\hat{\theta}$, by determining the value of

$$\text{trig}^{-1}(|r|) = \hat{\theta}$$

(Recall that *all* inverse trigonometric functions return angles in quadrant I if the argument of the inverse is positive.)

Once you have the reference angle, you can use the sign of r along with the knowledge of where your inverse trigonometric function returns angles to determine the value of $\theta = \text{trig}^{-1}(r)$.

The Invertibility Properties are tricky for most students. Just remember that the inverse trigonometric functions return restricted angles. That's why, for example, $\cos^{-1}(\cos(x))$ is not necessarily x.

Using a Calculator to Evaluate Inverse Trigonometric Functions

We have been working with the inverse trigonometric buttons on your calculator for quite some time; however, it is important to clarify how they are used once you encounter negative ratios as arguments of inverse trigonometric functions.

Example 7.1.5

Evaluate each of the following using a calculator. Your answers should be in radians.

a. $\sin^{-1}(0.97)$
b. $\cos^{-1}(-0.561)$
c. $\tan^{-1}(-3)$

Solutions available at the end of this section.

I want to be clear, your calculator would have returned the answers to Example $7.1.5b$ and $7.1.5c$ if you had just entered $\text{COS}^{-1}\ -0.561$ and $\text{TAN}^{-1}\ -3$, respectively; however, as we move forward, your calculator will become less reliable when it comes to giving you the answer you're looking for. This is why I recommend evaluating the inverse trigonometric functions at their arguments' absolute value and treating the result as a reference angle. You honestly cannot go wrong if you do this.

Checkpoint 7.1.5

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

Answer available at the end of this section.

Alternate Notations

The inverse sine function, $\sin^{-1}(x)$, is also called the **arcsine function** and denoted by $\arcsin(x)$. (This terminology reminds us that the output of the inverse sine function is an angle, or the arc on a unit circle determined by that angle, as shown in Figure 7.1.11.)

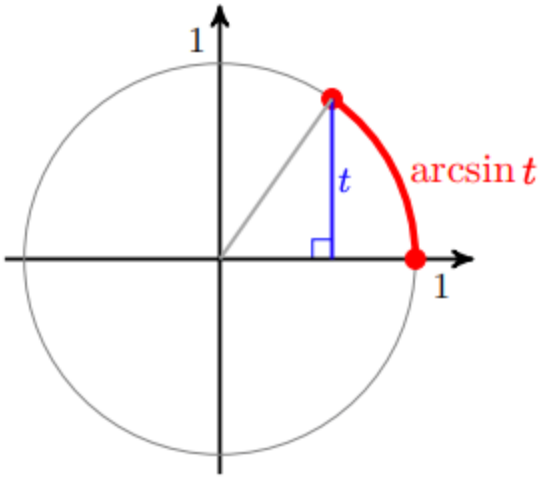


Figure 7.1.11

Similarly, the inverse cosine function is sometimes denoted by $\arccos(x)$, and the inverse tangent function by $\arctan(x)$.

Example 7.1.6

Simplify each expression.

a. $\cos\left(\arccos\left(\frac{1}{3}\right)\right)$
b. $\arcsin\left(\sin\left(\frac{3\pi}{2}\right)\right)$

Solutions available at the end of this section.

Checkpoint 7.1.6

Simplify each expression.

a. $\tan\left(\arctan\left(-\frac{5}{2}\right)\right)$
b. $\arctan\left(\tan\left(-\frac{5\pi}{6}\right)\right)$

Answers available at the end of this section.

Note: Do Arc Functions Return Angles?

Up to this point, we have been very clear that the inverse trigonometric functions return angles; however, very technically, the arcfunctions return *real numbers*—not angles. This is why many instructors prefer you to answer in radians when dealing with arcfunctions rather than degrees. Fortunately, this is a subtlety that you don't have to worry about. You can assume the arcfunctions, like the inverse trigonometric functions, return angles if that helps you remember them.

Simplifying Expressions Involving Different Trigonometric Functions

The key to simplifying expressions involving inverse trigonometric functions is to remember that the inverse sine, cosine, or tangent of a number can be treated as an angle. It can often clarify the computations if we assign a name such as θ or ϕ to the inverse trigonometric value. Additionally, drawing triangles on the Cartesian coordinate system can be immensely helpful.

Example 7.1.7

Evaluate $\sin\left(\cos^{-1}\left(\frac{3}{4}\right)\right)$.

Solution available at the end of this section.

Checkpoint 7.1.7

Evaluate $\cos\left(\tan^{-1}\left(\frac{2}{3}\right)\right)$.

Answer available at the end of this section.

We can verify the results of the previous example using a calculator, but the same technique can be applied to simplify similar expressions involving variables.

Example 7.1.8

Simplify $\tan\left(\sin^{-1}(x)\right)$, assuming that $-1 < x < 0$.

Solution available at the end of this section.

The answer to Example 7.1.8, $\frac{x}{\sqrt{1-x^2}}$, *looks* positive, but we know this cannot be the case because the tangent should be negative in QIV. Looks can be deceiving! While the denominator is definitely positive, the numerator, x, is negative. Therefore, the answer is, in fact, negative!

Checkpoint 7.1.8

Simplify $\sin\left(\tan^{-1}(z)\right)$, assuming that $z \geq 0$.

Answer available at the end of this section.

Simplifying Expressions Involving Identities

Now that we have learned all of the major identities in Trigonometry, we can use them with the inverse trigonometric functions. In doing so, we will demonstrate our fluency with inverse trigonometric functions and identities.

Example 7.1.9

a. Rewrite the following as an algebraic expression of x and state the domain on which the equivalence is valid.

$$f(x) = \cos\left(2\arcsin(x)\right)$$

b. Find the exact value of the expression.

$$\sin\left(\cos^{-1}\left(-\frac{1}{3}\right) + \sin^{-1}\left(\frac{2}{5}\right)\right)$$

Solutions available at the end of this section.

Checkpoint 7.1.9

Find the exact value of the expression.

$$\sin\left(2\cos^{-1}\left(-\frac{1}{3}\right)\right)$$

Answer available at the end of this section.

Modeling with Inverse Functions

The inverse trigonometric functions are used to model situations in which an angle is described in terms of one of its trigonometric ratios.

Example 7.1.10

The bottom of a 3-meter-tall tapestry on a chateau wall is at your eye level. The angle θ subtended vertically by the tapestry changes as you approach the wall.

a. Express your distance from the wall, d, as a function of θ.
b. Express θ as a function of d.

Solutions available at the end of this section.

In Example 7.1.10, we didn't have to worry about the quadrant in which θ belongs. Most (but not all) applications share this quality. If we are looking at the top of a tapestry on a wall, and if the bottom of the tapestry is at eye level, no matter how close or how far we are away from the tapestry, the angle of inclination from eye level to the top of the tapestry will always be acute (i.e., a quadrant I angle).

Checkpoint 7.1.10

The tapestry from the previous example includes a 2-meter-tall unicorn with its feet at the bottom of the tapestry.

a. Express α, the angle subtended vertically by the unicorn, as a function of d, your distance to the tapestry.
b. Express β, the angle subtended by the portion of the tapestry above the unicorn, as a function of d. (See the figure below).

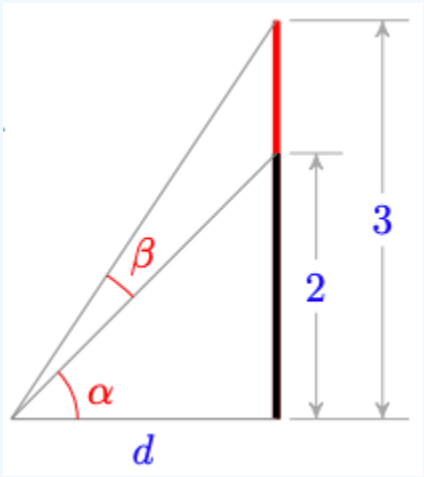


Figure 7.1.16

Answers available at the end of this section.

Success in Mathematics: Check Your Work

Checking your work is essential, but most students want to complete it and walk away from it. The fastest way to check your work in algebra is to plug your answers into the original problem. In trigonometry and calculus, this method may be a bit difficult. However, you can always check your solutions with graphs on the graphing calculator! It's a great idea.

1. Do you usually check your work?
2. Have you used the graphing calculator to check your work, or do you usually check "by hand?"

Miscellaneous Topics: Teachers and Role Models

Parents, teachers, and other role models influence our attitudes about math. What they say and do can change our view of the importance of math, how hard math is, and whether we are good at math. Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Describe the attitudes towards math of your parent, teacher (not me, of course), or another personal role model.
2. Describe how the opinion of this role model has influenced your attitudes about math.

Answers

Section Preview

When writing parts of this text, I was trying my best to stick with the mantra, "Don't introduce it, unless they are going to use it." While we did introduce the inverse trigonometric functions back when we covered Right Triangle Trigonometry, it was strictly from a "button-mashing on a calculator" perspective for *positive* ratios. Hence, from a student perspective, those pesky buttons on their calculator (or the arc-functions in Desmos) only returned angles between 0 and 90° (or, when we revisited them after learning about radian measure, between 0 and $\frac{\pi}{2}$). We had another visit of the inverse trigonometric functions when we discussed reference angles. At that time, I was very careful to tip-toe around the truth of inverse trigonometric functions and just told the reader to evaluate inverse trigonometric functions at absolute values of the given ratios. In doing so, I mentioned that this would return a reference angle. This section dives into the deep waters and discusses what inverse trigonometric functions truly are for the big three (sine, cosine, and tangent). We need this theory before diving into solving trigonometric equations.

Section Preview

- **An Overview of Functions**
  - Inverse Functions: This entire section assumes students understand the concept of an inverse function (and how to find the inverse of a function algebraically and graphically).
  - Restricting Domains and Forcing Invertibility: Since the trigonometric functions are not naturally invertible, we must use domain restrictions to help us create invertible versions of these functions. Students should have some experience with this already.

Section Preview

- **An Overview of Functions**
  - Function Composition and Domain: Function composition is needed for the composition evaluations in Examples 7–9.

Section Preview

- Find the exact value of an inverse trigonometric function.
- Use technology to approximate the value of an inverse trigonometric function.
- Evaluate compositions involving trigonometric functions and inverse trigonometric functions.
- Simplify expressions involving trigonometric functions and inverse trigonometric functions.

Example $7.1.1$

a. All three graphs pass the Vertical Line Test, so all three represent functions.
b. Only the function represented by Graph II passes the Horizontal Line Test, so it is the only one-to-one function.
c. Only function II is one-to-one, so it is the only function that has an inverse function.

Checkpoint $7.1.1$

I and III

Example $7.1.2$

a. By definition, $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\theta$ is an angle whose sine is $-\frac{\sqrt{3}}{2}$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. That is, $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\theta$ means $\sin(\theta)=-\frac{\sqrt{3}}{2}$ and $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. While you might be able to quickly state the value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ right now, we are going to take our time showcasing the thought process so that we can work with more complicated problems.

   Quadrant: Because $\sin(\theta)=-\frac{\sqrt{3}}{2}$ is negative, we further restrict θ to the interval $\left[-\frac{\pi}{2},0\right]$. In fact, since the value of the sine function is not ±1 or 0, we know θ is not a quadrantal angle, so $\theta\in\left(-\frac{\pi}{2},0\right)$. Hence, $\theta\in\text{QIV}$.

   Reference Angle: Since $\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$, the reference angle for θ is $\hat{\theta}=\frac{\pi}{3}$.

Now that we have the quadrant and reference angle, we can state the value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. $\theta \in (-\frac{\pi}{2}, 0)$ with a reference angle of $\hat{\theta} = \frac{\pi}{3}$ means

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}.$$

b. $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \theta$ means $\sin(\theta) = -\frac{1}{\sqrt{2}}$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Quadrant: Since the sine is negative, we need $\theta \in (-\frac{\pi}{2}, 0)$.

Reference Angle: We know the reference angle for $\sin(\theta) = -\frac{1}{\sqrt{2}}$ is $\hat{\theta} = \frac{\pi}{4}$.

A reference angle of $\frac{\pi}{4}$ in the interval $(-\frac{\pi}{2}, 0)$ means

$$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}.$$

c. As promised, the method we used in parts (a) and (b) can be used for more complicated problems (like this one).

$\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) = \theta$ means $\sin(\theta) = \sin\left(\frac{5\pi}{6}\right)$ where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. A lot of people would say that θ must therefore be $\frac{5\pi}{6}$; however, $\frac{5\pi}{6}$ is not in the interval between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Quadrants and reference angles to the rescue!

Quadrant: Since $\sin(\theta) = \sin\left(\frac{5\pi}{6}\right)$ is positive (because $\frac{5\pi}{6} \in$ QII), $\theta \in (0, \frac{\pi}{2})$. Hence, $\theta \in$ QI.

Reference Angle: Since $\sin(\theta) = \sin\left(\frac{5\pi}{6}\right)$, θ must have the same reference angle as $\frac{5\pi}{6}$. That is, $\hat{\theta} = \frac{\pi}{6}$.

So, we know $\theta \in$ QI with a reference angle of $\frac{\pi}{6}$. Hence,

$$\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) = \frac{\pi}{6}.$$

d. $\sin^{-1}(\sin(\pi)) = \theta$ means $\sin(\theta) = \sin(\pi)$, where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

This is one of those moments where quadrants and reference angles will not work. This is because π is a quadrantal angle. In these cases, we simply evaluate $\sin(\pi) = 0$. Therefore, $\sin^{-1}(\sin(\pi)) = \sin^{-1}(0)$. We want an angle whose sine is 0, and which lies in the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. This angle is 0. Therefore, $\sin^{-1}(\sin(\pi)) = 0$.

e. We finally reach an example where the inverse trigonometric function is *the argument* of another function. Our approach will be the same as before—we start by focusing on the inverse trigonometric function.

$\sin^{-1}(0.2) = \theta$ means $\sin(\theta) = 0.2$, where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Believe it or not, we have actually found the value of $\sin(\sin^{-1}(0.2))$. Reread the statement above, and then follow the next line:

$$\sin\left(\sin^{-1}(0.2)\right) = \sin(\theta) = 0.2.$$

This illustrates the fact that

$$\sin\left(\sin^{-1}(r)\right) = r$$

as long as $r \in [-1, 1]$.

Checkpoint 7.1.2

a. $\frac{\pi}{3}$
b. $\frac{\pi}{3}$
c. -0.389

Example 7.1.3

a. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \theta$ means $\cos(\theta) = -\frac{\sqrt{3}}{2}$, where $\theta \in [0, \pi]$.

Quadrant: Since $\cos(\theta) = -\frac{\sqrt{3}}{2}$ is negative, we further restrict θ so that $\theta \in (\frac{\pi}{2}, \pi)$. Thus, $\theta \in$ QII.

Reference Angle: To find the reference angle, we ask ourselves what acute angle, $\hat{\theta}$, results in

$$\cos\left(\hat{\theta}\right) = \left|-\frac{\sqrt{3}}{2}\right| = \frac{\sqrt{3}}{2}.$$

From our special triangles, we know this is $\hat{\theta} = \frac{\pi}{6}$.

The angle in the second quadrant with this reference angle is $\theta = \frac{5\pi}{6}$, so

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}.$$

b. $\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right) = \theta$ means $\cos(\theta) = \cos\left(\frac{7\pi}{4}\right)$, where $\theta \in [0, \pi]$.

Quadrant: Because $\frac{7\pi}{4} \in$ QIV, we know that $\cos(\theta) = \cos\left(\frac{7\pi}{4}\right)$ is positive. Therefore, $\theta \in (0, \frac{\pi}{2})$.

Reference Angle: Since $\cos(\theta) = \cos\left(\frac{7\pi}{4}\right)$, the reference angle for θ must be equal to the reference angle for $\frac{7\pi}{4}$. That is, $\hat{\theta} = \frac{\pi}{4}$.

We have a reference angle of $\frac{\pi}{4}$ in Quadrant I, this means

$$\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right)=\frac{\pi}{4}.$$

Checkpoint 7.1.3

π

Example 7.1.4

a. $\tan^{-1}(-\sqrt{3})=\theta$ means $\tan(\theta)=-\sqrt{3}$, where $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$.

Quadrant: Since $\tan(\theta)$ is negative, we further restrict the angle to $\theta\in(-\frac{\pi}{2},0)$, or Quadrant IV.

Reference Angle: Since $\tan\left(\frac{\pi}{3}\right)=\sqrt{3}$, $\hat{\theta}=\frac{\pi}{3}$.

Combining our results, we get

$$\tan^{-1}(-\sqrt{3})=-\frac{\pi}{3}.$$

b. $\tan^{-1}\left(\tan\left(\frac{7\pi}{4}\right)\right)=\theta$ means $\tan(\theta)=\tan\left(\frac{7\pi}{4}\right)$, where $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$.

Quadrant: Because $\frac{7\pi}{4}$ is in Quadrant IV, $\tan\left(\frac{7\pi}{4}\right)$ is negative. Hence, $\tan(\theta)$ is negative. As such, we further restrict the angle to $\theta\in(-\frac{\pi}{2},0)$.

Reference Angle: Since $\tan(\theta)=\tan\left(\frac{7\pi}{4}\right)$, both arguments must share the same reference angle. Therefore, $\hat{\theta}=\frac{\pi}{4}$.

Thus,

$$\tan^{-1}\left(\tan\left(\frac{7\pi}{4}\right)\right)=-\frac{\pi}{4}.$$

Checkpoint 7.1.4

$-\frac{\pi}{4}$

Example 7.1.5

a. This is a straightforward use of the calculator. At this point, you should be familiar enough with this to know we enter $\text{SIN}^{-1}\ 0.97$ in the calculator to get approximately 1.3252 radians; however, we are going to use this to link back to the *quadrant* and *reference angle* concept we have been building throughout this section.

$\sin^{-1}(0.97)=\theta$ means $\sin(\theta)=0.97$, where $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$.

Quadrant: Since the sine is positive, we further restrict our angle to $\theta\in(0,\frac{\pi}{2})$, or Quadrant I.

Reference Angle: The reference angle can be found by using your calculator (in radian mode) to evaluate $\sin^{-1}(|0.97|)\approx 1.3252$.

So our angle is in Quadrant I with a reference angle of $\hat{\theta}\approx 1.3252$. This means,

$$\sin^{-1}(0.97)\approx 1.3252.$$

b. $\cos^{-1}(-0.561)=\theta$ means $\cos(\theta)=-0.561$, where $\theta\in[0,\pi]$.

Quadrant: Since $\cos(\theta)$ is negative, we can guarantee that $\theta\in$ QII (more specifically, $\theta\in(\frac{\pi}{2},\pi)$).

Reference Angle: To find the *reference angle* for θ, we compute $\hat{\theta}=\cos^{-1}(|-0.561|)=\cos^{-1}(0.561)\approx 0.9752$ radians.

Now that we have a quadrant (II) and a reference angle (0.9752), we find the actual angle by subtracting the reference angle from π. Thus,

$$\cos^{-1}(-0.561)\approx\pi-0.9752\approx 2.1664.$$

c. $\tan^{-1}(-3)=\theta$ means $\tan(\theta)=-3$, where $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$.

Quadrant: Since the tangent is negative, we can restrict this interval to $\theta\in(-\frac{\pi}{2},0)$, or Quadrant IV.

Reference Angle: The reference angle is

$$\hat{\theta}=\tan^{-1}(|-3|)=\tan^{-1}(3)\approx 1.2490.$$

Thus,

$$\tan^{-1}(-3)\approx -1.2490.$$

Checkpoint 7.1.5

1.9823 radians or 113.578°

Example 7.1.6

a. $\arccos\left(\frac{1}{3}\right)=\theta$ means $\cos(\theta)=\frac{1}{3}$, where $0\le\theta\le\pi$. Since $\frac{1}{3}\in[-1,1]$, we can use the Invertibility Property to state

$$\cos\left(\arccos\left(\frac{1}{3}\right)\right)=\cos(\theta)=\frac{1}{3}.$$

b. $\arcsin\left(\sin\left(\frac{3\pi}{2}\right)\right)=\theta$ means $\sin(\theta)=\sin\left(\frac{3\pi}{2}\right)$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Noticing that $\frac{3\pi}{2}$ is a quadrantal angle, we go ahead and simplify the right side of that last equation.

$$\sin(\theta) = \sin\left(\frac{3\pi}{2}\right) = -1$$

Therefore, we require an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that the sine of that angle is -1. This would be $-\frac{\pi}{2}$. Thus,

$$\arcsin\left(\sin\left(\frac{3\pi}{2}\right)\right) = -\frac{\pi}{2}.$$

Checkpoint 7.1.6

a. $-\frac{5}{2}$

b. $\frac{\pi}{6}$

Example 7.1.7

Notice this is a mixture of trigonometric functions. Our process changes only slightly from before.

$\cos^{-1}\left(\frac{3}{4}\right) = \theta$ means $\cos(\theta) = \frac{3}{4}$, where $\theta \in [0, \pi]$.

Quadrant: Since the cosine of θ is positive, we know $\theta \in \text{QI}$. *Because we have a mixture of trigonometric functions*, it's best to draw a reference triangle in QI.

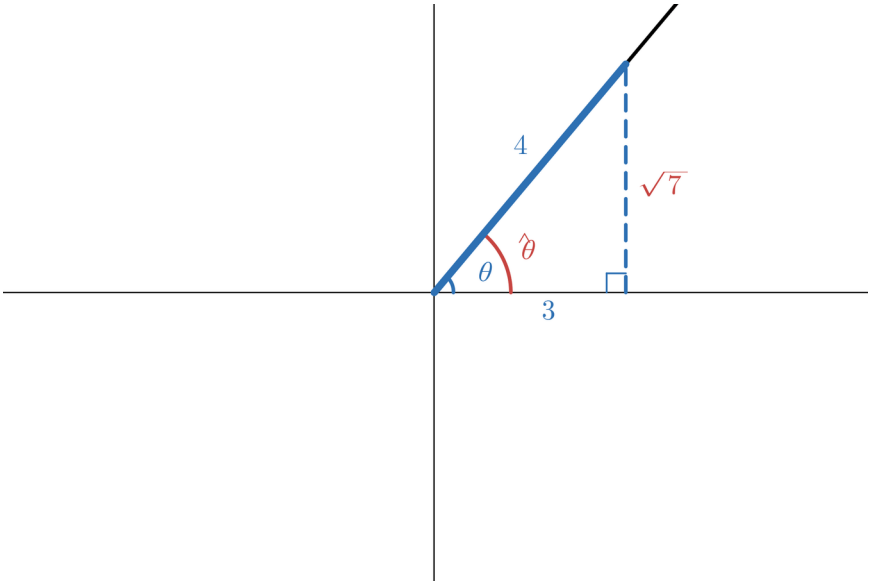


Figure 7.1.12

We now can use this reference triangle to answer the question (i.e., there is no need to find the reference angle).

$\sin\left(\cos^{-1}\left(\frac{3}{4}\right)\right)$ simplifies to $\sin(\theta)$. Using the Pythagorean Theorem and the definition for sine, we get

$$\sin\left(\cos^{-1}\left(\frac{3}{4}\right)\right) = \sin(\theta) = \frac{\sqrt{7}}{4}.$$

Checkpoint 7.1.7

$\frac{3}{\sqrt{13}}$

Example 7.1.8

Let $\theta = \sin^{-1}(x)$, so that $\sin(\theta) = x = \frac{x}{1}$, and the expression $\tan\left(\sin^{-1}(x)\right) = \tan(\theta)$. Since $-1 < x < 0$, sine is negative. Therefore, the inverse sine returns an angle in QIV. Specifically, $-\frac{\pi}{2} \leq \theta \leq 0$. Drawing a reference triangle in QIV, we get

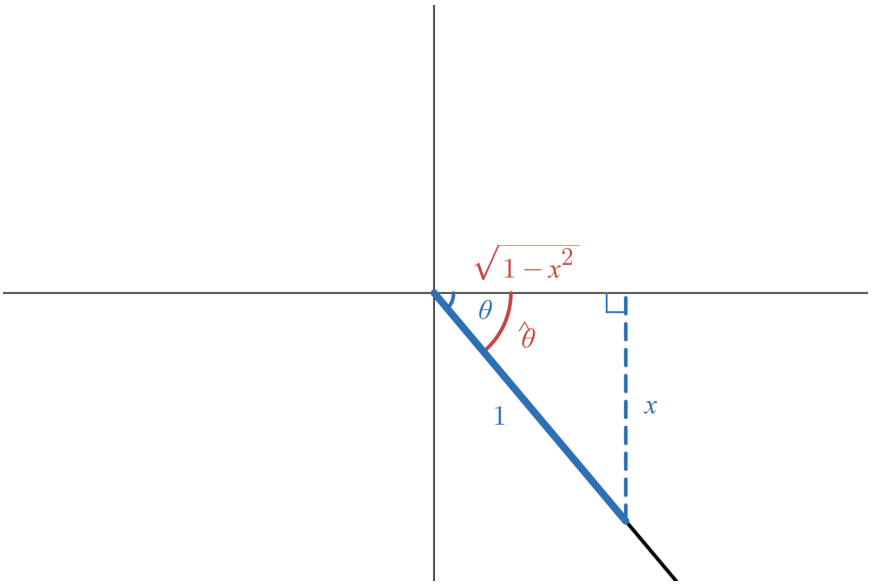


Figure 7.1.13

Now we use the fact that $\tan(\theta)$ is opposite over adjacent to get

$$\tan\left(\sin^{-1}(x)\right) = \tan(\theta) = \frac{x}{\sqrt{1-x^2}}.$$

Checkpoint 7.1.8

$$\frac{z}{\sqrt{1+z^2}}$$

Example 7.1.9

a. We start by writing $\theta = \arcsin(x)$ so that θ lies in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\sin(\theta) = x$. We want to express $\cos(2\arcsin(x)) = \cos(2\theta)$ in terms of x. Since $\cos(2\theta)$ is defined everywhere, we get no additional restrictions on θ. We have three choices for rewriting $\cos(2\theta)$:

$$\cos^2(\theta) - \sin^2(\theta), \quad 2\cos^2(\theta) - 1, \quad \text{and} \quad 1 - 2\sin^2(\theta).$$

Since we know $x = \sin(\theta)$, it is easiest to use the last form:

$$\cos(2\arcsin(x)) = \cos(2\theta) = 1 - 2\sin^2(\theta) = 1 - 2x^2.$$

To find the restrictions on x, we consider our substitution, $\theta = \arcsin(x)$. Since $\arcsin(x)$ is defined only for $-1 \le x \le 1$, the function, $f(x) = \cos(2\arcsin(x)) = 1 - 2x^2$, is valid only on $[-1, 1]$.

b. Let $\alpha = \cos^{-1}(-\frac{1}{3})$ and $\beta = \sin^{-1}(\frac{2}{5})$. Then we can rewrite the expression as

$$\sin\left(\cos^{-1}\left(-\frac{1}{3}\right) + \sin^{-1}\left(\frac{2}{5}\right)\right) = \sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta),$$

where

$$\cos(\alpha) = -\frac{1}{3}, \quad 0 \le \alpha \le \pi$$

and

$$\sin(\beta) = \frac{2}{5}, \quad -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}.$$

Since $\cos(\alpha)$ is negative, we immediately know $\alpha \in$ QII. Moreover, $\sin(\beta)$ being positive means that $\beta \in$ QI. Drawing reference triangles, we get the following figures:

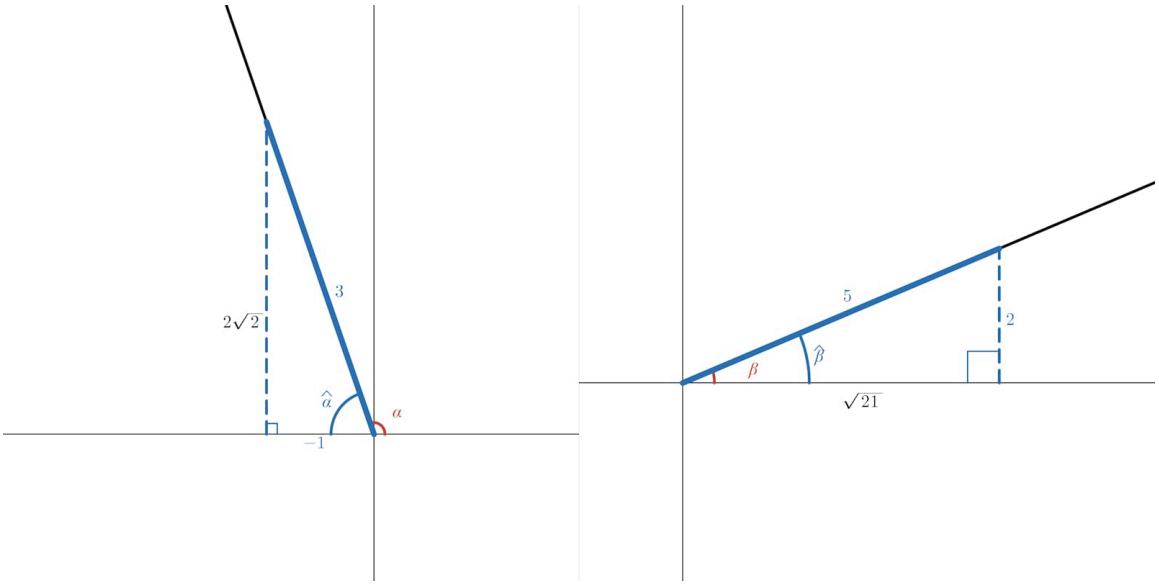


Figure 7.1.14

We now evaluate the equivalent expression, $\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$, using these reference triangles.

$$\begin{aligned}\sin\left(\cos^{-1}\left(-\frac{1}{3}\right)+\sin^{-1}\left(\frac{2}{5}\right)\right) &= \sin(\alpha+\beta)\\ &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\\ &= \left(\frac{2\sqrt{2}}{3}\right)\left(\frac{\sqrt{21}}{5}\right)+\left(-\frac{1}{3}\right)\left(\frac{2}{5}\right)\\ &= \frac{2\sqrt{42}}{15}-\frac{2}{15}\\ &= \frac{2\sqrt{42}-2}{15}\end{aligned}$$

Checkpoint 7.1.9

$-\frac{4\sqrt{2}}{9}$

Example 7.1.10

a. As shown below, we sketch the triangle formed by the tapestry and the lines of sight to its bottom and top. From the triangle we see that $\tan(\theta)=\frac{3}{d}$, so $d=\frac{3}{\tan(\theta)}$.

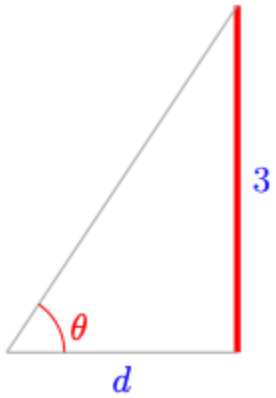


Figure 7.1.15

b. Because $\tan(\theta)=\frac{3}{d}$, $\theta=\tan^{-1}\left(\frac{3}{d}\right)$.

Checkpoint 7.1.10

a. $\alpha=\tan^{-1}\left(\frac{2}{d}\right)$

b. $\beta=\tan^{-1}\left(\frac{3}{d}\right)-\tan^{-1}\left(\frac{2}{d}\right)$

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7.1.1: Resources and Key Concepts

Resources

Prerequisite Topics

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **An Overview of Functions**
  - Function Composition and Domain: This skill is necessary for evaluating and simplifying composite functions involving trigonometric and inverse trigonometric functions, such as $\sin(\arccos(x))$.
  - Inverse Functions: This section is built on the concept of inverse functions. Finding a formula for an inverse, using the Horizontal Line Test to determine if a function has an inverse function, and understanding the relationship between the domain and range of a function and its inverse are all critical prerequisite skills. These are reviewed in the "Skills Refresher" exercises.
  - Restricting Domains and Forcing Invertibility: This section's primary theoretical basis is the idea of restricting the domains of the non-invertible trigonometric functions to create new functions that are one-to-one and thus have inverse functions.

Videos

- Inverse Trigonometric Functions
  - Implicit Differentiation - Derivatives of Inverse Trig Functions: Theorem, Proof, and Example
  - Inverse Trigonometric Functions - Defining the Inverse Trigonometric Functions
  - Inverse Trigonometric Functions - Evaluating Inverse Trigonometric Functions
  - Inverse Trigonometric Functions - Applications of Right Triangle Trigonometry (Part 2)

Key Concepts

Definitions

- **The Inverse Sine Function:** The function $f(x) = \sin^{-1}(x)$ (or $\arcsin(x)$) is defined as $\sin^{-1}(x) = \theta$ if and only if $\sin(\theta) = x$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
- **The Inverse Cosine Function:** The function $f(x) = \cos^{-1}(x)$ (or $\arccos(x)$) is defined as $\cos^{-1}(x) = \theta$ if and only if $\cos(\theta) = x$ and $0 \leq \theta \leq \pi$.
- **The Inverse Tangent Function:** The function $f(x) = \tan^{-1}(x)$ (or $\arctan(x)$) is defined as $\tan^{-1}(x) = \theta$ if and only if $\tan(\theta) = x$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Theorems

- **Lemma:** Let θ be an angle in standard position, and suppose the terminal side of θ intersects the unit circle at the point $P(x, y)$. Then $\cos(\theta) = x$ and $\sin(\theta) = y$.
- **Forcing Invertibility by Restricting Domain:** If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.
- **Properties of the Inverse Sine Function:** For $f(x) = \sin^{-1}(x)$:
  - Domain: $[-1, 1]$
  - Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$
  - $\sin(\sin^{-1}(x)) = x$ for all x in $[-1, 1]$
  - $\sin^{-1}(\sin(x)) = x$ provided x is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

- **Properties of the Inverse Cosine Function:** For $f(x) = \cos^{-1}(x)$:
  - Domain: $[-1, 1]$
  - Range: $[0, \pi]$
  - $\cos(\cos^{-1}(x)) = x$ for all x in $[-1, 1]$
  - $\cos^{-1}(\cos(x)) = x$ provided x is in $[0, \pi]$
- **Properties of the Inverse Tangent Function:** For $f(x) = \tan^{-1}(x)$:
  - Domain: $(-\infty, \infty)$
  - Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$
  - $\tan(\tan^{-1}(x)) = x$ for all x in $\mathbb{R}$
  - $\tan^{-1}(\tan(x)) = x$ provided x is in $(-\frac{\pi}{2}, \frac{\pi}{2})$

Common Mistakes

- **Inverse Composition Property:** Be cautious with compositions like $\sin^{-1}(\sin(\theta))$. This expression is not necessarily equal to θ. It is only equal to θ if θ is within the restricted range of the inverse sine function, $[-\frac{\pi}{2}, \frac{\pi}{2}]$. A similar caution applies to inverse cosine and inverse tangent.
- **Inverse Trigonometric Functions are NOT Reciprocals:** The notation $\sin^{-1}(x)$ denotes the inverse sine function, not the reciprocal of sine. The reciprocal of $\sin(x)$ is $\csc(x)$.
- **Forgetting Domain and Range Restrictions:** The inverse trigonometric functions have specific, restricted ranges. For example, the output of $\arccos(x)$ must be an angle between 0 and π, inclusive. Forgetting these restrictions is a common source of error.

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7.1.2: Homework

Reading Questions

1. What graphical test is used to determine if a function is one-to-one?
2. Why is it necessary to restrict the domain of the sine function before we can define its inverse function?
3. What is the restricted domain of the cosine function that is used to define the inverse cosine function?
4. What is the domain of the function $y = \arcsin(x)$?
5. What is the range of the function $y = \arccos(x)$?
6. What is the range of the function $y = \arctan(x)$?
7. The expression $\sin^{-1}(0.5)$ represents an angle. In which quadrant does this angle lie?
8. The expression $\cos^{-1}(-0.5)$ represents an angle. In which quadrant does this angle lie?
9. Is the statement $\sin^{-1}(\sin(3\pi)) = 3\pi$ true or false? Explain your reasoning.
10. Is the statement $\sin(\sin^{-1}(0.5)) = 0.5$ true or false? Explain your reasoning.
11. What is the key difference between the meaning of $\sin^{-1}(x)$ and $(\sin(x))^{-1}$?
12. When simplifying an expression like $\cos(\arctan(x))$, what is a helpful first step?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, find the inverse of the function, if it exists.

1. $f(x) = 2x^2 - 16x + 1$
2. $g(x) = (x-2)^3$
3. $f(x) = \dfrac{3x-1}{5x-1}$

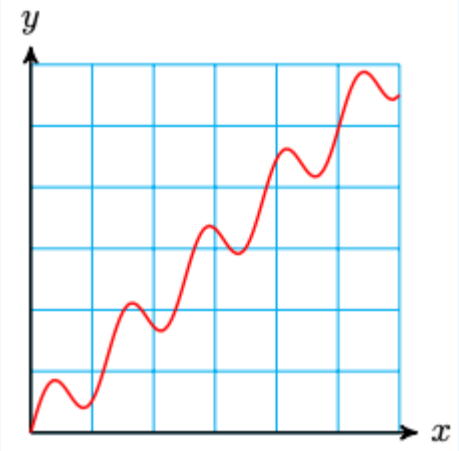
For the following exercises,
(a) Find a formula for the inverse function.
(b) State the domain and range of the inverse function.
(c) Graph the function and its inverse on the same grid.

4. $f(x) = \frac{1}{2}x - 4$
5. $g(x) = 3x + 6$
6. $F(x) = 2 + \frac{1}{x}$
7. $G(x) = \frac{1}{x+5}$
8. $h(x) = \sqrt{x+2}$
9. $H(x) = 3 + \sqrt[3]{x}$

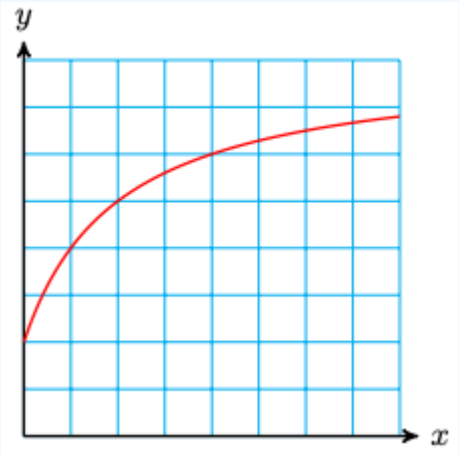
For the following exercises, which functions have an inverse function? Explain your answer.

10.

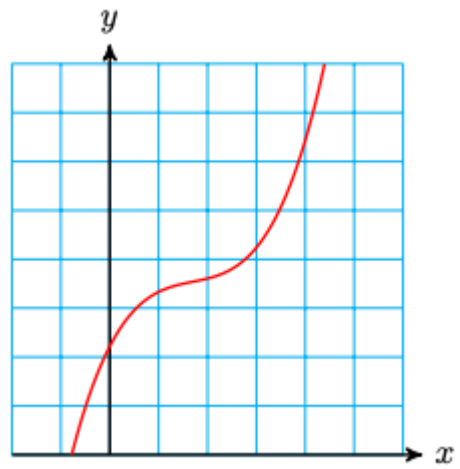
11.

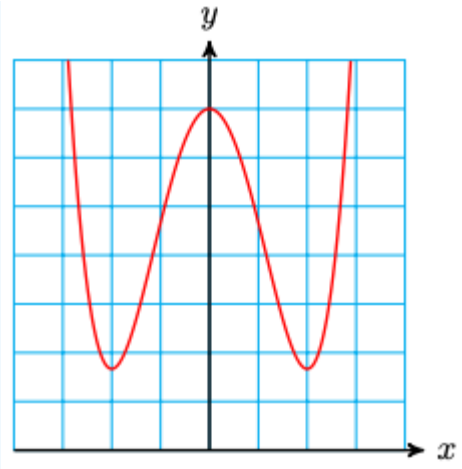


12.



13.





For the following exercises, use technology to graph the function and decide if it has an inverse function.

14. $f(x) = \sin(2x) - \cos(x)$
15. $g(x) = 4e^{-(x/4)^2}$
16. $G(x) = \sqrt{25 - x^2}$
17. $F(x) = \ln(x^3 + 8)$

Answers available at the end of this section.

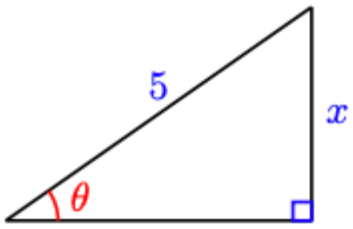
Homework

Concept Check

1. The following is a table of values defining a function f. Make a table of values for f^{-1}.

x	-3	-2	0	1	4
$f(x)$	6	3	1	0	-1

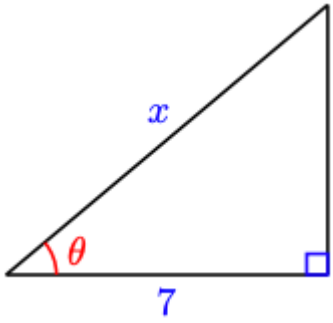
2. What does it mean for a function to be one-to-one? Give an example.
3. Why do we restrict the domains of the trigonometric functions when we define their inverse functions?
4. Which of the following expressions is undefined? Why?
   a. $\cos^{-1}(0)$
   b. $\arctan(-2)$
   c. $\arcsin(-2)$
5. Use your calculator to evaluate $\sin(118°)$, then evaluate $\sin^{-1}$ of your answer. Explain the result.
6.



a. Write x as a function of θ.

b. Write θ as a function of x.

7.



a. Write x as a function of θ.

b. Write θ as a function of x.

Basic Skills

For the following exercises,
(a) Evaluate each pair of angles to the nearest 0.1 radian, and show that they are supplements.
(b) Sketch both angles.
(c) Find the sine of each angle.

8. $\theta = \cos^{-1}\left(\frac{3}{4}\right), \phi = \cos^{-1}\left(-\frac{3}{4}\right)$

9. $\theta = \cos^{-1}\left(\frac{1}{5}\right), \phi = \cos^{-1}\left(-\frac{1}{5}\right)$

10. $\theta = \cos^{-1}(0.1525), \quad \phi = \cos^{-1}(-0.1525)$

11. $\theta = \cos^{-1}(0.6825), \quad \phi = \cos^{-1}(-0.6825)$

For the following exercises, use a calculator to evaluate. Round your answers to the nearest tenth of a radian.

12. $\sin^{-1}(0.2838)$

13. $\tan^{-1}(4.8972)$

14. $\cos^{-1}(0.6894)$

15. $\arccos(-0.8134)$

16. $\arctan(-1.2765)$

17. $\arcsin(-07493)$

For the following exercises, give exact values.

18. $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

19. $\tan^{-1}(-1)$

20. $\sin^{-1}\left(-\frac{1}{2}\right)$

21. $\arccos\left(\frac{\sqrt{3}}{2}\right)$

22. $\arctan\left(\frac{1}{\sqrt{3}}\right)$

23. $\arcsin(-1)$

24. $\arccos(-1)$

25. $\arctan\left(\sqrt{3}\right)$

26. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$

27. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

28. $\arctan\left(-\sqrt{3}\right)$

29. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

30. $\arccos(0)$

31. $\arctan(0)$

32. $\arcsin(0)$

33. $\arccos\left(\frac{1}{2}\right)$

34. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

35. $\arcsin\left(\frac{1}{2}\right)$

36. $\arccos\left(\frac{\sqrt{2}}{2}\right)$

37. $\arctan(1)$

38. $\arcsin\left(\frac{\sqrt{2}}{2}\right)$

39. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$

40. $\arcsin(1)$

41. $\arccos(1)$

For the following exercises, without using a calculator, find the exact value or state that it is undefined.

42. $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$

43. $\sin\left(\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right)$

44. $\sin\left(\arcsin\left(\frac{3}{5}\right)\right)$

45. $\sin\left(\arcsin\left(-0.42\right)\right)$

46. $\sin\left(\arcsin\left(\frac{5}{4}\right)\right)$

47. $\cos\left(\arccos\left(\frac{\sqrt{2}}{2}\right)\right)$

48. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$

49. $\cos\left(\arccos\left(\frac{5}{13}\right)\right)$

50. $\cos\left(\arccos\left(-0.998\right)\right)$

51. $\cos\left(\arccos\left(\pi\right)\right)$

52. $\tan\left(\arctan\left(-1\right)\right)$

53. $\tan\left(\arctan\left(\sqrt{3}\right)\right)$

54. $\tan\left(\arctan\left(\frac{5}{12}\right)\right)$

55. $\tan\left(\arctan\left(0.965\right)\right)$

56. $\tan\left(\arctan\left(3\pi\right)\right)$

For the following exercises, find the exact value without using a calculator.

57. $\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right)$

58. $\tan\left(\cos^{-1}\left(\frac{3}{4}\right)\right)$

59. $\cos\left[\tan^{-1}(-2)\right]$

60. $\sin\left[\tan^{-1}\left(-\frac{3}{\sqrt{5}}\right)\right]$

61. $\sin\left[\cos^{-1}\left(-\frac{2\sqrt{6}}{7}\right)\right]$

62. $\cos\left[\sin^{-1}\left(-\frac{2}{7}\right)\right]$

63. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right)$

64. $\sin\left(\arccos\left(\frac{3}{5}\right)\right)$

65. $\sin\left(\arctan\left(-2\right)\right)$

66. $\cos\left(\arcsin\left(-\frac{5}{13}\right)\right)$

67. $\cos\left(\arctan\left(\sqrt{7}\right)\right)$

68. $\tan\left(\arcsin\left(-\frac{2\sqrt{5}}{5}\right)\right)$

69. $\tan\left(\arccos\left(-\frac{1}{2}\right)\right)$

70. $\cot\left(\arcsin\left(\frac{12}{13}\right)\right)$

71. $\cot\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$

72. $\cot\left(\arctan\left(0.25\right)\right)$

73. $\sec\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$

74. $\sec\left(\arcsin\left(-\frac{12}{13}\right)\right)$

75. $\sec\left(\arctan\left(10\right)\right)$

76. $\csc\left(\arcsin\left(\frac{3}{5}\right)\right)$

77. $\csc\left(\arctan\left(-\frac{2}{3}\right)\right)$

For the following exercises, simplify the expression and state the domain on which the equivalence is valid.

78. $\tan\left(\cos^{-1}(x)\right)$

79. $\cos\left(\sin^{-1}(h)\right)$

80. $\sin\left(\arccos\left(x\right)\right)$

81. $\cos\left(\arctan\left(x\right)\right)$

82. $\tan\left(\arcsin\left(x\right)\right)$

83. $\sec\left(\arctan\left(x\right)\right)$

84. $\csc\left(\arccos\left(x\right)\right)$

85. $\sin\left(\tan^{-1}(2t)\right)$

86. $\tan\left(\sin^{-1}(3b)\right)$

For the following exercises, find the exact value or state that it is undefined.

87. $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right)$

88. $\arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right)$

89. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$

90. $\arcsin\left(\sin\left(\frac{11\pi}{6}\right)\right)$

91. $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right)$

92. $\arccos\left(\cos\left(\frac{\pi}{4}\right)\right)$

93. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right)$

94. $\arccos\left(\cos\left(\frac{3\pi}{2}\right)\right)$

95. $\arccos\left(\cos\left(-\frac{\pi}{6}\right)\right)$

96. $\arccos\left(\cos\left(\frac{5\pi}{4}\right)\right)$

97. $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right)$

98. $\arctan\left(\tan\left(-\frac{\pi}{4}\right)\right)$

99. $\arctan\left(\tan\left(\pi\right)\right)$

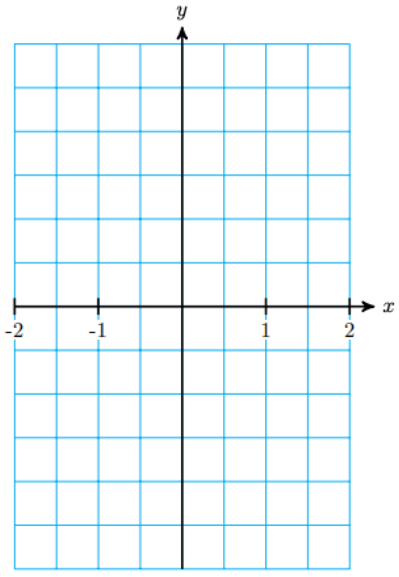
100. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$

101. $\arctan\left(\tan\left(\frac{2\pi}{3}\right)\right)$

For the following exercises, complete the table of values and sketch the function.

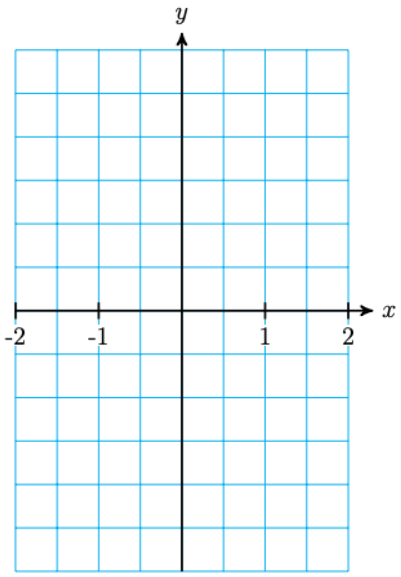
102.

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos^{-1} x$									



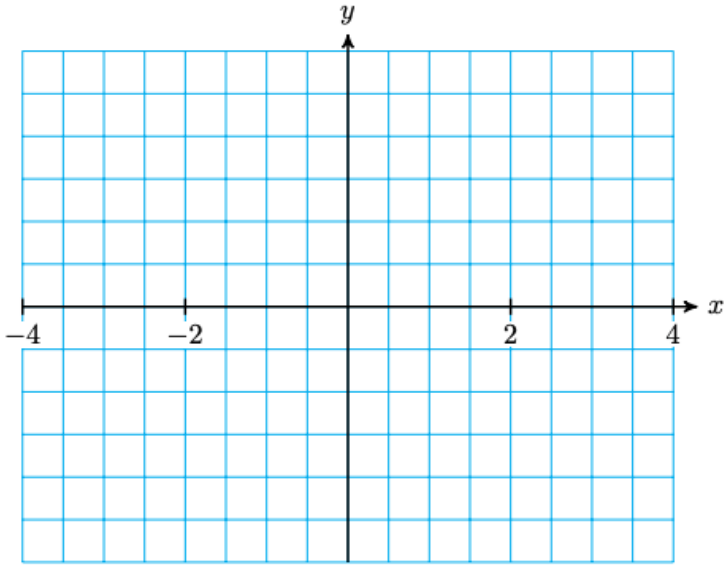
103.

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\sin^{-1} x$									



94.

x	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$\tan^{-1} x$							



95. Use a graphing calculator to answer each of the following questions. Then explain the results.

a. Does $\cos^{-1}(x) = \frac{1}{\cos(x)}$?

b. Does $\sin^{-1}(x) = \frac{1}{\sin(x)}$?

c. Does $\tan^{-1}(x) = \frac{1}{\tan(x)}$?

96. a. Sketch a graph of $y = \cos^{-1}(x)$, and label the scales on the axes.

b. Use transformations to sketch graphs of $y = 2\cos^{-1}(x)$ and $y = \cos^{-1}(2x)$.

c. Does $2\cos^{-1}(x) = \cos^{-1}(2x)$?

97. a. Sketch a graph of $y = \sin^{-1}(x)$, and label the scales on the axes.

b. Use transformations to sketch graphs of $y = \frac{1}{2}\sin^{-1}(x)$ and $y = \cos^{-1}\left(\frac{1}{2}x\right)$.

c. Does $\frac{1}{2}\sin^{-1}(x) = \cos^{-1}\left(\frac{1}{2}x\right)$?

98. a. Sketch a graph of $y = \tan^{-1}(x)$, and label the scales on the axes.

b. Use your calculator to graph $y = \frac{\sin^{-1}(x)}{\cos^{-1}(x)}$ on a suitable domain.

c. Does $\tan^{-1}(x) = \frac{\sin^{-1}(x)}{\cos^{-1}(x)}$?

99. a. Use your calculator to sketch $y = \sqrt[3]{x}$ and $\tan^{-1}(x)$ on $[-10, 10]$.

b. Describe the similarities and differences in the two graphs.

For the following exercises, find the exact value for the expression.

100. $\sin\left(\arcsin\left(\frac{5}{13}\right) + \frac{\pi}{4}\right)$

101. $\tan\left(\arctan(3) + \arccos\left(-\frac{3}{5}\right)\right)$

102. $\sin\left(2\arcsin\left(-\frac{4}{5}\right)\right)$

103. $\sin\left(2\arctan\left(2\right)\right)$

104. $\cos\left(2\arcsin\left(\frac{3}{5}\right)\right)$

105. $\sin\left(\frac{\arctan(2)}{2}\right)$

106. $\sin\left(2\tan^{-1}(4)\right)$

107. $\cos\left(2\sin^{-1}\left(\frac{5}{13}\right)\right)$

108. $\tan\left(2\cos^{-1}\left(\frac{3}{4}\right)\right)$

109. $\sin\left(2\cos^{-1}\left(-\frac{4}{5}\right)\right)$

110. $\tan\left(2\sin^{-1}\left(\frac{1}{3}\right)\right)$

111. $\cos\left(2\tan^{-1}\left(\frac{3}{2}\right)\right)$

112. Let $\alpha = \cos^{-1}\left(-\frac{4}{5}\right), \beta = \sin^{-1}\left(\frac{5}{13}\right)$. Find exact values for the following.

a. $\cos(\alpha + \beta)$

b. $\sin(\alpha + \beta)$

c. $\cos(\alpha - \beta)$

d. $\sin(\alpha - \beta)$

113. Let $\alpha = \sin^{-1}\left(-\frac{15}{17}\right), \beta = \tan^{-1}\left(\frac{4}{3}\right)$. Find exact values for the following.

a. $\cos(\alpha + \beta)$

b. $\sin(\alpha + \beta)$

c. $\cos(\alpha - \beta)$

d. $\sin(\alpha - \beta)$

24. Find an exact value for $\sin\left(\tan^{-1}\left(\frac{3}{4}\right) + \sin^{-1}\left(-\frac{4}{5}\right)\right)$.

25. Find an exact value for $\cos\left(\tan^{-1}\left(\frac{5}{12}\right) + \sin^{-1}\left(-\frac{3}{5}\right)\right)$.

For the following exercises, rewrite the expression as an algebraic expression without trigonometric functions.

26. $\sin\left(2\tan^{-1}(x)\right)$

27. $\cos\left(2\sin^{-1}(x)\right)$

28. $\sin\left(2\cos^{-1}(w)\right)$

29. $\cos\left(2\tan^{-1}(w)\right)$

30. $\sin(\arccos(2x))$

31. $\sin\left(\arccos\left(\frac{x}{5}\right)\right)$

32. $\cos\left(\arcsin\left(\frac{x}{2}\right)\right)$

33. $\cos\left(\arctan\left(3x\right)\right)$

34. $\sin(2\arcsin(7x))$

35. $\sin\left(2\arcsin\left(\frac{x\sqrt{3}}{3}\right)\right)$

36. $\cos(2\arcsin(4x))$

37. $\sec(\arctan(2x))\tan(\arctan(2x))$

38. $\sin\left(\arcsin(x) + \arccos(x)\right)$

39. $\cos\left(\arcsin(x) + \arctan(x)\right)$

40. $\tan\left(2\arcsin(x)\right)$

41. $\sin\left(\frac{1}{2}\arctan(x)\right)$

42. If $x = 5\sin(\theta), 0^\circ < \theta < 90^\circ$, express $\sin(2\theta)$ and $\cos(2\theta)$ in terms of x.

43. If $x - 1 = 2\cos(\theta), 0^\circ < \theta < 90^\circ$, express $\sin(2\theta)$ and $\cos(2\theta)$ in terms of x.

44. If $x = 3\tan(\theta)$, write $\theta + \frac{1}{4}\sin(2\theta)$ in terms of x.

45. If $x = 5\cos(\theta)$, write $\frac{\theta}{2} - \cos(2\theta)$ in terms of x.

46. If $\sin(\theta) = \dfrac{x}{2}$ for $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, find an expression for $\theta + \sin(2\theta)$ in terms of x.

47. If $\tan(\theta) = \dfrac{x}{7}$ for $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, find an expression for $\dfrac{1}{2}\theta - \dfrac{1}{2}\sin(2\theta)$ in terms of x.

48. If $\sec(\theta) = \dfrac{x}{4}$ for $0 < \theta < \dfrac{\pi}{2}$, find an expression for $4\tan(\theta) - 4\theta$ in terms of x.

49. a. For what values of x is the function $f(x) = \sin(\arcsin(x))$ defined?

b. Is $\sin(\arcsin(x)) = x$ for all x where it is defined? If not, for what values of x is the equation false?

c. For what values of x is the function $g(x) = \arcsin(\sin(x))$ defined?

d. Is $\arcsin(\sin(x)) = x$ for all x where it is defined? If not, for what values of x is the equation false?

50. a. For what values of x is the function $f(x) = \cos(\arccos(x))$ defined?

b. Is $\cos(\arccos(x)) = x$ for all x where it is defined? If not, for what values of x is the equation false?

c. For what values of x is the function $g(x) = \arccos(\cos(x))$ defined?

d. Is $\arccos(\cos(x)) = x$ for all x where it is defined? If not, for what values of x is the equation false?

Synthesis Questions

For the following exercises, state the domain of the function.

51. $f(x) = \arcsin(5x)$

52. $f(x) = \arccos\left(\frac{3x-1}{2}\right)$

53. $f(x) = \arcsin\left(2x^2\right)$

54. $f(x) = \arccos\left(\frac{1}{x^2-4}\right)$

55. $f(x) = \arctan(4x)$

Applications

For the following exercises, sketch a figure to help you model each problem.

56. Ivan is watching the launch of a satellite at Cape Canaveral. The viewing area is 500 yards from the launch site. The angle of elevation, θ, of Ivan's line of sight increases as the booster rocket rises.

a. Write a formula for the height, h, of the rocket as a function of θ.

b. Write a formula for θ as a function of h.

c. Evaluate the formula in part (b) for $h = 1000$, and interpret the result.

57. Camille's house lies under the flight path from the city airport, and commercial airliners pass overhead at an altitude of 15,000 feet. As Camille watches an airplane recede, its angle of elevation, θ, decreases.

a. Write a formula for the horizontal distance, d, to the airplane as a function of θ.

b. Write a formula for θ as a function of d.

c. Evaluate the formula in part (b) for $d = 20,000$, and interpret the result.

58. While driving along the interstate, you approach an enormous 50-foot-wide billboard that sits just beside the road. Your viewing angle, θ, increases as you get closer to the billboard.

a. Write a formula for your distance, d, from the billboard as a function of θ.

b. Write a formula for θ as a function of d.

c. Evaluate the formula in part (b) for $d = 200$, and interpret the result.

59. Sang is walking along the bank of a straight river toward a 200-meter-long bridge over the river. Let θ be the angle subtended horizontally by Sang's view of the bridge.

    a. Write a formula for Sang's distance from the bridge, d, as a function of θ.

    b. Write a formula for θ as a function of d.

    c. Evaluate the formula in part (b) for $d = 500$, and interpret the result.

60. Wyatt is viewing a 4-meter tall painting whose base is 1 meter above his eye level.

    a. Write a formula for α, the angle subtended from Wyatt's eye level to the bottom of the painting, when he stands x meters from the wall.

    b. Write a formula for β, the angle subtended by the painting, in terms of x.

    c. Evaluate the formula in part (b) for $x = 5$, and interpret the result.

61. A 5-foot-tall mirror is positioned so that its bottom is 1.5 feet below Kim's eye level.

    a. Write a formula for α, the angle subtended by the section of the mirror below Kim's eye level, when she stands x feet from the mirror.

    b. Write a formula for θ, the angle subtended by the entire mirror, in terms of x.

    c. Evaluate the formula in part (b) for $x = 10$, and interpret the result.

Challenge Problems

62. Use your calculator to graph $y = \sin^{-1}(x) + \cos^{-1}(x)$.

    a. State the domain and range of the graph.

    b. Explain why the graph looks as it does.

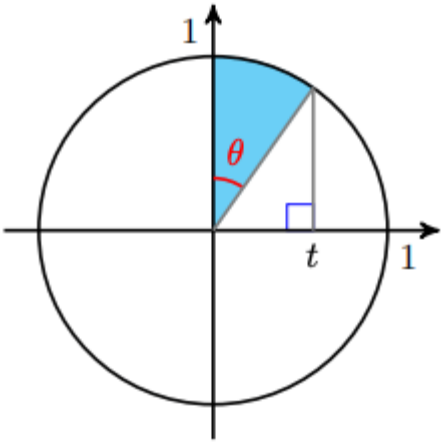
63. Use your calculator to graph $y = \tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)$.

    a. State the domain and range of the graph.

    b. Explain why the graph looks as it does.

64. Show that $\arcsin(x) + \arccos(x) = \dfrac{\pi}{2}$ for $-1 \leq x \leq 1$.

65. Use the figure of a unit circle to answer the following.

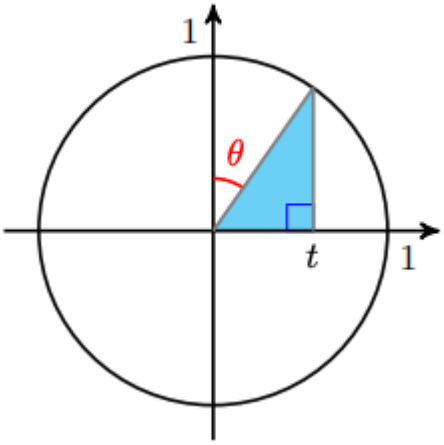


a. Write an expression for the area of the shaded sector in terms of θ.

b. How are θ and t related in the figure? (Hint: Write an expression for $\sin(\theta)$.)

c. Combine your answers to (a) and (b) to write an expression for the area of the sector in terms of t.

56. Use the figure of a unit circle to answer the following.



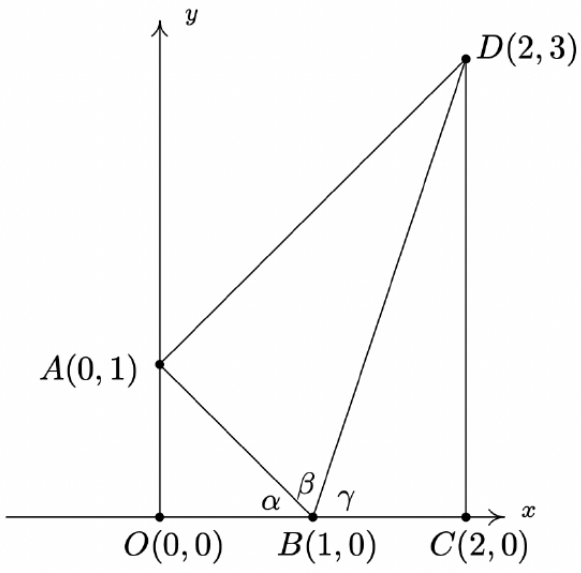
a. Write an expression for the height of the shaded triangle in terms of t. (Hint: Use the Pythagorean Theorem.)

b. Write an expression for the area of the triangle in terms of t.

c. Combine your answers to (b) and to the previous problem to write an expression for the area bounded above by the upper semicircle, below by the x-axis, on the left by the y-axis, and on the right by $x = t$, when $0 \leq t \leq 1$.

57. Use the following picture and the series of exercises on the next page to show that

$$\arctan(1) + \arctan(2) + \arctan(3) = \pi$$



a. Clearly $\triangle AOB$ and $\triangle BCD$ are right triangles because the line through O and A and the line through C and D are perpendicular to the x-axis. Use the distance formula to show that $\triangle BAD$ is also a right triangle (with $\angle BAD$ being the right angle) by showing that the sides of the triangle satisfy the Pythagorean Theorem.

b. Use $\triangle AOB$ to show that $\alpha = \arctan(1)$

c. Use $\triangle BAD$ to show that $\beta = \arctan(2)$

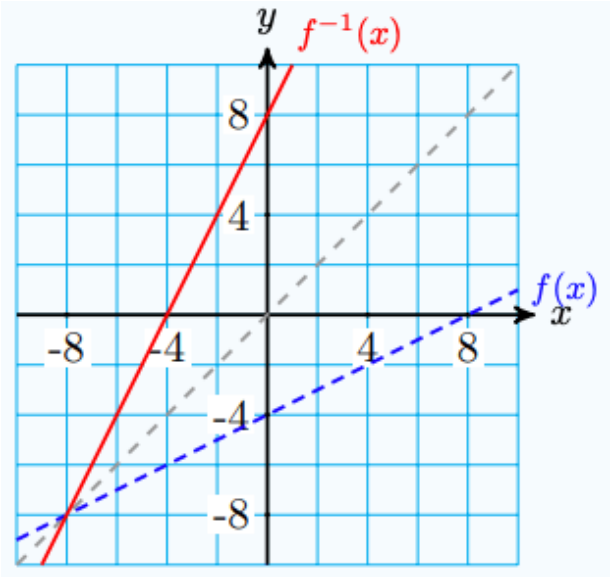
d. Use $\triangle BCD$ to show that $\gamma = \arctan(3)$

e. Use the fact that O, B and C all lie on the x-axis to conclude that $\alpha + \beta + \gamma = \pi$. Thus $\arctan(1) + \arctan(2) + \arctan(3) = \pi$.

Answers

Skills Refresher

1. No inverse exists for this function.
2. $g^{-1}(x) = \sqrt[3]{x} + 2$
3. $\dfrac{x-1}{5x-3}$
4.
   a. $f^{-1}(x) = 2x + 8$
   b. Dom: $(-\infty, \infty)$ Rge: $(-\infty, \infty)$
   c.

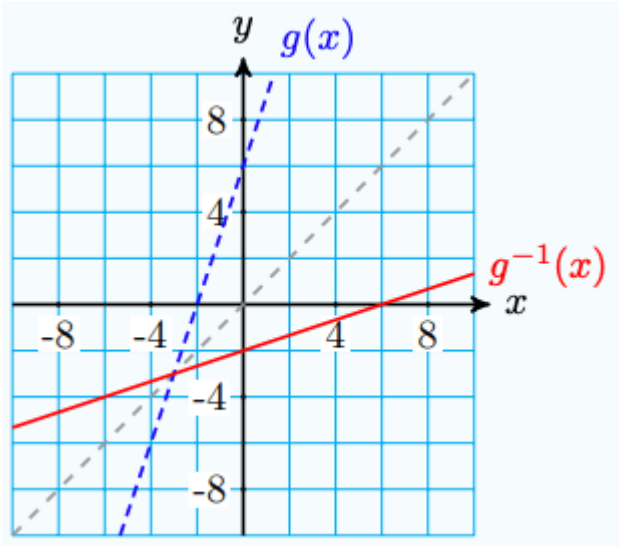


5.

a. $g^{-1}(x) = \frac{1}{3}x - 2$

b. Dom: $(-\infty, \infty)$ Rge: $(-\infty, \infty)$

c.

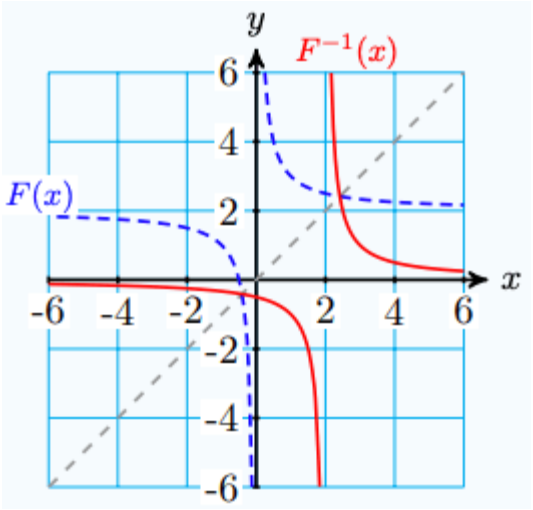


6.

a. $F^{-1}(x) = \frac{1}{x-2}$

b. Dom: $x \neq 0$ Rge: $y \neq 2$

c.

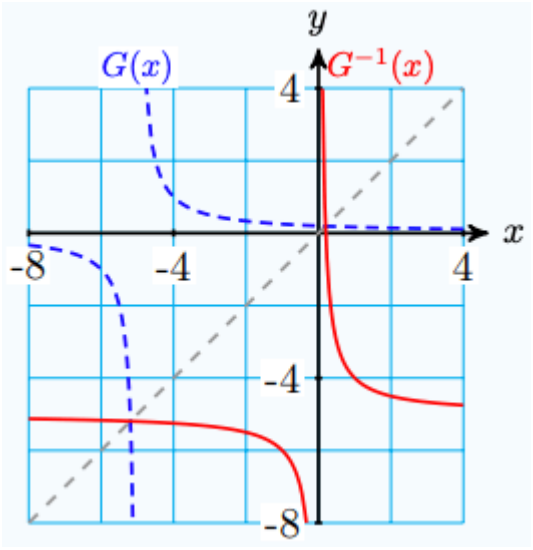


7.

a. $G^{-1}(x) = \frac{1}{x} - 5$

b. Dom: $x \neq -5$ Rge: $y \neq 0$

c.

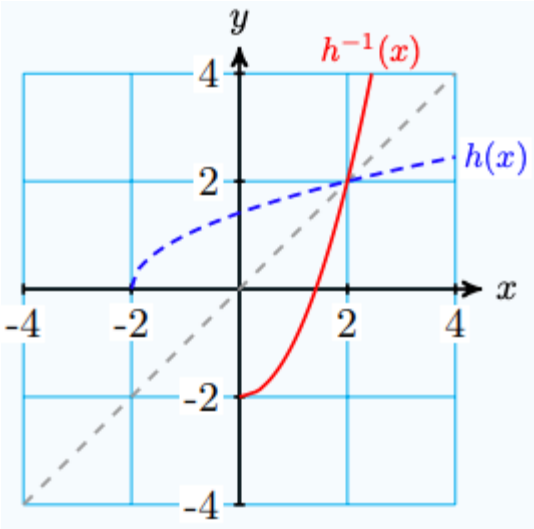


8.

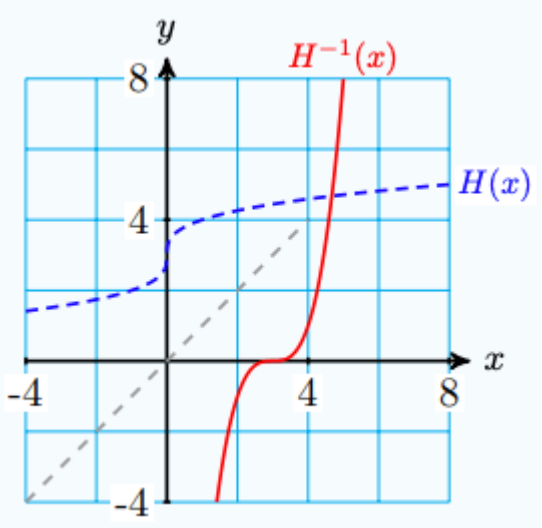
a. $h^{-1}(x) = x^2 - 2$

b. Dom: $x \geq 0$ Rge: $y \geq -2$

c.



9.
   a. $H^{-1}(x) = (x-3)^3$
   b. Dom: $(-\infty, \infty)$ Rge: $(-\infty, \infty)$
   c.



10. No. This graph does not pass the Horizontal Line Test.
11. Yes. This graph passes the Horizontal Line Test.
12. Yes. This graph passes the Horizontal Line Test.
13. No. This graph does not pass the Horizontal Line Test.
14. No.
15. No.
16. No.
17. Yes.

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7.2: The Remaining Inverse Trigonometric Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

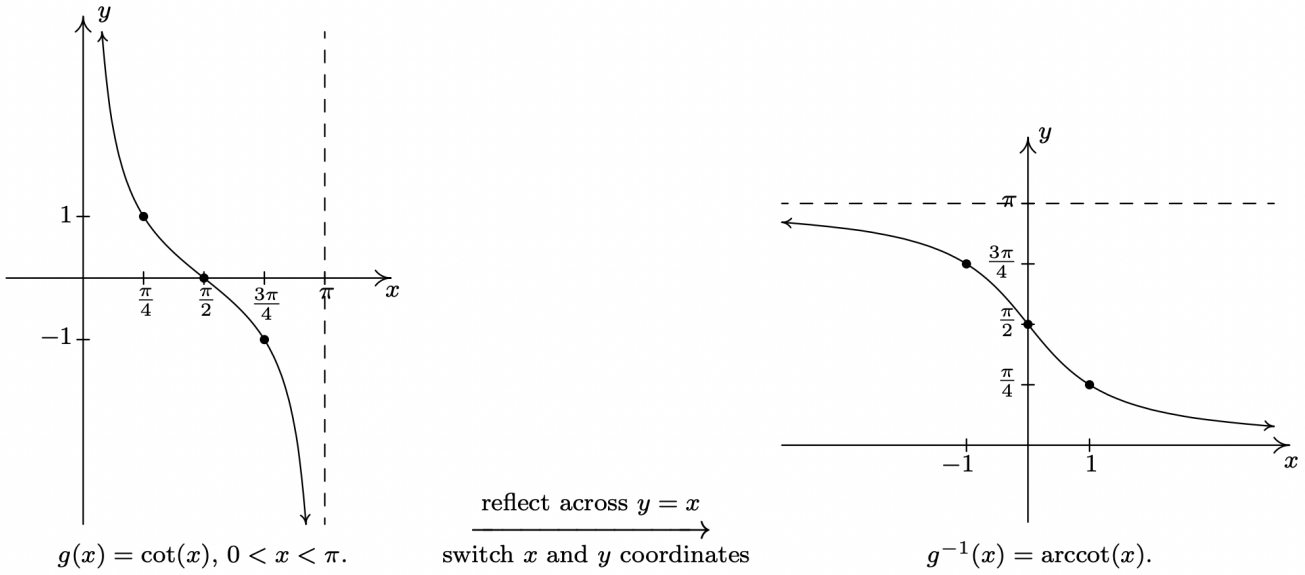
The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

In the previous section, we completely ignored the inverses of the secant, cosecant, and cotangent. This is not because they are unimportant—in fact, the inverses of the secant and cosecant are essential parts of Integral Calculus (also known as Calculus II). We have delayed their introduction because it's best to master the inverses of the fundamental trigonometric functions before jumping into the remaining inverse trigonometric functions.

The Inverse Cotangent Function

Like the rest of the trigonometric functions, the cotangent is not naturally one-to-one. Therefore, we restrict the domain of $g(x) = \cot(x)$ to its principal cycle on $(0, \pi)$ to obtain a one-to-one segment of the cotangent. The vertical asymptotes, $x = 0$ and $x = \pi$, of the graph of $g(x) = \cot(x)$ become the horizontal asymptotes, $y = 0$ and $y = \pi$, of the graph of $g^{-1}(x) = \operatorname{arccot}(x)$ (which, recall, is a reflection of the graph of $y = \cot(x)$ about the line $y = x$).



$g(x) = \cot(x),\ 0 < x < \pi.$ $\qquad$ $g^{-1}(x) = \operatorname{arccot}(x).$

Figure 7.2.1

The definition of the arccotangent follows the same pattern as the definitions of our other inverse trigonometric functions.

Definition: Inverse Cotangent Function

The function $f(x) = \cot^{-1}(x)$ is defined as follows:

$$\cot^{-1}(x) = \theta \quad \text{if and only if} \quad \cot(\theta) = x \quad \text{and} \quad 0 < \theta < \pi.$$

Again, I want to be clear: Very technically, the function $\cot^{-1}(x)$ returns an *angle* and $\operatorname{arccot}(x)$ returns a *real number*. This is true for all inverse trigonometric functions. Therefore, it is valid for $\cot^{-1}(x)$ to return an angle in degree measure or radian measure;

however, it is technically not valid for $\operatorname{arccot}(x)$ to return a number in degree measure (as this is an angle). Since radians is a unitless measure, a radian is just a real number, so $\operatorname{arccot}(x)$ can return a number in radian measure. The moral of the story is that you are always safe to work in radian measure when dealing with inverse trigonometric functions, but working in degree measure is a little risky.

Theorem: Properties of the Inverse Cotangent Function

Properties of $f(x) = \cot^{-1}(x)$

- $\theta = \cot^{-1}(x)$ if and only if $\cot(\theta) = x$ where $0 < \theta < \pi$
- Domain: $(-\infty, \infty)$
- Range: $(0, \pi)$
- As $x \to -\infty$, $\cot^{-1}(x) \to \pi^-$, and as $x \to \infty$, $\cot^{-1}(x) \to 0^+$
- $\cot(\cot^{-1}(x)) = x$ for all $x \in \mathbb{R}$
- $\cot^{-1}(\cot(x)) = x$ provided $0 < x < \pi$

Example 7.2.1

a. Find the exact values of the following.
   i. $\operatorname{arccot}(-\sqrt{3})$
   ii. $\cot(\operatorname{arccot}(-5))$

b. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.

$$\cos(\operatorname{arccot}(2x))$$

Solutions available at the end of this section.

Checkpoint 7.2.1

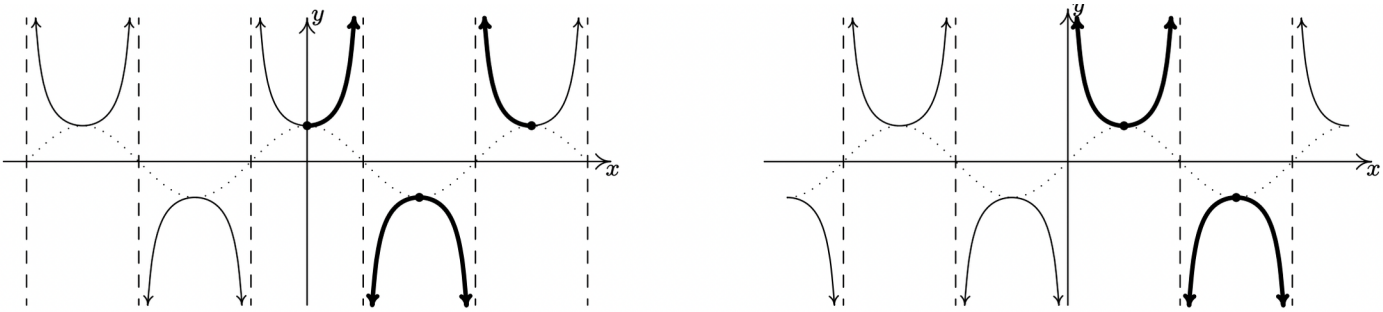
Find the exact value of the expression.

$$\sin\left(\cot^{-1}\left(-\frac{5}{2}\right)\right)$$

Answer available at the end of this section.

The Inverse Secant and Cosecant Functions

The last two functions to invert are secant and cosecant. The principal cycle for each is highlighted in the figure below.



The graph of $y = \sec(x)$. The graph of $y = \csc(x)$.

Figure 7.2.3

It is clear from the graph of secant that we cannot find one single, continuous piece of its graph which covers its entire range of

$(-\infty, -1] \cup [1, \infty)$, and which restricts the domain of the function so that it is one-to-one. The same is true for cosecant. Thus, in order to define the arcsecant and arccosecant functions, we must settle for a piecewise approach wherein we choose one piece to cover the top of the range, namely $[1, \infty)$, and another piece to cover the bottom, namely $(-\infty, -1]$. To make the definitions of the inverse secant and cosecant functions align with our previous work, we restrict their domains to coincide with the restrictions on cosine and sine, respectively.

It is critical to note that there are two ways to restrict the domains of the secant and cosecant functions. The method shown in this text aligns with all other texts in Trigonometry and Precalculus; however, there is a *much* better choice of domain restriction for these functions that makes Calculus (specifically, Integral Calculus) work much more smoothly. Despite this, we give the traditional definition here because it aligns with all of our previous work.

When defining the arccosine, we restricted the cosine to the interval $[0, \pi]$; however, such a restriction needs to be modified for the secant because the secant is undefined at $x = \frac{\pi}{2}$. Thus, we restrict the domain of the secant to $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ to help us define the arcsecant.

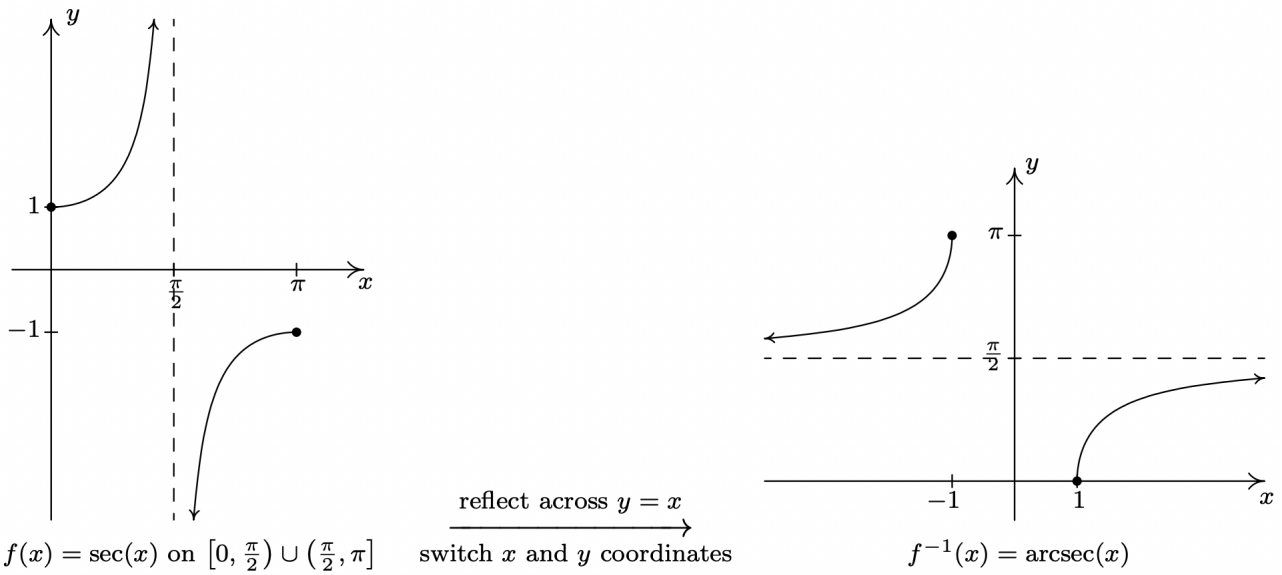


Figure 7.2.4

This gives us the following definition.

Definition: The Inverse Secant Function

The function $f(x) = \sec^{-1}(x)$ is defined as follows:

$$\sec^{-1}(x) = \theta \quad \text{if and only if} \quad \sec(\theta) = x \quad \text{and} \quad \theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right].$$

Likewise, we restricted the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ when defining the arcsine, but such a restriction will not work for the arccosecant because the cosecant is undefined at $x = 0$. Therefore, we restrict $g(x) = \csc(x)$ to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ to define the arccosecant.

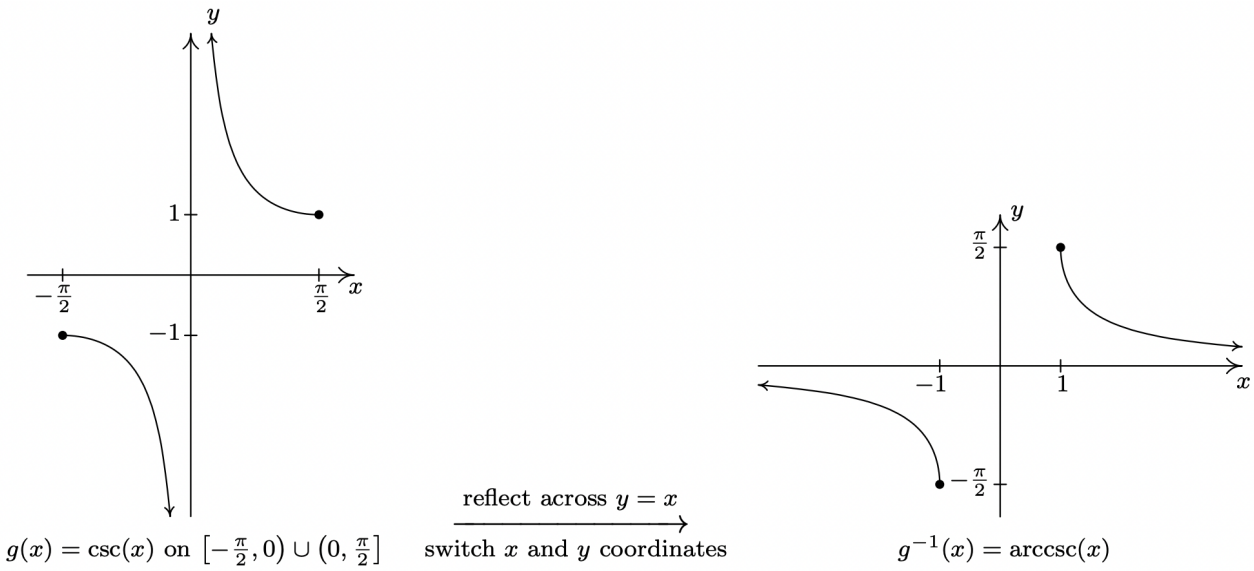


Figure 7.2.5

Note that, for both arcsecant and arccosecant, the domain is $(-\infty,-1]\cup[1,\infty)$. We can rewrite this using set-builder notation as $\{x \mid |x| \geq 1\}$. This is often done in Calculus textbooks, so we include it here for completeness.

Definition: The Inverse Cosecant Function

The function $f(x)=\csc^{-1}(x)$ is defined as follows:

$$\csc^{-1}(x)=\theta \quad \text{if and only if} \quad \csc(\theta)=x \quad \text{and} \quad \theta\in\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right].$$

Using these definitions, we get the following properties of the arcsecant and arccosecant functions.

Theorem: Properties of the Inverse Secant and Inverse Cosecant Functions

Properties of $f(x)=\operatorname{arcsec}(x)$

- $\sec^{-1}(x)=\theta$ if and only if $\sec(\theta)=x$ where $\theta\in[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$
- Domain: $\{x:|x|\geq 1\}=(-\infty,-1]\cup[1,\infty)$
- Range: $[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$
- As $x\to-\infty$, $\operatorname{arcsec}(x)\to\frac{\pi}{2}^{+}$, and as $x\to\infty$, $\operatorname{arcsec}(x)\to\frac{\pi}{2}^{-}$
- $\sec(\operatorname{arcsec}(x))=x$ provided $|x|\geq 1$
- $\operatorname{arcsec}(\sec(x))=x$ provided $0\leq x<\frac{\pi}{2}$ or $\frac{\pi}{2}<x\leq\pi$

Properties of $f(x)=\operatorname{arccsc}(x)$

- $\csc^{-1}(x)=\theta$ if and only if $\csc(\theta)=x$ where $\theta\in\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$
- Domain: $\{x:|x|\geq 1\}=(-\infty,-1]\cup[1,\infty)$
- Range: $[-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2}]$
- As $x\to-\infty$, $\operatorname{arccsc}(x)\to 0^{-}$, and as $x\to\infty$, $\operatorname{arccsc}(x)\to 0^{+}$
- $\csc(\operatorname{arccsc}(x))=x$ provided $|x|\geq 1$
- $\operatorname{arccsc}(\csc(x))=x$ provided $-\frac{\pi}{2}\leq x<0$ or $0<x\leq\frac{\pi}{2}$
- Additionally, arccosecant is an odd function.

Example 7.2.2

a. Find the exact values of the following.
   i. $\sec^{-1}(2)$
   ii. $\operatorname{arccsc}(-2)$
   iii. $\operatorname{arcsec}\left(\sec\left(\frac{8\pi}{7}\right)\right)$
   iv. $\cot\left(\operatorname{arccsc}\left(-3\right)\right)$

b. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.
   i. $\tan(\operatorname{arcsec}(x))$
   ii. $\cos(\operatorname{arccsc}(4x))$

Solutions available at the end of this section.

Checkpoint 7.2.2

Rewrite the function as one that does not involve trigonometric functions and state the domain.

$$f(x) = \sin\left(\sec^{-1}\left(2x\right)\right)$$

(Your answer might be a piecewise function or it might involve absolute values.)

Answer available at the end of this section.

Calculators and the Non-Fundamental Inverse Trigonometric Functions

In the sections to come, we will need to approximate the values of the inverse trigonometric functions. We have already discussed how to use technology to approximate values of the arcsine, arccosine, and arctangent functions; however, if we are asked for an approximation of the arccotangent, arcsecant, or arccosecant, how is this to be done when our calculators do not have these functions?

Example 7.2.3

a. Use a calculator to approximate the following values to four decimal places.
   i. $\cot^{-1}(2)$
   ii. $\operatorname{arcsec}(5)$
   iii. $\operatorname{arccot}(-2)$
   iv. $\csc^{-1}\left(-\frac{3}{2}\right)$

b. Find the domain and range of

$$f(x) = \frac{\pi}{2} - \sec^{-1}\left(\frac{x}{5}\right).$$

Solutions available at the end of this section.

Checkpoint 7.2.3

Use a calculator to approximate $\operatorname{arcsec}\left(-\frac{7}{3}\right)$.

Answer available at the end of this section.

The inverse trigonometric functions are typically found in applications whenever the measure of an angle is required. The following example presents one such scenario.

Example 7.2.4

The roof on the house below has a "$6/12$ pitch." This means that when viewed from the side, the roof line has a rise of 6 feet over a run of 12 feet. Find the angle of inclination from the bottom of the roof to the top of the roof. Express your answer in decimal degrees, rounded to the nearest hundredth of a degree.



Figure 7.2.8

Solution available at the end of this section.

Success in College: The Classroom and Learning Preferences

Read the strategies suggested for your learning preferences below.

Visual Learners

Strategies for visual learners include the following:

- take detailed notes during class
- reorganize and rewrite notes after class
- rework important examples
- watch the videos linked to your textbook
- use flashcards to study vocabulary

Auditory Learners

Strategies for auditory learners include the following:

- record your class, take brief notes of the big ideas and examples, and listen to the recording after class
- restate the main ideas from the lecture aloud to yourself
- talk to yourself as you do your homework, explaining each step to yourself
- study vocabulary by repeating definitions (listen to a recording of the words and definitions)

Kinesthetic Learners

Strategies for kinesthetic learners include the following:

- record the lectures, take brief notes of the big ideas and examples, and listen to the recording after class
- draw pictures, when possible
- with your finger, trace diagrams and graphs - don't just look at them
- imagine symbols as three-dimensional objects, or even cartoon characters
- study vocabulary as you exercise or walk to your car

- walk around your room as you repeat definitions

1. Do you use any of these strategies currently? If not, should you be?
2. Can you suggest any other ideas that I should add to this list?

Success in Mathematics: Why Study Mathematics? (Part 4)

Dr. Robert Cabin teaches biology in New York State. He writes,

> "... but many students are disillusioned by and uninterested in academics. They are on our campuses because they think they need a degree to get a good job, because their parents expect them to go to college, or because they just don't know what else to do."
>
> *Source: Chronicle of Higher Education, October 2003.*

1. Dr. Cabin states that many students are uninterested in academics. What do you think he means by academics? Explain.
2. Are you interested in academics? Explain.

Answers

Section Preview

For those who don't know, the Stewart Calculus textbook and, subsequently, Cosumnes River College's OER Calculus textbook use a different range definition for the secant and cosecant functions than introduced in this section. The redefinition that occurs in Calculus is to make some developments in Calculus II work more smoothly; however, I chose to stick with the conventional definitions in this text so students going to other campuses (or for those who are using the online homework) don't have to worry about things.

Section Preview

- Find the exact value of a non-fundamental inverse trigonometric function.
- Use technology to approximate the value of a non-fundamental inverse trigonometric function.
- Evaluate compositions involving trigonometric functions and non-fundamental inverse trigonometric functions.
- Simplify expressions involving trigonometric functions and non-fundamental inverse trigonometric functions.

Example 7.2.1

a. i. $\operatorname{arccot}(-\sqrt{3}) = \theta$ means $\cot(\theta) = -\sqrt{3}$, where $\theta \in (0, \pi)$.

Quadrant: Since the cotangent is negative, we further restrict the angle to $\theta \in (\frac{\pi}{2}, \pi)$, or Quadrant II.

Reference Angle: The reference angle for the cotangent that gives $\cot\left(\hat{\theta}\right) = \sqrt{3}$ is $\hat{\theta} = \frac{\pi}{6}$.

Given the quadrant and reference angle, we get

$$\operatorname{arccot}\left(-\sqrt{3}\right) = \frac{5\pi}{6}.$$

ii. It is helpful to remember that $\operatorname{trig}\left(\operatorname{trig}^{-1}(x)\right) = x$ for all of the trigonometric functions as long as x is in the domain of trig^{-1} (which is the range of trig). Therefore, $\operatorname{cot}(\operatorname{arccot}(-5)) = \operatorname{cot}\left(\operatorname{cot}^{-1}(-5)\right) = -5$ because -5 is definitely in the domain of the inverse cotangent.

Another way to think about this is to let $\theta = \operatorname{arccot}(-5)$. Then θ belongs to the interval $(0, \pi)$ and $\cot(\theta) = -5$. Hence, $\operatorname{cot}(\operatorname{arccot}(-5)) = \operatorname{cot}(\theta) = -5$.

b. To get started, we let $\theta = \operatorname{arccot}(2x)$ so that $\cot(\theta) = 2x$ where $0 < \theta < \pi$. We then sketch two reference triangles, one in QI and the other in QII.

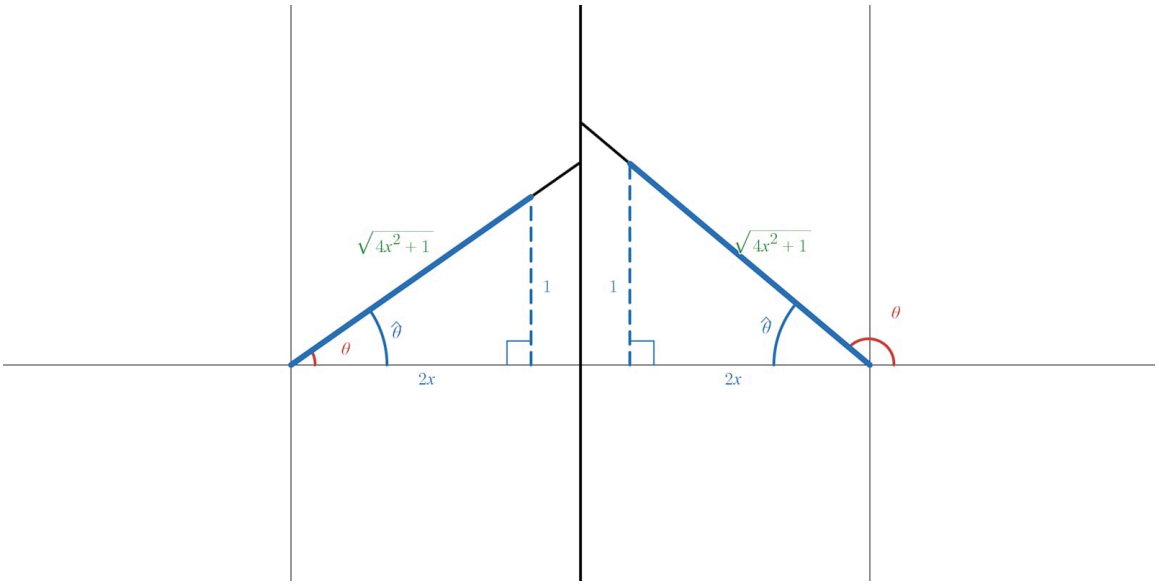


Figure 7.2.2

In terms of θ,

$$\cos(\operatorname{arccot}(2x)) = \cos(\theta).$$

Therefore, we can use our reference triangles to evaluate $\cos(\theta)$. Looking at the reference triangle in QI, $\cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}$. Likewise, the reference triangle from the QII sketch yields $\cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}$. Hence, it appears that

$$\cos(\operatorname{arccot}(2x)) = \cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}.$$

Being good mathematicians, we take a moment to reflect and make sure our logic is solid. What might give us pause is the fact that our second sketch, where $\theta \in$ QII, leads to cosine being negative. Still, our result, $\cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}$, looks to be positive; however, looks can be deceiving!

The expression, $2x$, in the second sketch, *is*, in fact, negative. The negativity of it is "hidden" within the variable x. That is, if the x-value of the point on the upper-left vertex of that reference triangle is -4, then $2x = -4 \implies x = -2$. Hence, when $\theta \in$ QII, $\cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}$ is negative (which is what we want).

We now concern ourselves with the domain of the function. We know that the domain of the arccotangent is all real numbers. Hence, the domain of $\operatorname{arccot}(2x)$ is also all real numbers. Moreover, the range of the arccotangent is $(0, \pi)$. Since the outputs of $\operatorname{arccot}(2x)$ are the inputs of $\cos(\operatorname{arccot}(2x))$ and the cosine can take any value as an input, the domain remains unchanged. That is, the domain of $y = \cos(\operatorname{arccot}(2x))$ is $(-\infty, \infty)$.

Checkpoint 7.2.1

$$\frac{2}{\sqrt{29}}$$

Example 7.2.2

a. i. Let $\theta = \sec^{-1}(2)$. Then $\sec(\theta) = 2$, where $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$. Of these two intervals, the only one where secant is positive is $[0, \frac{\pi}{2})$. Thus, $\theta \in$ QI. Moreover, $\sec\left(\hat{\theta}\right) = 2$ implies $\cos\left(\hat{\theta}\right) = \frac{1}{2}$. Hence, $\hat{\theta} = \frac{\pi}{3}$. Therefore,

$$\sec^{-1}(2) = \theta = \frac{\pi}{3}.$$

ii. We now start to streamline our arguments.

$$\begin{aligned}
& \operatorname{arccsc}(-2) = \theta && \text{(inverse trigonometric functions return angles,)} \\
\implies\ & \csc(\theta) = -2, \quad \text{where } \theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] && \text{(but the angle is restricted)} \\
\implies\ & \theta \in \left[-\frac{\pi}{2}, 0\right) && \text{(cosecant is negative)} \\
\text{and}\ & \hat{\theta} = \frac{\pi}{6} \\
\implies\ & \operatorname{arccsc}(-2) = -\frac{\pi}{6} && \left(\text{reference angle of } \frac{\pi}{6} \text{ in the interval } \left[-\frac{\pi}{2}, 0\right)\right)
\end{aligned}$$

iii. Our choice of non-standard angle for this example is purposeful. It's to force you to think about the process instead of focusing solely on computing.

$$\begin{aligned}
& \operatorname{arcsec}\left(\sec\left(\frac{8\pi}{7}\right)\right) = \theta, \quad \text{where } \theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] && \text{(inverse trigonometric functions return restricted angles)} \\
\implies\ & \sec(\theta) = \sec\left(\frac{8\pi}{7}\right)
\end{aligned}$$

Therefore, the reference angle for θ matches the reference angle for $\frac{8\pi}{7}$. That is,

$$\hat{\theta} = \frac{\pi}{7}.$$

We now need to determine in which quadrant θ belongs. Since $\frac{8\pi}{7} \in$ QIII, $\sec\left(\frac{8\pi}{7}\right)$ is negative. Moreover, since $\sec(\theta) = \sec\left(\frac{8\pi}{7}\right)$, $\sec(\theta)$ must be negative. Hence, $\theta \in \left(\frac{\pi}{2}, \pi\right]$. Thus, $\theta = \frac{6\pi}{7}$. This means that

$$\operatorname{arcsec}\left(\sec\left(\frac{8\pi}{7}\right)\right) = \theta = \frac{6\pi}{7}.$$

iv. As usual, we begin by addressing the fact that the inverse trigonometric function is an angle.

$$\begin{aligned} \operatorname{arccsc}(-3) &= \theta, \quad \text{where } \theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \quad \text{(inverse trigonometric functions return restricted angles)} \\ \Longrightarrow \quad \csc(\theta) &= -3 \end{aligned}$$

However, we are interested in

$$\cot\left(\operatorname{arccsc}(-3)\right) = \cot(\theta).$$

Since the cosecant is negative, we know that $\theta \in \left[-\frac{\pi}{2}, 0\right)$. Let us sketch a reference triangle in QIV to help us determine $\cot(\theta)$.

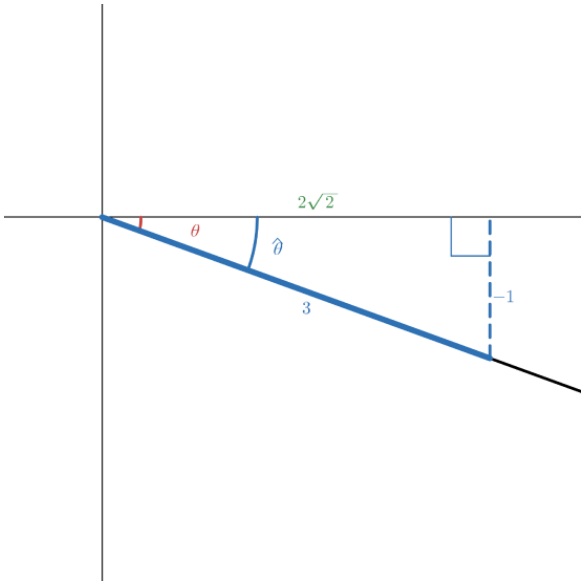


Figure 7.2.6

From the reference triangle, we see that

$$\cot\left(\operatorname{arccsc}(-3)\right) = \cot(\theta) = -2\sqrt{2}.$$

b. i. As per usual, let $\theta = \operatorname{arcsec}(x)$ so that $\sec(\theta) = x$, where we restrict θ to the interval $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. We then sketch two reference triangles, one for $\theta \in \left[0, \frac{\pi}{2}\right)$ and the other for $\theta \in \left(\frac{\pi}{2}, \pi\right]$.

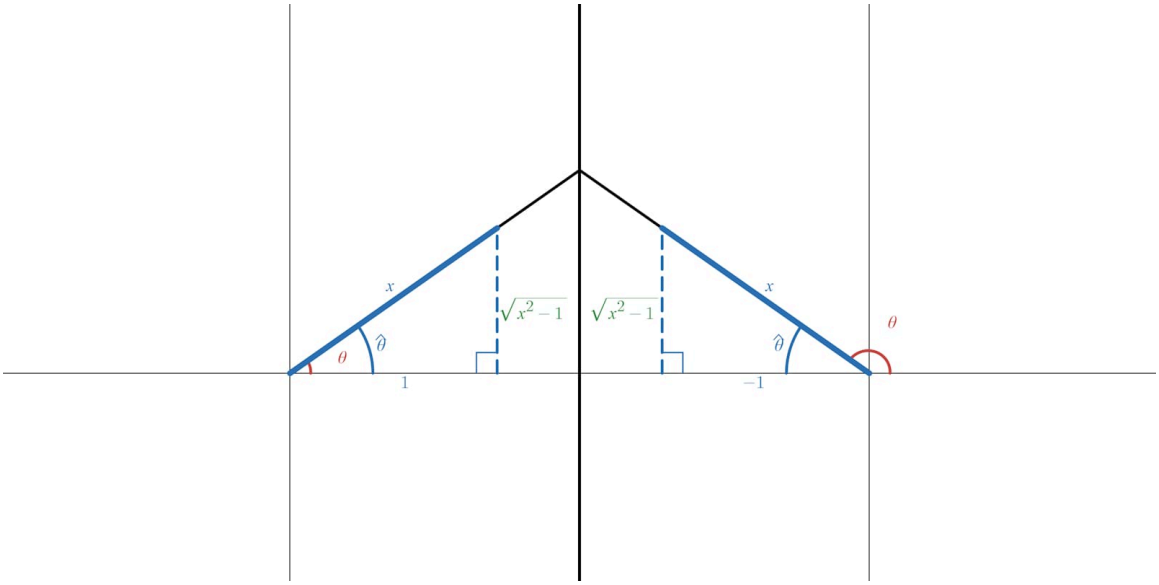


Figure 7.2.7

We now use these reference triangles to help us determine the value of $\tan(\operatorname{arcsec}(x)) = \tan(\theta)$. In this case, we run into a very interesting problem. In Quadrant I, $\tan(\theta) = \sqrt{x^2-1}$; however, in Quadrant II, $\tan(\theta) = -\sqrt{x^2-1}$. Pausing for a moment, let's think about this before going further. Does this make sense?

We know tangent is supposed to be positive in Quadrant I. Since $\sec(\theta) = x$, we know either $x \geq 1$ or $x \leq -1$. Luckily, $\sqrt{x^2-1}$ is definitely positive for these values of x (unless $x = \pm 1$, but that would mean $\tan(\theta) = 0 \implies \theta = 0$ which is not in Quadrant I). So the statement, $\tan(\theta) = \sqrt{x^2-1}$ when $\theta \in (0, \frac{\pi}{2})$ makes sense.

In Quadrant II, the tangent is supposed to be negative. It's easy to see that $-\sqrt{x^2-1}$ is negative whenever $\theta \in (\frac{\pi}{2}, \pi)$. Thus, the statement, $\tan(\theta) = -\sqrt{x^2-1}$ when $\theta \in (\frac{\pi}{2}, \pi)$ also makes sense. Therefore,

$$\tan(\theta) = \begin{cases} \sqrt{x^2-1}, & \text{if } 0 \leq \theta < \dfrac{\pi}{2} \\ -\sqrt{x^2-1}, & \text{if } \dfrac{\pi}{2} < \theta \leq \pi \end{cases}$$

Now we need to determine what these conditions on θ mean for x. Since $x = \sec(\theta)$, when $0 \leq \theta < \frac{\pi}{2}$, $x \geq 1$, and when $\frac{\pi}{2} < \theta \leq \pi$, $x \leq -1$. Since we encountered no further restrictions on θ, the equivalence below holds for all x in $(-\infty, -1] \cup [1, \infty)$.

$$\tan(\operatorname{arcsec}(x)) = \begin{cases} \sqrt{x^2-1}, & \text{if } x \geq 1 \\ -\sqrt{x^2-1}, & \text{if } x \leq -1 \end{cases}$$

ii. While I always use the graphing tactic, you do not have to think geometrically to work with inverse trigonometric functions. You can rely solely on identities.

For example, to simplify $\cos(\operatorname{arccsc}(4x))$, we start by letting $\theta = \operatorname{arccsc}(4x)$. Then $\csc(\theta) = 4x$ for $\theta \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.

We now set about finding an expression for $\cos(\operatorname{arccsc}(4x)) = \cos(\theta)$. Since $\cos(\theta)$ is defined for all θ, we do not encounter any additional restrictions on θ. From $\csc(\theta) = 4x$, we get $\sin(\theta) = \frac{1}{4x}$, so to find $\cos(\theta)$, we can make use of the Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$. Substituting $\sin(\theta) = \frac{1}{4x}$ gives

$$\cos^2(\theta) + \left(\frac{1}{4x}\right)^2 = 1.$$

Solving, we get

$$\cos(\theta) = \pm\sqrt{\frac{16x^2-1}{16x^2}} = \pm\frac{\sqrt{16x^2-1}}{4|x|}.$$

Since $\theta \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$, we know $\cos(\theta) \geq 0$. Therefore, we choose

$$\cos(\theta) = \frac{\sqrt{16x^2-1}}{4|x|}.$$

(The absolute values here are necessary, since x could be negative.) To find the values for which this equivalence is valid, we look back at our original substitution, $\theta = \operatorname{arccsc}(4x)$. Since the domain of $\operatorname{arccsc}(x)$ requires its argument x to satisfy $|x| \geq 1$, the domain of $\operatorname{arccsc}(4x)$ requires $|4x| \geq 1$. This implies $x \leq -\frac{1}{4}$ or $x \geq \frac{1}{4}$. Since we had no additional restrictions on θ, the equivalence

$$\cos(\operatorname{arccsc}(4x)) = \frac{\sqrt{16x^2-1}}{4|x|}$$

holds for all x in $\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \infty\right)$.

Checkpoint 7.2.2

There are two ways to write this function:

$$f(x) = \frac{\sqrt{4x^2-1}}{2|x|},$$

or

$$f(x) = \begin{cases} \dfrac{\sqrt{4x^2-1}}{2x}, & \text{if } x \geq \dfrac{1}{2} \\ -\dfrac{\sqrt{4x^2-1}}{2x}, & \text{if } x < -\dfrac{1}{2} \end{cases}$$

The domain of this function is all $x \in \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$.

Example 7.2.3

a. i. $\cot^{-1}(2) = \theta$ means $\cot(\theta) = 2$, where $\theta \in (0, \pi)$.

Quadrant: Since the cotangent is positive, we know $\theta \in (0, \frac{\pi}{2})$.

Reference Angle: 2 is not a special ratio for the cotangent, so we will need technology to help us find the reference angle; however, our calculators do not have an inverse cotangent button. What do we do? The answer is quite simple—we use the Reciprocal Identities. Since $\cot(\theta) = 2$, $\tan(\theta) = \frac{1}{2}$. To compute the reference angle, we evaluate

$$\hat{\theta} = \tan^{-1}\left(\left|\frac{1}{2}\right|\right) = \tan^{-1}\left(\frac{1}{2}\right) \approx 0.4636.$$

Now that we have a reference angle (≈ 0.4636) and a quadrant (QI), we get

$$\cot^{-1}(2) \approx 0.4636.$$

ii. $\operatorname{arcsec}(5) = \theta$ means $\sec(\theta) = 5$, where $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.

Quadrant: Since the secant is positive, we further restrict the angle to $\theta \in (0, \frac{\pi}{2})$ (Quadrant I).

Reference Angle: Again, since our technology does not have an inverse secant button, we use the Reciprocal Identities.

$$\sec(\theta) = 5 \implies \cos(\theta) = \frac{1}{5}.$$

Now we use a calculator to compute

$$\hat{\theta} = \cos^{-1}\left(\left|\frac{1}{5}\right|\right) = \cos^{-1}\left(\frac{1}{5}\right) \approx 1.3694.$$

Using our reference angle (≈ 1.3694) and quadrant (QI), we get

$$\operatorname{arcsec}(5) \approx 1.3694.$$

iii. $\operatorname{arccot}(-2) = \theta$ means $\cot(\theta) = -2$, where $\theta \in (0, \pi)$.

Quadrant: Since $\cot(\theta) < 0$, we know $\frac{\pi}{2} < \theta < \pi$. That is, $\theta \in$ QII.

Reference Angle: Once again, we do not have an appropriate inverse trigonometric function on the calculator, so we use the Reciprocal Identities.

$$\cot(\theta) = -2 \implies \tan(\theta) = -\frac{1}{2}.$$

Thus, we compute the reference angle as follows:

$$\hat{\theta} = \tan^{-1}\left(\left|-\frac{1}{2}\right|\right) = \tan^{-1}\left(\frac{1}{2}\right) \approx 0.4636.$$

We have a reference angle (≈ 0.4636) and a quadrant (QII). To find the actual angle, we subtract our reference angle from π to get

$$\operatorname{arccot}(-2) = \theta = \pi - \hat{\theta} = \pi - \arctan\left(\frac{1}{2}\right) \approx 2.6779.$$

iv. $\csc^{-1}\left(-\frac{3}{2}\right) = \theta$ means $\csc(\theta) = -\frac{3}{2}$, where $\theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

Quadrant: Since $\csc(\theta) < 0$, we further restrict the angle to the interval $\theta \in \left[-\frac{\pi}{2}, 0\right)$ (Quadrant IV).

Reference Angle: We do not have an appropriate inverse trigonometric button on our calculator for the cosecant. Therefore, we must use the Reciprocal Identities.

$$\csc(\theta) = -\frac{3}{2} \implies \sin(\theta) = -\frac{2}{3}.$$

Hence,

$$\hat{\theta} = \sin^{-1}\left(\left|-\frac{2}{3}\right|\right) = \sin^{-1}\left(\frac{2}{3}\right) \approx 0.7297.$$

Now that we have a reference angle (≈ 0.7297) and a quadrant (QIV), we get

$$\csc^{-1}\left(-\frac{3}{2}\right) \approx -0.7297.$$

b. The domain of the arcsecant is normally $(-\infty, -1] \cup [1, \infty)$. That is, if we were given $\sec^{-1}(r)$, we know that allowable values of r are $r \in (-\infty, -1] \cup [1, \infty)$. In this case, however, the argument of the arcsecant is $\frac{x}{5}$. Hence, $\frac{x}{5} \in (-\infty, -1] \cup [1, \infty)$. Rewriting this in interval notation will be helpful.

$$\begin{aligned} & \frac{x}{5} \leq -1 \quad \text{or} \quad \frac{x}{5} \geq 1 \\ \implies \quad & x \leq -5 \quad \text{or} \quad x \geq 5 \end{aligned}$$

Thus, the domain of f is $(-\infty, -5] \cup [5, \infty)$.

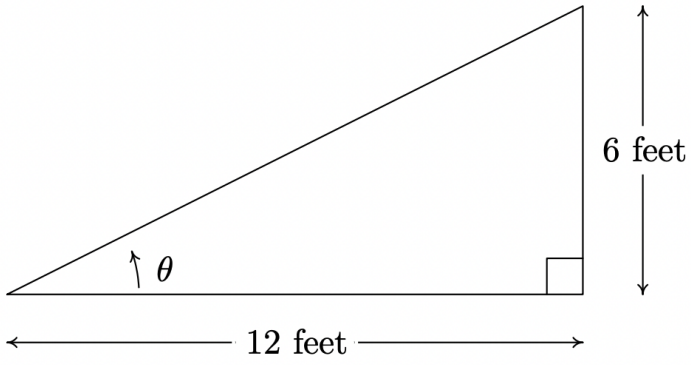
We know the range of the arcsecant is normally $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$. Given the function $f(x) = \frac{\pi}{2} - \sec^{-1}\left(\frac{x}{5}\right)$, the only transformation that affects the range is the addition of $\frac{\pi}{2}$ to the output values. Thus, the range of f is $[\frac{\pi}{2}, \pi) \cup (\pi, \frac{3\pi}{2}]$.

Checkpoint 7.2.3

≈ 2.0137

Example 7.2.4

If we divide the side view of the house down the middle, we find that the roof line forms the hypotenuse of a right triangle with legs of length 6 feet and 12 feet. We find the angle of inclination, labeled θ below, satisfies $\tan(\theta) = \frac{6}{12} = \frac{1}{2}$. Since θ is an acute angle, we can use the arctangent function and we find $\theta = \arctan\left(\frac{1}{2}\right)$ radians $\approx 26.56°$.



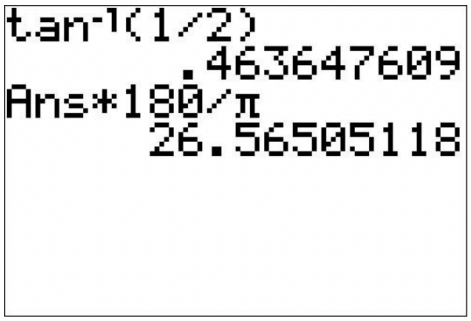


Figure 7.2.9

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7.2.1: Resources and Key Concepts

Resources

Videos

- Inverse Trigonometric Functions
  - Inverse Trigonometric Functions - Defining the Inverse Trigonometric Functions
  - Inverse Trigonometric Functions - Evaluating Inverse Trigonometric Functions

Key Concepts

Definitions

- **Inverse Cotangent Function:** The function $f(x) = \cot^{-1}(x)$ or $\operatorname{arccot}(x)$ is defined as $\operatorname{arccot}(x) = \theta$ if and only if $\cot(\theta) = x$ and $0 < \theta < \pi$.
- **The Inverse Secant Function:** The function $f(x) = \sec^{-1}(x)$ or $\operatorname{arcsec}(x)$ is defined as $\operatorname{arcsec}(x) = \theta$ if and only if $\sec(\theta) = x$ and $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.
- **The Inverse Cosecant Function:** The function $f(x) = \csc^{-1}(x)$ or $\operatorname{arccsc}(x)$ is defined as $\operatorname{arccsc}(x) = \theta$ if and only if $\csc(\theta) = x$ and $\theta \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.

Theorems

- **Properties of the Inverse Cotangent Function:**
  - Domain: $(-\infty, \infty)$
  - Range: $(0, \pi)$
  - As $x \to -\infty$, $\operatorname{arccot}(x) \to \pi^-$; as $x \to \infty$, $\operatorname{arccot}(x) \to 0^+$
  - $\cot(\operatorname{arccot}(x)) = x$ for all real numbers x
  - $\operatorname{arccot}(\cot(x)) = x$ provided $0 < x < \pi$
- **Properties of the Inverse Secant and Inverse Cosecant Functions:**
  - **arcsec(x):** Domain is $(-\infty, -1] \cup [1, \infty)$; Range is $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.
  - **arccsc(x):** Domain is $(-\infty, -1] \cup [1, \infty)$; Range is $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.
  - $\sec(\operatorname{arcsec}(x)) = x$ provided $|x| \geq 1$
  - $\csc(\operatorname{arccsc}(x)) = x$ provided $|x| \geq 1$
  - $\operatorname{arcsec}(\sec(x)) = x$ provided $x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
  - $\operatorname{arccsc}(\csc(x)) = x$ provided $x \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$
  - Arccosecant is an odd function.

Common Mistakes

- **Forgetting Domain and Range Restrictions:** The inverse reciprocal trigonometric functions have very specific ranges. For example, $\operatorname{arccot}(x)$ only returns angles in Quadrants I and II, while $\operatorname{arcsec}(x)$ returns angles in Quadrants I and II (excluding $\frac{\pi}{2}$). Forgetting these restrictions is a primary source of error, especially when simplifying compositions like $\operatorname{arcsec}(\sec(\frac{8\pi}{7}))$.
- **Calculator Usage:** Calculators do not have dedicated buttons for arccot, arcsec, or arccsc. To evaluate these, you must use the Reciprocal Identities first. For example, to find $\operatorname{arccot}(2)$, you must evaluate $\arctan(\frac{1}{2})$.
- **Ignoring the Sign of x in Algebraic Expressions:** When simplifying an expression like $\tan(\operatorname{arcsec}(x))$, the result may depend on whether x is positive or negative, as this determines the quadrant of the angle $\theta = \operatorname{arcsec}(x)$. The result can be a piecewise function.

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7.2.2: Homework

Reading Questions

1. Why must the domain of the cotangent function be restricted to define its inverse?
2. What is the restricted domain (principal cycle) used to define the inverse cotangent function?
3. What is the domain and range of the $\operatorname{arccot}(x)$ function?
4. Why is the domain of the $\operatorname{arcsec}(x)$ and $\operatorname{arccsc}(x)$ functions the set $(-\infty,-1]\cup[1,\infty)$?
5. What is the range of the $\operatorname{arcsec}(x)$ function? Why is $\frac{\pi}{2}$ excluded?
6. How would you use a calculator to find the value of $\operatorname{arccot}(5)$?
7. Is the statement $\operatorname{arcsec}(\sec(x))=x$ true for all values of x? Explain.
8. When simplifying an expression like $\sin(\operatorname{arccot}(x))$, what is a helpful first step?
9. What is the key difference between the range of $\arccos(x)$ and $\operatorname{arcsec}(x)$?

Homework

Basic Skills

For the following exercises, find the exact value.

1. $\operatorname{arccot}\left(\sqrt{3}\right)$
2. $\operatorname{arccot}\left(\frac{1}{\sqrt{3}}\right)$
3. $\operatorname{arcsec}\left(\sqrt{2}\right)$
4. $\operatorname{arccsc}\left(\sqrt{2}\right)$
5. $\operatorname{arcsec}\left(\frac{2}{\sqrt{3}}\right)$
6. $\operatorname{arccsc}\left(\frac{2}{\sqrt{3}}\right)$
7. $\operatorname{arcsec}(2)$
8. $\operatorname{arccsc}(2)$
9. $\operatorname{arccot}(1)$
10. $\operatorname{arcsec}(1)$
11. $\operatorname{arccsc}(1)$
12. $\operatorname{arccot}(0)$
13. $\operatorname{arccot}\left(-\sqrt{3}\right)$
14. $\operatorname{arccot}\left(-\frac{1}{\sqrt{3}}\right)$
15. $\operatorname{arcsec}(-2)$

16. $\operatorname{arccsc}(-2)$

17. $\operatorname{arcsec}\left(-\sqrt{2}\right)$

18. $\operatorname{arccsc}\left(-\sqrt{2}\right)$

19. $\operatorname{arcsec}\left(-\frac{2}{\sqrt{3}}\right)$

20. $\operatorname{arccsc}\left(-\frac{2}{\sqrt{3}}\right)$

21. $\operatorname{arcsec}(-1)$

22. $\operatorname{arccot}(-1)$

23. $\operatorname{arccsc}(-1)$

For the following exercises, find the exact value or state that it is undefined.

24. $\cot(\operatorname{arccot}(1))$

25. $\cot\left(\operatorname{arccot}\left(-\sqrt{3}\right)\right)$

26. $\cot\left(\operatorname{arccot}\left(-\frac{7}{24}\right)\right)$

27. $\cot(\operatorname{arccot}(-0.001))$

28. $\cot\left(\operatorname{arccot}\left(\frac{17\pi}{4}\right)\right)$

29. $\sec(\operatorname{arcsec}(2))$

30. $\sec(\operatorname{arcsec}(-1))$

31. $\sec\left(\operatorname{arcsec}\left(\frac{1}{2}\right)\right)$

32. $\sec(\operatorname{arcsec}(0.75))$

33. $\sec(\operatorname{arcsec}(117\pi))$

34. $\csc\left(\operatorname{arccsc}\left(\sqrt{2}\right)\right)$

35. $\csc\left(\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)\right)$

36. $\csc\left(\operatorname{arccsc}\left(\frac{\sqrt{2}}{2}\right)\right)$

37. $\csc(\operatorname{arccsc}(1.0001))$

38. $\csc\left(\operatorname{arccsc}\left(\frac{\pi}{4}\right)\right)$

For the following exercises, find the exact value or state that it is undefined.

39. $\operatorname{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)$

40. $\operatorname{arccot}\left(\cot\left(\frac{\pi}{3}\right)\right)$

41. $\operatorname{arccot}\left(\cot\left(-\frac{\pi}{4}\right)\right)$

42. $\operatorname{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right)$

43. $\operatorname{arccot}\left(\cot\left(\frac{\pi}{2}\right)\right)$

44. $\operatorname{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$

45. $\operatorname{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)$

46. $\operatorname{arccot}\left(\cot\left(\pi\right)\right)$

47. $\operatorname{arccot}\left(\cot\left(\frac{2\pi}{3}\right)\right)$

48. $\operatorname{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right)$

49. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right)$

50. $\operatorname{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right)$

51. $\operatorname{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)$

52. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)$

53. $\operatorname{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)$

54. $\operatorname{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right)$

55. $\operatorname{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)$

For the following exercises, find the exact value or state that it is undefined.

56. $\sin\left(\operatorname{arccot}\left(\sqrt{5}\right)\right)$

57. $\sin\left(\operatorname{arccsc}\left(-3\right)\right)$

58. $\cos\left(\operatorname{arccot}\left(3\right)\right)$

59. $\cos\left(\operatorname{arcsec}\left(5\right)\right)$

60. $\tan\left(\operatorname{arcsec}\left(\frac{5}{3}\right)\right)$

61. $\tan\left(\operatorname{arccot}\left(12\right)\right)$

62. $\cot\left(\operatorname{arccsc}\left(\sqrt{5}\right)\right)$

63. $\sec\left(\operatorname{arccot}\left(-\frac{\sqrt{10}}{10}\right)\right)$

64. $\csc\left(\operatorname{arccot}\left(9\right)\right)$

Synthesis Questions

For the following exercises, find the exact value or state that it is undefined.

65. $\cos\left(\operatorname{arcsec}(3)+\arctan(2)\right)$

66. $\sin\left(2\operatorname{arccsc}\left(\frac{13}{5}\right)\right)$

67. $\cos\left(2\operatorname{arcsec}\left(\frac{25}{7}\right)\right)$

68. $\cos\left(2\operatorname{arccot}\left(-\sqrt{5}\right)\right)$

For the following exercises, find the domain of the given function. Write your answers in interval notation.

69. $f(x)=\operatorname{arccot}\left(\frac{2x}{x^2-9}\right)$

70. $f(x)=\arctan(\ln(2x-1))$

71. $f(x)=\operatorname{arccot}(\sqrt{2x-1})$

72. $f(x)=\operatorname{arcsec}(12x)$

73. $f(x)=\operatorname{arccsc}(x+5)$

74. $f(x)=\operatorname{arcsec}\left(\frac{x^3}{8}\right)$

75. $f(x)=\operatorname{arccsc}\left(e^{2x}\right)$

Applications

76. A guy wire 1000 feet long is attached to the top of a tower. When pulled taut, it touches level ground 360 feet from the tower's base. What angle does the wire make with the ground? Express your answer using degree measure rounded to one decimal place.

77. At Cliffs of Insanity Point, The Great Sasquatch Canyon is 7117 feet deep. From that point, a fire is seen at a location known to be 10 miles away from the base of the sheer canyon wall. What angle of depression is made by the line of sight from the canyon edge to the fire? Express your answer using degree measure rounded to one decimal place.

78. Shelving is being built at the Utility Muffin Research Library, which is to be 14 inches deep. An 18-inch rod will be attached to the wall and the underside of the shelf at its edge away from the wall, forming a right triangle under the shelf to support it. What angle, to the nearest degree, will the rod make with the wall?

79. A parasailor is being pulled by a boat on Lake Ippizuti. The cable is 300 feet long, and the parasailor is 100 feet above the surface of the water. What is the angle of elevation from the boat to the parasailor? Express your answer using degree measure rounded to one decimal place.

80. A tag-and-release program to study the Sasquatch population of the eponymous Sasquatch National Park has begun. From a 200-foot-tall tower, a ranger spots a Sasquatch lumbering through the wilderness directly toward the tower. Let θ denote the angle of depression from the top of the tower to a point on the ground. If the range of the rifle with a tranquilizer dart is 300 feet, find the smallest value of θ for which the corresponding point on the ground is in range of the rifle. Round your answer to the nearest hundredth of a degree.

Challenge Problems

81. Show that $\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ for $|x| \geq 1$.

82. Show that $\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ for $|x| \geq 1$.

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7.3: Solving Trigonometric Equations - Algebraic Techniques

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The "Non-Uniqueness" of Solutions to Trigonometric Equations

Throughout Algebra, you constantly looked for "the" solution to an equation. For example, you might have been asked to solve

$$3x - \frac{1}{2} = 10 \quad \text{or} \quad 2e^{2x-1} = 5 \quad \text{or} \quad \sqrt{5x-1} = 3.$$

In each of these cases, the equation has a single solution; however, this was not the extent of your "equation solving" in Algebra. You also solved equations like

$$x^2 + 5x + 6 = 0 \quad \text{and} \quad \log_2\left(5x^2 - 1\right) = 2.$$

These last two equations each have two solutions. We now explore equations that often have an infinite number of solutions.

Definition: Trigonometric Equation

A **trigonometric equation** is an equation that involves trigonometric functions of a variable, typically an angle.

For example,

$$2\sec(x) - 1 = 0, \quad \sin^2(x) = 1, \quad \text{and} \quad 2\cos^2(x) = \sin(x) + 1$$

are all trigonometric equations.

Before we discuss how to solve trigonometric equations, it's best to recall the graphical meaning of solutions of equations.

In Algebra, when you were asked to solve an equation like $2e^{2x-1} = 5$, you learned all the mathematical manipulations—divide by 5, take the natural logarithm of both sides, add 1, then divide by 2. This results in the solution $x = \frac{1}{2}\ln\left(\frac{5}{2}\right) + \frac{1}{2}$; however, before you learned how to procedurally work to an answer to such an equation, you learned a way to interpret this solution graphically.

When asked to solve an equation, one interpretation is that you are being asked to find the x-values for which the graph of the left side of the equation meets the graph of the right side. For example, when asked to solve $2e^{2x-1} = 5$, we could view the graphs of $y = 2e^{2x-1}$ and $y = 5$ to visualize the solution.

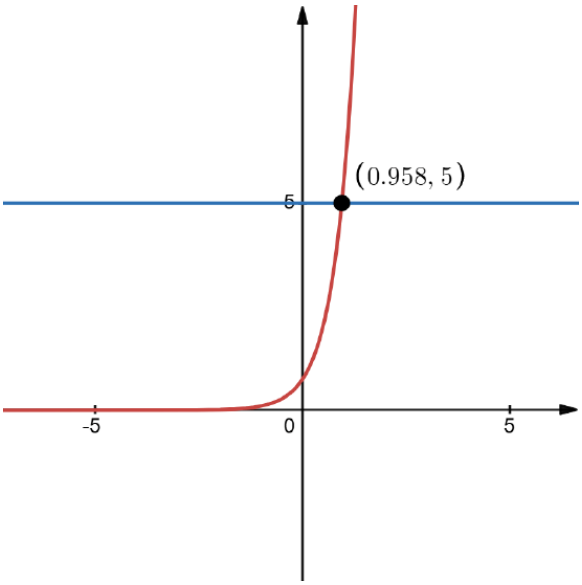


Figure 7.3.1**:** *The graphs of* $y = 2e^{2x-1}$ *(red) and* $y = 5$ *(blue)*

We can see in Figure 7.3.1 that these graphs meet at $x \approx 0.958 \approx \frac{1}{2}\ln\left(\frac{5}{2}\right) + \frac{1}{2}$.

Although not the best way (or even a good way) to solve an equation, the visual we get from the example above justifies why trigonometric equations often have infinite solutions. For example, Figure 7.3.2 illustrates that $\sin(x) = 0.5$ has five solutions on the interval $\left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$. These solutions correspond to the points on the unit circle where $y = 0.5$. Since the sine function oscillates in both directions forever, we can easily imagine that the sine wave intersects the line $y = 0.5$ an infinite number of times.

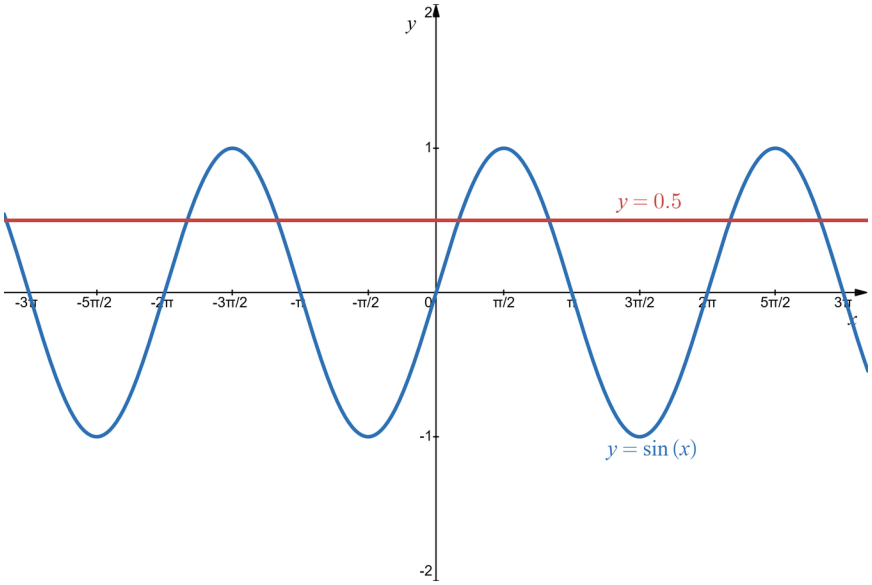


Figure 7.3.2**:** *Using the sine curve to visualize solutions to* $\sin(x) = 0.5$

Despite knowing there are an infinite number of solutions to $\sin(x) = 0.5$, we know from our recent discussions of the inverse trigonometric functions that $x = \sin^{-1}(0.5)$ will only give us one of these solutions. Therefore, we must create a method to obtain *all* solutions to a given trigonometric equation.

Solving Basic Trigonometric Equations

The keys to finding all solutions to a trigonometric equation can be summarized in two fundamental concepts from Trigonometry: quadrants and reference angles. We will begin with an example to illustrate our thought process.

Example 7.3.1

Find all degree solutions to the equation

$$\sin(\theta) = -0.6428,$$

and state the solutions occurring on the interval $[0^\circ, 360^\circ]$. Round your answers to the nearest tenth of a degree.

Solution available at the end of this section.

A few notes about Example 7.3.1 are in order. First, our equation had solutions derived from two different quadrants. This should not be a surprise when you consider Figure 7.3.2. The sine curve should attain a y-value of -0.6428 twice through any single cycle.

Second, since the given ratio, -0.6428, was not a special ratio, we had to grab a calculator to get an approximation of the reference angle by way of the arcsine. Moreover, we had to change our calculator to degree mode. You must remain aware of the angular measurement system in which you are being asked to work. You will frequently switch between degree and radian mode when solving trigonometric equations.

Third, we were told to round to the nearest tenth of a degree. If you are using online homework and need to answer with an approximation, pay attention to the level of rounding they request. If they don't state it explicitly, assume four decimal places (as this works well for most online homework systems).

Checkpoint 7.3.1

Solve the equation $\sin(\theta) = -0.4$ for angles between 0° and 360°. Round your answers to the nearest tenth of a degree.

Answer available at the end of this section.

Example 7.3.2

Find all solutions to the given equation, then state the solutions occurring on the interval $[0, 2\pi]$. Round your answers to the nearest tenth.

a. $\sin(x) = \frac{1}{2}$
b. $\cos(x) = -0.6$
c. $\tan(x) = -\sqrt{3}$
d. $\sec(x) = 3.7$

Solutions available at the end of this section.

Note: Combining Solutions for the Tangent

We found the set of all solutions to Example 7.3.2c to be

$$x = \frac{2\pi}{3} + 2\pi k$$

and

$$x = \frac{5\pi}{3} + 2\pi k.$$

However, the tangent and cotangent are special functions in the world of Trigonometry in that their natural periods are π. That is, when finding their solutions, we can add π instead of 2π.

For example, when solving $\tan(x) = -\sqrt{3}$, we could have written our solution associated with Quadrant II as

$$x = \frac{2\pi}{3} + \pi k, \text{ where } k \in \mathbb{Z}.$$

In doing so, we capture all of the Quadrant IV solutions as well because, when $k = 1$, $x = \frac{2\pi}{3} + \pi = \frac{5\pi}{3}$. This is precisely our Quadrant IV solution!

Thus, from this point forward, when solving trigonometric equations involving the tangent and cotangent functions, we will use this strategy to "combine" our solutions into a single statement.

Checkpoint 7.3.2

Find exact solutions to the given equation, and then state the solutions occurring on the interval $[0, 2\pi]$.

$$\cos(x) = -\frac{1}{\sqrt{2}}$$

Answer available at the end of this section.

Solving Trigonometric Equations Requiring Algebraic Techniques

The first Ferris wheel was built for the Chicago World's Fair in 1893. It had a diameter of 250 feet and could carry 2160 people in 36 carriages. From the top of the wheel, passengers could see into four states. After loading all the passengers, the wheel made one revolution in nine minutes.

If you are in the bottom carriage of the Ferris wheel at the start of its revolution, your height after t seconds is given by

$$h = f(t) = 139 - 125\cos\left(\frac{2t}{3}\right).$$

For how long are you more than 240 feet above the ground?

Figure 7.3.3 shows a graph of the height function and a horizontal line at $h = 240$.

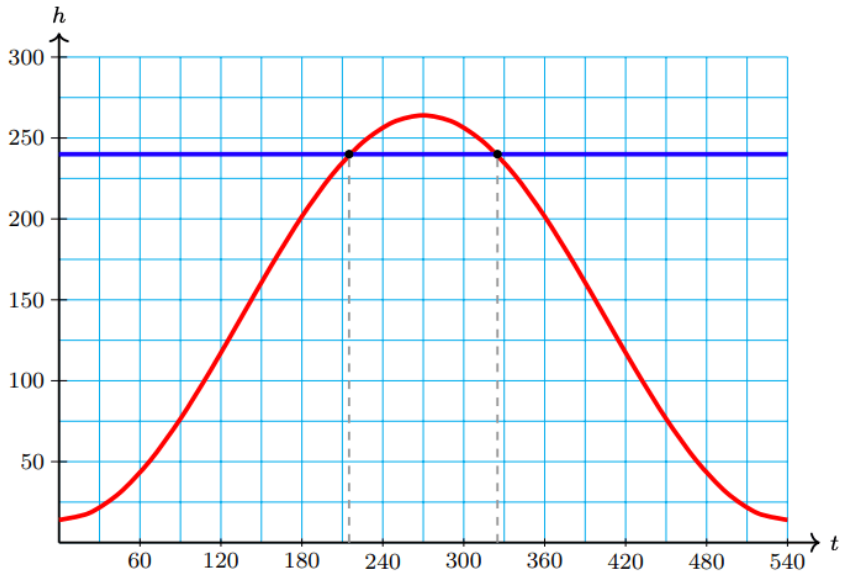


Figure 7.3.3

The graph shows that $h = 240$ at approximately 215 seconds and 325 seconds into the ride. Your height is more than 240 feet between those two times, or for about 110 seconds. To find a more precise solution, we can use algebraic methods. As an example, we'll solve the slightly simpler equation

$$139 - 125\cos(\theta) = 240.$$

We'll look for all solutions for θ between 0° and 360°. We begin by isolating the trigonometric ratio on one side of the equation.

$$\begin{aligned} 139 - 125\cos(\theta) &= 240 \\ \implies -125\cos(\theta) &= 101 \qquad \text{(subtract 139 from both sides)} \\ \implies \cos(\theta) &= -0.808 \qquad \text{(divide both sides by } -125\text{)} \end{aligned}$$

We have solved equations like this one before: we use the inverse cosine to solve for θ. Remember that there are two angles between 0° and 360° that have a cosine of -0.808, one in the second quadrant and one in the third quadrant. The reference angle is

$$\hat{\theta} = \cos^{-1}(|-0.808|) = \cos^{-1}(0.808) \approx 36.09901626^\circ.$$

Thus, the solution (rounded to the nearest tenth of a degree) in the second quadrant is

$$\theta = 180^\circ - \hat{\theta} \approx 143.9^\circ$$

and the solution in the third quadrant is

$$\theta = 180^\circ + \hat{\theta} \approx 216.1^\circ.$$

It's important to note that we have not discussed how to solve the original equation, $240 = 139 - 125\cos\left(\frac{2t}{3}\right)$. Instead, we considered the slightly simpler version, $240 = 139 - 125\cos(\theta)$. We will explore the original version a little bit later in this section.

In general, our old techniques of solving algebraic equations for an unknown still hold when solving trigonometric equations.

Example 7.3.3

Solve the equation $4\csc(\theta) - 1 = 7$ for $0^\circ \le \theta \le 360^\circ$.

Solution available at the end of this section.

Checkpoint 7.3.3

Solve the equation $5\tan(\theta) + 7 = 2$ for $0 \le \theta \le 2\pi$.

Answer available at the end of this section.

Example 7.3.3 and Checkpoint 7.3.3 illustrate an **implied angle measure**. That is, in Example 7.3.3, the statement, "for $0^\circ \le \theta \le 360^\circ$" implies our answer should be in degrees. Likewise, the statement, "for $0 \le \theta \le 2\pi$," in Checkpoint 7.3.3, implies our answer should be in radians.

Example 7.3.4

Solve the equation $3\tan(\beta) + 1 = -8$ for $0 \le \beta \le 2\pi$. Round your solutions to three decimal places.

Solution available at the end of this section.

It should not be surprising in Example 7.3.4 that the solutions are π radians apart. As we mentioned previously, the solutions of an equation $\tan(\theta) = k$ always differ by π radians (or 180°). Once we have found one solution, we can add π to find the other solution between 0 and 2π.

Checkpoint 7.3.4

Solve the equation $3\sec(C) - 6 = -2$, for $0^\circ \le C \le 360^\circ$. Round your solutions to three decimal places.

Answer available at the end of this section.

Example 7.3.5

Solve $\frac{\sin(\beta)}{5} - 3 = 1$ for $0 \leq \beta \leq 2\pi$.

Solution available at the end of this section.

Checkpoint 7.3.5

Solve $2\sec(\theta) - 5 = -4$ for $0^\circ \leq \theta \leq 360^\circ$.

Answer available at the end of this section.

Example 7.3.5 and Checkpoint 7.3.5 show us that not all trigonometric equations have solutions. Just as you did in Algebra, you should pause after each step to check the result for realism. Forming this into a habit can save you time and frustration in the future.

The equations in Examples 7.3.3 through 7.3.5 are **linear-in-form**. That is, if you substitute a variable, say t, for the trigonometric function, the equation would look like a linear equation in each case. The initial steps to solve the equation mimic those of a linear equation.

For example, the equation in Example 7.3.4 could be written as $3t + 1 = -8$. Solving for t, we get $t = -3$; however, t was a substitution for $\tan(\beta)$. Thus, $\tan(\beta) = -3$.

This tactic of simplifying the look of an equation using a substitution is beneficial as you move forward in Mathematics. While we will not use it in every example, Example 7.3.6 (below) shows how it can be helpful.

Example 7.3.6

Solve $4\tan^2(\theta) + 3 = 15$ for $0 \leq \theta \leq 2\pi$.

Solution available at the end of this section.

Checkpoint 7.3.6

Solve the equation $2\sin^2(\theta) = 1$ for $0^\circ \leq \theta \leq 360^\circ$.

Answer available at the end of this section.

Example 7.3.6 is **quadratic-in-form** because, with the substitution $t = \tan(\theta)$, it can be written as the quadratic equation $4t^2 + 3 = 15$.

Definition: Quadratic-in-Form

An equation in the variable x is **quadratic-in-form** if it can be rewritten as

$$au^2 + bu + c = 0,$$

where a, b, and c are constants with $a \neq 0$, and u represents an algebraic expression in x.

From this point forward in Mathematics, you will be using your techniques from Algebra to solve trigonometric equations that are

quadratic-in-form. If you are rusty with that material, I highly recommend reviewing it.

Example 7.3.7

Solve $2\cos^2(\theta) - \cos(\theta) - 1 = 0$ for $0 \leq \theta \leq 2\pi$.

Solution available at the end of this section.

Checkpoint 7.3.7

Solve $9\sin^2(\theta) - 3\sin(\theta) = 2$ for $0^\circ \leq \theta \leq 360^\circ$. Round your answers to the nearest degree.

Answer available at the end of this section.

Solving Trigonometric Equations with Non-Standard Arguments

If n is an integer, what can we say about the solutions of the equation

$$\sin(nx) = k?$$

For example, how many solutions are there for the equation $\sin(2x) = 0.5$? Truthfully, we already know there will be an infinite number of solutions; however, on the interval from 0 to 2π, how many solutions will this equation have?

As was stated earlier in the text, the coefficient of x can be thought of as the number of oscillations over an interval of length equal to the natural period of the function. That is, $\sin(2x)$ goes through two full oscillations on any interval of length 2π. Compare Figure 7.3.4 (below) to Figure 7.3.2.

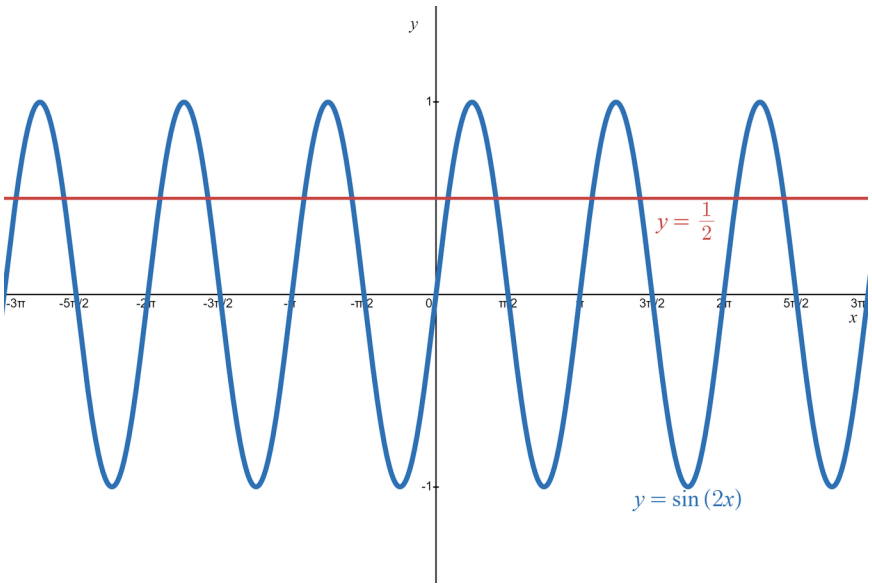


Figure 7.3.4: *Visualizing solutions to* $\sin(2x) = 0.5$

You can see there are twice as many solutions on the same interval as that of $\sin(x) = 0.5$. This visual helps us wrap our brains around the fact that the number of solutions to such equations over an interval like $[0, 2\pi]$ might be more (or less) than two.

Example 7.3.8

Find all solutions of $\sin(2x) = 0.5$ between 0 and 2π.

Solution available at the end of this section.

Notice in Example 7.3.8, that we had *twice* the number of solutions we would typically expect. Do you see why?

Checkpoint 7.3.8

a. Sketch a graph of $y = \cos(2x)$ for $0 \le \theta \le 2\pi$.
b. Find exact values for all solutions of $\cos(2x) = -\frac{1}{\sqrt{2}}$ between 0 and 2π.

Answer available at the end of this section.

As we observed earlier, equations involving the tangent function are easier to solve because there is only one solution in each cycle of the graph. Once we have found one solution, we can find all the others by adding multiples of the period.

Example 7.3.9

Find all solutions of $\tan(2t) = -1$ between 0 and 2π.

Solution available at the end of this section.

Checkpoint 7.3.9

Find all solutions of $\tan(2x) = \sqrt{3}$ between 0 and 2π.

Answer available at the end of this section.

The tactics shown in Examples 7.3.8 and 7.3.9 work for more complicated equations.

Example 7.3.10

Solve $\sin(2x + 1.5) = -0.3$ for $0 \le x \le 2\pi$. Round your answer to four decimal places.

Solution available at the end of this section.

Checkpoint 7.3.10

Solve $4\cos(3x - 0.5) = -3.2$ for $0 \le x \le 2\pi$.

Answer available at the end of this section.

Applications

Trigonometric equations often arise in the study of periodic models.

Example 7.3.11

A piston is pumping vertically at a rate of 1000 cycles per second. The distance between its lowest and highest position is 16 centimeters.

a. Suppose the piston is at its midline position at $t = 0$ and moving downwards. Find a formula for the sinusoidal function $h(t)$ that gives the piston's height.
b. Use the formula to find the first two times the piston is 14 centimeters above its lowest position.

Solutions available at the end of this section.

Checkpoint 7.3.11

A lever on an oil well is pumping vertically at a rate of 10 cycles per minute. The distance between its lowest position at the ground and its highest position is 1.8 meters.

a. Suppose the lever is at its midline position at $t = 0$ and moving downwards. Find a formula for the sinusoidal function $h(t)$ that gives the lever's height.
b. Find the first two times the lever is 1 meter above its lowest position.

Answers available at the end of this section.

Applications of Mathematics: Game Theory Could Save the World!

The following is from The Telegraph (U.K. newspaper):

> A team has been using "game theory" - where mathematics is used to capture how people deal with each other - to show that a classic problem that undermines the ability of individuals to cooperate can be overcome, if people are diverse enough, as is the case when it comes to the 6.5 billion citizens of planet Earth.
>
> Working together for the common good is crucial for progress in any society - not least for effectively addressing big issues such as recycling and tackling climate change. But there is a basic problem with how to make the public share responsibility for common problems, such as climate change.
>
> This was most vividly illustrated by Prof Garrett Hardin, of the University of California, Santa Barbara, in his highly influential 1968 paper 'The tragedy of the commons'.
>
> He used the example of a public pasture. Each herdsman will keep adding cows to a common field, because the benefit of an additional cow goes exclusively to the herdsman. Because the cost of overgrazing is shared by all, the pasture will end up ruined.
>
> Mathematical models have confirmed that this is a profound problem, whether for coming together to fight climate change or pay taxes, but today, in the journal Nature, hope for cooperation is raised by Prof Jorge Pacheco of the University of Lisbon, who did the work there with Ms Marta Santos and Dr Francisco Santos from the Free University of Brussels.
>
> Existing mathematical models treat individuals as equivalent, ignoring real-life diversity and population structure. So the team made the mathematics more realistic in this respect.
>
> The team shows that, contrary to expectations, the temptation to cheat ("Why shouldn't I let others pay more for being green?") declines as society becomes more diverse.
>
> Another discovery is that diversity also plays an important role in wealth distribution.
>
> "Cooperation blooms whenever the act of giving is more important than the amount given," said Prof Pacheco.
>
> The most important factor, he said, seems to be that people actually contribute to the public good.
>
> "In particular, the fact that the wealthy minority contributes stimulates the contribution from the rest, who take them as role models," he said.
>
> Prof Pacheco recalled the words of Anna Dreber and Prof Martin Nowak from Harvard University, leading game theorists,

who stated that "preserving the global climate is the biggest public good dilemma ever, the one we cannot afford to lose."

1. The basis of this article, game theory, is used to predict human behavior in many areas. Name one possible application of game theory.
2. It turns out that we, as humans, are reasonably predictable. We do things in patterns, and our behaviors can be modeled using complex functions. Do you believe there is a formula for how many heartbeats you will have (that may depend on how many cigarettes you smoke, steps you take, etc.)?

Success in College: Groupwork Learning Preferences

Most students are very intimidated by working in groups. It usually takes quite a while for students to realize that studying together can be very rewarding.

Think about the following questions regarding learning or practicing complicated math.

- Do you prefer to do homework by yourself, or do you like to do homework with others?
- Do you like to do homework at school, or do you like to do homework at home?
- If you need help, do you prefer to get it from friends or a spouse, or do you prefer to get it from your professor or a tutor?
- Do you concentrate on learning what you don't know by yourself, or do you prefer to help others because it helps you learn it better?

1. Do you work in conditions that match these preferences when you do your math homework? Why or why not?
2. Spending time with old and new friends is an integral part of college. However, studying with friends is not always practical. Do you ever feel peer pressure to study in ways that are not the best for you?

Answers

Section Preview

The equations in this section include everything *except for* those requiring trigonometric identities. Hence, students need to recall how to solve linear, quadratic, and rational equations. Moreover, they will need to recall what it means to factor out a GCF and the implications of dividing both sides of an equation by an algebraic expression (something I just tell students to not do, but to each their own).

Section Preview

- **Factoring Techniques**
  - Factoring Expressions that are Quadratic-in-Form: Needed to help solve many equations.
- **Solving Equations**
  - Solving Equations Quadratic-in-Form: Example 7 introduces quadratic-in-form equations solved by u-substitution.

Section Preview

- Solve a simple trigonometric equation.
- Use techniques from algebra to solve a trigonometric equation.

Example $7.3.1$

Since -0.6428 is not a special ratio, we will need to use technology to help us find the solution; however, technology should be reserved for the last step in any mathematical procedure.
Quadrants: Since $\sin(\theta)$ is negative, we know that θ is either in QIII or QIV.
Reference Angle: To find the reference angle, we compute

$$\sin^{-1}(|-0.6428|) = \sin^{-1}(0.6428) \approx 40.0^\circ.$$

So we have a reference angle, $\hat{\theta} \approx 40.0^\circ$, and *two* quadrants, QIII and QIV.
Solutions Associated with Quadrant III: The solution in QIII having reference angle $\hat{\theta} \approx 40.0^\circ$ is

$$\theta = 180^\circ + \hat{\theta} \approx 220.0^\circ.$$

Since $\sin(\theta) = \sin(\theta + 360^\circ k)$ (where $k \in \mathbb{Z}$), any angle coterminal to 220.0° will also lead to a solution of our equation. Thus, the general solution associated with Quadrant III is

$$\theta \approx 220.0^\circ + 360^\circ k, \quad \text{where } k \in \mathbb{Z}.$$

Solutions Associated with Quadrant IV: Taking a lesson from the previous paragraph, the solution in QIV having reference angle $\hat{\theta} \approx 40.0^\circ$ is

$$\theta = 360^\circ - \hat{\theta} + 360^\circ k \approx 320.0^\circ + 360^\circ k, \quad \text{where } k \in \mathbb{Z}.$$

Now that we have all solutions, let's restrict them to only the solutions occurring on the interval $[0^\circ, 360^\circ]$. To do this, we consider both sets of infinite solutions,

$$\theta \approx 220.0^\circ + 360^\circ k$$
$$\theta \approx 320.0^\circ + 360^\circ k,$$

and we let $k = 0$ (any other value of k will result in angles outside of the requested interval). Hence, the solutions on $[0^\circ, 360^\circ]$ are approximately 220.0° and 320.0°.

Checkpoint 7.3.1

$203.6^\circ, 336.4^\circ$

Example 7.3.2

a. Since $\frac{1}{2}$ is a special ratio for the sine function, we do not bother (and should not bother) grabbing a calculator.
Quadrants: Since $\sin(x) = \frac{1}{2}$ is positive, we know that x is either in QI or QII.
Reference Angle: From our special triangles, we know that $\hat{x} = \frac{\pi}{6}$.
Solutions Associated with Quadrant I: The angle in Quadrant I with reference angle $\frac{\pi}{6}$ is $x = \frac{\pi}{6}$. Any angle coterminal with this will also be a solution to the given equation, so our set of solutions associated with Quadrant I is

$$x = \frac{\pi}{6} + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

Solutions Associated with Quadrant II: The angle in Quadrant II with reference angle $\frac{\pi}{6}$ is $\frac{5\pi}{6}$. Thus,

$$x = \frac{5\pi}{6} + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

Finally, the solutions in the interval $[0, 2\pi]$ are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

b. -0.6 is not a special ratio for cosine, so we will eventually need to reach for technology to solve this equation.
Quadrants: We are given a negative cosine function, and we know this can only occur if the argument of the cosine terminates in either QII or QIII.
Reference Angle: To find the reference angle associated with the equation $\cos(x) = -0.6$, we approximate $\cos^{-1}(|-0.6|) = \cos^{-1}(0.6)$. It's important to note that we are looking for radian solutions to this equation, so we need to switch our calculator into radian mode.

$$\hat{x} = \cos^{-1}(0.6) \approx 0.927295218$$

Notice that I wrote down all of the digits to the approximation the calculator returned. This is because we should save our rounding until the very end of the problem. We want to work with as many digits as possible but *display* a solution rounded to the nearest tenth to the reader.
Solutions Associated with Quadrant II: The angle in Quadrant II having reference angle $\hat{x}$ is $\pi - \hat{x}$. Again, we list all coterminal angles as well.

$$x = \pi - \hat{x} + 2\pi k \approx 2.2 + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

Solutions Associated with Quadrant III: The angle in Quadrant III having reference angle $\hat{x}$ is $\pi + \hat{x}$. Therefore,

$$x = \pi + \hat{x} + 2\pi k \approx 4.1 + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

The solutions restricted to the interval $[0, 2\pi]$ are $x \approx 2.2$ and $x \approx 4.1$.

c. $\frac{\sqrt{3}}{1}$ is definitely a special ratio for the tangent function, so we will abstain from calculator usage here.
Quadrants: The given tangent function is negative, so we know the argument of the function terminates in either QII or QIV.
Reference Angle: From our special triangles, $\hat{x} = \frac{\pi}{3}$.
Solutions Associated with Quadrant II: The angle in Quadrant II having reference angle $\hat{x}$ is $\pi - \hat{x}$. Therefore,

$$x = \pi - \hat{x} + 2\pi k = \pi - \frac{\pi}{3} + 2\pi k = \frac{2\pi}{3} + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

Solutions Associated with Quadrant IV: The angle in Quadrant IV having reference angle $\hat{x}$ is $2\pi - \hat{x}$. Therefore,

$$x = 2\pi - \hat{x} + 2\pi k = 2\pi - \frac{\pi}{3} + 2\pi k = \frac{5\pi}{3} + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

The solutions restricted to the interval $[0, 2\pi]$ are $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$.

d. 3.7 is not a special ratio for the secant, so we know we will need a calculator at some point.
Quadrants: The given secant function is positive. Therefore, the argument terminates in either QI or QIV.
Reference Angle: Since our calculator doesn't have an inverse secant button, we rewrite $\sec(x) = 3.7$ using the Reciprocal Identities as

$$\cos(x) = \frac{1}{3.7}.$$

Therefore,

$$\hat{x} = \cos^{-1}\left(\left|\frac{1}{3.7}\right|\right) = \cos^{-1}\left(\frac{1}{3.7}\right) \approx 1.297122589.$$

Solutions Associated with Quadrant I: The angle in Quadrant I having reference angle $\hat{x}$ is $\hat{x}$. Thus,

$$x = \hat{x} + 2\pi k \approx 1.3 + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

Solutions Associated with Quadrant IV: The angle in Quadrant IV having reference angle $\hat{x}$ is $2\pi - \hat{x}$. Therefore,

$$x = 2\pi - \hat{x} + 2\pi k \approx 5.0 + 2\pi k, \text{ where } k \in \mathbb{Z}.$$

The solutions restricted to the interval $[0, 2\pi]$ are $x \approx 1.3$ and $x \approx 5.0$.

Checkpoint 7.3.2

$$\text{Solutions associated with QII}: \quad x = \frac{3\pi}{4} + 2\pi k, \text{ where } k \in \mathbb{Z}$$
$$\text{Solutions associated with QIII}: \quad x = \frac{5\pi}{4} + 2\pi k \text{ where } k \in \mathbb{Z}$$

On $[0, 2\pi]$, the solutions are $x = \frac{3\pi}{4}$ and $x = \frac{5\pi}{4}$.

Example 7.3.3

We begin by isolating the trigonometric ratio.

$$\begin{aligned} 4\csc(\theta) - 1 &= 7 \\ \Longrightarrow \quad 4\csc(\theta) &= 8 && \text{(add 1 to both sides)} \\ \Longrightarrow \quad \csc(\theta) &= 2 && \text{(divide both sides by 4)} \\ \Longrightarrow \quad \sin(\theta) &= \frac{1}{2} && \text{(Reciprocal Identity)} \end{aligned}$$

Quadrants: Sine is positive in QI and QII.
Reference Angle: $\frac{1}{2}$ is a special ratio for the sine function. Using our knowledge of the special triangles, we get the reference angle $\hat{\theta} = 30^\circ$.
Now that we know the angles terminate in QI and QII with reference angle $\hat{\theta} = 30^\circ$, we can confidently state the solutions are $\theta = 30^\circ$ and $\theta = 150^\circ$.

Checkpoint 7.3.3

$\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$

Example 7.3.4

First, we isolate the trigonometric ratio.

$$\begin{aligned} 3\tan(\beta) + 1 &= -8 \\ \Longrightarrow \quad 3\tan(\beta) &= -9 && \text{(subtract 1 from both sides)} \\ \Longrightarrow \quad \tan(\beta) &= -3 && \text{(divide both sides by 3)} \end{aligned}$$

There are two angles with tangent -3, one in the second quadrant and one in the fourth quadrant. The reference angle is $\hat{\beta} = \tan^{-1}(3) \approx 1.249045772$. Thus, we get the following:

$$\begin{aligned} \text{Solution Associated with QII}: \quad \beta &= \pi - \hat{\beta} \approx 1.893 \\ \text{Solution Associated with QIV}: \quad \beta &= 2\pi - \hat{\beta} \approx 5.034 \end{aligned}$$

Checkpoint 7.3.4

$C = 41.410^\circ$ or $C = 318.590^\circ$

Example 7.3.5

We begin by isolating $\sin(\beta)$.

$$\begin{aligned} \frac{\sin(\beta)}{5} - 3 &= 1 \\ \Longrightarrow \quad \frac{\sin(\beta)}{5} &= 4 && \text{(add 3 to both sides)} \\ \Longrightarrow \quad \sin(\beta) &= 20 && \text{(multiply both sides by 5)} \end{aligned}$$

Because $\sin(\beta)$ is never greater than 1, there is no angle β whose sine is 20. The equation has no solution.

Checkpoint 7.3.5

No solution.

Example 7.3.6

We start with the substitution $t = \tan(\theta)$ and solve $4t^2 + 3 = 15$ for t.

$$\begin{aligned} 4t^2 + 3 &= 15 \\ \Longrightarrow \quad 4t^2 &= 12 && \text{(subtract 3 from both sides)} \\ \Longrightarrow \quad t^2 &= 3 && \text{(divide both sides by 4)} \\ \Longrightarrow \quad t &= \pm\sqrt{3} && \text{(Extraction of Roots)} \end{aligned}$$

Resubstituting $\tan(\theta)$ for t, we get

$$\tan(\theta) = \pm\sqrt{3}.$$

There are two angles between 0 and 2π with tangent $\sqrt{3}$ and two angles with tangent $-\sqrt{3}$, making four solutions to this equation—one in each quadrant! Moreover, we know the reference angle such that $\tan\left(\hat{\theta}\right) = \sqrt{3}$ is $\hat{\theta} = \frac{\pi}{3}$.

Recall that tangent is the special function with period π. Therefore, we only need to list the solutions associated with QI and QII and add integer multiples of π to get all other solutions.

$$\begin{aligned} \text{Solutions associated with QI}: \quad \theta &= \frac{\pi}{3} + \pi k, \text{ where } k \in \mathbb{Z} \\ \text{Solutions associated with QII}: \quad \theta &= \pi - \hat{\theta} + \pi k = \pi - \frac{\pi}{3} + \pi k = \frac{2\pi}{3} + \pi k, \text{ where } k \in \mathbb{Z} \end{aligned}$$

To restrict this infinite set of solutions to those in the interval $[0, 2\pi]$, we let $k = 0$ and $k = 1$ to get

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ and } \frac{5\pi}{3}.$$

Checkpoint 7.3.6

$\theta = 45^\circ, \theta = 135^\circ, \theta = 225^\circ$, or $\theta = 315^\circ$

Example 7.3.7

Compare this equation to the algebraic equation $2t^2 - t - 1 = 0$. Our equation has the same algebraic form but with t replaced by $\cos(\theta)$.
We can solve the algebraic equation by factoring $2t^2 - t - 1$ as $(2t+1)(t-1)$, and we'll use the same strategy on the trigonometric equation.

$$\begin{aligned} 2\cos^2(\theta) - \cos(\theta) - 1 &= 0 \\ \implies \quad (2\cos(\theta)+1)(\cos(\theta)-1) &= 0 \qquad \text{(factor)} \end{aligned}$$

By the Zero Factor Property, this means

$$\begin{aligned} 2\cos(\theta)+1 = 0 \quad &\text{or} \quad \cos(\theta) - 1 = 0 \\ \implies \cos(\theta) = -\frac{1}{2} \quad &\text{or} \quad \cos(\theta) = 1 \end{aligned}$$

Now we solve each equation for θ.

For $\cos(\theta) = -\frac{1}{2}$: We know cosine is negative in QII and QIII and that $\hat{\theta} = \frac{\pi}{3}$. Thus,

$$\text{Solutions associated with QII}: \quad \theta = \pi - \hat{\theta} + 2\pi k = \pi - \frac{\pi}{3} + 2\pi k = \frac{2\pi}{3} + 2\pi k$$

$$\text{Solutions associated with QIII}: \quad \theta = \pi + \hat{\theta} + 2\pi k = \pi + \frac{\pi}{3} + 2\pi k = \frac{4\pi}{3} + 2\pi k$$

For $\cos(\theta) = 1$: This will result in a quadrantal angle. Visualizing the graph of the cosine, we can (hopefully) see that the cosine attains a value of 1 at $x = 0$ and every integer multiple of 2π after that. That is,

$$\theta = 0 + 2\pi k = 2\pi k.$$

Combining all of these solution sets and restricting our answers to those between 0 and 2π, we get

$$\theta = 0,\ \theta = \frac{2\pi}{3},\ \theta = \frac{4\pi}{3},\ \text{or}\ \theta = 2\pi.$$

Checkpoint 7.3.7

$\theta = 42^\circ, \theta = 138^\circ, \theta = 199^\circ$, or $\theta = 341^\circ$

Example 7.3.8

Treat the argument as if it were a single variable. The sine is positive, meaning the angle, $2x$, must terminate in QI or QII. Additionally, we know that the reference angle is $\widehat{2x} = \frac{\pi}{6}$. At this point, let's pause to make sure we understand what just happened. I am treating the argument of the sine, which is $2x$, as a single variable—sort of like calling it θ. Once I find a reference angle, I have found it for that single variable, which is $2x$. Like we traditionally do, we use the "hat" notation to denote the reference angle; however, in this case, the reference angle is for $2x$—not x. Therefore, our reference angle notation is $\widehat{2x}$.

Continuing as usual,

$$\text{Solutions associated with QI}: \quad 2x = \frac{\pi}{6} + 2\pi k$$

$$\text{Solutions associated with QII}: \quad 2x = \frac{5\pi}{6} + 2\pi k$$

where $k \in \mathbb{Z}$. We finally arrive at the reason why I write "Solutions **associated with**" rather than "Solutions **in**." Our goal is to solve for x. This requires us to divide both equations by 2 (including dividing the $2\pi k$). Therefore, we get

$$\text{Solutions associated with QI}: \quad x = \frac{\pi}{12} + \pi k$$

$$\text{Solutions associated with QII}: \quad x = \frac{5\pi}{12} + \pi k$$

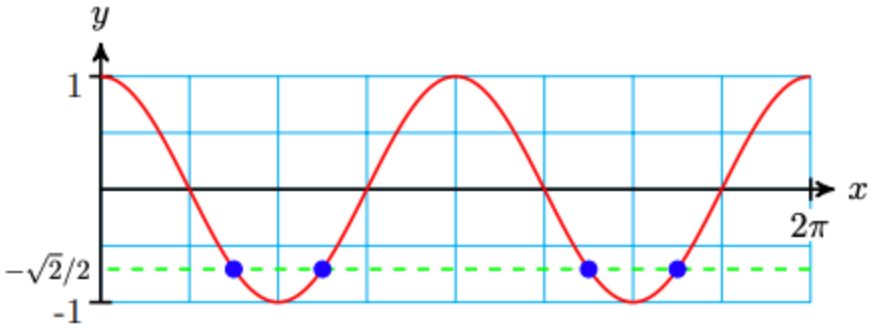
where $k \in \mathbb{Z}$. Do you notice that the "Solutions associated with QII" doesn't start with a Quadrant II angle? That's perfectly fine! The solution angles have been adjusted by the coefficient of x so they might not belong to the original quadrant they were derived from. This is why I use the words "associated with."

We now need to restrict our answers to the interval $[0, 2\pi]$. Allowing $k = 0$, we get $x = \frac{\pi}{12}$ and $x = \frac{5\pi}{12}$—both of which are in the requested interval. Trying $k = 1$, we get $x = \frac{13\pi}{12}$ and $x = \frac{17\pi}{12}$. Again, both of these solutions are still within the requested interval. Trying $k = 2$, however, gives $x = \frac{25\pi}{12}$ and $x = \frac{29\pi}{12}$. Neither of these are in the requested interval. Thus, the solutions on $[0, 2\pi]$ are

$$x = \frac{\pi}{12},\ x = \frac{5\pi}{12},\ \frac{13\pi}{12},\ \text{or}\ x = \frac{17\pi}{12}.$$

Checkpoint 7.3.8

a.



b. $\frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$

Example 7.3.9

Again, treating the argument, $2t$, as a single variable, we know the tangent is negative in QII and QIV; however, the tangent is that special case (along with the cotangent) where we can say the second solution is just π plus the first solution. So, we only need to concentrate on the solution in the second quadrant. The reference angle is $\widehat{2t} = \frac{\pi}{4}$. This is all we need!

$$\text{Solutions associated with QII}: \quad 2t = \frac{3\pi}{4} + \pi k$$

where $k \in \mathbb{Z}$. We then divide both sides of the equation by 2 to get

$$\text{Solutions associated with QII}: \quad t = \frac{3\pi}{8} + \frac{\pi}{2} k$$

where $k \in \mathbb{Z}$. Just so you can see a different approach, let's build a table to get all the solutions between 0 and 2π.



If we move to $k = 4$, the solutions will be outside of the requested interval. Thus, our solutions are $\frac{3\pi}{8}$, $\frac{7\pi}{8}$, $\frac{11\pi}{8}$ and $\frac{15\pi}{8}$. The following figure gives a visual of these solutions.

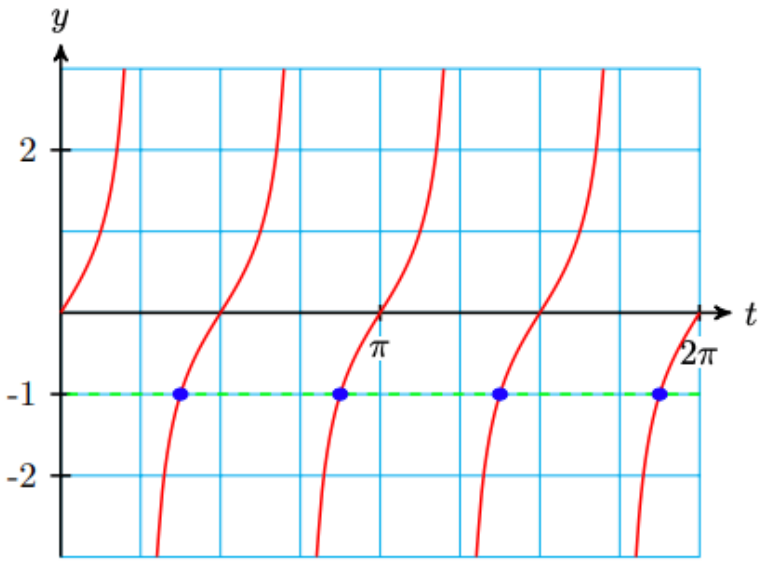


Figure 7.3.5

Checkpoint 7.3.9

$\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

Example 7.3.10

Again, we treat the entire argument as a single variable. The sine is negative in QIII and QIV. Since -0.3 is not a special ratio for the sine, we will have to use technology. The reference angle is given by

$$2\widehat{x+1.5} = \sin^{-1}(|-0.3|) = \sin^{-1}(0.3) \approx 0.304692654.$$

Therefore,

$$\begin{aligned} \text{Solutions associated with QIII}: \quad 2x+1.5 &= \pi + 2\widehat{x+1.5} + 2\pi k \\ \text{Solutions associated with QIV}: \quad 2x+1.5 &= 2\pi - 2\widehat{x+1.5} + 2\pi k \end{aligned}$$

where $k \in \mathbb{Z}$. Despite having an approximation for the reference angle, $2\widehat{x+1.5}$, I like to leave the equations in exact form (without decimal approximations) until I get to a final answer. Subtracting 1.5 from both sides of each equation and dividing by 2, we get the following:

$$\begin{aligned} \text{Solutions associated with QIII}: \quad x &= \frac{\pi}{2} + \frac{2\widehat{x+1.5}}{2} + \pi k - 0.75 \\ \text{Solutions associated with QIV}: \quad x &= \pi - \frac{2\widehat{x+1.5}}{2} + \pi k - 0.75 \end{aligned}$$

where $k \in \mathbb{Z}$. At this time, I will use technology to find an approximation for these solutions.

$$\begin{aligned} \text{Solutions associated with QIII}: \quad x &= 0.973142654 + \pi k \\ \text{Solutions associated with QIV}: \quad x &= 2.239246327 + \pi k \end{aligned}$$

where $k \in \mathbb{Z}$. Let's build a table like we did in Example 7.3.9.

[redacted]

Going to $k = 2$ gives solutions outside of the requested interval. Thus, the solutions on $[0, 2\pi]$ are 0.9731, 2.2392, 4.1147, and 5.3808.

Checkpoint 7.3.10

1.00, 1.43, 3.09, 3.52, 5.19, 5.62

Example 7.3.11

a. Because the piston starts at its midline and moves down, we will write $h(t)$ as a sine function:

$$h(t) = -A\sin(Bt) + D.$$

The amplitude is $A = \frac{16}{2} = 8$, and the midline is $D = 8$. The period is $\frac{1}{1000}$, so $\frac{B}{1000} = 2\pi$, and $B = 2000\pi$. Thus,

$$h(t) = -8\sin(2000\pi t) + 8.$$

The graph is shown in Figure 7.3.6.

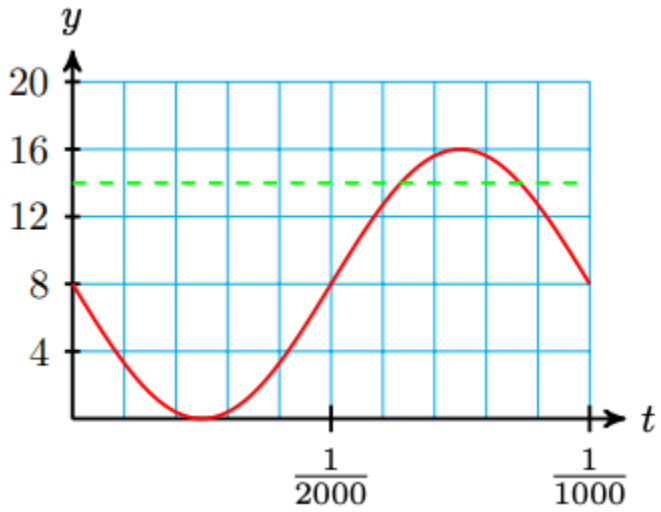


Figure 7.3.6

b. With $h(t) = 14$, we have

$$\begin{aligned} -8\sin(2000\pi t) + 8 &= 14 \\ \implies -8\sin(2000\pi t) &= 6 \qquad \text{(subtract 8 from both sides)} \\ \implies \sin(2000\pi t) &= -\frac{3}{4} \qquad \text{(divide both sides by } -8\text{)} \end{aligned}$$

Sine is negative in QIII and QIV. Moreover, we can compute the reference angle as follows:

$$\widehat{2000\pi t} = \sin^{-1}\left(\left|-\frac{3}{4}\right|\right) = \sin^{-1}\left(\frac{3}{4}\right) \approx 0.848062079.$$

We want the first two *positive* solutions to this equation.

$$\text{Solutions associated with QIII}: \quad 2000\pi t = \pi + \sin^{-1}\left(\frac{3}{4}\right) + 2\pi k \quad \Longrightarrow \quad t = \frac{1}{2000} + \frac{\sin^{-1}\left(\frac{3}{4}\right)}{2000\pi} + \frac{1}{1000}k$$

$$\text{Solutions associated with QIV}: \quad 2000\pi t = 2\pi - \sin^{-1}\left(\frac{3}{4}\right) + 2\pi k \quad \Longrightarrow \quad t = \frac{1}{1000} - \frac{\sin^{-1}\left(\frac{3}{4}\right)}{2000\pi} + \frac{1}{1000}k$$

where $k \in \mathbb{Z}$. Building a quick table, we get the following:



From our table, we can see the first two times the piston is at a height of 14 centimeters are approximately $t = 0.000865$ and $t = 0.000635$ seconds.

Checkpoint 7.3.11

a. $h(t) = -0.9\sin(20\pi t) + 0.9$
b. 0.0518 minutes and 0.9823 minutes

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7.3.1: Resources and Key Concepts

Resources

Prerequisite Topics

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Solving Equations**
  - Solving Quadratic Equations: This section requires solving equations that are quadratic-in-form. The "Skills Refresher" also includes basic quadratic equations.
  - Solving Equations Quadratic-in-Form: This section uses this skill frequently.
  - Solving Rational Equations: This skill is needed for the "Skills Refresher" exercises.

Videos

- Solving Trigonometric Equations
  - Solving Trigonometric Equations - The Concepts Behind Solving Trigonometric Equations
  - Solving Trigonometric Equations - Basic Equations

Key Concepts

Definitions

- **Trigonometric Equation:** An equation that involves one or more trigonometric functions of a variable.
- **Quadratic-in-Form:** An equation that can be written in the form $a[f(x)]^2 + b[f(x)] + c = 0$, where $f(x)$ is some function.

Common Mistakes

- **Forgetting to Find All Solutions:** Trigonometric equations typically have an infinite number of solutions due to the periodic nature of the functions. Forgetting to add multiples of the period ($+2\pi k$ or πk) will result in an incomplete solution set.
- **Stopping After the Inverse Function:** Using an inverse trigonometric function on a calculator provides only one solution (the principal value). You must use that angle as a reference angle to find all solutions in the required quadrants.
- **Incorrectly Handling Non-Standard Arguments:** When solving an equation like $\sin(2x) = k$, the solutions found are for the argument $2x$. A common mistake is forgetting the final step of dividing by 2 to solve for x.
- **Not Checking for Extraneous or Invalid Solutions:** It is important to check for realism. For example, an equation that simplifies to $\sin(\beta) = 20$ has no solution, since the range of the sine function is $[-1, 1]$.
- **Algebraic Errors:** Common algebraic mistakes, such as incorrectly applying the zero-factor principle or dividing by a variable that could be zero, can lead to incorrect or incomplete solutions.

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7.3.2: Homework

Reading Questions

1. Why do trigonometric equations typically have an infinite number of solutions?
2. What are the two fundamental concepts from trigonometry that are the keys to finding all solutions to a trigonometric equation?
3. If you are solving $\cos(\theta) = -0.6$, in which two quadrants will the solutions lie?
4. After using a calculator to find a reference angle, how do you find the general solution for all angles coterminal with that solution?
5. What does it mean for a trigonometric equation to be "quadratic-in-form"?
6. What algebraic technique is used to solve an equation like $2\cos^2(\theta) - \cos(\theta) - 1 = 0$?
7. When solving an equation involving $\tan(x)$ or $\cot(x)$, how does finding the general solution differ from finding it for sine or cosine?
8. When solving an equation with a non-standard argument, such as $\sin(2x) = 0.5$, what should you treat as the "angle" when finding the reference angle?
9. How does the coefficient on the variable, like the '2' in $\sin(2x)$, affect the number of solutions within the interval $[0, 2\pi]$?
10. Why does an equation like $\sin(x) = 5$ have no solution?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, solve the equation.

1. $(2x-6)^2 = 4$
2. $\sqrt{2x-6} = 4$
3. $\dfrac{1}{2x-6} = 4$
4. $|2x-6| = 4$
5. $\log 2x - 6 = 4$
6. $2^{2x-6} = 4$

For the following exercises, the "solution" contains an error. Find the error, and supply a correct solution.

7.

$$\begin{aligned} 3x^2 - 5x &= 0 \\ \implies 3x^2 &= 5x \\ \implies 3x &= 5 \\ \implies x &= \frac{5}{3} \end{aligned}$$

8.

$$\begin{aligned} 4x^2 - 5x &= 12 \\ \implies x^2 &= 3 \\ \implies x &= \sqrt{3} \end{aligned}$$

9.

$$\begin{aligned} (2x+1)^2 &= 4 \\ \implies 2x+1 &= 2 \\ \implies 2x &= 1 \\ \implies x &= \frac{1}{2} \end{aligned}$$

10.

$$\begin{aligned} 2x^2 - 4x &= 3 \\ \implies 2x(x-2) &= 3 \\ \implies 2x = 3 \quad &\text{or} \quad x - 2 = 3 \\ \implies x = \frac{3}{2} \quad &\text{or} \quad x = 5 \end{aligned}$$

Answers available at the end of this section.

Homework

Concept Check

1. How many solutions between 0° and 360° does the equation $\cos(\theta) = k$ have for each value of k between -1 and 0?
2. How many solutions between 0° and 360° does the equation $\cos(\theta) = k$ have for each value of k greater than 1?
3. How many solutions between 0° and 360° does the equation $\sin^2(\theta) = k$ have for each value of k between -1 and 0?
4. If $x = 0.68$ is one solution of the equation $\cos(x) = c$, what is other? Illustrate on a unit circle.
5. If $t = 2.45$ is one solution of the equation $\sin(t) = k$, what is the other? Illustrate on a unit circle.
6. If $\theta = 1.73$ is one solution of the equation $\tan(\theta) = C$, what is the other?
7. Explain why the equation $\cos(nx) = k, \quad 0 < k < 1$, has $2n$ solutions between 0 and 2π.
8. How many solutions are there to the equation $\cos(\theta) = 0.4$? How many solutions are there between 0 and 2π?
9. Find all angles between 0° and 360° for which the statement is true.
   a. $\cos(\theta) = -1$
   b. $\sin(\theta) = -1$
   c. $\tan(\theta) = -1$
10. a. Find two angles, $0 \le \theta \le 360^\circ$, with $\sin(\theta) = 0$.
    b. Find two angles, $0 \le \theta \le 360^\circ$, with $\cos(\theta) = 0$.

11. a. Find two angles, $0 \leq \theta \leq 360^\circ$, with $\sin(\theta) = \cos(\theta)$.

    b. Find two angles, $0 \leq \theta \leq 360^\circ$, with $\sin(\theta) = -\cos(\theta)$.

Basic Skills

For the following exercises,
(a) Evaluate the expression at the given values of the variable.
(b) Give one solution to the equation.

12. a. $\sin(\theta) + \cos(\theta)\,; \quad \theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ$

    b. $\sin(\theta) + \cos(\theta) = \sqrt{2}$

13. a. $\sin^2(\alpha) - \cos(\alpha)\,; \quad \alpha = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

    b. $\sin^2(\alpha) - \cos(\alpha) = 1$

14. a. $\sin(\beta) + 2\cos^2(\beta)\,; \quad \beta = 210^\circ, 225^\circ, 240^\circ, 270^\circ$

    b. $\sin(\beta) + 2\cos^2(\beta) = -1$

15. a. $3\cos^2(\phi) - \sin^2(\phi)\,; \quad \phi = \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{7\pi}{4}, \frac{11\pi}{6}$

    b. $3\cos^2(\phi) - \sin^2(\phi) = 2$

For the following exercises, find all degree solutions (in exact form) and then state the solutions for $0^\circ \leq \theta \leq 360^\circ$.

16. $7\sin(\theta) + 11 = 11$

17. $3 = 5 - 4\cos(\theta)$

18. $3\tan(\theta) = \sqrt{3}$

19. $8\csc(\theta) + 5 = -11$

20. $9\sec(\theta) + 15 = 6$

21. $6\cot(\theta) + 21 = 15$

22. $0 = \sqrt{2} + 2\sin(\theta)$

23. $\sqrt{3}\cos(\theta) = -\frac{3}{2}$

24. $\cos^2(\theta) - 1 = 0$

25. $1 - \sin^2(\theta) = 0$

26. $3\csc^2(\theta) - 4 = 0$

27. $0 = 4 - 2\sec^2(\theta)$

28. $1 - \tan^2(\theta) = 0$

29. $0 = 2\cot^2(\theta) - 6$

30. $2\sin^2(\theta) + \sin(\theta) - 1 = 0$

31. $4 - \cos(\theta) = 4\cos^2(\theta) + 3\cos(\theta) + 5$

32. $2\sin^2(\theta) - 11\sin(\theta) = -5$

33. $2 = \csc^2(\theta) - \csc(\theta)$

34. $\sec^2(\theta) + 3\sec(\theta) + 2 = 0$

35. $\sin(4x) = -1$

36. $\cos(3t) = 0$

37. $5\tan(2q) = 0$

38. $6\sin(4w) = -3\sqrt{2}$

39. $4\cos(3\phi) = -2$

For the following exercises, find all radian solutions (in exact form) and then state the solutions for $0 \leq \theta \leq 2\pi$.

40. $\sqrt{2}\sin(\theta) + 9 = 8$

41. $\tan(x) - 2\sin(x)\tan(x) = 0$

42. $13 = 13 - 4\cos(\theta)$

43. $4\tan(\theta) - 2 = -6$

44. $8\csc(\theta) + 5 = -3$

45. $2\cos^2(t) + \cos(t) = 1$

46. $\sec(x)\sin(x) - 2\sin(x) = 0$

47. $-6\sec(\theta) + 4 = -8$

48. $3\cot(\theta) + \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

49. $0 = 4\sqrt{2} + 8\sin(\theta)$

50. $\sqrt{3}\cos(\theta) = -\frac{3}{2}$

51. $\sin^2(\theta) - 1 = 0$

52. $1 - \cos^2(\theta) = 0$

53. $4\sec^2(\theta) - 3 = 0$

54. $0 = 1.5 - 0.75\csc^2(\theta)$

55. $1 - \cot^2(\theta) = 0$

56. $0 = -6 + 18\tan^2(\theta)$

57. $\tan^5(x) = \tan(x)$

58. $2\sin^2(\theta) + 3\sin(\theta) = -1$

59. $3 = 6\cos^2(\theta) - 3\cos(\theta)$

60. $8\sin^2(x) + 6\sin(x) + 1 = 0$

61. $\tan^2(x) = -1 + 2\tan(-x)$

62. $2\cos^2(\theta) + 7\cos(\theta) + 3 = 0$

63. $-2 = \csc^2(\theta) - 3\csc(\theta)$

64. $\sec^2(\theta) - 2 = -\sec(\theta)$

65. $\sqrt{3}\tan(2\alpha) = 3$

66. $2\sin(2\beta) = 1$

67. $-6\cos(2\theta) = 6$

68. $3\tan(3w) = \sqrt{3}$

69. $2\tan(3u) = -2$

70. $\frac{5}{7}\tan(\pi x) + 11 = 11$

71. $8\sin\left(2\theta - \frac{\pi}{6}\right) = -4$

72. $6\cos\left(3\theta - \frac{\pi}{2}\right) = -3\sqrt{2}$

73. $3\tan\left(\frac{w}{2} + \frac{\pi}{4}\right) + 4 = 1$

74. $50\sec\left(\frac{t}{\pi} + 6\right) - 10 = -110$

75. $2\tan(2\pi\beta) + 5 = 3$

76. $160\sin(\pi\phi - 1) + 10 = 90$

77. $2\cos\left(3t + \frac{\pi}{4}\right) = \sqrt{3}$

78. $2 - \tan\left(2x - \frac{\pi}{3}\right) = 2$

79. $7\csc\left(\frac{\phi}{2} + \frac{3\pi}{4}\right) + 3 = -4$

For the following exercises, solve the equation for $0^\circ \leq \theta \leq 360^\circ$. Round your answers to two decimal places.

80. $4 = 6\sin(\theta)$

81. $3 = 5\cos(\theta)$

82. $\frac{1}{2}\tan(\theta) - 1 = -3$

83. $7\csc(\theta) + 2 = -16$

84. $3\sec(\theta) - 2 = 5$

85. $2 = 5 - \frac{1}{3}\cot(\theta)$

86. $6\cos^2(\theta) = 2$

87. $2 - 7\sin^2(\phi) = 1$

88. $-4\tan^2(\theta) + 5 = -4$

89. $8 - 2\sec^2(\theta) = -2$

90. $3 = 9 - 2\csc^2(\theta)$

91. $9\cot^2(\theta) - 9 = -5$

92. $5\sin^2(\theta)+\sin(\theta)=0$

93. $10\cos^2(\theta)-7\cos(\theta)+1=0$

94. $4\tan^2(\theta)=\tan(\theta)$

95. $\cot^2(\theta)-2\cot(\theta)=15$

96. $5\sin(3\theta)-3=-4$

97. $9\cos(2\theta)+1=6$

98. $8\tan(4t)+1=-11$

99. $6\csc(2r)+2=13$

For the following exercises, solve the equation for $0\le\theta\le 2\pi$. Round your answers to two decimal places.

100. $5\sin(\theta)-4=-6$

101. $-4+3\cos(\theta)=-2$

102. $-2\tan(\theta)+5=6$

103. $-4=5-\csc(\theta)$

104. $3\sec(\theta)-2=-10$

105. $\frac{1}{3}\cot(\theta)+1=-2$

106. $5\sin^2(\theta)=2$

107. $3-6\cos^2(\phi)=-1$

108. $4-4\tan^2(\theta)=-5$

109. $4-\sec^2(\theta)=-1$

110. $-1+2\csc^2(\theta)=5$

111. $\frac{9}{2}\cot^2(\theta)-1=1$

112. $\sin^2(\theta)+8\sin(\theta)+7=0$

113. $\cos^2(\theta)-4\cos(\theta)+3=0$

114. $\tan^2(\theta)-5\tan(\theta)+6=0$

115. $6\cot^2(\theta)-\cot(\theta)-1=0$

116. $150\sin(3s)=27$

117. $7\cos(2t)-3=2$

118. $3=3\tan(4x)+4$

119. $2-8\sec(3t)=-10$

120. $9\sin\left(\frac{\pi}{2}(t-1)\right)+5=-1$

121. $3-5\cos(2\phi-1)=6$

22. $14\tan(\pi\beta-4)+31=10$

23. $0.3\csc\left(\frac{\pi}{2}(\alpha+0.1)\right)+0.4=1.9$

24. $2\sec\left(\frac{\pi}{4}\left(t+\frac{1}{4}\right)\right)+3=-7$

25. $6\cot\left(\frac{\pi}{3}(\theta-1)\right)+4=5$

26. $120\sin\left(\frac{\pi}{3}(t-0.2)\right)+21=-3$

27. $16\cos(3t-1)+4=-8$

28. $23-24\tan(\pi x+2)=17$

29. $3\csc\left(3w-\frac{\pi}{3}\right)+1=6$

30. $12\sec\left(\frac{\pi}{2}(t+0.3)\right)-7=9$

31. $8\cot\left(4t-\frac{\pi}{3}\right)-24=1$

Proofs

32. What is the angle in the first quadrant with reference angle θ? What is the angle in the third quadrant with reference angle θ ? What is the difference of these angles? Explain how this fact shows that the solutions of $\tan(\theta)=k$, for $k>0$, differ by π.

33. What is the angle in the second quadrant with reference angle θ? What is the angle in the fourth quadrant with reference angle θ? What is the difference of these angles? Explain how this fact shows that the solutions $\tan(\theta)=k$, for $k<0$, differ by π.

Applications

34. A sinusoidal function can approximate the population of deer in Marquette County over a typical year. The population reached a maximum of 50,000 deer on September 1st and a minimum of 42,000 on March 1st.

    a. Write a formula for the function $P(t)$ that gives the deer population on the first of each month, where $t=0$ is September 1.

    b. When is the deer population 45,000? Give exact expressions and approximations rounded to two decimal places.

    c. Graph your function over one period, and label the points that correspond to a deer population of 45,000. Is the population greater or less than 45,000 between the two solutions?

35. The percent of the moon visible from Earth is a sinusoidal function ranging from 0% to 100%, with a period of 29.5 days.

    a. Write a formula for the function $f(t)$ that gives the percent of the moon that is visible, if a new moon (0% visible) occurs at $t=0$ days.

    b. When is 25% of the moon visible? Give approximations rounded to two decimal places.

    c. Graph your function over one period, and label the points corresponding to a quarter moon. Is more or less than 25% of the moon visible between the two solutions you found in part (b)?

36. A Ferris wheel has a diameter of 20 meters and completes one revolution every 60 seconds. Jorge is at the lowest position of the Ferris wheel, 1 meter above the ground, at $t=0$ seconds.

    a. Write a formula for the function $h(t)$ that gives Jorge's altitude in meters after t seconds.

    b. When is Jorge at an altitude of 18 meters during his first revolution? Give exact expressions and approximations rounded

to two decimal places.

c. Graph your function over one period, labeling the points that correspond to an altitude of 18 meters. Is Jorge above or below 18 meters between the two solutions you found in part (b)?

37. High tides occur every 12.2 hours at Point Lookout. The depth of the water at the end of Roy's dock is 2.6 meters at high tide and 1.8 meters at low tide.

a. Write a formula for the function $d(t)$ that gives the depth of the water t hours after last night's high tide.

b. When is the water at the end of the dock 2 meters deep? Give exact approximations rounded to two decimal places.

c. Graph your function over one period, labeling the points that correspond to a depth of 2 meters. Is the water depth greater or less than 2 meters between the two solutions you found in part (b)?

Answers

Skills Refresher

1. 2, 4
2. 11
3. $\frac{25}{8}$
4. 1, 5
5. 5003
6. 4
7. We can't divide by x. $x = 0, \frac{5}{3}$
8. 3 has two square roots. $x = \pm\sqrt{3}$
9. 4 has two square roots. $x = \frac{-3}{2}, \frac{1}{2}$
10. One side must be zero to use the zero-factor principle. No real solutions.

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7.4: Solving Trigonometric Equations Using Identities

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Having a solid foundation of how trigonometric equations are solved, we now integrate our knowledge of trigonometric identities to further expand our equation-solving abilities.

Solving Equations Involving Different Trigonometric Functions Having the Same Arguments

The equations encountered in the previous section were ideal in that each featured only one trigonometric function, and all occurrences of that function within the equation shared the same argument. That is, we only encountered equations like the following:

$$\begin{aligned} \sin(\theta) &= -0.6428 \\ \tan(2x) &= \sqrt{3} \\ 4\tan^2(\theta) + 3 &= 15 \\ 2\cos^2(\theta) - \cos(\theta) - 1 &= 0 \end{aligned}$$

Each of these equations features only one trigonometric function. In the fourth equation, that trigonometric function (the cosine) occurs twice, but in both cases, the arguments are identical (θ).

How would we solve a trigonometric equation if it contains two *different* trigonometric functions? For example, if we were tasked with solving $\sec^2(x) = \tan(x) + 3$, what would we do?

Ultimately, the answer is that we use a mixture of the trigonometric identities learned in this course and algebraic manipulations learned in Algebra. The worst thing I could do at this point is throw a bunch of examples at you showing how to solve specific equations—that approach will make the *single* equation-solving tactic seem like a *dozen* different tactics. To avoid this, let's start by reviewing how we solved trigonometric equations in the previous section and codify this into a tactic.

When we were given an equation of the form

$$A\,\text{trig}(Bx + C) + D = E,$$

we first isolated the trigonometric function (using techniques from Algebra) to a form like

$$\text{trig}(Bx + C) = c,$$

where c is a real number. I also suggested a "substitution" technique if the argument of the trigonometric function wasn't a simple variable like θ or x. Let's suppose we use that technique to let $u = Bx + C$. Then our equation would simplify down to

$$\text{trig}(u) = c.$$

At this point, we used the sign of c to help us determine the quadrants and reference angle associated with our solution. Once those two pieces of information were found, we stated the answers.

This review leads to the following tactic for solving "basic" trigonometric equations.

How To... Solve a "Basic" Trigonometric Equation

Given *any* trigonometric equation of the form

$$\text{trig}(u) = c,$$

where "trig" could be $\sin, \cos, \tan, \csc, \sec,$ or $\cot$, do the following:

1. If the value of c is such that u is a quadrantal angle, skip to Step 5.
2. Use the sign of c to determine in which quadrants u terminates.
3. Find the reference angle, $\hat{u}$.
4. Using the reference angle from Step 3, write the solutions *associated* with the quadrants listed in Step 2. If possible, compress the solution set to a single equation (this is most often the case with equations involving the tangent or the cotangent functions).
5. If u is a function of another variable, solve the equations in Step 4 (or Step 1, if the angle is a quadrantal angle) for the "true" variable.
6. Check for extraneous solutions!

Therefore, our ultimate goal when solving *any* trigonometric equation is to get it into the form $\text{trig}\,(u) = c$.

Example 7.4.1

Solve the following equations and list the solutions which lie in the interval $[0, 2\pi)$.

a. $3\cos(x)\sin^2(x) = \sin^2(x)$
b. $\sec^2(x) = \tan(x) + 3$
c. $\cos^2(A) - \sin^2(A) - \sin(A) = 1$

Solutions available at the end of this section.

Advice: Squares on Trigonometric Functions Are Clues

In Examples $7.4.1b$ and $7.4.1c$, we used a Pythagorean Identity to transform the square of a trigonometric function into a different trigonometric function. This should convince you that squares on trigonometric functions are often desirable because they allow us to rewrite an expression in an equivalent form using a different trigonometric function.

Checkpoint 7.4.1

Solve each equation on the indicated interval.

a. $1 - \sin^2\theta + \cos\theta = 0$ for $0^\circ \le \theta < 360^\circ$
b. $2\tan(\theta)\sin(\theta) + \tan(\theta) = 1 + 2\sin(\theta)$ for $\theta \in [0, 2\pi)$

Answers available at the end of this section.

Solving Equations Involving Trigonometric Functions Having Different Angular Frequencies

Suppose an equation contains more than one trigonometric function (whether the same or not) but with different angular frequencies. In that case, we will need to use the Double-Angle Identities, Half-Angle Identities, or, in rare cases, the Sum-to-Product or Product-to-Sum Identities to help simplify the equation into the form $\text{trig}\,(u) = c$.

Example 7.4.2

Solve the following equations and list the solutions in the interval $[0, 2\pi)$.

a. $\cos(2x) = 3\cos(x) - 2$
b. $\cos(3x) = \cos(5x)$
c. $\sin(2x) = \sqrt{3}\cos(x)$
d. $\sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right) = 1$
e. $\sin\left(\frac{x}{2}\right) - \cos(x) = 0$

Solutions available at the end of this section.

Instructors will tell you to "check your solutions" after solving an equation. While this is always great advice, let's be realistic—most people don't check their solutions. With that sense of realism in mind, I want to clarify something from Algebra and tell you when you *must* check the validity of your solutions for the risk of them being wrong. There are three general cases when you *must* check solutions after solving an equation:

1. when the equation contains a function with a restricted domain,
2. when you multiply both sides of an equation by a variable expression, and
3. when you apply an even power (e.g., a square) to both sides of an equation.

When solving trigonometric equations, you can run into any of these issues. In Example $7.4.2e$, we ended up squaring both sides of the equation in the process of finding the solution (see the step with "**CHECK SOLUTIONS!**"). When doing so, you risk introducing "false" solutions—called **extraneous solutions**. Indeed, that is precisely what happened!

Checkpoint 7.4.2

Find the solutions to each equation on requested the interval.

a. $\tan(\theta) + \cot(\theta) = 4\sin(2\theta)$ on $[0, 2\pi)$
b. $\tan\left(\frac{\theta}{2}\right) - \sin(\theta) = 0$ for $x \in [0, 2\pi)$

Answers available at the end of this section.

Solving Other Trigonometric Equations

At any given step while solving an equation, the tactic I have continually modeled throughout this material is to do the Arithmetic first (i.e., performing simple operations on constants). Once this is done, you move on to see if there is any Algebra to be done (e.g., combining like terms, factoring, etc.). Once you have exhausted all possibilities to use Arithmetic and Algebra, then (and only then) you move on to looking for identities or concepts from Trigonometry.

But what happens when you exhaust all three and you're stuck?

In truth, this means one of two things—either the equation you were given involves a combination of trigonometric functions and non-trigonometric functions (in which case you have no choice but to use technology), or you missed some not-so-obvious Algebra or Trigonometry (Example $7.4.2b$, which required the abstract Sum-to-Product Identities, is an excellent example of this). There is no way to cover all possible "oddball" cases. Still, the following example covers a case that comes up often enough that we should investigate it.

Example 7.4.3

Solve the equation and list the solutions that lie in the interval $[0, 2\pi)$.

$$\cos(x) = \sin(x)$$

Solutions available at the end of this section.

Another option with Example 7.4.3 is to start by dividing both sides by $\cos(x)$. Doing so, we get

$$1 = \tan(x).$$

This is *much* easier to solve, however, by dividing by the cosine, we have introduced a possible issue—what if $\cos(x) = 0$?

A little thought goes a long way here. If $\cos(x) = 0$, then $x = \frac{\pi}{2} + \pi k$. If we go back to the original equation and plug in $\frac{\pi}{2}$, we would get

$$\begin{aligned} \sin\left(\frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{2}\right) \\ \Longrightarrow \qquad 1 &= 0 \end{aligned}$$

Since this is obviously not true, $x = \frac{\pi}{2}$ is not a solution to the equation. It is a simple task to verify that $x = \frac{3\pi}{2}$ is also not a solution. Therefore, we don't have to worry about $\cos(x) = 0$, and we can go back to solving

$$\tan(x) = 1.$$

Tangent is positive in QI and QIII, and $\hat{x} = \frac{\pi}{4}$. Hence, the solutions are $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Combining Waves Having Equal Angular Frequencies

Recall that a sinusoidal function of the form $A\sin(Bx + \phi)$ can be rewritten using the Sum of Angles Identity. That is,

$$\begin{aligned} A\sin(Bx+\phi) &= A[\sin(Bx)\cos(\phi) + \cos(Bx)\sin(\phi)] && \text{(Sum of Angles Identity)} \\ &= A[\cos(\phi)\sin(Bx) + \sin(\phi)\cos(Bx)] && \text{(rearrange terms)} \\ &= A\cos(\phi)\sin(Bx) + A\sin(\phi)\cos(Bx) && \text{(distribute)} \\ &= m\sin(Bx) + n\cos(Bx) && (\text{substitute } m = A\cos(\phi) \text{ and } n = A\sin(\phi)) \end{aligned} \tag{7.4.1}$$

Notice that the result of 7.4.1, $m\sin(Bx) + n\cos(Bx)$, is a vertical stretch (or compression) of the sine added to a different vertical stretch (or compression) of the cosine, but both have the same angular frequency.

Looking at the work in 7.4.1, but in reverse order, we reach a fascinating conclusion—if you have a function that is the sum (or difference) of a sine and cosine function having the same angular frequencies, the result is a single sine function with the same angular frequency, but having a phase shift. This is true even if the original sine and cosine functions have different amplitudes.

The question that we should have is if given a function of the form

$$f(x) = m\sin(Bx) + n\cos(Bx),$$

how do we find A and ϕ to write this as

$$f(x) = A\sin(Bx + \phi)?$$

The clue to answering this question is in the last line of 7.4.1. In that step, we let $m = A\cos(Bx)$ and $n = A\sin(Bx)$. Therefore, we can find the value of A by using the following ingenious move.

$$\begin{aligned} m^2 + n^2 &= A^2\cos^2(Bx) + n^2\sin^2(Bx) && \text{(substitute)} \\ &= A^2[\cos^2(Bx) + \sin^2(Bx)] && \text{(factor)} \\ &= A^2 \cdot [1] && \text{(Pythagorean Identity)} \\ &= A^2 \end{aligned}$$

Thus, we will choose

$$A = \sqrt{m^2 + n^2}.$$

Note that we will deliberately forgo the possibility of A being negative.

Now that we have a value for A, we still need to find the value of ϕ so that we can properly convert $f(x) = m\sin(Bx) + n\cos(Bx)$ into $f(x) = A\sin(Bx + \phi)$. Again, the clue to finding ϕ is in 7.4.1. In the last line, we made the substitutions

$$m = A\cos(\phi) \text{ and } n = A\sin(\phi).$$

This means

$$\cos(\phi) = \frac{m}{A} \text{ and } \sin(\phi) = \frac{n}{A}.$$

Since $A = \sqrt{m^2 + n^2}$ is positive, we can determine the quadrant in which ϕ terminates using the signs of m and n and how those affect the quadrant choices for $\cos(\phi)$ and $\sin(\phi)$, respectively. Once we have the proper quadrant for ϕ, we find the reference angle, $\widehat{\phi}$, using an inverse trigonometric function (arcsine, arccosine, or arctangent all work fine).

The following theorem summarizes these results.

Theorem: Same-Frequency Sinusoid Summation

$$m\sin(Bx) + n\cos(Bx) = A\sin(Bx + \phi),$$

where

$$A = \sqrt{m^2 + n^2}$$

and

$$\cos(\phi) = \frac{m}{A} \text{ and } \sin(\phi) = \frac{n}{A}.$$

It's time we showcase the method using a concrete example.

Example 7.4.4

Rewrite $4\sqrt{3}\sin(2x) - 4\cos(2x)$ as a single sinusoidal function.

Solution available at the end of this section.

Checkpoint 7.4.4

Rewrite $-3\sqrt{2}\sin(5x) + 3\sqrt{2}\cos(5x)$ as a single sinusoidal function.

Answer available at the end of this section.

Rewriting a combination of sine and cosine of equal periods as a single sinusoidal function provides an alternative approach for solving some equations.

Example 7.4.5

Find all solutions to the equation $3\sin(2x) + 4\cos(2x) = 1$ on the interval $[0, 2\pi)$.

Solution available at the end of this section.

Checkpoint 7.4.5

Find all solutions of the given equations on the specified intervals.

a. $\cos(2t) - \sqrt{3}\sin(2t) = 1$ for $t \in [0, 2\pi)$
b. $\frac{1}{2}\sin(\theta) + \frac{\sqrt{3}}{2}\cos(\theta) = 1$ for $0 \le \theta < 2\pi$
c. $\cos(\theta) + 1 = \sin(\theta)$ on $[0, 2\pi)$

Answers available at the end of this section.

Success in Mathematics: Why Study Mathematics? (Part 5)

Mathematics reveals hidden patterns that help us understand the world around us. Now, much more than arithmetic and geometry, mathematics today is a diverse discipline that deals with data, measurements, and observations from science; with inference, deduction, and proof; and with mathematical models of natural phenomena, human behavior, and social systems.

The process of "doing" mathematics is far more than just calculation or deduction; it involves observing patterns, testing conjectures, and estimating results. As a practical matter, mathematics is a science of pattern and order. Its domain is not molecules or cells but numbers, chance, form, algorithms, and change. As a science of abstract objects, mathematics relies on logic rather than observation as its standard of truth yet employs observation, simulation, and even experimentation to discover the truth.

The unique role of mathematics in education is a consequence of its universal applicability. The mathematics results - theorems and theories - are significant and valuable; the best results are also elegant and profound. Through its theorems, mathematics offers science both a foundation of truth and a standard of certainty. In addition to theorems and theories, mathematics offers distinctive modes of thought that are versatile and powerful, including modeling, abstraction, optimization, logical analysis, inference from data, and the use of symbols.

As a major intellectual tradition, mathematics is a subject appreciated as much for its beauty as for its power. The enduring qualities of such abstract concepts as symmetry, proof, and change have been developed through more than 3,000 years of intellectual effort. Like language, religion, and music, mathematics is a universal part of human culture.

1. What was your favorite moment or lecture in this course? Explain.
2. Has this course helped your appreciation for mathematics? Explain.

Answers

Section Preview

The equations in this section require students to use trigonometric identities. Additionally, students will learn how to rewrite a sum or difference of sine and cosine as a single sinusoidal function. If you don't recall, this is a necessary technique for Differential Equations.

Section Preview

- Use an identity to solve a trigonometric equation.
- Rewrite a sum or difference of sine and cosine as a single sinusoidal function.
- Use technology to approximate solutions to equations involving trigonometric functions.

Example 7.4.1

a. Recall from Algebra that when you divide both sides of an equation by a variable expression, you are assuming the expression you are dividing by is nonzero. Therefore, in the given equation, unless you can guarantee that $\sin^2(x) \neq 0$, you cannot divide both sides of the equation by $\sin^2(x)$. Thus, we resist the temptation to divide both sides of $3\cos(x)\sin^2(x) = \sin^2(x)$ by $\sin^2(x)$. Instead, we look at our desired destination—$\text{trig}(u) = c$. One thing is certain about that desired form: We need all trigonometric functions on one side. Let's start there.

$$\begin{aligned} & & 3\cos(x)\sin^2(x) &= \sin^2(x) & \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & \implies & 3\cos(x)\sin^2(x) - \sin^2(x) &= 0 & \text{(move all variable expressions to one side)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & \implies & \sin^2(x)\left[3\cos(x) - 1\right] &= 0 & \text{(factor out the GCF)} \end{aligned}$$

We now use the Zero Factor Property from Algebra to break the equation into two different (and simpler) equations.

$$\begin{aligned} & \sin^2(x) = 0 \quad \text{or} \quad 3\cos(x) - 1 = 0 \\ \implies & \sin(x) = 0 \quad \text{or} \quad \cos(x) = \frac{1}{3} \end{aligned}$$

These equations match the form $\text{trig}(u) = c$, so solving them will mirror what we did in the previous section.

For $\sin(x) = 0$

The right side of $\sin(x) = 0$ implies a quadrantal angle.

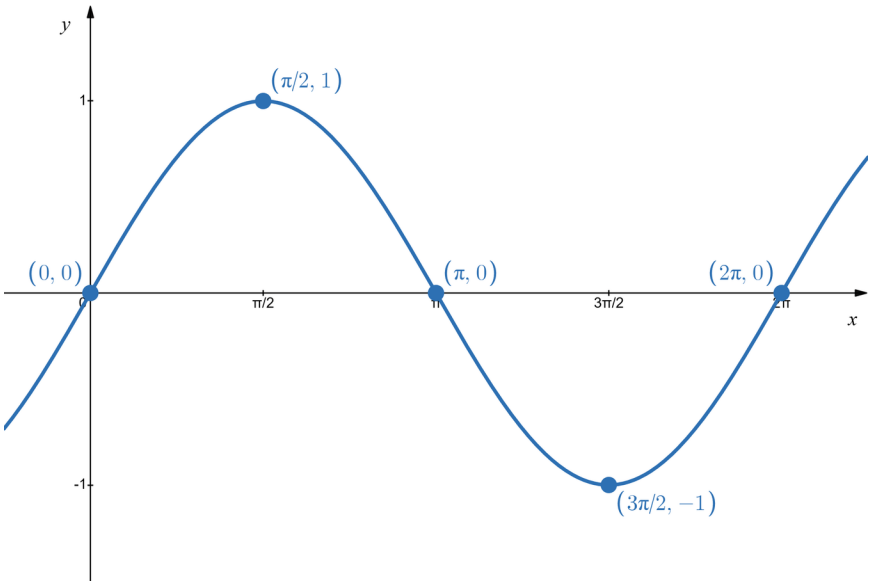


Figure 7.4.1

From the graph of $y = \sin(x)$, we know the solution to this equation is $x = \pi k$, with $x = 0$ and $x = \pi$ being the two solutions which lie in $[0, 2\pi)$.

For $\cos(x) = \frac{1}{3}$

Since the cosine is positive, its argument is an angle that terminates in QI or QIV. Additionally, the ratio, $\frac{1}{3}$, is not a special ratio, so we will eventually need the use of technology. For now, we note that the reference angle is $\hat{x} = \cos^{-1}\left(\left|\frac{1}{3}\right|\right) = \cos^{-1}\left(\frac{1}{3}\right)$. Hence, the two families of solutions are

$$\text{Solutions associated with QI}: \quad x = \cos^{-1}\left(\frac{1}{3}\right) + 2\pi k$$

$$\text{Solutions associated with QIV}: \quad x = 2\pi - \cos^{-1}\left(\frac{1}{3}\right) + 2\pi k$$

where $k \in \mathbb{Z}$. We find the two solutions which lie in $[0, 2\pi)$ to be

$$x = \cos^{-1}\left(\frac{1}{3}\right) \quad \text{and} \quad x = 2\pi - \cos^{-1}\left(\frac{1}{3}\right).$$

Thus, the solutions to the original equation on the interval $[0, 2\pi)$ are

$$x = 0,\ x = \cos^{-1}\left(\frac{1}{3}\right),\ x = \pi, \text{ and } x = 2\pi - \cos^{-1}\left(\frac{1}{3}\right).$$

Grabbing a calculator (and rounding to four decimal places), we get

$$x = 0,\ x = \pi,\ x \approx 1.2310, \text{ and } x \approx 5.0522.$$

b. Again, our ultimate goal is to get an equation (or equations) equivalent to $\text{trig}(u) = c$. At the very least, this means we need to move all the trigonometric functions to one side of the equation.

$$\begin{aligned} \sec^2(x) &= \tan(x) + 3 \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad \sec^2(x) - \tan(x) &= 3 \qquad \text{(move all variable expressions to one side)} \end{aligned}$$

Again, we have a mixture of different trigonometric functions on the left side, but this time, Algebra (specifically, factoring) is not going to help (do you see why?). Instead, we look for any trigonometric identities that might be of use. The $\sec^2(x)$ in an equation with $\tan(x)$ screams "Pythagorean Identity!"

$$\begin{aligned} \sec^2(x) - \tan(x) &= 3 \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} \quad \tan^2(x) + 1 - \tan(x) &= 3 \qquad \text{(Pythagorean Identities)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad \tan^2(x) - \tan(x) - 2 &= 0 \qquad \text{(subtract 3 from both sides)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad u^2 - u - 2 &= 0 \qquad \text{(substitute } u = \tan(x)\text{)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad (u+1)(u-2) &= 0 \qquad \text{(factor)} \end{aligned}$$

This gives $u = -1$ or $u = 2$. Since $u = \tan(x)$, we have $\tan(x) = -1$ or $\tan(x) = 2$.

For $\tan(x) = -1$

Since the tangent is negative, we know the argument terminates in either QII or QIV. Moreover, the reference angle is $\frac{\pi}{4}$. Hence,

$$\text{Solutions associated with QII and IV}: \quad x = \frac{3\pi}{4} + \pi k$$

where $k \in \mathbb{Z}$. Notice that we used the fact that the period of the tangent is π to compress our *two* solution sets into *one*. Letting $k = 0$ and then $k = 1$, we get the solutions $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$, respectively. Any further, and our values for x will go beyond the requested interval.

For $\tan(x) = 2$

The tangent is positive in QI and QIII. We should recognize that the right side is *not* a special ratio for the tangent function. Therefore, we set the reference angle to $\hat{x} = \tan^{-1}(|2|) = \tan^{-1}(2)$. Thus,

$$\text{Solutions associated with QI and III}: \quad x = \tan^{-1}(2) + \pi k$$

where $k \in \mathbb{Z}$. On $[0, 2\pi)$, $x = \tan^{-1}(2)$ and $x = \pi + \tan^{-1}(2)$ are the solutions.

Hence, the solution set to the original equation on the interval $[0, 2\pi)$ is

$$x \in \left\{\tan^{-1}(2),\ \frac{3\pi}{4},\ \pi + \tan^{-1}(2),\ \frac{7\pi}{4}\right\},$$

where $\tan^{-1}(2) \approx 1.1071$ and $\pi + \tan^{-1}(2) \approx 4.2487$.

c. Since this equation involves both the sine and cosine functions, we will need to either perform some Algebra or look for trigonometric identities to get to a point where we have a simple equation (or equations) like $\text{trig}(u) = c$. Factoring out the GCF is out of the question because the three terms on the left side of

$$\cos^2(A) - \sin^2(A) - \sin(A) = 1$$

do not share a common factor. Therefore, we immediately look for obvious trigonometric identities. We know by this point that any time we encounter the square of a trigonometric function, we can rewrite it using a Pythagorean Identity. In this case, we have *two* terms containing squares of trigonometric functions. We will rewrite $\cos^2(A)$ in terms of $\sin^2(A)$ so our equation will only have terms involving the sine function.

$$\begin{aligned}
& & \cos^2(A) - \sin^2(A) - \sin(A) &= 1 & \\
\xRightarrow{\text{Arithmetic} \to \text{Algebra} \to \text{Trigonometry}} & & 1 - \sin^2(A) - \sin^2(A) - \sin(A) &= 1 & \text{(Pythagorean Identity)} \\
\xRightarrow{\text{Arithmetic} \to \text{Algebra}} & & 1 - 2\sin^2(A) - \sin(A) &= 1 & \text{(combine like terms)} \\
\xRightarrow{\text{Arithmetic} \to \text{Algebra}} & & -2\sin^2(A) - \sin(A) &= 0 & \text{(subtract 1 from both sides)} \\
\xRightarrow{\text{Arithmetic} \to \text{Algebra}} & & 2\sin^2(A) + \sin(A) &= 0 & \text{(multiply both sides by } -1\text{)} \\
\xRightarrow{\text{Arithmetic} \to \text{Algebra}} & & \sin(A)\left[2\sin(A) + 1\right] &= 0 & \text{(factor out the GCF)}
\end{aligned}$$

Thus, by the Zero Factor Property,

$$\sin(A) = 0 \quad \text{or} \quad \sin(A) = -\frac{1}{2}.$$

For $\sin(A) = 0$

From part (a), we know the solutions on $[0, 2\pi)$ are $x = 0$ and $x = \pi$.

For $\sin(A) = -\frac{1}{2}$

The sine is negative in QIII and QIV, and the reference angle is $\frac{\pi}{6}$. Therefore, the solutions on $[0, 2\pi)$ are $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$.

Thus, the solution set to this equation on the interval $[0, 2\pi)$ is

$$x \in \left\{0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}.$$

Checkpoint 7.4.1

a. $90°, 180°, 270°$

b. $\frac{\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{6},\ \frac{11\pi}{6}$

Example 7.4.2

a. In the equation $\cos(2x) = 3\cos(x) - 2$, we have the same circular function, namely cosine, on both sides but the arguments differ. Using the Double-Angle Identity

$$\cos(2x) = 2\cos^2(x) - 1,$$

we obtain a "quadratic in disguise" and proceed as we have done in the past.

$$\begin{aligned}
& \cos(2x) &= 3\cos(x) - 2 & \\
\xRightarrow{\text{Arithmetic} \to \text{Algebra} \to \text{Trigonometry}} & 2\cos^2(x) - 1 &= 3\cos(x) - 2 & \text{(Double-Angle Identity)} \\
\xRightarrow{\text{Arithmetic} \to \text{Algebra}} & 2\cos^2(x) - 3\cos(x) + 1 &= 0 & \text{(gathering terms on the left side)} \\
\xRightarrow{\text{Arithmetic} \to \text{Algebra}} & 2u^2 - 3u + 1 &= 0 & (u - \text{Substitution}) \\
\xRightarrow{\text{Arithmetic} \to \text{Algebra}} & (2u - 1)(u - 1) &= 0 & \text{(factor)}
\end{aligned}$$

This gives $u = \frac{1}{2}$ or $u = 1$. Since $u = \cos(x)$, we get $\cos(x) = \frac{1}{2}$ or $\cos(x) = 1$.

For $\cos(x) = \frac{1}{2}$

Cosine is positive in QI and QIV and the reference angle is $\hat{x} = \frac{\pi}{3}$. Therefore, we have the following:

$$\text{Solutions associated with QI}: \quad x = \frac{\pi}{3} + 2\pi k$$

$$\text{Solutions associated with QIV}: \quad x = \frac{5\pi}{3} + 2\pi k,$$

where $k \in \mathbb{Z}$.

For $\cos(x) = 1$

This implies x is a quadrantal angle. Quickly sketching the cosine curve,

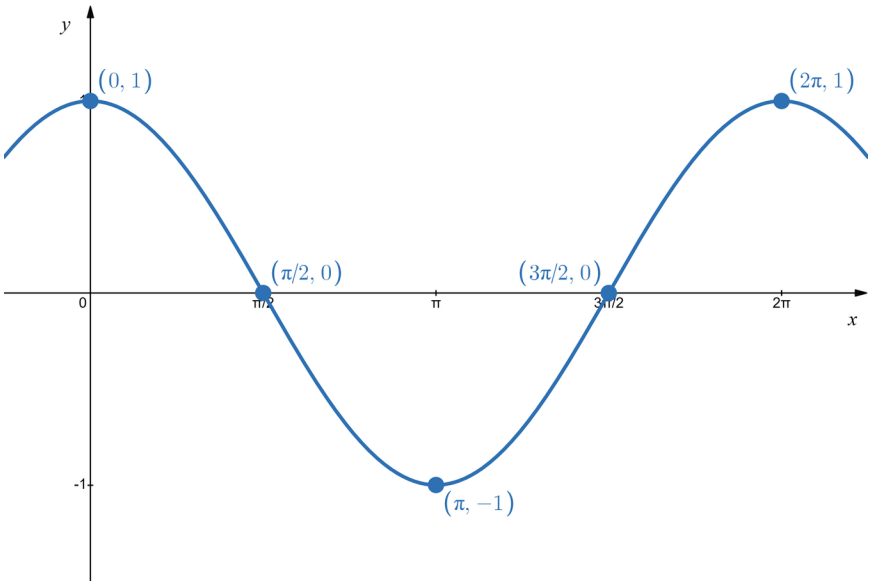


Figure 7.4.2

we see that $\cos(x) = 1$ when $x = 2\pi k$.

Thus, the answers which lie in $[0, 2\pi)$ are $x = 0$, $x = \frac{\pi}{3}$, and $x = \frac{5\pi}{3}$.

b. The given equation, $\cos(3x) = \cos(5x)$, is troublesome for a few reasons. First, while the equation involves a single trigonometric function (which happens to occur twice), the angular frequency of each function is different. The second issue is that, if we use our previous tactic of moving all trigonometric functions to one side, we get

$$\cos(3x) - \cos(5x) = 0,$$

and there is nothing from Algebra that can help us (e.g., there isn't a common factor). Finally, we cannot easily use the Double-Angle Identities for the cosine because the angular frequency of each function is not even. What do we do?

If you have exhausted all hope of rewriting your equation using algebraic manipulations and fundamental trigonometric identities, then you likely need to reach into your "abstract bag of tricks." Recall that I told you not to memorize the Sum-to-Product and Product-to-Sum Identities; however, I did state that you needed to know they exist. This is because, every once in a while, you will run into a need for them. Specifically, in this case, we need the following Sum-to-Product Identity:

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right).$$

Therefore, our solution process goes like this:

$$\begin{aligned}
& & \cos(3x) &= \cos(5x) & \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & \cos(3x) - \cos(5x) &= 0 & \text{(variables expressions to one side)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} & & -2\sin\left(\frac{3x+5x}{2}\right)\sin\left(\frac{3x-5x}{2}\right) &= 0 & \text{(Sum-To-Product Identities)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & -2\sin(4x)\sin(-x) &= 0 & \text{(simplify)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} & & 2\sin(4x)\sin(x) &= 0 & \text{(Symmetry Identities)}
\end{aligned}$$

Hence, the equation $\cos(3x) = \cos(5x)$ is equivalent to $2\sin(4x)\sin(x) = 0$. From this, we get $\sin(4x) = 0$ or $\sin(x) = 0$.

For $\sin(4x) = 0$

From the base graph for the sine,

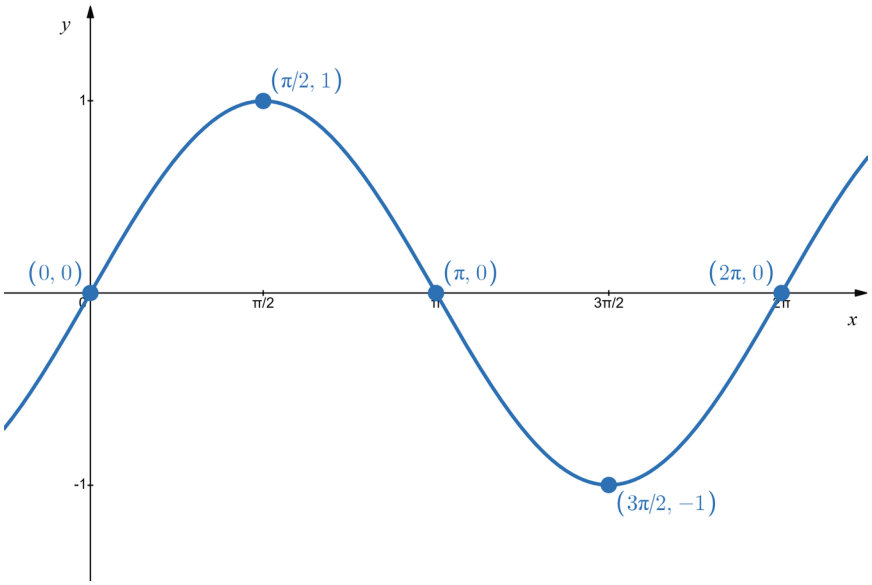


Figure 7.4.3

we know the sine is zero when its argument is an integer multiple of π. That is,

$$4x = \pi k,$$

where $k \in \mathbb{Z}$. This means

$$x = \frac{\pi}{4}k.$$

On the interval $[0, 2\pi)$, this leads to the solutions $x = 0$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π, $\frac{5\pi}{4}$, $\frac{3\pi}{2}$ and $\frac{7\pi}{4}$.

For $\sin(x) = 0$

We know the solutions on $[0, 2\pi)$ are $x = 0$ and π; however, these are already represented in the list of solutions for $\sin(4x) = 0$.

Thus, the solutions to the original equation on the interval $[0, 2\pi)$ are

$$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \text{ and } \frac{7\pi}{4}.$$

c. Here is where I shorten the discussion and rely on your ability to follow the logic we have been using up to this point.

$$\begin{aligned} \sin(2x) &= \sqrt{3}\cos(x) \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad \sin(2x) - \sqrt{3}\cos(x) &= 0 && \text{(move variable terms to one side)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} \quad 2\sin(x)\cos(x) - \sqrt{3}\cos(x) &= 0 && \text{(Double-Angle Identity)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad \cos(x)\left(2\sin(x) - \sqrt{3}\right) &= 0 && \text{(factor out the GCF)} \end{aligned}$$

From this, we get $\cos(x) = 0$ or $\sin(x) = \frac{\sqrt{3}}{2}$.

For $\cos(x) = 0$

Since 0 is a quadrantal ratio for the cosine, we quickly sketch the cosine function.

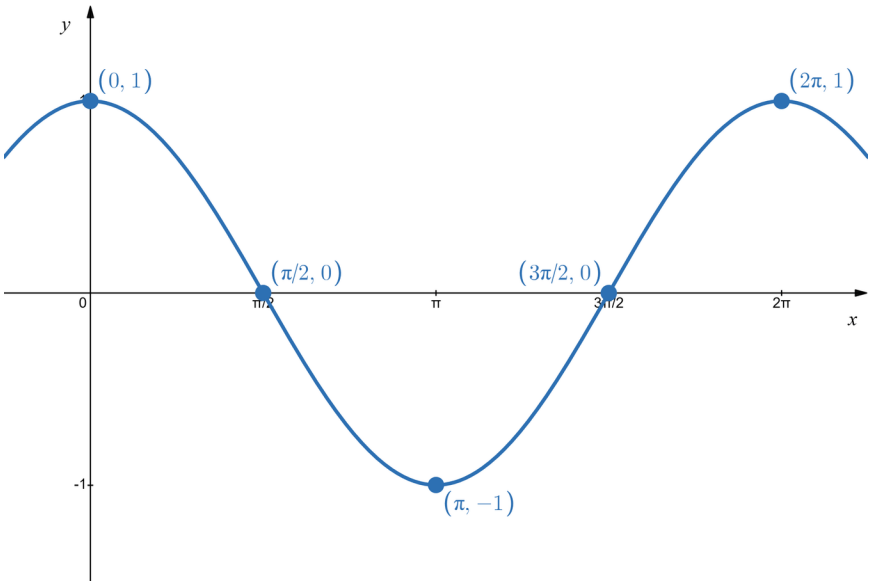


Figure 7.4.4

We see the solutions are $x = \frac{\pi}{2} + \pi k$ for $k \in \mathbb{Z}$. On this interval $[0, 2\pi)$, this gives the restricted solutions $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

For $\sin(x) = \frac{\sqrt{3}}{2}$

The sine is positive in QI and QII, and the reference angle for sine with the ratio $\frac{\sqrt{3}}{2}$ is $\hat{x} = \frac{\pi}{3}$. Thus, we get the solutions $x = \frac{\pi}{3} + 2\pi k$ or $x = \frac{2\pi}{3} + 2\pi k$ for $k \in \mathbb{Z}$.

The answers which lie in $[0, 2\pi)$ are $x = \frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

d. There seems to be no quick way to get the trigonometric functions or their arguments to match in the equation $\sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right) = 1$; however, if we stare at it long enough, we realize that the left side is the expanded form of the Sum of Angles Identity for the sine.

$$\sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right) = 1$$

$$\cancel{\text{Arithmetic}} \to \cancel{\text{Algebra}} \to \text{Trigonometry} \implies \sin\left(x + \frac{x}{2}\right) = 1 \qquad \text{(Sum of Angles Identity)}$$

$$\cancel{\text{Arithmetic}} \to \text{Algebra} \implies \sin\left(\frac{3}{2}x\right) = 1 \qquad \text{(combine like terms)}$$

Since 1 is a quadrantal ratio for the sine, we can quickly sketch $y = \sin(\theta)$

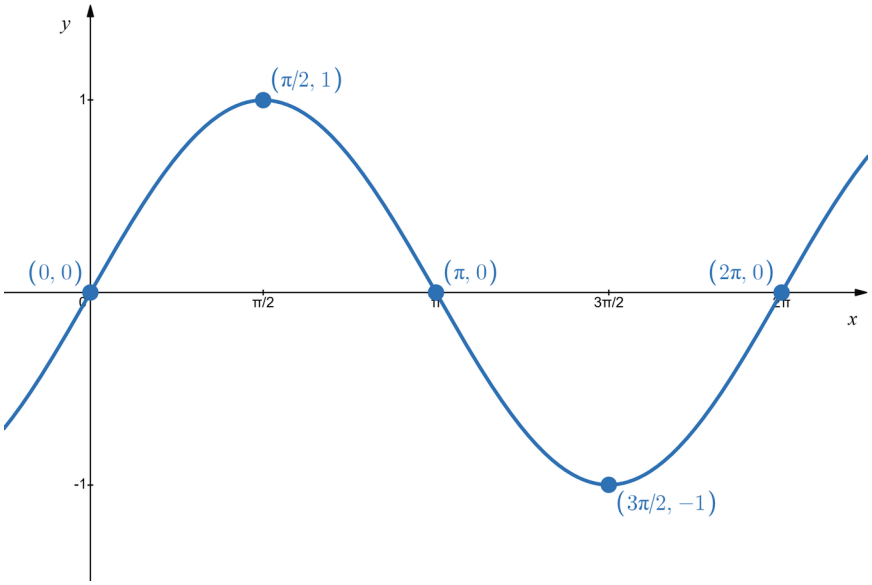


Figure 7.4.5

to see that $\sin(\theta) = 1$ when $\theta = \frac{\pi}{2} + 2\pi k$, where $k \in \mathbb{Z}$. Therefore,

$$\frac{3}{2}x = \frac{\pi}{2} + 2\pi k \implies x = \frac{\pi}{3} + \frac{4\pi}{3}k.$$

Two of these solutions lie in $[0, 2\pi)$: $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$.

e. As in part (c), we have two different trigonometric functions with different angular frequencies. Moreover, we immediately identify that we are locked from doing any impactful Arithmetic or Algebra; however, in this case, there is a fundamental identity we can use—the Half-Angle Identity for sine.

$$\begin{aligned}
& & \sin\left(\frac{x}{2}\right)-\cos(x) &= 0 & & \\
\xrightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} & & \pm\sqrt{\frac{1-\cos(x)}{2}}-\cos(x) &= 0 & & \text{(Half-Angle Identity)} \\
\xrightarrow{\text{Arithmetic}\to\text{Algebra}} & & \pm\sqrt{\frac{1-\cos(x)}{2}} &= \cos(x) & & \text{(isolating the radical)} \\
\xrightarrow{\text{Arithmetic}\to\text{Algebra}} & & \frac{1-\cos(x)}{2} &= \cos^2(x) & & \text{(squaring both sides} \\
& & & & & \implies \textbf{CHECK SOLUTIONS!)} \\
\xrightarrow{\text{Arithmetic}\to\text{Algebra}} & & 1-\cos(x) &= 2\cos^2(x) & & \text{(multiplying both sides by 2)} \\
\xrightarrow{\text{Arithmetic}\to\text{Algebra}} & & 0 &= 2\cos^2(x)+\cos(x)-1 & & \text{(recognizing the equation is quadratic-in-form)} \\
\xrightarrow{\text{Arithmetic}\to\text{Algebra}} & & 0 &= (2\cos(x)-1)(\cos(x)+1) & & \text{(factoring)}
\end{aligned}$$

Hence, either $\cos(x)=\frac{1}{2}$ or $\cos(x)=-1$.

For $\cos(x)=\frac{1}{2}$

The cosine function is positive in QI and QIV, and the reference angle is $\hat{x}=\frac{\pi}{3}$. Therefore, we have the following:

$$\begin{aligned}
\text{Solutions associated with QI:} \quad x &= \frac{\pi}{3}+2\pi k \\
\text{Solutions associated with QIV:} \quad x &= \frac{5\pi}{3}+2\pi k,
\end{aligned}$$

where $k\in\mathbb{Z}$. On the interval $[0,2\pi)$, this equates to $x=\frac{\pi}{3}$ and $x=\frac{5\pi}{3}$; *however*, since we squared both sides of the equation while solving (see the step with "**CHECK SOLUTIONS**"), we need to check to see if these solutions work with the *original* equation.

Letting $x=\frac{\pi}{3}$ in the original equation, we get

$$\begin{aligned}
\sin\left(\frac{\pi/3}{2}\right)-\cos\left(\frac{\pi}{3}\right) &= \sin\left(\frac{\pi}{6}\right)-\cos\left(\frac{\pi}{3}\right) \\
&= \frac{1}{2}-\frac{1}{2} \\
&= 0 \\
&= \textbf{RHS}
\end{aligned}$$

Thus, $x=\frac{\pi}{3}$ is a solution.

Trying $x=\frac{5\pi}{3}$, we get

$$\begin{aligned}
\sin\left(\frac{5\pi/3}{2}\right)-\cos\left(\frac{5\pi}{3}\right) &= \sin\left(\frac{5\pi}{6}\right)-\cos\left(\frac{5\pi}{3}\right) \\
&= \frac{1}{2}-\frac{1}{2} \\
&= 0 \\
&= \textbf{RHS}
\end{aligned}$$

Thus, $x=\frac{5\pi}{3}$ is also a solution.

For $\cos(x)=-1$

From the graph of the cosine,

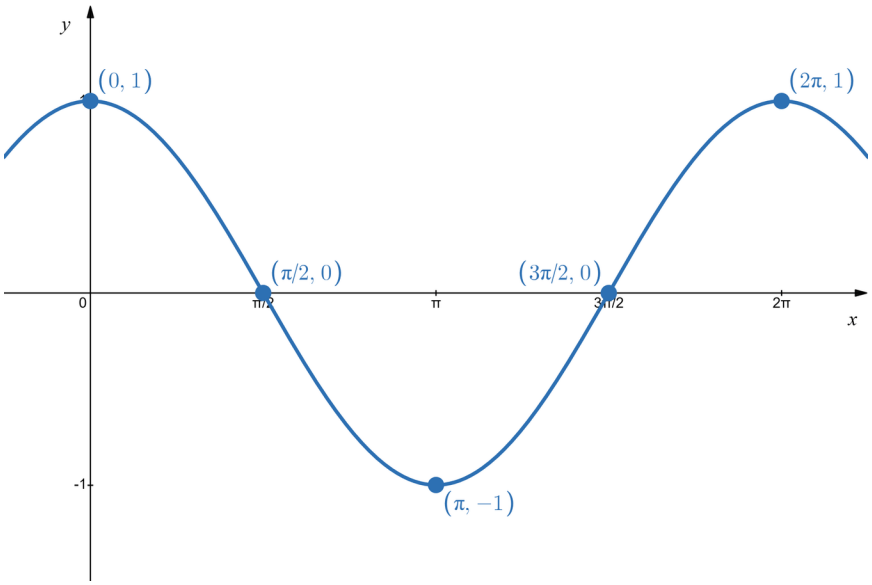


Figure 7.4.6

we see that the solutions to this equation occur at $x = \pi + 2\pi k$, where $k \in \mathbb{Z}$. On the interval $[0, 2\pi)$, this gives $x = \pi$. Again, we must check this solution.

Letting $x = \pi$ in the original equation, we get

$$\begin{aligned} \sin\left(\frac{\pi}{2}\right) - \cos(\pi) &= 1 - (-1) \\ &= 2 \\ &\neq \textbf{RHS} \end{aligned}$$

Thus, $x = \pi$ is an **extraneous solution**, and we throw it out.

In the end, our solutions on the interval $[0, 2\pi)$ to the original equation are $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$.

Checkpoint 7.4.2

a. $\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$
b. $\theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}$

Example 7.4.3

The angular frequencies are the same, but the circular functions are different. This has been fine up to this point because all the previous examples involving two different circular functions had at least one of these functions being squared. Squares of trigonometric functions are blessings in disguise, as we can always try to change them using the Pythagorean Identities. In this case, there are no squares. So what do we do? Since we would like squares, why don't we square both sides? Remember, when you square both sides of an equation, you run the risk of introducing extraneous solutions. Therefore, you must *always* check your solutions when you perform this action.

$$\begin{aligned} & & \cos(x) &= \sin(x) & \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & \cos^2(x) &= \sin^2(x) & \text{(square both sides} \\ & & & & \textbf{CHECK SOLUTIONS!)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} & & 1 - \sin^2(x) &= \sin^2(x) & \text{(Pythagorean Identity)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & 1 &= 2\sin^2(x) & \text{(move all variable terms to one side)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & \frac{1}{2} &= \sin^2(x) & \text{(divide both sides by 2)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & \pm\frac{1}{\sqrt{2}} &= \sin(x) & \text{(Extraction of Roots)} \end{aligned}$$

Therefore, $\hat{x} = \frac{\pi}{4}$ and we need to check *all* quadrants.

$$\begin{aligned} \text{Solutions associated with QI:} \quad & x = \hat{x} + 2\pi k = \frac{\pi}{4} + 2\pi k \\ \text{Solutions associated with QII:} \quad & x = \pi - \hat{x} + 2\pi k = \pi - \frac{\pi}{4} + 2\pi k = \frac{3\pi}{4} + 2\pi k \\ \text{Solutions associated with QIII:} \quad & x = \pi + \hat{x} + 2\pi k = \pi + \frac{\pi}{4} + 2\pi k = \frac{5\pi}{4} + 2\pi k \\ \text{Solutions associated with QIV:} \quad & x = 2\pi - \hat{x} + 2\pi k = 2\pi - \frac{\pi}{4} + 2\pi k = \frac{7\pi}{4} + 2\pi k \end{aligned}$$

where $k \in \mathbb{Z}$. This is a case where, rather than compressing these solutions into two families (which we *could* do), it's easier to check the four possibilities separately. In all four cases, the reference angles are $\hat{x} = \frac{\pi}{4}$. Therefore, the sine and cosine functions become $\pm\frac{1}{\sqrt{2}}$. We just require the signs to be the same.

In QI, $\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}=\sin\left(\frac{\pi}{4}\right)$. In QIII, $\cos\left(\frac{5\pi}{4}\right)=-\frac{1}{\sqrt{2}}=\sin\left(\frac{5\pi}{4}\right)$; however, we can throw out the other two possibilities because the signs of the functions are opposite in both QII and QIV. Thus, the solution set on the interval $[0,2\pi)$ is $x=\frac{\pi}{4}$ and $x=\frac{5\pi}{4}$.

Example 7.4.4

Let $m=4\sqrt{3}$ and $n=-4$. Therefore,

$$A=\sqrt{\left(4\sqrt{3}\right)^2+(-4)^2}=\sqrt{16\cdot 3+16}=\sqrt{64}=8.$$

We now turn to finding ϕ.We know

$$\cos(\phi)=\frac{m}{A}=\frac{4\sqrt{3}}{8}=\frac{\sqrt{3}}{2}\quad\text{and}\quad\sin(\phi)=\frac{n}{A}=\frac{-4}{8}=-\frac{1}{2}.$$

Cosine is positive in QI and QIV, and sine is negative in QIII and QIV. The intersection of these choices is QIV. Thus, $\phi\in$ QIV.

We can use either relation to find the reference angle, but I will use the cosine. The reference angle for cosine that returns $\frac{\sqrt{3}}{2}$ is $\widehat{\phi}=\frac{\pi}{6}$. Therefore, $\phi=\frac{11\pi}{6}$.
Combining these results gives us the expression

$$8\sin\left(2x+\frac{11\pi}{6}\right).$$

Checkpoint 7.4.4

$6\sin\left(5x+\frac{3\pi}{4}\right)$

Example 7.4.5

There are two ways to approach this problem: recognizing that you could isolate one of the trigonometric functions and square both sides or using what we just learned (combining sine and cosine into a single sinusoidal function). I encourage you to try the first method on your own (it slightly mimics Example 7.4.3, but it is more challenging). For now, we will pretend that we cannot see the possibility of using Algebra in the initial equation.

Not having any Arithmetic or Algebra to do, we then recall what we just learned. We let

$$A=\sqrt{(3)^2+(4)^2}=\sqrt{25}=5,$$

where

$$\cos(\phi)=\frac{3}{5}\quad\text{and}\quad\sin(\phi)=\frac{4}{5}.$$

The sine and cosine are both positive in QI. Therefore, $\phi\in$ QI. Finally,

$$\widehat{\phi}=\cos^{-1}\left(\left|\frac{3}{5}\right|\right)=\cos^{-1}\left(\frac{3}{5}\right).$$

Hence,

$$\phi=\cos^{-1}\left(\frac{3}{5}\right).$$

We now can rewrite the left side of the original equation as $A\ \sin(2x+\phi)$ and get

$$5\sin\left(2x+\cos^{-1}\left(\frac{3}{5}\right)\right)=1.$$

This is a much easier equation to solve.

$$\begin{aligned}5\sin\left(2x+\cos^{-1}\left(\frac{3}{5}\right)\right)&=1\\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}}\quad\sin\left(2x+\cos^{-1}\left(\frac{3}{5}\right)\right)&=\frac{1}{5}\qquad\text{(divide both sides by 5)}\end{aligned}$$

Sine is positive in QI and QII and the reference angle is $\hat{\theta}=\sin^{-1}\left(\frac{1}{5}\right)$, where $\theta=2x+\cos^{-1}\left(\frac{3}{5}\right)$. Therefore, we get:

$$\text{Solutions associated with QI:}\quad 2x+\cos^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{1}{5}\right)+2\pi k$$

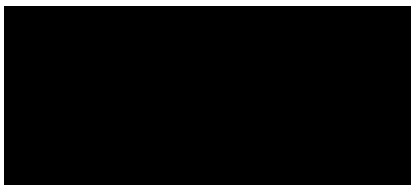
$$\text{Solutions associated with QII:}\quad 2x+\cos^{-1}\left(\frac{3}{5}\right)=\pi-\sin^{-1}\left(\frac{1}{5}\right)+2\pi k$$

for $k\in\mathbb{Z}$. Solving each equation for x, we get:

$$\text{Solutions associated with QI:}\quad x=\frac{1}{2}\sin^{-1}\left(\frac{1}{5}\right)+\pi k-\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)\approx -0.362968649+\pi k$$

$$\text{Solutions associated with QII:}\quad 2x+\cos^{-1}\left(\frac{3}{5}\right)=\frac{\pi}{2}-\frac{1}{2}\sin^{-1}\left(\frac{1}{5}\right)+\pi k-\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)\approx 1.006469757+\pi k$$

We only want solutions on $[0,2\pi)$, so we build a quick table.



Of these results, the solutions on the requested interval are $x\approx 1.0065$, $x\approx 2.7786$, $x\approx 4.1481$, and $x\approx 5.9202$.

Checkpoint 7.4.5

a. $t = 0$, $t = \frac{2\pi}{3}$, $t = \pi$, and $t = \frac{5\pi}{3}$
b. $\theta = \frac{\pi}{6}$
c. $\theta = \frac{\pi}{2}$ and $\theta = \pi$

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7.4.1: Resources and Key Concepts

Resources

Videos

- Solving Trigonometric Equations
  - Solving Trigonometric Equations - Equations Requiring the Use of Identities
  - Solving Trigonometric Equations - Multiple-Angle Equations
  - Solving Trigonometric Equations - Equations Requiring the Use of the Sum to Product Formulas
  - Solving Trigonometric Equations - Equations Requiring Technology and a Warning

Key Concepts

Theorems

- **Same-Frequency Sinusoid Summation:** An expression of the form $m\sin(Bx) + n\cos(Bx)$ can be rewritten as a single sinusoidal function $A\sin(Bx + \phi)$, where $A = \sqrt{m^2 + n^2}$, $\cos(\phi) = \frac{m}{A}$, and $\sin(\phi) = \frac{n}{A}$.

Common Mistakes

- **Dividing by a Variable Expression:** Do not divide both sides of an equation by an expression that could be equal to zero, such as $\sin^2(x)$. This can lead to losing solutions. Instead, move all terms to one side and factor.
- **Forgetting to Check for Extraneous Solutions:** When you square both sides of an equation, you risk introducing extraneous solutions. You must check all potential solutions in the original equation to ensure they are valid.
- **Ignoring Simpler Paths:** It is easy to get focused on one solution method. Always look for the simplest approach. For example, after finding $\sin(2\theta)$ and $\cos(2\theta)$, it's often easier to find $\tan(2\theta)$ using the Ratio Identity rather than a more complex identity.

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7.4.2: Homework

Reading Questions

1. What is the ultimate goal when solving a trigonometric equation, according to the text?
2. When an equation contains a squared trigonometric function, what type of identity is often useful for simplifying it?
3. When solving an equation like $3\cos(x)\sin^2(x) = \sin^2(x)$, what is the recommended first step instead of dividing by $\sin^2(x)$?
4. What algebraic principle allows you to solve a factored trigonometric equation like $\sin(A)[2\sin(A) + 1] = 0$?
5. If an equation contains trigonometric functions with different arguments, like $\cos(2x)$ and $\cos(x)$, what is the general strategy?
6. If an equation contains a sum or difference of trigonometric functions that cannot be easily factored, like $\cos(3x) - \cos(5x) = 0$, what type of identity might be required?
7. Under what circumstances must you check your solutions for extraneous solutions?
8. An expression of the form $m\sin(Bx) + n\cos(Bx)$ can be rewritten as a single sinusoidal function. What is the amplitude of this new function?

Homework

Concept Check

1. Will there always be solutions to trigonometric function equations? If not, describe an equation that would not have a solution. Explain why or why not.

2. When solving a trigonometric equation involving more than one trigonometric function, do we always want to try to rewrite it so it is expressed in one trigonometric function? Why or why not?

Basic Skills

For the following exercises, solve the equation, giving exact solutions on the interval $[0, 2\pi)$.

3. $\sin(2\theta) + \sqrt{2}\cos(\theta) = 0$

4. $\sin(2\alpha)\sin(\alpha) = \cos(\alpha)$

5. $\cos(2t) - 5\cos(t) + 3 = 0$

6. $\cos(2x) + 3\sin(x) = 2$

7. $\tan(2\beta) + 2\sin(\beta) = 0$

8. $\tan(2z) - 2\cos(z) = 0$

9. $3\cos(\phi) - \sin\left(\frac{\pi}{2} - \phi\right) = \sqrt{3}$

10. $\sin(w) + \cos\left(\frac{\pi}{2} - w\right) = 1$

11. $\sin(2\phi)\cos(\phi) + \cos(2\phi)\sin(\phi) = 1$

12. $\cos(\theta)\cos(3\theta) + \sin(\theta)\sin(3\theta) = \frac{\sqrt{2}}{2}$

13. $\sin(x) = \cos(x)$

14. $\sin(2x) = \sin(x)$

15. $\sin(2x) = \cos(x)$

16. $2\tan^2(t) = 3\sec(t)$

17. $\cos(2x) = \sin(x)$

18. $\cos(2x) = \cos(x)$

19. $\cos(2x) = 2 - 5\cos(x)$

20. $3\cos(2x) + \cos(x) + 2 = 0$

21. $\cos(2x) = 5\sin(x) - 2$

22. $3\cos(2x) = \sin(x) + 2$

23. $2\sec^2(x) = 3 - \tan(x)$

24. $\tan^2(x) = 1 - \sec(x)$

25. $\cot^2(x) = 3\csc(x) - 3$

26. $\sec(x) = 2\csc(x)$

27. $\cos(x)\csc(x)\cot(x) = 6 - \cot^2(x)$

28. $\sin(2x) = \tan(x)$

29. $\cot^4(x) = 4\csc^2(x) - 7$

30. $\cos(2x) + \csc^2(x) = 0$

31. $\tan^3(x) = 3\tan(x)$

32. $\cos(6x) - \cos(3x) = 0$

33. $\sin^2(x) - \cos^2(x) - \sin(x) = 0$

34. $\tan^2(x) = \frac{3}{2}\sec(x)$

35. $2\sin(x)\cos(x) - \sin(x) + 2\cos(x) - 1 = 0$

36. $\cos^3(x) = -\cos(x)$

37. $\tan(2x) - 2\cos(x) = 0$

38. $\csc^3(x) + \csc^2(x) = 4\csc(x) + 4$

39. $2\tan(x) = -1 - \tan^2(x)$

40. $\tan(x) = \sec(x)$

41. $\sin(6x)\cos(x) = -\cos(6x)\sin(x)$

42. $\sin(3x)\cos(x) = \cos(3x)\sin(x)$

43. $\cos(2x)\cos(x) + \sin(2x)\sin(x) = 1$

44. $\cos(5x)\cos(3x) - \sin(5x)\sin(3x) = \frac{\sqrt{3}}{2}$

45. $\sin(x) + \cos(x) = 1$

46. $\sin(x) + \sqrt{3}\cos(x) = 1$

47. $\sqrt{2}\cos(x) - \sqrt{2}\sin(x) = 1$

48. $\sqrt{3}\sin(2x) + \cos(2x) = 1$

49. $\cos(2x) - \sqrt{3}\sin(2x) = \sqrt{2}$

50. $9\cos(2\theta) = 9\cos^2(\theta) - 4$

51. $3\sqrt{3}\sin(3x) - 3\cos(3x) = 3\sqrt{3}$

52. $\cos(3x) = \cos(5x)$

53. $\cos(4x) = \cos(2x)$

54. $\sin(5x) = \sin(3x)$

55. $\cos(5x) = -\cos(2x)$

56. $\sin(6x) + \sin(x) = 0$

57. $\tan(x) = \cos(x)$

For the following exercises, solve the equation for $0^\circ \leq \theta \leq 360^\circ$. Round angles to three decimal places if necessary.

58. $\cos(\theta) - \sin^2(\theta) + 1 = 0$

59. $4\sin(\theta) + 2\cos^2(\theta) - 3 = -1$

60. $1 - \sin(\theta) - 2\cos^2(\theta) = 0$

61. $3\cos^2(\theta) - \sin^2(\theta) = 2$

62. $2\cos(\theta)\tan(\theta) + 1 = 0$

63. $\cos(\theta) - \sin(\theta) = 0$

64. $\frac{1}{3}\cos(\theta) = \sin(\theta)$

65. $5\sin(C) = 2\cos(C)$

For the following exercises, find all radian solutions to the equation.

66. $\sin(3x)\cos(6x) - \cos(3x)\sin(6x) = -0.9$

67. $\sin(6x)\cos(11x) - \cos(6x)\sin(11x) = -0.1$

68. $\cos(2x)\cos(x) + \sin(2x)\sin(x) = 1$

69. $\cos(5x)\cos(3x) - \sin(5x)\sin(3x) = \frac{\sqrt{3}}{2}$

70. $6\sin(2t) + 9\sin(t) = 0$

For the following exercises, find all radian solutions to the equation.

71. $\cos(5x) = -\cos(2x)$

72. $\sin(5x) = \sin(3x)$

73. $\cos(6\theta) - \cos(2\theta) = \sin(4\theta)$

74. $\cos(8\theta) - \cos(2\theta) = \sin(5\theta)$

For the following exercises, rewrite the expression as a single function of the form $A\ \sin(Bx+\phi)$.

75. $4\sin(x)-6\cos(x)$

76. $-\sin(x)-5\cos(x)$

77. $5\sin(3x)+2\cos(3x)$

78. $-3\sin(5x)+4\cos(5x)$

For the following exercises, find the first two positive (radian) solutions.

79. $-5\sin(x)+3\cos(x)=1$

80. $3\sin(x)+\cos(x)=2$

81. $3\sin(2x)-5\cos(2x)=3$

82. $-3\sin(4x)-2\cos(4x)=1$

For the following exercises, solve the equation for $0^\circ \le x < 360^\circ$.

83. $9\sin^2(\theta)-6\sin(\theta)=1$

84. $4\cos^2(\theta)+4\cos(\theta)=1$

85. $\sec^2(\alpha)-2\sec(\alpha)-3=0$

86. $\csc^2(\beta)+4\csc(\beta)-10=0$

87. $\csc^2(x)+4\csc(x)-7=0$

88. $3\cot^2(x)-3\cot(x)-1=0$

89. $2\sin^2(x)=1-\cos(x)$

90. $\cos^2(\alpha)+4=2\sin(\alpha)-3$

91. $\cos^2(\beta)-3\sin(\beta)+2\sin^2(\beta)=0$

92. $\sin^2(\theta)=2\cos(\theta)+3\cos^2(\theta)$

93. $\sec^2(x)=2\tan(x)+4$

94. $3\tan^2(x)=\sec(x)+2$

95. $\cos(\alpha)+1=2\cos(2\alpha)$

96. $\cos(2x)-3\sin(x)-2=0$

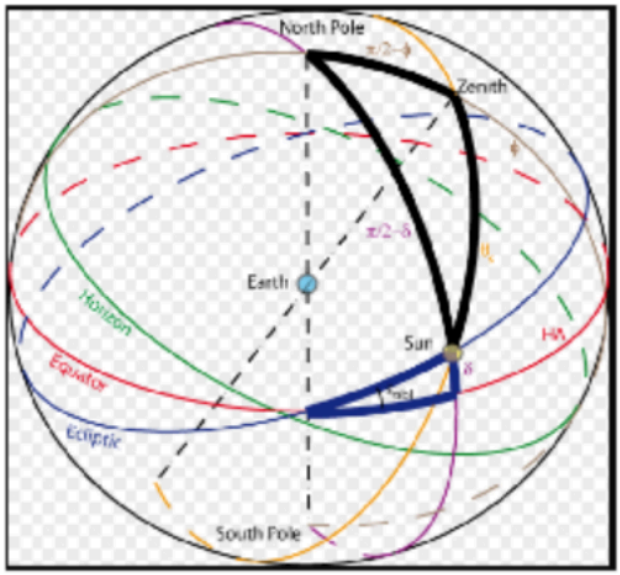
97. $\csc^2(\theta)=\cot(\theta)+5$

98. $\csc(\theta)+5=2\cot^2(\theta)+2$

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8: NON-RIGHT TRIANGLE TRIGONOMETRY

Figure 1

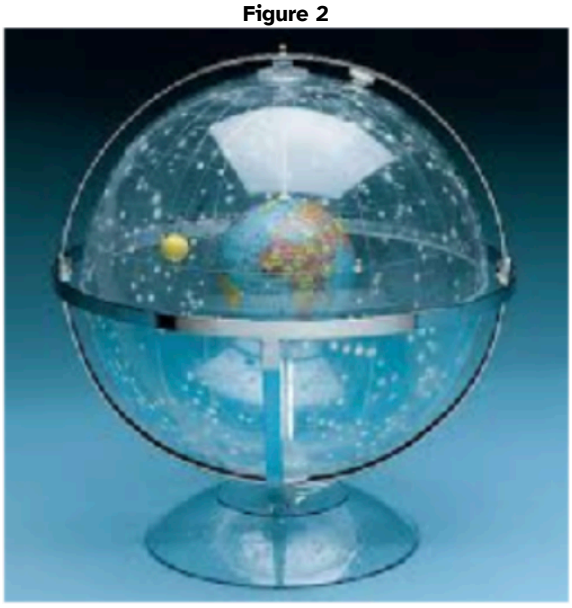


The first science developed by humans is probably Astronomy. Before the invention of clocks and calendars, early people looked to the night sky to help them keep track of time. What is the best time to plant crops, and when will they ripen? On what day exactly do important religious festivals fall?

By tracking the motions of the stars, early astronomers could identify the summer and winter solstices and the equinoxes. The rising and setting of certain stars marked the hours of the night.

If we think of the stars as traveling on a dome above the Earth, we create the **celestial sphere**. Actually, of course, the Earth itself rotates among the stars, but for calculating the motions of heavenly objects, this model works very well.

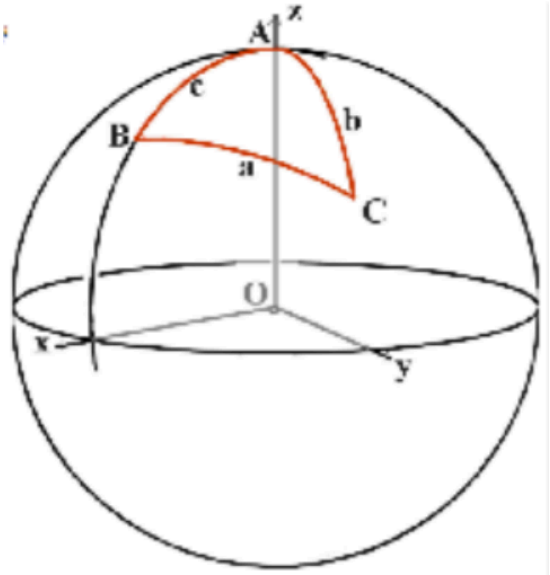
Figure 2



Babylonian astronomers kept detailed records on the motion of the planets, and were able to predict solar and lunar eclipses. All of this required familiarity with angular distances measured on the celestial sphere.

To find angles and distances on this imaginary sphere, astronomers invented techniques that are now part of **spherical trigonometry**. The laws of sines and cosines were first stated in this context, in a slightly different form than the laws for plane trigonometry.

Figure 3



On a sphere, a **great-circle** lies in a plane passing through the sphere's center. It gives the shortest distance between any two points on a sphere, and is the analogue of a straight line on a plane. A spherical angle is formed where two such arcs intersect, and a spherical triangle is made up of three arcs of great circles.

The spherical law of sines was first introduced in Europe in 1464 by Johann Muller, also known as Regiomontus, who wrote:

> "You, who wish to study great and wondrous things, who wonder about the movement of the stars, must read these theorems about triangles. ... For no one can bypass the science of triangles and reach a satisfying knowledge of the stars."

8.1: The Law of Sines

8.1.1: Resources and Key Concepts

8.1.2: Homework

8.2: The Law of Cosines

8.2.1: Resources and Key Concepts

8.2.2: Homework

8.3: Areas

8.3.1: Resources and Key Concepts

8.3.2: Homework

8.4: Vectors - A Geometric Approach

8.4.1: Resources and Key Concepts

8.4.2: Homework

8.5: Vectors - An Algebraic Approach

8.5.1: Resources and Key Concepts

8.5.2: Homework

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8.1: The Law of Sines

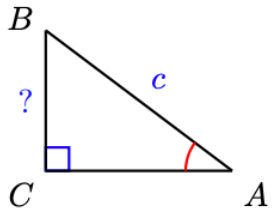
Section Preview

Note to the Instructor (click to expand) available at the end of this section.

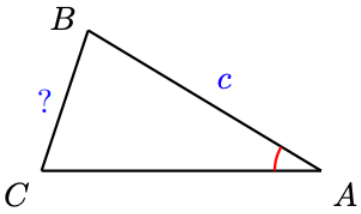
The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We have learned to use trigonometric ratios to solve right triangles. However, the ratios are only valid for the sides of right triangles. Can we find unknown sides or angles in an oblique triangle?



We can find a if we know A and c (and $C = 90°$).



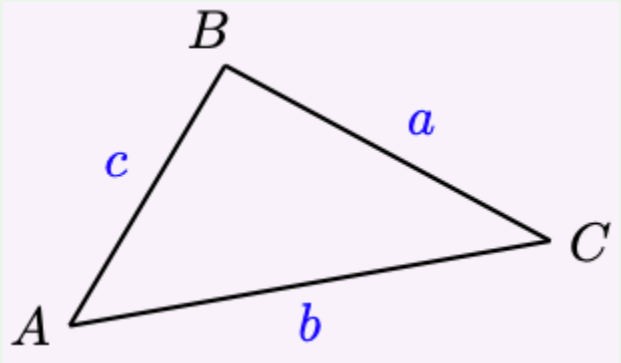
Can find a if we know A and c and C?

Figure 8.1.1

In this section and the next, we find relationships between the sides and angles of oblique triangles. These relationships are called the Law of Sines and the Law of Cosines. We use what we already know about right triangles to derive these new rules.

The Law of Sines

Theorem: Law of Sines



If the angles of a triangle are A, B, and C, and the opposite sides are a, b, and c, respectively, then

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c},$$

or equivalently,

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$

Proof available at the end of this section.

The Law of Sines is true for any triangle, whether acute, right, or obtuse. In the following example, we use the Law of Sines to find a distance.

Example 8.1.1

Two observers on the shore sight a ship at an unknown distance from the shoreline. The observers are 400 yards apart at points A and B, and they each measure the angle from the shoreline to the ship, as shown below. How far is the ship from the observer at A?

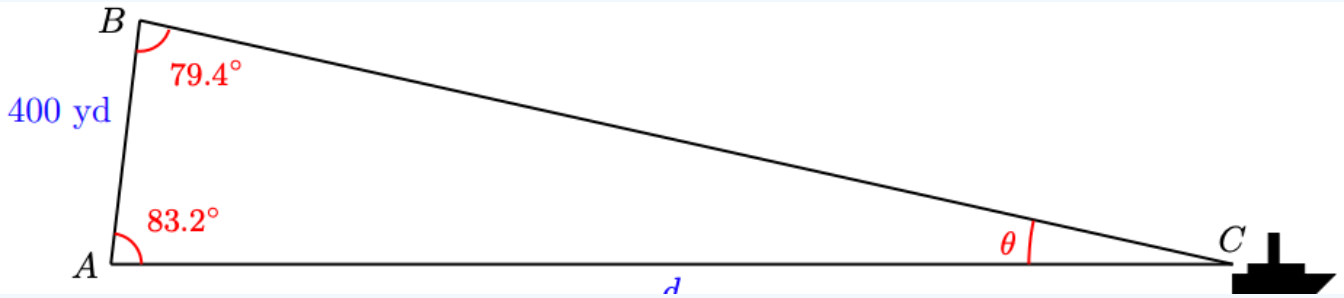


Figure 8.1.2

Solution available at the end of this section.

In Example 8.1.1, we saved all calculator work for the last step. That is, we did not approximate the values of $\sin(79.4^\circ)$ and $\sin(17.4^\circ)$, and then use those approximations to get a further approximation—doing so would propagate errors.

Advice: Save Calculator Work for the Last Step

As much as possible, only reach for technology in any mathematical work at the very last moment.

Another piece of advice is to collect information into tables when working through an application. This habit is especially helpful with the current material (and the material in the next section).

In Example 8.1.1, we could have collected the information into a table where the columns are the angles and their opposite side lengths. When missing information, we place a temporary question mark. Therefore, we would have started with the table

[redacted]

Looking at that first column, we realize these are the interior angles of a triangle and, as such, should sum to 180°. Hence,

$$\angle C = 180^\circ - 83.2^\circ - 79.4^\circ = 17.4^\circ.$$

Thus, our table becomes

[redacted]

That checkmark (✓) is not an accident. Recall, the Law of Sines tells us

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$

If we have the sine of an angle and the length of its opposite side, then we know the value of these ratios.

That is, once we have a complete row in our table, we can *try* to use the Law of Sines (I say "try" because sometimes solving a triangle using the Law of Sines will not work. We will see those cases momentarily). Since our goal is to find the side length opposite $\angle B$ (which, for the sake of building the table, I call b), we work with the equation

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$

We substitute the values from the table to get

$$\frac{b}{\sin(79.4^\circ)} = \frac{400}{\sin(17.4^\circ)}$$

and finish as we did in Example 8.1.1.

Checkpoint 8.1.1

Delbert and Francine are 40 feet apart on one side of a river. They make angle measurements to a pine tree on the opposite shore, as shown below. What is the distance from Francine to the pine tree?

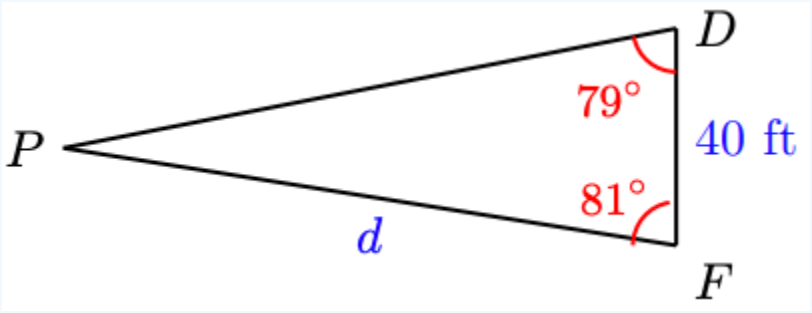


Figure 8.1.3

Answer available at the end of this section.

Solving Triangles with the Law of Sines

To apply the Law of Sines to find a side, we must know one angle of the triangle and its opposite side (either a and A, or b and B, or c and C), and one other angle. Then, we can find the side opposite that angle.

Example 8.1.2

In the triangle shown below, $A = 37^\circ$, $B = 54^\circ$, and $a = 11$.

a. Find b.
b. Solve the triangle.

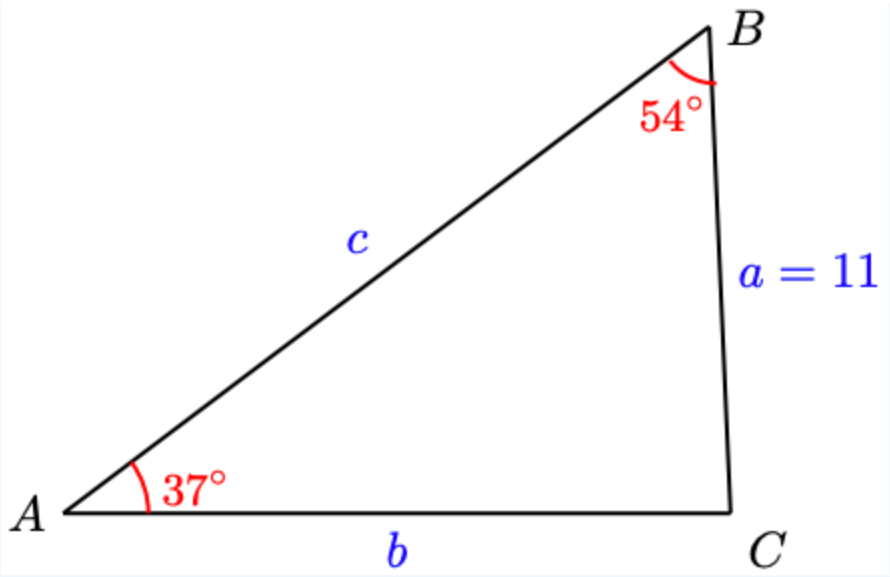


Figure 8.1.4

Solutions available at the end of this section.

Checkpoint 8.1.2

In the triangle below, $A = 65^\circ$, $C = 42^\circ$, and $c = 16$. Solve the triangle.

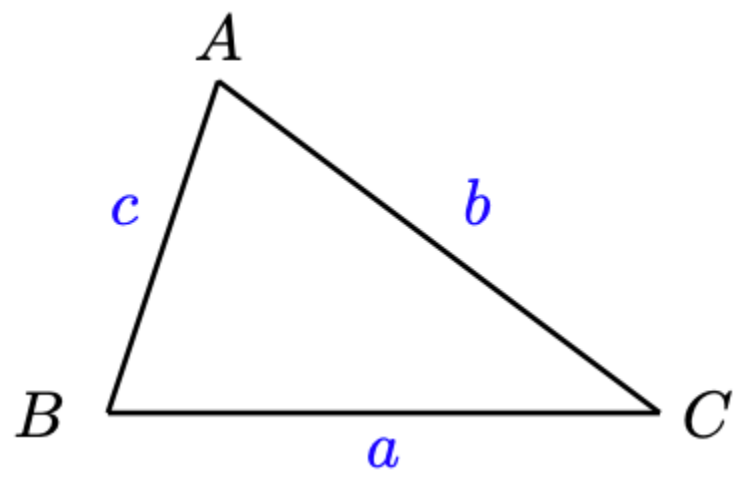


Figure 8.1.5

Answer available at the end of this section.

Finding an Angle

We can also use the Law of Sines to find an unknown angle of a triangle. We must know two sides of the triangle and the angle opposite one of them.

Example 8.1.3

In the triangle below, $B = 55°$, $a = 5$, and $b = 11$. Solve the triangle.

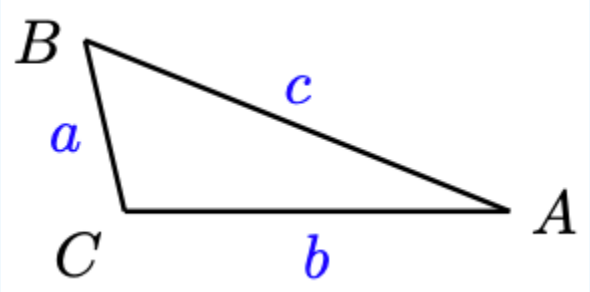


Figure 8.1.6

Solution available at the end of this section.

Caution: Do Not Use Rounded Values to Calculate

Example 8.1.3 has a lot going on; however, the simplest thing to cover at this point is proper etiquette when it comes to calculations.

While I am *displaying* the approximations in the tables within Example 8.1.3, any computations relying on these are done using the exact or stored values in the calculator. For example, when I do any computation relying upon A, I either use $\sin^{-1}\left(\frac{5\sin(55°)}{11}\right)$ or the unrounded approximation stored in my calculator. This is done to avoid the propagation of rounding errors. All modern calculators can store results, and I suggest you learn how to use that feature. For example, on the calculator I use (the TI-30XS), there is a button labeled $\boxed{\text{sto}\rightarrow}$ and another button labeled $\boxed{\text{X}^{\text{yzt}}_{\text{abc}}}$. To store the result of $\sin^{-1}\left(\frac{5\sin(55°)}{11}\right)$, I use the following button sequence:

$$\boxed{\text{sto}\rightarrow} \quad \boxed{\text{X}^{\text{yzt}}_{\text{abc}}} \text{ (tapping this a few times until } a \text{ is displayed)} \quad \boxed{\text{enter}}$$

When I need to access that stored value, I tap $\boxed{\text{X}^{\text{yzt}}_{\text{abc}}}$ until I get the variable I stored it as, and then hit $\boxed{\text{enter}}$.

Checkpoint 8.1.3

Sketch a triangle with $C = 93°$, $a = 7$, and $c = 11$.

a. Use the Law of Sines to find another angle of the triangle.
b. Solve the triangle, and label your sketch with the results.

Answers available at the end of this section.

In Examples 8.1.1 and 8.1.2, we used the Law of Sines to find missing side lengths, but we didn't need the Law of Sines to find angles. In Example 8.1.3 (and Checkpoint 8.1.3), on the other hand, we had no choice but to use the Law of Sines to find a missing angle—this is where we need to be careful when using the Law of Sines.

Note: Unpacking Law of Sines Subtleties

It's imperative to recognize that all angles within a triangle must be positive but less than $180°$. Thus, the sine of *any* angle within a triangle will always be positive. Since the side lengths are also positive, the arcsine we use to find the missing angle will always return an acute angle (i.e., an angle between $0°$ and $90°$).

"Why do we need to be aware of this?"

When using the Law of Sines to find a missing angle in a triangle, there could be two such triangles satisfying the given conditions—one containing the acute returned by the arcsine, and the other containing its supplement.

"How do we know when we have two triangles for a given set of information?"

When using the Law of Sines to find a missing angle, we consider the angle returned by the arcsine to be a *reference angle*. To help with this discussion, let's call this reference angle $\hat{\theta}$.

$\hat{\theta}$ gives two possibilities for the true value of the angle we seek. The first is $\theta = \hat{\theta}$. The second is its supplement, $\theta' = 180^\circ - \hat{\theta}$ (the "prime" notation is traditionally used for the supplementary angle). While θ is the solution that gives one triangle, θ' is a candidate for a second value (and, therefore, a second triangle). This is because the sine is positive and, therefore, its argument could also be between 90° and 180°.

With our candidate angles, θ and θ', in hand, we then check whether the other given angle works with each of these computed values. Sometimes, the provided information works with *both* candidate values; however, at the very least, the acute angle always works.

To illustrate this concept, let's look back at Example $8.1.3$. We found $\widehat{A} = \sin^{-1}\left(\frac{5\sin(55^\circ)}{11}\right) \approx 21.9^\circ$. Thus, we immediately knew $A \approx 21.9^\circ$ worked and would yield a triangle; however, what about the supplement of this angle?

In QII, our alternate option for A is $A' = 180^\circ - \widehat{A} \approx 158.1^\circ$; however, if we tried to fit this obtuse angle into a triangle, we would compute the final missing angle to be

$$C = 180^\circ - A - B \approx 180^\circ - 158.1^\circ - 55^\circ = -33.1^\circ.$$

This is impossible because angles within a triangle are never negative. Thus, we only have one triangle (given by the acute angle $A \approx 21.9^\circ$).

Again, sometimes both angles produce (different) triangles, and sometimes only the acute angle will work. You should always check whether both angles provide solutions.

Because there can be more than one solution for a triangle when using the Law of Sines to find a missing angle, we call this situation the **ambiguous case** for the Law of Sines.

Example $8.1.4$

Solve the triangle in which $B = 14.4^\circ$, $a = 8$, and $b = 3$. Sketch the resulting triangle.

Solution available at the end of this section.

Checkpoint $8.1.4$

Suppose that $C = 29.7^\circ$, $b = 8$, and $c = 5$.

a. Find two possible values for angle B.
b. Solve the triangle for both values of B, and sketch both solutions.

Answers available at the end of this section.

Example $8.1.5$

Solve the triangle in which $C = 73^\circ$, $a = 15$, and $c = 2$. Sketch your result.

Solution available at the end of this section.

Example $8.1.5$ illustrates the final possibility when using the Law of Sines to find a missing angle. In summary, when using the Law of Sines to find a missing angle within a triangle, one of three possibilities occurs:

1. There is a single triangle satisfying the given information. In this case, the angle will directly result from the arcsine when using the Law of Sines.
2. Two triangles satisfy the given information. In this case, one triangle will be the direct result of the arcsine, and the second triangle will contain the supplement of that angle.
3. There is no such triangle satisfying the given information. In this case, your calculator will return a domain error when computing the arcsine.

Checkpoint $8.1.5$

You are told that, in a triangle, $B = 30^\circ$, $b = 7$, and $c = 15$. Without a calculator, explain how you know no such triangle satisfies the given information.

Answer available at the end of this section.

Visualizing the Law of Sines Cases

Up to this point, we have computationally justified that the Law of Sines results in one, two, or no triangles for a given set of information; however, the visual of why this is true is often more impactful.

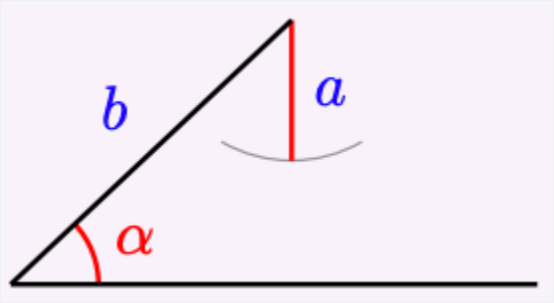
If we know two sides a and b of a triangle and the acute angle α opposite one of them, there may be one solution, two solutions, or no solution, depending on the size of a in relation to b and α, as shown below.

The Law of Sines - Ambiguous Case

In each of the following cases, $h = b\sin(\alpha)$ is the altitude of the triangle. Therefore, $b > h$.

The "No Solution" Case

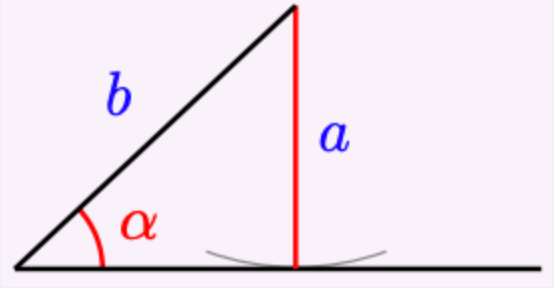
Condition: $a < h$



Reason: a is shorter than the altitude of the (non-existent) triangle. Therefore, it is impossible for side a to intersect the base of the "triangle."

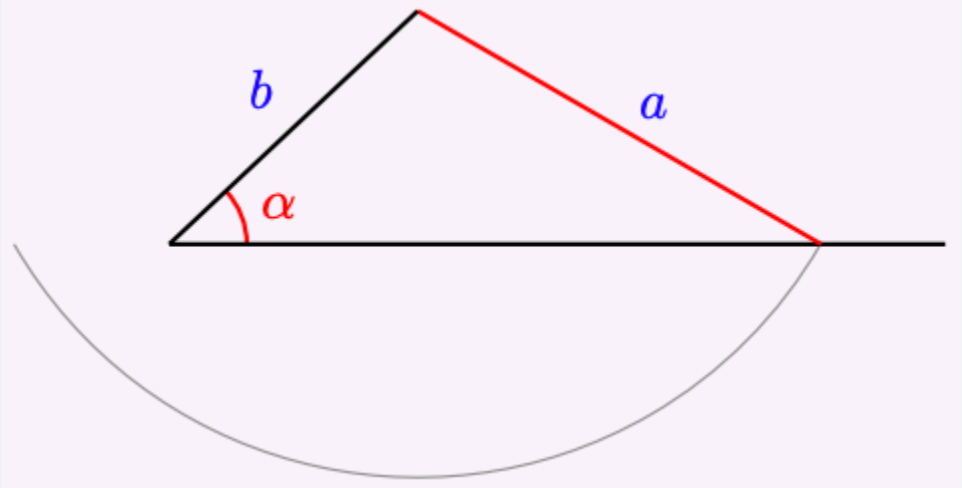
The "One Solution" Cases

Condition #1: $a = h$



Reason: a is exactly the right length to make a right triangle.

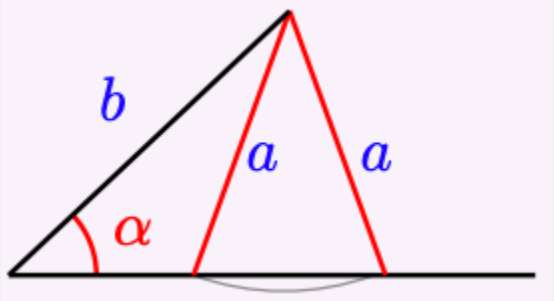
Condition #2: $a > b > h$



Reason: If $a > b$, then one triangle is created as side a swings downward to intersect the base; however, as that side continues to swing, it is too long to strike the base again. Hence, there can be only one triangle.

The "Two Solutions" Case

Condition: $b > a > h$



Reason: Since $a > h$, side a will strike the base a it swings downward, but before becoming vertical. Moreover, because $a < b$, side a will strike the base once more as it continues to swing clockwise.

Applications Requiring the Law of Sines

The Law of Sines is used quite a bit in applications within Trigonometry. While I could not cover all possible applications, I have chosen two that commonly occur.

Example 8.1.6

A vertical communications tower is located at the top of a steep hill, as shown below. The angle of inclination of the hill is 56°. A guy wire is to be attached to the top of the tower and the ground 175 yards downhill from the base of the tower. The angle formed by the guy wire is 26°. Find the length of the cable required for the guy wire. Round your answer to the nearest yard.

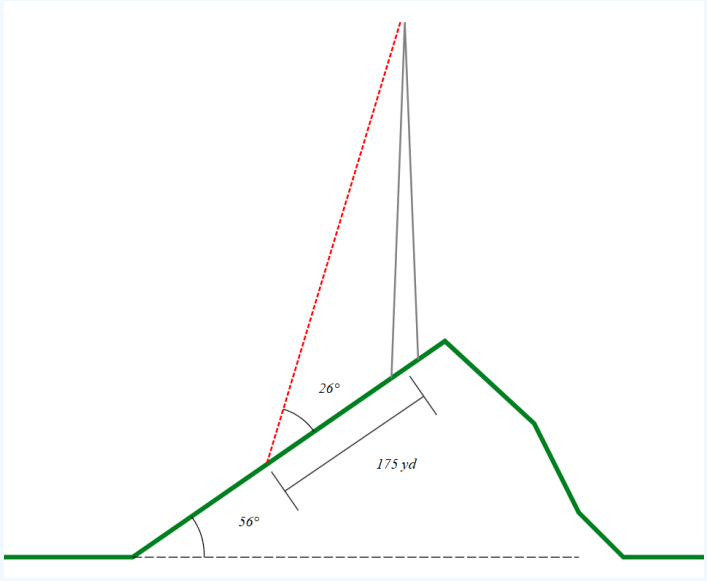


Figure 8.1.11

Solution available at the end of this section.

Checkpoint 8.1.6

Lap and Piyali are on the north side of a river that runs east to west. Lap is standing 225 meters due west of Piyali. They spot a treasure chest on the south shore of the river. For Lap, the angle between the chest and Piyali is 81°, and for Piyali, the angle between the chest and Lap is 58° (see Figure 8.1.14 below). How far is Lap from the chest? Round your answer to the nearest meter.

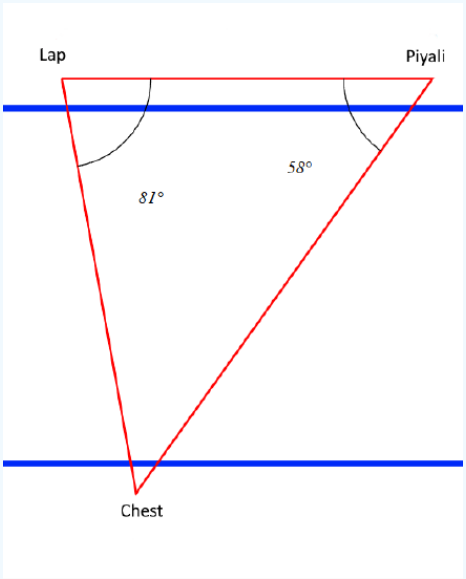


Figure 8.1.14

Answers available at the end of this section.

The following example (and the subsequent Checkpoint) illustrates that the Law of Sines can be used when you might otherwise use Right Triangle Trigonometry.

Example 8.1.7

Ron wants to measure the height of a castle controlled by hostile forces. When he is as close as he can get to the castle, the angle of elevation to the top of the wall is 18.5°. He then retreats 20 yards and measures the angle of elevation again; this time, it is 15.9°. How tall is the castle?

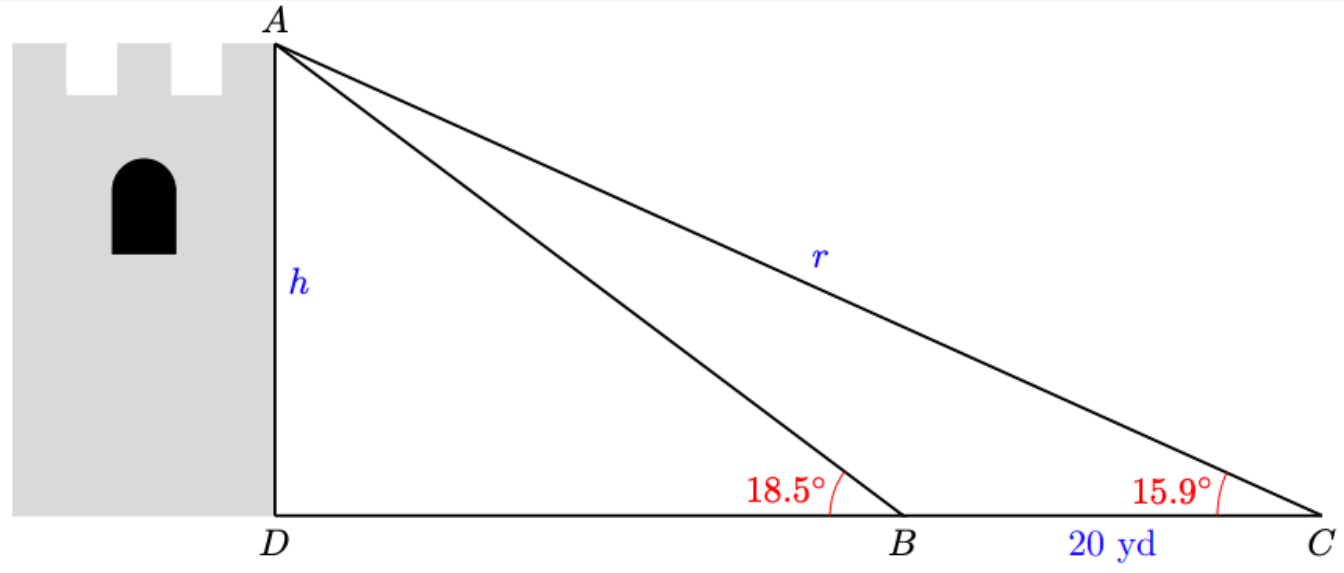


Figure 8.1.15

Solution available at the end of this section.

Checkpoint 8.1.7

Solve the problem in the previous example again, but instead of finding r, find the length AB, and then use $\triangle ABD$ to find h.

Answer available at the end of this section.

If you look at a nearby object and alternately close your left and right eyes, the object seems to jump in position. This apparent change occurs because your eyes view the object from two positions spaced several centimeters apart. If the object at point O is straight ahead of one eye, it appears to be at some angle p away from the line of sight of the other eye. The angle p is called the **parallax** of the object.

Use Figure 8.1.17 to see that p is also the angle between the directions to your two eyes when viewed from point O. (What fact from Geometry justifies this statement?)

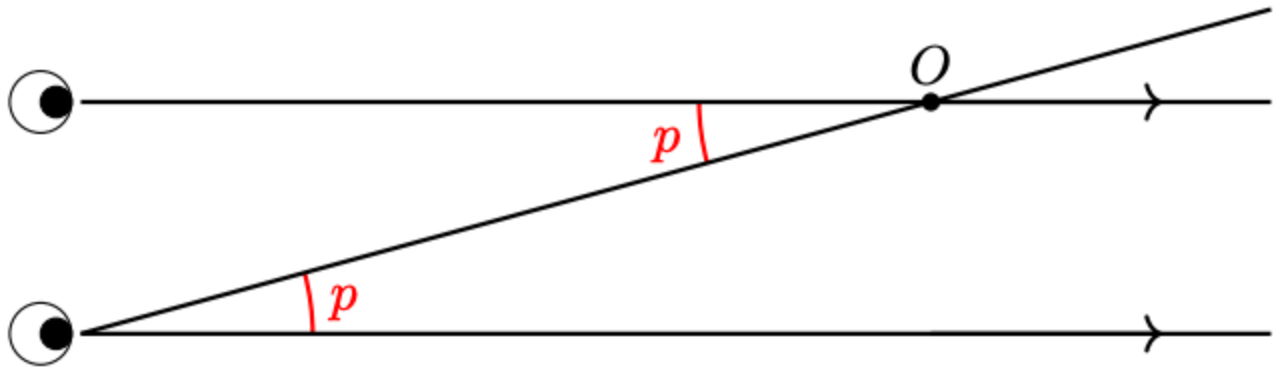


Figure 8.1.17

Astronomers use parallax to determine the distance from Earth to stars and other celestial objects. Two observers on Earth at a known distance apart both measure the direction to the star. The difference in angle between those two directions is the parallax.

Example 8.1.8

Astronomers 1000 kilometers apart observe an asteroid with a parallax of 0.001°. How far is the asteroid from Earth?

Solution available at the end of this section.

Checkpoint 8.1.8

Two observers 800 kilometers apart observe an object with a parallax of 0.0005°. How far is the object from Earth?

Answer available at the end of this section.

Answers

Section Preview

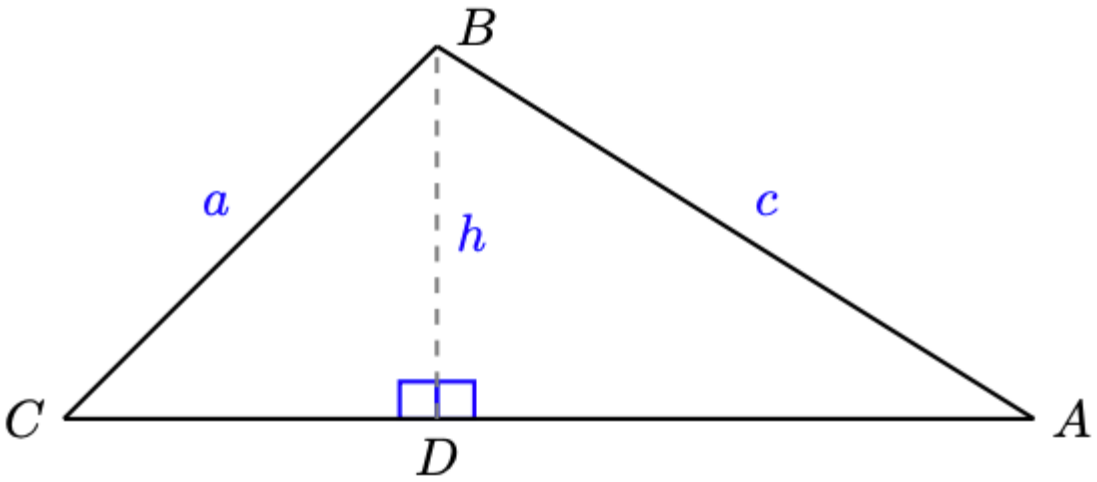
This is a very thorough section of all cases for the Law of Sines and their applications. Please pay close attention to my tabular method. While it is terribly algorithmic, students tend to do much better with it than other methods I have tried.

Section Preview

- Use the Law of Sines to find a missing side in an oblique triangle.
- Solve an application using the Law of Sines.
- Use the Law of Sines to find all possible solutions for the ambiguous case.
- Solve an application involving the ambiguous case.

Theorem: Law of Sines

Consider the oblique triangle in the figure below.



By drawing in the altitude h of the triangle, we create two right triangles, $\triangle BCD$ and $\triangle ABD$. We can write expressions in terms of h for $\sin(A)$ and for $\sin(C)$.

Looking at $\triangle BCD$, we see that

$$\frac{h}{a} = \sin(C).$$

Looking at $\triangle ABD$, we see that

$$\frac{h}{c} = \sin(A).$$

Now we solve each of these equations for h:

$$\frac{h}{a} = \sin(C) \quad \Longrightarrow \quad h = a\sin(C)$$

$$\text{and} \quad \frac{h}{c} = \sin(A) \quad \Longrightarrow \quad h = c\sin(A)$$

Therefore,

$$\begin{aligned} a\sin(C) &= c\sin(A) \\ \Longrightarrow \quad \frac{\sin(C)}{c} &= \frac{\sin(A)}{a} \qquad (\text{divide both sides by } ac) \end{aligned}$$

We have derived a relationship between the angles A and C and their opposite sides, a and c. If we know any three of these quantities, we can find the fourth.

In a similar way, by drawing in the altitude from the vertex C, we can show that

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}.$$

Putting both results together, we have proved the Law of Sines.

Example 8.1.1

First, note that $\triangle ABC$ is not a right triangle, so we cannot use the trigonometric ratios directly to find the sides of this triangle.

Let d be the unknown distance opposite $\angle B = 79.4^\circ$. To use the Law of Sines, we must know another angle and the side opposite that angle. Luckily, we know that side $c = 400$, and we can compute the angle at the ship, $\angle C$.

$$\angle C = 180^\circ - (79.4^\circ + 83.2^\circ) = 17.4^\circ.$$

Now we apply the Law of Sines, using angles B and C.

$$\begin{aligned} \frac{b}{\sin(B)} &= \frac{c}{\sin(C)} && (\text{Law of Sines}) \\ \Longrightarrow \quad \frac{d}{\sin(79.4^\circ)} &= \frac{400}{\sin(17.4^\circ)} && (\text{substitute}) \\ \Longrightarrow \quad d &= \frac{400\sin(79.4^\circ)}{\sin(17.4^\circ)} && (\text{multiply both sides by } \sin(79.4^\circ)) \\ \Longrightarrow \quad d &\approx 1315 \end{aligned}$$

The ship is about 1315 yards from the observer at A.

Checkpoint 8.1.1

About 114.8 feet

Example 8.1.2

We begin by building our table of given information (using question marks where we aren't explicitly given information).



We already have a complete row! This means we can immediately try using the Law of Sines; however, I cannot resist finding the value of C when it is so easy to compute.

$$C = 180^\circ - 54^\circ - 37^\circ = 89^\circ.$$

Hence, our table becomes



a. We use the Law of Sines with a and angle A to find b.

$$\begin{aligned} & \frac{a}{\sin(A)} = \frac{b}{\sin(B)} && \text{(Law of Sines)} \\ \Longrightarrow \quad & \frac{11}{\sin(37^\circ)} = \frac{b}{\sin(54^\circ)} && \text{(substitute)} \\ \Longrightarrow \quad & \frac{11\sin(54^\circ)}{\sin(37^\circ)} = b && \text{(multiply both sides by } \sin(54^\circ)) \end{aligned}$$

Evaluating this expression with a calculator, we find that $b \approx 14.79$. We fill this in our table for display purposes only (we will not use the rounded value for computing).



b. At this point, we are very close to having the triangle solved. We only need the length of side c. Using the given side, a, is safer than using the value we calculated for b.

$$\begin{aligned} & \frac{a}{\sin(A)} = \frac{c}{\sin(C)} && \text{(Law of Sines)} \\ \Longrightarrow \quad & \frac{11}{\sin(37^\circ)} = \frac{c}{\sin(89^\circ)} && \text{(substitute)} \\ \Longrightarrow \quad & \frac{11\sin(89^\circ)}{\sin(37^\circ)} = c && \text{(multiply both sides by } \sin(89^\circ)) \end{aligned}$$

So $c \approx 18.28$. Now our table is complete, and we have solved the triangle.



Checkpoint 8.1.2

$B = 73^\circ, b \approx 22.87, a \approx 21.68$

Example 8.1.3

Again, let's start by filling out a table.



This is the first time where we cannot simply find the remaining angles using the fact that the sum of angles in a triangle is 180°. Luckily, we still have a complete row so we can use the Law of Sines, and we will do so to find $\sin(A)$.

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} \qquad \text{(Law of Sines)}$$

$$\Rightarrow \frac{\sin(A)}{5} = \frac{\sin(55^\circ)}{11} \qquad \text{(substitute)}$$

$$\Rightarrow \sin(A) = \frac{5\sin(55^\circ)}{11} \qquad \text{(multiply both sides by 5)}$$

$$\Rightarrow \widehat{A} = \sin^{-1}\left(\frac{5\sin(55^\circ)}{11}\right) \qquad \text{(finding the reference angle - see } \textbf{Unpacking Law of Sines Subtleties}\text{)}$$

We have a *reference* angle, and we know that each angle in a triangle must be positive but less than 180°. Therefore, A is either in QI or QII. In all honesty, since A is an angle within a triangle and *not* an angle in standard position within a prescribed coordinate system, it is incorrect to say that $A \in$ QI or $A \in$ QII; however, I am okay with the "looseness" of this language.

Here's where a little extra logic comes into play. Since the length of side b is greater than the length of side a, it must be the case that $B > A$. Since $B = 55^\circ$, $0 < A < 55^\circ$. So

$$A = \sin^{-1}\left(\frac{5\sin(55^\circ)}{11}\right) \approx 21.9^\circ.$$

Let's update our table.

[redacted]

Now that we know two angles, we can find the third.

$$C = 180^\circ - B - A \approx 103.1^\circ.$$

Again, we update the table.

[redacted]

Finally, we use the Law of Sines to find side c.

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)} \qquad \text{(Law of Sines)}$$

$$\Rightarrow \frac{b\sin(C)}{\sin(B)} = c \qquad \text{(multiplying both sides by } \sin(C)\text{)}$$

Substituting in the values of b, B, and C (stored in the calculator) and evaluating this expression with a calculator gives $c \approx 13.1$. Thus, our final table is the following:

[redacted]

Checkpoint 8.1.3

a. $A \approx 39.5^\circ$
b. $b \approx 8.13, B \approx 47.5^\circ$

Example 8.1.4

We start, as usual, by building a table.

[redacted]

Other than the numbers, this looks like it's going to mimic what we did in Example 8.1.3.

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} \qquad \text{(Law of Sines)}$$

$$\Rightarrow \sin(A) = \frac{a\sin(B)}{b} \qquad \text{(multiplying both sides by } a\text{)}$$

$$\Rightarrow \widehat{A} = \sin^{-1}\left(\frac{a\sin(B)}{b}\right) \qquad \text{(finding the reference angle)}$$

Grabbing a calculator, substituting in the known values of B, a, and b, we get $\widehat{A} \approx 41.5^\circ$ (again, the exact value is stored in the calculator to be used for other calculations). As stated previously, the acute angle will always work because if $A \approx 41.5^\circ$, then $C = 180^\circ - A - B \approx 124.1^\circ$; however, before throwing all of this into a table, let's check to see if the supplementary (obtuse) angle works as well. Let

$$A' = 180^\circ - A \approx 138.5^\circ.$$

Then

$$C' = 180^\circ - A' - B \approx 27.1^\circ.$$

Since this is not a negative angle, this creates another workable triangle!

Thus, if $A \approx 41.5^\circ$, there is room in the triangle for the final, third angle $C \approx 124.1^\circ$, and if $A' \approx 138.5^\circ$, there is room in another triangle for the final, third angle C' $\approx 27.1^\circ$. In these ambiguous cases, we build *two* tables of information, one for the acute angle and one for the obtuse angle.




In either case, we need to compute the final missing sides, c and c', using the Law of Sines. Remember, when using the Law of Sines, we want to use as much of the given information as possible because it's cleaner than our approximated information. Moreover, we use the non-rounded values and save rounding for the end.

$$\begin{aligned} \frac{c}{\sin(C)} &= \frac{b}{\sin(B)} && \text{(Law of Sines)} \\ \Rightarrow \quad c &= \frac{b\sin(C)}{\sin(B)} && \text{(multiply both sides by } \sin(C)\text{)} \end{aligned}$$

This formula is the same for c' (just replace C with C'). Hence, we grab a calculator, substitute the necessary values, and get our approximations (listed in the tables below).




The following figure shows both triangles.

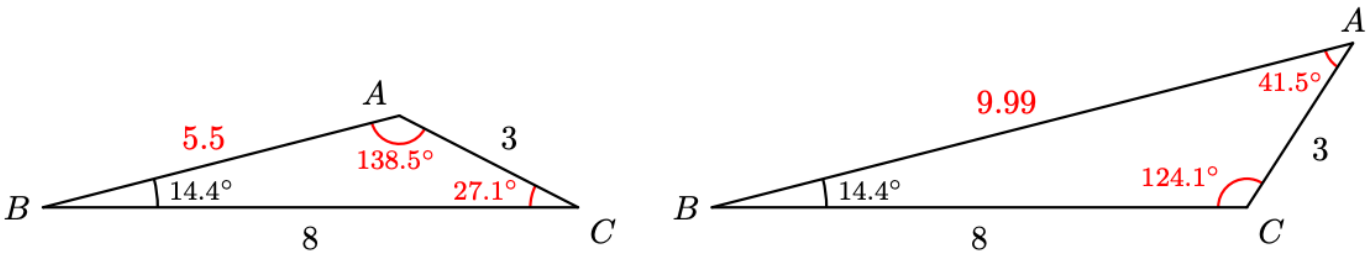


Figure 8.1.7

Checkpoint 8.1.4

a. $B = 52.4^\circ$ or $B = 127.6^\circ$
b. $A = 97.9^\circ, a = 10$

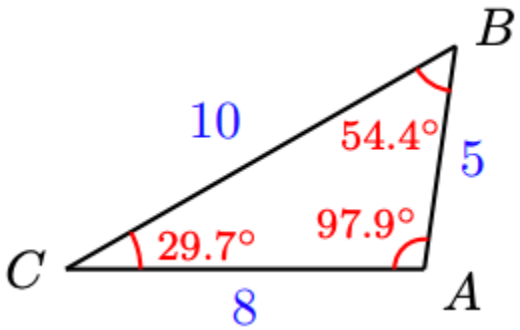


Figure 8.1.8

or $A = 22.7^\circ, a = 3.9$

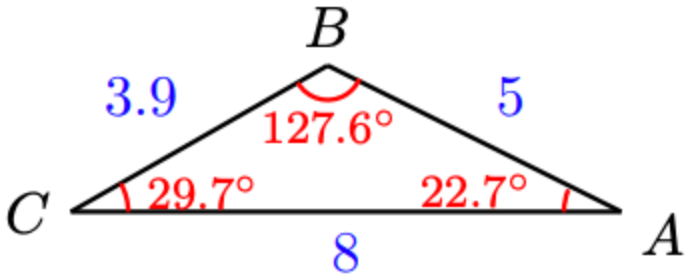


Figure 8.1.9

Example 8.1.5

Building a table, we get the following:



Seeing that we have a complete row, we use that to help us find the missing angle from the first row (angle A).

$$\frac{\sin(A)}{a} = \frac{\sin(C)}{c} \qquad \text{(Law of Sines)}$$

$$\implies \sin(A) = \frac{a\sin(C)}{c} \qquad \text{(multiplying both sides by } a\text{)}$$

$$\implies \widehat{A} = \sin^{-1}\left(\frac{a\sin(C)}{c}\right) \qquad \text{(finding the reference angle)}$$

Grabbing a calculator, substituting in the known values of C, a, and c, we find the calculator returns an error!

What happened?

Consider the following value:

$$\frac{a\sin(C)}{c} = \frac{15\sin(73^\circ)}{2} \approx 7.2$$

This is what $\sin(A)$ is supposed to equal; however, we know that the sine function is always between -1 and 1.

What does this mean?

This means there is no such triangle where $C = 73^\circ$, $a = 15$, and $c = 2$. We can see this from a sketch.

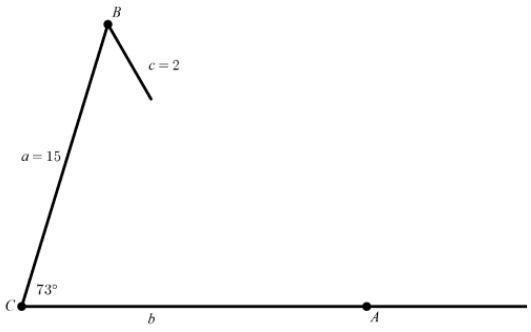


Figure 8.1.10

The angle C is large, but the length of its opposite side is relatively small (when compared to side length a). If we imagine side c being able to swing freely, we can see it will never intersect side b. Therefore, forming a triangle with the given information is impossible.

Checkpoint 8.1.5

While using the Law of Sines to find a missing angle in a triangle, you must compute

$$\sin^{-1}\left(\frac{15\sin(30^\circ)}{7}\right) = \sin^{-1}\left(\frac{15}{14}\right).$$

This means that $\sin(C) = \frac{15}{14}$, which is not possible.

Example 8.1.6

While it's nice that we have been given a sketch of the situation, I find it's best to redraw the sketch without the "fluff," labeling vertices in uppercase and side lengths in lowercase (as we have been doing this entire time).

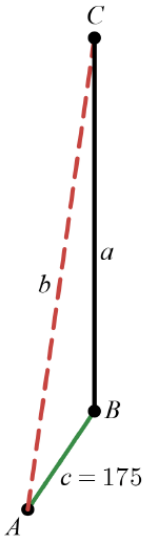


Figure 8.1.12

We now build our table.



We do not have a complete row in this table, so we might think the Law of Sines is not going to be useable; however, looking at the provided sketch, we see one additional piece of information that we have not used.

The green line (side c) makes an angle of 56° with the horizontal. If we draw a horizontal line at vertex B, we can label the angle between the horizontal and the extended green line as 56° (see Figure 8.1.13 below). Since the communications tower (side a) is vertical, the angle between the extended green line and the tower is the complement of 56° (again, see Figure 8.1.13 below). This is 34°.

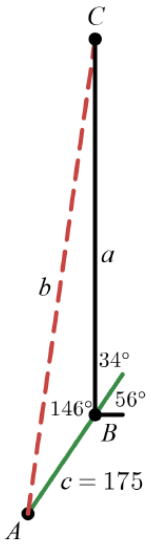


Figure 8.1.13

Finally, we can see that the angle within the triangle at vertex B is the supplement of this 34° angle. Hence, $B = 180^\circ - 34^\circ = 146^\circ$. Let's go ahead and place this information in our table.



Again, we do not have a complete row; however, now that we have two angles, we can easily find the third.

$$C = 180^\circ - A - B = 8^\circ$$

Filling in the table with this information, we get the following:



Finally, we have a complete row in the table, which tells us that the Law of Sines is now available. We are being asked to find side length b. Therefore,

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)} \qquad \text{(Law of Sines)}$$

$$\Longrightarrow \quad b = \frac{c\sin(B)}{\sin(C)} \qquad \text{(multiplying both sides by } \sin(B)\text{)}$$

Substituting in the values of B, C, and c, we get

$$b = \frac{175\sin(146^\circ)}{\sin(8^\circ)} \approx 703.$$

Hence, the length of the guy wire is approximately 703 yards.

Checkpoint 8.1.6

291 meters.

Example 8.1.7

Notice that h is one side of the right triangle $\triangle ADC$. If we find its hypotenuse, labeled r in Figure 8.1.15, we can use the sine ratio to find h. To find r, we consider a second triangle, $\triangle ABC$, as shown in Figure 8.1.16.

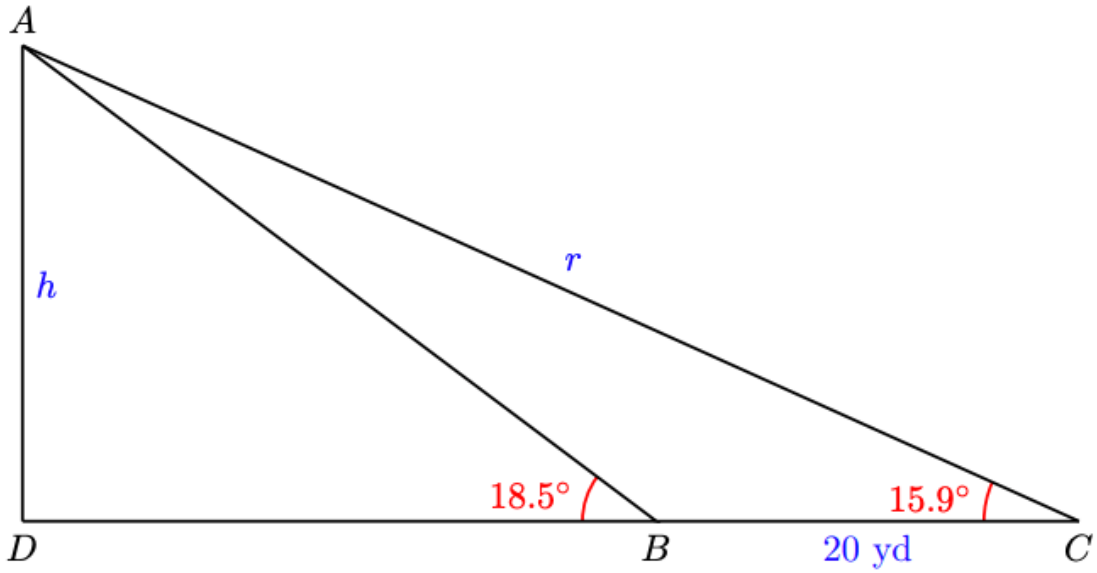


Figure 8.1.16

In this triangle, we know side $BC = 20$ and would like to find side $AC = r$. We can use the Law of Sines to find r, but first, we must calculate the other angles of the triangle.

Now, the angle opposite r, $\angle ABC$, is the complement of 18.5°, so

$$\angle ABC = 180^\circ - 18.5^\circ = 161.5^\circ$$

The angle opposite the 20-yard side, $\angle BAC$, is

$$\angle BAC = 180^\circ - (161.5^\circ + 15.9^\circ) = 2.6^\circ$$

Now we can apply the Law of Sines to find r. We have

$$\frac{r}{\sin(161.5^\circ)} = \frac{20}{\sin(2.6^\circ)} \qquad \text{(Law of Sines)}$$

$$\Longrightarrow \quad r = \frac{20\sin(161.5^\circ)}{\sin(2.6^\circ)} \qquad \text{(multiplying both sides by } \sin(161.5^\circ)\text{)}$$

Finally, using the right triangle $\triangle ADC$, we can write

$$\begin{aligned} \frac{h}{r} &= \sin(15.9^\circ) && \text{(right triangle definition of the sine)} \\ \Longrightarrow \quad h &= r\sin(15.9^\circ) && \text{(multiplying both sides by } r\text{)} \\ &= \frac{20\sin(161.5^\circ)}{\sin(2.6^\circ)} \cdot \sin(15.9^\circ) \\ &\approx 38.33 \end{aligned}$$

The castle is about 38.33 yards tall.

Checkpoint $8.1.7$

$AB = 120.79$, the castle is about 38.33 yards tall.

Example $8.1.8$

We let x represent the distance to the asteroid. The asteroid and the two observers make an isosceles triangle with a base of approximately 1000 km and equal sides of length x, as shown below.

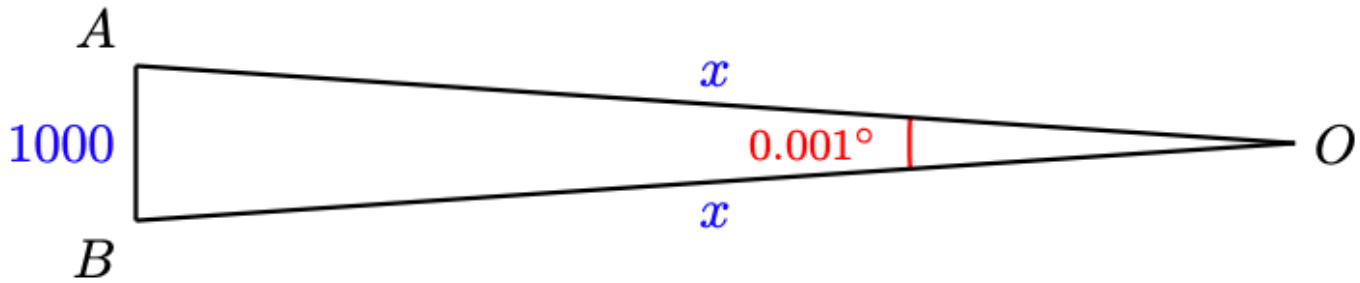


Figure $8.1.18$

The base angles of the triangle are both $\frac{180^\circ - 0.001^\circ}{2} = 89.999^\circ$. Thus,

$$\begin{aligned} \frac{x}{\sin(89.999^\circ)} &= \frac{1000}{\sin(0.001^\circ)} && \text{(Law of Sines)} \\ \Longrightarrow \quad x &= \frac{1000\sin(89.999^\circ)}{\sin(0.001^\circ)} && \text{(multiplying both sides by } \sin(89.999^\circ)\text{)} \\ \Longrightarrow \quad x &\approx 57,000,000 \end{aligned}$$

The asteroid is about 57 million kilometers from Earth (roughly one-third of the distance to the Sun).

Checkpoint $8.1.8$

About $91,673,247$ km

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8.1.1: Resources and Key Concepts

Resources

Videos

- Law of Sines
  - Non-Right Triangles - Law of Sines: Theorem and Proof
  - Non-Right Triangles - Law of Sines: A Conceptual Investigation of the Ambiguous Case
  - Non-Right Triangles - Law of Sines: Using the Law of Sines to Solve Triangles
  - Non-Right Triangles - Law of Sines: Applications of Non-Right Triangle Trigonometry (Part 1)
  - The Law of Sines - Review of the Law of Sines
  - The Law of Sines - Review of Applications of Non-Right Triangle Trigonometry (Part 1)

Key Concepts

Definitions

- **Ambiguous Case:** A situation that arises when using the Law of Sines to find a missing angle, where the given information (typically two sides and a non-included angle, or SSA) could result in one triangle, two different triangles, or no triangle at all.

Theorems

- **Law of Sines:** If the angles of a triangle are A, B, and C, and the opposite sides are a, b, and c, respectively, then

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \tag{8.1.1.1}$$

  or equivalently,

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \tag{8.1.1.2}$$

- **The Law of Sines - Ambiguous Case (SSA):** Given two sides a and b and an acute angle α opposite side a, let $h = b\sin(\alpha)$ be the altitude of the triangle.
  - **No Solution:** If $a < h$, no triangle can be formed.
  - **One Solution:** If $a = h$ (a right triangle) or if $a > b$, one triangle can be formed.
  - **Two Solutions:** If $h < a < b$, two distinct triangles can be formed.

Common Mistakes

- **Do Not Use Rounded Values to Calculate:** When solving a triangle, use the exact or stored values from your calculator for all intermediate steps. Using rounded approximations can propagate errors and lead to an inaccurate final answer.
- **Forgetting the Ambiguous Case:** When using the Law of Sines to solve for a missing angle (the SSA case), remember that there might be two possible triangles. The arcsine function on a calculator will only return an acute angle; you must also check its supplement as a potential second solution for the angle.
- **Misidentifying the Number of Solutions:** In the ambiguous case, failing to compare the side opposite the given angle to the altitude and the other given side can lead to incorrectly concluding there is one triangle when there are two or none, or vice versa.

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8.1.2: Homework

Reading Questions

1. What is an oblique triangle?
2. State the Law of Sines.
3. To use the Law of Sines to solve for a missing side or angle, what information must you already have about the triangle?
4. What does it mean to "solve a triangle"?
5. When using the Law of Sines to find an unknown angle, why must you be careful?
6. What is the "ambiguous case" for the Law of Sines? What given information (e.g., SSS, ASA, SSA) leads to this case?
7. If you use the Law of Sines to find that a missing angle θ could be $40°$, what is the other possible value for θ?
8. How do you determine if both possible angles in the ambiguous case lead to valid triangles?
9. If your calculator returns an error when you try to use the arcsin function to find a missing angle, what does this imply about the triangle?
10. Why is it better to use the given values in calculations rather than values you have already calculated and rounded?

Homework

Concept Check

1. Can we use the Law of Sines to solve a right triangle?

2. Explain why we cannot use the Law of Sines to solve the triangle with $a = 8$, $b = c$, and $C = 35°$.

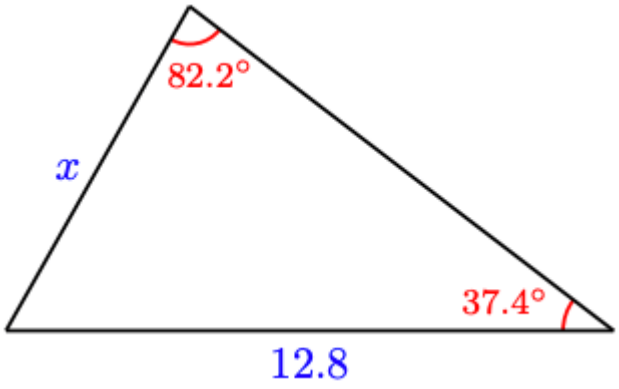
3. Linda says, "I'm thinking of an angle whose sine is 0.3420 (rounded to four decimal places)." Matt says, "The angle must be $20°$ (rounded to the nearest degree)." Is he correct? Why or why not?

4. Sketch two possible triangles with $A = 25°$, $b = 18$, and $a = 10$.

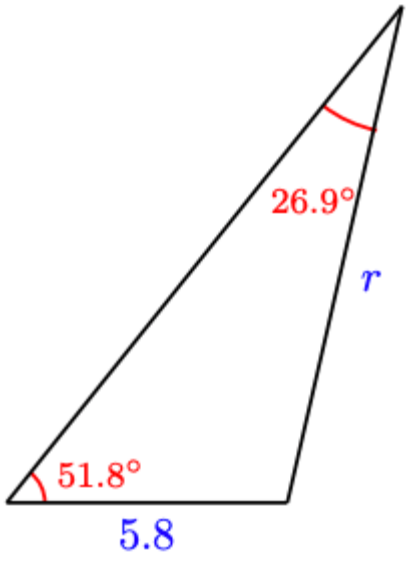
5. Try to sketch a triangle with $A = 65°$, $b = 18$, and $a = 10$. What went wrong?

Basic Skills

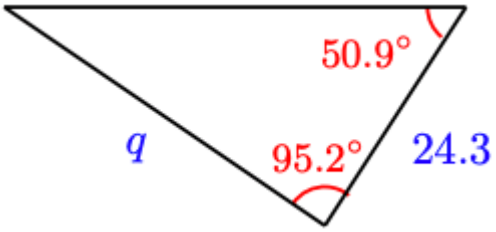
For the following exercises, find the indicated side. Round to two decimal places.

6. 

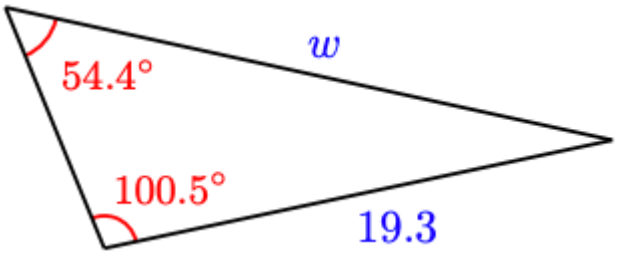
7.



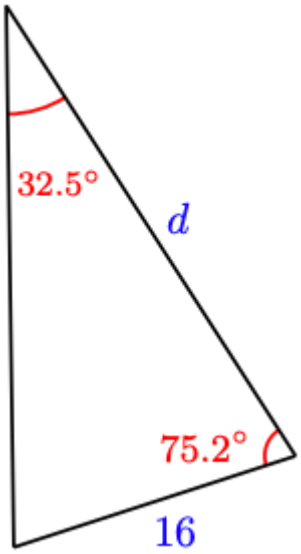
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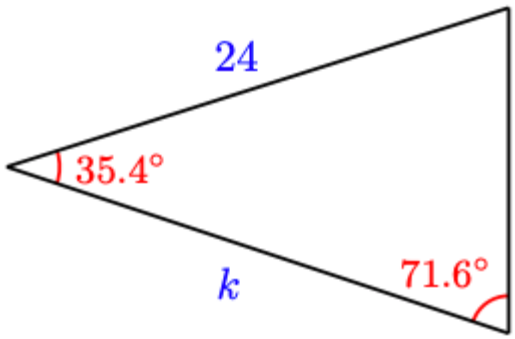


9.



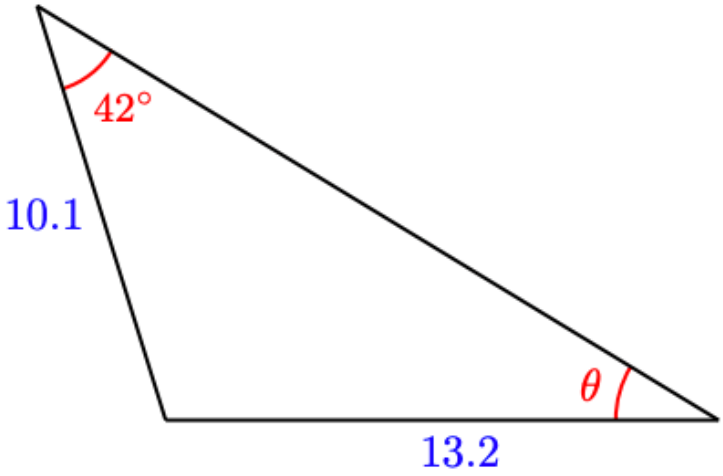
10.



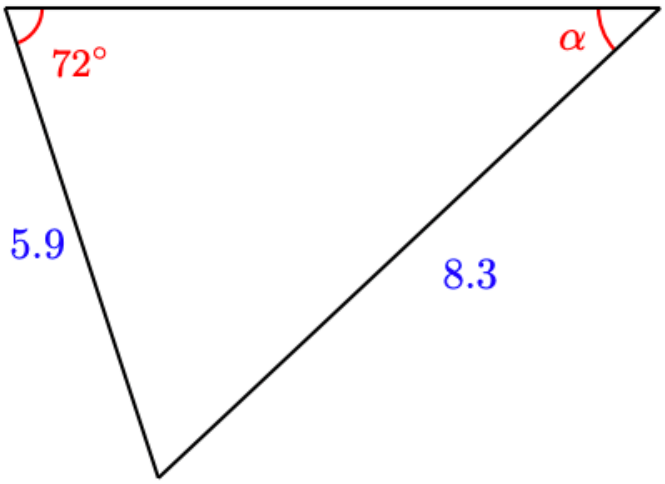


11.

For the following exercises, find the indicated angle. Round to two decimal places.

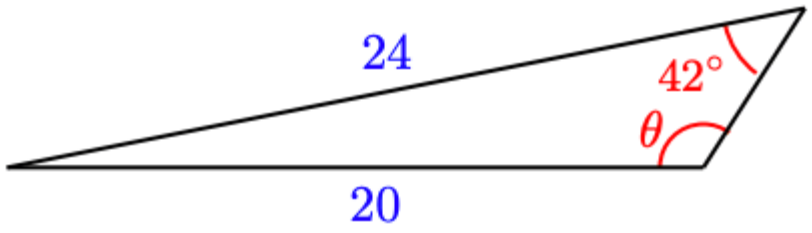


12. (θ is acute.)



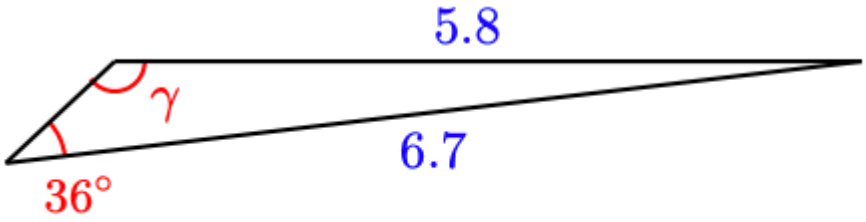
13. (α is acute.)

14.



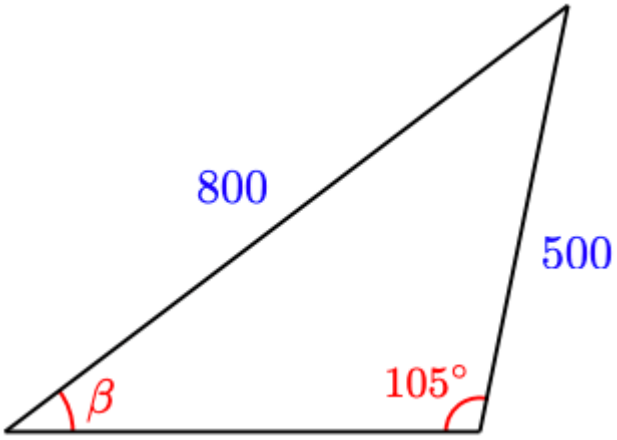
(θ is acute.)

15.

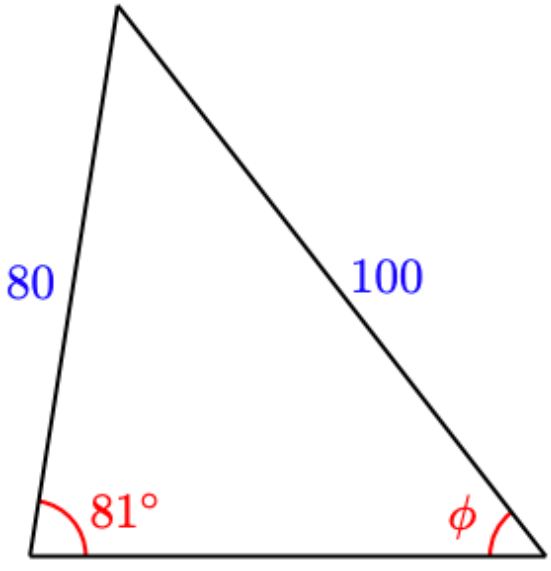


(γ is acute.)

16.



17.



For the following exercises, sketch the triangle and solve. Round answers to two decimal places.

18. $b = 7$, $A = 23^\circ$, $B = 42^\circ$

19. $c = 34$, $A = 53^\circ$, $C = 26^\circ$

20. $a = 1.8$, $c = 2.1$, $C = 44^\circ$

21. $b = 8.5$, $c = 6.8$, $B = 23^\circ$

22. $c = 75$, $A = 35^\circ$, $B = 46^\circ$

23. $a = 94$, $B = 29^\circ$, $C = 84^\circ$

24. In $\triangle ABC$, $\angle A = 30^\circ$ and $c = 12$. How many triangles are possible for each of the following lengths for side a? Sketch the solutions in each case.

    a. $a = 6$

    b. $a = 4$

    c. $a = 9$

    d. $a = 15$

For the following exercises, find the remaining angles of the triangle (or triangles, if this is the ambiguous case of the Law of Sines). Round answers to two decimal places.

25. $a = 66$, $c = 43$, $C = 25^\circ$

26. $b = 10$, $c = 14$, $B = 20^\circ$

27. $b = 100$, $c = 80$, $B = 49^\circ$

28. $b = 4.7$, $c = 6.3$, $C = 54^\circ$

29. a. Sketch a triangle with $A = 25^\circ$, $B = 35^\circ$, and $b = 16$.

    b. Use the Law of Sines to find a.

    c. Use the Law of Sines to find c.

    d. Find c without using the Law of Sines. (Hint: Sketch the altitude, h, from C to make two right triangles. Find h, then use h to find c.)

Synthesis Questions

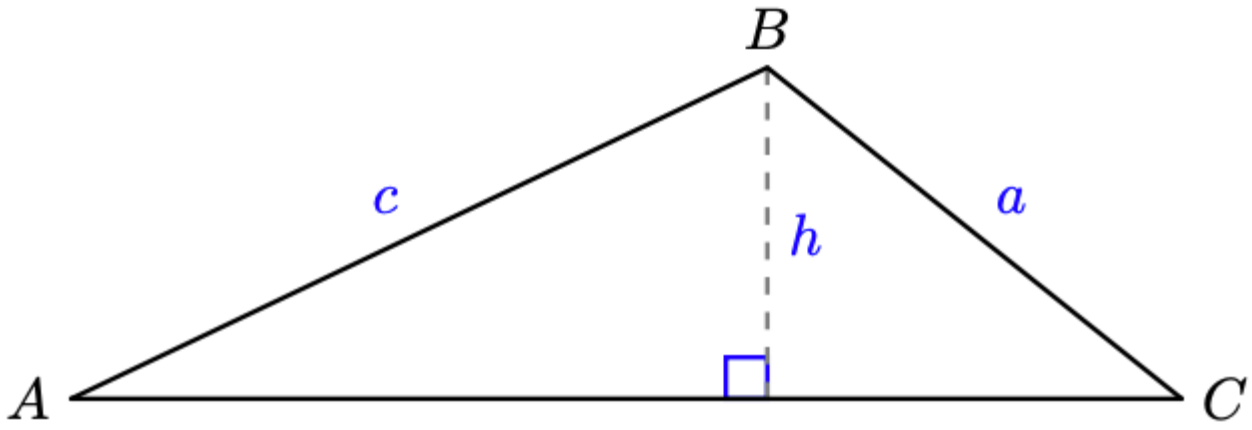
30. In this problem, we show that there are two different triangles $\triangle ABC$ with $A = 30^\circ$, $a = 2$, and $c = 3$.

    a. Use a protractor to draw an angle $A = 30^\circ$. Mark point B on one side of the angle so that $\overline{AB}$ is 3 inches long.

    b. Locate two distinct points on the other side of the angle that are each 2 inches from point B. These points are both possible locations for point C.

    c. Use the Law of Sines to find two distinct possible measures for $\angle C$.

31. Consider the triangle $\triangle ABC$ shown below.



a. Express the length of the altitude in terms of $\angle A$ and c.

b. Suppose we keep $\angle A$ and side c fixed but allow a to vary in length. What is the smallest value a can have and still be long enough to make a triangle?

c. What are the largest and smallest values that a can have in order to produce two distinct triangles $\triangle ABC$ (without changing $\angle A$ and side c)

32. For the triangle in the previous problem, suppose $A = 40^\circ$ and $c = 8$.

a. Sketch and solve the triangle if $a = 12$.

b. Sketch and solve the triangle if $a = 6$.

c. Sketch and solve the triangle if $a = 4$.

d. For what value of a is c the hypotenuse of a right triangle?

33. For the triangle in Problem 31, suppose $A = 70^\circ$ and $c = 20$.

a. For what value of a is the triangle a right triangle?

b. For what values of a are there two solutions for the triangle?

c. For what values of a is there one obtuse solution for the triangle?

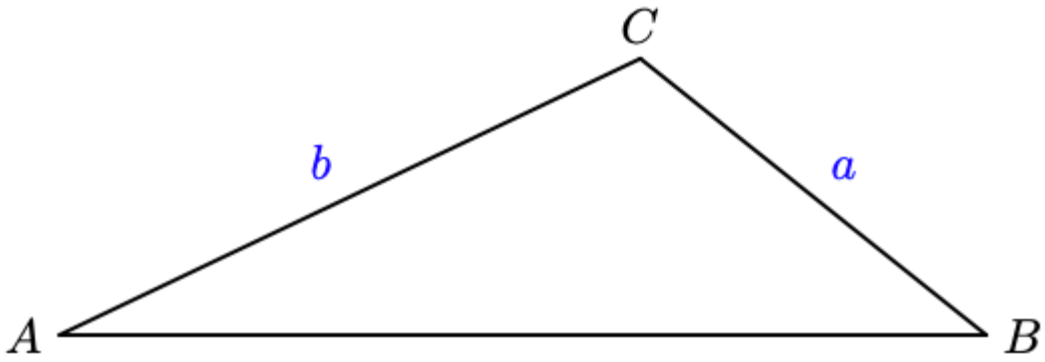
d. For what value of a is there no solution?

34. a. Sketch a triangle with $A = 75^\circ$, $a = 15$, and $b = 6$.

b. Use the Law of Sines to find c.

c. Find c without using the Law of Sines.

35. Here is a method for solving certain oblique triangles by dividing them into two right triangles. In the triangle shown, we know two angles, A and B, and the side opposite one of them, say a. We would like to find side b.



a. Draw the altitude h from angle C.

b. Write an expression for b in terms of h and angle A.

c. Write an expression for h in terms of angle B.

d. Substitute your expression for h into your expression for b.

e. Which of the following is equivalent to the formula you wrote in part (d)?

i. $a\sin(A) = b\sin(B)$

ii. $\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$

iii. $\frac{a}{\sin(B)} = \frac{b}{\sin(A)}$

Proofs

For the following exercises, you will prove the Law of Sines using the trigonometric formula for the area of a triangle. We will derive this formul later. For now, you can use the fact that the area of triangle $\triangle ABC$ is $\frac{1}{2}ab\sin(\theta)$, where θ is the angle between sides a and b.

36. Sketch a triangle with angles A, B, and C, and opposite sides of lengths a, b, and c, respectively.

a. Write the area of the triangle in terms of a, b, and angle C.

b. Write the area of the triangle in terms of a, c, and angle B.

c. Write the area of the triangle in terms of b, c, and angle A.

37. Equate the three different expressions from Problem 36 for the area of the triangle. Multiply through by $\frac{2}{abc}$ and simplify to deduce the Law of Sines.

Applications

For the following exercises,
(a) Sketch and label a triangle to illustrate the problem.
(b) Solve the problem.

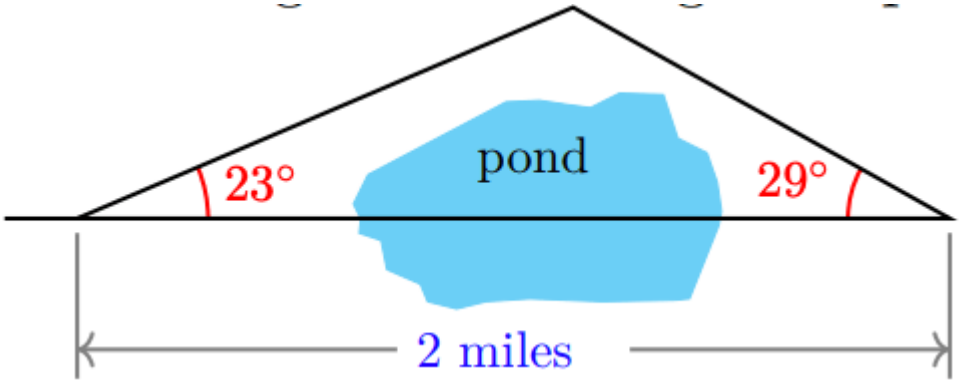
38. Phuong wants to know the height of a cliff on the other side of a ravine. The angle of elevation from the edge of the ravine to the cliff top is 84.6°. When she moves 30 feet back from the ravine, the angle of elevation is 82.5°. How tall is the cliff?

39. Nam wants to know the height of a tree in the median strip of a highway. The angle of elevation from the highway shoulder to the treetop is 43.5°. When he moves 10 feet farther away from the tree, the angle of elevation is 37.2°. How tall is the tree?

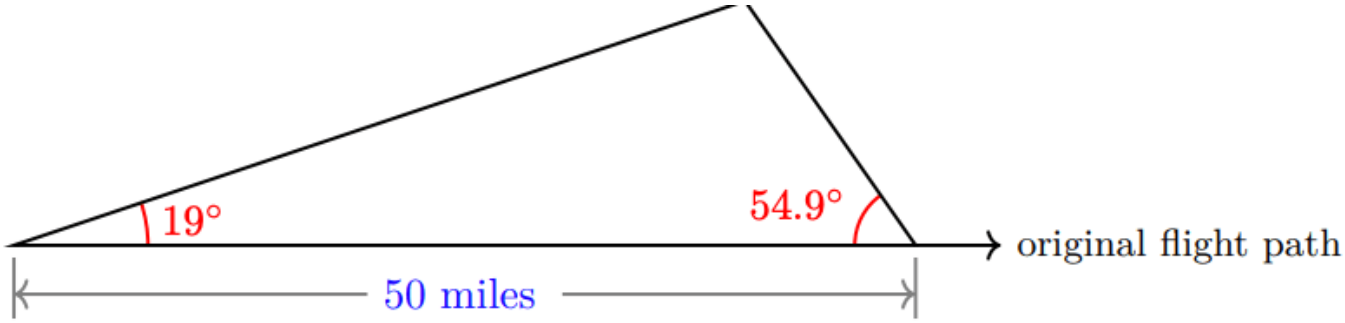
10. Mike and Loi are 10 kilometers apart, observing a satellite that passes directly over their heads. At a moment when the satellite is between them, Loi measures its angle of elevation as 84.6°, and Mike measures an angle of 87°. How far is the satellite from Mike?

11. Paige rows her kayak due east. When she began, she spotted a lighthouse 2000 meters in the distance at an angle of 14° south of east. After traveling for an hour, the lighthouse was at an angle of 83° south of east. How far did Paige travel, and what was her average speed?

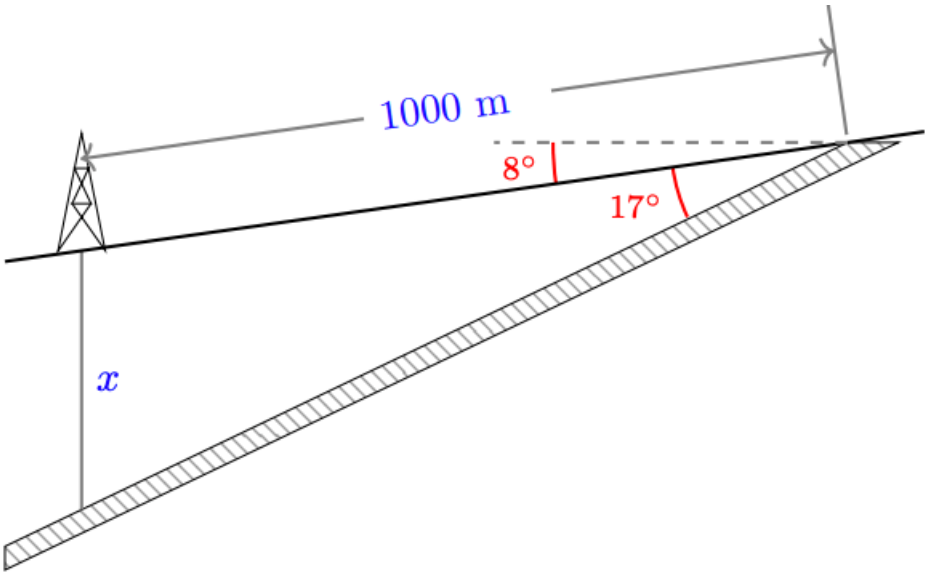
12. Ivan is hiking along a straight path but needs to detour around a large pond. He turns 23° from his path until clear of the pond, then walks back to his original path, intercepting it at an angle of 29° and at a distance of 2 miles from where he had left the path. How far did Ivan walk in each of the two segments of his detour, and how much farther did his detour require compared with a straight line through the pond?



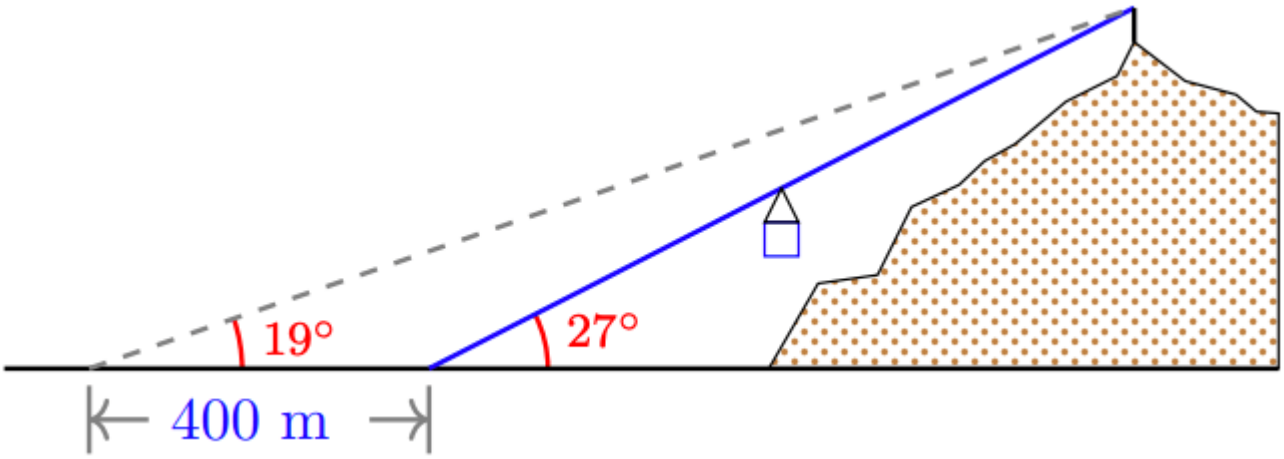
13. Wyatt is flying to Monterey but must change course to avoid a storm. He flies 19° off from his original direction until he clears the storm, then turns again to return to his original flight path, intercepting it at an angle of 54.9° and at a distance of 50 miles from where he had left it. How much farther did his detour require compared with his original course?



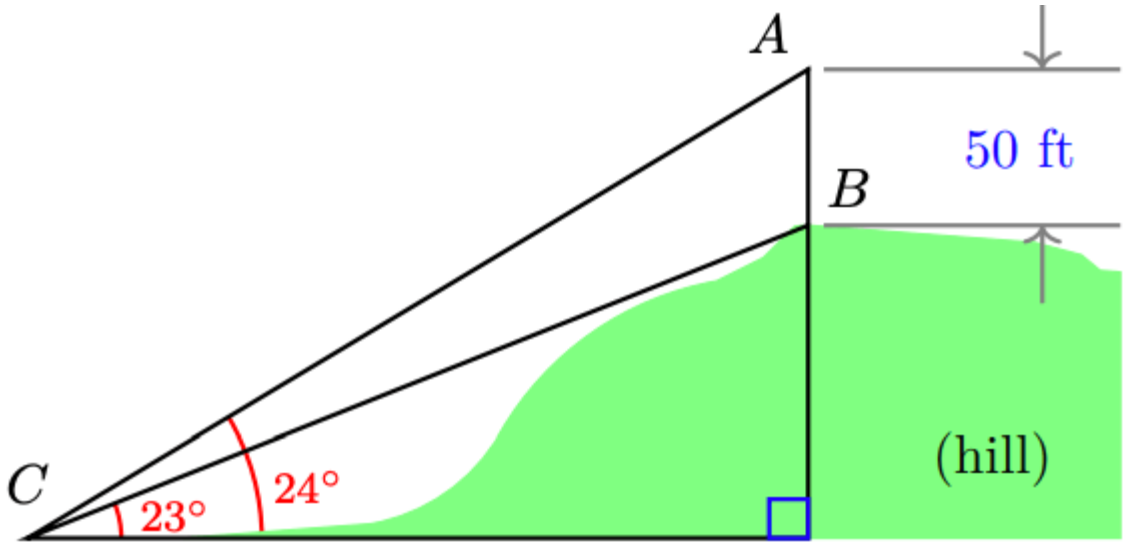
14. Geologists find an outcropping from an underground rock formation that usually indicates the presence of oil. The outcropping is on a hillside, and the formation dips another 17° from the surface. If an oil well is placed 1000 meters downhill from the outcropping, how far will they drill before reaching the formation?



15. A proposed ski lift will rise from a point near the base of the slope with an angle of 27°. At a distance of 400 meters further from the hill, the angle of elevation to the top of the ski lift is 19°. How long is the ski lift?



16. Nhat wants to measure the height of a hill. She first plants a 50-foot-tall antenna at the hill's peak. Then, she descends the hill and finds a point where she can see the antenna's top and bottom. The angle of elevation to the bottom of the antenna is 23°, and the angle of elevation to the top of the antenna is 24°.



a. Find $\angle ACB$.

b. Find $\angle CAB$ at the top of the antenna.

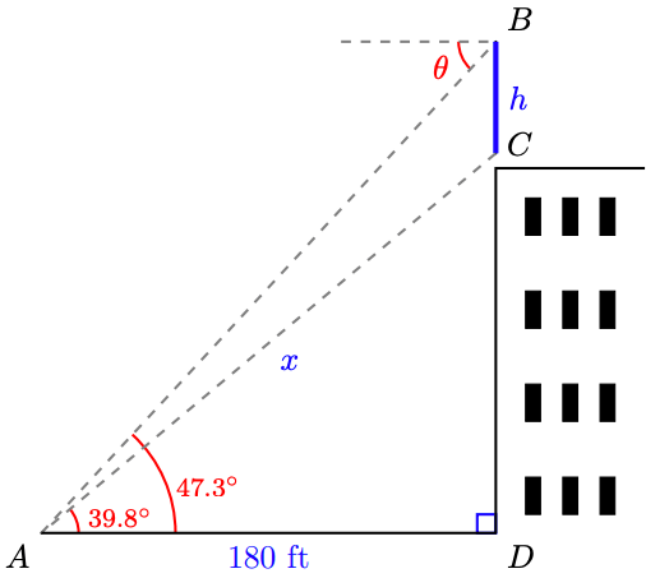
c. How long is BC, the distance from the bottom of the antenna to C?

d. How tall is the hill?

17. Jorge and Camille are 1000 yards apart. The angle Jorge sees between Camille and a particular tree is 38°. If the tree is 800 yards from Camille, how far is it from Jorge? (There are two possible answers.)

18. From the lookout point on Fabrick Rock, Sang can not only see the famous "Crooked Spire" in Chesterfield, which is 8 miles away but also the red phone box in the village of Alton. Chesterfield and Alton are 7 miles apart. Fabrick Rock has a plaque that shows directions to famous sites, and from the plaque, Sang determines that the angle between the lines to the spire and the phone box measures 19°. How far is Fabrick Rock from the phone box? (There are two possible answers.)

19. A billboard of California's gubernatorial candidate, Min, is located on the roof of a building. At a distance of 180 feet from the building, the angles of elevation to the bottom and top of the billboard are respectively 39.8° and 47.3°. How tall is the billboard?



For the following exercises, compute the distances in Astronomical Units. Then convert to kilometers, using the fact that 1 AU $\approx 1.5 \times 108$ km.

50. When observed from opposite sides of Earth's orbit, the star Alpha Centauri has a parallax of $0.76''$. How far from the Sun is Alpha Centauri?

51. How far from the Sun is Barnard's star, which has a parallax of $1.1''$ when observed at opposite ends of Earth's orbit?

52. How far from the Sun is Tau Ceti, which has a parallax of $0.55''$ when observed from opposite ends of Earth's orbit?

53. How far from the Sun is Sirius, which has a parallax of $0.75''$ when observed from opposite ends of Earth's orbit?

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8.2: The Law of Cosines

Section Preview

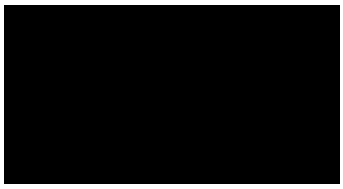
Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The Law of Cosines

When learning the Law of Sines, we used a table to help organize information and determine which unknown to find first. For example, suppose the information in the following table was what we were given.



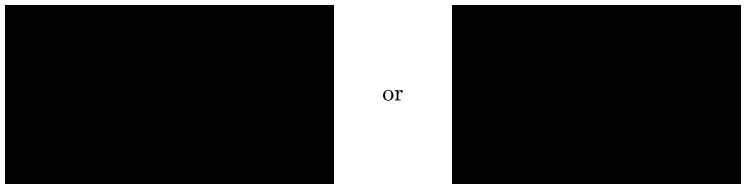
I mentioned in the previous section that a complete row in this table meant we could use the Law of Sines. Therefore, we can see that, given two angles and a side opposite one of those angles, we can use the Law of Sines to solve the triangle.

Suppose, instead, the given information looked as follows:



Again, we have a complete row, so we can use the Law of Sines. Therefore, given two sides and an angle opposite one of those sides, we could use the Law of Sines to solve the triangle (if one exists).

In contrast, given any other combination of sides or angles, the Law of Sines is not helpful. If you consider what is required for this to happen, we just need a table with no complete rows. The two major examples are the following:



Notice that there is not a complete row within either of these tables and, as of now, we have no simple way of completing a row. Not having an angle-side pairing presents an issue that the Law of Sines cannot surmount (unless you were given two angles, in which case you can find the third and proceed to use the Law of Sines).

To solve a triangle when we know two sides and the included angle, we will need a generalization of the Pythagorean Theorem.

In a right triangle, with $C = 90°$, the Pythagorean Theorem tells us that

$$c^2 = a^2 + b^2.$$

If we allow angle C to vary (rather than being locked in at a right angle) but keep a and b the same length, the side c will grow or

shrink, depending on whether we increase or decrease the angle C, as shown below.

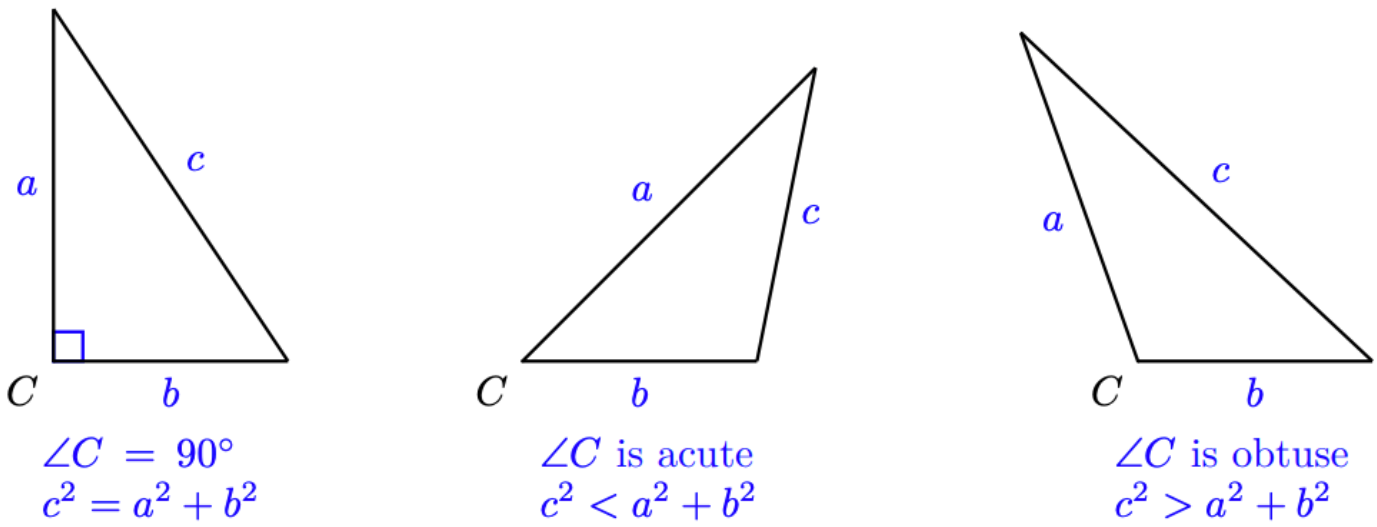


Figure 8.2.1

The Pythagorean Theorem is a special case of a more general law that applies to *all* triangles, no matter the size of angle C. This law is known as the **Law of Cosines**.

Theorem: Law of Cosines

If the angles of a triangle are A, B, and C, and the opposite sides are respectively a, b, and c, then

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos(A) \\ b^2 &= a^2 + c^2 - 2ac\cos(B) \\ c^2 &= a^2 + b^2 - 2ab\cos(C) \end{aligned}$$

Proof available at the end of this section.

The Law of Cosines is also known as the **Generalized Pythagorean Theorem**.

Rather than committing all three forms to memory, I find it much easier to think of these as modifications of the Pythagorean Theorem. We have a sum of squares of sides *almost* equal to the square of a quasi-hypotenuse. The adjustment we make to the sum of squares side is to subtract twice the product of the non-quasi-hypotenuse sides and the cosine of the angle opposite the quasi-hypotenuse.

Now that I have written that out, I can see that it might be confusing to most people, so let me say it symbolically. If s_1 and s_2 are playing the roles of the non-quasi-hypotenuse sides, and h is playing the role of the quasi-hypotenuse, then

$$s_1^2 + s_2^2 - 2s_1 s_2 \cos(H) = h^2,$$

where H is the angle opposite side h.

Finding a Side

Example 8.2.1

The town of Avery lies 48 miles due east of Baker, and Clio is 34 miles from Baker, in the direction 35° west of north. How far is it from Avery to Clio?

Solution available at the end of this section.

Caution: Follow the Order of Operations

When simplifying the Law of Cosines, follow the Order of Operations carefully. In Example 8.2.1, the right side of the equation

$$b^2 = 34^2 + 48^2 - 2(34)(48)\cos(125^\circ)$$

has three terms, and simplifies to

$$\begin{aligned} b^2 &= 1156 + 2304 - 3264\cos(125^\circ) \\ \Longrightarrow \quad b^2 &\approx 3460 - 3264(-0.573567364\ldots) \end{aligned}$$

Note that 3264 is the coefficient of $\cos(125^\circ)$, so it would be incorrect to subtract 3264 from 3460. Using a graphing calculator, you can enter the right side of the equation exactly as it is written.

Checkpoint 8.2.1

In $\triangle ABC$, $a = 11$, $c = 23$, and $B = 87^\circ$. Find b, and round your answer to two decimal places.

Answer available at the end of this section.

Finding an Angle

We can also use the Law of Cosines to find an angle when we know all three sides of a triangle. In the following example, pay close attention to the algebraic steps used to solve the equation.

Example 8.2.2

In the triangle below, $a = 6$, $b = 7$, and $c = 11$. Find angle C.

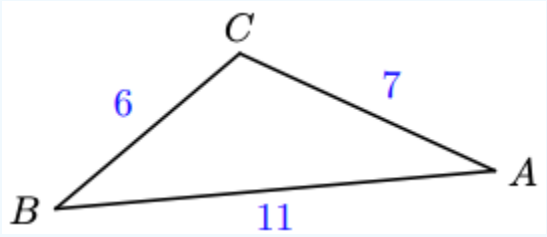


Figure 8.2.3

Solution available at the end of this section.

Checkpoint 8.2.2

In $\triangle ABC$, $a = 5.3$, $b = 4.7$, and $c = 6.1$. Find angle B, and round your answer to two decimal places.

Answer available at the end of this section.

Once we have calculated one of the angles in a triangle, we can use either the Law of Sines or the Law of Cosines to find a second angle. Here is how we would use the Law of Sines to find angle A in Example 8.2.2.

$$\begin{aligned} \frac{\sin(A)}{a} &= \frac{\sin(C)}{c} && \text{(Law of Sines)} \\ \Longrightarrow \quad \sin(A) &= \frac{a\sin(C)}{c} && \text{(multiplying both sides by } a\text{)} \\ \Longrightarrow \quad \widehat{A} &= \sin^{-1}\left(\frac{a\sin(C)}{c}\right) && \text{(finding the reference angle)} \end{aligned}$$

Substituting in the given values, we get $\widehat{A} \approx 29.5^\circ$. Since A is the shortest side of the triangle, A must be acute. Therefore, $A \approx 29.5^\circ$. Finally,

$$B = 180^\circ - A - C \approx 35.1^\circ.$$

Alternatively, we can use the Law of Cosines to find angle A.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos(A) && \text{(Law of Cosines)} \\ \Longrightarrow \quad a^2 - b^2 - c^2 &= -2bc\cos(A) && \text{(subtracting } b^2 \text{ and } c^2 \text{ from both sides)} \\ \Longrightarrow \quad \frac{a^2 - b^2 - c^2}{-2bc} &= \cos(A) && \text{(dividing both sides by } -2bc\text{)} \\ \Longrightarrow \quad \cos^{-1}\left(\left|\frac{a^2 - b^2 - c^2}{-2bc}\right|\right) &= \widehat{A} && \text{(finding the reference angle)} \end{aligned}$$

Substituting the given values into this result, we get the reference angle of $\widehat{A} \approx 29.5^\circ$. If we had computed the value of $\frac{a^2-b^2-c^2}{-2bc}$, we would have found that this fraction is positive. Since the arccosine returns in QI or QII, and, of those two quadrants, the cosine is positive in QI, our actual angle is $A \approx 29.5^\circ$.

From this conversation about finding a second angle once you have used the Law of Cosines to find an initial angle, I hope you agree that the amount of computational work is minimized when using the Law of Sines. You can choose either method to find a second angle, but I will always choose the Law of Sines over the Law of Cosines.

Advice: Choose Your Angle Wisely!

When you have used the Law of Cosines to find the first missing angle within a triangle, you can use the Law of Sines or the Law of Cosines to find the next unknown angle. If using the Law of Sines, however, *always* choose to find the angle opposite the smallest remaining side. This is because the angle opposite the smallest remaining side is *guaranteed* to be acute. Hence, the reference angle returned when using the Law of Sines will be the actual angle!

Applications Involving the Law of Cosines

The applications of the Law of Cosines are countless, more so than the Law of Sines; however, they all somewhat follow the structure of the following example.

Example 8.2.3

A researcher wishes to determine the width of a vernal pond, as drawn below. From a point P, he finds the distance to the eastern-most point of the pond to be 950 feet, while the distance to the western-most point of the pond from P is 1000 feet (obviously, the perspective of the provided image is such that due north is roughly facing the 4 o'clock position). If the angle between the two lines of sight is 60°, find the width of the pond.

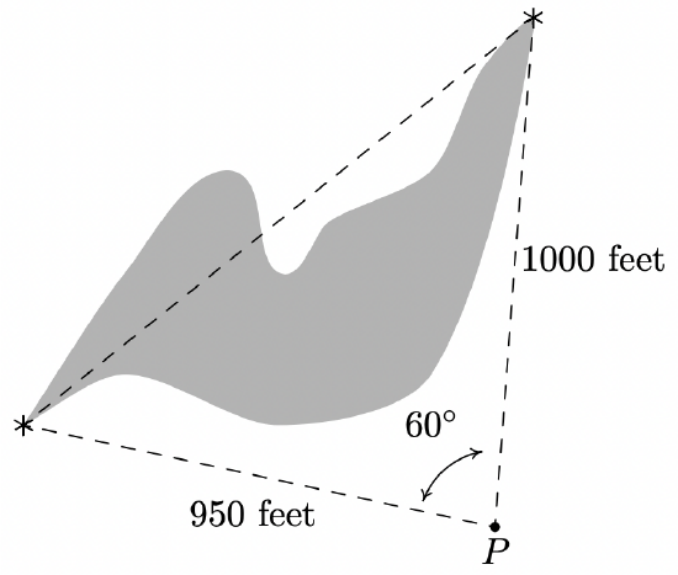


Figure 8.2.4

Solution available at the end of this section.

Checkpoint 8.2.3

The hour hand on Kim's antique clock is 4 inches long and the minute hand is 5.5 inches long. Find the distance between the ends of the hands when the clock reads four o'clock. Round your answer to the nearest hundredth of an inch.

Answer available at the end of this section.

Revisiting Navigation Problems

In a previous section, we were introduced to navigation problems (heading and bearing problems). At that time, however, we were only able to work with these styles of problems when they resulted in right triangles. We now have enough Trigonometry "under our belts" to approach these problems in *any* situation.

Example 8.2.4

The sailing club leaves the marina on a heading of 15° and sails for 18 miles. They then change course, and after traveling for 12 miles on a heading of 35°, they experience engine trouble and radio for help. The marina sends a speed boat to rescue them. How far should the speed boat go, and on what heading?

Solution available at the end of this section.

Checkpoint 8.2.4

Roy wants to fly from Anchorage to Nome, Alaska, a distance of 540 miles on a heading of 303°. After flying for some time, he discovers that his heading is in error and is actually flying at a heading of 313°. Roy corrects his flight plan and changes course when he is 200 miles from Anchorage. What is his new heading, and how far is he from Nome?

Answer available at the end of this section.

Success in College: Dealing With Failure (Part 4)

1. A first-year student fails her math class. She hates math. Her advisor wants her to retake the course the next term. The student wants to wait a year or so before trying again. Discuss the advantages and disadvantages of waiting to retake this math course.
2. Some colleges restrict the number of times a student can repeat a course. Describe any restrictions on repeating a course at your school.

Answers

Section Preview

This section is a very traditional treatment of the Law of Cosines and applications involving the Law of Cosines. You will likely spend a good chunk of time on navigation problems, as these are often difficult for students to master.

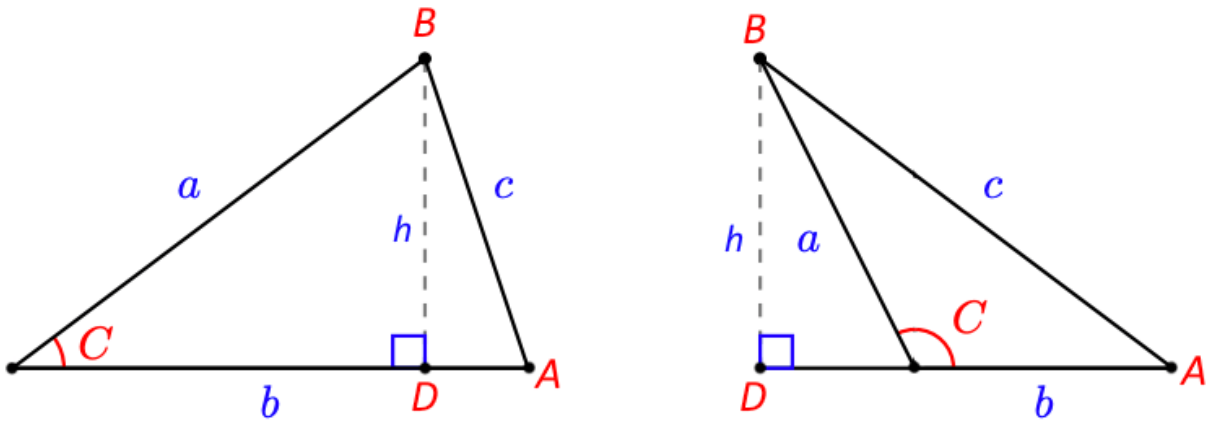
Section Preview

- Use the Law of Cosines to find a missing side or a missing angle in an oblique triangle.
- Use the Law of Sines to find the remaining angle once the Law of Cosines has been used to find one angle in an oblique triangle.
- Use the Law of Cosines to solve an application involving heading or bearing.
- Use the Law of Cosines to solve an application.

Theorem: Law of Cosines

Start with general triangles as seen in the figures below.

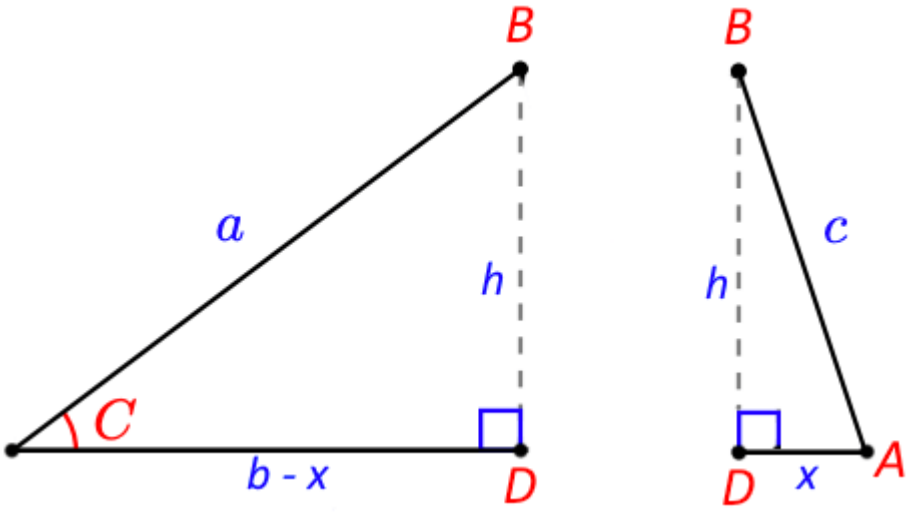
Two general triangles, one acute (left) and one obtuse (right), with altitudes labeled as h



We will perform the proof for the acute triangle (the proof associated with the obtuse triangle is similar).

The acute triangle is comprised of two right triangles - $\triangle ABD$ and $\triangle BCD$. Let x be the length of the line segment $\overline{AD}$. Then the length of the segment $\overline{CD}$ is $b - x$. We show these facts in the following figure, splitting the general right triangle into its two right triangle pieces.

"Splitting" the acute triangle into two right triangles



With the leftmost triangle, we have the following:

$$\begin{aligned} (b-x)^2 + h^2 &= a^2 && \text{(Pythagorean Theorem)} \\ \implies b^2 - 2bx + x^2 + h^2 &= a^2 && \text{(distributing)} \end{aligned}$$

With the rightmost triangle, we have

$$x^2 + h^2 = c^2 \quad \text{(Pythagorean Theorem)}$$

Therefore,

$$b^2 - 2bx + x^2 + h^2 = a^2 \quad \text{and} \quad x^2 + h^2 = c^2.$$

Replace $x^2 + h^2$ in the first equation with c^2. Doing so yields the following:

$$b^2 - 2bx + c^2 = a^2 \implies b^2 + c^2 - 2bx = a^2.$$

From the rightmost triangle in the figure above, we see that $\cos(A) = \frac{x}{c}$, and so $c\ \cos(A) = x$. Hence, we have developed the following relationship between the sides of *any* triangle.

$$\begin{aligned} b^2 + c^2 - 2bx &= a^2 \\ \implies b^2 + c^2 - 2b(c\cos(A)) &= a^2 && \text{(substitution)} \\ \implies b^2 + c^2 - 2bc\cos(A) &= a^2 \end{aligned}$$

When A is a right angle, $\cos(A) = \cos(90^\circ) = 0$, so the equation reduces to the Pythagorean Theorem.

We can write similar equations involving the angles B or C. In all cases, the angle within the cosine is opposite the side playing the role of the "hypotenuse" (really, a **quasi-hypotenuse**).

Example 8.2.1

We start with a simple sketch.

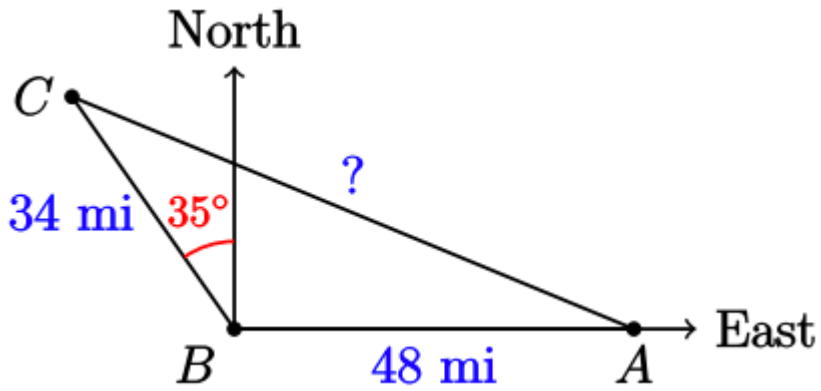


Figure 8.2.2

The angle $\angle ABC = 35^\circ + 90^\circ = 125^\circ$. Thus, in $\triangle ABC$ we have the following:



Since we do not have a complete row, we must use the Law of Cosines. We are interested in knowing the length of side b, so that will be our quasi-hypotenuse.

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos(B) && \text{(Law of Cosines)} \\ \implies b^2 &= 34^2 + 48^2 - 2(34)(48)\cos(125^\circ) && \text{(substitute)} \\ \implies b^2 &= 3460 - 3264\cos(125^\circ) \\ \implies b^2 &\approx 5332.153 \\ \implies b &\approx 73.02 \end{aligned}$$

Therefore, Avery is about 73 miles from Clio.

Checkpoint 8.2.1

24.97

Example 8.2.2

Our table looks like



Since there is no way to get a complete row, we must use the Law of Cosines. Moreover, because we are asked to find C, we choose the version of the Law of Cosines that uses angle C.

$$\begin{aligned} & c^2 = a^2 + b^2 - 2ab\cos(C) && \text{(Law of Cosines)} \\ \Longrightarrow\quad & 11^2 = 6^2 + 7^2 - 2(6)(7)\cos(C) && \text{(substitute)} \\ \Longrightarrow\quad & 121 = 36 + 49 - 84\cos(C) && \text{(apply powers and compute the product)} \\ \Longrightarrow\quad & 36 = -84\cos(C) && \text{(subtract 85 from both sides)} \\ \Longrightarrow\quad & -\frac{3}{7} = \cos(C) && \text{(divide both sides by } -84) \\ \Longrightarrow\quad & \widehat{C} = \cos^{-1}\left(\frac{3}{7}\right) && \text{(finding the reference angle)} \\ \Longrightarrow\quad & C \approx 115.4^\circ && \text{(arccosine is either in QI or QII.} \\ & && \text{Since the cosine is negative, } C \in \text{QII.} \\ & && \text{Thus, } C = 180^\circ - \widehat{C}\Big) \end{aligned}$$

Angle C is roughly 115.4°.

Checkpoint 8.2.2

48.07°

Example 8.2.3

Let p be the side length opposite angle P, $a = 1000$, and $b = 950$. Then our table looks like the following:



No complete rows (and no hope of getting a complete row using simple arithmetic) means we are using the Law of Cosines. We want to find length p.

$$\begin{aligned} & p^2 = a^2 + b^2 - 2ab\cos(P) && \text{(Law of Cosines)} \\ \Longrightarrow\quad & p = \sqrt{a^2 + b^2 - 2ab\cos(P)} && \text{(Extraction of Roots)} \end{aligned}$$

Substituting in the given values, we get

$$p = \sqrt{950^2 + 1000^2 - 2(950)(1000)\cos(60^\circ)} = \sqrt{952{,}500} \approx 976.$$

Thus, the pond is roughly 976 feet wide.

Checkpoint 8.2.3

8.26 inches.

Example 8.2.4

As we did when first introduced to heading problems, we sketch the situation, placing small axes at each pivot point.

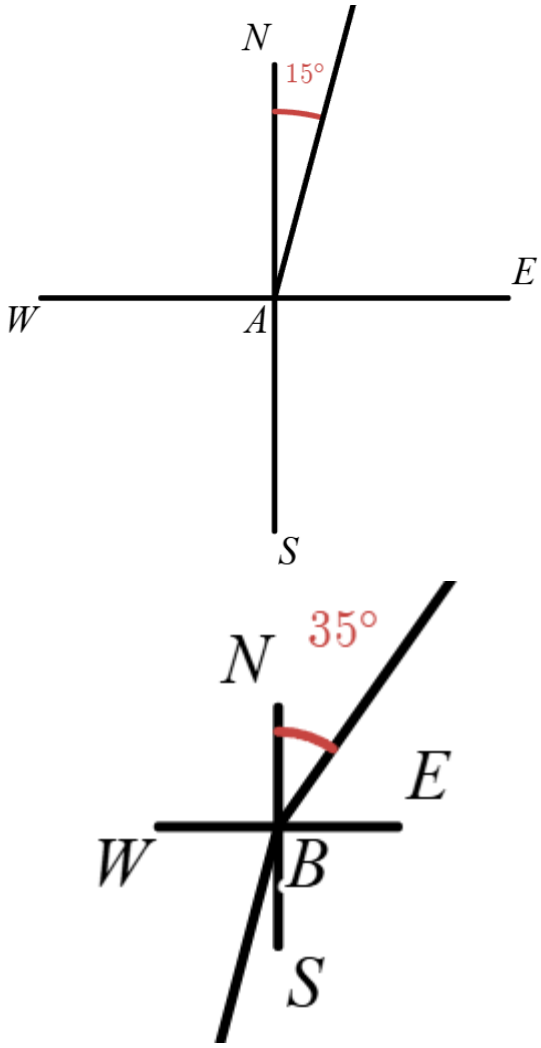


Figure 8.2.5

This results in the following figure.

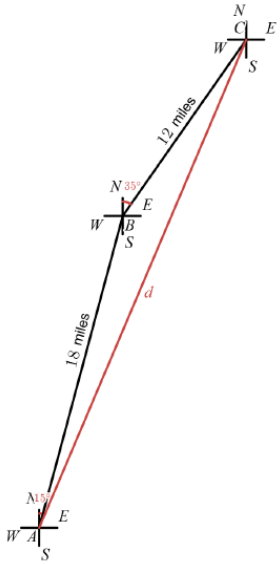


Figure 8.2.6

We want to find the distance d and the angle $\angle BAC$ shown in the figure. We use alternate interior angles to get further information about the angles within $\triangle ABC$.

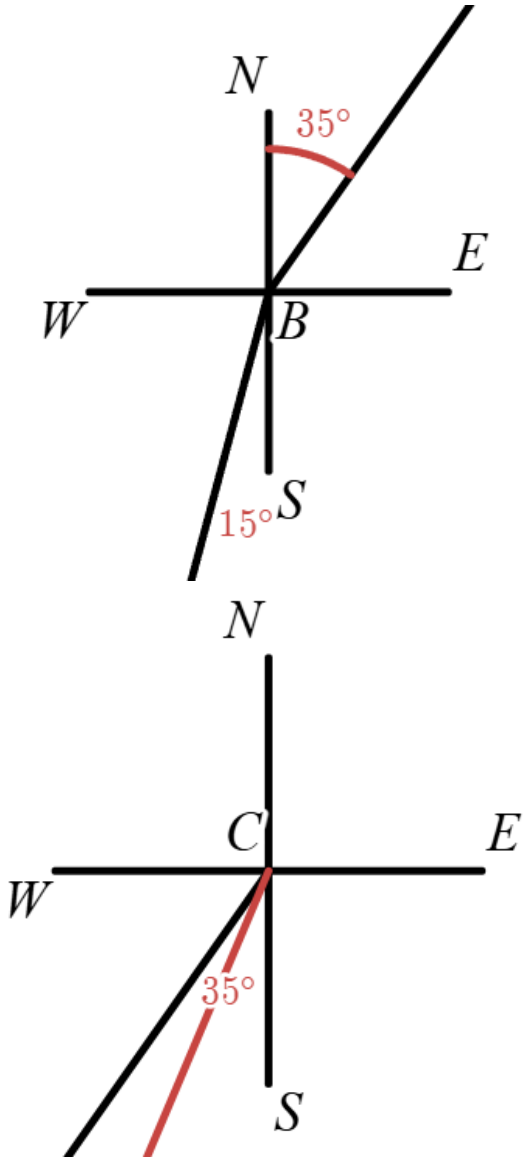


Figure 8.2.7

This results in the final, clean sketch.

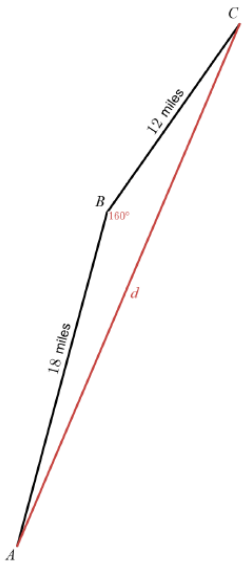


Figure 8.2.8

In $\triangle ABC$, we calculated the angle at point B where the sailing club changed course as follows:

$$B = 180^\circ - 35^\circ + 15^\circ = 160^\circ.$$

We now build our table by using the Law of Cosines to find b and $\angle A$.



Since there is not a complete row, and there is no way to get a complete row without Trigonometry, we use the Law of Cosines to find the length of side b (accidentally labeled as side d in the figure above).

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos(B) && \text{(Law of Cosines)} \\ \implies b &= \sqrt{a^2 + c^2 - 2ac\cos(B)} && \text{(Extraction of Roots)} \end{aligned}$$

Substituting in the given values, we get $b \approx 29.56$ miles. Taking the time to add this to our table, we get the following:



Now that we have a complete row, we can use the Law of Sines to find one of the remaining angles. The advice given previously was to find the angle opposite the smallest remaining side. This would be A (which is, in fact, the angle we need to find).

$$\begin{aligned} \frac{\sin(A)}{a} &= \frac{\sin(B)}{b} && \text{(Law of Sines)} \\ \implies \sin(A) &= \frac{a\sin(B)}{b} && \text{(multiplying both sides by } a\text{)} \\ \implies \widehat{A} &= \sin^{-1}\left(\frac{a\sin(B)}{b}\right) && \text{(finding the reference angle)} \end{aligned}$$

Substituting in the nonrounded values for a, b, and B, we get $\widehat{A} \approx 8.0^\circ$. Since A is opposite the smallest remaining side, A must be acute. Therefore, $A \approx 8.0^\circ$.

We finish this problem with a bit of extra logic. We know the speed boat must travel $b \approx 29.56$ miles; however, the heading of the speed boat is not 8.0°. This is because they must rotate through 15° and then an additional 8.0° to face the sailing club's boat. Thus, their heading is 23.0°.

Checkpoint 8.2.4

297.2°, 344.8 miles

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8.2.1: Resources and Key Concepts

Resources

Videos

- Law of Cosines
  - Non-Right Triangles - Law of Cosines: Theorem and Proof
  - Non-Right Triangles - Law of Cosines: Using the Law of Cosines to Solve Triangles
  - Non-Right Triangles - Law of Cosines: Applications of Non-Right Triangle Trigonometry (Part 2)
  - The Law of Cosines - Review of the Law of Cosines
  - The Law of Cosines - Applications of Non-Right Triangle Trigonometry (Part 2)
  - The Law of Cosines - Applications of Non-Right Triangle Trigonometry (Part 3): Navigation Problems

Key Concepts

Theorems

- **Law of Cosines:** If the angles of a triangle are A, B, and C, and the opposite sides are respectively a, b, and c, then:
  - $a^2 = b^2 + c^2 - 2bc\cos(A)$
  - $b^2 = a^2 + c^2 - 2ac\cos(B)$
  - $c^2 = a^2 + b^2 - 2ab\cos(C)$

Common Mistakes

- **Order of Operations:** When simplifying the Law of Cosines, follow the order of operations carefully. In an expression like $b^2 + c^2 - 2bc\cos(A)$, the term $2bc\cos(A)$ must be calculated before subtracting from the sum of squares. Do not subtract $2bc$ from $b^2 + c^2$ first.
- **Incorrectly Applying Law of Sines After Law of Cosines:** After using the Law of Cosines to find an angle, you can use the Law of Sines to find a second angle. To avoid the ambiguous case, you should always choose to find the angle opposite the smallest remaining side, as this angle is guaranteed to be acute.

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8.2.2: Homework

Reading Questions

1. The Law of Cosines is a generalization of what famous theorem?
2. Under what two conditions (e.g., SSS, ASA, etc.) is the Law of Cosines useful for solving a triangle when the Law of Sines is not?
3. State one of the three versions of the Law of Cosines.
4. When using the Law of Cosines to find a missing side, what information must you have?
5. When using the Law of Cosines to find a missing angle, what information must you have?
6. What is a common mistake regarding the order of operations when simplifying the right side of the Law of Cosines formula?
7. After using the Law of Cosines to find one angle in a triangle, what is the recommended method for finding a second angle?
8. When using the Law of Sines after using the Law of Cosines, which angle should you choose to find next to avoid the ambiguous case? Why?

Homework

Concept Check

1. The Law of Cosines is a generalization of what theorem?

2. If you know all three sides of a triangle and one angle, what might be the advantage of using the Law of Cosines to find another angle, instead of the Law of Sines?

3. Piyali is solving a triangle in which $a = 20$, $b = 16$, and $A = 26^\circ$. She finds that $\sin(B) \approx 0.3507$. How does she know that $B \approx 20.5^\circ$, and not 159.5°?

Basic Skills

4. a. Simplify $5^2 + 7^2 - 2(5)(7)\cos(\theta)$.

   b. Evaluate the expression in part (a) for $\theta = 29^\circ$.

   c. Evaluate the expression in part (a) for $\theta = 151^\circ$.

5. a. Simplify $26.1^2 + 32.5^2 - 2(26.1)(32.5)\cos(\phi)$.

   b. Evaluate the expression in part (a) for $\phi = 64^\circ$.

   c. Evaluate the expression in part (a) for $\phi = 116^\circ$.

6. a. Solve $b^2 = a^2 + c^2 - 2ac\cos(\beta)$ for $\cos(\beta)$.

   b. For the equation in part (a), find $\cos(\beta)$ if $a = 5$, $b = 11$, and $c = 8$.

7. a. Solve $a^2 = b^2 + c^2 - 2bc\cos(\alpha)$ for $\cos(\alpha)$.

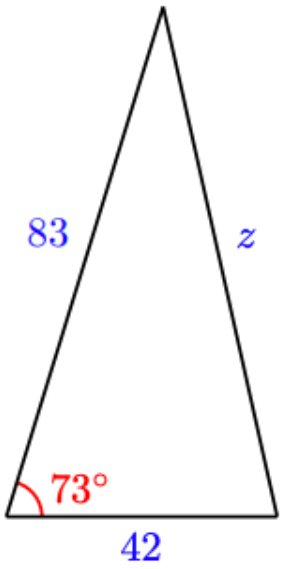
   b. For the equation in part (a), find $\cos(\alpha)$ if $a = 4.6$, $b = 7.2$, and $c = 9.4$.

For the following exercises, find the indicated side. Round to two decimal places.

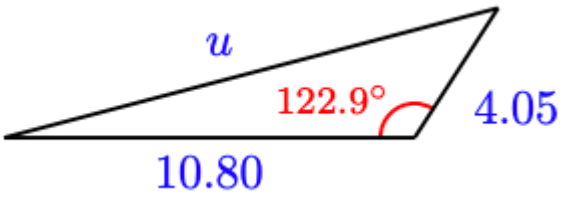
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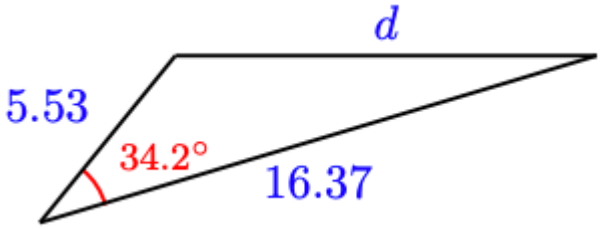
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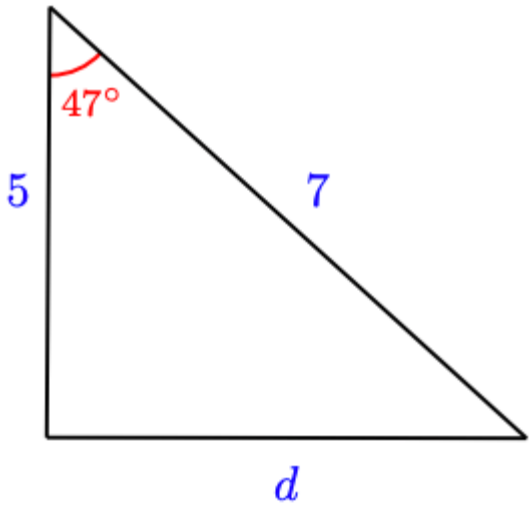
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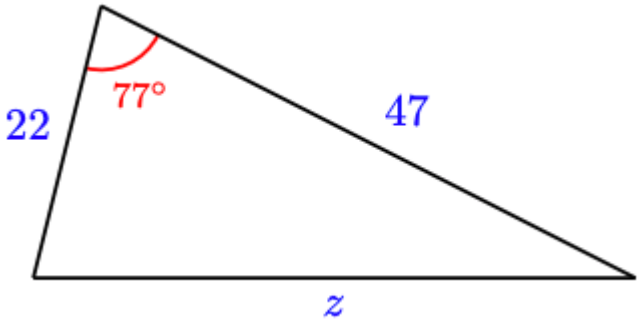
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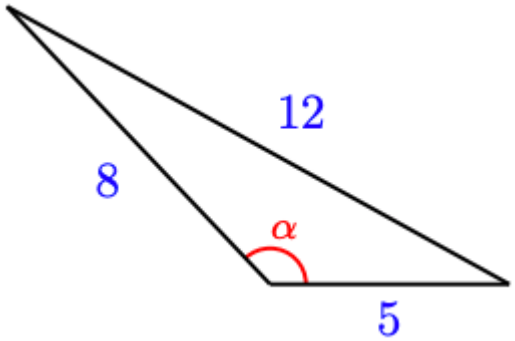


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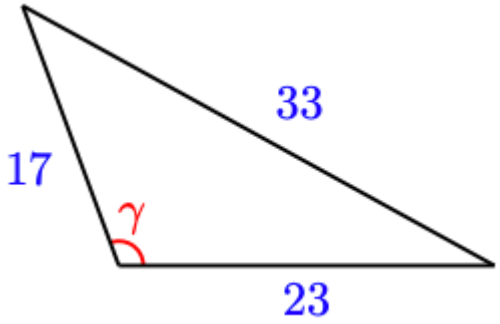


For the following exercises, find the indicated angle. Round to two decimal places.

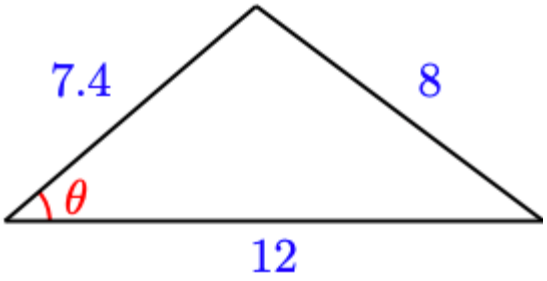
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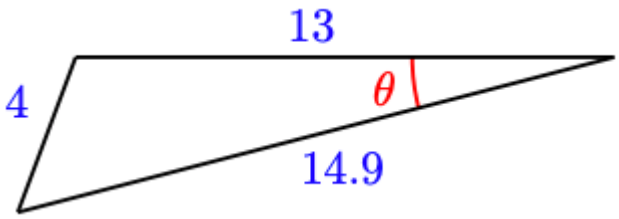
15.



16.



17.



For the following exercises, find the angles of the triangle. Round answers to two decimal places.

18. $a = 23, b = 14, c = 18$

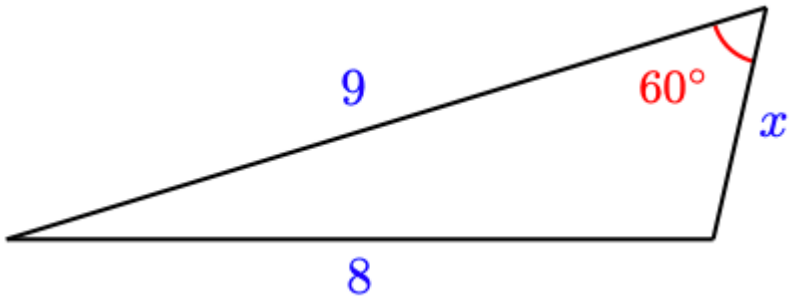
19. $a = 18, b = 25, c = 19$

20. $a = 16.3, b = 28.1, c = 19.4$

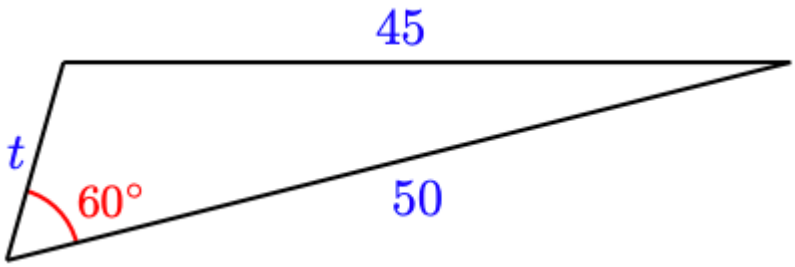
21. $a = 82.3, b = 22.5, c = 66.8$

For the following exercises, find the unknown side. Round your answers to two decimal places.

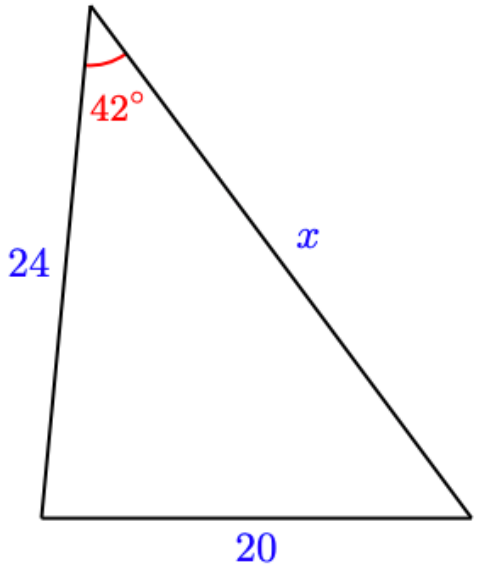
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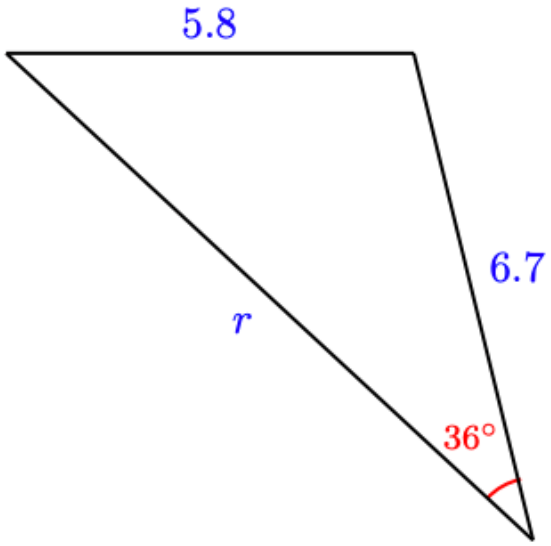
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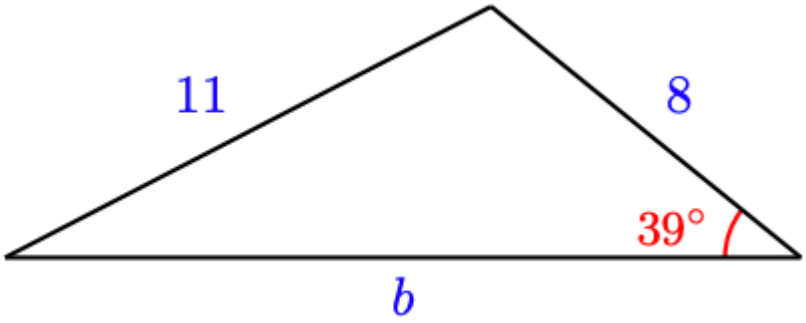
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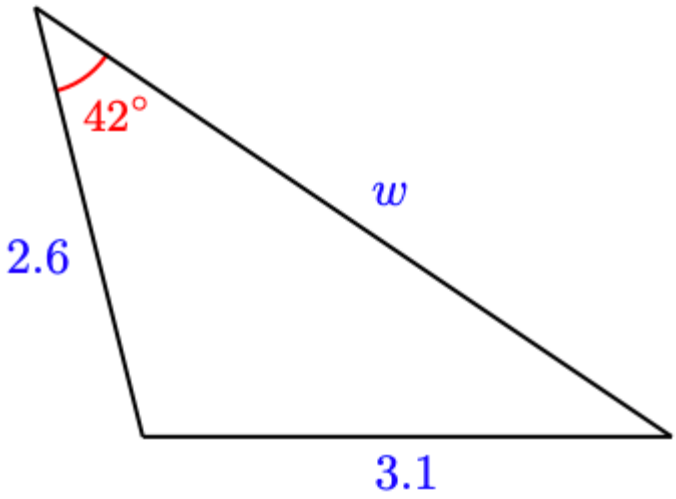
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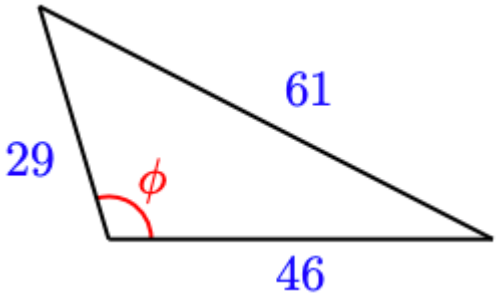


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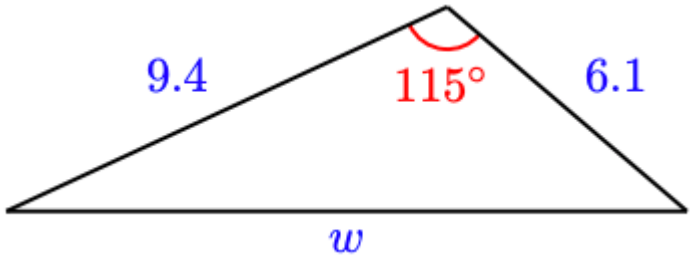


For the following exercises, which law should you use to find the labeled unknown value, the Law of Sines or the Law of Cosines? Write an equation you can solve to find the unknown value. For Problems 32 - 35, you may need two steps to find the unknown value.

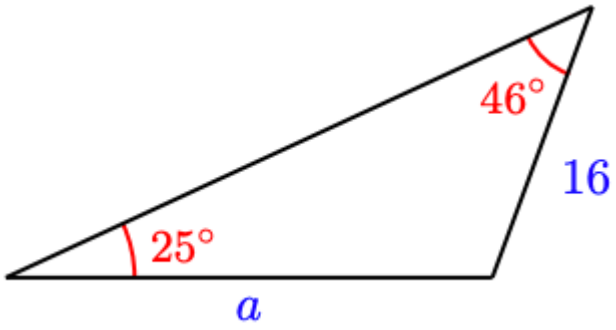
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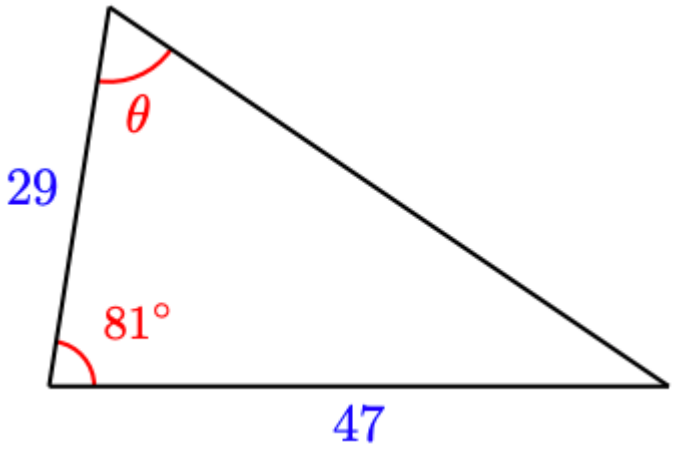
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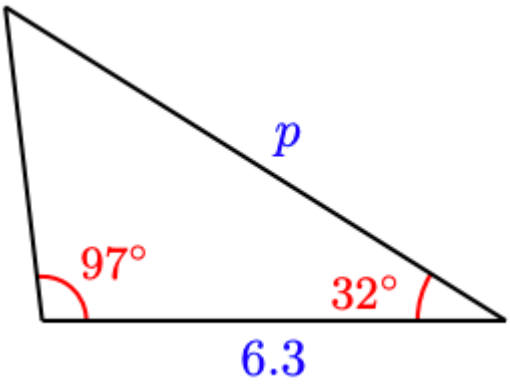
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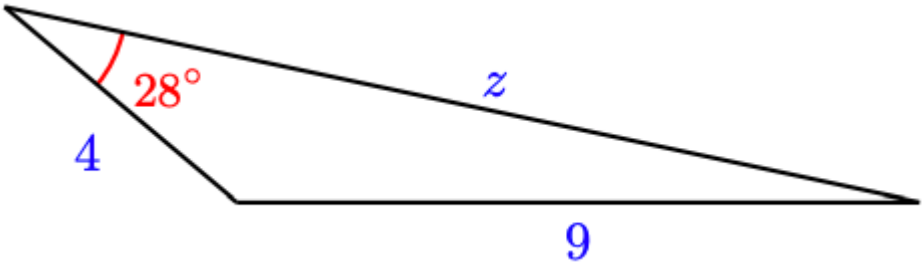
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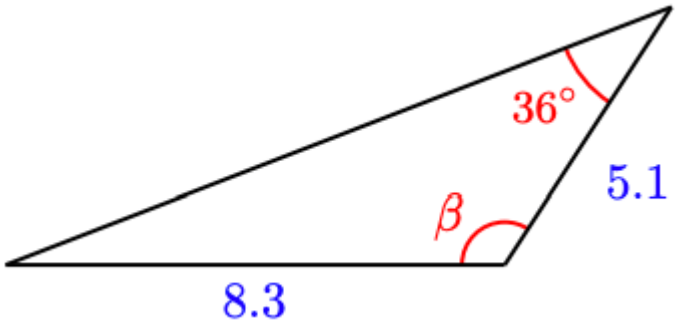
33.



34.



35.



For the following exercises,

(a) Sketch and label the triangle.
(b) Solve the triangle. Round answers to two decimal places.

36. $B = 47^\circ$, $a = 23$, $c = 17$

37. $C = 32^\circ$, $a = 14$, $b = 18$

38. $a = 8$, $b = 7$, $c = 9$

39. $a = 23$, $b = 34$, $c = 45$

40. $b = 72$, $c = 98$, $B = 38^\circ$

41. $a = 28$, $c = 41$, $A = 27^\circ$

42. $c = 5.7$, $A = 59^\circ$, $B = 82^\circ$

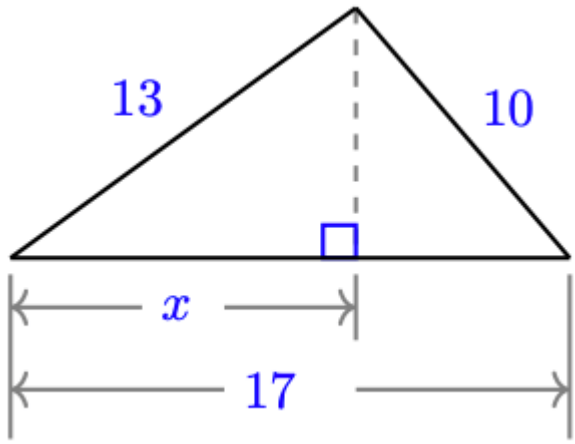
43. $b = 82$, $A = 11^\circ$, $C = 42^\circ$

Synthesis Questions

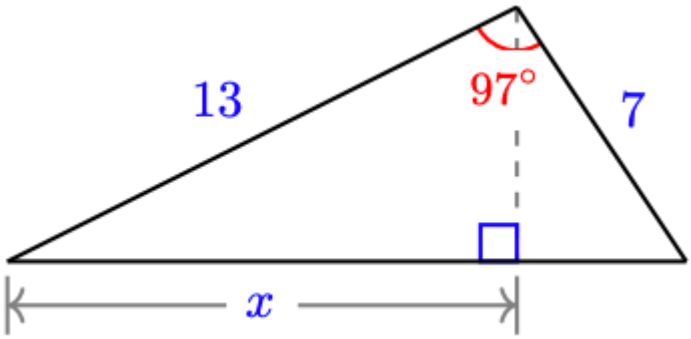
44. The sides of a parallelogram are 10 inches and 8 inches, and form an angle of 130°. Find the lengths of the diagonals of the parallelogram.

For the following exercises, find x, the distance from one vertex to the foot of the altitude.

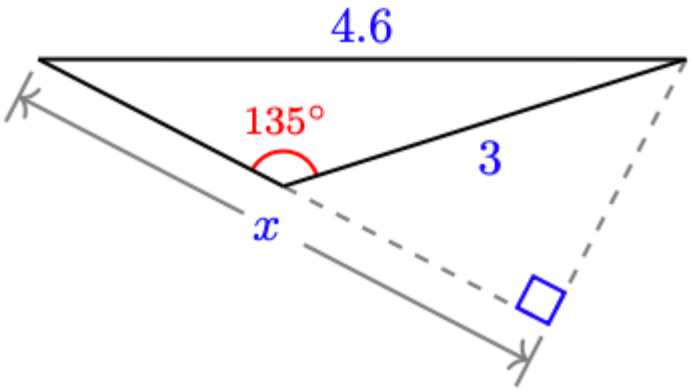
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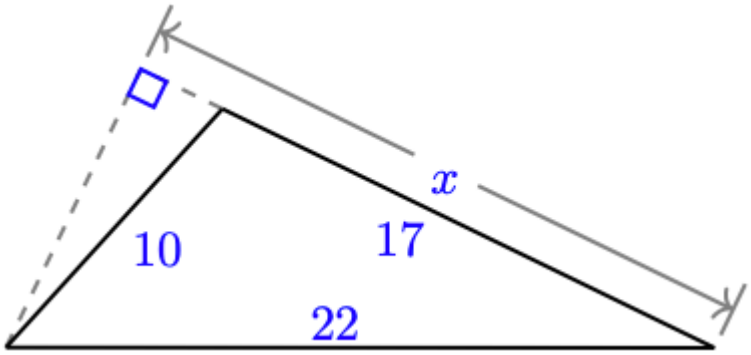
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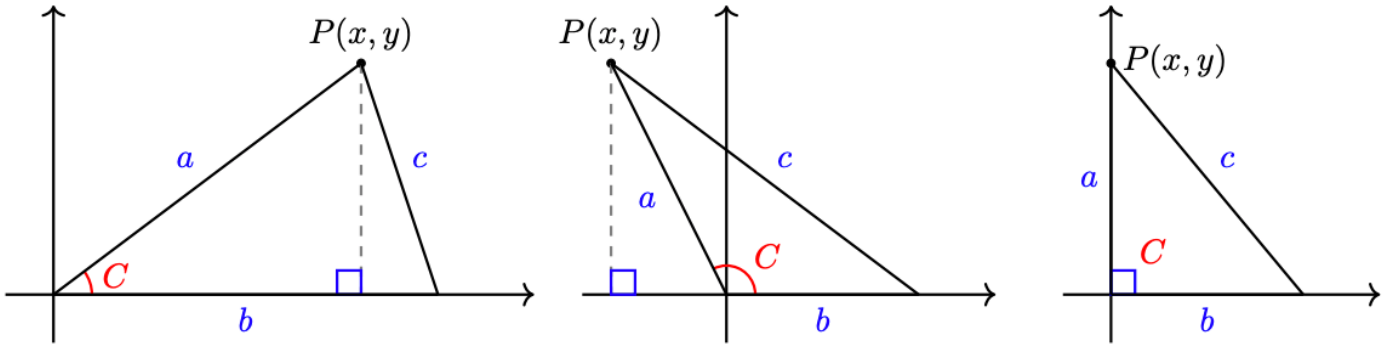
47.



48.



Proofs



49. a. Copy the three figures above showing the three possibilities for an angle C in a triangle: C is acute, obtuse, or a right angle. For each figure, explain why it is true that $c^2 = (b - x)^2 + y^2$, then rewrite the right side to get $c^2 = (x^2 + y^2) + b^2 - 2bx$.

b. For each figure, explain why it is true that $x^2 + y^2 = a^2$.

c. For all three figures, a is the distance from the origin to the point (x, y). Use the definition of cosine to write $\cos(C)$ in terms of a and x, then solve your equation for x.

d. Start with the last equation from (a), and substitute expressions from (b) and (c) to conclude one case of the Law of Cosines.

50. Demonstrate the other two cases of the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

and

$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

(Hint: See the previous problem and switch the roles of a and c, etc.)

51. Use the Law of Cosines to prove the projection laws:

$$\begin{aligned} a &= b\cos(C) + c\cos(B) \\ b &= c\cos(A) + a\cos(C) \\ c &= a\cos(B) + b\cos(A) \end{aligned}$$

Illustrate with a sketch. (Hint: Add together two of the versions of the Law of Cosines.)

52. If $\triangle ABC$ is isosceles with $a = b$, show that $c^2 = 2a^2(1 - \cos(C))$.

53. Use the Law of Cosines to prove:

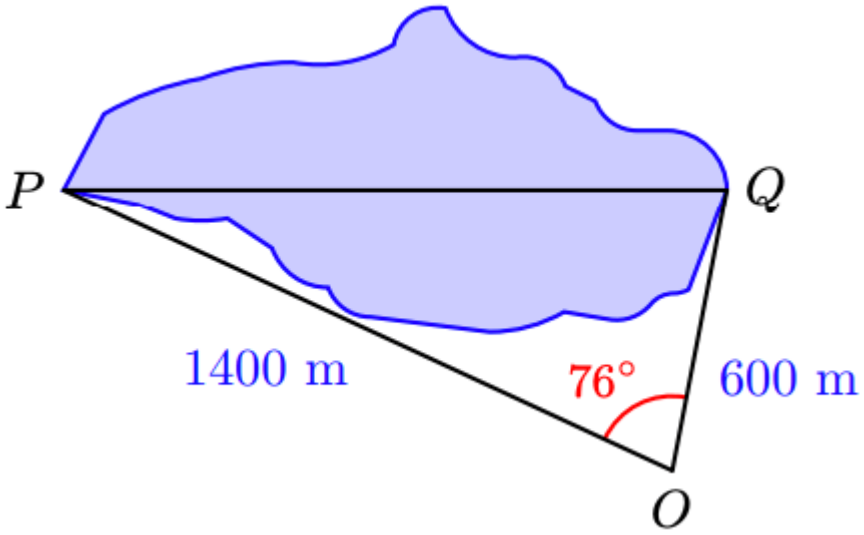
$$\begin{aligned} 1 + \cos(A) &= \frac{(a+b+c)(-a+b+c)}{2bc} \\ 1 - \cos(A) &= \frac{(a-b+c)(-a+b+c)}{2bc} \end{aligned}$$

54. Prove that

$$\frac{\cos(A)}{a} + \frac{\cos(B)}{b} + \frac{\cos(C)}{c} = \frac{a^2+b^2+c^2}{2abc}$$

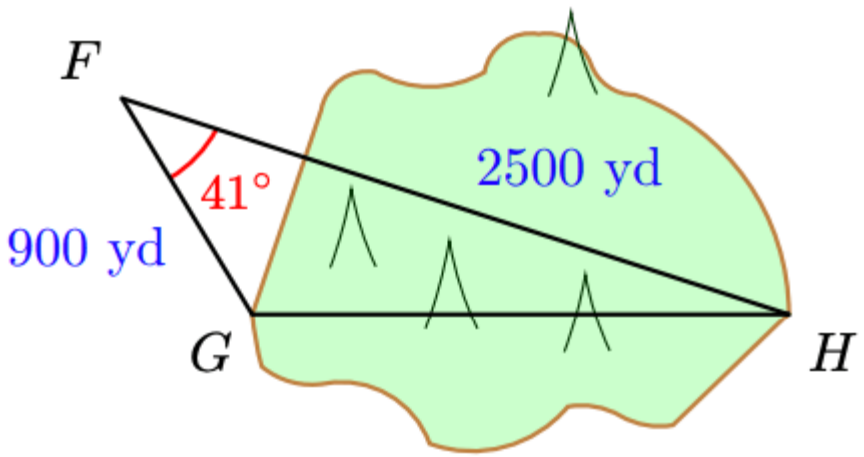
Applications

55. A surveyor would like to know the distance, PQ, across a small lake, as shown in the figure. She stands at point O and measures the angle between the lines of sight to points P and Q at 76°. She also finds $OP = 1400$ meters and $OQ = 600$ meters. Calculate the distance PQ.



56. A geologist wants to measure the diameter of a crater. From her camp, it is 4 miles to the northern-most point of the crater and 2 miles to the southern-most point. If the angle between the two lines of sight is 117°, what is the diameter of the crater? Round your answer to the nearest hundredth of a mile.

57. Highway engineers plan to drill a tunnel through Boney Mountain from G to H, as shown in the figure. The angle at point F is 41°, and the distances to G and H are 900 yards and 2500 yards, respectively. How long will the tunnel be?



58. From the Pedimaxus International Airport a tour helicopter can fly to Cliffs of Insanity Point by following a bearing of $\mathrm{N}\ 8.2^\circ\ \mathrm{E}$ for 192 miles and it can fly to Bigfoot Falls by following a bearing of $\mathrm{S}\ 68.5^\circ\ \mathrm{E}$ for 207 miles. Find the distance between Cliffs of Insanity Point and Bigfoot Falls. Round your answer to the nearest mile.

59. Cliffs of Insanity Point and Bigfoot Falls from Problem 58 above both lie on a straight stretch of the Great Sasquatch Canyon. What bearing would the tour helicopter need to follow to go directly from Bigfoot Falls to Cliffs of Insanity Point? Round your angle to the nearest tenth of a degree.

60. Two pilots leave an airport at the same time. One pilot flies at a heading of 3° and at a speed of 320 miles per hour, the other flies in the heading 157° at a speed of 406 miles per hour. How far apart are the two pilots after 3 hours? What is the heading from the first plane to the second plane at that time?

61. Two boats leave port at the same time. One boat sails due west at a speed of 17 miles per hour, the other moves in the heading 42° at a speed of 23 miles per hour. How far apart are the two boats after 2 hours? What is the heading from the first boat to the second boat at that time?

62. Linda wants to fly directly south from Indianapolis to Cancun, Mexico, a distance of 1290 miles. However, to avoid bad weather, she flies for 400 miles on a heading of 162°. What is the heading to Cancun from that location, and how far is it?

63. Lap sails 8 miles from Key West, Florida on a heading of 140°. He then changes course and sails for 10 miles due east. What is the heading back to Key West from that point, and how far is it?

64. A naturalist sets off on a hike from a lodge on a bearing of $\mathrm{S}\ 80^\circ\ \mathrm{W}$. After 1.5 miles, she changes her bearing to $\mathrm{S}\ 17^\circ\ \mathrm{W}$ and continues hiking for 3 miles. Find her distance from the lodge at this point. Round your answer to the nearest hundredth of a mile. What bearing should she follow to return to the lodge? Round your angle to the nearest degree.

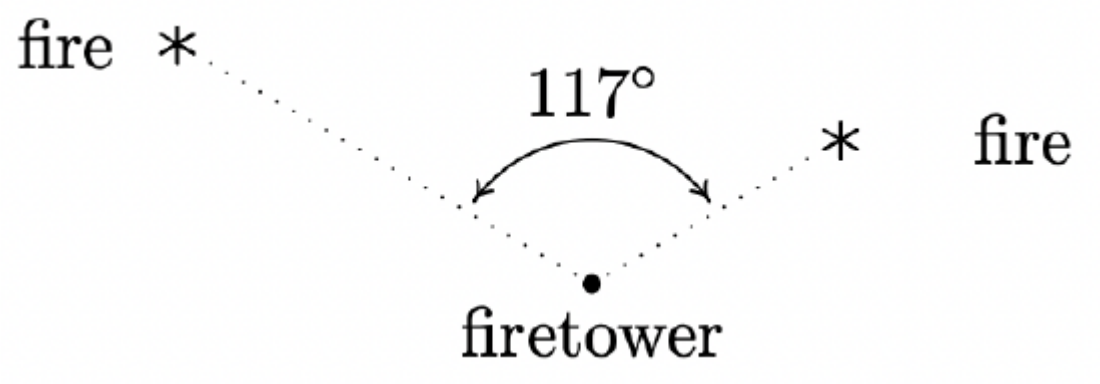
65. The HMS Sasquatch leaves port on a bearing of $\mathrm{N}\ 23^\circ\ \mathrm{E}$ and travels for 5 miles. It then changes course and follows a bearing of $\mathrm{S}\ 41^\circ\ \mathrm{E}$ for 2 miles. How far is it from port? Round your answer to the nearest hundredth of a mile. What is its bearing to port? Round your angle to the nearest degree.

66. The SS Bigfoot leaves a harbor bound for Nessie Island which is 300 miles away at a bearing of $\mathrm{N}\ 32^\circ\ \mathrm{E}$. A storm moves in and after 100 miles, the captain of the Bigfoot finds he has drifted off course. If his bearing to the harbor is now $\mathrm{S}\ 70^\circ\ \mathrm{W}$, how far is the SS Bigfoot from Nessie Island? Round your answer to the nearest hundredth of a mile. What course should the captain set to head to the island? Round your angle to the nearest tenth of a degree.

67. The phone company wants to erect a cell tower on a steep hill inclined 26° to the horizontal. The installation crew plans to run a guy wire from a point on the ground 20 feet uphill from the base of the tower and attach it to the tower at a height of 100 feet. How long should the guy wire be?

68. Sandstone Peak rises 3500 above the desert. The Park Service plans to run an aerial tramway up the north face, which is inclined at an angle of 68° to the horizontal. The base station will be located 500 feet from the foot of Sandstone Peak. Ignoring any slack in the cable, how long should it be?

59. From a point 300 feet above level ground in a firetower, a ranger spots two fires in the Yeti National Forest. The angle of depression made by the line of sight from the ranger to the first fire is 2.5° and the angle of depression made by line of sight from the ranger to the second fire is 1.3°. The angle formed by the two lines of sight is 117°. Find the distance between the two fires. Round your answer to the nearest foot. (Hint: In order to use the 117° angle between the lines of sight, you will first need to use right angle Trigonometry to find the lengths of the lines of sight. This will allow you to apply the Law of Cosines.)



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8.3: Areas

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The Area of a Triangle

Figure 8.3.1 shows part of the map for a new housing development, Pacific Shores. You are interested in the corner lot, number 86, and you would like to know the area of the lot in square feet. The sales representative for Pacific Shores provides you with the dimensions of the lot. Still, you need to know a formula for the area of an irregularly shaped quadrilateral.

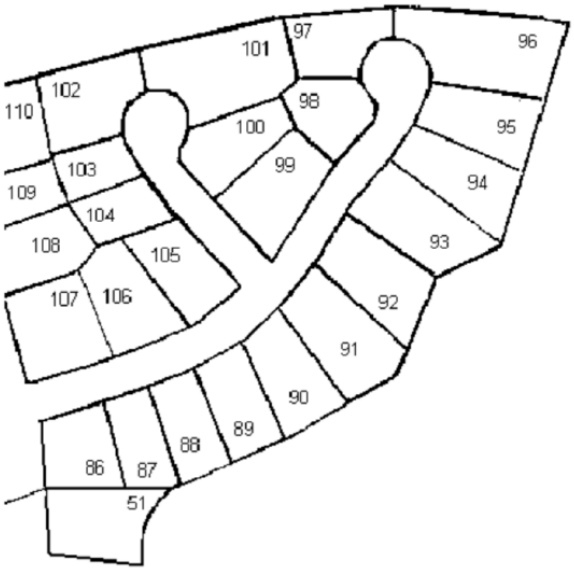


Figure 8.3.1**:** *Map of Pacific Shores development*

It occurs to you that you can divide the quadrilateral into two triangles and find the area of each.

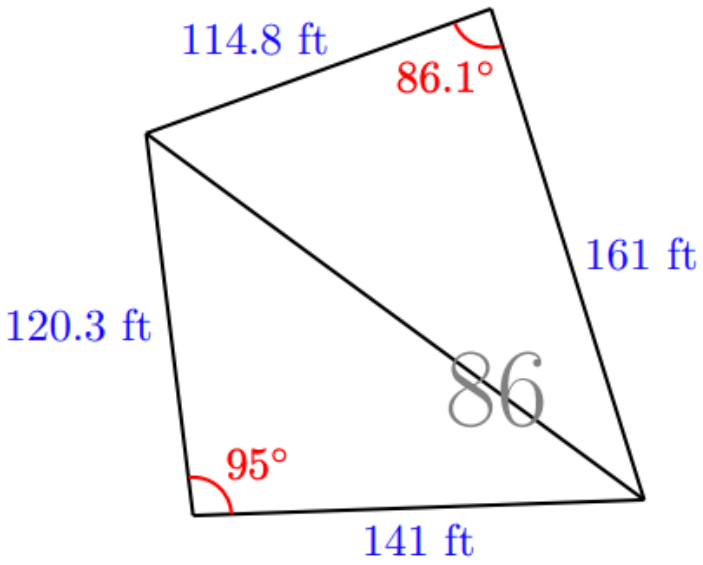


Figure 8.3.2**:** *Map of Lot 86*

You know a formula for the area of a triangle in terms of its base and height, namely, $A = \frac{1}{2}bh$, but unfortunately, you don't know the height of either triangle. You can, however, quickly measure the angles at the corners of the lot using the plot map and a protractor (see Figure 8.3.2). With this information, we can find the area of each triangle via the following theorem.

Theorem: Area of a Triangle

If a triangle has sides of length a and b, and the angle between those two sides is θ, then the area of the triangle is given by

$$A = \frac{1}{2}ab\sin(\theta).$$

Proof available at the end of this section.

Example 8.3.1

Find the area of lot 86.

Solution available at the end of this section.

Caution: θ is between the given sides

The formula, $A = \frac{1}{2}ab\sin(\theta)$, does not mean that we always use the sides labeled a and b to find the area of a triangle. The variables a and b represent the lengths of the sides that *include* the known angle. For example, the area of the triangle below is given by $A = \frac{1}{2}(5c)\sin(\phi)$.

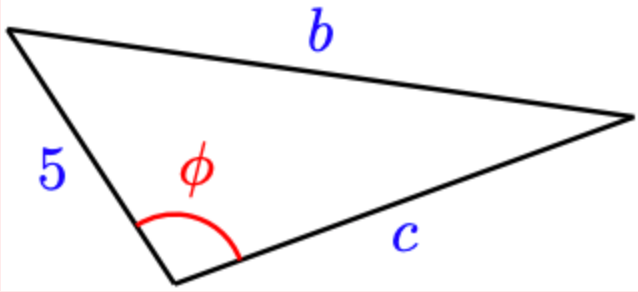


Figure 8.3.3

Checkpoint 8.3.1

A triangle has sides of length 6 and 7, and the angle between those sides is $150°$. Find the area of the triangle.

Answer available at the end of this section.

Area of a Triangle - Heron's Formula

To use the previous theorem when finding the area of a triangle, we must be given two sides and an included angle; however, is it possible to find the area of a triangle given all three sides but no angles? The answer to this question is yes, and the following theorem, Heron's Formula, explains how to find such an area.

Theorem: Heron's Formula

Suppose a, b, and c denote the lengths of the three sides of a triangle. Let s be the **semiperimeter** of the triangle. That is, let

$$s = \frac{1}{2}(a+b+c).$$

Then the area, A, enclosed by the triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

Proof available at the end of this section.

Example 8.3.2

Find the area of the given triangle.

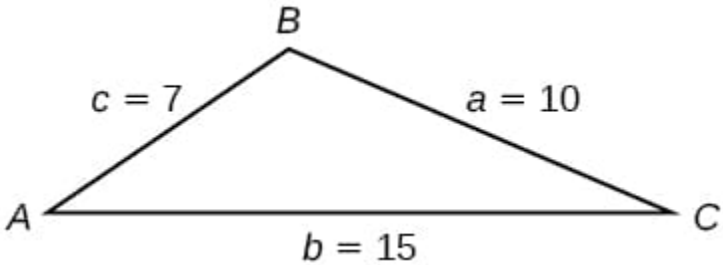


Figure 8.3.4

Solution available at the end of this section.

Checkpoint 8.3.2

Find the area of a triangle with sides of lengths $a = 29.7$ ft, $b = 42.3$ ft, and $c = 38.4$ ft.

Answer available at the end of this section.

Success in College: Learning From Tests (Part 5)

1. A math class includes one or two quizzes for each major test. A student is averaging about 85% on these quizzes. When he gets them back, he looks them over and drops them in the wastebasket as he leaves class. Explain why he should keep the quizzes.
2. A math professor tells her class to go over their tests so that they can learn from their mistakes. She does not require that students do this. Most do not. Do you think the professor should require the students to make corrections on the test and turn them back in? (Assume that the corrections will not change their original test grade.) Explain.

Answers

Section Preview

I deliberately chose to not cover the included angle formula for the area of a triangle until this section. It was not needed previously, and Heron's Formula, by itself, is too short of a topic for its own section. Thus, I waited until we had enough mathematical background to do both!

Section Preview

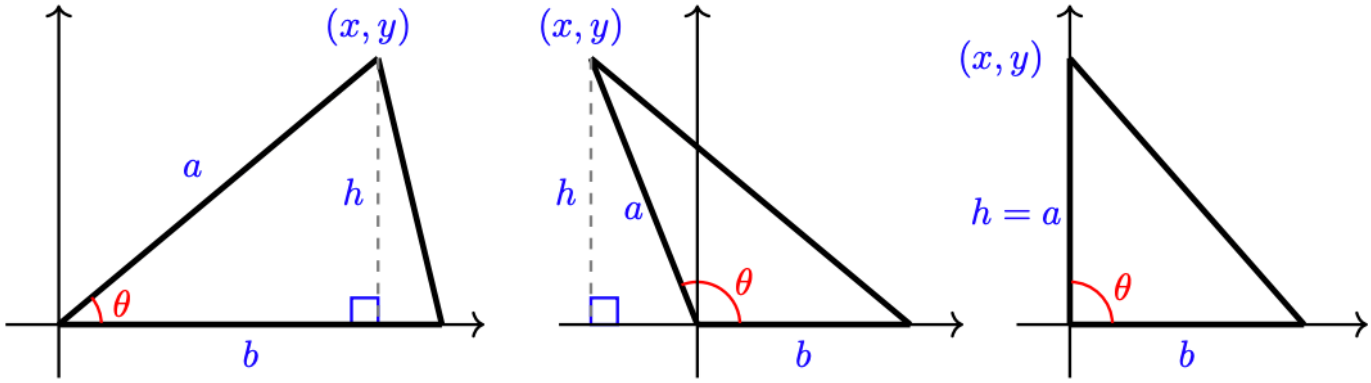
- **Geometry**
  - Area and Perimeter of Triangles and Quadrilaterals: Students must already know $A = \frac{1}{2}bh$ before the basic area proof in this section proceeds. The opening application also divides a quadrilateral into two triangles to find its total area — requiring the concept that area is additive and that a quadrilateral can be decomposed into triangles. This is geometric reasoning from prior coursework.

Section Preview

- Find the area of a triangle given two sides and the included angle.
- Find the area of a triangle using Heron's Formula.

Theorem: Area of a Triangle

The figure below shows three possibilities for triangles, depending on whether the angle θ is acute, obtuse, or 90°.



In each case, b is the base of the triangle, and the altitude is h. In all three triangles

$$\sin(\theta) = \frac{h}{a}.$$

(In the third triangle, if $h = a$, then $\frac{h}{a} = 1$. Since the sine of 90° is 1, $\sin(\theta) = 1 = \frac{h}{a}$)

Solving for h yields $h = a\sin(\theta)$. Substituting this expression for h into our triangle area formula from Geometry, we get

$$A = \frac{1}{2}bh = \frac{1}{2}ba\sin(\theta) = \frac{1}{2}ab\sin(\theta).$$

Example 8.3.1

For the triangle in the lower portion of lot 86, $a = 120.3$, $b = 141$, and $\theta = 95^\circ$. The area of that portion is

$$\begin{aligned} \text{First Area} &= \frac{1}{2}ab\sin(\theta) \\ &= \frac{1}{2}(120.3)(141)\sin(95^\circ) \\ &\approx 8448.88 \end{aligned}$$

For the triangle in the upper portion of the lot, $a = 161$, $b = 114.8$, and $\theta = 86.1°$. The area of that portion is

$$\begin{aligned} \text{Second Area} &= \frac{1}{2}ab\sin(\theta) \\ &= \frac{1}{2}(161)(114.8)\sin(86.1°) \\ &\approx 9220.00 \end{aligned}$$

The total area of the lot is the sum of the areas of the triangles. Therefore,

$$\text{Total area} = \text{First Area} + \text{Second Area} \approx 17668.88.$$

Lot 86 has an area of approximately 17,669 square feet.

Checkpoint 8.3.1

$\frac{21}{2}$

Theorem: Heron's Formula

Using the convention that the angle C is opposite the side c, and borrowing the area theorem from earlier in this section, we have

$$A = \frac{1}{2}ab\sin(C).$$

To simplify computations, we start by manipulating the expression for A^2.

$$
\begin{aligned}
& A^2 = \left(\tfrac{1}{2}ab\sin(C)\right)^2 && \text{(squaring both sides of the area formula)}\\
\Longrightarrow\ & A^2 = \frac{1}{4}a^2b^2\sin^2(C) && \text{(Laws of Exponents)}\\
\Longrightarrow\ & A^2 = \frac{a^2b^2}{4}\left(1-\cos^2(C)\right) && \text{(Pythagorean Identity)}\\
\Longrightarrow\ & A^2 = \frac{a^2b^2}{4}\left[1-(\cos(C))^2\right] &&\\
\Longrightarrow\ & A^2 = \frac{a^2b^2}{4}\left[1-\left(\frac{c^2-a^2-b^2}{-2ab}\right)^2\right] && \text{(Law of Cosines)}\\
\Longrightarrow\ & A^2 = \frac{a^2b^2}{4}\left[1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right] && \text{(multiplying numerator and denominator by } -1)\\
\Longrightarrow\ & A^2 = \frac{a^2b^2}{4}\left[1-\frac{\left(a^2+b^2-c^2\right)^2}{4a^2b^2}\right] && \text{(Laws of Exponents)}\\
\Longrightarrow\ & A^2 = \frac{a^2b^2}{4}\left[\frac{4a^2b^2}{4a^2b^2}-\frac{\left(a^2+b^2-c^2\right)^2}{4a^2b^2}\right] && \text{(making equivalent fractions)}\\
\Longrightarrow\ & A^2 = \frac{a^2b^2}{4}\left[\frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{4a^2b^2}\right] && \text{(combining like fractions)}\\
\Longrightarrow\ & A^2 = \frac{\cancel{a^2}^{1}\cancel{b^2}^{1}}{4}\left[\frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{4\cancel{a^2}^{1}\cancel{b^2}^{1}}\right] && \text{(cancel like factors)}\\
\Longrightarrow\ & A^2 = \frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{16} && \text{(arithmetic)}\\
\Longrightarrow\ & A^2 = \frac{\left(2ab-\left(a^2+b^2-c^2\right)\right)\left(2ab+\left(a^2+b^2-c^2\right)\right)}{16} && \text{(factoring via Difference of Squares)}\\
\Longrightarrow\ & A^2 = \frac{\left(2ab-a^2-b^2+c^2\right)\left(2ab+a^2+b^2-c^2\right)}{16} && \text{(distribution)}\\
\Longrightarrow\ & A^2 = \frac{\left(c^2-a^2+2ab-b^2\right)\left(a^2+2ab+b^2-c^2\right)}{16} && \text{(Commutative Property of Addition)}\\
\Longrightarrow\ & A^2 = \frac{\left[c^2-\left(a^2-2ab+b^2\right)\right]\left[\left(a^2+2ab+b^2\right)-c^2\right]}{16} && \text{(factoring out a negative and grouping)}\\
\Longrightarrow\ & A^2 = \frac{\left[c^2-(a-b)^2\right]\left[(a+b)^2-c^2\right]}{16} && \text{(factoring trinomials)}\\
\Longrightarrow\ & A^2 = \frac{[c-(a-b)]\,[c+(a-b)]\,[(a+b)-c]\,[(a+b)+c]}{16} && \text{(factoring via Difference of Squares)}\\
\Longrightarrow\ & A^2 = \frac{(c-a+b)\,(c+a-b)\,(a+b-c)\,(a+b+c)}{16} && \text{(distribution)}\\
\Longrightarrow\ & A^2 = \frac{c-a+b}{2}\cdot\frac{c+a-b}{2}\cdot\frac{a+b-c}{2}\cdot\frac{a+b+c}{2} && \text{(arithmetic)}\\
\Longrightarrow\ & A^2 = \frac{c-a+b}{2}\cdot\frac{c+a-b}{2}\cdot\frac{a+b-c}{2}\cdot s && \left(\text{let } s=\frac{a+b+c}{2}\right)\\
\Longrightarrow\ & A^2 = (s-a)\cdot\frac{c+a-b}{2}\cdot\frac{a+b-c}{2}\cdot s && \left(\text{then } s-a=\frac{a+b+c}{2}-a=\frac{b+c-a}{2}\right)\\
\Longrightarrow\ & A^2 = (s-a)\cdot(s-b)\cdot\frac{a+b-c}{2}\cdot s && \left(\text{and } s-b=\frac{a+b+c}{2}-b=\frac{a+c-b}{2}\right)\\
\Longrightarrow\ & A^2 = (s-a)\cdot(s-b)\cdot(s-c)\cdot s && \left(\text{and } s-c=\frac{a+b+c}{2}-c=\frac{a+b-c}{2}\right)\\
\Longrightarrow\ & A = \sqrt{s(s-a)(s-b)(s-c)} && \text{(Extraction of Roots)}
\end{aligned}
$$

Example 8.3.2

Let $s=\frac{1}{2}(a+b+c)=\frac{1}{2}(10+15+7)=\frac{1}{2}(32)=16$. Then, by Heron's Formula,

$$A=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{16(16-10)(16-15)(16-7)}=\sqrt{16(6)(1)(9)}=12\sqrt{6}\approx 29.4\,\text{square units.}$$

Checkpoint 8.3.2

≈ 552.3

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8.3.1: Resources and Key Concepts

Resources

Prerequisite Topics

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Geometry**
  - Common Areas: The section builds upon the basic formula for the area of a triangle, $A = \frac{1}{2}bh$, to derive the trigonometric formula for the area. This basic formula is also needed for the "Skills Refresher" exercises.

Videos

- Area of a Triangle
  - Trigonometric Functions of Angles - Areas of Triangles
  - Non-Right Triangles - Area of a Triangle (Part 1)
- Heron's Formula
  - The Law of Cosines - Heron's Formula: Theorem, Proof, and Examples

Key Concepts

Definitions

- **Semiperimeter:** Half the perimeter of a triangle. For a triangle with side lengths a, b, and c, the semiperimeter s is calculated as $s = \frac{1}{2}(a + b + c)$.

Theorems

- **Area of a Triangle:** If a triangle has two sides of length a and b, and the angle between those two sides is θ, then the area of the triangle is given by $A = \frac{1}{2}ab\sin(\theta)$.
- **Heron's Formula:** If a triangle has sides of length a, b, and c, and the semiperimeter is s, then the area of the triangle is given by $A = \sqrt{s(s-a)(s-b)(s-c)}$.

Common Mistakes

- **Using a Non-Included Angle:** The formula $A = \frac{1}{2}ab\sin(\theta)$ is only valid if the angle θ is the angle included between the sides of length a and b. Using an angle that is not between the two given sides will result in an incorrect area.
- **Confusing Perimeter and Semiperimeter:** Heron's formula requires the semiperimeter (s), which is half the perimeter. Using the full perimeter in the formula will produce an incorrect result.

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8.3.2: Homework

Reading Questions

1. What three pieces of information about a triangle are required to use the area formula $A = \frac{1}{2}ab\sin(\theta)$?
2. In the formula $A = \frac{1}{2}ab\sin(\theta)$, what is the specific relationship between the angle θ and the sides a and b?
3. What information about a triangle is required to use Heron's Formula?
4. What does the variable s represent in Heron's Formula, and how is it calculated?
5. How can you find the area of an irregular quadrilateral using the methods from this section?
6. The proof of Heron's Formula provided in the text relies on which two other major triangle formulas?

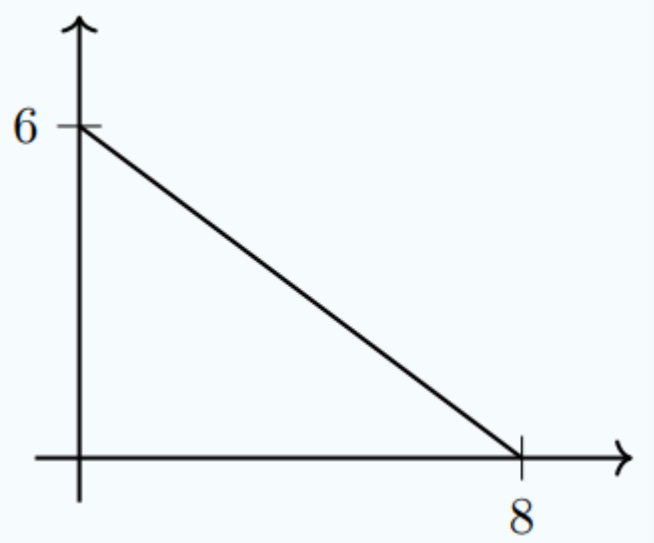
Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

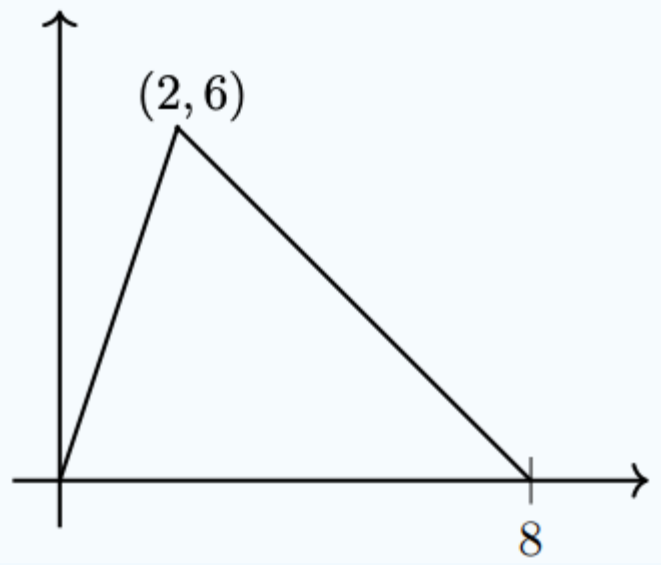
For the following exercises, find the area of the triangle.

1.

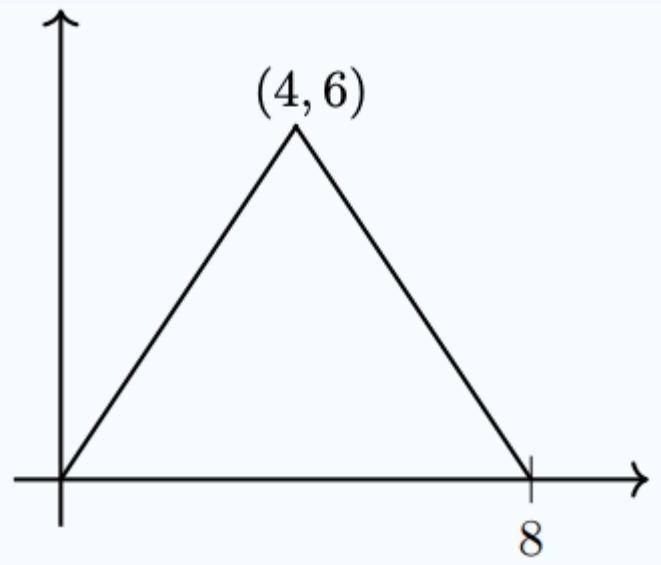


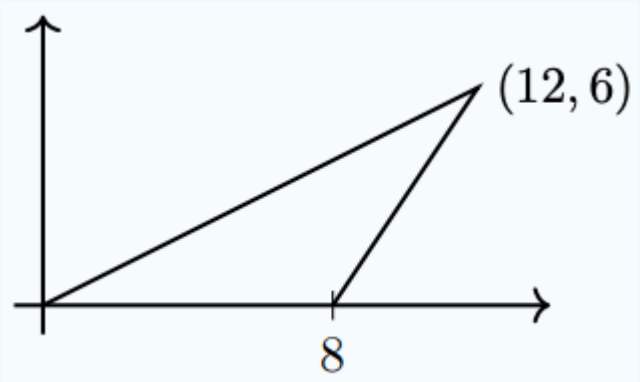
2.

3.



4.





Answers available at the end of this section.

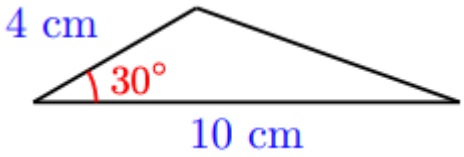
Homework

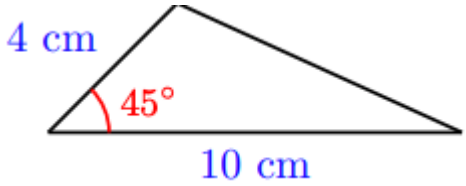
Concept Check

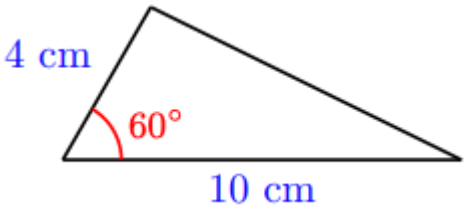
1. Explain what s represents in Heron's formula.

Basic Skills

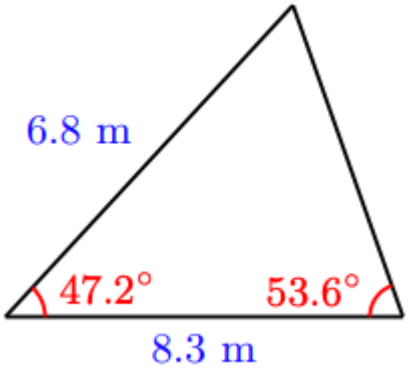
For the following exercises, find an exact value for the area of the triangle.

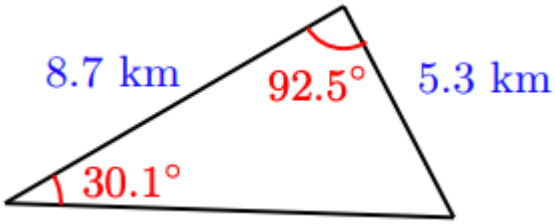
2. 

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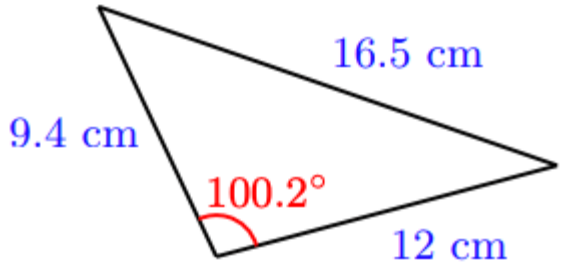
4. 

For the following exercises, find the area of the triangle. Round your answer to the nearest tenth.

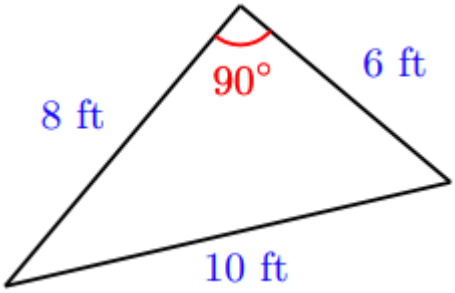
5. 

6. 

7.

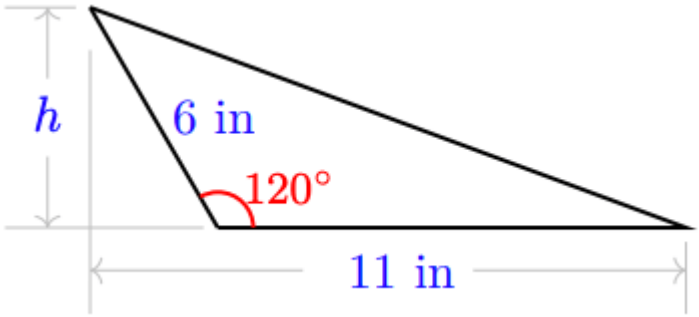


8.

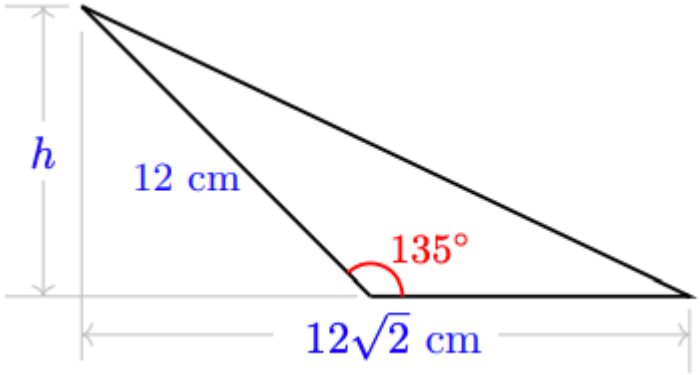


For the following exercises,
(a) Find exact values for the base and height of the triangle.
(b) Compute an exact value for the area of the triangle.

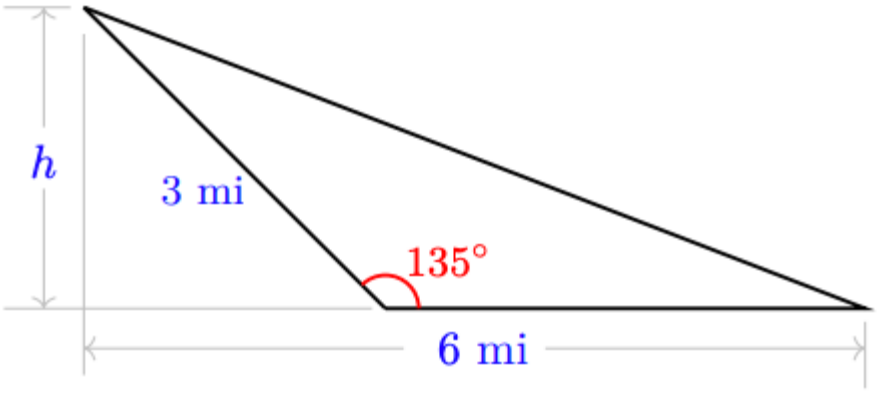
9.



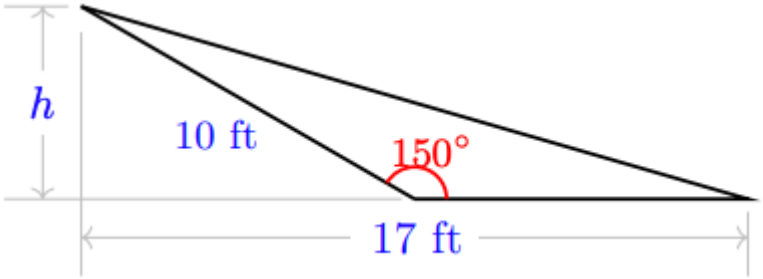
10.



11.



12.



For the following exercises, find the area of the triangle with the given properties. Round to the nearest tenth.

13. (MyOpenMath) one side has length 37 feet, another side has length 11 feet, and the included angle measures 51°.

14. $b = 2.5$ in, $c = 7.6$ in, $A = 138^\circ$

15. $a = 0.8 \text{ m}, c = 0.15 \text{ m}, B = 15^\circ$

16. the sides of the triangle are 27 cm, 15 cm, and 20 cm.

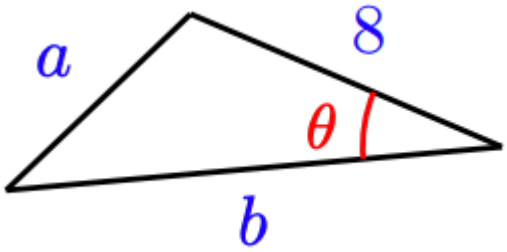
17. the side lengths are 18 in, 21 in, and 32 in.

18. the triangle has sides of length 20 cm, 26 cm, and 37 cm.

19. (MyOpenMath) An isosceles triangle has an area of 168 square feet, and the angle between the two sides of equal length is 37°. Find the length of one of the two equal sides. Round your answer to one decimal place.

20. (MyOpenMath) A triangle has an area of 352 square meters, and two sides of the triangle have lengths 42 meters and 19 meters. Find the angle included by these two sides. Find the length of one of the two equal sides. Round your answer to one decimal place.

21. Write an expression for the area of the triangle.



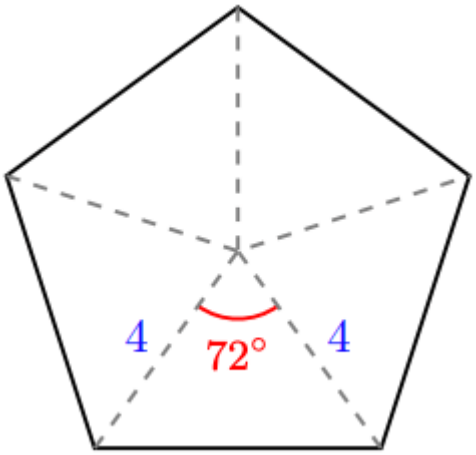
Synthesis Questions

19. A parallelogram has sides of length 15.4 units and 9.8 units. Its area is 72.9 square units. Find the measure of the longer diagonal.

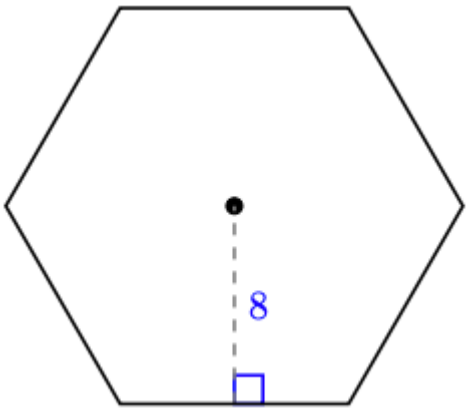
20. The four sequential sides of a quadrilateral have lengths 4.5 cm, 7.9 cm, 9.4 cm, and 12.9 cm. The angle between the two smallest sides is 117°. What is the area of this quadrilateral?

21. The four sequential sides of a quadrilateral have lengths 5.7 cm, 7.2 cm, 9.4 cm, and 12.8 cm. The angle between the two smallest sides is 106°. What is the area of this quadrilateral?

22. Find the area of the regular pentagon shown below. (Hint: The pentagon can be divided into five congruent triangles.)



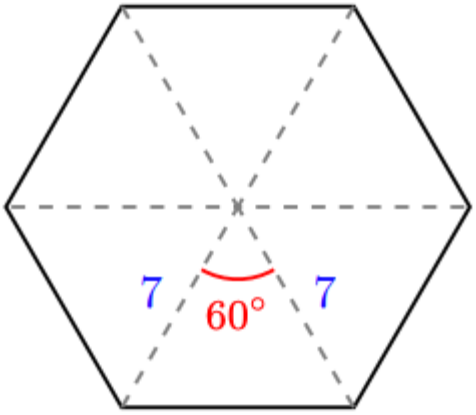
23.



    a. Find the perimeter of a regular hexagon if the apothegm is 8 cm long. (The apothegm is the segment from the center of the hexagon and perpendicular to one of its sides.)

    b. Find the area of the hexagon.

24. Find the area of the regular hexagon shown below. (Hint: The hexagon can be divided into six congruent triangles.)



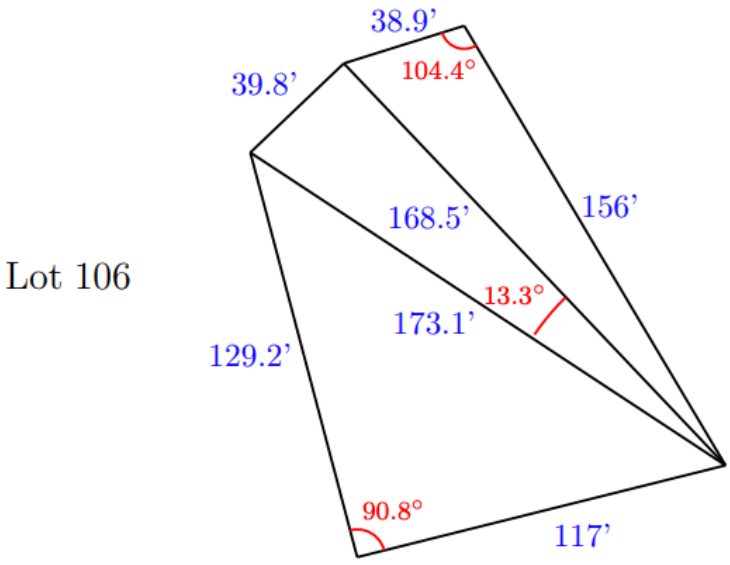
Applications

25. Find the area of a triangular piece of land that measures 30 feet on one side and 42 feet on another; the included angle measures 132°. Round to the nearest whole square foot.

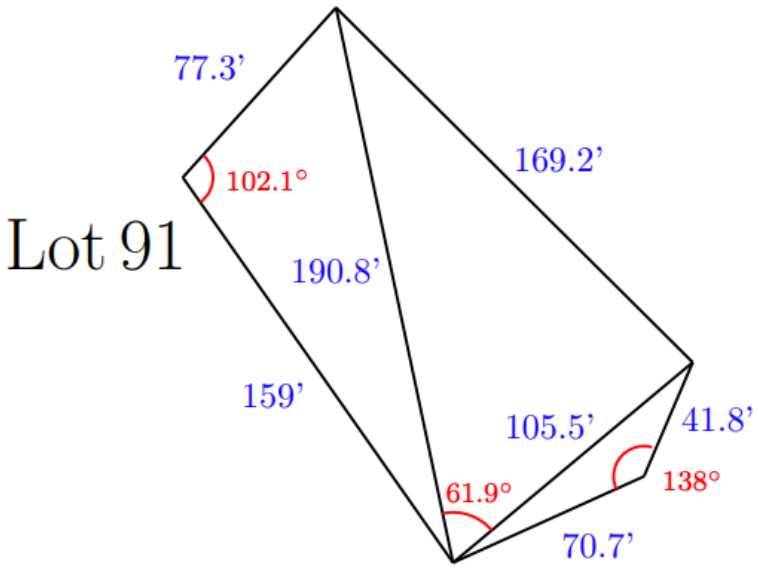
26. Find the area of a triangular piece of land that measures 110 feet on one side and 250 feet on another; the included angle measures 85°. Round to the nearest whole square foot.

For the following exercises, lots from a housing development have been subdivided into triangles. Find the total area of each lot by computing and adding the areas of each triangle.

27.



28.



Answers

Skills Refresher

1. 24
2. 24
3. 24
4. 24

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8.4: Vectors - A Geometric Approach

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Introduction

One way to specify a location is to give a direction and a distance from a fixed landmark. For example, the airport is 8 miles northeast of the town hall, or a ship has been sighted 20 miles from the lighthouse in the direction 10° west of north. It's not enough to give just the distance to the object or just the direction; we need both to describe the object's location. Thus, we have the following definition.

Definition: Vector

In Mathematics, a quantity defined by both a magnitude and a direction is called a **vector**.

Vectors are used to analyze motion and velocity in multiple dimensions, study forces such as gravity and electric fields, and design computer graphics and animation. They are also used throughout Engineering, Mathematics, and Physics.

The types of vectors we will concern ourselves with in this section are position vectors.

Definition: Position Vector

A **position vector** is a vector used to designate a location relative to a fixed landmark.

We often illustrate a vector with an arrow. The length of the arrow represents the magnitude (size) of the vector, and the head of the arrow indicates its direction.

Example 8.4.1

Draw an arrow to represent each vector.

a. The campsite is 3 miles away, in a direction 30° north of east.
b. The wind is blowing due west at 50 kilometers per hour.

Solutions available at the end of this section.

In both images in Figure 8.4.1, we intuitively understand that the direction of the vector is where the arrow is pointing; however, to be able to discuss vectors better, it's best to laboriously define the geometric parts of the graphical representation of a vector.

Definition: Head and Tail of a Vector

Given the graph of a vector, which is a finite-length ray, we define the **head** as the location of the terminal end of the ray (the arrowhead) and the **tail** as the initial point of the ray (the non-arrowhead end).

In Figure $8.4.1a$, the head of the vector is the campsite, and the tail is the origin.

Checkpoint $8.4.1$

Draw an arrow representing the velocity of an airplane traveling southeast at 300 miles per hour.

Answer available at the end of this section.

Notation for Vectors

Some texts use boldface characters such as $\mathbf{u}$ and $\mathbf{v}$ to represent vectors; however, this is difficult to replicate by hand. Therefore, we choose the alternative notation, $\vec{u}$ and $\vec{v}$, which uses arrows above a variable to indicate that it is a vector.

It is vital to distinguish vector quantities from ones with magnitude only, such as length or temperature.

Definition: Scalar

A quantity having only magnitude (and no direction) is called a **scalar**.

Therefore, scalars are the constants and variables you are used to dealing with, usually denoted by italic letters such as x or k. They have a magnitude (size), which we typically consider their absolute value; however, we refine the definition of magnitude when speaking of vectors.

Definition: Magnitude of a Vector (geometric)

The length of a vector $\vec{v}$ is called its **magnitude** and is denoted by $\norm{\vec{v}}$.

Caution: Magnitude is a Scalar Quantity

Note that the magnitude of a vector is a scalar quantity. Thus, $\vec{v}$ denotes a vector (a distance and direction), but $\norm{\vec{v}}$ represents a scalar (the length of the vector).

Figure $8.4.3$ shows two vectors, $\vec{u}$ and $\vec{v}$, that have the same magnitude but different directions. For this example, $\norm{\vec{u}} = \norm{\vec{v}}$ but $\vec{u} \neq \vec{v}$. It makes sense that two vectors may have the same length but point in different directions.

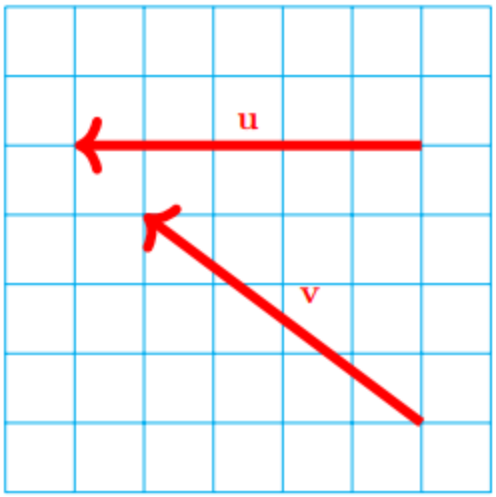


Figure 8.4.3

This begs the question: what does it mean for two vectors to be equal? Believe it or not, we have to define what this means!

Definition: Vector Equality (geometric)

Two vectors are **equal** if and only if they have the same length and direction.

Thus, we could have two equal vectors (same magnitude and direction) but with their tails (and, therefore, heads) at different locations. The fact that we can move vectors from one location to another is valuable.

Example 8.4.2

Which of the vectors are equal?

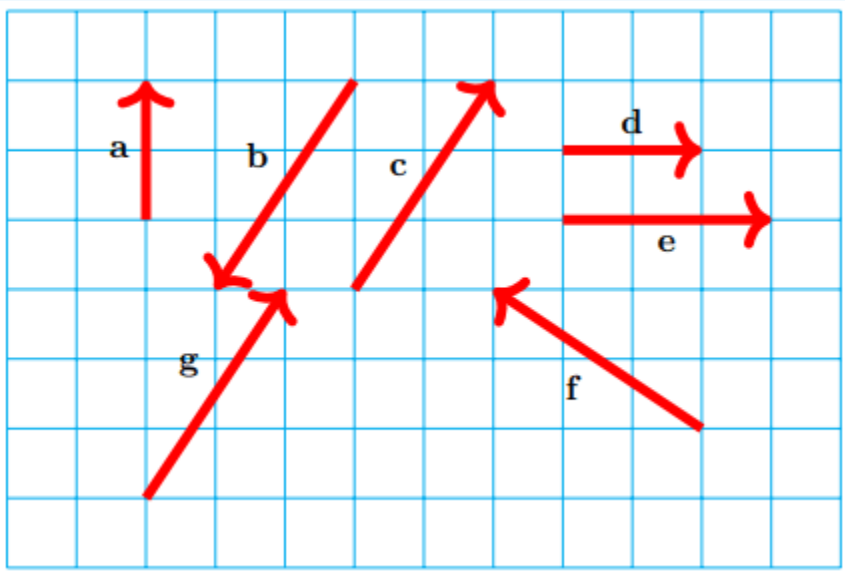


Figure 8.4.4

Solution available at the end of this section.

Checkpoint 8.4.2

Sketch a vector $\vec{w}$ that is equal to the vector $\vec{v}$, but with tail at the point $(-4, 2)$.

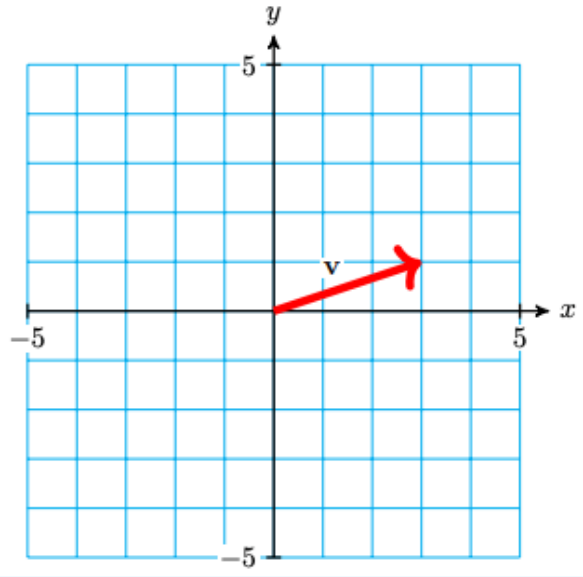


Figure 8.4.5

Answer available at the end of this section.

Scalar Multiplication of Vectors

We can define multiplication of a vector by a scalar.

Definition: Scalar Multiplication (geometric)

The notation $k\vec{v}$, where k is a scalar, is called the **scalar multiple** (or **scalar product**) of the vector $\vec{v}$ with the scalar k and it is defined to be the vector having same direction as $\vec{v}$, but magnitude k times the magnitude of $\vec{v}$.

Figure 8.4.7 shows the vector $\vec{w}$ and the vector $\vec{v} = 3\vec{w}$. Multiplying by a positive scalar changes a vector's length (magnitude) but not its direction.

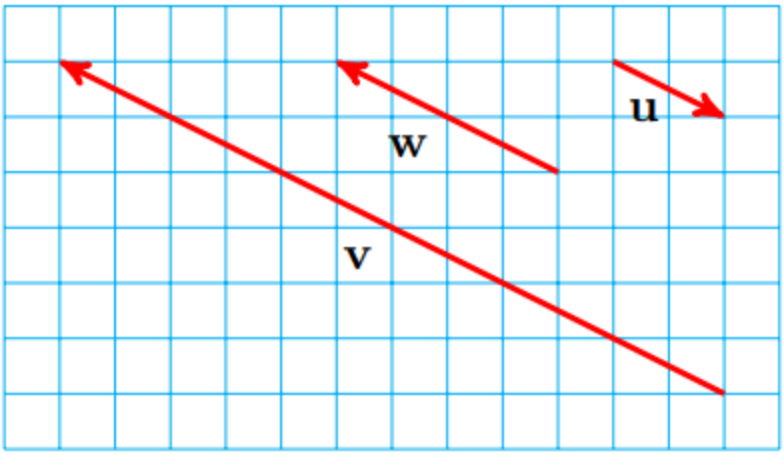


Figure 8.4.7

Thus, the vector $\vec{v}$ is 3 times as long as the vector $\vec{w}$. If we multiply a vector by a negative scalar, we alter its length and reverse its direction; that is, we change the direction by 180°. For example, the vector $\vec{u} = -\frac{1}{2}\vec{w}$. It is half the length of $\vec{w}$ and points in the opposite direction.

In general, if k is a real number, then $k\vec{v}$ represents the vector with magnitude k times the magnitude of $\vec{v}$. It points in the same direction as $\vec{v}$ when $k > 0$ and in the opposite direction from $\vec{v}$ when $k < 0$. Real numbers are called scalars because they "scale" vectors in this way.

Example 8.4.3

The figure below shows the vector $\vec{v}$ and two scalar multiples of $\vec{v}$.

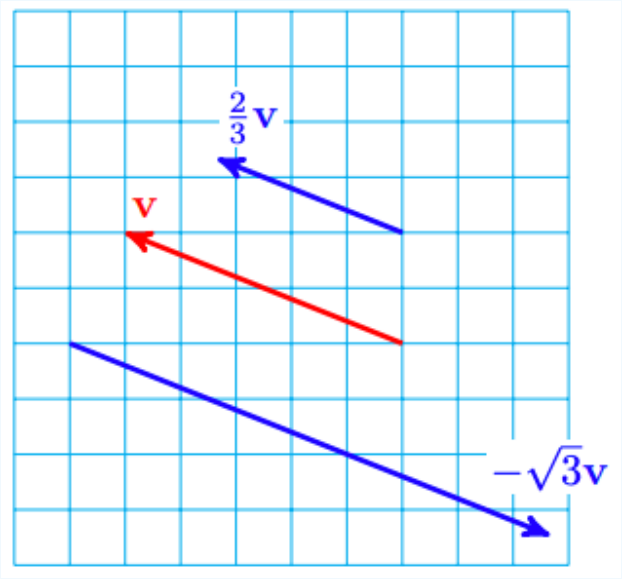


Figure 8.4.8

The vector $\frac{2}{3}\vec{v}$ points in the same direction as $\vec{v}$, but is only two-thirds as long as the vector $\vec{v}$. The vector $-\sqrt{3}\vec{v}$ points in the direction opposite to $\vec{v}$ and is $\sqrt{3}$ or approximately 1.7 times as long as $\vec{v}$.

Note that because $\vec{v}$ has a slope of $\frac{-2}{5}$, so does any nonzero multiple of $\vec{v}$.

Checkpoint 8.4.3

For the vector $\vec{w}$ shown below, draw the vectors $-0.6\vec{w}$ and $\sqrt{2}\vec{w}$.

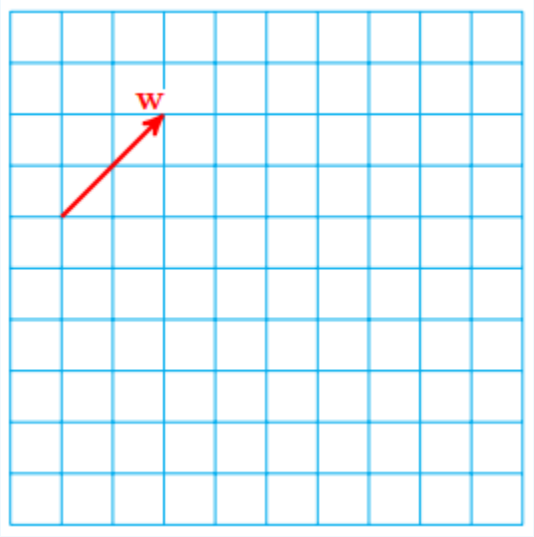


Figure 8.4.9

Answer available at the end of this section.

Addition of Vectors

Definition: Displacement Vector

A **displacement vector** represents the change in position from one point to another.

For example, you leave home and travel 6 miles east and 8 miles north. These two displacements are represented by the vectors $\vec{u}$ and $\vec{v}$, in Figure 8.4.11. When we follow one displacement vector with a second one, the net displacement is a new vector, $\vec{w}$, starting at the tail of the first vector and ending at the head of the second vector.

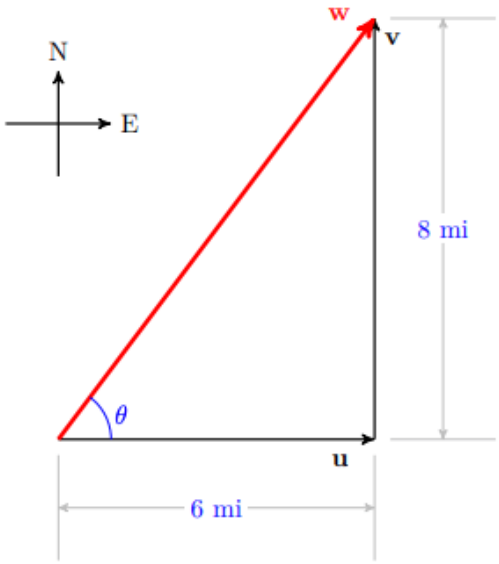


Figure 8.4.11

In this example, notice that $\vec{w}$ forms the hypotenuse of a right triangle, so we can calculate its magnitude and direction.

$$\norm{\vec{w}}^2 = \norm{\vec{u}}^2 + \norm{\vec{v}}^2 = 6^2 + 8^2 = 100.$$

Therefore,

$$\norm{\vec{w}} = 10.$$

To find the direction of the vector $\vec{w}$, we rely on the methods we have established up to this point in Trigonometry.

$$\begin{aligned} \tan(\theta) &= \frac{8}{6} \\ \Longrightarrow \quad \tan(\theta) &= \frac{4}{3} \\ \Longrightarrow \quad \hat{\theta} &= \tan^{-1}\left(\frac{4}{3}\right) \qquad \text{(finding the reference angle)} \end{aligned}$$

Since $0^\circ < \theta < 90^\circ$, $\theta = \hat{\theta} \approx 53.1^\circ$.

The net displacement gives your current position relative to home: 10 miles in the direction 53.1° north of east.

When we follow one vector by a second vector as described above, we are adding the two vectors. As Figure 8.4.11 illustrates, the sum of vectors results in a new vector. This new vector gets a special name.

Definition: Resultant Vector

The vector resulting from a sum of vectors is the **resultant vector**.

Example 8.4.4

For each pair of vectors, draw the resultant vector $\vec{w} = \vec{u} + \vec{v}$.

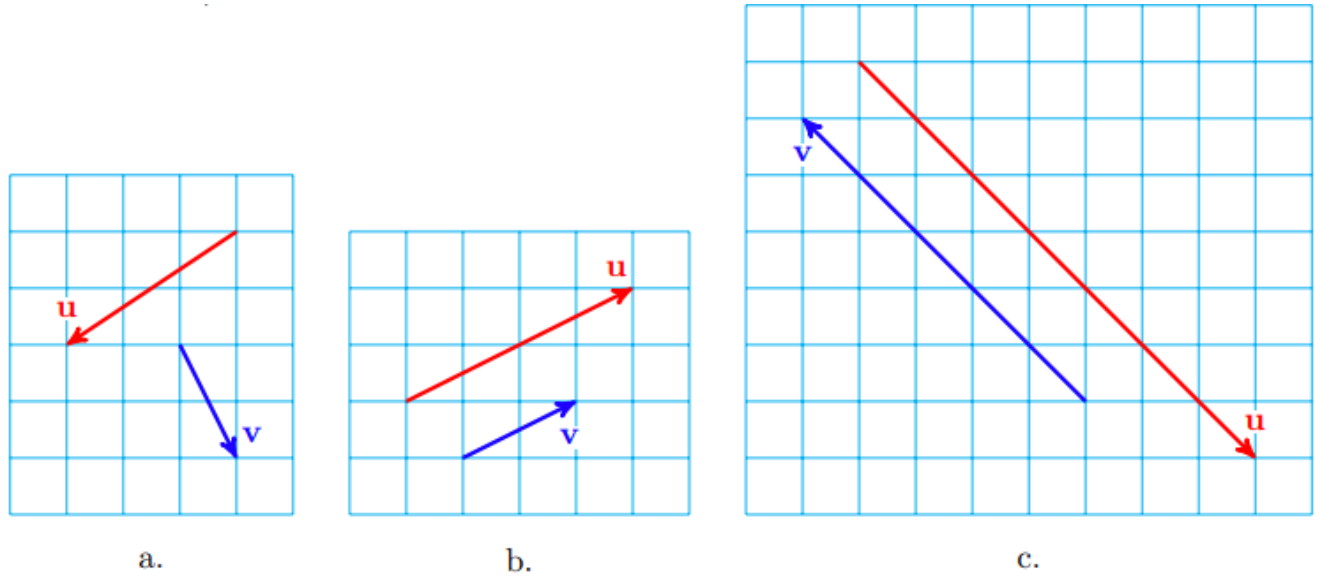


Figure 8.4.12

Solutions available at the end of this section.

Checkpoint 8.4.4

Draw the resultant vector $\vec{w} = \vec{u} + \vec{v}$

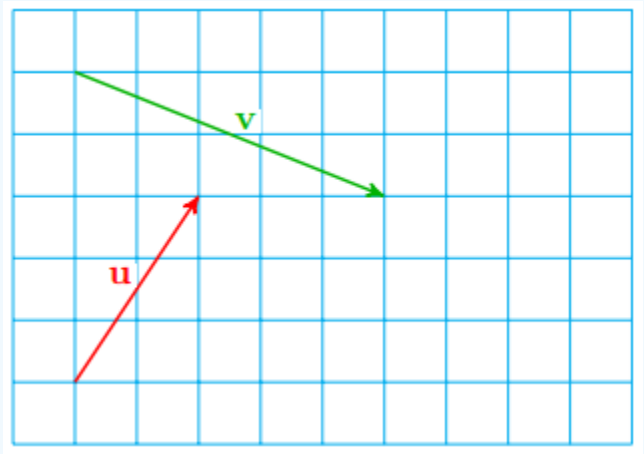


Figure 8.4.14

Answer available at the end of this section.

When the resultant vector forms the third side of a triangle, we can calculate its length and direction using the Law of Sines and the Law of Cosines.

Example 8.4.5

You are camping in a state park. In the morning, you start from your campground at point C and hike 4 miles bearing $\mathrm{S}\ 45^\circ\ \mathrm{W}$ to point B. After taking a break, you continue hiking, and this time you cover 3 miles in the bearing $\mathrm{N}\ 30^\circ\ \mathrm{E}$, and arrive at point A, as shown below. What is the position of the campground relative to your position?

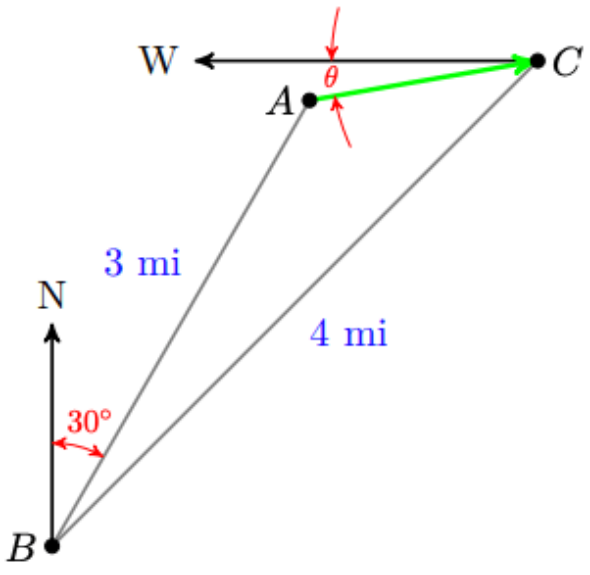


Figure 8.4.16

Solution available at the end of this section.

Checkpoint 8.4.5

Ron has gone sailing with friends. After leaving the marina, they sail 5 miles on a heading of 160° and stop at Gull Island for lunch. After lunch, they have sailed for 3 miles on a heading of 25° when they get a phone call to return home. What heading is the most direct route back to the marina, and how far is it?

Answer available at the end of this section.

It doesn't matter in which order we choose to add two vectors. As with the ordinary addition of scalars, the addition of vectors is commutative, so that $\vec{u}+\vec{v}=\vec{v}+\vec{u}$. To see this, first draw $\vec{u}$ and $\vec{v}$ starting at the same point.

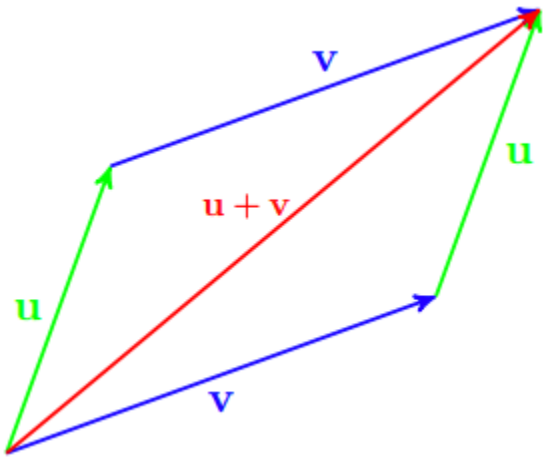


Figure 8.4.18

- To represent $\vec{u}+\vec{v}$, we place the tail of $\vec{v}$ at the head of $\vec{u}$, and draw the resultant vector $\vec{u}+\vec{v}$ as shown in Figure 8.4.18.
- To represent $\vec{v}+\vec{u}$, we place the tail of $\vec{u}$ at the head of $\vec{v}$, and the resultant vector $\vec{v}+\vec{u}$ is the same as the vector $\vec{u}+\vec{v}$.

Because the vector sum forms the diagonal of a parallelogram in this picture, the rule for adding vectors is sometimes called the **parallelogram rule**.

We summarize the operations on vectors as follows.

Summary: Operations on Vectors

- We can multiply a vector, $\vec{v}$, by a scalar, k.
  - If $k > 0$, the magnitude of $k\vec{v}$ is k times the magnitude of $\vec{v}$. The direction of $k\vec{v}$ is the same as the direction of $\vec{v}$.
  - If $k < 0$, the direction of $k\vec{v}$ is opposite the direction of $\vec{v}$.
- We can add two vectors $\vec{v}$ and $\vec{w}$ with the parallelogram rule.

Caution: Lengths of vectors don't add (in general)

Unless $\vec{u}$ and $\vec{v}$ are parallel vectors, it is not true that the length of $\vec{u} + \vec{v}$ is just the sum of the lengths of $\vec{u}$ and $\vec{v}$. Vector addition is a geometric operation; the length of $\vec{u} + \vec{v}$ depends on the lengths of $\vec{u}$ and $\vec{v}$ and on the angle between them. Be careful to distinguish between regular addition of scalars, such as $\text{norm}\vec{u} + \text{norm}\vec{v}$, and vector addition, $\vec{u} + \vec{v}$, which requires the parallelogram rule.

Velocity

Physicists and mathematicians use the word velocity to mean not simply speed but the combination of speed and direction of motion. The interesting thing about velocities is that they add like vectors; if an object's motion consists of two simultaneous components, the resulting displacement is the same as if the motions had occurred one after the other.

For example, imagine a beetle walking across a moving conveyor belt, as shown in the leftmost image in Figure 8.4.19. Suppose the beetle starts at the lower left corner. The conveyor belt is moving at a speed of 4 inches per second, and the beetle walks at right angles to the motion of the belt at 2 inches per second. After 1 second, the beetle has traveled upwards from its starting point, a total distance of 2 inches. Simultaneously, the conveyor belt has moved the beetle to the right a distance of 4 inches. Thus, the beetle has moved a distance of $\sqrt{2^2 + 4^2} = \sqrt{20}$, or about 4.47 inches. His actual velocity relative to the ground is approximately 4.47 inches per second at an angle of $\theta = \tan^{-1}\left(\frac{2}{4}\right) = 26.6^\circ$ from the direction of the conveyor belt.

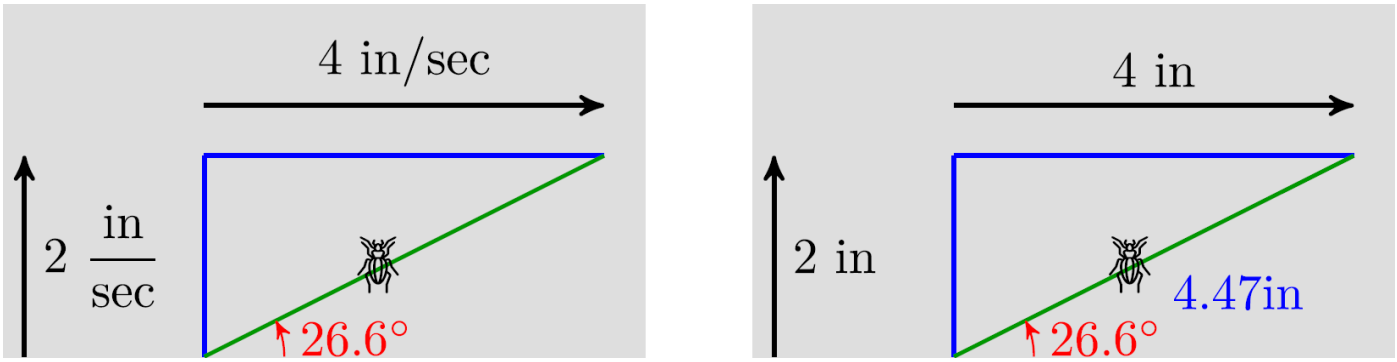


Figure 8.4.19

Notice that the resulting displacement would be the same if the two motions were performed in succession instead of simultaneously, as shown in the rightmost image in Figure 8.4.19. In other words, if the beetle had walked across the belt for 1 second before the belt started moving and then ridden the belt for 1 second without walking, he would still end up at point Q.

Thus, velocity is a vector quantity, and we can calculate the result of two simultaneous motions using the parallelogram rule. We treat the two motions as if they had occurred one after the other by starting one vector at the endpoint of the other. (Remember, we can move a vector from one location to another if preserve its length and direction.)

Example 8.4.6

A ship travels 15 miles per hour relative to the water on a heading of 280°. The water current flows 6 miles per hour on a heading of 160°. What is the actual speed and direction of the ship?

Solution available at the end of this section.

Checkpoint 8.4.6

A plane heads due north at an airspeed of 120 miles per hour. A 45 mph wind blows at a heading of 95°. What are the plane's actual speed and direction relative to the ground?

Answer available at the end of this section.

In some situations, we would like to find a vector to produce a particular sum instead of calculating a vector. That is, we know vectors $\vec{u}$ and $\vec{w}$, and we want to find a vector $\vec{v}$ so that $\vec{u} + \vec{v} = \vec{w}$.

Example 8.4.7

Phuong intends to travel west to an island at 15 miles per hour. However, she must compensate for a current running at a bearing of $\mathrm{N}\ 45^\circ\ \mathrm{E}$ at a speed of 3 miles per hour. In what direction and at what speed should Phuong move her boat?

Solution available at the end of this section.

Checkpoint 8.4.7

Matt would like to sail at 20 kilometers per hour due west towards a whale reported at that position. However, a steady ocean current is moving 52° east of north at 8 kilometers per hour. At what speed and heading should Matt sail?

Answer available at the end of this section.

Answers

Section Preview

This section introduces vectors from a geometric perspective. It is important to realize that all vectors in this section are represented either geometrically or through verbal description (in the case of applications). Components of a vector, $\vec{i}$-$\vec{j}$ notation, and the like are held off for the next section. The topics include:

- geometrically defining vectors, operations on vectors, vector magnitude, and resultant vectors;
- interpreting true heading and velocity as associated with the resultant vector; and
- computing the angle a vector makes with the positive x-axis and the magnitude of a vector (not given in component form).

Section Preview

- Sketch vectors and sums of vectors.
- Find the magnitude of the horizontal and vertical vector components for a vector.
- Find a vector's magnitude and its angle with the positive x-axis.
- Solve an applied problem using vectors.

Example 8.4.1

a. We draw an arrow making an angle of 30° from east. The length of the arrow is 3 units, representing 3 miles. See Figure $8.4.1a$.
b. We draw an arrow pointing due west. That is, making an angle of 180° from east. The arrow's length represents the wind's speed, 50 kilometers per hour. See Figure $8.4.1b$.

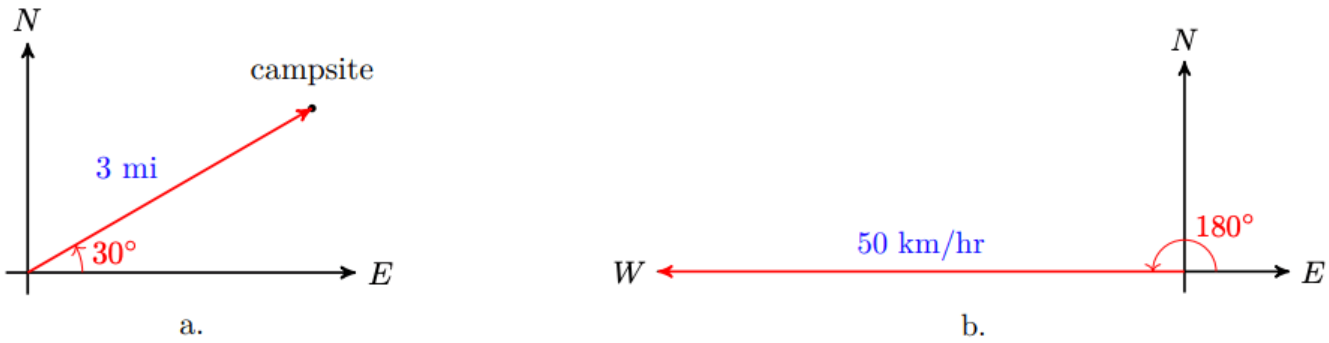


Figure 8.4.1

Checkpoint 8.4.1

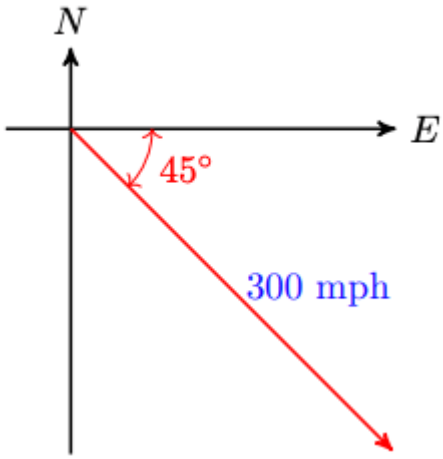


Figure 8.4.2

Example 8.4.2

Only vectors $\vec{c}$ and $\vec{g}$ are equal. Vectors $\vec{b}$, $\vec{c}$, and $\vec{f}$ are all the same length but have different directions. Vectors $\vec{a}$ and $\vec{d}$ also have equal lengths but different directions. Vectors $\vec{d}$ and $\vec{e}$ have the same direction but different lengths.

Checkpoint 8.4.2

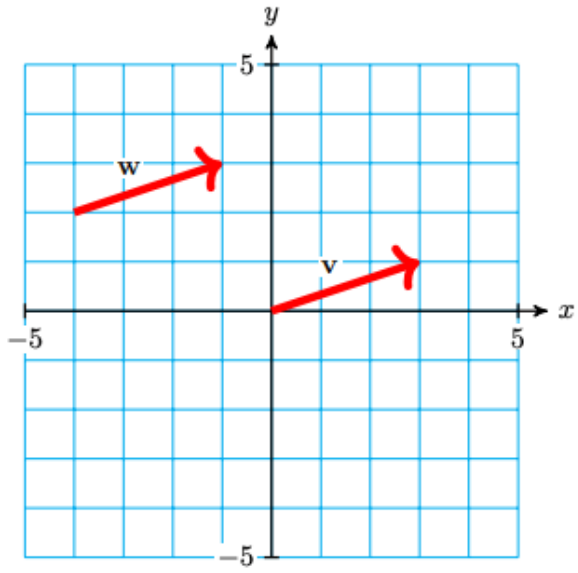


Figure 8.4.6

Checkpoint 8.4.3

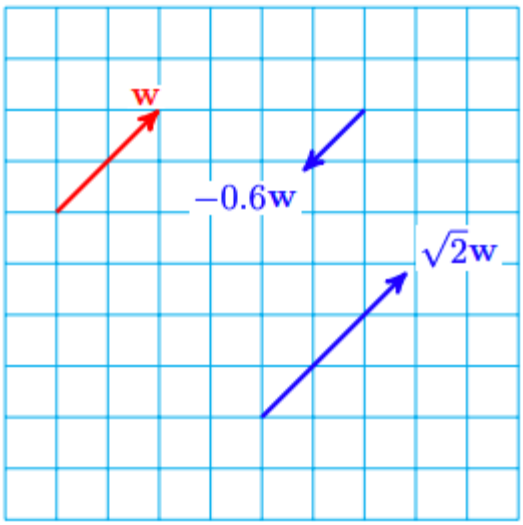


Figure 8.4.10

Example 8.4.4

We draw a copy of the vector $\vec{v}$ so that its tail (or starting point) is placed on the head (or ending point) of $\vec{u}$, and then we draw vector $\vec{w}$ from the tail of $\vec{u}$ to the head of $\vec{v}$.

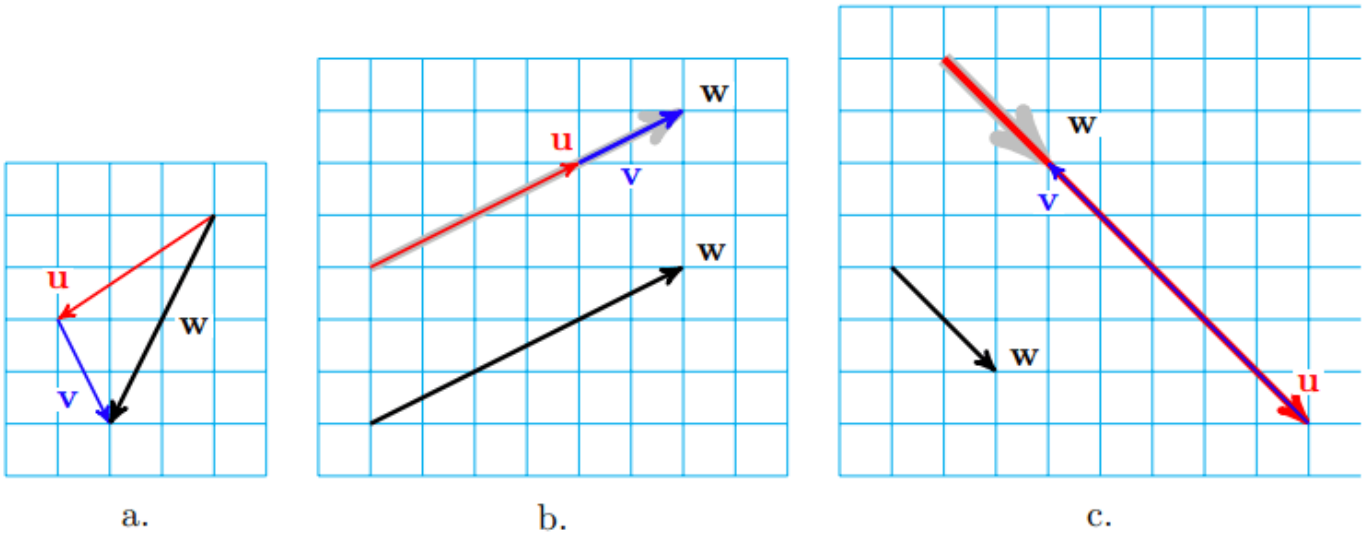


Figure 8.4.13

Checkpoint 8.4.4

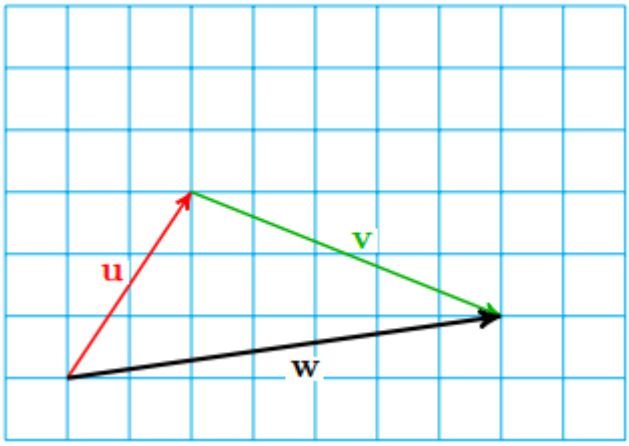


Figure 8.4.15

Example 8.4.5

The resultant vector, $\vec{w}$, forms the side opposite a 15° angle.

$$\begin{aligned} \norm{\vec{w}}^2 &= a^2 + c^2 - 2ac\cos(\angle ABC) && \text{(Law of Cosines)} \\ \implies \norm{\vec{w}} &= \sqrt{a^2 + c^2 - 2ac\cos(\angle ABC)} && \text{(Extraction of Roots)} \end{aligned}$$

In this case, $a = 4$, $c = 3$, and $\angle ABC = 15^\circ$. Substituting these values into our result, we get $\norm{\vec{w}} \approx 1.348$, so you are approximately 1.348 miles from the campground. To find the direction back to the campground, we must calculate the angle $\angle ACB$. Following the advice in the section covering the Law of Cosines, we find the angle opposite the smallest remaining side. Luckily, that angle is $\angle ACB$.

$$\begin{aligned} \frac{\sin(\angle ACB)}{c} &= \frac{\sin(\angle ABC)}{b} && \text{(Law of Sines)} \\ \implies \sin(\angle ACB) &= \frac{c\sin(\angle ABC)}{b} && \text{(multiplying both sides by } c\text{)} \\ \implies \angle\widehat{ACB} &= \sin^{-1}\left(\frac{c\sin(\angle ABC)}{b}\right) && \text{(finding the reference angle)} \end{aligned}$$

Since $\angle ACB < 90^\circ$, $\angle ACB = \angle\widehat{ACB} \approx 35.2^\circ$. It's best to get a visual of our angles, so we redraw the figure with the newly found angles.

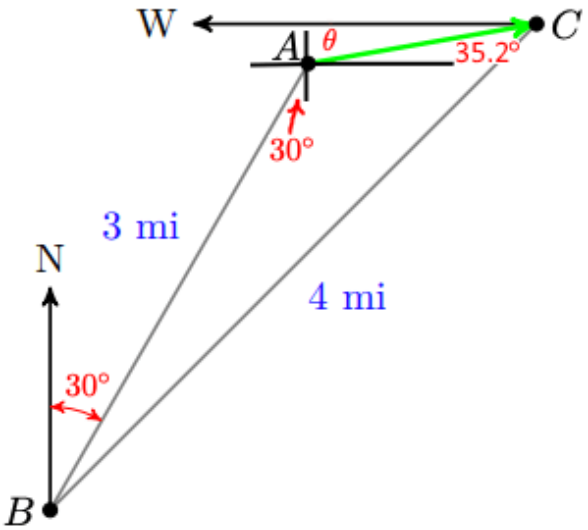


Figure 8.4.17

Since $\angle ABC = 15^\circ$ and $\angle ACB \approx 35.2^\circ$, $\angle BAC \approx 129.8^\circ$. However, this means

$$30^\circ + 90^\circ + (90^\circ - \theta) \approx 129.8^\circ \implies \theta \approx 80.2^\circ.$$

Thus, the campground is 1.348 miles away at a bearing of N 80.2° E.

Checkpoint 8.4.5

Heading of 303.6° for 3.58 miles

Example 8.4.6

We represent the ship's velocity by a vector $\vec{v}$ and the velocity of the water current by vector $\vec{w}$, as shown in Figure 8.4.20a. Note that the angle $\vec{v}$ makes with the horizontal is 10°, and angle $\vec{w}$ makes with the vertical is 20° — these will be important in a few moments.

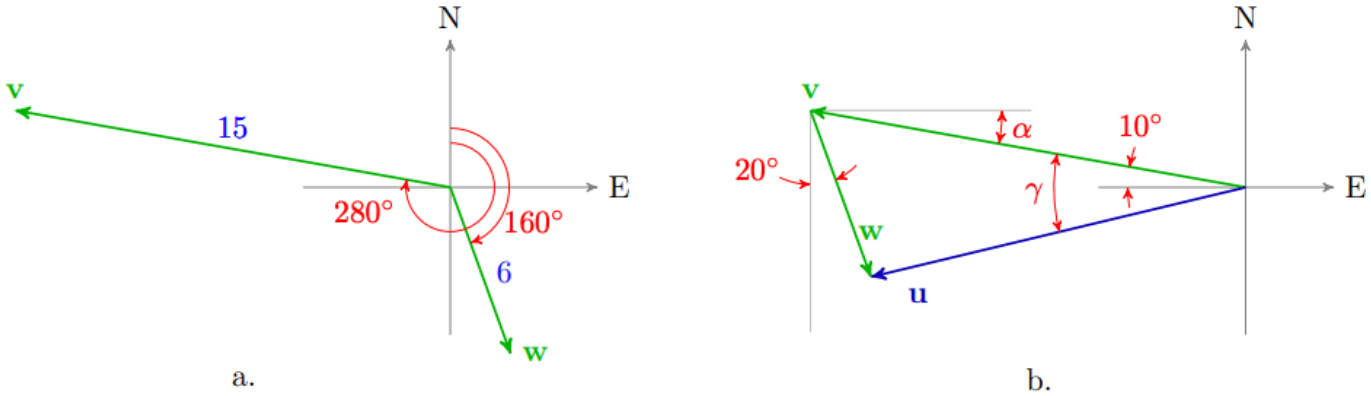


Figure 8.4.20

The actual motion of the ship is the sum of these two vectors. We can calculate the sum as if the two motions had occurred separately, one after the other, as shown in Figure 8.4.20b. So we will add $\vec{w}$ to $\vec{v}$ by placing the tail of $\vec{w}$ at the head of $\vec{v}$. The resultant vector, $\vec{u}$, represents the actual motion of the ship. Let's zoom in to Figure 8.4.20b.

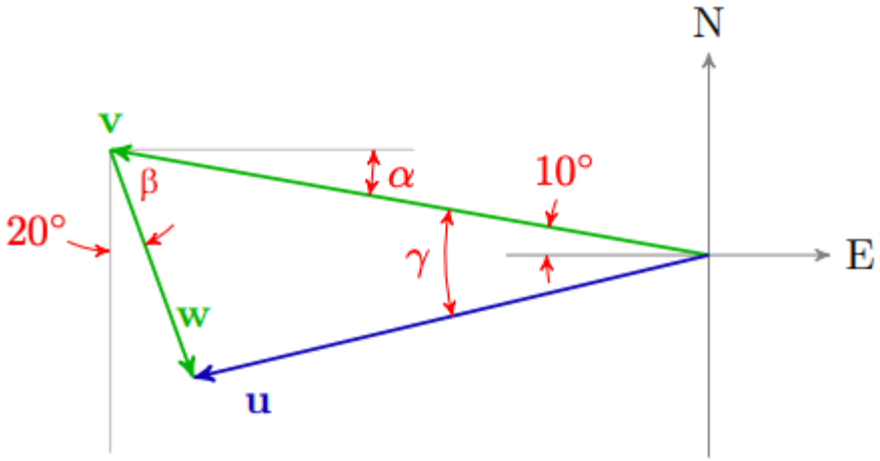


Figure 8.4.21

The angle α in Figure 8.4.21 is alternate interior to the 10° angle $\vec{v}$ makes with the horizontal, so $\alpha = 10^\circ$. Moreover, we already mentioned that $\vec{w}$ makes a 20° with the vertical. Therefore, if we let β be the angle between the head of $\vec{v}$ and the tail of $\vec{w}$ (as seen in Figure 8.4.21), we have $20^\circ + \beta + 10^\circ = 90^\circ$, so $\beta = 60^\circ$. Hence,

$$\begin{aligned} \norm{\vec{u}}^2 &= \norm{\vec{v}}^2 + \norm{\vec{w}}^2 - 2\norm{\vec{v}} \cdot \norm{\vec{w}} \cos(\beta) && \text{(Law of Cosines)} \\ \implies \norm{\vec{u}} &= \sqrt{\norm{\vec{v}}^2 + \norm{\vec{w}}^2 - 2\norm{\vec{v}} \cdot \norm{\vec{w}} \cos(\beta)} && \text{(Extraction of Roots)} \end{aligned}$$

Substitution in $\norm{\vec{v}} = 15$, $\norm{\vec{w}} = 6$, and $\beta = 60^\circ$, we find $\norm{\vec{u}} = \sqrt{171} \approx 13.1$. Thus, the ship's speed is approximately 13.1 miles per hour.

Next, we calculate the ship's heading (also known as its **true heading**). This means we need to calculate another angle within the triangle shown in Figure 8.4.21. We already know β. Of the remaining angles, we want to find the measure of γ (mostly because it is opposite the smallest remaining side, but also because this is the angle that will help us calculate the ship's true heading).

$$\begin{aligned} \frac{\sin(\gamma)}{\norm{\vec{w}}} &= \frac{\sin(\beta)}{\norm{\vec{u}}} && \text{(Law of Sines)} \\ \implies \sin(\gamma) &= \frac{\norm{\vec{w}} \sin(\beta)}{\norm{\vec{u}}} && \text{(multiplying both sides by } \norm{\vec{w}}) \\ \implies \hat{\gamma} &= \sin^{-1}\left(\frac{\norm{\vec{w}} \sin(\beta)}{\norm{\vec{u}}}\right) && \text{(finding the reference angle)} \end{aligned}$$

Since γ is opposite the smallest remaining side, we can guarantee $0^\circ < \gamma < 90^\circ$. Hence, $\gamma = \hat{\gamma} \approx 23.5^\circ$.

Looking back at Figure 8.4.21, we see the angle that $\vec{u}$ makes with the horizontal is $\gamma - 10^\circ = 13.4^\circ$. Hence, the ship is heading $270^\circ - 13.4^\circ = 256.6^\circ$ at 13.1 miles per hour.

Checkpoint 8.4.6

Heading of 21.1° moving at 124.4 mph.

Example 8.4.7

We draw a triangle using vectors to represent the desired velocity of Phuong's boat, $\vec{w}$, and the velocity of the current, $\vec{u}$, as shown in Figure 8.4.22. We want to find a vector $\vec{v}$ that represents the speed and bearing Phuong should take to compensate for the current.

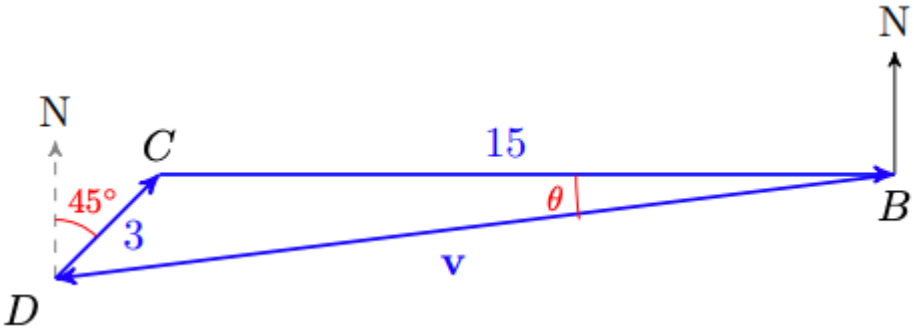


Figure 8.4.22

We first use the Law of Cosines to calculate the length or magnitude of the vector $\vec{v}$. The angle at C is 135°. (Do you see why?). Thus,

$$\begin{aligned} \norm{\vec{v}}^2 &= 3^2 + 15^2 - 2(3)(15)\cos(135^\circ) \\ &\approx 297.64 \\ \implies \norm{\vec{v}} &\approx 17.25 \end{aligned}$$

So, Phuong should travel at about 17.25 miles per hour. To find her bearing, we use the Law of Sines to calculate the angle θ.

$$\begin{aligned} \frac{\sin(\theta)}{3} &= \frac{\sin(45^\circ)}{15} \\ \Longrightarrow \quad \sin(\theta) &= \frac{3/\sqrt{2}}{15} \\ \Longrightarrow \quad \theta &= \sin^{-1}\left(\frac{1}{5\sqrt{2}}\right) \\ &\approx 8.1^\circ \end{aligned}$$

This means that the angle $\vec{v}$ makes with the vertical is approximately 81.9°. Therefore, Phuong should move her boat in the bearing of $\mathrm{S}\ 81.9^\circ\ \mathrm{W}$.

Checkpoint 8.4.7

$26.76\mathrm{kph}$ and $10.6^\circ\mathrm{S}$ of W

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8.4.1: Resources and Key Concepts

Resources

Keywords

- Vector
  - Vectors - A Geometric Approach

Key Concepts

Definitions

- **Vector:** A mathematical quantity defined by both a magnitude and a direction.
- **Position Vector:** A vector used to specify a location relative to a fixed landmark.
- **Head and Tail of a Vector:** For a vector represented by an arrow, the head is the terminal end (arrowhead), and the tail is the initial point.
- **Scalar:** A quantity that has only magnitude and no direction.
- **Magnitude of a Vector (geometric):** The length of a vector $\vec{v}$, denoted by $||\vec{v}||$.
- **Vector Equality (geometric):** Two vectors are equal if and only if they have the same magnitude and direction.
- **Scalar Multiplication (geometric):** The product of a scalar k and a vector $\vec{v}$, denoted $k\vec{v}$, is a vector with the same direction as $\vec{v}$ if $k > 0$ and the opposite direction if $k < 0$. Its magnitude is $|k|$ times the magnitude of $\vec{v}$.
- **Displacement Vector:** A vector that represents the change in position from one point to another.
- **Resultant Vector:** The vector that results from the sum of two or more vectors.

Common Mistakes

- **Confusing Vectors and Scalars:** A vector has both magnitude and direction, while a scalar has only magnitude. The magnitude of a vector, $||\vec{v}||$, is a scalar.
- **Adding Magnitudes Instead of Vectors:** The magnitude of a sum of vectors, $||\vec{u} + \vec{v}||$, is generally not equal to the sum of their magnitudes, $||\vec{u}|| + ||\vec{v}||$. Vector addition must be performed geometrically using the head-to-tail or parallelogram rule.

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8.4.2: Homework

Reading Questions

1. What two properties define a vector?
2. What is the difference between a vector and a scalar?
3. In a graphical representation of a vector, what does the length of the arrow represent? What does the arrowhead represent?
4. When are two vectors considered equal?
5. What is the term for the length of a vector?
6. Describe what happens to a vector when it is multiplied by a positive scalar $k > 1$.
7. Describe what happens to a vector when it is multiplied by a negative scalar, such as -1.
8. What is a resultant vector?
9. Explain the "head-to-tail" method for adding two vectors.
10. What is the "parallelogram rule" for vector addition?
11. Is the magnitude of the sum of two vectors, $||\vec{u} + \vec{v}||$, typically equal to the sum of their individual magnitudes, $||\vec{u}|| + ||\vec{v}||$? Why or why not?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

1. State two versions of the Law of Sines.
2. State two versions of the Law of Cosines.

For the following exercises, find the unknown part of the triangle. Round to two decimal places.

3. $A = 15°$, $B = 125°$, and $b = 12$ cm. Find a.
4. $C = 87°$, $b = 11$ inches, and $c = 13$ inches. Find B.
5. $A = 37°$, $b = 6$, and $c = 14$. Find a.
6. $a = 9$, $b = 4$, and $c = 7$. Find B.

Answers available at the end of this section.

Homework

Vocabulary Check

1. A quantity defined by both a magnitude (such as a distance) and a direction is called a ___.
2. Two vectors are equal if they have the same ___ and ___; it does not matter where the vector starts.
3. A quantity having only magnitude (and no direction) is called a ___.
4. The length of a vector $\vec{v}$ is called its ___, and is denoted by $\backslash\text{norm}\vec{v}$.

5. The sum of two vectors $\vec{u}$ and $\vec{v}$ is a new vector, $\vec{w}$, starting at the ___ of the first vector and ending at the ___ of the second vector.

6. The sum of two vectors is called the ___ vector.

7. The rule for adding vectors is sometimes called the ___ rule.

Concept Check

8. What is the difference between a scalar and a vector?

9. If velocity is represented by a vector, what is its magnitude called?

10. Does $\norm{\vec{u}+\vec{v}} = \norm{\vec{u}} + \norm{\vec{v}}$?

11. Does $\norm{k\vec{v}} = |k| \cdot \norm{\vec{v}}$?

12. What is the parallelogram rule?

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

13. If $k < 0$, the magnitude of $k\vec{v}$ is k times the magnitude of $\vec{v}$.

14. The direction of $k\vec{v}$ is the same as the direction of $\vec{v}$.

Basic Skills

For the following exercises, sketch a vector to represent the quantity.

15. The waterfall is 3 km away, at a bearing of $\mathrm{S}\ 75^\circ\ \mathrm{W}$.

16. The cave entrance is 450 meters away, at a bearing of $\mathrm{N}\ 45^\circ\ \mathrm{E}$.

17. The current is moving 6 feet per second at a heading of 60°.

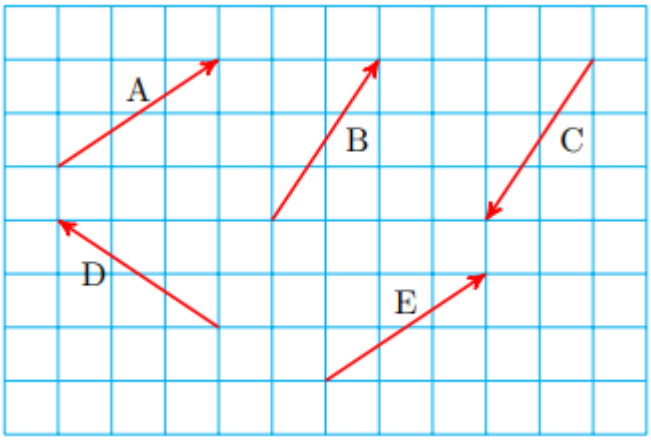
18. The bird is flying due south at 45 miles per hour.

19. The projectile was launched at a speed of 40 meters per second, at an angle of 30° above horizontal.

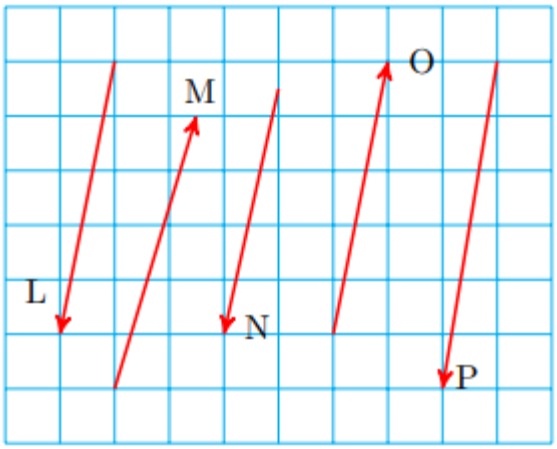
20. The baseball was hit straight up at a speed of 60 miles per hour.

For the following exercises, which vectors are equal?

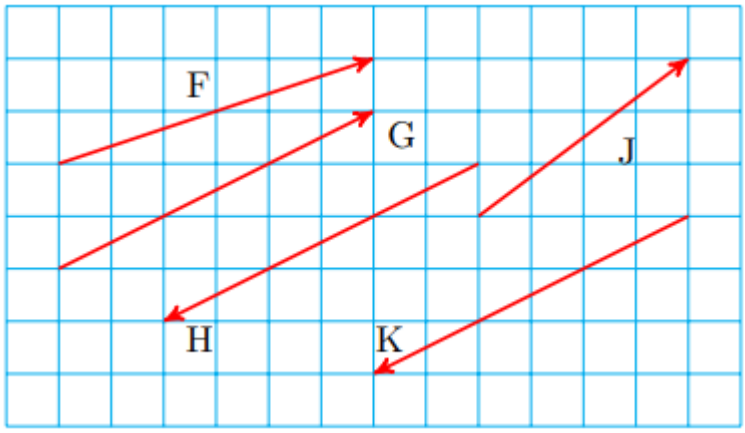
21.



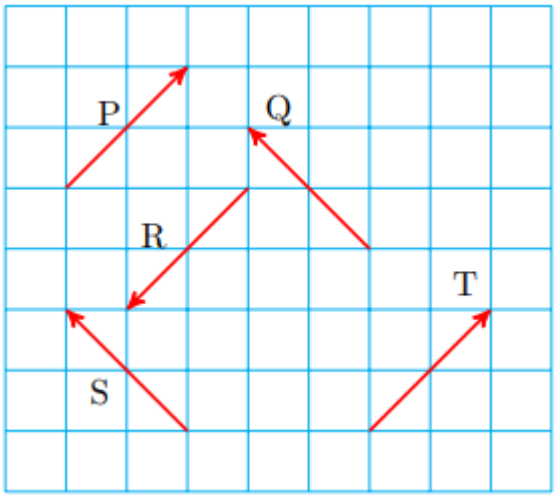
22.



23.

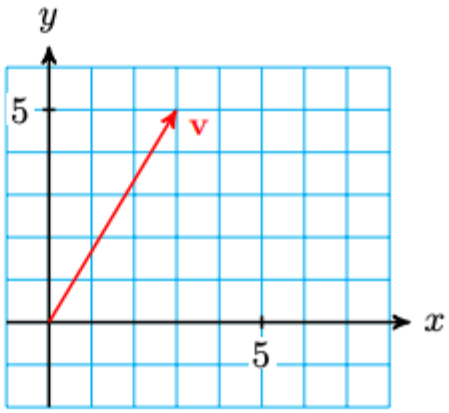


24.



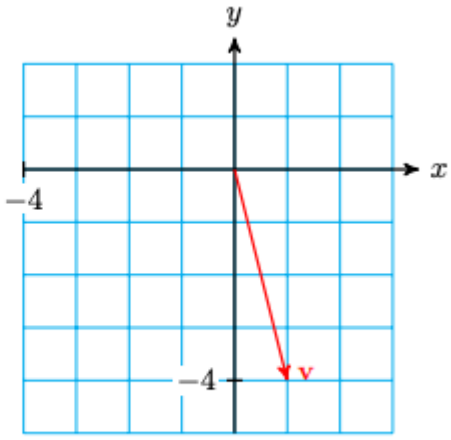
For the following exercises, sketch a vector equal to $\vec{v}$, but starting at the given point.

25.



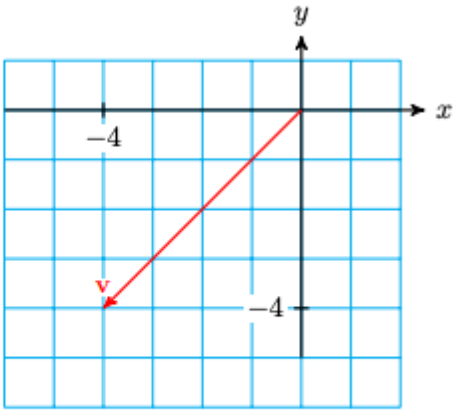
$(4, -1)$

26.

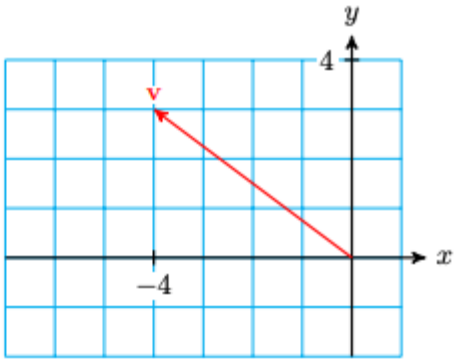


$(-3, 1)$

27.

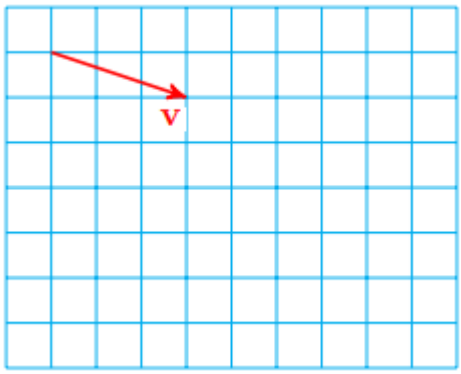


$(0, -2)$

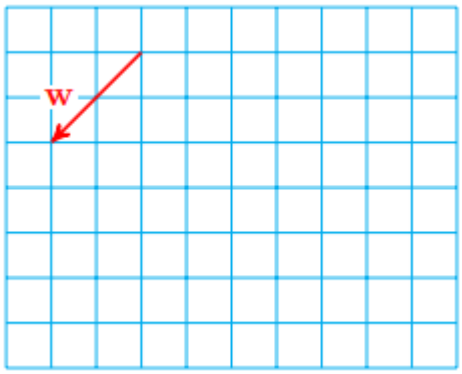


28. $(-3, -1)$

For the following exercises, draw the scalar multiples of the given vectors.

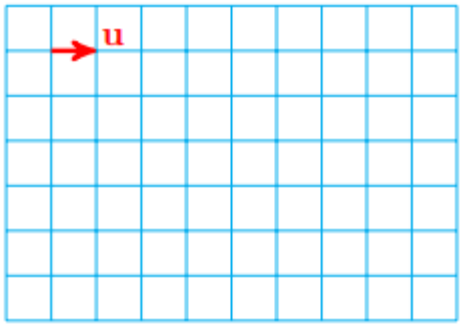


29. $-2v$ and $1.5v$



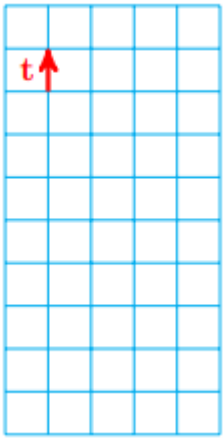
30. $-\frac{1}{2}w$ and $3w$

31.



$-2.5u$ and $\sqrt{2}u$

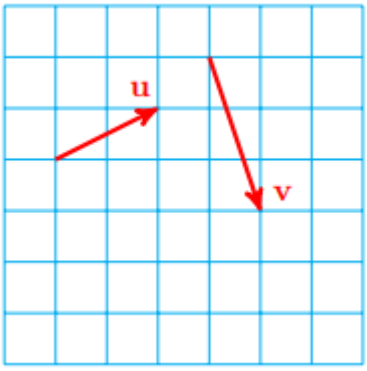
32.



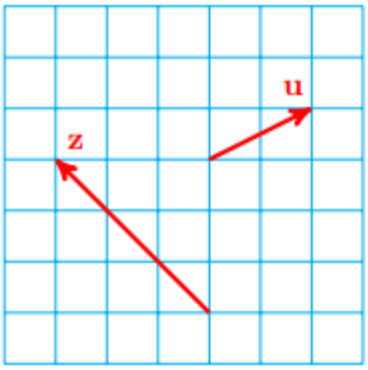
$-\sqrt{6}t$ and $5.4t$

For the following exercises,
(a) draw the resultant vector,
(b) calculate the length and direction of the resultant vector.

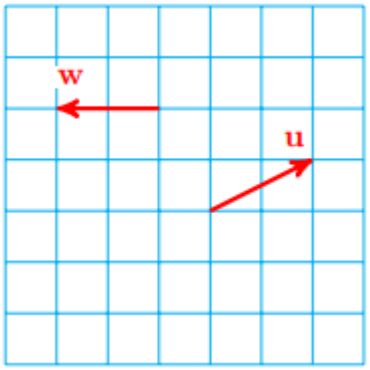
33. $\vec{A} = \vec{u} + \vec{v}$



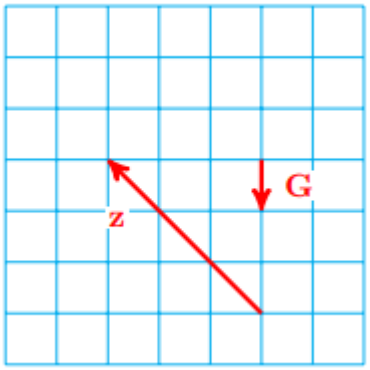
34. $\vec{B} = \vec{z} + \vec{u}$



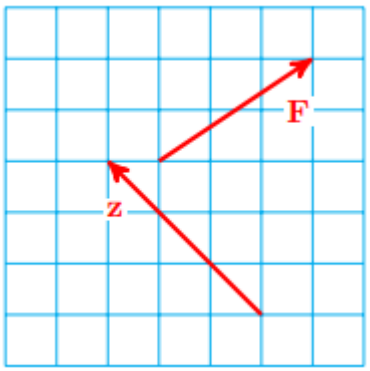
35. $\vec{C} = \vec{w} + \vec{u}$



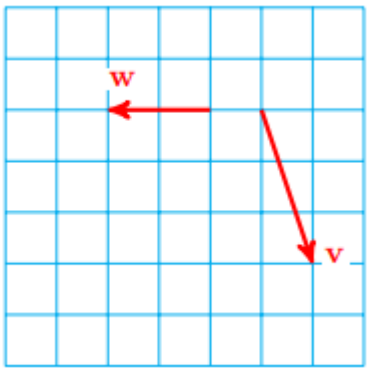
36. $\vec{D} = \vec{G} + \vec{z}$



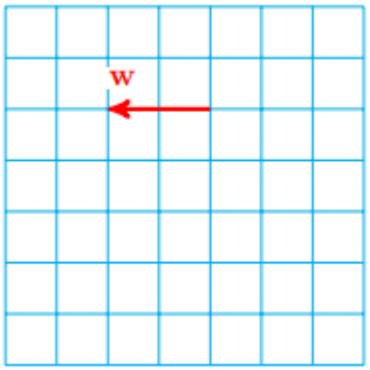
37. $\vec{E} = \vec{z} + \vec{F}$



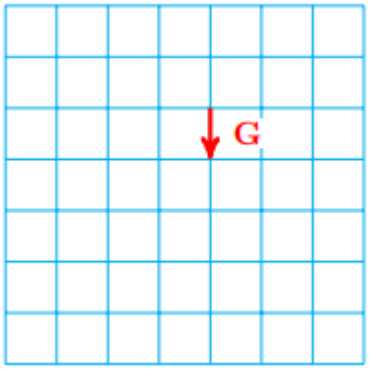
38. $\vec{F} = \vec{w} + \vec{v}$



39. $\vec{G} = \vec{w} + \vec{w}$



40. $\vec{H} = \vec{G} + \vec{G}$



For the following exercises, find the magnitude and direction of the vector.

41. $v_x = 5, v_y = -12$

42. $v_x = -8, v_y = 15$

43. $v_x = -6, v_y = -7$

44. $v_x = 1, v_y = -3$

For the following exercises, sketch the vectors, then calculate the resultant.

45. Add the vector $\vec{v}$ of length 45 pointing 26° east of north to the vector $\vec{w}$ of length 32 pointing 17° south of west.

46. Add the vector $\vec{v}$ of length 105 pointing 41° west of south to the vector w of length 77 pointing 8° west of north.

47. Let $\vec{v}$ have length 8 and point in the heading 10°. Let $\vec{w}$ have length 13 and point in the heading 250°. Find $\vec{v} + \vec{w}$.

48. Let a have length 43 and point in the heading 343°. Let $\vec{b}$ have length 19 and point in the heading 141°. Find $\vec{a} + \vec{b}$.

Synthesis Questions

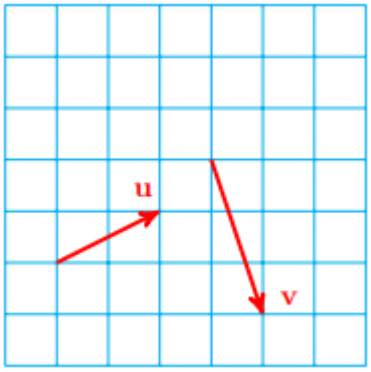
Multiplying a vector $\vec{v}$ by -1 gives a vector $-\vec{v}$ that has the same magnitude as $\vec{v}$ but points in the opposite direction. We define subtraction of two vectors the same way we define subtraction of integers:

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}).$$

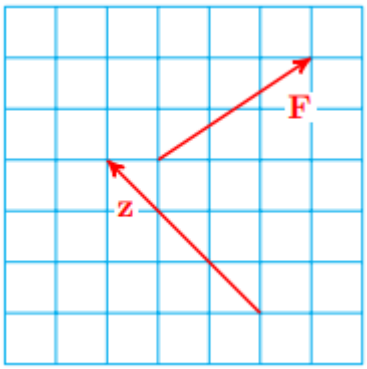
That is, to subtract a vector $\vec{v}$, we add its opposite.

For the following exercises, draw the resultant vector.

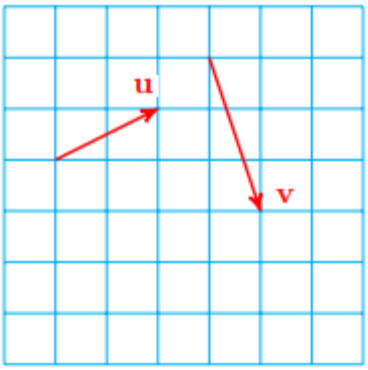
49. $\vec{A} = \vec{u} - \vec{v}$



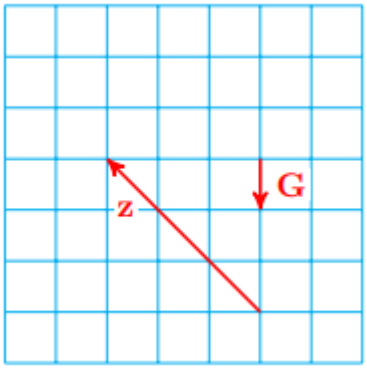
50. $\vec{B} = \vec{F} - \vec{z}$



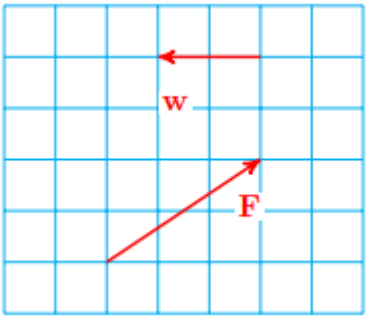
51. $\vec{C} = \vec{v} - \vec{u}$



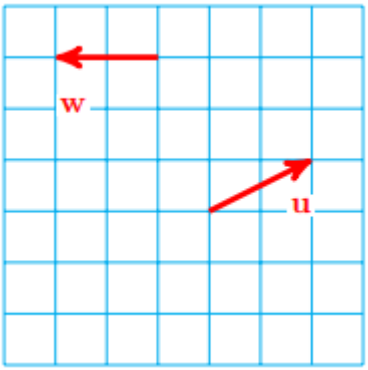
52. $\vec{D} = \vec{z} - \vec{G}$



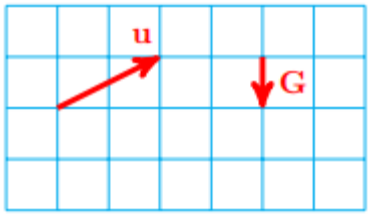
53. $\vec{P} = \vec{w} - \vec{F}$



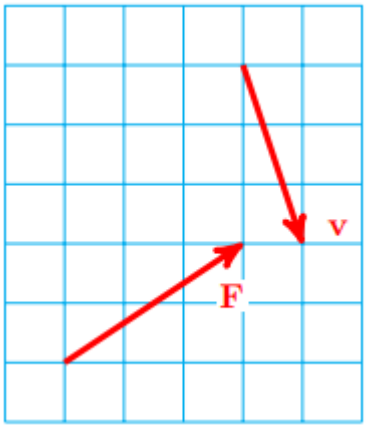
54. $\vec{Q} = \vec{u} - \vec{w}$



55. $\vec{R} = \vec{G} - \vec{u}$



56. $\vec{S} = \vec{v} - \vec{F}$



Applications

For the following exercises, sketch the vectors, then calculate the resultant.

57. Loi swam 3.6 miles at a bearing of $\mathrm{N}\ 20^\circ\ \mathrm{E}$. However, the water current displaced her by 0.9 miles at a bearing of $\mathrm{N}\ 37^\circ\ \mathrm{E}$. How far is Loi from her starting point, and at what bearing?

58. Paige paddles her canoe 4.5 miles at a bearing of $\mathrm{N}\ 12^\circ\ \mathrm{W}$. The water current pushes her 0.3 miles off course at a bearing of $\mathrm{N}\ 5^\circ\ \mathrm{E}$. How far is Paige from her starting point, and at what bearing?

59. Nhat wants to fly to an airport that is 103 miles due west in 1 hour. The prevailing winds blow at a heading of 112° at 28 miles per hour, so Nhat will head her plane somewhat north of due west to compensate. What airspeed and heading should Nhat take?

60. Nam wants to cross a 300 meter wide river, but the river is running due south at 80 meters per minute. There are rocks upstream and rapids downstream, so he wants to paddle straight across from east to west. At what heading should he point his kayak, and how fast should his water speed be in order to cross the river in 2 minutes? (Hint: The current will move him 160 meters due south compared with where his speed and heading would take him if the current stopped. Compute the distance he would have traveled, then divide by 2 minutes to get the speed.)

61. A ship maintains a heading of 30° and a speed of 20 miles per hour. There is a current in the water running at a heading of 135° and at a speed of 10 miles per hour. What is the actual heading and speed of the ship?

62. A plane is heading due south, with an airspeed of 180 kilometers per hour. The wind is blowing at 50 kilometers per hour at a heading of 225°. What is the actual heading and speed of the plane?

63. The campground is 3.6 kilometers from the trail head at a bearing of $\mathrm{N}\ 20^\circ\ \mathrm{W}$. A ranger station is located 2.3 kilometers from the campsite at a bearing of $\mathrm{S}\ 8^\circ\ \mathrm{W}$. What is the distance and bearing from the trail head to the ranger station?

64. There is treasure buried 40 paces due east from a dead tree. From the buried treasure, a hidden mine shaft is 100 paces at a bearing of $\mathrm{N}\ 58^\circ\ \mathrm{W}$. What is the distance and bearing from the dead tree to the mine shaft?

Answers

Skills Refresher

1. $\frac{\sin(A)}{a} = \frac{\sin(B)}{b}, \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
2. $a^2 = b^2 + c^2 - 2bc\cos(A), b^2 = a^2 + c^2 - 2ac\cos(B)$
3. 3.79
4. 57.67°
5. 9.89
6. 25.21°

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8.5: Vectors - An Algebraic Approach

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Components of a Vector

We have seen that we can add two vectors to get a third, or resultant, vector. We can also break down a vector into two or more component vectors. Breaking a vector into horizontal and vertical components is helpful for many applications.

For example, the beetle on the conveyor belt moved at 4.47 inches per second in the direction 26.6°. If we set up coordinate axes, as shown in Figure $8.5.1$, the vector $\vec{v}$ representing its velocity is the sum of the beetle's motion in the x-direction, $\vec{v}_x$, at 4 inches per second, and its motion in the y-direction, $\vec{v}_y$, at 2 inches per second.

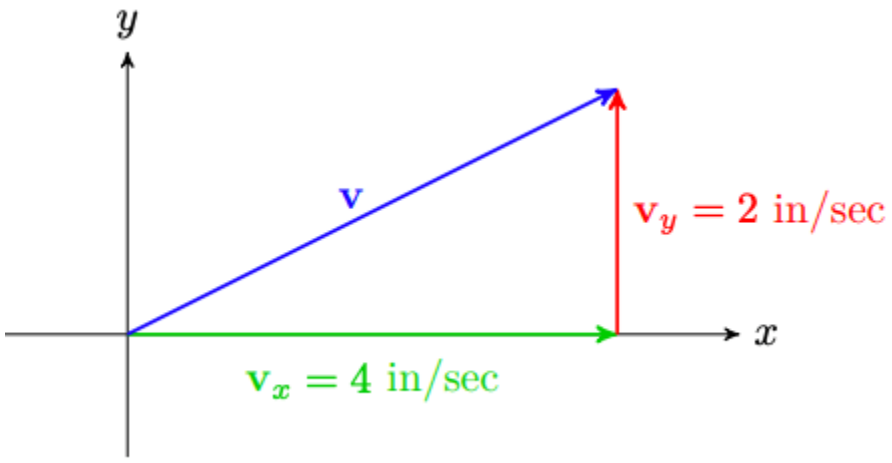


Figure $8.5.1$

We can break down any vector into its x- and y-components and the sum of those components is equal to the original vector. In other words,

$$\vec{v} = \vec{v}_x + \vec{v}_y.$$

Definition: Vector Components

A (two-dimensional) vector $\vec{v}$ has two **vector components** - a horizontal vector component and a vertical vector component. These vector components are themselves vectors. The horizontal vector component is denoted $\vec{v}_x$ and the vertical vector component is denoted $\vec{v}_y$. These are the only two vectors (of horizontal and vertical direction) whose sum is the original vector $\vec{v}$.

To find the component vectors, we use the following theorem.

Definition: Components

If θ is the angle measured counter-clockwise from the positive x-axis to the vector $\vec{v}$, where the tail of $\vec{v}$ is at the origin, then the scalar quantities given by

$$v_x = \norm\vec{v}\cos(\theta)$$

and

$$v_y = \norm\vec{v}\sin(\theta)$$

are called simply the **components** of the vector $\vec{v}$.

Note that the components v_x and v_y of a vector are scalars; they are not vectors. They can be either positive or negative (or zero), as shown in Figure 8.5.2.

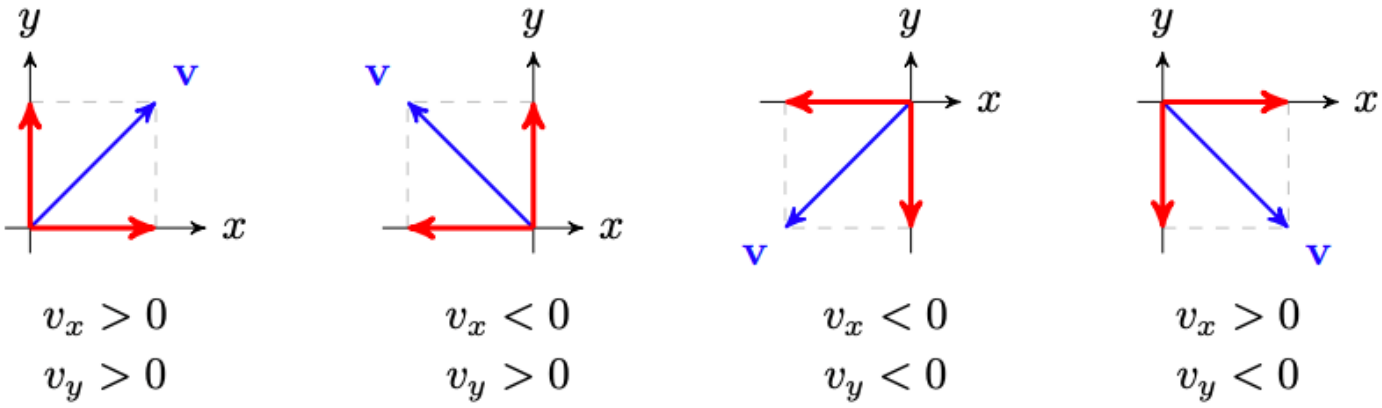


Figure 8.5.2

Again, for clarification, $\vec{v}_x$ is the red horizontal *vector* in each of the images in Figure 8.5.2. At the same time, v_x is a *scalar* representing the signed length of $\vec{v}_x$. A similar statement holds for the vertical *vector* $\vec{v}_y$.

Example 8.5.1

A plane flies 300 miles per hour at a heading of 300°. Find the x- and y-components of its velocity.

Solution available at the end of this section.

Checkpoint 8.5.1

The wind blows 50 kilometers per hour at a heading of 100°. Find the x- and y-components of its velocity.

Answer available at the end of this section.

We now know how to resolve a vector into horizontal and vertical vector components. Thus, every vector can be expressed as the sum of a horizontal vector and a vertical vector.

Example 8.5.2

A vector $\vec{w}$ has magnitude 4 and direction $\theta = 29^\circ$, where θ is measured counterclockwise from the positive x-axis. Express $\vec{w}$ as the sum of a horizontal vector, $\vec{w}_x$, and a vertical vector, $\vec{w}_y$.

Solution available at the end of this section.

Checkpoint 8.5.2

A vector $\vec{u}$ has magnitude 2 and direction $\theta = 116^\circ$, where θ is in standard position. Express $\vec{u}$ as the sum of a horizontal vector, $\vec{u}_x$, and a vertical vector, $\vec{u}_y$.

Answer available at the end of this section.

Using Components

We can describe a vector completely using either magnitude and direction, or components. Many calculations with vectors are more straightforward when we use components. For example, to add two vectors using components, we don't need the Law of Sines and the Law of Cosines. We resolve each vector into its horizontal and vertical components, add the corresponding components, and then compute the magnitude and direction of the resultant vector.

Before we get there, however, let's introduce a couple of theorems that should make sense to you.

Theorem: Vector Component Formula

$$\vec{v}_x = \norm{\vec{v}}\cos(\theta) \quad \text{and} \quad \vec{v}_y = \norm{\vec{v}}\sin(\theta).$$

This naturally leads to the next theorem.

Theorem: Vector Magnitude Formula

$$\norm{\vec{v}} = \sqrt{(v_x)^2 + (v_y)^2} \quad \text{and} \quad \tan(\theta) = \frac{v_y}{v_x}.$$

To calculate magnitude and direction from the components, we need only Right Triangle Trigonometry.

Example 8.5.3

After flying for some time, the airplane in the previous example encounters a steady wind blowing at 40 miles per hour at a heading of 100°. What are the actual speed and heading of the airplane relative to the ground?

Solution available at the end of this section.

Caution: Use Reference Angles and Quadrants

In Example 8.5.3, the airplane's heading is not $\tan^{-1}\left(-\frac{143.02}{220.42}\right) \approx -32.98^\circ$. Remember that there are always two angles with a given tangent. This is why we only consider reference angles when computing the values of inverse trigonometric functions. We can refer to a sketch of the vector or to the signs of its components to decide which of those two angles is appropriate.

Checkpoint 8.5.3

A plane is flying in a wind blowing 50 kilometers per hour in a direction 10° south of due west. The plane has an airspeed of 200kph and heading 142°. What is the ground speed and actual direction of the plane?

Answer available at the end of this section.

Unit Vectors

It is often helpful to describe a vector by giving its horizontal and vertical vector components instead of its magnitude and direction. To make the notation easier, we give names to the vectors of length 1 that point in the x- and y-directions.

Definition: Unit Vector

A vector of magnitude 1 is called a **unit vector**.

We can have unit vectors in any direction, but the unit vector in the x-direction is traditionally denoted by $\vec{i}$, and the unit vector in the y-direction is represented by $\vec{j}$, as shown in Figure 8.5.6.

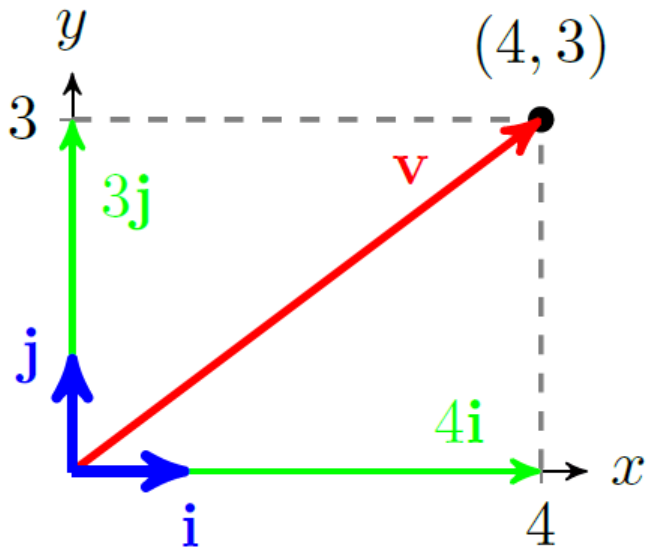


Figure 8.5.6

By taking scalar multiples of $\vec{i}$ and $\vec{j}$, we can describe any vector that lies in the directions of the coordinate axes. For example, $4\vec{i}$ represents the vector of magnitude 4 pointing in the x-direction, and $3\vec{j}$ represents the vector of magnitude 3 pointing in the y-direction. By adding multiples of $\vec{i}$ and $\vec{j}$, we can represent any vector we like. The vector $\vec{v} = 4\vec{i} + 3\vec{j}$ is shown in Figure 8.5.6.

Because the components of the vector are chosen to align with the coordinate system, we call this the vector's **coordinate form**.

Definition: Coordinate Form (of a vector)

The vector $\vec{v}$ is said to be wirtten in **coordinate form** when it is written as

$$\vec{v} = v_x\vec{i} + v_y\vec{j},$$

where v_x and v_y are the horizontal and vertical components of $\vec{v}$, respectively.

The coordinate form of the vector $\vec{w}$ in the previous example is $\vec{w} = 3.50\vec{i} + 1.94\vec{j}$.

Example 8.5.4

State the coordinate form of the vector shown below.

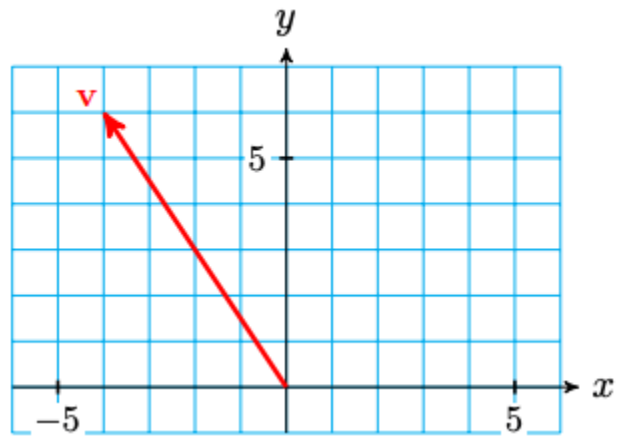


Figure 8.5.7

Solution available at the end of this section.

Checkpoint 8.5.4

State the coordinate form of the vector shown below.

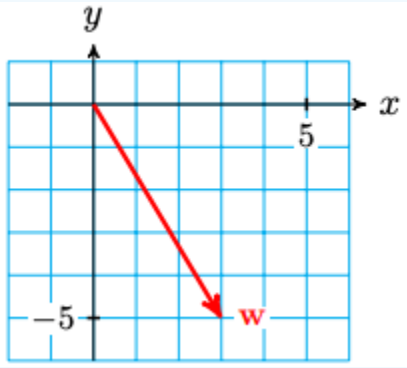


Figure 8.5.8

Answer available at the end of this section.

Converting Between Geometric and Coordinate Form

Finding the magnitude and direction of a vector given in coordinate form is a simple matter. The vector $\vec{v} = 4\vec{i} + 3\vec{j}$ has magnitude

$$\norm\vec{v} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

and direction $\theta = \tan^{-1}\left(\frac{3}{4}\right) \approx 36.9°$. Thus, we can readily convert vectors from geometric form to coordinate form or vice versa.

Theorem: Conversion Between Geometric and Coordinate Forms

Given the vector $\vec{v}$ with direction θ (in standard position) and coordinate form $\vec{v} = v_x\ \vec{i} + v_y\ \vec{j}$, then

$$v_x = \norm\vec{v}\cos(\theta)$$

and

$$v_y = \norm\vec{v}\sin(\theta)$$

where $\norm\vec{v} = \sqrt{v_x^2 + v_y^2}$ and $\tan(\theta) = \frac{v_y}{v_x}$.

Example 8.5.5

a. Find the geometric form of the vector $\vec{w} = -3\vec{i} - 2\vec{j}$.
b. Find the coordinate form of the vector whose magnitude is 5 and whose direction is the angle $\theta = 135°$.

Solutions available at the end of this section.

Checkpoint 8.5.5

Find the geometric form of the vector $\vec{F} = -6\vec{i} + 8\vec{j}$.

Answer available at the end of this section.

Scalar Multiples of Vectors in Coordinate Form

Scalar multiplication is easy to compute in coordinate form. This is because the sides of similar triangles are proportional.

Figure 8.5.9 shows the vectors $\vec{u} = 3\vec{i} + 2\vec{j}$ and $\vec{v} = 2\vec{u}$. The vector $\vec{v}$ is twice as long as $\vec{u}$ and points in the same direction as $\vec{u}$. You can see that each component of $\vec{v}$ is twice the corresponding component of $\vec{u}$, so that $\vec{v} = 6\vec{i} + 4\vec{j}$.

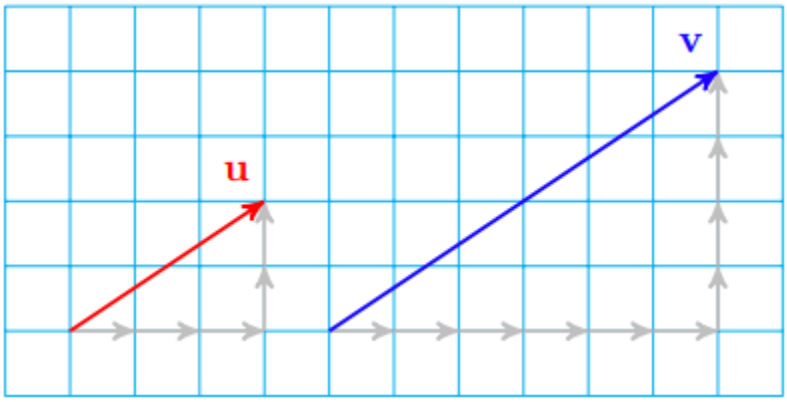


Figure 8.5.9

Theorem: Scalar Multiplication (coordinate form)

If $\vec{v} = v_x\ \vec{i} + v_y\ \vec{j}$ and k is a scalar, then

$$k\vec{v} = kv_x\vec{i} + kv_y\vec{j}.$$

In other words, to find a scalar multiple of a vector in coordinate form, we multiply each component by the scalar.

Example 8.5.6

The vector $\vec{v}$ has coordinate form $\vec{v} = 4\vec{i} - 7\vec{j}$. Find the coordinate form for the vector $\vec{w} = -\frac{1}{2}\vec{v}$.

Solution available at the end of this section.

Checkpoint 8.5.6

Find the coordinate form for the vector $\vec{u} = 3\vec{v}$, where $\vec{v} = -12\vec{i} + 15\vec{j}$.

Answer available at the end of this section.

If we divide a non-zero vector $\vec{v}$ by its length, we create a unit vector that points in the same direction as $\vec{v}$.

Definition: Normalizing (a vector)

The process of dividing a vector by its magnitude is called **normalizing**.

Normalizing vectors becomes incredibly important in Linear Algebra. Dividing a vector by a scalar k is the same as multiplying the vector by $\frac{1}{k}$.

Example 8.5.7

Find a unit vector pointing in the same direction as $\vec{v} = 2\vec{i} - 4\vec{j}$.

Solution available at the end of this section.

By computing its length, you can check that the vector $\vec{u}$ found in Example 8.5.7 is a unit vector.

$$\norm{\vec{u}} = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(-\frac{2}{\sqrt{5}}\right)^2} = \sqrt{\frac{1}{5} + \frac{4}{5}} = \sqrt{1} = 1.$$

Once we have a unit vector, $\vec{u}$, that points in a given direction, we can create a vector of any length in that direction, simply by scaling $\vec{u}$ by the length we desire. For example, the vector of length 10 pointing in the same direction as $\vec{v}$ in the previous example is

$$\begin{aligned} 10\vec{u} &= 10 \cdot \left(\frac{1}{\sqrt{5}}\vec{i} + \frac{-2}{\sqrt{5}}\vec{j}\right) \\ &= 10 \cdot \frac{1}{\sqrt{5}}\vec{i} + 10 \cdot \frac{-2}{\sqrt{5}}\vec{j} \\ &= 2\sqrt{5}\vec{i} - 4\sqrt{5}\vec{j} \end{aligned}$$

Theorem: Unit Vector Scaling

The unit vector $\vec{u}$ in the direction of $\vec{v}$ is given by $\vec{u} = \frac{1}{\norm{\vec{v}}}\vec{v}$.

A vector $\vec{w}$ of length k in the direction of $\vec{v}$ is given by $\vec{w} = \frac{k}{\norm{\vec{v}}}\vec{v}$.

Checkpoint 8.5.7

Find a unit vector $\vec{u}$ and a vector $\vec{w}$ of length 10 pointing in the same direction as $\vec{v} = -\vec{i} + 3\vec{j}$.

Answer available at the end of this section.

Adding Vectors in Coordinate Form

It is also easy to add vectors in coordinate form. Figure 8.5.10 shows the sum of $\vec{u} = \vec{i} + 2\vec{j}$ and $\vec{v} = 5\vec{i} + 3\vec{j}$. Remember that we add two vectors by following the first vector by the second.

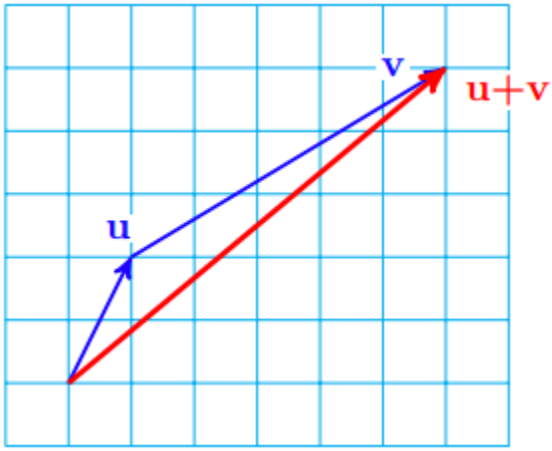


Figure 8.5.10

The vector sum describes the following path: 1 unit in the x-direction, then 2 units in the y-direction, then 5 units in the x-direction, then 3 units in the y-direction.

However, we arrive at the same endpoint by traveling 1 + 5 = 6 units in the x-direction, then 2 + 3 = 5 units in the y-direction. The resultant vector has components that are just the sums of the components of the vectors $\vec{u}$ and $\vec{v}$. To add two vectors in coordinate form, we add the corresponding components.

Theorem: Vector Addition (coordinate form)

If $\vec{u} = a\vec{i} + b\vec{j}$ and $\vec{v} = c\vec{i} + d\vec{j}$, then

$$\vec{u} + \vec{v} = (a + c)\vec{i} + (b + d)\vec{j}\,.$$

Example 8.5.8

Mike flew 15 miles southwest, followed by 20 miles on a heading of 260°. What is his current position relative to his starting point?

Solution available at the end of this section.

Checkpoint 8.5.8

Min's yacht moves at a heading of 350° at 20 kilometers per hour relative to the water. However, the water moves at a heading of 70° at 4 kilometers per hour. What is Min's velocity relative to land?

Answer available at the end of this section.

The coordinate form is especially efficient if we want to add more than two vectors.

Example 8.5.9

While exploring an abandoned cottage in the woods, you discover an old map showing the location of a buried treasure. According to the map, from the cottage, go 10 km bearing $\mathrm{N}\ 20^\circ\ \mathrm{E}$. From that point, go 12.8 km bearing $\mathrm{S}\ 78^\circ\ \mathrm{E}$. Finally, go 9.5 km bearing $\mathrm{N}\ 39^\circ\ \mathrm{W}$.

You wonder if the directions will lead you safely to the treasure. You draw vectors to illustrate the path described by the directions; however, instead of following the map directions, you plan to fly by helicopter from the abandoned cottage directly to the treasure. You will need to know the correct bearing and distance to fly. Draw the vector for the net displacement from the cottage to the treasure and give its length and direction.

Solution available at the end of this section.

Checkpoint 8.5.9

From your campsite you hike 1.6 km heading 175°, then 0.8 km at the heading of 65°, and finally 1.2 km in the heading 350°.

a. Draw a diagram for your hike, using vectors to represent each of the three segments. Use a grid in which 1 square represents 0.1 km.
b. Resolve each vector into components, and determine your location after your hike.

Answers available at the end of this section.

Force

When you push or pull on something, you exert a **force** on the object. For example, an object's weight is actually a force, the result of gravity pulling the object towards the earth. A force has magnitude (measured in pounds) and direction, so it is a vector quantity. A force applied to an object causes the object to accelerate in the direction of the force.

When two or more forces act simultaneously on an object, the object moves as if the sum of the individual force vectors acted on it. The sum of all the forces acting on an object is the **resultant force**.

Example 8.5.10

As shown below, Ivan and Wyatt are trying to move a refrigerator by pulling on it with forces of 150 pounds and 120 pounds, respectively. $\vec{A}$ represents the force with which Ivan is pulling, and $\vec{B}$ represents the force with which Wyatt is pulling. What is the resulting force on the refrigerator, and in what direction will it move?

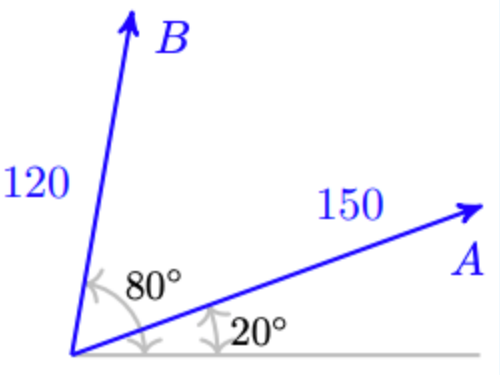


Figure 8.5.13

Solution available at the end of this section.

Checkpoint 8.5.10

Two tugboats pull on a barge in the river with forces $\vec{u} = 20\vec{i} + 8\vec{j}$ and $\vec{v} = 28\vec{i} - 6\vec{j}$, measured in thousands of pounds. In what direction will the barge move, and what is the magnitude of the force propelling it?

Answer available at the end of this section.

If vectors $\vec{u}$ and $\vec{v}$ have the same magnitude but opposite directions, then $\vec{u} + \vec{v}$ has zero magnitude and is called the **zero vector**.

Definition: Zero Vector

A **zero vector**, also known as a **null vector**, is a vector with a magnitude of zero and an undefined direction. It is denoted $\vec{0}$.

A zero displacement vector means that the object started and ended in the same place; a zero velocity vector means that the object is not moving.

The vector that has the same magnitude as $\vec{v}$ but the opposite direction is called the opposite of $\vec{v}$ and denoted by $-\vec{v}$. Then

$$\vec{v} + (-\vec{v}) = \vec{0}.$$

We now have enough language to tackle some of the most interesting (and important) applications of vectors—those involving **static equilibrium**.

Definition: Static Equilibrium

When the sum of the forces (as vectors) acting on a particle or object is $\vec{0}$, we say the object is in **static equilibrium** (or simply **equilibrium**).

Example 8.5.11

Recall the refrigerator in the previous example. Jorge does not want Ivan and Wyatt to move it. How hard must Jorge pull so that it remains motionless?

Solution available at the end of this section.

Checkpoint $8.5.11$

Suppose that $\vec{u}+\vec{v}+\vec{w}=\vec{0},$ and that $\vec{u}=-8\vec{i}+32\vec{j}$ and $\vec{v}=14\vec{i}-26\vec{j}$. What is the coordinate form of vector $\vec{w}$?

Answer available at the end of this section.

Success in College: Groupwork Learning Preferences

Most students are very intimidated by working in groups. It usually takes quite a while for students to realize that studying together can be very rewarding.

Think about the following questions regarding learning or practicing complicated math.

- Do you prefer to do homework by yourself, or do you like to do homework with others?
- Do you like to do homework at school, or do you like to do homework at home?
- If you need help, do you prefer to get it from friends or a spouse, or do you prefer to get it from your professor or a tutor?
- Do you concentrate on learning what you don't know by yourself, or do you prefer to help others because it helps you learn it better?

1. Do you work in conditions that match these preferences when you do your math homework? Why or why not?
2. Spending time with old and new friends is an integral part of college. However, studying with friends is not always practical. Do you ever feel peer pressure to study in ways that are not the best for you?

Answers

Section Preview

Now that students have (hopefully) a decent grasp of vectors from a geometric perspective, this section introduces vectors from an algebraic perspective. Topics include the component form of a vector, unit vectors, formulas for computing vector components and vector magnitude, normalizing a vector, and static equilibrium. In all, students might feel like this section is a repeat of the previous section; however, it will feel a bit "crunchier" because of the computations involved.

Section Preview

- **Simplifying Expressions**
  - Adding and Subtracting Polynomials: Adding vectors component-wise ($\langle a_1+a_2, b_1+b_2\rangle$) mirrors combining like terms in polynomial addition.
- **Linear Systems**
  - Solving 2×2 Systems of Linear Equations — Substitution Method: The static equilibrium examples set up a system of two equations (sum of x-components $=0$, sum of y-components $=0$) and solve simultaneously.

Section Preview

- Express a vector in coordinate form.
- Convert a vector between geometric and coordinate forms.
- Find the unit vector in the same direction as a given vector.
- Scale a vector.
- Perform arithmetic on vectors.
- Use vectors in force and static equilibrium problems.

Example $8.5.1$

We draw a triangle showing the plane's velocity, $\vec{v}$, as the sum of components in the x- and y-directions.

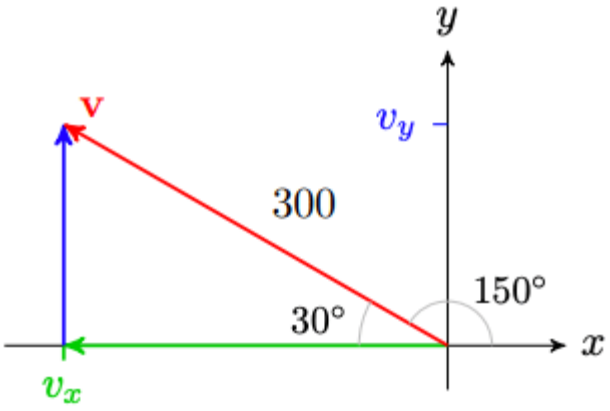


Figure 8.5.3

Since the *heading* is 300°, $\vec{v}$ makes an angle of 30° with the horizontal. The angle θ, on the other hand, is 150° because θ is *always* measured from the positive x -axis. Therefore,

$$\begin{aligned} v_x &= 300\cos(150^\circ) \approx -259.81 \\ &\text{and} \\ v_y &= 300\sin(150^\circ) = 150 \end{aligned}$$

Checkpoint 8.5.1

$v_x \approx 49.2$ kph and $v_y \approx 8.7$ kph

Example 8.5.2

The components of $\vec{w}$ are given by

$$\begin{aligned} w_x &= \norm\vec{w}\cos(\theta) \\ &= 4\cos(29^\circ) \\ &\approx 3.50 \\ &\text{and} \\ w_y &= \norm\vec{w}\sin(\theta) \\ &= 4\sin(29^\circ) \\ &\approx 1.94 \end{aligned}$$

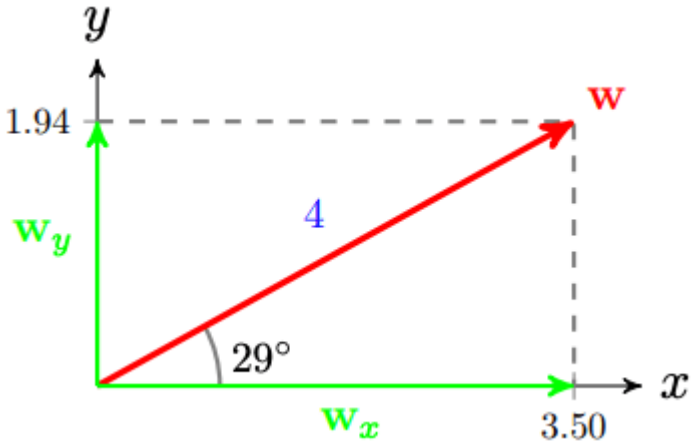


Figure 8.5.4

Then $\vec{w} = \vec{w}_x + \vec{w}_y$, where $\vec{w}_x$ is a horizontal vector of magnitude 3.50, and $\vec{w}_y$ is a vertical vector of magnitude 1.94. See the figure above.

Checkpoint 8.5.2

$\vec{u} = \vec{u}_x + \vec{u}_y$, where $\norm\vec{u}_x = -1.943, \norm\vec{u}_y = 0.473$

Example 8.5.3

We want to add the vectors $\vec{v}$, representing the plane's intended velocity, and $\vec{u}$, representing the velocity of the wind. We first resolve $\vec{u}$ into its components.

$$\begin{aligned} u_x &= 40\cos(-10^\circ) \approx 39.39 \\ &\text{and} \\ u_y &= 40\sin(-10^\circ) \approx -6.95 \end{aligned}$$

We find the components of the resultant, $\vec{w}$, by adding the components of $\vec{u}$ and $\vec{v}$, as shown below.

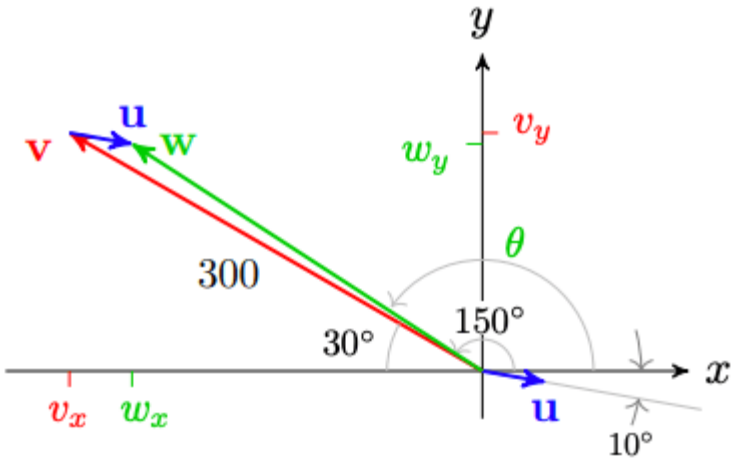


Figure 8.5.5

That is,

$$w_x = u_x + v_x \approx 39.39 - 259.81 = -220.42$$

and

$$w_y = u_y + v_y \approx -6.95 + 150 = 143.05$$

Finally, we compute the magnitude and direction of the resultant vector.

$$\norm{\vec{w}}^2 \approx (-220.49)^2 + (143.05)^2 \approx 69{,}079.14$$

Therefore, $\norm{\vec{w}} \approx 262.83$. So, the airplane's ground speed is approximately 262.83 miles per hour.

To find the plane's true *heading* we compute

$$\tan(\theta) \approx \frac{143.05}{-220.42} \implies \hat{\theta} \approx \tan^{-1}\left(\frac{143.05}{220.42}\right).$$

We don't need to compute the value of θ because we are going to state the true *heading* of the plane. We only need to know that θ is a second-quadrant angle. We will use the reference angle, $\hat{\theta}$, to get the heading.

Thus, the true heading of the plane is $270^\circ + \hat{\theta} \approx 302.98^\circ$.

Checkpoint 8.5.3

groundspeed $182\text{kph}, 66^\circ\text{S of E}$

Example 8.5.4

From its base to its tip, the vector extends 4 units in the negative $\vec{i}$ direction and 6 units in the $\vec{j}$ direction. Thus, $\vec{v} = -4\vec{i} + 6\vec{j}$.

Checkpoint 8.5.4

$\vec{w} = 3\vec{i} - 5\vec{j}$

Example 8.5.5

a. The magnitude of $\vec{w}$ is $\norm{\vec{w}} = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$. The direction satisfies the equation $\tan(\theta) = \frac{-2}{-3} = \frac{2}{3}$. Because the point $(-3, -2)$ lies in the third quadrant, we know that θ must be a third quadrant angle. Thus, $\theta = \arctan\left(\frac{2}{3}\right) + 180^\circ \approx 213.69^\circ$.

b. The coordinates of the vector are given by

$$a = 5\cos(135^\circ) = -\frac{5}{\sqrt{2}}, \quad b = 5\sin(135^\circ) = \frac{5}{\sqrt{2}},$$

so the coordinate form is $\vec{w} = -\frac{5}{\sqrt{2}}\vec{i} + \frac{5}{\sqrt{2}}\vec{j}$.

Checkpoint 8.5.5

magnitude 10, direction 127°

Example 8.5.6

We multiply each component of $\vec{v}$ by $-\frac{1}{2}$ to get $\vec{w} = -2\vec{i} + \frac{7}{2}\vec{j}$.

Checkpoint 8.5.6

$\vec{u} = -36\vec{i} + 45\vec{j}$

Example 8.5.7

We first compute the magnitude of $\vec{v}$.

$$\norm{\vec{v}} = \sqrt{2^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}.$$

Next we multiply $\vec{v}$ by the reciprocal of its length. The unit vector in the same direction as $\vec{v}$ is then

$$\begin{aligned}\vec{u} &= \frac{1}{\norm\vec{v}}\vec{v}\\ &= \frac{1}{2\sqrt{5}}(2\vec{i}-4\vec{j})\\ &= \frac{1}{2\sqrt{5}}\cdot 2\vec{i}+\frac{1}{2\sqrt{5}}\cdot(-4\vec{j})\\ &= \frac{1}{\sqrt{5}}\vec{i}-\frac{2}{\sqrt{5}}\vec{j}\end{aligned}$$

Checkpoint 8.5.7

$\vec{u}=-\frac{1}{\sqrt{10}}\vec{i}+\frac{3}{\sqrt{10}}\vec{j}, \quad \vec{w}=-\sqrt{10}\vec{i}+3\sqrt{10}\vec{j}$

Example 8.5.8

A sketch of Mike's journey is shown below. We begin by converting the vector for each leg of his journey into coordinate form.

$$\begin{aligned}v_x &= 15\cos(225^\circ)\approx -10.607\\ v_y &= 15\sin(225^\circ)\approx -10.607\\ w_x &= 20\cos(190^\circ)\approx -19.696\\ w_y &= 20\sin(190^\circ)\approx -3.473\end{aligned}$$

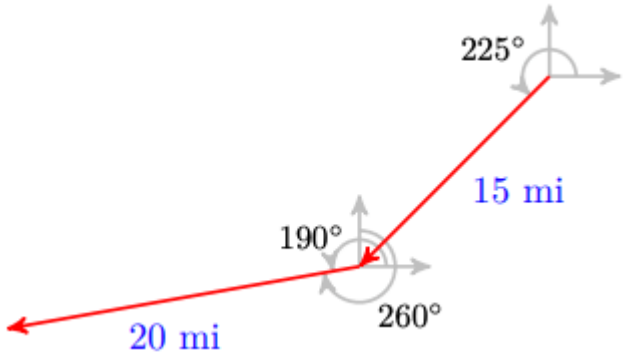


Figure 8.5.11

Thus,

$$\vec{v}=-10.607\vec{i}-10.607\vec{j} \quad \text{and} \quad \vec{w}=-19.696\vec{i}-3.473\vec{j}\,.$$

We find the resultant vector, $\vec{r}$, by adding $\vec{v}$ and $\vec{w}$.

$$\begin{aligned}\vec{r} &\approx (-10.607-19.696)\vec{i}+(-10.607-3.473)\vec{j}\\ &= -30.303\vec{i}-14.08\vec{j}\end{aligned}$$

Mike's position vector is $\vec{r}=30.303\vec{i}-14.08\vec{j}$, or about 30.3 miles west and 14.1 miles south of his starting position.

Checkpoint 8.5.8

22.7 kph due north

Example 8.5.9

The helicopter's flight plan is the vector sum of the three vectors $\vec{v}_1$ (for the first part of the journey), $\vec{v}_2$ (for the second part), and $\vec{v}_3$ (for the third part). We resolve each of the three vectors into its components to find their sum. We start by finding the angles each vector makes with the positive x-axis. From the beginning to the end of the journey, these angles are $\theta_1=70^\circ$, $\theta_2=-12^\circ$, and $\theta_3=129^\circ$ (You should verify this.). In coordinate form, the three displacement vectors are

$$\begin{aligned}\vec{v}_1 &= (10\cos(70^\circ))\,\vec{i}+(10\sin(70^\circ))\,\vec{j} &&\approx 3.42\vec{i}+9.40\vec{j}\\ \vec{v}_2 &= (12.8\cos(-12^\circ))\,\vec{i}+(12.8\sin(-12^\circ))\,\vec{j} &&\approx 12.52\vec{i}-2.66\vec{j}\\ \vec{v}_3 &= (9.5\cos(129^\circ))\,\vec{i}+(9.5\sin(129^\circ))\,\vec{j} &&\approx -5.98\vec{i}+7.38\vec{j}\end{aligned}$$

The net displacement is given by the resultant vector, $\vec{r}$. We add the corresponding components of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$\begin{aligned}\vec{r} &\approx (3.42\vec{i}+9.40\vec{j})+(12.52\vec{i}-2.66\vec{j})+(-5.98\vec{i}+7.38\vec{j})\\ &= (3.42+12.52-5.98)\vec{i}+(9.40-2.66+7.38)\vec{j}\\ &= 10\vec{i}+14.12\vec{j}\end{aligned}$$

The treasure is at the point 10 km east and 14.12 km north of the cottage. To find the flight plan for the helicopter, we compute the magnitude and direction of the vector $\vec{r}$.

$$\begin{aligned} \norm\vec{r} &= \sqrt{10^2 + 14.12^2} = 17.3 \\ &\text{and} \\ \tan(\theta) &= \frac{14.12}{10} \\ \implies \hat{\theta} &= \tan^{-1}\left(\frac{14.12}{10}\right) \\ &\approx 54.69^\circ \end{aligned}$$

However, since we know the treasure is to the east and north, we know $0^\circ < \theta < 90^\circ$. Hence, $\theta \approx 54.7^\circ$. This means the treasure is 17.3 km from the cottage at a bearing of $\text{N } 35.3^\circ \text{ E}$.

Checkpoint 8.5.9

a.

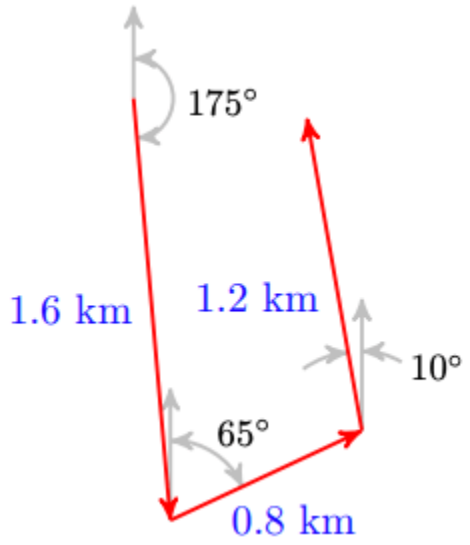


Figure 8.5.12

b. $-1.594\vec{i} + 0.139\vec{j}, 0.338\vec{i} + 0.725\vec{j}, 1.182\vec{i} - 0.208\vec{j}, \quad -0.074\vec{i} + 0.656\vec{j}$

Example 8.5.10

We write each force in its coordinate form.

$$\begin{aligned} \vec{A} &= 150\cos(20^\circ)\,\vec{i} + 150\sin(20^\circ)\,\vec{j} \approx 140.95\vec{i} + 51.30\vec{j} \\ \vec{B} &= 120\cos(80^\circ)\,\vec{i} + 120\sin(80^\circ)\,\vec{j} \approx 20.84\vec{i} + 118.18\vec{j} \end{aligned}$$

Next, we add the forces applied by Ivan and Wyatt to find the resultant force, $\vec{r} = \vec{A} + \vec{B}$.

$$\begin{aligned} \vec{r} &\approx (140.95\vec{i} + 51.30\vec{j}) + (20.84\vec{i} + 118.18\vec{j}) \\ &= 161.75\vec{i} + 169.48\vec{j} \end{aligned}$$

The magnitude of the resultant force is $\norm\vec{r} \approx \sqrt{161.75^2 + 169.48^2} \approx 234.28$ and the direction of the force is given by

$$\tan^{-1}\left(\frac{169.48}{161.75}\right) = 46.33^\circ.$$

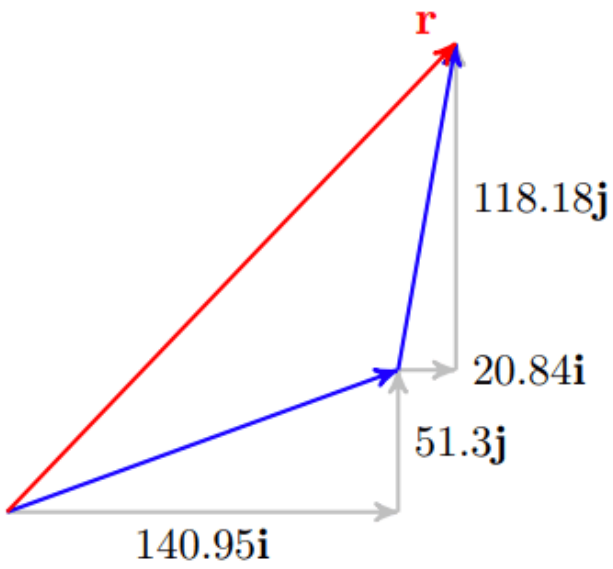


Figure 8.5.14

Therefore, the force exerted on the refrigerator is about 234.3 pounds, and it will move at an angle of approximately 46.33° from the horizontal.

Checkpoint 8.5.10

2.4°, 48.04 thousand lbs

Example 8.5.11

Let $\vec{C}$ be the force that Jorge applies. It must be equal in magnitude to the sum of the forces applied by Ivan and Wyatt but point in the opposite direction. That is,

$$\vec{C} = -\vec{r} = -(\vec{A} + \vec{B}).$$

So, Jorge must pull with 234.3 pounds in the direction $180^\circ + 46.33^\circ = 226.33^\circ$.

Checkpoint 8.5.11

$\vec{w} = -6\vec{i} - 6\vec{j}$

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8.5.1: Resources and Key Concepts

Resources

Videos

- Algebraic Vectors
  - Vectors - An Algebraic Approach
- Vector Arithmetic
  - Vectors - Vector Arithmetic and Properties of Vectors
- Component Form and Unit Vectors
  - Vectors - Unit Vectors and Component Form
- Applications
  - Vectors - Applications Involving Vectors

Key Concepts

Definitions

- **Vector Components:** A vector $\vec{v}$ can be broken down into a horizontal vector component, $\vec{v}_x$, and a vertical vector component, $\vec{v}_y$, such that $\vec{v} = \vec{v}_x + \vec{v}_y$. The scalar values v_x and v_y are called the components of the vector.
- **Unit Vector:** A vector with a magnitude of 1. The unit vectors in the horizontal and vertical directions are denoted by $\vec{i}$ and $\vec{j}$, respectively.
- **Coordinate Form (of a vector):** A vector $\vec{v}$ is in coordinate form when it is written as the sum of its scalar components multiplied by the standard unit vectors: $\vec{v} = v_x\vec{i} + v_y\vec{j}$.
- **Normalizing (a vector):** The process of creating a unit vector in the same direction as a given non-zero vector by dividing the vector by its magnitude.
- **Zero Vector:** A vector with a magnitude of zero and an undefined direction, denoted $\vec{0}$. It is also known as a null vector.
- **Static Equilibrium:** A state in which an object or particle is motionless because the sum of all forces acting on it is the zero vector.

Theorems

- **Components:** If θ is the angle measured counter-clockwise from the positive x-axis to the vector $\vec{v}$, then the scalar components of the vector are given by $v_x = ||\vec{v}||\cos(\theta)$ and $v_y = ||\vec{v}||\sin(\theta)$.
- **Vector Component Formula:** The vector components of $\vec{v}$ are $\vec{v}_x = ||\vec{v}||\cos(\theta)\vec{i}$ and $\vec{v}_y = ||\vec{v}||\sin(\theta)\vec{j}$.
- **Vector Magnitude Formula:** For a vector $\vec{v} = v_x\vec{i} + v_y\vec{j}$, the magnitude is $||\vec{v}|| = \sqrt{v_x^2 + v_y^2}$ and the direction angle θ satisfies $\tan(\theta) = \frac{v_y}{v_x}$.
- **Conversion Between Geometric and Coordinate Forms:** The formulas for components and magnitude/direction allow for conversion between the geometric representation (magnitude and direction) and the coordinate form of a vector.
- **Scalar Multiplication (coordinate form):** If $\vec{v} = v_x\vec{i} + v_y\vec{j}$ and k is a scalar, then $k\vec{v} = (kv_x)\vec{i} + (kv_y)\vec{j}$.
- **Unit Vector Scaling:** The unit vector $\vec{u}$ in the direction of $\vec{v}$ is given by $\vec{u} = \frac{1}{||\vec{v}||}\vec{v}$. A vector $\vec{w}$ of length k in the direction of $\vec{v}$ is given by $\vec{w} = k\vec{u}$.
- **Vector Addition (coordinate form):** If $\vec{u} = a\vec{i} + b\vec{j}$ and $\vec{v} = c\vec{i} + d\vec{j}$, then $\vec{u} + \vec{v} = (a + c)\vec{i} + (b + d)\vec{j}$.

Common Mistakes

- **Confusing Vector Components and Components:** The vector components ($\vec{v}_x, \vec{v}_y$) are vectors, while the components (v_x, v_y) are scalars that represent the signed lengths of the vector components.
- **Using Heading as the Standard Angle θ:** When finding components, the angle θ must be the angle in standard position

(measured counter-clockwise from the positive x-axis), not necessarily the heading or bearing angle given in a navigation problem.

- **Incorrectly Determining the Quadrant for Direction:** When finding the direction angle θ from components using $\arctan(\frac{v_y}{v_x})$, you must use the signs of v_x and v_y to determine the correct quadrant for θ, as the arctangent function only returns angles in Quadrants I or IV.

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8.5.2: Homework

Reading Questions

1. What are the two vector components of a two-dimensional vector $\vec{v}$?
2. If a vector $\vec{v}$ has magnitude $||\vec{v}||$ and direction angle θ, what are the formulas for its scalar components v_x and v_y?
3. What is a unit vector?
4. What is the process of "normalizing" a vector?
5. How do you write a vector $\vec{v}$ in coordinate form?
6. If you have a vector in coordinate form, $\vec{v} = v_x\vec{i} + v_y\vec{j}$, how do you find its magnitude?
7. How do you add two vectors when they are in coordinate form?
8. How do you perform scalar multiplication on a vector that is in coordinate form?
9. What does it mean for an object to be in static equilibrium?
10. What is the zero vector?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

1. State the sum of angles formula for sine.
2. State the sum of angles formula for cosine.
3. State the sum of angles formula for tangent.
4. Explain how the difference of angles formulas differ from the sum of angles formulas.

For the following exercises, let $\cos(\alpha) = -\frac{2}{\sqrt{5}}$, where $90^\circ < \alpha < 180^\circ$.

5. Find an exact value for $\cos(\alpha + 30^\circ)$.
6. Find an exact value for $\tan(\alpha - 30^\circ)$.

Answers available at the end of this section.

Homework

Vocabulary Check

1. A vector having a magnitude of 1 is called a ___ vector.
2. The ___ form of a vector expresses the vector in terms of $\vec{i}$ and $\vec{j}$.
3. Then ___ vector has magnitude 0.
4. $\vec{v}_x$ and $\vec{v}_y$ are called the ___ of the vector $\vec{v}$, while v_x and v_y are simply called the ___.

Concept Check

5. Is it easier to add vectors in geometric form or coordinate form? Why?
6. To find a unit vector in the direction of $\vec{v}$, multiply the coordinates of $\vec{v}$ by ___ .
7. To find a vector of length k in the direction of $\vec{v}$, multiply the coordinates of $\vec{v}$ by ___.
8. Name two physical quantities that are represented by vectors.
9. What are the components of a vector?
10. Does $\norm\vec{v} = \norm\vec{v}_x + \norm\vec{v}_y$?

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

11. Any vector can be written as the sum of its horizontal and vertical vector components, $\vec{v}_x$ and $\vec{v}_y$.
12. The component vectors of a vector $\vec{v}$ whose direction is given by the angle θ in standard position are the scalar quantities

$$\begin{aligned} \vec{v}_x &= \norm\vec{v}\sin(\theta) \\ \vec{v}_y &= \norm\vec{v}\cos(\theta) \end{aligned}$$

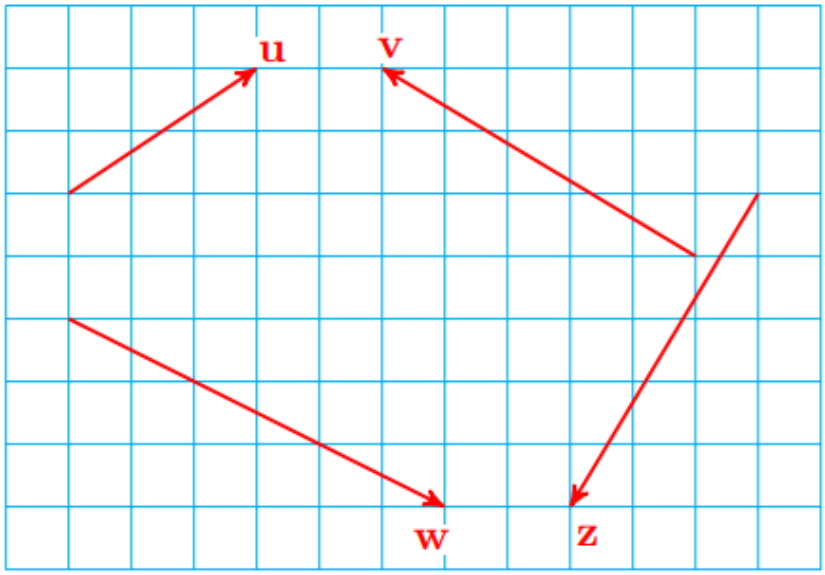
13. The magnitude and direction of a vector with components $\norm\vec{v}_x$ and $\norm\vec{v}_x$ are given by

$$\norm\vec{v} = \sqrt{(\norm\vec{v}_x)^2 + (\norm\vec{v}_y)^2} \quad \text{and} \quad \tan(\theta) = \frac{\norm\vec{v}_y}{\norm\vec{v}_x},$$

respectively.

Basic Skills

For the following exercises, use the coordinate form of each vector shown in the figure to answer the question.



14. a. $\norm\vec{u}$
    b. $2\vec{u}$
    c. $\norm2\vec{u}$

15. a. $\norm\vec{v}$

    b. $\frac{1}{2}\vec{v}$

    c. $\norm\frac{1}{2}\vec{v}$

16. a. $\norm\vec{w}$

    b. $-\vec{w}$

    c. $\norm-\vec{w}$

17. a. $\norm\vec{z}$

    b. $-3z$

    c. $\norm-3\vec{z}$

18. a. Calculate $\vec{u}+\vec{v}$ and $\norm\vec{u}+\vec{v}$.

    b. Which of the following statements is true?

        i. $\norm\vec{u}+\norm\vec{v}\leq\norm\vec{u}+\vec{v}$

        ii. $\norm\vec{u}+\norm\vec{v}=\norm\vec{u}+\vec{v}$

        iii. $\norm\vec{u}+\norm\vec{v}\geq\norm\vec{u}+\vec{v}$

19. a. Calculate $\vec{w}+\vec{z}$ and $\norm\vec{w}+\vec{z}$.

    b. Which of the following statements is true?

        i. $\norm\vec{w}+\norm\vec{z}\leq\norm\vec{w}+\vec{z}$

        ii. $\norm\vec{w}+\norm\vec{z}=\norm\vec{w}+\vec{z}$

        iii. $\norm\vec{w}+\norm\vec{z}\geq\norm\vec{w}+\vec{z}$

For the following exercises,
(a) Sketch the vector and give its coordinate form.
(b) Find the magnitude and direction of the vector.

20. The displacement vector from (1, –2) to (–4, 6).

21. The displacement vector from (–5, 2) to (4, 7).

22. The displacement vector from (–2, 9) to (–4, 8).

23. The displacement vector from (–6, 2) to (3, 0).

24. Camille is 12 meters east and 3 meters north of Roy. Lap is 6 meters east and 9 meters north of Camille.

    a. Calculate the displacement vector from Roy to Lap in coordinate form. Let $\vec{i}$ point east and $\vec{j}$ point north.

    b. Find the magnitude and direction of the displacement vector.

25. Ron and Sang are climbing a rock wall. Ron is 8 feet to the right and and 23 feet above their starting point. Sang is 2 feet to the right and 7 feet above Ron.

    a. Calculate the displacement vector from the starting point to Sang in coordinate form. Let $\vec{i}$ point right and $\vec{j}$ point up.

b. Find the magnitude and direction of the displacement vector.

For the following exercises, find the magnitude and direction of the vector.

26. $\vec{v} = -6\vec{i} + 6\vec{j}$

27. $\vec{p} = -12\vec{i} - 5\vec{j}$

28. $\vec{w} = 7\sqrt{3}\vec{i} - 7\vec{j}$

29. $z = -6\sqrt{2}\vec{i} + 6\sqrt{6}\vec{j}$

30. $\vec{q} = 52\vec{i} + 96\vec{j}$

31. $\vec{s} = 3.2\vec{i} - 1.8\vec{j}$

For the following exercises, find the coordinate form of the vector.

32. $\norm\vec{v} = 6, \quad \theta = -45^\circ$

33. $\norm\vec{v} = 200, \quad \theta = 240^\circ$

34. $\norm\vec{v} = 8.3, \quad \theta = 37^\circ$

35. $\norm\vec{v} = 23, \quad \theta = 200^\circ$

For the following exercises, sketch each vector and its components. Use the coordinate form to find the resultant vector $\vec{u} + \vec{v}$, and sketch it.

36. $\vec{u} = -3\vec{i} + 2\vec{j}, \quad \vec{v} = 4\vec{i} - 4\vec{j}$

37. $\vec{u} = 5\vec{i} + \vec{j}, \quad \vec{v} = 2\vec{i} - 3\vec{j}$

38. $\vec{u} = -5\vec{i} - 2\vec{j}, \quad \vec{v} = \vec{i} + 6\vec{j}$

39. $\vec{u} = 8\vec{i} - 3\vec{j}, \quad \vec{v} = -4\vec{i} - 2\vec{j}$

For the following exercises, find the sum $\vec{u} + \vec{v}$ of the given vectors.

40. $\vec{u} = 13\vec{i} - 8\vec{j}, \quad \vec{v} = -1\vec{i} + 11\vec{j}$

41. $\vec{u} = 3.7\vec{i} + 2.6\vec{j}, \quad \vec{v} = -1.3\vec{i} - 5.7\vec{j}$

42. $\vec{u} = -3\vec{i} + 9\vec{j}, \quad \vec{v} = 5.8\vec{i} - 7.1\vec{j}$

43. $\vec{u} = 6\vec{i} - 8\vec{j}, \quad \vec{v} = 23\vec{i} + 42\vec{j}$

For the following exercises, find the coordinate form of the vector, where

$$\vec{u} = 2\vec{i} + 3\vec{j}, \quad \vec{v} = -5\vec{i} + 4\vec{j}, \quad \vec{w} = -2\vec{i} - 5\vec{j}, \quad \vec{z} = 8\vec{i} - 3\vec{j}$$

44. $\vec{u} + \vec{v}$

45. $\vec{w} - \vec{z}$

46. $4\vec{w}$

47. $-3\vec{v}$

48. $2\vec{z} - \vec{u}$

49. $-\vec{w}+5\vec{u}$

50. $3\vec{v}-\vec{w}+2\vec{u}$

51. $\vec{z}-2(\vec{v}+\vec{w})$

For the following exercises, find a unit vector $\vec{u}$ in the same direction as the given vector.

52. $\vec{r}=-12\vec{i}+5\vec{j}$

53. $\vec{s}=7\vec{i}-24\vec{j}$

54. $\vec{t}=\vec{i}-\vec{j}$

55. $\vec{w}=-2\vec{i}-3\vec{j}$

For the following exercises, find a vector $\vec{v}$ in the same direction as $\vec{w}$, but with the given length.

56. $\vec{w}=8\vec{i}+15\vec{j}, \quad \norm\vec{v}=51$

57. $\vec{w}=-20\vec{i}-21\vec{j}, \quad \norm\vec{v}=58$

58. $\vec{w}=-3\vec{i}+\vec{j}, \quad \norm\vec{v}=4$

59. $\vec{w}=\vec{i}-2\vec{j}, \quad \norm\vec{v}=7$

For the following exercises,
(a) Draw a diagram using arrows to represent the vectors.
(b) Convert each vector to coordinate form.
(c) Use the coordinate form to add or subtract the vectors.

60. Find $\vec{u}+\vec{v}$, where $\vec{u}$ has magnitude 2.6 and direction $\theta=23^\circ, \vec{v}$ has magnitude 5.8 and direction $\theta=223^\circ$.

61. Find $\vec{u}+\vec{v}$, where $\vec{u}$ has magnitude 50 and direction $\theta=173^\circ, \vec{v}$ has magnitude 70 and direction $\theta=308^\circ$.

62. Find $\vec{u}-\vec{v}$, where $\vec{u}$ has magnitude 35 and direction $\theta=110^\circ, \vec{v}$ has magnitude 60 and direction $\theta=165^\circ$.

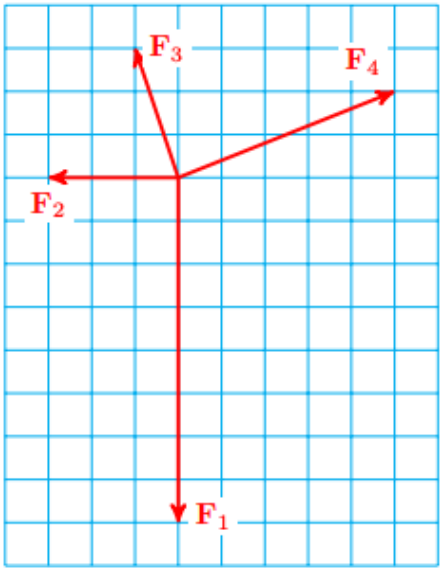
63. Find $\vec{u}-\vec{v}$, where $\vec{u}$ has magnitude 12.4 and direction $\theta=250^\circ, \vec{v}$ has magnitude 8.8 and direction $\theta=315^\circ$.

For the following exercises,
(a) Find the resultant force.
(b) Find the additional force needed for the system to be in equilibrium.

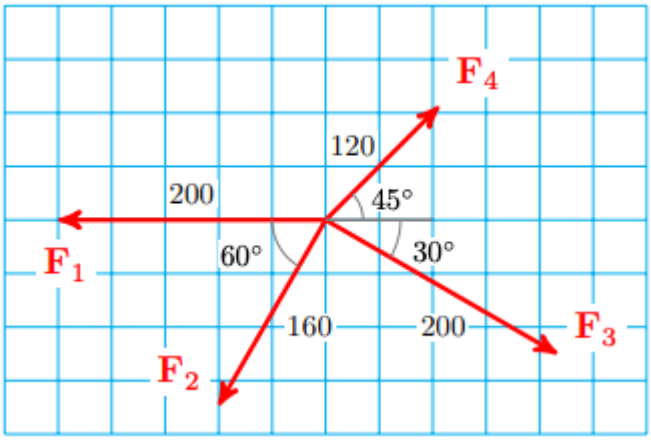
64. $\vec{F}_1=-3\vec{i}+\vec{j}, \quad \vec{F}_2=5\vec{i}-2\vec{j}, \quad \vec{F}_3=-6\vec{i}-4\vec{j}$

65. $\vec{F}_1=10\vec{i}+4\vec{j}, \quad \vec{F}_2=-12\vec{i}-9\vec{j}, \quad \vec{F}_3=-3\vec{i}+5\vec{j}$

56.



57.



58. a. Find the magnitude of the vector $\vec{v} = 6\vec{i} - 8\vec{j}$, and the magnitude of the vector $2\vec{v} = 12\vec{i} - 16\vec{j}$, and verify that $\norm{2\vec{v}} = 2\norm{\vec{v}}$.

    b. Let $\vec{v} = a\vec{i} + b\vec{j}$, and verify that $\norm{k\vec{v}} = k\norm{\vec{v}}$, for $k > 0$.

59. a. Find the magnitude of the vector $\vec{v} = 3\vec{i} + 5\vec{j}$, and verify that the vector $\frac{\vec{v}}{\norm{\vec{v}}}$ has magnitude 1.

    b. Let $\vec{v} = a\vec{i} + b\vec{j}$, where a and b are not both 0, and verify that $\frac{\vec{v}}{\norm{\vec{v}}}$ is a unit vector.

Synthesis Questions

70. Find the horizontal and vertical components of $\vec{u}, \vec{v}$, and $\vec{A}$ from Problem 55 . What do you notice when you compare the horizontal components of two vectors with the horizontal component of the difference?

71. Find the horizontal and vertical components of $\vec{z}, \vec{y}$, and $\vec{B}$ from Problem 56 . What do you notice when you compare the horizontal components of two vectors with the horizontal component of the difference?

Applications

For the following exercises,
(a) Make a sketch using vectors to illustrate the problem.
(b) Use the coordinate form of the vectors to solve problem.

72. The tornado displaced the trash bin to a spot 500 meters north and 800 meters east of its original position, and the flood later displaced the bin 2000 meters due south from there. How far and in what direction was the trash bin moved from it original position?

73. A radio-controlled model plane pointed due west with an airspeed of 15 miles per hour, but there was a crosswind from the north at a speed of 8 miles per hour. How fast and in what direction is the plane moving relative to the ground?

74. Matt flies 10 km in a direction 175° north from east, then turns and flies an additional 12 km due west. How far and in what direction is Matt's final position relative to his starting point?

75. Kim sails 500 yards due south, then turns and sails 350 yards in the direction 300° from east. How far and in what direction is Kim's final position relative to her starting point?

76. After leaving the airport, Piyali flew 30 miles at a heading 30° east of north, then 50 miles 70° east of north, and finally 12 miles 20° south of east. What is her current position relative to the airport?

77. On a whale-watching trip, the SS Dolphin sailed 15 miles from port on a heading of 40°, then 8 miles on a heading of 320°, and then 4 miles on a heading of 250°. What is the current position of the SS Dolphin relative to port?

Answers

Skills Refresher

1. $\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$
2. $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$
3. $\tan(\alpha+\beta)=\dfrac{\tan(\alpha)+\tan(\beta)}{1\tan(\alpha)\tan(\beta)}$
4. Change subtraction signs to addition and vice versa.
5. $\dfrac{-2\sqrt{3}-1}{2\sqrt{5}}$
6. $\dfrac{-\sqrt{3}-2}{1+2\sqrt{3}}$

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9: POLAR COORDINATES AND COMPLEX NUMBERS

The picture on the left below is a photo of a fern leaf. The picture on the right is a computer-generated image called the Barnsley fern. It is named after the British mathematician Michael Barnsley, who developed it in the 1980's.

Figure 1

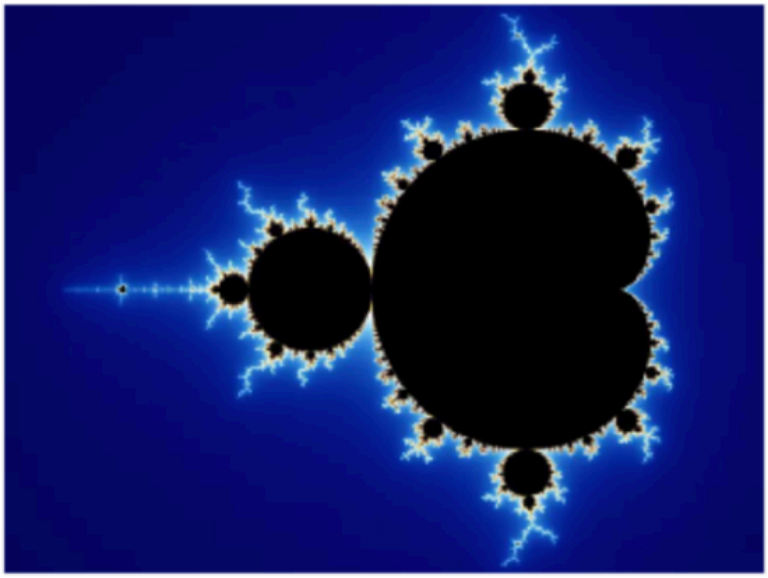




The Barnsley fern is an example of a **fractal**. A fractal is an infinitely complex pattern created by iteration: repeating a process over and over in an ongoing feedback loop. You can watch a Barnsley fern being created at https://www.youtube.com/watch?v=iGMGVpLMtMs.

One of the most famous and intriguing fractals is the Mandelbrot set, named after the French mathematician Benoît Mandelbrot, who lived from 1924 to 2010. The Mandelbrot set looks like a cardioid, or heart-shaped figure, studded with circles all around its boundary.

Figure 2



Like the Barnsley fern, we can create the Mandelbrot set by repeatedly evaluating a simple equation over and over. This equation

uses complex numbers.

The plane of complex numbers is similar to a two dimensional coordinate system. Every point on the plane is represented by a complex number $c = x + iy$, and we perform the following steps for every point:

Starting with $z_1 = 0$, we create an infinite sequence of complex numbers z_n according to the rule

$$z_{n+1} = (z_n)^2 + c$$

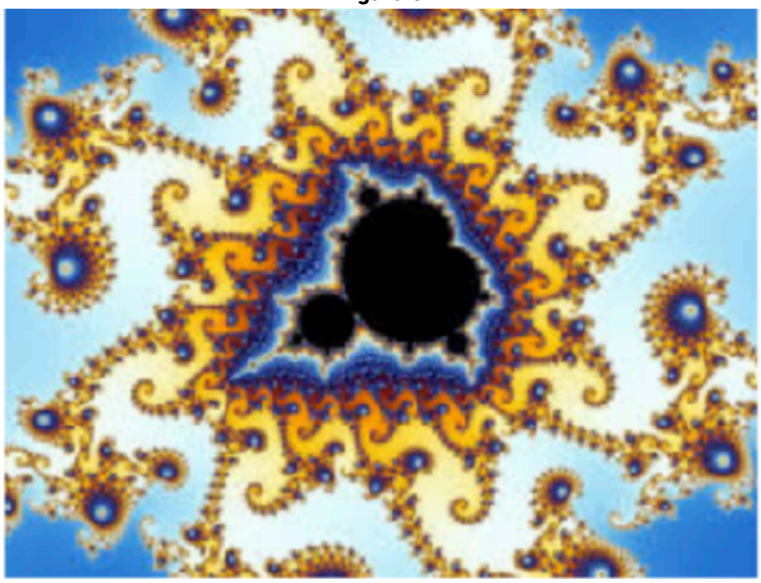
If this sequence increases to infinity (it diverges), we color the point c white. If the sequence does not increase beyond a certain limit (it is bounded), we color the point black.

We create a sequence for every point c in the plane. The collection of all the black points is the Mandelbrot set. It sits in the portion of the plane where x is between –1 and 2, and y is between –1.5 and 1.5. Outside the set, all the values of c cause the sequence to go to infinity, and the color of the point is proportional to the speed at which the sequence diverges.

Because the calculations must be performed thousands or millions of times, we need computers to study them. The higher the number of iterations, the better the quality of the image produced. Not coincidentally, the Mandelbrot set was discovered in 1980, shortly after the invention of the personal computer.

One of the intriguing features of fractal images is **self-similarity**. If we zoom in on a fractal, we see the same pattern repeated again and again, sometimes with interesting variations. Here is a portion of the boundary of the Mandelbrot set, blown up to many times its original size. You can see the copy of whole set embedded in the image.

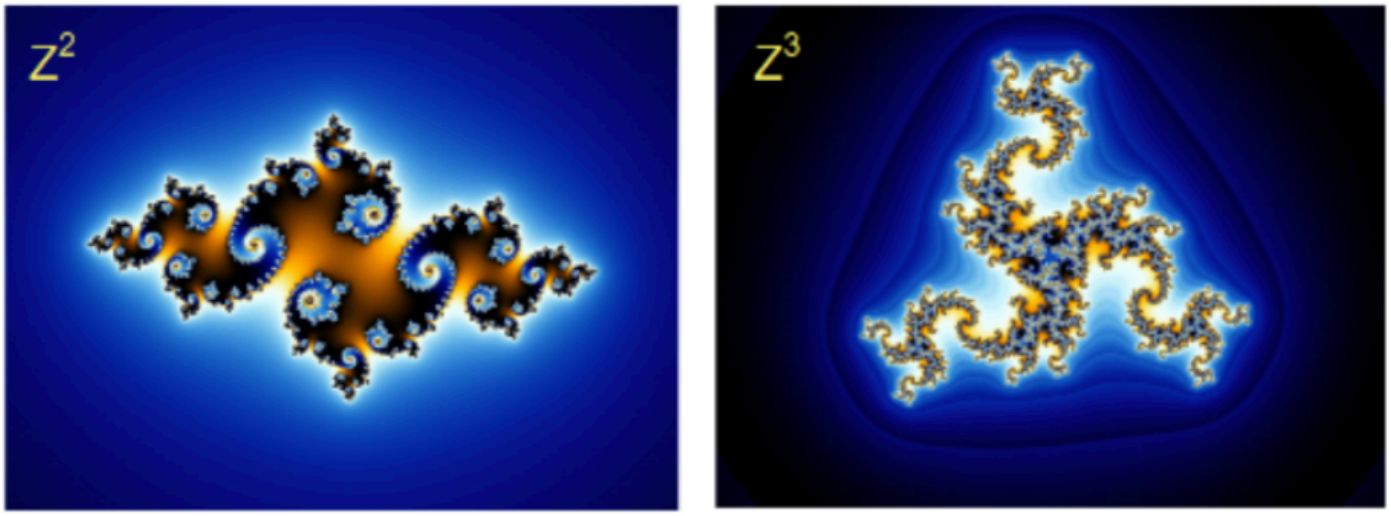
Figure 3



You can find a "Mandelbrot Zoomer" at https://mathigon.org/course/fractals/introduction.

Here are some more fractals, called Julia sets:

Figure 4



Fractal patterns seem familiar because the laws that govern the creation of fractals are found throughout the natural world. Tree branches, rivers, ice crystals, and seashells all form in fractal shapes. You can see some examples at https://www.treehugger.com/earth-matters/wilderness-resources/blogs.

9.1: Polar Coordinates

9.1.1: Resources and Key Concepts

9.1.2: Homework

9.2: Polar Graphs

9.2.1: Resources and Key Concepts

9.2.2: Homework

9.3: Complex Numbers

9.3.1: Resources and Key Concepts

9.3.2: Homework

9.4: Polar Form of Complex Numbers

9.4.1: Resources and Key Concepts

9.4.2: Homework

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9.1: Polar Coordinates

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Location by Coordinates

One of the most valuable applications of trigonometric ratios allows us to find distances or locations specified by angles. Starting with the definitions of sine and cosine,

$$\cos(\theta) = \frac{x}{r} \quad \text{and} \quad \sin(\theta) = \frac{y}{r}$$

we can solve for x and y, the coordinates of points on the terminal side of the angle, and obtain the following results.

Theorem: Cartesian Coordinates in Terms of Cosine and Sine

If point P is located at a distance r from the origin in the direction specified by angle θ in standard position, then the coordinates of P are

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta).$$

These formulas make sense when we think of the unit circle. On a unit circle, the coordinates of a point designated by an angle θ are $(\cos(\theta), \sin(\theta))$, as shown in Figure 9.1.1. On a circle of radius r, the angle θ forms a similar triangle whose dimensions are scaled up by a factor of r. In particular, the legs of the new triangle are r times larger than the original triangle.

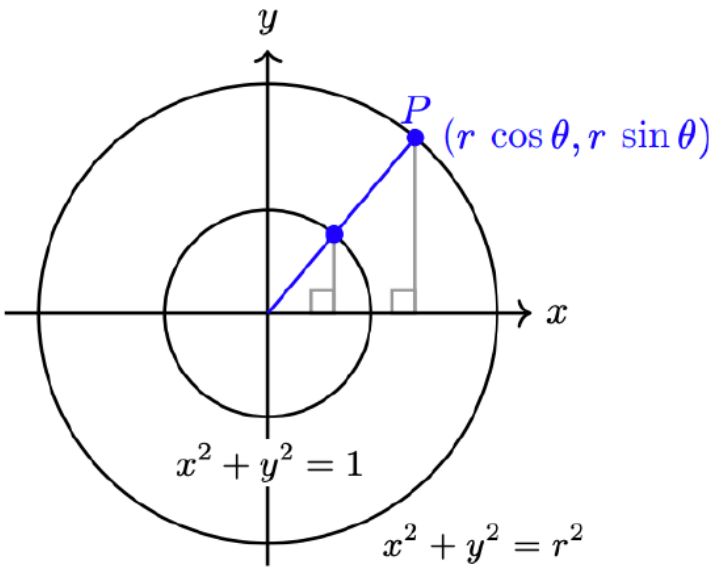


Figure 9.1.1

Example 9.1.1

Point P is located 6 centimeters from the origin in the direction of $292°$. Find the coordinates of P, rounded to the nearest hundredths.

Solution available at the end of this section.

Checkpoint 9.1.1

You find an old map that shows a buried treasure located 500 yards from the big oak tree in the direction $215°$, as shown below. You don't have anything with you to measure angles, but you have your calculator.

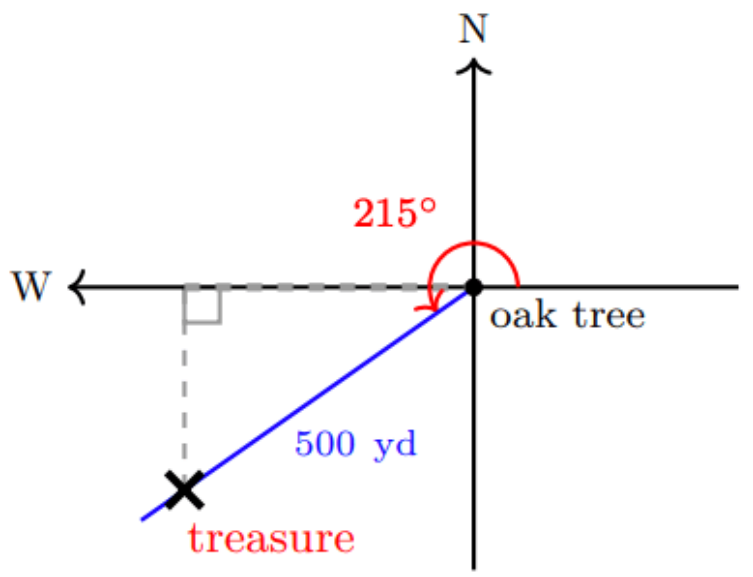


Figure 9.1.3

a. Find the cosine of $215°$. How far west should you walk from the big oak to be directly north of the treasure?
b. Find the tangent of $215°$. How far south should you walk from your present location before you begin digging?

Answers available at the end of this section.

The Cartesian coordinate system is also called the rectangular coordinate system because it describes a location in the plane as the vertex of a rectangle. To construct a rectangular coordinate system, we begin with two perpendicular axes intersecting at the origin. The x- and y-coordinates of a point indicate the length and width of a rectangle with one vertex at the origin. The point $P(x, y)$ sits at the opposite vertex of the rectangle, as shown in Figure 9.1.4.



Figure 9.1.4

Previously, we used vectors to specify a location by giving a distance and a direction. For example, we might say that the airport is located 5 miles southwest of the city center. This method of designating locations is so helpful that we will construct a new coordinate system using the same tools: distance and direction.

Definition: Polar Coordinate System

The **polar coordinate system** is a two-dimensional system for specifying points in a plane using a distance from a central point (called the **pole**) and an angle from a reference direction (called the **polar axis**). Typically, the pole is aligned with what we would consider the origin in the rectangular coordinate system, and the polar axis is aligned with the positive x-axis.

We describe the location of a point P in the plane by measuring the distance, $|r|$, from P to the pole, and the angle, θ, that $\vec{OP}$ makes with the polar axis (measured counterclockwise). The components of the ordered pair, (r, θ), are called the **polar coordinates** of the point P.

Definition: Polar Coordinates

The **polar coordinates** of a point P in the plane are (r, θ), where $|r|$ is the distance from P to the pole, and θ is the signed angle measured counterclockwise from the polar axis to the ray through P from the pole.

In our work with polar coordinates, we *always* use radians for the angle θ. For example, the point $P\left(2, \frac{\pi}{2}\right)$ is located 2 units from the pole at an angle of $\frac{\pi}{2}$ radians, and $Q(3, 4)$ is located 3 units from the pole at an angle of 4 radians. The graphs of P and Q are shown in Figure 9.1.5.

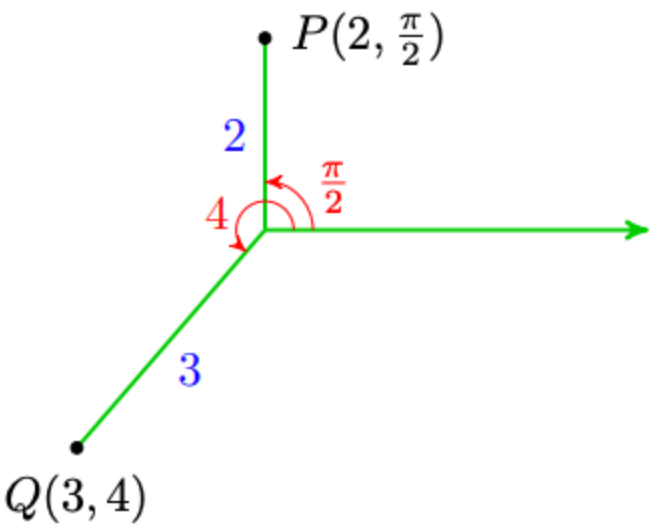


Figure 9.1.5: The graphs of $P\left(2, \frac{\pi}{2}\right)$ and $Q(3,4)$.

A piece of graph paper in rectangular coordinates consists of a grid of horizontal and vertical lines. These are the lines $x = k$ and $y = k$, for evenly spaced values of k. Each vertical grid line consists of points with the same x-coordinate, and each horizontal grid line is made up of points with the same y-coordinate. The grid lines are benchmarks to help us locate points with specific rectangular coordinates.

A piece of graph paper in polar coordinates consists of a grid of concentric circles and radial lines, as shown in Figure 9.1.6. Each circle consists of points with the same r-coordinate, all the same distance from the pole. All points with the same θ-coordinate lie on the same radial line. This grid helps us locate points with specific polar coordinates.

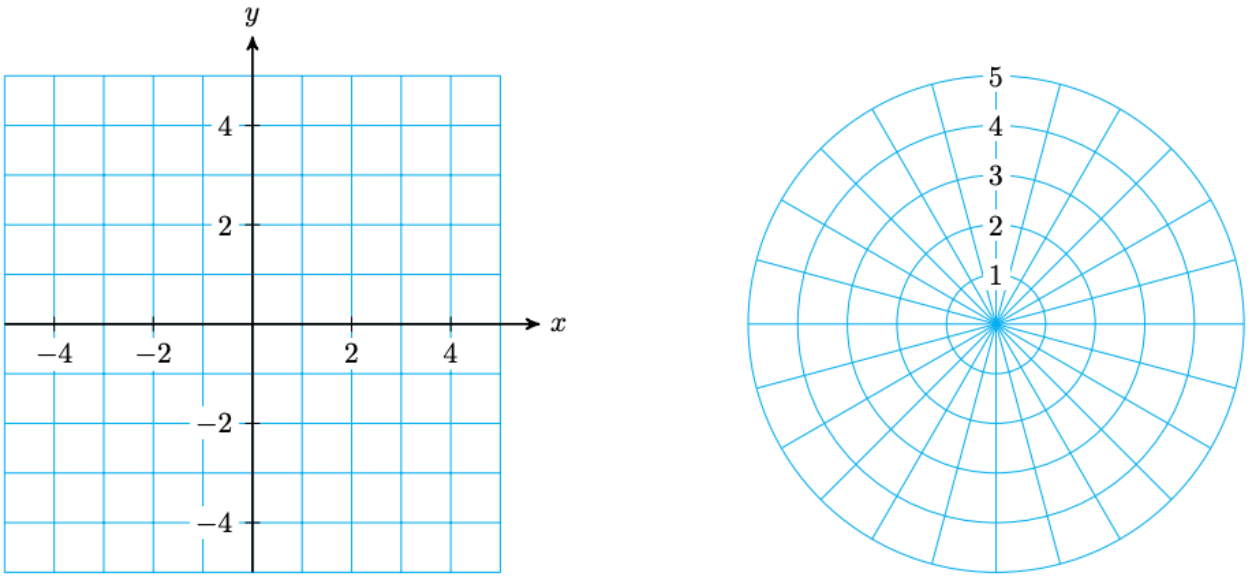


Figure 9.1.6

Plotting Points in the Polar Coordinate System

You are used to thinking in rectangular coordinates: to locate a point, we move so many units left or right and so many units up or down. When working in polar coordinates, we want to "think radially"—how far to move from the pole and in which direction.

Example 9.1.2

Plot the points whose polar coordinates are given: $A\left(1, \frac{\pi}{2}\right)$ and $B\left(2, \frac{7\pi}{4}\right)$.

Solution available at the end of this section.

Caution: Polar Backwardness

We are used to "plugging in" a value for x and using the given equation or relation to find the corresponding value of y. This is why most of us think of x as the default independent variable and y as depending on x (and, hence, is called the dependent variable). We get the ordered pair, (x, y), which is listed in the order $(\text{independent variable}, \text{dependent variable})$. This convention, however, is a bit backward with polar coordinates. Even though the independent, or input, variable in polar coordinates is almost always θ, it is customary to list the dependent or output variable first: (r, θ).

Checkpoint 9.1.2

Give the polar coordinates of the points S and T shown below, with $0 \leq \theta \leq 2\pi$.

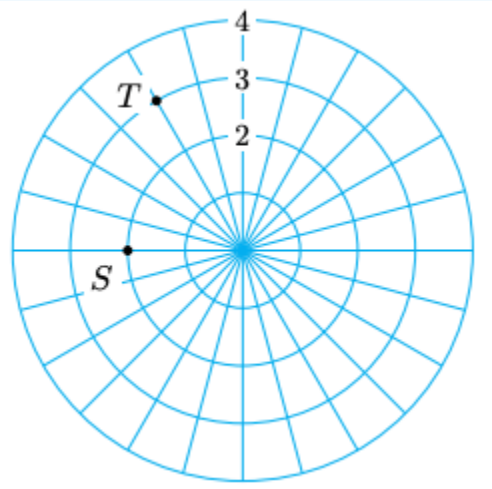


Figure 9.1.8

Answer available at the end of this section.

Each point has infinitely many polar coordinates because we can add multiples of 2π to the value of θ. For example, the point with polar coordinates $\left(1, \frac{\pi}{2}\right)$ also has polar coordinates $\left(1, \frac{5\pi}{2}\right)$ and $\left(1, \frac{-3\pi}{2}\right)$.

If we allow negative values of r, there are even more ways to write the coordinates of a point. Each angle designates a line through the pole, and each line has both a positive and a negative direction.

On the line at $\theta = \frac{\pi}{4}$, for instance, the positive direction lies in the first quadrant, so that the point $\left(2, \frac{\pi}{4}\right)$ is designated by P in the figure below. To plot the point $\left(-2, \frac{\pi}{4}\right)$, we move in the opposite direction from the pole, arriving at point Q. Note that the point Q can also be designated by the coordinates $\left(2, \frac{5\pi}{4}\right)$.

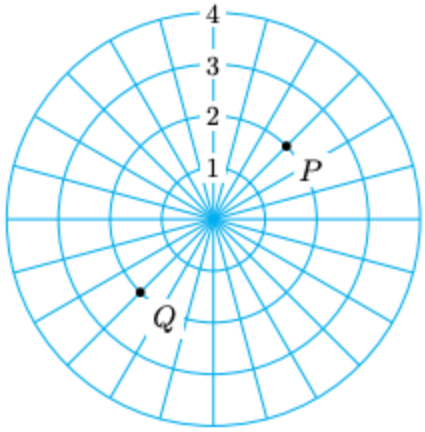


Figure 9.1.9

Theorem: Non-Uniqueness of Polar Coordinates

Any point with polar coordinates (r, θ) also has coordinates $(r, \theta + 2k\pi)$, where $k \in \mathbb{Z}$. The point can also be designated by $(-r, \theta + \pi)$. Finally, the pole has coordinates $(0, \theta)$, for any value of θ.

Example 9.1.3

Give polar coordinates with negative r-values for the points in the previous exercise.

Solution available at the end of this section.

Checkpoint 9.1.3

Give polar coordinates with positive r-values and $0 \leq \theta \leq 2\pi$ for each point given in polar coordinates.

a. $\left(-2.5, \frac{\pi}{6}\right)$
b. $(-0.7, 5)$

Answers available at the end of this section.

Regions in the Plane

In Cartesian coordinates, we use equations and inequalities to describe regions in the plane. For example, the shaded region in Figure $9.1.10a$ is the graph of $y \leq x + 2$. The pair of inequalities $x \geq -2$ and \(1

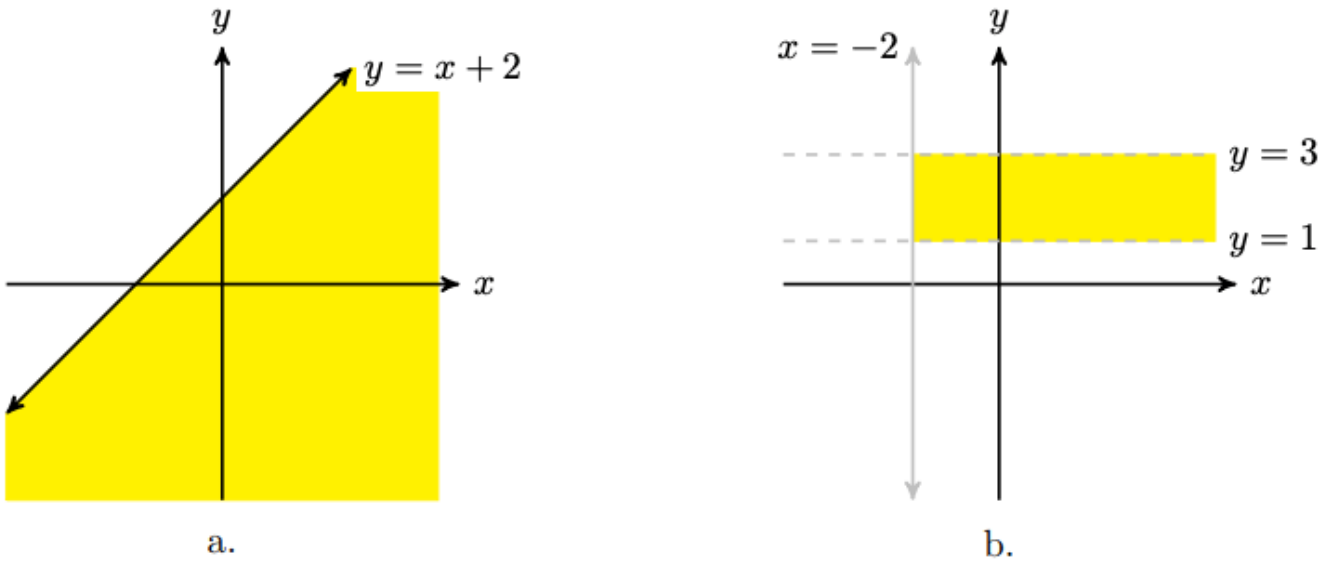


Figure 9.1.10

In the polar plane, the coordinate grid lines are circles centered at the pole, with equations $r = k$, and lines through the pole, with equations $\theta = k$.

Example 9.1.4

Sketch the region designated by each set of inequalities.

a. $1 \leq r < 3$
b. $0 \leq r \leq 2, \frac{\pi}{6} < \theta < \frac{\pi}{3}$

Solutions available at the end of this section.

Checkpoint 9.1.4

Write inequalities to describe each region shown below.

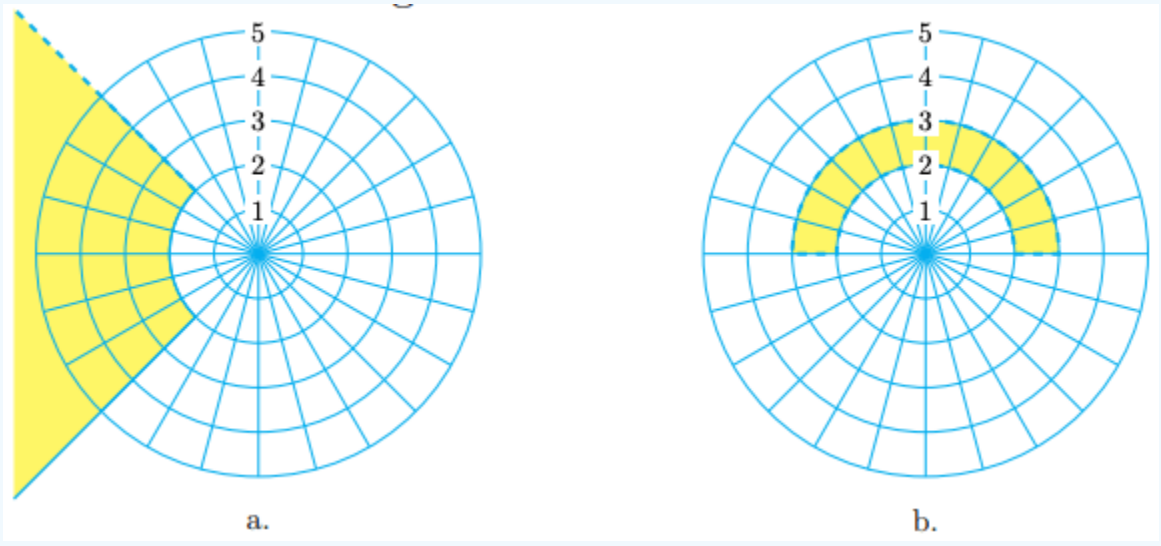


Figure 9.1.12

Answers available at the end of this section.

Converting Between Polar and Cartesian Coordinates

Using Trigonometry, converting from polar coordinates to Cartesian is easy, or vice versa. Figure $9.1.13$ shows how the two coordinates are related.

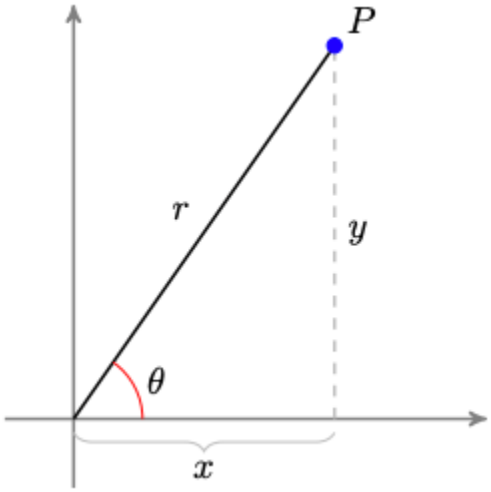


Figure $9.1.13$

If we know the polar coordinates (r, θ), we can compute the Cartesian coordinates (x, y) as the legs of a right triangle.

$$\begin{aligned} \frac{x}{r} &= \cos(\theta) \implies x = r\cos(\theta) \\ \frac{y}{r} &= \sin(\theta) \implies y = r\sin(\theta) \end{aligned}$$

Example $9.1.5$

Find the Cartesian coordinates of the point with polar coordinates $\left(\sqrt{2}, \frac{3\pi}{4}\right)$.

Solution available at the end of this section.

Checkpoint $9.1.5$

Find the Cartesian coordinates of the point with polar coordinates $\left(4, \frac{5\pi}{3}\right)$.

Answer available at the end of this section.

If we know the Cartesian coordinates of a point, we can compute the polar coordinates as follows.

$$\begin{aligned} x^2 + y^2 &= r^2 \\ \implies r &= \sqrt{x^2 + y^2} \\ \tan(\theta) &= \frac{y}{x} \end{aligned}$$

As we are used to at this point, to find the angle θ, we first compute its reference angle, $\hat{\theta} = \tan^{-1}\left(\left|\frac{y}{x}\right|\right)$. We use this reference angle to help find the value of θ by considering the given point's quadrant.

Example 9.1.6

Find the polar coordinates of the point with Cartesian coordinates $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Solution available at the end of this section.

Caution: Reference Angles and Quadrants!

In Example 9.1.6, the angle θ is not given by $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$, because the given point lies in the third quadrant. Remember that there are always two angles between 0 and 2π with a given tangent.

Checkpoint 9.1.6

Find polar coordinates for the points with Cartesian coordinates $(2, -5)$. Give decimal approximations rounded to hundredths.

Answer available at the end of this section.

Equations in Polar Coordinates

Polar coordinates are useful for studying objects or phenomena with radial symmetry, such as circles, spheres, and cylinders, or the central forces (those that act equally in all directions), such as gravity and electric charge. The equations describing these objects are often simpler in polar coordinates than in Cartesian coordinates.

It is not hard to convert an equation from Cartesian to polar coordinates: we replace each x with $r\cos(\theta)$ and each y with $r\sin(\theta)$.

Example 9.1.7

Convert the equation $2x + 3y = 6$ to polar coordinates.

Solution available at the end of this section.

Checkpoint 9.1.7

Convert the equation $x^2 + y^2 = 4$ to polar coordinates.

Answer available at the end of this section.

You may recognize the equation in Checkpoint 9.1.7 as a circle. Note also that its polar equation is more straightforward than the familiar Cartesian version.

Converting equations from polar to Cartesian form is more complex than the other direction (and is sometimes impossible). Here are some strategies to try:

- Expressions of the form $r\cos(\theta)$ or $r\sin(\theta)$ can be replaced by x and y, respectively.
- r^2 can be replaced by $x^2 + y^2$.
- $\tan(\theta)$ can be replaced by $\frac{y}{x}$.

Example $9.1.8$

Convert the equation $r = 3\cos(\theta)$ to Cartesian coordinates.

Solution available at the end of this section.

Checkpoint $9.1.8$

Convert the equation $r = \tan(\theta)$ to Cartesian coordinates.

Answer available at the end of this section.

Success in College: Learning From Tests (Part 6)

1. How complex are mathematics tests for you compared to exams in other subjects?
2. How hard do you try on the mathematics test compared to your efforts in other subjects?

Success in College: Preparing for the Final Exam (Part 1)

1. Describe what you will do to prepare for the final exam in this course.
2. Describe how well you think you are prepared right now to take the final exam.

Answers

Section Preview

The material in this section is the traditional introduction to polar coordinates. You will cover all the basics except graphing polar equations (which is done in the next section).

Section Preview

- Graph an ordered pair given in polar coordinates.
- Give multiple representations for a point in polar coordinates.
- Sketch a region in the polar coordinate system.
- Convert an ordered pair from polar to rectangular coordinates or vice versa.
- Express a polar equation in rectangular coordinates.
- Express a rectangular equation in polar coordinates.

Example $9.1.1$

The location of point P is shown below. We see that $r = 6$, and we can use a calculator to evaluate $\cos(292^\circ)$ and $\sin(292^\circ)$.

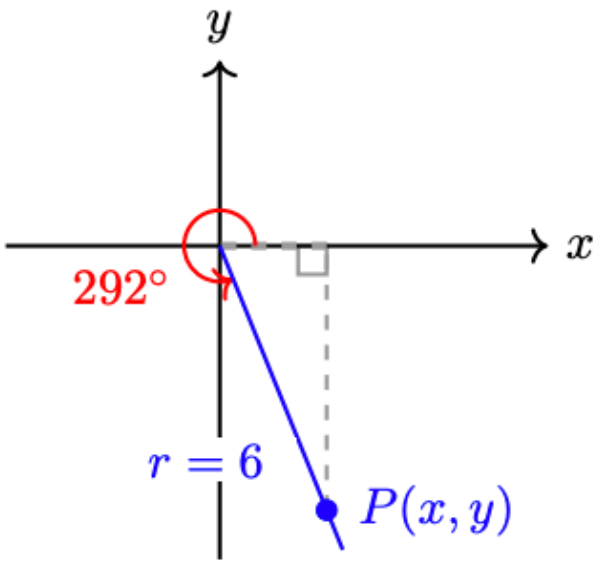


Figure 9.1.2

$$\begin{aligned} x &= r\cos(292^\circ) \quad \text{and} \quad y = r\sin(292^\circ) \\ &\approx 6(0.3746) \qquad\qquad \approx 6(-0.9272) \\ &\approx 2.2476 \qquad\qquad \approx -5.5632 \end{aligned}$$

The coordinates of P are approximately $(2.25, -5.56)$.

Checkpoint 9.1.1

a. 409.58 yds
b. 286.79 yds

Example 9.1.2

To plot point A, we move 1 unit away from the pole in the direction $\frac{\pi}{2}$, as shown below. To plot point B, we move 2 units away from the pole in the direction $\frac{7\pi}{4}$.

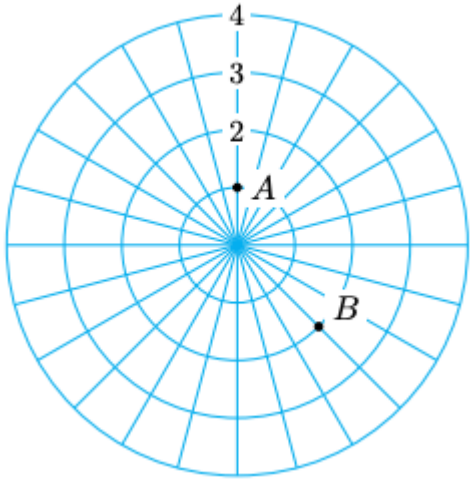


Figure 9.1.7

Checkpoint 9.1.2

$S(2,\pi),\ T\left(3, \frac{2\pi}{3}\right)$

Example 9.1.3

The point $S(2,\pi)$ is also $(-2, 2\pi)$ or $(-2, 0)$. The point $T\left(3, \frac{2\pi}{3}\right)$ can also be designated by $\left(-3, \frac{2\pi}{3} + \pi\right)$, or $\left(-3, \frac{5\pi}{3}\right)$.

Checkpoint 9.1.3

a. $\left(2.5, \frac{7\pi}{6}\right)$
b. $(0.7, 5 - \pi)$

Example 9.1.4

a. The region consists of all points between 1 (inclusive) and 3 units from the pole. No restriction is given on θ, so the region is the annular ring shown in Figure 9.1.11.

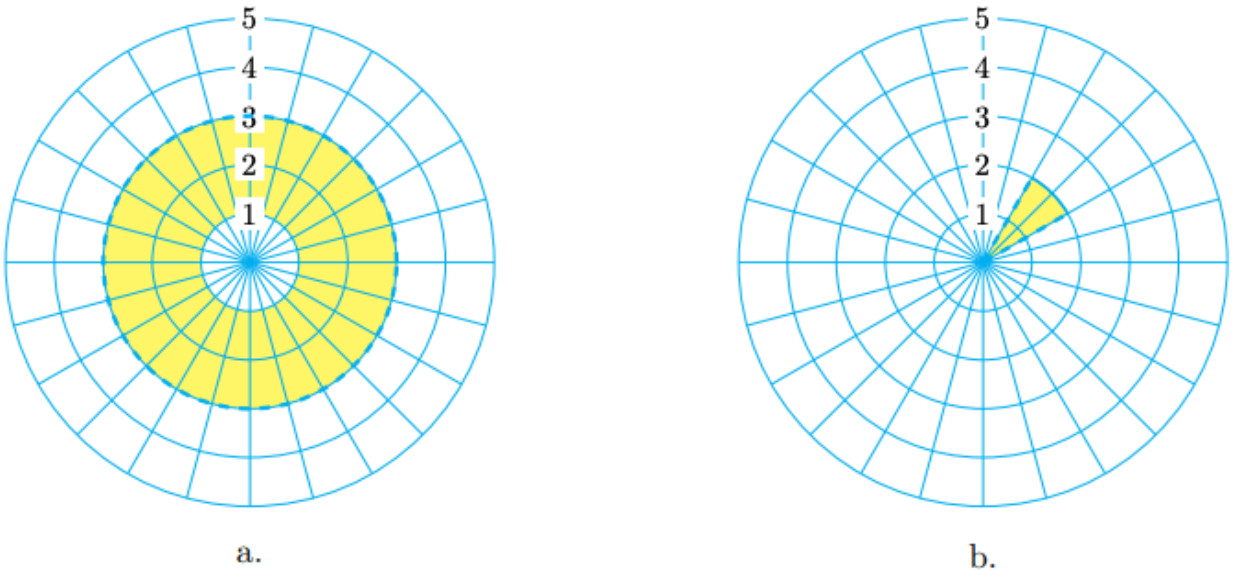


Figure 9.1.11

b. The region includes all points up to 2 units from the pole, but only between the lines $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{3}$. These curves bound the pie-shaped wedge (circular sector) shown in Figure 9.1.11*b*.

Checkpoint 9.1.4

a. $r \geq 2, \frac{3\pi}{4} < \theta \leq \frac{5\pi}{4}$
b. \(2

Example 9.1.5

The polar coordinates tell us that $r = \sqrt{2}$ and $\theta = \frac{3\pi}{4}$, as shown below. The x-coordinate of the point is

$$\begin{aligned} x &= r\cos(\theta) \\ &= \sqrt{2}\cos\left(\frac{3\pi}{4}\right) \\ &= \sqrt{2}\cdot\frac{-1}{\sqrt{2}} \\ &= -1 \end{aligned}$$

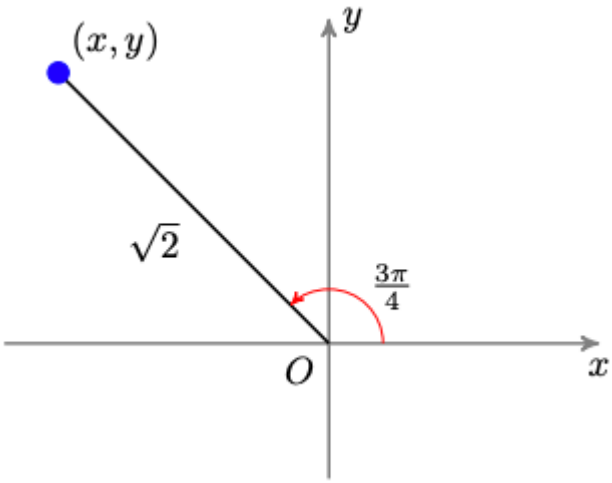


Figure 9.1.14

and its y-coordinate is

$$\begin{aligned} y &= r\sin(\theta) \\ &= \sqrt{2}\sin\left(\frac{3\pi}{4}\right) \\ &= \sqrt{2}\cdot\frac{1}{\sqrt{2}} \\ &= 1 \end{aligned}$$

Thus, the Cartesian coordinates are $(-1, 1)$.

Checkpoint 9.1.5

$(2, -2\sqrt{3})$

Example 9.1.6

We first compute the value of r.

$$r^2 = x^2 + y^2 = \left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1.$$

Thus, $r = \sqrt{1} = 1$. To find θ, we compute

$$\tan(\theta) = \frac{y}{x} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}.$$

Hence, $\hat{\theta} = \frac{\pi}{3}$. There are infinitely many values of θ that satisfy this equation, but our point lies in the third quadrant, so we take $\theta = \frac{4\pi}{3}$. Hence, the polar coordinates of the given point are $(1, \frac{4\pi}{3})$.

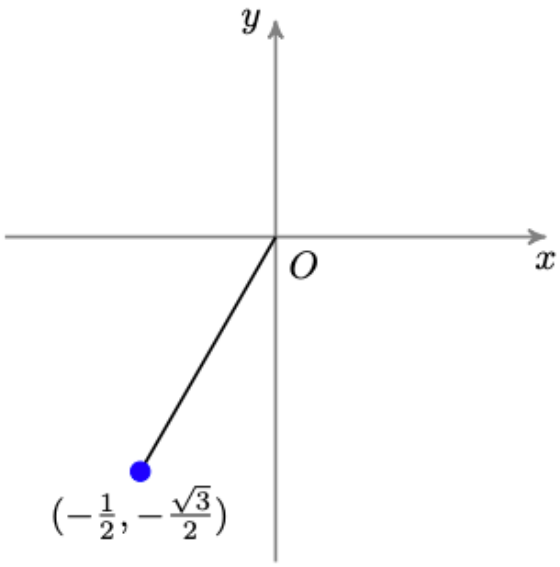


Figure 9.1.15

Checkpoint 9.1.6

(5.39, 5.09)

Example 9.1.7

Replacing x by $r\cos(\theta)$ and y by $r\sin(\theta)$, we get

$$2r\cos(\theta) + 3r\sin(\theta) = 6.$$

If we solve for r in terms of θ, we find

$$\begin{aligned} r(2\cos(\theta) + 3\sin(\theta)) &= 6 \\ \Longrightarrow \qquad r &= \frac{6}{2\cos(\theta) + 3\sin(\theta)} \end{aligned}$$

Neither of these equations is simpler than the Cartesian version; however, that should not surprise us because the original equation describes a line. Cartesian coordinates are better suited to rectilinear graphs.

Checkpoint 9.1.7

$r = 2$

Example 9.1.8

We multiply both sides of the equation by r to obtain

$$r^2 = 3r\cos(\theta).$$

Now we can replace the left side by $x^2 + y^2$ and the right side by $3x$ to obtain

$$x^2 + y^2 = 3x.$$

This is the equation of a circle. By completing the square in each variable, we can write the equation in standard form,

$$\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}.$$

The center of the circle is $\left(\frac{3}{2}, 0\right)$, and its radius is $\frac{3}{2}$.

Checkpoint 9.1.8

$y^2 = \dfrac{x^4}{1 - x^2}$

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9.1.1: Resources and Key Concepts

Resources

Prerequisite Topics

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Quadratic Functions**
  - The Basics of Quadratic Functions and Vertex Form: This skill is required to convert some polar equations into the standard form of a circle in rectangular coordinates, as seen in Example 8.
- **Graphing Inequalities**
  - Graphing Linear Inequalities in Two Variables: The concept of graphing regions in the plane defined by inequalities is extended from the Cartesian system to the polar system in this section (Example 4).

Videos

- Polar Coordinates
  - Polar Coordinates - The Polar Coordinate System
- Converting Coordinates
  - Polar Coordinates - Converting Ordered Pairs from Polar to Rectangular Coordinates
  - Polar Coordinates - Converting Ordered Pairs from Rectangular to Polar Coordinates
- Polar Equations
  - Polar Coordinates - Converting Equations from Polar to Rectangular Form
  - Polar Coordinates - Converting Equations from Rectangular to Polar Form

Key Concepts

Definitions

- **Polar Coordinate System:** A two-dimensional system for specifying points in a plane using a distance from a central point (the pole) and an angle from a reference direction (the polar axis).
- **Pole:** The central point in the polar coordinate system, equivalent to the origin in the rectangular coordinate system.
- **Polar Axis:** The reference direction from the pole, equivalent to the positive x-axis in the rectangular coordinate system.
- **Polar Coordinates:** The ordered pair (r, θ) that describes the location of a point, where $|r|$ is the distance from the point to the pole and θ is the signed angle from the polar axis to the ray passing through the point.

Theorems

- **Cartesian Coordinates in Terms of Cosine and Sine:** If a point P is located at a distance r from the origin in the direction specified by an angle θ in standard position, then the Cartesian coordinates of P are $x = r\cos(\theta)$ and $y = r\sin(\theta)$.
- **Non-Uniqueness of Polar Coordinates:** Any point with polar coordinates (r, θ) can also be represented by $(r, \theta + 2k\pi)$ for any integer k, or by $(-r, \theta + \pi)$. The pole has coordinates $(0, \theta)$ for any angle θ.

Common Mistakes

- **Polar Backwardness:** It is customary to write polar coordinates as (r, θ), which lists the dependent variable r before the independent variable θ. This is the reverse of the standard (x, y) convention for rectangular coordinates.
- **Reference Angles and Quadrants:** When converting from rectangular coordinates (x, y) to polar coordinates (r, θ), the angle

θ is not simply $\arctan(\frac{y}{x})$. The arctangent function only returns angles in Quadrants I or IV. You must use the quadrant of the point (x, y) to find the correct angle θ.

- **Plotting Negative r-values:** A negative value for r in polar coordinates, such as $(-2, \frac{\pi}{4})$, means you move in the direction opposite to the angle. In this case, you would face the direction $\frac{\pi}{4}$ but move 2 units in the opposite direction, ending up on the ray for $\frac{5\pi}{4}$.

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9.1.2: Homework

Reading Questions

1. What are the two components of an ordered pair in the polar coordinate system?
2. What is the "pole" and what is the "polar axis"?
3. If a point has polar coordinates (r, θ), what are the formulas to convert it to rectangular coordinates (x, y)?
4. If a point has rectangular coordinates (x, y), how do you find its distance from the origin, r?
5. Explain why a single point can be represented by infinitely many different polar coordinates.
6. If a point has polar coordinates $(3, \frac{\pi}{4})$, give another representation for the same point where r is negative.
7. What does the graph of the polar equation $r = 3$ look like?
8. What does the graph of the polar equation $\theta = \frac{\pi}{3}$ look like?
9. What is the general strategy for converting a rectangular equation into a polar equation?
10. What are some algebraic strategies you can use to convert a polar equation into a rectangular equation?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

Recall the standard form for the equation of a circle of radius r centered at (h, k):

$$(x - h)^2 + (y - k)^2 = r^2.$$

For the following exercises, use completing the square to write each equation in standard form.

1. $x^2 + y^2 - 4x + 2y - 4 = 0$
2. $x^2 + y^2 = 88 + 18y$
3. $x^2 + y^2 + 25 = 9 - 10x - 9y$
4. $5y^2 = 85 - 5x^2 - 10x$

Answers available at the end of this section.

Homework

Vocabulary Check

1. The origin in the polar coordinate system is called the ___.
2. The positive x-axis in the polar coordinate system is called the ___.

Concept Check

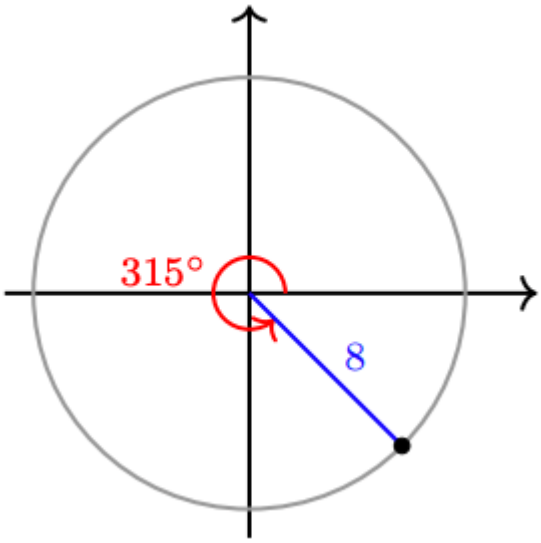
3. Why is the Cartesian coordinate system sometimes called the rectangular system?
4. What do the graphs of the polar equations $r = k$ and $\theta = k$, where k is a constant, look like?

5. Fill in the blank: the point with polar coordinates $(-6, \theta)$ also has coordinates $(6, ?)$.
6. What does the region described by $a \leq r \leq b$ look like? What does the region described by $a \leq \theta \leq b$ look like?
7. Sketch a figure that explains how to convert between polar and Cartesian coordinates.

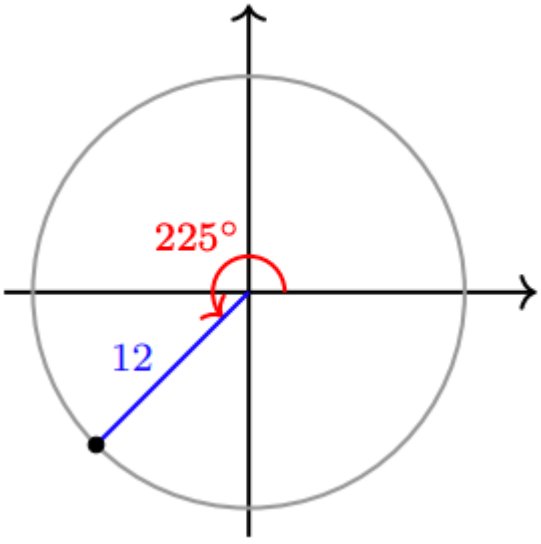
Basic Skills

For the following exercises, find exact values for the coordinates of the point.

8.

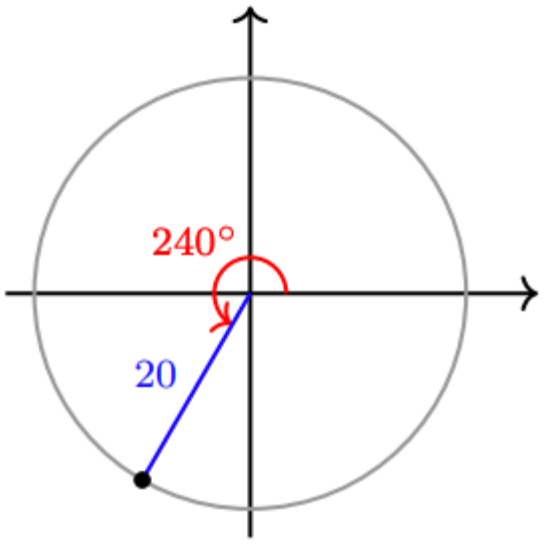


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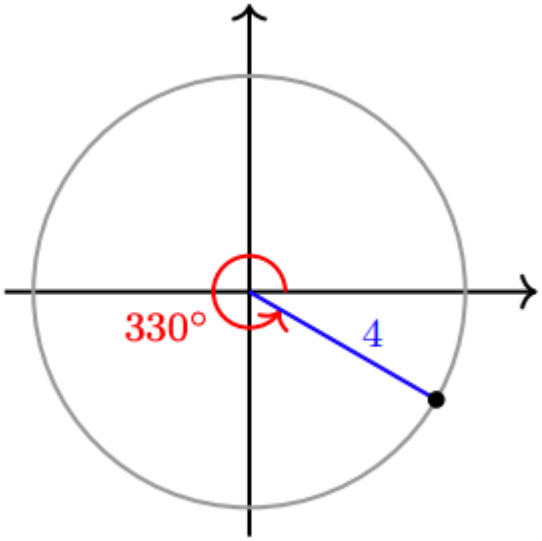


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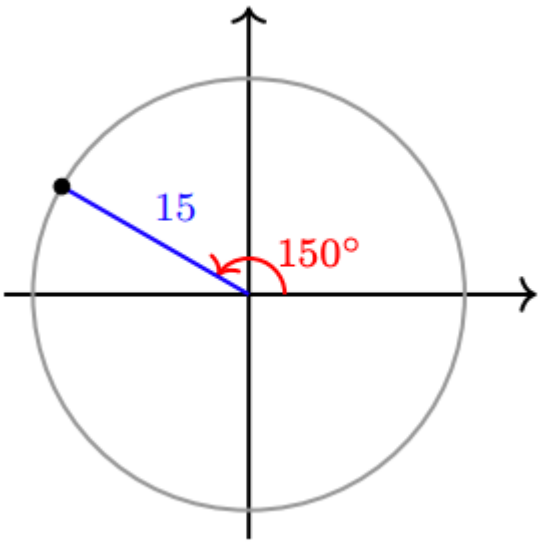
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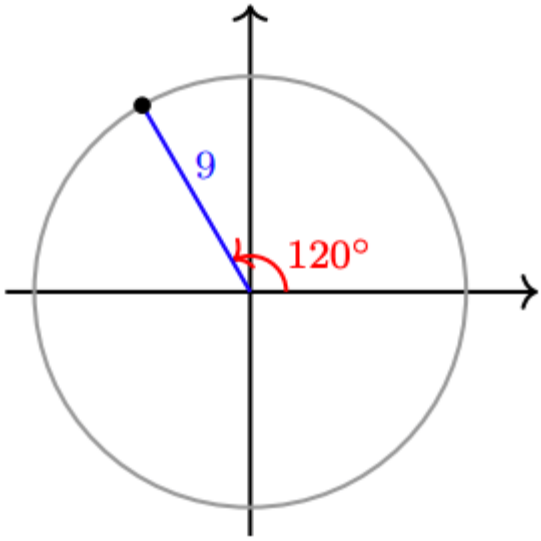


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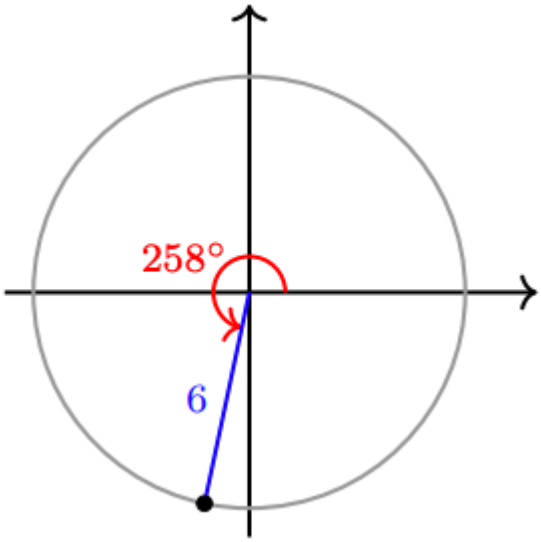
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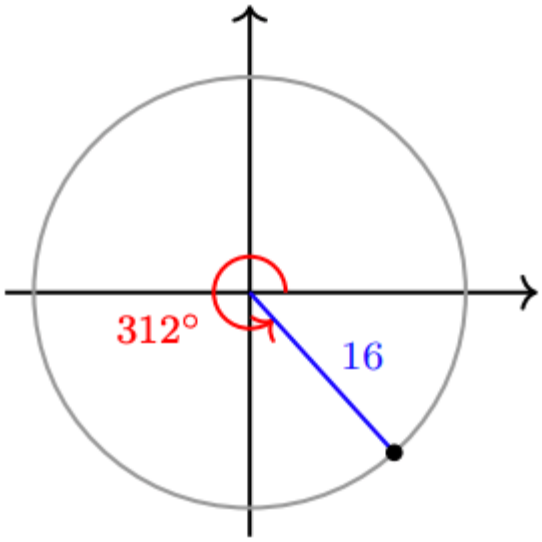
For the following exercises, find the coordinates of the point, rounded to hundredths.

14.

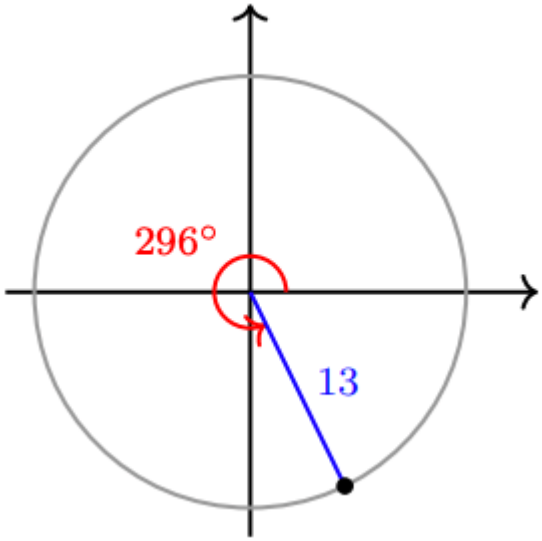


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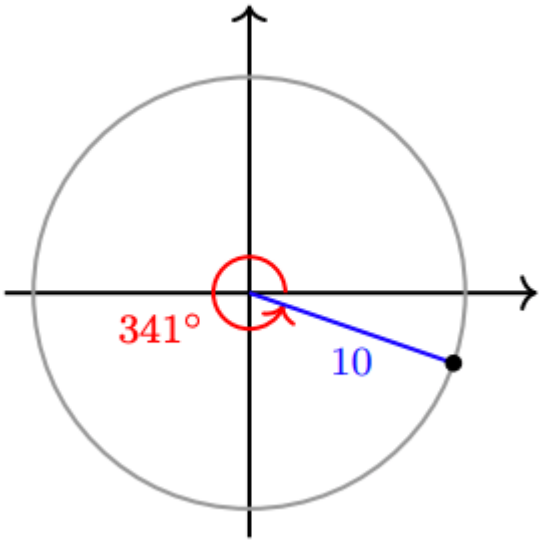
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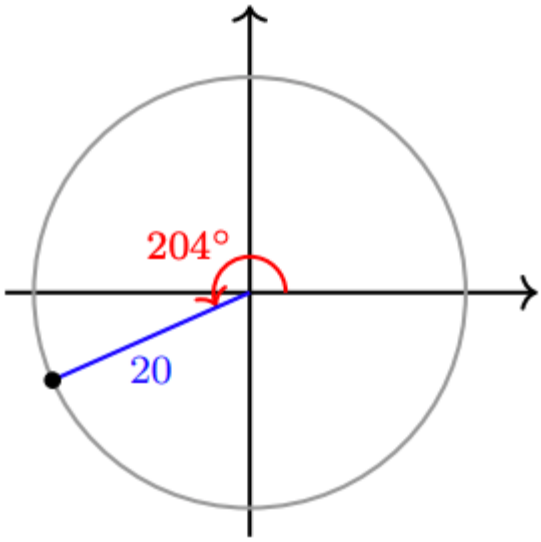


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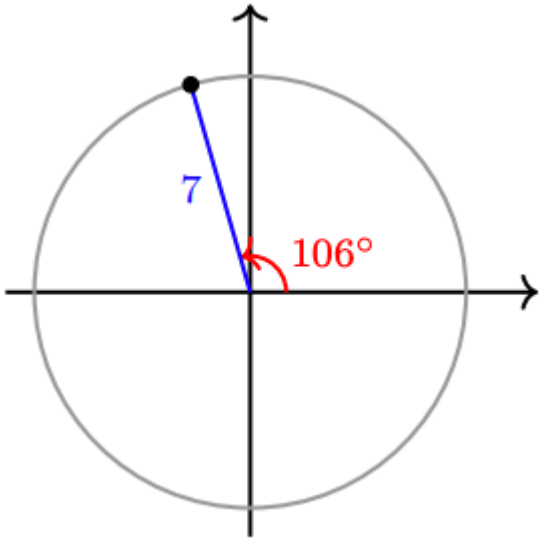


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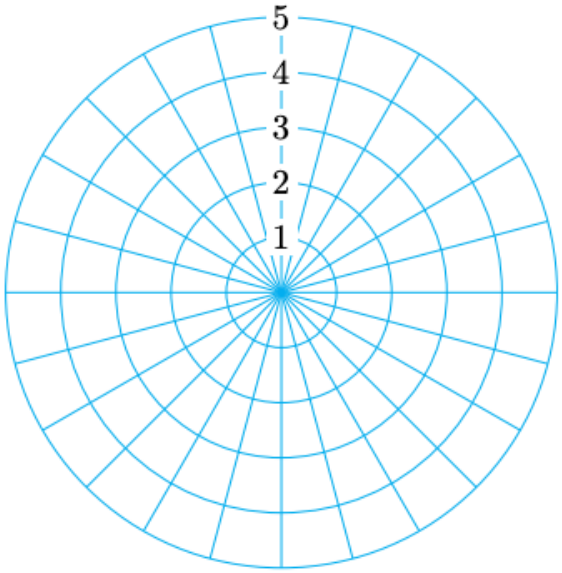




19.



For the following exercises, use the grid below to plot the points whose polar coordinates are given.



20. $\left(2, \frac{2\pi}{3}\right)$

21. $\left(3, \frac{5\pi}{4}\right)$

22. $\left(3, \frac{3\pi}{2}\right)$

23. $(4, \pi)$

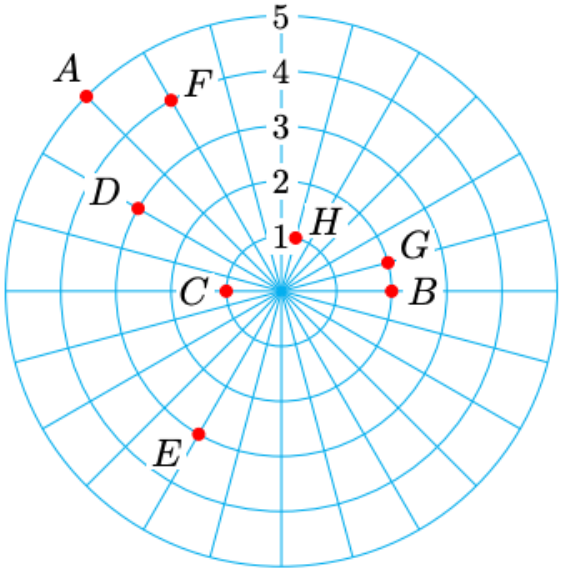
24. $\left(-4, \frac{\pi}{6}\right)$

25. $\left(-1, \frac{\pi}{2}\right)$

26. $\left(-1, \frac{7\pi}{4}\right)$

27. $\left(-2, \frac{5\pi}{3}\right)$

For the following exercises, give polar coordinates for each point shown below, with $r \geq 0$ and $0 \leq \theta \leq 2\pi$.



28. A

29. B

30. C

31. D

32. E

33. F

34. G

35. H

For the following exercises, convert the polar coordinates to Cartesian coordinates.

36. $\left(6, \frac{2\pi}{3}\right)$

37. $\left(5, \frac{7\pi}{6}\right)$

38. $\left(-3, \frac{3\pi}{4}\right)$

39. $\left(-4, \frac{5\pi}{3}\right)$

40. $(2.4, 3.6)$

41. $(1.7, 5.2)$

42. $(-2, 1.5)$

43. $(-3, 4.5)$

For the following exercises, convert the Cartesian coordinates to polar coordinates with $r \geq 0$ and $0 \leq \theta \leq 2\pi$. Give exact values for r and θ.

44. $(7, 7)$

45. $(-7,-7)$

46. $(\sqrt{6},-\sqrt{2})$

47. $(-\sqrt{3},3)$

48. $(-3,-2)$

49. $(1,4)$

50. $(-2,0)$

51. $(0,-5)$

For the following exercises, polar coordinates are given.
(a) Convert to polar coordinates with $r \leq 0, 0 \leq \theta < 2\pi$.
(b) Convert to polar coordinates with $r \geq 0, \quad -2\pi < \theta \leq 0$.

52. $\left(2, \frac{5\pi}{6}\right)$

53. $\left(1, \frac{4\pi}{3}\right)$

54. $(3,\pi)$

55. $(4,0)$

56. $(2.3,5.2)$

57. $(1.2,1.2)$

For the following exercises, sketch the region described by the inequalities.

58. $r \leq 3$

59. $2 \leq r \leq 3$

60. $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$

61. $0 \leq \theta \leq \frac{\pi}{2}$

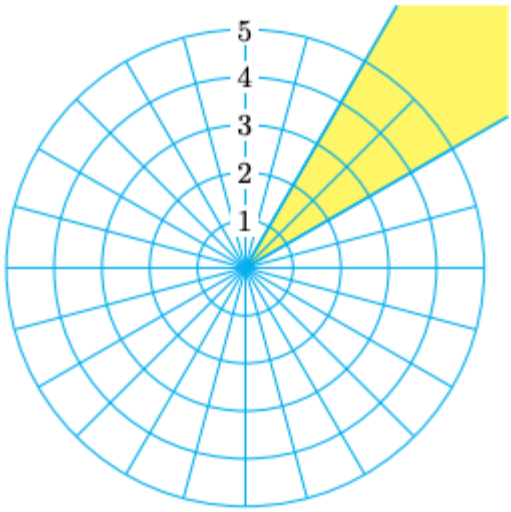
62. $r \geq 4, \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$

63. $2 \leq r \leq 3, \frac{5\pi}{3} \leq \theta \leq \frac{11\pi}{6}$

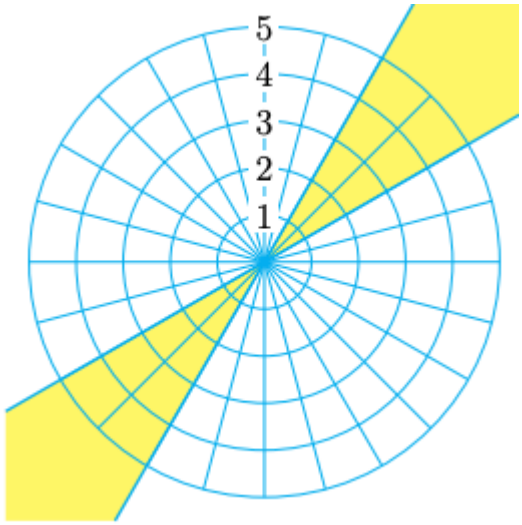
For the following exercises, write inequalities to describe the region.

64.

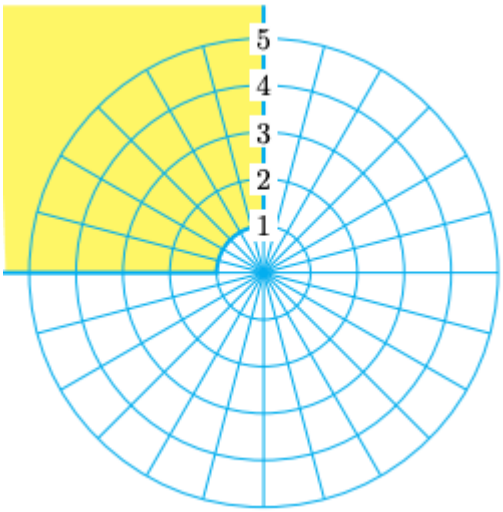
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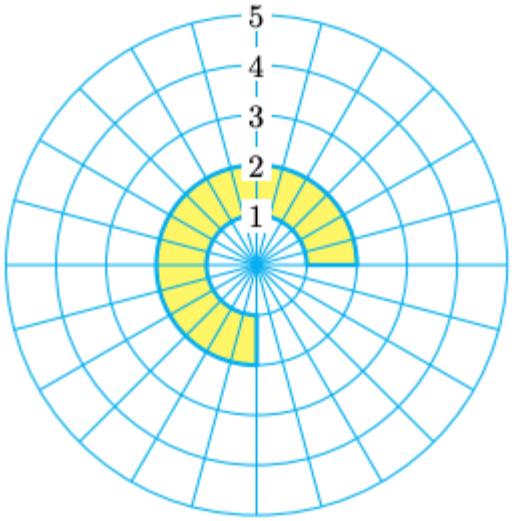


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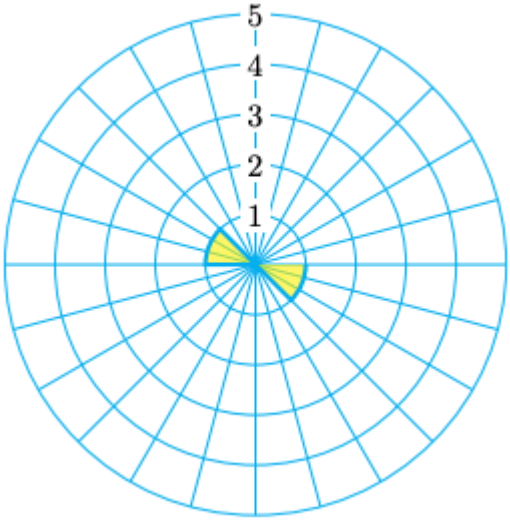


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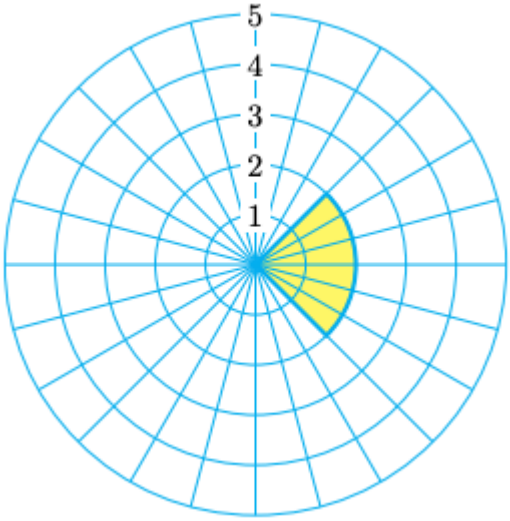




58.



59.



For the following exercises, convert the equation into Cartesian coordinates.

70. $r^2 = 2$

71. $r = 2$

72. $r = 4\cos(\theta)$

73. $r = \sin(\theta)$

74. $r = \csc(\theta)$

75. $r = 2\sec(\theta)$

76. $\tan(\theta) = 2$

77. $r^2 = \tan(\theta)$

78. $r\sec(\theta) = 3$

79. $\theta = \dfrac{\pi}{3}$

80. $r = \dfrac{2}{1+\sin(\theta)}$

81. $r = \dfrac{4}{4-\cos(\theta)}$

82. $2r\cos(\theta) + r\sin(\theta) = 1$

83. $\tan(\theta) = r\cos(\theta) - 2$

For the following exercises, convert the equation into polar coordinates.

84. $x = 2$

85. $y^2 = 2x - x^2$

86. $2xy = 1$

87. $y = -x$

88. $y^2 = 4x$

89. $x^2 - y^2 = 1$

90. $x^2 + y^2 = 4\sqrt{x^2+y^2}$

91. $\left(x^2+y^2\right)^3 = 9x^2y^2$

Synthesis Questions

92. Show that the graph of $r = a\cos(\theta) + b\sin(\theta)$ is a circle. Find its center and radius.

Proofs

93. Use the Law of Cosines to prove the Distance Formula in polar coordinates:

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$$

Answers

Skills Refresher

1. $(x-2)^2+(y+1)^2=3^2$
2. $x^2+(y-9)^2=13^2$
3. $(x-5)^2+(y-3)^2=(3\sqrt{2})^2$
4. $(x-1)^2+y^2=4^2$

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9.2: Polar Graphs

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Graphing in Polar Coordinates

When we plot points in Cartesian coordinates, we start at the origin and move a distance right or left given by the x-coordinate of the point, then move up or down according to the y-coordinate. When we sketch the graph of an equation or function, we think of drawing the graph from left to right, with the "height" of the graph at each x-value given by the function, as shown in Figure $9.2.1a$.

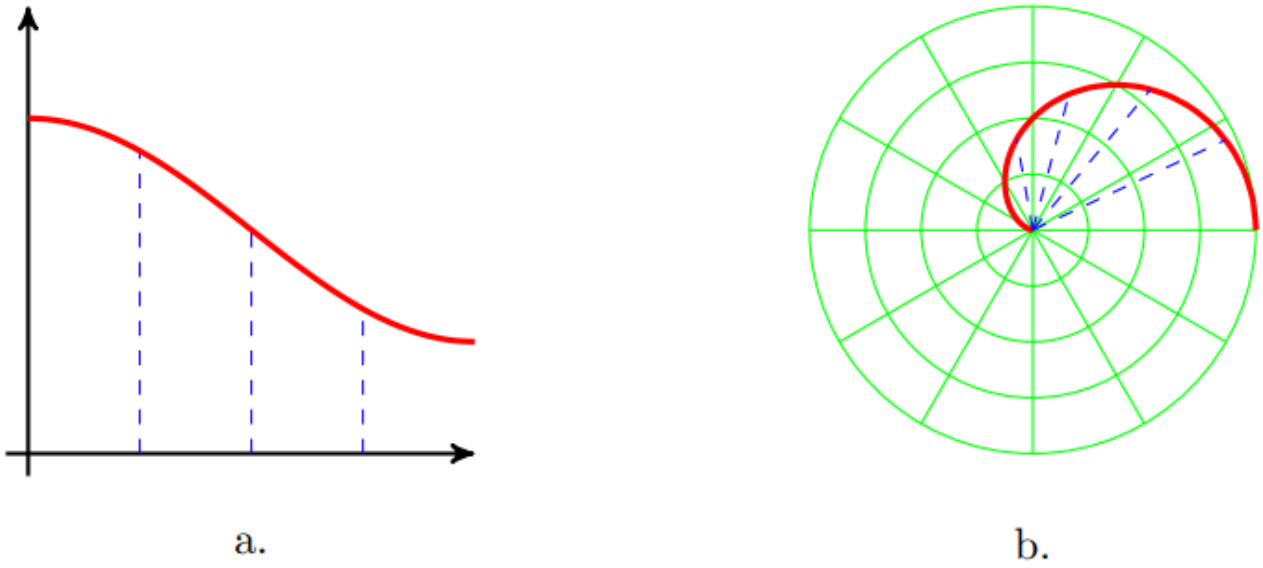


Figure $9.2.1$

In polar coordinates, however, the dependent variable, r, gives not a height but a distance from the pole in direction θ, as shown in Figure $9.2.1b$. When graphing an equation in polar coordinates, we think of sweeping around the pole in the counterclockwise direction. At each angle θ, the r-value tells us how far the graph is from the pole.

Caution: Coordinate Order in Polars vs Cartesians

We are used to listing Cartesian coordinates in alphabetical order, as (x, y)—in this respect, polar coordinates operate in a similar fashion. That is, the Greek letter θ starts with a t. Hence, polar coordinates are also written in alphabetical order, as (r, θ); however, the odd thing that we must get used to with polar coordinates is the placement of the dependent and independent variables.

In the Cartesian coordinate system, dependent and independent variables are listed in a coordinate pair as $(\text{Independent}, \text{Dependent}) = (x, y)$. This becomes more obvious when we use function notation, $(x, f(x))$.

In the polar coordinate system, r is often the *dependent* variable and, therefore, $theta$ is the *independent* variable. However, this means $(r, \theta) = (\text{Dependent}, \text{Independent})$. This swapping in roles might take some getting used to.

Example 9.2.1

Graph the polar equation $r = 2\sin(\theta)$.

Solution available at the end of this section.

To prove that the graph in Example 9.2.1 is a circle, we convert the equation $r = 2\sin(\theta)$ to Cartesian form. First, multiply both sides by r to obtain

$$r^2 = 2r\sin(\theta).$$

Next, replace r^2 by $x^2 + y^2$ and $r\sin(\theta)$ by y, to get

$$x^2 + y^2 = 2y.$$

This equation is quadratic in two variables. Hence, its graph is a conic section. We put the equation in standard form by completing the square in y.

$$\begin{aligned} & x^2 + y^2 = 2y \\ \Longrightarrow \quad & x^2 + y^2 - 2y = 0 && \text{(subtract } 2y \text{ from both sides)} \\ \Longrightarrow \quad & x^2 + (y^2 - 2y + 1) = 1 && \text{(complete the square)} \\ \Longrightarrow \quad & (x-0)^2 + (y-1)^2 = 1 && \text{(factor to write in standard form)} \end{aligned}$$

We have the equation of a circle with center $(0, 1)$ and radius 1.

Checkpoint 9.2.1

Graph the polar equation $r = 4\cos(\theta)$.

Answer available at the end of this section.

Using a Graphing Calculator

You are familiar with the graphs of many equations in Cartesian coordinates, including lines, parabolas, and other conic sections, as well as the graphs of basic functions. You should now become familiar with some standard graphs in polar coordinates. These include circles, **roses**, **cardioids**, **limaçons**, **lemniscates**, and **spirals**.

At the end of this section, you will find a Catalog of Polar Curves which contains the basic polar graphs and their properties. You can use your calculator, set in Polar mode, to experiment with these graphs. If you are using Desmos, there is no need to worry about Polar mode. Desmos is coded in such a way that it will graph a polar curve if you choose to use the variables r and θ.

Example 9.2.2

Graph the Archimedean spiral $r = 2\theta$, for $0 \le \theta \le 6\pi$.

Solution available at the end of this section.

Checkpoint 9.2.2

Use a calculator to graph the polar equation $r = e^{0.1\theta}$. Use the window settings

$$\theta\,\text{min} = 0 \quad \theta\,\text{max} = 12\pi \quad \theta\ \text{step} = 0.5$$
$$\text{Xmin} = -45 \quad \text{Xmax} = 45 \quad \text{Xstep} = 10$$
$$\text{Ymin} = -35 \quad \text{Ymax} = 35 \quad \text{Yscl} = 10$$

Answer available at the end of this section.

It is essential to connect the points on the graph in order of increasing θ.

Example 9.2.3

Graph the polar equation $r = 2\sin(3\theta)$.

Solution available at the end of this section.

Checkpoint 9.2.3

a. Use your calculator to graph the polar equation $r = 2\cos(3\theta)$.
b. Complete the table of values for the function.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r							

c. Sketch the graph by hand on the grid below.

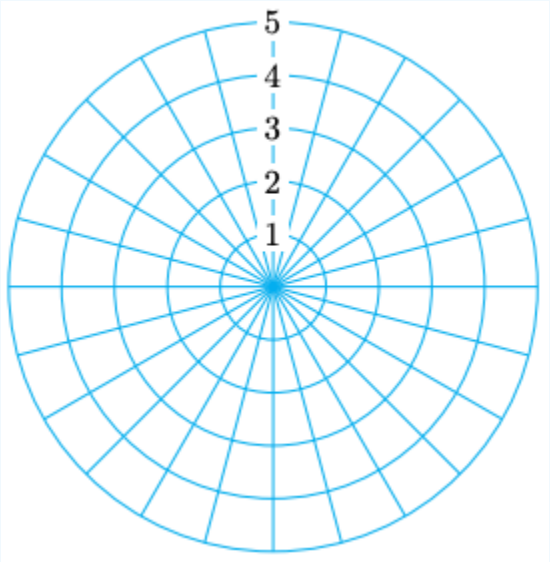


Figure 9.2.7

Answer available at the end of this section.

Sketching Familiar Equations

You should also be able to sketch the standard polar graphs by hand. Once you recognize an equation as a particular type of graph, say a rose or a limaçon, you can sketch the graph quickly by finding just a few well-chosen guide points. The following example demonstrates a technique for sketching a rose.

Example 9.2.4

Graph the polar equation $r = 3\sin(2\theta)$.

Solution available at the end of this section.

Checkpoint 9.2.4

Graph the polar equation $r = 4\cos(5\theta)$.

Answer available at the end of this section.

The **limaçons**, $r = a \pm \sin(\theta)$ and $r = a \pm \cos(\theta)$, are another family of polar graphs. In particular, the **cardioid** is a special case of a limaçon with $a = b$.

Example 9.2.5

Graph the polar equation $r = 3 + 2\cos(\theta)$.

Solution available at the end of this section.

Checkpoint 9.2.5

Graph the polar equation $r = 1 - 3\sin(\theta)$.

Answer available at the end of this section.

You should also be able to identify a polar graph and write its equation.

Example 9.2.6

Give a polar equation for each of the graphs below.

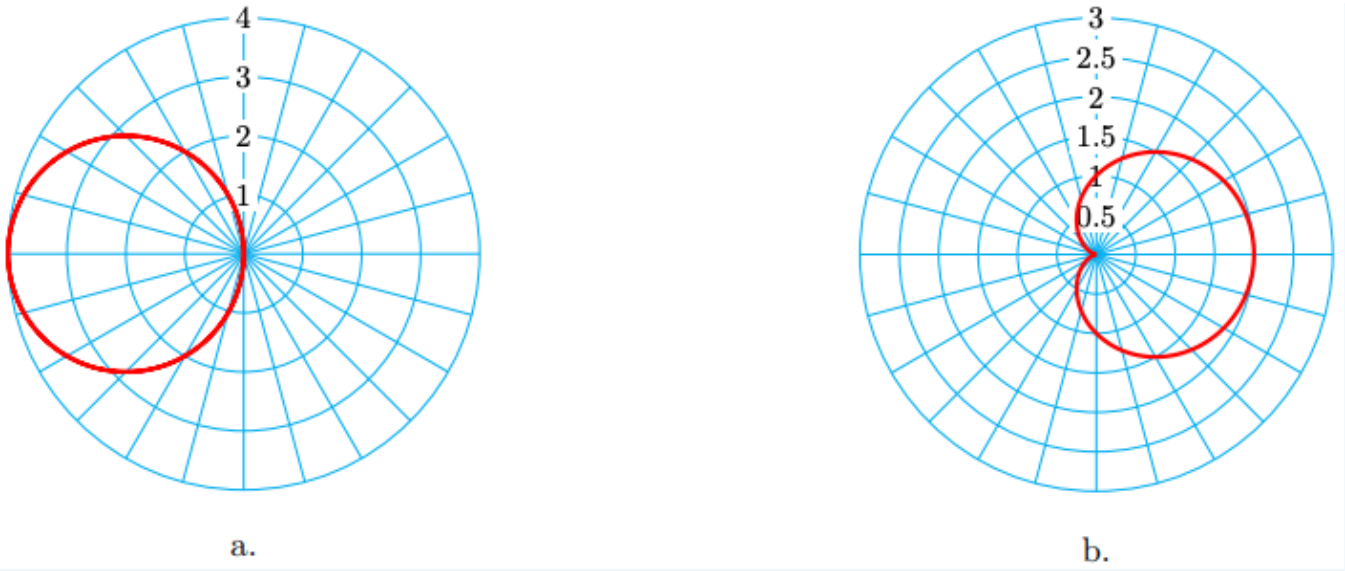


Figure 9.2.13

Solutions available at the end of this section.

Checkpoint 9.2.6

Give a polar equation for each of the graphs below.

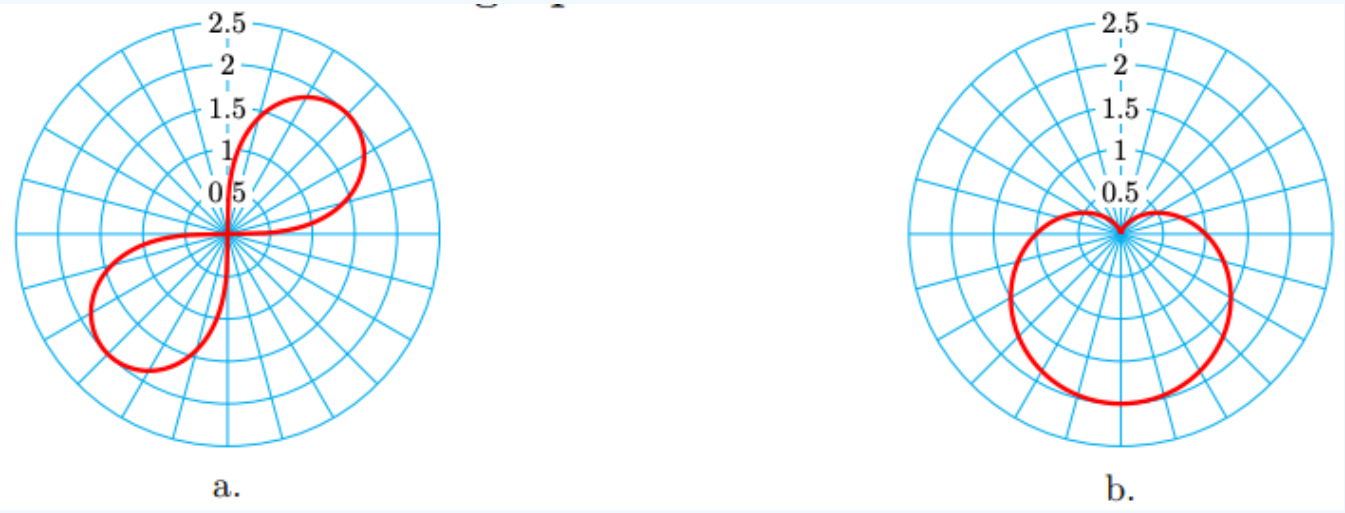


Figure 9.2.14

Answer available at the end of this section.

Finding Intersection Points

To find the intersection points of two graphs, we solve the system made up of their equations. If the equations are $y = f(x)$ and $y = g(x)$, we simply solve the equation $f(x) = g(x)$. For example, we find the intersection points of $y = x^2$ and $y = x + 2$ by solving the equation $x^2 - x - 2 = 0$ to get

$$\begin{aligned} x^2 - x - 2 &= 0 \\ \implies (x-2)(x+1) &= 0 \end{aligned}$$

Therefore, $x-2=0$ or $x+1=0$, which leads us to $x=2$ or $x=-1$. These are the x-coordinates of the intersection points, and we can find the y-coordinates by substituting these values into either equation.

- For $x=2$, we find $y=2^2=4$.
- For $x=-1$, we find $y=(-1)^2=1$.

Thus, the intersection points are $(2,4)$ and $(-1,1)$, as shown in Figure 9.2.15.

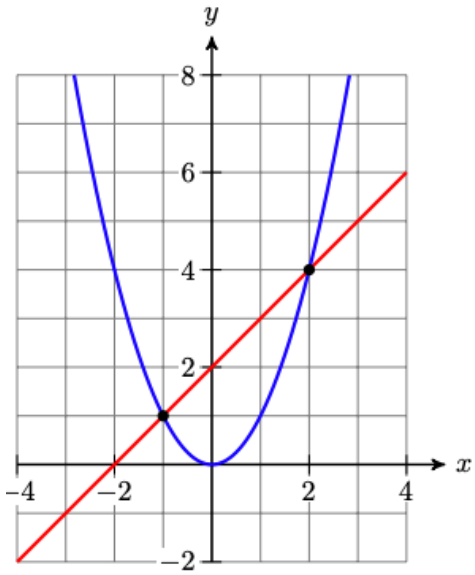


Figure 9.2.15

To find the intersection points of the polar graphs $r=f(\theta)$ and $r=g(\theta)$ we solve the equation $f(\theta)=g(\theta)$.

Example 9.2.7

Find all intersection points of the graphs of $r=2+2\sin(\theta)$ and $r=2+2\cos(\theta)$.

Solution available at the end of this section.

Caution: Check the Pole Separately!

As we saw in Example 9.2.7, solving a system of equations $r=f(\theta)$ and $r=g(\theta)$ will not always reveal an intersection at the pole, because r may be equal to zero for different values of θ in the two equations. We should always check separately whether the pole is a point on both graphs.

Checkpoint 9.2.7

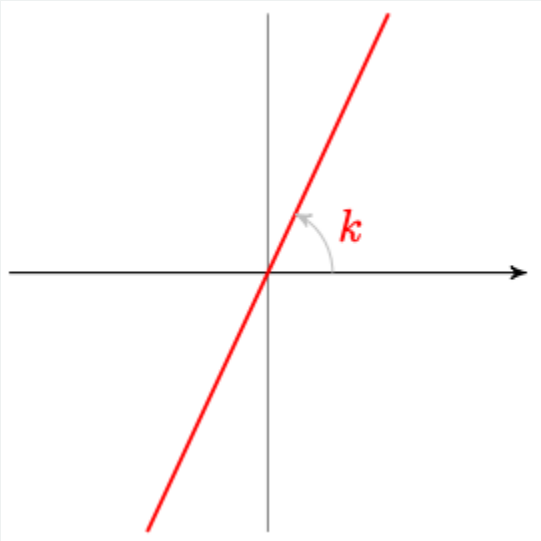
Find all intersection points of the graphs of $r=1$ and $r=2\cos(\theta)$.

Answer available at the end of this section.

Catalog of Polar Curves

Lines

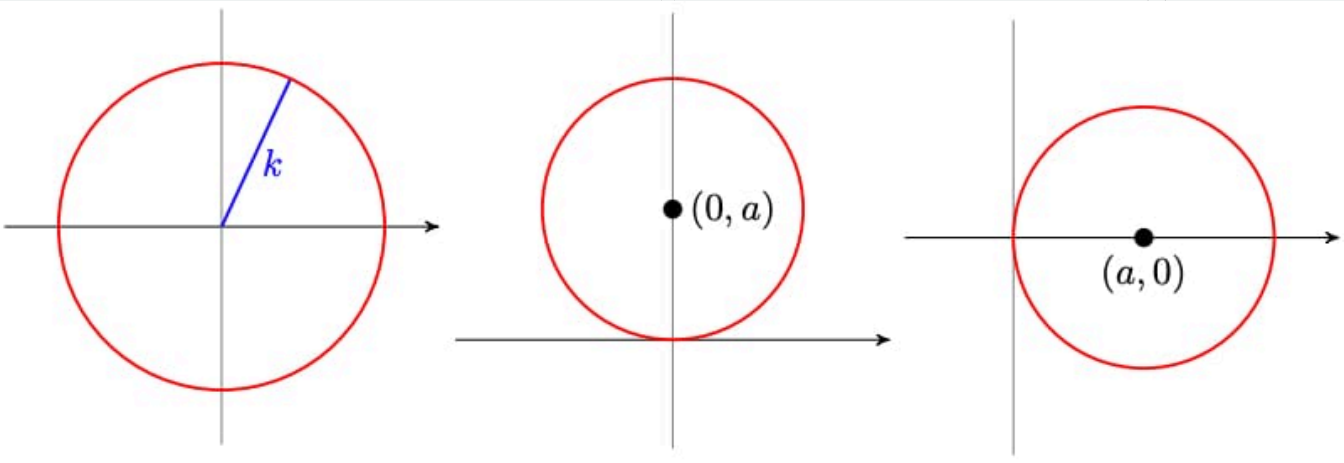
Line (through the pole)
$\theta = k$, *where* k *a constant*
k *gives the angle of inclination of the line (in radians)*



Circles

Circles

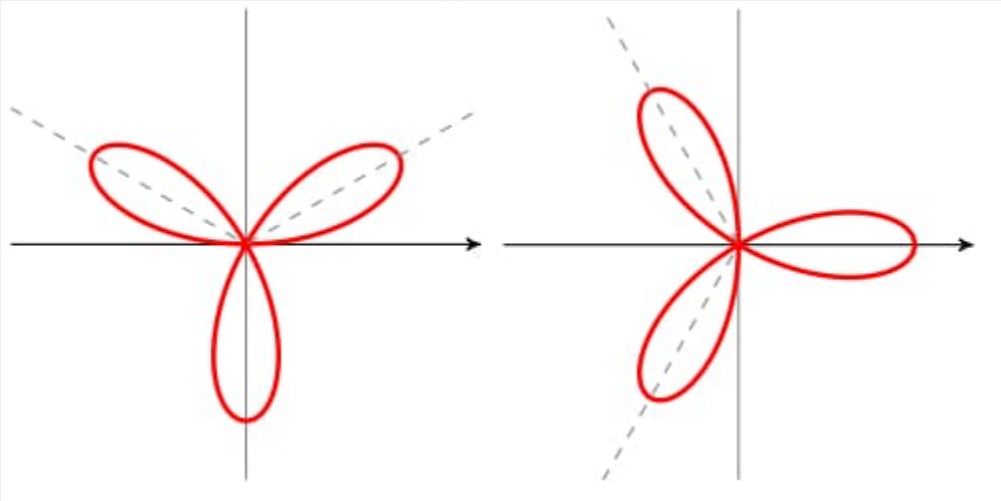
Left (Center at Pole): $r = k$, *where* k *a constant;* k *is the radius of the circle*
Middle (Centered on the vertical axis): $r = 2a\sin(\theta)$*; Center (0,* a*) on the vertical axis and radius* $|a|$
Right (centered on the horizontal axis): $r = 2a\cos(\theta)$*; Center (*a*, 0) on the horizontal axis and radius* $|a|$



Roses

Roses

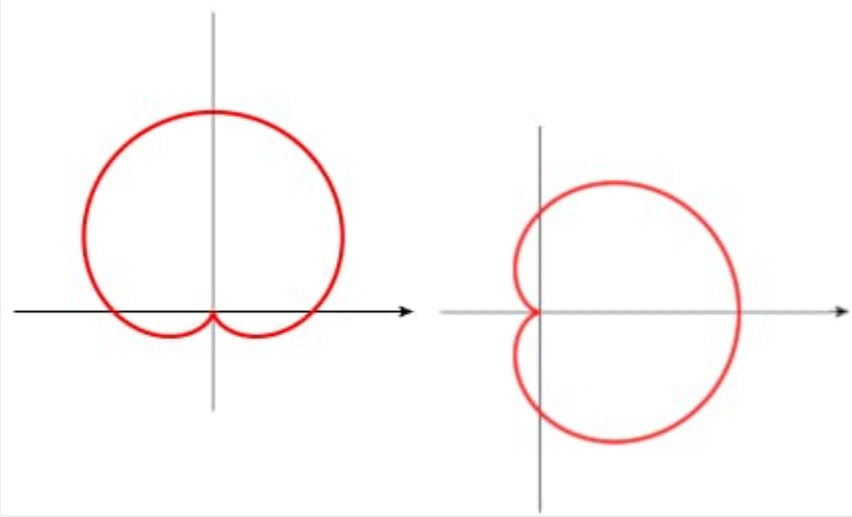
Left: $r = a\sin(n\theta)$; petal length $|a|$; n petals if n is odd; $2n$ petals if n is even
Right: $r = a\cos(n\theta)$; petal length $|a|$; n petals if n is odd; $2n$ petals if n is even



Cardioids

Cardioids

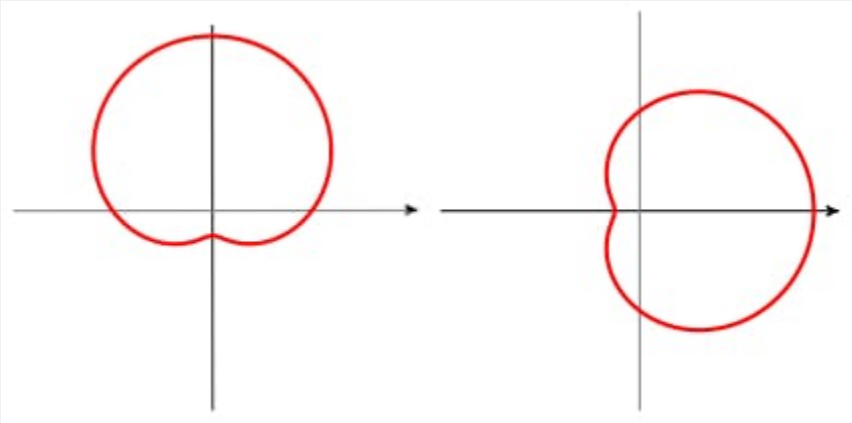
Left: $r = a \pm b\sin(\theta)$, where $|a| = |b|$; touches pole; "heavy" on positive vertical axis if $b > 0$; "heavy" on negative vertical axis if $b < 0$
Right: $r = a \pm b\cos(\theta)$, where $|a| = |b|$; touches pole; "heavy" on positive horizontal axis if $b > 0$; "heavy" on negative horizontal axis if $b < 0$



Limaçons

Limaçons

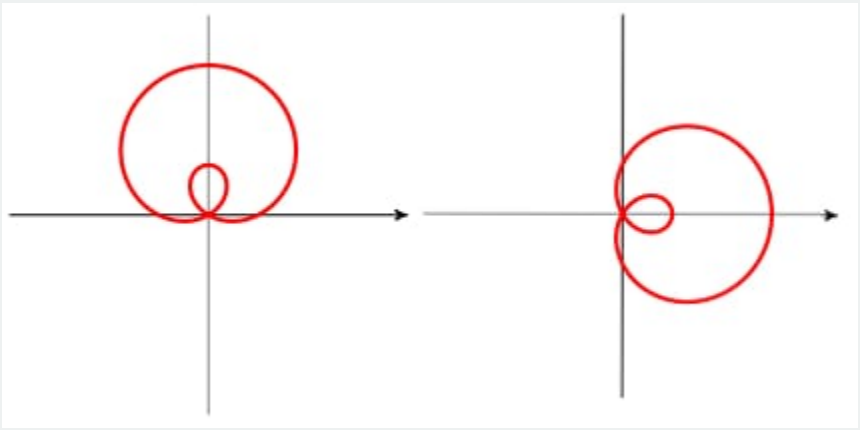
Left: $r = a \pm b\sin(\theta)$, *where* $|a| > |b|$*; doesn't touch pole; "heavy" on positive vertical axis if* $b > 0$*; "heavy" on negative vertical axis if* $b < 0$
Right: $r = a \pm b\cos(\theta)$, *where* $|a| > |b|$*; doesn't touch pole; "heavy" on positive horizontal axis if* $b > 0$*; "heavy" on negative horizontal axis if* $b < 0$



Limaçons with Inner Loops

Limaçons with Inner Loops

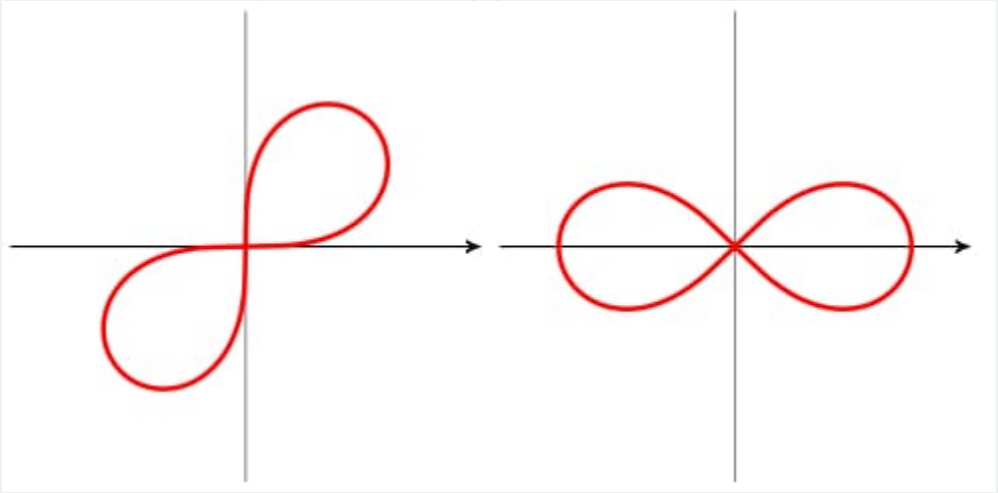
Left: $r = a \pm b\sin(\theta)$, *where* $|a| < |b|$*; goes through pole; "heavy" on positive vertical axis if* $b > 0$*; "heavy" on negative vertical axis if* $b < 0$
Right: $r = a \pm b\cos(\theta)$, *where* $|a| < |b|$*; goes through pole; "heavy" on positive horizontal axis if* $b > 0$*; "heavy" on negative horizontal axis if* $b < 0$



Lemniscates

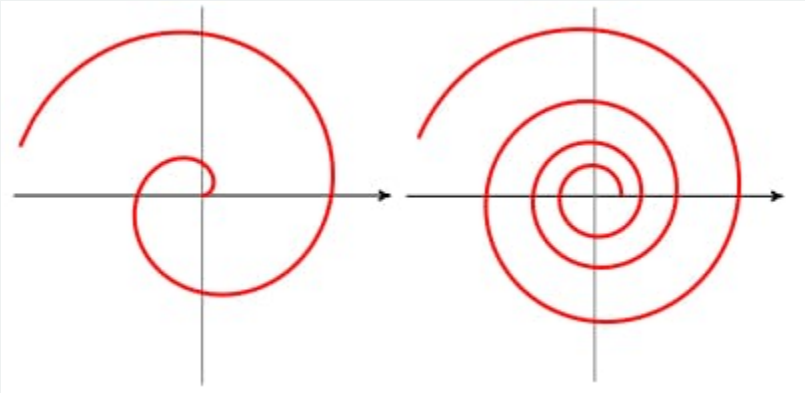
Lemniscates

Left: $r^2 = a^2\sin(2\theta)$*; goes through pole*
Right: $r^2 = a^2\cos(2\theta)$*; goes through pole*



Spirals

Spirals
*Left (**Archimedean Spiral**): $r = a\theta$*
*Right (**Logarithmic Spiral**): $r = ae^{b\theta}$*



Success in College: Preparing for the Final Exam (Part 2)

1. Think about the concepts and skills you have learned this semester. Describe two concepts or skills that were most difficult for you.
2. Describe what you are doing to ensure you can answer the questions about these concepts or skills on the final exam.

Answers

Section Preview

There's not much to say here other than that you will be covering the standard polar curves. Symmetry properties of polar curves are not focused on in this section for the sake of time.

Section Preview

- **Solving Equations**
  - Solving Quadratic Equations by Completing the Square: The algebraic verification that $r = 2\sin\theta$ is a circle uses completing the square.

Section Preview

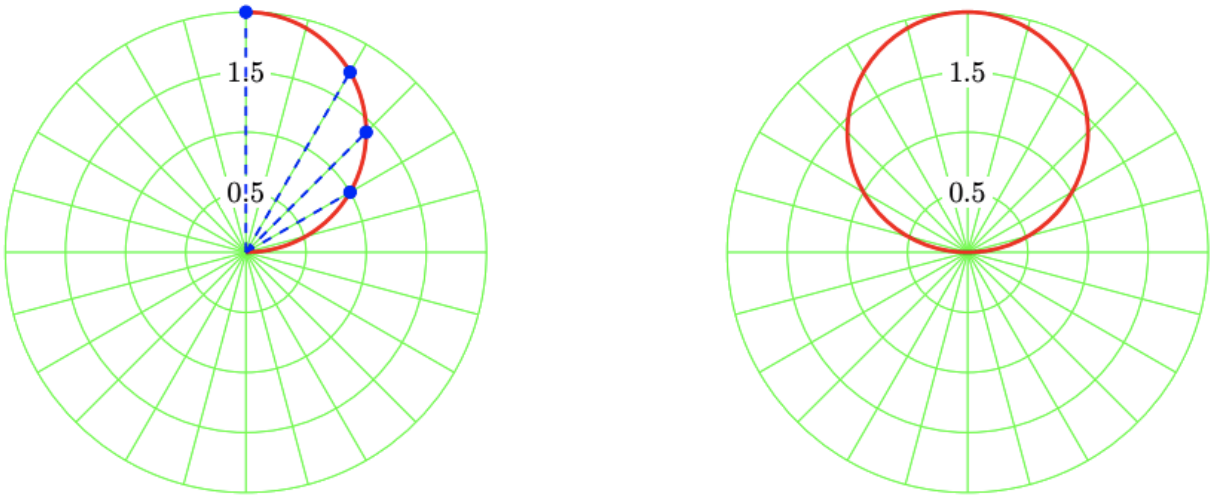
- Graph an equation in polar coordinates by plotting points.
- Graph an equation in polar coordinates using a graphing calculator.
- Identify the graph of a polar equation.
- Determine the intersection points of polar curves.

Example $9.2.1$

We begin by making a table of values. Since our independent variable is typically θ, we begin by choosing values for θ. For each value of θ, we evaluate $r = 2\sin(\theta)$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{2}$	1	0

Remember, when plotting the polar coordinates, they are ordered as $r, \theta)$. Let's plot the points in the first quadrant. Observe that as θ increases from 0 to $\frac{\pi}{2}$, r increases from 0 to 2. Starting at the pole, we connect the points in order of increasing θ. Imagine a radial line sweeping around the graph through the first quadrant: as the angle increases, the length of the segment increases so that its tip traces out the graph shown in Figure $9.2.2a$. Now continue plotting the points in the table as θ increases from $\frac{\pi}{2}$ to π. In the second quadrant, r decreases as θ increases, as shown in Figure $9.2.2b$. The graph we obtain is, in fact, a circle, which we will prove algebraically shortly.



a. b.

Figure 9.2.2

However, we have not yet plotted points for θ between 0 and 2π. Because $\sin(\theta)$ is negative in the third and fourth quadrants, all the r-values for these angles are negative.

θ	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
r	0	-1	$-\sqrt{2}$	$-\sqrt{3}$	-2	$-\sqrt{3}$	$-\sqrt{2}$	-1	0

Plotting the points in this table shows that the original graph is traced out again. For example, the point $\left(\frac{7\pi}{6}, -1\right)$ is the same as the point $\left(\frac{\pi}{6}, 1\right)$, the point $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$ is the same as the point $\left(\frac{\pi}{4}, \sqrt{2}\right)$, and so on, around the circle. Thus, the graph of $r = 2\sin(\theta)$ is a circle, traced twice for $0 \leq \theta \leq 2\pi$.

Checkpoint 9.2.1

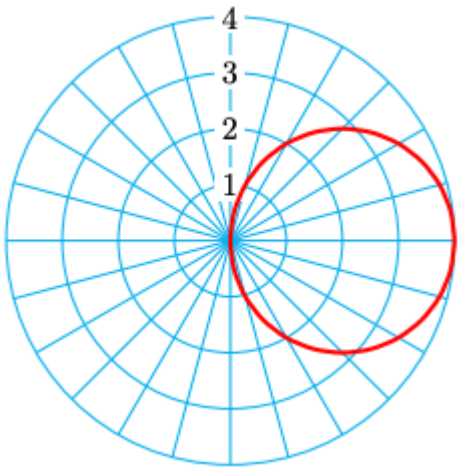


Figure 9.2.3

Example 9.2.2

After setting the calculator in Polar mode, we enter the equation $r = 2\theta$ and enter the window settings

$$\theta\min = 0 \quad \theta\max = 6\pi \quad \theta \text{ step } = 0.1$$
$$\text{Xmin} = -25 \quad \text{Xmax} = 25 \quad \text{X step } = 5$$
$$\text{Ymin} = -25 \quad \text{Ymax} = 25 \quad \text{Yscl} = 5$$

We then press Zoom 5 to set a square window.

The graph is shown below.

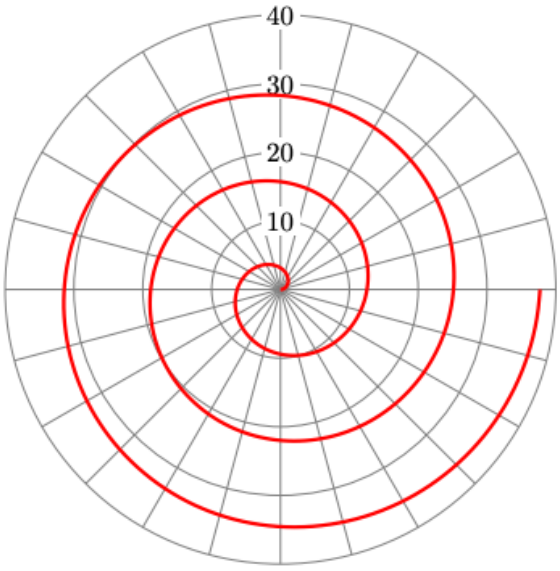


Figure 9.2.4

Studying a table of values can help us understand the shape of the graph.

θ	0.25	0.50	0.75	1.00	1.25
r	0.50	1.00	1.50	2.00	2.50

As θ increases, r also increases at a constant rate. We wind our way around the pole, steadily increasing our distance from the pole as we go. We spiral outwards, tracing the graph shown above.

Checkpoint 9.2.2

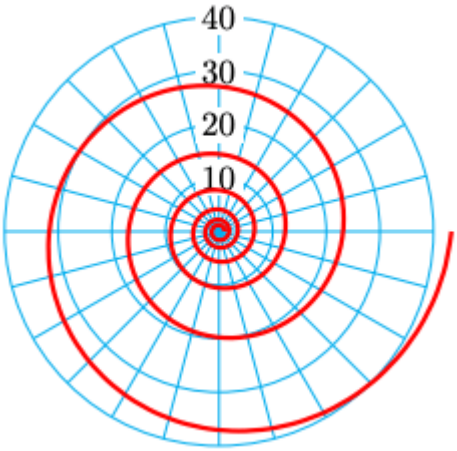


Figure 9.2.5

Example 9.2.3

We'll graph the equation in stages to see how the graph is traced out. Begin with the window settings

$$\theta\,\text{min} = 0 \quad \theta\,\text{max} = \tfrac{\pi}{3} \quad \theta\ \text{step} = 0.5$$
$$\text{Xmin} = -4.5 \quad \text{Xmax} = 4.5 \quad \text{Xstep} = 1$$
$$\text{Ymin} = -3 \quad \text{Ymax} = 3 \quad \text{Yscl} = 1$$

Watch as your calculator produces the graph shown in Figure 9.2.6a.

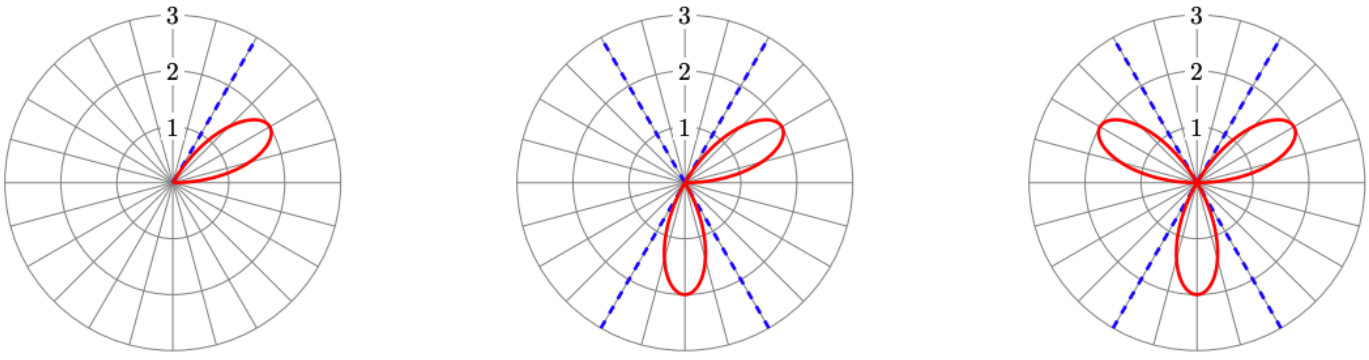


Figure 9.2.6

Observe that, as θ increases from 0 to $\frac{\pi}{3}$, r first increases, reaching its maximum value of 2 at $\theta = \frac{\pi}{6}$ and then decreases back to 0. You can verify the values in the table below, which shows the points at multiples of $\frac{\pi}{18}$. These points create the first loop of the graph.

θ	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{2\pi}{9}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$
3θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	0	1	$\sqrt{3}$	2	$\sqrt{3}$	1	0

Next, change $\theta \max$ to $\frac{2\pi}{3}$, and graph again. The calculator traces out two loops this time, as shown in Figure 9.2.6*b*. For θ between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, r is negative, so the second loop lies in the third and fourth quadrants.

θ	$\frac{\pi}{3}$	$\frac{7\pi}{18}$	$\frac{4\pi}{9}$	$\frac{\pi}{2}$	$\frac{5\pi}{9}$	$\frac{11\pi}{18}$	$\frac{2\pi}{3}$
3θ	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	0	-1	$-\sqrt{3}$	-2	$-\sqrt{3}$	-1	0

Finally, change $\theta \max$ to π, and graph again. For θ between $\frac{2\pi}{3}$ and π, the graph traces out a third loop, as shown in Figure 9.2.6*c*. For θ between π and 2π, the entire graph is traced a second time. The finished graph is a **rose** with 3 petals of length 2.

Checkpoint 9.2.3

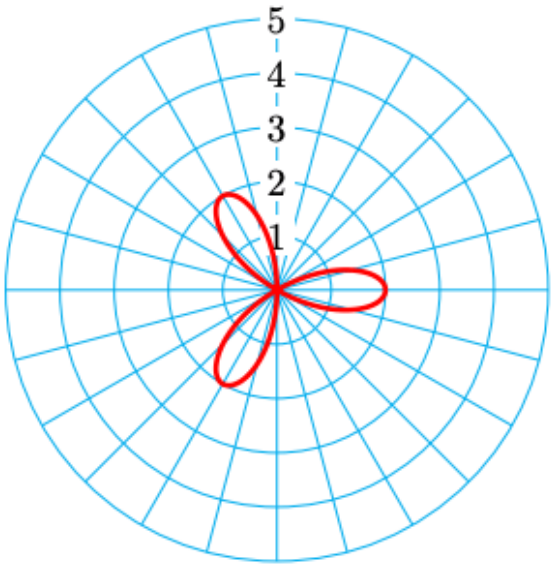


Figure 9.2.8

Example 9.2.4

Referring to the Catalog of Polar Curves, we see that the graph of this equation is a rose, with petal length $a = 3$ and four petals, because $2n = 4$. If we can locate the tips of the petals, we can use them as guide points to sketch the graph.

The points at the tips of the petals have $r = 3$, so we substitute $r = 3$ into the equation of the rose to find the values of θ at those points.

$$\begin{aligned} 3 &= 3\sin(2\theta) && \text{(letting } r=3\text{)} \\ \Longrightarrow \quad 1 &= \sin(2\theta) && \text{(dividing both sides by 3)} \\ \Longrightarrow \quad \frac{\pi}{2} &= 2\theta && \text{(using the arcsine to find } 2\theta\text{)} \\ \Longrightarrow \quad \frac{\pi}{4} &= \theta && \text{(dividing both sides by 2)} \end{aligned}$$

Thus, one of the petal tips is located at $\theta = \frac{\pi}{4}$. Because the 4 petals are evenly spaced around the pole, the angle between the petals is $\frac{2\pi}{4} = \frac{\pi}{2}$, and the other petal tips occur at

$$\begin{aligned} \theta &= \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4} \\ \theta &= \frac{\pi}{4} + 2\left(\frac{\pi}{2}\right) = \frac{5\pi}{4} \\ \theta &= \frac{\pi}{4} + 3\left(\frac{\pi}{2}\right) = \frac{7\pi}{4} \end{aligned}$$

We plot the tips of the petals as guide points at $(3, \frac{\pi}{4})$, $(3, \frac{3\pi}{4})$, $(3, \frac{5\pi}{4})$ and $(3, \frac{7\pi}{4})$. We can sketch a rose with 4 petals of length 3, as shown below.

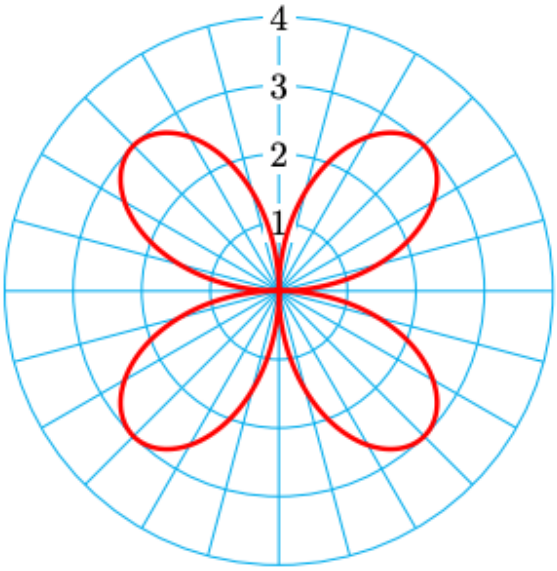


Figure 9.2.9

Checkpoint 9.2.4

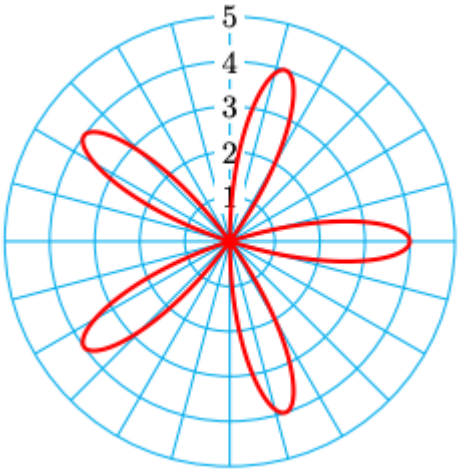


Figure 9.2.10

Example 9.2.5

This is the equation for a limaçon, with $a = 3$ and $b = 2$. Because $a > b$, the limaçon will have a dent, like a lima bean, rather than a loop. As guide points, we locate the points at the four quadrantal angles.

θ	$r = 3 + 2\cos(\theta)$
0	5

θ	$r = 3 + 2\cos(\theta)$
$\frac{\pi}{2}$	3
π	1
$\frac{3\pi}{2}$	3

We plot the guide points and connect them in order with a smooth curve, as shown below. Note that this limaçon involving cosine is symmetric about the x-axis; limaçons involving sine are symmetric about the y-axis.

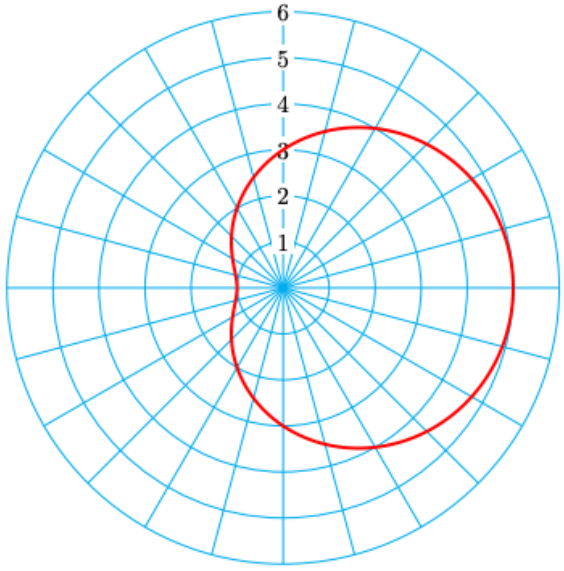


Figure 9.2.11

Checkpoint 9.2.5

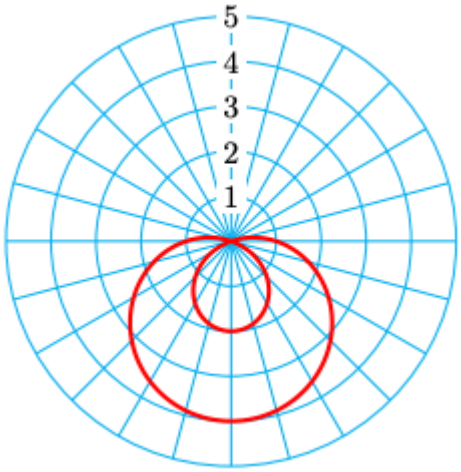


Figure 9.2.12

Example 9.2.6

a. The graph is a circle centered at $(-2, 0)$ and with radius 2, so we choose the equation $r = 2a\cos(\theta)$, with $a = -2$. Thus, $r = -4\cos(\theta)$.
b. The graph is a cardioid with its axis of symmetry on the x-axis, and the "bottom" of the heart points in the positive x direction, so its equation has the form $r = a + a\cos(\theta)$. At $\theta = 0$, $r = 2$, so we can solve for a:

$$\begin{aligned} 2 &= a + a\cos(0) \\ \Longrightarrow \quad 2 &= a + a(1) \\ \Longrightarrow \quad 2 &= 2a \end{aligned}$$

Thus, $a = 1$ and we choose the equation $r = 1 + \cos(\theta)$.

Checkpoint 9.2.6

a. $r^2 = 4\sin(2\theta)$
b. $r = 1 - \sin(\theta)$

Example 9.2.7

We equate the two expressions for r and solve for θ.

$$\begin{aligned} 2+2\sin(\theta) &= 2+2\cos(\theta) && \text{(equating the two equations)} \\ \Longrightarrow \quad \sin(\theta) &= \cos(\theta) && \text{(subtracting 2 from both sides and then dividing both sides by 2)} \\ \Longrightarrow \quad \tan(\theta) &= 1 && \text{(dividing both sides by } \cos(\theta)\text{)} \\ & && \textbf{WARNING: check solutions)} \\ \Longrightarrow \quad \theta &= \frac{\pi}{4} \text{ or } \frac{5\pi}{4} && \text{(solutions on } [0,2\pi)\text{)} \end{aligned}$$

We evaluate either expression for θ to find the other coordinate of the intersection point.

$$\text{when} \quad \theta = \frac{\pi}{4}, \quad r = 2+2\sin\left(\frac{\pi}{4}\right) = 2+\left(\frac{1}{\sqrt{2}}\right) = 2+\sqrt{2}$$

and

$$\text{when} \quad \theta = \frac{5\pi}{4}, \quad r = 2+2\sin\left(\frac{5\pi}{4}\right) = 2+\left(\frac{-1}{\sqrt{2}}\right) = 2-\sqrt{2}.$$

Thus, two of the intersection points are $\left(2+\sqrt{2}, \frac{\pi}{4}\right)$ and $\left(2-\sqrt{2}, \frac{5\pi}{4}\right)$, as shown in Figure 9.2.16.

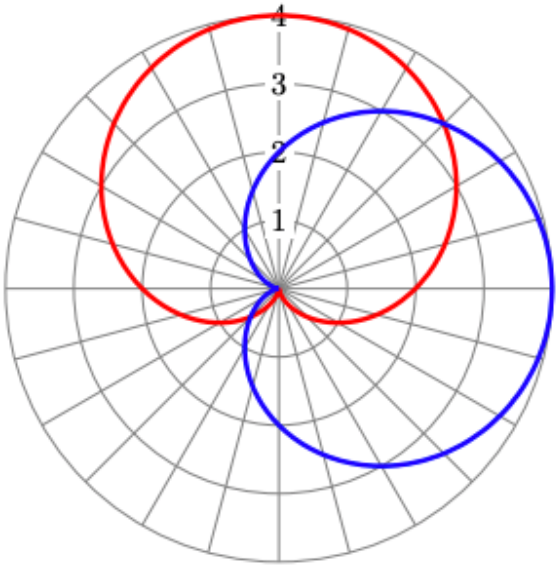


Figure 9.2.16

However, you can see in the figure that the graphs also appear to intersect at the pole. To verify that the pole indeed lies on both graphs, we can solve for θ in each equation when $r = 0$.

$$\begin{aligned} 0 &= 2+2\sin(\theta) & 0 &= 2+2\cos(\theta) \\ \Longrightarrow \quad -1 &= \sin(\theta) & -1 &= \cos(\theta) \\ \Longrightarrow \quad \theta &= \frac{3\pi}{2} & \theta &= \pi \end{aligned}$$

Both points, $\left(0, \frac{3\pi}{2}\right)$ and $(0, \pi)$, represent the pole. Thus, the graphs intersect at three points: $\left(2+\sqrt{2}, \frac{\pi}{4}\right)$, $\left(2-\sqrt{2}, \frac{5\pi}{4}\right)$, and the pole.

Checkpoint 9.2.7

$\left(1, \frac{\pi}{3}\right)$, $\left(1, \frac{5\pi}{3}\right)$

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9.2.1: Resources and Key Concepts

Resources

Prerequisite Topics

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Systems**
  - Solving Systems of Nonlinear Equations: Finding the intersection points of two polar curves is equivalent to solving a system of nonlinear equations.

Videos

- Polar Graphs
  - Graphs of Polar Equations - The General Idea Behind Graphing Polar Curves
  - Graphs of Polar Equations - Graphing Polar Functions of the Form $r(\theta) = A\sin(\theta)$ or $r(\theta) = A\cos(\theta)$
  - Graphs of Polar Equations - Graphing Polar Functions of the Form $r(\theta) = A \pm B\sin(\theta)$ or $r(\theta) = A \pm B\cos(\theta)$
  - Graphs of Polar Equations - Graphing Polar Functions of the Form $r(\theta) = A\sin(B\theta)$ or $r(\theta) = A\cos(B\theta)$
  - Graphs of Polar Equations - Graphing Polar Functions of the Form $r(\theta) = c$ or $\theta = c$
  - Graphs of Polar Equations - Naming the Polar Curves

Key Concepts

Definitions

- This section focuses on graphing polar equations and does not introduce any new vocabulary terms. The names of various polar curves (circles, roses, cardioids, etc.) are introduced as categories of graphs.

Common Mistakes

- **Incorrectly Tracing the Graph:** When plotting points for a polar graph, it is essential to connect them in order of increasing θ. Failing to do so can result in an incorrect shape, especially for curves like limaçons and roses that may trace over themselves or have loops.
- **Missing Intersection Points at the Pole:** When finding the intersection of two polar graphs $r = f(\theta)$ and $r = g(\theta)$, solving $f(\theta) = g(\theta)$ may not reveal an intersection at the pole. This is because the pole can be represented by $r = 0$ for many different values of θ. You must always check separately if the pole ($r = 0$) is a point on both graphs.
- **Assuming a Single Representation:** A single polar graph can often be represented by multiple different equations. For example, a circle centered on the y-axis is typically written with sine, but a phase-shifted cosine function could produce the same graph.

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9.2.2: Homework

Reading Questions

1. When graphing a polar equation, what does the variable r represent at each angle θ?
2. Explain why the graph of $r = 2\sin(\theta)$ for $0 \leq \theta \leq 2\pi$ traces the circle twice.
3. What is the general shape of the polar graph for an equation of the form $r = a\sin(n\theta)$ when n is an integer?
4. How do you determine the number of petals for a rose curve $r = a\cos(n\theta)$?
5. What is the difference between the graph of a cardioid and a limaçon with a dent?
6. What is the general strategy for finding the points of intersection of two polar graphs, $r = f(\theta)$ and $r = g(\theta)$?
7. Why must you check for an intersection at the pole separately from solving the equation $f(\theta) = g(\theta)$?
8. What is the shape of the graph for the polar equation $r = k$, where k is a constant?
9. What is the shape of the graph for the polar equation $\theta = k$, where k is a constant?
10. In the equation for a limaçon, $r = a + b\cos(\theta)$, what condition on a and b results in a cardioid? What condition results in a limaçon with an inner loop?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

Solve each equation. Give two solutions, $0 \leq \theta \leq 2\pi$.

1. $\cos(\theta) = \dfrac{\sqrt{3}}{2}$
2. $\sin(\theta) = \dfrac{1}{\sqrt{2}}$
3. $\tan(\theta) = \sqrt{3}$
4. $\tan(\theta) = \dfrac{1}{\sqrt{3}}$
5. $\sin(\theta) = \dfrac{11}{\sqrt{2}}$
6. $\cos(\theta) = \dfrac{-1}{2}$
7. $\tan(\theta) = -1$
8. $\cos(\theta) = 0$

Answers available at the end of this section.

Homework

Concept Check

1. Mike says that the graph of $r = 2\sin(\theta)$ in the Example $9.2.2.1$ cannot be correct because there are no points on the graph for

angles between π and 2π. How do you respond?

2. Is it possible to have a rose with only two petals? What would its equation be?

3. Linda says that a circle of the form $r = 2a\cos(\theta)$ is a special case of a limaçon. Support or refute her statement.

4. There are no points on the graph of $r^2 = 4\cos(2\theta)$ for angles between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ Why is that?

Basic Skills

For the following exercises, use your calculator to graph the equation.

5. a. Graph $r = k$, for $k = 1, 2, 3$. How do these graphs change for different values of k?

   b. Write a Cartesian equation for each graph in part (a).

6. a. Graph $r = k$, for $k = -1, -2, -3$. How do these graphs compare to the graphs in Problem 5?

   b. Write a Cartesian equation for each graph in part (a).

7. a. Graph $\theta = k$, for $k = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$. How do the graphs change for different values of k?

   b. Write a Cartesian equation for each graph in part (a).

8. a. Graph $\theta = k$, for $k = \frac{7\pi}{6}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{6}$. How do the graphs change for different values of k?

   b. Write a Cartesian equation for each graph in part (a).

9. Complete the table of values for each equation. Plot the points in order of increasing θ. What is different about the two graphs?

   a. $r = 2$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$
$r = 2$								

   b. $r = -2$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$
$r = -2$								

10. Graph each line and label the points with their coordinates. How are the points on the two lines related?

   a. $\theta = \frac{\pi}{4}$

$\theta = \frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$
r	-2	-1	0	1	2

   b. $\theta = \frac{5\pi}{4}$

$\theta = \frac{5\pi}{4}$	$\frac{5\pi}{4}$	$\frac{5\pi}{4}$	$\frac{5\pi}{4}$	$\frac{5\pi}{4}$	$\frac{5\pi}{4}$

r	-2	-1	0	1	2

11. a. Graph the circle $r = 4\cos(\theta)$. Label the points corresponding to $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, and π.

    b. Complete the table of values. What happens to the graph as θ increases from π to 2π?

θ	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r					

    c. Find the center and radius of the circle.

    d. Give the Cartesian equation of the circle.

12. a. Graph the circle $r = 4\sin(\theta)$. Label the points corresponding to $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, and π.

    b. Complete the table of values. What happens to the graph as θ increases from π to 2π?

θ	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r					

    c. Find the center and radius of the circle.

    d. Give the Cartesian equation of the circle.

13. a. Graph $r = 2a\sin(\theta)$ for $a = -2, -1, 1, 2$.

    b. How do the graphs change for different values of a?

14. a. Graph $r = 2a\cos(\theta)$ for $a = -2, -1, 1, 2$.

    b. How do the graphs change for different values of a?

15. Complete the table of values for each cardioid and graph the equation.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r					

    a. $r = 1 + \sin(\theta)$

    b. $r = -1 + \sin(\theta)$

    c. $r = 1 - \sin(\theta)$

    d. $r = -1 - \sin(\theta)$

16. Complete the table of values for each cardioid and graph the equation.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r					

    a. $r = 1 + \cos(\theta)$

b. $r = -1 + \cos(\theta)$

c. c $r = 1 - \cos(\theta)$

d. $r = -1 - \cos(\theta)$

17. Complete the table of values for each limaçon and graph the equation.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r					

a. $r = 2 + \cos(\theta)$

b. $r = 2 - \cos(\theta)$

c. $r = 1 + 2\cos(\theta)$

d. $r = 1 - 2\cos(\theta)$

18. Complete the table of values for each limaçon and graph the equation.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r					

a. $r = 2 + \sin(\theta)$

b. $r = 2 - \sin(\theta)$

c. $r = 1 + 2\sin(\theta)$

d. $r = 1 - 2\sin(\theta)$

19. a. Graph the following roses and compare. How is the number of petals related to the value of n in the equation $r = a\sin n\theta$?

$$r = \sin(2\theta), \quad r = \sin(3\theta), \quad r = \sin(4\theta), \quad r = \sin(5\theta) \backslash)$$

b. For each graph above, list the values of θ where the tips of the petals occur.

c. Graph $r = a\sin(3\theta)$ for $a = 1$, 2, and 3. How does the value of a affect the graph?

20. a. Graph the following roses and compare. How is the number of petals related to the value of n in the equation $r = a\cos n\theta$?

$$r = \cos(2\theta), \quad r = \cos(3\theta), \quad r = \cos(4\theta), \quad r = \cos(5\theta)$$

b. For each graph above, list the values of θ where the tips of the petals occur.

c. Graph $r = a\cos(3\theta)$ for $a = 1, 2$, and 3 . How does the value of a affect the graph?

21. a. Solve $r^2 = 9\cos(2\theta)$ for r. (You should get two equations for r.)

b. Graph both equations together. Change the θ step to 0.02 to see the whole graph.

c. How does the value of a affect the graph of $r^2 = a^2\cos(2\theta)$?

22. a. Solve $r^2 = 9\cos(2\theta)$ for r. (You should get two equations for r.)

b. Graph both equations together. Change the θ step to 0.02 to see the whole graph.

c. How does this graph differ from the graph in Problem 21?

23. Graph the Archimedean spiral $r = \theta$. Set your window to

$$\theta\min = 0 \quad \theta\max = 8\pi$$
$$\text{Xmin} = -20 \quad \text{X max} = 20$$
$$\text{Y min} = -20 \quad \text{Y max} = 20$$

Then graph by pressing Zoom 5.

24. Graph the logarithmic spiral $r = e^{0.2\theta}$. Set your window to

$$\theta\min = 0 \quad \theta\max = 8\pi$$
$$\text{Xmin} = -100 \quad \text{X max} = 100$$
$$\text{Ymin} = -100 \quad \text{Y max} = 100$$

Then graph by pressing Zoom 5.

25. a. Complete the table and graph the equation $y = \sin(3\theta)$ in Cartesian coordinates, for $0 \le \theta \le 2\pi$.

θ									
3θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
y									

b. Complete the table and graph the equation $r = \sin(3\theta)$ in polar coordinates, for $0 \le \theta \le 2\pi$.

θ									
3θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r									

26. a. Complete the table and graph the equation $y = \cos(2\theta)$ in Cartesian coordinates, for $0 \le \theta \le 2\pi$.

θ									
2θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
y									

b. Complete the table and graph the equation $r = \cos(2\theta)$ in polar coordinates, for $0 \le \theta \le 2\pi$.

θ									
2θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r									

27. a. Complete the table and graph the equation $y = 2 + 2\cos(\theta)$ in Cartesian coordinates, for $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
y									

b. Complete the table and graph the equation $r = 2 + 2\cos(\theta)$ in polar coordinates, for $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r									

28. a. Complete the table and graph the equation $y = 1 - \sin(\theta)$ in Cartesian coordinates, for $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
y									

b. Complete the table and graph the equation $r = 1 - \sin(\theta)$ in polar coordinates, for $0 \le \theta \le 2\pi$.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r									

For the following exercises, use the catalog of polar graphs to help you identify and sketch the curve. Check your work by graphing with a calculator.

29. $r = 3\cos(\theta)$

30. $r = 2\sin(\theta)$

31. $\theta = \frac{\pi}{4}$

32. $\theta = \frac{4\pi}{3}$

33. $r = 4$

34. $r = 2$

35. $r = 2 + 2\sin(\theta)$

36. $r = 3 + 3\cos(\theta)$

37. $r = 2 - \cos(\theta)$

38. $r = 1 - 3\sin(\theta)$

39. $r = 3\sin(2\theta)$

40. $r = 2\cos(3\theta)$

41. $r = 2\cos(5\theta)$

42. $r = 4\sin(4\theta)$

43. $r = 2 + 3\sin(\theta)$

44. $r = 3 + 2\sin(\theta)$

45. $r^2 = \cos(2\theta)$

46. $r^2 = 4\sin(2\theta)$

For the following exercises, identify the curve and graph it.

47. $r \csc \theta = 2$

48. $r = 2 \sec \theta$

49. $r^2 = 4, 0 \leq \theta \leq \dfrac{3\pi}{4}$

50. $\theta = \dfrac{\pi}{4}, |r| < 2$

51. $r = \sin(\theta), \dfrac{3\pi}{4} \leq \theta \leq \dfrac{5\pi}{4}$

52. $r = \cos(\theta), 0 \leq \theta \leq \dfrac{\pi}{2}$

53. $r = 2\sin(2\theta)\cos(2\theta)$

54. $r = \cos^2\theta - \sin^2\theta$

55. $r(1 - \cos(\theta)) = \sin^2\theta$

56. $r\sec\theta = \sec\theta - \tan(\theta)$

For the following exercises, graph the polar curve. Do you recognize it?

57. $r = \dfrac{2}{1 - \cos(\theta)}$

58. $r = \dfrac{6}{2 + \sin(\theta)}$

59. $r = \dfrac{2}{2 - \cos(\theta)}$

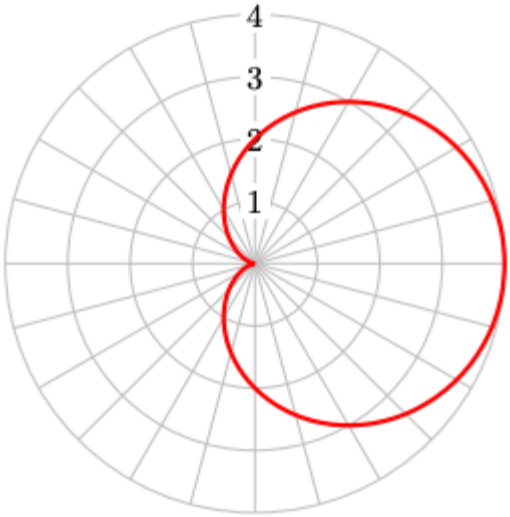
60. $r = \dfrac{1}{1 + \sin(\theta)}$

61. $r = \dfrac{1}{1 + 2\sin(\theta)}$

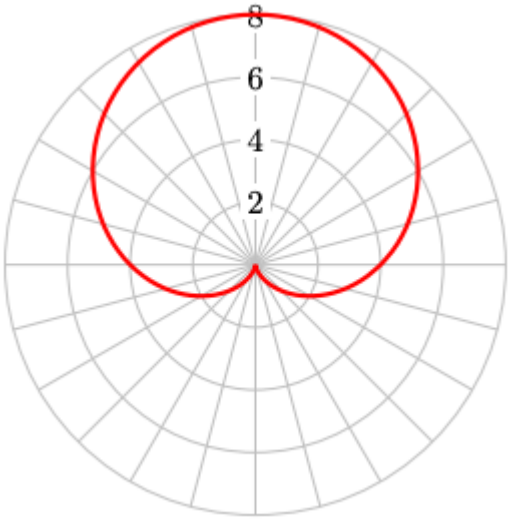
62. $r = \dfrac{3}{2 - 3\cos(\theta)}$

For the following exercises, write a polar equation for the graph.

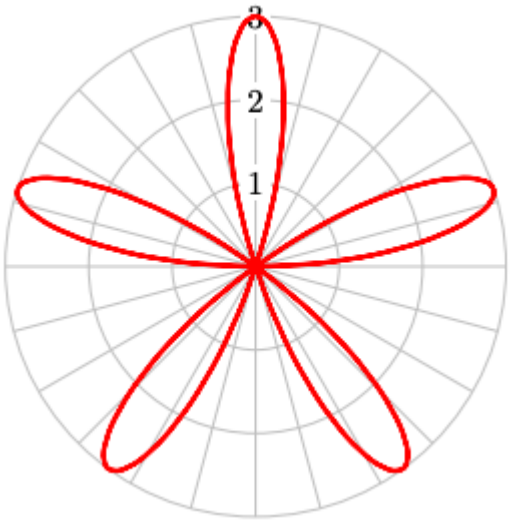
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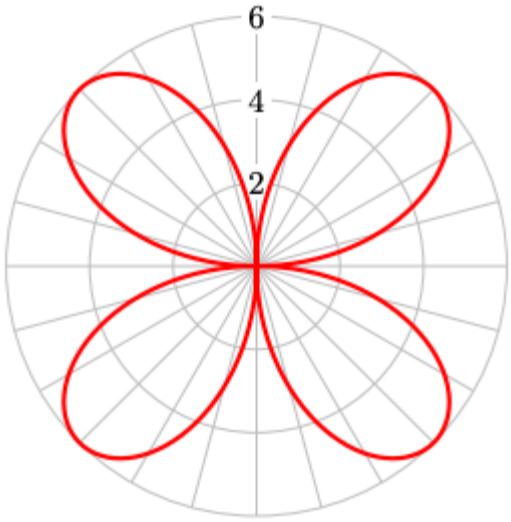
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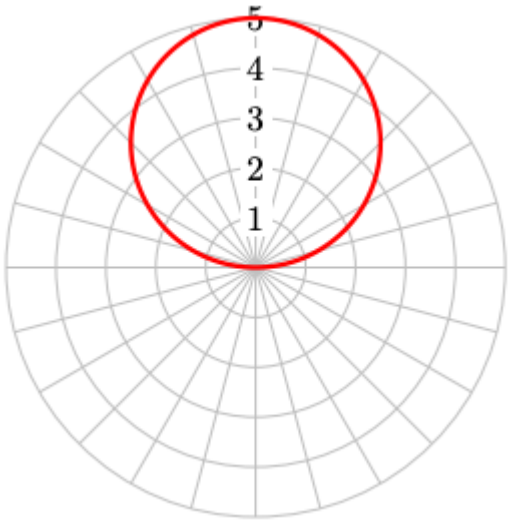
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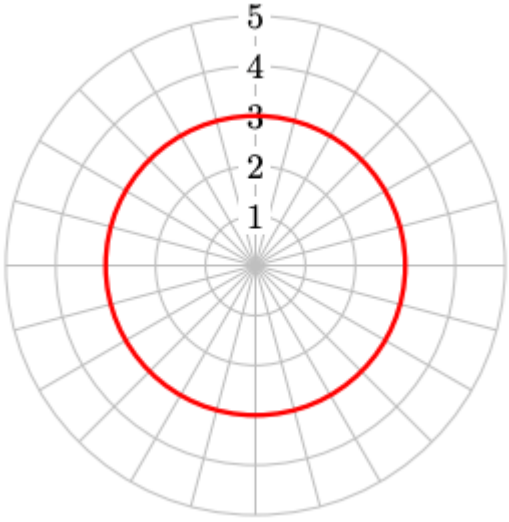
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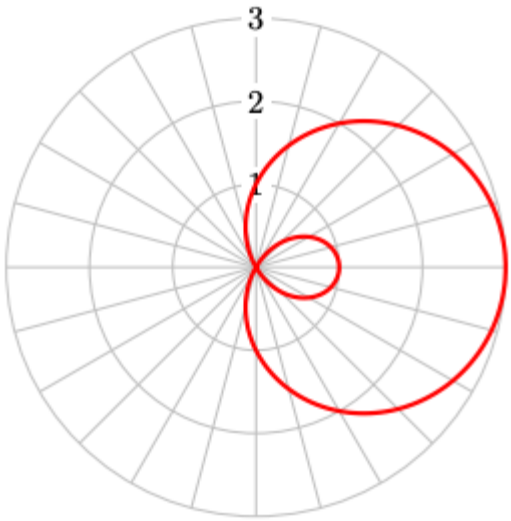
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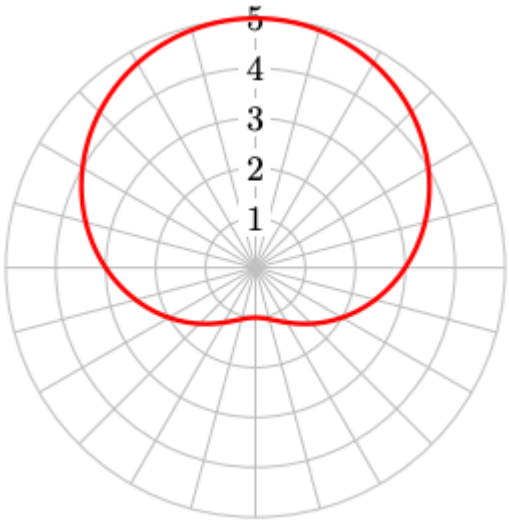
58.



59.



70.



For the following exercises, find the coordinates of the intersection points of the two curves analytically. Then, graph the curves to verify your answers.

71. $r = \cos(\theta), r = 1 - \cos(\theta)$

72. $r = \sin(\theta), r = \cos(\theta)$

73. $r = 3\sin(\theta), r = 3\cos(\theta)$

74. $r = \sin(2\theta), r = \cos(2\theta)$

75. $r = 1, r = 1 - \cos(\theta)$

76. $r = 3\cos(\theta), r = 1 + \cos(\theta)$

77. $r = 2 + \sin(\theta), r = 2 - \cos(\theta)$

78. $r = \sin(\theta), r = \sin(2\theta)$

For the following exercises, graph the polar curve.

79. $r^2 = \tan(\theta)$

80. $r^2 = \cot\theta$

81. $r = \csc\theta - 2$ (conchoid)

82. $r = \tan(\theta)$ (kappa curve)

83. $r = \cos(2\theta)\sec\theta$ (strophoid)

84. $r = \sin(\theta)\tan(\theta)$ (cissoid)

85. $r = \dfrac{1}{\sqrt{\theta}}$

86. $r = \cos\dfrac{\theta}{2}, 0 \le \theta \le 4\pi$

87. Graph the polar curves $r = 1 - 2\sin n\theta$ for $n = 2, 3, 4, 5, 6$. Explain how the value of the parameter n affects the curve.

88. Graph the polar curves $r = 1 - 3\cos n\theta$ for $n = 2, 3, 4, 5, 6$. Explain how the value of the parameter n affects the curve.

Answers

Skills Refresher

1. $\frac{\pi}{6}, \frac{11\pi}{6}$
2. $\frac{\pi}{4}, \frac{3\pi}{4}$
3. $\frac{\pi}{3}, \frac{4\pi}{3}$
4. $\frac{\pi}{6}, \frac{7\pi}{6}$
5. $\frac{5\pi}{4}, \frac{7\pi}{4}$
6. $\frac{2\pi}{3}, \frac{4\pi}{3}$
7. $\frac{3\pi}{4}, \frac{7\pi}{4}$
8. $\frac{\pi}{2}, \frac{3\pi}{2}$

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9.3: Complex Numbers

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

So far, most of your work in Mathematics has been done using the set of real numbers. We often represent the real numbers by a number line because they can be matched up one-for-one with the points on the line. Every real number is either rational or irrational. It can be expressed as a decimal number, although irrational numbers are non-repeating non-terminating decimals. However, the real numbers are a subset of a more extensive set of numbers called the **complex numbers**.

You may have first encountered complex numbers as solutions to certain quadratic equations. For example, the graph of

$$f(x) = x^2 - 2x + 2$$

has no x-intercepts (as shown in Figure $9.3.1$), because the equation $x^2 - 2x + 2 = 0$ has no real-valued solutions.

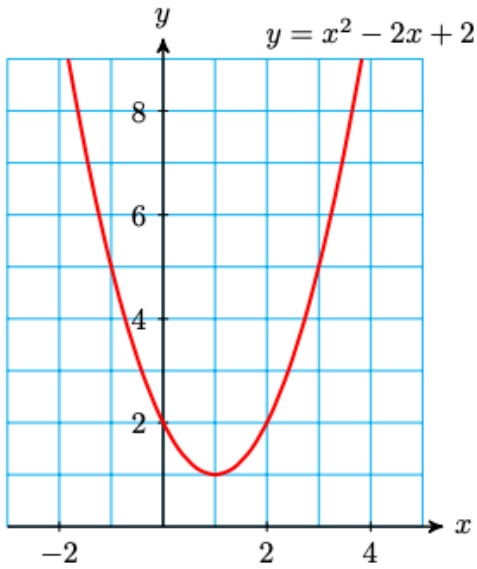


Figure $9.3.1$

Applying the Quadratic Formula, we find

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2}.$$

The solutions of the equation are $\frac{2+\sqrt{-4}}{2}$ and $\frac{2-\sqrt{-4}}{2}$, but they are not real numbers. Because $\sqrt{-4}$ is not a real number, the equation $x^2 - 2x + 2 = 0$ has no real solutions.

Imaginary Numbers

Although square roots of negative numbers such as $\sqrt{-4}$ are not real numbers, they occur often in Mathematics and its applications. Mathematicians began working with square roots of negative numbers in the sixteenth century in their attempts to solve quadratic and cubic equations. René Descartes gave them the name **imaginary numbers**, which reflected the mistrust with which mathematicians regarded them at the time. Today, however, such numbers are well understood and used routinely by scientists and engineers.

We begin by defining a new number, called i, whose square is -1.

Definition: Imaginary Unit

The **unit imaginary number**, denoted as i, is defined as the number whose square root is -1. That is,

$$\sqrt{-1} = i.$$

The letter i used in this way is *not* a variable; it is the name of a specific number and hence is a constant. We have seen this sort of designation before—π represents the approximation $3.14159\ldots$ and e represents the approximation $2.71828\ldots$.

The square root of any negative number can be written as the product of a real number and i. For example,

$$\begin{aligned}\sqrt{-4} &= \sqrt{-1 \cdot 4} \\ &= \sqrt{-1}\sqrt{4} \\ &= i \cdot 2 \\ &= 2i\end{aligned}$$

Any number that is the product of i and a real number is called an **imaginary number**.

Definition: Imaginary Number

Any number of the form bi, where $b \neq 0$, is called a **pure imaginary number** (or simply an **imaginary number**).

Examples of imaginary numbers are $3i$, $\frac{7}{8}i$, $-38i$, and $i\sqrt{5}$.

Example 9.3.1

Write each radical as an imaginary number.

a. $\sqrt{-25}$
b. $2\sqrt{-3}$

Solutions available at the end of this section.

Checkpoint 9.3.1

Write each radical as an imaginary number.

a. $\sqrt{-18}$
b. $-6\sqrt{-5}$

Answers available at the end of this section.

Just as each positive number has two real-valued square roots, every negative number has two imaginary square roots. For example, the two square roots of -9 are $3i$ and $-3i$.

Complex Numbers

Consider the quadratic equation

$$x^2 - 2x + 5 = 0.$$

Using the Quadratic Formula to solve the equation, we find

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2}.$$

If we now replace $\sqrt{-16}$ by $4i$, we have

$$x = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

The two solutions are $1 + 2i$ and $1 - 2i$. These are examples of **complex numbers**.

Definition: Complex Number

A **complex number** is a number that can be written in the form $a + bi$ where a is called the **real part** and b is called the **imaginary part** of the complex number. If $b = 0$, then $a + bi$ is simply a real number. If $a = 0$ and $b \neq 0$, the complex number is called an **imaginary number**. The form, $a + bi$ is called the **standard form** of the complex number.

Examples of complex numbers are

$$3 - 5i, \quad 2 + \sqrt{7}i, \quad \frac{4 - i}{3}, \quad 6i, \text{ and } \quad -9.$$

The real and imaginary parts of a complex number cannot be combined. Thus, two complex numbers z_1 and z_2 are equal if and only if their real parts are equal and their imaginary parts are equal.

Theorem: Equality of Complex Numbers

Two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ are said to be **equal** if and only if

$$a = c \text{ and } c = d.$$

Example 9.3.2

The solutions of the equation $x^2 - 2x + 2 = 0$ are $\frac{2 \pm \sqrt{-4}}{2}$. Write these solutions as complex numbers and simplify.

Solution available at the end of this section.

Checkpoint 9.3.2

Use Extraction of Roots to solve $(2x + 1)^2 + 9 = 0$. Write your answers as complex numbers.

Answer available at the end of this section.

Arithmetic of Complex Numbers

We add and subtract complex numbers by combining their real and imaginary parts separately. For example,

$$\begin{aligned}(4+5i)+(2-3i) &= (4+2)+(5-3)i \\ &= 6+2i\end{aligned}$$

The algebraic form of this rule can be stated as follows.

Definition: Addition and Subtraction of Complex Numbers

Given two complex numbers, $z_1 = a + bi$ and $z_2 = c + di$, we define addition and subtraction as follows:

Addition:

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i.$$

Subtraction:

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i.$$

Example 9.3.3

Subtract: $(8 - 6i) - (5 + 2i)$.

Solution available at the end of this section.

Checkpoint 9.3.3

Subtract: $(-3 + 2i) - (-3 - 2i)$

Answer available at the end of this section.

Products of Complex Numbers

We use the following theorem to help us find the product of two imaginary numbers.

Theorem: The Square of the Imaginary Unit

$$i^2 = -1$$

Proof available at the end of this section.

For example,

$$\begin{aligned}(3i)(4i) &= 3 \cdot 4i^2 \\ &= 12(-1) \\ &= -12\end{aligned}$$

To find the product of two complex numbers, we use the Distributive Law as if the numbers were binomials.

Definition: Multiplication of Complex Numbers

Given two complex numbers, $z_1 = a + bi$ and $z_2 = c + di$, we define multiplication as follows:

$$\begin{aligned} z_1 \cdot z_2 &= (z_1)(z_2) \\ &= (a+bi)(c+di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + (ad+bc)i + bc(-1) \\ &= (ac-bd) + (ad+bc)i \end{aligned}$$

Example 9.3.4

Multiply $(2+3i)(3-5i)$.

Solution available at the end of this section.

Checkpoint 9.3.4

Multiply $(-3+2i)(-3-2i)$.

Answer available at the end of this section.

Caution: A Property that Doesn't Carry Over from Arithmetic of Real Numbers

One property of real numbers that is *not* true of complex numbers is $\sqrt{ab} = \sqrt{a}\sqrt{b}$. This identity fails when a and b are negative. For example, if $a = b = -2$, we have

$$\sqrt{ab} = \sqrt{(-2)(-2)} = \sqrt{4} = 2,$$

but

$$\begin{aligned} \sqrt{a} \cdot \sqrt{b} &= \sqrt{(-2)} \cdot \sqrt{(-2)} \\ &= \sqrt{(-1 \cdot 2)} \cdot \sqrt{(-1 \cdot 2)} \\ &= i\sqrt{2} \cdot i\sqrt{2} \\ &= i^2(\sqrt{2})^2 \\ &= -2 \end{aligned}$$

so $\sqrt{ab} \neq \sqrt{a}\sqrt{b}$. We can avoid possible errors by writing the square roots of negative numbers as imaginary numbers before attempting any calculations.

Quotients of Complex Numbers

To find the quotient of two complex numbers, we use the technique of rationalizing the denominator. First, consider division by a pure imaginary number.

Example 9.3.5

Divide $\dfrac{3+4i}{2i}$.

Solution available at the end of this section.

Checkpoint 9.3.5

Divide $\frac{8+9i}{3i}$.

Answer available at the end of this section.

Perhaps you recall that to rationalize a binomial denominator, we multiply by its conjugate. For example, to rationalize the denominator of $\frac{5}{2+\sqrt{3}}$, we multiply numerator and denominator by $2-\sqrt{3}$. A similar technique works for dividing complex numbers. We first define the **conjugate** of a complex number.

Definition: Complex Conjugate

The **complex conjugate** of $z=a+bi$ is defined to be $\bar{z}=a-bi$. Likewise, the complex conjugate of $z=a-bi$ is defined to be $\bar{z}=a+bi$.

The conjugate of a complex number has several useful properties. In particular, the product of a nonzero complex number and its conjugate is always a positive real number.

Theorem: Products of Complex Conjugates

Let $z=a+bi$, so that its complex conjugate is $\bar{z}=a-bi$. Then

$$z\cdot\bar{z}=a^2+b^2.$$

Proof available at the end of this section.

Now that we have an idea of what a complex conjugate is, and what the product of a complex number and its conjugate become, we are ready to define division of complex numbers.

Definition: Division of Complex Numbers

Given two complex numbers, $z_1=a+bi$ and $z_2=c+di$, we define division as follows:

$$\begin{aligned} z_1 \div z_2 &= \frac{z_1}{z_2} \\ &= \frac{a+bi}{c+di} \\ &= \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} \\ &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} \\ &= \frac{ac-adi+bci-bdi^2}{c^2+d^2} \\ &= \frac{ac+(-ad+bc)i-bd(-1)}{c^2+d^2} \\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i \\ &= r+si \end{aligned}$$

where r and s are real numbers.

To illustrate, we calculate the quotient $\frac{5}{2+3i}$. We multiply the numerator and denominator by the complex conjugate of the denominator, $2-3i$, to obtain

$$\frac{5}{2+3i} \cdot \frac{2-3i}{2-3i}.$$

The denominator is then a real number, because

$$(2+3i)(2-3i) = 4-6i+6i-9i^2 = 4+9 = 13$$

and the quotient is

$$\frac{5}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{10-15i}{13} = \frac{10}{13} - \frac{15}{13}i.$$

Example 9.3.6

Divide $\frac{2+3i}{4-2i}$.

Solution available at the end of this section.

Checkpoint 9.3.6

Write the quotient $\frac{4-2i}{1+i}$ in the form $a+bi$.

Answer available at the end of this section.

Graphing Complex Numbers

Real numbers can be plotted on a number line, but we use the **complex plane** to graph a complex number.

Definition: Complex Plane

The **complex plane** is a coordinate system in which the horizontal axis represents the real part of a complex number and the vertical axis represents the imaginary part. In this sense, we define the horizontal axis to be the **real axis** and the vertical axis to be the **complex axis**. Complex numbers are the points on the plane, expressed as ordered pairs (a, b), where a represents the coordinate for the horizontal axis and b represents the coordinate for the vertical axis.

To plot a complex number $a + bi$, we move a units from the origin in the horizontal direction and b units in the vertical direction.

Example 9.3.7

Plot the numbers $z = 2 + 3i$ and $\bar{z} = 2 - 3i$ in the complex plane.

Solution available at the end of this section.

Checkpoint 9.3.7

Plot $z = -3 + 5i$ and its conjugate in the complex plane.

Answer available at the end of this section.

Caution: Points in the Complex Plane Represent Numbers

You are used to plotting ordered pairs (x, y) in the Cartesian plane so that one point represents two real numbers given by its coordinates. However, in the complex plane, one point represents a *single* complex number $z = a + bi$, and the coordinates of the point are the real and imaginary parts of z. For example, the point $(2, -6)$ in the real plane represents the pair of real numbers $x = 2$ and $y = -6$, but the point $(2, -6)$ in the complex plane represents the complex number $z = 2 - 6i$.

The **modulus**, or length, of a complex number is its distance from the origin in the complex plane. The modulus of a complex number is analogous to the absolute value of a real number. It is similarly denoted by $|z|$. If $z = a + bi$, we can use the Pythagorean Theorem to compute its modulus.

Definition: Modulus (of a complex number)

The **modulus** of a complex number $z = a + bi$ is given by

$$|z| = \sqrt{a^2 + b^2}.$$

For example, the modulus of $z = 2 + 3i$ is

$$|z| = \sqrt{2^2 + 3^2} = \sqrt{13}.$$

Example 9.3.8

Plot all points in the complex plane that satisfy the equation $|z| = 4$.

Solution available at the end of this section.

Checkpoint 9.3.8

Write an equation for the circle of radius 1 centered at the origin in the complex plane.

Answer available at the end of this section.

We can think of the graph of a complex number $z = a + bi$ as the vector that starts at the origin and ends at the point (a, b) in the complex plane, as shown in Figure $9.3.5a$.

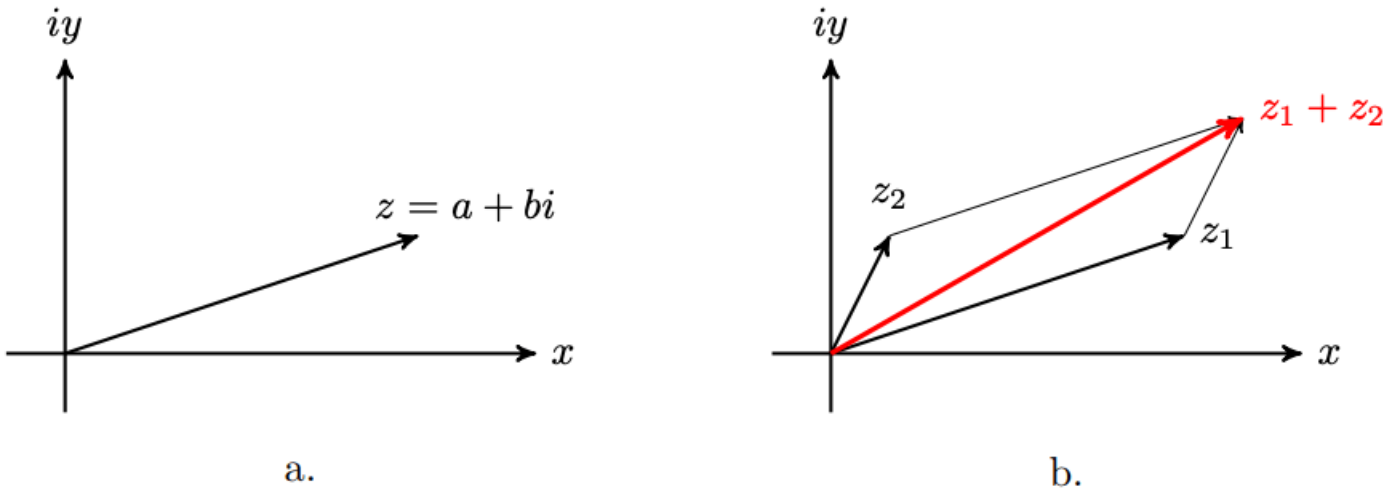


Figure 9.3.5

Then, the sum of two complex numbers corresponds to the sum of the two vectors representing them. Figure $9.3.5b$ illustrates the sum of two complex numbers, z_1 and z_2, by vector addition using the parallelogram law: we form a parallelogram with the vectors z_1 and z_2 as adjacent sides. Their sum is the vector that forms the diagonal of the parallelogram, starting at the origin.

Example 9.3.9

Illustrate the sum of $z_1 = 5 - i$ and $z_2 = -7 + 4i$ by vector addition.

Solution available at the end of this section.

To visualize subtraction, we add the opposite of the second vector because

$$z_1 - z_2 = z_1 + (-z_2)\,.$$

Recall that the opposite of a vector has the same length, but it points in the opposite direction.

Checkpoint 9.3.9

Illustrate $z_1 - z_2$, for $z_1 = 3 - 2i$ and $z_2 = -4 + 5i$.

Answer available at the end of this section.

Zeros of Polynomials

A polynomial with real-number coefficients may or may not have real-valued zeros. For example, the polynomial $x^4 + 4$ has no real-valued zeros. But a polynomial always has a zero if we allow complex numbers as inputs. Because we can add, subtract, and multiply any two complex numbers, we can evaluate a polynomial function at a complex number. Thus, we can extend the domain

of any polynomial to include all complex numbers.

Example 9.3.10

Evaluate the polynomial $f(x) = x^2 - 2x + 2$ for $x = 1 + i$, and simplify.

Solution available at the end of this section.

Checkpoint 9.3.10

If $f(x) = x^2 - 6x + 13$, evaluate $f(3 + 2i)$.

Answer available at the end of this section.

The zeros of a quadratic polynomial $ax^2 + bx + c$, of course, are the solutions of the quadratic equation $ax^2 + bx + c = 0$, and those solutions are given by the Quadratic Formula,

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

If the discriminant $D = b^2 - 4ac$ is negative, the two solutions are complex conjugates. For example, the solutions of the equation $x^2 - 4x + 5 = 0$ are

$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{-4}}{2}$$

or $z = 2 + i$ and $\bar{z} = 2 - i$. Thus, if we know that $z = a + bi$ is one complex solution of a quadratic equation, we know that $\bar{z} = a - bi$ is the other solution.

We can now write a quadratic polynomial, with real coefficients, having any complex number as one of its zeros. The factored form of the quadratic polynomial with zeros z and $\bar{z}$ is

$$p(x) = (x - z)(x - \bar{z}).$$

Expanding the right side, we find

$$p(x) = x^2 - (z + \bar{z})x + z\bar{z}.$$

Because $(z + \bar{z})$ and $z\bar{z}$ are both real numbers, this polynomial has real-valued coefficients.

Theorem: Complex Zeros Theorem

Let $f(x)$ be a polynomial with *real* coefficients. If the complex number $z = a + bi$ is a zero of $f(x)$, then $\overline{z} = a - bi$ is also a zero of $f(x)$, and z and $\overline{z}$ have the same multiplicity.

Proof available at the end of this section.

Example 9.3.11

a. Let $z = 7 - 5i$. Compute $(z + \bar{z})$ and $z\bar{z}$.
b. Find a quadratic polynomial with one zero being $z = 7 - 5i$.

Solutions available at the end of this section.

Checkpoint 9.3.11

a. Let $z = -3 + 4i$. Compute $(z + \bar{z})$ and $z\bar{z}$.
b. Find a quadratic polynomial with one zero being $z = -3 + 4i$.

Answers available at the end of this section.

One of the most important results in mathematics is the **Fundamental Theorem of Algebra**.

Theorem: Fundamental Theorem of Algebra

Every nonconstant polynomial with complex coefficients has at least one complex zero.

Proof available at the end of this section.

Despite the name, however, the Fundamental Theorem of Algebra needs one linked theorem (called a corollary) to be useful for our current conversation.

Corollary: Complete Factorization Theorem

Let $f(x)$ be a polynomial of degree $n \geq 1$ with complex coefficients and leading coefficient a. Then $f(x)$ has exactly n linear factors over the complex number system. That is, there exists complex numbers $z_1, z_2, \ldots, z_n$ such that

$$f(x) = a\,(x - z_1)\,(x - z_2)\ldots(x - z_n)\,.$$

Proof available at the end of this section.

Before we continue, let's define the **multiplicity** of a zero of a polynomial.

Definition: Multiplicity of a Zero

Let h be a zero of a nonzero polynomial function f. The **multiplicity** of h is the largest positive integer p for which $(x - h)^p$ is a factor of $f(x)$. We say that $x = h$ is a **zero of multiplicity** p. A zero of multiplicity 1 is called a **simple zero**; a zero of multiplicity greater than 1 is called a **repeated zero** (or **multiple zero**).

For example, the polynomial $p(x) = (x - 5)^2$ has a zero of multiplicity 2 at $x = 5$.

As a result of the Fundamental Theorem of Algebra and the Complete Factorization Theorem, every polynomial can be factored as the product of linear terms.

For example, although the graph of $y = x^4 + 4$ shown below has no x-intercepts, the Fundamental Theorem of Algebra tells us that there are four complex solutions to $x^4 + 4 = 0$, and that $x^4 + 4$ can be factored. You can check that the four solutions to $x^4 + 4 = 0$ are

$$1 + i,\ -1 + i,\ -1 - i,\ \text{and}\ 1 - i.$$

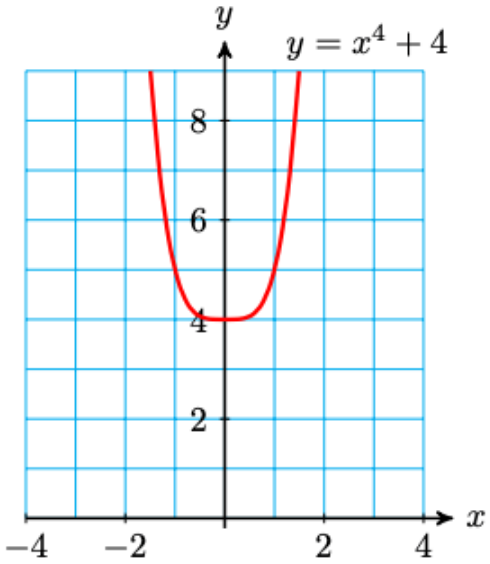


Figure 9.3.8

For example, if $x = 1 + i$, then

$$x^2 = (1 + i)^2 = 1 + 2i + i^2 = 2i$$

and

$$x^4 = \left(x^2\right)^2 = (2i)^2 = -4.$$

Because each zero corresponds to a factor of the polynomial, the factored form of $x^4 + 4$ is

$$x^4 + 4 = [x - (1 + i)][x - (-1 + i)][x - (-1 - i)][x - (1 - i)].$$

The four solutions to $x^4 + 4 = 0$ form two complex conjugate pairs, namely $1 \pm i$ and $-1 \pm i$. We see again that, for every polynomial with real coefficients, the nonreal zeros always occur in complex conjugate pairs.

Example 9.3.12

Find a fourth degree polynomial with real coefficients, two of whose zeros are $3i$ and $2 + i$.

Solution available at the end of this section.

Checkpoint 9.3.12

a. Find the zeros of the polynomial $x^4 + 15x^2 - 16$.
b. Write the polynomial in factored form.

Answers available at the end of this section.

Success in College: Preparing for the Final Exam (Part 3)

One suggestion I like to make concerning preparing for a final exam is to generate your own tests. This is pretty simple; however, you wouldn't believe how many students fail to do this. When studying, most people focus on the wrong things. You get sidetracked and work on problems that may only make up 5% of the entire exam. Try the following:

Let's assume that your final exam covers nine chapters. Let's also suppose that chapter four has the most sections, with eight. Finally, suppose the greatest number of problems within any given section is 82.

- Open Excel (or Google Sheets)
- In the first cell, type "=RANDBETWEEN(1,9)". This will generate a random chapter number.
- In the second cell, type "=RANDBETWEEN(1,8)". This will generate a random section between 1 and 8.
- In the third cell, type "=RANDBETWEEN(1,82)". This will generate a random problem number between 1 and 82.

Of course, it is entirely possible to generate problem 5.7.77 (chapter 5, section 7, problem 77) when section 7 of chapter 5 only has, for example, 50 problems. Just hit F9, and Excel will generate a new combination for you!

All you have to do is repeat this a few times, and you have a randomly generated test!

1. What do you think are the advantages of generating an exam this way? Explain.
2. How do you usually prepare for final exams?

Answers

Section Preview

You have arrived at a decision point. While this material is technically *not* optional, you need to weigh whether or not squeezing it into a short period of time is worth it. If you have roughly 2 or more hours of instructional time remaining, you can easily cover this and the next (and final) section—it is worth it because of the beauty of the material. If you have less than 2 hours of instructional time remaining, review the course; squeezing this material into a rapid-fire session is not going to be worth it.

Section Preview

- **Solving Equations**
  - Solving Quadratic Equations Using the Quadratic Formula: The zeros of polynomials subsection applies the Quadratic Formula to find complex zeros.

Section Preview

- Express the square root of a negative number as an imaginary number.
- Add and subtract complex numbers.
- Simplify powers of i.
- Multiply and divide complex numbers.
- Graph complex numbers.
- Use the Fundamental Theorem of Algebra to find all complex-valued zeros of a polynomial.

Example 9.3.1

a. $\sqrt{-25} = \sqrt{-1}\sqrt{25} = i\sqrt{25} = 5i$
b. $2\sqrt{-3} = \sqrt{-1}\sqrt{3} = 2i\sqrt{3}$

Checkpoint 9.3.1

a. $3i\sqrt{2}$
b. $-6i\sqrt{5}$

Example 9.3.2

Because $\sqrt{-4} = \sqrt{-1}\sqrt{4} = 2i$, we have $\frac{2\pm\sqrt{-4}}{2} = \frac{2\pm 2i}{2}$, or $1 \pm i$. The solutions are $1 + i$ and $1 - i$.

Checkpoint 9.3.2

$x = \dfrac{-1}{2} \pm \dfrac{3}{2}i$

Example 9.3.3

Combine the real and imaginary parts.

$$\begin{aligned}(8-6i)-(5+2i) &= (8-5)+(-6-2)i \\ &= 3+(-8)i = 3-8i\end{aligned}$$

Checkpoint 9.3.3

$4i$

Theorem: The Square of the Imaginary Unit

$$i^2 = (i)^2 = \left(\sqrt{-1}\right)^2 = -1$$

Example 9.3.4

We multiply each term of the first complex number by each term of the second one.

$$(2+3i)(3-5i) = 6-10i+9i-15i^2.$$

Because $i^2 = -1$, the last term, $-15i^2$, can be replaced by $-15(-1)$, or 15, to obtain

$$6-10i+9i+15.$$

Finally, we combine the real parts and imaginary parts to obtain

$$(6+15)+(-10i+9i) = 21-i.$$

Checkpoint 9.3.4

13

Example 9.3.5

Because i is a radical (remember that $i = \sqrt{-1}$), we multiply the numerator and denominator of the quotient by i to obtain

$$\begin{aligned} \frac{(3+4i)\cdot i}{2i\cdot i} &= \frac{3i+4i^2}{2i^2} && \text{(multiplying numerator and denominator by } i\text{)} \\ &= \frac{3i-4}{-2} && \text{(distributing (in the numerator) and recalling that } i^2=-1\text{)} \end{aligned}$$

To write the quotient in the form $a+bi$, we divide -2 into each term of the numerator.

$$\frac{3i}{-2} - \frac{4}{-2} = \frac{-3}{2}i + 2.$$

The quotient is $2 - \frac{3}{2}i$.

Checkpoint 9.3.5

$3 - \frac{8}{3}i$

Theorem: Products of Complex Conjugates

Let $z = a + bi$. Then

$$\begin{aligned} z\cdot\bar{z} &= (a+bi)(a-bi) \\ &= a^2 - abi + abi - b^2i^2 \\ &= a^2 - \cancel{abi} + \cancel{abi} - b^2(-1) \\ &= a^2 + b^2 \end{aligned}$$

Example 9.3.6

We multiply the numerator and denominator by $4+2i$, the complex conjugate of the denominator.

$$\begin{aligned} \frac{2+3i}{4-2i} &= \frac{(2+3i)(4+2i)}{(4-2i)(4+2i)} && \text{(multiplying both numerator and denominator by the complex conjugate } 4+2i\text{)} \\ &= \frac{8+4i+12i+6i^2}{16+8i-8i-4i^2} && \text{(distributing)} \\ &= \frac{8+16i-6}{16-(-4)} && \text{(combining like terms in the numerator and using } i^2=-1 \text{ in the denominator)} \\ &= \frac{2+16i}{20} && \text{(combining like terms)} \\ &= \frac{2}{20} + \frac{16i}{20} && \text{(splitting the fraction)} \\ &= \frac{1}{10} + \frac{4}{5}i && \text{(simplifying fractions)} \end{aligned}$$

Checkpoint 9.3.6

$1 - 3i$

Example 9.3.7

To plot $z = 2+3i$, we move from the origin 2 units to the right and 3 units up. To plot $\bar{z} = 2-3i$, we move from the origin 2 units to the right and 3 units down. The numbers $2+3i$ and $2-3i$ are plotted in the figure below.

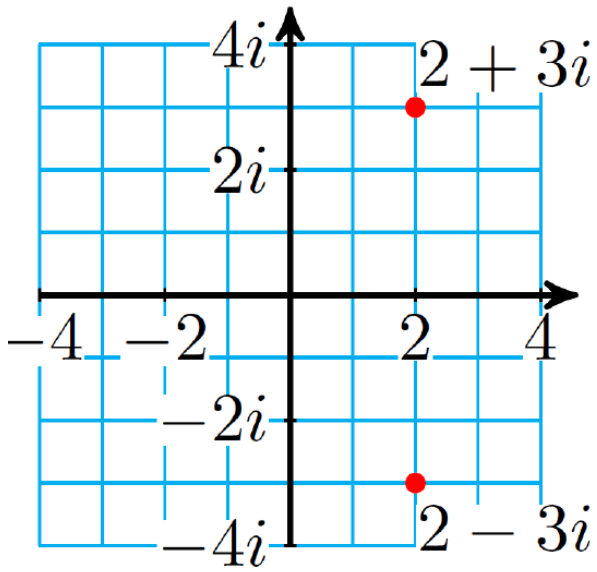


Figure 9.3.2

Checkpoint 9.3.7

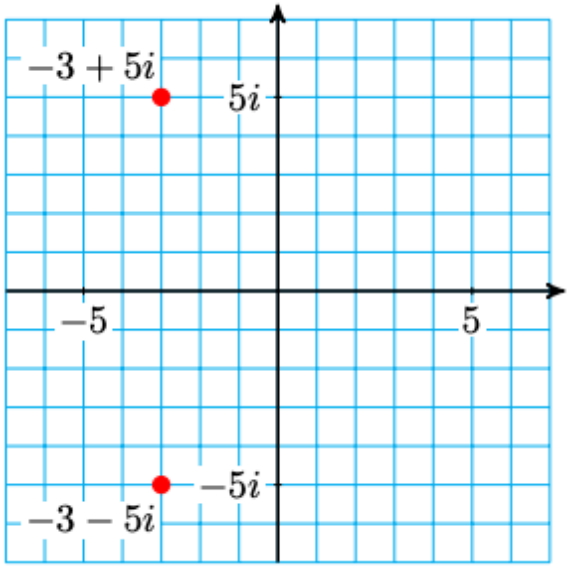


Figure 9.3.3

Example 9.3.8

The points that satisfy $|z|=4$ all lie 4 units from the origin. In other words, they lie on a circle of radius 4 centered at the origin, as shown below.

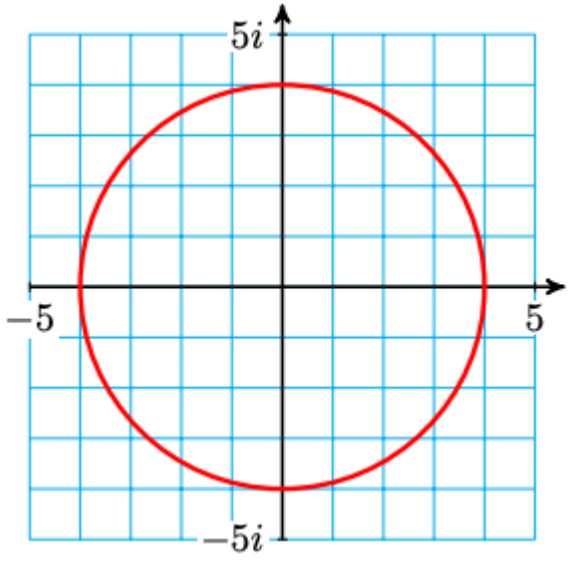


Figure 9.3.4

Checkpoint 9.3.8

$|z|=1$

Example 9.3.9

Draw a vector starting at the origin and ending at the point $(5,-1)$ in the complex plane to represent z_1, and another vector ending at $(-7,4)$ to represent z_2. Now sketch the opposite sides of a parallelogram and draw the diagonal starting at the origin, as shown below.

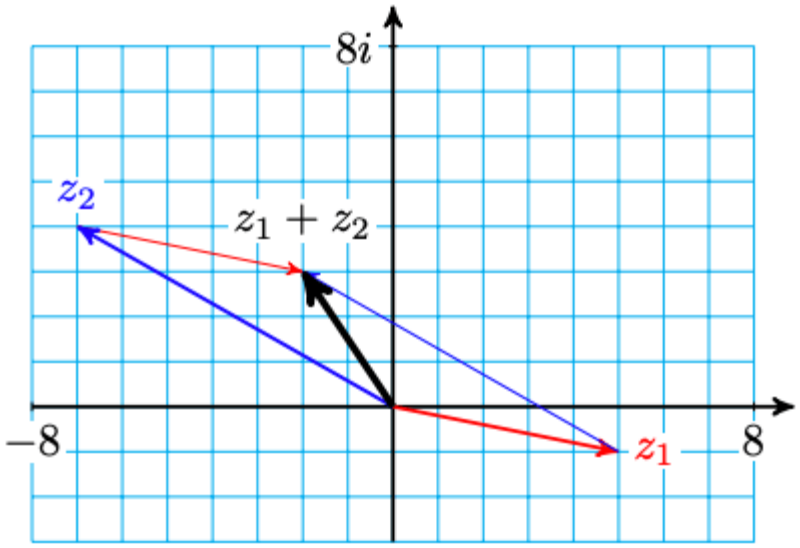


Figure 9.3.6

This vector represents the sum $z_1 + z_2$. You can check that the vector ends at the point

$$z_1 + z_2 = (5 - 7) + (-1 + 4)i = -2 + 3i.$$

Checkpoint 9.3.9

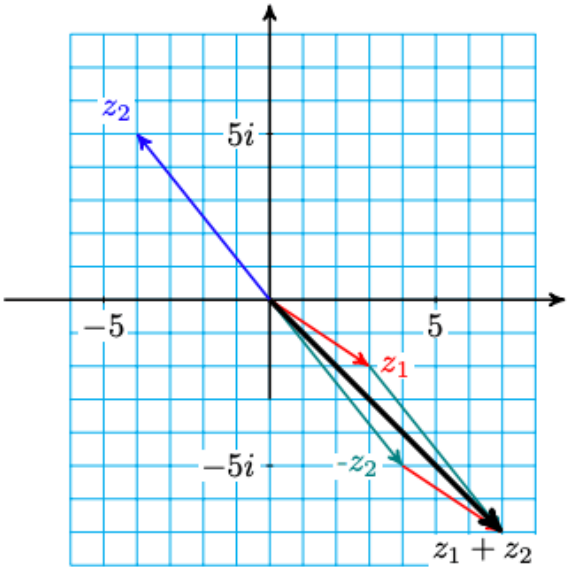


Figure 9.3.7

Example 9.3.10

We substitute $x = 1 + i$, to find

$$\begin{aligned} f(1+i) &= (1+i)^2 - 2(1+i) + 2 \\ &= 1^2 + 2i + i^2 - 2 - 2i + 2 \\ &= 1 + 2i + (-1) - 2 - 2i + 2 \end{aligned}$$

Thus, $f(1 + i) = 0$, so $1 + i$ is a zero of the polynomial.

Checkpoint 9.3.10

0

Theorem: Complex Zeros Theorem

The proof requires several theorems from complex numbers. These are mentioned within the proof. If $z = a + bi$ is a zero of a polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then

$$\begin{aligned}
f(\overline{z}) &= a_n(\overline{z})^n + a_{n-1}(\overline{z})^{n-1} + \cdots + a_1\overline{z} + a_0 && \text{(substituting)}\\
&= a_n\overline{z^n} + a_{n-1}\overline{z^{n-1}} + \cdots + a_1\overline{z} + a_0 && \left(\overline{z^n} = (\overline{z})^n\right)\\
&= \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} && (\text{if } a_i \in \mathbb{R},\ \text{then } a_i = \overline{a_i})\\
&= \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} && (\overline{z_1}\cdot\overline{z_2} = \overline{z_1\cdot z_2})\\
&= \overline{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0} && \left(\overline{z_1}+\overline{z_2}+\cdots+\overline{z_k} = \overline{z_1+z_2+\cdots+z_k}\right)\\
&= \overline{f(z)}\\
&= \overline{0} && (z \text{ is a zero of } f(x))\\
&= 0 && \left(\overline{k} = k \text{ for real numbers } k\right)
\end{aligned}$$

Hence, $\overline{z}$ (the conjugate) is also a zero of $f(x)$.

To prove that multiplicities must be the same, one can argue via contradiction. This part of the proof is left for the reader.

Example 9.3.11

a. The conjugate of $z = 7 - 5i$ is $\bar{z} = 7 + 5i$, so

$$(z + \bar{z}) = (7 - 5i) + (7 + 5i) = 14$$

and

$$\begin{aligned}
z\bar{z} &= (7 - 5i)(7 + 5i)\\
&= 49 - 25i^2\\
&= 49 + 25\\
&= 74
\end{aligned}$$

b. The polynomial is

$$\begin{aligned}
p(x) &= x^2 - (z + \bar{z})x + z\bar{z}\\
&= x^2 - 14x + 74
\end{aligned}$$

Checkpoint 9.3.11

a. $z + \bar{z} = -6, \quad z\bar{z} = 25$
b. $x^2 + 6x + 25 = 0$

Theorem: Fundamental Theorem of Algebra

The proof of this theorem is beyond the scope of lower-division mathematics.

Corollary: Complete Factorization Theorem

The proof of this theorem is done using mathematical induction and the Fundamental Theorem of Algebra. Since this topic is usually covered *prior* to learning proofs by mathematical induction, the proof is omitted.

Example 9.3.12

The other two zeros are $-3i$ and $2 - i$. The factored form of the polynomial is

$$(x - 3i)(x + 3i)[x - (2 + i)][x - (2 - i)].$$

We multiply together the factors to find the polynomial. The product of $(x - 3i)(x + 3i)$ is $x^2 + 9$, and

$$\begin{aligned}
[x - (2 + i)][x - (2 - i)] &= x^2 - (2 + i + 2 - i)x + (2 + i)(2 - i)\\
&= x^2 - 4x + 5
\end{aligned}$$

Finally, we multiply these two partial products to find the polynomial we seek,

$$\left(x^2 + 9\right)\left(x^2 - 4x + 5\right) = x^4 - 4x^3 + 14x^2 - 36x + 45.$$

Checkpoint 9.3.12

a. $\pm 1, \pm 4i$
b. $(x - 1)(x + 1)(x - 4i)(x + 4i)$

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9.3.1: Resources and Key Concepts

Resources

Prerequisite Topics

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Arithmetic**
  - Mathematical Operations - Powers and Square Roots: This section introduces the imaginary unit by considering the square root of negative numbers.
- **Polynomial Functions**
  - The Basics of Polynomial Functions: This section concludes with a discussion of the zeros of polynomials, requiring a basic understanding of what a polynomial is.

Videos

- Complex Numbers
  - Arithmetic - Sets - Complex Numbers - Definitions, Theorem, and Example
- Complex Plane
  - Arithmetic - Sets - Complex Numbers - The Complex Plane and Plotting Complex Numbers
- Arithmetic with Complex Numbers
  - Arithmetic - Sets - Complex Numbers - Adding and Subtracting Complex Numbers
  - Arithmetic - Sets - Complex Numbers - Multiplying Complex Numbers
  - Arithmetic - Sets - Complex Numbers - Dividing Complex Numbers
- Power of the Imaginary Unit
  - Arithmetic - Sets - Complex Numbers - Simplifying Powers of i
- Fundamental Theorem of Algebra
  - Complex Zeros and the Fundamental Theorem of Algebra - Intro to the Fundamental Theorem of Algebra
- Complete Factorization Theorem
  - Complex Zeros and the Fundamental Theorem of Algebra - The Complete Factorization Theorem
- Conjugate Zeros Theorem
  - Complex Zeros and the Fundamental Theorem of Algebra - The Zeros and Conjugate Zeros Theorems
  - Complex Zeros and the Fundamental Theorem of Algebra - Using the Fundamental Theorem of Algebra

Key Concepts

Definitions

- **Imaginary Unit:** The number i, defined as $i = \sqrt{-1}$.
- **Imaginary Number:** Any number of the form bi, where b is a non-zero real number.
- **Complex Number:** A number that can be written in the standard form $a + bi$, where a is the real part and b is the imaginary part.
- **Addition and Subtraction of Complex Numbers:** The process of adding or subtracting two complex numbers by combining their real and imaginary parts separately.
- **Multiplication of Complex Numbers:** The process of multiplying two complex numbers using the Distributive Law, as if they were binomials.
- **Complex Conjugate:** The complex conjugate of a complex number $z = a + bi$ is $\bar{z} = a - bi$.
- **Division of Complex Numbers:** The process of finding the quotient of two complex numbers, typically by multiplying the numerator and denominator by the complex conjugate of the denominator.

- **Complex Plane:** A coordinate system used to graph complex numbers, where the horizontal axis represents the real part and the vertical axis represents the imaginary part.
- **Modulus (of a complex number):** The distance of a complex number from the origin in the complex plane, given by $|z| = \sqrt{a^2 + b^2}$ for $z = a + bi$.
- **Multiplicity of a Zero:** If $(x - h)^p$ is the greatest power of the factor $(x - h)$ in a polynomial, then h is a zero of multiplicity p.

Theorems

- **Equality of Complex Numbers:** Two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ are equal if and only if their real parts are equal ($a = c$) and their imaginary parts are equal ($b = d$).
- **The Square of the Imaginary Unit:** $i^2 = -1$.
- **Products of Complex Conjugates:** For a complex number $z = a + bi$, the product of the number and its conjugate is a real number: $z\bar{z} = a^2 + b^2$.
- **Complex Zeros Theorem:** If a polynomial function f has real coefficients and a complex zero of the form $a + bi$, then the complex conjugate of the zero, $a - bi$, is also a zero of f.
- **Fundamental Theorem of Algebra:** If $f(x)$ is a polynomial of degree $n \geq 1$ with complex-valued coefficients, then $f(x)$ has at least one complex-valued zero.
- **Corollary: Complete Factorization Theorem:** If $f(x)$ is a polynomial of degree $n \geq 1$ with complex-valued coefficients, then $f(x)$ has exactly n linear factors over the complex number system (counting multiplicities).

Common Mistakes

- **A Property that Doesn't Carry Over from Arithmetic of Real Numbers:** The property $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is not true for complex numbers when both a and b are negative. Always write the square roots of negative numbers as imaginary numbers first before performing calculations.
- **Points in the Complex Plane Represent Numbers:** A single point (a, b) in the complex plane represents one complex number, $z = a + bi$, not two separate real numbers as in the Cartesian plane.

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9.3.2: Homework

Reading Questions

1. What is the imaginary unit, i, defined to be?
2. What are the real and imaginary parts of the complex number $3 - 5i$?
3. How do you add two complex numbers?
4. What is the complex conjugate of $z = a + bi$?
5. What is the result of multiplying a complex number by its conjugate?
6. When dividing by a complex number, what is the general technique used to eliminate the imaginary part from the denominator?
7. What do the horizontal and vertical axes represent in the complex plane?
8. What is the modulus of a complex number, and what does it represent geometrically?
9. What does the Fundamental Theorem of Algebra guarantee for any polynomial of degree 1 or higher?
10. If a polynomial with real coefficients has a zero of $4 - i$, what is another zero that it must have?
11. What is the multiplicity of the zero at $x = 2$ for the polynomial $p(x) = (x - 2)^3(x + 1)$?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, solve.

1 $3x^2 = 6x$

2 $16x^2 = 9$

3 $x^2 + 4x = 12$

4 $2(x - 3)^2 = 8$

5 $3x^2 - 8 = 10x$

6 $x^2 - 3x + 1 = 0$

Answers available at the end of this section.

Homework

Concept Check

1. What are imaginary numbers, and why were they invented?
2. Simplify the following powers of i:

   $$i^2, i^3, i^4, i^5, i^6, i^7, i^8.$$

   What do you notice?
3. Explain how the complex conjugate is used in dividing complex numbers.

4. If one solution of a quadratic equation is $3+i\sqrt{2}$, what is the other solution?

5. If $P(x)$ is a polynomial of degree 7, how many zeros does $P(x)$ have? How many x-intercepts could its graph have? How many complex zeros could $P(x)$ have?

Basic Skills

For the following exercises, write the complex number in the form $a+bi$, where a and b are real numbers.

6. a. $\sqrt{-25}-4$

   b. $\dfrac{-8+\sqrt{-4}}{2}$

   c. $\dfrac{-5-\sqrt{-2}}{6}$

7. a. $\sqrt{-9}+3$

   b. $\frac{6-\sqrt{-36}}{2}$

   c. $\frac{7+\sqrt{-3}}{4}$

For the following exercises, find the zeros of the quadratic polynomial. Write each zero in the form $a+bi$, where a and b are real numbers.

8. $x^2+6x+13$

9. $x^2-2x+10$

10. $3x^2-x+1$

11. $5x^2+2x+2$

For the following exercises, add or subtract.

12. $(11-4i)-(-2-8i)$

13. $(7i-2)+(6-4i)$

14. $(2.1+5.6i)+(-1.8i-2.9)$

15. $\left(\frac{1}{5}i-\frac{2}{5}\right)-\left(\frac{4}{5}-\frac{3}{5}i\right)$

For the following exercises, multiply.

16. $5i(2-4i)$

17. $-7i(-1+4i)$

18. $(4-i)(-6+7i)$

19. $(2-3i)(2-3i)$

20. $(7+i\sqrt{3})^2$

21. $(5-i\sqrt{2})^2$

22. $(7+i\sqrt{3})(7-i\sqrt{3})$

23. $(5-i\sqrt{2})(5+i\sqrt{2})$

24. $(1-i)^3$

25. $(2+i)^3$

For the following exercises, divide.

26. $\dfrac{12+3i}{-3i}$

27. $\dfrac{12+4i}{8i}$

28. $\dfrac{10+15i}{2+i}$

29. $\dfrac{4-6i}{1-i}$

30. $\dfrac{5i}{2-5i}$

31. $\dfrac{-2i}{7+2i}$

32. $\dfrac{\sqrt{3}}{\sqrt{3}+i}$

33. $\dfrac{2\sqrt{2}}{1-i\sqrt{2}}$

34. $\dfrac{1+i\sqrt{5}}{1-i\sqrt{5}}$

35. $\dfrac{\sqrt{2}-i}{\sqrt{2}+i}$

36. $\dfrac{3+2i}{2-3i}$

37. $\dfrac{4-6i}{-3-2i}$

38. Simplify.

    a. i^6

    b. i^{12}

    c. i^{15}

    d. i^{102}

39. Express with a positive exponent and simplify.

    a. i^{-1}

    b. i^{-2}

    c. i^{-3}

    d. i^{-6}

For the following exercises, evaluate the polynomial for the given values of the variable.

40. $z^2 + 9$

    a. $z = 3i$

    b. $z = -3i$

41. $2y^2 - y - 2$

    a. $y = 2 - i$

    b. $y = -2 - i$

42. $x^2 - 2x + 2$

    a. $x = 1 - i$

    b. $x = 1 + i$

43. $3w^2 + 5$

    a. $w = 2i$

    b. $w = -2i$

44. $q^2 + 4q + 13$

    a. $q = -2 + 3i$

    b. $q = -2 - 3i$

45. $v^2 + 2v + 3$

    a. $v = 1 + i$

    b. $v = -1 + i$

For the following exercises, expand the product of polynomials.

46. $(2z + 7i)(2z - 7i)$

47. $(5w + 3i)(5w - 3i)$

48. $[x + (3 + i)][x + (3 - i)]$

49. $[s - (1 + 2i)][s - (1 - 2i)]$

50. $[v - (4 + i)][v - (4 - i)]$

51. $[Z + (2 + i)][Z + (2 - i)]$

For the following exercises, plot the number and its complex conjugate in the complex plane. What is the geometric relationship between complex conjugates?

52. $z = -3 + 2i$

53. $z = 4 - 3i$

54. $z = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

55. $z = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

For the following exercises, sketch the set of points in the complex plane.

56. $|z| \leq 2$

57. $1 < |z| < 3$

58. The set of all $z = a + bi$ for which $a \geq b$

59. The set of all $z = a + bi$ for which $a + b < 2$

For the following exercises, illustrate addition using the parallelogram rule in the complex plane.

60. $(1 - 4i) + (-3 + 2i)$

61. $(-4 + 2i) + (-2 + i)$

62. $(2 + 6i) - (3 + 3i)$

63. $(-5 - 2i) - (-3 + 2)$

For the following exercises,
(a) Given one solution of a quadratic equation with rational coefficients, find the other solution.
(b) Write a quadratic equation that has those solutions.

64. $2 + \sqrt{5}$

65. $3 - \sqrt{2}$

66. $4 - 3i$

67. $5 + i$

For the following exercises, find a fourth degree polynomial with real coefficients that has the given complex numbers as two of its zeros.

68. $1 - 3i, 2 - i$

69. $5 - 4i, -i$

70. $\frac{1}{2} - \frac{\sqrt{3}}{2}i, 3 + 2i$

71. $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, 4 - i$

Synthesis Questions

72. Suppose t, w, and z are complex numbers. If $w = t + z$, is it necessarily true that $|w| = |t| + |z|$? Provide examples to support your conclusion.

Proofs

73. Prove that the product of two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ is

$$z_1 z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

74. Prove that the quotient of two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ is

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i.$$

75. Prove the commutative laws for addition and multiplication of complex numbers:

$$z_1 + z_2 = z_2 + z_1$$

and

$$z_1 z_2 = z_2 z_1$$

76. Prove the Distributive Law for complex numbers:

$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

77. a. Prove that $z + \bar{z}$ is a real number, and $z - \bar{z}$ is an imaginary number.

    b. Prove that $z\bar{z} = |z|^2$

78. Prove that $\overline{zw} = \bar{z} \cdot \bar{w}$

79. Prove the triangle inequality for complex numbers:

$$|w + z| \leq |w| + |z|.$$

Answers

Skills Refresher

1 $0, 2$
2 $\frac{\pm 3}{4}$
3 $-6, 2$
4 $1, 5$
5 $\frac{-2}{3}, 4$
6 $\frac{3 \pm \sqrt{5}}{2}$

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9.4: Polar Form of Complex Numbers

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

"God made the integers; all else is the work of man."

This rather famous quote by nineteenth-century German mathematician Leopold Kronecker sets the stage for this section on the polar form of a complex number. Complex numbers were invented by people and represent over a thousand years of continuous investigation and struggle by mathematicians such as Pythagoras, Descartes, De Moivre, Euler, Gauss, and others. Complex numbers answered questions that for centuries had puzzled the greatest minds in science.

Writing Complex Numbers in Polar Form

Given a complex number expressed as $z = x + y\ i$, we can use the material from the previous section to visualize z as the point (x, y) on the Cartesian plane.[1] We then observe that we can rewrite both x and y in trigonometric form:

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ r &= \sqrt{x^2 + y^2} \end{aligned}$$

where θ is the angle in standard position with terminal side being the line segment connecting the origin to the point (x, y). We review these relationships in Figure $9.4.1$.

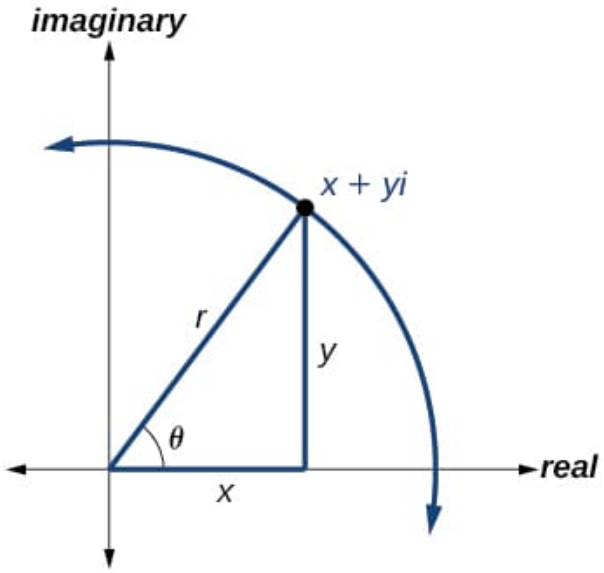


Figure $9.4.1$

Thus, we can rewrite the complex number, $z = x + y\ i$, in the form

$$z = r\cos(\theta) + (r\sin(\theta))\, i = r\left(\cos(\theta) + i\ \sin(\theta)\right).$$

This form is known as the **polar form** of the complex number.

Definition: Polar Form of a Complex Number

The complex number, $z = x + y\ i$, can be written in the form

$$z = r\left(\cos(\theta) + i\ \sin(\theta)\right) \equiv r\,\mathrm{cis}(\theta),$$

where

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ r &= \sqrt{x^2 + y^2} \end{aligned}$$

r is the modulus of z, and θ is the argument. This form is called the **polar form** of the complex number.

The notation $z = r\ \mathrm{cis}(\theta)$ is read as "cosine plus i sine of theta."

As we did in the previous section, we use the term modulus to represent the "size" of a complex number, or the distance from the origin to the point (x, y) on the complex plane. The modulus, therefore, is the same as r, the radius in polar form.

Theorem: Modulus in Polar Form

The modulus of the complex number z, written in polar form $z = r\left(\cos(\theta) + i\ \sin(\theta)\right)$, is r.

Example 9.4.1

Express the complex number $4i$ in polar form.

Solution available at the end of this section.

Checkpoint 9.4.1

Express $z = 3i$ as $r\ \mathrm{cis}(\theta)$ in polar form.

Answer available at the end of this section.

Example 9.4.2

Find the polar form of $-4 + 4i$.

Solution available at the end of this section.

It's very tempting to evaluate the trigonometric functions at $\frac{3\pi}{4}$ in Example 9.4.2. I would not blame you for wanting to do that because that's what we have been doing this entire course; however, in doing so, you would be reverting back to the standard (rectangular) form of the complex number (thereby undoing all of your work to get the number in polar form).

Checkpoint 9.4.2

Write $z = \sqrt{3} + i$ in polar form.

Answer available at the end of this section.

Converting a Complex Number from Polar to Rectangular Form

Converting a complex number from polar form to rectangular form is a matter of evaluating what is given and using the distributive property. In other words, given

$$z = r\left(\cos\left(\theta\right) + i\,\sin\left(\theta\right)\right),$$

we first evaluate the trigonometric functions $\cos\left(\theta\right)$ and $\sin\left(\theta\right)$, then multiply through by r.

Example 9.4.3

Convert the polar form of the given complex number to rectangular form:

$$z = 12\left(\cos\left(\frac{\pi}{6}\right) + i\,\sin\left(\frac{\pi}{6}\right)\right)$$

Solution available at the end of this section.

Checkpoint 9.4.3

Convert the complex number to rectangular form:

$$z = 4\left(\cos\left(\frac{11\pi}{6}\right) + i\,\sin\left(\frac{11\pi}{6}\right)\right)$$

Answer available at the end of this section.

Finding Products of Complex Numbers in Polar Form

Now that we can convert complex numbers to polar form we will learn how to perform operations on complex numbers in polar form. For the rest of this section, we will work with formulas developed by French mathematician Abraham de Moivre (1667-1754). These formulas have made working with products, quotients, powers, and roots of complex numbers much simpler than they appear. The rules are based on multiplying the moduli and adding the arguments.

Theorem: Products of Complex Numbers Written in Polar Form

If $z_1 = r_1(\cos\left(\theta_1\right) + i\ \sin\left(\theta_1\right))$ and $z_2 = r_2(\cos\left(\theta_2\right) + i\ \sin\left(\theta_2\right))$, then the product of these numbers is given as

$$z_1 z_2 = r_1 r_2\left[\cos\left(\theta_1 + \theta_2\right) + i\,\sin\left(\theta_1 + \theta_2\right)\right] = r_1 r_2\,\mathrm{cis}\left(\theta_1 + \theta_2\right).$$

Notice that the product calls for multiplying the moduli and adding the angles. The proof of this theorem is left as a homework exercise.

Example 9.4.4

Find the product of $z_1 z_2$, given $z_1 = 4\left(\cos\left(80^\circ\right) + i\,\sin\left(80^\circ\right)\right)$ and $z_2 = 2\ \mathrm{cis}\left(145^\circ\right)$.

Solution available at the end of this section.

Checkpoint 9.4.4

Let $z_1 = -\sqrt{3} + i$ and $z_2 = 3 + 3i$.

a. Find the product of z_1 and z_2 in rectangular form (without converting to polar form).
b. Without converting to polar form, determine the angle $z_1 z_2$ makes with the positive x-axis.
c. Rewrite z_1 and z_2 in polar form.
d. Find the product of z_1 and z_2 in polar form.
e. Using your answer from part (d), determine the angle $z_1 z_2$ makes with the positive x-axis.

Answers available at the end of this section.

Finding Quotients of Complex Numbers in Polar Form

The quotient of two complex numbers in polar form follows the same patterns set for products.

Theorem: Quotients of Complex Numbers Written in Polar Form

If $z_1 = r_1 (\cos(\theta_1) + i \sin(\theta_1))$ and $z_2 = r_2 (\cos(\theta_2) + i \sin(\theta_2))$, then the quotient of these numbers is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2),$$

where $z_2 \neq 0$.

The proof of this theorem is left as a homework exercise.

Example 9.4.5

Find the quotient of $z_1 = 2(\cos(213^\circ) + i \sin(213^\circ))$ and $z_2 = 4(\cos(33^\circ) + i \sin(33^\circ))$.

Solution available at the end of this section.

Checkpoint 9.4.5

Find the product and the quotient of $z_1 = 2\sqrt{3}(\cos(150^\circ) + i \sin(150^\circ))$ and $z_2 = 2(\cos(30^\circ) + i \sin(30^\circ))$.

Answer available at the end of this section.

Finding Powers of Complex Numbers in Polar Form

Finding powers of complex numbers is *greatly* simplified when they are written in polar form. For example, let $z = 1 + i$. This is a fairly innocuous complex number; however, let's try to compute $z^5 = (1 + i)^5$.

$$\begin{aligned} z^5 &= (1+i)^5 \\ &= (1+i)^2 (1+i)^2 (1+i) \\ &= (1+2i+i^2)(1+2i+i^2)(1+i) \\ &= (1+2i-1)(1+2i-1)(1+i) \\ &= (2i)(2i)(1+i) \\ &= 4i^2(1+i) \\ &= -4(1+i) \\ &= -4-4i \end{aligned}$$

This is only the fifth power of a somewhat simple complex number. Imagine working with the tenth power of something like $z = -\sqrt{3}+i$. If our complex number is written in polar form, however, something fantastic happens.

(Note: The following derivation of *why* De Moivre's Theorem works requires a proof method called Mathematical Induction. If you have already taken College Algebra, then you have worked with, and performed, proofs using Mathematical Induction; however, since College Algebra is not a prerequisite for this course, you might not have seen it. As such, I provide the proof only for those who are interested.)

Theorem: De Moivre's Theorem

If $z = r(\cos(\theta) + i\sin(\theta))$ is a complex number and $n \in \mathbb{N}$, then

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)] = r^n \operatorname{cis}(n\theta).$$

Proof available at the end of this section.

De Moivre's Theorem states that, for a positive integer n, z^n is found by raising the modulus to the nth power and multiplying the argument by n. It is the standard method used in modern Mathematics.

Example 9.4.6

Evaluate the expression $(1+i)^5$ using De Moivre's Theorem.

Solution available at the end of this section.

Checkpoint 9.4.6

Compute $\left(-\sqrt{3}+i\right)^{10}$.

Answer available at the end of this section.

Finding Roots of Complex Numbers in Polar Form

DeMoivre's Theorem also works for fractional values of n, so we can compute roots of complex numbers. For example, by applying the theorem with $n = \frac{1}{2}$, we see that one of the square roots of $z = r(\cos(\theta) + i\sin(\theta))$ is

$$z^{1/2} = r^{1/2}\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right).$$

Now, every number, whether real or complex, has two square roots. To find the other root, remember that we can add a multiple of 2π to the argument of z, that is, we can also write the polar form of z as

$$z = r(\cos(\theta + 2\pi) + i\sin(\theta + 2\pi)).$$

The second square root of z is thus

$$r^{1/2}\left(\cos\left(\frac{\theta}{2}+\pi\right)+i\,\sin\left(\frac{\theta}{2}+\pi\right)\right).$$

As an example, consider $w=-8+8i\sqrt{3}$, whose polar form is $16\left(\cos\left(\frac{2\pi}{3}\right)+i\,\sin\left(\frac{2\pi}{3}\right)\right)$, or by adding 2π to the argument, $16\left(\cos\left(\frac{8\pi}{3}\right)+i\,\sin\left(\frac{8\pi}{3}\right)\right)$. The two square roots of w are

$$\begin{aligned} z_1 &= 16^{1/2}\left(\cos\left(\frac{\pi}{3}\right)+i\,\sin\left(\frac{\pi}{3}\right)\right) & z_2 &= 16^{1/2}\left(\cos\left(\frac{4\pi}{3}\right)+i\,\sin\left(\frac{4\pi}{3}\right)\right)\\ &= 4\left(\frac{1}{2}+i\,\frac{\sqrt{3}}{2}\right) & &= 4\left(\frac{-1}{2}+i\frac{-\sqrt{3}}{2}\right)\\ &= 2+2i\sqrt{3} & &= -2-2i\sqrt{3}\end{aligned}$$

You can verify that both of these numbers are square roots of $w=-8+8i\sqrt{3}$, for instance,

$$\begin{aligned}(2+2i\sqrt{3})^2 &= 4+2(2)(2i\sqrt{3})+12i^2\\ &= -8+8i\sqrt{3}\end{aligned}$$

The graphs of w and its two square roots are shown below.

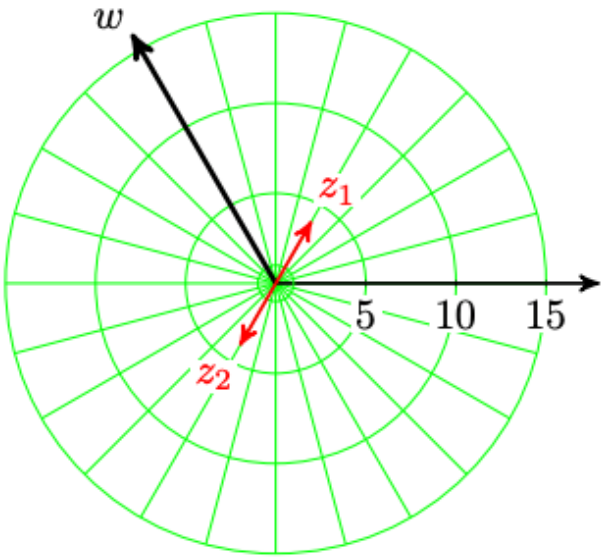


Figure 9.4.3

It is not hard to show that every number has three complex cube roots, four complex fourth roots, and so on.

In general, to find the nth root of a complex number in polar form, we use the nth Root Theorem, or we can use De Moivre's Theorem as we did in our work above. The following theorem is presented without proof.

Theorem: n^{th} Root Theorem

If $z=r\left(\cos\left(\theta\right)+i\,\sin\left(\theta\right)\right)$ is a complex number, the nth root of z is

$$z^{\frac{1}{n}}=r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right].$$

where $k=0,1,2,3,\ldots,n-1$. We add $\frac{2k\pi}{n}$ to $\frac{\theta}{n}$ in order to obtain the periodic roots.

Example 9.4.7

Find the cube roots of $z = 8\left(\cos\left(\frac{2\pi}{3}\right) + i\,\sin\left(\frac{2\pi}{3}\right)\right)$.

Solution available at the end of this section.

Checkpoint 9.4.7

Find the four fourth roots of $16(\cos(120^\circ) + i\ \sin(120^\circ))$.

Answer available at the end of this section.

Time-Sensitive Advice: Pass/No Pass Application Deadline

Within the Los Rios Community College District, the last day you can apply to switch the grading scheme on your class to Pass/No Pass is the last day of finals week.[1]

Why should you not apply for this for your math classes?

Generally, applying for a Pass/No Pass grade in mathematics looks sloppy on a transcript. When you apply for transfer to a four-year university, you are competing with thousands of other applicants, most of whom have letter grades. Individuals who exercise the Pass/No Pass grade scheme notoriously do so for fear of failing a class, and the application reviewers at four-year institutions know this! Do yourself a favor and stick with letter grades for math courses.

[1] Strangely, this was recently changed from Friday of the fifth week of class without any notice to faculty.

Success in College: Reflecting on the Course

1. Henry Ford said, "Failure is only the opportunity to begin again more intelligently." If you could start this course over, would you do anything differently? Explain.
2. Barack Obama said, "Making your mark on the world is hard. If it were easy, everybody would do it. But it's not. It takes patience, it takes commitment, and it comes with plenty of failure along the way. The real test is not whether you avoid this failure, because you won't. It's whether you let it harden or shame you into inaction, or whether you learn from it; whether you choose to persevere." Do you find it challenging to persevere, keep going, and get your college education? Explain.

Answers

Section Preview

If you made it to (and through) this section, congratulate yourself and your students. You have performed a substantial accomplishment.

Section Preview

- **Arithmetic**
  - Mathematical Operations — Higher-Index Roots: The nth Root Theorem requires computing $r^{1/n}$ as a prior skill.

Section Preview

- Write complex numbers in polar form.
- Convert a complex number from polar to rectangular form.
- Find products of complex numbers in polar form.
- Find quotients of complex numbers in polar form.
- Find powers of complex numbers in polar form.
- Find roots of complex numbers in polar form.

Example 9.4.1

On the complex plane, the number $z = 4i$ is the same as $z = 0 + 4i$. To write this in polar form, we must first calculate r.

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{0^2+4^2} \\ &= \sqrt{16} \\ &= 4 \end{aligned}$$

Next, we look at x. If $x = r\cos(\theta)$ and $x = 0$, then $\theta = \frac{\pi}{2}$. Therefore, in polar form, the complex number $z = 0 + 4i$ can be written as

$$z = 4\left(\cos\left(\frac{\pi}{2}\right) + i\,\sin\left(\frac{\pi}{2}\right)\right) = 4\,\text{cis}\left(\frac{\pi}{2}\right).$$

See Figure 9.4.2.

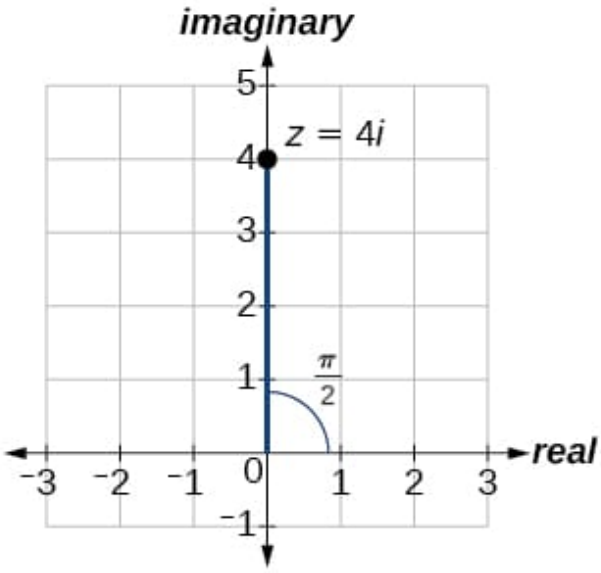


Figure 9.4.2

Checkpoint 9.4.1

$z = 3\left(\cos\left(\frac{\pi}{2}\right) + i\,\sin\left(\frac{\pi}{2}\right)\right)$ or $z = 3\ \text{cis}\left(\frac{\pi}{2}\right)$

Example 9.4.2

We begin by finding the value of r.

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{(-4)^2+4^2} \\ &= \sqrt{32} \\ &= 4\sqrt{2} \end{aligned}$$

We now find the angle θ.

$$\begin{aligned} \cos(\theta) &= \frac{x}{r} \\ \Longrightarrow \quad \cos(\theta) &= -\frac{4}{4\sqrt{2}} \\ \Longrightarrow \quad \cos(\theta) &= -\frac{1}{\sqrt{2}} \\ \Longrightarrow \quad \hat{\theta} &= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ \Longrightarrow \quad \hat{\theta} &= \frac{\pi}{4} \end{aligned}$$

Using the fact that $z = -4 + 4i$ is in the second quadrant, we get $\theta = \frac{3\pi}{4}$. Thus, the polar form is

$$z = 4\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right).$$

Checkpoint 9.4.2

$z = 2\left(\cos\left(\frac{\pi}{6}\right) + i\,\sin\left(\frac{\pi}{6}\right)\right) = 2\text{cis}\left(\frac{\pi}{6}\right)$

Example 9.4.3

We begin by evaluating the trigonometric expressions.

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

After substitution, the complex number is

$$z = 12\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\,i\right) = 6\sqrt{3} + 6i$$

Hence, the rectangular form of the given point in polar form is $6\sqrt{3} + 6i$.

Checkpoint 9.4.3

$z = 2\sqrt{3} - 2i$

Example 9.4.4

Follow the formula:

$$\begin{aligned} z_1 z_2 &= 4 \cdot 2\left[\cos\left(80^\circ + 145^\circ\right) + i\,\sin\left(80^\circ + 145^\circ\right)\right] \\ &= 8\left[\cos\left(225^\circ\right) + i\,\sin\left(225^\circ\right)\right] \\ &= 8\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right] \\ &= -4\sqrt{2} - 4i\sqrt{2} \end{aligned}$$

Checkpoint 9.4.4

a. $z_1 z_2 = (-3 - 3\sqrt{3}) + (3 - 3\sqrt{3})i$
b. 195°
c. $z_1 = 2\left(\cos\left(150^\circ\right) + i\,\sin\left(150^\circ\right)\right)$ and $z_2 = 3\left(\cos\left(45^\circ\right) + i\,\sin\left(45^\circ\right)\right)$
d. $z_1 z_2 = 6\left(\cos\left(195^\circ\right) + i\,\sin\left(195^\circ\right)\right)$
e. 195°

Example 9.4.5

Using the formula, we have

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2}{4}\left[\cos(213^\circ - 33^\circ) + i\,\sin(213^\circ - 33^\circ)\right] \\ &= \frac{1}{2}\left[\cos(180^\circ) + i\,\sin(180^\circ)\right] \\ &= \frac{1}{2}[-1 + 0i] \\ &= -\frac{1}{2} + 0i \\ &= -\frac{1}{2} \end{aligned}$$

Checkpoint 9.4.5

$z_1 z_2 = -4\sqrt{3}$; $\frac{z_1}{z_2} = -\frac{\sqrt{3}}{2} + \frac{3}{2}i$

Theorem: De Moivre's Theorem

Let $z = r\left(\cos\left(\theta\right) + i\,\sin\left(\theta\right)\right)$ and $n \in \mathbb{N}$.

CLAIM: $z^n = r^n\left[\cos(n\theta) + i\,\sin(n\theta)\right]$.

For $n = 1$,

$$\begin{aligned} z^1 &= \left(r\left(\cos\left(\theta\right) + i\,\sin\left(\theta\right)\right)\right)^1 \\ &= r^1\left(\cos\left(\theta\right) + i\,\sin\left(\theta\right)\right)^1 && \text{(Laws of Exponents)} \\ &= r^1\left(\cos\left(1 \cdot \theta\right) + i\,\sin\left(1 \cdot \theta\right)\right) && \text{(Multiplicative Identity)} \end{aligned}$$

Thus the claim is true for $n = 1$.

Induction Hypothesis: Suppose the claim is true for $n = k$. That is, suppose $z^k = r^k\left[\cos(k\theta) + i\,\sin(k\theta)\right]$.

Assuming this Induction Hypothesis,

$$\begin{aligned} z^{k+1} &= z^k \cdot z^1 && \text{(Laws of Exponents)} \\ &= r^k\left[\cos(k\theta) + i\,\sin(k\theta)\right] \cdot z^1 && \text{(Induction Hypothesis)} \\ &= r^k\left[\cos(k\theta) + i\,\sin(k\theta)\right] \cdot r\left[\cos\left(\theta\right) + i\,\sin\left(\theta\right)\right] && \text{(substituting } z = r\left(\cos\left(\theta\right) + i\,\sin\left(\theta\right)\right)\text{)} \\ &= r^{k+1}\left[\cos(k\theta) + i\,\sin(k\theta)\right]\left[\cos\left(\theta\right) + i\,\sin\left(\theta\right)\right] && \text{(Commutative Property of Multiplication)} \\ &= r^{k+1}\left[\cos(k\theta)\cos\left(\theta\right) + i\,\cos(k\theta)\sin\left(\theta\right) + i\,\sin\left(k\theta\right)\cos\left(\theta\right) + i^2\sin\left(k\theta\right)\sin\left(\theta\right)\right] && \text{(distributing)} \\ &= r^{k+1}\left[\cos(k\theta)\cos\left(\theta\right) + i\,\cos(k\theta)\sin\left(\theta\right) + i\,\sin\left(k\theta\right)\cos\left(\theta\right) - \sin\left(k\theta\right)\sin\left(\theta\right)\right] && \left(i^2 = -1\right) \\ &= r^{k+1}\left[\cos(k\theta)\cos\left(\theta\right) - \sin\left(k\theta\right)\sin\left(\theta\right) + i\,\cos(k\theta)\sin\left(\theta\right) + i\,\sin\left(k\theta\right)\cos\left(\theta\right)\right] && \text{(Commutative Property of Addition)} \\ &= r^{k+1}\left[\left(\cos(k\theta)\cos\left(\theta\right) - \sin\left(k\theta\right)\sin\left(\theta\right)\right) + i\left(\cos(k\theta)\sin\left(\theta\right) + \sin\left(k\theta\right)\cos\left(\theta\right)\right)\right] && \text{(grouping and factoring)} \\ &= r^{k+1}\left[\cos\left((k+1)\theta\right) + i\sin\left((k+1)\theta\right)\right] && \text{(Sum of Angles Identities)} \end{aligned}$$

Thus the claim is true for $n = k + 1$.

Since the claim holds for $n = 1$, and subsequently for $n = k + 1$ when it is assumed it works for $n = k$, the claim must be true for all $n \in \mathbb{N}$.

Example 9.4.6

Since De Moivre's Theorem applies to complex numbers written in polar form, we must first write $(1 + i)$ in polar form. Let us find r.

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{(1)^2+(1)^2} \\ &= \sqrt{2} \end{aligned}$$

Then we find θ. Using the formula $\tan\theta = \frac{y}{x}$ gives

$$\begin{aligned} \tan(\theta) &= \frac{1}{1} \\ \Longrightarrow \quad \tan(\theta) &= 1 \\ \Longrightarrow \quad \hat{\theta} &= \frac{\pi}{4} \end{aligned}$$

Since $(1+i)$ is in the first quadrant, this means $\theta = \frac{\pi}{4}$.

Now we use De Moivre's Theorem to evaluate the expression.

$$\begin{aligned} (1+i)^5 &= (\sqrt{2})^5\left[\cos\left(5\cdot\frac{\pi}{4}\right)+i\sin\left(5\cdot\frac{\pi}{4}\right)\right] \\ &= 4\sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right)+i\sin\left(\frac{5\pi}{4}\right)\right] \\ &= 4\sqrt{2}\left[-\frac{\sqrt{2}}{2}+i\left(-\frac{\sqrt{2}}{2}\right)\right] \\ &= -4-4i \end{aligned}$$

Checkpoint 9.4.6

$2^{10}\ \text{cis}\left(\frac{25\pi}{3}\right) = 1024\ \text{cis}\left(\frac{\pi}{3}\right) = 1024\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 512+512\ i\sqrt{3}$

Example 9.4.7

We have

$$\begin{aligned} z^{\frac{1}{3}} &= 8^{\frac{1}{3}}\left[\cos\left(\frac{\frac{2\pi}{3}}{3}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{\frac{2\pi}{3}}{3}+\frac{2k\pi}{3}\right)\right] \\ &= 2\left[\cos\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)\right] \end{aligned}$$

There will be three roots - one for each of $k = 0$, 1, and 2.

When $k = 0$, we have

$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right).$$

When $k = 1$, we have

$$\begin{aligned} z^{\frac{1}{3}} &= 2\left[\cos\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)\right] \qquad \left(\text{adding } \frac{2(1)\pi}{3} \text{ to each angle}\right) \\ &= 2\left(\cos\left(\frac{8\pi}{9}\right)+i\sin\left(\frac{8\pi}{9}\right)\right) \end{aligned}$$

When $k = 2$, we have

$$\begin{aligned} z^{\frac{1}{3}} &= 2\left[\cos\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)\right] \qquad \left(\text{adding } \frac{2(2)\pi}{3} \text{ to each angle}\right) \\ &= 2\left(\cos\left(\frac{14\pi}{9}\right)+i\sin\left(\frac{14\pi}{9}\right)\right) \end{aligned}$$

Remember to find the common denominator to simplify fractions in situations like this one. For $k = 1$, the angle simplification is

$$\begin{aligned} \frac{\frac{2\pi}{3}}{3}+\frac{2(1)\pi}{3} &= \frac{2\pi}{3}\left(\frac{1}{3}\right)+\frac{2(1)\pi}{3}\left(\frac{3}{3}\right) \\ &= \frac{2\pi}{9}+\frac{6\pi}{9} \\ &= \frac{8\pi}{9} \end{aligned}$$

Checkpoint 9.4.7

$z_0 = 2(\cos(30^\circ)+i\ \sin(30^\circ))$

$z_1 = 2(\cos(120^\circ)+i\ \sin(120^\circ))$

$z_2 = 2(\cos(210^\circ)+i\ \sin(210^\circ))$

$z_3 = 2(\cos(300^\circ)+i\ \sin(300^\circ))$

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9.4.1: Resources and Key Concepts

Key Concepts

Definitions

- **Polar Form of a Complex Number:** A way of writing a complex number $z = x + yi$ using its modulus r and argument θ. The form is $z = r(\cos(\theta) + i\sin(\theta))$, which is often abbreviated as $z = r\text{cis}(\theta)$.
- **Modulus:** The distance of the point (x, y) from the origin in the complex plane, given by $r = \sqrt{x^2 + y^2}$. It is the same as the radius r in polar form.
- **Argument:** The angle θ in standard position whose terminal side passes through the point (x, y) in the complex plane.

Theorems

- **Cartesian Coordinates in Terms of Cosine and Sine:** If a point P is located at a distance r from the origin in the direction specified by an angle θ in standard position, then the coordinates of P are $x = r\cos(\theta)$ and $y = r\sin(\theta)$.
- **Modulus in Polar Form:** The modulus of the complex number z, written in polar form $z = r(\cos(\theta) + i\sin(\theta))$, is r.
- **Products of Complex Numbers Written in Polar Form:** If $z_1 = r_1\text{cis}(\theta_1)$ and $z_2 = r_2\text{cis}(\theta_2)$, then their product is $z_1 z_2 = r_1 r_2\text{cis}(\theta_1 + \theta_2)$.
- **Quotients of Complex Numbers Written in Polar Form:** If $z_1 = r_1\text{cis}(\theta_1)$ and $z_2 = r_2\text{cis}(\theta_2)$, then their quotient is $\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$.
- **De Moivre's Theorem:** If $z = r(\cos(\theta) + i\sin(\theta))$ is a complex number and n is a positive integer, then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$, or $z^n = r^n\text{cis}(n\theta)$.
- **nth Root Theorem:** If $z = r(\cos(\theta) + i\sin(\theta))$ is a complex number, then its n distinct nth roots are given by $z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$, where $k = 0, 1, 2, \ldots, n-1$.

Common Mistakes

- **Incorrectly Determining the Argument:** When converting a complex number from rectangular form to polar form, the argument θ is not simply $\arctan(\frac{y}{x})$. You must consider the quadrant of the point (x, y) to find the correct angle.
- **Evaluating Trigonometric Functions Prematurely:** When converting a complex number *to* polar form, do not evaluate the trigonometric functions. The goal is to leave the number in the form $r(\cos\theta + i\sin\theta)$. Evaluating them would convert the number back to rectangular form.
- **Forgetting to Find All Roots:** A complex number has n distinct nth roots. A common mistake is to only find the principal root (for $k = 0$) and forget to find the other roots by adding multiples of $\frac{2\pi}{n}$ to the angle.

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9.4.2: Homework

Reading Questions

1. What is the polar form of a complex number $z = x + yi$?
2. In the polar form $z = r(\cos\theta + i\sin\theta)$, what is another name for r? What is another name for θ?
3. What does the notation $r\text{cis}(\theta)$ represent?
4. When multiplying two complex numbers in polar form, what do you do with their moduli and what do you do with their arguments?
5. When dividing two complex numbers in polar form, what do you do with their moduli and what do you do with their arguments?
6. State De Moivre's Theorem for finding the power of a complex number.
7. How many distinct cube roots does a complex number have?
8. According to the nth Root Theorem, how do you find all the nth roots of a complex number?
9. When converting $z = -4 + 4i$ to polar form, how do you determine that the argument θ is $\frac{3\pi}{4}$ and not its reference angle $\frac{\pi}{4}$?

Homework

Basic Skills

For the following exercises, simplify and plot each complex number as a point on the complex plane.

1. $1, i, i^2, i^3$ and i^4
2. $-1, -i, -i^2, -i^3$ and $-i^4$
3. $1 + 2i$ and $i(1 + 2i)$
4. $3 - 4i$ and $i(3 - 4i)$

For the following exercises, write the complex numbers in standard form. Give exact values for your answers.

5. $6\left(\cos\frac{2\pi}{3} + i\,\sin\frac{2\pi}{3}\right)$
6. $4\left(\cos\frac{7\pi}{4} + i\,\sin\frac{7\pi}{4}\right)$
7. $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\,\sin\frac{3\pi}{4}\right)$
8. $\frac{3}{2}\left(\cos\frac{5\pi}{6} + i\,\sin\frac{5\pi}{6}\right)$

For the following exercises, write the complex numbers in standard form. Round your answers to hundredths.

9. $5(\cos 5.2 + i\ \sin 5.2)$
10. $3(\cos 3.5 + i\ \sin 3.5)$
11. $12\,(\cos 115^\circ + i\,\sin 115^\circ)$
12. $20\,(\cos 250^\circ + i\,\sin 250^\circ)$

For the following exercises, write the complex numbers in polar form. Give exact values for your answers.

13. $3i$ and $-3i$

14. $2+2i$ and $2-2i$

15. $-3-\sqrt{3}i$ and $3-\sqrt{3}i$

16. $2\sqrt{3}+2i$ and $-2\sqrt{3}+2i$

For the following exercises, write the complex numbers in polar form. Round your answers to hundredths.

17. $-4+2i$ and $4-2i$

18. $-3-8i$ and $3+8i$

19. $9-5i$ and $9+5i$

20. $2+6i$ and $2-6i$

21. $3+4i, 3-4i, -3+4i$, and $-3-4i$

22. $1+3i, 1-3i, -1+3i$, and $-1-3i$

23. What can you conclude about the polar forms of z and $\bar{z}$?

24. What can you conclude about the polar forms of z and $-z$?

For the following exercises, find the product $z_1 z_2$ and the quotient $\frac{z_1}{z_2}$.

25. $z_1 = 4\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$ and $z_2 = \frac{1}{2}\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

26. $z_1 = 6\left(\cos\frac{5\pi}{8}+i\sin\frac{5\pi}{8}\right)$ and $z_2 = \frac{3}{2}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)$

27. $z_1 = 3\left(\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}\right)$ and $z_2 = 2\left(\cos\frac{3\pi}{10}+i\sin\frac{3\pi}{10}\right)$

28. $z_1 = 4\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$ and $z_2 = 6\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

For the following exercises, convert the complex number to polar form, then find the product $z_1 z_2$ and the quotient $\frac{z_1}{z_2}$.

29. $z_1 = 2i, z_2 = 4i$

30. $z_1 = -2i, z_2 = 3i$

31. $z_1 = 2\sqrt{3}-2i, \quad z_2 = -1+i$

32. $z_1 = \sqrt{3}+i, \quad z_2 = -1+\sqrt{3}$

For the following exercises, find the power.

33. $(2+2i)^5$

34. $(\sqrt{2}-\sqrt{2}i)^6$

35. $(-1+\sqrt{3}i)^8$

36. $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{12}$

37. $(\sqrt{3}-i)^{10}$

38. $(1-i)^{20}$

For the following exercises, use De Moivre's theorem to find the reciprocal.

39. $2-2i$

40. $3+\sqrt{3}i$

41. $-\sqrt{2}+\sqrt{6}i$

42. $-1-i$

For the following exercises,
(a) Find the roots and plot them in the complex plane.
(b) Write the roots in standard form.

43. The square roots of $9i$.

44. The fourth roots of -81.

45. The fifth roots of 32.

46. The cube roots of i.

47. The cube roots of $4\sqrt{3}+4i$.

48. The square roots of $-2+2\sqrt{3}i$.

49. Show that any complex number of the form $z=\cos\theta+i\ \sin\theta$ lies on the unit circle in the complex plane.

50. Show that if $|z|=1$, then $\dfrac{1}{z}=\bar{z}$.

51. a. Find three distinct cube roots of 1.

    b. Find four distinct fourth roots of 1.

    c. Find five distinct fifth roots of 1.

    d. Find six distinct sixth roots of 1.

52. a. Find the sum of the three distinct cube roots of 1. (Hint: Plot the roots.)

    b. Find the sum of the four distinct fourth roots of 1.

    c. Find the sum of the five distinct fifth roots of 1.

    d. Find the sum of the six distinct sixth roots of 1.

53. If n is a positive integer, define

$$\omega_k=\cos\frac{2\pi k}{n}+i\ \sin\frac{2\pi k}{n}$$

for $k=0,1,2,\cdots,n-1$. Show that $(\omega_k)^n=1$. (We call ω_k an nth **root of unity**.)

54. Let $\omega=\cos\frac{2\pi}{n}+i\ \sin\frac{2\pi}{n}$, where n is a positive integer. Show that the n distinct n^{th} roots of unity are $\omega,\omega^2,\omega^3,\cdots,\omega^{n-1}$.

For the following exercises, solve the equation.

55. $z^4+4z^2+8=0$

56. $z^6+4z^3+8=0$

57. $z^6 - 8 = 0$

58. $z^4 - 9i = 0$

59. $z^4 + 2z^2 + 4 = 0$

60. $z^4 - 2z^2 + 4 = 0$

61. a. Let $z = \cos\theta + i\ \sin\theta$. Compute z^2 by expanding the product.

    b. Use DeMoivre's theorem to compute z^2.

    c. Compare your answers to (a) and (b) to write identities for $\sin 2\theta$ and $\cos 2\theta$.

62. a. Let $z = \cos\theta + i\ \sin\theta$. Compute z^3 by expanding the product.

    b. Use DeMoivre's theorem to compute z^3.

    c. Compare your answers to (a) and (b) to write identities for $\sin 3\theta$ and $\cos 3\theta$.

For the following exercises, show that multiplication by i results in a rotation of $90°$.

63. Suppose that $z = a + bi$ and that the real numbers a and b are both nonzero.

    a. What is the slope of the segment in the complex plane joining the origin to z?

    b. What is the slope of the segment in the complex plane joining the origin to zi?

    c. What is the product of the slopes of the two segments from parts (a) and (b)? What can you conclude about the angle between the two segments?

64. Suppose that $z = a + bi$ and that a and b are both real numbers.

    a. If $a \neq 0$ and $b = 0$, then what is the slope of the segment in the complex plane joining the origin to z? What is the slope of the segment joining the origin to zi?

    b. If $a = 0$ and $b \neq 0$, then what is the slope of the segment in the complex plane joining the origin to z? What is the slope of the segment joining the origin to zi?

    c. What can you conclude about the angle between the two segments from parts (a) and (b)?

Proofs

65. Prove the product rule by following the steps.

    a. Suppose $z_1 = a + bi$ and $z_2 = c + di$. Compute $z_1 z_2$.

    b. Now suppose that $z_1 = r(\cos\alpha + i\ \sin\alpha)$ and $z_2 = R(\cos\beta + i\ \sin\beta)$. Write a, b, c and d in terms of r, R, α and β.

    c. Substitute your expressions for a, b, c and d into your formula for $z_1 z_2$.

    d. Use the laws of sines and cosines to simplify your answer to part (c).

66. Let $z_1 = r(\cos\alpha + i\ \sin\alpha)$ and $z_2 = R(\cos\beta + i\ \sin\beta)$. Prove the quotient rule as follows: Set $w = \frac{r}{R}(\cos(\alpha - \beta) + i\ \sin(\alpha - \beta))$, and show that $z_1 = wz_2$.

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10: THE LANGUAGE OF MATHEMATICS

The development of mathematics from the ground up is an astounding feat that is still unfinished and will never be. There is more that goes into creating the concepts and structures of the subject than most people could ever fathom. To complicate matters, the historical and conceptual development of mathematics have followed completely different paths.

The historical development of mathematics is full of odd leaps in knowledge and vast lengths of time with little development. Some very advanced mathematical concepts came about long before some elementary ones. For example, Calculus, arguably a very challenging field of study relying on mastery of Arithmetic, Algebra, and Trigonometry, was developed during the 17th century. However, this was nearly 200 years before the development of the complex number system and almost 300 years before the modern multiplication symbol, $\times$, was created (1902).

This nonlinear development is because mathematics has historically been driven by application and necessity rather than philosophy. This ebb and flow of mathematical concepts throughout time makes writing a textbook based upon *when* an idea was discovered impossible.

The conceptual development of mathematics is how we will approach learning Trigonometry. We will develop mathematics at a pace and in such a way that concepts will naturally lead to the next set of consequences. Upon this framework, we will place the usefulness of our concepts. That is, we will apply our knowledge to real-world problems.

It will seem, at times, that Trigonometry is full of definitions and theorems;[1] however, the definitions will be used enough to become part of our natural language, and we will concentrate on understanding *why* mathematics works the way it does so that theorems become "obvious" results rather than obscure mysteries. We will not allow ourselves to become intimidated by what many believe to be a "cold, lifeless subject." Instead, we will embrace how we can use this knowledge to open ourselves to a better, more robust understanding of our world. I hope this textbook enhances your worldview and enables you to apply critical thinking and analysis to many facets of your life.

With all that said, let us put on our thinking caps and prepare to become philosophers. We are going to figure out what makes mathematics so beautiful and true. This exercise will require brain power and creativity.

Footnotes

[1] We will talk about what theorems and definitions are momentarily.

10.1: Definitions

10.2: Axioms, Theorems, and Proofs

10.3: Operations

10.4: Simplifying

10.5: Theory versus Application

10.6: The Mathematical Mantra

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10.1: Definitions

Our first task in understanding mathematics is to imagine "truth." We all think we know what this word means, but the "truth" I am talking about here is likely a little more stringent than most people imagine. So what does "truth" mean for us (at least, us reading this textbook)?

It means that we must ignore everything around us that is supposition and stubborn, ego-driven prejudice – none of that truly matters. We must be devoid of falsehoods and lies. There is no judgment. There is no right or wrong. There is no such thing as politics, history, religion, war, wealth, feast, famine, TikTok, or Facebook.

We should have a clean slate – a blank canvas. There is only truth, which is now bare and waiting to be filled in. We must look around ourselves as if we are newborns – as if we have no idea what anything is or what anything does. This is essential to understanding mathematics.

Now that we are floating in this void, we must form "truth." How is this done? What is our first step?

To interact with the world around us, we must devise a method to refer to things in our environment. We need a set of names to give to things. The object you are reading we shall call a **book**. The object holding liquid next to you we shall call a **glass**. The liquid inside of it we shall call **tea**.

On we go, defining everything around us. Someone may tell us that a **table** is the surface on which we are leaning our elbows. Yet another person may tell us that the thing beeping in the background is an **alarm clock**. In some way, our young, naive minds get filled with a dizzying list of names attributed to objects around us.

We further classify these objects into subgroups. Whenever you see a flat, vertical surface with a handle, you know it is called a **door**. Interestingly, we can now have a group of doors that all share the same properties but look completely different (e.g., my father's house has a maroon door, while mine is white).

Why does starting with definitions like the ones discussed above make sense? It's best explained by example. Suppose your friend came to your house and said,

"*I thought you may like to see the zart downtown so I bought a troda and put on some fofels so we can ginal.*"

Without the knowledge of what a zart, troda, fofel, or a ginal are, that sentence is completely useless. We can try to guess the meaning of each word, but we are likely to get something completely wrong. For anything to make sense, we first must have things defined.

Key to Success

You probably have already noticed that this textbook is written in such a way that you need to thoughtfully process information as you read. This is on purpose. I do not want you to just skim through and memorize my words. Instead, you should think carefully and critically about the information being presented to you.

Consistent Language

I have always had issues with words having multiple meanings. It makes life more confusing when a word has too many implications. It is like having two siblings with the same name. Who would be so cruel?[1] It's bad enough having been raised with the same name as my father. I cannot recall how many times that caused confusion.

From a student's standpoint, having a word or phrase mean several different things is a complete disaster. Unfortunately, it occurs all too often. Consider the following example.

Example 10.1.1: A tale of too many meanings

How many **distinct** interpretations can you think of for the word "set?"

Answer available at the end of this section.

Clarification

In mathematics, the word **distinct** is frequently used. The literal meaning of distinct is "different in nature or quality."

Example 10.1.1 illustrates the difficulties that ambiguous language can present. The English language is full of ambiguities, which is why many second-language learners find English difficult to master. Often, ambiguous language leads to disastrous interpretations.

The following example is an actual incident experienced by my former philosophy professor at Sacramento City College.

Example 10.1.2: Accidental Ambiguity

As a student walked by his former professor, he turned to his friend and said (loud enough for the professor to hear), "There goes my old professor." Explain why the professor may take offense to this statement.

Answer available at the end of this section.

To avoid ambiguities like these, our language, at least in mathematics, must be precise. Every definition should be scrutinized, and contrary meanings for a single term should be removed from our dictionary. Despite this, some words in mathematics will seemingly have multiple meanings. Whenever such a word is encountered in this textbook, I will clarify the supposed double meaning.

Subsection Footnotes

[1] No offense to George Foreman and his five sons... all named George.

The Definition of... Definition

For the rest of your college mathematical career, you will be continually presented with definitions, theorems, lemmas, corollaries, axioms, and many other mathematical knick-knacks.[2] Each has its place in the mathematical collage and the more familiar you are with these distinctions, the better you will do in this, and future, mathematics courses.

A **definition** is something that we state concerning the meaning of a word or phrase. In general, when someone defines something (by giving it a name or a "naming phrase"), there is little else you can say to refute that definition. For example, a "cell phone" is called a cell phone because someone else did not call it a "tark phone" first. Unfortunately, as mentioned previously, some definitions are ambiguous.

Definition: Chair

A **chair** is an object consisting of a seat, legs, back, and often arms, designed to accommodate one person in the sitting position.

Some may argue that they have a chair with no back or legs, so this definition is flawed. I completely agree. This is the problem with many definitions in the "real world" – they can be left to interpretation. A well-built definition can withstand this scrutiny, but

building such a definition is complex and can lead to very long definitions if you are not careful.

Definition: Red

The hue of the long-wave end of the visible spectrum, evoked in the human observer by radiant energy with wavelengths of approximately 630 to 750 nanometers, is defined to be the color **red**.

Even if you tackle the philosophical question of whether your version of red is the same as my version of red, this definition holds no ambiguity. A strong definition has this quality of precision.

It is imperative to note that definitions work both ways. Therefore, by the last definition, if we have a red light, our definition tells us it must emit wavelengths of approximately 630 to 750 nanometers. On the other hand, if we have energy in the form of light at wavelengths of approximately 630 to 750 nanometers, our definition states that it must emit the color red.

Definitions in mathematics are similar to those elsewhere in that they are statements that we accept because we must agree to some common language to communicate effectively. However, mathematical definitions differ from traditional ones in that multiple definitions for the same concept are generally not allowed.

Caution

Mathematics is not entirely devoid of multiple terms for the same definition. Whenever a definition has several different words associated with it, I will mention each word form and clarify what the most popular form is.

In the "real world," we have the luxury of being able to ask what the definition of something is a few times, and, in general, people forgive our ignorance. For example, most people would forgive you if you had to ask repeatedly for the definition of the Higgs Boson (a.k.a. the Higgs particle). It's a tricky definition to grasp.

In contrast, when taking a mathematics course, you must familiarize yourself with the definitions as soon as they are presented. The rest of the material is built upon a strong knowledge of these definitions. I will constantly use the terms we define to help you get used to mathematical definitions as quickly as possible. Moreover, I will make every effort to be consistent with my language.

In all honesty, we could spend much more time discussing the structure of a definition; however, what we have developed thus far will suffice for our needs.

Subsection Footnotes

[2] At this point, don't worry too much about theorems, lemmas, corollaries, and axioms. We will get to these topics in the next section.

Answers

Example 10.1.1**: A tale of too many meanings**

Here are a few definitions I found for the word "set" - this is not an exhaustive list.

1. to put into another place or location (e.g., "Set it down over there.")
2. to determine or fix conclusively (e.g., "Let's set the rules for this game.")
3. to establish as the highest level (e.g., "Set a new world record!")
4. to ready in advance (e.g., "Is everyone set to go?")
5. to locate (e.g., "The book is set in Europe.")
6. to apply or start (e.g., "The vandals set fire to the building.")
7. a group of objects (e.g., "This is my set of dolls.")
8. a predetermined time period in a game (e.g., "They played one set of tennis.")
9. a slang term used to reference a television (e.g., "Did you watch the commercial on your TV set?")
10. a musician's term for a list of songs to be played (e.g., "We have a set list for the gig tonight at Old Ironsides.")

Example 10.1.2**: Accidental Ambiguity**

The student was likely pointing out to some of his friends that the professor was his previous instructor; however, the professor could have had a very sensitive ego and felt the student was calling him old. In this case, "old" has two meanings: previous and elderly.

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10.2: Axioms, Theorems, and Proofs

Axioms

> "A day without sunshine is like... well... night."

There is a strange creature in mathematics, not typically mentioned in lower division texts, called an **axiom** (or, in some texts, a **postulate**).

Definition: Axiom

An **axiom** is a self-evident or universally recognized truth. It is accepted as true, without proof, as the basis for argument.

Like definitions, the truthfulness of any axiom is taken for granted; however, axioms do not define things – instead, they describe a fundamental, underlying quality about something. The opening quote of this section can be considered an axiom. It is not defining sunshine; it is just stating something obvious.

An axiom has the feel of something that should be justifiable or proved.[a] Oddly enough, though, axioms cannot be proved. They are jumping points for future logical deductions, but nothing exists to state that the axiom is true initially. The foundation of everything we know in mathematics comes from a simple set of axioms.

Mathematical Vignette

In the study of advanced mathematics, specifically Number Theory, axioms are further divided into logical and non-logical axioms. The remainder of this textbook, as well as most undergraduate mathematics courses, is based on non-logical axioms.

Example 10.2.1

Each of the following statements is an example of an axiom.

a. If I am wearing a red shirt and blue jeans, I am wearing something red.
b. If I have $30 and you have $30, then we have the same amount.
c. It is possible to draw a line between any two points.

The first statement in Example 10.2.1 is almost trivial. There is no arguing that I would be wearing red because I already told you I was wearing red. The second is also obvious. Finally, it's easy to see that the third statement is a universally recognized truth. It is one of those statements that students, and most professors, would likely say, "It's true because... well... it's obvious."

When we began our exploration of numbers back in arithmetic, we were introduced to the ten fundamental axioms upon which almost all of mathematics is built. These included such catchy phrases as the "commutative property of addition," the "distributive property," and the "multiplicative identity." It takes some creative thinking to develop axioms; however, that is not the goal of this section, nor will we even approach that conversation in this textbook. For now, it's enough that you know there is an object in mathematics called an axiom.

Subsection Footnotes

[a] We will cover what a proof is in a moment.

Theorems

In essence, mathematics develops new knowledge by logical deduction from old knowledge. The new knowledge is called the **conclusion**, and the old knowledge is called the **premise** (or **assumption**). The following definition introduces the vehicle we will use to conclude new knowledge from old premises (or assumptions).

Definition: Theorem

A **theorem** is a proposition that has been, or is to be, proved based on explicit assumptions.

Like many definitions, this seems a little difficult to understand without some more straightforward explanation. In layperson's terms, theorems are claims that can be proven using previous information given to us. Let's take a look at a simple theorem which is often cited in textbooks on logic.

Theorem

If it is raining, then it must be cloudy.

This is a theorem because someone, namely me, is proposing that for it to rain, we must have clouds. Can this be proven? Since I am not an expert in meteorology, I can only back my claim using a common sense approach. Nonetheless, this theorem would hold true as long as my argument has no flaws.

In mathematics, the theorem is king! It is the basis of correctness, and no statement is more potent than the theorem (as long as it has been proved). I often tell my students they must only believe something if it has been proven to them. While this is admittedly extreme, it is a wise suggestion for mathematics.

There is structure to a theorem. Theorems have two parts, both of which we have already mentioned. These are the assumption and the conclusion (also called the **antecedent** and the **consequent**, respectively).

Definition: Assumption and Conclusion of a Theorem

The **assumption** (or **antecedent**) of a theorem is a formal set of conditions that we accept as truth for the statement. A theorem's **conclusion** (or **consequent**) is the result that can be derived from the assumption according to a logical string of arguments.

This definition essentially states that the first part of a theorem is some assumption. This assumption is often considered the "if" statement. The second part of a theorem is what we think should happen as long as our assumption holds. This conclusion is often considered the "then" statement. To determine the assumption and conclusion of a theorem, it is recommended to rewrite the theorem in the form "If..., then...". When written in this form, the statement following the word "if" is the assumption, and the statement following the word "then" is the conclusion.

Example 10.2.2

In the following theorem, what is the assumption, and what is the conclusion?

"If it is raining, then it must be cloudy."

Answer available at the end of this section.

Just a single example is not enough to do justice to the idea of assumptions and conclusions. Let's try another one.

Example 10.2.3

In the following theorem, what is the assumption, and what is the conclusion?

"If I drink 14 gallons of water within 10 minutes, I will die."

Answer available at the end of this section.

The following example illustrates that critical thinking will be necessary to reword sentences or claims properly to see the assumptions and conclusions.

Example 10.2.4

The following is a statement a mother made to her son.

"Eating ice cream causes polio."

What is the assumption, and what is the conclusion?

Answer available at the end of this section.

Historical Note

The claim in Example 10.2.4 is not fictitious. A long-term study released in the late 1940s claimed this statement to be true. While some still cling to this belief, it is widely accepted that the interpretation of the data was flawed. First, polio is a communicable viral infection. Second, the high values of polio cases and ice cream consumption were from data collected during the summer. Of course, kids play in groups more during the warmer weather months, and warmer seasons will increase ice cream consumption. Finally, the low values of polio cases and ice cream consumption were from data collected during the winter. Again, this is pretty obvious – ice cream sales drop during colder months, and children tend to play less in groups when cold outside.

In other fields, terms like **conjecture** or **hypothesis** are used instead of theorem; however, each implies that the statement has not yet been proved and, therefore, the conclusion should be suspect. Only in mathematics do we have the pleasure of the theorem.

Since I am not speaking "math-speak" right now, the examples we have seen so far cannot officially be called theorems – they are technically conjectures or hypotheses. This is because someone, possibly you, could find a reason why the conjecture is false (remember, theorems have already been proved). Despite this, to keep our language less confusing, I will continue to use the word theorem to describe these and all future examples.

Example 10.2.5

In each of the following theorems, state the assumption and the conclusion. Then, state a possible reason why the theorem could be false.

a. If Nam eats ice cream cake, he will gain weight.
b. Camille will get ticketed if she speeds down this street.
c. Calculus II is a course that all engineering students at Cosumnes River College must take.

Answer available at the end of this section.

As you can see from these examples, "theorems" outside of mathematics often have logical loopholes. Fortunately, true theorems (those within the field of mathematics) are generally flawless.

How important are theorems? You might say that without theorems, there is no thought. Without theorems, the world is just a bunch of statements with no conclusions. The idea of a theorem allows us to say,

"If it is raining, then it must be cloudy."

This statement makes sense, and almost serves as advice or warning. Without the theorem structure, either the conclusion or the assumption is gone, and we would only say,

"It is raining."

or

"It is cloudy."

These are just exclamations without any conclusion. It sounds more like we are stating a fact rather than suggesting what would happen "if" a condition was met.

As another example, consider the following "theorem."

"Camille will get ticketed if she speeds down this street."

As we stated in Example $10.2.5b$, the assumption here is that Camille is speeding down the street, and the conclusion is that she will get ticketed. Removing the assumption leads to the odd statement,

"Camille will get ticketed."

Removing the conclusion leads to an even more nonsensical statement,

"Camille speeds down this street."

In either case, these are just statements without any assumptions or conclusions. They offer no advice, no warning, and no way out. With the assumption removed, Camille is getting ticketed – period! With the conclusion removed, Camille is definitely speeding down the street.

The following example illustrates a method to help determine whether or not a statement is a theorem. We essentially are trying to put it in the "if..., then..." form. While more complicated theorems are challenging to place into this form, the examples we see in this textbook are nice to work with and can be manipulated into this form somewhat easily.

Example $10.2.6$

Which of the following has the structure to be a theorem?

a. Ron has the winning lottery ticket.
b. When Linda goes to the store, she spends too much money.
c. Who are you?

Answer available at the end of this section.

Proofs

The fact that a theorem (in mathematics) has been proved allows us to rely on its results without fear of exceptions. But what does it mean for a theorem to be proved? What is a proof?

Definition: Proof

A **proof**, in mathematics, is the validation of a proposition or theorem by the application of specified rules in a series of logical steps.

In layperson's terms, this means that a proof is a carefully constructed set of arguments laid out so that nobody, no matter how smart or creative, could refute the logic.

Let's show how a proof is done by using a series of logical arguments to prove the following theorem.

Theorem

If it is raining, then it must be cloudy.

Proof available at the end of this section.

Okay, I am not a meteorologist, so I may have butchered that last bit of the proof; however, if I did start using more technical terms from meteorological science, I may bore you to death. I hope that example delivers the idea of a proof.

A subtle note about proofs is required here. The proof is necessary for a conjecture to be classified as a theorem; however, the proof is not considered part of the theorem. This means that proofs and theorems are different beasts. When someone in mathematics states a theorem, you have the right to request a proof of their statement; however, it is not necessary.

Every effort is made to prove all theorems stated in this textbook, and you should genuinely try to understand how each proof works; however, there may be times when a proof will be beyond the scope of our skills. For these theorems, I will mention that more advanced work is needed to derive the proof, or I may include their proofs in the appendix.

Note

I have always thought of the theorem-proof relationship as a challenge. It's as if someone walks up to you and says,

"*If it is raining, then it must be cloudy. Prove it!*"

You meet the challenger eye-to-eye and begin to explain why it has to be cloudy, given the fact that it is raining. Every statement you make is precise and clear. Each word is meant as another nail in the coffin of the challenger. Eventually, when all doubt as to the weakness of the conjecture has been erased, you claim victory. The challenger walks away, knowing that you are the wiser.

This is the essence of the proof.

Let's try another proof to get the hang of it.

Theorem

If you owned the car called the Aurora, you would own the only Aurora in the world.

Proof available at the end of this section.

Historical Note

Designed in 1957 by an eccentric New York priest as the world's safest car, the Aurora flopped, and only one was ever made. It had foam-filled bumpers, a roll cage, and seats that swiveled 180 degrees before a crash.

We could go for one more while we are at it. Prove the following theorem.

Theorem

Every president with the last name of Adams finished his presidency in the 19th century.

Proof available at the end of this section.

Before walking away from our discussion of theorems, I would like to mention that we encounter two subtypes of theorems in mathematics: the lemma and the corollary. At this point, the distinction between a theorem, a corollary, and a lemma is somewhat arbitrary. We will define these when we encounter them later in the textbook; however, you could start a new skill by looking in the index for the word lemma and reading its definition right now.

Answers

Example 10.2.2

The words following "if" are "it is raining." Therefore, the assumption is that it is raining. The words following "then" are "it must be cloudy." Hence, the conclusion is that it must be cloudy.

Example 10.2.3

This is nearly written in "If..., then..." form. Rewriting it as such, we get, "If I drink 14 gallons of water within 10 minutes, then I will die." The assumption is that I drink 14 gallons of water in 10 minutes. The conclusion, based on that assumption, is that I will die.

Example 10.2.4

In this statement (theorem), we are entirely missing the words "if" and "then." However, we can always reword it to contain these key words.

"If you eat ice cream, then you will get polio."

Now we can easily see that the assumption is that the son is eating ice cream, and the conclusion is that doing so will cause polio.

Example 10.2.5

a. The assumption is that Nam is going to eat ice cream cake. The conclusion is the consequence of that action - he will gain weight. Is this theorem necessarily true? If Nam plans on exercising a lot after he eats the ice cream cake, then it could be a false statement.
b. This is an example where rewriting as an "if..., then..." statement will help immensely. The statement says, "If Camille speeds down this street, then she will get ticketed." Thus, the assumption is that Camille is speeding down the street, and the conclusion is the consequence of that action, which is that she will be ticketed. Can you think of a way that this does not need to be true?
c. The way this is written can throw some students off, but this is a theorem. The assumption is that we have an engineering student at Cosumnes River College. The conclusion is that she must take Calculus II ("If you are an engineering student at Cosumnes River College, then you must take Calculus II"). There is an easy way around this, though. If the student reads a Calculus textbook over the summer and tests out of Calculus II, she successfully avoids taking the course. So the theorem has a flaw.

Example 10.2.6

a. This cannot be written in the "if..., then..." form, so it is *not* a theorem. It would be a theorem if it were rewritten as, "If Ron has the winning lottery ticket, then he has at least a million dollars."
b. This is a theorem because we can rewrite this as, "If Linda goes to the store, then she spends too much money."
c. This is a far cry from a theorem. How can you fit an "if..., then..." statement here?

Theorem

Let us first assume that it is raining. Then, by the definition of rain (which we will assume we know), we have water falling from the sky. However, if water is falling from the sky, there must be moisture in the sky (since water is moisture). For there to be enough moisture in the sky to cause this moisture to coalesce into raindrops, there must be clouds. Therefore, if it is raining, it must be cloudy.

Theorem

Suppose that you owned the car called the Aurora. Since the designer only built one of these cars, you would own the only Aurora in the world.

Theorem

Rewriting the theorem as an "if..., then..." statement, we have the equivalent statement, "If a president has the last name of Adams, then he finished his presidency in the 19th century."

The presidents with the last name of Adams are John Adams and John Quincy Adams. John Adams finished his presidency in 1801, and John Quincy Adams finished his presidency in 1829. Therefore, all presidents with the last name of Adams finished their presidency in the 19th century.

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10.3: Operations

Having something is not the same as *using* something. I understand that most readers will find that statement obvious, but remember that we are building mathematics up from the ground - we need to be meticulous to understand the structures with which we are playing.

You might own books, electronics, clothing, or a car. However, those are just a pile of things unless you intend to use them - it's a veritable junkyard of...

> "For every action, there is an equal and opposite reaction."

What is done can be undone... sometimes

Have you ever sent an email or said something to someone that you wished you could immediately take back? If you haven't, then you likely are the only one on this planet who is that perfect. Most of us suffer from "foot-in-mouth disease" at one point or another. Repairing the damage done from these unfortunate situations is often difficult, if not impossible.

In mathematics, the ability to "undo" an action is crucial. Before we...

The key to success when solving equations is to remember that you are inverting actions.

As was mentioned in the section on definitions, language lends itself to ambiguous definitions. The definition of the word inverse is, unfortunately, one of those with many meanings; however, the only meaning we need in mathematics is the following.

Definition: Operations and Operators

In mathematics, any action that can possibly transform one or more objects into another object is called an **operation**. The symbol used for the operation, which completely depends on the action being done, is called the **operator**.

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10.4: Simplifying

The Meaning of Simplification

Life is complex - not just the biological processes

What is simplifying?

Definition (Simplified)
A statement is considered to be simplified if it is completely diminished in complexity. That is, it has been reduced to basic essentials and any further action would result in a more complicated statement.

Example 1.4.1: Simplify the following statement:
Cary is wearing blue shoes, Susan is wearing blue shoes, Khojesta is wearing blue shoes, Dillon is wearing blue shoes, Alok is wearing blue shoes, Nicholas is wearing blue shoes, Jason is wearing blue shoes, and Jessica is wearing blue shoes.

Solution 1.4.1: The easiest way to simplify that statement is to mention at the beginning what each of these people have in common.
The following people are each wearing blue shoes: Cary, Susan, Khojesta, Dillon, Alok, Nicholas, Jason, and Jessica.

The importance of simplification
It is important to know that everything you learn in mathematics exists for a reason. You are responsible to use all of your previous knowledge to simplify current problems.

This is a tricky question. If we define a "simplified" form as one involving the fewest number of operations once the unknown has been given, then $(\cos(\theta)+\sin(\theta))^2$ would require two operations to evaluate the trigonometric functions (one for each function), an addition of those values, and then squaring that result. This is a total of four operations. On the other hand, $1+2\cos(\theta)\sin(\theta)$ requires two operations to evaluate the trigonometric functions, two multiplications (one for the product of the values of the trigonometric functions and one multiplying that result by 2), and an addition by 1. This is a total of five operations. Hence, the original expression could be considered a more simplified form. However, most mathematicians would consider the opposite to be true.

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10.5: Theory versus Application

For years, I have heard people make many claims about the connections between mathematics and other seemingly unrelated facets of life. Only when I began writing this section did I find a direct correlation between the music business and mathematics; however, this correlation can be seen in many fields. Having spent years playing music professionally and as a hobby, I can honestly state that there are generally two types of musicians: those who make music for themselves and those who make music for others.

The former have little concern over the public's opinion of their music and instead revel in the process of creating what they find to be beautiful and inspirational art. For these musicians, it is neither necessary for their creations to be consumed en masse, nor is it a desire that their music be used in any way to create a new style. The musicians who create for themselves are often ignored by society or thought of as too esoteric; however, some are moved into the public eye when another, more mainstream, musician cites their unique style as a "huge influence." Others might be thrust into the spotlight when there is the rare intersection between their experimental artistry and the publics' musical palate. After such forays into the public domain, these artists generally go back to their experimental roots, unconcerned with public opinion.[1]

On the other hand, the musician who writes for a massive audience is dedicated to developing music that will withstand critique and be looked upon favorably in the public eye. It is often the case that artistic flare is sacrificed to make a song marketable or understandable to a broader audience. This type of musician creates most of the music played on the radio or streamed online. Many in the underground music scenes call them "sellouts" because their aim is to sell as much music as possible. Whether this is true or not does not diminish their music's usefulness or appeal.

So, where is the comparison to mathematics?

The first kind of musician, the true artist, is the equivalent of the **theoretical**, or **pure**, **mathematician**. These mathematicians are not necessarily focused on the applicability of their theorems or the impact of the results of their countless hours of work. For them, the journey is important - the fact that a philosophical question may have been answered for all of eternity. Many results in pure mathematics are considered unusable in applications. However, they are still admired for the intrinsic beauty that they embody. This is not to say that all results in pure mathematics are useless. Quite to the contrary, applied mathematics would not exist without the multitude of theoretical mathematicians who developed and proved theorems that once seemed "pointless."

The second musician, who creates music for the masses, is akin to the **applied mathematician**. They are solely concerned with providing a use for the mathematics they, or pure mathematicians, have created. Generally, this type of mathematics is used to advance science, engineering, and technology (to name a few fields). However, applied mathematics would be nonexistent without pure mathematics and vice versa.

When talking about the language of mathematics, the distinction between theory and application is inescapable.[2] In this textbook, we will venture into both areas. A solid conceptual foundation is critical. Therefore, we will approach most topics from a conceptual standpoint. However, once we grasp the theoretical structure, we showcase applications to reinforce the usability of the concept.

Footnotes

[1] George Harrison from the Beatles is an excellent example of this type of musician. His track, "Within You, Without You," is an anomaly on the hugely popular album, "Sgt. Pepper's Lonely Hearts Club Band." Harrison's writing was often cited as the most esoteric of the Beatles' work.

[2] To be honest, the distinction between applied and pure mathematics is more complicated than this. According to the Mathematics Subject Classification system developed, in part, by the American Mathematical Society, there are currently over 60 recognized subfields in mathematics.

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10.6: The Mathematical Mantra

Traditionally, when you learn mathematics, you start with **Arithmetic**. You learn the axioms that define the "game of mathematics" (e.g., the Associative Property of Addition and the Commutative Property of Multiplication) and you use those, along with the operations of addition, subtraction, multiplication, and division, to perform simple computations.

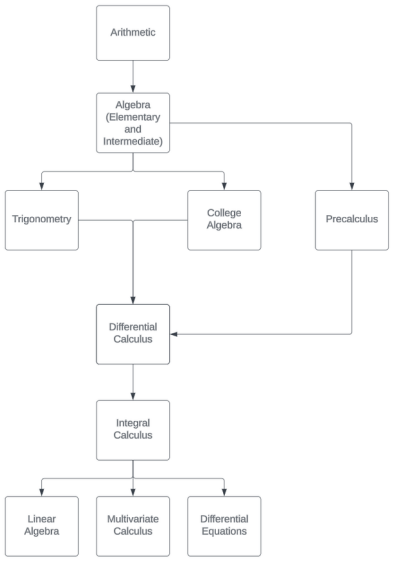
Once you attain mastery of Arithmetic, you move on to **Basic Algebra** (this is usually split into Elementary and Intermediate Algebra). You learn how to simplify expressions involving unknown quantities. It is also here that you begin to solve equations involving unknown quantities. You spend time visualizing the infinite number of solutions to equations using graphs, and you apply your knowledge to find solutions to complex real-world problems.

Once you have completed Basic Algebra, you reach a fork in the road of your mathematics education - you could take **Trigonometry** (this course) or **College Algebra**. In either case, you *must* complete both subjects before moving into **Differential Calculus** (also known as Calculus I), but they can be done in either order. In fact, some institutions combine them into a single course called Precalculus.[1]

Trigonometry is the study of angles within triangles and is necessary to master before you can begin Calculus. In Trigonometry, you spend a good deal of time refining your critical-thinking skills by practicing proofs. You also develop skills to help you tackle applications requiring spatial interpretations.

College Algebra, on the other hand, consists of the advanced topics in Algebra that are not covered in previous Algebra courses, but required for a student to understand and succeed in Calculus. You spend time working with transcendental functions like logarithms and exponentials. You further refine your critical-thinking skills by proving statements using both direct logic and mathematical induction. It is also within College Algebra that you investigate conic sections, vectors, and parametric curves.

On and on you travel the path of mathematics, adding to your ever-expanding knowledge-base at each stop. At some point, you have such an abundance of tools within your "mathematical toolkit" that it seems daunting to know what to do when facing problems within mathematics, physics, engineering, other complex fields. It is in these instances that the **Mathematical Mantra** will be incredibly helpful.



Mathematical Mantra

Perform mathematics in the order you learned mathematics. That is, perform

Arithmetic before Basic Algebra,

Basic Algebra before College Algebra,

College Algebra before Trigonometry, and

Trigonometry before Calculus.

Always be willing to ask yourself, "Is there something I learned previously that can simplify my work here?"

I will try to reinforce these concepts in the worked out examples using the following notations



If you spend a moment reminding yourself of the Mathematical Mantra before jumping into most math problems, you will (hopefully) be surprised at how it simplifies your work into nice little logical chunks.

Footnotes

[1] As mathematics is a subject that requires time to "digest," I do not recommend taking the "single-step" Precalculus course. It contains too much material in too little time and you end up getting a superficial treatment of the necessary mathematics with very little time to make sense of any of it.

For those interesting in learning more about the path of mathematics, I recommend watching this wonderful video created by Dr. Dominic Walliman (Domain of Science).

YouTube Video

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

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