COLLEGE TECHNICAL MATH 1B (NWTC)



LibreTexts™

College Technical Math 1B (NWTC)

This text is disseminated via the Open Education Resource (OER) LibreTexts Project (https://LibreTexts.org) and like the thousands of other texts available within this powerful platform, it is freely available for reading, printing, and "consuming."

The LibreTexts mission is to bring together students, faculty, and scholars in a collaborative effort to provide and accessible, and comprehensive platform that empowers our community to develop, curate, adapt, and adopt openly licensed resources and technologies; through these efforts we can reduce the financial burden born from traditional educational resource costs, ensuring education is more accessible for students and communities worldwide.

Most, but not all, pages in the library have licenses that may allow individuals to make changes, save, and print this book. Carefully consult the applicable license(s) before pursuing such effects. Instructors can adopt existing LibreTexts texts or Remix them to quickly build course-specific resources to meet the needs of their students. Unlike traditional textbooks, LibreTexts' web based origins allow powerful integration of advanced features and new technologies to support learning.



LibreTexts is the adaptable, user-friendly non-profit open education resource platform that educators trust for creating, customizing, and sharing accessible, interactive textbooks, adaptive homework, and ancillary materials. We collaborate with individuals and organizations to champion open education initiatives, support institutional publishing programs, drive curriculum development projects, and more.

The LibreTexts libraries are Powered by NICE CXone Expert and was supported by the Department of Education Open Textbook Pilot Project, the California Education Learning Lab, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

Have questions or comments? For information about adoptions or adaptions contact info@LibreTexts.org or visit our main website at https://LibreTexts.org.

This text was compiled on 06/10/2025

TABLE OF CONTENTS

Licensing

1: Law of Sines and Cosines

- 1.1: Non-right Triangles - Law of Sines
- 1.2: Non-right Triangles - Law of Cosines

2: Polynomials

- 2.1: Add and Subtract Polynomials
- 2.2: Multiply Polynomials
- 2.3: Special Products
- 2.4: Divide Polynomials

3: Factoring

- 3.1: Greatest Common Factor and Factor by Grouping
- 3.2: Factor Quadratic Trinomials with Leading Coefficient 1
- 3.3: Factor Quadratic Trinomials with Leading Coefficient Other than 1
- 3.4: Factor Special Products
- 3.5: General Strategy for Factoring Polynomials

4: Quadratic Equations

- 4.1: Quadratic Equations and Solving by Factoring
- 4.2: Solve Quadratic Equations Using the Square Root Property
- 4.3: Solve Quadratic Equations Using the Quadratic Formula
- 4.4: Solve Applications Modeled by Quadratic Equations

5: Rational Expressions and Equations

- 5.1: Simplify Rational Expressions
- 5.2: Multiply and Divide Rational Expressions
- 5.3: Add and Subtract Rational Expressions with a Common Denominator
- 5.4: Add and Subtract Rational Expressions with Unlike Denominators
- 5.5: Simplify Complex Rational Expressions
- 5.6: Solve Rational Equations
- 5.7: Solve Uniform Motion and Work Applications

6: Systems of Equations

- 6.1: Solve Systems of Equations by Graphing
- 6.2: Solve Systems of Equations by Substitution
- 6.3: Solve Systems of Equations by Elimination
- 6.4: Solve Mixture Applications with Systems of Equations
- 6.5: Solve Systems of Equations with Three Variables

Index

Glossary

Detailed Licensing

Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

1: Law of Sines and Cosines

1.1: Non-right Triangles - Law of Sines
1.2: Non-right Triangles - Law of Cosines

1: Law of Sines and Cosines is shared under a CC BY-SA license and was authored, remixed, and/or curated by LibreTexts.

1.1: Non-right Triangles - Law of Sines

Learning Objectives

- Use the Law of Sines to solve oblique triangles.
- Solve applied problems using the Law of Sines.

Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station is 35 degrees, whereas the **angle of elevation** measured by the second station is 15 degrees. How can we determine the altitude of the aircraft? We see in Figure 1.1.1 that the triangle formed by the aircraft and the two stations is not a right triangle, so we cannot use what we know about right triangles. In this section, we will find out how to solve problems involving **non-right triangles**.

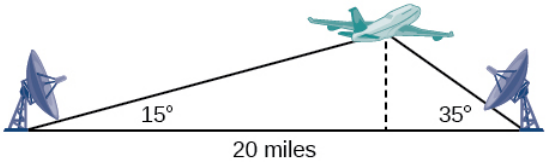


***Figure* 1.1.1**

Using the Law of Sines to Solve Oblique Triangles

In any triangle, we can draw an **altitude**, a perpendicular line from one vertex to the opposite side, forming two right triangles. It would be preferable, however, to have methods that we can apply directly to non-right triangles without first having to create right triangles.

Any triangle that is not a right triangle is an **oblique triangle**. Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

ASA (angle-side-angle) We know the measurements of two angles and the included side. See Figure 1.1.2.

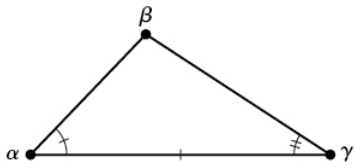


***Figure* 1.1.2**

AAS (angle-angle-side) We know the measurements of two angles and a side that is not between the known angles. See Figure 1.1.3.

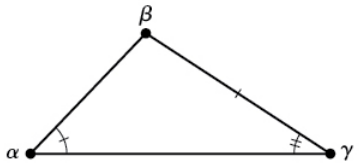


***Figure* 1.1.3**

SSA (side-side-angle) We know the measurements of two sides and an angle that is not between the known sides. See Figure 1.1.4.

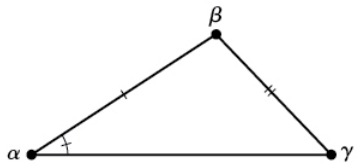


***Figure* 1.1.4**

Knowing how to approach each of these situations enables us to solve oblique triangles without having to drop a perpendicular to form two right triangles. Instead, we can use the fact that the ratio of the measurement of one of the angles to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. Let's see how this statement is derived by considering the triangle shown in Figure 1.1.5.

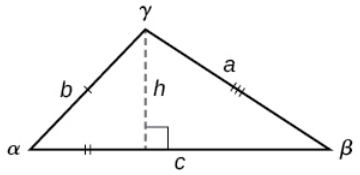


***Figure* 1.1.5**

Using the right triangle relationships, we know that $\sin\alpha = \dfrac{h}{b}$ and $\sin\beta = \dfrac{h}{a}$. Solving both equations for h gives two different expressions for h.

$$h = b\sin\alpha \text{ and } h = a\sin\beta$$

We then set the expressions equal to each other.

$$\begin{aligned} b\sin\alpha &= a\sin\beta \\ \left(\frac{1}{ab}\right)(b\sin\alpha) &= (a\sin\beta)\left(\frac{1}{ab}\right) \qquad \text{Multiply both sides by } \frac{1}{ab} \\ \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \end{aligned}$$

Similarly, we can compare the other ratios.

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \text{ and } \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Collectively, these relationships are called the **Law of Sines**.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Note the standard way of labeling triangles: angle α (alpha) is opposite side a; angle β (beta) is opposite side b; and angle γ (gamma) is opposite side c. See Figure 1.1.6.

While calculating angles and sides, be sure to carry the exact values through to the final answer. Generally, final answers are rounded to the nearest tenth, unless otherwise specified.

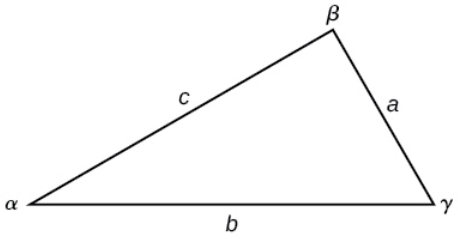


***Figure* 1.1.6**

Note: LAW OF SINES

Given a triangle with angles and opposite sides labeled as in Figure 1.1.6, the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. All proportions will be equal. The **Law of Sines** is based on proportions and is presented symbolically two ways.

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c} \tag{1.1.1}$$

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma} \tag{1.1.2}$$

To solve an oblique triangle, use any pair of applicable ratios.

Example 1.1.1: Solving for Two Unknown Sides and Angle of an AAS Triangle

Solve the triangle shown in Figure 1.1.7 to the nearest tenth.

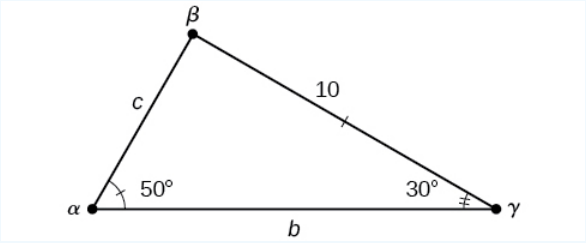


***Figure* 1.1.7**

Solution

The three angles must add up to 180 degrees. From this, we can determine that

$$\begin{aligned}\beta &= 180^\circ - 50^\circ - 30^\circ\\ &= 100^\circ\end{aligned}$$

To find an unknown side, we need to know the corresponding angle and a known ratio. We know that angle $\alpha = 50^\circ$ and its corresponding side $a = 10$. We can use the following proportion from the Law of Sines to find the length of c.

$$\begin{aligned}\frac{\sin(50^\circ)}{10} &= \frac{\sin(30^\circ)}{c}\\ c\frac{\sin(50^\circ)}{10} &= \sin(30^\circ) \qquad \text{Multiply both sides by } c\\ c &= \sin(30^\circ)\frac{10}{\sin(50^\circ)} \qquad \text{Multiply by the reciprocal to isolate } c\\ c &\approx 6.5\end{aligned}$$

Similarly, to solve for b, we set up another proportion.

$$\begin{aligned}\frac{\sin(50^\circ)}{10} &= \frac{\sin(100^\circ)}{b}\\ b\sin(50^\circ) &= 10\sin(100^\circ) \qquad \text{Multiply both sides by } b\\ b &= \frac{10\sin(100^\circ)}{\sin(50^\circ)} \qquad \text{Multiply by the reciprocal to isolate } b\\ b &\approx 12.9\end{aligned}$$

Therefore, the complete set of angles and sides is

$$\begin{aligned}\alpha &= 50^\circ \quad & a &= 10\\ \beta &= 100^\circ \quad & b &\approx 12.9\\ \gamma &= 30^\circ \quad & c &\approx 6.5\end{aligned}$$

Exercise 1.1.1

Solve the triangle shown in Figure 1.1.8 to the nearest tenth.

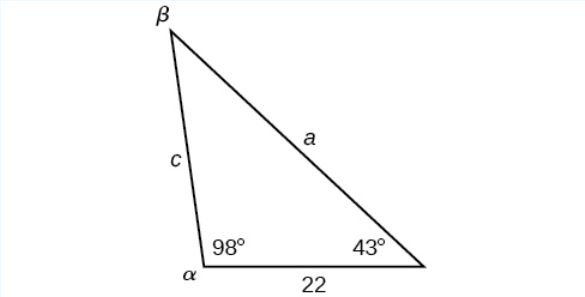


***Figure* 1.1.8**

Answer

$\alpha = 98^\circ \quad a = 34.6$
$\beta = 39^\circ \quad b = 22$
$\gamma = 43^\circ \quad c = 23.8$

Using The Law of Sines to Solve SSA Triangles

We can use the Law of Sines to solve any oblique triangle, but some solutions may not be straightforward. In some cases, more than one triangle may satisfy the given criteria, which we describe as an ambiguous case. Triangles classified as SSA, those in which we know the lengths of two sides and the measurement of the angle opposite one of the given sides, may result in one or two solutions, or even no solution.

Note: POSSIBLE OUTCOMES FOR SSA TRIANGLES

Oblique triangles in the category SSA may have four different outcomes. Figure 1.1.9 illustrates the solutions with the known sides a and b and known angle α.

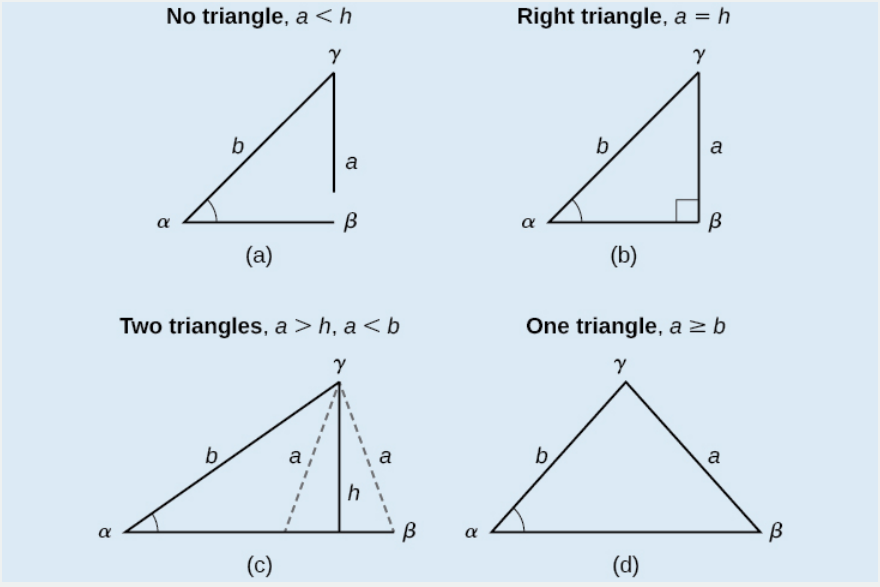


***Figure* 1.1.9**

Example 1.1.2: Solving an Oblique SSA Triangle

Solve the triangle in Figure 1.1.10 for the missing side and find the missing angle measures to the nearest tenth.

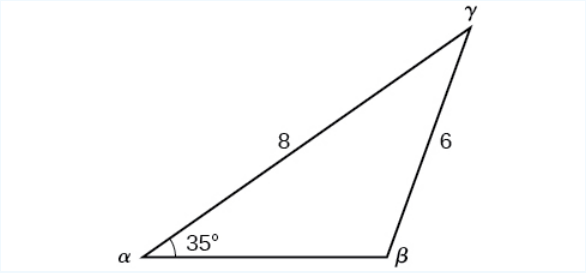


***Figure* 1.1.10**

Solution

Use the Law of Sines to find angle β and angle γ, and then side c. Solving for β, we have the proportion

$$\begin{aligned}\frac{\sin\alpha}{a} &= \frac{\sin\beta}{b}\\ \frac{\sin(35^\circ)}{6} &= \frac{\sin\beta}{8}\\ \frac{8\sin(35^\circ)}{6} &= \sin\beta\\ 0.7648 &\approx \sin\beta\\ \sin^{-1}(0.7648) &\approx 49.9^\circ\\ \beta &\approx 49.9^\circ\end{aligned}$$

However, in the diagram, angle β appears to be an obtuse angle and may be greater than 90°. How did we get an acute angle, and how do we find the measurement of β? Let's investigate further. Dropping a perpendicular from γ and viewing the triangle from a right angle perspective, we have Figure 1.1.11. It appears that there may be a second triangle that will fit the given criteria.

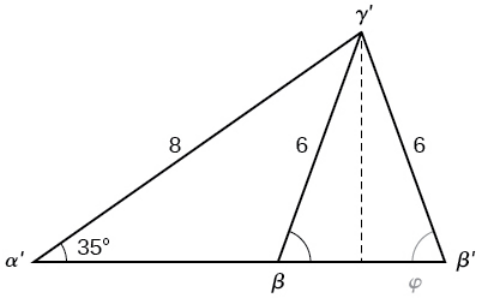


***Figure* 1.1.11**

The angle supplementary to β is approximately equal to 49.9°, which means that $\beta = 180^\circ - 49.9^\circ = 130.1^\circ$. (Remember that the sine function is positive in both the first and second quadrants.) Solving for γ, we have

$$\begin{aligned}\gamma &= 180^\circ - 35^\circ - 130.1^\circ\\ &\approx 14.9^\circ\end{aligned}$$

We can then use these measurements to solve the other triangle. Since β' is supplementary to β, we have

$$\begin{aligned}\gamma' &= 180^\circ - 35^\circ - 49.5^\circ\\ &\approx 95.1^\circ\end{aligned}$$

Now we need to find c and c'.

We have

$$\frac{c}{\sin(14.9^\circ)} = \frac{6}{\sin(35^\circ)}$$
$$c = \frac{6\sin(14.9^\circ)}{\sin(35^\circ)}$$
$$\approx 2.7$$

Finally,

$$\frac{c'}{\sin(95.1^\circ)} = \frac{6}{\sin(35^\circ)}$$
$$c' = \frac{6\sin(95.1^\circ)}{\sin(35^\circ)}$$
$$\approx 10.4$$

To summarize, there are two triangles with an angle of 35°, an adjacent side of 8, and an opposite side of 6, as shown in Figure 1.1.12

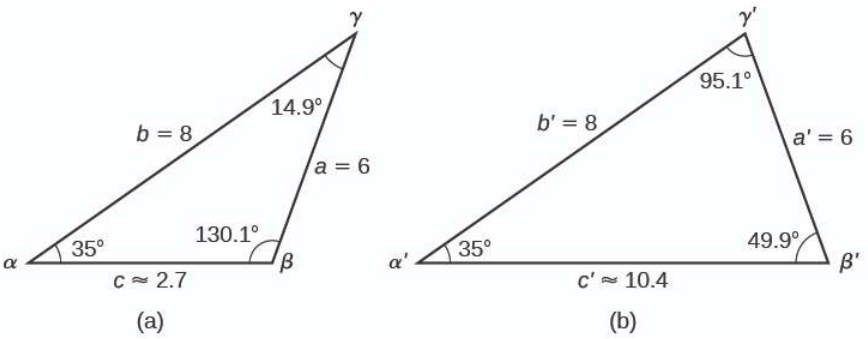


***Figure* 1.1.12**

However, we were looking for the values for the triangle with an obtuse angle β. We can see them in the first triangle (a) in Figure 1.1.12

? Exercise 1.1.2

Given $\alpha = 80^\circ$, $a = 120$, and $b = 121$, find the missing side and angles. If there is more than one possible solution, show both.

Answer

Solution 1

$\alpha = 80^\circ$ $\quad a = 120$
$\beta \approx 83.2^\circ$ $\quad b = 121$
$\gamma \approx 16.8^\circ$ $\quad c \approx 35.2$

Solution 2

$\alpha' = 80^\circ$ $\quad a' = 120$
$\beta' \approx 96.8^\circ$ $\quad b' = 121$
$\gamma' \approx 3.2^\circ$ $\quad c' \approx 6.8$

✓ Example 1.1.3: Solving for the Unknown Sides and Angles of a SSA Triangle

In the triangle shown in Figure 1.1.13, solve for the unknown side and angles. Round your answers to the nearest tenth.

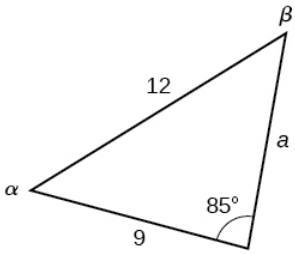


***Figure* 1.1.13**

Solution

In choosing the pair of ratios from the Law of Sines to use, look at the information given. In this case, we know the angle, $\gamma = 85^\circ$, and its corresponding side $c = 12$, and we know side $b = 9$. We will use this proportion to solve for β.

$$\frac{\sin(85^\circ)}{12} = \frac{\sin\beta}{9} \qquad \text{Isolate the unknown.}$$
$$\frac{9\sin(85^\circ)}{12} = \sin\beta$$

To find β, apply the inverse sine function. The inverse sine will produce a single result, but keep in mind that there may be two values for β. It is important to verify the result, as there may be two viable solutions, only one solution (the usual case), or no solutions.

$$\beta = \sin^{-1}\left(\frac{9\sin(85^\circ)}{12}\right)$$
$$\beta \approx \sin^{-1}(0.7471)$$
$$\beta \approx 48.3^\circ$$

In this case, if we subtract β from 180°, we find that there may be a second possible solution. Thus, $\beta = 180^\circ - 48.3^\circ \approx 131.7^\circ$. To check the solution, subtract both angles, 131.7° and 85°, from 180°. This gives

$$\alpha = 180^\circ - 85^\circ - 131.7^\circ$$
$$\approx -36.7^\circ$$

which is impossible, and so $\beta \approx 48.3^\circ$.

To find the remaining missing values, we calculate $\alpha = 180^\circ - 85^\circ - 48.3^\circ \approx 46.7^\circ$. Now, only side a is needed. Use the Law of Sines to solve for a by one of the proportions.

$$\frac{\sin(85^\circ)}{12} = \frac{\sin(46.7^\circ)}{a}$$
$$a\frac{\sin(85^\circ)}{12} = \sin(46.7^\circ)$$
$$a = \frac{12\sin(46.7^\circ)}{\sin(85^\circ)}$$
$$\approx 8.8$$

The complete set of solutions for the given triangle is

$$\alpha \approx 46.7^\circ \quad a \approx 8.8$$
$$\beta \approx 48.3^\circ \quad b = 9$$
$$\gamma = 85^\circ \quad c = 12$$

? Exercise 1.1.3

Given $\alpha = 80^\circ$, $a = 100$, $b = 10$, find the missing side and angles. If there is more than one possible solution, show both. Round your answers to the nearest tenth.

Answer

$\beta \approx 5.7^\circ, \gamma \approx 94.3^\circ, c \approx 101.3$

Example 1.1.4: Finding the Triangles That Meet the Given Criteria

Find all possible triangles if one side has length 4 opposite an angle of 50°, and a second side has length 10.

Solution

Using the given information, we can solve for the angle opposite the side of length 10. See Figure 1.1.14

$$\frac{\sin\alpha}{10} = \frac{\sin(50^\circ)}{4}$$
$$\sin\alpha = \frac{10\sin(50^\circ)}{4}$$
$$\sin\alpha \approx 1.915$$

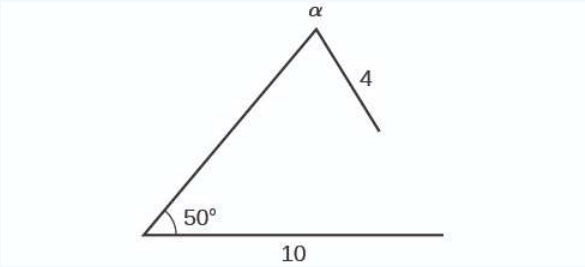


***Figure* 1.1.14**

We can stop here without finding the value of α. Because the range of the sine function is $[-1, 1]$, it is impossible for the sine value to be 1.915. In fact, inputting $\sin^{-1}(1.915)$in a graphing calculator generates an ERROR DOMAIN. Therefore, no triangles can be drawn with the provided dimensions.

Exercise 1.1.4

Determine the number of triangles possible given $a = 31, b = 26, \beta = 48^\circ$.

Answer

two

Solving Applied Problems Using the Law of Sines

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering, and physics involve three dimensions and motion.

Example 1.1.6: Finding an Altitude

Find the altitude of the aircraft in the problem introduced at the beginning of this section, shown in Figure 1.1.16. Round the altitude to the nearest tenth of a mile.

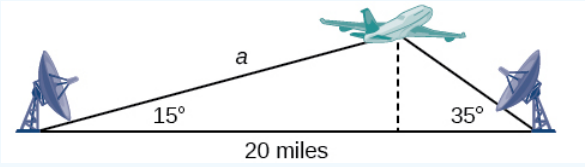


***Figure* 1.1.16**

Solution

To find the elevation of the aircraft, we first find the distance from one station to the aircraft, such as the side a, and then use right triangle relationships to find the height of the aircraft, h.

Because the angles in the triangle add up to 180 degrees, the unknown angle must be $180^\circ - 15^\circ - 35^\circ = 130^\circ$. This angle is opposite the side of length 20, allowing us to set up a Law of Sines relationship.

$$\begin{aligned} \frac{\sin(130^\circ)}{20} &= \frac{\sin(35^\circ)}{a} \\ a\sin(130^\circ) &= 20\sin(35^\circ) \\ a &= \frac{20\sin(35^\circ)}{\sin(130^\circ)} \\ a &\approx 14.98 \end{aligned}$$

The distance from one station to the aircraft is about 14.98 miles.

Now that we know a, we can use right triangle relationships to solve for h.

$$\begin{aligned} \sin(15^\circ) &= \frac{opposite}{hypotenuse} \\ \sin(15^\circ) &= \frac{h}{a} \\ \sin(15^\circ) &= \frac{h}{14.98} \\ h &= 14.98\sin(15^\circ) \\ h &\approx 3.88 \end{aligned}$$

The aircraft is at an altitude of approximately 3.9 miles.

? Exercise 1.1.6

The diagram shown in Figure $1.1.17$ represents the height of a blimp flying over a football stadium. Find the height of the blimp if the angle of elevation at the southern end zone, point A, is 70°, the angle of elevation from the northern end zone, point B, is 62°, and the distance between the viewing points of the two end zones is 145 yards.

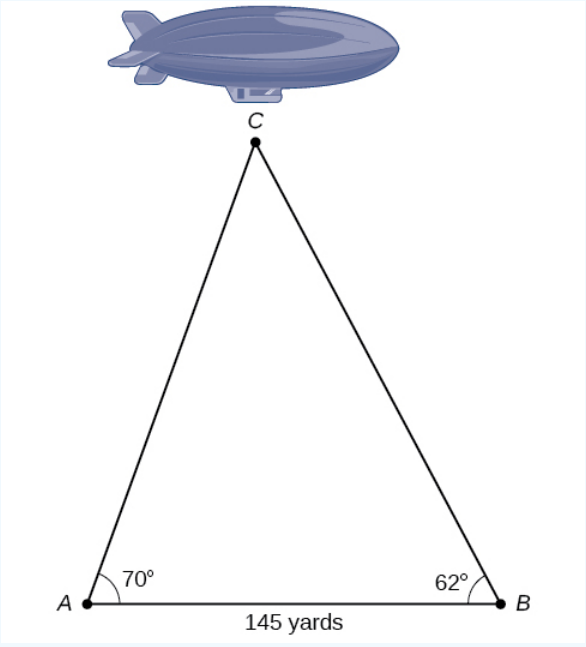


***Figure* 1.1.17**

Answer

161.9yd.

Media

Access these online resources for additional instruction and practice with trigonometric applications.

- Law of Sines: The Basics
- Law of Sines: The Ambiguous Case

Key Equations

Law of Sines	$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$ $\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$
Area for oblique triangles	$Area=\frac{1}{2}bc\sin\alpha$ $=\frac{1}{2}ac\sin\beta$ $=\frac{1}{2}ab\sin\gamma$

Key Concepts

- The Law of Sines can be used to solve oblique triangles, which are non-right triangles.
- According to the Law of Sines, the ratio of the measurement of one of the angles to the length of its opposite side equals the other two ratios of angle measure to opposite side.
- There are three possible cases: ASA, AAS, SSA. Depending on the information given, we can choose the appropriate equation to find the requested solution. See Example 1.1.1.
- The ambiguous case arises when an oblique triangle can have different outcomes.
- There are three possible cases that arise from SSA arrangement—a single solution, two possible solutions, and no solution. See Example 1.1.2 and Example 1.1.3.
- The Law of Sines can be used to solve triangles with given criteria. See Example 1.1.4.
- The general area formula for triangles translates to oblique triangles by first finding the appropriate height value. See Example 1.1.5.
- There are many trigonometric applications. They can often be solved by first drawing a diagram of the given information and then using the appropriate equation. See Example 1.1.6.

Contributors and Attributions

- Jay Abramson (Arizona State University) with contributing authors. Textbook content produced by OpenStax College is licensed under a Creative Commons Attribution License 4.0 license. Download for free at https://openstax.org/details/books/precalculus.

This page titled 1.1: Non-right Triangles - Law of Sines is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

1.2: Non-right Triangles - Law of Cosines

Learning Objectives

In this section, you will:

- Use the Law of Cosines to solve oblique triangles.
- Solve applied problems using the Law of Cosines.

Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles as shown in Figure 1.2.1 How far from port is the boat?

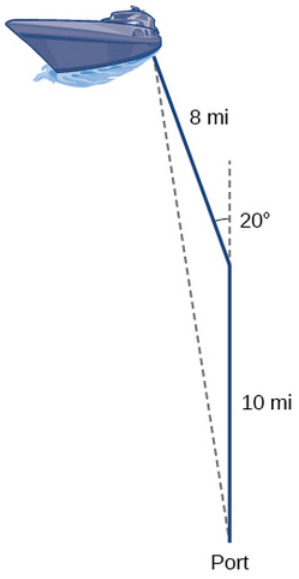


***Figure* 1.2.1**

Unfortunately, while the Law of Sines enables us to address many non-right triangle cases, it does not help us with triangles where the known angle is between two known sides, a SAS (side-angle-side) triangle, or when all three sides are known, but no angles are known, a SSS (side-side-side) triangle. In this section, we will investigate another tool for solving oblique triangles described by these last two cases.

Using the Law of Cosines to Solve Oblique Triangles

The tool we need to solve the problem of the boat's distance from the port is the *Law of Cosines*, which defines the relationship among angle measurements and side lengths in oblique triangles. Three formulas make up the Law of Cosines. At first glance, the formulas may appear complicated because they include many variables. However, once the pattern is understood, the Law of Cosines is easier to work with than most formulas at this mathematical level.

Understanding how the Law of Cosines is derived will be helpful in using the formulas. The derivation begins with the Generalized Pythagorean Theorem, which is an extension of the Pythagorean Theorem to non-right triangles. Here is how it works: An arbitrary non-right triangle ABC is placed in the coordinate plane with vertex A at the origin, side c drawn along the x-axis, and vertex C located at some point (x, y) in the plane, as illustrated in Figure 1.2.2. Generally, triangles exist anywhere in the plane, but for this explanation we will place the triangle as noted.

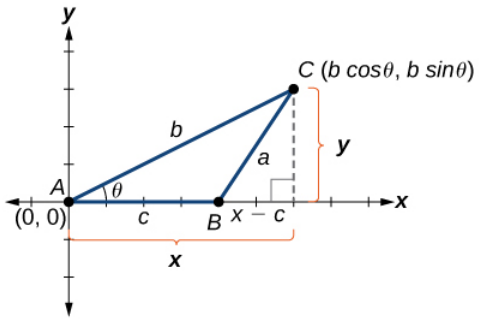


***Figure* 1.2.2**

We can drop a perpendicular from C to the x-axis (this is the altitude or height). Recalling the basic trigonometric identities, we know that

$$\cos\theta = \frac{x(adjacent)}{b(hypotenuse)} \text{ and } \sin\theta = \frac{y(opposite)}{b(hypotenuse)}$$

In terms of θ, $x = b\cos\theta$ and $y = b\sin\theta$. The (x, y) point located at C has coordinates $(b\cos\theta, b\sin\theta)$. Using the side $(x - c)$ as one leg of a right triangle and y as the second leg, we can find the length of hypotenuse a using the Pythagorean Theorem. Thus,

$$\begin{aligned}
a^2 &= (x-c)^2 + y^2 \\
&= (b\cos\theta - c)^2 + (b\sin\theta)^2 && \text{Substitute } (b\cos\theta) \text{ for } x \text{ and } (b\sin\theta) \text{ for } y \\
&= (b^2\cos^2\theta - 2bc\cos\theta + c^2) + b^2\sin^2\theta && \text{Expand the perfect square.} \\
&= b^2\cos^2\theta + b^2\sin^2\theta + c^2 - 2bc\cos\theta && \text{Group terms noting that } \cos^2\theta + \sin^2\theta = 1 \\
&= b^2(\cos^2\theta + \sin^2\theta) + c^2 - 2bc\cos\theta && \text{Factor out } b^2 \\
a^2 &= b^2 + c^2 - 2bc\cos\theta
\end{aligned}$$

The formula derived is one of the three equations of the Law of Cosines. The other equations are found in a similar fashion.

Keep in mind that it is always helpful to sketch the triangle when solving for angles or sides. In a real-world scenario, try to draw a diagram of the situation. As more information emerges, the diagram may have to be altered. Make those alterations to the diagram and, in the end, the problem will be easier to solve.

The LAW OF COSINES

The Law of Cosines states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle.

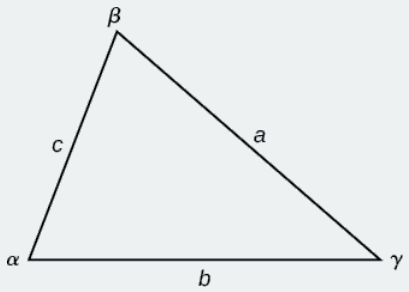


***Figure* 1.2.3**

For triangles labeled as in Figure 1.2.3, with angles α, β and γ, and opposite corresponding sides a, b, and c, respectively, the Law of Cosines is given as three equations.

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \tag{1.2.1}$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \tag{1.2.2}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \tag{1.2.3}$$

To solve for a missing side measurement, the corresponding opposite angle measure is needed.

When solving for an angle, the corresponding opposite side measure is needed. We can use another version of the Law of Cosines to solve for an angle.

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} \tag{1.2.4}$$

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac} \tag{1.2.5}$$

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab} \tag{1.2.6}$$

How to: Given two sides and the angle between them (SAS), find the measures of the remaining side and angles of a triangle

1. Sketch the triangle. Identify the measures of the known sides and angles. Use variables to represent the measures of the unknown sides and angles.
2. Apply the Law of Cosines to find the length of the unknown side or angle.
3. Apply the Law of Sines or Cosines to find the measure of a second angle.
4. Compute the measure of the remaining angle.

Example 1.2.1: Finding the Unknown Side and Angles of a SAS Triangle

Find the unknown side and angles of the triangle in Figure 1.2.4.

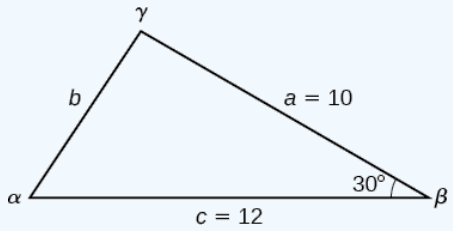


***Figure* 1.2.4**

Solution

First, make note of what is given: two sides and the angle between them. This arrangement is classified as SAS and supplies the data needed to apply the Law of Cosines.

Each one of the three laws of cosines begins with the square of an unknown side opposite a known angle. For this example, the first side to solve for is side b, as we know the measurement of the opposite angle β.

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$b^2 = 10^2 + 12^2 - 2(10)(12)\cos(30^\circ) \quad \text{Substitute the measurements for the known quantities.}$$

$$b^2 = 100 + 144 - 240\left(\frac{\sqrt{3}}{2}\right) \quad \text{Evaluate the cosine and begin to simplify.}$$

$$b^2 = 244 - 120\sqrt{3}$$

$$b = \sqrt{244 - 120\sqrt{3}} \quad \text{Use the square root property.}$$

$$b \approx 6.013$$

Because we are solving for a length, we use only the positive square root. Now that we know the length b, we can use the Law of Sines to fill in the remaining angles of the triangle. Solving for angle α, we have

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

$$\frac{\sin\alpha}{10} = \frac{\sin(30^\circ)}{6.013}$$

$$\sin\alpha = \frac{10\sin(30^\circ)}{6.013} \qquad \text{Multiply both sides of the equation by } 10.$$

$$\alpha = \sin^{-1}\left(\frac{10\sin(30^\circ)}{6.013}\right) \qquad \text{Find the inverse sine of } \frac{10\sin(30^\circ)}{6.013}.$$

$$\alpha \approx 56.3^\circ$$

The other possibility for α would be $\alpha = 180^\circ - 56.3^\circ \approx 123.7^\circ$. In the original diagram,α is adjacent to the longest side, so α is an acute angle and, therefore, 123.7° does not make sense. Notice that if we choose to apply the Law of Cosines, we arrive at a unique answer. We do not have to consider the other possibilities, as cosine is unique for angles between 0° and 180°. Proceeding with $\alpha \approx 56.3^\circ$, we can then find the third angle of the triangle.

$$\begin{aligned}\gamma &= 180^\circ - 30^\circ - 56.3^\circ \\ &\approx 93.7^\circ\end{aligned}$$

The complete set of angles and sides is

$$\alpha \approx 56.3^\circ \; a = 10$$

$$\beta = 30^\circ \; b \approx 6.013$$

$$\gamma \approx 93.7^\circ \; c = 12$$

? Exercise 1.2.1

Find the missing side and angles of the given triangle: $\alpha = 30^\circ$, $b = 12$, $c = 24$.

Answer

$a \approx 14.9$, $\beta \approx 23.8^\circ$, $\gamma \approx 126.2^\circ$.

✓ Example 1.2.2: Solving for an Angle of a SSS Triangle

Find the angle α for the given triangle if side $a = 20$, side $b = 25$, and side $c = 18$.

Solution

For this example, we have no angles. We can solve for any angle using the Law of Cosines. To solve for angle α, we have

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$20^2 = 25^2 + 18^2 - 2(25)(18)\cos\alpha \qquad \text{Substitute the appropriate measurements.}$$

$$400 = 625 + 324 - 900\cos\alpha \qquad \text{Simplify in each step.}$$

$$400 = 949 - 900\cos\alpha$$

$$-549 = -900\cos\alpha \qquad \text{Isolate } \cos\alpha.$$

$$0.61 \approx \cos\alpha \qquad \text{Find the inverse cosine.}$$

$$\alpha \approx 52.4^\circ$$

See Figure 1.2.5.

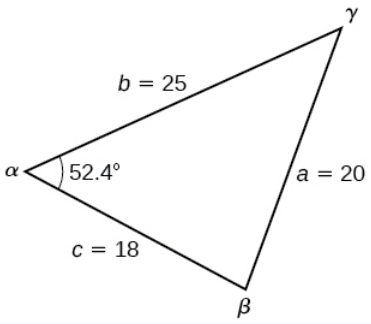


***Figure* 1.2.5**

Analysis

Because the inverse cosine can return any angle between 0 and 180 degrees, there will not be any ambiguous cases using this method.

? Exercise 1.2.2

Given $a = 5$, $b = 7$, and $c = 10$, find the missing angles.

Answer

$\alpha \approx 27.7^{\circ}, \beta \approx 40.5^{\circ}, \gamma \approx 111.8^{\circ}$

Solving Applied Problems Using the Law of Cosines

Just as the Law of Sines provided the appropriate equations to solve a number of applications, the Law of Cosines is applicable to situations in which the given data fits the cosine models. We may see these in the fields of navigation, surveying, astronomy, and geometry, just to name a few.

✔ Example 1.2.3*A*: Using the Law of Cosines to Solve a Communication Problem

On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is accomplished through a process called triangulation, which works by using the distances from two known points. Suppose there are two cell phone towers within range of a cell phone. The two towers are located 6000 feet apart along a straight highway, running east to west, and the cell phone is north of the highway. Based on the signal delay, it can be determined that the signal is 5050 feet from the first tower and 2420 feet from the second tower. Determine the position of the cell phone north and east of the first tower, and determine how far it is from the highway.

Solution

For simplicity, we start by drawing a diagram similar to Figure $1.2.6$ and labeling our given information.

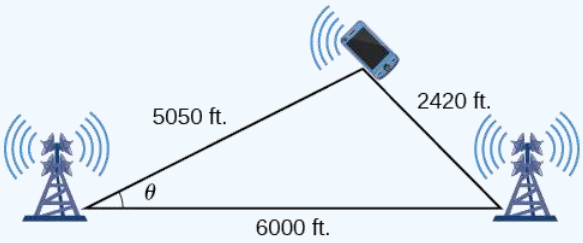


***Figure* 1.2.6**

Using the Law of Cosines, we can solve for the angle θ. Remember that the Law of Cosines uses the square of one side to find the cosine of the opposite angle. For this example, let $a = 2420$, $b = 5050$, and $c = 6000$. Thus, θ corresponds to the opposite side $a = 2420$.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\theta \\ (2420)^2 &= (5050)^2 + (6000)^2 - 2(5050)(6000)\cos\theta \\ \cos\theta &\approx 0.9183 \\ \theta &\approx \cos^{-1}(0.9183) \\ \theta &\approx 23.3^\circ \end{aligned}$$

To answer the questions about the phone's position north and east of the tower, and the distance to the highway, drop a perpendicular from the position of the cell phone, as in Figure 1.2.7. This forms two right triangles, although we only need the right triangle that includes the first tower for this problem.

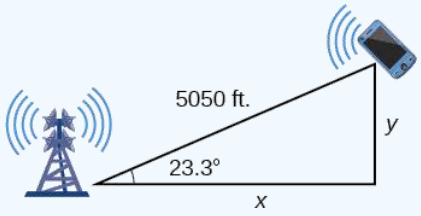


***Figure* 1.2.7**

Using the angle $\theta = 23.3°$ and the basic trigonometric identities, we can find the solutions. Thus

$$\begin{aligned} \cos(23.3^\circ) &= \frac{x}{5050} \\ x &= 5050\cos(23.3^\circ) \\ x &\approx 4638.15 \; feet \\ \sin(23.3^\circ) &= \frac{y}{5050} \\ y &= 5050\sin(23.3^\circ) \\ y &\approx 1997.5 \; feet \end{aligned}$$

The cell phone is approximately 4638 feet east and 1998 feet north of the first tower, and 1998 feet from the highway.

Example 1.2.3B: Calculating Distance Traveled Using a SAS Triangle

Returning to our problem at the beginning of this section, suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. How far from port is the boat? The diagram is repeated here in Figure 1.2.8.

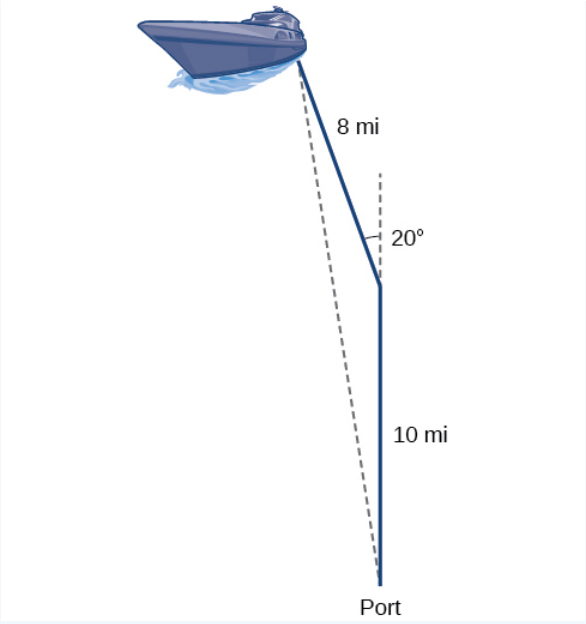


***Figure* 1.2.8**

Solution

The boat turned 20 degrees, so the obtuse angle of the non-right triangle is the supplemental angle, $180^\circ - 20^\circ = 160^\circ$. With this, we can utilize the Law of Cosines to find the missing side of the obtuse triangle—the distance of the boat to the port.

$$\begin{aligned} x^2 &= 8^2 + 10^2 - 2(8)(10)\cos(160^\circ) \\ x^2 &= 314.35 \\ x &= \sqrt{314.35} \\ x &\approx 17.7\, miles \end{aligned}$$

The boat is about 17.7 miles from port.

Media

Access these online resources for additional instruction and practice with the Law of Cosines.

- Law of Cosines
- Law of Cosines: Applications
- Law of Cosines: Applications 2

Key Equations

Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos\alpha$ $b^2 = a^2 + c^2 - 2ac\cos\beta$ $c^2 = a^2 + b^2 - 2ab\cos\gamma$

Key Concepts

- The Law of Cosines defines the relationship among angle measurements and lengths of sides in oblique triangles.
- The Generalized Pythagorean Theorem is the Law of Cosines for two cases of oblique triangles: SAS and SSS. Dropping an imaginary perpendicular splits the oblique triangle into two right triangles or forms one right triangle, which allows sides to be related and measurements to be calculated. See Example $1.2.1$ and Example $1.2.2$.
- The Law of Cosines is useful for many types of applied problems. The first step in solving such problems is generally to draw a sketch of the problem presented. If the information given fits one of the three models (the three equations), then apply the Law

of Cosines to find a solution. See Example 1.2.3 and Example 1.2.4.

Contributors and Attributions

- Jay Abramson (Arizona State University) with contributing authors. Textbook content produced by OpenStax College is licensed under a Creative Commons Attribution License 4.0 license. Download for free at https://openstax.org/details/books/precalculus.

This page titled 1.2: Non-right Triangles - Law of Cosines is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

CHAPTER OVERVIEW

2: Polynomials

2.1: Add and Subtract Polynomials
2.2: Multiply Polynomials
2.3: Special Products
2.4: Divide Polynomials

2: Polynomials is shared under a CC BY-SA license and was authored, remixed, and/or curated by LibreTexts.

2.1: Add and Subtract Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Identify polynomials, monomials, binomials, and trinomials
- Determine the degree of polynomials
- Add and subtract monomials
- Add and subtract polynomials
- Evaluate a polynomial for a given value

Quiz

Before you get started, take this readiness quiz.

1. Simplify: $8x+3x$.
   If you missed this problem, review Example 1.3.37.
2. Subtract: $(5n+8)-(2n-1)$.
   If you missed this problem, review Example 1.10.52.
3. Write in expanded form: a^5.
   If you missed this problem, review Example 1.3.7.

Identify Polynomials, Monomials, Binomials and Trinomials

You have learned that a *term* is a constant or the product of a constant and one or more variables. When it is of the form ax^m, where a is a constant and m is a whole number, it is called a *monomial*. Some examples of monomial are 8, $-2x^2$, $4y^3$, and $11z^7$.

Definition: Monomials

A **monomial** is a term of the form ax^m, where a is a constant and m is a positive whole number.

A monomial, or two or more monomials combined by addition or subtraction, is a polynomial. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly one term. A binomial has exactly two terms, and a trinomial has exactly three terms. There are no special names for polynomials with more than three terms.

Definitions: Polynomials

- **polynomial**—A monomial, or two or more monomials combined by addition or subtraction, is a *polynomial*.
- **monomial**—A polynomial with exactly one term is called a *monomial*.
- **binomial**—A polynomial with exactly two terms is called a *binomial*.
- **trinomial**—A polynomial with exactly three terms is called a *trinomial*.

Here are some examples of polynomials.

Polynomial	$b+1$	$4y^2-7y+2$	$4x^4+x^3+8x^2-9x+1$	
Monomial	14	$8y^2$	$-9x^3y^5$	-13
Binomial	$a+7$	$4b-5$	y^2-16	$3x^3-9x^2$
Trinomial	$x^2-7x+12$	$9y^2+2y-8$	$6m^4-m^3+8m$	z^4+3z^2-1

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the “family” of polynomials and so they have special names. We use the words *monomial*, *binomial*, and *trinomial* when referring to these special polynomials and just call all the rest *polynomials*.

Example 2.1.1

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial.

a. $4y^2 - 8y - 6$
b. $-5a^4b^2$
c. $2x^5 - 5x^3 - 3x + 4$
d. $13 - 5m^3$
e. q

Solution

	Polynomial	Number of terms	Type
(a)	$4y^2 - 8y - 6$	3	Trinomial
(b)	$-5a^4b^2$	1	Monomial
(c)	$2x^5 - 5x^3 - 9x^2 + 3x + 4$	5	Ponomial
(d)	$13 - 5m^3$	2	Binomial
(e)	q	1	Monomial

Try It 2.1.2

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial:

a. 5b
b. $8y^3 - 7y^2 - y - 3$
c. $-3x^2 - 5x + 9$
d. $81 - 4a^2$
e. $-5x^6$

Answer

1. monomial
2. polynomial
3. trinomial
4. binomial
5. monomial

Try It 2.1.3

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial:

a. $27z^3 - 8$
b. $12m^3 - 5m^2 - 2m$
c. $\frac{5}{6}$
d. $8x^4 - 7x^2 - 6x - 5$
e. $-n^4$

Answer

a. binomial
b. trinomial
c. monomial
d. polynomial
e. monomial

Determine the Degree of Polynomials

The degree of a polynomial and the degree of its terms are determined by the exponents of the variable. A **monomial** that has no variable, just a constant, is a special case. The degree of a constant is 0, i.e., it has no variable.

Definition: Degree of a Polynomial

- The **degree of a term** is the sum of the exponents of its variables.
- The **degree of a constant** is 0.
- The **degree of a polynomial** is the highest degree of all its terms.

Let's see how this works by looking at several polynomials. We'll take it step by step, starting with monomials, and then progressing to polynomials with more terms.

Monomial	14	$8y^2$	$-9x^3y^6$	$-13a$
Degree	0	2	8	1
Binomial	$a+7$	$4b^2-5b$	x^2y^2-16	$3n^3-9n^2$
Degree of each term	1 0	2 1	4 0	3 2
Degree of polynomial	1	2	4	3
Trinomial	$x^2-7x+12$	$9a^2+6ab+b^2$	$6m^4-m^3n^2+8mn^5$	z^4+3z^2-1
Degree of each term	2 1 0	2 2 2	4 5 6	4 2 0
Degree of polynomial	2	2	6	4
Polynomial	$b+1$	$4y^2-7y+2$	$4x^4+x^3+8x^2-9x+1$	
Degree of each term	1 0	2 1 0	4 3 2 1 0	
Degree of polynomial	1	2	4	

A polynomial is in **standard form** when the terms of a polynomial are written in descending order of degrees. Get in the habit of writing the term with the highest degree first.

Example 2.1.4

Find the degree of the following polynomials.

a. 10y
b. $4x^3-7x+5$
c. -15
d. $-8b^2+9b-2$
e. $8xy^2+2y$

Solution

a. $10y$ The exponent of y is one. $y=y^1$ The degree is 1.
b. $4x^3-7x+5$ The highest degree of all the terms is 3. The degree is 3.
c. -15 The degree of a constant is 0. The degree is 0.
d. $-8b^2+9b-2$ The highest degree of all the terms is 2. The degree is 2.

e.

The highest degree of all the terms is 3.	$8xy^2+2y$
	The degree is 3.

Try It 2.1.5

Find the degree of the following polynomials:

a. $-15b$
b. $10z^4+4z^2-5$
c. $12c^5d^4+9c^3d^9-7$
d. $3x^2y-4x$
e. -9

Answer

a. 1
b. 4
c. 12
d. 3
e. 0

Try It 2.1.6

Find the degree of the following polynomials:

a. 52
b. a^4b-17a^4
c. $5x+6y+2z$
d. $3x^2-5x+7$
e. $-a^3$

Answer

a. 0
b. 5
c. 1
d. 2
e. 3

Add and Subtract Monomials

You have learned how to simplify expressions by combining like terms. Remember, like terms must have the same variables with the same exponent. Since monomials are terms, adding and subtracting monomials is the same as combining like terms. If the monomials are like terms, we just combine them by adding or subtracting the coefficient.

Example 2.1.7

Add: $25y^2+15y^2$

Solution

	$25y^2+15y^2$
Combine like terms.	$40y^2$

Try It 2.1.8

Add: $12q^2 + 9q^2$

Answer

$21q^2$

Try It 2.1.9

Add: $-15c^2 + 8c^2$

Answer

$-7c^2$

Example 2.1.10

Subtract: 16p−(−7p)

Solution

	$16p - (-7p)$
Combine like terms.	$23p$

Try It 2.1.11

Subtract: 8m−(−5m).

Answer

13m

Try It 2.1.12

Subtract: $-15z^3 - \left(-5z^3\right)$

Answer

$-10z^3$

Remember that like terms must have the same variables with the same exponents.

Example 2.1.13

Simplify: $c^2 + 7d^2 - 6c^2$

Solution

	$c^2 + 7d^2 - 6c^2$
Combine like terms.	$-5c^2 + 7d^2$

Try It 2.1.14

Add: $8y^2 + 3z^2 - 3y^2$

Answer

$5y^2 + 3z^2$

Try It 2.1.15

Add: $3m^2+n^2-7m^2$

Answer

$-4m^2+n^2$

Example 2.1.16

Simplify: $u^2v+5u^2-3v^2$

Solution

	$u^2v+5u^2-3v^2$
There are no like terms to combine.	$u^2v+5u^2-3v^2$

Try It 2.1.17

Simplify: $m^2n^2-8m^2+4n^2$

Answer

There are no like terms to combine.

Try It 2.1.18

Simplify: $pq^2-6p-5q^2$

Answer

There are no like terms to combine.

Add and Subtract Polynomials

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like terms—those with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

Example 2.1.19

Find the sum: $\left(5y^2-3y+15\right)+\left(3y^2-4y-11\right)$

Solution

Identify like terms.	$(\underline{5y^2}-\underline{3y}+\underline{15})+(\underline{3y^2}-\underline{4y}-\underline{11})$
Rearrange to get the like terms together.	$\underline{5y^2}+\underline{3y^2}-\underline{3y}-\underline{4y}+\underline{15}-\underline{11}$
Combine like terms.	$8y^2-7y+4$

Try It 2.1.20

Find the sum: $\left(7x^2-4x+5\right)+\left(x^2-7x+3\right)$

Answer

$8x^2-11x+1$

Try It 2.1.21

Find the sum: $(14y^2+6y-4)+(3y^2+8y+5)$

Answer

$17y^2+14y+1$

Example 2.1.22

Find the difference: $(9w^2-7w+5)-(2w^2-4)$

Solution

	$(9w^2-7w+5)-(2w^2-4)$
Distribute and identify like terms.	$9w^2-7w+5-2w^2+4$
Rearrange the terms.	$9w^2-2w^2-7w+5+4$
Combine like terms.	$7w^2-7w+9$

Try It 2.1.23

Find the difference: $(8x^2+3x-19)-(7x^2-14)$

Answer

$15x^2+3x-5$

Try It 2.1.24

Find the difference: $(9b^2-5b-4)-(3b^2-5b-7)$

Answer

$6b^2+3$

Example 2.1.25

Subtract: (c^2-4c+7) from $(7c^2-5c+3)$

Solution

	Subtract (c^2-4c+7) from $(7c^2-5c+3)$.
	$(7c^2-5c+3)-(c^2-4c+7)$
Distribute and identify like terms.	$7c^2-5c+3-c^2+4c-7$
Rearrange the terms.	$7c^2-c^2-5c+4c+3-7$
Combine like terms.	$6c^2-c-4$

Try It 2.1.26

Subtract: $(5z^2-6z-2)$ from $(7z^2+6z-4)$

Answer

$2z^2+12z-2$

Try It 2.1.27

Subtract: $\left(x^2-5x-8\right)$ from $\left(6x^2+9x-1\right)$

Answer

$5x^2+14x+7$

Example 2.1.28

Find the sum: $\left(u^2-6uv+5v^2\right)+\left(3u^2+2uv\right)$

Solution

	$\left(u^2-6uv+5v^2\right)+\left(3u^2+2uv\right)$
Distribute.	$u^2-6uv+5v^2+3u^2+2uv$
Rearrange the terms, to put like terms together	$u^2+3u^2-6uv+2uv+5v^2$
Combine like terms.	$4u^2-4uv+5v^2$

Try It 2.1.29

Find the sum: $\left(3x^2-4xy+5y^2\right)+\left(2x^2-xy\right)$

Answer

$5x^2-5xy+5y^2$

Try It 2.1.30

Find the sum: $\left(2x^2-3xy-2y^2\right)+\left(5x^2-3xy\right)$

Answer

$7x^2-6xy-2y^2$

Example 2.1.31

Find the difference: $\left(p^2+q^2\right)-\left(p^2+10pq-2q^2\right)$

Solution

	$\left(p^2+q^2\right)-\left(p^2+10pq-2q^2\right)$
Distribute.	$p^2+q^2-p^2-10pq+2q^2$
Rearrange the terms, to put like terms together	$p^2-p^2-10pq+q^2+2q^2$
Combine like terms.	$-10pq+3q^2$

Try It 2.1.32

Find the difference: $\left(a^2+b^2\right)-\left(a^2+5ab-6b^2\right)$

Answer

$-5ab-5b^2$

Try It 2.1.33

Find the difference: $(m^2+n^2)-(m^2-7mn-3n^2)$

Answer

$4n^2+7mn$

Example 2.1.34

Simplify: $(a^3-a^2b)-(ab^2+b^3)+(a^2b+ab^2)$

Solution

	$(a^3-a^2b)-(ab^2+b^3)+(a^2b+ab^2)$
Distribute.	$a^3-a^2b-ab^2-b^3+a^2b+ab^2$
Rearrange the terms, to put like terms together	$a^3-a^2b+a^2b-ab^2+ab^2-b^3$
Combine like terms.	a^3-b^3

Try It 2.1.35

Simplify: $(x^3-x^2y)-(xy^2+y^3)+(x^2y+xy^2)$

Answer

x^3-y^3

Try It 2.1.36

Simplify: $(p^3-p^2q)+(pq^2+q^3)-(p^2q+pq^2)$

Answer

$p^3-2p^2q+q^3$

Evaluate a Polynomial for a Given Value

We have already learned how to evaluate expressions. Since polynomials are expressions, we'll follow the same procedures to evaluate a **polynomial**. We will substitute the given value for the variable and then simplify using the order of operations.

Example 2.1.37

Evaluate $5x^2-8x+4$ when

1. x=4
2. x=−2
3. x=0

Solution

1. x=4	
	$5x^2-8x+4$
Substitute 4 for x.	$5(4)^2-8(4)+4$
Simplify the exponents.	$5\cdot16-8(4)+4$
Multiply.	$80-32+4$

Simplify.	52

2. x=−2	
	$5x^2 - 8x + 4$
Substitute –2 for *x*.	$5(-2)^2 - 8(-2) + 4$
Simplify the exponents.	$5 \cdot 4 - 8(-2) + 4$
Multiply.	$20 + 16 + 4$
Simplify.	40

3. x=0	
	$5x^2 - 8x + 4$
Substitute 0 for *x*.	$5(0)^2 - 8(0) + 4$
Simplify the exponents.	$5 \cdot 0 - 8(0) + 4$
Multiply.	$0 + 0 + 4$
Simplify.	4

? Try It 2.1.38

Evaluate: $3x^2 + 2x - 15$ when

1. x=3
2. x=−5
3. x=0

Answer

1. 18
2. 50
3. −15

? Try It 2.1.39

Evaluate: $5z^2 - z - 4$ when

1. z=−2
2. z=0
3. z=2

Answer

1. 18
2. −4
3. 14

✓ Example 2.1.40

The polynomial $-16t^2 + 250$ gives the height of a ball tt seconds after it is dropped from a 250 foot tall building. Find the height after t=2 seconds.

Solution

	$-16t^2+250$
Substitute t = 2.	$-16(2)^2+250$
Simplify	$-16\cdot 4+250$
Simplify	$-64+250$
Simplify	186
	After 2 seconds the height of the ball is 186 feet.

Try It 2.1.41

The polynomial $-16t^2+250$ gives the height of a ball tt seconds after it is dropped from a 250 foot tall building. Find the height after t=0 seconds.

Answer

250

Try It 2.1.42

The polynomial $-16t^2+250$ gives the height of a ball tt seconds after it is dropped from a 250 foot tall building. Find the height after t=3 seconds.

Answer

106

Example 2.1.43

The polynomial $6x^2+15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and sides of height y feet. Find the cost of producing a box with x=4 feet and y=6y=6 feet.

Solution

	$6x^2+15xy$
Substitute $x=4, y=6$.	$6(4)^2+15(4)(6)$
Simplify.	$6\cdot 16+15(4)(6)$
Simplify.	$96+360$
Simplify.	456
	The cost of producing the box is $456.

Try It 2.1.43

The polynomial $6x^2+15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and sides of height y feet. Find the cost of producing a box with x=6 feet and y=4 feet.

Answer

$576

? Try It 2.1.44

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side *x* feet and sides of height *y* feet. Find the cost of producing a box with x=5 feet and y=8 feet.

Answer

$750

Key Concepts

- **Monomials**
  - A monomial is a term of the form ax^m, where aa is a constant and mm is a whole number
- **Polynomials**
  - **polynomial**—A monomial, or two or more monomials combined by addition or subtraction is a polynomial.
  - **monomial**—A polynomial with exactly one term is called a monomial.
  - **binomial**—A polynomial with exactly two terms is called a binomial.
  - **trinomial**—A polynomial with exactly three terms is called a trinomial.
- **Degree of a Polynomial**
  - The **degree of a term** is the sum of the exponents of its variables.
  - The **degree of a constant** is 0.
  - The **degree of a polynomial** is the highest degree of all its terms.

Glossary

binomial

A binomial is a polynomial with exactly two terms.

degree of a constant

The degree of any constant is 0.

degree of a polynomial

The degree of a polynomial is the highest degree of all its terms.

degree of a term

The degree of a term is the exponent of its variable.

monomial

A monomial is a term of the form ax^m, where a is a constant and m is a whole number; a monomial has exactly one term.

polynomial

A polynomial is a monomial, or two or more monomials combined by addition or subtraction.

standard form

A polynomial is in standard form when the terms of a polynomial are written in descending order of degrees.

trinomial

A trinomial is a polynomial with exactly three terms.

This page titled 2.1: Add and Subtract Polynomials is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **6.1: Add and Subtract Polynomials** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

2.2: Multiply Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Multiply a polynomial by a monomial
- Multiply a binomial by a binomial
- Multiply a trinomial by a binomial

Note

Before you get started, take this readiness quiz.

1. Distribute: $2(x+3)$.
   If you missed this problem, review Example 1.10.31.
2. Combine like terms: $x^2+9x+7x+63$.
   If you missed this problem, review Example 1.3.37.

Multiply a Polynomial by a Monomial

We have used the Distributive Property to simplify expressions like $2(x-3)$. You multiplied both terms in the parentheses, x and 3, by 2, to get $2x-6$. With this chapter's new vocabulary, you can say you were multiplying a binomial, $x-3$, by a monomial, 2.

Multiplying a **binomial** by a **monomial** is nothing new for you! Here's an example:

Example 2.2.1

Multiply: $4(x+3)$.

Solution

	$4(x+3)$
Distribute.	$4\cdot x+4\cdot 3$
Simplify.	$4x+12$

Try It 2.2.2

Multiply: $5(x+7)$.

Answer

5x+35

Try It 2.2.3

Multiply: $3(y+13)$.

Answer

3y+39

Example 2.2.4

Multiply: y(y−2).

Solution

	
Distribute.	$y \cdot y - y \cdot 2$
Simplify.	$y^2 - 2y$

Try It 2.2.5

Multiply: $x(x-7)$.

Answer

$x^2 - 7x$

Try It 2.2.6

Multiply: $d(d-11)$.

Answer

$d^2 - 11d$

Example 2.2.7

Multiply: $7x(2x+y)$

Solution

	
Distribute.	$7x \cdot 2x + 7x \cdot y$
Simplify.	$14x^2 + 7xy$

Try It 2.2.8

Multiply: $5x(x+4y)$

Answer

$5x^2 + 20xy$

Try It 2.2.9

Multiply: $2p(6p+r)$

Answer

$12p^2 + 2pr$

Example 2.2.10

Multiply: $-2y\left(4y^2+3y-5\right)$

Solution

	
Distribute.	$-2y \cdot 4y^2+(-2y) \cdot 3y-(-2y) \cdot 5$
Simplify.	$-8y^3-6y^2+10y$

Try It 2.2.11

Multiply: $-3y\left(5y^2+8y-7\right)$

Answer

$-15y^3-24y^2+21y$

Try It 2.2.12

Multiply: $4x^2\left(2x^2-3x+5\right)$

Answer

$8x^4-24x^3+20x^2$

Example 2.2.13

Multiply: $2x^3\left(x^2-8x+1\right)$

Solution

	
Distribute.	$2x^3 \cdot x^2+\left(2x^3\right) \cdot(-8x)+\left(2x^3\right) \cdot 1$
Simplify.	$2x^5-16x^4+2x^3$

Try It 2.2.14

Multiply: $4x\left(3x^2-5x+3\right)$

Answer

$12x^3-20x^2+12x$

Try It 2.2.15

Multiply: $-6a^3\left(3a^2-2a+6\right)$

Answer

$-18a^5+12a^4-36a^3$

Example 2.2.16

Multiply: $(x+3)p$

Solution

The monomial is the second factor.	
Distribute.	$x \cdot p + 3 \cdot p$
Simplify.	$xp + 3p$

Try It 2.2.17

Multiply: $(x+8)p$

Answer

$xp + 8p$

Try It 2.2.18

Multiply: $(a+4)p$

Answer

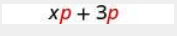
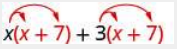
$ap + 4p$

Multiply a Binomial by a Binomial

Just like there are different ways to represent multiplication of numbers, there are several methods that can be used to multiply a **binomial** times a binomial. We will start by using the Distributive Property.

Multiply a Binomial by a Binomial Using the Distributive Property

Look at Example 2.2.16, where we multiplied a binomial by a **monomial**.

Instructions	Expression
Starting Expressions	
We distributed the p to get:	
What if we have $(x+7)$ instead of p?	
Distribute $(x+7)$.	
Distribute again.	$x^2 + 7x + 3x + 21$
Combine like terms.	$x^2 + 10x + 21$

Notice that before combining like terms, you had four terms. You multiplied the two terms of the first binomial by the two terms of the second binomial—four multiplications.

Example 2.2.19

Multiply: $(y+5)(y+8)$

Solution

	
Distribute $(y+8)$.	$y(y+8)+5(y+8)$
Distribute again	$y^2+8y+5y+40$
Combine like terms.	$y^2+13y+40$

Try It 2.2.20

Multiply: $(x+8)(x+9)$

Answer

$x^2+17x+72$

Try It 2.2.21

Multiply: $(5x+9)(4x+3)$

Answer

$20x^2+51x+27$

Example 2.2.22

Multiply: $(2y+5)(3y+4)$

Solution

	
Distribute $(3y+4)$.	$2y(3y+4)+5(3y+4)$
Distribute again	$6y^2+8y+15y+20$
Combine like terms.	$6y^2+23y+20$

Try It 2.2.23

Multiply: $(3b+5)(4b+6)$

Answer

$12b^2+38b+30$

Try It 2.2.24

Multiply: $(a+10)(a+7)$

Answer

$a^2+17a+70$

Example 2.2.25

Multiply: $(4y+3)(2y-5)$

Solution

	$(4y+3)(2y-5)$
Distribute.	$4y(2y-5)+3(2y-5)$
Distribute again.	$8y^2-20y+6y-15$
Combine like terms.	$8y^2-14y-15$

Try It 2.2.26

Multiply: $(5y+2)(6y-3)$

Answer

$30y^2-3y-6$

Try It 2.2.27

Multiply: $(3c+4)(5c-2)$

Answer

$15c^2+14c-8$

Example 2.2.28

Multiply: $(x-2)(x-y)$

Solution

	$(x-2)(x-y)$
Distribute.	$x(x-y)-2(x-y)$
Distribute again.	$x^2-xy-2x+2y$
There are no like terms to combine.	

Try It 2.2.29

Multiply: $(a+7)(a-b)$

Answer

$a^2-ab+7a-7b$

Try It 2.2.30

Multiply: $(x+5)(x-y)$

Answer

$x^2 - xy + 5x - 5y$

Multiply a Binomial by a Binomial Using the FOIL Method

Remember that when you multiply a binomial by a binomial you get four terms. Sometimes you can combine like terms to get a **trinomial**, but sometimes, like in Example 2.2.28, there are no like terms to combine.

Let's look at the last example again and pay particular attention to how we got the four terms.

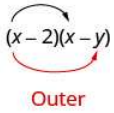
$$(x-2)(x-y)$$
$$x^2 - xy - 2x + 2y$$

Where did the first term, x^2, come from?

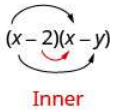
It is the product of *x* and *x*, the *first* terms in (*x* – 2) and (*x* – *y*).



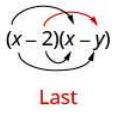
The next term, –*xy*, is the product of *x* and –*y*, the two *outer* terms.



The third term, –2*x*, is the product of –2 and *x*, the two *inner* terms.



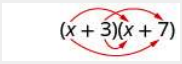
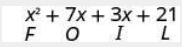
And the last term, +2*y*, came from multiplying the two *last* terms, –2 and –*y*.



We abbreviate "First, Outer, Inner, Last" as FOIL. The letters stand for '**F**irst, **O**uter, **I**nner, **L**ast'. The word FOIL is easy to remember and ensures we find all four products.

$$\begin{array}{c} (x-2)(x-y) \\ x^2 - xy - 2x + 2y \\ F \quad O \quad I \quad L \end{array} \tag{2.2.1}$$

Let's look at (x+3)(x+7).

Distributive Property	FOIL
$(x+3)(x+7)$	
$x(x+7)+3(x+7)$	
$x^2+7x+3x+21$ F O I L	
$x^2+10x+21$	$x^2+10x+21$

Notice how the terms in third line fit the FOIL pattern.

Now we will do an example where we use the FOIL pattern to multiply two binomials.

Example 2.2.31: How to Multiply a Binomial by a Binomial using the FOIL Method

Multiply using the FOIL method: $(x+5)(x+9)$

Solution

Step 1. Multiply the *First* terms.	$(x+5)(x+9)$	$x^2+\_+\_+\_$ F O I L
Step 2. Multiply the *Outer* terms.	$(x+5)(x+9)$	$x^2+9x+\_+\_$ F O I L
Step 3. Multiply the *Inner* terms.	$(x+5)(x+9)$	$x^2+9x+5x+\_$ F O I L
Step 4. Multiply the *Last* terms.	$(x+5)(x+9)$	$x^2+9x+5x+45$ F O I L
Step 5. Combine like terms, when possible.		$x^2+14x+45$

Try It 2.2.32

Multiply using the FOIL method: $(x+6)(x+8)$

Answer

$x^2+14x+48$

Try It 2.2.33

Multiply using the FOIL method: $(y+17)(y+3)$

Answer

$y^2+20y+51$

We summarize the steps of the FOIL method below. The FOIL method only applies to multiplying binomials, not other polynomials!

MULTIPLY TWO BINOMIALS USING THE FOIL METHOD

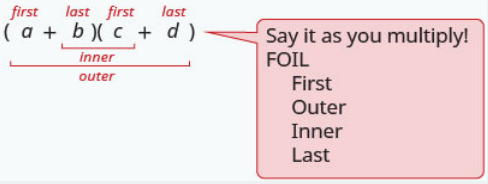
Step 1. Multiply the *First* terms.

Step 2. Multiply the *Outer* terms.

Step 3. Multiply the *Inner* terms.

Step 4. Multiply the *Last* terms.

Step 5. Combine like terms, when possible.



When you multiply by the FOIL method, drawing the lines will help your brain focus on the pattern and make it easier to apply.

Example 2.2.34

Multiply: $(y-7)(y+4)$.

Solution

		$(y-7)(y+4)$
Multiply the *First* terms.	$(y-7)(y+4)$	$y^2+\_+\_+\_$ F O I L
Multiply the *Outer* terms.	$(y-7)(y+4)$	$y^2+4y+\_+\_$ F O I L
Multiply the *Inner* terms.	$(y-7)(y+4)$	$y^2+4y-7y+\_$ F O I L
Multiply the *Last* terms.	$(y-7)(y+4)$	$y^2+4y-7y-28$ F O I L
Combine like terms.		$y^2-3y-28$

Try It 2.2.35

Multiply: $(x-7)(x+5)$.

Answer

$x^2-2x-35$

Try It 2.2.36

Multiply: (b−3)(b+6).

Answer

$b^2+3b-18$

Example 2.2.37

Multiply: $(4x+3)(2x-5)$.

Solution

	$(4x-3)(2x-5)$ $(4x+3)(2x-5)$
Multiply the *First* terms, $4x \cdot 2x$.	$8x^2+\_+\_+\_$ F O I L
Multiply the *Outer* terms, $4x \cdot (-5)$.	$8x^2-20x+\_+\_$ F O I L
Multiply the *Inner* terms, $3 \cdot 2x$.	$8x^2-2x+6x+\_$ F O I L
Multiply the *Last* terms, $3 \cdot (-5)$.	$8x^2-20x+6x-15$ F O I L
Combine like terms.	$8x^2-14x-15$

Try It 2.2.38

Multiply: $(3x+7)(5x-2)$.

Answer

$15x^2+29x-14$

Try It 2.2.39

Multiply: $(4y+5)(4y-10)$.

Answer

$16y^2-20y-50$

The final products in the last four examples were trinomials because we could combine the two middle terms. This is not always the case.

Example 2.2.40

Multiply: $(3x-y)(2x-5)$.

Solution

	$(3x-y)(2x-5)$
	
Multiply the *First*.	$6x^2+\_+\_+\_$ F O I L
Multiply the *Outer*.	$6x^2-15x+\_+\_$ F O I L
Multiply the *Inner*.	$6x^2-15x-2xy+\_$ F O I L
Multiply the *Last*.	$6x^2-15x-2xy+5y$ F O I L
Combine like terms—there are none.	$6x^2-15x-2xy+5y$

Try It 2.2.41

Multiply: (10c−d)(c−6).

Answer

$10c^2-60c-cd+6d$

Try It 2.2.42

Multiply: (7x−y)(2x−5).

Answer

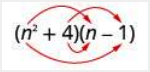
$14x^2-35x-2xy+10y$

Be careful of the exponents in the next example.

Example 2.2.43

Multiply: $\left(n^2+4\right)(n-1)$

Solution

	$(n^2+4)(n-1)$
	
Multiply the *First*.	$n^3+\_+\_+\_$
Multiply the *Outer*.	$n^3-n^2+\_+\_$
Multiply the *Inner*.	$n^3-n^2+4n+\_$
Multiply the *Last*.	n^3-n^2+4n-4
Combine like terms—there are none.	n^3-n^2+4n-4

Try It 2.2.44

Multiply: $(x^2+6)(x-8)$

Answer

$x^3-8x^2+6x-48$

Try It 2.2.45

Multiply: $(y^2+7)(y-9)$

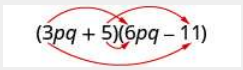
Answer

$y^3-9y^2+7y-63$

Example 2.2.46

Multiply: $(3pq+5)(6pq-11)$

Solution

	$(3pq+5)(6pq-11)$	
Multiply the *First*.	$18p^2q^2+\_+\_+\_$	
Multiply the *Outer*.	$18p^2q^2-33pq+\_+\_$	
Multiply the *Inner*.	$18p^2q^2-33pq+30pq+\_$	
Multiply the *Last*.	$18p^2q^2-33pq+30pq-55$	
Combine like terms—there are none.	$18p^2q^2-3pq-55$	

Try It 2.2.47

Multiply: $(2ab+5)(4ab-4)$

Answer

$8a^2b^2+12ab-20$

Try It 2.2.48

Multiply: $(2xy+3)(4xy-5)$

Answer

$8x^2y^2+2xy-15$

Multiply a Binomial by a Binomial Using the Vertical Method

The FOIL method is usually the quickest method for multiplying two binomials, but it *only* works for binomials. You can use the Distributive Property to find the product of any two polynomials. Another method that works for all polynomials is the Vertical Method. It is very much like the method you use to multiply whole numbers. Look carefully at this example of multiplying two-digit numbers.

23	
×46	
138 partial product	Start by multiplying 23 by 6 to get 138.
92 partial product	Next, multiply 23 by 4, lining up the partial product in the correct columns.
1058 product	Last you add the partial products.

Now we'll apply this same method to multiply two binomials.

Example 2.2.49

Multiply using the Vertical Method: $(3y-1)(2y-6)$

Solution

It does not matter which binomial goes on the top.

Multiply $3y-1$ by -6.
Multiply $3y-1$ by $2y$.

$$\begin{array}{rl} 3y-1 & \\ \times\ 2y-6 & \\ \hline -18y+6 & \text{partial product} \\ 6y^2-2y & \text{partial product} \\ \hline 6y^2-20y+6 & \text{product} \end{array} \qquad (2.2.2)$$

Add like terms.

Notice the partial products are the same as the terms in the FOIL method.

$$\begin{array}{rl} (3y-1)(2y-6) & \\ 6y^2-2y-18y+6 & \\ 6y^2-20y+6 & \end{array} \qquad \begin{array}{r} 3y-1 \\ \times\ 2y-6 \\ \hline -18y+6 \\ 6y^2-2y \\ \hline 6y^2-20x+6 \end{array}$$

Try It 2.2.50

Multiply using the Vertical Method: $(5m-7)(3m-6)$

Answer

$15m^2-51m+42$

Try It 2.2.51

Multiply using the Vertical Method: $(6b-5)(7b-3)$

Answer

$42b^2-53b+15$

We have now used three methods for multiplying binomials. Be sure to practice each method, and try to decide which one you prefer. The methods are listed here all together, to help you remember them.

MULTIPLYING TWO BINOMIALS

To multiply binomials, use the:

- Distributive Property
- FOIL Method
- Vertical Method

Remember, FOIL only works when multiplying two binomials.

Multiply a Trinomial by a Binomial

We have multiplied monomials by monomials, monomials by polynomials, and binomials by binomials. Now we're ready to multiply a **trinomial** by a **binomial**. Remember, FOIL will not work in this case, but we can use either the Distributive Property or the Vertical Method. We first look at an example using the Distributive Property.

Example 2.2.52

Multiply using the Distributive Property: $(b+3)\left(2b^2-5b+8\right)$

Solution

	
Distribute.	
Multiply.	$2b^3-5b^2+8b+6b^2-15b+24$
Combine like terms.	$2b^3+b^2-7b+24$

Try It 2.2.53

Multiply using the Distributive Property: $(y-3)\left(y^2-5y+2\right)$

Answer

$y^3-8y^2+17y-6$

Try It 2.2.54

Multiply using the Distributive Property: $(x+4)\left(2x^2-3x+5\right)$

Answer

$2x^3+5x^2-7x+20$

Now let's do this same multiplication using the Vertical Method.

Example 2.2.55

Multiply using the Vertical Method: $(b+3)\left(2b^2-5b+8\right)$

Solution

It is easier to put the polynomial with fewer terms on the bottom because we get fewer partial products this way.

Multiply $(2b^2-5b+8)$ by 3.	$2b^2-5b+8$ $\times \quad b+3$ $6b^2-15b+24$
	$2b^3-5b^2+8b$
Multiply $(2b^2-5b+8)$ by b.	$2b^3+b^2-7b+24$
Add like terms.	

Try It 2.2.56

Multiply using the Vertical Method: $(y-3)\left(y^2-5y+2\right)$

Answer

$y^3-8y^2+17y-6$

Try It 2.2.57

Multiply using the Vertical Method: $(x+4)\left(2x^2-3x+5\right)$

Answer

$2x^3+5x^2-7x+20$

We have now seen two methods you can use to multiply a **trinomial** by a binomial. After you practice each method, you'll probably find you prefer one way over the other. We list both methods are listed here, for easy reference.

MULTIPLYING A TRINOMIAL BY A BINOMIAL

To multiply a trinomial by a binomial, use the:

- Distributive Property
- Vertical Method

Note

Access these online resources for additional instruction and practice with multiplying polynomials:

- Multiplying Exponents 1
- Multiplying Exponents 2
- Multiplying Exponents 3

Key Concepts

- **FOIL Method for Multiplying Two Binomials**—To multiply two binomials:
  1. Multiply the **First** terms.
  2. Multiply the **Outer** terms.

3. Multiply the **Inner** terms.
4. Multiply the **Last** terms.

- **Multiplying Two Binomials**—To multiply binomials, use the:
  - Distributive Property (Example)
  - FOIL Method (Example)
  - Vertical Method (Example)
- **Multiplying a Trinomial by a Binomial**—To multiply a trinomial by a binomial, use the:
  - Distributive Property (Example)
  - Vertical Method (Example)

This page titled 2.2: Multiply Polynomials is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **6.3: Multiply Polynomials** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

2.3: Special Products

Learning Objectives

By the end of this section, you will be able to:

- Square a binomial using the Binomial Squares Pattern
- Multiply conjugates using the Product of Conjugates Pattern
- Recognize and use the appropriate special product pattern

Note

Before you get started, take this readiness quiz.

Simplify: a. 9^2 b. $(-9)^2$ c. -9^2.

If you missed this problem, review Exercise 1.5.13.

Square a Binomial Using the Binomial Squares Pattern

Mathematicians like to look for patterns that will make their work easier. A good example of this is squaring binomials. While you can always get the product by writing the **binomial** twice and using the methods of the last section, there is less work to do if you learn to use a pattern.

Let's start by looking at $(x+9)^2$.	
What does this mean?	$(x+9)^2$
It means to multiply $(x+9)$ by itself.	$(x+9)(x+9)$
Then, using FOIL, we get:	$x^2+9x+9x+81$
Combining like terms gives:	$x^2+18x+81$
Here's another one:	$(y-7)^2$
Multiply $(y-7)$ by itself.	$(y-7)(y-7)$
Using FOIL, we get:	$y^2-7y-7y+49$
And combining like terms:	$y^2-14y+49$
And one more:	$(2x+3)^2$
Multiply.	$(2x+3)(2x+3)$
Use FOIL:	$4x+6x+6x+9$
Combine like terms.	$4x^2+12x+9$

Look at these results. Do you see any patterns?

What about the number of terms? In each example we squared a binomial and the result was a **trinomial**.

$$(a+b)^2 = \_\_\_\_ + \_\_\_\_ + \_\_\_\_$$

Now look at the ***first term*** in each result. Where did it come from?

$(x+9)^2$	$(y-7)^2$	$(2x+3)^2$
$(x+9)(x+9)$	$(y-7)(y-7)$	$(2x+3)(2x+3)$
$x^2+9x+9x+81$	$y^2-7y-7y+49$	$4x^2+6x+6x+9$
$x^2+18x+81$	$y^2-14y+49$	$4x^2+12x+9$

The first term is the product of the first terms of each binomial. Since the binomials are identical, it is just the square of the first term!

$$(a+b)^2 = a^2 + \_\_\_\_ + \_\_\_\_$$

To get the ***first term*** of the product, ***square the first term***.

Where did the ***last term*** come from? Look at the examples and find the pattern.

The last term is the product of the last terms, which is the square of the last term.

$$(a+b)^2 = \_\_\_\_ + \_\_\_\_ + b^2$$

*To get the **last term** of the product, **square the last term**.*

Finally, look at the ***middle term***. Notice it came from adding the "outer" and the "inner" terms—which are both the same! So the middle term is double the product of the two terms of the binomial.

$$(a+b)^2 = \_\_\_\_ + 2ab + \_\_\_\_$$

$$(a+b)^2 = \_\_\_\_ - 2ab + \_\_\_\_$$

*To get the **middle term** of the product, **multiply the terms and double their product**.*

Putting it all together:

Binomial Square Pattern

If a and b are real numbers, the binomial square pattern is

$$\underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\times(\text{product of terms})} + \underbrace{a^2}_{(\text{last term})^2}$$

Applying this to two binomial types:

$$\begin{aligned}(a+b)^2 &= a^2+2ab+b^2\\ (a-b)^2 &= a^2-2ab+b^2\end{aligned}$$

To square a binomial:

1. square the first term
2. square the last term
3. double their product

A number example helps verify the pattern.

	$(10+4)^2$
Square the first term.	$10^2 + \_\_\_\_ + \_\_\_\_$
Square the last term.	$10^2 + \_\_\_\_ + \frac{1}{4^2}$
Double their product.	$10^2 + 2\cdot 10\cdot 4 + 4^2$
Simplify.	$100 + 80 + 16$
Simplify.	196

To multiply $(10+4)^2$ usually you'd follow the Order of Operations.

$$\begin{gathered}(10+4)^2\\ (14)^2\\ 196\end{gathered}$$

The pattern works!

Example 2.3.1

Multiply: $(x+5)^2$

Solution:

	$(x+5)^2$
Square the first term.	$x^2+\_\_\_+\_\_\_$
Square the last term.	$x^2+\_\_\_+5^2$
Double the product.	$x^2+2\cdot x\cdot 5+5^2$
Simplify.	$x^2+10x+25$

Try It! 2.3.1

Multiply: $(x+9)^2$

Answer

$x^2+18x+81$

Try It! 2.3.2

Multiply: $(y+11)^2$

Answer

$y^2+22y+121$

Example 2.3.2

Multiply: $(y-3)^2$

Solution:

	$(y-3)^2$
Square the first term.	$y^2-\_\_\_+\_\_\_$
Square the last term.	$y^2-\_\_\_+3^2$
Double the product.	$y^2-2\cdot y\cdot 3+3^2$
Simplify.	y^2-6y+9

Try It! 2.3.3

Multiply: $(x-9)^2$

Answer

$x^2-18x+81$

Try It! 2.3.4

Multiply: $(p-13)^2$

Answer

$p^2-26p+169$

Example 2.3.3

Multiply: $(4x+6)^2$

Solution:

	$(4x+6)^2$
Use the pattern.	$(4x)^2+2\cdot 4x\cdot 6+6^2$
Simplify.	$16x^2+48x+36$

Try It! 2.3.5

Multiply: $(6x+3)^2$

Answer

$36x^2+36x+9$

Try It! 2.3.6

Multiply: $(4x+9)^2$

Answer

$16x^2+72x+81$

Example 2.3.4

Multiply:$(2x-3y)^2$

Solution:

	$(2x-3y)^2$
Use the pattern.	$(2x)^2-2\cdot 2x\cdot 3y+(3y)^2$
Simplify.	$4x^2-12xy+9y^2$

Try It! 2.3.7

Multiply: $(2c-d)^2$

Answer

$4c^2-4cd+d^2$

Try It! 2.3.8

Multiply: $(4x-5y)^2$

Answer

$16x^2-40xy+25y^2$

Example 2.3.5

Multiply: $\left(4u^3+1\right)^2$

Solution:

	$\left(4u^3+1\right)^2$
Use the pattern.	$(4u^3)^2+2\cdot 4u^3\cdot 1+(1)^2$
Simplify.	$16u^6+8u^3+1$

Try It! 2.3.9

Multiply: $\left(2x^2+1\right)^2$

Answer

$4x^4+4x^2+1$

Try It! 2.3.10

Multiply: $\left(3y^3+2\right)^2$

Answer

$9y^6+12y^3+4$

Multiply Conjugates Using the Product of Conjugates Pattern

We just saw a pattern for squaring binomials that we can use to make multiplying some binomials easier. Similarly, there is a pattern for another product of binomials. But before we get to it, we need to introduce some vocabulary.

What do you notice about these pairs of binomials?

$$(x-9)(x+9) \qquad (y-8)(y+8) \qquad (2x-5)(2x+5)$$

Look at the first term of each **binomial** in each pair.

$$(x-9)(x+9) \qquad (y-8)(y+8) \qquad (2x-5)(2x+5)$$

Notice the first terms are the same in each pair.

Look at the last terms of each binomial in each pair.

$$(x-9)(x+9) \qquad (y-8)(y+8) \qquad (2x-5)(2x+5)$$

Notice the last terms are the same in each pair.

Notice how each pair has one sum and one difference.

$(x-9)(x+9)$ (Difference, Sum) $(y-8)(y+8)$ (Difference, Sum) $(2x-5)(2x+5)$ (Difference, Sum)

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference has a special name. It is called a *conjugate pair* and is of the form (a−b),(a+b).

Definition: Conjugate Pair

A **conjugate pair** is two binomials of the form

$$(a-b),(a+b)$$

The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

There is a nice pattern for finding the product of conjugates. You could, of course, simply FOIL to get the product, but using the pattern makes your work easier.

Let's look for the pattern by using FOIL to multiply some conjugate pairs.

$(x-9)(x+9)$	$(y-8)(y+8)$	$(2x-5)(2x+5)$
$x^2+9x-9x-81$	$y^2+8y-8y-64$	$4x^2+10x-10x-25$
x^2-81	y^2-64	$4x^2-25$

Each **first term** is the product of the first terms of the binomials, and since they are identical it is the square of the first term.

$$(a+b)(a-b)=a^2-\_\_\_\_$$

To get the **first term, square the first term.**

The **last term** came from multiplying the last terms, the square of the last term.

$$(a+b)(a-b)=a^2-b^2$$

To get the **last term, square the last term.**

What do you observe about the products?

The product of the two binomials is also a binomial! Most of the products resulting from FOIL have been trinomials.

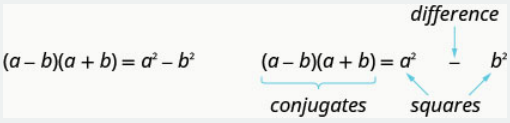
Why is there no middle term? Notice the two middle terms you get from FOIL combine to 0 in every case, the result of one addition and one subtraction.

The product of conjugates is always of the form a^2-b^2. This is called a difference of squares.

This leads to the pattern:

PRODUCT OF CONJUGATES PATTERN

If a and b are real numbers,



The product is called a **difference of squares**.

To multiply conjugates, square the first term, square the last term, and write the product as a difference of squares.

Let's test this pattern with a numerical example.

	$(10-2)(10+2)$
It is the product of conjugates, so the result will be the difference of two squares.	
	$\_\_\_\_ - \_\_\_\_$
Square the first term.	$10^2 - \_\_\_\_$
Square the last term.	$10^2 - 2^2$
Simplify.	$100 - 4$
Simplify.	96
What do you get using the Order of Operations?	

$$(10-2)(10+2)$$
$$(8)(12)$$
$$96$$

Notice, the result is the same!

Example 2.3.6

Multiply: $(x-8)(x+8)$

Solution:

First, recognize this as a product of conjugates. The binomials have the same first terms, and the same last terms, and one binomial is a sum and the other is a difference.

It fits the pattern.	$(a-b)(a+b)$ $(x-8)(x+8)$
Square the first term, x.	$a^2 - b^2$ $x^2 - \_\_$
Square the last term, 8.	$a^2 - b^2$ $x^2 - 8^2$
The product is a difference of squares.	$a^2 - b^2$ $x^2 - 64$

Try It! 2.3.11

Multiply: $(x-5)(x+5)$

Answer

$x^2 - 25$

Try It! 2.3.12

Multiply: $(w-3)(w+3)$

Answer

$w^2 - 9$

Example 2.3.7

Multiply: $(2x+5)(2x-5)$

Solution:

Are the binomials conjugates?

It is the product of conjugates.	$(a+b)(a-b)$ $(2x+5)(2x-5)$
Square the first term, $2x$.	a^2-b^2 $(2x)^2-\_\_$
Square the last term, 5.	a^2-b^2 $(2x)^2-5^2$
Simplify. The product is a difference of squares.	a^2-b^2 $4x^2-25$

Try It! 2.3.13

Multiply: $(6x+5)(6x-5)$

Answer

$36x^2-25$

Try It! 2.3.14

Multiply: $(2x+7)(2x-7)$

Answer

$4x^2-49$

The binomials in the next example may look backwards – the variable is in the second term. But the two binomials are still conjugates, so we use the same pattern to multiply them.

Example 2.3.8

Find the product:$(3+5x)(3-5x)$

Solution:

It is the product of conjugates.	$(a-b)(a+b)$ $(3+5x)(3-5x)$
Use the pattern.	a^2-b^2 $3^2-(5x)^2$
Simplify.	$9-25x^2$

Try It! 2.3.15

Multiply: $(7+4x)(7-4x)$

Answer

$49-16x^2$

Try It! 2.3.16

Multiply: $(9-2y)(9+2y)$

Answer

$81-4y^2$

Now we'll multiply conjugates that have two variables.

Example 2.3.9

Find the product: $(5m-9n)(5m+9n)$

Solution:

This fits the pattern.	$(\underset{5m}{a}-\underset{9n}{b})(\underset{5m}{a}+\underset{9n}{b})$
Use the pattern.	$\underset{(5m)^2}{a^2}-\underset{(9n)^2}{b^2}$
Simplify.	$25m^2-81n^2$

Try It! 2.3.17

Find the product: $(4p-7q)(4p+7q)$

Answer

$16p^2-49q^2$

Try It! 2.3.18

Find the product: $(3x-y)(3x+y)$

Answer

$9x^2-y^2$

Example 2.3.10

Find the product: $(cd-8)(cd+8)$

Solution:

This fits the pattern.	$(\underset{cd}{a}-\underset{8}{b})(\underset{cd}{a}+\underset{8}{b})$
Use the pattern.	$\underset{(cd)^2}{a^2}-\underset{(8)^2}{b^2}$
Simplify.	c^2d^2-64

Try It! 2.3.19

Find the product: $(xy-6)(xy+6)$

Answer

x^2y^2-36

Try It! 2.3.20

Find the product: $(ab-9)(ab+9)$

Answer

a^2b^2-81

Example 2.3.11

Find the product: $\left(6u^2-11v^5\right)\left(6u^2+11v^5\right)$

Solution:

This fits the pattern.	$\left(\underset{6u^2}{a}-\underset{11v^5}{b}\right)\left(\underset{6u^2}{a}+\underset{11v^5}{b}\right)$
Use the pattern.	$\underset{(6u^2)^2}{a^2}-\underset{(11v^5)^2}{b^2}$
Simplify.	$36u^4-121v^{10}$

Try It! 2.3.21

Find the product: $\left(3x^2-4y^3\right)\left(3x^2+4y^3\right)$

Answer

$9x^4-16y^6$

Try It! 2.3.22

Find the product: $\left(2m^2-5n^3\right)\left(2m^2+5n^3\right)$

Answer

$4m^4-25n^6$

Recognize and Use the Appropriate Special Product Pattern

We just developed special product patterns for Binomial Squares and for the Product of Conjugates. The products look similar, so it is important to recognize when it is appropriate to use each of these patterns and to notice how they differ. Look at the two patterns together and note their similarities and differences.

COMPARING THE SPECIAL PRODUCT PATTERNS

Binomial Squares	**Product of Conjugates**
$(a+b)^2=a^2+2ab+b^2$	$(a-b)(a+b)=a^2-b^2$
$(a-b)^2=a^2+2ab+b^2$	
- Squaring a binomial	- Multiplying conjugates
- Product is a trinomial	- Product is a binomial
- Inner and outer terms with FOIL are the same.	- Inner and outer terms with FOIL are opposites.
- Middle term is double the product of the terms.	- There is no middle term.

Example 2.3.12

Choose the appropriate pattern and use it to find the product:

1. $(2x-3)(2x+3)$
2. $(8x-5)^2$
3. $(6m+7)^2$
4. $(5x-6)(6x+5)$

Solution:

1. $(2x-3)(2x+3)$ These are conjugates. They have the same first numbers, and the same last numbers, and one binomial is a sum and the other is a difference. It fits the Product of Conjugates pattern.

This fits the pattern.	$(\underset{2x}{a}-\underset{3}{b})(\underset{2x}{a}+\underset{3}{b})$
Use the pattern.	$\underset{(2x)^2}{a^2}-\underset{3^2}{b^2}$
Simplify.	$4x^2-9$

2. $(8x-5)^2$ We are asked to square a binomial. It fits the **binomial squares** pattern.

	$(\underset{8x}{a}-\underset{5}{b})^2$
Use the pattern.	$\underset{(8x)^2}{a^2}-\underset{2\cdot 8x\cdot 5}{2ab}+\underset{5^2}{b^2}$
Simplify.	$64x^2-80x+25$

3. $(6m+7)^2$ Again, we will square a binomial so we use the **binomial squares** pattern.

	$(\underset{6m}{a}+\underset{7}{b})^2$
Use the pattern.	$\underset{(6m)^2}{a^2}+\underset{2\cdot 6m\cdot 7}{2ab}+\underset{7^2}{b^2}$
Simplify.	$36m^2+84m+49$

4. $(5x-6)(6x+5)$ This product does not fit the patterns, so we will use FOIL.

	$(5x-6)(6x+5)$
Use FOIL.	$30x^2+25x-36x-30$
Simplify.	$30x^2-11x-30$

? Try It! 2.3.23

Choose the appropriate pattern and use it to find the product:

1. $(9b-2)(2b+9)$
2. $(9p-4)2$
3. $(7y+1)^2$
4. $(4r-3)(4r+3)$

Answer

1. FOIL; $18b^2+77b-18$
2. Binomial Squares; $81p^2-72p+16$
3. Binomial Squares; $49y^2+14y+1$
4. Product of Conjugates; $16r^2-9$

? Try It! 2.3.24

Choose the appropriate pattern and use it to find the product:

1. $(6x+7)^2$
2. $(3x-4)(3x+4)$
3. $(2x-5)(5x-2)$
4. $(6n-1)^2$

Answer

1. Binomial Squares; $36x^2+84x+49$
2. Product of Conjugates; $9x^2-16$
3. FOIL; $10x^2-29x+10$
4. Binomial Squares; $36n^2-12n+1$

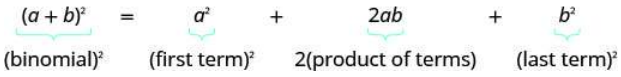
Note

Access these online resources for additional instruction and practice with special products:

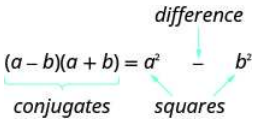
- Special Products

Key Concepts

- **Binomial Squares Pattern**
  - If a,b are real numbers,



  - $(a+b)^2=a^2+2ab+b^2$
  - $(a-b)^2=a^2-2ab+b^2$
  - To square a binomial: square the first term, square the last term, double their product.
- **Product of Conjugates Pattern**
  - If a,ba,b are real numbers,



  - $(a-b)(a+b)=a^2-b^2$
  - The product is called a difference of squares.
- **To multiply conjugates:**
  - **square the first term square the last term** write it as a difference of squares

Glossary

conjugate pair

A conjugate pair is two binomials of the form $(a-b)$ and $(a+b)$; the pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

This page titled 2.3: Special Products is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **6.4: Special Products** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

2.4: Divide Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Divide a polynomial by a monomial
- Divide a polynomial by a binomial

Note

Before you get started, take this readiness quiz.

1. Add: $\frac{3}{d}+\frac{x}{d}$
   If you missed this problem, review Example 1.7.1.
2. Simplify: $\frac{30xy^3}{5xy}$
   If you missed this problem, review Example 6.5.37.
3. Combine like terms: $8a^2+12a+1+3a^2-5a+4$
   If you missed this problem, review Example 1.3.37.

Divide a Polynomial by a Monomial

In the last section, you learned how to divide a monomial by a monomial. As you continue to build up your knowledge of polynomials the next procedure is to divide a polynomial of two or more terms by a **monomial**.

The method we'll use to divide a polynomial by a monomial is based on the properties of fraction addition. So we'll start with an example to review fraction addition.

The sum, $\frac{y}{5}+\frac{2}{5}$

simplifies to $\frac{y+2}{5}$

Now we will do this in reverse to split a single fraction into separate fractions.

We'll state the fraction addition property here just as you learned it and in reverse.

FRACTION ADDITION

If a,b, and c are numbers where $c\neq 0$, then

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \quad \text{and} \quad \frac{a+b}{c}=\frac{a}{c}+\frac{b}{c} \tag{2.4.1}$$

We use the form on the left to add fractions and we use the form on the right to divide a polynomial by a monomial.

We use this form of fraction addition to divide polynomials by monomials.

DIVISION OF A POLYNOMIAL BY A MONOMIAL

To divide a polynomial by a monomial, divide each term of the polynomial by the **monomial**.

Example 2.4.1

Find the quotient: $\frac{7y^2+21}{7}$

Solution

	$\frac{7y^2+21}{7}$
Divide each term of the numerator by the denominator.	$\frac{7y^2}{7}+\frac{21}{7}$
Simplify each fraction.	y^2+3

Try It 2.4.2

Find the quotient: $\frac{8z^2+24}{4}$

Answer

$2z^2+6$

Try It 2.4.3

Find the quotient: $\frac{18z^2-27}{9}$

Answer

$2z^2-3$

Remember that division can be represented as a fraction. When you are asked to divide a polynomial by a monomial and it is not already in fraction form, write a fraction with the polynomial in the numerator and the monomial in the denominator.

Example 2.4.4

Find the quotient: $\left(18x^3-36x^2\right)\div 6x$

Solution

	$\left(18x^3-36x^2\right)\div 6x$
Rewrite as a fraction.	$\frac{18x^3-36x^2}{6x}$
Divide each term of the numerator by the denominator.	$\frac{18x^3}{6x}-\frac{36x^2}{6x}$
Simplify.	$3x^2-6x$

Try It 2.4.5

Find the quotient: $\left(27b^3-33b^2\right)\div 3b$

Answer

$9b^2-11b$

Try It 2.4.6

Find the quotient: $\left(25y^3-55y^2\right)\div 5y$

Answer

$5y^2-11y$

When we divide by a negative, we must be extra careful with the signs.

Example 2.4.7

Find the quotient: $\frac{12d^2-16d}{-4}$

Solution

	$\frac{12d^2-16d}{-4}$
Divide each term of the numerator by the denominator.	$\frac{18x^3-36x^2}{6x}$
Simplify. Remember, subtracting a negative is like adding a positive!	$-3d^2+4d$

Try It 2.4.8

Find the quotient: $\frac{25y^2-15y}{-5}$

Answer

$-5y^2+3y$

Try It 2.4.9

Find the quotient: $\frac{42b^2-18b}{-6}$

Answer

$-7b^2+3b$

Example 2.4.10

Find the quotient: $\frac{105y^5+75y^3}{5y^2}$

Solution

	$\frac{105y^5+75y^3}{5y^2}$
Separate the terms.	$\frac{105y^5}{5y^2}+\frac{75y^3}{5y^2}$
Simplify.	$21y^3+15y$

Try It 2.4.11

Find the quotient: $\frac{60d^7+24d^5}{4d^3}$

Answer

$15d^4+6d^2$

Try It 2.4.12

Find the quotient: $\frac{216p^7-48p^5}{6p^3}$

Answer

$36p^4 - 8p^2$

Example 2.4.13

Find the quotient: $\left(15x^3y - 35xy^2\right) \div (-5xy)$

Solution

	$\left(15x^3y - 35xy^2\right) \div (-5xy)$
Rewrite as a fraction.	$\dfrac{15x^3y - 35xy^2}{-5xy}$
Separate the terms. Be careful with the signs!	$\dfrac{15x^3y}{-5xy} - \dfrac{35xy^2}{-5xy}$
Simplify.	$-3x^2 + 7y$

Try It 2.4.14

Find the quotient: $\left(32a^2b - 16ab^2\right) \div (-8ab)$

Answer

$-4a + 2b$

Try It 2.4.15

Find the quotient: $\left(-48a^8b^4 - 36a^6b^5\right) \div \left(-6a^3b^3\right)$

Answer

$8a^5b + 6a^3b^2$

Example 2.4.16

Find the quotient: $\dfrac{36x^3y^2 + 27x^2y^2 - 9x^2y^3}{9x^2y}$

Solution

	$\dfrac{36x^3y^2 + 27x^2y^2 - 9x^2y^3}{9x^2y}$
Separate the terms.	$\dfrac{36x^3y^2}{9x^2y} + \dfrac{27x^2y^2}{9x^2y} - \dfrac{9x^2y^3}{9x^2y}$
Simplify.	$4xy + 3y - y^2$

Try It 2.4.17

Find the quotient: $\dfrac{40x^3y^2 + 24x^2y^2 - 16x^2y^3}{8x^2y}$

Answer

$5xy + 3y - 2y^2$

Try It 2.4.18

Find the quotient: $\frac{35a^4b^2+14a^4b^3-42a^2b^4}{7a^2b^2}$

Answer

$5a^2+2a^2b-6b^2$

Example 2.4.19

Find the quotient: $\frac{10x^2+5x-20}{5x}$

Solution

	$\frac{10x^2+5x-20}{5x}$
Separate the terms.	$\frac{10x^2}{5x}+\frac{5x}{5x}-\frac{20}{5x}$
Simplify.	$2x+1-\frac{4}{x}$

Try It 2.4.20

Find the quotient: $\frac{18c^2+6c-9}{6c}$

Answer

$3c+1-\frac{3}{2c}$

Try It 2.4.21

Find the quotient: $\frac{10d^2-5d-2}{5d}$

Solution

$2d-1-\frac{2}{5d}$

Divide a Polynomial by a Binomial

To divide a polynomial by a **binomial**, we follow a procedure very similar to long division of numbers. So let's look carefully the steps we take when we divide a 3-digit number, 875, by a 2-digit number, 25.

We write the long division	$25\overline{)875}$
We divide the first two digits, 87, by 25.	$\begin{array}{r} 3 \\ 25\overline{)875} \end{array}$
We multiply 3 times 25 and write the product under the 87.	$\begin{array}{r} 3 \\ 25\overline{)875} \\ \underline{75} \end{array}$

Now we subtract 75 from 87.	$\begin{array}{r} 3 \\ 25\overline{)875} \\ -75 \\ \hline 12 \end{array}$ $$\begin{array}{r} 3 \\ 25\longdiv{875} \\ -75 \\ \hline 125 \end{array} \quad (2.4.2)$$
Then we bring down the third digit of the dividend, 5.	$\begin{array}{r} 3 \\ 25\overline{)875} \\ -75 \\ \hline 125 \end{array}$
Repeat the process, dividing 25 into 125.	$\begin{array}{r} 35 \\ 25\overline{)875} \\ -75 \\ \hline 125 \\ -125 \\ \hline \end{array}$

We check division by multiplying the quotient by the divisor.

If we did the division correctly, the product should equal the dividend.

$$\begin{array}{l} 35 \cdot 25 \\ 875\checkmark \end{array} \quad (2.4.3)$$

Now we will divide a **trinomial** by a binomial. As you read through the example, notice how similar the steps are to the numerical example above.

Example 2.4.22

Find the quotient: $\left(x^2+9x+20\right) \div (x+5)$

Solution

	$(x^2 + 9x + 20) \div (x + 5)$
Write it as a long division problem.	
Be sure the dividend is in standard form.	$x+5\overline{)\,x^2+9x+20}$
Divide x^2 by x. It may help to ask yourself, "What do I need to multiply x by to get x^2?"	
Put the answer, x, in the quotient over the x term.	$\begin{array}{r} x \\ x+5\overline{)\,x^2+9x+20} \end{array}$
Multiply x times $x + 5$. Line up the like terms under the dividend.	$\begin{array}{r} x \\ x+5\overline{)\,x^2+9x+20} \\ x^2+5x \end{array}$
Subtract $x^2 + 5x$ from $x^2 + 9x$.	
You may find it easier to change the signs and then add. Then bring down the last term, 20.	$\begin{array}{r} x \\ x+5\overline{)\,x^2+9x+20} \\ -x^2+(-5x) \\ \hline 4x+20 \end{array}$
Divide $4x$ by x. It may help to ask yourself, "What do I need to multiply x by to get $4x$?"	

Put the answer, 4, in the quotient over the constant term.	$x+4$ $x+5 \overline{)\, x^2+9x+20}$ $-x^2+(-5x)$ $4x+20$
Multiply 4 times x + 5.	$x+4$ $x+5 \overline{)\, x^2+9x+20}$ $-x^2+(-5x)$ $4x+20$ $4x+20$
Subtract 4x + 20 from 4x + 20.	$x+4$ $x+5 \overline{)\, x^2+9x+20}$ $-x^2+(-5x)$ $4x+20$ $-4x+(-20)$ 0
Check:	
Multiply the quotient by the divisor.	
(x + 4)(x + 5)	
You should get the dividend.	
x^2 + 9x + 20✓	

Try It 2.4.23

Find the quotient: $\left(y^2+10y+21\right) \div (y+3)$

Answer

y+7

Try It 2.4.24

Find the quotient: $\left(m^2+9m+20\right) \div (m+4)$

Answer

m+5

When the divisor has subtraction sign, we must be extra careful when we multiply the partial quotient and then subtract. It may be safer to show that we change the signs and then add.

Example 2.4.25

Find the quotient: $\left(2x^2-5x-3\right) \div (x-3)$

Solution

	$(2x^2-5x-3) \div (x-3)$
Write it as a long division problem.	
Be sure the dividend is in standard form.	$x-3 \overline{)\, 2x^2-5x-3}$
Divide 2x^2 by x. Put the answer, 2x, in the quotient over the x term.	$2x$ $x-3 \overline{)\, 2x^2-5x-3}$

Multiply $2x$ times $x-3$. Line up the like terms under the dividend.	$\begin{array}{r} 2x \\ x-3\overline{)2x^2-5x-3} \\ 2x^2-6x \end{array}$
Subtract $2x^2-6x$ from $2x^2-5x$. Change the signs and then add. Then bring down the last term.	$\begin{array}{r} 2x \\ x-3\overline{)2x^2-5x-3} \\ -2x^2+6x \\ \hline x-3 \end{array}$
Divide x by x. Put the answer, 1, in the quotient over the constant term.	$\begin{array}{r} 2x+1 \\ x-3\overline{)2x^2-5x-3} \\ -2x^2-(-6x) \\ \hline x-3 \end{array}$
Multiply 1 times $x-3$.	$\begin{array}{r} 2x+1 \\ x-3\overline{)2x^2-5x-3} \\ -2x^2+6x \\ \hline x-3 \\ x-3 \end{array}$
Subtract $x-3$ from $x-3$ by changing the signs and adding.	$\begin{array}{r} 2x+1 \\ x-3\overline{)2x^2-5x-3} \\ -2x^2+6x \\ \hline x-3 \\ -x+3 \\ \hline 0 \end{array}$
To check, multiply $(x-3)(2x+1)$.	
The result should be $2x^2-5x-3$.	

Try It 2.4.26

Find the quotient: $\left(2x^2-3x-20\right)\div(x-4)$

Answer

2x+5

Try It 2.4.27

Find the quotient: $\left(3x^2-16x-12\right)\div(x-6)$

Answer

3x+2

When we divided 875 by 25, we had no remainder. But sometimes division of numbers does leave a remainder. The same is true when we divide polynomials. In Example 2.4.25, we'll have a division that leaves a remainder. We write the remainder as a fraction with the divisor as the denominator.

Example 2.4.28

Find the quotient: $\left(x^3-x^2+x+4\right)\div(x+1)$

Solution

	$(x^3-x^2+x+4)\div(x+1)$
Write it as a long division problem.	

Be sure the dividend is in standard form.	$x+1\overline{)\,x^3-x^2+x+4}$
Divide x^3 by x. Put the answer, x^2, in the quotient over the x^2 term. Multiply x^2 times $x + 1$. Line up the like terms under the dividend.	$\begin{array}{r} x^2 \\ x+1\overline{)\,x^3-x^2+x+4} \\ \underline{x^3+x^2} \end{array}$
Subtract $x^3 + x^2$ from $x^3 - x^2$ by changing the signs and adding. Then bring down the next term.	$\begin{array}{r} x^2 \\ x+1\overline{)\,x^3-x^2+x+4} \\ \underline{-x^3+(-x^2)} \\ -2x^2+x \end{array}$
Divide $-2x^2$ by x. Put the answer, $-2x$, in the quotient over the x term. Multiply $-2x$ times $x + 1$. Line up the like terms under the dividend.	$\begin{array}{r} x^2-2x \\ x+1\overline{)\,x^3-x^2+x+4} \\ \underline{-x^3+(-x^2)} \\ -2x^2+x \\ \underline{-2x^2-2x} \end{array}$
Subtract $-2x^2 - 2x$ from $-2x^2 + x$ by changing the signs and adding. Then bring down the last term.	$\begin{array}{r} x^2-2x \\ x+1\overline{)\,x^3-x^2+x+4} \\ \underline{-x^3+(-x^2)} \\ -2x^2+x \\ \underline{+2x^2+2x} \\ 3x+4 \end{array}$
Divide $3x$ by x. Put the answer, 3, in the quotient over the constant term. Multiply 3 times $x + 1$. Line up the like terms under the dividend.	$\begin{array}{r} x^2-2x+3 \\ x+1\overline{)\,x^3-x^2+x+4} \\ \underline{-x^3+(-x^2)} \\ -2x^2+x \\ \underline{+2x^2+2x} \\ 3x+4 \\ \underline{3x+3} \end{array}$
Subtract $3x + 3$ from $3x + 4$ by changing the signs and adding. Write the remainder as a fraction with the divisor as the denominator.	$\begin{array}{r} x^2-2x+3+\dfrac{1}{x+1} \\ x+1\overline{)\,x^3-x^2+x+4} \\ \underline{-x^3+(-x^2)} \\ -2x^2+x \\ \underline{+2x^2+2x} \\ 3x+4 \\ \underline{-3x+(-3)} \\ 1 \end{array}$
To check, multiply $(x+1)\left(x^2-2x+3+\dfrac{1}{x+1}\right)$ The result should be x^3-x^2+x+4	

? Try It 2.4.29

Find the quotient: $\left(x^3+5x^2+8x+6\right)\div(x+2)$

Answer

$$x^2+3x+2+\frac{2}{x+2}$$

Try It 2.4.30

Find the quotient: $\left(2x^3+8x^2+x-8\right)\div(x+1)$

Answer

$$2x^2+6x-5-\frac{3}{x+1}$$

Look back at the dividends in Example, Example, and Example. The terms were written in descending order of degrees, and there were no missing degrees. The dividend in Example will be x^4-x^2+5x-2 . It is missing an x^3 term. We will add in $0x^3$ as a placeholder.

Example 2.4.31

Find the quotient: $\left(x^4-x^2+5x-2\right)\div(x+2)$

Solution

Notice that there is no x^3 term in the dividend. We will add $0x^3$ as a placeholder.

	$(x^4 - x^2 + 5x - 2) \div (x + 2)$
Write it as a long division problem. Be sure the dividend is in standard form with placeholders for missing terms.	$x+2\overline{)\,x^4 - 0x^3 - x^2 + 5x - 2}$
Divide x^4 by x. Put the answer, x^3, in the quotient over the x^3 term. Multiply x^3 times $x + 2$. Line up the like terms. Subtract and then bring down the next term.	$\begin{array}{r} x^3 \\ x+2\overline{)\,x^4+0x^3-x^2+5x-2} \\ \underline{-(x^4+2x^3)} \\ -2x^3-x^2 \end{array}$ It may be helpful to change the signs and add.
Divide $-2x^3$ by x. Put the answer, $-2x^2$, in the quotient over the x^2 term. Multiply $-2x^2$ times $x + 1$. Line up the like terms. Subtract and bring down the next term.	$\begin{array}{r} x^3-2x^2 \\ x+2\overline{)\,x^4+0x^3-x^2+5x-2} \\ \underline{-(x^4+2x^3)} \\ -2x^3-x^2 \\ \underline{-(-2x^3-4x^2)} \\ 3x^2+5x \end{array}$ It may be helpful to change the signs and add.
Divide $3x^2$ by x. Put the answer, $3x$, in the quotient over the x term. Multiply $3x$ times $x + 1$. Line up the like terms. Subtract and bring down the next term.	$\begin{array}{r} x^3-2x^2+3x \\ x+2\overline{)\,x^4+0x^3-x^2+5x-2} \\ \underline{-(x^4+2x^3)} \\ -2x^3-x^2 \\ \underline{-(-2x^3-4x^2)} \\ 3x^2+5x \\ \underline{-(3x^2+6x)} \\ -x-2 \end{array}$ It may be helpful to change the signs and add.
Divide $-x$ by x. Put the answer, -1, in the quotient over the constant term. Multiply -1 times $x + 1$. Line up the like terms. Change the signs, add.	$\begin{array}{r} x^3-2x^2+3x-1 \\ x+2\overline{)\,x^4+0x^3-x^2+5x-2} \\ \underline{-(x^4+2x^3)} \\ -2x^3-x^2 \\ \underline{-(-2x^3-4x^2)} \\ 3x^2+5x \\ \underline{-(3x^2+6x)} \\ -x-2 \\ \underline{-(-x-2)} \\ 0 \end{array}$ It may be helpful to change the signs and add.
To check, multiply $(x+2)\left(x^3-2x^2+3x-1\right)$	
The result should be x^4-x^2+5x-2	

Try It 2.4.32

Find the quotient: $\left(x^3+3x+14\right) \div (x+2)$

Answer

x^2-2x+7

Try It 2.4.33

Find the quotient: $\left(x^4-3x^3-1000\right) \div (x+5)$

Answer

$x^3-8x^2+40x-200$

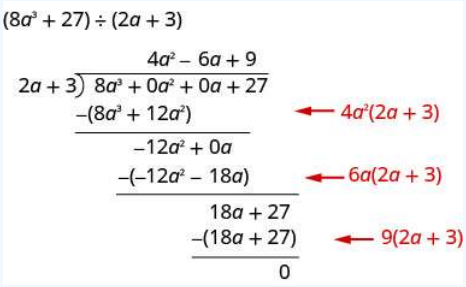
In Example 2.4.31, we will divide by $2a-3$. As we divide we will have to consider the constants as well as the variables.

Example 2.4.34

Find the quotient: $\left(8a^3+27\right) \div (2a+3)$

Solution

This time we will show the division all in one step. We need to add two placeholders in order to divide.



To check, multiply $(2a+3)\left(4a^2-6a+9\right)$

The result should be $8a^3+27$

Try It 2.4.35

Find the quotient: $\left(x^3-64\right) \div (x-4)$

Answer

$x^2+4x+16$

Try It 2.4.36

Find the quotient: $\left(125x^3-8\right) \div (5x-2)$

Answer

$25x^2+10x+4$

Note

Access these online resources for additional instruction and practice with dividing polynomials:

- Divide a Polynomial by a Monomial
- Divide a Polynomial by a Monomial 2
- Divide Polynomial by Binomial

Key Concepts

- **Fraction Addition**
  - If a,b, and c are numbers where $c \neq 0$, then $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$ and $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$
- **Division of a Polynomial by a Monomial**
  - To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

This page titled 2.4: Divide Polynomials is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **6.6: Divide Polynomials** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

CHAPTER OVERVIEW

3: Factoring

3.1: Greatest Common Factor and Factor by Grouping
3.2: Factor Quadratic Trinomials with Leading Coefficient 1
3.3: Factor Quadratic Trinomials with Leading Coefficient Other than 1
3.4: Factor Special Products
3.5: General Strategy for Factoring Polynomials

3: Factoring is shared under a CC BY-SA license and was authored, remixed, and/or curated by LibreTexts.

3.1: Greatest Common Factor and Factor by Grouping

Learning Objectives

By the end of this section, you will be able to:

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial
- Factor by grouping

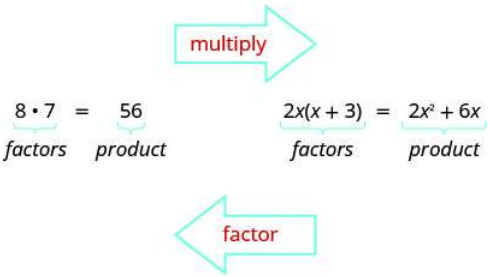
BE PREPARED

Before you get started, take this readiness quiz.

1. Factor 56 into primes.
   If you missed this problem, review Example 1.2.19.
2. Find the least common multiple of 18 and 24.
   If you missed this problem, review Example 1.2.28.
3. Simplify $-3(6a+11)$.
   If you missed this problem, review Example 1.10.40.

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a product. Now, we will be reversing this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.



We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the **greatest common factor** of two or more expressions. The method we use is similar to what we used to find the LCM.

GREATEST COMMON FACTOR

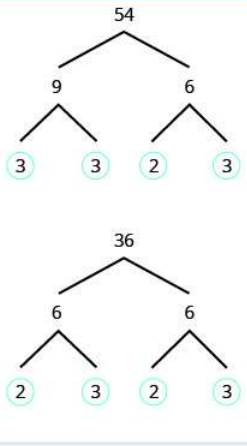
The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

First we'll find the GCF of two numbers.

Example 3.1.1: HOW TO FIND THE GREATEST COMMON FACTOR OF TWO OR MORE EXPRESSIONS

Find the GCF of 54 and 36.

Solution

Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.	Factor 54 and 36.	
Step 2. In each column, circle the common factors.	Circle the 2, 3, and 3 that are shared by both numbers.	$36 = 2 \cdot 2 \cdot 3 \cdot 3$ $18 = 2 \cdot 3 \cdot 3 \cdot 3$
Step 3. Bring down the common factors that all expressions share.	Bring down the 2, 3, and 3, and then multiply.	$GCF = 2 \cdot 3 \cdot 3$
Step 4. Multiply the factors.		$GCF = 18$ The GCF of 54 and 36 is 18.

Notice that, because the GCF is a factor of both numbers, 54 and 36 can be written as multiples of 18.

$$\begin{aligned} 54 &= 18 \cdot 3 \\ 36 &= 18 \cdot 2 \end{aligned} \tag{3.1.1}$$

? Try It 3.1.2

Find the GCF of 48 and 80.

Answer

16

? Try It 3.1.3

Find the GCF of 18 and 40.

Answer

2

We summarize the steps we use to find the GCF below.

HOW TO

Find the Greatest Common Factor (GCF) of two expressions.

1. Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
2. Step 2. List all factors—matching common factors in a column. In each column, circle the common factors.
3. Step 3. Bring down the common factors that all expressions share.
4. Step 4. Multiply the factors.

In the first example, the GCF was a constant. In the next two examples, we will get variables in the greatest common factor.

Example 3.1.4

Find the greatest common factor of $27x^3$ and $18x^4$.

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	$27x^3 = 3 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x$ $18x^4 = 2 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x \cdot x$
Bring down the common factors.	$\text{GCF} = 3 \cdot 3 \cdot x \cdot x \cdot x$
Multiply the factors.	$\text{GCF} = 9x^3$
	The GCF of $27x^3$ and $18x^4$ is $9x^3$.

Try It 3.1.5

Find the GCF: $12x^2, 18x^3$

Answer

$6x^2$

Try It 3.1.6

Find the GCF: $16y^2, 24y^3$

Answer

$8y^2$

Example 3.1.7

Find the GCF of $4x^2y, 6xy^3$

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	$4x^2y = 2 \cdot 2 \cdot x \cdot x \cdot y$ $6xy^3 = 2 \cdot 3 \cdot x \cdot y \cdot y \cdot y$
Bring down the common factors.	$\text{GCF} = 2 \cdot x \cdot y$
Multiply the factors.	$\text{GCF} = 2xy$
	The GCF of $4x^2y$ and $6xy^3$ is $2xy$.

Try It 3.1.8

Find the GCF: $6ab^4, 8a^2b$

Answer

$2ab$

Try It 3.1.9

Find the GCF: $9m^5n^2, 12m^3n$

Answer

$3m^3n$

Example 3.1.10

Find the GCF of: $21x^3, 9x^2, 15x$

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	$21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x$ $9x^2 = 3 \cdot 3 \cdot x \cdot x$ $15x = 3 \cdot 5 \cdot x$
Bring down the common factors.	GCF = $3 \cdot x$
Multiply the factors.	GCF = $3x$
	The GCF of $21x^3, 9x^2$ and $15x$ is $3x$

Try It 3.1.11

Find the greatest common factor: $25m^4, 35m^3, 20m^2$

Answer

$5m^2$

Try It 3.1.12

Find the greatest common factor: $14x^3, 70x^2, 105x$

Answer

$7x$

Factor the Greatest Common Factor from a Polynomial

Just like in arithmetic, where it is sometimes useful to represent a number in factored form (for example, 12 as 2·6or3·4),2·6or3·4), in algebra, it can be useful to represent a polynomial in factored form. One way to do this is by finding the GCF of all the terms. Remember, we multiply a polynomial by a monomial as follows:

$$\begin{aligned} &2(x+7) && \text{factors} \\ &2\cdot x+2\cdot 7 && \\ &2x+14 && \text{product} \end{aligned} \tag{3.1.2}$$

Now we will start with a product, like $2x+14$, and end with its factors, $2(x+7)$. To do this we apply the Distributive Property "in reverse."

We state the Distributive Property here just as you saw it in earlier chapters and "in reverse."

DISTRIBUTIVE PROPERTY

If a, b, c are real numbers, then

$$a(b+c) = ab+ac \quad \text{and} \quad ab+ac = a(b+c) \tag{3.1.3}$$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a polynomial? You just find the GCF of all the terms and write the polynomial as a product!

Example 3.1.13: HOW TO FACTOR THE GREATEST COMMON FACTOR FROM A POLYNOMIAL

Factor: $4x+12$

Solution

Step 1. Find the GCF of all the terms of the polynomial.	Find the GCF of $4x$ and 12.	$4x = 2 \cdot 2 \cdot \cdot x$ $12 = 2 \cdot 2 \cdot 3$ $\text{GCF} = 2 \cdot 2$ $\text{GCF} = 4$
Step 2. Rewrite each term as a product using the GCF.	Rewrite $4x$ and 12 as products of their GCF, 4. $4x = 4 \cdot x$ $12 = 4 \cdot 3$	$4x + 12$ $4 \cdot x + 4 \cdot 3$
Step 3. Use the "reverse" Distributive Property to factor the expression.		$4(x + 3)$
Step 4. Check by multiplying the factors.		$4(x + 3)$ $4 \cdot x + 4 \cdot 3$ $4x + 12$ ✓

Try It 3.1.14

Factor: $6a+24$

Answer

$6(a+4)$

Try It 3.1.15

Factor: $2b+14$

Answer

$2(b+7)$

HOW TO

Factor the greatest common factor from a polynomial.

Step 1. Find the GCF of all the terms of the polynomial.

Step 2. Rewrite each term as a product using the GCF.

Step 3. Use the "reverse" Distributive Property to factor the expression.

Step 4. Check by multiplying the factors.

FACTOR AS A NOUN AND A VERB

We use “factor” as both a noun and a verb.

Noun	7 is a factor of 14
Verb	factor 3 from $3a+3$

Example 3.1.16

Factor: $5a+5$

Solution

Find the GCF of $5a$ and 5.	$5a = 5 \cdot a$ $5 = 5$ $\text{GCF} = 5$
	$5a+5$
Rewrite each term as a product using the GCF.	$5 \cdot a + 5 \cdot 1$
Use the Distributive Property "in reverse" to factor the GCF.	$5(a+1)$
Check by multiplying the factors to get the original polynomial.	
$5(a+1)$	
$5 \cdot a + 5 \cdot 1$	
$5a+5$✓	

Try It 3.1.17

Factor: $14x+14$

Answer

$14(x+1)$

Try It 3.1.18

Factor: $12p+12$

Answer

$12(p+1)$

The expressions in the next example have several factors in common. Remember to write the GCF as the product of all the common factors.

Example 3.1.19

Factor: $12x-60$

Solution

Find the GCF of $12x$ and 60.	$12x = 2 \cdot 2 \cdot 3 \cdot x$ $60 = 2 \cdot 2 \cdot 3 \cdot 5$ $GCF = 2 \cdot 2 \cdot 3$ $GCF = 12$
	$12x - 60$
Rewrite each term as a product using the GCF.	$12 \cdot x - 12 \cdot 5$
Factor the GCF.	$12(x - 5)$
Check by multiplying the factors.	
$12(x-5)$	
$12 \cdot x - 12 \cdot 5$	
$12x - 60$✓	

Try It 3.1.20

Factor: $18u - 36$

Answer

$8(u - 2)$

Try It 3.1.21

Factor: $30y - 60$

Answer

$30(y - 2)$

Now we'll factor the greatest common factor from a trinomial. We start by finding the GCF of all three terms.

Example 3.1.22

Factor: $4y^2 + 24y + 28$

Solution

We start by finding the GCF of all three terms.

Find the GCF of $4y^2$, $24y$ and 28	$4y^2 = 2 \cdot 2 \cdot y \cdot y$ $24y = 2 \cdot 2 \cdot 2 \cdot 3 \cdot y$ $28 = 2 \cdot 2 \cdot 7$ $GCF = 2 \cdot 2$ $GCF = 4$
	$4y^2 + 24y + 28$
Rewrite each term as a product using the GCF.	$4 \cdot y^2 + 4 \cdot 6y + 4 \cdot 7$
Factor the GCF.	$4(y^2 + 6y + 7)$
Check by multiplying.	
$4\left(y^2 + 6y + 7\right)$	
$4 \cdot y^2 + 4 \cdot 6y + 4 \cdot 7$	

$4y^2+24y+28$✓

? Try It 3.1.23

Factor: $5x^2-25x+15$

Answer

$5\left(x^2-5x+3\right)$

? Try It 3.1.24

Factor: $3y^2-12y+27$

Answer

$3\left(y^2-4y+9\right)$

✓ Example 3.1.25

Factor: $5x^3-25x^2$

Solution

Find the GCF of $5x^3$ and $25x^2$	$5x^3 = 5 \cdot x \cdot x \cdot x$ $25x^2 = 5 \cdot 5 \cdot x \cdot x$ $GCF = 5 \cdot x \cdot x$ $GCF = 5x^2$
	$5x^3-25x^2$
Rewrite each term.	$5x^2 \cdot x - 5x^2 \cdot 5$
Factor the GCF.	$5x^2(x-5)$
Check.	
$5x^2(x-5)$	
$5x^2 \cdot x - 5x^2 \cdot 5$	
$5x^3-25x^2$✓	

? Try It 3.1.26

Factor: $2x^3+12x^2$

Answer

$2x^2(x+6)$

? Try It 3.1.27

Factor: $6y^3-15y^2$

Answer

$3y^2(2y-5)$

Example 3.1.28

Factor: $21x^3 - 9x^2 + 15x$

Solution

In a previous example we found the GCF of $21x^3, 9x^2, 15x$ to be $3x$.

	$21x^3 - 9x^2 + 15x$
Rewrite each term using the GCF, $3x$.	$3x \cdot 7x^2 - 3x \cdot 3x + 3x \cdot 5$
Factor the GCF.	$3x(7x^2 - 3x + 5)$
Check.	
$3x\left(7x^2 - 3x + 5\right)$	
$3x \cdot 7x^2 - 3x \cdot 3x + 3x \cdot 5$	
$21x^3 - 9x^2 + 15x$✓	

Try It 3.1.29

Factor: $20x^3 - 10x^2 + 14x$

Answer

$2x(10x^2 - 5x + 7)$

Try It 3.1.30

Factor: $24y^3 - 12y^2 - 20y$

Answer

$4y(6y^2 - 3y - 5)$

Example 3.1.31

Factor: $8m^3 - 12m^2n + 20mn^2$

Solution

Find the GCF of $8m^3, 12m^2n, 20mn^2$	$8m^3 = 2 \cdot 2 \cdot 2 \cdot m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot 5 \cdot m \cdot n \cdot n$ $GCF = 2 \cdot 2 \cdot m$ $GCF = 4m$
	$8m^3 - 12m^2n + 20mn^2$
Rewrite each term.	$4m \cdot 2m^2 - 4m \cdot 3m\,n + 4m \cdot 5n^2$
Factor the GCF.	$4m(2m^2 - 3m\,n + 5n^2)$
Check.	
$4m\left(2m^2 - 3mn + 5n^2\right)$	
$4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$	

$8m^3 - 12m^2n + 20mn^2$ ✓

Try It 3.1.32

Factor: $9xy^2 + 6x^2y^2 + 21y^3$

Answer

$3y^2(3x + 2x^2 + 7y)$

Try It 3.1.33

Factor: $3p^3 - 6p^2q + 9pq^3$

Answer

$3p(p^2 - 2pq + 3q^2$

When the leading coefficient is negative, we factor the negative out as part of the GCF.

Example 3.1.34

Factor: $-8y - 24$

Solution

When the leading coefficient is negative, the GCF will be negative.

Ignoring the signs of the terms, we first find the GCF of 8*y* and 24 is 8. Since the expression −8*y* − 24 has a negative leading coefficient, we use −8 as the GCF.	$8y = 2 \cdot 2 \cdot 2 \cdot y$ $24 = 2 \cdot 2 \cdot 2 \cdot 3$ $GCF = 2 \cdot 2 \cdot 2$ $GCF = 8$
Rewrite each term using the GCF.	$-8y - 24$ $-8 \cdot y + (-8) \cdot 3$
Factor the GCF.	$-8(y + 3)$
Check.	
$-8(y + 3)$	
$-8 \cdot y + (-8) \cdot 3$	
$-8y - 24$ ✓	

Try It 3.1.35

Factor: $-16z - 64$

Answer

$-16(z + 4)$

Try It 3.1.36

Factor: $-9y-27$

Answer

$-9(y+3)$

Example 3.1.37

Factor: $-6a^2+36a$

Solution

The leading coefficient is negative, so the GCF will be negative.?

Since the leading coefficient is negative, the GCF is negative, $-6a$.	$6a^2 = 2 \cdot 3 \cdot a \cdot a$ $36a = 2 \cdot 2 \cdot 3 \cdot 3 \cdot a$ $GCF = 2 \cdot 3 \cdot a$ $GCF = 6a$ $-6a^2 + 36a$
Rewrite each term using the GCF.	$-6a \cdot a - (-6a) \cdot 6$
Factor the GCF.	$-6a(a - 6)$
Check.	
$-6a(a-6)$	
$-6a \cdot a + (-6a)(-6)$	
$-6a^2+36av$	

Try It 3.1.38

Factor: $-4b^2+16b$

Answer

$-4b(b-4)$

Try It 3.1.39

Factor: $-7a^2+21a$

Answer

$-7a(a-3)$

Example 3.1.40

Factor: $5q(q+7)-6(q+7)$

Solution

The GCF is the binomial q+7.

	$5q(q + 7) - 6(q + 7)$

Factor the GCF, $(q + 7)$.	$(q + 7)(5q - 6)$
Check on your own by multiplying.	

Try It 3.1.41

Factor: $4m(m+3)-7(m+3)$

Answer

$(m+3)(4m-7)$

Try It 3.1.42

Factor: $8n(n-4)+5(n-4)$

Answer

$(n-4)(8n+5)$

Factor by Grouping

When there is no common factor of all the terms of a polynomial, look for a common factor in just some of the terms. When there are four terms, a good way to start is by separating the polynomial into two parts with two terms in each part. Then look for the GCF in each part. If the polynomial can be factored, you will find a common factor emerges from both parts.

(Not all polynomials can be factored. Just like some numbers are prime, some polynomials are prime.)

Example 3.1.43

Factor: $xy+3y+2x+6$

Solution

Step 1. Group terms with common factors.	Is there a greatest common factor of all four terms? No, so let's separate the first two terms from the second two.	$xy + 3y + 2x + 6$ $\underline{xy + 3y}\underline{+ 2x + 6}$
Step 2. Factor out the common factor in each group.	Factor the GCF from the first two terms. Factor the GCF from the second two terms.	$y(x + 3)\underline{+ 2x + 6}$ $y(x + 3) + 2(x + 3)$
Step 3. Factor the common factor from the expression.	Notice that each term has a common factor of $(x + 3)$. Factor out the common factor.	$y(x + 3) + 2(x + 3)$ $(x + 3)\,(y + 2)$
Step 4. Check.	Multiply $(x + 3)(y + 2)$. Is the product the original expression?	$(x + 3)\,(y + 2)$ $xy + 2x + 3y + 6$ $xy + 3y + 2x + 6$ ✓

Try It 3.1.44

Factor: $xy+8y+3x+24$

Answer

$(x+8)(y+3)$

Try It 3.1.45

Factor: $ab+7b+8a+56$

Answer

$(a+7)(b+8)$

HOW TO

Factor by grouping.

Step 1. Group terms with common factors.

Step 2. Factor out the common factor in each group.

Step 3. Factor the common factor from the expression.

Step 4. Check by multiplying the factors.

Example 3.1.46

Factor: $x^2+3x-2x-6$

Solution

There is no GCF in all four terms. $x^2+3x-2x-6$

Separate into two parts. $\underbrace{x^2+3x}\ \underbrace{-2x-6}$

Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.

$x(x+3)-2(x+3)$

$(x+3)(x-2)$

Check on your own by multinlying.

Try It 3.1.47

Factor: $x^2+2x-5x-10$

Answer

$(x-5)(x+2)$

Try It 3.1.48

Factor: $y^2+4y-7y-28$

Answer

$(y+4)(y-7)$

MEDIA ACCESS ADDITIONAL ONLINE RESOURCES

Access these online resources for additional instruction and practice with greatest common factors (GFCs) and factoring by grouping.

- Greatest Common Factor (GCF)
- Factoring Out the GCF of a Binomial
- Greatest Common Factor (GCF) of Polynomials

Glossary

factoring

Factoring is splitting a product into factors; in other words, it is the reverse process of multiplying.

greatest common factor

The greatest common factor is the largest expression that is a factor of two or more expressions is the greatest common factor (GCF).

This page titled 3.1: Greatest Common Factor and Factor by Grouping is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **7.1: Greatest Common Factor and Factor by Grouping** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

3.2: Factor Quadratic Trinomials with Leading Coefficient 1

Learning Objectives

By the end of this section, you will be able to:

- Factor trinomials of the form x^2+bx+c
- Factor trinomials of the form $x^2+bxy+cy^2$

BE PREPARED

Before you get started, take this readiness quiz.

1. Multiply: (x+4)(x+5).
   If you missed this problem, review Example 6.3.31.
2. Simplify: (a) $-9+(-6)$ (b) $-9+6$.
   If you missed this problem, review Example 1.4.18.
3. Simplify: (a) $-9(6)$ (b) $-9(-6)$.
   If you missed this problem, review Example 1.5.1.
4. Simplify: (a) $|-5|$ (b) $|3|$.
   If you missed this problem, review Example 1.4.2.

Factor Trinomials of the Form x^2+bx+c

You have already learned how to multiply binomials using FOIL. Now you'll need to "undo" this multiplication—to start with the product and end up with the factors. Let's look at an example of multiplying binomials to refresh your memory.

$(x+2)(x+3)$ factors
F O I L
$x^2+3x+2x+6$
x^2+5x+6 product

To factor the trinomial means to start with the product, x^2+5x+6, and end with the factors, $(x+2)(x+3)$. You need to think about where each of the terms in the trinomial came from.

The *first term* came from multiplying the first term in each binomial. So to get x^2 in the product, each binomial must start with an *x*.

$$\begin{array}{l} x^2+5x+6 \\ (x\quad)(x\quad) \end{array} \tag{3.2.1}$$

The *last term* in the trinomial came from multiplying the last term in each binomial. So the last terms must multiply to 6.

What two numbers multiply to 6?

The factors of 6 could be 1 and 6, or 2 and 3. How do you know which pair to use?

Consider the *middle term*. It came from adding the outer and inner terms.

So the numbers that must have a product of 6 will need a sum of 5. We'll test both possibilities and summarize the results in Table 3.2.1—the table will be very helpful when you work with numbers that can be factored in many different ways.

Factors of 6	Sum of factors
1,6	$1+6=7$
2,3	$2+3=5$

Table 3.2.1

We see that 2 and 3 are the numbers that multiply to 6 and add to 5. So we have the factors of x^2+5x+6. They are $(x+2)(x+3)$.

$$\begin{array}{ll} x^2+5x+6 & \text{product} \\ (x+2)(x+3) & \text{factors} \end{array} \tag{3.2.2}$$

You should check this by multiplying.

Looking back, we started with x^2+5x+6, which is of the form x^2+bx+c, where b=5 and c=6. We factored it into two binomials of the form (x+m) and (x+n).

$$\begin{array}{ll} x^2+5x+6 & x^2+bx+c \\ (x+2)(x+3) & (x+m)(x+n) \end{array} \tag{3.2.3}$$

To get the correct factors, we found two numbers *m* and *n* whose product is *c* and sum is *b*.

Example 3.2.1: HOW TO FACTOR TRINOMIALS OF THE FORM x^2+bx+c

Factor: $x^2+7x+12$

Solution

Step 1. Write the factors as two binomials with first terms *x*.	Write two sets of parentheses and put *x* as the first term.	$x^2+7x+12$ $(x\quad)(x\quad)$
Step 2. Find two numbers *m and n* that multiply to *c*, $m \cdot n = c$ add to *b*, $m+n=b$	Find two numbers that multiply to 12 and add to 7. **Factors of 12** / **Sum of factors** 1, 12 / 1 + 12 = 13 2, 6 / 2 + 6 = 8 3, 4 / 3 + 4 = 7*	
Step 3. Use *m and n* as the last terms of the factors.	Use 3 and 4 as the last terms of the binomials.	$(x+3)(x+4)$
Step 4. Check by multiplying the factors.		$(x+3)(x+4)$ $x^2+4x+3x+12$ $x^2+7x+12$ ✓

Try It 3.2.2

Factor: x^2+6x+8

Try It 3.2.3

Factor: $y^2+8y+15$

Let's summarize the steps we used to find the factors.

HOW TO

Factor trinomials of the form x^2+bx+c.

Step 1. Write the factors as two binomials with first terms *x*: $(x\quad)(x\quad)$

Step 2. Find two numbers *m* and *n* that
Multiply to *c*, $m \cdot n = c$
Add to *b*, $m+n=b$

Step 3. Use *m* and *n* as the last terms of the factors: $(x+m)(x+n)$

Step 4. Check by multiplying the factors.

Example 3.2.4

Factor: $u^2+11u+24$

Solution

Notice that the variable is *u*, so the factors will have first terms *u*.

Write the factors as two binomials with first terms u .

$u^2+11u+24$
$(u\quad)(u\quad)$

Find two numbers that: multiply to 24 and add to 11.

Factors of 24	**Sum of factors**
1,24	1+24=25
2,12	2+12=14
3,8	3+8=11*
4,6	4+6=10

Use 3 and 8 as the last terms of the binomials. $(u+3)(u+8)$

Check.

$(u+3)(u+8)$
$u^2+3u+8u+24$
$u^2+11u+24v$✓

Try It 3.2.5

Factor: $q^2+10q+24$

Try It 3.2.6

Factor: $t^2+14t+24$

Example 3.2.7

Factor: $y^2+17y+60$

Solution

Write the factors as two binomials with first terms y.

$y^2+17y+60$
$(y\quad)(y\quad)$

Find two numbers that multiply to 60 and add to 17.

Factors of 60	**Sum of factors**
1,60	1+60=61
2,30	2+30=32
3,20	3+20=23
4,15	4+15=19

Factors of 60	Sum of factors
5,12	5+12=17*
6,10	

Use 5 and 12 as the last terms. $(y+5)(y+12)$
Check.

$(y+5)(y+12)$
$\left(y^2+12y+5y+60\right)$
$\left(y^2+17y+60\right)\checkmark$

Try It 3.2.8

Factor: $x^2+19x+60$

Try It 3.2.9

Factor: $v^2+23v+60$

Factor Trinomials of the Form $x^2 + bx + c$ with b Negative, c Positive

In the examples so far, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

Remember: To get a negative sum and a positive product, the numbers must both be negative.

Again, think about FOIL and where each term in the trinomial came from. Just as before,

- the first term, x^2, comes from the product of the two first terms in each binomial factor, *x* and *y*;
- the positive last term is the product of the two last terms
- the negative middle term is the sum of the outer and inner terms.

How do you get a *positive product* and a *negative sum*? With two negative numbers.

Example 3.2.10

Factor: $t^2-11t+28$

Solution

Again, with the positive last term, 28, and the negative middle term, −11t, we need two negative factors. Find two numbers that multiply 28 and add to −11.

Write the factors as two binomials with first terms t $t^2-11t+28$ $(t\quad)(t\quad)$

Find two numbers that: multiply to 28 and add to −11.

Factors of 28	Sum of factors
−1,−28	−1+(−28)=−29
−2,−14	−2+(−14)=−16
−4,−7	$-4+(-7)=-11^*$

Use $-4, -7$ as the last terms of the binomials. $(t-4)(t-7)$
Check.

$(t-4)(t-7)$
$t^2-7t-4t+28$
$t^2-11t+28$✓

Try It 3.2.11

Factor: $u^2-9u+18$

Try It 3.2.12

Factor: $y^2-16y+63$

Factor Trinomials of the Form x2+bx+c with *c* Negative

Now, what if the last term in the trinomial is negative? Think about FOIL. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

Remember: To get a negative product, the numbers must have different signs.

Example 3.2.13

Factor: z^2+4z-5

Solution

To get a negative last term, multiply one positive and one negative. We need factors of −5 that add to positive 4.

Factors of −5	Sum of factors
1,−5	1+(−5)=−4
−1,5	−1+5=4*

Notice: We listed both 1,−5 and −1,5 to make sure we got the sign of the middle term correct.

	z^2+4z-5
Factors will be two binomials with first terms z.	$(z\quad)(z\quad)$
Use $-1, 5$ as the last terms of the binomials.	$(z-1)(z+5)$
Check.	

$(z-1)(z+5)$
$z^2+5z-1z-5$
z^2+4z-5✓

Try It 3.2.14

Factor: $h^2+4h-12$

Answer

$(h-2)(h+6)$

Try It 3.2.15

Factor: : $2^2 + k - 20$

Answer

$(k-4)(k+5)$

Let's make a minor change to the last trinomial and see what effect it has on the factors.

Example 3.2.16

Factor: $z^2 - 4z - 5$

Solution

Solution

This time, we need factors of -5 that add to -4.

Factors of -5	Sum of factors
$1, -5$	$1+(-5)=-4$*
$-1, 5$	$-1+5=4$

	$z^2 - 4z - 5$
Factors will be two binomials with first terms z.	$(z \quad)(z \quad)$
Use $1, -5$ as the last terms of the binomials.	$(z+1)(z-5)$
Check.	

$(z+1)(z-5)$
$z^2 - 5z + 1z - 5$
$z^2 - 4z - 5$✓

Try It 3.2.17

Factor: $x^2 - 4x - 12$

Answer

$(x+2)(x-6)$

Try It 3.2.18

Factor: $y^2 - y - 20$

Answer

$(y+4)(y-5)$

Example 3.2.19

Factor: $q^2 - 2q - 15$

Solution

Factors will be two binomials with first terms q.	$q^2-2q-15$ $(q\quad)(q\quad)$
You can use $3,-5$ as the last terms of the binomials.	$(q+3)(q-5)$

Factors of −15	Sum of factors
1,−15	1+(−15)=−14
−1,15	−1+15=14
3,−5	3+(−5)=−2*
−3,5	

Check.

$(q+3)(q-5)$
$q^2-5q+3z-15$
$q^2-2q-15$✓

Try It 3.2.20

Factor: $r^2-3r-40$

Answer

$(r+5)(r-8)$

Try It 3.2.21

Factor: $s^2-3s-10$

Answer

$(s+2)(s-5)$

Some trinomials are prime. The only way to be certain a trinomial is prime is to list all the possibilities and show that none of them work.

Example 3.2.22

Factor: $y^2-6y+15$

Solution

Factors will be two binomials with first terms y.	$y^2-6y+15$ $(y\quad)(y\quad)$

Factors of 15	Sum of factors
−1,−15	−1+(−15)=−16
−3,−5	−3+(−5)=−8

As shown in the table, none of the factors add to −6; therefore, the expression is prime.

Try It 3.2.23

Factor: $m^2+4m+18$

Answer

prime

Try It 3.2.24

Factor: $n^2-10n+12$

Answer

prime

Example 3.2.25

Factor: $2x+x^2-48$

Solution

	$2x+x^2-48$
First we put the terms in decreasing degree order.	$x^2+2x-48$
Factors will be two binomials with first terms x .	$(x\quad)(x\quad)$

As shown in the table, you can use $-6,8$ as the last terms of the binomials.

$$(x-6)(x+8) \tag{3.2.4}$$

Factors of -48	Sum of factors
$-1,48$	$-1+48=47$
$-2,24$ $-3,16$ $-4,12$ $-6,8$	$-2+24=22$ $-3+16=13$ $-4+12=8$ $-6+8=2$

Check.

$(x-6)(x+8)$

$x^2-6q+8q-48$

$x^2+2x-48$✓

Try It 3.2.26

Factor: $9m+m^2+18$

Answer

$(m+3)(m+6)$

Try It 3.2.27

Factor: $-7n+12+n^2$

Answer

$(n-3)(n-4)$

Let's summarize the method we just developed to factor trinomials of the form x^2+bx+c

Note

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

$$\begin{array}{c} x^2+bx+c \\ (x+m)(x+n) \end{array} \tag{3.2.5}$$

When *c* is positive, *m* and *n* have the same sign.

$$\begin{array}{cc} \text{b positive} & \text{b negative} \\ m, n \text{ positive} & m, n \text{ negative} \\ x^2+5x+6 & x^2-6x+8 \\ (x+2)(x+3) & (x-4)(x-2) \\ \text{same signs} & \text{same signs} \end{array} \tag{3.2.6}$$

When *c* is negative, *m* and *n* have opposite signs.

$$\begin{array}{cc} x^2+x-12 & x^2-2x-15 \\ (x+4)(x-3) & (x-5)(x+3) \\ \text{opposite signs} & \text{opposite signs} \end{array} \tag{3.2.7}$$

Notice that, in the case when *m* and *n* have opposite signs, the sign of the one with the larger absolute value matches the sign of *b*.

Factor Trinomials of the Form $x^2 + bxy + cy^2$

Sometimes you'll need to factor trinomials of the form $x^2+bxy+cy^2$ with two variables, such as $x^2+12xy+36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x \cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain *y*. To get the coefficients *b* and *c*, you use the same process summarized in the previous objective.

Example 3.2.28

Factor: $x^2+12xy+36y^2$

Solution

	$x^2+12xy+36y^2$
Note that the first terms are x, last terms contain y	$(x\_y)\,(x\_y)$

Find the numbers that multiply to 36 and add to 12.

Factors of 36	Sum of factors
1, 36	1+36=37
2, 18	2+18=20
3, 12	3+12=15
4, 9	4+9=13
6, 6	6+6=12*

Use 6 and 6 as the coefficients of the last terms.	$(x+6y)(x+6y)$
Check your answer.	

$(x+6y)(x+6y)$
$x^2+6xy+6xy+36y^2$
$x^2+12xy+36y^2$ ✓

Try It 3.2.29

Factor: $u^2+11uv+28v^2$

Answer

$(u+4v)(u+7v)$

Try It 3.2.30

Factor: $x^2+13xy+42y^2$

Answer

$(x+6y)(x+7y)$

Example 3.2.31

Factor: $r^2-8rs-9s^2$

Solution

We need r in the first term of each binomial and s in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

	$r^2-8rs-9s^2$
Note that the first terms are r, last terms contain s	$(r\_s)\,(r\_s)$

Factors of −9	Sum of factors
1,−9	1+(−9)=−8*
−1,9	−1+9=8
3,−3	3+(−3)=0

Use $1,-9$ as coefficients of the last terms.	$(r+s)(r-9s)$

Check your answer.

$(r-9s)(r+s)$
$r^2+rs-9rs-9s^2$
$r^2-8rs-9s^2$ ✓

Try It 3.2.32

Factor: $a^2-11ab+10b^2$

Answer

$(a-b)(a-10b)$

Try It 3.2.33

Factor: $m^2 - 13mn + 12n^2$

Answer

$(m - n)(m - 12n)$

Example 3.2.34

Factor: $u^2 - 9uv - 12v^2$

Solution

We need *u* in the first term of each binomial and v in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

	$u^2 - 9uv - 12v^2$
Note that the first terms are u, last terms contain v	$(u\_v)\,(u\_v)$

Find the numbers that multiply to −12 and add to −9.

Factors of −12	Sum of factors
1,−12	1+(−12)=−11
−1,12	−1+12=11
2,−6	2+(−6)=−4
−2,6	−2+6=4
3,−4	3+(−4)=−1
−3,4	−3+4=1

Note there are no factor pairs that give us −9 as a sum. The trinomial is prime.

Try It 3.2.35

Factor: $x^2 - 7xy - 10y^2$

Answer

prime

Try It 3.2.36

Factor: $p^2 + 15pq + 20q^2$

Answer

prime

Key Concepts

- **Factor trinomials of the form $x^2 + bx + c$**
  1. Write the factors as two binomials with first terms x: $(x \quad)(x \quad)$
  2. Find two numbers m and n that
     Multiply to c, $m \cdot n = c$
     Add to b, $m + n = b$

3. Use m and n as the last terms of the factors: $(x+m)(x+n)$.
4. Check by multiplying the factors.

This page titled 3.2: Factor Quadratic Trinomials with Leading Coefficient 1 is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **7.2: Factor Quadratic Trinomials with Leading Coefficient 1** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

3.3: Factor Quadratic Trinomials with Leading Coefficient Other than 1

Learning Objectives

By the end of this section, you will be able to:

- Recognize a preliminary strategy to factor polynomials completely
- Factor trinomials of the form ax^2+bx+c with a GCF
- Factor trinomials using trial and error
- Factor trinomials using the 'ac' method

Note

Before you get started, take this readiness quiz.

1. Find the GCF of $45p^2$ and $30p^6$
   If you missed this problem, review Example 7.1.4.
2. Multiply $(3y+4)(2y+5)$
   If you missed this problem, review Example 6.3.37.
3. Combine like terms $12x^2+3x+5x+9$
   If you missed this problem, review Example 1.3.37.

Recognize a Preliminary Strategy for Factoring

Let's summarize where we are so far with factoring polynomials. In the first two sections of this chapter, we used three methods of factoring: factoring the GCF, factoring by grouping, and factoring a trinomial by "undoing" FOIL. More methods will follow as you continue in this chapter, as well as later in your studies of algebra.

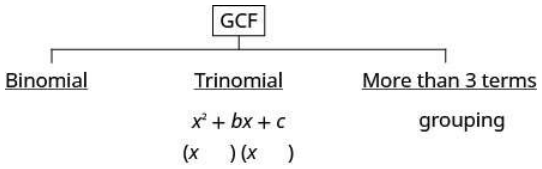
How will you know when to use each factoring method? As you learn more methods of factoring, how will you know when to apply each method and not get them confused? It will help to organize the factoring methods into a strategy that can guide you to use the correct method.

As you start to factor a polynomial, always ask first, "Is there a greatest common factor?" If there is, factor it first.

The next thing to consider is the type of polynomial. How many terms does it have? Is it a binomial? A trinomial? Or does it have more than three terms?

- If it is a trinomial where the leading coefficient is one, x^2+bx+c, use the "undo FOIL" method.
- If it has more than three terms, try the grouping method. This is the only method to use for polynomials of more than three terms.

Some polynomials cannot be factored. They are called "prime." Below we summarize the methods we have so far.



CHOOSE A STRATEGY TO FACTOR POLYNOMIALS COMPLETELY.

1. Is there a greatest common factor?
   - Factor it out.
2. Is the polynomial a binomial, trinomial, or are there more than three terms?
   - If it is a binomial, right now we have no method to factor it.
   - If it is a trinomial of the form x^2+bx+c: Undo FOIL $(x\quad)(x\quad)$
   - If it has more than three terms: Use the grouping method.

3. Check by multiplying the factors.

Use the preliminary strategy to completely factor a polynomial. A polynomial is factored completely if, other than monomials, all of its factors are prime.

Example 3.3.1

Identify the best method to use to factor each polynomial.

a. $6y^2-72$
b. $r^2-10r-24$
c. $p^2+5p+pq+5q$

Answer a

	$6y^2-72$
Is there a greatest common factor?	Yes, 6.
Factor out the 6	$6\left(y^2-12\right)$
Is it a binomial, trinomial, or are there more than 3 terms?	Binomial, we have no method to factor binomials yet.

Answer b

	$r^2-10r-24$
Is there a greatest common factor?	No, there is no common factor.
Is it a binomial, trinomial, or are there more than three terms?	Trinomial, with leading coefficient 1, so "undo" FOIL.

Answer c

	$p^2+5p+pq+5q$
Is there a greatest common factor?	No, there is no common factor.
Is it a binomial, trinomial, or are there more than three terms?	More than three terms, so factor using grouping.

Try It 3.3.2

Identify the best method to use to factor each polynomial:

a. $4y^2+32$
b. $y^2+10y+21$
c. $yz+2y+3z+6$

Answer a

no method

Answer b

undo using FOIL

Answer c

factor with grouping

Try It 3.3.3

Identify the best method to use to factor each polynomial:

a. $ab+a+4b+4$

b. $3k^2+15$
c. p^2+9p+8

Answer a

factor using grouping

Answer b

no method

Answer c

undo using FOIL

Factor Trinomials of the form $ax^2 + bx + c$ with a GCF

Now that we have organized what we've covered so far, we are ready to factor trinomials whose leading coefficient is not 1, trinomials of the form ax^2+bx+c . Remember to always check for a GCF first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods in the last section. Let's do a few examples to see how this works. Watch out for the signs in the next two examples.

Example 3.3.4

Factor completely: $2n^2-8n-42$.

Solution

Use the preliminary strategy.

Is there a greatest common factor?	$2n^2-8n-42$
Yes, GCF $=2$. Factor it out.	$2\left(n^2-4n-21\right)$
Inside the parentheses, is it a binomial, trinomial, or are there more than three terms?	
It is a trinomial whose coefficient is 1, so undo FOIL.	$2(n\quad)(n\quad)$
Use 3 and -7 as the last terms of the binomials.	$2(n+3)(n-7)$

Factors of −21	Sum of factors
1,−21	1+(−21)=−20
3,−7	3+(−7)=−4*

Check.
$2(n+3)(n-7)$
$2\left(n^2-7n+3n-21\right)$
$2\left(n^2-4n-21\right)$
$2n^2-8n-42$✓

Try It 3.3.5

Factor completely: $4m^2-4m-8$

Answer

$4(m+1)(m-2)$

Try It 3.3.6

Factor completely: $5k^2-15k-50$

Answer

$5(k+2)(k-5)$

Example 3.3.7

Factor completely: $4y^2-36y+56$

Solution

Use the preliminary strategy.

Is there a greatest common factor?	$4y^2-36y+56$
Yes, GCF $=4$. Factor it out.	$4\left(y^2-9y+14\right)$
Inside the parentheses, is it a binomial, trinomial, or are there more than three terms?	
It is a trinomial whose coefficient is 1, so undo FOIL.	$4(y\quad)(y\quad)$
Use a table like the one below to find two numbers that multiply to 14 and add to -9	
Both factors of 14 must be negative.	$4(y-2)(y-7)$

Factors of 14	Sum of factors
−1,−14	−1+(−14)=−15
−2,−7	−2+(−7)=−9*

Check.

$4(y-2)(y-7)$

$4\left(y^2-7y-2y+14\right)$

$4\left(y^2-9y+14\right)$

$4y^2-36y+42$✓

Try It 3.3.8

Factor completely: $3r^2-9r+6$

Answer

$3(r-1)(r-2)$

Try It 3.3.9

Factor completely: $2t^2-10t+12$

Answer

$2(t-2)(t-3)$

In the next example the GCF will include a variable.

Example 3.3.10

Factor completely: $4u^3+16u^2-20u$

Solution

Use the preliminary strategy.

Is there a greatest common factor?	$4u^3+16u^2-20u$
Yes, GCF $=4u$. Factor it.	$4u\left(u^2+4u-5\right)$
Binomial, trinomial, or more than three terms? more than three terms?	
It is a trinomial. So "undo FOIL."	$4u(u\quad)(u\quad)$
Use a table like the table below to find two numbers that multiply to -5 and add to 4	$4u(u-1)(u+5)$

Factors of −5	Sum of factors
−1,5	−1+5=4*
1,−5	1+(−5)=−4

Check.

$4u(u-1)(u+5)$

$4u\left(u^2+5u-u-5\right)$

$4u\left(u^2+4u-5\right)$

$4u^3+16u^2-20u$✓

Try It 3.3.11

Factor completely: $5x^3+15x^2-20x$

Answer

$5x(x-1)(x+4)$

Try It 3.3.12

Factor completely: $6y^3+18y^2-60y$

Answer

$6y(y-2)(y+5)$

Factor Trinomials using Trial and Error

What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let's factor the trinomial $3x^2+5x+2$

From our earlier work we expect this will factor into two binomials.

$$\begin{array}{c} 3x^2+5x+2 \\ (\quad)(\quad) \end{array} \tag{3.3.1}$$

We know the first terms of the binomial factors will multiply to give us $3x^2$. The only factors of $3x^2$ are $1x, 3x$. We can place them in the binomials.

$3x^2 + 5x + 2$
1x, 3x
$(x\ \ )(3x\ \ )$

Check. Does $1x \cdot 3x = 3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1 and 2. But we now have two cases to consider as it will make a difference if we write 1, 2, or 2, 1.

$3x^2 + 5x + 2$ (1x, 3x; 1, 2): $(x + 1)(3x + 2)$ or $3x^2 + 5x + 2$ (1x, 3x; 1, 2): $(x + 2)(3x + 1)$

Which factors are correct? To decide that, we multiply the inner and outer terms.

$3x^2 + 5x + 2$ (1x, 3x; 1, 2): $(x + 1)(3x + 2)$ (3x, 2x, 5x) or $3x^2 + 5x + 2$ (1x, 3x; 1, 2): $(x + 2)(3x + 1)$ (6x, 1x, 7x)

Since the middle term of the trinomial is $5x$, the factors in the first case will work. Let's FOIL to check.

$$\begin{aligned} &(x+1)(3x+2) \\ &3x^2+2x+3x+2 \\ &3x^2+5x+2\checkmark \end{aligned} \tag{3.3.2}$$

Our result of the factoring is:

$$\begin{aligned} &3x^2+5x+2 \\ &(x+1)(3x+2) \end{aligned} \tag{3.3.3}$$

Example 3.3.13: How to Factor Trinomials of the Form $ax^2 + bx + c$ Using Trial and Error

Factor completely: $3y^2+22y+7$

Solution

Step 1. Write the trinomial in descending order.	The trinomial is already in descending order.	$3y^2 + 22y + 7$
Step 2. Find all the factor pairs of the first term.	The only factors of $3y^2$ are $1y$, $3y$ Since there is only one pair, we can put them in the parentheses.	$3y^2 + 22y + 7$ (1y, 3y) $3y^2 + 22y + 7$ (1y, 3y) $(y\ \ )(3y\ \ )$

Step 3. Find all the factor pairs of the third term.	The only factors of 7 are 1, 7.	$3y^2 + 22y + 7$ 1y, 3y 1, 7 $(y \quad)(3y \quad)$
Step 4. Test all the possible combinations of the factors until the correct product is found.	$3y^2 + 22y + 7$ 1y, 3y 1, 7 $(y + 1)(3y + 7)$ $3y$ $7y$ $10y$ No. We need $22y$ $3y^2 + 22y + 7$ 1y, 3y 1, 7 $(y + 7)(3y + 1)$ $21y$ y $22y$	$3y^2 + 22y + 7$ Possible factors / Product $(y + 1)(3y + 7)$ / $3y^2 + 10y + 7$ $(y + 7)(3y + 1)$ / $3y^2 + 22y + 7$ $(y + 7)(3y + 1)$
Step 5. Check by multiplying.		$(y + 7)(3y + 1)$ $3y^2 + 22y + 7$ ✓

Try It 3.3.14

Factor completely: $2a^2 + 5a + 3$

Answer

$(a + 1)(2a + 3)$

Try It 3.3.15

Factor completely: $4b^2 + 5b + 1$

Answer

$(b + 1)(4b + 1)$

FACTOR TRINOMIALS OF THE FORM $ax^2 + bx + c$ USING TRIAL AND ERROR.

1. Write the trinomial in descending order of degrees.
2. Find all the factor pairs of the first term.
3. Find all the factor pairs of the third term.
4. Test all the possible combinations of the factors until the correct product is found.
5. Check by multiplying.

When the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

Example 3.3.16

Factor completely: $6b^2 - 13b + 5$

Solution

The trinomial is already in descending order.	$6b^2 - 13b + 5$
Find the factors of the first term.	$6b^2 - 13b + 5$ $1b \cdot 6b$ $2b \cdot 3b$

Find the factors of the last term. Consider the signs. Since the last term, 5 is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.	$6b^2 - 13b + 5$ $1b \cdot 6b$ $\quad -1, -5$ $2b \cdot 3b$

Consider all the combinations of factors.

$6b^2 - 13b + 5$

Possible factors	Product
(b−1)(6b−5)	$6b^2 - 11b + 5$
(b−5)(6b−1)	$6b^2 - 31b + 5$
(2b−1)(3b−5)	$6b^2 - 13b + 5$ *
(2b−5)(3b−1)	$6b^2 - 17b + 5$

The correct factors are those whose product is the original trinomial. $(2b - 1)(3b - 5)$

Check by multiplying.

$(2b - 1)(3b - 5)$
$6b^2 - 10b - 3b + 5$
$6b^2 - 13b + 5v$✓

Try It 3.3.17

Factor completely: $8x^2 - 14x + 3$

Answer

$(2x - 3)(4x - 1)$

Try It 3.3.18

Factor completely: $10y^2 - 37y + 7$

Answer

$(2y - 7)(5y - 1)$

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

Example 3.3.19

Factor completely: $14x^2 - 47x - 7$

Solution

The trinomial is already in descending order.	$14x^2 - 47x - 7$
Find the factors of the first term.	$14x^2 - 47x - 7$ $1x \cdot 14x$ $2x \cdot 7x$

Find the factors of the last term. Consider the signs. Since it is negative, one factor must be positive and one negative.	$14x^2 - 47x - 7$ $1x \cdot 14x$ $\quad$ $1, -7$ $2x \cdot 7x$ $\quad$ $-1, 7$

Consider all the combinations of factors. We use each pair of the factors of $14x^2$ with each pair of factors of -7.

Factors of $14x^2$	Pair with	Factors of -7
$x, 14x$		$11, -7$ $-7, 11$ (reverse order)
$x, 14x$		$-1, 77$ $77, -1$ (reverse order)
$2x, 7x$		$11, -7$ $-7, 11$ (reverse order)
$2x, 7x$		$-1, 77$ $77, -1$ (reverse order)

These pairings lead to the following eight combinations.

$14x^2 - 47x - 7$	
Possible factors	**Product**
$(x + 1)(14x - 7)$	Not an option
$(x - 7)(14x + 1)$	$14x^2 - 97x - 7$
$(x - 1)(14x + 7)$	Not an option
$(x + 7)(14x - 1)$	$14x^2 + 97x - 7$
$(2x + 1)(7x - 7)$	Not an option
$(2x - 7)(7x + 1)$	$14x^2 - 47x - 7$*
$(2x - 1)(7x + 7)$	Not an option
$(2x + 7)(7x - 1)$	$14x^2 + 47x - 7$

If the trinomial has no common factors, then neither factor can contain a common factor. That means each of these combinations is not an option.

The correct factors are those whose product is the original trinomial. $(2x - 7)(7x + 1)$

Check by multiplying.

$(2b - 1)(3b - 5)$
$6b^2 - 10b - 3b + 5$
$6b^2 - 13b + 5$✓

? Try It 3.3.20

Factor completely: $8a^2 - 3a - 5$

Answer

$(a - 1)(8a + 5)$

? Try It 3.3.21

Factor completely: $6b^2 - b - 15$

Answer

$(2b + 3)(3b - 5)$

Example 3.3.22

Factor completely: $18n^2 - 37n + 15$

Solution

The trinomial is already in descending order.	$18n^2 - 37n + 15$
Find the factors of the first term.	$18n^2 - 37n + 15$ $1n \cdot 18n$ $2n \cdot 9n$ $3n \cdot 6n$
Find the factors of the last term. Consider the signs. Since 15 is positive and the coefficient of the middle term is negative, we use the negative factors.	$18n^2 - 37n + 15$ $1n \cdot 18n$ $\quad -1(-15)$ $2n \cdot 9n$ $\quad -3(-5)$ $3n \cdot 6n$

Consider all the combinations of factors.

$18n^2 - 37n + 15$	
Possible factors	**Product**
$(n-1)(18n-15)$	Not an option
$(n-15)(18n-1)$	$18n^2 - 271n + 15$
$(n-3)(18n-5)$	$18n^2 - 59n + 15$
$(n-5)(18n-3)$	Not an option
$(2n-1)(9n-15)$	Not an option
$(2n-15)(9n-1)$	$18n^2 - 137n + 15$
$(2n-3)(9n-5)$	$18n^2 - 37n + 15$*
$(2n-5)(9n-3)$	Not an option
$(3n-1)(6n-15)$	Not an option
$(3n-15)(6n-1)$	Not an option
$(3n-3)(6n-5)$	Not an option
$(3n-5)(6n-3)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. $(2n-3)(9n-5)$

Check by multiplying.

$(2n-3)(9n-5)$
$18n^2 - 10n - 27n + 15$
$18n^2 - 37n + 15$✓

Try It 3.3.23

Factor completely: $18x^2 - 3x - 10$

Answer

$(3x+2)(6x-5)$

Try It 3.3.24

Factor completely: $30y^2 - 53y - 21$

Answer

$(3y+1)(10y-21)$

Don’t forget to look for a GCF first.

Example 3.3.25

Factor completely: $10y^4+55y^3+60y^2$

Solution

	$10y^4+55y^3+60y^2$
Notice the greatest common factor, and factor it first.	$5y^2\left(2y^2+11y+12\right)$
Factor the trinomial.	$5y^2(2y^2+11y+12)$ $y\cdot 2y$ $\quad$ $1\cdot 12$, $2\cdot 6$, $3\cdot 4$

Consider all the combinations.

$2y^2+11y+12$	
Possible factors	**Product**
$(y+1)(2y+12)$	Not an option
$(y+12)(2y+1)$	$2y^2+25y+12$
$(y+2)(2y+6)$	Not an option
$(y+6)(2y+2)$	Not an option
$(y+3)(2y+4)$	Not an option
$(y+4)(2y+3)$	$2y^2+11y+12$*

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. Remember to include the factor $5y^2$ $\quad 5y^2(y+4)(2y+3)$

Check by multiplying.

$5y^2(y+4)(2y+3)$
$5y^2\left(2y^2+8y+3y+12\right)$
$10y^4+55y^3+60y^2$ ✓

Try It 3.3.26

Factor completely: $15n^3-85n^2+100n$

Answer

$5n(n-4)(3n-5)$

Try It 3.3.27

Factor completely: $56q^3+320q^2-96q$

Answer

$8q(q+6)(7q-2)$

Factor Trinomials using the “ac” Method

Another way to factor trinomials of the form ax^2+bx+c is the “ac” method. (The “ac” method is sometimes called the grouping method.) The “ac” method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

Example 3.3.28: How to Factor Trinomials Using the “ac” Method

Factor: $6x^2+7x+2$

Solution

Step 1. Factor any GCF.	Is there a greatest common factor? No.	$6x^2+7x+2$
Step 2. Find the product ac.	$a \cdot c$ $6 \cdot 2$ 12	ax^2+bx+c $6x^2+7x+2$
Step 3. Find two numbers m and n that: Multiply to ac $m \cdot n = a \cdot c$ Add to b $m+n=b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12$ $3+4=7$	
Step 4. Split the middle term using m, and n ax^2+bx+c $ax^2+\underbrace{mx+nx}_{bx}+c$	Rewrite $7x$ as $3x+4x$. Notice that $6x^2+3x+4x+2$ is equal to $6x^2+7x+2$. We just split the middle term to get a more useful form.	$6x^2+7x+2$ $6x^2+\underbrace{3x+4x}_{7x}+2$
Step 5. Factor by grouping.		$3x(2x+1)+2(2x+1)$ $(2x+1)(3x+2)$
Step 6. Check by multiplying.		$(2x+1)(3x+2)$ $6x^2+4x+3x+2$ $6x^2+7x+2$ ✓

Try It 3.3.29

Factor: $6x^2+13x+2$

Answer

$(x+2)(6x+1)$

Try It 3.3.30

Factor: $4y^2+8y+3$

Answer

$(2y+1)(2y+3)$

FACTOR TRINOMIALS OF THE FORM USING THE “AC” METHOD.

1. Factor any GCF.
2. Find the product ac.
3. Find two numbers m and n that:
   Multiply to ac $m \cdot n = a \cdot c$
   Add to b $m+n=b$

4. Split the middle term using m and n: ax^2+bx+c → $ax^2+mx+nx+c$
5. Factor by grouping.
6. Check by multiplying the factors.

When the third term of the trinomial is negative, the factors of the third term will have opposite signs.

Example 3.3.31

Factor: $8u^2-17u-21$

Solution

		ax^2+bx+c $8u^2-17u-21$
Is there a greatest common factor? No.		
Find $a\cdot c$	$a\cdot c$	
	$8(-21)$	
	-168	

Find two numbers that multiply to -168 and add to -17. The larger factor must be negative.

Factors of -168	Sum of factors
$1,-168$	$1+(-168)=-167$
$2,-84$	$2+(-84)=-82$
$3,-56$	$3+(-56)=-53$
$4,-42$	$4+(-42)=-38$
$6,-28$	$6+(-28)=-22$
$7,-24$	$7+(-24)=-17$*
$8,-21$	$8+(-21)=-13$

Split the middle term using $7u$ and $-24u$	$8u^2-17u-21$ $\underbrace{8u^2+7u}\ \underbrace{-24u-21}$
Factor by grouping.	$u(8u+7)-3(8u+7)$ $(8u+7)(u-3)$
Check by multiplying.	$(8u+7)(u-3)$ $8u^2-24u+7u-21$ $8u^2-17u-21$✓

Try It 3.3.32

Factor: $20h^2+13h-15$

Answer

$(4h-5)(5h+3)$

Try It 3.3.33

Factor: $6g^2+19g-20$

Answer

$(q+4)(6q-5)$

Example 3.3.34

Factor: $2x^2+6x+5$

Solution

Is there a greatest common factor? No.	ax^2+bx+c $2x^2+6x+5$
Find a·c	a·c
	2(5)
	10

Find two numbers that multiply to 10 and add to 6.

Factors of 10	Sum of factors
1,10	1+10=11
2, 5	2+5=7

There are no factors that multiply to 10 and add to 6. The polynomial is prime.

Try It 3.3.35

Factor: $10t^2+19t-15$

Answer

$(2t+5)(5t-3)$

Try It 3.3.36

Factor: $3u^2+8u+5$

Answer

$(u+1)(3u+5)$

Don’t forget to look for a common factor!

Example 3.3.37

Factor: $10y^2-55y+70$

Solution

Is there a greatest common factor? Yes. The GCF is 5.	$10y^2-55y+70$
Factor it. Be careful to keep the factor of 5 all the way through the solution!	$5(2y^2-11y+14)$

The trinomial inside the parentheses has a leading coefficient that is not 1.	$ax^2 + bx + c$ $5(2y^2 - 11y + 14)$
Factor the trinomial.	$5(y-2)(2y-7)$
Check by multiplying all three factors.	
$5(2y^2 - 2y - 4y + 14)$	
$5(2y^2 - 11y + 14)$	
$10y^2 - 55y + 70$ ✓	

? Try It 3.3.38

Factor: $16x^2 - 32x + 12$

Answer

$4(2x-3)(2x-1)$

? Try It 3.3.39

Factor: $18w^2 - 39w + 18$

Answer

$3(3w-2)(2w-3)$

We can now update the Preliminary Factoring Strategy, as shown in Figure 3.3.1 and detailed in Choose a strategy to factor polynomials completely (updated), to include trinomials of the form $ax^2 + bx + c$. Remember, some polynomials are prime and so they cannot be factored.

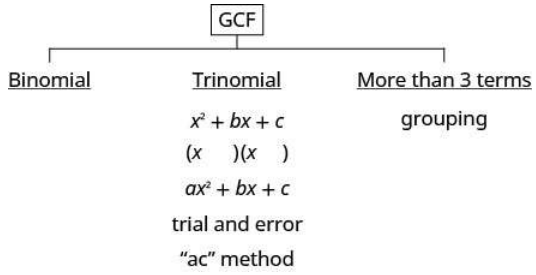


Figure 3.3.1

CHOOSE A STRATEGY TO FACTOR POLYNOMIALS COMPLETELY (UPDATED).

1. Is there a greatest common factor?
   - Factor it.
2. Is the polynomial a binomial, trinomial, or are there more than three terms?
   - If it is a binomial, right now we have no method to factor it.
   - If it is a trinomial of the form $x^2 + bx + c$
     Undo FOIL \(x)(x).
   - If it is a trinomial of the form $ax^2 + bx + c$
     Use Trial and Error or the "ac" method.
   - If it has more than three terms
     Use the grouping method.
3. Check by multiplying the factors.

Note

Access these online resources for additional instruction and practice with factoring trinomials of the form ax^2+bx+c

- Factoring Trinomials, a is not 1

Key Concepts

- **Factor Trinomials of the Form ax^2+bx+c using Trial and Error:** See Example.
  1. Write the trinomial in descending order of degrees.
  2. Find all the factor pairs of the first term.
  3. Find all the factor pairs of the third term.
  4. Test all the possible combinations of the factors until the correct product is found.
  5. Check by multiplying.
- **Factor Trinomials of the Form ax^2+bx+c Using the "ac" Method:** See Example.
  1. Factor any GCF.
  2. Find the product ac.
  3. Find two numbers *m* and *n* that: $\begin{array}{ll}\text{Multiply to } ac & m\cdot n=a\cdot c\\ \text{Add to } b & m+n=b\end{array}$
  4. Split the middle term using *m* and *n*:
     ax^2+bx+c
     $ax^2+mx+nx+c$
  5. Factor by grouping.
  6. Check by multiplying the factors.
- **Choose a strategy to factor polynomials completely (updated):**
  1. Is there a greatest common factor? Factor it.
  2. Is the polynomial a binomial, trinomial, or are there more than three terms?
     If it is a binomial, right now we have no method to factor it.
     If it is a trinomial of the form x^2+bx+c
     Undo FOIL $(x\quad)(x\quad)$.
     If it is a trinomial of the form ax^2+bx+c
     Use Trial and Error or the "ac" method.
     If it has more than three terms
     Use the grouping method.
  3. Check by multiplying the factors.

Glossary

prime polynomials

Polynomials that cannot be factored are prime polynomials.

This page titled 3.3: Factor Quadratic Trinomials with Leading Coefficient Other than 1 is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **7.3: Factor Quadratic Trinomials with Leading Coefficient Other than 1** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

3.4: Factor Special Products

Learning Objectives

By the end of this section, you will be able to:

- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes
- Choose method to factor a polynomial completely

Note

Before you get started, take this readiness quiz.

1. Simplify: $(12x)^2$
   If you missed this problem, review Example 6.2.22.
2. Multiply: $(m+4)^2$
   If you missed this problem, review Example 6.4.1.
3. Multiply: $(p-9)^2$
   If you missed this problem, review Example 6.4.4.
4. Multiply: $(k+3)(k-3)$
   If you missed this problem, review Example 6.4.16.

The strategy for factoring we developed in the last section will guide you as you factor most binomials, trinomials, and polynomials with more than three terms. We have seen that some binomials and trinomials result from special products—squaring binomials and multiplying conjugates. If you learn to recognize these kinds of polynomials, you can use the special products patterns to factor them much more quickly.

Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself. You can square a binomial by using FOIL, but using the Binomial Squares pattern you saw in a previous chapter saves you a step. Let's review the Binomial Squares pattern by squaring a binomial using FOIL.

$$(3x+4)^2$$
$$(3x+4)(3x+4)$$
$$\underset{9x^2}{F} + \underset{12x}{O} + \underset{12x}{I} + \underset{16}{L}$$
$$9x^2+24x+16$$

The first term is the square of the first term of the binomial and the last term is the square of the last. The middle term is twice the product of the two terms of the binomial.

$$\begin{aligned}(3x)^2+2(3x\cdot 4)+4^2\\ 9x^2+24x+16\end{aligned} \tag{3.4.1}$$

The trinomial $9x^2+24+16$ is called a perfect square trinomial. It is the square of the binomial 3*x*+4.

We'll repeat the Binomial Squares Pattern here to use as a reference in factoring.

BINOMIAL SQUARES PATTERN

If *a* and *b* are real numbers,

$$(a+b)^2=a^2+2ab+b^2 \qquad (a-b)^2=a^2-2ab+b^2 \tag{3.4.2}$$

When you square a binomial, the product is a perfect square trinomial. In this chapter, you are learning to factor—now, you will start with a perfect square trinomial and factor it into its prime factors.

You could factor this trinomial using the methods described in the last section, since it is of the form ax^2+bx+c. But if you recognize that the first and last terms are squares and the trinomial fits the **perfect square trinomials pattern**, you will save yourself a lot of work.

Here is the pattern—the reverse of the binomial squares pattern.

PERFECT SQUARE TRINOMIALS PATTERN

If a and b are real numbers,

$$a^2+2ab+b^2=(a+b)^2 \qquad a^2-2ab+b^2=(a-b)^2 \tag{3.4.3}$$

To make use of this pattern, you have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square, a^2. Next check that the last term is a perfect square, b^2. Then check the middle term—is it twice the product, $2ab$? If everything checks, you can easily write the factors.

Example 3.4.1: How to Factor Perfect Square Trinomials

Factor: $9x^2+12x+4$

Solution

Step 1. Does the trinomial fit the perfect square trinomials pattern, $a^2+2ab+b^2$?		
• Is the first term a perfect square? Write it as a square, a^2.	Is $9x^2$ a perfect square? Yes—write it as $(3x)^2$.	$9x^2+12x+4$ $(3x)^2$
• Is the last term a perfect square? Write it as a square, b^2.	Is 4 a perfect square? Yes—write it as $(2)^2$.	$(3x)^2 \qquad (2)^2$
• Check the middle term. Is it $2ab$?	Is $12x$ twice the product of $3x$ and 2? Does it match? Yes, so we have a perfect square trinomial!	$(3x)^2 \qquad (2)^2$ $2(3x)(2)$ $12x$
Step 2. Write the square of the binomial.	Write it as the square of a binomial.	$9x^2+12x+4$ $a^2 + 2 \cdot a \cdot b + b^2$ $(3x)^2+2\cdot 3x\cdot 2+2^2$ $(a+b)^2$ $(3x+2)^2$
Step 3. Check. $(3x+2)^2$ $(3x)^2+2\cdot 3x\cdot 2+2^2$ $9x^2+12x+4$ ✓		

Try It 3.4.2

Factor: $4x^2+12x+9$

Answer

$(2x+3)^2$

? Try It 3.4.3

Factor: $9y^2+24y+16$

Answer

$(3y+4)^2$

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2-2ab+b^2$, which factors to $(a-b)^2$.

The steps are summarized here.

FACTOR PERFECT SQUARE TRINOMIALS.

Step 1. Does the trinomial fit the pattern? $a^2+2ab+b^2$ $a^2-2ab+b^2$

- Is the first term a perfect square? $(a)^2$ $(a)^2$
  Write it as a square.
- Is the last term a perfect square? $(a)^2$ $(b)^2$ $(a)^2$ $(b)^2$
  Write it as a square.
- Check the middle term. Is it $2ab$? $(a)^2$ $\searrow 2\cdot a\cdot b \swarrow$ $(b)^2$ $(a)^2$ $\searrow 2\cdot a\cdot b \swarrow$ $(b)^2$

Step 2. Write the square of the binomial. $(a+b)^2$ $(a-b)^2$

Step 3. Check by multiplying.

We'll work one now where the middle term is negative.

✓ Example 3.4.4

Factor: $81y^2-72y+16$

Solution

The first and last terms are squares. See if the middle term fits the pattern of a perfect square trinomial. The middle term is negative, so the binomial square would be $(a-b)^2$.

	$81y^2-72y+16$
Are the first and last terms perfect squares?	$(9y)^2$ $(4)^2$
Check the middle term.	$(9y)^2$ $(4)^2$ $2(9y)(4)$ $72y$
Does is match $(a-b)^2$? Yes.	a^2 -2 a $b+b^2$ $(9y)^2-2\cdot 9y\cdot 4+4^2$
Write the square of a binomial.	$(9y-4)^2$
Check by multiplying.	
$(9y-4)^2$	
$(9y)^2-2\cdot 9y\cdot 4+4^2$	
$81y^2-72y+16$✓	

Try It 3.4.5

Factor: $64y^2 - 80y + 25$

Answer

$(8y - 5)^2$

Try It 3.4.6

Factor: $16z^2 - 72z + 81$

Answer

$(4z - 9)^2$

The next example will be a perfect square trinomial with two variables.

Example 3.4.7

Factor: $36x^2 + 84xy + 49y^2$

Solution

	$36x^2 + 84xy + 49y^2$
Test each term to verify the pattern.	$a^2 \quad +2 \quad a \quad b \quad + \quad b^2$ $(6x)^2 + 2 \cdot 6x \cdot 7y + (7y)^2$
Factor.	$(6x + 7y)^2$
Check by multiplying.	
$(6x + 7y)^2$	
$(6x)^2 + 2 \cdot 6x \cdot 7y + (7y)^2$	
$36x^2 + 84xy + 49y^2$ ✓	

Try It 3.4.8

Factor: $49x^2 + 84xy + 36y^2$

Answer

$(7x + 6y)^2$

Try It 3.4.9

Factor: $64m^2 + 112mn + 49n^2$

Answer

$(8m + 7n)^2$

Example 3.4.10

Factor: $9x^2 + 50x + 25$

Solution

	$9x^2+50x+25$
Are the first and last terms perfect squares?	$(3x)^2 \qquad (5)^2$
Check the middle term-is it 2ab?	$(3x)^2 \searrow_{2(3x)(5)} \swarrow (5)^2.$
	$30x$
No! $30x \neq 50x$	This does not fit the pattern!
Factor using the "ac" method.	$9x^2+50x+25$
ac	
Notice: $9 \cdot 25$ and $5 \cdot 45 = 225$	
225	
	$9x^2+5x+45x+25$
Split the middle term.	$x(9x+5)+5(9x+5)$
	$(9x+5)(x+5)$
Factor by grouping.	
Check.	
$(9x+5)(x+5)$	
$9x^2+45x+5x+25$	
$9x^2+50x+25$✓	

? Try It 3.4.11

Factor: $16r^2+30rs+9s^2$

Answer

$(8r+3s)(2r+3s)$

? Try It 3.4.12

Factor: $9u^2+87u+100$

Answer

$(3u+4)(3u+25)$

Remember the very first step in our Strategy for Factoring Polynomials? It was to ask "is there a greatest common factor?" and, if there was, you factor the GCF before going any further. Perfect square trinomials may have a GCF in all three terms and it should be factored out first. And, sometimes, once the GCF has been factored, you will recognize a perfect square trinomial.

✓ Example 3.4.13

Factor: $36x^2y-48xy+16y$

Solution

	$36x^2y-48xy+16y$
Is there a GCF? Yes, $4y$, so factor it out.	$4y\left(9x^2-12x+4\right)$
Is this a perfect square trinomial?	
Verify the pattern.	$4y[(3x)^2-2 \cdot 3x \cdot 2+2^2]$ (a^2 -2 a $b+b^2$)
Factor.	$4y(3x-2)^2$
Remember: Keep the factor $4y$ in the final product.	

Check.
$4y(3x-2)^2$
$4y[(3x)^2-2\cdot 3x\cdot 2+2^2]$
$4y(9x)^2-12x+4$
$36x^2y-48xy+16y$✓

? Try It 3.4.14

Factor: $8x^2y-24xy+18y$

Answer

$2y(2x-3)^2$

? Try It 3.4.15

Factor: $27p^2q+90pq+75q$

Answer

$3q(3p+5)^2$

Factor Differences of Squares

The other special product you saw in the previous was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Here's an example:

$$\begin{array}{c}(3x-4)(3x+4)\\ 9x^2-16\end{array} \tag{3.4.4}$$

Remember, when you multiply conjugate binomials, the middle terms of the product add to 0. All you have left is a binomial, the difference of squares.

Multiplying conjugates is the only way to get a binomial from the product of two binomials.

PRODUCT OF CONJUGATES PATTERN

If *a* and *b* are real numbers

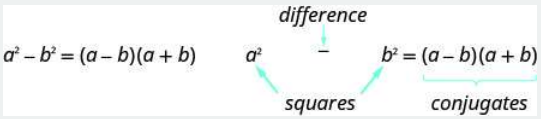
$$(a-b)(a+b)=a^2-b^2 \tag{3.4.5}$$

The product is called a difference of squares.

To factor, we will use the product pattern "in reverse" to factor the difference of squares. A **difference of squares** factors to a product of conjugates.

DIFFERENCE OF SQUARES PATTERN

If *a* and *b* are real numbers,



Remember, “difference” refers to subtraction. So, to use this pattern you must make sure you have a binomial in which two squares are being subtracted.

Example 3.4.16: How to Factor Differences of Squares

Factor: $x^2 - 4$

Solution

Step 1. Does the binomial fit the pattern?		$x^2 - 4$
• Is this a difference?	Yes	$x^2 - 4$
• Are the first and last terms perfect squares?	Yes	
Step 2. Write them as squares.	Write them as x^2 and 2^2.	$a^2 - b^2$ $(x)^2 - 2^2$
Step 3. Write the product of conjugates.		$(a - b)(a + b)$ $(x - 2)(x + 2)$
Step 4. Check.		$(x - 2)(x + 2)$ $x^2 - 4$ ✓

Try It 3.4.17

Factor: $h^2 - 81$

Answer

$(h-9)(h+9)$

Try It 3.4.18

Factor: $k^2 - 121$

Answer

$(k-11)(k+11)$

FACTOR DIFFERENCES OF SQUARES.

Step 1. Does the binomial fit the pattern? $a^2 - b^2$
- Is this a difference? $\_\_ - \_\_$
- Are the first and last terms perfect squares?

Step 2. Write them as squares. $(a)^2 - (b)^2$

Step 3. Write the product of conjugates. $(a-b)(a+b)$

Step 4. Check by multiplying.

It is important to remember that *sums of squares do not factor into a product of binomials*. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression $a^2 + b^2$ is prime!

Don’t forget that 1 is a perfect square. We’ll need to use that fact in the next example.

Example 3.4.19

Factor: $64y^2 - 1$

Solution

	$64y^2 - 1$
Is this a difference? Yes.	$64y^2 - 1$
Are the first and last terms perfect squares?	
Yes - write them as squares.	$a^2 \quad - b^2$ $(8y)^2 - 1^2$
Factor as the product of conjugates.	$(a \quad - b)\ (a \quad + b)$ $(8y - 1)(8y + 1)$
Check by multiplying.	
$(8y - 1)(8y + 1)$	
$64y^2 - 1$✓	

Try It 3.4.20

Factor: $m^2 - 1$

Answer

$(m - 1)(m + 1)$

Try It 3.4.21

Factor: $81y^2 - 1$

Answer

$(9y - 1)(9y + 1)$

Example 3.4.22

Factor: $121x^2 - 49y^2$

Solution

	$121x^2 - 49y^2$
Is this a difference of squares? Yes.	$(11x)^2 - (7y)^2$
Factor as the product of conjugates.	$(11x - 7y)(11x + 7y)$
Check by multiplying.	
$(11x - 7y)(11x + 7y)$	
$121x^2 - 49y^2$✓	

Try It 3.4.23

Factor: $196m^2 - 25n^2$

Answer

$(16m - 5n)(16m + 5n)$

Try It 3.4.24

Factor: $144p^2 - 9q^2$

Answer

$(12p - 3q)(12p + 3q)$

The binomial in the next example may look "backwards," but it's still the difference of squares.

Example 3.4.25

Factor: $100 - h^2$

Solution

	$100 - h^2$
Is this a difference of squares? Yes.	$(10)^2 - (h)^2$
Factor as the product of conjugates.	$(10 - h)(10 + h)$

Check by multiplying.

$(10 - h)(10 + h)$

$100 - h^2 \checkmark$

Be careful not to rewrite the original expression as $h^2 - 100$.

Factor $h^2 - 100$ on your own and then notice how the result differs from $(10 - h)(10 + h)$.

Try It 3.4.26

Factor: $144 - x^2$

Answer

$(12 - x)(12 + x)$

Try It 3.4.27

Factor: $169 - p^2$

Answer

$(13 - p)(13 + p)$

To completely factor the binomial in the next example, we'll factor a difference of squares twice!

Example 3.4.28

Factor: $x^4 - y^4$

Solution

Is this a difference of squares? Yes.	x^4-y^4
Factor it as the product of conjugates.	$\left(x^2\right)^2-\left(y^2\right)^2$
Notice the first binomial is also a difference of squares!	$\left(x^2-y^2\right)\left(x^2+y^2\right)$
Factor it as the product of conjugates. The last factor, the sum of squares, cannot be factored.	$(x-y)(x+y)\left(x^2+y^2\right)$

Check by multiplying.

$(x-y)(x+y)\left(x^2+y^2\right)$

$[(x-y)(x+y)]\left(x^2+y^2\right)$

$\left(x^2-y^2\right)\left(x^2+y^2\right)$

x^4-y^4✓

Try It 3.4.29

Factor: a^4-b^4

Answer

$\left(a^2+b^2\right)(a+b)(a-b)$

Try It 3.4.30

Factor: x^4-16

Answer

$\left(x^2+4\right)(x+2)(x-2)$

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may "disguise" the difference of squares and you won't recognize the perfect squares until you factor the GCF.

Example 3.4.31

Factor: $8x^2y-98y$

Solution

	$8x^2y-98y$
Is there a GCF? Yes, $2y-$ factor it out!	$2y\left(4x^2-49\right)$
Is the binomial a difference of squares? Yes.	$2y\left((2x)^2-(7)^2\right)$
Factor as a product of conjugates.	$2y(2x-7)(2x+7)$
Check by multiplying.	

$2y(2x-7)(2x+7)$

$2y[(2x-7)(2x+7)]$

$2y\left(4x^2-49\right)$

$8x^2y-98y$✓

Try It 3.4.32

Factor: $7xy^2 - 175x$

Answer

$7x(y-5)(y+5)$

Try It 3.4.33

Factor: $45a^2b - 80b$

Answer

$5b(3a-4)(3a+4)$

Example 3.4.34

Factor: $6x^2 + 96$

Solution

	$6x^2 + 96$
Is there a GCF? Yes, 6 — factor it out!	$6\left(x^2 + 16\right)$
Is the binomial a difference of squares? No, it is a sum of squares. Sums of squares do not factor!	
Check by multiplying.	

$6\left(x^2 + 16\right)$

$6x^2 + 96$✓

Try It 3.4.35

Factor: $8a^2 + 200$

Answer

$8\left(a^2 + 25\right)$

Try It 3.4.36

Factor: $36y^2 + 81$

Answer

$9\left(4y^2 + 9\right)$

Factor Sums and Differences of Cubes

There is another special pattern for factoring, one that we did not use when we multiplied polynomials. This is the pattern for the sum and difference of cubes. We will write these formulas first and then check them by multiplication.

$$\begin{aligned} a^3 + b^3 &= (a+b)\left(a^2 - ab + b^2\right) \\ a^3 - b^3 &= (a-b)\left(a^2 + ab + b^2\right) \end{aligned}$$

We'll check the first pattern and leave the second to you.

	$(a+b)\left(a^2-ab+b^2\right)$
Distribute.	$a\left(a^2-ab+b^2\right)+b\left(a^2-ab+b^2\right)$
Multiply.	$a^3-a^2b+ab^2+a^2b-ab^2+b^3$
Combine like terms.	a^3+b^3

SUM AND DIFFERENCE OF CUBES PATTERN

$$\begin{aligned} a^3+b^3&=(a+b)\left(a^2-ab+b^2\right)\\ a^3-b^3&=(a-b)\left(a^2+ab+b^2\right)\end{aligned} \tag{3.4.6}$$

The two patterns look very similar, don't they? But notice the signs in the factors. The sign of the binomial factor matches the sign in the original binomial. And the sign of the middle term of the trinomial factor is the opposite of the sign in the original binomial. If you recognize the pattern of the signs, it may help you memorize the patterns.

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ — same sign; opposite signs

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ — same sign; opposite signs

The trinomial factor in the **sum and difference of cubes pattern** cannot be factored.

It can be very helpful if you learn to recognize the cubes of the integers from 1 to 10, just like you have learned to recognize squares. We have listed the cubes of the integers from 1 to 10 in Figure 3.4.1.

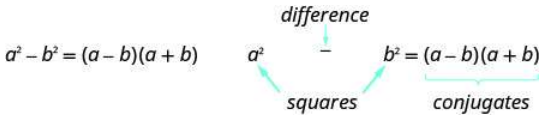


Figure 3.4.1

Example 3.4.37: How to Factor the Sum or Difference of Cubes

Factor: x^3+64

Solution

Step 1. Does the binomial fit the sum or difference of cubes pattern?		$x^3 + 64$
• Is it a sum or difference?	This is a sum.	$x^3 + 64$
• Are the first and last terms perfect cubes?	Yes	
Step 2. Write the terms as cubes.	Write them as x^3 and 4^3	$a^3 + b^3$ $x^3 + 4^3$
Step 3. Use either the sum or difference of cubes pattern.	This is a sum of cubes.	$(a + b)(a^2 - ab + b^2)$ $(x + 4)(x^2 - 4x + 4^2)$

Step 4. Simplify inside the parentheses.	Simplify 4^2.	$(x+4)(x^2-4x+16)$
Step 5. Check by multiplying the factors.		$\begin{array}{r} x^2-4x+16 \\ x+4 \\ \hline 4x^2-16x+64 \\ x^3-4x^2+16x \\ \hline x^3 \qquad +64 \checkmark \end{array}$

Try It 3.4.38

Factor: x^3+27

Answer

$(x+3)\left(x^2-3x+9\right)$

Try It 3.4.39

Factor: y^3+8

Answer

$(y+2)\left(y^2-2y+4\right)$

FACTOR THE SUM OR DIFFERENCE OF CUBES.

To factor the sum or difference of cubes:

1. Does the binomial fit the sum or difference of cubes pattern?
   - Is it a sum or difference?
   - Are the first and last terms perfect cubes?
2. Write them as cubes.
3. Use either the sum or difference of cubes pattern.
4. Simplify inside the parentheses
5. Check by multiplying the factors.

Example 3.4.40

Factor: : x^3-1000

Solution

	x^3-1000
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\begin{array}{c} a^3 - b^3 \\ x^3 - 10^3 \end{array}$
Use the difference of cubes pattern.	$\begin{array}{c} a - b \\ x - 10 \end{array} \left(\begin{array}{c} a^2 + ab + b^2 \\ x^2 + 10 \cdot x + 10^2 \end{array} \right)$
Simplify.	$\begin{array}{c} a - b \\ x - 10 \end{array} \left(\begin{array}{c} a^2 + ab + b^2 \\ x^2 + 10x + 100 \end{array} \right)$
Check by multiplying.	

$(x-10)\ \ (x^2+10x+100)$

$$\begin{array}{r} x^2+10x+100 \\ \times\quad x-100 \\ \hline x^3+10x^2+100x \\ -10x^2-100x-1000\ \checkmark \\ \hline x^3\qquad\qquad -1000 \end{array}$$

Try It 3.4.41

Factor: u^3-125

Answer

$(u-5)\left(u^2+5u+25\right)$

Try It 3.4.42

Factor: v^3-343

Answer

$(v-7)\left(v^2+7v+49\right)$

Be careful to use the correct signs in the factors of the sum and difference of cubes.

Example 3.4.43

Factor: $512-125p^3$

Solution

	$512-125p^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{8^3}-\overset{b^3}{(5p)^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{8}-\overset{b}{5p}\right)\left(\overset{a^2}{8^2}+\overset{ab}{8\cdot 5p}+\overset{b^2}{(5p)^2}\right)$
Simplify.	$\left(\overset{a}{8}-\overset{b}{5p}\right)\left(\overset{a^2}{64}+\overset{ab}{40p}+\overset{b^2}{25p^2}\right)$
Check by multiplying.	We'll leave the check to you.

Try It 3.4.44

Factor: $64-27x^3$

Answer

$(4-3x)\left(16+12x+9x^2\right)$

Try It 3.4.45

Factor: $27-8y^3$

Answer

$(3-2y)\left(9+6y+4y^2\right)$

Example 3.4.46

Factor: $27u^3-125v^3$

Solution

	$27u^3-125v^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$(3u)^3-(5v)^3$
Use the difference of cubes pattern.	$(3u-5v)\left((3u)^2+3u\cdot 5v+(5v)^2\right)$
Simplify.	$(3u-5v)\left(9u^2+15uv+25v^2\right)$
Check by multiplying.	We'll leave the check to you.

Try It 3.4.47

Factor: $8x^3-27y^3$

Answer

$(2x-3y)\left(4x^2+6xy+9y^2\right)$

Try It 3.4.48

Factor: $1000m^3-125n^3$

Answer

$(10m-5n)\left(100m^2+50mn+25n^2\right)$

In the next example, we first factor out the GCF. Then we can recognize the sum of cubes.

Example 3.4.49

Factor: $5m^3+40n^3$

Solution

	$5m^3+40n^3$
Factor the common factor.	$5(m^3+8n^3)$
This binomial is a sum. The first and last terms are perfect cubes.	

Write the terms as cubes.	$5\left(m^3+(2n)^3\right)$
Use the sum of cubes pattern.	$5(m+2n)\left(m^2-m\cdot 2n+(2n)^2\right)$
Simplify.	$5(m+2n)\left(m^2-2mn+4n^2\right)$

Check. To check, you may find it easier to multiply the sum of cubes factors first, then multiply that product by 5. We'll leave the multiplication for you.

$5(m+2n)\left(m^2-2mn+4n^2\right)$

Try It 3.4.50

Factor: $500p^3+4q^3$

Answer

$4(5p+q)\left(25p^2-5pq+q^2\right)$

Try It 3.4.51

Factor: $432c^3+686d^3$

Answer

$2(6c+7d)\left(36c^2-42cd+49d^2\right)$

Note

Access these online resources for additional instruction and practice with factoring special products.

- Sum of Difference of Cubes
- Difference of Cubes Factoring

Key Concepts

- **Factor perfect square trinomials** See Example.

  Step 1 . Does the trinomial fit the pattern? $a^2+2ab+b^2$ $a^2-2ab+b^2$

  - Is the first term a perfect square? $(a)^2$ $(a)^2$

    Write it as a square.

  - Is the last term a perfect square? $(a)^2$ $(b)^2$ $(a)^2$ $(b)^2$

    Write it as a square.

  - Check the middle term. Is it $2ab$? $(a)^2$ $\searrow 2\cdot a\cdot b \swarrow$ $(b)^2$ $(a)^2$ $\searrow 2\cdot a\cdot b \swarrow$ $(b)^2$

  Step 2. Write the square of the binomial. $(a+b)^2$ $(a-b)^2$

  Step 3. Check by multiplying.

- **Factor differences of squares** See Example.

  Step 1. Does the binomial fit the pattern? a^2-b^2

  - Is this a difference? $\_\_-\_\_$
  - Are the first and last terms perfect squares?

  Step 2. Write them as squares. $(a)^2-(b)^2$

  Step 3. Write the product of conjugates. $(a-b)(a+b)$

  Step 4. Check by multiplying.

- **Factor sum and difference of cubes** To factor the sum or difference of cubes: See Example.

  1. Does the binomial fit the sum or difference of cubes pattern? Is it a sum or difference? Are the first and last terms perfect cubes?

2. Write them as cubes.
3. Use either the sum or difference of cubes pattern.
4. Simplify inside the parentheses
5. Check by multiplying the factors.

Glossary

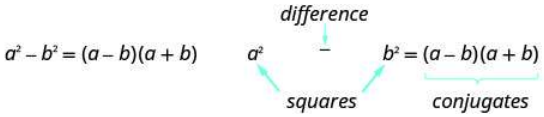
perfect square trinomials pattern

If a and b are real numbers,

$$a^2+2ab+b^2=(a+b)^2 \qquad a^2-2ab+b^2=(a-b)^2$$

difference of squares pattern

If a and b are real numbers,



sum and difference of cubes pattern

$$a^3+b^3=(a+b)(a^2-ab+b^2) \qquad a^3-b^3=(a-b)(a^2+ab+b^2)$$

This page titled 3.4: Factor Special Products is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **7.4: Factor Special Products** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

3.5: General Strategy for Factoring Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Recognize and use the appropriate method to factor a polynomial completely

Note

Before you get started, take this readiness quiz.

1. Factor $y^2 - 2y - 24$.
   If you missed this problem, review Example 7.2.19.
2. Factor $3t^2 + 17t + 10$.
   If you missed this problem, review Example 7.3.28.
3. Factor $36p^2 - 60p + 25$.
   If you missed this problem, review Example 7.4.1.
4. Factor $5x^2 - 80$.
   If you missed this problem, review Example 7.4.31.

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

You have now become acquainted with all the methods of factoring that you will need in this course. (In your next algebra course, more methods will be added to your repertoire.) The figure below summarizes all the factoring methods we have covered. Figure 3.5.1 outlines a strategy you should use when factoring polynomials.

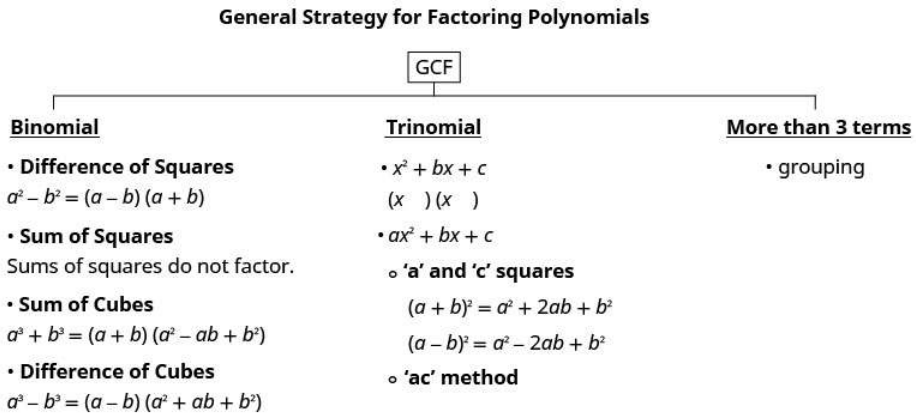


Figure 3.5.1

FACTOR POLYNOMIALS.

1. Is there a greatest common factor?
   - Factor it out.
2. Is the polynomial a binomial, trinomial, or are there more than three terms?
   - If it is a binomial:
     Is it a sum?
     - Of squares? Sums of squares do not factor.
     - Of cubes? Use the sum of cubes pattern.

     Is it a difference?
     - Of squares? Factor as the product of conjugates.
     - Of cubes? Use the difference of cubes pattern.

- If it is a trinomial:
  Is it of the form x^2+bx+c? ? Undo FOIL.
  Is it of the form ax^2+bx+c ?
  - If aa and cc are squares, check if it fits the trinomial square pattern.
  - Use the trial and error or “ac” method.
- If it has more than three terms:
  Use the grouping method.

3. Check.
   - Is it factored completely?
   - Do the factors multiply back to the original polynomial?

Remember, a polynomial is completely factored if, other than monomials, its factors are prime!

Example 3.5.1

Factor completely: $4x^5+12x^4$

Solution

Is there a GCF?	Yes, $4x^4$	$4x^5+12x^4$
Factor out the GCF.		$4x^4(x+3)$
In the parentheses, is it a binomial, a trinomial, or are there more than three terms?	Binomial.	
Is it a sum?		Yes.
Of squares? Of cubes?		No.
Check.		
Is the expression factored completely?		Yes.
Multiply.		

$4x^4(x+3)$
$4x^4\cdot x+4x^4\cdot 3$
$4x^5+12x^4$✓

Try It 3.5.2

Factor completely: $3a^4+18a^3$

Answer

$3a^3(a+6)$

Try It 3.5.3

Factor completely: $45b^6+27b^5$

Answer

$9b^5(5b+3)$

Example 3.5.4

Factor completely: $12x^2-11x+2$

Solution

		$12x^2 - 11x + 2$
Is there a GCF?	No.	
Is it a binomial, trinomial, or are there more than three terms?	Trinomial.	
Are a and c perfect squares?	No, $a = 12$, not a perfect square.	
Use trial and error or the "ac" method. We will use trial and error here.		$12x^2 - 11x + 2$ $1x, 12x$ $\quad -1, -2$ $2x, 6x$ $3x, 4x$

$12x^2 - 11x + 2$	
Possible factors	**Product**
$(x - 1)(12x - 2)$	Not an option
$(x - 2)(12x - 1)$	$12x^2 - 25x + 2$
$(2x - 1)(6x - 2)$	Not an option
$(2x - 2)(6x - 1)$	Not an option
$(3x - 1)(4x - 2)$	Not an option
$(3x - 2)(4x - 1)$	$12x^2 - 11x + 2$

If the trinomial has no common factors, then neither factor can contain a common factor. That means each of these combinations is not an option.

$(3x - 2)(4x - 1)$

Check. $12x^2 - 3x - 8x + 2$

$12x^2 - 11x + 2$✓

Try It 3.5.5

Factor completely: $10a^2 - 17a + 6$

Answer

$(5a - 6)(2a - 1)$

Try It 3.5.6

Factor completely: $8x^2 - 18x + 9$

Answer

$(2x - 3)(4x - 3)$

Example 3.5.7

Factor completely: $g^3 + 25g$

Solution

Is there a GCF?	Yes, g.	g^3+25g
Factor out the GCF.		$g\left(g^2+25\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum? Of squares?	Yes.	Sums of squares are prime.
Check.		
Is the expression factored completely?	Yes.	
Multiply.		
$g\left(g^2+25\right)$		
g^3+25g✓		

Try It 3.5.8

Factor completely: x^3+36x

Answer

$x\left(x^2+36\right)$

Try It 3.5.9

Factor completely: $27y^2+48$

Answer

$3\left(9y^2+16\right)$

Example 3.5.10

Factor completely: $12y^2-75$

Solution

Is there a GCF?	Yes, 3.	$12y^2-75$
Factor out the GCF.		$3\left(4y^2-25\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum?	No.	
Is it a difference? Of squares or cubes?	Yes, squares.	$3\left((2y)^2-(5)^2\right)$
Write as a product of conjugates.		$3(2y-5)(2y+5)$
Check.		
Is the expression factored completely?	Yes.	
Neither binomial is a difference of squares.		
Multiply.		
$3(2y-5)(2y+5)$		
$3\left(4y^2-25\right)$		
$12y^2-75$✓		

Try It 3.5.11

Factor completely: $16x^3 - 36x$

Answer

$4x(2x-3)(2x+3)$

Try It 3.5.12

Factor completely: $27y^2 - 48$

Answer

$3(3y-4)(3y+4)$

Example 3.5.13

Factor completely: $4a^2 - 12ab + 9b^2$

Solution

Is there a GCF?	No.	$4a^2 - 12ab + 9b^2$
Is it a binomial, trinomial, or are there more terms?		
Trinomial with $a \neq 1$. But the first term is a perfect square.		
Is the last term a perfect square?	Yes.	$(2a)^2 - 12ab + (3b)^2$
Does it fit the pattern, $a^2 - 2ab + b^2$?	Yes.	$(2a)^2 - 12ab + (3b)^2$ $-2(2a)(3b)$ $12ab$
Write it as a square.		$(2a - 3b)^2$
Check your answer.		
Is the expression factored completely?		
Yes.		
The binomial is not a difference of squares.		
Multiply.		
$(2a-3b)^2$		
$(2a)^2 - 2 \cdot 2a \cdot 3b + (3b)^2$		
$4a^2 - 12ab + 9b^2$ ✓		

Try It 3.5.14

Factor completely: $4x^2 + 20xy + 25y^2$

Answer

$(2x+5y)^2$

Try It 3.5.15

Factor completely: $9m^2+42mn+49n^2$

Answer

$(3m+7n)^2$

Example 3.5.16

Factor completely: $6y^2-18y-60$

Solution

Is there a GCF?	Yes, 6.	$6y^2-18y-60$
Factor out the GCF.	Trinomial with leading coefficient 1	$6\left(y^2-3y-10\right)$
In the parentheses, is it a binomial, trinomial, or are there more terms?		
"Undo' FOIL.	$6(y\quad)(y\quad)$	$6(y+2)(y-5)$
Check your answer.		
Is the expression factored completely?		Yes.
Neither binomial is a difference of squares.		
Multiply.		

$$6(y+2)(y-5)$$
$$6\left(y^2-5y+2y-10\right)$$
$$6\left(y^2-3y-10\right)$$
$$6y^2-18y-60\checkmark$$

Try It 3.5.17

Factor completely: $8y^2+16y-24$

Answer

$8(y-1)(y+3)$

Try It 3.5.18

Factor completely: $5u^2-15u-270$

Answer

$5(u-9)(u+6)$

Example 3.5.19

Factor completely: $24x^3+81$

Solution

Is there a GCF?	Yes, 3.	$24x^3+81$
Factor it out.		$3\left(8x^3+27\right)$

In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum or difference?	Sum.	
Of squares or cubes?	Sum of cubes.	$3\left((2x)^3+(3)^3\right)$
Write it using the sum of cubes pattern.		$3\left(2x+3\right)\left((2x)^2-2x\cdot 3+3^2\right)$
Is the expression factored completely?	Yes.	$3(2x+3)\left(4x^2-6x+9\right)$
Check by multiplying.		We leave the check to you.

Try It 3.5.20

Factor completely: $250m^3+432$

Answer

$2(5m+6)\left(25m^2-30m+36\right)$

Try It 3.5.21

Factor completely: $81q^3+192$

Answer

$3(3q+4)\left(9q^2-12q+16\right)$

Example 3.5.22

Factor completely: $2x^4-32$

Solution

Is there a GCF?	Yes, 2.	$2x^4-32$
Factor out the GCF.		$2\left(x^4-16\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum or difference?	Yes.	
Of squares or cubes?	Difference of squares.	$2\left(\left(x^2\right)^2-(4)^2\right)$
Write it as a product of conjugates.		$2\left(x^2-4\right)\left(x^2+4\right)$
The first binomial is again a difference of squares.		$2\left((x)^2-(2)^2\right)\left(x^2+4\right)$
Write it as a product of conjugates.		$2(x-2)(x+2)\left(x^2+4\right)$
Is the expression factored completely?	Yes.	

None of these binomials is a difference of squares.
Check your answer.
Multiply.

$$2(x-2)(x+2)\left(x^2+4\right)$$
$$2(x-2)(x+2)\left(x^2+4\right)$$
$$2(x-10)$$
$$2x^4-32\checkmark$$

? Try It 3.5.23

Factor completely: $4a^4-64$

Answer

$4\left(a^2+4\right)(a-2)(a+2)$

? Try It 3.5.24

Factor completely: $7y^4-7$

Answer

$7\left(y^2+1\right)(y-1)(y+1)$

✓ Example 3.5.25

Factor completely: $3x^2+6bx-3ax-6ab$

Solution

Is there a GCF?	Yes, 3.	$3x^2+6bx-3ax-6ab$
Factor out the GCF.		$3\left(x^2+2bx-ax-2ab\right)$
In the parentheses, is it a binomial, trinomial, or are there more terms?	More than 3 terms.	
Use grouping.		$3[x(x+2b)-a(x+2b)]$
		$3(x+2b)(x-a)$
Check your answer.		

Is the expression factored completely? Yes.

Multiply.

$3(x+2b)(x-a)$

$3\left(x^2-ax+2bx-2ab\right)$

$3x^2-3ax+6bx-6ab$✓

? Try It 3.5.26

Factor completely: $6x^2-12xc+6bx-12bc$

Answer

$6(x+b)(x-2c)$

? Try It 3.5.27

Factor completely: $16x^2+24xy-4x-6y$

Answer

$2(4x-1)(x+3y)$

✓ Example 3.5.28

Factor completely: $10x^2-34x-24$

Solution

Is there a GCF?	Yes, 2.	$10x^2-34x-24$
Factor out the GCF.		$2\left(5x^2-17x-12\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Trinomial with $a\neq 1$	
Use trial and error or the "ac" method.		$2\left(5x^2-17x-12\right)$
		$2(5x+3)(x-4)$
Check your answer. Is the expression factored completely? Yes.		

Multiply.

$2(5x+3)(x-4)$

$2\left(5x^2-20x+3x-12\right)$

$2\left(5x^2-17x-12\right)$

$10x^2-34x-24$✓

? Try It 3.5.29

Factor completely: $4p^2 - 16p + 12$

Answer

$4(p-1)(p-3)$

? Try It 3.5.30

Factor completely: $6q^2 - 9q - 6$

Answer

$3(q-2)(2q+1)$

Key Concepts

- **General Strategy for Factoring Polynomials See Figure 3.5.1.**
- **How to Factor Polynomials**
  1. Is there a greatest common factor? Factor it out.
  2. Is the polynomial a binomial, trinomial, or are there more than three terms?
     - If it is a binomial:
       Is it a sum?
       - Of squares? Sums of squares do not factor.
       - Of cubes? Use the sum of cubes pattern.

       Is it a difference?
       - Of squares? Factor as the product of conjugates.
       - Of cubes? Use the difference of cubes pattern.
     - If it is a trinomial:
       Is it of the form $x^2 + bx + c$? Undo FOIL.
       Is it of the form $ax^2 + bx + c$?
       - If 'a' and 'c' are squares, check if it fits the trinomial square pattern.
       - Use the trial and error or 'ac' method.
     - If it has more than three terms:
       Use the grouping method.
  3. Check. Is it factored completely? Do the factors multiply back to the original polynomial?

This page titled 3.5: General Strategy for Factoring Polynomials is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **7.5: General Strategy for Factoring Polynomials** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

CHAPTER OVERVIEW

4: Quadratic Equations

4.1: Quadratic Equations and Solving by Factoring
4.2: Solve Quadratic Equations Using the Square Root Property
4.3: Solve Quadratic Equations Using the Quadratic Formula
4.4: Solve Applications Modeled by Quadratic Equations

4: Quadratic Equations is shared under a CC BY-SA license and was authored, remixed, and/or curated by LibreTexts.

4.1: Quadratic Equations and Solving by Factoring

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations by using the Zero Product Property
- Solve quadratic equations factoring
- Solve applications modeled by quadratic equations

Note

Before you get started, take this readiness quiz.

1. Solve: $5y-3=0$.
   If you missed this problem, review Example 2.3.1.
2. Solve: $10a=0$.
   If you missed this problem, review Example 2.2.1.
3. Combine like terms: $12x^2-6x+4x$.
   If you missed this problem, review Example 1.3.37.
4. Factor n^3-9n^2-22n completely.
   If you missed this problem, review Example 7.3.10.

We have already solved linear equations, equations of the form $ax+by=c$. In linear equations, the variables have no exponents. Quadratic equations are equations in which the variable is squared. Listed below are some examples of quadratic equations:

$$x^2+5x+6=0 \quad 3y^2+4y=10 \quad 64u^2-81=0 \quad n(n+1)=42 \tag{4.1.1}$$

The last equation doesn't appear to have the variable squared, but when we simplify the expression on the left we will get n^2+n.

The general form of a quadratic equation is $ax^2+bx+c=0$,with $a\neq 0$.

QUADRATIC EQUATION

An equation of the form $ax^2+bx+c=0$ is called a quadratic equation.

$$a, b, \text{ and } c \text{ are real numbers and } a\neq 0 \tag{4.1.2}$$

To solve **quadratic equations** we need methods different than the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

Solve Quadratic Equations Using the Zero Product Property

We will first solve some quadratic equations by using the Zero Product Property. The **Zero Product Property** says that if the product of two quantities is zero, it must be that at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

ZERO PRODUCT PROPERTY

If $a\cdot b=0$, then either $a=0$ or $b=0$ or both.

We will now use the **Zero Product Property**, to solve a quadratic equation.

Example 4.1.1: How to Use the Zero Product Property to Solve a Quadratic Equation

Solve: $(x+1)(x-4)=0$

Solution

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(x+1)(x-4)=0$ $x+1=0$ or $x-4=0$
Step 2. Solve the linear equations.	Solve each equation.	$x=-1$ or $x=4$

Step 3. Check.	Substitute each solution separately into the original equation.	$x=-1$ $(x+1)(x-4)=0$ $(-1+1)(-1-4)\stackrel{?}{=}0$ $(0)(-5)\stackrel{?}{=}0$ $0=0$ ✓ $x=4$ $(x+1)(x-4)=0$ $(4+1)(4-4)\stackrel{?}{=}0$ $(5)(0)\stackrel{?}{=}0$ $0=0$ ✓

Try It 4.1.2

Solve: $(x-3)(x+5)=0$

Answer

$x=3, x=-5$

Try It 4.1.3

Solve: $(y-6)(y+9)=0$

Answer

$y=6, y=-9$

We usually will do a little more work than we did in this last example to solve the linear equations that result from using the Zero Product Property.

Example 4.1.4

Solve: $(5n-2)(6n-1)=0$

Solution

	$(5n-2)(6n-1)=0$	
Use the Zero Product Property to set each factor to 0.	$5n-2=0$	$6n-1=0$
Solve the equations.	$n=\frac{2}{5}$	$n=\frac{1}{6}$
Check your answers.		
$n=\frac{2}{5}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{2}{5}-2\right)\left(6\cdot\frac{2}{5}-1\right)\stackrel{?}{=}0$ $(2-2)\left(\frac{12}{5}-\frac{5}{5}\right)\stackrel{?}{=}0$ $(0)\left(\frac{7}{5}\right)\stackrel{?}{=}0$ $0=0$ ✓	$n=\frac{1}{6}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{1}{6}-2\right)\left(6\cdot\frac{1}{6}-1\right)\stackrel{?}{=}0$ $\left(\frac{5}{6}-\frac{12}{6}\right)(1-1)\stackrel{?}{=}0$ $\left(-\frac{7}{6}\right)(0)\stackrel{?}{=}0$ $0=0$ ✓	

Try It 4.1.5

Solve: $(3m-2)(2m+1)=0$

Answer

$m=\frac{2}{3}, m=-\frac{1}{2}$

Try It 4.1.6

Solve: $(4p+3)(4p-3)=0$

Answer

$p=-\frac{3}{4}, p=\frac{3}{4}$

Notice when we checked the solutions that each of them made just one factor equal to zero. But the product was zero for both solutions.

Example 4.1.7

Solve: $3p(10p+7)=0$

Solution

	$3p(10p+7)=0$	
Use the Zero Product Property to set each factor to 0.	3p=0	10p+7=0
Solve the equations.	p=0	10p=−7
		$p=-\frac{7}{10}$
Check your answers.		

$p=0$	$p=-\frac{7}{10}$
$3p(10p+7)=0$	$3p(10p+7)=0$
$3\cdot 0(10\cdot 0+7)\stackrel{?}{=}0$	$3\left(-\frac{7}{10}\right)10\left(-\frac{7}{10}\right)+7\stackrel{?}{=}0$
$0(0+7)\stackrel{?}{=}0$	$\left(-\frac{21}{10}\right)(-7+7)\stackrel{?}{=}0$
$0(7)\stackrel{?}{=}0$	$\left(-\frac{21}{10}\right)(0)\stackrel{?}{=}0$
$0=0$ ✓	$0=0$ ✓

Try It 4.1.8

Solve: $2u(5u-1)=0$

Answer

$u=0, u=\frac{1}{5}$

Try It 4.1.9

Solve: $w(2w+3)=0$

Answer

$w=0, w=-\frac{3}{2}$

It may appear that there is only one factor in the next example. Remember, however, that $(y-8)^2$ means $(y-8)(y-8)$.

Example 4.1.10

Solve: $(y-8)^2=0$

Solution

	$(y-8)^2=0$	
Rewrite the left side as a product.	(y−8)(y−8)=0	
Use the Zero Product Property and set each factor to 0.	y−8=0	y−8=0
Solve the equations.	y=8	y=8

When a solution repeats, we call it a double root.		
Check your answer.		
$y = 8$ $(y-8)^2 = 0$ $(8-8)^2 \stackrel{?}{=} 0$ $(0)^2 \stackrel{?}{=} 0$ $0 = 0$ ✓		

? Try It 4.1.11

Solve: $(x+1)^2 = 0$

Answer

$x = 1$

? Try It 4.1.12

Solve: $(v-2)^2 = 0$

Answer

$v = 2$

Solve Quadratic Equations by Factoring

Each of the equations we have solved in this section so far had one side in factored form. In order to use the Zero Product Property, the quadratic equation must be factored, with zero on one side. So we be sure to start with the quadratic equation in standard form, $ax^2 + bx + c = 0$. Then we factor the expression on the left.

✓ Example 4.1.13

Solve: $x^2 + 2x - 8 = 0$

Solution

Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.	The equation is already in standard form.	$x^2 + 2x - 8 = 0$
Step 2. Factor the quadratic expression.	Factor $x^2 + 2x - 8$ $(x+4)(x-2)$	$(x+4)(x-2) = 0$
Step 3. Use the Zero Product Property.	Set each factor equal to zero.	$x + 4 = 0$ or $x - 2 = 0$
Step 4. Solve the linear equations.	We have two linear equations.	$x = -4$ or $x = 2$
Step 5. Check.	Substitute each solution separately into the original equation.	$x^2 + 2x - 8 = 0$ $x = -4$ $(-4)^2 - 2(-4) - 8 \stackrel{?}{=} 0$ $16 + (-8) - 8 \stackrel{?}{=} 0$ $0 = 0$ ✓ $x^2 + 2x - 8 = 0$ $x = 2$ $2^2 - 2(2) - 8 \stackrel{?}{=} 0$ $4 + 4 - 8 \stackrel{?}{=} 0$ $0 = 0$ ✓

Try It 4.1.14

Solve: $x^2 - x - 12 = 0$

Answer

$x = 4, x = -3$

Try It 4.1.15

Solve: $b^2 + 9b + 14 = 0$

Answer

$b = -2, b = -7$

SOLVE A QUADRATIC EQUATION BY FACTORING.

1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
2. Factor the quadratic expression.
3. Use the Zero Product Property.
4. Solve the linear equations.
5. Check.

Before we factor, we must make sure the quadratic equation is in standard form.

Example 4.1.16

Solve: $2y^2 = 13y + 45$

Solution

	$2y^2 = 13y + 45$	
Write the quadratic equation in standard form.	$2y^2 - 13y - 45 = 0$	
Factor the quadratic expression.	$(2y + 5)(y - 9) = 0$	
Use the Zero Product Property to set each factor to 0.	$2y + 5 = 0$	$y - 9 = 0$
Solve each equation.	$y = -\frac{5}{2}$	$y = 9$
Check your answers.		

$y = -\frac{5}{2}$	$y = 9$
$2y^2 = 13y + 45$	$2y^2 = 13y + 45$
$2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right) + 45$	$2(9)^2 \stackrel{?}{=} 13(9) + 45$
$2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right) + \frac{90}{2}$	$2(81) \stackrel{?}{=} 117 + 45$
$\frac{25}{2} = \frac{25}{2}$ ✓	$162 = 162$ ✓

Try It 4.1.17

Solve: $3c^2 = 10c - 8$

Answer

$c = 0, c = \frac{4}{3}$

Try It 4.1.18

Solve: $2d^2 - 5d = 3$

Answer

$d = 3, d = -\frac{1}{2}$

Example 4.1.19

Solve: $5x^2 - 13x = 7x$

Solution

	$5x^2 - 13x = 7x$	
Write the quadratic equation in standard form.	$5x^2 - 20x = 0$	
Factor the left side of the equation.	$5x(x-4) = 0$	
Use the Zero Product Property to set each factor to 0.	$5x = 0$	$x - 4 = 0$
Solve each equation.	$x = 0$	$x = 4$
Check your answers.		

$x = 0$

$5x^2 - 13x = 7x$

$5(0)^2 - 13(0) \stackrel{?}{=} 7(0)$

$0 - 0 \stackrel{?}{=} 0$

$0 = 0$ ✓

$x = 4$

$5x^2 - 13x = 7x$

$5(4)^2 - 13(4) \stackrel{?}{=} 7(4)$

$5(16) - 52 \stackrel{?}{=} 28$

$28 = 28$ ✓

Try It 4.1.20

Solve: $6a^2 + 9a = 3a$

Answer

$a = 0, a = -1$

Try It 4.1.21

Solve: $45b^2 - 2b = -17b$

Answer

$b = 0, b = -\frac{1}{3}$

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

Example 4.1.22

Solve: $144q^2 = 25$

Solution

	$144q^2$	$=$	25			
Write the quadratic equation in standard form.	$144q^2 - 25$	$=$	0			
Factor. It is a difference of squares.	$(12q-5)(12q+5)$	$=$	0			
Use the Zero Product Property to set each factor to 0.	$12q - 5$	$=$	0	$12q + 5$	$=$	0
Solve each equation.	$12q$	$=$	5	$12q$	$=$	-5
	q	$=$	$\frac{5}{12}$	q	$=$	$-\frac{5}{12}$

Check your answers.

Try It 4.1.23

Solve: $25p^2 = 49$

Answer

$p = \frac{7}{5}, p = -\frac{7}{5}$

Try It 4.1.24

Solve: $36x^2 = 121$

Answer

$x = \frac{11}{6}, x = -\frac{11}{6}$

The left side in the next example is factored, but the right side is not zero. In order to use the Zero Product Property, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

Example 4.1.25

Solve: $(3x-8)(x-1) = 3x$

Solution

	$(3x-8)(x-1) = 3x$
Multiply the binomials.	$3x^2 - 11x + 8 = 3x$
Write the quadratic equation in standard form.	$3x^2 - 14x + 8 = 0$
Factor the trinomial.	$(3x-2)(x-4) = 0$
Use the Zero Product Property to set each factor to 0.	$3x-2=0 \quad x-4=0$
Solve each equation.	$3x = 2 \quad x = 4$
	$x = \frac{2}{3}$
Check your answers.	The check is left to you!

Try It 4.1.26

Solve: $(2m+1)(m+3) = 12m$

Answer

$m = 1, m = \frac{3}{2}$

Try It 4.1.27

Solve: $(k+1)(k-1) = 8$

Answer

$k = 3, k = -3$

The Zero Product Property also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree more than two by using the Zero Product Property, just like we solved quadratic equations.

Example 4.1.28

Solve: $9m^3 + 100m = 60m^2$

Solution

	$9m^3 + 100m$	$=$	$60m^2$					
Bring all the terms to one side so that the other side is zero.	$9m^3 - 60m^2 + 100m$	$=$	0					
Factor the greatest common factor first.	$m\left(9m^2 - 60m + 100\right)$	$=$	0					
Factor the trinomial.	$m(3m-10)(3m-10)$	$=$	0					
Use the Zero Product Property to set each factor to 0.	m	$=$	0	$3m-10$	$=$	0	$3m-10$	$=$ 0
Solve each equation.	m	$=$	0	m	$=$	$\frac{10}{3}$	m	$=$ $\frac{10}{3}$
Check your answers.	The check is left to you.							

Try It 4.1.29

Solve: $8x^3 = 24x^2 - 18x$

Answer

$x = 0, x = \frac{3}{2}$

Try It 4.1.30

Solve: $16y^2 = 32y^3 + 2y$

Answer

$y = 0, y = \frac{1}{4}$

When we factor the quadratic equation in the next example we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

Example 4.1.31

Solve: $4x^2 = 16x + 84$

Solution

	$4x^2$	$=$	$16x+84$					
Write the quadratic equation in standard form.	$4x^2-16x-84$	$=$	0					
Factor the greatest common factor first.	$4\left(x^2-4x-21\right)$	$=$	0					
Factor the trinomial.	$4(x-7)(x+3)$	$=$	0					
Use the Zero Product Property to set each factor to 0.	4	$\neq$	0	$x-7$	$=$	0	$x+3$	$=$ 0
Solve each equation.	4	$\neq$	0	x	$=$	7	x	$=$ -3
Check your answers.	The check is left to you.							

Try It 4.1.32

Solve: $18a^2 - 30 = -33a$

Answer

$a = -\frac{5}{2}, a = \frac{2}{3}$

Try It 4.1.33

Solve: $123b = -6 - 60b^2$

Answer

$b = 2, b = \frac{1}{20}$

Solve Applications Modeled by Quadratic Equations

The problem solving strategy we used earlier for applications that translate to linear equations will work just as well for applications that translate to quadratic equations. We will copy the problem solving strategy here so we can use it for reference.

USE A PROBLEM-SOLVING STRATEGY TO SOLVE WORD PROBLEMS

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We will start with a number problem to get practice translating words into a quadratic equation.

Example 4.1.34

The product of two consecutive integers is 132. Find the integers.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two consecutive integers.
Step 3. Name what we are looking for.	Let $n =$ the first integer $n+1 =$ the next consecutive integer
Step 4. Translate into an equation. Restate the problem in a sentence.	The product of the two consecutive integers is 132.
Translate to an equation.	The first integer times the next integer is 132. $n(n+1) = 132$
Step 5. Solve the equation.	$n^2+n = 132$
Bring all the terms to one side.	$n^2+n-132 = 0$
Factor the trinomial.	$(n-11)(n+12) = 0$
Use the zero product property.	$n-11 = 0 \quad n+12 = 0$
Solve the equations.	$n = 11 \quad n = -12$

Try It 4.1.35

The product of two consecutive integers is 240. Find the integers.

Answer

$-15, -16$ and $15, 16$

Try It 4.1.36

The product of two consecutive integers is 420. Find the integers.

Answer

$-21, -20$ and $20, 21$

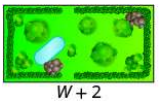
Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give 132.

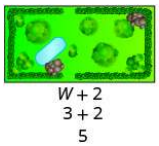
In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

Example 4.1.37

A rectangular garden has an area of 15 square feet. The length of the garden is two feet more than the width. Find the length and width of the garden.

Solution

Step 1. Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation.		
Step 2. Identify what you are looking for.	We are looking for the length and width.	
Step 3. Name what you are looking for. The length is two feet more than width.	Let W = the width of the garden. $W + 2$ = the length of the garden	
Step 4. Translate into an equation. Restate the important information in a sentence.	The area of the rectangular garden is 15 square feet.	
Use the formula for the area of a rectangle.	$A = L \cdot W$	
Substitute in the variables.	$15 = (W+2)W$	
Step 5. Solve the equation. Distribute first.	$15 = W^2 + 2W$	
Get zero on one side.	$0 = W^2 + 2W - 15$	
Factor the trinomial.	$0 = (W+5)(W-3)$	
Use the Zero Product Property.	$0 = W+5$	$0 = W-3$
Solve each equation.	$-5 = W$	$3 = W$

Since W is the width of the garden, it does not make sense for it to be negative. We eliminate that value for W.	$W = -5$ cannot be the width, since it's negative. $W = 3$	$3 = W$ Width is 3 feet.
Find the value of the length.	$\text{length} = W + 2$	
	$\text{length} = 3 + 2$	
	$\text{length} = 5$	Length is 5 feet.
Step 6. Check the answer. Does the answer make sense?		
 $A = L \cdot W$ $A = 3 \cdot 5$ $A = 15$		
	Yes, this makes sense.	
Step 7. Answer the question.	The width of the garden is 3 feet and the length is 5 feet.	

Try It 4.1.38

A rectangular sign has an area of 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

Answer

55 feet and 66 feet

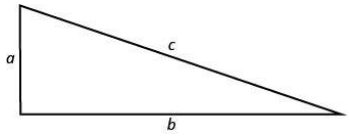
Try It 4.1.39

A rectangular patio has an area of 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

Answer

12 feet and 15 feet

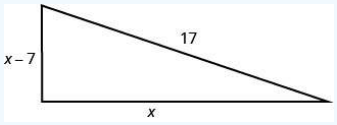
In an earlier chapter, we used the Pythagorean Theorem $\left(a^2 + b^2 = c^2\right)$. It gave the relation between the legs and the hypotenuse of a right triangle.



We will use this formula to in the next example.

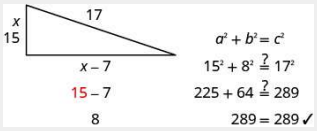
Example 4.1.40

Justine wants to put a deck in the corner of her backyard in the shape of a right triangle, as shown below. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the deck.



Solution

Step 1. Read the problem.			
Step 2. Identify what you are looking for.	We are looking for the lengths of the sides of the deck.		
Step 3. Name what you are looking for. One side is 7 less than the other.	Let x = length of a side of the deck $x - 7$ = length of other side		

Step 4. Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.	$a^2+b^2=c^2$		
Substitute in the variables.	$x^2+(x-7)^2=17^2$		
Step 5. Solve the equation.	$x^2+x^2-14x+49=289$		
Simplify.	$2x^2-14x+49=289$		
It is a quadratic equation, so get zero on one side.	$2x^2-14x-240=0$		
Factor the greatest common factor.	$2\left(x^2-7x-120\right)=0$		
Factor the trinomial.	$2(x-15)(x+8)=0$		
Use the Zero Product Property.	$2\neq 0$	$x-15=0$	$x+8=0$
Solve.	$2\neq 0$	$x=15$	$x=-8$
Since x is a side of the triangle, $x=-8$ does not make sense.	$2\neq 0$	$x=15$	$x=-8$
Find the length of the other side.			
If the length of one side is		$x=15$	
then the length of the other side is		$x-7$	
		$15-7=8$	
	8 is the length of the other side.		
Step 6. Check the answer. Do these numbers make sense?			
			
Step 7. Answer the question.	The sides of the deck are 8, 15, and 17 feet.		

? Try It 4.1.41

A boat's sail is a right triangle. The length of one side of the sail is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the sail.

Answer

5 feet and 12 feet

? Try It 4.1.42

A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of one of the other legs. Find the lengths of the hypotenuse and the other leg.

Answer

24 feet and 25 feet

Key Concepts

- **Zero Product Property** If $a\cdot b=0$, then either a=0 or b=0 or both. See Example.
- **Solve a quadratic equation by factoring** To solve a quadratic equation by factoring: See Example.
  1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.
  2. Factor the quadratic expression.
  3. Use the Zero Product Property.

4. Solve the linear equations.
5. Check.

- **Use a problem solving strategy to solve word problems** See Example.
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

Glossary

quadratic equations

are equations in which the variable is squared.

Zero Product Property

The Zero Product Property states that, if the product of two quantities is zero, at least one of the quantities is zero.

This page titled 4.1: Quadratic Equations and Solving by Factoring is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **7.6: Quadratic Equations** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

4.2: Solve Quadratic Equations Using the Square Root Property

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x-h)^2 = k$ using the Square Root Property

Quadratic equations are equations of the form $ax^2 + bx + c = 0$, where $a \neq 0$. They differ from linear equations by including a term with the variable raised to the second power. We use different methods to solve **quadratic equation**s than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will use three other methods to solve quadratic equations.

Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

	$x^2 = 9$
Put the equation in standard form.	$x^2 - 9 = 0$
Factor the left side.	$(x-3)(x+3) = 0$
Use the Zero Product Property.	$(x-3) = 0, (x+3) = 0$
Solve each equation.	$x = 3, x = -3$
Combine the two solutions into $\pm$ form	$x = \pm 3$

(The solution is read 'x is equal to positive or negative 3.')

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

These equations are all of the form $x^2 = k$.
We defined the square root of a number in this way:

If $n^2 = m$, then n is a square root of m.

This leads to the **Square Root Property**.

Definition: SQUARE ROOT PROPERTY

If $x^2 = k$, and $k \geq 0$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

Notice that the **Square Root Property** gives two solutions to an equation of the form $x^2 = k$: the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$

Now, we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

	$x^2 = 9$
Use the Square Root Property.	$x = \pm\sqrt{9}$
Simplify the radical.	$x = \pm 3$
Rewrite to show the two solutions.	$x = 3, x = -3$

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

Use the Square Root Property.	$x = \pm\sqrt{7}$
Rewrite to show two solutions.	$x = \sqrt{7}, x = -\sqrt{7}$

We cannot simplify $\sqrt{7}$ This is considered an exact answer.

We can use a calculator to get an approximate answer. For this example our answers would be 2.65, -2.65 rounded to the hundredth.

Example 4.2.1

Solve: $x^2 = 169$

Answer

	$x^2 = 169$
Use the Square Root Property.	$x = \pm\sqrt{169}$
Simplify the radical.	$x = \pm 13$
Rewrite to show two solutions.	$x = 13, x = -13$

Example 4.2.2

Solve: $x^2 = 81$

Answer

x=9, x=−9

Example 4.2.3

Solve: $y^2 = 121$

Answer

y = 11, y = −11

Definition:SOLVE A QUADRATIC EQUATION USING THE SQUARE ROOT PROPERTY.

1. Isolate the quadratic term and make its coefficient one.
2. Use Square Root Property.
3. Simplify the radical.
4. Check the solutions.

To use the Square Root Property, the coefficient of the variable term must equal 1. In the next example, we must divide both sides of the equation by 5 before using the Square Root Property.

Example 4.2.7

Solve: $5m^2 = 80$

Answer

The quadratic term is isolated.	$5m^2 = 80$
Divide by 5 to make its cofficient 1.	$\frac{5m^2}{5} = \frac{80}{5}$
Simplify.	$m^2 = 16$
Use the Square Root Property.	$m = \pm\sqrt{16}$

Simplify the radical.	$m = \pm 4$
Rewrite to show two solutions.	m=4,m=−4
Check the solutions. $5m^2 = 80$ $\quad$ $5m^2 = 80$ $5(4)^2 \stackrel{?}{=} 80$ $\quad$ $5(-4)^2 \stackrel{?}{=} 80$ $5 \cdot 16 \stackrel{?}{=} 80$ $\quad$ $5 \cdot 16 \stackrel{?}{=} 80$ $80 = 80$ ✓ $\quad$ $80 = 80$ ✓	

Example 4.2.8

Solve: $2x^2 = 98$.

Answer

x=7, x=−7

Example 4.2.9

Solve: $3z^2 = 108$.

Answer

z=6, z=−6

The Square Root Property started by stating, 'If $x^2 = k$, and $k \geq 0$'. What will happen if $k < 0$? This will be the case in the next example.

Example 4.2.10

Solve: $q^2 + 24 = 0$.

Answer

	$q^2 = 24$
Isolate the quadratic term.	$q^2 = -24$
Use the Square Root Property.	$q = \pm\sqrt{-24}$
The $\sqrt{-24}$ is not a real number	There is no real solution

Example 4.2.11

Solve: $c^2 + 12 = 0$.

Answer

no real solution

Example 4.2.12

Solve: $d^2 + 81 = 0$.

Answer

no real solution

Remember, we first isolate the quadratic term and then make the coefficient equal to one.

Solve Quadratic Equations of the Form $a(x-h)^2=k$ Using the Square Root Property

We can use the Square Root Property to solve an equation like $(x-3)^2=16$, too. We will treat the whole binomial, (x−3), as the quadratic term.

Example 4.2.19

Solve: $(x-3)^2=16$.

Answer

	$(x-3)^2=16$
Use the Square Root Property.	$x-3=\pm\sqrt{16}$
Simplify.	$x-3=\pm 4$
Write as two equations.	$x-3=4$, $x-3=-4$
Solve.	x=7,x=−1
Check. $(7-3)^2=16$ $(4)^2=16$ $16=16$ ✓ $(-1-3)^2=16$ $(-4)^2=16$ $16=16$ ✓	

Example 4.2.20

Solve: $(q+5)^2=1$.

Answer

q=−6, q=−4

Example 4.2.21

Solve: $(r-3)^2=25$.

Answer

r=8, r=−2

We will start the solution to the next example by isolating the binomial.

Example 4.2.28

Solve: $(x-2)^2+3=30$.

Answer

	$(x-2)^2+3=30$
Isolate the binomial term.	$(x-2)^2=27$
Use the Square Root Property.	$x-2=\pm\sqrt{27}$
Calculate the radical.	$x-2=\pm 5.1961524\ldots$
Solve for x.	$x=2\pm 5.1961524$

Let's approximate our final answer to the thousandth place.	$x = 7.196, x = -3.196$
Check. We leave the check for you	

Example 4.2.29

Solve: $(a-5)^2+4=24$.

Answer

$a = 9.472, a = 0.528$

Example 4.2.30

Solve: $(b-3)^2-8=24$.

Answer

$b = 8.657, b = -2.657$

Example 4.2.31

Solve:$(3v-7)^2=-12$.

Answer

	$(3v-7)^2=-12$
Use the Square Root Property.	$3v-7=\pm\sqrt{-12}$
The $\sqrt{-12}$ is not a real number.	There is no real solution.

Example 4.2.32

Solve: $(3r+4)^2=-8$.

Answer

no real solution

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

Example 4.2.33

Solve: $p^2-10p+25=18$.

Answer

The left side of the equation is a perfect square trinomial. We will factor it first.

	$p^2-10p+25=18$
Factor the perfect square trinomial.	$(p-5)^2=18$
Use the Square Root Property.	$p-5=\pm\sqrt{18}$
Calculate the radical.	$p-5=\pm4.242640687$

Solve for p.	$p = 5 \pm 4.242640687$
Calculate to show the two solutions to the thousandth place.	$p = 9.243, p = 0.757$
Check. We leave the check for you.	

Example 4.2.34

Solve: $x^2 - 6x + 9 = 12$.

Answer

$x = 6.464, x = -0.464$

Example 4.2.36

Solve: $4n^2 + 4n + 1 = 16$.

Answer

Again, we notice the left side of the equation is a perfect square trinomial. We will factor it first.

	$4n^2 + 4n + 1 = 16$
Factor the perfect square trinomial.	$(2n + 1)^2 = 16$
Use the Square Root Property.	$(2n + 1) = \pm\sqrt{16}$
Simplify the radical.	$(2n + 1) = \pm 4$
Solve for n.	$2n = -1 \pm 4$
Divide each side by 2.	$\frac{2n}{2} = \frac{-1 \pm 4}{2}$ $n = \frac{-1 \pm 4}{2}$
Rewrite to show two solutions.	$n = \frac{-1+4}{2}, n = \frac{-1-4}{2}$
Simplify each equation.	$n = \frac{3}{2}, n = -\frac{5}{2}$
Check. $4n^2 + 4n + 1 = 16$ $4\left(\frac{3}{2}\right)^2 + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16$ $4\left(\frac{9}{4}\right) + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16$ $9 + 6 + 1 \stackrel{?}{=} 16$ $16 = 16$ ✓ $4n^2 + 4n + 1 = 16$ $4\left(-\frac{5}{2}\right)^2 + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16$ $4\left(\frac{25}{4}\right) + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16$ $25 - 10 + 1 \stackrel{?}{=} 16$ $16 = 16$ ✓	

Example 4.2.37

Solve: $9m^2 - 12m + 4 = 25$.

Answer

$m = \frac{7}{3}, m = -1$

? Example 4.2.38

Solve: $16n^2+40n+25=4$.

Answer

$n=-\frac{3}{4}$, $n=-\frac{7}{4}$

Access these online resources for additional instruction and practice with solving quadratic equations:

- Solving Quadratic Equations: Solving by Taking Square Roots
- Using Square Roots to Solve Quadratic Equations
- Solving Quadratic Equations: The Square Root Method

Key Concepts

- Square Root Property
  If $x^2=k$, and $k\geq 0$, then $x=\sqrt{k}$ or $x=-\sqrt{k}$.

Glossary

quadratic equation

A quadratic equation is an equation of the form $ax^2+bx+c=0 \quad$ where $a\neq 0$.

Square Root Property

The Square Root Property states that, if $x^2=k$, and $k\geq 0$, then $x=\sqrt{k}$ or $x=-\sqrt{k}$.

This page titled 4.2: Solve Quadratic Equations Using the Square Root Property is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **10.1: Solve Quadratic Equations Using the Square Root Property** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

4.3: Solve Quadratic Equations Using the Quadratic Formula

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations using the quadratic formula
- Use the discriminant to predict the number of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering 'isn't there an easier way to do this?' The answer is 'yes.' In this section, we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable 'in general' so that we would do the algebraic steps only once and then use the new formula to find the value of the specific variable. Now, we will go through the steps of completing the square in general to solve a quadratic equation for *x*. It may be helpful to look at one of the examples at the end of the last section where we solved an equation of the form$ax^2+bx+c=0$ as you read through the algebraic steps below, so you see them with numbers as well as 'in general.'

We start with the standard form of a quadratic equation and solve it for x by completing the square.	$ax^2+bx+c=0$
Isolate the variable terms on one side.	$ax^2+bx=-c$
Make leading coefficient 1, by dividing by a.	$\frac{ax^2}{a}+\frac{b}{a}x=-\frac{c}{a}$
Simplify.	$x^2+\frac{b}{a}x=-\frac{c}{a}$
To complete the square, find$(\frac{1}{2}\cdot\frac{b}{a})^2$ and add it to both sides of the equation. $(\frac{1}{2}\frac{b}{a})^2=\frac{b^2}{4a^2}$	$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$(x+\frac{b}{2a})^2=-\frac{c}{a}+\frac{b^2}{4a^2}$
Find the common denominator of the right side and writeequivalent fractions with the common denominator.	$(x+\frac{b}{2a})^2=-\frac{c\cdot 4a}{a\cdot 4a}+\frac{b^2}{4a^2}$
Simplify.	$(x+\frac{b}{2a})^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
Combine to one fraction.	$(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}$
Use the square root property.	$(x+\frac{b}{2a})=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$
Simplify.	$(x+\frac{b}{2a})=\pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$to both sides of the equation.	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

This last equation is the Quadratic Formula.

Definition: QUADRATIC FORMULA

The solutions to a quadratic equation of the form $ax^2+bx+c=0$, $a\geq 0$ are given by the formula:

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

To use the Quadratic Formula, we substitute the values of a, b, and c into the expression on the right side of the formula. Then, we do all the math to simplify the expression. The result gives the solution(s) to the quadratic equation.

How to Solve a Quadratic Equation Using the Quadratic Formula

Example 4.3.1

Solve $2x^2 + 9x - 5 = 0$ by using the Quadratic Formula.

Answer

Step 1. Write the quadratic equation in standard form. Identify the *a*, *b*, *c* values.	This equation is in standard form.	$ax^2 + bx + c = 0$ $2x^2 + 9x - 5 = 0$ $a = 2, b = 9, c = -5$
Step 2. Write the quadratic formula. Then substitute in the values of *a*, *b*, *c*.	Substitute in $a = 2, b = 9, c = -5$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-9 \pm \sqrt{9^2 - 4 \cdot 2 \cdot (-5)}}{2 \cdot 2}$
Step 3. Simplify the fraction, and solve for *x*.		$x = \frac{-9 \pm \sqrt{81 - (-40)}}{4}$ $x = \frac{-9 \pm \sqrt{121}}{4}$ $x = \frac{-9 \pm 11}{4}$ $x = \frac{-9 + 11}{4}$ $\quad x = \frac{-9 - 11}{4}$ $x = \frac{2}{4}$ $\quad x = \frac{-20}{4}$ $x = \frac{1}{2}$ $\quad x = -5$
Step 4. Check the solutions.	Put each answer in the original equation to check. Substitute $x = \frac{1}{2}$.	$2x^2 + 9x - 5 = 0$ $2\left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $\frac{1}{2} + \frac{9}{2} - 5 \stackrel{?}{=} 0$ $\frac{10}{2} - 5 \stackrel{?}{=} 0$ $5 - 5 \stackrel{?}{=} 0$ $0 = 0$ ✓
	Substitute $x = -5$.	$2x^2 + 9x - 5 = 0$ $2(-5)^2 + 9(-5) - 5 \stackrel{?}{=} 0$ $2 \cdot 25 - 45 - 5 \stackrel{?}{=} 0$ $50 - 45 - 5 \stackrel{?}{=} 0$ $0 = 0$ ✓

Example 4.3.2

Solve $3y^2 - 5y + 2 = 0$ by using the Quadratic Formula.

Answer

$y = \frac{2}{3}, y = 1$

Example 4.3.3

Solve $4z^2 + 2z - 6 = 0$ by using the Quadratic Formula.

Answer

$z = -\frac{3}{2}, z = 1$

Definition: SOLVE A QUADRATIC EQUATION USING THE QUADRATIC FORMULA

1. Write the Quadratic Formula in standard form. Identify the a, b, and c values.
2. Write the Quadratic Formula. Then substitute in the values of a, b, and c.
3. Simplify.
4. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time. And remember, the Quadratic Formula is an equation. Be sure you start with '$x =$'.

Example 4.3.4

Solve $x^2 - 6x + 5 = 0$ by using the Quadratic Formula.

Answer

	$x^2 - 6x + 5 = 0$
This equation is in standard form.	$ax^2 + bx + c = 0$ $x^2 - 6x + 5 = 0$
Identify the a, b, c values.	$a = 1, b = -6, c = 5$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (5)}}{2 \cdot 1}$
Simplify.	$x = \frac{6 \pm \sqrt{36 - 20}}{2}$ $x = \frac{6 \pm \sqrt{16}}{2}$ $x = \frac{6 \pm 4}{2}$
Rewrite to show two solutions.	$x = \frac{6+4}{2}, x = \frac{6-4}{2}$
Simplify.	$x = \frac{10}{2}, x = \frac{2}{2}$ $x = 5, x = 1$
Check. $x^2 - 6x + 5 = 0$ $\quad$ $x^2 - 6x + 5 = 0$ $5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} 0$ $\quad$ $1^2 - 6 \cdot 1 + 5 \stackrel{?}{=} 0$ $25 - 30 + 5 \stackrel{?}{=} 0$ $\quad$ $1 - 6 + 5 \stackrel{?}{=} 0$ $0 = 0$ ✓ $\quad$ $0 = 0$ ✓	

Example 4.3.5

Solve $a^2 - 2a - 15 = 0$ by using the Quadratic Formula.

Answer

$a = -3, a = 5$

Example 4.3.6

Solve $b^2 + 10b + 24 = 0$ by using the Quadratic Formula.

Answer

$b = -6, b = -4$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the Quadratic Formula. If we get a radical as a solution, this is the exact answer. We will get the approximate answer using our calculator to calculate the radical.

Example 4.3.7

Solve $4y^2 - 5y - 3 = 0$ by using the Quadratic Formula.

Answer

We can use the Quadratic Formula to solve for the variable in a quadratic equation, whether or not it is named '*x*'.

	$4y^2 - 5y - 3 = 0$
This equation is in standard form.	$ax^2 + bx + c = 0$ $4y^2 - 5y - 3 = 0$
Identify the *a*, *b*, *c* values.	$a = 4, b = -5, c = -3$
Write the Quadratic Formula.	$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$y = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 4 \cdot (-3)}}{2 \cdot 4}$
Simplify.	$y = \frac{5 \pm \sqrt{25 + 48}}{8}$ $y = \frac{5 \pm \sqrt{73}}{8}$
Calculate the radical.	$y = \frac{5 \pm 8.544003745}{8}$
Calculate each answer, rounding to the nearest thousandth.	y = 1.693, -0.443
Check. We leave the check to you.	

Example 4.3.8

Solve $2p^2 + 8p + 5 = 0$ by using the Quadratic Formula.

Answer

$p = -0.775, -3.225$

Example 4.3.9

Solve $5q^2 - 11q + 3 = 0$ by using the Quadratic Formula.

Answer

$q = 1.881, 0.319$

We cannot take the square root of a negative number. So, when we substitute a, b, and c into the Quadratic Formula, if the quantity inside the radical is negative, the quadratic equation has no real solution. We will see this in the next example.

Example 4.3.13

Solve $3p^2 + 2p + 9 = 0$ by using the Quadratic Formula.

Answer

This equation is in standard form.	$ax^2 + bx + c = 0$ $3p^2 + 2p + 9 = 0$
Identify the *a*, *b*, *c* values.	$a = 3,\ b = 2,\ c = 9$
Write the Quadratic Formula.	$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$p = \frac{-(2) \pm \sqrt{(2)^2 - 4 \cdot 3 \cdot (9)}}{2 \cdot 3}$
Simplify.	$p = \frac{-2 \pm \sqrt{4 - 108}}{6}$
Simplify the radical.	$p = \frac{-2 \pm \sqrt{-104}}{6}$
We cannot take the square root of a negative number.	There is no real solution.

Example 4.3.14

Solve $4a^2 - 3a + 8 = 0$ by using the Quadratic Formula.

Answer

no real solution

Exeample 4.3.15

Solve $5b^2 + 2b + 4 = 0$ by using the Quadratic Formula.

Answer

no real solution

When we solved linear equations, if an equation had too many fractions we 'cleared the fractions' by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions—to solve. We can use the same strategy with quadratic equations.

Example 4.3.19

Solve $\frac{1}{2}u^2 + \frac{2}{3}u = \frac{1}{3}$ by using the Quadratic Formula.

Answer

	$\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2+\frac{2}{3}u\right)=6\left(\frac{1}{3}\right)$
Multiply.	$3u^2+4u=2$
Subtract 2 to get the equation in standard form.	$\underset{ax^2\ +\ bx\ +\ c\ =\ 0}{3u^2+4u-2=0}$
Identify the *a*, *b*, *c* values.	$a=3,\ b=4,\ c=-2$
Write the Quadratic Formula.	$u=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$u=\frac{-(4)\pm\sqrt{(4)^2-4\cdot 3\cdot(-2)}}{2\cdot 3}$
Simplify.	$u=\frac{-4\pm\sqrt{16+24}}{6}$ $u=\frac{-4\pm\sqrt{40}}{6}$
Calculate the radical.	$u=\frac{-4\pm 6.32455532}{6}$
Calculate to show two solutions to the thousandth.	u = 0.387, -1.721
Check. We leave the check to you.	

? Example 4.3.20

Solve $\frac{1}{4}c^2-\frac{1}{3}c=\frac{1}{12}$ by using the Quadratic Formula.

Answer

$c=1.549,-0.215$

? Example 4.3.21

Solve $\frac{1}{9}d^2-\frac{1}{2}d=-\frac{1}{2}$ by using the Quadratic Formula.

Answer

$d=\frac{3}{2}$, $d=3$

Think about the equation $(x-3)^2=0$. We know from the Zero Products Principle that this equation has only one solution: $x=3$.

We will see in the next example how using the Quadratic Formula to solve an equation with a perfect square also gives just one solution.

? Example 4.3.22

Solve $4x^2-20x=-25$ by using the Quadratic Formula.

Answer

	$4x^2-20x=-25$
Add 25 to get the equation in standard form.	$\underset{ax^2\ +\ bx\ +\ c\ =\ 0}{4x^2-20x+25=0}$
Identify the *a*, *b*, *c* values.	$a=4,\ b=-20,\ c=25$

Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$
Simplify.	$x = \frac{20 \pm \sqrt{400 - 400}}{8}$ $x = \frac{20 \pm \sqrt{0}}{8}$
Simplify the radical.	$x = \frac{20}{8}$
Simplify the fraction.	$x = \frac{5}{2}$
Check. We leave the check to you.	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square?

Example 4.3.23

Solve $r^2 + 10r + 25 = 0$ by using the Quadratic Formula.

Answer

$r = -5$

Example 4.3.24

Solve $25t^2 - 40t = -16$ by using the Quadratic Formula.

Answer

$t = \frac{4}{5}$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two solutions, sometimes one solution, sometimes no real solutions. Is there a way to predict the number of solutions to a quadratic equation without actually solving the equation?

Yes, the quantity inside the radical of the Quadratic Formula makes it easy for us to determine the number of solutions. This quantity is called the **discriminant**.

Definition: DISCRIMINANT

In the Quadratic Formula$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the **discriminant**.

Let's look at the discriminant of the equations in Example, Example, and Example, and the number of solutions to those quadratic equations.

	Quadratic Equation (in standard form)	Discriminant $b^2 - 4ac$	Sign of the Discriminant	Number of real solution
Example	$2x^2 + 9x - 5 = 0$	$9^2 - 4 \cdot 2(-5) = 121$	+	2
Example	$4x^2 - 20x + 25 = 0$	$(-20)^2 - 4 \cdot 4 \cdot 25 = 0$	0	1
Example	$3p^2 + 2p + 9 = 0$	$2^2 - 4 \cdot 3 \cdot 9 = -104$	−	0

When the discriminant is **positive** $x = \frac{-b \pm \sqrt{+}}{2a}$ the quadratic equation has **two solutions**.

When the discriminant is **zero** $x = \frac{-b \pm \sqrt{0}}{2a}$ the quadratic equation has **one solution**.

When the discriminant is **negative** $x = \frac{-b \pm \sqrt{-}}{2a}$ the quadratic equation has **no real solutions**.

Definition:USE THE DISCRIMINANT, $b^2 - 4ac$, TO DETERMINE THE NUMBER OF SolutionS OF A QUADRATIC EQUATION

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \geq 0$,

- if $b^2 - 4ac > 0$, the equation has two solutions.
- if $b^2 - 4ac = 0$, the equation has one solution.
- if $b^2 - 4ac < 0$, the equation has no real solutions.

Example 4.3.25

Determine the number of solutions to each quadratic equation:

1. $2v^2 - 3v + 6 = 0$
2. $3x^2 + 7x - 9 = 0$
3. $5n^2 + n + 4 = 0$
4. $9y^2 - 6y + 1 = 0$

Answer

1.

	$2v^2 - 3v + 6 = 0$
The equation is in standard form, identify a, b, c.	$a = 2, b = -3, c = 6$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of a, b, c.	$(3)^2 - 4 \cdot 2 \cdot 6$
Simplify.	$9 - 48$ -39
Because the discriminant is negative, there are no real solutions to the equation.	

2.

	$3x^2 + 7x - 9 = 0$
The equation is in standard form, identify a, b, c.	$a = 3, b = 7, c = -9$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of a, b, c.	$(7)^2 - 4 \cdot 3 \cdot (-9)$
Simplify.	$49 + 108$ 157
Because the discriminant is positive, there are two solutions to the equation.	

3.

	$5n^2 + n + 4 = 0$
The equation is in standard form, identify a, b, c.	$a = 5, b = 1, c = 4$
Write the discriminant.	$b^2 - 4ac$

Substitute in the values of a, b, c.	$(1)^2-4\cdot 5\cdot 4$
Simplify.	$1-80$ -79
Because the discriminant is negative, there are no real solutions to the equation.	

4.

	$9y^2-6y+1=0$
The equation is in standard form, identify a, b, c.	$a=9, b=-6, c=1$
Write the discriminant.	b^2-4ac
Substitute in the values of a, b, c.	$(-6)^2-4\cdot 9\cdot 1$
Simplify.	$36-36$ 0
Because the discriminant is 0, there is one solution to the equation.	

? Example 4.3.26

Determine the number of solutions to each quadratic equation:

1. $8m^2-3m+6=0$
2. $5z^2+6z-2=0$
3. $9w^2+24w+16=0$
4. $9u^2-2u+4=0$

Answer

1. no real solutions
2. 2
3. 1
4. no real solutions

? Example 4.3.27

Determine the number of solutions to each quadratic equation:

1. $b^2+7b-13=0$
2. $5a^2-6a+10=0$
3. $4r^2-20r+25=0$
4. $7t^2-11t+3=0$

Answer

1. 2
2. no real solutions
3. 1
4. 2

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We have used four methods to solve quadratic equations:

- Factoring

- Square Root Property
- Completing the Square
- Quadratic Formula

You can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method to use.

Definition: IDENTIFY THE MOST APPROPRIATE METHOD TO SOLVE A QUADRATIC EQUATION.

1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.
2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.
3. Use the **Quadratic Formula**. Any quadratic equation can be solved by using the Quadratic Formula.

What about the method of completing the square? Most people find that method cumbersome and prefer not to use it. We needed to include it in this chapter because we completed the square in general to derive the Quadratic Formula. You will also use the process of completing the square in other areas of algebra.

Example 4.3.28

Identify the most appropriate method to use to solve each quadratic equation:

1. $5z^2 = 17$
2. $4x^2 - 12x + 9 = 0$
3. $8u^2 + 6u = 11$

Answer

1. $5z^2 = 17$

   Since the equation is in the $ax^2 = k$, the most appropriate method is to use the Square Root Property.

2. $4x^2 - 12x + 9 = 0$

   We recognize that the left side of the equation is a perfect square trinomial, and so Factoring will be the most appropriate method.

3. $8u^2 + 6u = 11$

   Put the equation in standard form. $8u^2 + 6u - 11 = 0$

   While our first thought may be to try Factoring, thinking about all the possibilities for trial and error leads us to choose the Quadratic Formula as the most appropriate method.

Example 4.3.29

Identify the most appropriate method to use to solve each quadratic equation:

1. $x^2 + 6x + 8 = 0$
2. $(n-3)^2 = 16$
3. $5p^2 - 6p = 9$

Answer

1. factor
2. Square Root Property
3. Quadratic Formula

? Example 4.3.30

Identify the most appropriate method to use to solve each quadratic equation:

1. $8a^2+3a-9=0$
2. $4b^2+4b+1=0$
3. $5c2=125$

Answer

1. Quadratic Formula
2. factoring
3. Square Root Property

Access these online resources for additional instruction and practice with using the Quadratic Formula:

- Solving Quadratic Equations: Solving with the Quadratic Formula
- How to solve a quadratic equation in standard form using the Quadratic Formula (example)
- Solving Quadratic Equations using the Quadratic Formula—Example 3
- Solve Quadratic Equations using Quadratic Formula

Key Concepts

- **Quadratic Formula** The solutions to a quadratic equation of the form $ax^2+bx+c=0$, $a\geq 0$ are given by the formula:

  $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
- **Solve a Quadratic Equation Using the Quadratic Formula**
  To solve a quadratic equation using the Quadratic Formula.
  1. Write the quadratic formula in standard form. Identify the a, b, c values.
  2. Write the quadratic formula. Then substitute in the values of a, b, c.
  3. Simplify.
  4. Check the solutions.
- **Using the Discriminant, b^2-4ac, to Determine the Number of Solutions of a Quadratic Equation**
  For a quadratic equation of the form $ax^2+bx+c=0$, $a\geq 0$,
  - if $b^2-4ac>0$, the equation has 2 solutions.
  - if $b^2-4ac=0$, the equation has 1 solution.
  - if $b^2-4ac<0$, the equation has no real solutions.
- **To identify the most appropriate method to solve a quadratic equation:**
  1. Try Factoring first. If the quadratic factors easily this method is very quick.
  2. Try the Square Root Property next. If the equation fits the form $ax^2=k$ or $a(x-h)^2=k$, it can easily be solved by using the Square Root Property.
  3. Use the Quadratic Formula. Any other quadratic equation is best solved by using the Quadratic Formula.

Glossary

discriminant

In the Quadratic Formula,$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ the quantity b^2-4ac is called the discriminant.

This page titled 4.3: Solve Quadratic Equations Using the Quadratic Formula is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **10.3: Solve Quadratic Equations Using the Quadratic Formula** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

4.4: Solve Applications Modeled by Quadratic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve applications modeled by Quadratic Equations

Be Prepared

Before you get started, take this readiness quiz.

1. The sum of two consecutive odd numbers is −100. Find the numbers.
   If you missed this problem, review [link].
2. The area of triangular mural is 64 square feet. The base is 16 feet. Find the height.
   If you missed this problem, review [link].
3. Find the length of the hypotenuse of a right triangle with legs 5 inches and 12 inches.
   If you missed this problem, review [link].

Solve Applications of the Quadratic Formula

We solved some applications that are modeled by quadratic equations earlier, when the only method we had to solve them was factoring. Now that we have more methods to solve quadratic equations, we will take another look at applications. To get us started, we will copy our usual Problem Solving Strategy here so we can follow the steps.

Definition: USE THE PROBLEM SOLVING STRATEGY

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We have solved number applications that involved **consecutive even integers** and **consecutive odd integers** by modeling the situation with linear equations. Remember, we noticed each even integer is 2 more than the number preceding it. If we call the first one n, then the next one is $n+2$. The next one would be $n+2+2$ or $n+4$. This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.

Consecutive even integers		**Consecutive odd integers**		
	$64, 66, 68$		$77, 79, 81$	
n	1st even number	n	1st odd number	(4.4.1)
$n+2$	2nd even number	$n+2$	2nd odd number	
$n+4$	3rd even number	$n+4$	3rd odd number	

Some applications of **consecutive odd integers** or **consecutive even integers** are modeled by quadratic equations. The notation above will be helpful as you name the variables.

Example 4.4.1

The product of two consecutive odd integers is 195. Find the integers.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two consecutive odd integers.
Step 3. Name what we are looking for.	Let $n =$ the first odd integer. $n + 2 =$ the next odd integer
Step 4. Translate into an equation. State the problem in one sentence.	"The product of two consecutive odd integers is 195." The product of the first odd integer and the second odd integer is 195.
Translate into an equation.	$n(n + 2) = 195$
Step 5. Solve the equation. Distribute.	$n^2 + 2n = 195$
Subtract 195 to get the equation in standard form.	$ax^2 + bx + c = 0$ $n^2 + 2n - 195 = 0$
Identify the a, b, c values.	$a = 1,\ b = 2,\ c = -195$
Write the quadratic equation.	$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c..	$n = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-195)}}{2 \cdot 1}$
Simplify.	$n = \frac{-2 \pm \sqrt{4 + 780}}{2}$ $n = \frac{-2 \pm \sqrt{784}}{2}$
Simplify the radical.	$n = \frac{-2 \pm 28}{2}$
Rewrite to show two solutions.	$n = \frac{-2 + 28}{2}, n = \frac{-2 - 28}{2}$
Solve each equation.	$n = \frac{26}{2}, n = \frac{-30}{2}$ $n = 13\ \ n = -15$
There are two values of n that are solutions. This will give us two pairs of consecutive odd integers for our solution.	First odd integer n=13 next odd integer n+2 13+2 15
	First odd integer n=−15 next odd integer n+2 −15+2 −13
Step 6. Check the answer. Do these pairs work? Are they consecutive odd integers? Is their product 195?	13, 15, yes −13, −15, yes 13·15=195, yes −13(−15)=195, yes
Step 7. Answer the question.	The two consecutive odd integers whose product is 195 are 13, 15, and −13, −15.

Example 4.4.2

The product of two consecutive odd integers is 99. Find the integers.

Answer

Two consecutive odd numbers whose product is 99 are 9 and 11, and −9 and −11.

Example 4.4.3

The product of two consecutive odd integers is 168. Find the integers.

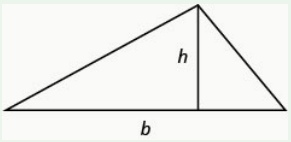
Answer

Two consecutive even numbers whose product is 168 are 12 and 14,and −12 and −14.

We will use the formula for the area of a triangle to solve the next example.

Definition: AREA OF A TRIANGLE

For a triangle with base b and height h, the area, A, is given by the formula $A = \frac{1}{2}bh$.

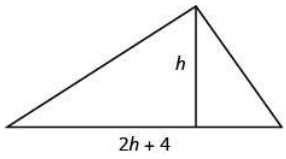


Recall that, when we solve geometry applications, it is helpful to draw the figure.

Example 4.4.4

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can have an area of 120 square feet and the architect wants the width to be 4 feet more than twice the height. Find the height and width of the window.

Answer

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the height and width.
Step 3. Name what we are looking for.	Let $h =$ the height of the triangle. $2h + 4 =$ the width of the triangle
Step 4. Translate.	We know the area. Write the formula for the area of a triangle. $A = \frac{1}{2}bh$
Step 5. Solve the equation. Substitute in the values.	$120 = \frac{1}{2}(2h + 4)h$
Distribute.	$120 = h^2 + 2h$
This is a quadratic equation, rewrite it in standard form.	$ax^2 + bx + c = 0$ $h^2 + 2h - 120 = 0$

Solve the equation using the Quadratic Formula. Identify the a, b, c values.	$a=1,\ b=2,\ c=-120$
Write the quadratic equation.	$h=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$h=\frac{-2\pm\sqrt{2^2-4\cdot 1\cdot(-120)}}{2\cdot 1}$
Simplify.	$h=\frac{-2\pm\sqrt{4+480}}{2}$ $h=\frac{-2\pm\sqrt{484}}{2}$
Simplify the radical.	$h=\frac{-2\pm 22}{2}$
Rewrite to show two solutions.	$h=\frac{-2+22}{2},\ h=\frac{-2-22}{2}$
Simplify.	$h=\frac{20}{2},\ h=\frac{-24}{2}$
Since h is the height of a window, a value of $h=-12$ does not make sense.	$h=10$ ~~$h=-12$~~
	The height of the triangle: $h=10$ The width of the triangle: $2h+4$ $2\cdot 10+4$ 24
Step 6. Check the answer. Does a triangle with a height 10 and width 24 have area 120? Yes.	
Step 7. Answer the question.	The height of the triangular window is 10 feet and the width is 24 feet.

Notice that the solutions were integers. That tells us that we could have solved the equation by factoring.

When we wrote the equation in standard form, $h^2+2h-120=0$, we could have factored it. If we did, we would have solved the equation $(h+12)(h-10)=0$.

? Example 4.4.5

Find the dimensions of a triangle whose width is four more than six times its height and has an area of 208 square inches.

Answer

The height of the triangle is 8 inches and the width is 52 inches.

? Example 4.4.6

If a triangle that has an area of 110 square feet has a height that is two feet less than twice the width, what are its dimensions?

Answer

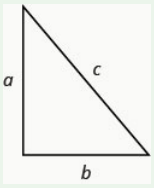
The height of the triangle is 20 feet and the width is 11 feet.

In the two preceding examples, the number in the radical in the Quadratic Formula was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

The Pythagorean Theorem gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

Definition: PYTHAGOREAN THEOREM

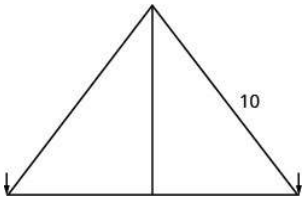
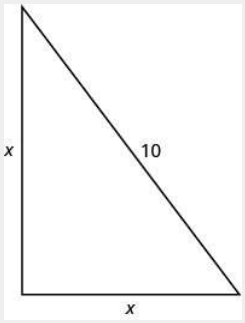
In any right triangle, where a and b are the lengths of the legs and c is the length of the hypotenuse, $a^2 + b^2 = c^2$



Example 4.4.7

Rene is setting up a holiday light display. He wants to make a 'tree' in the shape of two right triangles, as shown below, and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

Answer

Step 1. Read the problem. Draw a picture	
Step 2. Identify what we are looking for.	We are looking for the height of the pole.
Step 3. Name what we are looking for.	The distance from the base of the pole to either stake is the same as the height of the pole. Let $x =$ the height of the pole. $x =$ the distance from the pole to stake
Each side is a right triangle. We draw a picture of one of them.	
Step 4. Translate into an equation. We can use the Pythagorean Theorem to solve for *x*.	
Write the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Step 5. Solve the equation. Substitute.	$x^2 + x^2 = 10^2$
Simplify.	$2x^2 = 100$
Divide by 2 to isolate the variable.	$\frac{2x^2}{2} = \frac{100}{2}$
Simplify.	$x^2 = 50$

Use the Square Root Property.	$x = \pm\sqrt{50}$
Simplify the radical.	$x = \pm 5\sqrt{2}$
Rewrite to show two solutions.	$x = 5\sqrt{2}$ $x = -5\sqrt{2}$
Approximate this number to the nearest tenth with a calculator.	$x \approx 7.1$
Step 6. Check the answer. Check on your own in the Pythagorean Theorem.	
Step 7. Answer the question.	The pole should be about 7.1 feet tall.

Example 4.4.8

The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth of a foot.

Answer

The length of the shadow is 6.3 feet and the length of the flag pole is 18.9 ft.

Example 4.4.9

The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

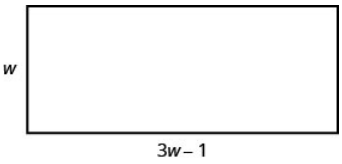
Answer

The distance to the opposite corner is 3.2.

Example 4.4.10

Mike wants to put 150 square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot less than three times the width. Find the length and width. Round to the nearest tenth of a foot.

Answer

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the length and width.
Step 3. Name what we are looking for.	Let $w =$ the width of the rectangle. $3w - 1 =$ the length of the rectangle
Step 4. Translate into an equation. We know the area. Write the formula for the area of a rectangle.	$A = L \cdot W$
Step 5. Solve the equation. Substitute in the values.	$150 = (3w - 1)w$
Distribute.	$150 = 3w^2 - w$

This is a quadratic equation, rewrite it in standard form.	$ax^2+bx+c=0$ $3w^2-w-150=0$
Solve the equation using the Quadratic Formula.	
Identify the a, b, c values.	$a=3,\ b=-1,\ c=-150$
Write the Quadratic Formula.	$w=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$w=\frac{-(-1)\pm\sqrt{(-1)^2-4\cdot 3\cdot(-150)}}{2\cdot 3}$
Simplify.	$w=\frac{1\pm\sqrt{1+1800}}{6}$ $w=\frac{1\pm\sqrt{1801}}{6}$
Rewrite to show two solutions.	$w=\frac{1+\sqrt{1801}}{6},\ w=\frac{1-\sqrt{1801}}{6}$
Approximate the answers using a calculator. We eliminate the negative solution for the width.	$w\approx 7.2,\quad w\approx -6.9$ Width $w\approx 7.2$ Length $\approx 3w-1$ $\approx 3(7.2)-1$ ≈ 20.6
Step 6. Check the answer. Make sure that the answers make sense.	
Step 7. Answer the question.	The width of the rectangle is approximately 7.2 feet and the length 20.6 feet.

Example 4.4.11

The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden. Round to the nearest tenth of a foot..

Answer

The width of the garden is 11 feet and the length is 18 feet.

Example 4.4.12

A rectangular tablecloth has an area of 80 square feet. The width is 5 feet shorter than the length. What are the length and width of the tablecloth? Round to the nearest tenth of a foot.

Answer

The width of the tablecloth is 6.8 feet and the length is 11.8 feet.

The height of a projectile shot upwards is modeled by a quadratic equation. The initial velocity, \(v_{0}\), propels the object up until gravity causes the object to fall back down.

Definition: PROJECTILE MOTION

The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula:

$h = -16t^2 + v_0 t$

We can use the formula for projectile motion to find how many seconds it will take for a firework to reach a specific height.

Example 4.4.13

A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of seconds, which is time.
Step 3. Name what we are looking for.	Let $t =$ the number of seconds.
Step 4. Translate into an equation.	Use the formula.
	$h = -16t^2 + v_0 t$
Step 5. Solve the equation. We know the velocity v_0 is 130 feet per second.	
The height is 260 feet. Substitute the values.	$260 = -16t^2 + 130t$
This is a quadratic equation, rewrite it in standard form.	$\underset{ax^2}{16t^2} + \underset{bx}{130t} + \underset{c}{260} = \underset{0}{0}$
Solve the equation using the Quadratic Formula.	
Identify the a, b, c values.	$a = 16,\ b = -130,\ c = 260$
Write the Quadratic Formula.	$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$t = \frac{-(-130) \pm \sqrt{(-130)^2 - 4 \cdot 16 \cdot (260)}}{2 \cdot 16}$
Simplify.	$t = \frac{130 \pm \sqrt{16{,}900 - 16{,}640}}{32}$ $t = \frac{130 \pm \sqrt{260}}{32}$
Rewrite to show two solutions.	$t = \frac{130 + \sqrt{260}}{32},\ t = \frac{130 - \sqrt{260}}{32}$
Approximate the answers with a calculator.	$t \approx 4.6$ seconds, $t \approx 3.6$
Step 6. Check the answer. The check is left to you.	

Step 7. Answer the question.	The firework will go up and then fall back down. As the firework goes up, it will reach 260 feet after approximately 3.6 seconds. It will also pass that height on the way down at 4.6 seconds. 

? Example 4.4.14

An arrow is shot from the ground into the air at an initial speed of 108 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the arrow will be 180 feet from the ground. Round the nearest tenth of a second.

Answer

The arrow will reach 180 on its way up in 3 seconds, and on the way down in 3.8 seconds.

? Example 4.4.15

A man throws a ball into the air with a velocity of 96 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the height of the ball will be 48 feet. Round to the nearest tenth of a second.

Answer

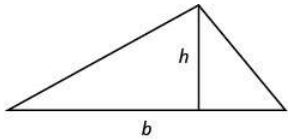
The ball will reach 48 feet on its way up in .6 seconds and on the way down in 5.5 seconds.

Access these online resources for additional instruction and practice with solving word problems using the quadratic equation:

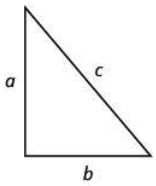
- General Quadratic Word Problems
- Word problem: Solve a projectile problem using a quadratic equation

Key Concepts

- **Area of a Triangle** For a triangle with base, b, and height, h, the area, A, is given by the formula: $A = \frac{1}{2}bh$



- **Pythagorean Theorem** In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypothenuse, $a^2 + b^2 = c^2$



- **Projectile motion** The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after tt seconds can be modeled by the formula

  $h = -16t^2 + v_0 t$

Glossary

consecutive even integers

Consecutive even integers are even integers that follow right after one another. If an even integer is represented by n, the next consecutive even integer is $n+2$, and the next after that is $n+4$.

consecutive odd integers

Consecutive odd integers are odd integers that follow right after one another. If an odd integer is represented by n, the next consecutive odd integer is $n+2$, and the next after that is $n+4$.

This page titled 4.4: Solve Applications Modeled by Quadratic Equations is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **10.4: Solve Applications Modeled by Quadratic Equations** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

CHAPTER OVERVIEW

5: Rational Expressions and Equations

5.1: Simplify Rational Expressions
5.2: Multiply and Divide Rational Expressions
5.3: Add and Subtract Rational Expressions with a Common Denominator
5.4: Add and Subtract Rational Expressions with Unlike Denominators
5.5: Simplify Complex Rational Expressions
5.6: Solve Rational Equations
5.7: Solve Uniform Motion and Work Applications

5: Rational Expressions and Equations is shared under a CC BY-SA license and was authored, remixed, and/or curated by LibreTexts.

5.1: Simplify Rational Expressions

Summary

By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Evaluate rational expressions
- Simplify rational expressions
- Simplify rational expressions with opposite factors

Are you Ready?

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Simplify: $\frac{90y}{15y^2}$.
   If you missed this problem, review [link].
2. Factor: $6x^2-7x+2$.
   If you missed this problem, review [link].
3. Factor: n^3+8.
   If you missed this problem, review [link].

In Chapter 1, we reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers, and the denominator is not zero.

In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call these rational expressions.

Definition: RATIONAL EXPRESSION

A **rational expression** is an expression of the form $\frac{p(x)}{q(x)}$, where p and q are polynomials and $q \neq 0$.

Remember, division by 0 is undefined.

Here are some examples of rational expressions:

$$-\frac{13}{42} \quad \frac{7y}{8z} \quad \frac{5x+2}{x^2-7} \quad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{13}{42}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will perform same operations with rational expressions that we do with fractions. We will simplify, add, subtract, multiply, divide, and use them in applications.

Determine the Values for Which a Rational Expression is Undefined

When we work with a numerical fraction, it is easy to avoid dividing by zero, because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

Definition: DETERMINE THE VALUES FOR WHICH A RATIONAL EXPRESSION IS UNDEFINED.

1. Set the denominator equal to zero.
2. Solve the equation in the set of reals, if possible.

Example 5.1.1

Determine the values for which the rational expression is undefined:

1. $\frac{9y}{x}$
2. $\frac{4b-3}{2b+5}$
3. $\frac{x+4}{x^2+5x+6x}$

Solution

The expression will be undefined when the denominator is zero.

1.	$\frac{9y}{x}$
Set the denominator equal to zero. Solve for the variable.	x=0
	$\frac{9y}{x}$ is undefined for x=0.
2.	$\frac{4b-3}{2b+5}$
Set the denominator equal to zero. Solve for the variable.	2b+5=0
	2b=−5
	$b=-\frac{5}{2}$
	$\frac{4b-3}{2b+5}$ is undefined for $b=-\frac{5}{2}$.
3.	$\frac{x+4}{x^2+5x+6x}$
Set the denominator equal to zero. Solve for the variable.	$x^2+5x+6x=0$
	$(x+2)(x+3)=0$
	x+2=0 or x+3=0
	x=−2 or x=−3
	$\frac{x+4}{x^2+5x+6x}$ is undefined for x=−2 or x=−3.

Saying that the rational expression $\frac{x+4}{x^2+5x+6x}$ is undefined for x=−2orx=−3 is similar to writing the phrase “void where prohibited” in contest rules.

Try It 5.1.2

Determine the values for which the rational expression is undefined:

1. $\frac{3y}{x}$
2. $\frac{8n-5}{3n+1}$
3. $\frac{a+10}{a^2+4a+3a}$

Answer

1. x=0
2. $n=-\frac{1}{3}$
3. a=−1, a=−3

Try It 5.1.3

Determine the values for which the rational expression is undefined:

1. $\frac{4p}{5q}$
2. $\frac{y-1}{3y+2}$
3. $\frac{m-5}{m^2+m-6}$

Answer

1. q=0
2. $y=-\frac{2}{3}$
3. m=2, m=−3

Evaluate Rational Expressions

To evaluate a rational expression, we substitute values of the variables into the expression and simplify, just as we have for many other expressions in this book.

Example 5.1.4

Evaluate $\frac{2x+3}{3x-5}$ for each value:

1. x=0
2. x=2
3. x=−3

Solution

1.	$\frac{2x+3}{3x-5}$
Substitute 0 for x.	$\frac{2(0)+3}{3(0)-5}$
Simplify.	$-\frac{3}{5}$
2.	$\frac{2x+3}{3x-5}$
Substitute 2 for x.	$\frac{2(2)+3}{3(2)-5}$
Simplify.	$\frac{4+3}{6-5}$
	$\frac{7}{1}$
	7
3.	$\frac{2x+3}{3x-5}$
Substitute –3 for x.	$\frac{2(-3)+3}{3(-3)-5}$
Simplify.	$\frac{-6+3}{-9-5}$
	$\frac{-3}{-14}$
	$\frac{3}{14}$

Try It 5.1.5

Evaluate $\frac{y+1}{2y-3}$ for each value:

1. y=1
2. y=−3
3. y=0

Answer

1. −2
2. $\frac{2}{9}$
3. $-\frac{1}{3}$

Try It 5.1.6

Evaluate $\frac{5x-1}{2x+1}$ for each value:

1. x=1
2. x=−1
3. x=0

Answer

1. $\frac{4}{3}$
2. 6
3. −1

Example 5.1.7

Evaluate $\frac{x^2+8x+7}{x^2-4}$ for each value:

1. x=0
2. x=2
3. x=−1

Solution

1.	$\frac{x^2+8x+7}{x^2-4}$
Substitute 0 for x.	$\frac{(0)^2+8(0)+7}{(0)^2-4}$
Simplify.	$\frac{7}{-4}$
	$-\frac{7}{4}$
2.	$\frac{x^2+8x+7}{x^2-4}$
Substitute 2 for x.	$\frac{(2)^2+8(2)+7}{(2)^2-4}$
Simplify.	$\frac{4+16+7}{4-4}$
	$\frac{27}{0}$
This rational expression is undefined for $x = 2$.	

3.	$\frac{x^2+8x+7}{x^2-4}$
Substitute -1 for x.	$\frac{(-1)^2+8(-1)+7}{(-1)^2-4}$
Simplify.	$\frac{1-8+7}{1-4}$
	$\frac{-7+7}{-3}$
	$\frac{0}{-3}$
	0

Try It 5.1.8

Evaluate $\frac{x^2+1}{x^2-3x+2}$ for each value:

1. x=0
2. x=−1
3. x=3

Answer

1. $\frac{1}{2}$
2. $\frac{1}{3}$
3. 2

Try It 5.1.9

Evaluate $\frac{x^2+x-6}{x^2-9}$ for each value.

1. x=0
2. x=−2
3. x=1

Answer

1. $\frac{2}{3}$
2. $\frac{4}{5}$
3. $\frac{1}{2}$

Remember that a fraction is simplified when it has no common factors, other than 1, in its numerator and denominator. When we evaluate a rational expression, we make sure to simplify the resulting fraction.

Example 5.1.10

Evaluate $\frac{a^2+2ab+b^2}{3ab}$ for each value.

1. a=1, b=2
2. a=−2, b=−1
3. $a=\frac{1}{3}$, b=0

Solution

1.	$\frac{a^2+2ab+b^2}{3ab}$ when a=1, b=2

Substitute 1 for a and 2 for b.	$\frac{(1)^2+2(1)(2)+(2)^2}{3(1)(2)^2}$
Simplify.	$\frac{1+4+4}{3(4)}$
	$\frac{9}{12}$
	$\frac{3}{4}$
2.	$\frac{a^2+2ab+b^2}{3ab}$ when a=−2, b=−1
Substitute –2 for a and –1 for b.	$\frac{(-2)^2+2(-2)(-1)+(-1)^2}{3(-2)(-1)^2}$
Simplify.	$\frac{4+4+1}{-6}$
	$-\frac{9}{6}$
	$-\frac{3}{2}$
3.	$\frac{a^2+2ab+b^2}{3ab}$ when $a=\frac{1}{3}$, b=0
Substitute $\frac{1}{3}$ for a and 0 for b.	$\frac{\left(\frac{1}{3}\right)^2+2\left(\frac{1}{3}\right)(0)+(0)^2}{3\left(\frac{1}{3}\right)(0)^2}$
Simplify.	$\frac{\frac{1}{9}+0+0}{0}$
	$\frac{\frac{1}{9}}{0}$

Try It 5.1.11

Evaluate $\frac{2a^3b}{a^2+2ab+b^2}$ for each value.

1. a=−1, b=2
2. a=0, b=−1
3. a=1, $b=\frac{1}{2}$

Answer

1. −4
2. 0
3. $\frac{4}{9}$

Try It 5.1.12

Evaluate $\frac{a^2-b^2}{8ab^3}$ for each value:

1. a=1, b=−1
2. $a=\frac{1}{2}$, b=−1
3. a=−2, b=1

Answer

1. 0
2. $\frac{3}{16}$

3. $\frac{3}{16}$

Simplify Rational Expressions

Just like a fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator, a rational expression is *simplified* if it has no common factors, other than 1, in its numerator and denominator.

Definition: SIMPLIFIED RATIONAL EXPRESSION

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example:

- $\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.
- $\frac{2x}{3x}$ is not simplified because x is a common factor of $2x$ and $3x$.

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify **rational expressions**.

Definition: EQUIVALENT FRACTIONS PROPERTY

If a, b, and c are numbers where $b \neq 0$, $c \neq 0$, then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0$, $c \neq 0$ clearly stated. Every time we write a rational expression, we should make a similar statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

Let's start by reviewing how we simplify numerical fractions.

Example 5.1.13

Simplify: $-\frac{36}{63}$.

Solution

	$-\frac{36}{63}$
Rewrite the numerator and denominator showing the common factors.	$-\frac{4 \cdot 9}{7 \cdot 9}$
Simplify using the Equivalent Fractions Property.	$-\frac{4}{7}$

Notice that the fraction $-\frac{4}{7}$ is simplified because there are no more common factors.

Try It 5.1.14

Simplify: $-\frac{45}{81}$.

Answer

$-\frac{5}{9}$

Try It 5.1.15

Simplify: $-\frac{42}{54}$.

Answer

$-\frac{7}{9}$

Throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$ and $y \neq 0$.

Example 5.1.16

Simplify: $\frac{3xy}{18x^2y^2}$.

Solution

	$\frac{3xy}{18x^2y^2}$
Rewrite the numerator and denominator showing the common factors.	$\frac{1 \cdot 3xy}{6xy \cdot 3xy}$
Simplify using the Equivalent Fractions Property.	$\frac{1}{6xy}$

Did you notice that these are the same steps we took when we divided monomials in Polynomials?

Try It 5.1.17

Simplify: $\frac{4x^2y}{12xy^2}$.

Answer

$\frac{x}{3y}$

Try It $PageIndex18$

Simplify: $\frac{16x^2y}{2xy^2}$.

Answer

$\frac{8x}{y}$

To simplify rational expressions we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

$\frac{2 \cdot \cancel{3} \cdot \cancel{7}}{\cancel{3} \cdot 5 \cdot \cancel{7}}$	$\frac{3x(x-9)}{5(x-9)}$	$\frac{x+5}{x}$
$\frac{2}{5}$	$\frac{3x}{5}$	NO COMMON FACTORS
We removed the common factors 3 and 7. They are factors of the product.	We removed the common factor $(x-9)$. It is a factor of the product.	While there is an x in both the numerator and denominator, the x in the numerator is a term of a sum!

Note that removing the x's from $\frac{x+5}{x}$ would be like cancelling the 2's in the fraction $\frac{2+5}{2}$!

Example 5.1.19

Simplify: $\frac{2x+8}{5x+20}$.

Solution

Step 1. Factor the numerator and denominator completely.	Factor $2x + 8$ and $5x - 20$.	$\frac{2x+8}{5x+20}$ $\frac{2(x+4)}{5(x+4)}$
Step 2. Simplify by dividing out common factors.	Divide out the common factors.	$\frac{2(x+4)}{5(x+4)}$ $\frac{2}{5}$

Try It 5.1.20

Simplify: $\frac{3x-6}{2x-4}$.

Answer

$\frac{3}{2}$

Try It 5.1.21

Simplify: $\frac{7y+35}{5y+25}$.

Answer

$\frac{7}{5}$

We now summarize the steps you should follow to simplify rational expressions.

Definition: SIMPLIFY A RATIONAL EXPRESSION.

1. Factor the numerator and denominator completely.
2. Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way it is easy to check that we have removed all the common factors!

We'll use the methods we covered in Factoring to factor the polynomials in the numerators and denominators in the following examples.

Example 5.1.22

Simplify: $\frac{x^2+5x+6}{x^2+8x+12}$.

Solution

	$\frac{x^2+5x+6}{x^2+8x+12}$
Factor the numerator and denominator.	$\frac{(x+2)(x+3)}{(x+2)(x+6)}$

Remove the common factor x+2 from the numerator and the denominator.	$\frac{x+3}{x+6}$

Can you tell which values of *x* must be excluded in this example?

Try It 5.1.23

Simplify: $\frac{x^2-x-2}{x^2-3x+2}$.

Answer

$\frac{x+1}{x-1}$

Try It 5.1.24

Simplify: $\frac{x^2-3x-10}{x^2+x-2}$.

Answer

$\frac{x-5}{x-1}$

Example 5.1.25

Simplify: $\frac{y^2+y-42}{y^2-36}$.

Solution

	$\frac{y^2+y-42}{y^2-36}$.
Factor the numerator and denominator.	$\frac{(y+7)(y-6)}{(y+6)(y-6)}$
Remove the common factor y−6 from the numerator and the denominator.	$\frac{y+7}{y+6}$

Try It 5.1.26

Simplify: $\frac{x^2+x-6}{x^2-4}$.

Answer

$\frac{x+3}{x+2}$

Try It 5.1.27

Simplify: $\frac{x^2+8x+7}{x^2-49}$.

Answer

$\frac{x+1}{x-7}$

Example 5.1.28

Simplify: $\frac{p^3-2p^2+2p-4}{p^2-7p+10}$.

Solution

	$\frac{p^3-2p^2+2p-4}{p^2-7p+10}$
Factor the numerator and denominator, using grouping to factor the numerator.	$\frac{p^2(p-2)+2(p-2)}{(p-5)(p-2)}$
	$\frac{(p^2+2)(p-2)}{(p-5)(p-2)}$
Remove the common factor p−2 from the numerator and the denominator.	$\frac{p^2+2}{p-5}$

Try It 5.1.29

Simplify: $\frac{y^3-3y^2+y-3}{y^2-y-6}$.

Answer

$\frac{y^2+1}{y+2}$

Try It 5.1.30

Simplify: $\frac{p^3-p^2+2p-2}{p^2+4p-5}$.

Answer

$\frac{p^2+2}{p+5}$

Example 5.1.31

Simplify: $\frac{2n^2-14n}{4n^2-16n-48}$.

Solution

	$\frac{2n^2-14n}{4n^2-16n-48}$
Factor the numerator and denominator, first factoring out the GCF.	$\frac{2n(n-7)}{4(n^2-4n-12)}$
	$\frac{2n(n-7)}{4(n-6)(n+2)}$
Remove the common factor, 2.	$\frac{n(n-7)}{2(n-6)(n+2)}$

Try It 5.1.32

Simplify: $\frac{2n^2-10n}{4n^2-16n-20}$.

Answer

$\frac{n}{2(n+1)}$

Try It 5.1.33

Simplify: $\frac{4x^2-16x}{8x^2-16x-64}$.

Answer

$\frac{x}{2(x+2)}$

Example 5.1.34

Simplify: $\frac{3b^2-12b+12}{6b^2-24}$.

Solution

	$\frac{3b^2-12b+12}{6b^2-24}$
Factor the numerator and denominator, first factoring out the GCF.	$\frac{3(b^2-4b+4)}{6(b^2-4)}$
	$\frac{3(b-2)(b-2)}{6(b-2)(b+2)}$
Remove the common factors of b−2 and 3.	$\frac{3(b-2)}{2(b+2)}$

Try It 5.1.35

Simplify: $\frac{2x^2-12x+18}{3x^2-27}$.

Answer

$\frac{2(x-3)}{3(x+3)}$

Try It $PageIndex36$

Simplify: $\frac{5y^2-30y+25}{2y^2-50}$.

Answer

$\frac{5(x-1)}{2(x+5)}$

Example 5.1.37

Simplify: $\frac{m^3+8}{m^2-4}$.

Solution

	$\frac{m^3+8}{m^2-4}$
Factor the numerator and denominator, using the formulas for sum of cubes and difference of squares.	$\frac{(m+2)(m^2-2m+4)}{(m+2)(m-2)}$

Remove the common factors of m+2.	$\frac{m^2-2m+4}{m-2}$

Try It 5.1.38

Simplify: $\frac{p^3-64}{p^2-16}$.

Answer

$\frac{p^2+4p+16}{p+4}$

Try It 5.1.39

Simplify: $\frac{x^3+8}{x^2-4}$.

Answer

$\frac{x^2-2x+4}{x-2}$

Simplify Rational Expressions with Opposite Factors

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. Let's start with a numerical fraction, say $\frac{7}{-7}$.

We know this fraction simplifies to −1. We also recognize that the numerator and denominator are opposites.

In Foundations, we introduced opposite notation: the opposite of a is −a. We remember, too, that −a=−1·a

We simplify the fraction $\frac{a}{-a}$

	$\frac{a}{-a}$
We could rewrite this.	$\frac{1\cdot a}{-1\cdot a}$
Remove the common factors.	$\frac{1}{-1}$
Simplify.	-1

So, in the same way, we can simplify the fraction $\frac{x-3}{-(x-3)}$

	$\frac{x-3}{-(x-3)}$
We could rewrite this.	$\frac{1\cdot(x-3)}{-1\cdot(x-3)}$
Remove the common factors.	$\frac{1}{-1}$
Simplify.	-1

But the opposite of x−3 could be written differently:

	$-(x-3)$
Distribute.	$-x+3$
Rewrite.	$3-x$

This means the fraction $\frac{x-3}{3-x}$ simplifies to −1.

In general, we could write the opposite of a−b as b−a. So the rational expression $\frac{a-b}{b-a}$ simplifies to −1.

Definition: OPPOSITES IN A RATIONAL EXPRESSION

The opposite of a−b is b−a

$\frac{a-b}{b-a}=-1, a \neq b$

An expression and its opposite divide to −1

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators.

Example 5.1.40

Simplify: $\frac{x-8}{8-x}$.

Solution

	$\frac{x-8}{8-x}$.
Recognize that x−8 and 8−x are opposites	−1

Try It 5.1.41

Simplify: $\frac{y-2}{2-y}$.

Answer

−1

Try It 5.1.42

Simplify: $\frac{n-9}{9-n}$.

Answer

−1

Remember, the first step in simplifying a rational expression is to factor the numerator and denominator completely.

Example 5.1.43

Simplify: $\frac{14-2x}{x^2-49}$.

Solution

	$\frac{14-2x}{x^2-49}$
Factor the numerator and denominator.	$\frac{2(7-x)}{(x+7)(x-7)}$
Recognize 7−x and x−7 are opposites.	$\frac{2(7-x)}{(x+7)(x-7)}(-1)$
Simplify.	$-\frac{2}{x+7}$

Try It 5.1.44

Simplify: $\frac{10-2y}{y^2-25}$.

Answer

$-\frac{2}{y+5}$

Try It 5.1.45

Simplify: $\frac{3y-27}{81-y^2}$.

Answer

$-\frac{3}{9+y}$

Example 5.1.46

Simplify: $\frac{x^2-4x-32}{64-x^2}$.

Solution

	$\frac{x^2-4x-32}{64-x^2}$
Factor the numerator and denominator.	$\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Recognize the factors that are opposites.	$(-1)\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Simplify.	$-\frac{x+4}{x+8}$

Try It 5.1.47

Simplify: $\frac{x^2-4x-5}{25-x^2}$.

Answer

$-\frac{x+1}{x+5}$

Try It 5.1.48

Simplify: $\frac{x^2+x-2}{1-x^2}$.

Answer

$-\frac{x+2}{x+1}$

Key Concepts

- **Determine the Values for Which a Rational Expression is Undefined**
  1. Set the denominator equal to zero.
  2. Solve the equation, if possible.
- **Simplified Rational Expression**

- A rational expression is considered simplified if there are no common factors in its numerator and denominator.

- **Simplify a Rational Expression**
  1. Factor the numerator and denominator completely.
  2. Simplify by dividing out common factors.

- **Opposites in a Rational Expression**
  - The opposite of a−b is b−a
    $\frac{a-b}{b-a} = -1 \; a \neq b, b \neq 0, a \neq b$

Glossary

rational expression

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

This page titled 5.1: Simplify Rational Expressions is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **8.1: Simplify Rational Expressions** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

5.2: Multiply and Divide Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Multiply rational expressions
- Divide rational expressions

Note

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Multiply: $\frac{14}{15} \cdot \frac{6}{35}$.
   If you missed this problem, review Exercise 1.6.13.
2. Divide: $\frac{14}{15} \div \frac{6}{35}$.
   If you missed this problem, review Exercise 1.6.22.
3. Factor completely: $2x^2 - 98$.
   If you missed this problem, review Exercise 7.5.10.
4. Factor completely: $10n^3 + 10$.
   If you missed this problem, review Exercise 7.5.19.
5. Factor completely: $10p^2 - 25pq - 15q^2$.
   If you missed this problem, review Exercise 7.5.28.

Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

Definition: MULTIPLICATION OF RATIONAL EXPRESSIONS

If p,q,r,s are polynomials where $q \neq 0$ and $s \neq 0$

$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$

To multiply rational expressions, multiply the numerators and multiply the denominators.

We'll do the first example with numerical fractions to remind us of how we multiplied fractions without variables.

Example 5.2.1

Multiply: $\frac{10}{28} \cdot \frac{8}{15}$.

Answer

	$\frac{10}{28} \cdot \frac{8}{15}$
Multiply the numerators and denominators.	$\frac{10 \cdot 8}{28 \cdot 15}$
Look for common factors, and then remove them.	$\frac{2 \cdot \cancel{5} \cdot 2 \cdot \cancel{4}}{7 \cdot \cancel{4} \cdot 3 \cdot \cancel{5}}$
Simplify.	$\frac{4}{21}$

Example 5.2.2

Mulitply: $\frac{6}{10} \cdot \frac{15}{12}$.

Answer

$\frac{3}{4}$

Example 5.2.3

Mulitply: $\frac{20}{15} \cdot \frac{6}{8}$.

Answer

1

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$ and $y \neq 0$.

Example 5.2.4

Mulitply: $\frac{2x}{3y^2} \cdot \frac{6xy^3}{x^2y}$.

Answer

	$\frac{2x}{3y^2} \cdot \frac{6xy^3}{x^2y}$
Multiply.	$\frac{2x \cdot 6xy^3}{3y^2 \cdot x^2y}$
Factor the numerator and denominator completely, and then remove common factors.	$\frac{2 \cdot x \cdot 2 \cdot 3 \cdot x \cdot y \cdot y \cdot y}{3 \cdot y \cdot y \cdot x \cdot x \cdot y}$
Simplify.	4

Example 5.2.5

Mulitply: $\frac{3pq}{q^2} \cdot \frac{5p^2q}{6pq}$.

Answer

$\frac{5p^2}{2q}$

Example 5.2.6

Mulitply: $\frac{6x^3y}{7x^2} \cdot \frac{2xy}{3x^2y}$.

Answer

$\frac{12y^3}{7}$

How to Multiply Rational Expressions

Example 5.2.7

Mulitply: $\frac{2x}{x^2-7x+12} \cdot \frac{x^2-9}{6x^2}$.

Answer

Step 1. Factor the numerator and denominator completely.	Factor $x^2 - 9$ and $x^2 + x + 12$.	$\frac{2x}{x^2-7x+12} \cdot \frac{x^2-9}{6x^2}$ $\frac{2x}{(x-3)(x-4)} \cdot \frac{(x-3)(x+3)}{6x^2}$
Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$\frac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$
Step 3. Simplify by dividing out common factors.	Divide out the common factors. Leave the denominator in factored form.	$\frac{\cancel{2x}\,\cancel{(x-3)}(x+3)}{\cancel{2}\cdot 3\cdot\cancel{x}\cdot x\,\cancel{(x-3)}(x-4)}$ $\frac{(x+3)}{3x(x-4)}$

Example 5.2.8

Mulitply: $\frac{5x}{x^2+5x+6} \cdot \frac{x^2-4}{10x}$.

Answer

$\frac{x-2}{2(x+3)}$

Example 5.2.9

Mulitply: $\frac{9x^2}{x^2+11x+30} \cdot \frac{x^2-36}{3x^2}$.

Answer

$\frac{3(x-6)}{x+5}$

Definition: MULTIPLY A RATIONAL EXPRESSION.

1. Factor each numerator and denominator completely.
2. Multiply the numerators and denominators.
3. Simplify by dividing out common factors.

Example 5.2.10

Multiply: $\frac{n^2-7n}{n^2+2n+1} \cdot \frac{n+1}{2n}$.

Answer

	$\frac{n^2-7n}{n^2+2n+1} \cdot \frac{n+1}{2n}$
Factor each numerator and denominator.	$\frac{n(n-7)}{(n+1)(n+1)} \cdot \frac{n+1}{2n}$
Multiply the numerators and the denominators.	$\frac{n(n-7)(n+1)}{(n+1)(n+1)2n}$
Simplify.	$\frac{n-7}{2(n+1)}$

Example 5.2.11

Multiply: $\frac{x^2-25}{x^2-3x-10} \cdot \frac{x+2}{x}$.

Answer

$\frac{x+5}{x}$

Example 5.2.12

Multiply: $\frac{x^2-4x}{x^2+5x+6} \cdot \frac{x+2}{x}$.

Answer

$\frac{x-4}{x+3}$

Example 5.2.13

Multiply: $\frac{16-4x}{2x-12} \cdot \frac{x^2-5x-6}{x^2-16}$.

Answer

	$\frac{16-4x}{2x-12} \cdot \frac{x^2-5x-6}{x^2-16}$
Factor each numerator and denominator.	$\frac{4(4-x)}{2(x-6)} \cdot \frac{(x-6)(x+1)}{(x-4)(x+4)}$
Multiply the numerators and the denominators.	$\frac{4(4-x)(x-6)(x+1)}{2(x-6)(x-4)(x+4)}$
Simplify.	$-\frac{2(x+1)}{(x+4)}$

Example 5.2.14

Multiply: $\frac{12x-6x^2}{x^2+8x} \cdot \frac{x^2+11x+24}{x^2-4}$.

Answer

$-\frac{6(x+3)}{x+2}$

Example 5.2.15

Multiply: $\frac{9v-3v^2}{9v+36} \cdot \frac{v^2+7v+12}{v^2-9}$.

Answer

$-\frac{v}{3}$

Example 5.2.16

Multiply: $\frac{2x-6}{x^2-8x+15} \cdot \frac{x^2-25}{2x+10}$.

Answer

	$\frac{2x-6}{x^2-8x+15} \cdot \frac{x^2-25}{2x+10}$
Factor each numerator and denominator.	$\frac{2(x-3)}{(x-3)(x-5)} \cdot \frac{(x-5)(x+5)}{2(x+5)}$
Multiply the numerators and denominators.	$\frac{2(x-3)(x-5)(x+5)}{2(x-3)(x-5)(x+5)}$
Remove common factors.	$\frac{\cancel{2}\cancel{(x-3)}\cancel{(x-5)}\cancel{(x+5)}}{\cancel{2}\cancel{(x-3)}\cancel{(x-5)}\cancel{(x+5)}}$
Simplify.	1

Example 5.2.17

Multiply: $\frac{3a-21}{a^2-9a+14} \cdot \frac{a^2-4}{3a+6}$.

Answer

1

Example 5.2.18

Multiply: $\frac{b^2-b}{b^2+9b-10} \cdot \frac{b^2-100}{b^2-10b}$.

Answer

1

Divide Rational Expressions

To divide rational expressions we multiply the first fraction by the reciprocal of the second, just like we did for numerical fractions.

Remember, the **reciprocal** of $\frac{a}{b}$ is $\frac{b}{a}$. To find the reciprocal we simply put the numerator in the denominator and the denominator in the numerator. We "flip" the fraction.

Definition: DIVISION OF RATIONAL EXPRESSIONS

If p,q,r,s are polynomials where $q \neq 0$, $r \neq 0$, $s \neq 0$

$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$

To divide rational expressions multiply the first fraction by the reciprocal of the second.

How to Divide Rational Expressions

Example 5.2.19

Divide: $\frac{x+9}{6-x} \div \frac{x^2-81}{x-6}$.

Answer

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	"Flip" the second fraction and change the division sign to multiplication.	$\frac{x+9}{6-x} \div \frac{x^2-81}{x-6}$ $\frac{x+9}{6-x} \cdot \frac{x-6}{x^2-81}$
Step 2. Factor the numerators and denominators completely.	Factor x^2-81.	$\frac{x+9}{6-x} \cdot \frac{x-6}{(x-9)(x+9)}$
Step 3. Multiply the numerators and denominators.		$\frac{(x+9)(x-6)}{(6-x)(x-9)(x+9)}$
Step 4. Simplify by dividing out common factors.	Divide out the common factors. Remember opposites divide to –1.	$(-1)\frac{\cancel{(x+9)}\cancel{(x-6)}}{\cancel{(6-x)}(x-9)\cancel{(x+9)}}$ $-\frac{1}{(x-9)}$

Example 5.2.20

Divide: $\frac{c+3}{5-c} \div \frac{c^2-9}{c-5}$.

Answer

$-\frac{1}{c-3}$

Example 5.2.21

Divide: $\frac{2-d}{d-4} \div \frac{4-d^2}{4-d}$.

Answer

$-\frac{1}{2+d}$

Definition: DIVIDE RATIONAL EXPRESSIONS.

1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
2. Factor the numerators and denominators completely.
3. Multiply the numerators and denominators together.
4. Simplify by dividing out common factors.

Example 5.2.22

Divide: $\frac{3n^2}{n^2-4n} \div \frac{9n^2-45n}{n^2-7n+10}$.

Answer

	$\frac{3n^2}{n^2-4n} \div \frac{9n^2-45n}{n^2-7n+10}$
Rewrite the division as the product of the first rational expression and the reciprocal of the second.	$\frac{3n^2}{n^2-4n} \cdot \frac{n^2-7n+10}{9n^2-45n}$
Factor the numerators and denominators and then multiply.	$\frac{3 \cdot n \cdot n \cdot (n-5)(n-2)}{n(n-4) \cdot 3 \cdot 3 \cdot n \cdot (n-5)}$
Simplify by dividing out common factors.	$\frac{\cancel{3} \cdot \cancel{n} \cdot \cancel{n(n-5)}(n-2)}{\cancel{n}(n-4)\cancel{3} \cdot 3 \cdot \cancel{n(n-5)}}$

	$\frac{n-2}{3(n-4)}$

Example 5.2.23

Divide: $\frac{2m^2}{m^2-8m} \div \frac{8m^2+24m}{m^2+m-6}$.

Answer

$\frac{(m-2)}{4(m-8)}$

Example 5.2.24

Divide: $\frac{15n^2}{3n^2+33n} \div \frac{5n-5}{n^2+9n-22}$.

Answer

$\frac{n(n-2)}{n-1}$

Remember, first rewrite the division as multiplication of the first expression by the reciprocal of the second. Then factor everything and look for common factors.

Example 5.2.25

Divide: $\frac{2x^2+5x-12}{x^2-16} \div \frac{2x^2-13x+15}{x^2-8x+16}$.

Answer

	$\frac{2x^2+5x-12}{x^2-16} \div \frac{2x^2-13x+15}{x^2-8x+16}$
Rewrite the division as the product of the first rational expression and the reciprocal of the second.	$\frac{2x^2+5x-12}{x^2-16} \cdot \frac{x^2-8x+16}{2x^2-13x+15}$
Factor the numerators and denominators and then multiply.	$\frac{(2x-3)(x+4)(x-4)(x-4)}{(x-4)(x+4)(2x-3)(x-5)}$
Simplify.	$\frac{(x-4)}{(x-5)}$

Example 5.2.26

Divide: $\frac{3a^2-8a-3}{a^2-25} \div \frac{3a^2-14a-5}{a^2+10a+25}$.

Answer

$\frac{(a-3)(a+5)}{(a-5)(a-5)}$

Exercise 5.2.27

Divide: $\frac{4b^2+7b-2}{1-b^2} \div \frac{4b^2+15b-4}{b^2-2b+1}$.

Answer

$-\frac{(b+2)(b-1)}{(1+b)(b+4)}$

Example 5.2.28

Divide: $\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$.

Answer

	$\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$
Rewrite the division as the product of the first rational expression and the reciprocal of the second.	$\frac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \frac{6}{p^2-q^2}$
Factor the numerators and denominators and then multiply.	$\frac{(p+q)(p^2-pq+q^2)6}{2(p^2+pq+q^2)(p-q)(p+q)}$
Simplify.	$\frac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

Example 5.2.29

Divide: $\frac{x^3-8}{3x^2-6x+12} \div \frac{x^2-4}{6}$.

Answer

$\frac{2(x^2+2x+4)}{(x+2)(x^2-2x+4)}$

Example 5.2.30

Divide: $\frac{2z^2}{z^2-1} \div \frac{z^3-z^2+z}{z^3-1}$.

Answer

$\frac{2z(z^2+z+1)}{(z+1)(z^2-z+1)}$

Before doing the next example, let's look at how we divide a fraction by a whole number. When we divide $\frac{3}{5} \div 4$

$$\frac{3}{5} \div 4$$
$$\frac{3}{5} \div \frac{4}{1}$$
$$\frac{3}{5} \cdot \frac{1}{4}$$

We do the same thing when we divide rational expressions.

Example 5.2.31

$\frac{a^2-b^2}{3ab} \div (a^2+2ab+b^2)$.

Answer

	$\frac{a^2-b^2}{3ab} \div (a^2+2ab+b^2)$
Write the second expression as a fraction.	$\frac{a^2-b^2}{3ab} \div \frac{a^2+2ab+b^2}{1}$
Rewrite the division as the first expression times the reciprocal of the second expression.	$\frac{a^2-b^2}{3ab} \cdot \frac{1}{a^2+2ab+b^2}$
Factor the numerators and the denominators, and then multiply.	$\frac{(a-b)(a+b)1}{3ab \cdot (a+b)(a+b)}$

Simplify.	$\frac{a-b}{3ab(a+b)}$

Example 5.2.32

$\frac{2x^2-14x-16}{4} \div (x2+2x+1)$.

Answer

$\frac{x-8}{2(x+1)}$

Example 5.2.33

$\frac{y^2-6y+8}{y^2-4y} \div (3y2-12y)$.

Answer

$\frac{y-2}{3y(y-4)}$

Example 5.2.34

$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$.

Answer

	$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$
Rewrite with a division sign.	$\frac{6x^2-7x+2}{4x-8} \div \frac{2x^2-7x+3}{x^2-5x+6}$
Rewrite as product of first times reciprocal of second.	$\frac{6x^2-7x+2}{4x-8} \cdot \frac{x^2-5x+6}{2x^2-7x+3}$
Factor the numerators and the denominators, and then multiply	$\frac{(2x-1)(3x-2)(x-2)(x-3)}{4(x-2)(2x-1)(x-3)}$
Simplify.	$\frac{3x-2}{4}$

Example 5.2.35

$\frac{\frac{3x^2+7x+2}{4x+24}}{\frac{3x^2-14x-5}{x^2+x-30}}$.

Answer

$\frac{x+2}{4}$

Example 5.2.36

$\frac{\frac{y^2-36}{2y^2+11y-6}}{\frac{2y^2-2y-60}{8y-4}}$.

Answer

$\frac{2}{y+5}$

If we have more than two rational expressions to work with, we still follow the same procedure. The first step will be to rewrite any division as multiplication by the reciprocal. Then we factor and multiply.

Example 5.2.37

$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$.

Answer

	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$
Rewrite the division as multiplication by the reciprocal.	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \cdot \frac{8x+16}{2x+12}$
Factor the numerators and the denominators, and then multiply.	$\frac{3 \cdot 8(x-2)(x+3)(x-1)(x+2)}{4 \cdot 2(x-1)(x+2)(x-5)(x+6)}$
Simplify by dividing out common factors.	$\frac{3 \cdot \cancel{8}(x-2)(x+3)\cancel{(x-1)(x+2)}}{\cancel{4} \cdot \cancel{2}\cancel{(x-1)(x+2)}(x-5)(x+6)}$
Simplify.	$\frac{3(x-2)(x+3)}{(x-5)(x+6)}$

Example 5.2.38

$\frac{4m+4}{3m-15} \cdot \frac{m^2-3m-10}{m^2-4m-32} \div \frac{12m-36}{6m-48}$.

Answer

$\frac{2(m+1)(m+2)}{3(m+4)(m-3)}$

Example 5.2.39

$\frac{2n^2+10n}{n-1} \div \frac{n^2+10n+24}{n^2+8n-9} \cdot \frac{n+4}{8n^2+12n}$.

Answer

$\frac{(n+5)(n+9)}{2(n+6)(2n+3)}$

Key Concepts

- **Multiplication of Rational Expressions**
  - If p,q,r,s are polynomials where $q \neq 0$ and $s \neq 0$, then $\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$
  - To multiply rational expressions, multiply the numerators and multiply the denominators
- **Multiply a Rational Expression**
  1. Factor each numerator and denominator completely.
  2. Multiply the numerators and denominators.
  3. Simplify by dividing out common factors.
- **Division of Rational Expressions**
  - If p,q,r,s are polynomials where $q \neq 0$, $r \neq 0$, $s \neq 0$, then $\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$
  - To divide rational expressions multiply the first fraction by the reciprocal of the second.
- **Divide Rational Expressions**

1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
2. Factor the numerators and denominators completely.
3. Multiply the numerators and denominators together.
4. Simplify by dividing out common factors.

This page titled 5.2: Multiply and Divide Rational Expressions is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **8.2: Multiply and Divide Rational Expressions** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

5.3: Add and Subtract Rational Expressions with a Common Denominator

Learning Objectives

By the end of this section, you will be able to:

- Add rational expressions with a common denominator
- Subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites

Note

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Add: $\frac{y}{3}+\frac{9}{3}$.
   If you missed this problem, review Exercise 1.7.1.
2. Subtract: $\frac{10}{x}-\frac{2}{x}$.
   If you missed this problem, review Exercise 1.7.7.
3. Factor completely: $8n^5-20n^3$.
   If you missed this problem, review Exercise 7.5.1.
4. Factor completely: $45a^3-5ab^2$.
   If you missed this problem, review Exercise 7.5.10.

Add Rational Expressions with a Common Denominator

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, you add the numerators and place the sum over the common denominator.

Definition: RATIONAL EXPRESSION ADDITION

If p, q, and r are polynomials where $r\neq 0$, then

$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r}$

To add rational expressions with a common denominator, add the numerators and place the sum over the common denominator.

We will add two numerical fractions first, to remind us of how this is done.

Example 5.3.1

Add: $\frac{5}{18}+\frac{7}{18}$.

Answer

	$\frac{5}{18}+\frac{7}{18}$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{5+7}{18}$
Add in the numerator.	$\frac{12}{18}$
Factor the numerator and denominator to show the common factors.	$\frac{6\cdot 2}{6\cdot 3}$

Simplify.	$\frac{2}{3}$

Example 5.3.2

Add: $\frac{7}{16}+\frac{5}{16}$.

Answer

$\frac{3}{4}$

Example 5.3.3

Add: $\frac{3}{10}+\frac{1}{10}$.

Answer

$\frac{2}{5}$

Remember, we do not allow values that would make the denominator zero. What value of yy should be excluded in the next example?

Example 5.3.4

Add: $\frac{3y}{4y-3}+\frac{7}{4y-3}$.

Answer

	$\frac{3y}{4y-3}+\frac{7}{4y-3}$.
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{3y+7}{4y-3}$
The numerator and denominator cannot be factored. The fraction is simplified.	

Example 5.3.5

Add: $\frac{5x}{2x+3}+\frac{2}{2x+3}$.

Answer

$\frac{5x+2}{2x+3}$.

Example 5.3.6

Add: $\frac{x}{x-2}+\frac{1}{x-2}$.

Answer

$\frac{x+1}{x-2}$

Example 5.3.7

Add: $\frac{7x+12}{x+3}+\frac{x^2}{x+3}$.

Answer

	$\frac{7x+12}{x+3}+\frac{x^2}{x+3}$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{7x+12+x^2}{x+3}$
Write the degrees in descending order.	$\frac{x^2+7x+12}{x+3}$
Factor the numerator.	$\frac{(x+3)(x+4)}{x+3}$
Simplify.	x+4

Example 5.3.8

Add: $\frac{9x+14}{x+7}+\frac{x^2}{x+7}$.

Answer

x+2

Example 5.3.9

Add: $\frac{x^2+8x}{x+5}+\frac{15}{x+5}$.

Answer

x+3

Subtract Rational Expressions with a Common Denominator

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, you subtract the numerators and place the difference over the common denominator.

Definition: RATIONAL EXPRESSION SUBTRACTION

If p, q, and r are polynomials where $r \neq 0$

$\frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$

To subtract rational expressions, subtract the numerators and place the difference over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after you subtract the numerators so you can identify any common factors.

Example 5.3.10

Subtract: $\frac{n^2}{n-10}-\frac{100}{n-10}$.

Answer

	$\frac{n^2}{n-10}-\frac{100}{n-10}$

The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{n^2-100}{n-10}$
Factor the numerator.	$\frac{(n-10)(n+10)}{n-10}$
Simplify.	n+10

Example 5.3.11

Subtract: $\frac{x^2}{x+3}-\frac{9}{x+3}$.

Answer

x−3

Example 5.3.12

Subtract: $\frac{4x^2}{2x-5}-\frac{25}{2x-5}$.

Answer

2x+5

Be careful of the signs when you subtract a binomial!

Example 5.3.13

Subtract: $\frac{y^2}{y-6}-\frac{2y+24}{y-6}$.

Answer

	$\frac{y^2}{y-6}-\frac{2y+24}{y-6}$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{y^2-(2y+24)}{y-6}$
Distribute the sign in the numerator.	$\frac{y^2-2y-24}{y-6}$
Factor the numerator.	$\frac{(y-6)(y+4)}{y-6}$
Simplify.	y+4

Example 5.3.14

Subtract: $\frac{n^2}{n-4}-\frac{n+12}{n-4}$.

Answer

n+3

Example 5.3.15

Subtract: $\frac{y^2}{y-1}-\frac{9y-8}{y-1}$.

Answer

y−8

Example 5.3.16

Subtract: $\frac{5x^2-7x+3}{x^2-3x-18} - \frac{4x^2+x-9}{x^2-3x-18}$.

Answer

	$\frac{5x^2-7x+3}{x^2-3x+18} - \frac{4x^2+x-9}{x^2-3x+18}$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{5x^2-7x+3-(4x^2+x-9)}{x^2-3x+18}$
Distribute the sign in the numerator.	$\frac{5x^2-7x+3-4x^2-x+9}{x^2-3x+18}$
Combine like terms.	$\frac{x^2-8x+12}{x^2-3x+18}$
Factor the numerator and the denominator.	$\frac{(x-2)(x-6)}{(x+3)(x-6)}$
Simplify.	$\frac{x-2}{x+3}$

Example 5.3.17

Subtract: $\frac{4x^2-11x+8}{x^2-3x+2} - \frac{3x^2+x-3}{x^2-3x+2}$.

Answer

$\frac{x-11}{x-2}$

Example 5.3.18

Subtract: $\frac{6x^2-x+20}{x^2-81} - \frac{5x^2+11x-7}{x^2-81}$.

Answer

$\frac{x-3}{x+9}$

Add and Subtract Rational Expressions whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $\frac{-1}{-1}$

Let's see how this works.

	$\frac{7}{d} + \frac{5}{-d}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{7}{d} + \frac{(-1)5}{(-1)(-d)}$
The denominators are the same.	$\frac{7}{d} + \frac{-5}{d}$
Simplify.	$\frac{2}{d}$

Example 5.3.19

Add: $\frac{4u-1}{3u-1}+\frac{u}{1-3u}$.

Answer

	$\frac{4u-1}{3u-1}+\frac{u}{1-3u}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{4u-1}{3u-1}+\frac{(-1)u}{(-1)(1-3u)}$
Simplify the second fraction.	$\frac{4u-1}{3u-1}+\frac{-u}{3u-1}$
The denominators are the same. Add the numerators.	$\frac{4u-1-u}{3u-1}$
Simplify.	$\frac{3u-1}{3u-1}$
Simplify.	1

Example 5.3.20

Add: $\frac{8x-15}{2x-5}+\frac{2x}{5-2x}$.

Answer

3

Example 5.3.21

Add: $\frac{6y^2+7y-10}{4y-7}+\frac{2y^2+2y+11}{7-4y}$.

Answer

y+3

Example 5.3.22

Subtract: $\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$.

Answer

	$\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{m^2-6m}{m^2-1}-\frac{-1(3m+2)}{-1(1-m^2)}$
Simplify the second fraction.	$\frac{m^2-6m}{m^2-1}-\frac{-3m-2}{m^2-1}$
The denominators are the same. Subtract the numerators.	$\frac{m^2-6m-(-3m-2)}{m^2-1}$
Distribute.	m2−6m+3m+2m2−1
Combine like terms.	$\frac{m^2-3m+2}{m^2-1}$
Factor the numerator and denominator.	$\frac{(m-1)(m-2)}{(m-1)(m+1)}$
Simplify by removing common factors.	$\frac{\cancel{(m-1)}(m-2)}{\cancel{(m-1)}(m+1)}$

Simplify.	$\frac{m-2}{m+1}$

? Example 5.3.23

Subtract: $\frac{y^2-5y}{y^2-4} - \frac{6y-6}{4-y^2}$.

Answer

$\frac{y+3}{y+2}$

? Example 5.3.24

Subtract: $\frac{2n^2+8n-1}{n^2-1} - \frac{n^2-7n-1}{1-n^2}$.

Answer

$\frac{3n-2}{n-1}$

Key Concepts

- **Rational Expression Addition**
  - If p, q, and r are polynomials where $r \neq 0$, then

    $\frac{p}{r} + \frac{q}{r} = \frac{p+q}{r}$
  - To add rational expressions with a common denominator, add the numerators and place the sum over the common denominator.
- **Rational Expression Subtraction**
  - If p, q, and r are polynomials where $r \neq 0$

    $\frac{p}{r} - \frac{q}{r} = \frac{p-q}{r}$
  - To subtract rational expressions, subtract the numerators and place the difference over the common denominator.

This page titled 5.3: Add and Subtract Rational Expressions with a Common Denominator is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **8.3: Add and Subtract Rational Expressions with a Common Denominator** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

5.4: Add and Subtract Rational Expressions with Unlike Denominators

Learning Objectives

By the end of this section, you will be able to:

- Find the least common denominator of rational expressions
- Find equivalent rational expressions
- Add rational expressions with different denominators
- Subtract rational expressions with different denominators

Note

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Add: $\frac{7}{10}+\frac{8}{15}$.
   If you missed this problem, review Exercise 1.7.13.
2. Subtract: $6(2x+1)-4(x-5)$.
   If you missed this problem, review Exercise 1.10.52.
3. Find the Greatest Common Factor of $9x^2y^3$ and $12xy^5$
   If you missed this problem, review Exercise 7.1.7.
4. Factor completely −48n−12
   If you missed this problem, review Exercise 7.1.31.

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let's look at the example $\frac{7}{12}+\frac{5}{18}$ from Foundations. Since the denominators are not the same, the first step was to find the least common denominator (LCD). Remember, the LCD is the least common multiple of the denominators. It is the smallest number we can use as a common denominator.

To find the LCD of 12 and 18, we factored each number into primes, lining up any common primes in columns. Then we "brought down" one prime from each column. Finally, we multiplied the factors to find the LCD.

12=2·2·3

18=2·3·3

LCD=2·2·3·3

LCD=36

We do the same thing for rational expressions. However, we leave the LCD in factored form.

Definition: FIND THE LEAST COMMON DENOMINATOR OF RATIONAL EXPRESSIONS.

1. Factor each expression completely.
2. List the factors of each expression. Match factors vertically when possible.
3. Bring down the columns.
4. Multiply the factors.

Remember, we always exclude values that would make the denominator zero. What values of *xv*should we exclude in this next example?

Example 5.4.1

Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$

Answer

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each expression completely, lining up common factors. Bring down the columns.	$x^2-2x-3=(x-3)(x+1)$
	$x^2+4x+3=(x+1)(x+3)$
Multiply the factors.	LCD=(x+1)(x−3)(x+3)
	The LCD is (x+1)(x−3)(x+3).

Example 5.4.2

Find the LCD for $\frac{2}{x^2-x-12}, \frac{1}{x^2-16}$

Answer

(x−4)(x+4)(x+3)

Example 5.4.3

Find the LCD for $\frac{x}{x^2+8x+15}, \frac{5}{x^2+9x+18}$

Answer

(x+3)(x+6)(x+5)

Find Equivalent Rational Expressions

When we add numerical fractions, once we find the LCD, we rewrite each fraction as an equivalent fraction with the LCD.

$\frac{7}{12}+\frac{5}{18}$

$\frac{7\cdot 3}{12\cdot 3}+\frac{5\cdot 2}{18\cdot 2}$

$\frac{21}{36}+\frac{10}{36}$

$12=2\cdot 2\cdot 3$
$18=2\cdot \quad 3\cdot 3$
$LCD=2\cdot 2\cdot 3\cdot 3$
$LCD=36$

We will do the same thing for rational expressions.

Example 5.4.4

Rewrite as equivalent rational expressions with denominator (x+1)(x−3)(x+3): $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.

Answer

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each denominator.	$\frac{8}{(x+1)(x-3)}, \frac{3x}{(x+1)(x+3)}$
Find the LCD. $x^2-2x-3=(x+1)(x-3)$, $x^2+4x+3=(x+1)(x+3)$, LCD $=(x+1)(x-3)(x+3)$	

Multiply each denominator by the 'missing' factor and multiply each numerator by the same factor.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}, \frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}, \frac{3x^2-9x}{(x+1)(x+3)(x-3)}$

Example 5.4.5

Rewrite as equivalent rational expressions with denominator (x+3)(x−4)(x+4):
$\frac{2}{x^2-x-12}, \frac{1}{x^2-16}$.

Answer

$\frac{2x+8}{(x-4)(x+3)(x+4)}, \frac{x+3}{(x-4)(x+3)(x+4)}$

Example 5.4.6

Rewrite as equivalent rational expressions with denominator (x+3)(x+5)(x+6)
$\frac{x}{x^2+8x+15}, \frac{5}{x^2+9x+18}$.

Answer

$\frac{x^2+6x}{(x+3)(x+5)(x+6)}, \frac{x+3}{(x+3)(x+5)(x+6)}$

Add Rational Expressions with Different Denominators

Now we have all the steps we need to add rational expressions with different denominators. As we have done previously, we will do one example of adding numerical fractions first.

Example 5.4.7

Add: $\frac{7}{12}+\frac{5}{18}$.

Answer

	$\frac{7}{12}+\frac{5}{18}$
Find the LCD of 12 and 18. $\begin{array}{l} 12=2\cdot 2\cdot 3 \\ 18=2\cdot \quad 3\cdot 3 \\ \hline \text{LCD}=2\cdot 2\cdot 3\cdot 3 \\ \text{LCD}=36 \end{array}$	
Rewrite each fraction as an equivalent fraction with the LCD.	$\frac{7\cdot 3}{12\cdot 3}+\frac{5\cdot 2}{18\cdot 2}$
Add the fractions.	$\frac{21}{36}+\frac{10}{36}$
The fraction cannot be simplified.	$\frac{31}{36}$

Example 5.4.8

Add: $\frac{11}{30}+\frac{7}{12}$.

Answer

$\frac{19}{20}$

Example 5.4.9

Add: $\frac{3}{8}+\frac{9}{20}$.

Answer

$\frac{33}{40}$

Now we will add rational expressions whose denominators are monomials.

Example 5.4.10

Add: $\frac{5}{12x^2y}+\frac{4}{21xy^2}$.

Answer

	$\frac{5}{12x^2y}+\frac{4}{21xy^2}$
Find the LCD of $12x^2y$ and $21xy^2$ $12x^2y = 2 \cdot 2 \cdot 3 \cdot x \cdot x \cdot y$ $21xy^2 = 3 \cdot 7 \cdot x \cdot y \cdot y$ LCD $= 2 \cdot 2 \cdot 3 \cdot 7 \cdot x \cdot x \cdot y \cdot y$ LCD $= 84x^2y^2$	
	$\frac{5}{12x^2y}+\frac{4}{21xy^2}$
Rewrite each rational expression as an equivalent fraction with the LCD.	$\frac{5 \cdot 7y}{12x^2y \cdot 7y}+\frac{4 \cdot 4x}{21xy^2 \cdot 4x}$
Simplify.	$\frac{35y}{84x^2y^2}+\frac{16x}{84x^2y^2}$
Add the rational expressions.	$\frac{16x+35y}{84x^2y^2}$
There are no factors common to the numerator and denominator. The fraction cannot be simplified.	

Example 5.4.11

Add: $\frac{2}{15a^2b}+\frac{5}{6ab^2}$.

Answer

$\frac{4b+25a}{30a^2b^2}$

Example 5.4.12

Add: $\frac{5}{16c}+\frac{3}{8cd^2}$.

Answer

$\frac{5d^2+6}{16cd^2}$

Now we are ready to tackle polynomial denominators.

How to Add Rational Expressions with Different Denominators

Example 5.4.13

Add: $\frac{3}{x-3}+\frac{2}{x-2}$.

Answer

Step 1. Determine if the expressions have a common denominator. • Yes – Go to step 2. • No – Rewrite each rational expression with the LCD. ○ Find the LCD. ○ Rewrite each rational expression as an equivalent rational expression with the LCD.	No Find the LCD of $(x-3), (x-2)$ Change into equivalent rational expressions with the LCD, $(x-3)(x-2)$. Keep the denominators factored!	$x-3 : (x-3)$ $x-2 : (x-2)$ LCD $: (x-3)(x-2)$ $\frac{3}{x-3}+\frac{2}{x-2}$ $\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}$ $\frac{3x-6}{(x-3)(x-2)}+\frac{2x-6}{(x-2)(x-3)}$
Step 2. Add the rational expressions.	Add the numerators and place the sum over the common denominator.	$\frac{3x-6+2x-6}{(x-3)(x-2)}$ $\frac{5x-12}{(x-3)(x-2)}$
Step 3. Simplify, if possible.	Because $5x-12$ cannot be factored, the answer is simplified.	

Example 5.4.14

Add: $\frac{2}{x-2}+\frac{5}{x+3}$.

Answer

$\frac{7x-4}{(x+3)(x-2)}$

Example 5.4.15

Add: $\frac{4}{m+3}+\frac{3}{m+4}$.

Answer

$\frac{7m+25}{(m+3)(m+4)}$

The steps to use to add rational expressions are summarized in the following procedure box.

Definition: ADD RATIONAL EXPRESSIONS.

1. Determine if the expressions have a common denominator.
   Yes – go to step 2.
   No – Rewrite each rational expression with the LCD.

Find the LCD.
Rewrite each rational expression as an equivalent rational expression with the LCD.
2. Add the rational expressions.
3. Simplify, if possible.

ExAMPLe 5.4.16

Add: $\frac{2a}{2ab+b^2}+\frac{3a}{4a^2-b^2}$.

Answer

	$\frac{2a}{2ab+b^2}+\frac{3a}{4a^2-b^2}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $\begin{array}{rl} 2ab+b^2 &= b(2a+b) \\ 4a^2-b^2 &= (2a+b)(2a-b) \\ \text{LCD} &= b(2a+b)(2a-b) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2a(2a-b)}{b(2a+b)(2a-b)}+\frac{3a\cdot b}{(2a+b)(2a-b)\cdot b}$
Simplify the numerators.	$\frac{4a^2-2ab}{b(2a+b)(2a-b)}+\frac{3ab}{b(2a+b)(2a-b)}$
Add the rational expressions.	$\frac{4a^2-2ab+3ab}{b(2a+b)(2a-b)}$
Simplify the numerator.	$\frac{4a^2+ab}{b(2a+b)(2a-b)}$
Factor the numerator.	$\frac{a(4a+b)}{b(2a+b)(2a-b)}$
There are no factors common to the numerator and denominator. The fraction cannot be simplified.	

Example 5.4.17

Add: $\frac{5x}{xy-y^2}+\frac{2x}{x^2+y^2}$.

Answer

$\frac{x(5x+7y)}{y(x-y)(x+y)}$

Example 5.4.18

Add: $\frac{7}{2m+6}+\frac{4}{m^2+4m+3}$.

Answer

$\frac{7m+15}{2(m+3)(m+1)}$

Avoid the temptation to simplify too soon! In the example above, we must leave the first rational expression as $\frac{2a(2a-b)}{b(2a+b)(2a-b)}$ to be able to add it to $\frac{3a\cdot b}{(2a+b)(2a-b)\cdot b}$.

Example 5.4.19

Add: $\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$.

Answer

	$\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $\begin{array}{l} x^2-2x-3=(x+1)(x-3) \\ x^2+4x+3=(x+1)\quad\quad(x+3) \\ \text{LCD}\quad\quad=(x+1)(x-3)(x+3) \end{array}$	
Rewrite each rational expression as an equivalent fraction with the LCD.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}+\frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}+\frac{3x^2-9x}{(x+1)(x+3)(x-3)}$
Add the rational expressions.	$\frac{8x+24+3x^2+9x}{(x+1)(x-3)(x+3)}$
Simplify the numerator.	$\frac{3x^2-x^2+24}{(x+1)(x-3)(x+3)}$
The numerator is prime, so there are no common factors.	

Example 5.4.20

Add: $\frac{1}{m^2-m-2}+\frac{5m}{m^2+3m+2}$.

Answer

$\frac{5m^2-9m+2}{(m-2)(m+1)(m+2)}$

Example 5.4.21

Add: $\frac{2n}{n^2-3n-10}+\frac{6}{n^2+5n+6}$.

Answer

$\frac{2(n2+6n-15)}{(n+2)(n-5)(n+3)}$

Subtract Rational Expressions with Different Denominators

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

How to Subtract Rational Expressions with Different Denominators

Example 5.4.22

Subtract: $\frac{x}{x-3}-\frac{x-2}{x+3}$.

Answer

Step 1. Determine if the expressions have a common denominator. • Yes – Go to step 2. • No – Rewrite each rational expression with the LCD. ○ Find the LCD. ○ Rewrite each rational expression as an equivalent rational expression with the LCD.	No Find the LCD of $(x-3), (x+3)$. Change into equivalent fractions with the LCD, $(x-3)(x+3)$. Keep the denominators factored!	$x-3$: $(x-3)$ $x+3$: $(x+3)$ LCD: $(x-3)(x+3)$ $\frac{x}{x-3} - \frac{x-2}{x+3}$ $\frac{x(x+3)}{(x-3)(x+3)} - \frac{(x-2)(x-3)}{(x+3)(x-3)}$ $\frac{x^2+3x}{(x-3)(x+3)} - \frac{x^2-5x+6}{(x-3)(x+3)}$
Step 2. Subtract the rational expressions.	Subtract the numerators and place the difference over the common denominator. Be careful with the signs!	$\frac{x^2+3x-(x^2-5x+6)}{(x-3)(x+3)}$ $\frac{x^2+3x-x^2+5x-6}{(x-3)(x+3)}$ $\frac{8x-6}{(x-3)(x+3)}$
Step 3. Simplify, if possible.	The numerator and denominator have no factors in common. The answer is simplified.	$\frac{2(4x-3)}{(x-3)(x+3)}$

Example 5.4.23

Subtract: $\frac{y}{y+4} - \frac{y-2}{y-5}$.

Answer

$\frac{-7y+8}{(y+4)(y-5)}$

Example 5.4.24

Subtract: $\frac{z+3}{z+2} - \frac{z}{z+3}$.

Answer

$\frac{4z+9}{(z+2)(z+3)}$

The steps to take to subtract rational expressions are listed below.

Definition: SUBTRACT RATIONAL EXPRESSIONS.

1. Determine if they have a common denominator.
   Yes – go to step 2.
   No – Rewrite each rational expression with the LCD.
   Find the LCD.
   Rewrite each rational expression as an equivalent rational expression with the LCD.
2. Subtract the rational expressions.
3. Simplify, if possible.

Example 5.4.25

Subtract: $\frac{8y}{y^2-16} - \frac{4}{y-4}$.

Answer

	$\frac{8y}{y^2-16} - \frac{4}{y-4}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $\begin{array}{l} y^2-16=(y-4)(y+4) \\ y-4 \quad =y-4 \\ \hline \text{LCD} \quad =(y-4)(y+4) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8y}{(y-4)(y+4)} - \frac{4(y+4)}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{8y}{(y-4)(y+4)} - \frac{4y+16}{(y-4)(y+4)}$
Subtract the rational expressions.	$\frac{8y-4y-16}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{4y-16}{(y-4)(y+4)}$
Factor the numerator to look for common factors.	$\frac{4(y-4)}{(y-4)(y+4)}$
Remove common factors.	$\frac{4\cancel{(y-4)}}{\cancel{(y-4)}(y+4)}$
Simplify.	$\frac{4}{(y+4)}$

Example 5.4.26

Subtract: $\frac{2x}{x^2-4} - \frac{1}{x+2}$.

Answer

$\frac{1}{x-2}$

Example 5.4.27

Subtract: $\frac{3}{z+3} - \frac{6z}{z^2-9}$.

Answer

$\frac{-3}{z-3}$

There are lots of negative signs in the next example. Be extra careful!

Example 5.4.28

Subtract: $\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$.

Answer

	$\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$

Factor the denominator.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{n+3}{2-n}$
Since n−2 and 2−n are opposites, we will multiply the second rational expression by $\frac{-1}{-1}$.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{(-1)(n+3)}{(-1)(2-n)}$
Simplify.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)}{(n-2)}$
Do the expressions have a common denominator? No.	
Find the LCD. $\begin{array}{r} n^2+n-6=(n-2)(n+3) \\ n-2=(n-2) \\ \hline \text{LCD}=(n-2)(n+3) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)(n+3)}{(n-2)(n+3)}$
Simplify the numerators.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{n^2+6n+9}{(n-2)(n+3)}$
Simplify the rational expressions.	$\frac{-3n-9+n^2+6n+9}{(n-2)(n+3)}$
Simplify the numerator.	$\frac{n^2+3n}{(n-2)(n+3)}$
Factor the numerator to look for common factors.	$\frac{n(n+3)}{(n-2)(n+3)}$
Simplify.	$\frac{n}{(n-2)}$

Example 5.4.29

Subtract: $\frac{3x-1}{x^2-5x-6} - \frac{2}{6-x}$.

Answer

$\frac{1}{x-6}$

Example 5.4.30

Subtract: $\frac{-2y-2}{y^2+2y-8} - \frac{y-1}{2-y}$.

Answer

$\frac{y+3}{y+4}$

When one expression is not in fraction form, we can write it as a fraction with denominator 1.

Example 5.4.31

Subtract: $\frac{5c+4}{c-2} - 3$.

Answer

	$\frac{5c+4}{c-2} - 3$
Write 3 as $\frac{3}{1}$ to have 2 rational expressions.	$\frac{5c+4}{c-2} - \frac{3}{1}$
Do the rational expressions have a common denominator? No.	

Find the LCD of c−2 and 1. LCD = c−2.	
Rewrite $\frac{3}{1}$ as an equivalent rational expression with the LCD.	$\frac{5c+4}{c-2}-\frac{3(c-2)}{1(c-2)}$
Simplify.	$\frac{5c+4}{c-2}-\frac{3c-6}{(c-2)}$
Subtract the rational expressions.	$\frac{5c+4-(3c-6)}{c-2}$
Simplify.	$\frac{2c+10}{c-2}$
Factor to check for common factors.	$\frac{2(c+5)}{c-2}$
There are no common factors; the rational expression is simplified.	

Example 5.4.32

Subtract: $\frac{2x+1}{x-7}-3$.

Answer

$\frac{-x+22}{x-7}$

Example 5.4.33

Subtract: $\frac{4y+3}{2y-1}-5$.

Answer

$\frac{-2(3y-4)}{2y-1}$

Definition: ADD OR SUBTRACT RATIONAL EXPRESSIONS.

1. Determine if the expressions have a common denominator.
   Yes – go to step 2.
   No – Rewrite each rational expression with the LCD.
   Find the LCD.
   Rewrite each rational expression as an equivalent rational expression with the LCD.
2. Add or subtract the rational expressions.
3. Simplify, if possible.

We follow the same steps as before to find the LCD when we have more than two rational expressions. In the next example we will start by factoring all three denominators to find their LCD.

Example 5.4.34

Simplify: $\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u^2-u}$.

Answer

	$\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u^2-u}$
Do the rational expressions have a common denominator? No.	

$u-1=\quad u-1$ $u\quad\ =u$ $u^2-u=u(u-1)$ Find the LCD. $\text{LCD}=u(u-1)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2u\cdot u}{(u-1)u}+\frac{1\cdot(u-1)}{u\cdot(u-1)}-\frac{2u-1}{u(u-1)}$
	$\frac{2u^2}{(u-1)u}+\frac{u-1}{u\cdot(u-1)}-\frac{2u-1}{u(u-1)}$
Write as one rational expression.	$\frac{2u^2+u-1-2u+1}{u(u-1)}$
Simplify.	$\frac{2u^2-u}{u(u-1)}$
Factor the numerator, and remove common factors.	$\frac{u(2u-1)}{u(u-1)}$
Simplify.	$\frac{2u-1}{u-1}$

? Example 5.4.35

Simplify: $\frac{v}{v+1}+\frac{3}{v-1}-\frac{6}{v^2-1}$.

Answer

$\frac{v+3}{v+1}$

? Example 5.4.36

Add exercises text here.Simplify: $\frac{3w}{w+2}+\frac{2}{w+7}-\frac{17w+4}{w^2+9w+14}$.

Answer

$\frac{3w}{w+7}$

Key Concepts

- **Find the Least Common Denominator of Rational Expressions**
  1. Factor each expression completely.
  2. List the factors of each expression. Match factors vertically when possible.
  3. Bring down the columns.
  4. Multiply the factors.
- **Add or Subtract Rational Expressions**
  1. Determine if the expressions have a common denominator.
     Yes – go to step 2.
     No – Rewrite each rational expression with the LCD.
     - Find the LCD.
     - Rewrite each rational expression as an equivalent rational expression with the LCD.
  2. Add or subtract the rational expressions.
  3. Simplify, if possible.

This page titled 5.4: Add and Subtract Rational Expressions with Unlike Denominators is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **8.4: Add and Subtract Rational Expressions with Unlike Denominators** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

5.5: Simplify Complex Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Note

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Simplify: $\frac{\frac{3}{5}}{\frac{9}{10}}$.
   If you missed this problem, review Exercise 1.6.25.
2. Simplify: $\frac{1-\frac{1}{3}}{4^2+4\cdot 5}$.
   If you missed this problem, review Exercise 1.6.31.

Complex fractions are fractions in which the numerator or denominator contains a fraction. In Chapter 1 we simplified complex fractions like these:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \quad \frac{\frac{x}{2}}{\frac{xy}{6}}$$

In this section we will simplify *complex rational expressions*, which are rational expressions with rational expressions in the numerator or denominator.

Definition: COMPLEX RATIONAL EXPRESSION

A **complex rational expression** is a rational expression in which the numerator or denominator contains a rational expression.

Here are a few complex rational expressions:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}}$$

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$$

$$\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

Simplify a Complex Rational Expression by Writing it as Division

We have already seen this complex rational expression earlier in this chapter.

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-8x+3}{x^2-5x+6}}$$

We noted that fraction bars tell us to divide, so rewrote it as the division problem

$$\left(\frac{6x^2-7x+2}{4x-8}\right) \div \left(\frac{2x^2-8x+3}{x^2-5x+6}\right)$$

Then we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify rational expressions. We write it as if we were dividing two fractions.

Example 5.5.1

$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}}$.

Answer

	$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}}$
Rewrite the complex fraction as division.	$\frac{4}{y-3} \div \frac{8}{y^2-9}$
Rewrite as the product of first times the reciprocal of the second.	$\frac{4}{y-3} \cdot \frac{y^2-9}{8}$
Multiply.	$\frac{4(y^2-9)}{8(y-3)}$
Factor to look for common factors.	$\frac{4(y-3)(y+3)}{8(y-3)}$
Simplify.	$\frac{y+3}{2}$

Are there any value(s) of y that should not be allowed? The simplified rational expression has just a constant in the denominator. But the original **complex rational expression** had denominators of y−3 and y^2-9. This expression would be undefined if y=3 or y=−3

Example 5.5.2

$\frac{\frac{2}{x^2-1}}{\frac{3}{x+1}}$.

Answer

$\frac{2}{3(x-1)}$

Example 5.5.3

$\frac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}$.

Answer

$\frac{1}{2(x-3)}$

Example 5.5.4

$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Answer

	$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
Simplify the numerator and denominator.	

Find the LCD and add the fractions in the numerator. Find the LCD and add the fractions in the denominator.	$\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}$ $\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}$
Simplify the numerator and denominator.	$\frac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$
Simplify the numerator and denominator, again.	$\frac{\frac{3}{6}}{\frac{1}{6}}$
Rewrite the complex rational expression as a division problem.	$\frac{3}{6}\div\frac{1}{6}$
Multiply the first times by the reciprocal of the second.	$\frac{3}{6}\cdot\frac{6}{1}$
Simplify.	3

? Example 5.5.5

$\frac{\frac{1}{2}+\frac{2}{3}}{\frac{5}{6}+\frac{1}{12}}$.

Answer

$\frac{14}{11}$

? Example 5.5.6

$\frac{\frac{3}{4}-\frac{1}{3}}{\frac{1}{8}+\frac{5}{6}}$.

Answer

$\frac{10}{23}$

How to Simplify a Complex Rational Expression by Writing it as Division

? Example 5.5.7

$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Answer

Step 1. Simplify the numerator and denominator.	We will simplify the sum in the numerator and difference in the denominator.	$\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
	Find a common denominator and add the fractions in the numerator.	$\dfrac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{x\cdot x}{y\cdot x}-\frac{y\cdot y}{x\cdot y}}$
	Find a common denominator and subtract the fractions in the numerator.	$\dfrac{\frac{y}{xy}+\frac{x}{xy}}{\frac{x^2}{xy}-\frac{y^2}{xy}}$
	We now have just one rational expression in the numerator and one in the denominator.	$\dfrac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$
Step 2. Rewrite the complex rational expression as a division problem.	We write the numerator divided by the denominator.	$\left(\frac{y+x}{xy}\right)\div\left(\frac{x^2-y^2}{xy}\right)$
Step 3. Divide the expressions.	Multiply the first by the reciprocal of the second.	$\left(\frac{y+x}{xy}\right)\cdot\left(\frac{xy}{x^2-y^2}\right)$
	Factor any expressions if possible.	$\frac{xy(y+x)}{xy(x-y)(x+y)}$
	Remove common factors.	$\frac{\cancel{xy}\cancel{(y+x)}}{\cancel{xy}(x-y)\cancel{(x+y)}}$
	Simplify.	$\frac{1}{x-y}$

Example 5.5.8

$\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$.

Answer

$\frac{y+x}{y-x}$

Example 5.5.9

$\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}$.

Answer

$\frac{ab}{b-a}$

Definition: SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY WRITING IT AS DIVISION.

1. Simplify the numerator and denominator.
2. Rewrite the complex rational expression as a division problem.
3. Divide the expressions.

Example 5.5.10

$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$

Answer

	$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$
Simplify the numerator and denominator.	
Find the LCD and add the fractions in the numerator. Find the LCD and add the fractions in the denominator.	$\frac{\frac{n(n+5)}{1(n+5)}-\frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)}+\frac{1(n+5)}{(n-5)(n+5)}}$
Simplify the numerators.	$\frac{\frac{n^2+5n}{n+5}-\frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)}+\frac{n+5}{(n-5)(n+5)}}$
Subtract the rational expressions in the numerator and add in the denominator.	$\frac{\frac{n^2+n}{n+5}}{\frac{2n}{(n+5)(n-5)}}$
Rewrite as fraction division.	$\frac{n^2+n}{n+5}\div\frac{2n}{(n+5)(n-5)}$
Multiply the first times the reciprocal of the second.	$\frac{n^2+n}{n+5}\cdot\frac{(n+5)(n-5)}{2n}$
Factor any expressions if possible.	$\frac{n(n+1)(n+5)(n-5)}{(n+5)2n}$
Remove common factors.	$\frac{\cancel{n}(n+1)\cancel{(n+5)}(n-5)}{\cancel{(n+5)}2\cancel{n}}$
Simplify.	$\frac{(n+1)(n-5)}{2}$

Example 5.5.11

$\frac{b-\frac{3b}{b+5}}{\frac{2}{b+5}+\frac{1}{b-5}}$.

Answer

b(b+2)

Example 5.5.12

$\frac{1-\frac{3}{c+4}}{\frac{1}{c+4}+\frac{c}{3}}$.

Answer

3c+3

Simplify a Complex Rational Expression by Using the LCD

We “cleared” the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by LCD of all the rational expressions.

Let’s look at the complex rational expression we simplified one way in Example. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\frac{LCD}{LCD}$ we are multiplying by 1, so the value stays the same.

Example 5.5.13

Simplify: $\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Answer

	$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
The LCD of all the fractions in the whole expression is 6.	
Clear the fractions by multiplying the numerator and denominator by that LCD.	$\frac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$
Distribute.	$\frac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$
Simplify.	$\frac{2+1}{3-2}$
	$\frac{3}{1}$
	3

Example 5.5.14

Simplify: $\frac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{10}+\frac{1}{5}}$.

Answer

$\frac{7}{3}$

Example 5.5.15

Simplify: $\frac{\frac{1}{4}+\frac{3}{8}}{\frac{1}{2}-\frac{5}{16}}$.

Answer

$\frac{7}{3}$

How to Simplify a Complex Rational Expression by Using the LCD

Example 5.5.16

Simplify: $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Answer

Step 1. Find the LCD of all fractions in the complex rational expression.	The LCD of all the fractions is xy.	$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
Step 2. Multiply the numerator and denominator by the LCD.	Multiply both the numerator and denominator by xy.	$\frac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$
Step 3. Simplify the expression.	Distribute.	$\frac{xy\cdot\frac{1}{x}+xy\cdot\frac{1}{y}}{xy\cdot\frac{x}{y}-xy\cdot\frac{y}{x}}$
		$\frac{y+x}{x^2-y^2}$
	Simplify.	$\frac{(y+x)}{(x-y)(x+y)}$
	Remove common factors.	$\frac{1}{x-y}$

Example 5.5.17

Simplify: $\frac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}-\frac{b}{a}}$.

Answer

$\frac{b+a}{a^2+b^2}$

Example 5.5.18

Simplify: $\frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}-\frac{1}{y}}$.

Answer

$\frac{y-x}{xy}$

Definition: SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY USING THE LCD.

1. Find the LCD of all fractions in the complex rational expression.
2. Multiply the numerator and denominator by the LCD.
3. Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

Example 5.5.19

Simplify: $\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$.

Answer

	$\dfrac{\dfrac{2}{x+6}}{\dfrac{4}{x-6}-\dfrac{4}{x^2-36}}$
Find the LCD of all fractions in the complex rational expression. The LCD is (x+6)(x−6)	
Multiply the numerator and denominator by the LCD.	$\dfrac{(x+6)(x-6)\dfrac{2}{x+6}}{(x+6)(x-6)\left(\dfrac{4}{x-6}-\dfrac{4}{(x+6)(x-6)}\right)}$
Simplify the expression.	
Distribute in the denominator.	$\dfrac{(x+6)(x-6)\dfrac{2}{x+6}}{(x+6)(x-6)\left(\dfrac{4}{x-6}\right)-(x+6)(x-6)\left(\dfrac{4}{(x+6)(x-6)}\right)}$
Simplify.	$\dfrac{\cancel{(x+6)}(x-6)\dfrac{2}{\cancel{x+6}}}{\cancel{(x+6)}\cancel{(x-6)}\left(\dfrac{4}{\cancel{x-6}}\right)-\cancel{(x+6)}\cancel{(x-6)}\left(\dfrac{4}{\cancel{(x+6)}\cancel{(x-6)}}\right)}$
Simplify.	$\dfrac{2(x-6)}{4(x+6)-4}$
To simplify the denominator, distribute and combine like terms.	$\dfrac{2(x-6)}{4x+20}$
Remove common factors.	$\dfrac{\cancel{2}(x-6)}{\cancel{2}(2x+10)}$
Simplify.	$\dfrac{x-6}{2x+10}$
Notice that there are no more factors common to the numerator and denominator.	

Example 5.5.20

Simplify: $\dfrac{\frac{3}{x+2}}{\frac{5}{x-2}-\frac{3}{x^2-4}}$.

Answer

$\frac{3x-6}{5x+7}$

Example 5.5.21

Simplify: $\dfrac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{x^2-49}}$.

Answer

$\frac{x+21}{6x+43}$

Example 5.5.22

Simplify: $\dfrac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$.

Answer

	$\dfrac{\dfrac{4}{m^2-7m+12}}{\dfrac{3}{m-3}-\dfrac{2}{m-4}}$
Find the LCD of all fractions in the complex rational expression. The LCD is (m−3)(m−4)	
Multiply the numerator and denominator by the LCD.	$\dfrac{(m-3)(m-4)\dfrac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\dfrac{3}{m-3}-\dfrac{2}{m-4}\right)}$
Simplify.	$\dfrac{\cancel{(m-3)(m-4)}\dfrac{4}{\cancel{(m-3)(m-4)}}}{\cancel{(m-3)}(m-4)\left(\dfrac{3}{\cancel{m-3}}\right)-(m-3)\cancel{(m-4)}\left(\dfrac{2}{\cancel{m-4}}\right)}$
Simplify.	$\dfrac{4}{3(m-4)-2(m-3)}$
Distribute.	$\dfrac{4}{3m-12-2m+6}$
Combine like terms.	$\dfrac{4}{m-6}$

Example 5.5.23

Simplify: $\dfrac{\frac{3}{x^2+7x+10}}{\frac{4}{x+2}+\frac{1}{x+5}}$.

Answer

$\dfrac{3}{5x+22}$

Example 5.5.24

Simplify: $\dfrac{\frac{4y}{y+5}+\frac{2}{y+6}}{\frac{3y}{y^2+11y+30}}$.

Answer

$\dfrac{6y+34}{3y}$

Example 5.5.25

Simplify: $\dfrac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$.

Answer

	$\dfrac{\dfrac{y}{y+1}}{1+\dfrac{1}{y-1}}$
Find the LCD of all fractions in the complex rational expression.	
The LCD is (y+1)(y−1)	

Multiply the numerator and denominator by the LCD.	$\frac{(y+1)(y-1)\frac{y}{y+1}}{(y+1)(y-1)\left(1+\frac{1}{y-1}\right)}$
Distribute in the denominator and simplify.	$\frac{\cancel{(y+1)}(y-1)\left(\frac{y}{\cancel{y+1}}\right)}{(y+1)(y-1)(1)+(y+1)\cancel{(y-1)}\left(\frac{1}{\cancel{y-1}}\right)}$
Simplify.	$\frac{(y-1)y}{(y+1)(y-1)+(y+1)}$
Simplify the denominator, and leave the numerator factored.	$\frac{y(y-1)}{y^2-1+y+1}$
	$\frac{y(y-1)}{y^2+y}$
Factor the denominator, and remove factors common with the numerator.	$\frac{\cancel{y}(y-1)}{\cancel{y}(y+1)}$
Simplify.	$\frac{y-1}{y+1}$

? Example 5.5.26

Simplify: $\frac{\frac{x}{x+3}}{1+\frac{1}{x+3}}$.

Answer

$\frac{x}{x+4}$

? Example 5.5.27

Simplify: $\frac{1+\frac{1}{x-1}}{\frac{3}{x+1}}$.

Answer

$\frac{x(x+1)}{3(x-1)}$

Key Concepts

- **To Simplify a Rational Expression by Writing it as Division**
  1. Simplify the numerator and denominator.
  2. Rewrite the complex rational expression as a division problem.
  3. Divide the expressions.
- **To Simplify a Complex Rational Expression by Using the LCD**
  1. Find the LCD of all fractions in the complex rational expression.
  2. Multiply the numerator and denominator by the LCD.
  3. Simplify the expression.

Glossary

complex rational expression

A complex rational expression is a rational expression in which the numerator or denominator contains a rational expression.

This page titled 5.5: Simplify Complex Rational Expressions is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **8.5: Simplify Complex Rational Expressions** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

5.6: Solve Rational Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve rational equations
- Solve a rational equation for a specific variable

Note

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Solve: $\frac{1}{6}x+\frac{1}{2}=\frac{1}{3}$.
   If you missed this problem, review Exercise 2.5.1.
2. Solve: $n^2-5n-36=0$.
   If you missed this problem, review Exercise 7.6.13.
3. Solve for y in terms of x: 5x+2y=10 for y.
   If you missed this problem, review Exercise 2.6.22.

After defining the terms *expression* and *equation* early in Foundations, we have used them throughout this book. We have *simplified* many kinds of *expressions* and *solved* many kinds of *equations*. We have simplified many rational expressions so far in this chapter. Now we will solve rational equations.

The definition of a rational equation is similar to the definition of equation we used in Foundations.

Definition: RATIONAL EQUATION

A **rational equation** is two rational expressions connected by an equal sign.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational Expression	Rational Equation
$\frac{1}{8}x+\frac{1}{2}$	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36}=y+1$
$\frac{1}{n-3}+\frac{1}{n+4}$	$\frac{1}{n-3}+\frac{1}{n+4}=\frac{15}{n^2+n-12}$

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to “clear” the fractions.

Here is an example we did when we worked with linear equations:

	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$	LCD = 8
We multiplied both sides by the LCD.	$8\left(\frac{1}{8}x+\frac{1}{2}\right)=8\left(\frac{1}{4}\right)$	
Then we distributed.	$8\cdot\frac{1}{8}x+8\cdot\frac{1}{2}=8\cdot\frac{1}{4}$	
We simplified—and then we had an equation with no fractions.	$x+4=2$	
Finally, we solved that equation.	$x+4-4=2-4$	

	$x = -2$	

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then we will have an equation that does not contain rational expressions and thus is much easier for us to solve.

But because the original equation may have a variable in a denominator we must be careful that we don't end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a **rational equation** we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an *extraneous solution*.

Definition: EXTRANEOUS Solution TO A RATIONAL EQUATION

An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, c, by writing $x \neq c$ next to the equation.

How to Solve Equations with Rational Expressions

Example 5.6.1

Solve: $\frac{1}{x} + \frac{1}{3} = \frac{5}{6}$.

Answer

Step 1. Note any value of the variable that would make any denominator zero.	If $x = 0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation.	$\frac{1}{x} + \frac{1}{3} = \frac{5}{6}, x \neq 0$
Step 2. Find the least common denominator of *all* denominators in the equation.	Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$.	The LCD is $6x$.
Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.	Multiply both sides of the equation by the LCD, $6x$. Use the Distributive Property. Simplify – and notice, no more fractions!	$6x \cdot \left(\frac{1}{x} + \frac{1}{3}\right) = 6x \cdot \left(\frac{5}{6}\right)$ $6x \cdot \frac{1}{x} + 6x \cdot \frac{1}{3} = 6x \cdot \left(\frac{5}{6}\right)$ $6 + 2x = 5x$
Step 4. Solve the resulting equation.	Simplify.	$6 = 3x$ $2 = x$
Step 5. Check. • If any values found in Step 1 are algebraic solutions, discard them. • Check any remaining solutions in the original equation.	We did not get 0 as an algebraic solution. We substitute $x = 2$ into the original equation.	$\frac{1}{x} + \frac{1}{3} = \frac{5}{6}$ $\frac{1}{2} + \frac{1}{3} \stackrel{?}{=} \frac{5}{6}$ $\frac{3}{6} + \frac{2}{6} \stackrel{?}{=} \frac{5}{6}$ $\frac{5}{6} = \frac{5}{6}$ ✓

Example 5.6.2

Solve: $\frac{1}{y}+\frac{2}{3}=\frac{1}{5}$.

Answer

$-\frac{15}{7}$

Example 5.6.3

Solve: $\frac{2}{3}+\frac{1}{5}=\frac{1}{x}$.

Answer

$\frac{15}{13}$

The steps of this method are shown below.

Definition: SOLVE EQUATIONS WITH RATIONAL EXPRESSIONS.

1. Note any value of the variable that would make any denominator zero.
2. Find the least common denominator of *all* denominators in the equation.
3. Clear the fractions by multiplying both sides of the equation by the LCD.
4. Solve the resulting equation.
5. Check.
   - If any values found in Step 1 are algebraic solutions, discard them.
   - Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

Example 5.6.4

Solve: $1-\frac{5}{y}=-\frac{6}{y^2}$.

Answer

	$1-\frac{5}{y}=-\frac{6}{y^2}$
Note any value of the variable that would make any denominator zero.	$1-\frac{5}{y}=-\frac{6}{y^2}, y\neq 0$
Find the least common denominator of all denominators in the equation. The LCD is y^2	
Clear the fractions by multiplying both sides of the equation by the LCD.	$y^2\left(1-\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$
Distribute.	$y^2\cdot 1-y^2\left(\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$
Multiply.	$y^2-5y=-6$
Solve the resulting equation. First write the quadratic equation in standard form.	$y^2-5y+6=0$
Factor.	$(y-2)(y-3)=0$
Use the Zero Product Property.	$y-2=0$ or $y-3=0$

Solve.	$y = 2$ or $y = 3$
Check.	
We did not get 0 as an algebraic solution.	
Check $y = 2$ and $y = 3$ in the original equation. $1 - \frac{5}{y} = -\frac{6}{y^2}$ $1 - \frac{5}{2} \stackrel{?}{=} -\frac{6}{2^2}$ $1 - \frac{5}{2} \stackrel{?}{=} -\frac{6}{4}$ $\frac{2}{2} - \frac{5}{2} \stackrel{?}{=} -\frac{6}{4}$ $-\frac{3}{2} \stackrel{?}{=} -\frac{6}{4}$ $-\frac{3}{2} = -\frac{3}{2}$ ✓ $1 - \frac{5}{y} = -\frac{6}{y^2}$ $1 - \frac{5}{3} \stackrel{?}{=} -\frac{6}{3^2}$ $1 - \frac{5}{3} \stackrel{?}{=} -\frac{6}{9}$ $\frac{3}{3} - \frac{5}{3} \stackrel{?}{=} -\frac{6}{9}$ $-\frac{2}{3} \stackrel{?}{=} -\frac{6}{9}$ $-\frac{2}{3} = -\frac{2}{3}$ ✓	

Example 5.6.5

Solve: $1 - \frac{2}{a} = \frac{15}{a^2}$.

Answer

5, −3

Example 5.6.6

Solve: $1 - \frac{4}{b} = \frac{12}{b^2}$.

Answer

6, −2

Example 5.6.7

Solve: $\frac{5}{3u-2} = \frac{3}{2u}$.

Answer

	$\frac{5}{3u-2} = \frac{3}{2u}$
Note any value of the variable that would make any denominator zero.	$\frac{5}{3u-2} = \frac{3}{2u}$, $u \neq \frac{2}{3}$, $u \neq 0$
Find the least common denominator of all denominators in the equation. The LCD is 2u(3u−2).	
Clear the fractions by multiplying both sides of the equation by the LCD.	$2u(3u-2)\left(\frac{5}{3u-2}\right) = 2u(3u-2)\left(\frac{3}{2u}\right)$
Remove common factors.	$2u\cancel{(3u-2)}\left(\frac{5}{\cancel{3u-2}}\right) = \cancel{2u}(3u-2)\left(\frac{3}{\cancel{2u}}\right)$
Simplify.	$2u(5) = (3u-2)(3)$
Multiply.	$10u = 9u - 6$

Solve the resulting equation.	$u = -6$
We did not get 0 or $\frac{2}{3}$ as algebraic solutions.	
Check $u = -6$ in the original equation. $\frac{5}{3u-2} = \frac{3}{2u}$ $\frac{5}{3(-6)-2} \stackrel{?}{=} \frac{3}{2(-6)}$ $\frac{5}{-20} \stackrel{?}{=} \frac{3}{-12}$ $-\frac{1}{4} = -\frac{1}{4}$ ✓	

Example 5.6.8

Solve: $\frac{1}{x-1} = \frac{2}{3x}$.

Answer

−2

Example 5.6.9

Solve: $\frac{3}{5n+1} = \frac{2}{3n}$.

Answer

−2

When one of the denominators is a quadratic, remember to factor it first to find the LCD.

Example 5.6.10

Solve: $\frac{2}{p+2} + \frac{4}{p-2} = \frac{p-1}{p^2-4}$.

Answer

	$\frac{2}{p+2} + \frac{4}{p-2} = \frac{p-1}{p^2-4}$
Note any value of the variable that would make any denominator zero.	$\frac{2}{p+2} + \frac{4}{p-2} = \frac{p-1}{(p+2)(p+2)}, p \neq -2, p \neq 2$
Find the least common denominator of all denominators in the equation. The LCD is (p+2)(p−2).	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(p+2)(p-2)\left(\frac{2}{p+2} + \frac{4}{p-2}\right) = (p+2)(p-2)\left(\frac{p-1}{p^2-4}\right)$
Distribute.	$(p+2)(p-2)\frac{2}{p+2} + (p+2)(p-2)\frac{4}{p-2} = (p+2)(p-2)\left(\frac{p-1}{p^2-4}\right)$
Remove common factors.	$\cancel{(p+2)}(p-2)\frac{2}{\cancel{p+2}} + (p+2)\cancel{(p-2)}\frac{4}{\cancel{p-2}} = \cancel{(p+2)}\cancel{(p-2)}\left(\frac{p-1}{\cancel{p^2-4}}\right)$
Simplify.	$2(p-2) + 4(p+2) = p-1$
Distribute.	$2p - 4 + 4p + 8 = p - 1$
Solve.	$6p + 4 = p - 1$

	$5p = -5$
	$p = -1$
We did not get 2 or −2 as algebraic solutions.	
Check $p = -1$ in the original equation. $\frac{2}{p+2} + \frac{4}{p-2} = \frac{p-1}{p^2-4}$ $\frac{2}{(-1)+2} + \frac{4}{(-1)-2} \stackrel{?}{=} \frac{(-1)-1}{(-1)^2-4}$ $\frac{2}{1} + \frac{4}{-3} \stackrel{?}{=} \frac{-2}{-3}$ $\frac{6}{3} - \frac{4}{-3} \stackrel{?}{=} \frac{-2}{-3}$ $\frac{2}{3} = \frac{2}{3}$ ✓	

Example 5.6.11

Solve: $\frac{2}{x+1} + \frac{1}{x-1} = \frac{1}{x^2-1}$.

Answer

$\frac{2}{3}$

Example 5.6.12

Solve: $\frac{5}{y+3} + \frac{2}{y-3} = \frac{5}{y^2-9}$

Answer

2

Example 5.6.13

Solve: $\frac{4}{q-4} - \frac{3}{q-3} = 1$.

Answer

	$\frac{4}{q-4} - \frac{3}{q-3} = 1$
Note any value of the variable that would make any denominator zero.	$\frac{4}{q-4} + \frac{3}{q-3} = 1, \quad q \neq 4, q \neq 3$
Find the least common denominator of all denominators in the equation. The LCD is (q−4)(q−3).	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(q-4)(q-3)\left(\frac{4}{q-4} - \frac{3}{q-3}\right) = (q-4)(q-3)(1)$
Distribute.	$(q-4)(q-3)\left(\frac{4}{q-4}\right) - (q-4)(q-3)\left(\frac{3}{q-3}\right) = (q-4)(q-3)(1)$
Remove common factors.	$\cancel{(q-4)}(q-3)\left(\frac{4}{\cancel{q-4}}\right) - (q-4)\cancel{(q-3)}\left(\frac{3}{\cancel{q-3}}\right) = (q-4)(q-3)(1)$
Simplify.	$4(q-3) - 3(q-4) = (q-4)(q-3)$
Simplify.	$4q - 12 - 3q + 12 = q^2 - 7q + 12$

Combine like terms.	$q = q^2 - 7q + 12$
Solve. First write in standard form.	$0 = q^2 - 8q + 12$
Factor.	$0 = (q-2)(q-6)$
Use the Zero Product Property.	$q = 2$ or $q = 6$
We did not get 4 or 3 as algebraic solutions.	
Check $q = 2$ and $q = 6$ in the original equation.	

$$\frac{4}{q-4} - \frac{3}{q-3} = 1 \qquad \frac{4}{q-4} - \frac{3}{q-3} = 1$$

$$\frac{4}{2-4} - \frac{3}{2-3} \stackrel{?}{=} 1 \qquad \frac{4}{6-4} - \frac{3}{6-3} \stackrel{?}{=} 1$$

$$\frac{4}{-2} - \frac{3}{-1} \stackrel{?}{=} 1 \qquad \frac{4}{2} - \frac{3}{1} \stackrel{?}{=} 1$$

$$-2 - (-3) \stackrel{?}{=} 1 \qquad 2 - 1 \stackrel{?}{=} 1$$

$$1 = 1 \checkmark \qquad 1 = 1 \checkmark$$

Example 5.6.14

Solve: $\frac{2}{x+5} - \frac{1}{x-1} = 1$.

Answer

$-1, -2$

Example 5.6.15

Solve: $\frac{3}{x+8} - \frac{2}{x-2} = 1$.

Answer

$-2, -3$

Example 5.6.16

Solve: $\frac{m+11}{m^2-5m+4} = \frac{5}{m-4} - \frac{3}{m-1}$.

Answer

	$\frac{m+11}{m^2-5m+4} = \frac{5}{m-4} = \frac{3}{m-1}$
Factor all the denominators, so we can note any value of the variable the would make any denominator zero.	$\frac{m+11}{(m-4)(m-1)} = \frac{5}{m-4} - \frac{3}{m-1},\ m \neq 4, m \neq 1$
Find the least common denominator of all denominators in the equation. The LCD is (m−4)(m−1)	
Clear the fractions.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right) = (m-4)(m-1)\left(\frac{5}{m-4} - \frac{3}{m-1}\right)$
Distribute.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right) = (m-4)(m-1)\frac{5}{m-4} - (m-4)(m-1)\frac{3}{m-1}$
Remove common factors.	$\cancel{(m-4)(m-1)}\left(\frac{m+11}{\cancel{(m-4)(m-1)}}\right) = \cancel{(m-4)}(m-1)\frac{5}{\cancel{m-4}} - (m-4)\cancel{(m-1)}\frac{3}{\cancel{m-1}}$
Simplify.	$m + 11 = 5(m-1) - 3(m-4)$
Solve the resulting equation.	$m + 11 = 5m - 5 - 3m + 12$

	$4 = m$
Check. The only algebraic solution was 4, but we said that 4 would make a denominator equal to zero. The algebraic solution is an extraneous solution. There is no solution to this equation.	

Example 5.6.17

Solve: $\frac{x+13}{x^2-7x+10} = \frac{6}{x-5} - \frac{4}{x-2}$.

Answer

no solution

Example 5.6.18

Solve: $\frac{y-14}{y^2+3y-4} = \frac{2}{y+4} + \frac{7}{y-1}$.

Answer

no solution

The equation we solved in Example had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. Some equations have no solution.

Example 5.6.19

Solve: $\frac{n}{12} + \frac{n+3}{3n} = \frac{1}{n}$.

Answer

	$\frac{n}{12} + \frac{n+3}{3n} = \frac{1}{n}$
Note any value of the variable that would make any denominator zero.	$\frac{n}{12} + \frac{n+3}{3n} = \frac{1}{n}, n \neq 0$
Find the least common denominator of all denominators in the equation. The LCD is 12n.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$12n\left(\frac{n}{12} + \frac{n+3}{3n}\right) = 12n\left(\frac{1}{n}\right)$
Distribute.	$12n\left(\frac{n}{12}\right) + 12n\left(\frac{n+3}{3n}\right) = 12n\left(\frac{1}{n}\right)$
Remove common factors.	$\cancel{12}n\left(\frac{n}{\cancel{12}}\right) + 4 \cdot \cancel{3}n\left(\frac{n+3}{\cancel{3}n}\right) = 12\cancel{n}\left(\frac{1}{\cancel{n}}\right)$
Simplify.	$n \cdot n + 4(n+3) = 12 \cdot 1$
Solve the resulting equation.	$n^2 + 4n + 12 = 12$
	$n^2 + 4n = 0$
	$n(n+4) = 0$
	$n = 0$ or $n = -4$
Check.	

n=0 is an extraneous solution.	
Check $n=-4$ in the original equation. $\frac{n}{12}+\frac{n+3}{3n}=\frac{1}{n}$ $\frac{-4}{12}+\frac{-4+3}{3(-4)}\stackrel{?}{=}\frac{1}{-4}$ $-\frac{4}{12}+\frac{1}{12}\stackrel{?}{=}-\frac{1}{4}$ $-\frac{3}{12}\stackrel{?}{=}-\frac{1}{4}$ $-\frac{1}{4}=-\frac{1}{4}$ ✓	

? Example 5.6.20

Solve: $\frac{x}{18}+\frac{x+6}{9x}=\frac{2}{3x}$.

Answer

−2

? Example 5.6.21

Solve: $\frac{y+5}{5y}+\frac{y}{15}=\frac{1}{y}$.

Answer

−3

? Example 5.6.22

Solve: $\frac{y}{y+6}=\frac{72}{y^2-36}+4$.

Answer

	$\frac{y}{y+6}=\frac{72}{y^2-36}+4$
Factor all the denominators, so we can note any value of the variable that would make any denominator zero.	$\frac{y}{y+6}=\frac{72}{(y-6)(y+6)}+4, y\neq 6, y\neq -6$
Find the least common denominator. The LCD is (y−6)(y+6).	
Clear the fractions.	$(y-6)(y+6)\left(\frac{y}{y+6}\right)=(y-6)(y+6)\left(\frac{72}{(y-6)(y+6)}+4\right)$
Simplify.	$(y-6)\cdot y=72+(y-6)(y+6)\cdot 4$
Simplify.	$y(y-6)=72+4(y^2-36)$
Solve the resulting equation.	$y^2-6y=72+4y^2-144$
	$0=3y^2+6y-72$
	$0=3(y^2+2y-24)$
	$0=3(y+6)(y-4)$
	$y=-6,\ y=4$

Check.	
y=−6 is an extraneous solution.	
Check $y = 4$ in the original equation. $\frac{y}{y+6} = \frac{72}{y^2-36} + 4$ $\frac{4}{4+6} \stackrel{?}{=} \frac{72}{4^2-36} + 4$ $\frac{4}{10} \stackrel{?}{=} \frac{72}{-20} + 4$ $\frac{4}{10} \stackrel{?}{=} -\frac{36}{10} + \frac{40}{10}$ $\frac{4}{10} = \frac{4}{10}$ ✓	

Example 5.6.23

Solve: $\frac{x}{x+4} = \frac{32}{x^2-16} + 5$.

Answer

−4, 3

Example 5.6.24

Solve: $\frac{y}{y+8} = \frac{128}{y^2-64} + 9$.

Answer

7

Example 5.6.25

Solve: $\frac{x}{2x-2} - \frac{2}{3x+3} = \frac{5x^2-2x+9}{12x^2-12}$.

Answer

	$\frac{x}{2x-2} - \frac{2}{3x+3} = \frac{5x^2-2x+9}{12x^2-12}$
We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.	$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2-2x+9}{12(x-1)(x+1)}$
Note any value of the variable that would make any denominator zero.	$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2-2x+9}{12(x-1)(x+1)}, x \neq 1, x \neq -1$
Find the least common denominator.The LCD is 12(x−1)(x+1)	
Clear the fractions.	$12(x-1)(x+1)\left(\frac{x}{2(x-1)} - \frac{2}{3(x+1)}\right) = 12(x-1)(x+1)\left(\frac{5x^2-2x+9}{12(x-1)(x+1)}\right)$
Simplify.	$6(x+1) \cdot x - 4(x-1) \cdot 2 = 5x^2 - 2x + 9$
Simplify.	$6x(x+1) - 4 \cdot 2(x-1) = 5x^2 - 2x + 9$
Solve the resulting equation.	$6x^2 + 6x - 8x + 8 = 5x^2 - 2x + 9$
	$x^2 - 1 = 0$
	$(x-1)(x+1) = 0$

	$x = 1$ or $x = -1$
Check.	
x=1 and x=−1 are extraneous solutions. The equation has no solution.	

Example 5.6.26

Solve: $\frac{y}{5y-10} - \frac{5}{3y+6} = \frac{2y^2-19y+54}{15y^2-60}$.

Answer

no solution

Example 5.6.27

Solve: $\frac{z^2}{z+8} - \frac{3}{4z-8} = \frac{3z^2-16z-68}{z^2+8z-64}$.

Answer

no solution

Solve a Rational Equation for a Specific Variable

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

We'll start with a formula relating distance, rate, and time. We have used it many times before, but not usually in this form.

Example 5.6.28

Solve: $\frac{D}{T} = R$ for T.

Answer

	$\frac{D}{T} = R$ for T
Note any value of the variable that would make any denominator zero.	$\frac{D}{T} = R, T \neq 0$
Clear the fractions by multiplying both sides of the equations by the LCD, T.	$T\left(\frac{D}{T}\right) = T(R)$
Simplify.	$D = T \cdot R$
Divide both sides by R to isolate T.	$\frac{D}{R} = \frac{RT}{R}$
Simplify.	$\frac{D}{R} = T$

Example 5.6.29

Solve: $\frac{A}{L} = W$ for L.

Answer

$L = \frac{A}{W}$

Example 5.6.30

Solve: $\frac{F}{A} = M$ for A.

Answer

$A = \frac{F}{M}$

Example uses the formula for slope that we used to get the point-slope form of an equation of a line.

Example 5.6.31

Solve: $m = \frac{x-2}{y-3}$ for y.

Answer

	$m = \frac{x-2}{y-3}$ for y
Note any value of the variable that would make any denominator zero.	$m = \frac{x-2}{y-3},\ y \neq 3$
Clear the fractions by multiplying both sides of the equations by the LCD, y−3.	$(y-3)m = (y-3)\left(\frac{x-2}{y-3}\right)$
Simplify.	$ym - 3m = x - 2$
Isolate the term with y.	$ym = x - 2 + 3m$
Divide both sides by m to isolate y.	$\frac{ym}{m} = \frac{x-2+3m}{m}$
Simplify.	$y = \frac{x-2+3m}{m}$

Example 5.6.32

Solve: $\frac{y-2}{x+1} = \frac{2}{3}$ for x.

Answer

$x = \frac{3y-8}{2}$

Example 5.6.33

Solve: $x = \frac{y}{1-y}$ for y.

Answer

$y = \frac{x}{1+x}$

Be sure to follow all the steps in Example. It may look like a very simple formula, but we cannot solve it instantly for either denominator.

Example 5.6.34

Solve $\frac{1}{c}+\frac{1}{m}=1$ for c.

Answer

	$\frac{1}{c}+\frac{1}{m}=1$ for c
Note any value of the variable that would make any denominator zero.	$\frac{1}{c}+\frac{1}{m}=1, c\neq 0, m\neq 0$
Clear the fractions by multiplying both sides of the equations by the LCD, cm	$cm\left(\frac{1}{c}+\frac{1}{m}\right)=cm(1)$
Distribute.	$cm\left(\frac{1}{c}\right)+cm\,\frac{1}{m}=cm(1)$
Simplify.	$m+c=cm$
Collect the terms with c to the right.	$m=cm-c$
Factor the expression on the right.	$m=c(m-1)$
To isolate c, divide both sides by m−1.	$\frac{m}{m-1}=\frac{c(m-1)}{m-1}$
Simplify by removing common factors.	$\frac{m}{m-1}=c$

Notice that even though we excluded c=0 and m=0 from the original equation, we must also now state that $m\neq 1$.

Example 5.6.35

Solve: $\frac{1}{a}+\frac{1}{b}=c$ for a.

Answer

$a=\frac{b}{cb-1}$

Example 5.6.36

Solve: $\frac{2}{x}+\frac{1}{3}=\frac{1}{y}$ for y.

Answer

$y=\frac{3x}{6+x}$

Key Concepts

- **Strategy to Solve Equations with Rational Expressions**
  1. Note any value of the variable that would make any denominator zero.
  2. Find the least common denominator of *all* denominators in the equation.
  3. Clear the fractions by multiplying both sides of the equation by the LCD.
  4. Solve the resulting equation.
  5. Check.
     - If any values found in Step 1 are algebraic solutions, discard them.
     - Check any remaining solutions in the original equation.

Glossary

rational equation

A rational equation is two rational expressions connected by an equal sign.

extraneous solution to a rational equation

An extraneous solution to a rational equation is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

This page titled 5.6: Solve Rational Equations is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **8.6: Solve Rational Equations** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

5.7: Solve Uniform Motion and Work Applications

Learning Objectives

By the end of this section, you will be able to:

- Solve uniform motion applications
- Solve work applications

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. An express train and a local bus leave Chicago to travel to Champaign. The express bus can make the trip in 2 hours and the local bus takes 5 hours for the trip. The speed of the express bus is 42 miles per hour faster than the speed of the local bus. Find the speed of the local bus.
   If you missed this problem, review [link].
2. Solve $\frac{1}{3}x+\frac{1}{4}x=\frac{5}{6}$.
   If you missed this problem, review [link].
3. Solve: $18t^2-30=-33t$.
   If you missed this problem, review [link].

Solve Uniform Motion Applications

We have solved uniform motion problems using the formula D=rt in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate •	Time =	Distance

The formula D=rt assumes we know *r* and *t* and use them to find D. If we know D and *r* and need to find *t*, we would solve the equation for *t* and get the formula $t=\frac{D}{r}$

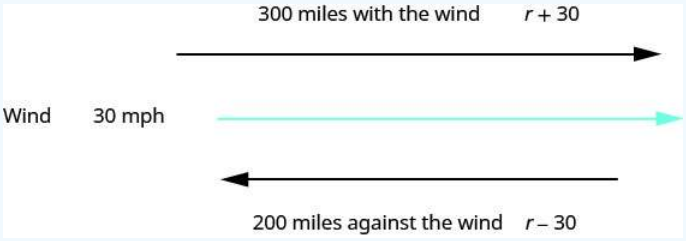
We have also explained how flying with or against a current affects the speed of a vehicle. We will revisit that idea in the next example.

Example 5.7.1

An airplane can fly 200 miles into a 30 mph headwind in the same amount of time it takes to fly 300 miles with a 30 mph tailwind. What is the speed of the airplane?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for the speed of the airplane.	Let r= the speed of the airplane.

When the plane flies with the wind, the wind increases its speed and the rate is r+30.	
When the plane flies against the wind, the wind decreases its speed and the rate is r−30.	
Write in the rates. Write in the distances. Since D=r·t , we solve for t and get $\frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.	<table><tr><th></th><th>Rate • Time = Distance</th><th></th><th></th></tr><tr><td>Headwind</td><td>$r-30$</td><td>$\frac{200}{r-30}$</td><td>200</td></tr><tr><td>Tailwind</td><td>$r+30$</td><td>$\frac{300}{r+30}$</td><td>300</td></tr></table>
We know the times are equal and so we write our equation.	$\frac{200}{r-30}=\frac{300}{r+30}$
We multiply both sides by the LCD. 200(r+30)=300(r−30)	
Simplify.	(r+30)(200)=(r−30)(300)
	200r+6000=300r−9000
Solve.	15000=100r 150=r
Check.	
Is 150 mph a reasonable speed for an airplane? Yes. If the plane is traveling 150 mph and the wind is 30 mph:	
Tailwind 150+30=180 mph $\frac{300}{180}=\frac{5}{3}$ hours	
Headwind 150−30=120 mph $\frac{200}{120}=\frac{5}{3}$ hours	
The times are equal, so it checks.	The plane was traveling 150 mph.

? Try It 5.7.2

Link can ride his bike 20 miles into a 3 mph headwind in the same amount of time he can ride 30 miles with a 3 mph tailwind. What is Link's biking speed?

Answer

15 mph

? Try It 5.7.3

Judy can sail her boat 5 miles into a 7 mph headwind in the same amount of time she can sail 12 miles with a 7 mph tailwind. What is the speed of Judy's boat without a wind?

Answer

17 mph

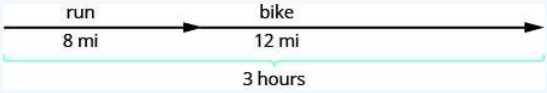
In the next example, we will know the total time resulting from travelling different distances at different speeds.

Example 5.7.4

Jazmine trained for 3 hours on Saturday. She ran 8 miles and then biked 24 miles. Her biking speed is 4 mph faster than her running speed. What is her running speed?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Jazmine's running speed.	Let r= Jazmine's running sp
Her biking speed is 4 miles faster than her running speed.	r+4= her biking speed
The distances are given, enter them into the chart.	
Since D=r·t , we solve for t and get $t = \frac{D}{r}$ We divide the distance by the rate in each row, and place the expression in the time column.	<table><tr><th></th><th>Rate • Time = Distance</th></tr><tr><td>Run</td><td>r	$\frac{8}{r}$	8</td></tr><tr><td>Bike</td><td>$r+4$	$\frac{24}{r+4}$	24</td></tr><tr><td></td><td></td><td>3</td><td></td></tr></table>
Write a word sentence.	Her time plus the time biking
Translate the sentence to get the equation.	$\frac{8}{r} + \frac{24}{r+4} = 3$
Solve.	$r(r+4)(\frac{8}{r} + \frac{24}{r+4}) = 3r(r+4)$ $8(r+4) + 24r = 3r^2 + 12r$ $8r + 32 + 24r = 3r^2 + 12r$ $32 + 32r = 3r^2 + 12r$ $0 = 3r^2 - 20r - 32$ $0 = (3r+4)(r-8)$
	(3r+4)=0, (r−8)=0
	$r = -\frac{4}{3}$, r=8
Check. r=8	
A negative speed does not make sense in this problem, so r=8 is the solution.	
Is 8 mph a reasonable running speed? Yes.	
Run 8 mph, $\frac{8miles}{8mph} = 1hour$ Bike 12 mph, $\frac{24miles}{12mph} = 2hours$ Total 3 hours. Jazmine's running speed is 8 mph.	

Try It 5.7.5

Dennis went cross-country skiing for 6 hours on Saturday. He skied 20 mile uphill and then 20 miles back downhill, returning to his starting point. His uphill speed was 5 mph slower than his downhill speed. What was Dennis' speed going uphill and his speed going downhill?

Answer

10 mph

Try It 5.7.6

Tony drove 4 hours to his home, driving 208 miles on the interstate and 40 miles on country roads. If he drove 15 mph faster on the interstate than on the country roads, what was his rate on the country roads?

Answer

50 mph

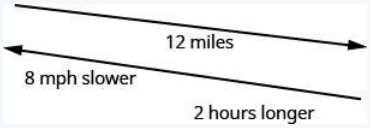
Once again, we will use the uniform motion formula solved for the variable *t*.

Example 5.7.7

Hamilton rode his bike downhill 12 miles on the river trail from his house to the ocean and then rode uphill to return home. His uphill speed was 8 miles per hour slower than his downhill speed. It took him 2 hours longer to get home than it took him to get to the ocean. Find Hamilton's downhill speed.

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Hamilton's downhill speed.	Let r= Hamilton's downhill speed.
His uphill speed is 8 miles per hour slower. Enter the rates into the chart.	h−8= Hamilton's uphill speed
The distance is the same in both directions, 12 miles. Since D=r·t , we solve for *t* and get $t = \frac{D}{r}$ We divide the distance by the rate in each row, and place the expression in the time column.	<table><tr><th></th><th>Rate • Time = Distance</th></tr><tr><td>Downhill</td><td>h	$\frac{12}{h}$	12</td></tr><tr><td>Uphill</td><td>$h-8$	$\frac{12}{h-8}$	12</td></tr></table>
Write a word sentence about the time.	He took 2 hours longer uphill than downhill. The uphill time is 2 more than the downhill time.
Translate the sentence to get the equation. Solve.	$\frac{12}{h-8} = \frac{12}{h} + 2$ $h(h-8)(\frac{12}{h-8}) = h(h-8)(\frac{12}{h} + 2)$ 12h=12(h−8)+2h(h−8) $12h = 12h - 96 + 2h^2 - 16h$ $0 = 2h^2 - 16h - 96$ $0 = 2(h^2 - 8h - 48)$ 0=2(h−12)(h+4) h−12=0, h+4=0 h=12, h=−4
Check. Is 12 mph a reasonable speed for biking downhill? Yes.	
Downhill 12 mph, $\frac{12miles}{12mph} = 1hour$	

Uphill 12−8=4 mph $\frac{12miles}{4mph} = 3hours$	
	The uphill time is 2 hours more than the downhill time. Hamilton's downhill speed is 12 mph.

Try It 5.7.8

Kayla rode her bike 75 miles home from college one weekend and then rode the bus back to college. It took her 2 hours less to ride back to college on the bus than it took her to ride home on her bike, and the average speed of the bus was 10 miles per hour faster than Kayla's biking speed. Find Kayla's biking speed.

Answer

15 mph

Try It 5.7.9

Victoria jogs 12 miles to the park along a flat trail and then returns by jogging on an 18 mile hilly trail. She jogs 1 mile per hour slower on the hilly trail than on the flat trail, and her return trip takes her two hours longer. Find her rate of jogging on the flat trail.

Answer

6 mph

Solve Work Applications

Suppose Pete can paint a room in 10 hours. If he works at a steady pace, in 1 hour he would paint $\frac{1}{10}$ of the room. If Alicia would take 8 hours to paint the same room, then in 1 hour she would paint $\frac{1}{8}$ of the room. How long would it take Pete and Alicia to paint the room if they worked together (and didn't interfere with each other's progress)?

This is a typical 'work' application. There are three quantities involved here – the time it would take each of the two people to do the job alone and the time it would take for them to do the job together.

Let's get back to Pete and Alicia painting the room. We will let t be the number of hours it would take them to paint the room together. So in 1 hour working together they have completed $\frac{1}{t}$ of the job.

In one hour Pete did $\frac{1}{10}$ of the job. Alicia did $\frac{1}{8}$ of the job. And together they did $\frac{1}{t}$ of the job.

We can model this with the word equation and then translate to a rational equation. To find the time it would take them if they worked together, we solve for t.

	$\underbrace{\text{Pete's part}}_{\frac{1}{10}} + \underbrace{\text{Alicia's part}}_{\frac{1}{8}} = \underbrace{\text{part of total}}_{\frac{1}{t}}$ $\frac{1}{10} + \frac{1}{8} = \frac{1}{t}$
	$\frac{1}{10}+\frac{1}{8}=\frac{1}{t}$
Multiply by the LCD, 40t	$40t\left(\frac{1}{10}+\frac{1}{8}\right)=40t\left(\frac{1}{t}\right)$
Distribute.	$40t\cdot\frac{1}{10}+40t\cdot\frac{1}{8}=40t\left(\frac{1}{t}\right)$
Simplify and solve.	$4t+5t=40$

	$9t = 40$
	$t = \frac{40}{9}$
We'll write as a mixed number so that we can convert it to hours and minutes.	$t = 4\frac{4}{9}$ hours
Remember, 1 hour = 60 minutes.	$t = 4$ hours $+ \frac{4}{9}(60$ minutes$)$
Multiply, and then round to the nearest minute.	$t = 4$ hours $+ 27$ minutes
	It would take Pete and Alica about 4 hours and 27 minutes to paint the room.

Keep in mind, it should take less time for two people to complete a job working together than for either person to do it alone.

Example 5.7.10

The weekly gossip magazine has a big story about the Princess' baby and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours to do the job and Press #2 takes 12 hours to do the job. How long will it take the printer to get the magazine printed with both presses running together?

Solution

This is a work problem. A chart will help us organize the information.

Let t= the number of hours needed to complete the job together.	
Enter the hours per job for Press #1, Press #2 and when they work together. If a job on Press #1 takes 6 hours, then in 1 hour $\frac{1}{6}$ of the job is completed. Similarly find the part of the job completed/hours for Press #2 and when they both work together.	<table><tr><th></th><th>Number of hours to complete the job</th><th>Part of job completed/ hour</th></tr><tr><td>Press #1</td><td>6</td><td>$\frac{1}{6}$</td></tr><tr><td>Press #2</td><td>12</td><td>$\frac{1}{12}$</td></tr><tr><td>Together</td><td>t</td><td>$\frac{1}{t}$</td></tr></table>
Write a word sentence.	
The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.	
Translate to an equation.	Work completed by Press #1 + Press #2 = Together $\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$
Solve.	$\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$
Multiply by the LCD, 12t.	$12t\left(\frac{1}{6} + \frac{1}{12}\right) = 12t\left(\frac{1}{t}\right)$
Simplify.	$2t + t = 12$
	$3t = 12$
	$t = 4$
	When both presses are running it takes 4 hours to do the job.

Try It 5.7.11

One gardener can mow a golf course in 4 hours, while another gardener can mow the same golf course in 6 hours. How long would it take if the two gardeners worked together to mow the golf course?

Answer

2 hours and 24 minutes

Try It 5.7.12

Carrie can weed the garden in 7 hours, while her mother can do it in 3. How long will it take the two of them working together?

Answer

2 hours and 6 minutes

Example 5.7.13

Corey can shovel all the snow from the sidewalk and driveway in 4 hours. If he and his twin Casey work together, they can finish shoveling the snow in 2 hours. How many hours would it take Casey to do the job by himself?

Solution

This is a work application. A chart will help us organize the information.	
We are looking for how many hours it would take Casey to complete the job by himself.	
Let t= the number of hours needed for Casey to complete.	
Enter the hours per job for Corey, Casey, and when they work together. If Corey takes 4 hours, then in 1 hour $\frac{1}{4}$ of the job is completed. Similarly find the part of the job completed/hours for Casey and when they both work together.	<table><tr><th></th><th>Number of hours needed to complete the job</th><th>Part of job completed/ hour</th></tr><tr><td>Corey</td><td>4</td><td>$\frac{1}{4}$</td></tr><tr><td>Casey</td><td>t</td><td>$\frac{1}{t}$</td></tr><tr><td>Together</td><td>2</td><td>$\frac{1}{2}$</td></tr></table>
Write a word sentence.	
The part completed by Corey plus the part completed by Casey equals the amount completed together.	
Translate to an equation:	$\overbrace{\text{Corey} + \text{Casey}}^{} = \text{Together}$ $\frac{1}{4} + \frac{1}{t} = \frac{1}{2}$
Solve.	$\frac{1}{4}+\frac{1}{t}=\frac{1}{2}$
Multiply by the LCD, 4t.	$4t\left(\frac{1}{4}+\frac{1}{t}\right)=4t\left(\frac{1}{2}\right)$
Simplify.	$t+4=2t$
	$4=t$
	It would take Casey 4 hours to do the job alone.

Try It 5.7.14

Two hoses can fill a swimming pool in 10 hours. It would take one hose 26 hours to fill the pool by itself. How long would it take for the other hose, working alone, to fill the pool?

Answer

16.25 hours

Try It 5.7.15

Cara and Cindy, working together, can rake the yard in 4 hours. Working alone, it takes Cindy 6 hours to rake the yard. How long would it take Cara to rake the yard alone?

Answer

12 hours

This page titled 5.7: Solve Uniform Motion and Work Applications is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **8.8: Solve Uniform Motion and Work Applications** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

CHAPTER OVERVIEW

6: Systems of Equations

6.1: Solve Systems of Equations by Graphing
6.2: Solve Systems of Equations by Substitution
6.3: Solve Systems of Equations by Elimination
6.4: Solve Mixture Applications with Systems of Equations
6.5: Solve Systems of Equations with Three Variables

6: Systems of Equations is shared under a CC BY-SA license and was authored, remixed, and/or curated by LibreTexts.

6.1: Solve Systems of Equations by Graphing

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of equations
- Solve a system of linear equations by graphing
- Determine the number of solutions of linear system
- Solve applications of systems of equations by graphing

Note

Before you get started, take this readiness quiz.

1. For the equation $y = \frac{2}{3}x - 4$
   ⓐ is (6,0) a solution? ⓑ is (−3,−2) a solution?
   If you missed this problem, review Example 2.1.1.
2. Find the slope and y-intercept of the line 3x−y=12.
   If you missed this problem, review Example 4.5.7.
3. Find the x- and y-intercepts of the line 2x−3y=12.
   If you missed this problem, review Example 4.3.7.

Determine Whether an Ordered Pair is a Solution of a System of Equations

In the section on Solving Linear Equations and Inequalities we learned how to solve linear equations with one variable. Remember that the solution of an equation is a value of the variable that makes a true statement when substituted into the equation. Now we will work with **systems of linear equations**, two or more linear equations grouped together.

Definition: SYSTEM OF LINEAR EQUATIONS

When two or more linear equations are grouped together, they form a system of linear equations.

We will focus our work here on systems of two linear equations in two unknowns. Later, you may solve larger systems of equations.

An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases} \tag{6.1.1}$$

A linear equation in two variables, like $2x + y = 7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line.

To solve a system of two linear equations, we want to find the values of the variables that are solutions to both equations. In other words, we are looking for the ordered pairs (x, y) that make both equations true. These are called the **solutions to a system of equations**.

Definition: SolutionS OF A SYSTEM OF EQUATIONS

Solutions of a system of equations are the values of the variables that make all the equations true. A solution of a system of two linear equations is represented by an ordered pair (x, y).

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

Let's consider the system below:

$$\begin{cases} 3x-y=7 \\ x-2y=4 \end{cases} \tag{6.1.2}$$

Is the ordered pair (2,−1) a solution?

We substitute $x=2$ and $y=-1$ into both equations.

$$\begin{aligned} 3x-y &= 7 & \qquad x-2y &= 4 \\ 3(2)-(-1) &\stackrel{?}{=} 7 & \qquad 2-2(-1) &\stackrel{?}{=} 4 \\ 7 &= 7 \text{ true} & \qquad 4 &= 4 \text{ true} \end{aligned}$$

The ordered pair (2, −1) made both equations true. Therefore (2, −1) is a solution to this system.

Let's try another ordered pair. Is the ordered pair (3, 2) a solution?

We substitute $x=3$ and $y=2$ into both equations.

$$\begin{aligned} 3x-y &= 7 & \qquad x-2y &= 4 \\ 3(3)-2 &\stackrel{?}{=} 7 & \qquad 2-2(2) &\stackrel{?}{=} 4 \\ 7 &= 7 \text{ true} & \qquad -2 &= 4 \text{ false} \end{aligned}$$

The ordered pair (3, 2) made one equation true, but it made the other equation false. Since it is not a solution to **both** equations, it is not a solution to this system.

Example 6.1.1

Determine whether the ordered pair is a solution to the system: $\begin{cases} x-y=-1 \\ 2x-y=-5 \end{cases}$

1. (−2,−1)
2. (−4,−3)

Solution

1.

$$\begin{cases} x-y=-1 \\ 2x-y=-5 \end{cases}$$

We substitute $x=-2$ and $y=-1$ into both equations

$$\begin{aligned} x-y &= 1 & \qquad 2x-y &= -5 \\ -2-(-1) &= -1 & \qquad 2(-2)-(-1) &\stackrel{?}{=} -5 \\ -1 &= -1 \checkmark & \qquad -3 &= -5 \times \end{aligned}$$

(−2, −1) does not make both equations true. (−2, −1) is not a solution.

2.

We substitute $x=-4$ and $y=-3$ into both equations.

$$\begin{aligned} x-y &= -1 & \qquad 2x-y &= -5 \\ -4-(-3) &\stackrel{?}{=} -1 & \qquad 2(-4)-(-3) &\stackrel{?}{=} -5 \\ -1 &= -1 \checkmark & \qquad -5 &= -5 \checkmark \end{aligned}$$

(−4, −3) makes both equations true. (−4, −3) is a solution.

Try It 6.1.2

Determine whether the ordered pair is a solution to the system: $\begin{cases} 3x+y=0 \\ x+2y=-5 \end{cases}$

1. (1,−3)

2. (0,0)

Answer

1. yes
2. no

? Try It 6.1.3

Determine whether the ordered pair is a solution to the system: $\begin{cases} x-3y=-8 \\ -3x-y=4 \end{cases}$

1. (2,−2)
2. (−2,2)

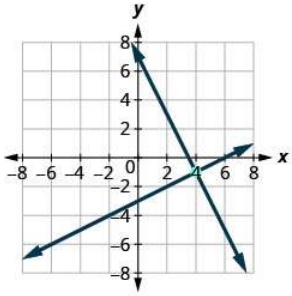
Answer

1. no
2. yes

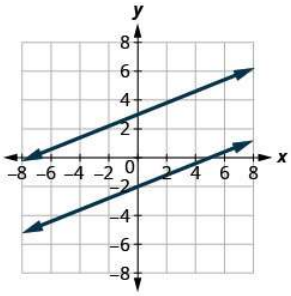
Solve a System of Linear Equations by Graphing

In this chapter we will use three methods to solve a system of linear equations. The first method we'll use is graphing. The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what the lines have in common, we'll find the solution to the system.

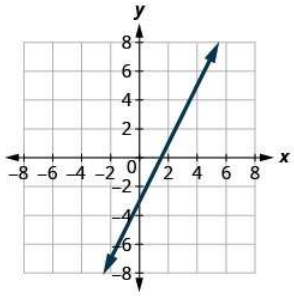
Most linear equations in one variable have one solution, but we saw that some equations, called **contradictions**, have no solutions and for other equations, called **identities**, all numbers are solutions. Similarly, when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown in Figure 6.1.1:



The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.



The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.



Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

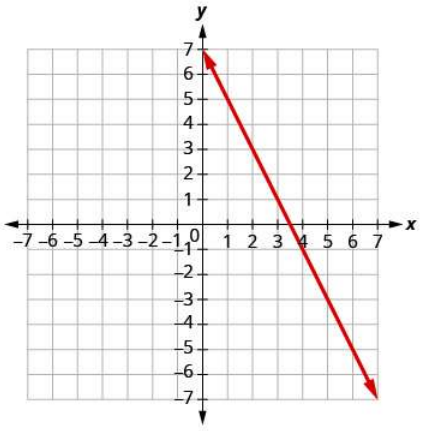
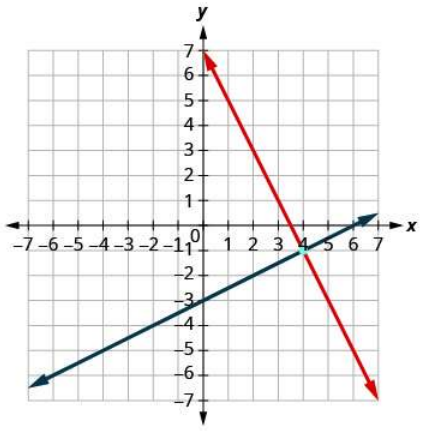
Figure 6.1.1

For the first example of solving a system of linear equations in this section and in the next two sections, we will solve the same system of two linear equations. But we'll use a different method in each section. After seeing the third method, you'll decide which method was the most convenient way to solve this system.

✔ Example 6.1.4: How to Solve a System of Linear Equations by Graphing

Solve the system by graphing: $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Solution

Step 1. Graph the first equation.	To graph the first line, write the equation in slope–intercept form. $2x + y = 7$ $y = -2x + 7$ $m = -2$ $b = 7$	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ 
Step 2. Graph the second equation on the same rectangular coordinate system.	To graph the second line, use intercepts. $x - 2y = 6$ $(0, -3)$ $(6, 0)$	
Step 3. Determine whether the lines intersect, are parallel, or are the same line.	Look at the graph of the lines.	The lines intersect.
Step 4. Identify the solution to the system. If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. If the lines are parallel, the system has no solution If the lines are the same, the system has an infinite number of solutions.	Since the lines intersect, find the point of intersection. Check the point in both equations.	The lines intersect at $(4, -1)$. $2x + y = 7$ $2(4) + (-1) \stackrel{?}{=} 7$ $8 - 1 \stackrel{?}{=} 7$ $7 = 7$ ✓ $x - 2y = 6$ $4 - 2(-1) \stackrel{?}{=} 6$ $6 = 6$ ✓ The solution is $(4, -1)$.

? Try It 6.1.5

Solve each system by graphing: $\begin{cases} x - 3y = -3 \\ x + y = 5 \end{cases}$

Answer

(3,2)

Try It 6.1.6

Solve each system by graphing: $\begin{cases} -x+y=1 \\ 3x+2y=12 \end{cases}$

Answer

(2,3)

The steps to use to solve a system of linear equations by graphing are shown below.

TO SOLVE A SYSTEM OF LINEAR EQUATIONS BY GRAPHING.

1. Graph the first equation.
2. Graph the second equation on the same rectangular coordinate system.
3. Determine whether the lines intersect, are parallel, or are the same line.
4. Identify the solution to the system.
   - If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system.
   - If the lines are parallel, the system has no solution.
   - If the lines are the same, the system has an infinite number of solutions.

Example 6.1.7

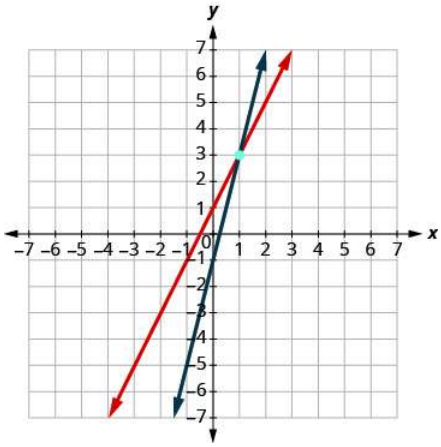
Solve the system by graphing: $\begin{cases} y=2x+1 \\ y=4x-1 \end{cases}$

Solution

Both of the equations in this system are in slope-intercept form, so we will use their slopes and y-intercepts to graph them.

$\begin{cases} y=2x+1 \\ y=4x-1 \end{cases}$

Find the slope and y-intercept of the first equation.	$y=2x+1$ $m=2$ $b=1$
Find the slope and y-intercept of the first equation.	$y=4x-1$ $m=4$ $b=-1$
Graph the two lines.	
Determine the point of intersection.	The lines intersect at (1, 3).

	
Check the solution in both equations.	$y = 2x + 1$ $\quad$ $y = 4x - 1$ $3 \stackrel{?}{=} 2 \cdot 1 + 1$ $\quad$ $3 \stackrel{?}{=} 4 \cdot 1 - 1$ $3 = 3\checkmark$ $\quad$ $3 = 3\checkmark$
	The solution is (1, 3).

Try It 6.1.8

Solve the system by graphing: $\begin{cases} y = 2x + 2 \\ y = -x - 4 \end{cases}$

Answer

(−2,−2)

Try It 6.1.9

Solve the system by graphing: $\begin{cases} y = 3x + 3 \\ y = -x + 7 \end{cases}$

Answer

(1,6)

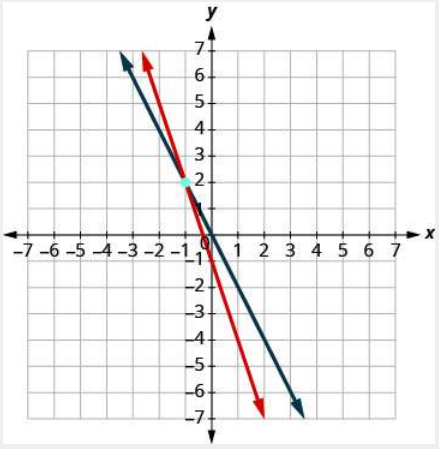
Both equations in Example 6.1.7 were given in slope–intercept form. This made it easy for us to quickly graph the lines. In the next example, we'll first re-write the equations into slope–intercept form.

Example 6.1.10

Solve the system by graphing: $\begin{cases} 3x + y = -1 \\ 2x + y = 0 \end{cases}$

Solution

We'll solve both of these equations for y so that we can easily graph them using their slopes and y-intercepts. $\begin{cases} 3x + y = -1 \\ 2x + y = 0 \end{cases}$

Solve the first equation for y.	
Find the slope and y-intercept.	$3x+y=-1$ $y=-3x-1$ $m=-3$ $b=-1$
Solve the second equation for y.	$2x+y=0$ $y=-2x$
Find the slope and y-intercept.	$b=0$
Graph the lines.	
Determine the point of intersection.	The lines intersect at $(-1, 2)$.
Check the solution in both equations.	$3x+y=-1$ $\quad 3(-1)+2 \stackrel{?}{=} -1 \quad -1=-1\checkmark$ $2x+y=0$ $\quad 2(-1)+2 \stackrel{?}{=} 0 \quad 0=0\checkmark$
	The solution is $(-1, 2)$.

? Try It 6.1.11

Solve each system by graphing: $\begin{cases} -x+y=1 \\ 2x+y=10 \end{cases}$

Answer

(3,4)

? Try It 6.1.12

Solve each system by graphing: $\begin{cases} 2x+y=6 \\ x+y=1 \end{cases}$

Answer

(5,−4)

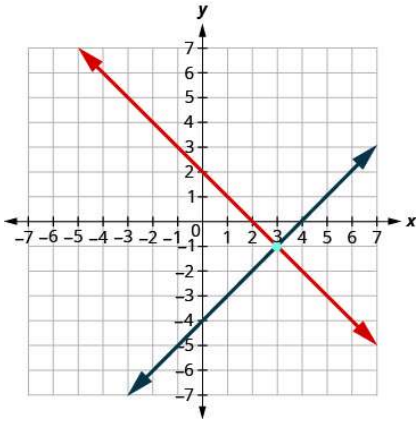
Usually when equations are given in standard form, the most convenient way to graph them is by using the intercepts. We'll do this in Example 6.1.13.

Example 6.1.13

Solve the system by graphing: $\begin{cases} x+y=2 \\ x-y=4 \end{cases}$

Solution

We will find the *x*- and *y*-intercepts of both equations and use them to graph the lines.

	$x+y=2$	
To find the intercepts, let $x = 0$ and solve for y, then let $y = 0$ and solve for x.	$x+y=2 \quad x+y=2$ $0+y=2 \quad x+0=2$ $y=2 \quad x=2$	x | y 0 | 2 2 | 0
	$x-y=4$	
To find the intercepts, let $x = 0$ then let $y = 0$.	$x-y=4 \quad x-y=4$ $0-y=4 \quad x-0=4$ $-y=4 \quad x=4$ $y=-4$	x | y 0 | –4 4 | 0
Graph the line.		
Determine the point of intersection.	The lines intersect at (3, −1).	
Check the solution in both equations.	$x+y = 2 \quad x-y = 4$ $3+(-1) \stackrel{?}{=} 2 \quad 3-(-1) \stackrel{?}{=} 4$ $2 = 2\checkmark \quad 4 = 4\checkmark$ The solution is (3, −1).	

(6.1.3)

Try It 6.1.14

Solve each system by graphing: $\begin{cases} x+y=6 \\ x-y=2 \end{cases}$

Answer

(4,2)

Try It 6.1.15

Solve each system by graphing: $\begin{cases} x+y=2 \\ x-y=8 \end{cases}$

Answer

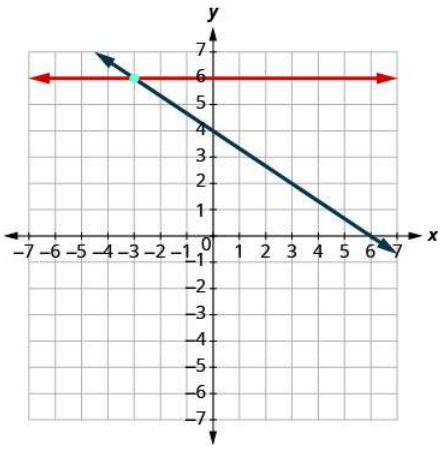
(5,−3)

Do you remember how to graph a linear equation with just one variable? It will be either a vertical or a horizontal line.

Example 6.1.16

Solve the system by graphing: $\begin{cases} y=6 \\ 2x+3y=12 \end{cases}$

Solution

	$y=6$ $2x+3y=12$
We know the first equation represents a horizontal line whose y-intercept is 6.	$y=6$
The second equation is most conveniently graphed using intercepts.	$2x+3y=12$
To find the intercepts, let $x=0$ and then $y=0$.	x | y 0 | 4 6 | 0
Graph the lines.	
Determine the point of intersection.	The lines intersect at (−3, 6).
Check the solution to both equations.	$y = 6$, $6 \stackrel{?}{=} 6$, $6 = 6\checkmark$ $2x+3y = 12$, $2(-3)+3(6) \stackrel{?}{=} 12$, $-6+18 \stackrel{?}{=} 12$, $12 = 12\checkmark$
	The solution is (−3, 6).

Try It 6.1.17

Solve each system by graphing: $\begin{cases} y = -1 \\ x + 3y = 6 \end{cases}$

Answer

(9,−1)

Try It 6.1.18

Solve each system by graphing: $\begin{cases} x = 4 \\ 3x - 2y = 24 \end{cases}$

Answer

(4,−6)

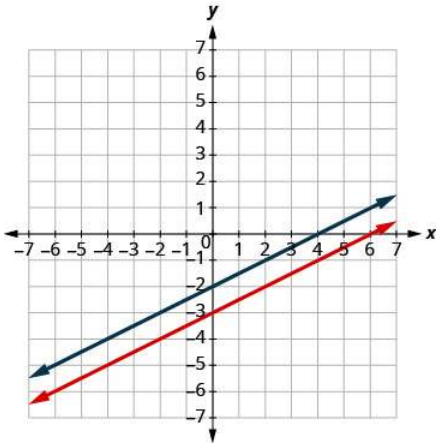
In all the systems of linear equations so far, the lines intersected and the solution was one point. In the next two examples, we'll look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

Example 6.1.19

Solve the system by graphing: $\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$

Solution

	$y = \frac{1}{2}x - 3$ $x - 2y = 4$
To graph the first equation, we will use its slope and y-intercept.	$y = \frac{1}{2}x - 3$
	$m = \frac{1}{2}$
	$b = -3$
To graph the second equation, we will use the intercepts.	$x - 2y = 4$
	$\begin{array}{\|c\|c\|} \hline x & y \\ \hline 0 & -2 \\ \hline 4 & 0 \\ \hline \end{array}$

Graph the lines.	
Determine the point of intersection.	The lines are parallel.
	Since no point is on both lines, there is no ordered pair that makes both equations true. There is no solution to this system.

Try It 6.1.20

Solve each system by graphing: $\begin{cases} y = -\frac{1}{4}x + 2 \\ x + 4y = -8 \end{cases}$

Answer

no solution

Try It 6.1.21

Solve each system by graphing: $\begin{cases} y = 3x - 1 \\ 6x - 2y = 6 \end{cases}$

Answer

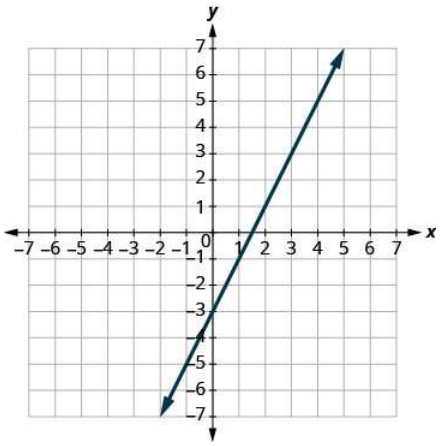
no solution

Example 6.1.22

Solve the system by graphing: $\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$

Solution

	$y = 2x - 3$ $-6x + 3y = -9$
Find the slope and y-intercept of the first equation.	$y = 2x - 3$ $m = 2$ $b = -3$
Find the intercepts of the second equation.	$-6x + 3y = -9$

	$\begin{array}{\|c\|c\|}\hline x & y \\ \hline 0 & -3 \\ \hline \frac{3}{2} & 0 \\ \hline \end{array}$
Graph the lines.	
Determine the point of intersection.	The lines are the same!
	Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true.
	There are infinitely many solutions to this system.

Try It 6.1.23

Solve each system by graphing: $\begin{cases} y = -3x - 6 \\ 6x + 2y = -12 \end{cases}$

Answer

infinitely many solutions

Try It 6.1.24

Solve each system by graphing: $\begin{cases} y = \frac{1}{2}x - 4 \\ 2x - 4y = 16 \end{cases}$

Answer

infinitely many solutions

If you write the second equation in Example 6.1.22 in slope-intercept form, you may recognize that the equations have the same slope and same *y*-intercept.

When we graphed the second line in the last example, we drew it right over the first line. We say the two lines are coincident. Coincident lines have the same slope and same *y*-intercept.

COINCIDENT LINES

Coincident lines have the same slope and same *y*-intercept.

Determine the Number of Solutions of a Linear System

There will be times when we will want to know how many solutions there will be to a system of linear equations, but we might not actually have to find the solution. It will be helpful to determine this without graphing.

We have seen that two lines in the same plane must either intersect or are parallel. The systems of equations in Example 6.1.4 through Example 6.1.18 all had two intersecting lines. Each system had one solution.

A system with parallel lines, like Example 6.1.19, has no solution. What happened in Example 6.1.22? The equations have **coincident lines**, and so the system had infinitely many solutions.

We'll organize these results in Figure 6.1.2 below:

Graph	Number of solutions
2 intersecting lines	1
Parallel lines	None
Same line	Infinitely many

Figure 6.1.2

Parallel lines have the same slope but different *y*-intercepts. So, if we write both equations in a system of linear equations in slope–intercept form, we can see how many solutions there will be without graphing! Look at the system we solved in Example 6.1.19.

$$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$$

The first line is in slope–intercept form. If we solve the second equation for y, we get

$$y = \frac{1}{2}x - 3 \qquad x - 2y = 4$$
$$x - 2y = -x + 4$$
$$y = \frac{1}{2}x - 2$$
$$m = \frac{1}{2}, b = -3 \qquad m = \frac{1}{2}, b = -2$$

The two lines have the same slope but different *y*-intercepts. They are parallel lines.

Figure 6.1.3 shows how to determine the number of solutions of a linear system by looking at the slopes and intercepts.

Number of Solutions of a Linear System of Equations			
Slopes	**Intercepts**	**Type of Lines**	**Number of Solutions**
Different		Intersecting	1 point
Same	Different	Parallel	No solution
Same	Same	Coincident	Infinitely many solutions

Figure 6.1.3

Let's take one more look at our equations in Example 6.1.19 that gave us parallel lines.

$$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases} \tag{6.1.4}$$

When both lines were in slope-intercept form we had:

$$y = \frac{1}{2}x - 3 \quad y = \frac{1}{2}x - 2 \tag{6.1.5}$$

Do you recognize that it is impossible to have a single ordered pair (x,y) that is a solution to both of those equations?

We call a system of equations like this **an inconsistent system**. It has no solution.

A system of equations that has at least one solution is called a **consistent system**.

CONSISTENT AND INCONSISTENT SYSTEMS

A **consistent system** of equations is a system of equations with at least one solution.

An **inconsistent system** of equations is a system of equations with no solution.

We also categorize the equations in a system of equations by calling the equations *independent* or *dependent*. If two equations are **independent equations**, they each have their own set of solutions. Intersecting lines and parallel lines are independent.

If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two **dependent equations**, we get coincident lines.

INDEPENDENT AND DEPENDENT EQUATIONS

Two equations are **independent** if they have different solutions.

Two equations are **dependent** if all the solutions of one equation are also solutions of the other equation.

Let's sum this up by looking at the graphs of the three types of systems. See Figure 6.1.4 and Figure 6.1.5.

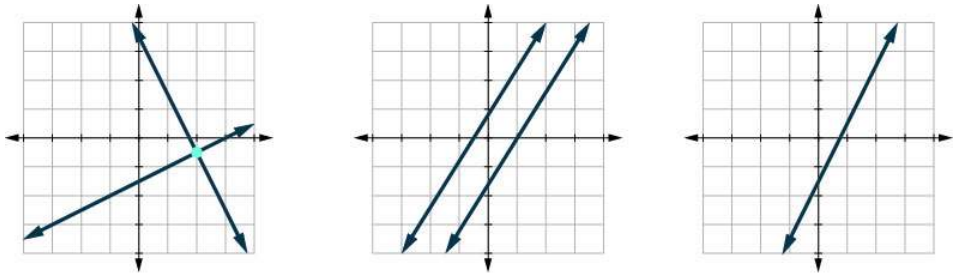


Figure 6.1.4

Lines	Intersecting	Parallel	Coincident
Number of solutions	1 point	No solution	Infinitely many
Consistent/inconsistent	Consistent	Inconsistent	Consistent
Dependent/independent	Independent	Independent	Dependent

Figure 6.1.5

Example 6.1.25

Without graphing, determine the number of solutions and then classify the system of equations: $\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$

Solution

We will compare the slopes and intercepts of the two lines.

$$\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$$

The first equation is already in slope-intercept form.

$$y = 3x - 1$$

Write the second equation inslope–intercept form.

$$\begin{aligned} 6x - 2y &= 12 \\ -2y &= -6x - 12 \\ \tfrac{-2y}{-2} &= \tfrac{-6x+12}{-2} \\ y &= 3x - 6 \end{aligned}$$

Find the slope and intercept of each line.

$$\begin{aligned} y &= 3x - 1 & y &= 3x - 6 \\ m &= 3 & m &= 3 \\ b &= -1 & b &= -6 \end{aligned}$$

Since the slopes are the same and y-intercepts are different, the lines are parallel.

A system of equations whose graphs are parallel lines has no solution and is inconsistent and independent.

Try It 6.1.26

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} y = -2x - 4 \\ 4x + 2y = 9 \end{cases}$$

Answer

no solution, inconsistent, independent

Try It 6.1.27

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} y = \frac{1}{3}x - 5 \\ x - 3y = 6 \end{cases}$$

Answer

no solution, inconsistent, independent

Example 6.1.28

Without graphing, determine the number of solutions and then classify the system of equations: $\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}$

Solution

We will compare the slopes and intercepts of the two lines.

$$\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}$$

Write the second equation in slope–intercept form.

$$\begin{aligned} 2x + y &= -3 \\ y &= -2x - 3 \end{aligned} \qquad \begin{aligned} x - 5y &= 5 \\ -5y &= -x + 5 \\ \tfrac{-5y}{-5} &= \tfrac{-x+5}{-5} \\ y &= \tfrac{1}{5}x - 1 \end{aligned}$$

Find the slope and intercept of each line.

$$\begin{aligned} y &= -2x - 3 \\ m &= -2 \\ b &= -3 \end{aligned} \qquad \begin{aligned} y &= \tfrac{1}{5}x - 1 \\ m &= \tfrac{1}{5} \\ b &= -1 \end{aligned}$$

Since the slopes are different, the lines intersect.

A system of equations whose graphs are intersect has 1 solution and is consistent and independent.

Try It 6.1.29

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} 3x + 2y = 2 \\ 2x + y = 1 \end{cases}$$

Answer

one solution, consistent, independent

Try It 6.1.30

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} x+4y=12 \\ -x+y=3 \end{cases}$$

Answer

one solution, consistent, independent

Example 6.1.31

Without graphing, determine the number of solutions and then classify the system of equations. $\begin{cases} 3x-2y=4 \\ y=\frac{3}{2}x-2 \end{cases}$

Solution

We will compare the slopes and intercepts of the two lines. $\begin{cases} 3x-2y &= 4 \\ y &= \frac{3}{2}x-2 \end{cases}$

Write the second equation in slope–intercept form.

$$\begin{aligned} 3x-2y &= 4 \\ -2y &= -3x+4 \\ \frac{-2y}{-2} &= \frac{-3x+4}{-2} \\ y &= \frac{3}{2}x-2 \end{aligned}$$

Find the slope and intercept of each line. $y = \frac{3}{2}x-2$

Since the equations are the same, they have the same slope and same y-intercept and so the lines are coincident.

A system of equations whose graphs are coincident lines has infinitely many solutions and is consistent and dependent.

Try It 6.1.32

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} 4x-5y=20 \\ y=\frac{4}{5}x-4 \end{cases}$$

Answer

infinitely many solutions, consistent, dependent

Try It 6.1.33

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} -2x-4y=8 \\ y=-\frac{1}{2}x-2 \end{cases}$$

Answer

infinitely many solutions, consistent, dependent

Solve Applications of Systems of Equations by Graphing

We will use the same problem solving strategy we used in Math Models to set up and solve applications of systems of linear equations. We'll modify the strategy slightly here to make it appropriate for systems of equations.

USE A PROBLEM SOLVING STRATEGY FOR SYSTEMS OF LINEAR EQUATIONS.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose variables to represent those quantities.
4. **Translate** into a system of equations.
5. **Solve** the system of equations using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Step 5 is where we will use the method introduced in this section. We will graph the equations and find the solution.

Example 6.1.34

Sondra is making 10 quarts of punch from fruit juice and club soda. The number of quarts of fruit juice is 4 times the number of quarts of club soda. How many quarts of fruit juice and how many quarts of club soda does Sondra need?

Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for.

We are looking for the number of quarts of fruit juice and the number of quarts of club soda that Sondra will need.

Step 3. Name what we are looking for. Choose variables to represent those quantities.

Let f= number of quarts of fruit juice.
c= number of quarts of club soda

Step 4. Translate into a system of equations.

The number of quarts of fruit juice and the number of quarts of club soda is 10

$$f + c = 10$$

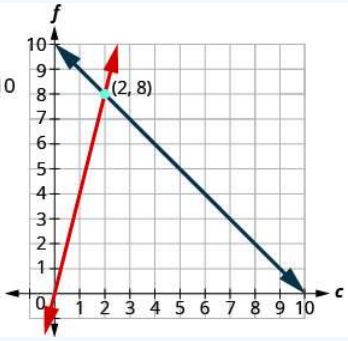
The number of quarts of fruit juice is four times the number of quarts of club soda

$$f = 4c$$

We now have the system. $\begin{cases} f + c = 10 \\ f = 4c \end{cases}$

Step 5. Solve the system of equations using good algebra techniques.

$f = 4c$ $\quad$ $f + c = 10$
$m = 4$ $\quad$ $f = -c + 10$
$b = 0$ $\quad$ $m = -1$
$\quad$ $\quad$ $b = 10$



The point of intersection (2, 8) is the solution. This means Sondra needs 2 quarts of club soda and 8 quarts of fruit juice.

Step 6. Check the answer in the problem and make sure it makes sense.

Does this make sense in the problem?

Yes, the number of quarts of fruit juice, 8 is 4 times the number of quarts of club soda, 2.

Yes, 10 quarts of punch is 8 quarts of fruit juice plus 2 quarts of club soda.

Step 7. Answer the question with a complete sentence.

Sondra needs 8 quarts of fruit juice and 2 quarts of soda.

Try It 6.1.35

Manny is making 12 quarts of orange juice from concentrate and water. The number of quarts of water is 3 times the number of quarts of concentrate. How many quarts of concentrate and how many quarts of water does Manny need?

Answer

Manny needs 3 quarts juice concentrate and 9 quarts water.

Try It 6.1.36

Alisha is making an 18 ounce coffee beverage that is made from brewed coffee and milk. The number of ounces of brewed coffee is 5 times greater than the number of ounces of milk. How many ounces of coffee and how many ounces of milk does Alisha need?

Answer

Alisha needs 15 ounces of coffee and 3 ounces of milk.

Note

Access these online resources for additional instruction and practice with solving systems of equations by graphing.

- Instructional Video Solving Linear Systems by Graphing
- Instructional Video Solve by Graphing

Key Concepts

- **To solve a system of linear equations by graphing**
  1. Graph the first equation.
  2. Graph the second equation on the same rectangular coordinate system.
  3. Determine whether the lines intersect, are parallel, or are the same line.
  4. Identify the solution to the system.
     If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system.
     If the lines are parallel, the system has no solution.
     If the lines are the same, the system has an infinite number of solutions.
  5. Check the solution in both equations.
- Determine the number of solutions from the graph of a linear system

Graph	Number of solutions
2 intersecting lines	1
Parallel lines	None
Same line	Infinitely many

- Determine the number of solutions of a linear system by looking at the slopes and intercepts

Number of Solutions of a Linear System of Equations			
Slopes	**Intercepts**	**Type of Lines**	**Number of Solutions**
Different		Intersecting	1 point
Same	Different	Parallel	No solution
Same	Same	Coincident	Infinitely many solutions

- Determine the number of solutions and how to classify a system of equations

Lines	**Intersecting**	**Parallel**	**Coincident**
Number of solutions	1 point	No solution	Infinitely many
Consistent/inconsistent	Consistent	Inconsistent	Consistent
Dependent/independent	Independent	Independent	Dependent

- **Problem Solving Strategy for Systems of Linear Equations**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose variables to represent those quantities.
  4. **Translate** into a system of equations.
  5. **Solve** the system of equations using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

Glossary

coincident lines

Coincident lines are lines that have the same slope and same y-intercept.

consistent system

A consistent system of equations is a system of equations with at least one solution.

dependent equations

Two equations are dependent if all the solutions of one equation are also solutions of the other equation.

inconsistent system

An inconsistent system of equations is a system of equations with no solution.

independent equations

Two equations are independent if they have different solutions.

solutions of a system of equations

Solutions of a system of equations are the values of the variables that make all the equations true. A solution of a system of two linear equations is represented by an ordered pair (x, y).

system of linear equations

When two or more linear equations are grouped together, they form a system of linear equations.

This page titled 6.1: Solve Systems of Equations by Graphing is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **5.1: Solve Systems of Equations by Graphing** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

6.2: Solve Systems of Equations by Substitution

Learning Objectives

By the end of this section, you will be able to:

- Solve a system of equations by substitution
- Solve applications of systems of equations by substitution

Note

Before you get started, take this readiness quiz.

1. Simplify −5(3−x).
   If you missed this problem, review Exercise 1.10.43.
2. Simplify 4−2(n+5).
   If you missed this problem, review Exercise 1.10.41.
3. Solve for y. 8y−8=32−2y
   If you missed this problem, review Exercise 2.3.22.
4. Solve for x. 3x−9y=−3
   If you missed this problem, review Exercise 2.6.22.

Solving systems of linear equations by graphing is a good way to visualize the types of solutions that may result. However, there are many cases where solving a system by graphing is inconvenient or imprecise. If the graphs extend beyond the small grid with x and y both between −10 and 10, graphing the lines may be cumbersome. And if the solutions to the system are not integers, it can be hard to read their values precisely from a graph.

In this section, we will solve systems of linear equations by the substitution method.

Solve a System of Equations by Substitution

We will use the same system we used first for graphing.

$$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$$

We will first solve one of the equations for either x or y. We can choose either equation and solve for either variable—but we'll try to make a choice that will keep the work easy.

Then we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true.

We'll fill in all these steps now in Exercise 6.2.1.

Exercise 6.2.1: How to Solve a System of Equations by Substitution

Solve the system by substitution. $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Answer

Step 1. Solve one of the equations for either variable.	We'll solve the first equation for y.	$2x + y = 7$ $y = 7 - 2x$
Step 2. Substitute the expression from Step 1 into the other equation.	We replace y in the second equation with the expression $7 - 2x$.	$x - 2y = 6$ $x - 2(7 - 2x) = 6$
Step 3. Solve the resulting equation.	Now we have an equation with just 1 variable. We know how to solve this!	$x - 2(7 - 2x) = 6$ $x - 14 + 4x = 6$ $5x = 20$ $x = 4$
Step 4. Substitute the solution in Step 3 into one of the original equations to find the other variable.	We'll use the first equation and replace x with 4.	$2x + y = 7$ $2(4) + y = 7$ $8 + y = 7$ $y = -1$
Step 5. Write the solution as an ordered pair.	The ordered pair is (x, y).	$(4, -1)$
Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute $(4, -1)$ into both equations and make sure they are both true.	$2x + y = 7$ $\quad$ $x - 2y = 6$ $2(4) + (-1) \stackrel{?}{=} 7$ $\quad$ $4 - 2(-1) \stackrel{?}{=} 6$ $7 = 7$ ✓ $\quad$ $6 = 6$ ✓ Both equations are true. $(4, -1)$ is the solution to the system.

Exercise 6.2.2

Solve the system by substitution. $\begin{cases} -2x + y = -11 \\ x + 3y = 9 \end{cases}$

Answer

(6,1)

Exercise 6.2.3

Solve the system by substitution. $\begin{cases} x + 3y = 10 \\ 4x + y = 18 \end{cases}$

Answer

(4,2)

SOLVE A SYSTEM OF EQUATIONS BY SUBSTITUTION.

1. Solve one of the equations for either variable.
2. Substitute the expression from Step 1 into the other equation.
3. Solve the resulting equation.
4. Substitute the solution in Step 3 into one of the original equations to find the other variable.

5. Write the solution as an ordered pair.
6. Check that the ordered pair is a solution to **both** original equations.

If one of the equations in the system is given in slope–intercept form, Step 1 is already done! We'll see this in Exercise 6.2.4.

Exercise 6.2.4

Solve the system by substitution. $\begin{cases} x+y=-1 \\ y=x+5 \end{cases}$

Answer

The second equation is already solved for y. We will substitute the expression in place of y in the first equation.

	$x+y=-1$ $y=x+5$
The second equation is already solved for y. We will substitute into the first equation.	
Replace the y with $x+5$.	$y=x+5$ $x+y=-1$
Solve the resulting equation for x.	$x+x+5=-1$
	$2x+5=-1$
	$2x=-6$
Substitute $x=-3$ into $y=x+5$ to find y.	$x=-3$ $y=x+5$
	$y=-3+5$
The ordered pair is $(-3, 2)$.	$y=2$
Check the ordered pair in both equations: $x+y=-1 \quad y=x+5$ $-3+2 \stackrel{?}{=} -1 \quad 2 \stackrel{?}{=} -3+5$ $-1=-1\checkmark \quad 2=2\checkmark$	
	The solution is $(-3, 2)$.

Exercise 6.2.5

Solve the system by substitution. $\begin{cases} x+y=6 \\ y=3x-2 \end{cases}$

Answer

(2,4)

Exercise 6.2.6

Solve the system by substitution. $\begin{cases} 2x-y=1 \\ y=-3x-6 \end{cases}$

Answer

$(-1,-3)$

If the equations are given in standard form, we'll need to start by solving for one of the variables. In this next example, we'll solve the first equation for y.

Exercise 6.2.7

Solve the system by substitution. $\begin{cases} 3x+y=5 \\ 2x+4y=-10 \end{cases}$

Answer

We need to solve one equation for one variable. Then we will substitute that expression into the other equation.

Solve for y. Substitute into the other equation.	$3x+y=5$ $y=-3x+5$ $2x+4y=-10$
Replace the y with $-3x+5$.	$2x+4(-3x+5)=-10$
Solve the resulting equation for x.	$2x-12x+20=-10$
	$-10x+20=-10$ $-10x=-30$
Substitute $x=3$ into $3x+y=5$ to find y.	$x=3$ $3x+y=5$
	$3(3)+y=5$ $9+y=5$
The ordered pair is $(3,-4)$.	$y=-4$
Check the ordered pair in both equations: $3x+y=5$: $3\cdot 3+(-4) \stackrel{?}{=} 5$, $9-4 \stackrel{?}{=} 5$, $5=5\checkmark$ $2x+4y=-10$: $2\cdot 3+4(-4) \stackrel{?}{=} -10$, $6-16 \stackrel{?}{=} -10$, $-10=-10\checkmark$	
	The solution is $(3,-4)$.

Exercise 6.2.8

Solve the system by substitution. $\begin{cases} 4x+y=2 \\ 3x+2y=-1 \end{cases}$

Answer

$(1,-2)$

Exercise 6.2.9

Solve the system by substitution. $\begin{cases} -x+y=4 \\ 4x-y=2 \end{cases}$

Answer

$(2,6)$

In Exercise 6.2.7 it was easiest to solve for y in the first equation because it had a coefficient of 1. In Exercise 6.2.10 it will be easier to solve for x.

Exercise 6.2.10

Solve the system by substitution. $\begin{cases} x-2y=-2 \\ 3x+2y=34 \end{cases}$

Answer

We will solve the first equation for xx and then substitute the expression into the second equation.

	$x-2y=-2$
Solve for x. Substitute into the other equation.	$x=2y-2$ $3x+2y=34$
Replace the x with $2y-2$.	$3(2y-2)+2y=34$
Solve the resulting equation for y.	$6y-6+2y=34$
Substitute $y=5$ into $x-2y=-2$ to find x.	$8y-6=34$ $8y=40$ $y=5$ $x-2y=-2$ $x-2\cdot 5=-2$ $x-10=-2$ $x=8$
The ordered pair is (8, 5).	
Check the ordered pair in both equations: $x-2y=-2$ $8-2\cdot 5 \stackrel{?}{=} -2$ $8-10 \stackrel{?}{=} -2$ $-2=-2\checkmark$ $3x+2y=34$ $3\cdot 8+2\cdot 5 \stackrel{?}{=} 34$ $24+10 \stackrel{?}{=} 34$ $34=34\checkmark$	
	The solution is (8, 5).

Exercise 6.2.11

Solve the system by substitution. $\begin{cases} x-5y=13 \\ 4x-3y=1 \end{cases}$

Answer

$(-2,-3)$

Exercise 6.2.12

Solve the system by substitution. $\begin{cases} x-6y=-6 \\ 2x-4y=4 \end{cases}$

Answer

(6,2)

When both equations are already solved for the same variable, it is easy to substitute!

Exercise 6.2.13

Solve the system by substitution. $\begin{cases} y = -2x + 5 \\ y = \frac{1}{2}x \end{cases}$

Answer

Since both equations are solved for y, we can substitute one into the other.

Substitute $\frac{1}{2}x$ for y in the first equation.	$y = \frac{1}{2}x$ $y = -2x + 5$
Replace the y with $\frac{1}{2}x$	$\frac{1}{2}x = -2x + 5$
Solve the resulting equation. Start by clearing the fraction.	$2\left(\frac{1}{2}x\right) = 2(-2x + 5)$
Solve for x.	$x = -4x + 10$
	$5x = 10$
Substitute $x = 2$ into $y = \frac{1}{2}x$ to find y.	$x = 2$ $y = \frac{1}{2}x$ $y = \frac{1}{2} \cdot 2$ $y = 1$
The ordered pair is (2,1).	
Check the ordered pair in both equations: $\begin{array}{rcl} y & = & \frac{1}{2}x \\ 1 & \stackrel{?}{=} & \frac{1}{2} \cdot 2 \\ 1 & = & 1\checkmark \end{array}$ $\begin{array}{rcl} y & = & -2x + 5 \\ 1 & \stackrel{?}{=} & -2 \cdot 2 + 5 \\ 1 & = & -4 + 5 \\ 1 & = & 1\checkmark \end{array}$	
	The solution is (2,1).

Exercise 6.2.14

Solve the system by substitution. $\begin{cases} y = 3x - 16 \\ y = \frac{1}{3}x \end{cases}$

Answer

(6,2)

Exercise 6.2.15

Solve the system by substitution. $\begin{cases} y = -x + 10 \\ y = \frac{1}{4}x \end{cases}$

Answer

(8,2)

Be very careful with the signs in the next example.

Exercise 6.2.16

Solve the system by substitution. $\begin{cases} 4x+2y=4 \\ 6x-y=8 \end{cases}$

Answer

We need to solve one equation for one variable. We will solve the first equation for y.

	$4x + 2y = 4$
Solve the first equation for y.	$2y = -4x + 4$
Substitute $-2x + 2$ for y in the second equation.	$y = -2x + 2$ $6x - y = 8$
Replace the y with $-2x + 2$.	$6x - (-2x + 2) = 8$
Solve the equation for x.	$6x + 2x - 2 = 8$
	$8x - 2 = 8$ $8x = 10$
Substitute $x = \frac{5}{4}$ into $4x + 2y = 4$ to find y.	$x = \frac{5}{4}$ $4x + 2y = 4$ $4\left(\frac{5}{4}\right) + 2y = 4$ $5 + 2y = 4$ $2y = -1$ $y = -\frac{1}{2}$
The ordered pair is $(\frac{5}{4}, -\frac{1}{2})$.	
Check the ordered pair in both equations.	
$\begin{aligned} 4x+2y &= 4 \\ 4(\tfrac{5}{4})+2(-\tfrac{1}{2}) &\stackrel{?}{=} 4 \\ 5-1 &\stackrel{?}{=} 4 \\ 4 &= 4\checkmark \end{aligned}$	$\begin{aligned} 6x-y &= 8 \\ 6(\tfrac{5}{4})-(-\tfrac{1}{2}) &\stackrel{?}{=} 8 \\ \tfrac{15}{4}-(-\tfrac{1}{2}) &\stackrel{?}{=} 8 \\ \tfrac{16}{2} &\stackrel{?}{=} 8 \\ 8 &= 8\checkmark \end{aligned}$
	The solution is (54,−12).

Exercise 6.2.17

Solve the system by substitution. $\begin{cases} x-4y=-4 \\ -3x+4y=0 \end{cases}$

Answer

$(2, \frac{3}{2})$

Exercise 6.2.18

Solve the system by substitution. $\begin{cases} 4x - y = 0 \\ 2x - 3y = 5 \end{cases}$

Answer

$(-\frac{1}{2}, -2)$

In Example, it will take a little more work to solve one equation for x or y.

Exercise 6.2.19

Solve the system by substitution. $\begin{cases} 4x - 3y = 6 \\ 15y - 20x = -30 \end{cases}$

Answer

We need to solve one equation for one variable. We will solve the first equation for x.

	$4x - 3y = 6$
Solve the first equation for x.	$4x = 3y + 6$
Substitute $\frac{3}{4}y + \frac{3}{2}$ for x in the second equation.	$x = \frac{3}{4}y + \frac{3}{2}$ $15y - 20x = -30$
Replace the x with $\frac{3}{4}y + \frac{3}{2}$	$15y - 20\left(\frac{3}{4}y + \frac{3}{2}\right) = -30$
Solve for y.	$15y - 15y - 30 = -30$
	$0 - 30 = -30$
	$0 = 0$

Since 0 = 0 is a true statement, the system is consistent. The equations are dependent. The graphs of these two equations would give the same line. The system has infinitely many solutions.

Exercise 6.2.20

Solve the system by substitution. $\begin{cases} 2x - 3y = 12 \\ -12y + 8x = 48 \end{cases}$

Answer

infinitely many solutions

Exercise 6.2.21

Solve the system by substitution. $\begin{cases} 5x + 2y = 12 \\ -4y - 10x = -24 \end{cases}$

Answer

infinitely many solutions

Look back at the equations in Exercise 6.2.22 Is there any way to recognize that they are the same line?

Let's see what happens in the next example.

Exercise 6.2.22

Solve the system by substitution. $\begin{cases} 5x-2y=-10 \\ y=\frac{5}{2}x \end{cases}$

Answer

The second equation is already solved for *y*, so we can substitute for *y* in the first equation.

Substitute *x* for *y* in the first equation.	$y=\left(\frac{5}{2}x\right)$ $5x-2y=-10$
Replace the *y* with $\frac{5}{2}x$.	$5x-2\left(\frac{5}{2}x\right)=-10$
Solve for *x*.	$5x-5x=-10$
	$0\neq-10$

Since 0 = −10 is a false statement the equations are inconsistent. The graphs of the two equation would be parallel lines. The system has no solutions.

Exercise 6.2.23

Solve the system by substitution. $\begin{cases} 3x+2y=9 \\ y=-\frac{3}{2}x+1 \end{cases}$

Answer

no solution

Exercise 6.2.24

Solve the system by substitution. $\begin{cases} 5x-3y=2 \\ y=\frac{5}{3}x-4 \end{cases}$

Answer

no solution

Solve Applications of Systems of Equations by Substitution

We'll copy here the problem solving strategy we used in the Solving Systems of Equations by Graphing section for solving systems of equations. Now that we know how to solve systems by substitution, that's what we'll do in Step 5.

HOW TO USE A PROBLEM SOLVING STRATEGY FOR SYSTEMS OF LINEAR EQUATIONS.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose variables to represent those quantities.
4. **Translate** into a system of equations.
5. **Solve** the system of equations using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.

7. **Answer** the question with a complete sentence.

Some people find setting up word problems with two variables easier than setting them up with just one variable. Choosing the variable names is easier when all you need to do is write down two letters. Think about this in the next example—how would you have done it with just one variable?

? Exercise 6.2.25

The sum of two numbers is zero. One number is nine less than the other. Find the numbers.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name what we are looking for.	Let n= the first number Let m= the second number
Step 4. Translate into a system of equations.	The sum of two numbers is zero.
	$n+m=0$
	One number is nine less than the other.
	$n=m-9$
The system is:	$\begin{aligned} n+m&=0\\ n&=m-9\end{aligned}$
Step 5. Solve the system of equations. We will use substitution since the second equation is solved for n.	
Substitute $m-9$ for n in the first equation.	$n=m-9$ $n+m=0$
Solve for m.	$m-9+m=0$
	$2m-9=0$
	$2m=9$
Substitute $m=\frac{9}{2}$ into the second equation and then solve for n.	$m=\frac{9}{2}$ $n=m-9$
	$m=\frac{9}{2}-9$
	$m=\frac{9}{2}-\frac{18}{2}$
	$n=-\frac{9}{2}$
Step 6. Check the answer in the problem.	Do these numbers make sense in the problem? We will leave this to you!
Step 7. Answer the question.	The numbers are $\frac{9}{2}$ and $-\frac{9}{2}$.

Exercise 6.2.26

The sum of two numbers is 10. One number is 4 less than the other. Find the numbers.

Answer

The numbers are 3 and 7.

Exercise 6.2.27

The sum of two number is −6. One number is 10 less than the other. Find the numbers.

Answer

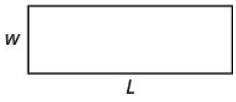
The numbers are 2 and −8.

In the Exercise 6.2.28, we'll use the formula for the perimeter of a rectangle, $P = 2L + 2W$.

Exercise 6.2.28

Find the dimensions of a rectangle in which the length is five more than twice the width.

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what we are looking for.	Let L= the length W= the width
Step 4. Translate into a system of equations.	The perimeter of a rectangle is 88.
	$2L + 2W = P$ $2L + 2W = 88$
	The length is five more than twice the width.
	$L = 2W + 5$
The system is:	$2L + 2W = 88$ $L = 2W + 5$
Step 5. Solve the system of equations. We will use substitution since the second equation is solved for L. Substitute $2W + 5$ for L in the first equation.	$L = 2W + 5$ $2L + 2W = 88$
Solve for W.	$2(2W + 5) + 2W = 88$
	$4W + 10 + 2W = 88$
	$6W + 10 = 88$
	$6W = 78$
Substitute $W = 13$ into the second equation and then solve for L.	$W = 13$ $L = 2W + 5$
	$L = 2 \cdot 13 + 5$

	$L = 31$
Step 6. Check the answer in the problem.	Does a rectangle with length 31 and width 13 have perimeter 88? Yes.
Step 7. Answer the equation.	The length is 31 and the width is 13.

? Exercise 6.2.29

The perimeter of a rectangle is 40. The length is 4 more than the width. Find the length and width of the rectangle.

Answer

The length is 12 and the width is 8.

? Exercise 6.2.30

The perimeter of a rectangle is 58. The length is 5 more than three times the width. Find the length and width of the rectangle.

Answer

The length is 23 and the width is 6.

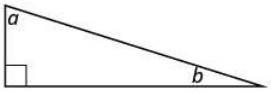
For Exercise 6.2.31 we need to remember that the sum of the measures of the angles of a triangle is 180 degrees and that a right triangle has one 90 degree angle.

? Exercise 6.2.31

The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle. Find the measures of both angles.

Answer

We will draw and label a figure.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the measures of the angles.
Step 3. Name what we are looking for.	Let a= the measure of the 1st angle b= the measure of the 2nd angle
Step 4. Translate into a system of equations.	The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle.
	$a = 3b + 10$
	The sum of the measures of the angles of a triangle is 180.
	$a + b + 90 = 180$
The system is:	$\begin{aligned} a &= 3b + 10 \\ a + b + 90 &= 180 \end{aligned}$

Step 5. Solve the system of equations. We will use substitution since the first equation is solved for a.	$a = 3b + 10$ $a + b + 90 = 180$
Substitute $3b + 10$ for a in the second equation.	$(3b + 10) + b + 90 = 180$
Solve for b.	$4b + 100 = 180$
	$4b = 80$
	$b = 20$ $a = 3b + 10$
Substitute $b = 20$ into the first equation and then solve for a.	$a = 3 \cdot 20 + 10$ $a = 70$
Step 6. Check the answer in the problem.	We will leave this to you!
Step 7. Answer the question.	The measures of the small angles are 20 and 70.

Exercise 6.2.32

The measure of one of the small angles of a right triangle is 2 more than 3 times the measure of the other small angle. Find the measure of both angles.

Answer

The measure of the angles are 22 degrees and 68 degrees.

Exercise 6.2.33

The measure of one of the small angles of a right triangle is 18 less than twice the measure of the other small angle. Find the measure of both angles.

Answer

The measure of the angles are 36 degrees and 54 degrees.

Exercise 6.2.34

Heather has been offered two options for her salary as a trainer at the gym. Option A would pay her $25,000 plus $15 for each training session. Option B would pay her $10,000 + $40 for each training session. How many training sessions would make the salary options equal?

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the number of training sessions that would make the pay equal.
Step 3. Name what we are looking for.	Let s= Heather's salary. n= the number of training sessions
Step 4. Translate into a system of equations.	Option A would pay her $25,000 plus $15 for each training session.

	$s = 25{,}000 + 15n$
	Option B would pay her \$10,000 + \$40 for each training session
	$s = 10{,}000 + 40n$
The system is:	$s = 25{,}000 + 15n$ $s = 10{,}000 + 40n$
Step 5. Solve the system of equations. We will use substitution.	$s = 25{,}000 + 15n$ $s = 10{,}000 + 40n$
Substitute $25{,}000 + 15n$ for s in the second equation.	$25{,}000 + 15n = 10{,}000 + 40n$
Solve for n.	$25{,}000 = 10{,}000 + 25n$
	$15{,}000 = 25n$
	$600 = n$
Step 6. Check the answer.	Are 600 training sessions a year reasonable? Are the two options equal when $n = 600$?
Step 7. Answer the question.	The salary options would be equal for 600 training sessions.

Exercise 6.2.35

Geraldine has been offered positions by two insurance companies. The first company pays a salary of \$12,000 plus a commission of \$100 for each policy sold. The second pays a salary of \$20,000 plus a commission of \$50 for each policy sold. How many policies would need to be sold to make the total pay the same?

Answer

There would need to be 160 policies sold to make the total pay the same.

Exercise 6.2.36

Kenneth currently sells suits for company A at a salary of \$22,000 plus a \$10 commission for each suit sold. Company B offers him a position with a salary of \$28,000 plus a \$4 commission for each suit sold. How many suits would Kenneth need to sell for the options to be equal?

Answer

Kenneth would need to sell 1,000 suits.

Note

Access these online resources for additional instruction and practice with solving systems of equations by substitution.

- Instructional Video-Solve Linear Systems by Substitution
- Instructional Video-Solve by Substitution

Key Concepts

- **Solve a system of equations by substitution**
  1. Solve one of the equations for either variable.
  2. Substitute the expression from Step 1 into the other equation.
  3. Solve the resulting equation.

4. Substitute the solution in Step 3 into one of the original equations to find the other variable.
5. Write the solution as an ordered pair.
6. Check that the ordered pair is a solution to both original equations.

This page titled 6.2: Solve Systems of Equations by Substitution is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **5.2: Solve Systems of Equations by Substitution** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

6.3: Solve Systems of Equations by Elimination

Learning Objectives

By the end of this section, you will be able to:

- Solve a system of equations by elimination
- Solve applications of systems of equations by elimination
- Choose the most convenient method to solve a system of linear equations

Note

Before you get started, take this readiness quiz.

1. Simplify −5(6−3a).
   If you missed this problem, review Example 1.10.43.
2. Solve the equation $\frac{1}{3}x+\frac{5}{8}=\frac{31}{24}$.
   If you missed this problem, review Example 2.5.1.

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the Elimination Method. When we solved a system by substitution, we started with two equations and two variables and reduced it to one equation with one variable. This is what we'll do with the elimination method, too, but we'll have a different way to get there.

Solve a System of Equations by Elimination

The **Elimination Method** is based on the Addition Property of Equality. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal.

For any expressions *a*, *b*, *c*, and *d*,

$$\begin{array}{ll}\text{if} & a=b\\ \text{and} & c=d\\ \text{then} & a+c=b+d\end{array} \tag{6.3.1}$$

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable.

Notice how that works when we add these two equations together:

$$\begin{array}{r}3x+y=5\\ 2x-y=0\\ \hline 5x\quad\quad=5\end{array} \tag{6.3.2}$$

The *y*'s add to zero and we have one equation with one variable.

Let's try another one:

$$\begin{cases}x+4y=2\\ 2x+5y=-2\end{cases} \tag{6.3.3}$$

This time we don't see a variable that can be immediately eliminated if we add the equations.

But if we multiply the first equation by −2, we will make the coefficients of *x* opposites. We must multiply every term on both sides of the equation by −2.

$$\begin{cases} -2(x+4y) = -2(2) \\ 2x+5y = -2 \end{cases}$$

$$\begin{cases} -2x-8y = -4 \\ 2x+5y = -2 \end{cases}$$

Now we see that the coefficients of the x terms are opposites, so x will be eliminated when we add these two equations.

Add the equations yourself—the result should be $-3y = -6$. And that looks easy to solve, doesn't it? Here is what it would look like.

$$\begin{array}{r} -2x-8y = -4 \\ 2x+5y = -2 \\ \hline -3y = -6 \end{array}$$

We'll do one more:

$$\begin{cases} 4x-3y = 10 \\ 3x+5y = -7 \end{cases} \tag{6.3.4}$$

It doesn't appear that we can get the coefficients of one variable to be opposites by multiplying one of the equations by a constant, unless we use fractions. So instead, we'll have to multiply both equations by a constant.

We can make the coefficients of x be opposites if we multiply the first equation by 3 and the second by -4, so we get $12x$ and $-12x$.

$$3(4x-3y) = 3(10)$$
$$-4(3x+5y) = -4(-7)$$

This gives us these two new equations:

$$\begin{cases} 12x-9y = 30 \\ -12x-20y = 28 \end{cases}$$

When we add these equations,

$$\begin{array}{r} 12x-9y = 30 \\ -12x-20y = 28 \\ \hline -29y = 58 \end{array} \tag{6.3.5}$$

the x's are eliminated and we just have $-29y = 58$.

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations.

Now we'll see how to use elimination to solve the same system of equations we solved by graphing and by substitution.

Example 6.3.1: How to Solve a System of Equations by Elimination

Solve the system by elimination. $\begin{cases} 2x+y = 7 \\ x-2y = 6 \end{cases}$

Solution

Step 1. Write both equations in standard form. If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax + By = C$. There are no fractions.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$
Step 2. Make the coefficients of one variable opposites. Decide which variable you will eliminate. Multiply one or both equations so that the coefficients of that variable are opposites.	We can eliminate the y's by multiplying the first equation by 2. Multiply both sides of $2x + y = 7$ by 2.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ $\begin{cases} 2(2x + y) = 2(7) \\ x - 2y = 6 \end{cases}$
Step 3. Add the equations resulting from Step 2 to eliminate one variable.	We add the x's, y's, and constants.	$\begin{array}{r} 4x + 2y = 14 \\ x - 2y = 6 \\ \hline 5x \quad\quad = 20 \end{array}$
Step 4. Solve for the remaining variable.	Solve for x.	$x = 4$
Step 5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.	Substitute $x = 4$ into the second equation, $x - 2y = 6$. Then solve for y.	$x - 2y = 6$ $4 - 2y = 6$ $-2y = 2$ $y = -1$
Step 6. Write the solution as an ordered pair.	Write it as (x, y).	$(4, -1)$
Step 7. Check that the ordered pair is a solution to **both** original equations.	Substitute $(4, -1)$ into $2x + y = 7$ and $x - 2y = 6$ Do they make both equations true? Yes!	$2x + y = 7$: $2(4) + (-1) \stackrel{?}{=} 7$: $7 = 7$ ✓ $x - 2y = 6$: $4 - 2(-1) \stackrel{?}{=} 6$: $6 = 6$ ✓ The solution is $(4, -1)$.

Try It 6.3.2

Solve the system by elimination. $\begin{cases} 3x + y = 5 \\ 2x - 3y = 7 \end{cases}$

Answer

$(2, -1)$

Try It 6.3.3

Solve the system by elimination. $\begin{cases} 4x + y = -5 \\ -2x - 2y = -2 \end{cases}$

Answer

$(-2, 3)$

The steps are listed below for easy reference.

HOW TO SOLVE A SYSTEM OF EQUATIONS BY ELIMINATION.

1. Write both equations in standard form. If any coefficients are fractions, clear them.
2. Make the coefficients of one variable opposites.
   - Decide which variable you will eliminate.
   - Multiply one or both equations so that the coefficients of that variable are opposites.
3. Add the equations resulting from Step 2 to eliminate one variable.
4. Solve for the remaining variable.
5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
6. Write the solution as an ordered pair.
7. Check that the ordered pair is a solution to **both** original equations.

First we'll do an example where we can eliminate one variable right away.

Example 6.3.4

Solve the system by elimination. $\begin{cases} x+y=10 \\ x-y=12 \end{cases}$

Solution

	$\begin{array}{r} x+y=10 \\ x-y=12 \end{array}$
Both equations are in standard form.	
The coefficients of y are already opposites.	
Add the two equations to eliminate y. The resulting equation has only 1 variable, x.	$\begin{array}{r} x+y=10 \\ x-y=12 \\ \hline 2x \quad =22 \end{array}$
Solve for x, the remaining variable. Substitute $x = 11$ into one of the original equations.	$x=11$ $x+y=10$
	$11+y=10$
Solve for the other variable, y.	$y=-1$
Write the solution as an ordered pair.	The ordered pair is (11, −1).
Check that the ordered pair is a solution to **both** original equations. $\begin{array}{rcl} x+y & = & 10 \\ 11+(-1) & \stackrel{?}{=} & 10 \\ 10 & = & 10\checkmark \end{array}$ $\quad$ $\begin{array}{rcl} x-y & = & 12 \\ 11-(-1) & \stackrel{?}{=} & 12 \\ 12 & = & 12\checkmark \end{array}$	
	The solution is (11, −1).

Try It 6.3.5

Solve the system by elimination. $\begin{cases} 2x+y=5 \\ x-y=4 \end{cases}$

Answer

(3,−1)

Try It 6.3.6

Solve the system by elimination. $\begin{cases} x+y=3 \\ -2x-y=-1 \end{cases}$

Answer

$(-2,5)$

In Example 6.3.7, we will be able to make the coefficients of one variable opposites by multiplying one equation by a constant.

Example 6.3.7

Solve the system by elimination. $\begin{cases} 3x-2y=-2 \\ 5x-6y=10 \end{cases}$

Solution

	$\begin{aligned} 3x-2y&=-2\\ 5x-6y&=10 \end{aligned}$
Both equations are in standard form.	
None of the coefficients are opposites.	
We can make the coefficients of y opposites by multiplying the first equation by -3.	$\begin{aligned} -3(3x-2y)&=-3(-2)\\ 5x-6y&=10 \end{aligned}$
Simplify.	$\begin{aligned} -9x+6y&=6\\ 5x-6y&=10 \end{aligned}$
Add the two equations to eliminate y.	$\begin{aligned} -9x+6y&=6\\ 5x-6y&=10\\ \hline -4x&=16 \end{aligned}$
Solve for the remaining variable, x. Substitute $x=-4$ into one of the original equations.	$\begin{aligned} x&=-4\\ 3x-2y&=-2 \end{aligned}$
	$3(-4)-2y=-2$
Solve for y.	$\begin{aligned} -12-2y&=-2\\ -2y&=10\\ y&=-5 \end{aligned}$
Write the solution as an ordered pair.	The ordered pair is $(-4,-5)$.
Check that the ordered pair is a solution to both original equations.	$\begin{aligned} 3x-2y&=-2 \\ 3(-4)-2(-5)&\stackrel{?}{=}-2 \\ -12+10&\stackrel{?}{=}-2 \\ -2&=-2\checkmark \end{aligned}$ $\begin{aligned} 5x-6y&=10 \\ 5(-4)-6(-5)&\stackrel{?}{=}10 \\ -20+30&\stackrel{?}{=}10 \\ 10&=10\checkmark \end{aligned}$
	The solution is $(-4,-5)$.

Try It 6.3.8

Solve the system by elimination. $\begin{cases} 4x-3y=1 \\ 5x-9y=-4 \end{cases}$

Answer

(1,1)

Try It 6.3.9

Solve the system by elimination. $\begin{cases} 3x+2y=2 \\ 6x+5y=8 \end{cases}$

Answer

(−2,4)

Now we'll do an example where we need to multiply both equations by constants in order to make the coefficients of one variable opposites.

Example 6.3.10

Solve the system by elimination. $\begin{cases} 4x-3y=9 \\ 7x+2y=-6 \end{cases}$

Solution

In this example, we cannot multiply just one equation by any constant to get opposite coefficients. So we will strategically multiply both equations by a constant to get the opposites.

	$\begin{aligned} 4x-3y&=9 \\ 7x+2y&=-6 \end{aligned}$
Both equations are in standard form. To get opposite coefficients of y, we will multiply the first equation by 2 and the second equation by 3.	$\begin{aligned} 2(4x-3y)&=2(9) \\ 3(7x+2y)&=3(-6) \end{aligned}$
Simplify.	$\begin{aligned} 8x-6y&=18 \\ 21x+6y&=-18 \end{aligned}$
Add the two equations to eliminate y.	$\begin{aligned} 8x-6y&=18 \\ 21x+6y&=-18 \\ \hline 39x&=0 \end{aligned}$
Solve for x. Substitute $x = 0$ into one of the original equations.	$x=0$ $7x+2y=-6$
	$7\cdot 0+2y=-6$
Solve for y.	$2y=-6$
	$y=-3$
Write the solution as an ordered pair.	The ordered pair is (0, −3).
Check that the ordered pair is a solution to **both** original equations. $\begin{aligned} 4x-3y &= 9 \\ 4(0)-3(-3) &\stackrel{?}{=} 9 \\ 9 &= 9\checkmark \end{aligned}$ $\quad$ $\begin{aligned} 7x+2y &= -6 \\ 7(0)+2(-3) &\stackrel{?}{=} -6 \\ -6 &= -6\checkmark \end{aligned}$	
	The solution is (0, −3).

Try It 6.3.11

Solve the system by elimination. $\begin{cases} 3x-4y=-9 \\ 5x+3y=14 \end{cases}$

Answer

(1,3)

Try It 6.3.12

Solve the system by elimination. $\begin{cases} 7x+8y=4 \\ 3x-5y=27 \end{cases}$

Answer

(4,−3)

When the system of equations contains fractions, we will first clear the fractions by multiplying each equation by its LCD.

Example 6.3.13

Solve the system by elimination. $\begin{cases} x+\frac{1}{2}y=6 \\ \frac{3}{2}x+\frac{2}{3}y=\frac{17}{2} \end{cases}$

Solution

In this example, both equations have fractions. Our first step will be to multiply each equation by its LCD to clear the fractions.

	$x+\frac{1}{2}y=6$ $\frac{3}{2}x+\frac{2}{3}y=\frac{17}{2}$
To clear the fractions, multiply each equation by its LCD.	$2\left(x+\frac{1}{2}y\right)=2(6)$ $6\left(\frac{3}{2}x+\frac{2}{3}y\right)=6\left(\frac{17}{2}\right)$
Simplify.	$2x+y=12$ $9x+4y=51$
Now we are ready to eliminate one of the variables. Notice that both equations are in standard form.	
We can eliminate y multiplying the top equation by −4.	$-4(2x+y)=-4(12)$ $9x+4y=51$
Simplify and add. Substitute $x = 3$ into one of the original equations.	$-8x-4y=-48$ $9x+4y=51$ $x=3$ $x+\frac{1}{2}y=6$
Solve for y.	$3+\frac{1}{2}y=6$
	$\frac{1}{2}y=3$
	$y=6$
Write the solution as an ordered pair.	The ordered pair is (3, 6).

Check that the ordered pair is a solution to **both** original equations.

$$\begin{aligned} x+\tfrac{1}{2}y &= 6 \\ 3+\tfrac{1}{2}(6) &\stackrel{?}{=} 6 \\ 3+3 &\stackrel{?}{=} 6 \\ 6 &= 6\checkmark \end{aligned} \qquad \begin{aligned} \tfrac{3}{2}x+\tfrac{2}{3}y &= \tfrac{17}{2} \\ \tfrac{3}{2}(3)+\tfrac{2}{3}(6) &\stackrel{?}{=} \tfrac{17}{2} \\ \tfrac{9}{2}+4 &\stackrel{?}{=} \tfrac{17}{2} \\ \tfrac{9}{2}+\tfrac{8}{2} &\stackrel{?}{=} \tfrac{17}{2} \\ \tfrac{17}{2} &= \tfrac{17}{2}\checkmark \end{aligned}$$

The solution is (3, 6).

Try It 6.3.14

Solve the system by elimination. $\begin{cases} \frac{1}{3}x-\frac{1}{2}y=1 \\ \frac{3}{4}x-y=\frac{5}{2} \end{cases}$

Answer

(6,2)

Try It 6.3.15

Solve the system by elimination. $\begin{cases} x+\frac{3}{5}y=-\frac{1}{5} \\ -\frac{1}{2}x-\frac{2}{3}y=\frac{5}{6} \end{cases}$

Answer

(1,−2)

In the Solving Systems of Equations by Graphing we saw that not all systems of linear equations have a single ordered pair as a solution. When the two equations were really the same line, there were infinitely many solutions. We called that a consistent system. When the two equations described parallel lines, there was no solution. We called that an inconsistent system.

Example 6.3.16

Solve the system by elimination. $\begin{cases} 3x+4y=12 \\ y=3-\frac{3}{4}x \end{cases}$

Solution

$$\begin{cases} 3x+4y=12 \\ y=3-\frac{3}{4}x \end{cases}$$

Write the second equation in standard form.

$$\begin{cases} 3x+4y=12 \\ \frac{3}{4}x+y=3 \end{cases}$$

Clear the fractions by multiplying thesecond equation by 4.

$$\begin{cases} 3x+4y=12 \\ 4\left(\frac{3}{4}x+y\right)=4(3) \end{cases}$$

Simplify.

$$\begin{cases} 3x+4y=12 \\ 3x+4y=12 \end{cases}$$

To eliminate a variable, we multiply thesecond equation by -1.

$$\begin{cases} 3x+4y=12 \\ \underline{-3x-4y=-12} \end{cases}$$

$$0=0$$

Simplify and add.

This is a true statement. The equations are consistent but dependent. Their graphs would be the same line. The system has infinitely many solutions.

After we cleared the fractions in the second equation, did you notice that the two equations were the same? That means we have coincident lines.

Try It 6.3.17

Solve the system by elimination. $\begin{cases} 5x-3y=15 \\ y=-5+\frac{5}{3}x \end{cases}$

Answer

infinitely many solutions

Try It 6.3.18

Solve the system by elimination. $\begin{cases} x+2y=6 \\ y=-\frac{1}{2}x+3 \end{cases}$

Answer

infinitely many solutions

Example 6.3.19

Solve the system by elimination. $\begin{cases} -6x+15y=10 \\ 2x-5y=-5 \end{cases}$

Solution

The equations are in standard form. $\begin{cases} -6x+15y=10 \\ 2x-5y=-5 \end{cases}$

Multiply the second equation by 3 to eliminate a variable. $\begin{cases} -6x+15y=10 \\ 3(2x-5y)=3(-5) \end{cases}$

Simplify and add. $\begin{cases} -6x+15y=10 \\ 6x-15y=-15 \\ \hline 0 \neq 5 \end{cases}$

This statement is false. The equations are inconsistent and so their graphs would be parallel lines.

The system does not have a solution.

Try It 6.3.20

Solve the system by elimination. $\begin{cases} -3x+2y=8 \\ 9x-6y=13 \end{cases}$

Answer

no solution

Try It 6.3.21

Solve the system by elimination. $\begin{cases} 7x-3y=-2 \\ -14x+6y=8 \end{cases}$

Answer

no solution

Solve Applications of Systems of Equations by Elimination

Some applications problems translate directly into equations in standard form, so we will use the elimination method to solve them. As before, we use our Problem Solving Strategy to help us stay focused and organized.

Example 6.3.22

The sum of two numbers is 39. Their difference is 9. Find the numbers.

Solution

Step 1. Read the problem	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name what we are looking for.	Let n = the first number. m = the second number
Step 4. Translate into a system of equations.	
	The sum of two numbers is 39. $n+m=39$ Their difference is 9. $n-m=9$
The system is:	$\begin{cases} n+m=39 \\ n-m=9 \end{cases}$
Step 5. Solve the system of equations. To solve the system of equations, use elimination. The equations are in standard form and the coefficients of m are opposites. Add.	$\begin{cases} n+m=39 \\ \underline{n-m=9} \\ 2n \quad\quad =48 \end{cases}$
Solve for n.	$n=24$
Substitute n=24 into one of the original equations and solve form.	$n+m=39$ $24+m=39$ $m=15$
Step 6. Check the answer.	Since 24+15=39 and 24−15=9, the answers check.
Step 7. Answer the question.	The numbers are 24 and 15.

? Try It 6.3.23

The sum of two numbers is 42. Their difference is 8. Find the numbers.

Answer

The numbers are 25 and 17.

? Try It 6.3.24

The sum of two numbers is −15. Their difference is −35. Find the numbers.

Answer

The numbers are −25 and 10.

✓ Example 6.3.25

Joe stops at a burger restaurant every day on his way to work. Monday he had one order of medium fries and two small sodas, which had a total of 620 calories. Tuesday he had two orders of medium fries and one small soda, for a total of 820 calories. How many calories are there in one order of medium fries? How many calories in one small soda?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of calories in one order of medium fries and in one small soda.
Step 3. Name what we are looking for.	Let f = the number of calories in 1 order of medium fries. s = the number of calories in 1 small soda.
Step 4. Translate into a system of equations:	one medium fries and two small sodas had a total of 620 calories
	$f+2s=620$
	two medium fries and one small soda had a total of 820 calories.
	$2f+s=820$
Our system is:	$\begin{cases} f+2s=620 \\ 2f+s=820 \end{cases}$
Step 5. Solve the system of equations. To solve the system of equations, use elimination. The equations are in standard form. To get opposite coefficients of f, multiply the top equation by -2.	$\begin{cases} -2(f+2s)=-2(620) \\ 2f+s=820 \end{cases}$
Simplify and add.	$\begin{array}{r} -2f-4s=-1240 \\ 2f+s=820 \\ \hline -3s=-420 \end{array}$
Solve for s.	$s=140$
Substitute $s=140$ into one of the original equations and then solve for f.	$f+2s=620$
	$f+2\cdot 140=620$
	$f+280=620$
	$f=340$
Step 6. Check the answer.	Verify that these numbers make sense in the problem and that they are solutions to both equations. We leave this to you!
Step 7. Answer the question.	The small soda has 140 calories and the fries have 340 calories.

? Try It 6.3.26

Malik stops at the grocery store to buy a bag of diapers and 2 cans of formula. He spends a total of $37. The next week he stops and buys 2 bags of diapers and 5 cans of formula for a total of $87. How much does a bag of diapers cost? How much is one can of formula?

Answer

The bag of diapers costs $11 and the can of formula costs $13.

Try It 6.3.27

To get her daily intake of fruit for the day, Sasha eats a banana and 8 strawberries on Wednesday for a calorie count of 145. On the following Wednesday, she eats two bananas and 5 strawberries for a total of 235 calories for the fruit. How many calories are there in a banana? How many calories are in a strawberry?

Answer

There are 105 calories in a banana and 5 calories in a strawberry.

Choose the Most Convenient Method to Solve a System of Linear Equations

When you will have to solve a system of linear equations in a later math class, you will usually not be told which method to use. You will need to make that decision yourself. So you'll want to choose the method that is easiest to do and minimizes your chance of making mistakes.

Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved for one variable.	Use when the equations are in standard form.

Example 6.3.28

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

1. $\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$
2. $\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$

Solution

1. $\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$

Since both equations are in standard form, using elimination will be most convenient.

2. $\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$

Since one equation is already solved for y, using substitution will be most convenient.

Try It 6.3.29

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

1. $\begin{cases} 4x-5y=-32 \\ 3x+2y=-1 \end{cases}$
2. $\begin{cases} x=2y-1 \\ 3x-5y=-7 \end{cases}$

Answer

1. Since both equations are in standard form, using elimination will be most convenient.
2. Since one equation is already solved for xx, using substitution will be most convenient.

Try It 6.3.30

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

1. $\begin{cases} y = 2x - 1 \\ 3x - 4y = -6 \end{cases}$
2. $\begin{cases} 6x - 2y = 12 \\ 3x + 7y = -13 \end{cases}$

Answer

1. Since one equation is already solved for y, using substitution will be most convenient;
2. Since both equations are in standard form, using elimination will be most convenient.

Note

Access these online resources for additional instruction and practice with solving systems of linear equations by elimination.

- Instructional Video-Solving Systems of Equations by Elimination
- Instructional Video-Solving by Elimination
- Instructional Video-Solving Systems by Elimination

Key Concepts

- **To Solve a System of Equations by Elimination**
  1. Write both equations in standard form. If any coefficients are fractions, clear them.
  2. Make the coefficients of one variable opposites.
     - Decide which variable you will eliminate.
     - Multiply one or both equations so that the coefficients of that variable are opposites.
  3. Add the equations resulting from Step 2 to eliminate one variable.
  4. Solve for the remaining variable.
  5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
  6. Write the solution as an ordered pair.
  7. Check that the ordered pair is a solution to **both** original equations.

This page titled 6.3: Solve Systems of Equations by Elimination is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **5.3: Solve Systems of Equations by Elimination** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

6.4: Solve Mixture Applications with Systems of Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve mixture applications
- Solve interest applications

Note

Before you get started, take this readiness quiz.

1. Multiply 4.025(1,562).
   If you missed this problem, review Example 1.8.22.
2. Write 8.2% as a decimal.
   If you missed this problem, review Example 1.8.46.
3. Earl's dinner bill came to $32.50 and he wanted to leave an 18% tip. How much should the tip be?
   If you missed this problem, review Example 3.2.10.

Solve Mixture Applications

When we solved mixture applications with coins and tickets earlier, we started by creating a table so we could organize the information. For a coin example with nickels and dimes, the table looked like this:

Type	Number	• Value($)	= Total Value($)
nickels		0.05	
dimes		0.10	

Using one variable meant that we had to relate the number of nickels and the number of dimes. We had to decide if we were going to let n be the number of nickels and then write the number of dimes in terms of n, or if we would let d be the number of dimes and write the number of nickels in terms of d.

Now that we know how to solve systems of equations with two variables, we'll just let n be the number of nickels and d be the number of dimes. We'll write one equation based on the total value column, like we did before, and the other equation will come from the number column.

For the first example, we'll do a ticket problem where the ticket prices are in whole dollars, so we won't need to use decimals just yet.

Example 6.4.1

Translate to a system of equations and solve:

The box office at a movie theater sold 147 tickets for the evening show, and receipts totaled $1,302. How many $11 adult and how many $8 child tickets were sold?

Solution

Step 1. Read the problem.	We will create a table to organize the information.
Step 2. Identify what we are looking for.	We are looking for the number of adult tickets and the number of child tickets sold.
Step 3. Name what we are looking for.	Let a= the number of adult tickets. c= the number of child tickets
A table will help us organize the data. We have two types of tickets: adult and child.	Write a and c for the number of tickets.

Write the total number of tickets sold at the bottom of the Number column.	Altogether 147 were sold.
Write the value of each type of ticket in the Value column.	The value of each adult ticket is $11. The value of each child tickets is $8.
The number times the value gives the total value, so the total value of adult tickets is $a \cdot 11 = 11a$, and the total value of child tickets is $c \cdot 8 = 8c$.	<table><tr><th>Type</th><th>Number</th><th>•</th><th>Value ($)</th><th>=</th><th>Total Value ($)</th></tr><tr><td>adult</td><td>a</td><td></td><td>11</td><td></td><td>$11a$</td></tr><tr><td>child</td><td>c</td><td></td><td>8</td><td></td><td>$8c$</td></tr><tr><td></td><td>147</td><td></td><td></td><td></td><td>1302</td></tr></table>
Altogether the total value of the tickets was $1,302.	Fill in the Total Value column.
Step 4. Translate into a system of equations.	
The Number column and the Total Value column give us the system of equations. We will use the elimination method to solve this system.	$a + c = 147$ $11a + 8c = 1302$
Multiply the first equation by −8.	$-8(a + c) = -8(147)$ $11a + 8c = 1302$
Simplify and add, then solve for a.	$-8a - 8c = -1176$ $11a + 8c = 1302$ $3a = 126$
	$a = 42$
Substitute $a = 42$ into the first equation, then solve for c.	$42 + c = 147$
	$c = 105$
Step 5. Check the answer in the problem. 42 adult tickets at $11 per ticket makes $462 105 child tickets at $8 per ticket makes $840. The total receipts are $1,302.✓	
Step 6. Answer the question.	The movie theater sold 42 adult tickets and 105 child tickets.

Try It 6.4.2

Translate to a system of equations and solve:

The ticket office at the zoo sold 553 tickets one day. The receipts totaled $3,936. How many $9 adult tickets and how many $6 child tickets were sold?

Answer

There were 206 adult tickets sold and 347 children tickets sold.

Try It 6.4.3

Translate to a system of equations and solve:

A science center sold 1,363 tickets on a busy weekend. The receipts totaled $12,146. How many $12 adult tickets and how many $7 child tickets were sold?

Answer

There were 521 adult tickets sold and 842 children tickets sold.

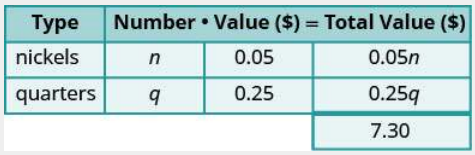
In Example 6.4.4 we'll solve a coin problem. Now that we know how to work with systems of two variables, naming the variables in the 'number' column will be easy.

Example 6.4.4

Translate to a system of equations and solve:

Priam has a collection of nickels and quarters, with a total value of $7.30. The number of nickels is six less than three times the number of quarters. How many nickels and how many quarters does he have?

Solution

Step 1. Read the problem.	We will create a table to organize the information.
Step 2. Identify what we are looking for.	We are looking for the number of nickels and the number of quarters.
Step 3. Name what we are looking for.	Let n= the number of nickels. q= the number of quarters
A table will help us organize the data. We have two types of coins, nickels and quarters.	Write n and q for the number of each type of coin.
Fill in the Value column with the value of each type of coin.	The value of each nickel is $0.05. The value of each quarter is $0.25.
The number times the value gives the total value, so, the total value of the nickels is n (0.05) = 0.05n and the total value of quarters is q(0.25) = 0.25q. Altogether the total value of the coins is $7.30.	
Step 4. Translate into a system of equations.	
The Total value column gives one equation.	$0.05n + 0.25q = 7.30$
We also know the number of nickels is six less than three times the number of quarters. Translate to get the second equation.	$n = 3q - 6$
Now we have the system to solve.	$0.05n + 0.25q = 7.30$ $n = 3q - 6$
Step 5. Solve the system of equations We will use the substitution method. Substitute $n = 3q - 6$ into the first equation. Simplify and solve for q.	$0.05n + 0.25q = 7.30$
	$0.05(3q - 6) + 0.25q = 7.3$

	$0.15q - 0.3 + 0.25q = 7.3$
	$0.4q - 0.3 = 7.3$
	$0.4q = 7.6$
	$q = 19$
To find the number of nickels, substitute $q = 19$ into the second equation.	$n = 3q - 6$
	$n = 3 \cdot 19 - 6$
	$n = 51$
Step 6. Check the answer in the problem. $19 \text{ quarters at } \$0.25 = \$4.75$ $51 \text{ ickels at } \$0.05 = \$2.55$ $\text{Total } = \$7.30\checkmark$ $3 \cdot 19 - 16 = 51\checkmark$	
Step 7. Answer the question.	Priam has 19 quarters and 51 nickels.

Try It 6.4.5

Translate to a system of equations and solve:

Matilda has a handful of quarters and dimes, with a total value of $8.55. The number of quarters is 3 more than twice the number of dimes. How many dimes and how many quarters does she have?

Answer

Matilda has 13 dimes and 29 quarters.

Try It 6.4.6

Translate to a system of equations and solve:

Juan has a pocketful of nickels and dimes. The total value of the coins is $8.10. The number of dimes is 9 less than twice the number of nickels. How many nickels and how many dimes does Juan have?

Answer

Juan has 36 nickels and 63 dimes.

Some mixture applications involve combining foods or drinks. Example situations might include combining raisins and nuts to make a trail mix or using two types of coffee beans to make a blend.

Example 6.4.7

Translate to a system of equations and solve:

Carson wants to make 20 pounds of trail mix using nuts and chocolate chips. His budget requires that the trail mix costs him $7.60 per pound. Nuts cost $9.00 per pound and chocolate chips cost $2.00 per pound. How many pounds of nuts and how many pounds of chocolate chips should he use?

Solution

Step 1. Read the problem.	We will create a table to organize the information.

Step 2. Identify what we are looking for.	We are looking for the number of pounds of nuts and the number of pounds of chocolate chips.
Step 3. Name what we are looking for.	Let n= the number of pound of nuts. c= the number of pounds of chips
Carson will mix nuts and chocolate chips to get trail mix. Write in n and c for the number of pounds of nuts and chocolate chips. There will be 20 pounds of trail mix. Put the price per pound of each item in the Value column. Fill in the last column using	

Type	Number of pounds	• Value ($)	= Total Value ($)
nuts	n	9.00	$9n$
chocolate chips	c	2.00	$2c$
trail mix	20	7.60	$7.60(20) = 152$

Number · Value = Total Value	
Step 4. Translate into a system of equations. We get the equations from the Number and Total Value columns.	$\begin{aligned} n + c &= 20 \\ 9n + 2c &= 152 \end{aligned}$
Step 5. Solve the system of equations We will use elimination to solve the system.	
Multiply the first equation by -2 to eliminate c.	$\begin{aligned} -2(n + c) &= -2(20) \\ 9n + 2c &= 152 \end{aligned}$
Simplify and add. Solve for n.	$\begin{aligned} -2n - 2c &= -40 \\ 9n + 2c &= 152 \\ \hline 7n \quad &= 112 \end{aligned}$
	$n = 16$
To find the number of pounds of chocolate chips, substitute n = 16 into the first equation, then solve for c.	$\begin{aligned} n + c &= 20 \\ 16 + c &= 20 \end{aligned}$
	c=4
Step 6. Check the answer in the problem. $16 + 4 = 20$✓ $9 \cdot 16 + 2 \cdot 4 = 152$✓	
Step 7. Answer the question.	Carson should mix 16 pounds of nuts with 4 pounds of chocolate chips to create the trail mix.

? Try It 6.4.8

Translate to a system of equations and solve:

Greta wants to make 5 pounds of a nut mix using peanuts and cashews. Her budget requires the mixture to cost her $6 per pound. Peanuts are $4 per pound and cashews are $9 per pound. How many pounds of peanuts and how many pounds of cashews should she use?

Answer

Greta should use 3 pounds of peanuts and 2 pounds of cashews.

Try It 6.4.9

Translate to a system of equations and solve:

Sammy has most of the ingredients he needs to make a large batch of chili. The only items he lacks are beans and ground beef. He needs a total of 20 pounds combined of beans and ground beef and has a budget of $3 per pound. The price of beans is $1 per pound and the price of ground beef is $5 per pound. How many pounds of beans and how many pounds of ground beef should he purchase?

Answer

Sammy should purchase 10 pounds of beans and 10 pounds of ground beef.

Another application of mixture problems relates to concentrated cleaning supplies, other chemicals, and mixed drinks. The concentration is given as a percent. For example, a 20% concentrated household cleanser means that 20% of the total amount is cleanser, and the rest is water. To make 35 ounces of a 20% concentration, you mix 7 ounces (20% of 35) of the cleanser with 28 ounces of water.

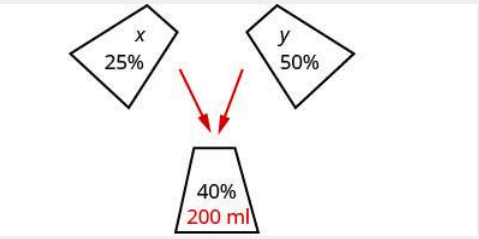
For these kinds of mixture problems, we'll use percent instead of value for one of the columns in our table.

Example 6.4.10

Translate to a system of equations and solve:

Sasheena is a lab assistant at her community college. She needs to make 200 milliliters of a 40% solution of sulfuric acid for a lab experiment. The lab has only 25% and 50% solutions in the storeroom. How much should she mix of the 25% and the 50% solutions to make the 40% solution?

Solution

Step 1. Read the problem.	A figure may help us visualize the situation, then we will create a table to organize the information.
Sasheena must mix some of the 25% solution and some of the 50% solution together to get 200 ml of the 40% solution.	
Step 2. Identify what we are looking for.	We are looking for how much of each solution she needs.
Step 3. Name what we are looking for.	Let x= number of ml of 25% solution. y= number of ml of 50% solution
A table will help us organize the data. She will mix *x* ml of 25% with *y* ml of 50% to get 200 ml of 40% solution. We write the percents as decimals in the chart. We multiply the number of units times the concentration to get the total amount of sulfuric acid in each solution.	

Type	Number of units	• Concentration %	= Amount
25%	*x*	0.25	0.25*x*
50%	*y*	0.50	0.50*y*
40%	200	0.40	0.40(200)

Step 4. Translate into a system of equations. We get the equations from the Number column and the Amount column.	
Now we have the system.	$\begin{cases} x+y=200 \\ 0.25x+0.50y=0.40(200) \end{cases}$
Step 5. Solve the system of equations. We will solve the system by elimination. Multiply the first equation by −0.5 to eliminate y.	$\begin{cases} -0.5(x+y)=-0.5(200) \\ 0.25x+0.50y=80 \end{cases}$
Simplify and add to solve for x.	$\begin{cases} -0.5x-0.5y=-100 \\ 0.25x+0.5y=80 \end{cases}$ $-0.25x = -20$ $x = 80$
To solve for y, substitute $x = 80$ into the first equation.	$x+y=200$
	$80+y=200$
	$y=120$
Step 6. Check the answer in the problem. $80+120 = 120$✓ $0.25(80)+0.50(120) = 80$✓ Yes!	
Step 7. Answer the question.	Sasheena should mix 80 ml of the 25% solution with 120 ml of the 50% solution to get the 200 ml of the 40% solution.

Try It 6.4.11

Translate to a system of equations and solve:

LeBron needs 150 milliliters of a 30% solution of sulfuric acid for a lab experiment but only has access to a 25% and a 50% solution. How much of the 25% and how much of the 50% solution should he mix to make the 30% solution?

Answer

LeBron needs 120 ml of the 25% solution and 30 ml of the 50% solution.

Try It 6.4.12

Translate to a system of equations and solve:

Anatole needs to make 250 milliliters of a 25% solution of hydrochloric acid for a lab experiment. The lab only has a 10% solution and a 40% solution in the storeroom. How much of the 10% and how much of the 40% solutions should he mix to make the 25% solution?

Answer

Anatole should mix 125 ml of the 10% solution and 125 ml of the 40% solution.

Solve Interest Applications

The formula to model interest applications is $I = Prt$. Interest, I, is the product of the principal, P, the rate, r, and the time, t. In our work here, we will calculate the interest earned in one year, so t will be 1.

We modify the column titles in the mixture table to show the formula for interest, as you'll see in Example 6.4.13.

Example 6.4.13

Translate to a system of equations and solve:

Adnan has $40,000 to invest and hopes to earn 7.1% interest per year. He will put some of the money into a stock fund that earns 8% per year and the rest into bonds that earns 3% per year. How much money should he put into each fund?

Solution

Step 1. Read the problem.	A chart will help us organize the information.
Step 2. Identify what we are looking for.	We are looking for the amount to invest in each fund.
Step 3. Name what we are looking for.	Let s= the amount invested in stocks. b= the amount invested in bonds.
Write the interest rate as a decimal for each fund. Multiply: Principal · Rate · Time to get the Interest.	<table><tr><th>Account</th><th>Principal</th><th>Rate</th><th>Time</th><th>Interest</th></tr><tr><td>stock fund</td><td>s</td><td>0.08</td><td>1</td><td>0.08s</td></tr><tr><td>bonds</td><td>b</td><td>0.03</td><td>1</td><td>0.03b</td></tr><tr><td>Total</td><td>40,000</td><td>0.071</td><td></td><td>0.071(40,000)</td></tr></table>
Step 4. Translate into a system of equations. We get our system of equations from the Principal column and the Interest column.	$$\begin{cases} s + b = 40{,}000 \\ 0.08s + 0.03b = 0.071(40{,}000) \end{cases}$$
Step 5. Solve the system of equations Solve by elimination. Multiply the top equation by −0.03.	$$\begin{cases} -0.03(s + b) = -0.03(40{,}000) \\ 0.08s + 0.03b = 2{,}840 \end{cases}$$
Simplify and add to solve for s.	$$\begin{cases} -0.03s - 0.03b = -1{,}200 \\ 0.08s + 0.03b = 2{,}840 \end{cases}$$ $$0.05s = 1{,}640$$
	$$s = 32{,}800$$
To find b, substitute s = 32,800 into the first equation.	$$s + b = 40{,}000$$ $$32{,}800 + b = 40{,}000$$
	$$b = 7{,}200$$
Step 6. Check the answer in the problem.	We leave the check to you.
Step 7. Answer the question.	Adnan should invest $32,800 in stock and $7,200 in bonds.

Did you notice that the Principal column represents the total amount of money invested while the Interest column represents only the interest earned? Likewise, the first equation in our system, $s + b = 40{,}000$, represents the total amount of money invested and the second equation, $0.08s + 0.03b = 0.071(40{,}000)$, represents the interest earned.

Try It 6.4.14

Translate to a system of equations and solve:

Leon had $50,000 to invest and hopes to earn 6.2 % interest per year. He will put some of the money into a stock fund that earns 7% per year and the rest in to a savings account that earns 2% per year. How much money should he put into each fund?

Answer

Leon should put $42,000 in the stock fund and $8000 in the savings account.

Try It 6.4.15

Translate to a system of equations and solve:

Julius invested $7,000 into two stock investments. One stock paid 11% interest and the other stock paid 13% interest. He earned 12.5% interest on the total investment. How much money did he put in each stock?

Answer

Julius invested $1,750 at 11% and $5,250 at 13%.

Example 6.4.16

Translate to a system of equations and solve:

Rosie owes $21,540 on her two student loans. The interest rate on her bank loan is 10.5% and the interest rate on the federal loan is 5.9%. The total amount of interest she paid last year was $1,669.68. What was the principal for each loan?

Solution

Step 1. Read the problem.	A chart will help us organize the information.
Step 2. Identify what we are looking for.	We are looking for the principal of each loan.
Step 3. Name what we are looking for.	Let b= the principal for the bank loan. f= the principal on the federal loan
The total loans are $21,540.	
Record the interest rates as decimals in the chart.	
Multiply using the formula $I = Prt$ to get the Interest.	
Step 4. Translate into a system of equations. The system of equations comes from the Principal column and the Interest column.	$b + f = 21{,}540$ $0.105b + 0.059f = 1669.68$
Step 5. Solve the system of equations We will use substitution to solve. Solve the first equation for b.	$b + f = 21{,}540$ $b = -f + 21{,}540$
Substitute $b = -f + 21{,}540$ into the second equation.	$0.105b + 0.059f = 1669.68$ $0.105(-f + 21{,}540) + 0.059f = 1669.68$

Account	Principal	• Rate	• Time	= Interest
bank	b	0.105	1	$0.105b$
federal	f	0.059	1	$0.059f$
Total	21,540			1669.68

Simplify and solve for f.	$-0.105f + 2261.70 + 0.059f = 1669.68$
	$-0.046f + 2261.70 = 1669.68$
	$-0.046f = -592.02$
	$f = 12{,}870$
To find b, substitute $f = 12{,}870$ into the first equation.	$b + f = 21{,}540$ $12{,}870 + f = 21{,}540$
	$f = 8{,}670$
Step 6. Check the answer in the problem.	We leave the check to you.
Step 7. Answer the question.	The principal of the bank loan is \$12,870 and the principal for the federal loan is \$8,670.

Try It 6.4.17

Translate to a system of equations and solve:

Laura owes \$18,000 on her student loans. The interest rate on the bank loan is 2.5% and the interest rate on the federal loan is 6.9 %. The total amount of interest she paid last year was \$1,066. What was the principal for each loan?

Answer

The principal amount for the bank loan was \$4,000. The principal amount for the federal loan was \$14,000.

Try It 6.4.18

Translate to a system of equations and solve:

Jill's Sandwich Shoppe owes \$65,200 on two business loans, one at 4.5% interest and the other at 7.2% interest. The total amount of interest owed last year was \$3,582. What was the principal for each loan?

Answer

The principal amount for was \$41,200 at 4.5%. The principal amount was, \$24,000 at 7.2%.

Note

Access these online resources for additional instruction and practice with solving application problems with systems of linear equations.

- Cost and Mixture Word Problems
- Mixture Problems

Key Concepts

- **Table for coin and mixture applications**

Type	Number	• Value(\$)	= Total Value(\$)
Total			

- **Table for concentration applications**

Type	Number of units	•	Concentration %	=	Amount
Total					

- **Table for interest applications**

Account	Principal	•	Rate	•	Time	=	Interest
					1		
					1		
Total							

This page titled 6.4: Solve Mixture Applications with Systems of Equations is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **5.5: Solve Mixture Applications with Systems of Equations** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/elementary-algebra-2e.

6.5: Solve Systems of Equations with Three Variables

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered triple is a solution of a system of three linear equations with three variables
- Solve a system of linear equations with three variables
- Solve applications using systems of linear equations with three variables

Before you get started, take this readiness quiz.

1. Evaluate $5x-2y+3z$ when $x=-2, y=-4$, and $z=3$.
   If you missed this problem, review [link].
2. Classify the equations as a conditional equation, an identity, or a contradiction and then state the solution.
   $\begin{cases} -2x+y=-11 \\ x+3y=9 \end{cases}$
   If you missed this problem, review [link].
3. Classify the equations as a conditional equation, an identity, or a contradiction and then state the solution. $\begin{cases} 7x+8y=4 \\ 3x-5y=27 \end{cases}$
   If you missed this problem, review [link].

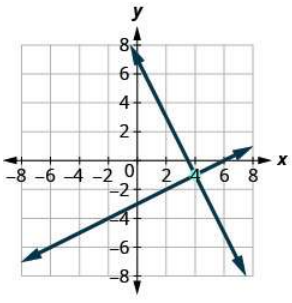
Determine Whether an Ordered Triple is a Solution of a System of Three Linear Equations with Three Variables

In this section, we will extend our work of solving a system of linear equations. So far we have worked with **systems of equations** with two equations and two variables. Now we will work with systems of three equations with three variables. But first let's review what we already know about solving equations and systems involving up to two variables.

We learned earlier that the graph of a **linear equation**, $ax+by=c$, is a line. Each point on the line, an ordered pair (x,y), is a solution to the equation. For a system of two equations with two variables, we graph two lines. Then we can see that all the points that are solutions to each equation form a line. And, by finding what the lines have in common, we'll find the solution to the system.

Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions

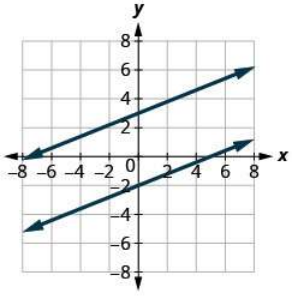
We know when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown.



One solution

The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.

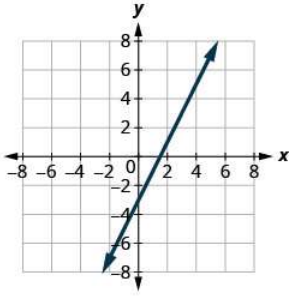
Consistent
Independent



No solution

The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.

Inconsistent



Infinitely many solutions

Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

Consistent
Dependent

Similarly, for a linear equation with three variables ax+by+cz=d,ax+by+cz=d, every solution to the equation is an ordered triple, (x,y,z)(x,y,z)that makes the equation true.

LINEAR EQUATION IN THREE VARIABLES

A linear equation with three variables, where a, b, c, and d are real numbers and a, b, and c are not all 0, is of the form

$$ax + by + cz = d$$

Every solution to the equation is an ordered triple, (x, y, z) that makes the equation true.

All the points that are solutions to one equation form a plane in three-dimensional space. And, by finding what the planes have in common, we'll find the solution to the system.

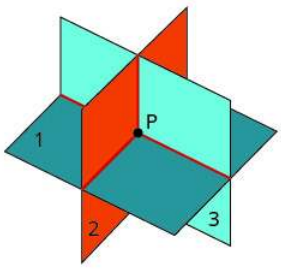
When we solve a system of three linear equations represented by a graph of three planes in space, there are three possible cases.

One solution

Consistent system and Independent equations

The 3 planes intersect.

The three intersecting planes have one point in common.

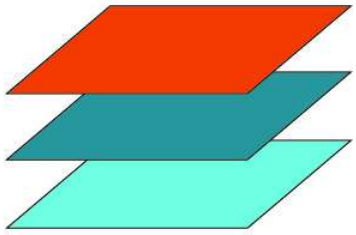


No solution

Inconsistent system

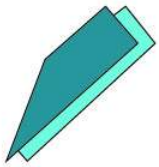
The planes are parallel.

Parallel planes have no points in common.



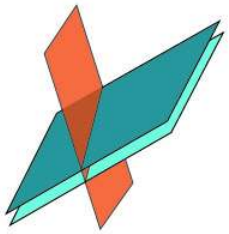
Two planes are coincident and parallel to the third plane.

The planes have no points in common.



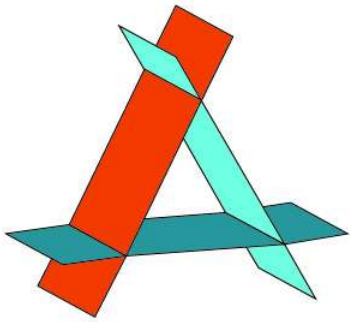
Two planes are parallel and each intersect the third plane.

The planes have no points in common.



Each plane intersects the other two, but all three share no points.

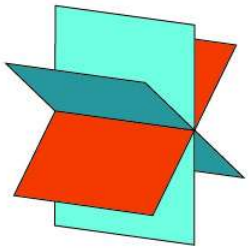
The planes have no points in common.



Infinitely many solutions
Consistent system and dependent equations

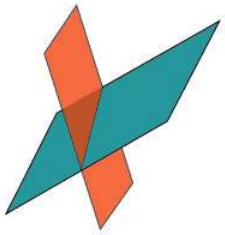
Three planes intersect in one line.

There is just one line, so there are infinitely many solutions.



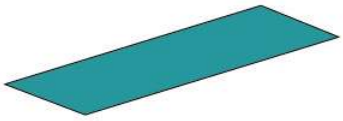
Two planes are coincident and intersect the third plane in a line.

There is just one line, so there are infinitely many solutions.



Three planes are coincident.

There is just one plane, so there are infinitely many solutions.



To solve a system of three linear equations, we want to find the values of the variables that are solutions to all three equations. In other words, we are looking for the ordered triple (x, y, z) that makes all three equations true. These are called the **solutions of the system of three linear equations with three variables**.

SolutionS OF A SYSTEM OF LINEAR EQUATIONS WITH THREE VARIABLES

Solutions of a system of equations are the values of the variables that make all the equations true. A solution is represented by an **ordered triple** (x, y, z).

To determine if an ordered triple is a solution to a system of three equations, we substitute the values of the variables into each equation. If the ordered triple makes all three equations true, it is a solution to the system.

Example 6.5.1

Determine whether the ordered triple is a solution to the system: $\begin{cases} x-y+z=2 \\ 2x-y-z=-6 \\ 2x+2y+z=-3 \end{cases}$

ⓐ $(-2,-1,3)$ ⓑ $(-4,-3,4)$

Answer

ⓐ

$$\begin{cases} x-y+z=2 \\ 2x-y-z=-6 \\ 2x+2y+z=-3 \end{cases}$$

We substitute $x=-2$ and $y=-1$ and $z=3$ into all three equations.

$x-y+z=2$	$2x-y-z=-6$	$2x+2y+z=-3$
$-2-(-1)+3 \stackrel{?}{=} 2$	$2\cdot(-2)-(-1)-3 \stackrel{?}{=} -6$	$2\cdot(-2)+2(-1)+3 \stackrel{?}{=} -3$
$2=2$ ✓	$-6=-6$ ✓	$-3=-3$ ✓
(–2, –1, 3) does make all three equations true.		(–2, –1, 3) is a solution.

ⓑ

$$\begin{cases} x - y + z = 2 \\ 2x - y - z = -6 \\ 2x + 2y + z = -3 \end{cases}$$

We substitute $x = -4$ and $y = -3$ and $z = 4$ into all three equations.

$x - y + z = 2$	$2x - y - z = -6$	$2x + 2y + z = -3$
$-4 - (-3) + 4 \stackrel{?}{=} 2$	$2 \cdot (-4) - (-3) - 4 \stackrel{?}{=} -6$	$2 \cdot (-4) + 2(-3) + 4 \stackrel{?}{=} -3$
$3 \neq 2$	$-9 \neq -6$	$-10 \neq -3$
$(-4, -3, 4)$ does not make all three equations true.		$(-4, -3, 4)$ is not a solution.

Example 6.5.2

Determine whether the ordered triple is a solution to the system: $\begin{cases} 3x + y + z = 2 \\ x + 2y + z = -3 \\ 3x + y + 2z = 4 \end{cases}$

ⓐ $(1, -3, 2)$ ⓑ $(4, -1, -5)$

Answer

ⓐ yes ⓑ no

Example 6.5.3

Determine whether the ordered triple is a solution to the system: $\begin{cases} x - 3y + z = -5 \\ -3x - y - z = 1 \\ 2x - 2y + 3z = 1 \end{cases}$

ⓐ $(2, -2, 3)$ ⓑ $(-2, 2, 3)$

Answer

ⓐ no ⓑ yes

Solve a System of Linear Equations with Three Variables

To solve a system of linear equations with three variables, we basically use the same techniques we used with systems that had two variables. We start with two pairs of equations and in each pair we eliminate the same variable. This will then give us a system of equations with only two variables and then we know how to solve that system!

Next, we use the values of the two variables we just found to go back to the original equation and find the third variable. We write our answer as an ordered triple and then check our results.

Example 6.5.4: How to Solve a System of Equations With Three Variables by Elimination

Solve the system by elimination: $\begin{cases} x - 2y + z = 3 \\ 2x + y + z = 4 \\ 3x + 4y + 3z = -1 \end{cases}$

Answer

Step 1. Write the equations in standard form. • If any coefficients are fractions, clear them.	The equations are in standard form. There are no fractions.	$\begin{cases} x-2y+z=3 & (1)\\ 2x+y+z=4 & (2)\\ 3x+4y+3z=-1 & (3)\end{cases}$
Step 2. Eliminate the same variable from two equations. • Decide which variable you will eliminate. • Work with a pair of equations to eliminate the chosen variable. • Multiply one or both equations so that the coefficients of that variable are opposites. • Add the equations to eliminate one variable.	We can eliminate the y's from equations (1) and (2) by multiplying equation (2) by 2. Multiply. We add the x's, y's, and constants. The new equation, (4), has only x and z.	$x-2y+z=3 \quad (1)$ $2(2x+y+z)=(4)2 \quad (2)$ $x-2y+z=3 \quad (1)$ $4x+2y+2z=8 \quad (2)$ $x-2y+z=3 \quad (1)$ $4x+2y+2z=8 \quad (2)$ $5x+3z=11 \quad (4)$
Step 3. Repeat Step 2 using two other equations and eliminate the same variable as in Step 2.	We can again eliminate the y's using the equations (1) and (3) by multiplying equation (1) by 2. Add the new equations and the result will be equation (5).	$2(x-2y+z)=(3)2 \quad (1)$ $3x+4y+3z=-1 \quad (3)$ $2x-4y+2z=6 \quad (1)$ $3x+4y+3z=-1 \quad (3)$ $5x+5z=5 \quad (5)$
Step 4. The two new equations form a system of two equations with two variables. Solve this system.	Use equations (4) and (5) to eliminate either x or z. Multiply (5) by –1. Use either (4) or (5) to solve for x using substitution.	$\begin{cases} 5x+3z=11 & (4)\\ 5x+5z=5 & (5)\end{cases}$ $5x+3z=11 \quad (4)$ $-1(5x+5z)=5(-1) \quad (5)$ $5x+3z=11 \quad (4)$ $-5x-5z=-5 \quad (5)$ $-2z=6$ $z=-3$ $5x+3z=11 \quad (4)$ $5x+3(-3)=11$ $5x-9=11$ $5x=20$ $x=4$
Step 5. Use the values of the two variables found in Step 4 to find the third variable.	Use an original equation to find y.	$2x+y+z=4 \quad (2)$ $2\cdot 4+y+z=4$ $8+y-3=4$ $y=-1$
Step 6. Write the solution as an ordered triple.	Write it as (x, y, z).	$(4, -1, -3)$
Step 7. Check that the ordered triple is a solution to **all three** original equations.	Substitute $x=4$, $y=-1$ and $z=-3$ into all three equations. Do they make the equations true?	We'll leave the checks to you. The solution is $(4, -1, -3)$.

Example 6.5.5

Solve the system by elimination: $\begin{cases} 3x+y-z=2 \\ 2x-3y-2z=1 \\ 4x-y-3z=0 \end{cases}$

Answer

$(2,-1,3)$

Example 6.5.6

Solve the system by elimination: $\begin{cases} 4x+y+z=-1 \\ -2x-2y+z=2 \\ 2x+3y-z=1 \end{cases}$

Answer

$(-2,3,4)$

The steps are summarized here.

SOLVE A SYSTEM OF LINEAR EQUATIONS WITH THREE VARIABLES.

1. Write the equations in standard form
   - If any coefficients are fractions, clear them.
2. Eliminate the same variable from two equations.
   - Decide which variable you will eliminate.
   - Work with a pair of equations to eliminate the chosen variable.
   - Multiply one or both equations so that the coefficients of that variable are opposites.
   - Add the equations resulting from Step 2 to eliminate one variable
3. Repeat Step 2 using two other equations and eliminate the same variable as in Step 2.
4. The two new equations form a system of two equations with two variables. Solve this system.
5. Use the values of the two variables found in Step 4 to find the third variable.
6. Write the solution as an ordered triple.
7. Check that the ordered triple is a solution to **all three** original equations.

Example 6.5.7

Solve: $\begin{cases} 3x-4z=0 \\ 3y+2z=-3 \\ 2x+3y=-5 \end{cases}$

Answer

$$\begin{cases} 3x-4z=0 & (1) \\ 3y+2z=-3 & (2) \\ 2x+3y=-5 & (3) \end{cases}$$

We can eliminate z from equations (1) and (2) by multiplying equation (2) by 2 and then adding the resulting equations.

$$\begin{array}{l} 3x \quad -4z=0 \quad (1) \\ \quad 3y+2z=-3 \quad (2) \end{array} \longrightarrow \begin{array}{l} 3x \quad -4z=0 \\ \quad 2(3y+2z)=(-3)2 \end{array} \longrightarrow \begin{array}{l} 3x \quad -4z=0 \\ \quad 6y+4z=-6 \\ \hline 3x+6y \quad =-6 \quad (4) \end{array}$$

Notice that equations (3) and (4) both have the variables x and y. We will solve this new system for x and y.

$$\begin{array}{l} 2x+3y=-5 \quad (3) \\ 3x+6y=-6 \quad (4) \end{array} \longrightarrow \begin{array}{l} -2(2x+3y)=-2(-5) \\ 3x+6y=-6 \end{array} \longrightarrow \begin{array}{r} -4x-6y=10 \\ 3x+6y=-6 \\ \hline -x \quad = \quad 4 \\ x=-4 \end{array}$$

To solve for y, we substitute $x=-4$ into equation (3).

$$\begin{array}{r} 2x+3y=-5 \quad (3) \\ 2(-4)+3y=-5 \\ -8+3y=-5 \\ 3y=3 \\ y=1 \end{array}$$

We now have $x=-4$ and $y=1$. We need to solve for z. We can substitute $x=-4$ into equation (1) to find z.

$$\begin{array}{r} 3x-4z=0 \quad (1) \\ 3(-4)-4z=0 \\ -12-4z=0 \\ -4z=12 \\ z=-3 \end{array}$$

We write the solution as an ordered triple. $(-4,1,-3)$

We check that the solution makes all three equations true.

$3x-4z=0\ (1)$ $\quad$ $3y+2z=-3\ (2)$ $\quad$ $2x+3y=-5\ (3)$

$3(-4)-4(-3)\stackrel{?}{=}0$ $\quad$ $3(1)+2(-3)\stackrel{?}{=}-3$ $\quad$ $2(-4)+3(1)\stackrel{?}{=}-5$

$0=0$✓ $\quad$ $-3=-3$✓ $\quad$ $-5=-5$✓

The solution is $(-4,1,-3)$

Example 6.5.8

Solve: $\begin{cases} 3x-4z=-1 \\ 2y+3z=2 \\ 2x+3y=6 \end{cases}$

Answer

$(-3,4,-2)$

Example 6.5.9

Solve: $\begin{cases} 4x-3z=-5 \\ 3y+2z=7 \\ 3x+4y=6 \end{cases}$

Answer

$(-2,3,-1)$

When we solve a system and end up with no variables and a false statement, we know there are no solutions and that the system is inconsistent. The next example shows a system of equations that is inconsistent.

Example 6.5.10

Solve the system of equations: $\begin{cases} x+2y-3z=-1 \\ x-3y+z=1 \\ 2x-y-2z=2 \end{cases}$

Answer

$$\begin{cases} x+2y-3z=-1 & (1) \\ x-3y+z=1 & (2) \\ 2x-y-2z=2 & (3) \end{cases}$$

Use equation (1) and (2) to eliminate z.

$$\begin{array}{l} x+2y-3z=-1 \quad (1) \\ x-3y+\ z=\ 1 \quad (2) \end{array} \xrightarrow{\text{multiply by 3}} \begin{array}{r} x+2y-3z=-1 \\ 3x-9y+3z=\ 3 \\ \hline 4x-7y\ \ =\ 2 \quad (4) \end{array}$$

Use (2) and (3) to eliminate z again.

$$\begin{array}{l} x-3y+\ z=1 \quad (2) \\ 2x-\ y-2z=2 \quad (3) \end{array} \xrightarrow{\text{multiply by 2}} \begin{array}{r} 2x-6y+2z=2 \\ 2x-\ y-2z=2 \\ \hline 4x-7y\ \ =4 \quad (5) \end{array}$$

Use (4) and (5) to eliminate a variable.

$$\begin{array}{l} 4x-7y=2 \quad (4) \\ 4x-7y=4 \quad (5) \end{array} \xrightarrow{\text{multiply by } -1} \begin{array}{r} 4x-7y=2 \\ -4x+7y=-4 \\ \hline 0=-2\ \textit{false} \end{array}$$

There is no solution.

We are left with a false statement and this tells us the system is inconsistent and has no solution.

Example 6.5.11

Solve the system of equations: $\begin{cases} x+2y+6z=5 \\ -x+y-2z=3 \\ x-4y-2z=1 \end{cases}$

Answer

no solution

Example 6.5.12

Solve the system of equations: $\begin{cases} 2x-2y+3z=6 \\ 4x-3y+2z=0 \\ -2x+3y-7z=1 \end{cases}$

Answer

no solution

When we solve a system and end up with no variables but a true statement, we know there are infinitely many solutions. The system is consistent with dependent equations. Our solution will show how two of the variables depend on the third.

Example 6.5.13

Solve the system of equations: $\begin{cases} x+2y-z=1 \\ 2x+7y+4z=11 \\ x+3y+z=4 \end{cases}$

Answer

$$\begin{cases} x+2y-z=1 & (1) \\ 2x+7y+4z=11 & (2) \\ x+3y+z=4 & (3) \end{cases}$$

Use equation (1) and (3) to eliminate x.

$$\begin{array}{rl} x+2y-z=1 & (1) \\ x+3y+z=4 & (3) \end{array} \xrightarrow{\text{multiply by } -1} \begin{array}{rl} -x-2y+z=-1 & (1) \\ x+3y+z=4 & (3) \\ \hline y+2z=3 & (4) \end{array}$$

Use equation (1) and (2) to eliminate x again.

$$\begin{array}{rl} x+2y-z=1 & (1) \\ 2x+7y+4z=11 & (2) \end{array} \xrightarrow{\text{multiply by } -2} \begin{array}{rl} -2x-4y+2z=-2 & (1) \\ 2x+7y+4z=11 & (2) \\ \hline 3y+6z=9 & (5) \end{array}$$

Use equation (4) and (5) to eliminate y.

$$\begin{array}{rl} y+2z=3 & (4) \\ 3y+6z=9 & (5) \end{array} \xrightarrow{\text{multiply by } -3} \begin{array}{rl} -3y-6z=-9 & (4) \\ 3y+6z=9 & (5) \\ \hline 0=0 & \end{array}$$

	There are infinitely many solutions.
Solve equation (4) for y.	Represent the solution showing how x and y are dependent on z. $y+2z=3$ $y=-2z+3$
Use equation (1) to solve for x.	$x+2y-z=1$
Substitute $y=-2z+3$.	$x+2(-2z+3)-z=1$ $x-4z+6-z=1$ $x-5z+6=1$ $x=5z-5$

The true statement $0=0$ tells us that this is a dependent system that has infinitely many solutions. The solutions are of the form (x,y,z)(x,y,z) where $x=5z-5;\ y=-2z+3$ and z is any real number.

Example 6.5.14

Solve the system by equations: $\begin{cases} x+y-z=0 \\ 2x+4y-2z=6 \\ 3x+6y-3z=9 \end{cases}$

Answer

infinitely many solutions $(x,3,z)$ where $x=z-3;\ y=3;\ z$ is any real number

Example 6.5.15

Solve the system by equations: $\begin{cases} x-y-z=1 \\ -x+2y-3z=-4 \\ 3x-2y-7z=0 \end{cases}$

Answer

infinitely many solutions (x, y, z) where $x=5z-2;\ y=4z-3;\ z$ is any real number

Solve Applications using Systems of Linear Equations with Three Variables

Applications that are modeled by a systems of equations can be solved using the same techniques we used to solve the systems. Many of the application are just extensions to three variables of the types we have solved earlier.

Example 6.5.16

The community college theater department sold three kinds of tickets to its latest play production. The adult tickets sold for $15, the student tickets for $10 and the child tickets for $8. The theater department was thrilled to have sold 250 tickets and brought in $2,825 in one night. The number of student tickets sold is twice the number of adult tickets sold. How many of each type did the department sell?

Answer

We will use a chart to organize the information.

Type	Number	Value	Total Value
adult	x	15	$15x$
student	y	10	$10y$
child	z	8	$8z$
	250		2825

Number of students is twice number of adults.

Rewrite the equation in standard form.

$$\begin{aligned} y &= 2x \\ 2x-y &= 0 \end{aligned}$$

Write the system of equations.

$$\begin{cases} x+y+z=250 & (1) \\ 15x+10y+8z=2825 & (2) \\ -2x+y=0 & (3) \end{cases}$$

Use equations (1) and (2) to eliminate z.

$$\begin{matrix} x+y+z=250 & (1) \\ 15x+10y+8z=2825 & (2) \end{matrix} \xrightarrow{\text{Multiply by } -8} \begin{matrix} -8x-8y-8z=250 & (1) \\ 15x+10y+8z=2825 & (2) \\ \hline 7x+2y=825 & (4) \end{matrix}$$

Use (3) and (4) to eliminate y.

$$\begin{matrix} -2x+y=0 & (3) \\ 7x+2y=825 & (4) \end{matrix} \xrightarrow{\text{Multiply by } -2} \begin{matrix} 4x-2y=0 & (3) \\ 7x+2y=825 & (4) \\ \hline 11x=825 \end{matrix}$$

Solve for x. $x=75$ adult tickets

Use equation (3) to find y. $-2x+y=0$

Substitute $x=75$.

$$\begin{aligned} -2(75)+y &= 0 \\ -150+y &= 0 \\ y &= 150 \text{ student tickets} \end{aligned}$$

Use equation (1) to find z. $x+y+z=250$

Substitute in the values $x = 75,\ y = 150$.	$\begin{aligned} 75 + 150 + z &= 250 \\ 225 + z &= 250 \\ z &= 25 \text{ child tickets} \end{aligned}$
Write the solution.	The theater department sold 75 adult tickets, 150 student tickets, and 25 child tickets.

Example 6.5.17

The community college fine arts department sold three kinds of tickets to its latest dance presentation. The adult tickets sold for $20, the student tickets for $12 and the child tickets for $10.The fine arts department was thrilled to have sold 350 tickets and brought in $4,650 in one night. The number of child tickets sold is the same as the number of adult tickets sold. How many of each type did the department sell?

Answer

The fine arts department sold 75 adult tickets, 200 student tickets, and 75 child tickets.

Example 6.5.18

The community college soccer team sold three kinds of tickets to its latest game. The adult tickets sold for $10, the student tickets for $8 and the child tickets for $5. The soccer team was thrilled to have sold 600 tickets and brought in $4,900 for one game. The number of adult tickets is twice the number of child tickets. How many of each type did the soccer team sell?

Answer

The soccer team sold 200 adult tickets, 300 student tickets, and 100 child tickets.

Access this online resource for additional instruction and practice with solving a linear system in three variables with no or infinite solutions.

- Solving a Linear System in Three Variables with No or Infinite Solutions
- 3 Variable Application

Key Concepts

- **Linear Equation in Three Variables:** A linear equation with three variables, where *a*, *b*, *c*, and *d* are real numbers and *a*, *b*,and *c* are not all 0, is of the form

$$ax + by + cz = d$$

  Every solution to the equation is an ordered triple, (x, y, z) that makes the equation true.
- **How to solve a system of linear equations with three variables.**
  1. Write the equations in standard form
     If any coefficients are fractions, clear them.
  2. Eliminate the same variable from two equations.
     Decide which variable you will eliminate.
     Work with a pair of equations to eliminate the chosen variable.
     Multiply one or both equations so that the coefficients of that variable are opposites.
     Add the equations resulting from Step 2 to eliminate one variable
  3. Repeat Step 2 using two other equations and eliminate the same variable as in Step 2.
  4. The two new equations form a system of two equations with two variables. Solve this system.
  5. Use the values of the two variables found in Step 4 to find the third variable.
  6. Write the solution as an ordered triple.

7. Check that the ordered triple is a solution to **all three** original equations.

Glossary

solutions of a system of linear equations with three variables

The solutions of a system of equations are the values of the variables that make all the equations true; a solution is represented by an ordered triple (x,y,z).

This page titled 6.5: Solve Systems of Equations with Three Variables is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- **4.5: Solve Systems of Equations with Three Variables** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/intermediate-algebra-2e.

Index

A

Area of oblique triangles

1.1: Non-right Triangles - Law of Sines

D

Difference of Squares

2.3: Special Products

dividing polynomials

2.4: Divide Polynomials

E

elimination

6.3: Solve Systems of Equations by Elimination

F

Factor by Grouping

3.1: Greatest Common Factor and Factor by Grouping

factoring polynomials

3.5: General Strategy for Factoring Polynomials

G

greatest common factor

3.1: Greatest Common Factor and Factor by Grouping

H

Heron's Formula

1.2: Non-right Triangles - Law of Cosines

L

law of cosines

1.2: Non-right Triangles - Law of Cosines

Law of Sines

1.1: Non-right Triangles - Law of Sines

M

multiply polynomials

2.2: Multiply Polynomials

Q

quadratic formula

4.3: Solve Quadratic Equations Using the Quadratic Formula

S

Solutions of a system of equations

6.1: Solve Systems of Equations by Graphing

Glossary

Sample Word 1 | Sample Definition 1

Detailed Licensing

Overview

Title: College Technical Math 1B (NWTC)

Webpages: 44

All licenses found:

- CC BY-SA 4.0: 97.7% (43 pages)
- Undeclared: 2.3% (1 page)

By Page

- College Technical Math 1B (NWTC) - *CC BY-SA 4.0*
  - Front Matter - *CC BY-SA 4.0*
    - TitlePage - *CC BY-SA 4.0*
    - InfoPage - *CC BY-SA 4.0*
    - Table of Contents - *Undeclared*
    - Licensing - *CC BY-SA 4.0*
  - 1: Law of Sines and Cosines - *CC BY-SA 4.0*
    - 1.1: Non-right Triangles - Law of Sines - *CC BY-SA 4.0*
    - 1.2: Non-right Triangles - Law of Cosines - *CC BY-SA 4.0*
  - 2: Polynomials - *CC BY-SA 4.0*
    - 2.1: Add and Subtract Polynomials - *CC BY-SA 4.0*
    - 2.2: Multiply Polynomials - *CC BY-SA 4.0*
    - 2.3: Special Products - *CC BY-SA 4.0*
    - 2.4: Divide Polynomials - *CC BY-SA 4.0*
  - 3: Factoring - *CC BY-SA 4.0*
    - 3.1: Greatest Common Factor and Factor by Grouping - *CC BY-SA 4.0*
    - 3.2: Factor Quadratic Trinomials with Leading Coefficient 1 - *CC BY-SA 4.0*
    - 3.3: Factor Quadratic Trinomials with Leading Coefficient Other than 1 - *CC BY-SA 4.0*
    - 3.4: Factor Special Products - *CC BY-SA 4.0*
    - 3.5: General Strategy for Factoring Polynomials - *CC BY-SA 4.0*
  - 4: Quadratic Equations - *CC BY-SA 4.0*
    - 4.1: Quadratic Equations and Solving by Factoring - *CC BY-SA 4.0*
    - 4.2: Solve Quadratic Equations Using the Square Root Property - *CC BY-SA 4.0*
    - 4.3: Solve Quadratic Equations Using the Quadratic Formula - *CC BY-SA 4.0*
    - 4.4: Solve Applications Modeled by Quadratic Equations - *CC BY-SA 4.0*
  - 5: Rational Expressions and Equations - *CC BY-SA 4.0*
    - 5.1: Simplify Rational Expressions - *CC BY-SA 4.0*
    - 5.2: Multiply and Divide Rational Expressions - *CC BY-SA 4.0*
    - 5.3: Add and Subtract Rational Expressions with a Common Denominator - *CC BY-SA 4.0*
    - 5.4: Add and Subtract Rational Expressions with Unlike Denominators - *CC BY-SA 4.0*
    - 5.5: Simplify Complex Rational Expressions - *CC BY-SA 4.0*
    - 5.6: Solve Rational Equations - *CC BY-SA 4.0*
    - 5.7: Solve Uniform Motion and Work Applications - *CC BY-SA 4.0*
  - 6: Systems of Equations - *CC BY-SA 4.0*
    - 6.1: Solve Systems of Equations by Graphing - *CC BY-SA 4.0*
    - 6.2: Solve Systems of Equations by Substitution - *CC BY-SA 4.0*
    - 6.3: Solve Systems of Equations by Elimination - *CC BY-SA 4.0*
    - 6.4: Solve Mixture Applications with Systems of Equations - *CC BY-SA 4.0*
    - 6.5: Solve Systems of Equations with Three Variables - *CC BY-SA 4.0*
  - Back Matter - *CC BY-SA 4.0*
    - Index - *CC BY-SA 4.0*
    - Glossary - *CC BY-SA 4.0*
    - Detailed Licensing - *CC BY-SA 4.0*
    - Detailed Licensing - *CC BY-SA 4.0*

Detailed Licensing

Overview

Title: College Technical Math 1B (NWTC)

Webpages: 44

All licenses found:

- CC BY-SA 4.0: 97.7% (43 pages)
- Undeclared: 2.3% (1 page)

By Page

- College Technical Math 1B (NWTC) - *CC BY-SA 4.0*
  - Front Matter - *CC BY-SA 4.0*
    - TitlePage - *CC BY-SA 4.0*
    - InfoPage - *CC BY-SA 4.0*
    - Table of Contents - *Undeclared*
    - Licensing - *CC BY-SA 4.0*
  - 1: Law of Sines and Cosines - *CC BY-SA 4.0*
    - 1.1: Non-right Triangles - Law of Sines - *CC BY-SA 4.0*
    - 1.2: Non-right Triangles - Law of Cosines - *CC BY-SA 4.0*
  - 2: Polynomials - *CC BY-SA 4.0*
    - 2.1: Add and Subtract Polynomials - *CC BY-SA 4.0*
    - 2.2: Multiply Polynomials - *CC BY-SA 4.0*
    - 2.3: Special Products - *CC BY-SA 4.0*
    - 2.4: Divide Polynomials - *CC BY-SA 4.0*
  - 3: Factoring - *CC BY-SA 4.0*
    - 3.1: Greatest Common Factor and Factor by Grouping - *CC BY-SA 4.0*
    - 3.2: Factor Quadratic Trinomials with Leading Coefficient 1 - *CC BY-SA 4.0*
    - 3.3: Factor Quadratic Trinomials with Leading Coefficient Other than 1 - *CC BY-SA 4.0*
    - 3.4: Factor Special Products - *CC BY-SA 4.0*
    - 3.5: General Strategy for Factoring Polynomials - *CC BY-SA 4.0*
  - 4: Quadratic Equations - *CC BY-SA 4.0*
    - 4.1: Quadratic Equations and Solving by Factoring - *CC BY-SA 4.0*
    - 4.2: Solve Quadratic Equations Using the Square Root Property - *CC BY-SA 4.0*
    - 4.3: Solve Quadratic Equations Using the Quadratic Formula - *CC BY-SA 4.0*
    - 4.4: Solve Applications Modeled by Quadratic Equations - *CC BY-SA 4.0*
  - 5: Rational Expressions and Equations - *CC BY-SA 4.0*
    - 5.1: Simplify Rational Expressions - *CC BY-SA 4.0*
    - 5.2: Multiply and Divide Rational Expressions - *CC BY-SA 4.0*
    - 5.3: Add and Subtract Rational Expressions with a Common Denominator - *CC BY-SA 4.0*
    - 5.4: Add and Subtract Rational Expressions with Unlike Denominators - *CC BY-SA 4.0*
    - 5.5: Simplify Complex Rational Expressions - *CC BY-SA 4.0*
    - 5.6: Solve Rational Equations - *CC BY-SA 4.0*
    - 5.7: Solve Uniform Motion and Work Applications - *CC BY-SA 4.0*
  - 6: Systems of Equations - *CC BY-SA 4.0*
    - 6.1: Solve Systems of Equations by Graphing - *CC BY-SA 4.0*
    - 6.2: Solve Systems of Equations by Substitution - *CC BY-SA 4.0*
    - 6.3: Solve Systems of Equations by Elimination - *CC BY-SA 4.0*
    - 6.4: Solve Mixture Applications with Systems of Equations - *CC BY-SA 4.0*
    - 6.5: Solve Systems of Equations with Three Variables - *CC BY-SA 4.0*
  - Back Matter - *CC BY-SA 4.0*
    - Index - *CC BY-SA 4.0*
    - Glossary - *CC BY-SA 4.0*
    - Detailed Licensing - *CC BY-SA 4.0*
    - Detailed Licensing - *CC BY-SA 4.0*