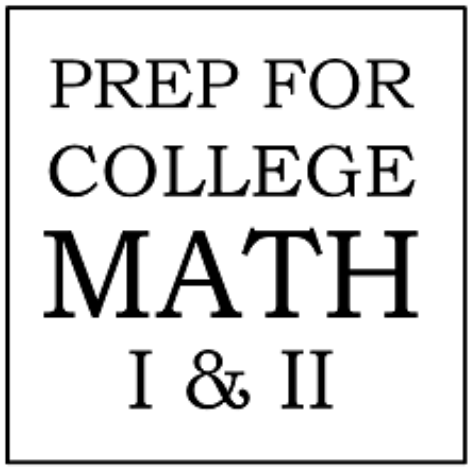
PREPARATION FOR COLLEGE MATHEMATICS



Lydia de Wolf
Siena College

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Siena College

Preparation for College Mathematics

Lydia de Wolf

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This text was compiled on 06/04/2026

About This Book

This text has been written as a resource to prepare students for college mathematics at Siena College. The textbook covers select essential skills, starting from the very beginning (such as working with fractions, estimation skills, and basic algebra) and progressing through to a unit-circle-based crash course on trigonometry. Courses should select material from across the text according to the needs of the class. An Intermediate Algebra course, for example, could focus more on Chapters 1-6, spending more time shoring up fundamental arithmetic and algebra skills needed for all science majors, and spend less time on exponential, logarithmic, and trigonometric functions. A more traditional College Algebra/Precalc course, on the other hand, could assume the foundational skills and cover mainly Chapters 3-8. Suggested section lists to use for each course are given below.

If not indicated otherwise, I have made all graphics myself, using Microsoft Visio, Desmos, Geogebra, Overleaf, Google Workspace, etc.

With questions or corrections (this text was created in the course of a single summer, so it will certainly undergo live adjustments over time), please email ldewolf@siena.edu.

Suggested sections for Level I course:

1. 1.1 Real Numbers
2. 1.2 Working With Fractions
3. 1.3 Working With Decimals
4. 1.4 Powers and Radicals
5. 1.5 Estimation, Common Sense, and Intuition
6. 2.1 Algebraic Expressions
7. 2.2 Factoring Techniques
8. 2.3 Rational Expressions
9. 2.4 Algebravaganza (Exam 1 here)
10. 3.1 Solving for a Variable
11. 3.2 Solving Inequalities
12. 3.3 Lines and Circles
13. 3.4 Modeling With Equations
14. 3.5 Solving Systems of Linear Equations (Exam 2 here)
15. 4.1 Functions, Domain, and Notation
16. 4.2 Graphing Functions With Transformations
17. 4.3 Compositions of Functions
18. 4.4 Inverse Functions
19. 5.1 Linear Functions and Rate of Change
20. 5.2 Quadratic Functions
21. 5.3 Polynomial Functions
22. 5.4 Roots of Polynomial Functions (Exam 3 here)
23. 6.1 Rational Functions
24. 6.3 Basics of Exponential Functions
25. 6.4 Basics of Logarithmic Functions
26. 8.1 Geometry Essentials
27. 8.2 Solving Problems With Geometry (Final exam here)

Suggested sections for Level II course:

1. 1.5 Estimation, Common Sense, and Intuition
2. 2.2 Factoring Techniques
3. 2.4 Algebravaganza
4. 3.3 Lines and Circles
5. 3.4 Modeling With Equations
6. 3.5 Solving Systems of Linear Equations
7. 3.6 Vectors and Matrices (Exam 1 here)
8. 4.1 Functions, Domain, and Notation

9. 4.2 Graphing Functions With Transformations
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24. 8.3 Trigonometry Via Triangles
25. 8.4 Trigonometric Functions
26. 8.5 More (and Inverse) Trig Functions
27. 8.6 Trigonometric Identities
28. 8.7 Solving Trigonometric Equations (Final exam here)

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How to Use This Book as a Student

Hey hey, welcome to Prep for College Math! Here are some basic tips that you should keep in mind as we get started.

- When you come across "Examples," it's a good idea to pause and think about them a little bit before reading on, but I am intending to show you something via the solution! Meanwhile, when you see something labeled "Exercise," I am expecting you to give it a shot on your own **before** opening the "Answer" tab.
- Similarly, in the exercise list marked "E" for each section, you should be doing these problems **on your own,** without looking back at notes or Google or anything, the majority of the time! Whenever a student is like, "I get all the homework problems right, but when I get to the exam, I can't do it," that's code for, "I looked up how to do every homework problem and it was a total waste of my time because that doesn't effectively integrate the skills into my mental toolbox." Treat exercises like mini-exams, and you will be fine when real exams come along. See below for a more detailed step-by-step on homework and learning techniques.
- I do intend for you to eventually do **every** problem in the Exercises section, because sometimes a problem is **didactic** and there is something for you to ponder and learn from it. Sometimes concepts don't make it into the section, but I still want to expose you to them, so they appear in the exercises. Challenge problems will have more explanation in the "Answer" tab than others, but you should stretch your critical thinking skills before resorting to reading that.
- Almost everything in this book is designed to be done **without a calculator.** When I want you to use calculators, I'll alert you and tell you how, but the basic skills and knowledge covered here is all completely reasonable to expect students to do without technology. You'll notice that none of the numbers turn out devastatingly gross, because I expect you to be using mental math or very quick scratch work. You'll also notice that we learn a lot about what functions tend to look like, because you need to have a sense going forward of how they behave. If your professor tells you, "This population exhibits exponential growth," they expect you to be aware of what an exponential function looks like without pulling out your TI-84. By the end of this class, you should be able to sketch graphs of simple functions using basic knowledge and transformation tricks.
- I've provided a study guide at the end of each chapter, but it is much more valuable for you to **write your own.** Mainly I'm trying to compile a quick reference for you to flip back to later on if you want to recall a detail without clicking through the whole chapter. I highly recommend that you use it as a skeleton to construct your own study guides, in which you should definitely write out example problems! Staring at my lists and bullet points with glassy eyes will do you almost no good compared to doing problems.
- I'm writing this book just the way I talk when I'm teaching, which is to say, I'm leaning heavily on what I think is important for you to take away in the end, and I'm goofing off while I'm doing it. There are more "rigorous" ways to say everything in math, and you'll find them in other textbooks, and eventually you will probably have to cope with reading more jargon. I just intend this book to be like a "No Tears" math crash course. Like those "No Tears" kids' shampoo bottles, the ones that look like fish? You remember.



- If you're using this book for MATH040 at Siena College, you will be starting from the beginning and mostly not doing Chapters 7 and 8. If you intend to go straight on into MATH110 (Calculus I) afterwards, you should take some time during the winter or summer break to work through those final chapters on your own, so you're not totally in the dark on those final subjects. I'll put a study schedule in the Appendix for you.
- On the other hand, if you're using this book for MATH101 at Siena, you will be skipping significant portions of Chapters 1-3 and going all the way through the rest of the material. If you're rusty on anything basic, you can always look back in detail at

the sections we skip. If you want a quick reference on the topics, I recommend just looking at the study guides at the end of each chapter.

How to Succeed in Math

The process of studying math is really unique compared to studying other subjects. The most effective and time-efficient thing you can do is homework/practice problems! Here is how to do homework:

- Problems in the Exercises sections come in type "blocks." In each one, start by doing all the odd numbered problems. On whatever type you're struggling with, go back and do the evens for more practice, or use them later when reviewing for exams. Eventually, all problems should be done in some way (as review, used in a practice exam, etc.) as there aren't a TON of them.
- After each class session where you cover a section, sit down for a 1-2 hr block of time and go through any examples/exercises from the section that weren't covered in class, and then move to the related Exercises section. For each problem,
  1. First attempt to figure out the problem BY YOURSELF. Try for a few minutes. DO NOT look at the answers.
  2. If you're totally lost, review the section and look at the examples and exercises for clues (open in a separate tab so you can flip back and forth).
  3. Come back to the problem and try again to do it without looking at anything else!
  4. When you're finished with a whole block, open the answer and compare. If any of your answers don't match, go back and figure out what you did wrong. This is where the real learning happens!
  5. Make a note of any problems you can't figure out completely, and go to office hours to have them specifically explained. That's what office hours are for!!! Every single successful student that I poll after a course says that the big factor in their success was coming to office hours.

Here are other learning strategies to try during the rest of your study time:

- When reading the textbook, preview by scanning and making up questions related to the material so you can actively look for the answers.
- As you read, paraphrase material back to yourself.
- Preview before class (if only for 10 min); go to class and take notes by hand; review after class (if only for 5 min).
- Teach the material to a real or imagined audience.
- Work in pairs or groups.
- Hand-write a study guide or "cheat sheet" even if you're not allowed to use it on the exam. The magic is in making it yourself.
- Make up practice exams and simulate testing environment for yourself.

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CHAPTER OVERVIEW

Chapter 1: Computation, Estimation, and Common Sense

This chapter will take you through a recap on numbers, basically: what a real number is, how numbers compare to each other, how to do computations and calculations with numbers, working with fractions and decimals, representing numbers with letters (don't panic, we gotta do it to do algebra later), dealing with powers and radicals, and simplification. Then we will do a very important dive into other aspects of numeracy such as: being able to estimate expressions involving fractions and irrational numbers at a glance, telling whether expressions are bigger or smaller than others at a glance, using common sense checks to see if an answer is even reasonable, and the usefulness of intuition versus when it might lie to you(!! this is why we actually need rigorous mathematics).

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Chapter 1: Computation, Estimation, and Common Sense

Lydia de Wolf

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1.1 Real Numbers

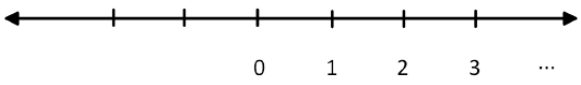
Learning Objectives

By the end of this section, you will:

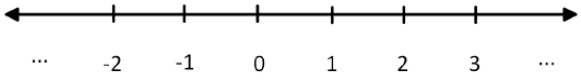
- know the definitions of different types of numbers (natural, rational, real, positive, negative)
- visualize real numbers on a number line and understand greater than/less than relationships
- know the arithmetic operations and in what order to perform them

Numbers! Can't live with 'em, can't live without 'em, or something like that? Well, they're everywhere at any rate, so we're going to learn how to use them. Let's start out with counting. (A-one...a-two-whoo...a-three...)

We're familiar with the counting numbers, aka whole numbers, aka **natural numbers** denoted by the symbol $\mathbb{N}$, which are $0, 1, 2, 3, ..., 17, ..., 10001, ...$ etc. They go on forever getting bigger, and you can tick them off on a number line like the one below.



Why are there extra ticks and an arrow indicating the number line also goes off forever to the left? Glad you asked. Those are for the rest of the **integers** (denoted by $\mathbb{Z}$), which include **negative** whole numbers as well as the **(positive)** natural numbers. Check it out.



Addition and **subtraction** are two arithmetic operations we can use on numbers. If you **add** two integers, the result is their **sum,** and it will also be an integer. You can think of **subtraction** as adding a negative number, and the resulting **difference** will also be an integer. Addition and subtraction are **inverse operations**, in the sense that they undo each other. Also, it doesn't matter what order you add in, because addition is a **commutative** operation, but order does matter with subtraction! Consider the examples below illustrating these properties.

Example 1

1. $3 + 2 = 2 + 3 = 5$
2. $4000 + 101 = 4101$
3. $-3 + (-4) = -7$
4. $8 - 6 = 8 + (-6) = 2$
5. $300 - 10 = 290$, but $10 - 300 = -290$
6. $2 + 4 - 4 = 2$

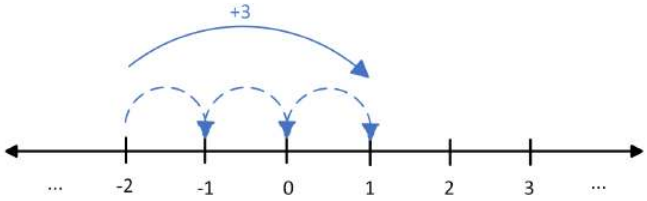
Exercise 1

1. Think of two integers whose sum is 13.
2. Think of two integers whose difference is 5.
3. Think of an integer such that subtracting 7 will result in -2.

Answer

1. 8 and 5 will work, and so will 10 and 3, or even $20 + (-7)$.
2. 20 and 15 (in the order $20 - 15$) will work, and so will 47 and 42, etc.
3. For this one, only positive 5 will work. That's $5 - 7 = -2$.

By the way, you can think of addition (and subtraction) as hopping forwards (or backwards) a certain amount of times along the number line. For example, you can represent $-2 + 3 = 1$ with the picture below.



The next arithmetic operations to look at are **multiplication** and **division.** We multiply two numbers (sometimes called **factors**) using a dot symbol, and the result is called their **product.** Multiplying is code for adding a bunch of copies of a number, and multiplication is also a commutative operation. For example, $5 \cdot 3 = 5 + 5 + 5 = 15$ and equivalently $3 \cdot 5 = 3 + 3 + 3 + 3 + 3 = 15$. This interpretation of multiplication should make clear why...

- any number times 0 is zero!
- any number times 1 is just itself!

However, you do NOT want to be calculating products by writing out a gazillion additions, so you must learn your "times tables" if you don't already have them memorized. You can find multiplication drill sheets in the appendix! Set a timer and speed run those until they're second nature. You should be able to do things like the list below using mental math pretty quickly.

? Exercise 2

1. $8 \cdot 4 =$
2. $10 \cdot 3 =$
3. $7 \cdot 9 =$
4. $13 \cdot 2 =$
5. $200 \cdot 400 =$
6. Think of two integers whose product is 18.

Answer

1. 32
2. 30
3. 63
4. 26
5. 80000 (Any time you're multiplying things with zeros on the end, just look at the non-zero parts on the front, and then tack on the total number of zeros in the numbers. Two zeros came from the 200 and two more came from the 400.)
6. 6 and 3 will work, or 9 and 2, or 18 and 1.

We use the dot symbol $\cdot$ so we don't mix up $\times$ and x later, but if you see things in parentheses right next to each other, that also means multiplication, such as $(2)(4) = 8$, or just $2(6) = 12$.

One thing to watch out for is when negative numbers are multiplied. Sometimes the signs will change, and sometimes they won't. Here's the rundown:

Ingredients	Product	Example
positive times positive	positive	$(7)(3) = 21$
positive times negative	negative	$(-2)(3) = -6$
negative times negative	positive	$(-11)(-2) = 22$
(-1) times a negative, or "minus a minus"	positive	$-(-4) = (-1)(-4) = 4$

Try using these rules when you practice in the Exercises section.

Division is the inverse operation to multiplication. Intuitively, it's taking a number or quantity and splitting it into a certain number of equal pieces. Like $\frac{12}{3} = 4$ because splitting 12 candy bars among 3 children will result in each child getting 4 candy bars. Order matters in division; that is, $\frac{12}{3} \neq \frac{3}{12}$. You might also see division written with the symbol $\div$, but that's gross and I won't use it often. You'll want to know how to do simple divisions in your head as well, so if that's rusty, check out the drill sheets in the appendix.

? Exercise 3

1. Divide 50 by 2.
2. Compute $\frac{9}{3}$.
3. Compute $\frac{24}{6}$.
4. Take the number 4 and multiply it by 8. Then divide the result by 8. What happened?

Answer

1. 25
2. 3
3. 4
4. $4 \cdot 8 = 32$ and then $\frac{32}{8} = 4$. We got back to the same number. This is because multiplication and division are inverse operations.

The number you start with is called the **dividend,** and the number you divide BY is the **divisor,** and the result is called the **quotient.** You'll also want to watch out for negative numbers involved in division, just like with multiplication above.

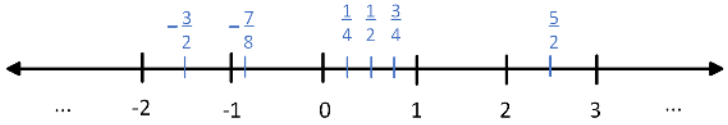
Ingredients	Quotient	Example
positive divided by positive	positive	$\frac{25}{5} = 5$
positive divided by negative or vice versa	negative	$\frac{-12}{3} = -4$ or $\frac{36}{-12} = -3$
negative divided by negative	positive	$\frac{-16}{-8} = 2$

Try using these rules when you practice in the Exercises section.

Did you notice that $\frac{3}{12}$ up above?? I can't split 3 candy bars among 12 children without cutting the candy bars into pieces! In fact, I need to cut them into four pieces each to have enough... So you see, introducing the division operation is leading us to talk about **fractions.** Goodbye integers and hello **rational numbers!** A rational number is one that can be written as a **ratio** of two integers, and the set of all such numbers is denoted by the symbol $\mathbb{Q}$. Since any whole number can be written as itself over 1, the rational numbers include all the integers. We're talking about things like

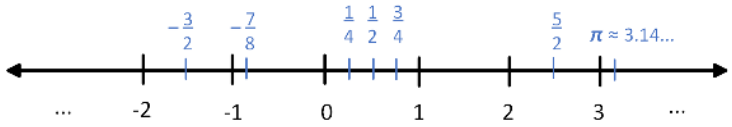
$$\frac{3}{2}, \quad \frac{4}{1}, \quad \frac{37}{9}, \quad \frac{-2}{5}, \quad \frac{1}{100}, \quad \cdots$$

and so on and so forth. You can visualize rational numbers on the number line between the whole number ticks.



Rational numbers can be written as **decimals** instead of fractions, and these decimals will always "end" somewhere, like how $\frac{3}{2} = 1.5$ or $\frac{1}{8} = 0.125$ end after a certain number of decimal places. We will work with fractions and decimals in more detail in later sections, so for now let's get a move on toward those real numbers we're supposed to be learning about. If a decimal representation of a number never ends and never settles into a repeating pattern, like the special number $\pi = 3.141592653589793...$, it's called an **irrational number,** meaning it canNOT be written as a ratio of integers. Other examples we will meet later are things like $\sqrt{2}$

$= 1.41421356237...$ or the special number $e = 2.718281828...$. We could still toss π on our number line to visualize him using his approximation, though, right thereabouts...



Finally, the set of numbers that includes all integers, all rational numbers, AND all irrational numbers is called the set of **real numbers.**

The **real numbers**, denoted by $\mathbb{R}$, contain all integers (positive and negative whole numbers), all rational numbers (numbers that can be written as a ratio of two integers or a decimal that ends or settles into a repeating pattern eventually), and all irrational numbers (numbers whose decimal representation never ends and doesn't repeat, like $\pi, e,$ or $\sqrt{2}$).

Now let's learn how to compare numbers. I want to start representing numbers using letters so it's easier to talk about general ideas. I'll say things like, "Let m be an integer," which is code for, "I have some whole number nicknamed m, maybe he's 2, maybe he's 17, maybe he's -4000, it doesn't matter." Try to see what I mean by the following description of possible situations:

Given two real numbers a and b, we fall into one of the following three cases...

1. $a > b$. We say a is **greater than** b, aka a is further to the **right** on the number line than b, aka the difference $a - b$ is a **positive** number.
2. $a < b$. We say a is **less than** b, aka a is further to the **left** on the number line than b, aka the difference $a - b$ is a **negative** number.
3. $a = b$. We say a and b are **equal**, aka they are the same number, aka the difference $a - b$ is **zero.**

The symbols involved here are called **inequality** symbols, and if you picture them as an alligator opening his mouth, he's always opening his mouth toward the bigger number. Because he's greedy. Here's the whole list:

- $a > b$ means a is **strictly** greater than b, Case 1 above. To be clear, "strict" means a cannot equal b. For example, $5 > 2$.
- $a < b$ means a is **strictly** less than b. For example, $-2 < 9$
- $a \geq b$ means a is **either** greater than **or** equal to b. For example, if $a \geq 7$, a could be 7, or 8, or 32.99, or anything as long as it is at least 7.
- $a \leq b$ means a is **either** less than **or** equal to b. If $a \leq 4$, then a could be 4, or 3, or -100, as long as it's not more than 4. You get the idea.

? Exercise 4

True or False?

1. $8 < 12$
2. $-5 > -2$
3. $1.8 \geq 1.5$
4. $\pi \leq 3$

Answer

1. True, 8 is smaller than 12.
2. False! Note that -5 is further to the left on the number line than -2, which makes it "more negative," which makes it technically a smaller number! You can check this way: $-5 - (-2) = -5 + 2 = -3$. Yep, the difference is negative.
3. True, 1.8 is greater than 1.5, which satisfies one of the options for the $\geq$ symbol. The difference $1.8 - 1.5 = 0.3$ is positive.
4. False, we know $\pi \approx 3.14159...$ which is actually a bit bigger than 3.

Order of Operations

All those arithmetic operations we covered above can be done one after another all in a row, to make a big combination of operations. For example, if I say, "Take 2 and add 4, then subtract 8, then multiply by 3, then add 9, then divide by 3," all that garbage translates to

$$2 \longrightarrow 2+4=6 \longrightarrow 6-8=-2 \longrightarrow (-2)(3) = -6 \longrightarrow -6+9=3 \longrightarrow \frac{3}{3}=1.$$

But good lord, I don't want to have to type that stuff ever again! The good news is we can write the whole process using math notation as long as we're careful about what order we're doing things in. See if you can understand why the expression below is the equivalent to the long-winded instruction I gave above.

$$\frac{3((2+4)-8)+9}{3}$$

All of those parentheses are doing some heavy lifting there to indicate the order. We always look at parts in parentheses first, and we always work from the inside out, so the first step is that inner $(2+4)$. Once that's done, we look at the next layer of parentheses, which include the subtraction, so that happens next. Outside all of that, we see the multiplication by 3. Moving along to the right, we see the addition of 9 step. And all of that is finally being divided by the 3 downstairs. Woof.

Computations follow the conventional **Order of Operations: PEMDAS**

1. **Parentheses:** Start with the innermost parentheses, ignore everything else going on outside, and proceed. Each time you finish working out the contents of some parentheses, move to the next innermost layer and repeat.
2. **Exponents:** This is when things are raised to powers, but we're going to deal with powers and radicals in detail later. If these were around, you would do them next.
3. **Multiplication/Division:** Perform any multiplications and divisions, moving from left to right (which may not matter, but it's convention). Note that if a bunch of stuff is happening in the numerator of a fraction, it's as if there are parentheses around the numerator, and you should do all of it before performing the division. That is,

$$\frac{a+b}{c} = \frac{(a+b)}{c}$$

4. **Addition/Subtraction:** Perform any additions and subtractions, again moving from left to right.

These rules allow us to know for sure whether $2+3\cdot 5$ is supposed to equal 25 or 17. There are no parentheses, so we look for multiplications and do those first, getting $2+15$. We now finish by adding to get 17. Give the practice problems below a shot, and then check your answers.

? Exercise 5

Compute:

1. $7-2\cdot 4$
2. $\frac{2+7}{3}-2$
3. $4(15+(2-3))-6$
4. $2(3-(-3))+2\cdot 8-\frac{10}{5}$

Answer

1. $7-2\cdot 4 = 7-8=-1$
2. $\frac{2+7}{3}-2=\frac{9}{3}-2=3-2=1$
3. $4(15+(2-3))-6=4(15+(-1))-6=4(14)-6=56-6=50$
4. $2(3-(-3))+2\cdot 8-\frac{10}{5}=2(6)+2\cdot 8-\frac{10}{5}=12+16-2=26$

Order of operations is important, because if you mess it up, you could get a totally different answer!

Exercise 6

Which is the correct math translation of the instructions, "Take the number 6 and subtract 2, then divide by 4, then add 9, then multiply by 5," and what is the correct answer?

$$\frac{6-2}{4}+9\cdot 5 \quad \text{or} \quad 5\left(\frac{6-2}{4}+9\right)$$

Answer

The first one is wrong, because if you follow the order of operations, you will multiply the 9 and 5 too early, and only later perform the addition! If we want to add 9 first and then multiply by 5, we must use the second option, which performs the addition inside the parentheses, and multiplies by 5 last of all. The result of the whole fiasco should be 50. (The wrong expression gives an answer of 46.)

Okay, that's all, folks. Scroll back up to the Learning Objectives and make sure you feel good about the skills you were aiming to pick up from this section! Then move on to trying the exercises in the next section.

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1.1E Exercises

How to Do Homework: Start by doing all the odd numbered problems. On whatever type you're struggling with, go back and do the evens for more practice, or use them later when reviewing for exams.

1. First attempt to figure out the problem BY YOURSELF. Try for a few minutes. DO NOT look at the answers.
2. If you're totally lost, review the section and look at the examples and exercises for clues.
3. Come back to the problem and do it without looking at anything else!
4. When you're finished, open the answer and compare. If any of your answers don't match, go back and figure out what you did wrong. This is where the real learning happens!

? Adding/Subtracting Integers

Compute.

1. $12 + 8$
2. $3 + 4$
3. $6 - 5$
4. $6 - 9$
5. $5 + (-3)$
6. $(-8) + 4$
7. $15 - (-9)$
8. $2 - (-1)$
9. $4 - (-2)$
10. $36 + 24$
11. $25 + 45$
12. $175 - 50$
13. $180 - 30$
14. $1200 + (-200)$
15. $1000 + 400$
16. $200 - (-150)$
17. $360 + 30$
18. $29 + (-47)$
19. $-34 + (-54)$
20. $-40 - 32$

Answer

1. 20
2. 7
3. 1
4. -3
5. 2
6. -4
7. 24
8. 3
9. 6
10. 60
11. 70
12. 125
13. 120
14. 1000
15. 1400
16. 350
17. 390

18. -18
19. -88
20. -72

Multiplying/Dividing Integers

Compute.

1. $3 \cdot 6$
2. $12 \cdot 2$
3. $5 \cdot (-3)$
4. $(-8)(4)$
5. $(-100) \cdot 33$
6. $(-9)(-9)$
7. $\frac{12}{4}$
8. $\frac{-28}{7}$
9. $\frac{36}{12}$
10. $\frac{-100}{-4}$
11. $\frac{26}{2}$
12. $\frac{40}{-5}$
13. $16 \div (-4)$
14. $-18 \div 3$

Answer

1. 18
2. 24
3. -15
4. -32
5. -3300
6. 81
7. 3
8. -4
9. 3
10. 25
11. 13
12. -8
13. -4
14. -6

Order of Operations

Compute.

1. $5 + 3 \cdot 2$
2. $\frac{6}{2} - 1$
3. $4(9 - 3) + 7$
4. $12(1 + 2) - 14$
5. $6 - 2(3 \cdot 9 - 20)$
6. $-4(15 - 3 \cdot 5)$
7. $3 \cdot \frac{1 - (2 + 3)}{2} + 5(-1)$
8. $(-1)\left(\frac{2(1 + 8)}{3} - 4\right)$

Answer

1. 11
2. 2
3. 31
4. 22
5. -8
6. 0
7. -11
8. -2

? Translation Challenge

Translate from instructions into math notation.

1. To cut out fabric with a seam allowance added, I must take the measurement 60 inches, divide it in half, and then add 2 inches. Write an expression to perform this calculation.
2. I'm trying to calculate my score in the game Whack-Bat. First, I get one score-down point because I hit the cedar stick. Also, there are five twig runners, and they each get two score-downs, so we need to add that total in as well. Finally, we must divide the total number of score-downs by nine. Write an expression to perform this calculation but do not evaluate.
3. Whenever I order cupcakes, they charge me a certain rate per cupcake needed, as well as a flat fee on top for the packaging. I need two dozen cupcakes, at $2 each, and the packaging fee is $5. Write an expression for the total cost of my order.
4. I buy six peaches at $2 each, but I forgot my shopping bag at home, so at the register they charge me another dollar. Write an expression for the total cost of my grocery store trip.

Answer

1. $\frac{60}{2} + 2 = 32$ inches.
2. $\frac{1+(5)(2)}{9}$.
3. $2 \cdot 24 + 5 = 53$ dollars.
4. $6 \cdot 2 + 1 = 13$ dollars.

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1.2 Working With Fractions

Learning Objectives

By the end of this section, you will be able to:

- break down a number into prime factors
- add, subtract, multiply, divide, take reciprocals, and simplify fractions

By fraction, we mean an expression that has some guy divided by some other guy. As long as that second guy isn't zero. We're never allowed to divide by zero. If I ever catch you dividing by zero, you're grounded for a month.

A **fraction** is written $\frac{a}{b}$, where $b \neq 0$. The top a is called the **numerator**, and the bottom b is called the **denominator.**

Clearly, working with fractions is intimately related to worrying about division. Sometimes a fraction is equivalent to a whole number, like how $\frac{12}{3} = 4$, in which case we say "12 **is divisible by** 3," or "3 divides evenly into 12." Sometimes it's not, like $\frac{2}{37}$, where we would say "2 is not divisible by 37 and everything is terrible." Sometimes a fraction will even have a negative sign in it, and we already learned what happens when negatives come into division. For example,

$$\frac{-6}{2} = -3, \quad \frac{6}{-2} = -3, \quad -\left(\frac{6}{2}\right) = -3$$

all turn out to be the same. In general...

For positive numbers a and b, we have

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}.$$

Another vocabulary word we will need is **reciprocal.** The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$. Just flip him over. We'll come back to that later.

We need to be able to do arithmetic and algebra on fractions, and there are rules about what is legal to do when computing with fractions. Before we get into those, however, let's take a quick detour that will help us later.

Factors of Numbers

When you **factor** a number or expression, you are splitting it into pieces (also called factors) that you can multiply together to get the original number. For example, $2 \cdot 12, 3 \cdot 8, 6 \cdot 4,$ and $2 \cdot 2 \cdot 2 \cdot 3$ are all ways to break 24 down into factors. In the last part there, I broke it down all the way into pieces that were **prime numbers** (a prime number is only divisible by 1 and itself, like 2, 3, 5, 7, 11, 13, ...). The ability to recognize what factors are "hiding" inside a particular integer will make your life much easier!

Example 1

Factor 48 all the way down to prime factors.

Solution

You can systematically break 48 into pieces and then break those pieces into pieces and then break those pieces...until we find:

$$48 = 2 \cdot 24 = 2 \cdot 2 \cdot 12 = 2 \cdot 2 \cdot 2 \cdot 6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

Exercise 1

1. Find two numbers that multiply to give -27.
2. Factor 51.
3. Find three pairs of numbers that multiply to give 36.
4. Factor 81 all the way down to prime factors.

Answer

1. $(-3)(9)$ or $3(-9)$.
2. This one is sneaky! It "feels prime," but actually $51 = 3 \cdot 17$. Did I get you? 51 is my favorite number.
3. $36 = 3 \cdot 12 = 9 \cdot 4 = 6 \cdot 6$ all work.
4. $81 = 9 \cdot 9 = 3 \cdot 3 \cdot 3 \cdot 3$.

Okay now we're ready to get to work...

Multiplying and Dividing Fractions

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \qquad \text{and} \qquad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Translation: To multiply two fractions, just multiply across the top and multiply across the bottom. That's it! To divide fractions, you leave the first as it is and **multiply** by the divisor's **reciprocal** instead. A fraction's reciprocal is its "multiplicative inverse," meaning if you multiply a fraction by its reciprocal, you will get 1.

These formulas also work backwards, such as $\frac{a \cdot c}{b \cdot d} = \frac{a}{b} \cdot \frac{c}{d}$, and now that we know about factoring, we can use this fact to learn how to **cancel.** Consider the fraction $\frac{35}{21}$. Using factoring, and the multiplication rule backwards, we can write

$$\frac{35}{21} = \frac{5 \cdot 7}{3 \cdot 7} = \frac{5}{3} \cdot \frac{7}{7} = \frac{5}{3} \cdot 1 = \frac{5}{3}.$$

Omg that's so much better! We call this fraction **fully reduced** because the numerator and denominator have no common factors left. Those 7s that 35 and 21 had in common came together to form a 1.

Caution

Cancelling is NOT a magic wand that you can wave and make numbers disappear. "I saw a 4 on a billboard once, so I'm gonna cancel this 4 right over here—" NO! Grounded. Straight to jail. You may NOT cancel in a fraction unless you can make the 1 appear as in the process above. I'm watching you.

Exercise 2

Simplify the fractions to fully reduced form.

1. $\frac{16}{8}$
2. $\frac{10}{15}$
3. $\frac{-42}{48}$
4. $\frac{26}{13}$
5. $\frac{50}{-125}$

Answer

1. 2
2. $\frac{2}{3}$
3. $-\frac{7}{8}$
4. 2
5. $-\frac{2}{5}$

Canceling as much as possible **first** can be very helpful before you do any real multiplying or dividing.

Example 2

Compute $\frac{12}{16} \cdot \frac{8}{18}$ and simplify.

Solution

I don't want to multiply $12 \cdot 8$ in my head... Let's clean it up first.

$$\frac{12}{16} \cdot \frac{8}{18} = \frac{3 \cdot 4}{4 \cdot 4} \cdot \frac{2 \cdot 4}{2 \cdot 9} = \frac{3}{4} \cdot \frac{4}{9} = \frac{3 \cdot 4}{4 \cdot 9} = \frac{3}{9} = \frac{1}{3}.$$

I've written out a lot of steps here to try to make clear what is happening, but we're going to start taking these skills for granted eventually, so you don't have to write this much detail if you don't need it!

Division involving fractions is more likely to appear in the form of **compound fractions**, or fractions within fractions, or Evil Condominiums of Fractions. They look like this:

$$\frac{4}{\frac{3}{2}}, \quad \frac{\frac{1}{2}}{3}, \quad \frac{\frac{2}{25}}{\frac{4}{3}}$$

To deal with these, remember it's just a big ol' division and **multiply the top of the big fraction by the reciprocal of the bottom.** Like this:

$$\frac{4}{\frac{3}{2}} = 4 \cdot \frac{2}{3} = \frac{4}{1} \cdot \frac{2}{3} = \frac{8}{3},$$

$$\frac{\frac{1}{2}}{-3} = -\frac{\frac{1}{2}}{\frac{3}{1}} = -\frac{1}{2} \cdot \frac{1}{3} = -\frac{1}{6},$$

$$\frac{\frac{2}{25}}{\frac{4}{3}} = \frac{2}{25} \cdot \frac{3}{4} = \frac{2 \cdot 3}{25 \cdot 2 \cdot 2} = \frac{3}{50}$$

Bring it all together with some exercises...

Exercise 3

Compute and simplify.

1. $\frac{3}{-4} \cdot \frac{-12}{27}$
2. $\frac{\frac{2}{5}}{10}$
3. $\frac{15}{30} \cdot \frac{9}{12}$
4. $\frac{\frac{1}{2}}{-\frac{3}{8}}$

Answer

1. $\frac{1}{3}$
2. $\frac{1}{25}$
3. $\frac{3}{8}$
4. $-\frac{4}{3}$

Adding and Subtracting Fractions

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \qquad \text{and} \qquad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

Translation: If you have the same denominator, you can combine the fractions by just adding or subtracting the numerators! These rules also work backwards, by the way. That is, you can split up a fraction over a plus in the numerator like so: $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$. **Watch out!** This only works when the plus or minus is in the **numerator**!

Great, I hear you saying, fine and dandy and all, but what if I don't have the same denominator? Then you must bake one from scratch! Let's demonstrate with an example.

Example 3

Add $\frac{4}{9} + \frac{2}{12}$.

Solution

The denominators currently don't match! Here's the process:

1. Factor the denominators into their prime factors: $9 = 3 \cdot 3$ and $12 = 2 \cdot 2 \cdot 3$. These two numbers share a factor 3, but the 9 has another 3, while the 12 has two 2s.
2. "Merge" the collections of factors into a set $3, 3, 2, 2$ that has all the factors they need. Notice that the number $2 \cdot 2 \cdot 3 \cdot 3 = 36$ is divisible by **both** 9 and 12. This is called the **least common multiple (LCM)** of 9 and 12. It's the smallest number that both 9 and 12 divide cleanly into, which we like because



3. Now that we have the LCM, we multiply top and bottom on each fraction by whatever is necessary to make the denominator match the LCM:

$$\frac{4 \cdot 4}{9 \cdot 4} + \frac{2 \cdot 3}{12 \cdot 3} = \frac{16}{36} + \frac{6}{36}$$

4. Now that the denominators do match, we just use the addition rule and simplify:

$$\frac{16}{36} + \frac{6}{36} = \frac{16+6}{36} = \frac{22}{36} = \frac{2 \cdot 11}{2 \cdot 18} = \frac{11}{18}.$$

Here's the summary and one more example, and then you can try some challenge exercises.

Getting a Common Denominator:

1. Determine the prime factors of each of the denominators involved.
2. Merge the collections of prime factors to find the LCM.
3. Make each denominator match the LCM by multiplying the fraction top and bottom by whatever is needed.
4. Now that there is a common denominator, add or subtract as usual.

Example 4

Compute $\frac{1}{3} \cdot \frac{9}{4} - \frac{5}{12} + \frac{3}{\frac{2}{7}}$ and simplify.

Solution

It looks scary, but we just have to start somewhere! Let's do that first term's multiplication and simplify.

$$\frac{1}{3} \cdot \frac{9}{4} - \frac{5}{12} + \frac{3}{\frac{2}{7}} = \frac{1 \cdot 3 \cdot 3}{3 \cdot 4} - \frac{5}{12} + \frac{3}{\frac{2}{7}} = \frac{3}{4} - \frac{5}{12} + \frac{3}{\frac{2}{7}}$$

7 7 7

Now let's clean up that compound fraction too.

$$= \frac{3}{4} - \frac{5}{12} + 3 \cdot \frac{7}{2} = \frac{3}{4} - \frac{5}{12} + \frac{21}{2}$$

We need a common denominator now, and the LCM of 4, 12, and 2 is just 12. We make everybody match...

$$= \frac{3 \cdot 3}{4 \cdot 3} - \frac{5}{12} + \frac{21 \cdot 6}{2 \cdot 6} = \frac{9}{12} - \frac{5}{12} + \frac{126}{12}$$

And finally do the addition and subtraction!

$$= \frac{9 - 5 + 126}{12} = \frac{130}{12} = \frac{2 \cdot 65}{2 \cdot 6} = \frac{65}{6}$$

? Exercise 4

Compute and simplify.

1. $\frac{-5}{15} \cdot \frac{3}{2} - \frac{1}{2}$
2. $\frac{\frac{6}{4}}{\frac{3}{12}} + 4 \cdot \frac{9}{16}$
3. $\frac{\frac{2}{3} + \frac{1}{6}}{4}$
4. $\frac{1}{2} + \frac{1}{3} - \frac{1}{5}$

Answer

1. -1. Be careful with your negative signs: $-\frac{1}{2} - \frac{1}{2} = \frac{-1-1}{2} = \ldots$
2. $\frac{33}{4}$. If you got stuck with what to do with the 6, remember that $6 = \frac{6}{1}$ and you can get a common denominator.
3. $\frac{5}{24}$. Clean up the numerator first and then worry about the big division downstairs.
4. $\frac{19}{30}$.

Fraction Formats

Let's wrap up this section with a note on ways you can write fractions. When the numerator is a larger number than the denominator, such as $\frac{7}{2}$, it's called an **improper fraction.** This is rude, because this format is generally the easiest to work with if you're going along doing math problems. I pretty much always leave fractions in this form.

You may be familiar with **mixed numbers,** which is a way of writing an improper fraction by figuring out how many "wholes" are included and pulling them out front. Like this: $\frac{7}{2} = \frac{6+1}{2} = \frac{6}{2} + \frac{1}{2} = 3\frac{1}{2}$. I have "seven halves," but "six halves" is just a whole three, and there's one extra "half" left over. You see these written with a whole number squashed up next to a small fraction, and that displeases me philosophically since squashing things next to each other often means multiplication. I won't use this notation, BUT it's still a good idea to get your head around it, because it will be very helpful when a) working with decimals in about five seconds here and b) estimating numbers with our brains later.

? Exercise 5

Find the mixed number equivalent of improper fractions, and find the improper fraction equivalent of mixed numbers.

1. $\frac{3}{2}$
2. $\frac{4}{3}$
3. $\frac{11}{2}$
4. $1\frac{1}{4}$
5. $2\frac{1}{3}$

6. $5\frac{3}{4}$

Answer

1. $1\frac{1}{2}$
2. $\frac{3+1}{3} = \frac{3}{3} + \frac{1}{3} = 1\frac{1}{3}$
3. $5\frac{1}{2}$
4. $1\frac{1}{4} = 1 + \frac{1}{4} = \frac{4}{4} + \frac{1}{4} = \frac{5}{4}$
5. $2 + \frac{1}{3} = \frac{6}{3} + \frac{1}{3} = \frac{7}{3}$
6. $\frac{23}{4}$

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1.2E Exercises

These exercises will combine everything you've learned about working with fractions, which means finding common denominators, adding and subtracting, multiplying and dividing, cancelling and simplifying, and in some cases remembering how Order of Operations works!

Simplifying and Cancelling

Write the fraction in simplest form (meaning the numerator and denominator have no common factors).

1. $\frac{8}{4}$
2. $\frac{6}{2}$
3. $\frac{2}{16}$
4. $\frac{17}{34}$
5. $\frac{25}{30}$
6. $\frac{42}{35}$
7. $\frac{-8}{40}$
8. $-\frac{36}{12}$

Answer

1. 2
2. 3
3. $\frac{1}{8}$
4. $\frac{1}{2}$
5. $\frac{5}{6}$
6. $\frac{6}{5}$
7. $-\frac{1}{5}$
8. -3

Compute and Simplify

Simplify:

1. $\frac{3}{4}+\frac{1}{2}$
2. $\frac{5}{6}-\frac{1}{3}$
3. $\frac{2}{5}\cdot\frac{3}{4}$
4. $\frac{\frac{3}{8}}{\frac{1}{2}}$
5. $\frac{2}{3}+\frac{4}{9}-\frac{1}{6}$
6. $\frac{\frac{5}{12}}{\frac{2}{3}}\cdot\frac{4}{5}$
7. $\frac{3}{4}-\frac{1}{4}\cdot\frac{2}{3}$
8. $\frac{\frac{5}{6}}{\frac{2}{5}}+\frac{1}{3}$
9. $\frac{3}{5}\cdot\frac{4}{7}\cdot\frac{2}{3}$
10. $\frac{2}{3}-\frac{5}{9}\cdot\frac{3}{4}$
11. $\frac{7}{-5}-\frac{-5}{4}$

12. $-\frac{7}{2}+\frac{6}{-7}$

Answer

1. $\frac{5}{4}$
2. $\frac{1}{2}$
3. $\frac{3}{10}$
4. $\frac{3}{4}$
5. $\frac{17}{18}$
6. $\frac{1}{2}$
7. $\frac{7}{12}$
8. $\frac{29}{12}$
9. $\frac{8}{35}$
10. $\frac{1}{4}$
11. $-\frac{3}{20}$
12. $-\frac{61}{14}$

Compound Fractions

Simplify:

1. $\frac{\frac{3}{4}}{\frac{1}{2}}$
2. $\frac{\frac{5}{6}+\frac{1}{3}}{\frac{2}{3}}$
3. $\frac{\frac{2}{5}\cdot\frac{3}{4}}{\frac{3}{8}}$
4. $\frac{\frac{1}{2}}{\frac{3}{4}-\frac{1}{3}}$
5. $\frac{\frac{3}{4}-\frac{1}{4}}{\frac{2}{3}}\cdot\frac{5}{6}$
6. $\frac{\frac{5}{6}}{\frac{2}{5}\cdot\frac{1}{3}}$
7. $\frac{\frac{1}{3}+\frac{4}{5}\cdot\frac{10}{3}}{2}$

Answer

1. $\frac{3}{2}$
2. $\frac{7}{4}$
3. $\frac{4}{5}$
4. $\frac{6}{5}$
5. $\frac{5}{16}$
6. $\frac{25}{4}$
7. $\frac{3}{2}$

Mixed Numbers and Improper Fractions

1. Convert $\frac{5}{2}$ to a mixed number.
2. Convert $3\frac{1}{4}$ to an improper fraction.

3. Convert $6\frac{3}{5}$ to an improper fraction.

4. Convert $\frac{11}{3}$ to a mixed number.

5. Convert $4\frac{2}{7}$ to an improper fraction.

6. Convert $\frac{7}{2}$ to a mixed number.

7. Convert $2\frac{5}{8}$ to an improper fraction.

8. Convert $\frac{17}{4}$ to a mixed number.

Answer

1. $2\frac{1}{2}$
2. $\frac{13}{4}$
3. $\frac{33}{5}$
4. $3\frac{2}{3}$
5. $\frac{30}{7}$
6. $3\frac{1}{2}$
7. $\frac{21}{8}$
8. $4\frac{1}{4}$

? Investigation

1. Remember I said you can only split up a fraction like $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ because the addition is in the numerator? Simplify the expression $\frac{a}{b} + \frac{a}{c}$ and see what you get. Conclude that $\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$.

2. We have to be careful with negative signs when working with messier fractions. For example, this statement is **NOT TRUE:** $\frac{a}{d} - \frac{b+c}{d} \stackrel{?}{=} \frac{a-b+c}{d}$. What is the correct result of $\frac{a}{d} - \frac{b+c}{d}$? Try choosing some numbers for $a, b, c,$ and d to get the juices flowing or confirm...

Answer

1. Hint: you need to get a common denominator, and since you don't know what b and c are, just use bc for that.

2. Let's choose $a = 2, b = 1, c = 3,$ and $d = 4$ and see what happens. The left hand side is $\frac{a}{d} - \frac{b+c}{d} = \frac{2}{4} - \frac{1+3}{4} = \frac{1}{2} - \frac{4}{4} = -\frac{1}{2}$, but the hypothesized right hand side would be $\frac{2-1+3}{4} = \frac{4}{4} = 1.$ Not the same! Try rewriting

$$\frac{a}{d} - \frac{b+c}{d} = \frac{a}{d} - \frac{(b+c)}{d}$$

and proceeding from there.

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1.3 Working With Decimals

Learning Objectives

By the end of this section, you will be able to:

- interpret decimal notation and round decimals correctly
- add, subtract, and multiply decimals
- convert decimals to and from common fractions
- use decimals to calculate percents

Decimals are just another way we can express real numbers that are in between the whole numbers, and in fact they are basically shorthand for fractions whose denominators are powers of 10. See what I mean below...

Decimal	Fraction Equivalent	In Words
0.1	$\frac{1}{10}$	"one-tenth"
0.01	$\frac{1}{100}$	"one-hundredth"
0.001	$\frac{1}{1000}$	"one-thousandth"

When we learn about whole numbers, we talk about **place values** in terms of powers of 10 as well. For a number like 1,357 we know that the 5 actually represents 5 *tens*, the 3 actually represents 3 *hundreds*, etc. Here's an example:

...	ten-thousands	thousands	hundreds	tens	ones	. POINT	tenths	hundredths	thousandths	ten-thousandths	...
	3	1	4	2	6	.	2	8	7	9	

So you see we can think of the number in the image as listing out a bunch of components,

$$\begin{aligned} &30000 + 1000 + 400 + 20 + 6 + \frac{2}{10} + \frac{8}{100} + \frac{7}{1000} + \frac{9}{10000} \\ &= 31426 + 0.2 + 0.08 + 0.007 + 0.0009 = 31426.2879 \end{aligned}$$

which can all be written in a single decimal representation of the value.

When I calculate final course grades for students, I generally **round** each number to the nearest integer, meaning if you got an 87.6, I just consider it an 88. In that case, I'm rounding to the ones place, but you can round to any place value you want. For example, when performing calculations about money, we round to two decimal places, because we only work with dollars and cents.

Rounding to a desired place value: If you want to round to the n^{th} place value, look at the digit directly to the **right** of that position. If it is **less than** 5, keep the n^{th} place digit as is and rewrite the number, deleting all digits that come after. If it is **greater than or equal to** 5, add 1 to the n^{th} place digit before deleting all digits that come after.

Exercise 1

Round 201.765832904 to the...

1. nearest integer
2. nearest hundred
3. nearest tenth
4. nearest hundredth

5. nearest ten-thousandth (four decimal places)
6. sixth decimal place

Answer

1. Find the ones digit (it's 1) and look to the right. We see a 7 next to it, which is bigger than 5, so we round up and give the answer as 202.
2. Look at the hundreds digit (it's 2) and look to the right. We see a 0, which is less than 5, so we keep the 2 and write 200.
3. Look at the tenths place (it's a 7). To the right, there is a 6, which is bigger than 5, so we round up to 201.8 as our answer.
4. The hundredths place is the 6, and next to it we see a 5, so we round up and give 201.77 as our answer.
5. The fourth decimal place is the 8, and next to it we see a 3, so we give 201.7658 as our answer.
6. The sixth decimal place is the 2, and next to it is a 9, so we round up to 201.765833 as our answer.

Decimals are great, because in real life problems, the numbers don't generally stay cute and clean and whole. Decimals also, in my opinion, are way better to work with than mixed numbers, while giving a similar sort of intuition about the size of a quantity. However, I don't generally expect students to work with decimals much without using a calculator. I want you to know how to do reasonably simple arithmetic in your head, so we'll go over it briefly, but in your science classes you'll use technology to perform calculations with decimals.

Arithmetic With Decimals

To **add or subtract** two decimals,

1. Determine the sign of the answer (positive or negative),
2. Write them vertically with the decimal points aligned, using placeholder 0s if necessary, then...
3. Add or subtract down columns as usual.

For example, to add $21.5 + 0.25$, line them up by tacking a 0 on the end of 21.5 (this doesn't change its value as a number), and add.

$$\begin{array}{r} 21.50 \\ +\ 0.25 \\ \hline 21.75 \end{array}$$

To subtract $17.503 - 45.001$, we will have a negative answer since we are subtracting a larger number.

$$\begin{array}{r} 17.503 \\ -\ 45.001 \\ \hline -32.502 \end{array}$$

To **multiply** two decimals,

1. Determine the sign of the answer (positive or negative),
2. Multiply them as if they were whole numbers, ignoring the decimal points for the moment, then...
3. The new decimal point position is the sum of the decimal places in the two factors.

This sounds weird, but all I want you to take away from this is common sense like the following examples:

Example 1

Multiply.

1. $2(3.01)$
2. $0.5(0.5)$

3. $1.5(30)$

Solution

1. I just want to double 3.01. Intuitively, I double the whole part ($3 \to 6$) and double the decimal part ($0.01 \to 0.02$), and then add them back together to get 6.02. I could also ignore the decimal point and say $2 \cdot 301 = 602$, and then since there are two decimal places in 3.01 and none in 2, I write the point two places in to get the 6.02.

2. Ignoring the decimal points, $5 \cdot 5 = 25$. Both decimals have a single decimal place, so the new point goes a total of two decimal places in to yield 0.25. Or you can realize that $0.5 = \frac{1}{2}$, so I'm looking for "half of one-half," which we know is $\frac{1}{4} = 0.25$.

3. I can think of this as wanting "one-and-a-half of a 30." So $30 + 15 = 45$ is the right answer. Or, I can ignore the decimal point, multiply $15 \cdot 30 = 450$, and then put the new point one place in to get 45.0.

There is also a special trick you can use when multiplying or dividing by a power of 10 that you should be aware of.

When **multiplying** a decimal by a **power of 10**, 10^k for some whole number k, the result will be the original decimal with the point moved k times to the **right,** with zeros inserted in as needed!

Example 2

Compute.

1. $(1.23)(10)$
2. $(1.23)(100)$
3. $(1.23)(1000)$

Solution

1. We multiply $123 \cdot 10 = 1230$ and see a total of two decimal places from the factor (1.23), so the final answer is $12.30 = 12.3$. We could have just moved the decimal point to the right by one jump!

2. We get 123.00, which is the same as if we moved the point two places right to get 123. The factor (100) is 10^2 so two jumps makes sense!

3. The factor (1000) is 10^3, so we move the decimal place three jumps to the right, filling in with a 0 after the 3 to get 1230. The long method will give you 1230.00 which is the same!

When **dividing** a decimal by a 10^k, the result will be the same number with the decimal point moved k times to the **left,** with zeros inserted in as needed.

Example 3

Compute.

1. $\frac{1.23}{10}$
2. $\frac{1.23}{100}$
3. $\frac{1.23}{1000}$

Solution

1. You don't really have to know anything about dividing here actually, if you remember that $0.1 = \frac{1}{10}$. That means $\frac{1.23}{10} = 1.23 \cdot \frac{1}{10} = (1.23)(0.1)$ can be seen as a multiplication. In that case, we notice there are two decimal places from the first factor and one from the second, for a total of three. We take the product $123 \cdot 1 = 123$ and put the point three places in to get 0.123, which is the same as if we moved the point one position to the left from the beginning! Let's do that instead for the other two, now that we've convinced ourselves the trick works.

2. Again, $100 = 10^2$, so we move the point two places to the left this time, to get 0.0123.

3. Here we need to move the point three places to the left, to get 0.00123 as the answer.

Converting Between Fractions and Decimals

For your health and happiness in daily life and future math and science courses, you should have a grasp of the decimal versions of commonly used fractions. Your average welder (I'm told by my brother, who builds rocket pads) knows all of his powers-of-2 fractions by heart: $\frac{1}{2} = 0.5$, $\frac{1}{4} = 0.25$, $\frac{1}{8} = 0.125$, etc. So have a gander at the list below, and then try to answer the exercise without looking!

Fraction	Decimal
$\frac{1}{2}$	0.5
$\frac{1}{3}$	$0.\overline{3}$, which means the 3s go on forever like $0.33333...$
$\frac{2}{3}$	$0.\overline{6}$, which due to rounding rules rounds to things like 0.6667
$\frac{1}{4}$	0.25, like how 25 cents is a quarter of a dollar
$\frac{3}{4}$	0.75
$\frac{1}{5}$	0.2, which you can remember as 0.20 because 20 is one-fifth of a 100
$\frac{1}{8}$	0.125
$\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$	$1.5, 2.5, 3.5, \ldots$, which are just like finding mixed numbers from improper fractions

Exercise 2

Translate from decimal to fraction or vice versa, as appropriate.

1. $1.5 =$
2. $\frac{7}{2} =$
3. $0.75 =$
4. $\frac{2}{100}$
5. $0.2 =$
6. $0.4 =$
7. $\frac{1}{3}$

Answer

1. $\frac{3}{2}$
2. 3.5
3. $\frac{3}{4}$
4. 0.02 because that's "two one-hundredths" if you say it in words.
5. $\frac{1}{5}$ or if you forget, think "two tenths," which is the fractions $\frac{2}{10}$, and then simplify!
6. $\frac{2}{5}$ I know, this one isn't on the list, but notice that it's just two copies of 0.2 and proceed...
7. $0.3333... = 0.\overline{3}$

While we're here, let's expand a bit on that "0.3 repeating" situation for $\frac{1}{3}$. It turns out that some rational numbers have a decimal version that goes on forever, but it follows a pattern. Some examples:

$$\frac{1}{3} = 0.333333..., \quad \frac{1}{6} = 0.16666666..., \quad \frac{1}{7} = 0.142857142857142857\ldots, \quad \frac{1}{11} = 0.09090909...$$

This fun fact works the other direction too, meaning that if you have a repeating decimal, it can be written as a ratio of two integers! If you want to learn how to translate back and forth, check out the end of the exercises section.

If a decimal never stops and there is no repeating pattern to its digits continuing forever, that's an irrational number.

Decimals and Percents

You paid for a math course, but you're getting some Latin today for free. The ending "-cent" comes from the Latin *centum*, which means a hundred, so the word "percent" means "out of a hundred." If I say 85% of my students like my jokes, that means out of every hundred students I teach, 85 of them like my jokes. This proportion can also be written as a fraction, $\frac{85}{100}$, or its equivalent decimal, 0.85. Whenever you want to compute a percentage, you will want to convert it to a decimal by following this pattern and moving the decimal point two places to the **left**. (Where was the decimal point in 85? It was hiding, right there in 85.0...)

Here are the basic types of percentage computation you might be expected to know how to do. You would probably use a calculator for this type of problem.

Example 4

I had 32 students last semester, and I know 85% of my students like my jokes. How many students liked my jokes last semester?

Solution

I need to answer the question, "What is 85% of 32?" We already know how to translate "85%" to math, and the word "of" is usually code for multiplication. I calculate $0.85 \cdot 32 = 27.2$ with my calculator. I can't have 0.2 of a student (without a mess, anyway) so I round down to 27 students.

Example 5

I took a poll of my 32 students, and actually only 22 thought my jokes were funny. :(What percentage of my students like my jokes?

Solution

I know that $\frac{22}{32}$ of my students liked my jokes. My calculator says the decimal equivalent of that is 0.6875. To find the percent, I just move the decimal point two places to the **right** to get 68.75%. Yikes.

Okay, that's the gist on what you should be able to do with decimals! Take a stab at the exercises in the next section.

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1.3E Exercises

Place Value & Rounding

Consider the number 2024.05041339 and answer the following questions. (I made this number up because I wrote this question at 13:39 on 05/04/2024 lol.)

1. What digit is in the tens place?
2. What digit is in the hundreds place?
3. What digit is in the tenths place?
4. What digit is in the hundredths place?
5. What digit is in the thousandths place?
6. Round to the nearest integer.
7. Round to the nearest tenth (round to one decimal place).
8. Round to the nearest hundredth (two decimal places).
9. Round to five decimal places.
10. Round to seven decimal places.

Answer

1. 2
2. 0
3. 0
4. 5
5. 0
6. 2024
7. 2024.1
8. 2024.05
9. 2024.05041
10. 2024.0504134

Decimal Arithmetic

Compute the following using mental math for practice.

1. $3(0.\overline{3})$
2. $0.25 \cdot 4$
3. $0.25 + 0.5$
4. $0.2 + 0.3$
5. $1 - 0.5$
6. $0.5 - 1$
7. $2(0.4)$
8. $3(0.1)$
9. $2(0.6)$
10. $2(0.9)$
11. $2.4(100)$
12. $3.35 \cdot 1000$
13. $\frac{2.4}{100}$
14. $\frac{14.8}{10}$

Answer

1. That's one third times three, so 1.
2. 1
3. 0.75

4. 0.5
5. 0.5
6. -0.5
7. 0.8
8. 0.3
9. 1.2 (12 with the point moved one place in)
10. 1.8
11. 240
12. 3350
13. 0.024
14. 1.48

Converting Between Decimals & Fractions

1. Write $\frac{1}{3}$ as a decimal. Then round to four decimal places.
2. Write $\frac{2}{3}$ as a decimal. Then round to four decimal places.
3. Write 2.5 as a fraction.
4. Write -1.75 as a fraction
5. Write the mixed number $3\frac{1}{5}$ as a decimal.
6. Write the mixed number $4\frac{1}{2}$
7. Write -0.125 as a fraction.
8. Write 0.42 as a fraction.

Answer

1. $0.33333... \approx 0.3333$
2. $0.66666... \approx 0.6667$
3. That's 2 and a half, so $2 + \frac{1}{2} = \frac{4}{2} + \frac{1}{2} = \frac{5}{2}$
4. $-\frac{7}{4}$
5. 3.2
6. 4.5
7. $-\frac{1}{8}$
8. $\frac{42}{100} = \frac{21}{50}$

Percents

1. Convert 75% to a decimal.
2. Convert 110% to a decimal.
3. What is 50% of 24?
4. What is 10% of 300?

Answer

1. 0.75
2. 1.10
3. 12
4. 30

Calculating Tip

You should tip 20% and your bill is $12.50, and your phone is dead so you can't get a calculator out. Luckily, you know you can find 10% of 15.50 in your head and then just double it! What is the tip amount? What is the bill total with tip?

Answer

The tip is $2.50, so your total bill will be $15.

? (Challenge) Finding a Rational Number From Repeating Decimal

Let's figure out how to write $0.123123123123... = 0.\overline{123}$ as a fraction $\frac{m}{n}$, where m and n are integers.

1. First nickname the decimal as $x = 0.123123123123...$ and notice that its repeating pattern is three digits long.
2. Since the pattern is three digits long, we will multiply x by the third power of 10, so compute $1000x$ using the decimal moving trick.
3. Now subtract $1000x - x = ?$.
4. We'll see more about factoring later, but $1000x - x = (1000 - 1)x = 999x$. Using this fact, we can write $x = \frac{?}{999}$. That's a rational number, technically!

Answer

1. Duly noted.
2. $1000x = 1000(0.123123123...) = 123.123123123...$
3. $1000x - x = 123.123123123... - 0.123123123...$ calculated the vertical way looks like this:
4. So $999x = 123$ and dividing both sides by 999 gives $x = \frac{123}{999}$, a ratio of two integers!

$$\begin{array}{r} 123.123123... \\ -\ 0.123123... \\ \hline 123.000000... \end{array}$$

? Gas Station Games

This is a true story. The other day, I was putting gas in my car, and I managed to stop the pump at *exactly* 5.00 gallons and *exactly* $18.00 total cost. Find (a) the price per gallon at that particular Stewart's on that particular day.

It made me curious about what other prices lead to nice whole number gallons and cost possibilities... (b) Investigate this idea and see if you can figure out what integer number of gallons I would need to get to have my total cost come out to an integer, if the price is $2.75 /gal. And (c) is it possible for me to win this "game" if the price per gallon is $3.239? I can only get up to 9 whole gallons of gas at a time, because my car has a 10-gallon tank.

Answer

(a) We can find the price per gallon by just dividing the total cost by the number of gallons I got. That is, $\frac{\$18}{5}$ is 3 and $\frac{3}{5}$. I can write that as a decimal as $3.60 per gallon. (Gas stations try to make you feel better by saying the price is $3.599, but it's all money so it's rounded to two decimal places anyway, lol.)

(b) We just computed the price by dividing the gallons by the cost. In letters, we could nickname price p, number of gallons n, and total cost m. The whole point of the game is to have $p = \frac{m}{n}$ with the m and n *integers*. That means we're just looking for the *rational number* p. Or, we're looking for a way to write the price (given to us in a decimal) as rational number so we can see the integer top and integer bottom! If the price is $2.75 per gallon, we convert from two-and-three-fourths to $\frac{11}{4}$. The top integer is m, the total cost, and the bottom integer is n, the total gallons. So at the price $2.75, I can get exactly 4.00 gallons for exactly $11.00.

(c) I don't know what fraction 3.239 or even 3.24 is equivalent to off the top of my head. You can always figure it out, though, by writing

$$3.24 = 3 + \frac{24}{100} = \frac{324}{100}$$

and trying to cancel/simplify that fraction to some nicer integers. I first say, 4 goes into both the top and bottom, so I cancel that away.

$$\frac{324}{100} = \frac{4 \cdot 81}{4 \cdot 25} = \frac{81}{25}$$

At this point, there's no more cancelling I can do, as the top and bottom have no more factors in common. So my only option for whole number gallons and cost would be to buy 25.00 gallons for $81.00, but I can't put that much in my car, so if the price is $3.239 I can't win the game. :(

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1.4 Powers and Radicals

Learning Objectives

By the end of this section, you will be able to:

- compute and simplify expressions with powers in them
- compute and simplify expressions with radicals in them, and relate them to powers
- deal with negative numbers in these contexts

Working With Powers (Exponents)

Powers (also called **exponents**) are when we write a lil guy upstairs to indicate that we're multiplying a number by itself multiple times. They look like this:

$$10^2 = 10 \cdot 10 = 100, \quad 2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32, \quad 5^3 = 5 \cdot 5 \cdot 5 = 125,$$

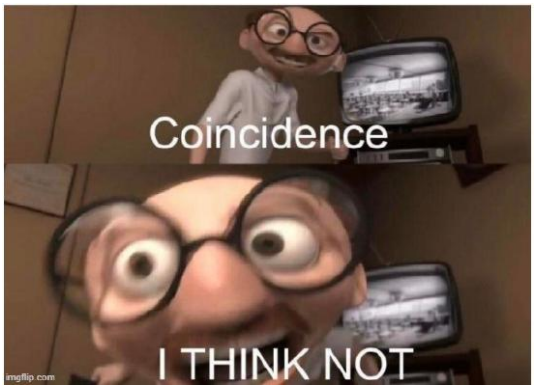
In words, we say 10^2 as "ten squared" or "ten to the power of two" or if you're lazy like me, maybe "ten to the two." You'll hear 5^3 said as "five cubed," but for numbers bigger than 3, we just say "to the power of" whatever. The downstairs number is called the **base**, and the upstairs number is called the **exponent**. In all those examples (for this "repeated multiplication" interpretation of a power) the exponent was a positive integer. It's also possible to have negative powers, as well as fractional powers. In fact, any real number can be used as a power, even irrational numbers, but we won't see those much at this point. First, let's explore powers using a nice security blanket plain 'ol number. What's the most comforting number? Two? I agree. (Plus if you're in computer science, you have to memorize these anyway.) Let's raise him to some powers.

2 to the power of...	Result
1	$2^1 = 2$
2	$2^2 = 2 \cdot 2 = 4$
3	$2^3 = 2 \cdot 2 \cdot 2 = 8$
4	$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$
5	$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$
k, a positive integer	$2^k = 2 \cdot 2 \cdot \ldots \cdot 2$ (k times)

Okay, not too bad. I noticed something interesting up there while I was calculating the results. I didn't have to do all the multiplications every time, I could just look at the previous answer and multiply that by 2 again. So I observe that $2^k = 2^{k-1} \cdot 2$. Cool. What happens if I multiply two powers of 2 together?

$$2^2 \cdot 2^3 = (2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$$

Since it doesn't matter what order I multiply numbers in, I didn't need those parentheses at all! What do you observe? The new exponent turned out to be 5, which just so happens to be 2+3.



What happens if I divide two powers of 2?

$$\frac{2^3}{2^2} = \frac{2 \cdot 2 \cdot 2}{2 \cdot 2} = \frac{2 \cdot 2}{2 \cdot 2} \cdot \frac{2}{1} = 1 \cdot 2 = 2$$

What's the new power? Remember that $2 = 2^1$ and notice that $1 = 3 - 2$... Iiiinterestinggggggg...

Finally, let's take a power of 2 and raise that whole thing to a power.

$$(2^3)^2 = (2^3)(2^3) = (2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2) = 2^6$$

The new power is 6, which just so happens to be the product of 2 and 3. Guess what? Ain't nothing special about the choice of base 2.

Rules for Exponents

Let $a, b,$ and c be real numbers. Then we have

$$a^b a^c = a^{b+c} \quad \text{and} \quad \frac{a^b}{a^c} = a^{b-c} \quad \text{and} \quad (a^b)^c = a^{bc}.$$

ALERT: The base a must be the same for this to work! Also, you must subtract the bottom power from the top power—order matters!

In caveman speak, "Multiplication in the base translates to addition in the exponent. Division in the base translates to subtraction in the exponent. Power to a power translates to multiplication in the exponent." You must memorize these rules!

Maybe you already noticed that 2^1 just turned out to be itself, based on the multiplication idea. That works for any base. Another special case is that any number raised to the power of zero is defined to be 1. Sound like black magic? Consider $\frac{a^b}{a^b} = a^{b-b} = a^0$ and look back at the beginning fraction. Whoop, it was the same on top and bottom. That's just 1!

Special Powers

- Let a be any real number. Then $a^0 = 1$ and $a^1 = a$.
- For any power n, we have $0^n = 0$ and $1^n = 1$.

Okay so we're feeling good about positive integers in exponents. What about negative integers? Well, we saw subtraction appear when we divided $\frac{2^3}{2^2} = 2^{3-2}$. If we wrote $2^{3-2} = 2^{3+(-2)} = 2^3 \cdot 2^{-2}$, it becomes clear that using a negative exponent means dividing by 2 a certain number of times! We can write division as a negative exponent, and we can write negative exponents as division, and as a consequence, we can move factors between numerators and denominators just by changing the sign of the exponent.

$$2^{-2} = \frac{1}{2^2}, \qquad \frac{3}{2^2} = 3 \cdot 2^{-2}, \qquad \frac{2^3}{3^{-2}} = 2^3 3^2$$

Negative Powers

Let a and b be real numbers. Then $a^{-b} = \frac{1}{a^b}$. Note that this also means $\frac{1}{a^{-b}} = a^b$.

Let's shake things up and change some bases instead of exponents. Check it out:

$$3^2 \cdot 2^2 = 3 \cdot 3 \cdot 2 \cdot 2 = 3 \cdot 2 \cdot 3 \cdot 2 = 6 \cdot 6 = 6^2.$$

Same power on different bases being multiplied? Turns out you can multiply the bases and then raise it to the power! And look, it works backwards!

$$(2 \cdot 3)^2 = (2 \cdot 3)(2 \cdot 3) = 2^2 3^2$$

Power of a Product

Let $a, b,$ and c be real numbers. Then $(ab)^c = a^c b^c$. **ALERT:** The power c must be the same to use this rule backwards.

Caution

You **cannot** pass an exponent into a set of parentheses with a plus or minus! Aka, $(a+b)^c \neq a^c + b^c$. This is illegal!!!! Do not do it.

Now let's confront the elephant in the room: negatives. Consider $(-2)^2 = (-2)(-2) = 4$. Squaring a negative number means multiplying a negative by a negative, which results in a positive number. Meanwhile, $(-2)^3 = (-2)(-2)(-2) = (4)(-2) = -8$ stays negative. **Raising a negative number to an even power gives a positive result, and raising a negative number to an odd power gives a negative result.**

Parentheses are extremely important here. Note that $-5^2 = -(5 \cdot 5) = -25$ while $(-5)^2 = (-5)(-5) = 25$.

Let a be a positive base.

- When n is even, $(-a)^n = a^n$.
- When n is odd, $(-a)^n = -a^n$.

Let's bring it all together by doing some examples, and then you will try an exercise. We're going to start using some letters instead of numbers, so don't panic! Just push them around according to the rules and pretend they're your favorite numbers if it makes you feel better for now.

Example 1

Simplify the expressions.

1. $\frac{(-5)^3}{5^4}$
2. $2^2 3^3 2^{-1} (2 \cdot 3)^6$
3. $((-a)^2 b^3)^4$
4. $\frac{x^2 y^{-3}}{x^3 y}$

Solution

1. $\frac{(-5)^3}{5^4} = \frac{-5^3}{5^4} = -\frac{5^3}{5^4} = -5^{3-4} = -5^{-1} = -\frac{1}{5^1} = -\frac{1}{5}$. You do not need to write all these steps out once these simplifications become second nature. I'm just trying to emphasize which rules are being used at each step.

2. I'm going to highlight in red the pieces I'm working on from step to step. Notice how these bases match: $2^2 3^3 2^{-1} (2 \cdot 3)^6$. Bring them together to get $2^{2-1} 3^3 (2 \cdot 3)^6 = 2^1 3^3 (2 \cdot 3)^6$. Let's use the power of a product rule next: $2^1 3^3 (2 \cdot 3)^6 = 2^1 3^3 2^6 3^6$. Now

we have a lot of matching bases we can combine by adding the exponents. Our final answer is $2^7 3^9$. I won't ask you to calculate that by hand, it's a huge number. This is as clean as it gets!

3. Uh oh, letters! It's okay, we still follow the same rules. First note that $(-a)^2 = a^2$, so we actually have $(a^2 b^3)^4$. Then by the power of a product rule, we rewrite: $(a^2)^4 (b^3)^4$. Then by the power to a power rule, we have: $a^{2(4)} b^{3(4)} = a^8 b^{12}$.

4. We move everybody to the numerator by changing the sign of the exponent, and rewrite: $\frac{x^2 y^{-3}}{x^3 y} = x^2 y^{-3} x^{-3} y^{-1} = x^{2-3} y^{-3-1} = x^{-1} y^{-4} = \frac{1}{xy^4}$.

? Exercise 1

Simplify.

1. $(2^3 \cdot (-2)^5)$
2. $\frac{x^6}{(-x)^2}$
3. $(3^4)^2$
4. $\frac{-2^4 \cdot 3^2}{2^2 \cdot 3^3}$
5. $\frac{x^3 y^{-3}}{x^5 y^2}$
6. $\left(\frac{a^5}{a^2}\right)^3 \left(\frac{b^4}{b^3}\right)^2$

Answer

1. -2^8
2. x^4
3. 3^8
4. $-\frac{2^2}{3} = \frac{4}{3}$
5. $\frac{x^3}{y^5}$ or $x^3 y^{-5}$ is fine
6. $a^9 b^2$

Working With Radicals (Roots)

Here's a question... If there was such thing as a fractional power, say like $4^{\frac{1}{2}}$, the power rules would imply that

$$(4^{\frac{1}{2}})^2 = 4^{\frac{1}{2} \cdot 2} = 4^1 = 4.$$

In other words, $4^{\frac{1}{2}}$ should be a number that, when squared, results in 4. Who do we know who works like that? *(Dora the Explorer pause here.)*

Did you say 2? Nice. Did you say -2 too? You iconoclast! We say that 2 and -2 are **square roots** of 4. You will have seen this written $\sqrt{4} = \pm 2$, and now we know that the root symbol can be written using a fractional power, $\sqrt{4} = 4^{\frac{1}{2}}$. When you're given the **radical** symbol $\sqrt{\ }$ in a problem, by convention it is defined to report the **positive** square root (the **principal root**).

If $a^2 = b$, then a is a **square root of** b. For $a \geq 0$, we write $\sqrt{b} = a$.

You can ask the same question about $8^{\frac{1}{3}}$, whose answer would be 2 since $2^3 = 8$, and we write $\sqrt[3]{8} = 2$, saying "2 is the cube root of 8."

If $a^n = b$, then a is an **n$^{\text{th}}$ root of** b. The **principal** n$^{\text{th}}$ root of b is written $\sqrt[n]{b} = a$, where $a \geq 0$.

Convince yourself of the following fun fact.

Special Roots

For any n, we have $\sqrt[n]{0} = 0$ and $\sqrt[n]{1} = 1$.

You will want to get really good at writing roots as fractional powers and combining them with exponents, because you do it all the time in Calculus I! So here's that stuff.

Roots as Fractional Powers

We write n^{th} roots as fractional powers, $\sqrt[n]{a} = a^{\frac{1}{n}}$, and we have

$$(\sqrt[n]{a})^m = \left(a^{\frac{1}{n}}\right)^m = a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Notice that line is saying that it doesn't matter if you take a root first and then a power, or take the power first and then the root! It all follows from the power to a power rule. All you have to do is place any powers into the numerator of a fractional exponent, and place any roots into its denominator.

Example 2

Simplify.

1. $(\sqrt[3]{a^2})^6$
2. $\sqrt{(a^3\sqrt{b})}$

Solution

1. Looking at the inside first, we have $(\sqrt[3]{a^2})^6 = (a^{\frac{2}{3}})^6$. Then using power rules, we have $(a^{\frac{2}{3}})^6 = a^{\frac{2\cdot 6}{3}} = a^4$.

2. We convert the outer square root into a fractional power first, so $\sqrt{a^3\sqrt{b}} = (a^3\sqrt{b})^{\frac{1}{2}}$. We bring it through the product to get $(a^3)^{\frac{1}{2}}(\sqrt{b})^{\frac{1}{2}} = a^{\frac{3}{2}}(b^{\frac{1}{2}})^{\frac{1}{2}} = a^{\frac{3}{2}}b^{\frac{1}{4}}$.

Regarding negatives under radicals, we run into some legality issues. For example, any number will give a positive result when it gets squared, right? So there's no way to have a number who gets squared to produce a negative result. This means there is no real number answer to $\sqrt{-4}$. (There is a world beyond the real numbers, but we don't live there right now.) On the other hand, it does make sense to ask about $\sqrt[3]{-8}$, because we know $(-2)^3 = -8$, so that'll work as the answer!

> You are **not allowed** to take **even roots** ($\sqrt{}, \sqrt[4]{}, \ldots$) of **negative numbers!** But you **can** take **odd roots** ($\sqrt[3]{}, \sqrt[5]{}, \ldots$) of negative numbers, and the result will be negative.

First practice computing some roots:

Exercise 2

Simplify (without using a calculator).

1. $\sqrt{16}$
2. $\sqrt[3]{125}$
3. $\sqrt{3^2}$
4. $\sqrt[3]{-27}$
5. $\sqrt[8]{1}$
6. $\sqrt{400}$

Answer

1. This is 4.
2. This is 5.
3. $\sqrt{3^2} = \sqrt{9} = 3 = (\sqrt{3})^2$.

4. This is 1.
5. Notice that $400 = 20 * 20 = 20^2$ so this is 20.

In working with radicals, we need a very important rule.

Root of a Product/Products of Roots

You can break up a root over a multiplication or division:

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

These rules also work backwards!

Caution

You **cannot** break up a root over a plus or minus! Aka, $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$. It is a lie. Straight to jail.

Whenever you take a square root of some number a and the answer is a whole number, we call a a **perfect square.** Examples would be 4, 9, 16, 25, etc. Sometimes, a root doesn't work out nice and clean, but you can still do a little simplifying. This utilizes the rule above. For example, $\sqrt{32}$ isn't a whole number, but we can write

$$\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16}\sqrt{2} = 4\sqrt{2}.$$

It all boils down to finding any perfect squares under the square root, like the 16 hidden inside the 32. Another aspect to cleaning up expressions with radicals in them is the topic of "rationalizing the denominator." I don't really believe in it personally (we haven't needed it since we stopped approximating radicals by hand like peasants), but it's the process of multiplying top and bottom of a fraction by whatever is needed to remove radicals from the denominator. For example,

$$\frac{3}{\sqrt{5}} = \frac{3 \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = \frac{3\sqrt{5}}{5}.$$

Example 3

Simplify.

1. $\sqrt{\frac{1}{25}}$
2. $\sqrt[3]{x^6}$
3. $\frac{\sqrt[4]{16}}{\sqrt{4}}$
4. $\frac{\sqrt{2}+\sqrt{18}}{2}$
5. $\sqrt{x^3y^4z^5}$
6. $(\sqrt[4]{ab^2})^8$

Solution

1. $\sqrt{\frac{1}{25}} = \frac{\sqrt{1}}{\sqrt{25}} = \frac{1}{5}$. You don't have to write the middle step if you don't want to.
2. Using power rules, $\sqrt[3]{x^6} = x^{\frac{6}{3}} = x^2$.
3. $\frac{\sqrt[4]{16}}{\sqrt{4}} = \frac{2}{2} = 1$
4. Cleaning up that $\sqrt{18}$ as much as possible, $\frac{\sqrt{2}+\sqrt{18}}{2} = \frac{\sqrt{2}+\sqrt{2 \cdot 3^2}}{2} = \frac{\sqrt{2}+3\sqrt{2}}{2}$. Then we combine the single $\sqrt{2}$ with the other three of 'em in the numerator to get $\frac{4\sqrt{2}}{2} = 2\sqrt{2}$.
5. Collecting as many perfect squares as we can and bringing them outside the radical sign, $\sqrt{x^3y^4z^5} = \sqrt{x \cdot x^2 \cdot (y^2)^2 \cdot z \cdot (z^2)^2}$ $= xy^2z^2\sqrt{xz}$
6. Using fractional powers, rewrite: $(\sqrt[4]{ab^2})^8 = (ab^2)^{\frac{8}{4}} = (ab^2)^2$. Then using power rules, simplify to a^2b^4.

Now you try the exercises in the next section!

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1.4E Exercises

Computation

Compute without using a calculator.

1. 4^2
2. 3^3
3. $\sqrt{25}$
4. $\sqrt{100}$
5. 0^4
6. 1^{13}
7. $(-1)^{13}$
8. $(-1)^{2024}$
9. $\sqrt{\frac{9}{25}}$
10. $\sqrt{\frac{1}{4}}$
11. $\sqrt[3]{-27}$
12. $\sqrt[3]{-125}$
13. $7^{-100} \cdot 7^{101}$
14. $\frac{3^{12}}{9^2}$
15. 16^0
16. 1337^0
17. $(\sqrt{18})^2$
18. $(\sqrt[3]{8})^4$

Answer

1. 16
2. 27
3. 5
4. 10
5. 0
6. 1
7. -1
8. 1
9. $\frac{3}{5}$
10. $\frac{1}{2}$
11. -3
12. -5
13. $7^{-100+101} = 7^1 = 7$
14. Since $9 = 3^2$, we have $\frac{3^{12}}{(3^2)^2} = \frac{3^{12}}{3^4} = 3^8$.
15. 1
16. 1
17. 18
18. 16

Converting Between Radicals and Fractional Powers

1. Write $\sqrt{z^7}$ as a fractional power.
2. Write $(\sqrt[5]{6})^2$ as a fractional power.
3. Write $27^{\frac{4}{3}}$ using a radical and simplify.
4. Write $\left(\frac{1}{4}\right)^{\frac{1}{2}}$ using a radical and simplify.

Answer

1. $z^{\frac{7}{2}}$
2. $6^{\frac{2}{5}}$
3. $(\sqrt[3]{27})^4 = 3^4 = 81$
4. $\sqrt{\frac{1}{4}} = \frac{1}{2}$

? Combo Simplification Practice

Simplify.

1. $\sqrt{\frac{4}{9}} - \frac{5}{3}$
2. $(2x^2)^6$
3. $\left(\frac{2}{7} \cdot \frac{7}{\sqrt{4}} + 1\right)^{11}$
4. $\left(\frac{\sqrt{3}+\sqrt{12}}{6}\right)^2$
5. $\frac{a^7 b^{-3} c}{a^3 b^2 c^{-1}}$
6. $(\frac{2xy}{3})^2 + \frac{4}{3}$

Answer

1. -1
2. $2^6 x^{12}$
3. 2^{11}
4. $\frac{3}{4}$ (Hint: see the last example in the previous section, part 4.)
5. $\frac{a^4 c^2}{b^5}$ or $a^4 b^{-5} c^2$
6. $\frac{4x^2y^2+12}{9}$

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1.5 Estimation, Common Sense, and Intuition

Learning Objectives

By the end of this section, you will be able to:

- estimate computations without performing them, including using special numbers π and e.
- compare numbers and tell which is bigger or smaller
- get the gist of scientific notation
- assess possible answers to questions to see which are reasonable
- recognize that intuition can fail us

One of the major complaints we hear from science departments about their students' math skill gaps is that students can't estimate numbers or tell when their answers are unreasonable. Maybe this is due to the national infatuation with calculators, but for whatever reason, students are arriving at college without a fluent sense of how numbers relate to each other. A physics student is tasked with calculating the amount of time it takes for a ball to fall to Earth, and after typing a few numbers into their calculator, they write down, "$-304,000$" and move on. They don't stop to realize that a) without checking their units, it's impossible to know for sure if that number is a reasonable amount of time (but I doubt it!!) and b) they shouldn't have gotten a negative answer in the first place! Unless this ball is a time machine. In which case, okay fine, I guess.

In this section, I want to start curing you of some of those blind spots. Hi, meet the real numbers. Get to know them, get a sense of their sizes, and how they relate to each other. Get in the habit of stopping after every problem or even every step and asking yourself, "Is that what I expected to get?" In fact, that's maybe the most important habit you can get into while studying math and science.

While working problems, adopt this routine:

- Ask yourself, "What do I expect to see coming next, based on context or my goal for this problem? How do I expect the answer to look?"
- Proceed with your attempt.
- Pause again and ask yourself, "Was that what I expected to happen? Is my answer even feasible? Does the physical context give me any clues as to what would be reasonable?"

We need to start developing our estimation and common sense skills in order to answer these questions, so let's practice.

Estimation

I want you to read through this material, thinking about each problem on your own as much as possible, before moving on to the answer. It's a no-stakes situation, and you need to start developing the "guts" to take a stab at a problem without a clue. Let's go.

Example 1

If I'm hosting a banquet for 212 guests and the dinner costs $37 a plate, about how much do I need to budget for the catering?

Solution

To calculate the exact cost, I would need to multiply 212 and 37. For a rough estimate, I say 212 is pretty close to 200, and 37 is pretty close to 40. Those numbers are much easier to multiply in my head. I estimate I need about $8000 in my food budget. I could also round 37 down to 35, since I can multiply 2 and 35 in my head pretty easily. My new estimate would be $7000, but I think that's low because I've now rounded down both my numbers! At least I've got a range that gives me a sense of the cost. The actual cost is $7844, so I'm not far off!

Exercise 1

1. I'm about to type $21,300 \times 11$ into my phone calculator, but my screen is broken. My phone says the answer is $256,300$, but I'm not sure the touchscreen was working right. About how much do I think the answer should be, so I'll know if I fatfingered it?

2. I'm buying a car for $4,667 in cash, and also buying an aftermarket stereo from my buddy to put in it for $140. I'm withdrawing cash at the bank right now and they're about to close! About how much should I ask for so I'll be sure to have enough?

Answer

1. Again, I say 11 is close to 10, and I can do $21300 \cdot 10$ in my head knowing that I just need to tack on another zero to get $213,000$ as my estimate. This gives me a clue that I must have typed something in wrong when my calculator gives me an answer of $256,300$.

2. I round up to $4700 and $150 and ask the teller for $4850, as that will certainly cover my bases. When the pressure is off, I realize the exact total is $4807, but now I have some cash to hide in my glove box for emergencies. :)

It's also important to me that you get a sense of the most common **irrational** numbers that come up, namely π (the Greek letter **"pi"**) and e (**"Euler's number,"** a special lil guy you will learn about in logarithms). You don't have to memorize a hundred digits of π unless you have a really nerdy date you're trying to impress, but you should know:

$$\pi \approx 3.14\text{ish} \quad \text{and} \quad e \approx 2.72\text{ish}.$$

It's enough sometimes for our purposes just to think, "π is a bit more than 3," or "e is between 2 and 3." The next example might actually prove useful to you if you ever play the game Worldle and struggle to interpret its clues...

Example 2

Fun fact: the radius of the Earth is about 3,959 miles. Using the fact that the circumference of a circle is $2\pi r$, where r is the radius of the circle, estimate the circumference of the Earth. (About how many miles you would travel if you followed the equator all the way around.)

Here's another fun fact: The point that is exactly on the opposite side of the globe from a location (you've heard of "digging a hole to China"?) is called its **antipode.** The antipode of New York City is in Australia. About how far, in miles, is it from NYC to Australia?

Solution

Let's round 3,959 up to 4,000 and work with that. Knowing that π is not much more than 3, I can start with a very rough estimate of $2 \cdot 3 \cdot 4000 = 24,000$ miles. Maybe I even feel pretty good about my mental math doubling skills and approximate π as 3.1 instead, to get $2 \cdot 3.1 \cdot 4000 = 3.1 \times 8000 = (3 + 0.1)(8000) = 24000 + 800.0 = 24,800$ miles. (The more you practice the decimal computations in Section 1.3, the more you will get comfortable with these ideas.) Anyway, the real value is around 24,901.55 miles.

If Australia is the antipode of NYC, it's halfway around the world! If the circumference of the Earth is around 25,000 miles, halfway would be about 12,500 miles.

Exercise 2

My investment is expected to make 10% interest, compounded continuously, and I have a formula that computes the amount of money I'll have after 10 years. The formula gives an answer of $100e^{0.1 \cdot 10} = 100e$, but about how much money is that?

Answer

Knowing that $e \approx 2.72$, I need to multiply that by 100 using the trick of moving the decimal two places to the right. So I expect to have about $272. Since we round money to two decimal places, the real answer is $100(2.71828\ldots) = \$271.83$

We should have some easy roots memorized, say at least the list below:

$\sqrt{4}=2$	$\sqrt{400}=20$
$\sqrt{9}=3$	$\sqrt{3600}=60$, etc.
$\sqrt{16}=4$	$\sqrt{10000}=100$, etc.
$\sqrt{25}=5$	$\sqrt[3]{8}=2$
$\sqrt{36}=6$	$\sqrt[3]{27}=3$
$\sqrt{49}=7$	$\sqrt[3]{64}=4$
$\sqrt{64}=8$	$\sqrt[3]{125}=5$
$\sqrt{81}=9$	$\sqrt[3]{1000}=10$
$\sqrt{100}=10$	$\sqrt[4]{16}=2$
$\sqrt{121}=11$	$\sqrt[4]{81}=3$
$\sqrt{144}=12$	$\sqrt[4]{10000}=10$

Roots of other numbers, though, can at least be estimated between two integers, using our knowledge of the aforementioned perfect squares.

Example 3

Estimate the square roots between two integers.

1. $\sqrt{2}$
2. $\sqrt{5}$
3. $\sqrt{30}$

Solution

1. Well, 2 is between the perfect squares 1 and 4, so $\sqrt{2}$ must be between their roots, 1 and 2. And since 2 is closer to 1 than it is to 4, I expect $\sqrt{2}$ to be closer to 1 than to 2. In fact, $\sqrt{2}=1.41421456...$.
2. Again, 5 is between the perfect squares 4 and 9, and closer to 4. Then $\sqrt{5}$ must be between 2 and 3, and closer to 2. In fact, $\sqrt{5}=2.2360679...$.
3. Since 30 is between the perfect squares 25 and 36, and pretty much in the middle, I expect $\sqrt{30}$ to be something like 5.5, and in fact, $\sqrt{30}=5.477225575...$.

Exercise 3

Estimate the square roots between two integers.

1. $\sqrt{12}$
2. $\sqrt{78}$
3. $\sqrt{52}$
4. $\sqrt{150}$

Answer

1. Between 3 and 4, maybe 3.5. Real value: 3.464101...
2. 78 is almost 81, so I expect its square root to be a bit less than 9. Real value: 8.8317608...
3. Between 7 and 8 and closer to 7, maybe 7.2. Real value: 7.2111025...
4. 150 is a bit more than 144, whose root is 12, so let's say 12.2ish. Real value: 12.247448...

Bigger or Smaller?

When you get to Calc I, there's a whole section in which you're looking for the maximum and minimum values a function takes on an interval. You'll learn the "Closed Interval Method," during which you'll calculate some values and have to compare them to see which are the max and min. But if those values are expressions with fractions or irrational numbers in them, you'll need to have a sense of what makes a number bigger or smaller than another!

Example 4

Which is bigger?

1. $\frac{5}{2}$ vs $\frac{7}{2}$
2. $\frac{7}{4}$ vs $\frac{7}{5}$
3. $\frac{5}{4}$ vs $\frac{13}{12}$

Solution

1. If two fractions have the same denominator, look at the numerators. A bigger numerator makes a **bigger** number as a whole. The answer is $\frac{7}{2} > \frac{5}{2}$.
2. If two fractions have the same numerator, look at the denominators. A bigger denominator makes a **smaller** number as a whole. Say I have a pizza that I'm cutting into pieces. Certainly a half of a pizza is bigger than a quarter of a pizza, so $\frac{1}{2} > \frac{1}{4}$. Great. Now I'm hungry. No more pizzas! The answer is $\frac{7}{4} > \frac{7}{5}$.
3. In the second fraction, both the numerator and the denominator are larger, so I can't tell if the number as a whole is bigger or smaller... But if I **get a common denominator** by multiplying the first fraction top and bottom by 3, I can compare instead $\frac{15}{12}$ and $\frac{13}{12}$, at which point it's clear the first fraction is a bigger number.

Exercise 4

Which is bigger?

1. $\frac{37}{100}$ vs $\frac{41}{100}$
2. $\frac{9}{11}$ vs $\frac{9}{13}$
3. $\frac{7}{6}$ vs $\frac{5}{4}$

Answer

1. $\frac{41}{100}$
2. $\frac{9}{11}$
3. Hint: change **both** fractions so they have a common denominator and then compare. It's $\frac{5}{4}$.

Reviewing the section on converting between fractions and decimals will help you with the next exercise.

Exercise 5

Which is bigger?

1. 2 vs $\frac{3}{2}$
2. $\frac{5}{4}$ vs 1.5
3. $\frac{105}{50}$ vs 2

Answer

1. 2 is bigger since the other is 1.5
2. 1.5 is bigger since $\frac{5}{4}$ is a whole $\frac{4}{4} = 1$ plus an extra fourth, so 1.25
3. Hint: Compare $\frac{105}{50}$ to $\frac{100}{50}$. It's bigger.

Let's go back to π for a challenge round that came up last time I taught Calc I...

Example 5

Which is bigger, $2\pi - 1$ or $\frac{3}{2}\pi$?

Solution

Let's estimate the value of each option and then compare. If π is 3.14ish, then 2π is 6.28ish, and subtracting 1 puts me around 5.28ish. Meanwhile, $\frac{3}{2}\pi = \frac{3\pi}{2}$, and I know that 3π is bigger than 9 but less than 10... Well, then half of 3π is certainly less than 5.28ish! The true values are $2\pi - 1 = 5.283185...$ and $\frac{3}{2}\pi = 4.712338....$

Now you try a combo of ideas...

Exercise 6

Which is bigger, $3\sqrt{5}$ or 5?

Answer

5 is a tad more than 4, so we know $\sqrt{5}$ is a bit more than 2, which means $3\sqrt{5}$ is a bit more than 6. Certainly bigger than 5.

Scientific Notation

If you're doing anything in the sciences, you are going to run into scientific notation. This is a technique we use for writing gigantic numbers or super tiny numbers without getting a hand cramp from writing zeros, like the approximate number of oxygen atoms in a cubic centimeter of air,

$$5,300,000,000,000,000,000 \text{ (yikes)}$$

It all comes down to the multiplying/dividing by powers of 10 trick. Observe:

$$1.2345 \cdot 10^5 = 1.2345 \cdot 100,000 = 123,450.$$

This means I could write 123,450 in terms of a number with a ones digit and decimal places, multiplying by however many tens I need to be equivalent. This makes my life way easier when dealing with those oxygen atoms, because I can just count up the times I would move the decimal point to get it next to the 5 (it's 18 steps), and write $5.3 \cdot 10^{18}$ instead. Finally, notice that as the powers of ten go up, the numbers are increasing fast. $1 \cdot 10^7 = 10,000,000$ is a lot bigger than $1 \cdot 10^5 = 100,000$.

In the US, the likelihood that a person who goes to beaches will be attacked by a shark is 1 in 11.5 million (according to Wikipedia), or

$$\frac{1}{11,500,000} = 0.0000000869565$$

but that can also be written as $8.69565 \cdot 10^{-8}$, by counting how many times I would need to move the decimal to the right to get it to the right of 8. Notice that the more negative the power, the smaller the number, like how $1 \cdot 10^{-6} = 0.000001$ is smaller than $1 \cdot 10^{-3} = 0.001$. You will work with scientific notation in your science classes, but what I want you take away for now is this:

- Something times 10 to a **positive** power is a **large** number, like the first example. When the power increases, the number increases rapidly!
- Something times 10 to a **negative** power is a **small** number, like the second example. The more negative the power, the tinier the number!

Common Sense

Always, always, always, ALWAYS, ALWAYS, **ALWAYS**, I'M BEGGING YOU, always!!! ask yourself if your answer makes common sense at the end of a problem.

Example 6

My washing machine manufacturing business (De Wolf's De Washing Machines) made $120,000 in profits last year. My total revenue was $700,000 and I'm using a spreadsheet to total up my monthly costs because they varied according to season and demand... It says the costs I paid for the year total $650,000. Is that reasonable?

Solution

No, I must have gone wrong somewhere... Revenue is money coming in, and cost is money going out, and profit is what's left over, so Profit = Revenue - Cost. If I had $700,000 in revenue and $650,000 in costs, I would have only made a profit of $700,000-$650,000 = $50,000. That's not what my books say! My spreadsheet must have an error in its entries or formulas.

Exercise 7

Water drains out of a full 5-gallon drum at a rate of 0.16 gallons per minute. I ask you calculate the amount of water left in the tub after half an hour, and you try a first time and get an answer of 6.18 gallons, but you try a second time and get an answer of 4.8 gallons. Are either of these answers reasonable?

Answer

The first one can't be the right number, because we started with 5 gallons and we're losing water, so we sure as heck can't magically have more than we started with! The second answer is at least less than 5, but look at that leaking rate. I'm losing more than 0.1 gal/min, so more than a tenth of a gallon. That means every 10 minutes I'm losing more than a whole gallon. After 30 minutes, I should lose more than 3 gallons, so I must be lower than 4.8! Neither is reasonable.

(What mistake probably happened here? Maybe you said, hey $0.16\frac{\text{gal}}{\text{min}} \cdot 30 \text{ min } = 4.8$ so that's the answer! You forgot that's the total amount of water that has flowed out, so the amount left in the drum is actually $5 - 4.8 = 0.2$ gallons.)

Intuition (a cautionary tale)

You may have been reading through this section and saying to yourself, "Well dagnab hecky durn, De Wolf! You're doing so well with all this rounding and mental math, what do I even need a math class for if I can just learn to guesstimate everything in my head??" If intuition is such a handy tool when solving problems and going through life, what is rigorous detailed math reasoning for?

Great question. Sometimes, intuition is a lie, Steven. A LIE! Let me give you some examples.

You and I are in prison (for being too funny, we're in a dystopia) and there's nothing to do but flip a quarter we found and see if it's heads or tails. I flip it six times in a row and get tails each time. Six tails in a row! You feel in your bones we're "due" for a heads, like it's more likely that the next time I flip it, we'll get heads. In fact, the odds of getting heads are 50% every single time, regardless of previous flips. If I describe the results of six flips by "TTTTTT" for all tails, or "THHTHT" for tails-heads-heads-tails-heads-tails, it ~feels~ like it should be "rarer" to get the TTTTTT outcome. But actually, the odds of getting *exactly* the TTTTTT outcome are the same as the odds of getting *exactly* the THHTHT outcome. (My intuition is even worse: after six tails in a row, I start expecting it to be tails every time, assuming I'm sitting in some kind of magical magnetic field singularity or something where everything is tails.)

Or another scenario: I have 25,000 miles worth of rope lying around so I decide to wrap it all the way around the Earth, right on the surface of the Earth. Then I decide I want to raise the rope 1 meter above the ground everywhere, all the way around. How much more rope do I need? I feel like a lot! The Earth is big! But no... The radius of the circle of rope, when it was lying on the ground, was some number of meters, call it r. If I want to raise the rope 1 meter above the ground, the new radius would be $r + 1$. Then the circumference of the circle changes from $2\pi r$ to $2\pi(r + 1) = 2\pi r + 2\pi$, for a change of only about 6.28 meters!

Hopefully in this section you've gained some skills that help you check your answers and impress your friends. Check out the exercises section next!

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1.5E Exercises

Estimation

1. According to College Board at the time of writing this, the average cost per year to attend Siena College, after aid, is $32,810. In Fall 2022, Siena College had a total undergraduate enrollment of 3,519 students. Assuming everything stays about the same going forward, estimate how much money Siena will receive each year from its students' payments.

2. One of my washing machine models is sold for the low low price of $789.99. If I sold 32 of them last month, about how much revenue did I take in?

3. Approximately 10% of people are left-handed. About 19.62 million people live in the New York-Newark-Jersey City metropolitan area. Roughly how many of them are left-handed?

4. Approximately 2% of people are redheads (in the northern hemisphere, anyway). Roughly how many redheads are there in the metropolitan area?

Answer

1. We just need to multiply the average amount by the total number of students. 32810 is pretty close to 33000 and 3519 is pretty close to 3500. With quick scratch work, $33 \times 35 = 1155$, to which I tack on a total of five zeros to get $115,500,000. With a calculator, $32810 \times 3519 = \$115,458,390$.

2. 32 is about 30, and the price is about $800 each. I estimate about $24,000 in revenue, and I could even say hey, the other 2 washers on top of that would bring in about $1600 as well. All together, I estimate something between $24,000 and $25,600. A calculator gives the exact answer $25,279.68.

2. 19.62 million means 19,620,000 people. The decimal version of 10% is 0.1, so we just need to multiply the two numbers. We use the trick of moving the invisible decimal one place to the left, to get 1,962,000.0 people. Dang, almost two million lefties!

4. Convert from 2% to 0.02. That's double 0.01, so let's think about that first. We can find $0.01(19,620,000) = 196200.00$ by moving the decimal point. So 1% is 196,200 people, which I could round up to 200,000ish and finally double. I expect a bit under 400,000 redheads.

Estimating for Common Sense

Mohammad bought a box of dates in preparation for Ramadan. The box's dimensions are 2 inches by 6 inches by 10 inches. On the back, it a serving size is 2 dates and there are about 35 servings per container. Is that a reasonable claim?

Answer

If these are some big ol' Medjool dates, each one might have a volume of 2 cubic inches (approximating their oblong shape with a 1x1x2 inch rectangular prism). The total volume of the box is $2 \cdot 6 \cdot 10 = 120$ cubic inches. The question is, can a total of 70 dates fit in that box? If we divide the total volume by the estimated volume of a date, we expect 60 dates to fit in. But dates are squishy and not actually rectangular, so it seems reasonable that an extra 10 could be crammed in, especially if some are smaller than others. It's probably not false advertising!

Estimating to Evaluate Claims

Driving home just now, I heard a guy on the radio claim that "the average American now spends $5000 a year on gas." I was immediately skeptical because I feel pretty average and I've never spent remotely that much on gas per year. I started doing some mental math to entertain myself while driving and evaluate this claim.

1. Assuming that it costs $50 on average to fill up your gas tank, how many times a year would you have to fill up to reach a total of $5000?
2. Filling up that many times, how many times a week would you be getting gas (and not just topping up, a full tank!)?

3. Conservatively assuming I get 20 miles to the gallon on average (both cars that I've owned in my life did much better, in fact), how many miles a week would I be driving to use that much gas?
4. There are 52 weeks in a year, so even rounding down to 50 weeks, how many miles driven per year would that be?
5. What possible motivations could someone have for exaggerating the amount of money Americans spend on gas per year?

I now believe that the radio guy meant to say that the average HOUSEHOLD spends $5000 a year on gas. That is certainly more believable! But a quick search showed that the $5000/household figure comes from 2022, and the guy is announcing it in late 2024? Meanwhile household gasoline yearly costs have been dropping between 2022 and 2024, possibly even by about half!

Answer

1. That's 100 fill-ups per year.

2. Roughly twice a week! Meanwhile, I think I have pretty normal driving habits and I only fill up maybe two to three per month...

3. Say that filling up twice a week is buying about 25 gallons of gas. If I'm using 25 gallons at 20 miles per gallon, I must be driving 500 miles a week.

4. That would be putting 25,000 miles a year on my car. I went home and looked up the average miles driven per year, and it's actually only about 15,000.

5. Hmmm, makes ya wonder... Maybe someone wants to make current gas prices sound like much more of a crisis than they are... Maybe someone wants to influence more people to buy electric cars... What do you think?

? Bigger or Smaller?

Without actually evaluating anything or using a calculator, which is bigger?

1. $\frac{6}{51}$ vs $\frac{6}{49}$
2. $\frac{7}{11}$ vs $\frac{8}{11}$
3. $\frac{5}{4}$ vs $\frac{9}{7}$
4. 3π vs 9
5. 6 vs $2e$
6. $\sqrt{8}$ vs $\sqrt{7}$
7. 5 vs $\sqrt{29}$
8. $\sqrt{101}$ vs $\frac{48}{5}$
9. $\frac{1}{2}$ vs $\left(\frac{1}{2}\right)^2$
10. $\frac{1}{n}$ vs $\frac{1}{n+1}$

Answer

1. $\frac{6}{49}$
2. $\frac{8}{11}$
3. $\frac{9}{7}$ (Hint: get a common denominator.)
4. 3π
5. 6
6. $\sqrt{8}$
7. $\sqrt{29}$
8. Notice that $\sqrt{101}$ must be a tad more than 10, but $\frac{48}{5}$ is a bit less than $\frac{50}{5}=10$. So $\sqrt{101}$ is bigger.
9. $\frac{1}{2}$ is bigger than $\frac{1}{4}$. This is important! If you take a lil fraction less than 1 and raise it to a positive integer power, the result will get smaller! But with numbers bigger than 1, raising them to a positive integer power makes a bigger number.
10. The denominator $n+1$ is bigger than n, so its fraction is a smaller number. Don't believe me? Pick $n=2$ for example and see what happens.

A Begrudging Nod to Rationalizing the Denominator

The only benefit I see to the practice of rationalizing denominators is because in some cases it's easier to grasp the rough size of a number. For example, I don't look at $\frac{1}{\sqrt{2}}$ and immediately get a sense of how much it is. But if you rationalize the denominator, it becomes much easier to estimate. Give it a shot.

Answer

By writing $\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, it becomes clear that this is *half* of the number $\sqrt{2}$. I've seen that $\sqrt{2} \approx 1.414$ish, and half of that is 0.7ish. That wasn't immediately clear to me from the $\frac{1}{\sqrt{2}}$ version.

Intuition

I'm selling two washing machine models for the same retail price. With Model A, I mark it as 30% off on Monday, and then on Tuesday, I mark it down again, 15% off the sale price! Meanwhile, with Model B, I put it on sale for 15% off on Monday, and then mark it down again Tuesday, 30% off the sale price! Which one is the better deal?

Answer

Maybe you realize that when you calculate percentages, 50% of a larger number is bigger than 50% of a smaller number. Perhaps you thought, well if I take off 30% first on Model A, that's a lot of money, compared to taking 30% off a lower price on Model B... Or maybe you had another thought! Either way, let's shine the inexorable light of math on the situation.

Whatever the original price is, let's call it x. For Model A, the price on Monday has 30% removed, so it's $(1-0.3)x = 0.7x$, and on Tuesday there is an additional markdown of 15%. That gives an ultimate sale price of $(1-0.15)(0.7x) = 0.85 \cdot 0.7 \cdot x$. For Model B, we start with x and mark it down by 15%, $(1-0.15)x = 0.85x$, and then take the new price and take off 30%, to get $(1-0.3)(0.85x) = 0.7 \cdot 0.85 \cdot x$. But wait, multiplication is commutative, so it doesn't matter what order you do it in! Those final prices are exactly the same!

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1.6 Chapter 1 Study Guide

Definitions

- **natural numbers ($\mathbb{N}$):** The "counting" numbers, nonnegative whole numbers, aka $0, 1, 2, 3, \ldots$ forever.
- **integers ($\mathbb{Z}$):** All of the whole numbers, including negative numbers. These also go on forever but in both directions, $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.
- **rational numbers ($\mathbb{Q}$):** All numbers that can be written as a ratio of two integers, so things like $\frac{1}{2}, -\frac{7}{5}, \frac{27}{4001}$, etc. These include integers, because we can write $4 = \frac{4}{1}$ if we want. Decimal representations of rational numbers either end, like 1.5, or repeat forever, like $1.33333\ldots$.
- **real numbers ($\mathbb{R}$):** All of the above numbers, together with **irrational numbers**, such as $\sqrt{2}, \sqrt{5}, \pi, e$, etc. These numbers' decimal representations never end or have a repeating pattern.
- **commutative:** An operation is commutative if it doesn't matter what order you do it in. Addition and multiplication are commutative since $a + b = b + a$ and $ab = ba$, but subtraction and division aren't!
- **factor:** A number's factors are the numbers that can be multiplied to produce it. For example, factors of 12 are $\pm 1, \pm 12, \pm 2, \pm 6, \pm 3, \pm 4$.
- **sum:** the result of an addition.
- **difference:** the result of a subtraction.
- **product:** the result of a multiplication.
- **quotient:** the result of a division. The number you divide BY is the **divisor.** The number getting divided is the **dividend**.
- **strictly less than / less than or equal to:** symbols are $<$ and $\leq$ respectively. For example, $3 < \pi$.
- **strictly greater than / greater than or equal to:** symbols are $>$ and $\geq$ respectively. For example, if we say $x \geq 4$, that means x could be equal to 4 or anything larger than that.
- **numerator / denominator:** The top of a fraction is the numerator. The bottom of a fraction is the denominator.
- **reciprocal:** The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.
- **compound fraction:** A fraction that has fractions inside of its numerator or denominator or both.
- **least common multiple (LCM):** The smallest number that all of a set of numbers divide evenly into. Example: the LCM of $3, 4$, and 6 is 12. This is used when getting a **common denominator.**
- **percent:** a proportion out of 100. Example: 20% is $20/100 = \frac{1}{5} = 0.2$.
- **power / exponent:** a^k is "a raised to the power k" and can be thought of as multiplying a by itself a total of k times. The **base** is a, and the **exponent** is k.
- **radical / root:** $\sqrt{a} = b$ means b is the positive number such that $b^2 = a$. Similarly, $\sqrt[3]{a} = b$ means $b^3 = a$, and so on.
- **scientific notation:** a number is in scientific notation if it is written as X.XXXX times an appropriate power of 10. Examples: $1,234,000 = 1.234 \times 10^6$ and $0.00001234 = 1.234 \times 10^{-5}$. If the exponent is positive, it's a large number. If the exponent is negative, it's a tiny number.

Arithmetic With Negatives

- Subtraction is the same as adding a negative number, and adding negative numbers is the same as taking the difference (the sign of the answer matches the sign of the bigger number). For example, $2 - 3 = 2 + (-3) = -1$.
- Multiplication distributes over sums and differences, like $2(1 + 3) = 2 \cdot 1 + 2 \cdot 3 = 2 + 6 = 8$, but you have to be careful when distributing a negative number. For example, $-3(a + b) = -3a - 3b$.
- Positive $\times$ positive = positive. Negative $\times$ negative = positive. Positive $\times$ negative = negative. The same applies to division!

Order of Operations (PEMDAS)

0. Parentheses first. Working from the innermost outward, completely take care of expressions inside parentheses.
1. Exponents (powers). If numbers are being raised to a power, compute those first.
2. Multiplication/Division. Working left to right, compute any multiplications and divisions.
3. Addition/Subtraction. Working left to right, compute any sums and differences.

Example:

$$2(3+(4-2)^2)-\frac{(3+1)}{4}=2(3+(2)^2)-\frac{4}{4}=2(3+4)-\frac{4}{4}=2(7)-\frac{4}{4}=14-1=13$$

Arithmetic With Fractions

- To **multiply** fractions, multiply across the top and multiply across the bottom: $\frac{a}{b}\cdot\frac{c}{d}=\frac{ab}{cd}$.
- To **divide** fractions, multiply by the reciprocal: $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$.
- To **add or subtract** fractions, get a common denominator for all fractions, and then add or subtract (using parentheses if necessary).

Place Values in Decimals

...	ten-thousands	thousands	hundreds	tens	ones	. POINT	tenths	hundredths	thousandths	ten-thousandths	...
	3	1	4	2	6	.	2	8	7	9	

When rounding to a certain place, look at the digit directly to the right. If it's less than 5, round down. If it's 5 and up, round up.

Arithmetic With Decimals

- **Adding and subtracting:** arrange vertically **with decimal points lined up** and add/subtract down columns as usual:

$$\begin{array}{r} 21.50 \\ +\ 0.25 \\ \hline 21.75 \end{array}$$

- **Multiplying:** ignore the decimal points and multiply the numbers, then count up the total number of decimal places and put the new point that far into the product. Example: for $1.5\cdot 0.4$, first multiply $15\cdot 4=60$, then each number had one decimal place, so put the point two places in to get 0.60 as the answer.
- **Powers of 10 trick:** If multiplying by 10^k, move the decimal point k steps to the right, adding zeros into the spaces. Example: $4.2\cdot 10^3=4200$. If dividing by 10^k (aka multiplying by 10^{-k}), move the decimal point k steps to the left, adding zeros if needed. Example: $4.2\cdot 10^{-3}=0.0042$.

Equivalent Fractions/Decimals to Know

Fraction	Decimal
$\frac{1}{2}$	0.5
$\frac{1}{3}$	$0.\overline{3}$, which means the 3s go on forever like $0.33333...$
$\frac{2}{3}$	$0.\overline{6}$, which due to rounding rules rounds to things like 0.6667
$\frac{1}{4}$	0.25, like how 25 cents is a quarter of a dollar
$\frac{3}{4}$	0.75
$\frac{1}{5}$	0.2, which you can remember as 0.20 because 20 is one-fifth of a 100
$\frac{1}{8}$	0.125
$\frac{3}{2},\frac{5}{2},\frac{7}{2},\ldots$	$1.5, 2.5, 3.5, \ldots$, which are just like finding mixed numbers from improper fractions

Percents

- To find XX% of a quantity, convert to a decimal by moving the point two steps to the left, getting 0.XX, and then multiply. Example: 10% of 30 would be $0.1(30) = 3$.
- To convert from a ratio to a percent, just divide $\frac{\text{part}}{\text{whole}}$ and multiply by 100. Example: 2 out of 5 students come to office hours means $\frac{2}{5} \cdot 100 = 0.4 \cdot 100 = 40\%$ of students come to office hours.

Power Rules

$a^b a^c = a^{b+c}$	$\frac{a^b}{a^c} = a^b a^{-c} = a^{b-c}$	$(a^b)^c = a^{bc}$	$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
$a^0 = 1$	$a^1 = a$	$0^n = 0$	$1^n = 1$
$(ab)^c = a^c b^c$	$(a+b)^c \neq a^c + b^c$	$\text{negative}^{\text{even power}} = \text{positive}$	$\text{negative}^{\text{odd power}} = \text{negative}$

Root Rules

$\sqrt[n]{0} = 0$	$\sqrt[n]{1} = 1$	$(\sqrt[n]{a})^n = a$	$\sqrt[n]{a} = a^{\frac{1}{n}}$
$(\sqrt[n]{a})^m = \sqrt[n]{a^m} = a^{\frac{m}{n}}$	$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$	$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$

Common Roots to Know

$\sqrt{4} = 2$	$\sqrt{400} = 20$
$\sqrt{9} = 3$	$\sqrt{3600} = 60$, etc.
$\sqrt{16} = 4$	$\sqrt{10000} = 100$, etc.
$\sqrt{25} = 5$	$\sqrt[3]{8} = 2$
$\sqrt{36} = 6$	$\sqrt[3]{27} = 3$
$\sqrt{49} = 7$	$\sqrt[3]{64} = 4$
$\sqrt{64} = 8$	$\sqrt[3]{125} = 5$
$\sqrt{81} = 9$	$\sqrt[3]{1000} = 10$
$\sqrt{100} = 10$	$\sqrt[4]{16} = 2$
$\sqrt{121} = 11$	$\sqrt[4]{81} = 3$
$\sqrt{144} = 12$	$\sqrt[4]{10000} = 10$

Comparing Fractions

- If two fractions have the same denominator, compare the numerators. The **bigger numerator** gives a **larger fraction**. Example: $\frac{2}{3} > \frac{1}{3}$.
- If they have the same numerator, compare the denominators. The **bigger denominator** gives a **smaller fraction**. Example: $\frac{1}{4} < \frac{1}{2}$.
- If they have different numerators **and** denominators, find a **common denominator** and then compare the new numerators. Example: $\frac{5}{4} = \frac{15}{12}$ is smaller than $\frac{4}{3} = \frac{16}{12}$.

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CHAPTER OVERVIEW

Chapter 2: Algebra Skills

In this chapter, we review all the basic algebra skills that we need in order to manipulate and simplify expressions that have variables in them. The last chapter focused on dealing with literal numbers, but the #1 skill you need for future math and science courses is fluency in algebra when letters are used to represent quantities instead. We'll be going over concepts such as like terms, FOILing, factoring techniques of all varieties, and simplifying rational expressions *legally*. All of this is—and I cannot say this enough—absolutely vital to your ability to work with equations and functions. The reason I made you do everything by hand without a calculator in the last chapter, even though we were mostly just dealing with numbers, is because your calculator can't help you with a lot of these operations once letters get involved.

One thing to note is that the types of expressions I give you to practice on are inspired by exactly what we do in Calculus I or applications of math. I want you to be ready to simplify difference quotients before you get to Calc I. I want you to be able to solve for a variable stuck in an exponent before you study time value of money in your actuarial science class. You have to factor the heck out of polynomials for a solid chunk of Calc I, so we're going to do a lot of those here too. Don't worry, the tools I'm forcing you to store in your toolbox will actually come in handy later. (I'm not that much of a masochist.)

2.1 Algebraic Expressions

2.1E Exercises

2.2 Factoring Techniques

2.2E Exercises

2.3 Rational Expressions

2.3E Exercises

2.4 Algebravaganza (Chapter 2 Study Guide)

2.4E Exercises

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2.1 Algebraic Expressions

Learning Objectives

By the end of this section, you will be able to:

- identify the important features of a polynomial
- add and subtract algebraic expressions
- multiply algebraic expressions using distribution techniques such as FOIL
- simplify expressions containing like terms

Vocab tiiiiime! Let's learn the language we need to talk about algebraic expressions.

Say I'm building a doghouse and I'm trying to calculate how much wood I need to buy. Everything depends on the size of the doghouse, right? And I could start describing size by talking about the height I want. But I don't know what height I want yet! It could be 2ft, or maybe 2.5ft, or maybe 2.789ft. Any of those real numbers could be good options, and anything I choose is going to affect how the rest of my calculations go. So instead of making the decision now, I use a nickname for the quantity "height of the doghouse," maybe oh I dunno like an h. Then since h could be various options from a set of reasonable real numbers, it's called a **variable.** Then I could work with that nickname h during my calculations.

A **variable** is a symbol (letter) that represents a quantity whose exact value we don't know yet, or whose exact value may be allowed to change (vary). If we take variables and real numbers and combine them using addition, subtraction, multiplication, division, powers, and radicals, the result is an **algebraic expression.**

Say I have a spherical balloon, and I represent the radius with the variable r. I can write an algebraic expression calculating the volume inside the balloon, $4\pi r^3$, using the variable, real numbers, and arithmetic. Algebraic expressions can look like all sorts of things:

$$4x^3 - 5x^2 + 1, \qquad \frac{\sqrt{xyz}}{10}, \qquad (1+r)^{10t}$$

In that first example above, various pieces getting added or subtracted are called **terms.** The 4 multiplied on the front of the x^3 term is called the **coefficient** of x^3. Likewise, -5 is the coefficient of the x^2 term.

Let's keep looking at $4x^3 - 5x^2 + 1$. Expressions of the form ax^n, some real number coefficient times a power of x, are called **monomials,** and if you add two such terms, you get a **binomial.** Add up a bunch, and you get a **polynomial.** (*Poly-* is Latin for "many." Okay fine, it's originally Greek. But also Latin.) Polynomials are our bestest friends. We love them.

Definition: Polynomial

A polynomial in the variable x is an expression of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

or equivalently,

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

where $a_0, a_1, a_2, ..., a_n$ are real number coefficients, and n is a nonnegative integer.

The **degree** of the polynomial is the highest power you see x raised to. That term, a_nx^n, is the **leading term**, and the coefficient a_n is the **leading coefficient.** The plain ol' real number term, a_0, is the **constant term.**

Exercise 1

For each expression, identify whether it is a polynomial, and if so, tell its degree and leading coefficient.

1. $3x + 2$

2. $x^2 + 9x + 17$
3. $6e^x - x$
4. $\frac{x^2-2}{x^3+1}$

Answer

1. Yes, it is composed of powers of x with coefficients on them added together. That 2 is the constant term a_0, and the highest power of x is 1, making this a degree one polynomial. The leading coefficient is the 3 on the x term.
2. Yes, this is a degree two polynomial, and the leading coefficient is a secret 1, because $x^2 = 1x^2$.
3. Nope! That $6e^x$ ruined everything.
4. Nope! The expressions in the numerator and denominator are themselves polynomials, but since they're in a fraction, this is not a polynomial. It's a rational expression, which we'll deal with later.

We can do all sorts of stuff with polynomials, like adding and subtracting them. For this, we're going to need a fun fact:

Distributive Property of Addition

For numbers (or variables) $a, b,$ and c, we have

$$a(b + c) = ab + ac.$$

Note: going from the left hand side to the right hand side is called **distributing** the a through the parentheses, and going backwards is called **factoring out** the a from both terms.

You should buy that this works for regular numbers. Anybody wanna fight that $2(3 + 5) = 2 \cdot 3 + 2 \cdot 5 = 6 + 10 = 16$ is the same as $2(3 + 5) = 2(8) = 16$? Didn't think so. I can use this property on variable terms as well, like this:

$$11x + 7x = (11 + 7)x = 18x, \quad \text{or} \quad 4x - 6x = (4 - 6)x = -2x.$$

This particular technique is called combining **like terms**, aka terms that have the same exact variable(s) raised to the same exact power(s). So I can combine $2xy$ with $3xy$ to make $5xy$, but I can't combine $2xy$ with $3x^2y$ because I don't have the same variables and same powers exactly.

Example 1

1. Distribute: $4(a + b)$
2. Distribute: $-7(2x + 11 + z)$
3. Factor: $3x + 6y$
4. Factor: $12x^2 - 12x$
5. Combine like terms: $3x - 4x^2 + 2 + x + 5x^2$

Solution

1. Apply the multiplication by 4 to both terms inside the parentheses: $4(a + b) = 4a + 4b$.
2. **ALERT!!!** Be very careful when multiplying by that **negative** 7! $-7(2x + 11 + z) = (-7)(2x) + (-7)(11) + (-7)(z) = -14x - 77 - 7z$.
3. Both terms have a factor of 3 in them. If I take the 3 from the $3x$, I have a leftover x in that term. If I take 3 from the $6y$, I have $2y$ left. So $3x + 6y = 3(x + 2y)$.
4. Both terms have the factors 12 and x in common. If I take $12x$ out of $12x^2$, there is a single x left over. How can I take $12x$ out of $12x$? That's the whole thing?? But wait, $12x = 12 \cdot x \cdot 1$, so there is only a 1 left over. Thus, $12x^2 - 12x = 12x(x - 1)$.
5. I look for the terms that have the same power on the x... Bring the $3x$ and x together, and then bring the $-4x^2$ and $5x^2$ together:

$$3x + x - 4x^2 + 5x^2 + 2 = (3 + 1)x + (-4 + 5)x^2 + 2 = 4x + x^2 + 2.$$

(You don't have to write the middle part there going forward, I'm just emphasizing why it works.)

? Exercise 2

1. Distribute: $-3(x+y)$
2. Factor: $4x^2+2xy+2y^2$
3. Combine like terms: $x^3+2x^2+5x-x^3-4x^2+x+6$
4. Combine like terms: $2a+3ab-b^2+5a-ab+b-8$

Answer

1. $-3x-3y$
2. $2(2x^2+xy+y^2)$
3. $-2x^2+6x+6$
4. $7a+2ab-b^2+b-8$

This is all we need to learn how to add and subtract polynomials!

Adding and Subtracting Expressions

✔ Example 2

1. Add the polynomials $(2x^3+x^2-4)$ and (x^4-x^2+x+5) and simplify

2. Subtract the polynomials (x^4-x+1) and $(2x^4-3x^2-x+2)$ and simplify.

Solution

1. We put the polynomials together with an addition:

$$(2x^3+x^2-4)+(x^4-x^2+x+5)$$

We can ignore those parentheses and combine all the like terms we see:

$$2x^3+x^2-4+x^4-x^2+x+5=x^4+2x^3+1$$

2. We put the polys together with a minus:

$$(x^4-x+1)-(2x^4-3x^2-x+2)$$

Before losing those parentheses, notice the subtraction needs to apply to **everything** inside the second polynomial! "Distribute" that minus throughout, and **then** combine like terms:

$$x^4-x+1-2x^4+3x^2+x-2=x^4-2x^4+3x^2-x+x+1-2=-x^4+3x^2-1$$

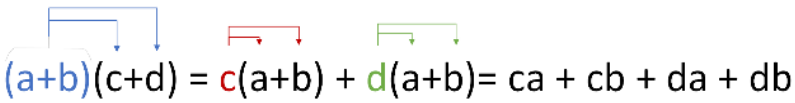
I want to open up the floor to more than just polynomial expressions for this concept. The Distributive Property of Addition is an equal-opportunity employer.

$$2e^x+3e^x=(2+3)e^x=5e^x, \qquad \sqrt{2}+3\sqrt{2}=(1+3)\sqrt{2}=4\sqrt{2},$$
$$4xy+8xy=(4+8)xy=12xy$$

In this sense, the phrase "like terms" feels a little looser, but I hope morally and spiritually you can see how the rule still applies.

Multiplying Expressions

Armed with the Distributive Property (and color coding), we valiantly sally forth against this uggo: $(a+b)(c+d)$.



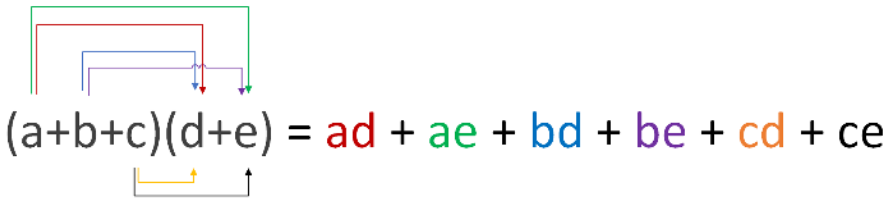
First, I treat $(a+b)$ as if it's just a number and distribute through the $(c+d)$. Then I distribute the c... and then I distribute the d. I never heard this growing up because I'm an aged lady of 30 years, but the kids these days call it **FOIL**, meaning you multiply the

First terms from each expression together, then the **Outer** terms, then the **Inner** terms, and finally the **Last** terms of each expression.

FOIL - First, Outer, Inner, Last

$$(a+b)(c+d) = ac + ad + bc + bd$$

Back in my day, I just thought of it as making everybody shake hands with everybody, which I like because it extrapolates well to dealing with more than just binomials. See below. Just make a shake hands with d and e, and then make b shake hands with d and e, and so on. As if they're two soccer teams passing by each other being sportsmanlike. This is called **expanding,** by the way.



Anyways, analogies aside, do some practice below.

Example 3

Expand and simplify.

1. $(x-2)(x+3)$
2. $(3x+1)(y-2)$
3. $(a^2+2a-b)(3c+4a)$
4. $4(x+1)(x+2)$

Solution

1. FOIL gives $x^2+3x-2x-6$ and we combine like terms to get x^2+x-6.
2. FOIL gives $3xy-6x+y-2$ and there are no like terms to combine.
3. Expanding yields

$$3a^2c + 4a^3 + 6ac + 8a^2 - 3bc - 4ab$$

4. FOILing and distributing, $4(x+1)(x+2) = 4(x^2+3x+2) = 4x^2+12x+8$.

Exercise 3

Expand and simplify.

1. $(z+5)(z-7)$
2. $(x-1)(x^2+2x+1)$
3. $(a+b)(a+b)$

Answer

1. $z^2-2z-35$
2. x^3+x^2-x-1
3. $a^2+2ab+b^2$

That last part in the exercise...wait a second... That's right, $(a+b)(a+b) = (a+b)^2$! Remember how I said if I ever see you trying to bring a power through parentheses like $(a+b)^c \longrightarrow a^c + b^c$ I would mail you a dead fish? (Did I say that? Well, I will.) This is why it's illegal!!

$$(a+b)^2 = (a+b)(a+b), \quad (a+b)^3 = (a+b)(a+b)(a+b),$$
$$(a+b)^4 = (a+b)(a+b)(a+b)(a+b), \text{ etc.}$$

Computing powers like that requires a lot of expanding, and it is NOT the same as just applying the power to each term! However, for the most commonly used products and powers, you can become familiar with some handy formulas. Here's a table. You can check every single one of these by expanding by hand.

Expression	Expansion
$(A+B)(A-B)$	A^2-B^2 (This is called a **difference of squares**)
$(A+B)^2=(A+B)(A+B)$	$A^2+2AB+B^2$
$(A-B)^2=(A-B)(A-B)$	$A^2-2AB+B^2$
$(A+B)^3=(A+B)(A+B)(A+B)$	$A^3+3A^2B+3AB^2+B^3$
$(A-B)^3=(A-B)(A-B)(A-B)$	$A^3-3A^2B+3AB^2-B^3$

(For a good time, ask your professor about Pascal's Triangle if you're interested in how symmetrical those last couple formulas came out.)

The nice thing here is that in those formulas, A and B are stand-ins for anything you want, be it variable or expression. Let me demonstrate.

Example 4

Expand and simplify.

1. $(x+1)(x-1)$
2. $(a+7)^2$
3. $(3z-\sqrt{2})^2$
4. $(x+2y)^3$
5. $(4^x+2)(4^x-2)$

Solution

1. Using the first formula with $A=x$ and $B=1$, we have $x^2-1^2=x^2-1$.
2. Using $A=a$ and $B=7$, we get $a^2+2(7)(a)+7^2=a^2+14a+49$.
3. Using $A=(3z)$ and $B=\sqrt{2}$, we get $(3z)^2-2(3z)(\sqrt{2})+(\sqrt{2})^2=9z^2-6z\sqrt{2}+2$.
4. Using $A=4^x$ and $B=2$, we get $(4^x)^2-2^2=4^{2x}-4$.

All right, that's what I got for you. Try these and then go for the exercise section!

Exercise 4

Simplify.

1. $(x+2)(x-4)+(x^2-1)$
2. $(2x+\sqrt{3})^2$
3. $(x^3+x^2)-(7x^2+3x)$
4. $(a+b)(c+d)^2$
5. $2(x+y)^2$

Answer

1. $3x^2-2x-9$
2. $4x^2+4x\sqrt{3}+3$
3. x^3-6x^2-3x (This one didn't require any FOILing.)
4. Watch out, that's $(a+b)(c+d)(c+d)$, which becomes $(a+b)(c^2+2cd+d^2)=ac^2+2acd+ad^2+bc^2+2bcd+bd^2$.
5. $2x^2+4xy+2y^2$

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2.1E Exercises

Combining Like Terms

Simplify.

1. $3x + 2 - x + 4$
2. $-11x + 3 + 8x - 3$
3. $2a^2b - 3ab + 5a^2b - ab$
4. $4x^2 - 2xy + 3xy - x^2 + 5y$

Answer

Hint: What does "Simplify" mean when a problem looks like this? You are aiming to combine like terms as much as possible. Go try them and come back.

1. $2x + 6$
2. $-3x$
3. $7a^2b - 4ab$
4. $3x^2 + 3xy + 5y$

Adding and Subtracting Polynomials

Simplify.

1. $(3x^2 - 2x + 5) + (4x^2 + 3x - 1)$
2. $(5a^3 + 2a^2 - 4a) - (a^2 + 6a)$
3. $(4y^2 - 3y + 2) - (2y^2 + 5y - 1) + (y^3 + 3)$
4. $(100x + 1) - (0.5x^2 - 30x) + 0.25x^2$

Answer

Hint: What does "Simplify" mean when a problem looks like this? The parentheses should be going away (watch out for subtractions) and like terms should all be combined. Go try them and come back.

1. $7x^2 + x + 4$
2. $5a^3 + a^2 - 10a$
3. $y^3 + 2y^2 - 8y + 6$
4. $-0.25x^2 + 130x + 1$

Basic FOIL Practice

Expand and simplify.

1. $(x + 2)(x + 3)$
2. $(2x - 1)(x + 1)$
3. $(2y - 5)(3y + 4)$
4. $(a - 4)(a + 4)$
5. $\left(3x - \frac{1}{2}\right)(x - 1)$
6. $\left(\frac{1}{2}x + 1\right)^2$
7. $(2a + \sqrt{2})(a - 2)$
8. $(e^x + 1)(7x + 2)$

Answer

Hint: "Expand" is the math word for multiplying everything out, like FOIL. Then "simplify" will mean combining any like terms.

1. $x^2 + 5x + 6$
2. $2x^2 + x - 1$
3. $6y^2 - 7y - 20$
4. $a^2 - 4$
5. $3x^2 - \frac{7}{2}x + \frac{1}{2}$
6. $\frac{1}{4}x^2 + x + 1$
7. $2a^2 - 4a + a\sqrt{2} - 2\sqrt{2}$
8. $7xe^x + 2e^x + 7x + 2$

? Advanced Expanding

Expand and simplify.

1. $(a + 2)(a^2 - 3a + 1)$
2. $(x - 1)(2x^2 + x + 1)$
3. $(a - b - 1)(a^2 + 2ab + b^2)$
4. $(4x - 2y + 3)(2x + 3y - 1)$
5. $(x + 3)^3$
6. $(a - b)^3$
7. $(x + 1)(x - 2)^2$
8. $(1 + u)(2 + u)^2$

Answer

1. $a^3 - a^2 - 5a + 2$
2. $2x^3 - x^2 - 1$
3. $8x^2 + 8xy - 6y^2 + 2x + 11y - 3$
4. $-a^2 + a^3 - 2ab + a^2b - b^2 - ab^2 - b^3$
5. $x^3 + 9x^2 + 27x + 27$
6. $a^3 - 3a^2b + 3ab^2 - b^3$
7. $x^3 - 3x^2 + 4$
8. $4 + 8u + 5u^2 + u^3$

? Combo Simplification

Simplify.

1. $3(x + h) - 3x$
2. $(x + h)^2 - x^2$
3. $(x - 1)^2 + (y + 1)^2$
4. $\sqrt{x}(\sqrt{x} + 2) - (x^2 + \sqrt{x})$
5. $(3 + b)(3 - b) - (3 + b)(3 + b)$
6. $(3x^2)(x^2 + 1) + (2x)(x^3 - 2)$

Answer

1. $3h$
2. $2xh + h^2$
3. $x^2 - 2x + y^2 + 2y + 2$
4. $x + \sqrt{x} - x^2$
5. $-2b^2 - 6b$ (Be careful with that subtraction!)
6. $-4x + 3x^2 + 5x^4$

Interpreting Expressions With Parentheses

Identify whether the expressions as presented are algebraically equivalent. That is, by using legal algebraic manipulations, can you make them match? When are parentheses important? Investigate by expanding and simplifying each expression as much as possible.

1.	$5x + (x^2 + 4)$	$5x + x^2 + 4$	**6.**	$(2x^2 + 2) + (x - 1)$	$x - 1 + (2x^2 + 2)$
2.	$x^2 + (-3)$	$x^2 - 3$	**7.**	$3 - (a^2 + b^2)$	$3 - a^2 + b^2$
3.	$(6x^2 - 3) - 2$	$(6x^2 - 3)(-2)$	**8.**	$3(x + y)^2$	$(3x + 3y)^2$
4.	$(-2)(6x^2 - 3)$	$-2(6x^2 - 3)$	**9.**	$(-2a + b)^2$	$-2(a + b)^2$
5.	$(9)(3)(x + y)$	$9(3)(x + y)$	**10.**	$(x + 3)^2$	$(x + 3)(x + 3)$

Answer

1. Equivalent.
2. Equivalent.
3. Not equivalent.
4. Equivalent.
5. Equivalent.
6. Equivalent.
7. Not equivalent.
8. Not equivalent! Expand the perfect square **before** distributing the 3.
9. Not equivalent.
10. Equivalent.

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2.2 Factoring Techniques

Learning Objectives

By the end of this section, you will be able to:

- identify when terms have factors in common and write the factored form
- factor simple trinomials with and without nontrivial leading coefficients
- complete the square
- factor by grouping

Common Factors

We practiced some basic factoring in the previous section just by virtue of reverse engineering the distribution process, but let's codify some things here. Let's consider the factors of all the terms in this algebraic expression:

$$3x^2y + 12xy - 9xy^2 \quad \rightarrow \quad 3 \cdot x \cdot x \cdot y + 3 \cdot 4 \cdot x \cdot y - 3 \cdot 3 \cdot x \cdot y \cdot y$$

Notice what they all have in common...

$$3 \cdot x \cdot x \cdot y + 3 \cdot 4 \cdot x \cdot y - 3 \cdot 3 \cdot x \cdot y \cdot y$$

All three terms have at least one 3, x, and y in them. In this case, we say that $3xy$ is the **greatest common factor** of these terms, and we can **factor out** $3xy$ by writing it in front of a set of parentheses. Inside the parentheses, we leave all the leftover factors from each term.

$$3x^2y + 12xy - 9xy^2 = 3xy(x + 4 - 3y)$$

There are lots of reasons you want to be able to recognize when factoring like this is possible. You'll see stuff that looks like this all the time in Calc I...

Example 1

Simplify the **difference quotient** $\dfrac{(x+h)^2 - x^2}{h}$.

Solution

In the numerator, we start by expanding the squared binomial and combine like terms (this should look familiar if you did the homework).

$$\frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h}$$

Then we notice that both terms in the numerator have at least one h, so we factor it out. Once the h is out in front being multiplied by *everything*, it's legal to cancel!

$$\frac{h(2x+h)}{h} = \frac{h}{h} \cdot \frac{2x+h}{1} = 2x + h$$

Exercise 1

Factor out the greatest common factor from these algebraic expressions.

1. $9m^3 + 27m^2 - 3m$
2. $12ab^2 + 24a^2b - 6ab^3$
3. $4p^2 - 2p^4 + 8p^6$

Answer

1. $3m(3m^2 + 9m - 1)$
2. $6ab(2b + 4a - b^2)$

3. $2p^2(2-p^2+4p^4)$

Factoring Trinomials With Leading Coefficient 1

All we're doing in this section is rewinding all of the FOILing we just did. In the equality

$$(x+2)(x-3)=x^2-x-6$$

FOILing is going from left to right, while factoring is going from right to left. It's not terrible, but it takes practice to get a good feel for it.

To factor an expression of the form x^2+bx+c, where b, c are real numbers, and the leading coefficient on x^2 is (secretly) 1:

1. Set up two factors with blank spaces: $(x\quad)(x\quad)$
2. Think of pairs of numbers that multiply to give c, and check what their sums are. Check different options of positive and negative factors as well.
3. If you find a pair that multiply to give c and add to give the number b, place them in the blanks, using $+$ for positive and $-$ for negative.
4. If you have time, check your answer by FOILing and make sure it yields the original expression.

Example 2

Factor the expressions.

1. x^2+6x+8
2. x^2-x-20
3. $x^2-18x+81$
4. (Important!) x^4-4x^2+4

Solution

1. Identify that $c=8$ and $b=6$. I think of pairs of numbers that multiply to give 8: $(1)(8), (-1)(-8), (2)(4), (-2)(-4)$. I want the pair that, when added, will give me a positive 6. The option that works is 2 with 4, so I write the factors $(x+2)(x+4)$. You can FOIL and check.

2. Identify that $c=-20$ and $b=-1$. Pairs of numbers that multiply to give -20 are: $(1)(-20), (-1)(20), (-4)(5), (4)(-5), (-2)(10), (2)(-10)$. You don't have to think of all of them if you find one that works! Notice that $(-5)+4=-1$ like I want, so I can stop there and write $(x+4)(x-5)$.

3. Identify that $c=81$ and $b=-18$. I want a pair of number that multiply to get 81 and add to get -18. I quickly see that $(-9)(-9)$ will do the trick. The answer is $(x-9)(x-9)=(x-9)^2$.

4. This one is tricky, because at first glance it doesn't look like the form I know. But look closer... if I give x^2 the nickname "y,"

$$x^4-4x^2+4=(x^2)^2-4(x^2)+4=y^2-4y+4,$$

suddenly I know what to do with this! I want two numbers that multiply to get 4 and add to get -4. I take -2 and -2 and write $(y-2)(y-2)$. But then remember that y is just a nickname, so replace it with x^2. The final answer is $(x^2-2)(x^2-2)$.

Keep an eye out for those special products we saw in the last section! You can always factor them this way to save time:

"Difference of Squares" : A^2-B^2	$(A+B)(A-B)$
"Perfect Square" : $A^2+2AB+B^2$	$(A+B)(A+B)$

Factoring by Grouping

Sometimes there isn't a greatest common factor among all the terms in an expression, but individually some groups of terms do have things in common. For example, there's no single common factor for all the terms here,

$$x^3+x^2+4x+4$$

$x^3 + x^2 + 4x + 4$

but if I look at the first two and the last two separately, I notice

$$x^3 + x^2 + 4x + 4 = x^2(x+1) + 4(x+1).$$

Then when I compare the black piece and the red piece, I see they both have $(x+1)$ in common, and I can factor *that* guy out, to get $(x+1)(x^2+4)$. Cute.

? Exercise 2

Factor $x^3 - 2x^2 - 9x + 18$.

Answer

$$x^3 - 2x^2 - 9x + 18 = x^2(x-2) - 9(x-2) = (x-2)(x^2-9) = (x-2)(x+3)(x-3)$$

Factoring Trinomials With Nontrivial Leading Coefficient

We know how to handle expressions that start with x^2, but what about something like $2x^2 + 3x + 1$? Not to "uphill both ways" on you, but when I learned algebra, we factored these by "Guess & Check," setting up blanks $(2x\quad)(x\quad)$ and trying different options to see what worked. I still do it that way, but students have shared with me this thing called **The ac Method.**

To factor an expression of the form $ax^2 + bx + c$ using the ac method,

1. Compute the product ac.
2. Find a pair of numbers m and n that multiply to give ac and add to give b.
3. Replace the b with $(m+n)$ and split up the terms: $ax^2 + bx + c = ax^2 + (m+n)x + c = ax^2 + mx + nx + c$.
4. Now that there are four terms, factor by grouping.

✓ Example 3

Factor $2x^2 - x - 3$.

Solution

Identify $a = 2, b = -1, c = -3$ and compute the product $ac = -6$. I think of some numbers that multiply to give -6: -2 and 3, or 2 and -3... Wait, $2 + (-3) = -1$ which is b, so that's what I want. Now I split the middle term,

$$2x^2 + (-1)x - 3 = 2x^2 + (2-3)x - 3 = 2x^2 + 2x - 3x - 3$$

and factor by grouping:

$$2x^2 + 2x - 3x - 3 = 2x(x+1) - 3(x+1) = (x+1)(2x-3).$$

? Exercise 3

Factor $3x^2 + 14x + 8$.

Answer

Identify $a = 3, b = 14, c = 8$ and compute the product $ac = 24$. The m and n should be 12 and 2. Now I split my bx term,

$$3x^2 + 14x + 8 = 3x^2 + (12+2)x + 8 = 3x^2 + 12x + 2x + 8$$

and factor by grouping:

$$(3x^2 + 12x) + (2x + 8) = 3x(x+4) + 2(x+4) = (x+4)(3x+2).$$

Caution

Don't make extra work for yourself if it's not necessary! If you see ax^2+bx+c with $a \neq 1$, **first check** for common factors from all terms and pull those out! For example, you don't need the ac method for $5x^2+20x-60$. All of the terms have a factor of 5 in common! If you take that out first to get $5(x^2+4x-12)$, you can use the un-FOIL-ing method on the expression inside the parentheses.

Completing the Square

Completing the square is a technique that you will need if you ever want to take the equation of a circle and figure out its center and radius, or take the equation of a parabola and figure out its vertex, or solve a certain type of quadratic equation, or use substitution tricks in Calculus II, or... It's called "completing the square" because we're going to force a perfect square like $(a+b)^2$ to appear. Let's introduce it here and get it simmering on the back burner.

I'm starting with the polynomial expression x^2-4x+7, and I want to end up with an equivalent expression in the form $(x-h)^2+k$, where h and k are some constants (plain ol' numbers). Why? I'll tell you when you're older. I just really wish I had a perfect square like $(x-h)^2$ in my life. First, I write my expression down with a big space, putting the 7 in time out and ignoring him for now:

$$x^2-4x \qquad +7$$

I do a little scratch work, taking the middle term's coefficient -4, cutting him in half, and squaring the result. That is, $-4 \xrightarrow{\div 2} -2 \xrightarrow[\text{squared}]{} 4$. I **add and subtract** 4 to my expression, which technically changes nothing!

$$x^2-4x+4+7-4$$

Now I look at the first three terms and factor them as usual, and simplify the constants hanging out at the end.

$$(x^2-4x+4)+7-4=(x-2)(x-2)+3=(x-2)^2+3$$

Hey presto! It worked!

Completing the Square

Starting with the form x^2+bx+c, and aiming for the form $(x-h)^2+k$, where $b, c, h,$ and k are constants...

1. Write out $x^2+bx \qquad +c$, leaving space.
2. Compute $\frac{b}{2}$ and $\left(\frac{b}{2}\right)^2$.
3. Add $\left(\frac{b}{2}\right)^2$ in the blank space and subtract $\left(\frac{b}{2}\right)^2$ from the end:

$$x^2+bx+\left(\frac{b}{2}\right)^2+c-\left(\frac{b}{2}\right)^2$$

4. Factor the first three terms as a perfect square $\left(x+\frac{b}{2}\right)^2$, and simplify the constants at the end.

People don't generally love completing the square the first eight thousand or so times they see it, but it will come in handy if you remember it.



Head over to the exercises section and practice your factoring skills!

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2.2E Exercises

Greatest Common Factors

Factor out the greatest common factor from all terms.

1. $2x^2 + 4x + 16$
2. $12x^3 + 36x^2 + 24x$
3. $m^2n + mn^2 + mn$
4. $5a + 25ab + 50a^2$

Answer

1. $2(x^2 + 2x + 8)$
2. $12x(x^2 + 3x + 2)$
3. $mn(m + n + 1)$
4. $5a(1 + 5b + 10a)$

unFOILing

Factor.

1. $x^2 + 8x + 15$
2. $x^2 - 5x - 24$
3. $b^2 + 12b + 35$
4. $x^2 + 96x - 400$
5. $x^4 - 5x^2 + 6$
6. $y^6 - 3y^3 + 2$

Answer

Hint: When a math problem just says "Factor," it means your final answer should be in a form like (____)(____). You shouldn't just factor something out of a few terms and leave other terms added/subtracted. Go try them and come back.

1. $(x + 3)(x + 5)$
2. $(x - 8)(x + 3)$
3. $(b + 7)(b + 5)$
4. $(x - 4)(x + 100)$
5. Hint: nickname trick... $(x^2 - 3)(x^2 - 2)$
6. $(y^3 - 1)(y^3 - 2)$

The ac Method

Factor.

1. $2x^2 + 7x - 15$
2. $3x^2 - 5x - 2$
3. $4x^2 - 4x - 3$
4. $3 + 13x + 4x^2$

Answer

1. $(x + 5)(2x - 3)$
2. $(x - 2)(3x + 1)$ Hint: if you're stuck at the grouping stage, realize that $\ldots + x - 2$ can be written as $\ldots + 1(x - 2)$.
3. $(2x - 3)(2x + 1)$
4. $(4x + 1)(x + 3)$

Factoring by Grouping

Factor.

1. $3x^3 - 6x^2 + 5x - 10$
2. $2x^3 + 3x^2 - 4x - 6$
3. $x^4 + 2x^3 - 3x^2 - 6x$
4. $2x^3 + 3x^2 + 2x + 3$

Answer

1. $(x-2)(3x^2+5)$
2. $(2x+3)(x^2-2)$
3. $(x+2)(x^3-3x)$ or $x(x+2)(x^2-3)$
4. $(x^2+1)(2x+3)$

Completing the Square

Complete the square to write in the form $(x-h)^2 + k$, where h, k are constants.

1. $x^2 - 6x - 1$
2. $x^2 - 10x + 26$
3. $x^2 + 2x + 4$
4. $x^2 + 4x + 8$

Answer

1. $(x-3)^2 - 10$
2. $(x-5)^2 + 1$
3. $(x-(-1))^2 + 3$ (The factors look like $(x+1)$, but we wanted a subtraction, so we write $x+1 = x-(-1)$ with "minus a minus.")
4. $(x+2)^2 + 4$ or $(x-(-2))^2 + 4$

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2.3 Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- assess the domain of an algebraic expression and describe it with various notations
- add, subtract, multiply, and divide rational expressions
- simplify and cancel legally in rational expressions

A First Look at Domain

A super important question to keep in the back of your mind while looking at an expression (or later, a function) is, "What kinds of values are actually legal to plug in for the variable here?" For example, I've told you two illegal actions already: 1.) Dividing by zero, and 2.) Taking an even root of a negative number. We'll see more legality concerns when we learn about logarithmic functions and trig functions.

> The **domain** of an algebraic expression is the set of all numbers that **are legal** to plug in for the variable.

When we're reporting back information about domain, we use a variety of **notation** styles. These are all ways of saying, "The set of all real numbers, except for the number 0."

$$\{x \mid x \neq 0\}, \qquad \{x \ : \ x \neq 0\}, \qquad (-\infty, 0) \cup (0, \infty), \qquad ``x \in \mathbb{R}, x \neq 0''$$

Here's a rundown on the symbols involved:

- A pair of **braces**, { }, is code for "I'm about to give you a bunch of objects, such as a set of numbers." It starts with a nickname for the elements, like the variable name x.
- Inside the braces, a **pipe symbol** | is code for **"such that"** and it's telling you the rules for x are coming next. You might also see a colon : used for this purpose.
- A pair of numbers (or including the ∞ symbol) with parentheses (,) or **brackets** [,] or a mix of both like (,] or [,) are telling you about an **interval:** a chunk of the number line starting at the first number and ending at the second number. A round parenthesis indicates that the endpoint number is **not included,** and a square bracket indicates that it **is included.**
- The symbol $\cup$ stand for **"union"** and it means "this set **and** that set." So $(-\infty, 0) \cup (0, \infty)$ means, "All real numbers from negative infinity up to but not including 0, and also all the numbers strictly bigger than zero, forever to infinity." If you see the symbol $\cap$ between two sets, it stands for **"intersection"** and means "take all the numbers that appear in **both** of these sets."
- The symbol $\in$ means **"in"** and the symbol $\mathbb{R}$ means "the set of real numbers," so $x \in \mathbb{R}$ means "x is in the set of real numbers," aka "x is any real number."

You can illustrate some sets using a number line. The picture below is showing the set $(-4, \infty)$.



The next picture shows $[-2, 4)$.



Make sure to read the next example carefully.

Example 1

What is the domain of the expression?

1. $\frac{1}{x}$
2. $\sqrt{x}$
3. x^2+2
4. $\frac{1}{\sqrt{x}}$
5. $\frac{1}{x-2}$
6. $\sqrt[4]{x+1}$

Solution

1. In this expression, if we let x be 0, terrible things will happen. So we say the domain is "all real numbers except zero," written $\{x \mid x \neq 0\}$ or even just "$x \neq 0$" if it's clear what we mean.
2. We're not allowed to take square roots of negative numbers, so we can only let x be 0 or bigger. We write $[0, \infty)$. You can also write this as $\{x \mid x \geq 0\}$.
3. Trick question! Everything is legal for x because I'm not at risk of dividing by zero or taking a bad root. The domain is "all real numbers," written $x \in \mathbb{R}$ or $(-\infty, \infty)$.
4. Because of the square root, I can't let x be negative, BUT ALSO, because I'm dividing here, I can't let x be 0 either. So the domain is "x strictly greater than 0," written $\{x \mid x > 0\}$ or $(0, \infty)$.
5. If I plug in a number for x that causes the denominator to be 0, it's illegal! What number would have that effect? Yeah, if x is 2. So he's not allowed. The domain is $\{x \mid x \neq 2\}$. Notice that $x = 0$ is fine, because that would lead to dividing by -2, which is legal.
6. I can't let whatever is under the fourth root be negative, so I have to only choose x-values that result in 0 or positive numbers. As long as x is at least -1, the expression $x+1$ is at least 0. So the domain is $[-1, \infty)$.

Exercise 1

1. Translate "the set of real numbers starting at but not including 2, ending at and including 12" into math.
2. Translate $(-\infty, 18]$ into English.
3. What is the domain of the expression $\frac{1}{x+4}$?

Answer

1. $(2, 12]$ or even $\{x \mid 2 < x \leq 12\}$.
2. "The set of all real numbers less than or equal to 18."
3. The only problem child here would be plugging in -4, which would cause division by 0. So $\{x \mid x \neq -4\}$.

Rational Expressions

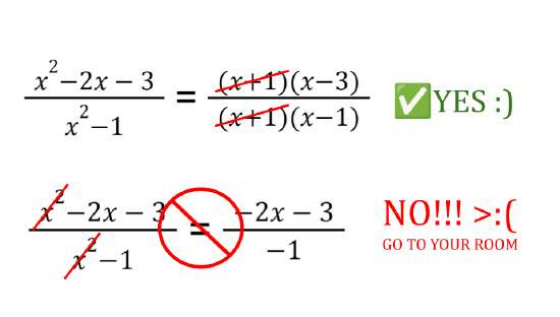
A **rational expression** is just a quotient (get it, ratio-nal) of polynomials. We need to be able to deal with these expressions in order to deal with rational functions and simplify stuff later in Calculus. Here are some rational expressions:

$$\frac{x}{x^2-1}, \qquad \frac{x^3-x^2+4}{x-3}, \qquad \frac{3a^{12}-6a^7}{a^4+a+1}$$

Something like $\frac{x^{3/2}+1}{x}$ would not count as a rational expression because the top is not a polynomial (it doesn't have only positive integer powers of x).

Caution

Be very careful with cancelling in rational expressions. You can only cancel a factor if it is being multiplied by **everything,** aka the numerator and denominator have a **common factor!**



(please learn this now—if i see one more calc student do this on an exam i'm gonna jump off a bridge)

Now that I've screamed my head off about simplifying, let's learn multiplication and division. The good news is, this is just a combo of the working with fractions section and the multiplying polynomials section.

To **multiply** rational expressions, use the usual fraction multiplication rule $\frac{A}{B} \cdot \frac{C}{D} = \frac{AB}{CD}$, cancelling any common factors. In my class, you can leave the final answer in factored form (do not FOIL top and bottom for no reason).

Example 2

Simplify.

1. $\frac{3x^2}{4y} \cdot \frac{2y}{5x^3}$
2. $\frac{1}{a+2} \cdot \frac{a-2}{a^2-4}$
3. $\frac{x^2-7x+6}{x-3} \cdot \frac{x^2-9}{x-6}$

Solution

1. Nothing crazy is going on here, just letters involved. We have

$$\frac{3x^2}{4y} \cdot \frac{2y}{5x^3} = \frac{6x^2y}{20x^3y} = \frac{3}{10x}$$

2. Now there are polynomials involved with multiple terms, so we have to be careful with products and cancelling. Factor everything as much as possible (look for shortcuts like "difference of squares") beforehand to see if there are common factors to kill off, and use the heck outta parentheses.

$$\frac{1}{a+2} \cdot \frac{a-2}{a^2-4} = \frac{1}{(a+2)} \cdot \frac{(a-2)}{(a+2)(a-2)} = \frac{(a-2)}{(a+2)(a+2)(a-2)} = \frac{1}{(a+2)^2}$$

3. Factor everything beforehand to see if anything magical happens, then multiply and cancel common factors.

$$\frac{x^2-7x+6}{x-3} \cdot \frac{x^2-9}{x-6} = \frac{(x-1)(x-6)}{(x-3)} \cdot \frac{(x-3)(x+3)}{(x-6)} = \frac{(x-1)(x+3)}{1} = (x-1)(x+3)$$

As for **dividing** rational expressions, if you ever see something like $\frac{x^2+2x+1}{x-1} \div \frac{x+1}{x-1}$ or a compound fraction $\frac{\frac{x^2+2x+1}{x-1}}{\frac{x+1}{x-1}}$ (sweet sister Frances!), just do the old **"multiply by the reciprocal"** trick. Get your practice here:

Exercise 2

1. Simplify $\frac{x^2+2x+1}{x-1} \div \frac{x+1}{x-1}$.
2. (Also practicing the ac method...) Simplify $\frac{\frac{y^2-36}{2y^2+11y-6}}{\frac{2y^2-2y-60}{8y-4}}$.

Answer

1. Flip over the second fraction and multiply instead:

$$\frac{x^2+2x+1}{x-1}\cdot\frac{x-1}{x+1}=\frac{(x+1)^2}{(x-1)}\cdot\frac{(x-1)}{(x+1)}=\frac{(x+1)}{1}=x+1$$

2. For this one, take the reciprocal of the downstairs fraction and multiply by that. This is your factoring practice challenge as well.

$$\frac{(y+6)(y-6)}{(2y-1)(y+6)}\cdot\frac{4(2y-1)}{(2y+10)(y-6)}=\frac{4}{2y+10}=\frac{4}{2(y+5)}=\frac{2}{y+5}$$

To **add and subtract** rational expressions, first factor everything as much as possible. Get a common denominator by multiplying each fraction top and bottom by any factors needed (use parentheses!). Then use the fraction rule $\frac{A}{C}+\frac{B}{C}=\frac{A+B}{C}$ and simplify.

Example 3

Simplify $\frac{1}{x^2-1}-\frac{2}{(x+1)^2}$.

Solution

The denominator of the first fraction is a difference of squares and factors as $(x+1)(x-1)$. The least common denominator for these two expressions must then contain two copies of $(x+1)$ and one copy of $(x-1)$. I multiply each term top and bottom by what it's lacking:

$$\frac{1\cdot(x+1)}{(x+1)(x-1)\cdot(x+1)}-\frac{2\cdot(x-1)}{(x+1)^2\cdot(x-1)}=\frac{(x+1)}{(x+1)^2(x-1)}-\frac{2(x-1)}{(x+1)^2(x-1)}$$

Now that the denominators match, we subtract (be careful and use parentheses!) and simplify.

$$\frac{(x+1)-2(x-1)}{(x+1)^2(x-1)}=\frac{x+1-2x+2}{(x+1)^2(x-1)}=\frac{-x+3}{(x+1)^2(x-1)}$$

Exercise 3

Simplify $\frac{8x}{x^2+4x+4}+\frac{2}{x-2}$

Answer

$\frac{10x^2-8x+8}{(x+2)^2(x-2)}$ or $\frac{2(5x^2-4x+4)}{(x+2)^2(x-2)}$

Another type of compound fraction you will encounter (esp in Calc I) combines addition or subtraction of rational expressions in the numerator of a giant fraction. There are two ways of dealing with this situation.

Example 4

Simplify the difference quotient $\frac{\frac{1}{1+x+h}-\frac{1}{1+x}}{h}$.

Solution

Option 1 is to look just at the numerator for the moment and use a common denominator to combine the terms. These denominators are distinct factors, so we have to multiply top and bottom to make them match. I could move the overall division by h out front of some parentheses to make this easier to write:

$$\frac{1}{\square}\left(\frac{1\cdot(1+x)}{\square}-\frac{1\cdot(1+x+h)}{\square}\right)$$

$$= \frac{1}{h}\left(\frac{(1+x)-(1+x+h)}{(1+x+h)(1+x)}\right) = \frac{1}{h}\left(\frac{1+x-1-x-h}{(1+x+h)(1+x)}\right)$$

Then I notice that when I simplify the numerator, everything kills each other off but the h, which I can cancel with the $\frac{1}{h}$ out front. This cleans up nice to

$$\frac{1}{h}\left(\frac{h}{(1+x+h)\cdot(1+x)}\right) = \frac{1}{(1+x+h)\cdot(1+x)}.$$

Option 2 is to use a technique called **clearing denominators.** To do this, you multiply the compound fraction top and bottom by whatever would be needed to cancel every denominator involved. This is the least common multiple of all the denominators you see. Watch what happens when we distribute.

$$\frac{\left(\frac{1}{1+x+h}-\frac{1}{1+x}\right)}{h}\cdot\frac{(1+x+h)(1+x)}{(1+x+h)(1+x)} = \frac{\left(\frac{(1+x+h)(1+x)}{(1+x+h)}-\frac{(1+x+h)(1+x)}{(1+x)}\right)}{h}$$
$$= \frac{(1+x)-(1+x+h)}{h(1+x+h)(1+x)}$$

This is the same expression we found during Option 1, and simplifies the same way! Clearing denominators can be a great way to clean up compound fractions in general.

Rationalizing the Denominator

This isn't about "rational expressions" per se, but while we're talking about fractiony stuff, let me show you a special technique. Any time you see an expression of the form $A+B\sqrt{C}$, "something plus somebody with a square root," there is a trick we can do to make the square root go away. Multiplying $A+B\sqrt{C}$ by his **conjugate radical,** $A-B\sqrt{C}$, causes something magical to happen. FOIL this out...

$$(A+B\sqrt{C})(A-B\sqrt{C}) = A^2+AB\sqrt{C}-AB\sqrt{C}-(B\sqrt{C})^2 = A^2-B^2C$$

Et voilà, no radicals in sight. To make the conjugate of an expression, just copy it but change the sign in the middle. We describe this in math like so: The conjugate of $A\pm B\sqrt{C}$ is $A\mp B\sqrt{C}$. This is a trick that comes in handy in Calc I when you're evaluating limits, usually applied to a numerator or denominator of a fractional expression. Just make sure to multiply on top **and** bottom, so you're not illegally changing the expression!

✔ Example 5

Rationalize the numerator of $\frac{\sqrt{9+h}-3}{h}$.

Solution

The conjugate of the numerator is $\sqrt{9+h}+3$, so we multiply top and bottom by that:

$$\frac{(\sqrt{9+h}-3)}{h}\cdot\frac{(\sqrt{9+h}+3)}{(\sqrt{9+h}+3)} = \frac{(\sqrt{9+h})^2-3\sqrt{9+h}+3\sqrt{9+h}-3^2}{h(\sqrt{9+h}+3)}$$

Now simplify...

$$(\frac{\sqrt{9+h})^2-9}{h(\sqrt{9+h}+3)} = \frac{(9+h)-9}{h(\sqrt{9+h}+3)} = \frac{h}{h(\sqrt{9+h}+3)} = \frac{1}{(\sqrt{9+h}+3)}$$

? Exercise 4

Rationalize the denominator of $\frac{3}{x+\sqrt{1-x}}$.

Answer

$$\frac{3(x-\sqrt{1-x})}{x^2+x-1}$$

To wrap up this section, I'd like to call out some mistakes I see students make ALL THE TIME so you can avoid them from the beginning.

YES	NO!!!!!
$A-(B+C)=A-B-C$	$A-(B+C)\neq A-B+C$
$(AB)^2=A^2B^2$	$(A+B)^2\neq A^2+B^2$
$\frac{A+B}{C}=\frac{A}{C}+\frac{B}{C}$	$\frac{A}{B+C}\neq\frac{A}{B}+\frac{A}{C}$
$\sqrt{AB}=\sqrt{A}\sqrt{B}$	$\sqrt{A+B}\neq\sqrt{A}+\sqrt{B}$
$\frac{AB}{AC}=\frac{B}{C}$	$\frac{A+B}{A}\neq B$
$\sqrt{a^2b^2}=\sqrt{(ab)^2}=ab, \quad \text{for } a,b\geq 0$	$\sqrt{a^2+b^2}\neq a+b$

And one last time, for the people in the back...



$$\frac{x^2-2x-3}{x^2-1}=\frac{-2x-3}{-1}$$

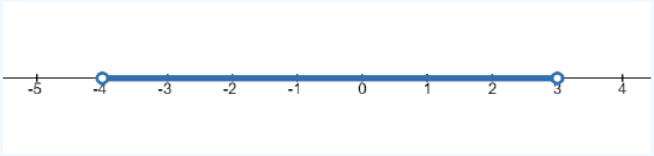
$$\frac{x^2-2x-3}{x^2-1}=\frac{(x+1)(x-3)}{(x+1)(x-1)}$$

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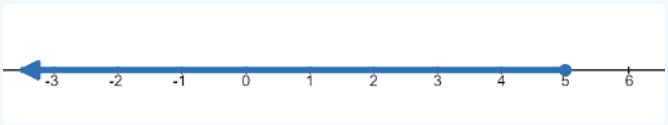
2.3E Exercises

Notation

1. Translate $(-8, 2] \cup [3, 5)$ into English.
2. Translate $[3, \infty)$ into English.
3. Translate "the set of all real numbers except 7" into math.
4. Translate "the set of all real numbers strictly between 1 and 4" into math.
5. Translate $\{x \mid x > 0\}$ into English.
6. Translate $\{x \mid x \neq 0\}$ into English.
7. Draw a number line representing the set $(-\infty, 5]$.
8. What set does the number line below represent?



Answer

1. "The set of real numbers strictly greater than -8 but less than or equal to 2, together with the set of real numbers greater than or equal to 3 but strictly less than 5."
2. "The set of all real numbers greater than or equal to 3."
3. $\{x \mid x \neq 7\}$, or $x \in \mathbb{R}, x \neq 7$, etc.
4. $\{x \mid 1 < x < 4\}$, or in interval notation, $(1, 4)$.
5. "The set of positive real numbers."
6. "All real numbers x except for $x = 0$."
7. 

8. $\{x \mid -4 < x < 3\}$ or in interval notation, $(-4, 3)$.

Domain

What is the domain of the expression?

1. $\dfrac{1}{3-x}$
2. $\dfrac{2}{(x-1)(x-2)}$
3. $\dfrac{3}{x^2-2x-3}$
4. $\dfrac{1}{x^2+5x+6}$
5. $1+x+x^2+x^3$
6. $\sqrt{x+4}$
7. $\dfrac{1}{\sqrt{x}}$
8. $(x-7)(x+2)$
9. $\sqrt[3]{x+8}$

Answer

1. $\{x \mid x \neq 3\}$
2. $\{x \mid x \neq 1, x \neq 2\}$
3. $\{x \mid x \neq -1, 3\}$

https://math.libretexts.org/@go/page/152902

4. $\{x \mid x \neq -2, -3\}$
5. No legality issues, domain is all real numbers.
6. $\{x \mid x \geq -4\}$
7. $\{x \mid x > 0\}$
8. No legality issues, domain is all real numbers.
9. Careful! No legality issues because we are allowed to put negative numbers into odd roots like cube roots! All real numbers.

? Adding and Subtracting Rational Expressions

Simplify.

1. $\frac{8y}{y^2-16} - \frac{4}{y-4}$
2. $\frac{3}{x-3} + \frac{2}{x-2}$
3. $\frac{x^2-6x}{x^2-1} - \frac{3x+2}{1-x^2}$
4. $\frac{5}{2x-10} - \frac{7}{4x-12}$

Answer

Hint: What does "simplify" mean in this type of problem? When you see rational expressions added or subtracted, your aim is to get them combined into a single fractional term (factor first to make life easier), expand and simplify the numerator as much as possible, re-factor everything you can, and see if any common factors cancel. Only then are you finished.

1. $\frac{4}{y+4}$
2. $\frac{5x-12}{(x-3)(x-2)}$
3. $\frac{x-2}{x+1}$
4. $\frac{3x+5}{4(x-5)(x-3)}$

? Multiplying and Dividing Rational Expressions

Simplify.

1. $\frac{3x}{x^2-5x+6} \cdot \frac{9x-27}{x^3}$
2. $\frac{2x+1}{x^2-4} \cdot \frac{x^3-4x}{2x^2-5x-3}$
3. $\frac{t^2+2t-3}{t^2-2t-3} \cdot \frac{3-t}{3+t}$
4. $\frac{\frac{x^3}{x+1}}{\frac{x}{x^2+2x+1}}$

Answer

Hint: What does "simplify" mean for this type of problem? Your aim is to factor as much as possible, do any cancelling you can, perform the multiplication or division, and then simplify.

1. $\frac{27}{x^2(x-2)}$
2. $\frac{x}{x-3}$
3. $-\frac{t-1}{t+1}$ or $\frac{1-t}{t+1}$

4. $x^2(x+1)$

Compound Fraction Combo Practice

In the compound fractions, get common denominators and add or subtract as needed in numerator and denominator until you have a single fraction of rational expressions. Then compute and simplify if necessary.

1. $\dfrac{1+\frac{1}{x}}{\frac{1}{x}-1}$
2. $\dfrac{1+\frac{1}{x-2}}{1-\frac{1}{x+3}}$

Answer

1. $\dfrac{x+1}{1-x}$
2. $\dfrac{(x-1)(x+3)}{(x-2)(x+2)}$

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2.4 Algebravaganza (Chapter 2 Study Guide)

Learning Objectives

In this section, you will:

- bring together all of the techniques we have learned from Chapter 1 and Chapter 2 to perform multi-step simplifications of algebraic expressions, especially expressions commonly seen in Calculus.

This section is a big ol' algebra skills extravaganza (algebravaganza, if you will) that is also going to serve as the Chapter 2 study guide, because it is synthesizing everything we've just learned into frankenproblems. Your aim at this point is to become so fluent in algebra skills that you can FOIL and factor in your sleep, and your future professors call me in a few months and say, "Great work on these students, De Wolf! Let me send you an edible arrangement!" To which I'll say, "Aw shucks, it's all the students' hard work and dedication, plus also I don't like cantaloupe." First, a quick highlights reel from this chapter. You will also need to remember all the rules about exponents and radicals from Chapter 1, and general arithmetic with numbers of course.

- A **polynomial** in variable x is an expression that looks like $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, having real number coefficients and positive integer powers of x. The **degree** is the highest power n. The **leading coefficient** is a_n. The **constant term** is a_0.
- You can add and subtract polynomials by combining **like terms,** i.e. terms that have the same exact variables to the same exact powers in them.
- You can distribute multiplication through a sum by the **distributive property of addition:** $a(b+c) = ab + ac$. This allows you to multiply polynomials by expanding, or **FOILing:** $(a+b)(c+d) = ac + ad + bc + bd$.
- Un-FOILing or un-distributing is **factoring.** We covered several techniques.
  - Factoring $x^2 + bx + c$ by finding two numbers that multiply to give c and add to give b.
  - Factoring $ax^2 + bx + c$ using the ac method.
  - Factoring by grouping.
  - Completing the square.
  - Special common situations like "difference of squares" $A^2 - B^2 = (A+B)(A-B)$ or "perfect squares" $A^2 + 2AB + B^2 = (A+B)^2$.
- A **rational expression** is a fraction with a polynomial in the numerator and a polynomial in the denominator. Because of the division, a rational expression could have a **domain** issue; i.e. there might be restrictions on what can be plugged in for the variable. Talking about domain is just saying what values are allowable for x, and there are two things to watch out for so far:
  - Any number(s) that, when plugged in for x, would cause division by zero.
  - Any number(s) that, when plugged in for x, would cause us to take the square root of a negative number.
- Multiplying rational expressions follows fraction multiplication rules, it just requires FOILing and factoring sometimes. Division is done by multiplying by the reciprocal.
- Adding and subtracting rational expressions requires a common denominator, which means some factoring and multiplications, but essentially follows the addition and subtractions rules for basic fractions.
- **Rationalizing** is when you see an expression like $A + B\sqrt{C}$ and multiply it by its **conjugate**, $A - B\sqrt{C}$. If you see $A - B\sqrt{C}$ instead, the conjugate is $A + B\sqrt{C}$. The point of this technique is to make radicals go away.

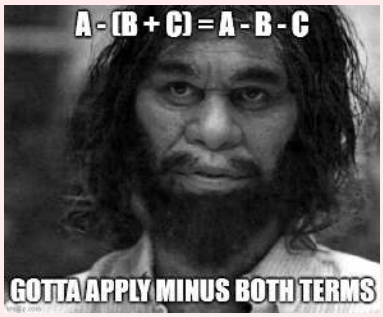
Caution

Be on your guard against these common errors!

YES	NO!!!!!
$A-(B+C) = A-B-C$	$A-(B+C) \neq A-B+C$
$(AB)^2 = A^2B^2$	$(A+B)^2 \neq A^2+B^2$

$\frac{A+B}{C}=\frac{A}{C}+\frac{B}{C}$	$\frac{A}{B+C}\neq\frac{A}{B}+\frac{A}{C}$
$\sqrt{AB}=\sqrt{A}\sqrt{B}$	$\sqrt{A+B}\neq\sqrt{A}+\sqrt{B}$
$\frac{AB}{AC}=\frac{B}{C}$	$\frac{A+B}{A}\neq B$
$\sqrt{a^2b^2}=\sqrt{(ab)^2}=ab,\quad a,b\geq 0$	$\sqrt{a^2+b^2}\neq a+b$

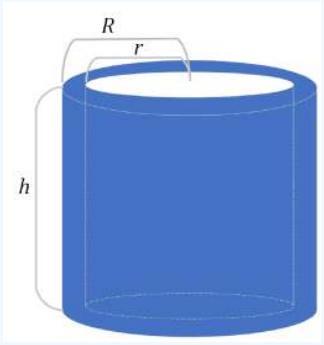
I recommend that you go through each line and say out loud to yourself in caveman-speak what is going on. For example, the first line is



Now let's get into practice problems. I'm taking these from actual literal expressions that arose in my past courses that students needed to simplify as part of doing calculus or applied math problems. This way, when you get to those later classes, you'll be like, *"In the immortal words of Dr. Taylor Swift...I think I've seen this film before."* And everyone will look impressed!

Example 1

Drainage culverts are made from large cylindrical shells like the one shown below, cast in concrete. If h represents the height of the cylinder, r represents the radius from the center to the inside of the shell, and R represents the outer radius, then you can calculate the volume of concrete needed using the formula $\pi R^2 h-\pi r^2 h$. Factor this expression.



Another way to compute the volume of concrete needed is to use the formula $2\pi\cdot\frac{(R+r)}{2}\cdot h\cdot(R-r)$. Expand and simplify this expression to confirm that the results are equivalent!

Solution

For the first formula, observe that both terms have the factors π and h in common, so pull those out to get $\pi h(R^2-r^2)$. Then notice that the parentheses contain a difference of squares! So the final answer is $\pi h(R+r)(R-r)$.

Next, we need to simplify the second option. We first see that there is a multiplication by 2 out front and later a division by 2, so we can cancel those. Then we just FOIL (or remember the difference of squares identity) and simplify.

$$2\pi\cdot\frac{(R+r)}{2}\cdot h\cdot(R-r)=\pi h(R+r)(R-r)$$

This matches the factored form we found for the first expression.

Exercise 1

If an expression is in factored form, expand it completely. If it is in expanded form, factor it completely.

1. $x(20-2x)(10-2x)$
2. $(x-5)^2+3$
3. $1600+40x-2x^2$
4. $2x^2+4xh$

Answer

1. $4x^3-60x^2+200x$. This expression calculates the volume of an open box created by cutting the corners from a 20×10 rectangle of cardboard, and then folding up the sides.
2. $x^2-10x+28$. This expression appears in an equation whose graph is a parabola, in what is called vertex form. Expanding takes the equation from vertex form to what is called standard form.
3. $-2(x+20)(x-40)$. This expression could appear in the equation of a downward-opening parabola and could be useful for determining where it crosses the x-axis.
4. $2x(x+2h)$. This expression calculates the surface area of a closed box with square $x\times x$ base and height h.

Example 2

1. FOIL and simplify: $(3x+\sqrt{x})(3x-\sqrt{x})$
2. Simplify: $\dfrac{x^2y^{-1}}{x^{-5}}$
3. Write as x to a single power: $\sqrt[3]{x\sqrt{x}}$

Solution

1. FOIL or notice that this is a difference of squares! $(3x)^2-(\sqrt{x})^2=9x^2-x$.
2. We get rid of negative powers by moving from numerator to denominator or vice versa as needed, then use the power rules. Remember "multiplication in the base translates to addition in the exponent." We get $\dfrac{x^2y^{-1}}{x^{-5}}=\dfrac{x^2x^5}{y}=\dfrac{x^7}{y}$.
3. This is all about writing the cube root and square root as fractional powers and being very careful with power rules. Remember that you can bring an exponent inside a product, and "power to a power translates to multiplication in the exponent," and "multiplication in the base translates to addition in the exponent." Use parentheses!

$$\sqrt[3]{x\sqrt{x}}=(x\sqrt{x})^{\frac{1}{3}}=(x(x)^{\frac{1}{2}})^{\frac{1}{3}}=x^{\frac{1}{3}}(x^{\frac{1}{2}})^{\frac{1}{3}}=x^{\frac{1}{3}}x^{\frac{1}{6}}=x^{\frac{1}{3}+\frac{1}{6}}=x^{\frac{1}{2}}$$

Another valid path is

$$\sqrt[3]{x\sqrt{x}}=(x\sqrt{x})^{\frac{1}{3}}=(x(x)^{\frac{1}{2}})^{\frac{1}{3}}=(x^{1+\frac{1}{2}})^{\frac{1}{3}}=(x^{\frac{3}{2}})^{\frac{1}{3}}=x^{\frac{1}{2}}$$

Exercise 2

Simplify.

1. $\dfrac{4x^2y^{-3}}{(2x)^3y}$
2. $(10+\sqrt{9x})^2$
3. $x\sqrt{x}+\left(\dfrac{x}{\sqrt[4]{x}}\right)^2$

Answer

1. $\dfrac{1}{2xy^4}$
2. $100+60\sqrt{x}+9x$
3. $2x^{\frac{3}{2}}$

Example 3

Simplify the rational expression $\dfrac{2x^2-2x-4}{x^2+2x-8}$.

Solution

We need to factor top and bottom completely and cancel any common factors. We will need the ac method (or guess and check) to handle the numerator since it's in the form ax^2+bx+c. Let's do that scratch work first.

The product ac is -8. We need two numbers that multiply to -8 and add to -2. These would be -4 and 2, so we split the middle term using that fact and then factor by grouping:

$$2x^2+(-2)x-4 = 2x^2+(-4+2)x-4 = 2x^2-4x+2x-4$$

$$= 2x(x-2)+2(x-2) = (2x+2)(x-2)$$

Are we done? No! That was just the numerator, lol. The denominator factors as $(x+4)(x-2)$. So we have

$$\frac{(2x+2)(x-2)}{(x+4)(x-2)} = \frac{2x+2}{x+4}$$

Okay, psych! I just wanted to make you practice the ac method. I could have noticed from the beginning that every term in the numerator has a factor of 2 in it, and pulled that out first. Then I would have done this:

$$\frac{2(x^2-x-2)}{x^2+2x-8} = \frac{2(x-2)(x+1)}{(x+4)(x-2)} = \frac{2(x+1)}{(x+4)}$$

Exercise 3

Simplify the rational expression $\dfrac{3x^2-3x}{x^2-2x+1}$.

Answer

$\dfrac{3x}{x-1}$

Example 4

The following expression comes from the equation of an ellipse in standard form. Perform the addition, expand the products, and simplify if possible.

$$\frac{(x-2)^2}{2^2}+\frac{(y-1)^2}{3^2}$$

Solution

The least common denominator for the two fractions is $9\cdot 4 = 36$. We multiply top and bottom by what is needed for each fraction and add:

$$\frac{(x-2)^2\cdot 9}{4\cdot 9}+\frac{(y-1)^2\cdot 4}{9\cdot 4} = \frac{9(x-2)^2+4(y-1)^2}{36}$$

Then it's just a matter of FOILing and simplifying to get $\dfrac{9x^2-36x+4y^2-8y+40}{36}$. You could also split the fraction over the pluses and minuses and simplify like so: $\frac{1}{4}x^2-x+\frac{1}{9}y^2-\frac{2}{9}y+\frac{10}{9}$.

Exercise 4

The following expression comes from the equation of a hyperbola in standard form. Perform the subtraction, expand the products, and simplify if possible.

$$\frac{(x+1)^2}{1^2}-\frac{(y-1)^2}{2^2}$$

Answer

$\frac{4x^2+8x-y^2+2y+3}{4}$ or $x^2+2x-\frac{1}{4}y^2+\frac{1}{2}y+\frac{3}{4}$.

Example 5

Complete the square to write x^2+6x+3 in the form $(x+a)^2+b$.

Solution

This is part of the process needed to find a useful form of an equation whose graph is a circle. We write $x^2+6x+\quad\quad+3$, leaving a space. Then we take the coefficient from the x term, divide it by 2, and square the result. That is, $6 \xrightarrow{\div 2} 3 \xrightarrow{\text{squared}} 9$. Add this in the blank, and subtract it from the end of the expression so that we have effectively not changed the expression. Then factor the perfect square we created.

$$(x^2+6x+9)+3-9=(x+3)^2-6$$

Exercise 5

Complete the square to write y^2-2y+3 in the form $(y-a)^2+b$.

Answer

$(y-1)^2+2$

Example 6

Practice clearing denominators and rationalizing the numerator by simplifying the difference quotient $\dfrac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}$.

Solution

These difference quotients appear when you study the limit definition of a derivative in Calc I. The division by h is a problem child because we want to be see what happens if h becomes 0, so the goal is generally to get him to go away. We start by clearing the denominators.

$$\frac{\left(\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}\right)}{h}\frac{\cdot\sqrt{x}\sqrt{x+h}}{\cdot\sqrt{x}\sqrt{x+h}}=\frac{\frac{\sqrt{x}\sqrt{x+h}}{\sqrt{x+h}}-\frac{\sqrt{x}\sqrt{x+h}}{\sqrt{x}}}{h\sqrt{x}\sqrt{x+h}}=\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$$

Now, I told you the goal is to get rid of the h factor in the denominator, so let's try rationalizing the numerator and see if that helps. We multiply top and bottom by the **conjugate** $\sqrt{x}+\sqrt{x+h}$.

$$\frac{(\sqrt{x}-\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}}\frac{\cdot(\sqrt{x}+\sqrt{x+h})}{\cdot(\sqrt{x}+\sqrt{x+h})}=\frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$$

$$=\frac{-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$$

Now I can cancel the h's! You can distribute in the denominator and simplify as well if you want.

$$-\frac{1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}=-\frac{1}{x\sqrt{x+h}+(x+h)\sqrt{x}}$$

I did that one with you because it's lowkey a challenge problem. The one below is more chill.

Exercise 6

Practice clearing denominators to simplify the difference quotient $\dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$.

Answer

$-\dfrac{2x+h}{x^2(x+h)^2}$.

Exercise 7

I did some simplifications but I think I goofed up somewhere. Identify the error in my work. No need to find the correct answer, just find the mistake and I'll fix it later.

1. $\dfrac{(x+2)^2-(x+1)(x+2)}{3x-2}=\dfrac{x^2+4-(x^2+3x+2)}{3x-2}$

   $=\dfrac{x^2+4-x^2-3x-2}{3x-2}=\dfrac{-3x+2}{3x-2}=-1.$

2. $(x+1)(x-1)-(3x-2)(x+1)=x^2-1-(3x^2+x-2)$

   $=x^2-1-3x^2+x-2=-2x^2+x-3.$

Answer

1. I made the error right out of the gate with saying that $(x+2)^2=x^2+4^2$. That's illegal! It's $(x+2)(x+2)=x^2+4x+4$.

2. Everything looks okay until I cross over the second equals sign. I didn't apply the subtraction to all terms in the parentheses! It should be x^2-1-3x^2-x+2.

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2.4E Exercises

Identifying and Factoring Polynomials

For each expression, identify whether it is a polynomial, and if so, tell its degree and leading coefficient, and factor it completely.

1. $x^2 + 8x - 20$
2. $6x - 2\sqrt{x} + 4$
3. $x^3 + 3^x$
4. $2x^2 - 5x - 7$
5. $-x^4 + 16$
6. $4x^4 - 9$

Answer

1. Yes, degree 2 and leading coefficient 1, and factored form is $(x + 10)(x - 2)$.
2. No, the $\sqrt{x}$ term is not an x to a positive integer power.
3. No, the 3^x is the problem.
4. Yes, degree 2 and leading coefficient 2, and factored form is $(2x - 7)(x + 1)$.
5. Yes, degree 4 and leading coefficient -1, and seeing this as a difference of squares, $4^2 - (x^2)^2 = (4 + x^2)(4 - x^2) = (4 + x^2)(2 - x)(2 + x)$.
6. Yes, degree 4 and leading coefficient 4, and $(2x^2 + 3)(2x^2 - 3)$.

Expanding & Like Terms

Expand and simplify.

1. $(2 - 7y)(y + 3) - 6$
2. $a(a + 1) + (a + 2)(a - 5) + 11$
3. $(x + 3)(x + 5) - (x - 1)^2$
4. $(x^4 - x^3) + x^2(x + 1)(x - 1)$
5. $(a + b)^4$
6. $x^3(x^2 + 3x) - 2x(x^4 - 3x^2)$

Answer

1. $-7y^2 - 19y$
2. $2a^2 - 2a + 1$
3. $10x + 14$
4. $2x^4 - x^3 - x^2$
5. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
6. $-x^5 + 3x^4 + 6x^3$

Mixed Factoring Practice

Factor completely. Remember the technique of factoring by grouping, and the ac method!

1. $x^2 + 130x + 3000$
2. $3m^3 + 6m^2 + 3m$
3. $x^3 + 2x^2 - x - 2$
4. $(a + b)^2 + 2(a + b) + 1$
5. $5x^2 - 7x - 6$

Answer

1. $(x + 30)(x + 100)$

2. $3m(m+1)^2$
3. $(x-1)(x+1)(x+2)$
4. $((a+b)+1)^2=(a+b+1)^2$
5. $(5x+3)(x-2)$

Completing the Square

Complete the square to write the degree 2 polynomials in the form $(x+a)^2+b$.

1. $x^2+4x-10$
2. $x^2-3x+\frac{5}{2}$

Answer

1. $(x+2)^2-14$
2. $\left(x-\frac{3}{2}\right)^2+\frac{1}{4}$

Simplifying Rational Expressions

Simplify.

1. $\dfrac{x^2+6x}{x^2+x-30}$
2. $\dfrac{2x^2-4x-6}{x^2-1}$
3. $\dfrac{x^2+8x-20}{x^2+20x+100}$
4. $\dfrac{(x+h)^2+3(x+h)-x^2-3x}{h}$

Answer

1. $\dfrac{x}{x-5}$
2. $\dfrac{2(x-3)}{(x-1)}$
3. $\dfrac{x-2}{x+10}$
4. $2x+h+3$

Arithmetic With Rational Expressions

Simplify into a single rational expression.

1. $\dfrac{h}{h+1}+\dfrac{h^2}{h-1}$
2. $\dfrac{2}{x}-\dfrac{2}{x+1}+\dfrac{2}{x^2+x}$
3. $\dfrac{3x+6}{16-x^4}\cdot\dfrac{8x^2+32}{9}$

Answer

1. $\dfrac{h^3+2h^2-h}{h^2-1}$
2. $\dfrac{4}{x(x+1)}$ or $\dfrac{4}{x^2+x}$

3. $\dfrac{8}{3(2-x)}$

Compound Fractions

Simplify.

1. $\dfrac{\dfrac{x+1}{x^2+5x+6}}{\dfrac{2x^2-2}{x+3}}$
2. $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$

Answer

1. $\dfrac{1}{2(x-1)(x+2)}$ (Hint: multiplying by the reciprocal works best here.)
2. $-\dfrac{1}{x(x+h)}$ (Hint: clearing the denominators is quick.)

Simplifying Difference Quotients

Simplify.

1. $\dfrac{(x+h)^3-x^3}{h}$
2. $\dfrac{(3x^2+1)-28}{x-3}$
3. $\dfrac{\frac{1}{2}(x+h)^2-\frac{1}{2}x^2}{h}$

Answer

Hint: What does "simplify" mean for this type of problem? Remember that in "difference quotient" problems, the goal is to make the denominator go away. This means first simplifying the numerator, which may involve expanding and/or combining like terms, and then re-factoring to see if anything cancels.

1. $3x^2+3xh+h^2$
2. $3(x+3)$
3. $x+\frac{1}{2}h$

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CHAPTER OVERVIEW

Chapter 3: Equations and Inequalities

Whyyyyyyyyy isn't it "equations and inequations" or even "equalities and inequalities"?????? I don't make the rules. I just suffer. Anyways, enough goofing around in the algebraic expressions sandbox. The whole point of getting good at manipulating expressions is so that we can make equations and functions out of them. In this chapter, we will figure out how to solve equations and inequalities, which just boils down to isolating a variable, and then look at some equations' graphs, especially the super important shapes like lines and circles. We'll also use equations to describe ("model") real life situations and how quantities relate to each other. Finally, we'll solve simultaneous systems of linear equations using basic methods you might be familiar with, as well as a method using vectors and matrices (oh my).

- 3.1 Solving for a Variable
  - 3.1E Exercises
- 3.2 Solving Inequalities
  - 3.2E Exercises
- 3.3 Lines and Circles
  - 3.3E Exercises
- 3.4 Modeling With Equations
  - 3.4E Exercises
- 3.5 Solving Systems of Linear Equations
  - 3.5E Exercises
- 3.6 Vectors and Matrices
  - 3.6E Exercises
- 3.7 Chapter 3 Study Guide

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3.1 Solving for a Variable

Learning Objectives

By the end of this section, you will be able to:

- solve a linear equation for a variable
- work with absolute value bars and relate them to powers
- solve quadratic equations by factoring, completing the square, or using the quadratic formula

It's time for order of operations, arithmetic, algebra rules, and our affection for variables to come home to roost. In equations.

An **equation** is a statement that two expressions are equal (shocking). For example, "$4+17=21$" is an equation. On the other hand, "$1+1=3$" is a dirty rotten lie, so we shouldn't take equals signs for granted. What about something like this, though?

$$x+3=12$$

Ooh, a man of mystery! This could be true or false, depending on what value the variable x is taking. What value would make the statement true? Who do we need to add to 3 to get 12? In this case, 9 is a **solution** to the equation. By figuring that out, we **solved** the equation. Whenever we come up with a solution, we can check our answer by just **plugging in for x.** If the statement is true, we're golden. Here, $9+3=12$ checks out!

Starting with one equation and manipulating it algebraically produces **equivalent equations,** equations that have exactly the same solutions. For example, there's no moral, philosophical, spiritual, or grammatical difference between these two equations:

$$x+3=12 \quad \text{and} \quad x+2=11$$

The solution to the first one is 9 as we saw, and plugging 9 into the second gives $9+2=11$, a true statement! Can you tell what I did to create the second equation? I subtracted 1 from both sides of the first equation. What happens if I multiply both sides of the original by 3? The result, $3x+9=36$, still has the solution $x=9$.

- **Adding or subtracting the same thing on both sides of an equation** gives an equivalent equation.

$$A=B \quad \Longleftrightarrow \quad A+C=B+C$$

- **Multiplying or dividing the same thing on both sides of an equation** gives an equivalent equation.

$$A=B \quad \Longleftrightarrow \quad AC=BC$$

Some very chill dudes we will learn a lot about later are **linear equations:** equations of the form $ax+b=0$, where x is the variable, and a, b are some constants (plain ol' numbers). Things like $x+1=0$ or $12x-5=0$. You might notice that the variable x is raised only to the secret first power, x^1. This counts as a polynomial, if a very boring one.

To solve a linear equation, we don't need to know anything more than the green box up there. **Our aim is always to isolate the variable on one side of the equals sign.** We're doing whatever we need to do to get x by himself.

Example 1

Solve the linear equation $17x-51=0$.

Solution

I know what x has to be because 51 is my favorite number, but I'm a deep thinker. Let's do this systematically by isolating x. Currently, there is a 51 cluttering the left hand side. If we add 51 to both sides, this happens:

$$17x-51=0 \quad \rightarrow \quad 17x-51+51=0+51 \quad \rightarrow \quad 17x=51$$

Now the only issue is the 17 on the x, so let's multiply both sides of the equation by $\frac{1}{17}$ to get rid of him.

$$\frac{1}{17}\cdot(17x)=(51)\cdot\frac{1}{17} \quad \rightarrow \quad x=\frac{51}{17}=3$$

The solution is $x = 3$, and we check by plugging in: $17(3) - 51 = 51 - 51 = 0$, yeppp!

Exercise 1

Solve the linear equation $4x + 12 = 0$.

Answer

$x = -3$

How did I know to take care of the 51 before worrying about the 17? This is where understanding the order of operations is important, because we need to think about **undoing** everything that the algebraic expression was doing to the variable x. So it's all about asking, "What happened?? What happened??" (*"He spoke French!"*) And then whatever happened, we undo it, **working backwards.** It's like we're peeling off layers of an onion, **working from the outside in**.

To undo an operation, we use the **inverse operation.** To undo addition by a number, subtract the same number. To undo multiplication by a number, divide by the same number.

Example 2

Solve the equation $\dfrac{3(x-1)+4}{2} = 1$ for x.

Solution

I say to myself, "Self, what happened to x on the left hand side?" And by understanding order of operations, I can give these instructions:

1. Subtract 1 from x.
2. Multiply the result by 3.
3. Add 4 to that.
4. Divide that result by 2.

Now to isolate x, all I have to do is work backwards from Step 4 to Step 1, undoing each action. To undo the division by 2, I multiply both sides by 2:

$$2 \cdot \left(\frac{3(x-1)+4}{2}\right) = 1 \cdot 2 \quad \rightarrow \quad 3(x-1)+4 = 2$$

To undo the addition of 4, I subtract 4 from both sides.

$$3(x-1)+4-4 = 2-4 \quad \rightarrow \quad 3(x-1) = -2$$

To undo the multiplication by 3, divide both sides by 3 (or multiply by $\frac{1}{3}$).

$$\frac{1}{3} \cdot 3(x-1) = -2 \cdot \frac{1}{3} \quad \rightarrow \quad x - 1 = -\frac{2}{3}$$

Finally, to undo the subtraction of 1, add 1 to both sides.

$$x - 1 + 1 = -\frac{2}{3} + 1 \quad \rightarrow \quad x = \frac{1}{3}$$

The solution is $x = \frac{1}{3}$.

Exercise 2

Describe the operations that were done to x on the left hand side **in order**, and then use that information to solve the equation.

1. $2x + 3 = 0$
2. $\dfrac{x+1}{2} = 2$

3. $\frac{x}{3}+6=0$

Answer

1. "Multiply x by 2, and then add 3." Solution: $x=-\frac{3}{2}$.
2. "Add 1 to x, then divide that by 2." Solution: $x=3$.
3. "Divide x by 3, and then add 6." Solution: $x=-18$.

What about the other stuff we know how to do, you ask? Exponents and radicals? Well, raising a base to the n^{th} power and taking an n^{th} root are also inverse operations. You can use one to undo the other. But real quick, I want to talk about the concept of **absolute value.**

Absolute Value

When you see absolute value bars placed around an expression like this, $|A|$, they are a set of instructions with two cases. If the interior A is a **positive** quantity, they do **nothing,** and the instructions are "spit A right back out." But if the interior A is a **negative** quantity, the instructions are "flip the sign on A so that he's positive and spit that out instead." In other words, we say the absolute value will return the **opposite** of A, which is $-A$. Be careful! If $A=-2$ is a negative number, then note that $-A=-(-2)=2$ is a positive number! Exactly how we wanted it. When a function or set of instructions has two cases like this, it can be written using a brace:

$$|A|=\begin{cases}A & \text{if } A\geq 0\\ -A & \text{if } A<0\end{cases}$$

I'm bringing this up now, because when you take roots of variables without knowing what they are, interesting situations can arise that absolute value helps us describe. Say I'm trying to solve $x^2=9$. Both 3 and -3 work as solutions! I would like to apply a square root on both sides to undo the square power on x. I describe this in math like this:

$$\sqrt{x^2}=\sqrt{9} \quad\rightarrow\quad |x|=3 \quad \text{aka } x=\pm 3$$

The statement "$|x|=3$" contains the information "x can be 3 or -3, because both have absolute value result 3." This is a handy way to a) efficiently describe why taking a square root in the course of solving results in a "$\pm$" answer, and b) help you not forget that there are two solutions.

Example 3

Solve the equations.

1. $(x+4)(x-4)=9$
2. $2-\sqrt{1+2x}=0$

Solution

1. Yikes, there are two copies of x, which we haven't seen before. Let's try FOILing to see if anything good happens. We get $x^2-16=9$, which looks more manageable. Whathappenedwhathappened?? Looks like x got squared, and then 16 was subtracted from that. To get rid of the -16, add 16 to both sides. Now we have $x^2=9+16=25$. To undo the squaring, we take a square root on both sides.

$$\sqrt{x^2}=\sqrt{25} \quad\longrightarrow\quad |x|=5 \quad\longrightarrow\quad x=\pm 5$$

2. Whathappenedwhathappened?? Looks like x got multiplied by 2, then 1 was added, then a square root was taken, and then the whole thing was subtracted from 2. To undo that last subtraction, we add $\sqrt{1+2x}$ to both sides, effectively "moving it over" to the other side.

$$2-\sqrt{1+2x}+\sqrt{1+2x}=0+\sqrt{1+2x} \quad\longrightarrow\quad 2=\sqrt{1+2x}$$

It doesn't matter what order you write an equation in, so I'll flip it around as we keep working with it. Now we're trying to undo the square root, so let's square both sides.

$$(\sqrt{1+2x})^2=2^2 \quad\longrightarrow\quad 1+2x=4$$

Now just continue solving by subtracting away the 1 and dividing away the 2.

$$1 + 2x = 4 \quad \longrightarrow \quad 2x = 3 \quad \longrightarrow \quad x = \frac{3}{2}$$

? Exercise 3

Solve the equations.

1. $\dfrac{x^2 - 3}{3} = 2$
2. $\sqrt{x + 4} = 3$

Answer

1. $x = \pm 3$
2. $x = 5$

The next level-up we need is situations where the variable appears in multiple places. We saw this in the example above when x appeared in two factors. We got around it by FOILing, because it conveniently gave us a single x^2 term. Most of the time, though, if we FOILed two factors like that, we would end up with a degree 2 polynomial in the equation. We now call this a **quadratic equation.** Any time you see an equation with an x^2 in it, your first move is to add or subtract **all** terms over to one side and get 0 on the other. Then proceed to the important block below.

To **solve a quadratic equation** in the **standard form** $ax^2 + bx + c = 0$, there are three main techniques. Ranked from most convenient to least convenient, on average, they are:

1. Basic factoring when $a = 1$. This is the "find two numbers that multiply to c and add to b" method. If $a \neq 1$, try the ac method to factor by grouping. After factoring, you have an expression containing factors in the form $(x - c)$ multiplied together, with zero on the other side. Each such factor has the potential to wipe out the entire expression were it to equal 0, and $x = c$ would cause that! Then each factor gives a solution to the equation.
2. Completing the square on the first two terms, while moving the constant term c to the other side, gives an equation of the form $(x + m)^2 = n$. Then by taking a square root of both sides and subtracting the m, we can isolate x.
3. Solutions can always be found using **the quadratic formula,** which you should memorize. Take the coefficients $a, b,$ and c from the quadratic equation's standard form and plug them in below:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Read the following example very carefully, because I'm going to show you a trick for completing the square when the x^2 term has a coefficient besides 1!

✓ Example 4

Solve the quadratic equations.

1. $x^2 - 14x = 51$
2. $5x^2 + 10x = 1$

Solution

1. Step one, always always always, is to get 0 on one side of the equation. We subtract 51 from both sides to achieve this. Next, the first resort is always to try factoring in case it's easy. This one works!

$$x^2 - 14x - 51 = 0 \quad \longrightarrow \quad (x - 17)(x + 3) = 0$$

Finally, we realize that if **either** $(x - 17) = 0$ **or** $(x + 3) = 0$, the equation will be **true**, because anything times 0 is 0. In each mini equation there, you need to isolate x to find that $x = 17$ and $x = -3$ are the solutions. In practice, you do not want to write

all of this out! You can look straight at the factors and take the opposite of whatever you see: the opposite of -17 is 17, and the opposite of $+3$ is -3. Go directly to writing down $x = -3, 17$.

2. Even if we get a 0 on the right hand side, the left side will not factor nicely. Sad face. We will try completing the square, but notice the 5 coefficient on the x^2... The way to deal with that is to factor out 5 from both terms and *then* complete the square *inside the parentheses*. So start by factoring 5 out on the left hand side, getting $5(x^2 + 2x) = 1$, and then...

$$5(x^2 + 2x + \quad) = 1 \quad \longrightarrow \quad 5(x^2 + 2x + 1 - 1) = 1 \quad \longrightarrow \quad 5\left((x+1)^2 - 1\right) = 1$$

To finish completing the square of the left hand side as a whole, simply distribute the 5 back into the parentheses, yielding $5(x+1)^2 - 5 = 1$. This process was how you **complete the square on a quadratic with nontrivial leading coefficient** a. Now we continue solving...

$$5(x+1)^2 = 6 \quad \longrightarrow \quad (x+1)^2 = \frac{6}{5}$$

$$\longrightarrow \quad \sqrt{(x+1)^2} = \sqrt{\frac{6}{5}} \quad \longrightarrow \quad |x+1| = \sqrt{\frac{6}{5}}$$

Now this statement is saying that as long as the absolute value of $(x+1)$ turns out to be $\sqrt{\frac{6}{5}}$, the equation is true. There are two ways this could happen:

- if $x+1$ is positive, absolute value does nothing, so $|x+1| = x+1 = \sqrt{\frac{6}{5}}$ and we isolate x to get a solution $x = -1 + \sqrt{\frac{6}{5}}$,
- or if $x+1$ is negative, absolute value results in its *opposite*. In that case, $|x+1| = -(x+1) = \sqrt{\frac{6}{5}}$. When we isolate x in this case, we have

$$-(x+1) = \sqrt{\frac{6}{5}} \quad \longrightarrow \quad x+1 = -\sqrt{\frac{6}{5}} \quad \longrightarrow \quad x = -1 - \sqrt{\frac{6}{5}}$$

Now, if you hate completing the square (ngl, I almost never do it, but some students love this method) we could do the same problem using the **quadratic formula**. We first get into standard form $ax^2 + bx + c = 0$ by moving over the 1 to get $5x^2 + 10x - 1 = 0$. We identify that $a = 5, b = 10$, and $c = -1$. Then we just plug into the formula and get simplifying!

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-10 \pm \sqrt{10^2 - 4(5)(-1)}}{2(5)} =$$

$$= \frac{-10}{10} \pm \frac{\sqrt{100+20}}{10} = -1 \pm \frac{\sqrt{120}}{10} = -1 \pm \frac{2\sqrt{30}}{10} = -1 \pm \frac{\sqrt{30}}{5}$$

Hang on! Hadn't we better get the same solution with both methods? Ahaaa, these are actually the same. Look what happens if I rationalize the denominator on our first answer:

$$-1 \pm \sqrt{\frac{6}{5}} = -1 \pm \frac{\sqrt{6}}{\sqrt{5}} = -1 \pm \frac{\sqrt{6} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = -1 \pm \frac{\sqrt{30}}{5}$$

All checks out. If I'm your professor, I don't care which way you write it. If I'm not your professor, ask if they prefer you to rationalize the denominator.

? Exercise 4

1. Solve by factoring: $x^2 + 8x + 7 = 0$
2. Solve by completing the square: $x^2 - 2x = 3$
3. Check your answers by solving using the quadratic formula instead.

Answer

1. $x = -1, -7$
2. $x = -1, 3$

When using the quadratic formula, there is a problem child that sometimes arises when calculating $\sqrt{b^2-4ac}$. If the inside (called the **discriminant**) b^2-4ac turns out to be a negative number, the square root is illegal in the real numbers. If you expand your horizons to what is called the **complex numbers**, denoted $\mathbb{C}$, we can deal with this situation. I'll tell you a bit more about this when we get to graphs of parabolas, but we don't really live there in this course.

Possible Outcomes When Solving Quadratic Equations

- **Distinct** real roots: either something like "$(x-3)(x-2)=0 \quad \Longrightarrow \quad x=2,3$," or something like $x=2\pm\sqrt{3}$ when the quadratic formula gets involved.
- **Repeated** real root: two copies of the same number. This happens like if you factor and get $(x-1)^2=0$.
- No real roots, but a **complex conjugate pair**: when the discriminant b^2-4ac turns out to be negative, these will look like $x=a\pm bi$, but don't worry about it yet.

Enough of that. There's one more scenario involving multiple x terms that I want to cover: equations that involve rational expressions. When these come up, it's extremely important to **check your answers for validity.**

Example 5

Solve the equations:

1. $\frac{v^2}{27}=\frac{1}{v}$
2. $\dfrac{2}{x-1}-\dfrac{1}{x}=\dfrac{4}{x^2-1}$

Solution

1. When you see something like this, use the **cross multiplying** technique. This is the same as multiplying on both sides by the two denominators.

$$\frac{v^2}{27}=\frac{1}{v} \quad \longrightarrow \quad v\cdot v^2=1\cdot 27 \quad \longrightarrow \quad v^3=27$$

Either take a cube root of both sides or just recognize by common sense that $v=3$ is the solution! When we plug back into the original equation to check, everything works out!

2. Cross multiplying is not an option here at the moment, because I don't have a single fraction on each side. Step one is to **clear the denominators** by multiplying both sides of the equation by the LCM, which is $x(x+1)(x-1)$.

$$x(x+1)(x-1)\cdot\left(\frac{2}{(x-1)}-\frac{1}{x}\right)=\left(\frac{4}{(x+1)(x-1)}\right)x(x+1)(x-1)$$
$$\longrightarrow \quad 2x(x+1)-1(x+1)(x-1)=4x$$

Much better. Now expand fully and combine all like terms to get 0 on one side.

$$2x^2+2x-x^2+1=4x \quad \longrightarrow \quad x^2-2x+1=0$$

This perfect square factors as $(x-1)^2=0$, and square rooting both sides (square root of 0 is 0), we get

$$|x-1|=0 \quad \longrightarrow \quad x-1=0 \text{ and } -(x-1)=0.$$

Oh, but when we isolate x in both cases, we get the same solution, $x=1$. BUT WAIT, we must check to see if it's even valid. Trying to plug $x=1$ into the original equation, we discover that it would cause division by zero! There are no valid solutions to this equation. Sometimes that happens—there is **no value** for x that will make the statement **true**.

Exercise 5

Solve the equations.

1. $\dfrac{x-1}{x}=\dfrac{x}{4}$
2. $\dfrac{2}{x+2}-\dfrac{1}{x}=\dfrac{1}{x+3}$

Answer

1. $x = 2$ (repeated root)
2. $x = -6$

Finally, let's revisit equations with radicals in them. The big takeaway from the example below is that **you MUST check your solutions when it comes to radical equations.** Sometimes in the course of the algebra, extraneous solutions are introduced. If you have time, you should always check all your solutions, but when it comes to these it is NOT optional!

Example 6

Solve the equation $2x = 1 - \sqrt{2-x}$.

Solution

In order to get rid of the radical, I need to isolate it and then square both sides. This will involve FOILing!

$$2x = 1 - \sqrt{2-x} \quad \longrightarrow \sqrt{2-x} = 1 - 2x \quad \longrightarrow (\sqrt{2-x})^2 = (1-2x)^2 = 1 - 4x + 4x^2$$

Now I have $2 - x = 1 - 4x + 4x^2$, and we combine all like terms until we get 0 on one side, and then factor.

$$4x^2 - 3x - 1 = 0 \quad \longrightarrow \quad (4x+1)(x-1) = 0$$

Setting each factor equal to 0 and isolating x gives us two **potential** solutions, $x = 1, -\frac{1}{4}$. However, we must plug these back into the **original** equation to see if they work.

$$2(1) \stackrel{?}{=} 1 - \sqrt{2-1} \quad \longrightarrow \quad 2 \stackrel{?}{=} 1 - \sqrt{1}$$

Does 2 equal 0? Nope, we must throw out the option $x = 1$.

$$2\left(-\frac{1}{4}\right) \stackrel{?}{=} 1 - \sqrt{2 + \frac{1}{4}} \quad \longrightarrow \quad -\frac{1}{2} \stackrel{?}{=} 1 - \sqrt{\frac{9}{4}}$$

The right hand side simplifies as $1 - \frac{3}{2} = -\frac{1}{2}$, so this one checks out! The solution to the original equation is only $x = -\frac{1}{4}$.

Exercise 6

Solve the equation $\sqrt{2x+1} + 1 = x$.

Answer

$x = 4$ only.

Phew, did that section feel like a marathon to anyone else? Just me? Welp.



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3.1E Exercises

Solving Linear Equations

Solve the equations.

1. $4x + 2 = 0$
2. $6y - 7 = 17$
3. $x\sqrt{2} - 1 = 1$
4. $ax + b = 0$, where a, b are constants.
5. $2x + 3 = x - 4$
6. $5x = 2 - x$

Answer

1. $x = -\frac{1}{2}$
2. $y = 4$
3. $x = \frac{2}{\sqrt{2}} = \sqrt{2}$
4. $x = -\frac{b}{a}$
5. $x = -7$
6. $x = \frac{1}{3}$

Inverse Operations

Describe, in order, the operations performed on the left side of the equation, and use that information to solve the equation for the variable.

1. $2(3v - 1) + 4 = 1$
2. $\frac{x+2}{5} + 2 = 0$
3. $3 - \frac{x^2}{3} = 0$
4. $\dfrac{\sqrt{x} + 1}{3} = 1$

Answer

1. "multiply by 3, then subtract 1, then multiply by 2, then add 4," $v = -\frac{1}{6}$
2. "add 2, then divide by 5, then add 2," $x = -12$
3. "square x, divide by 3, then subtract result from 3," $x = \pm 3$
4. "take square root, then add 1, then divide by 3," $x = 4$

Quadratic Equations

Solve the quadratic equations using any method you prefer.

1. $x^2 - 13x + 12 = 0$
2. $4t^2 - 9 = 0$
3. $2x^2 = 5x - 3$
4. $x^2 + 30x - 400 = 0$
5. $3x^2 + 4x = 2$
6. $(x + 1)^2 - 4 = 0$
7. $2x^2 - 7x - 3 = 0$

Answer

1. $x = 1, 12$
2. $t = \pm\frac{3}{2}$
3. $x = 1, \frac{3}{2}$

4. $x = -40, 10$
5. $x = -\frac{2}{3} \pm \frac{\sqrt{10}}{3}$
6. $x = -3, 1$
7. $x = \frac{7}{4} \pm \frac{\sqrt{73}}{4}$

Other Equations

Solve the equations.

1. $\dfrac{x^2}{x+100} = 50$
2. $\dfrac{2t+1}{t} = \dfrac{t-2}{t+2}$
3. $\sqrt{2x-1} = \sqrt{3x-5}$
4. $x^4 - 13x^2 + 36 = 0$

Answer

1. $x = -50, 100$
2. $t = \dfrac{-7 \pm \sqrt{41}}{2}$
3. $x = 4$
4. $x = \pm 2, \pm 3$ (Hint: use a nickname $y = x^2$ to transform this to a quadratic to get started.)

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3.2 Solving Inequalities

Learning Objectives

By the end of this section, you will be able to:

- solve basic linear inequalities
- solve quadratic and rational inequalities using positive/negative zone analysis
- solve simultaneous inequalities and ones involving absolute value

An **inequality** is also a statement that we want to be true, it's just looser than an equation. Instead of an equals sign, these will have inequality symbols in them, namely, $<, \leq, >, \geq$, which mean "(strictly) less than, less than or equal to, (strictly) greater than, greater than or equal to," respectively. Here are some examples.

- $x + 7 > 3$ means "The quantity $x + 7$ is a bigger number than 3."
- $x^2 + x - 2 \leq 0$ means "The quantity $x^2 + x - 2$ is less than or equal to 0, aka is 0 or a negative number."

Solving an inequality is still just about isolating x on one side by performing legal inverse operations across the whole inequality.

- You can **add** or **subtract** the same thing on both sides of an inequality, and it won't change:

$$A < B \iff A + C < B + C$$

- You can **multiply** or **divide** something **positive** on both sides of an inequality, and it won't change: if $C \geq 0$,

$$A < B \iff AC < BC$$

- **ALERT!!!** If you multiply or divide both sides by a **negative** number, you must **flip the direction of the inequality sign:** if $C < 0$,

$$A < B \iff AC > BC$$

Solving an inequality results in a whole slew of solutions. The statement "$x \geq 3$" will be true if x takes the value 3, or the value 5, or the value 1000, or any of the infinitely many values bigger than 3. I want to show you the weird stuff that can happen when you solve inequalities via some examples, so read all of these carefully. First, the basic situation.

Example 1

Solve the inequalities.

1. $2x + 8 < 4$
2. $9x - 3 \geq 0$
3. $6 - x \leq 1$
4. $-\frac{x}{2} + 1 > 10$

Solution

1. We undo all the stuff that happened to x using inverse operations just like with equations. He was multiplied by 2, and then added to 8, so we subtract 8 from both sides, and then divide by 2. Nothing needs to change on the inequality symbol.

$$2x + 8 < 4 \quad \longrightarrow \quad 2x < 4 - 8 \quad \longrightarrow \quad \frac{2x}{2} < \frac{-4}{2} \quad \longrightarrow \quad x < -2$$

Check your answers by taking some options from the solution statement and plugging them back in. For example, plugging in -3 would give $2(-3) + 8 = 2$, which is indeed less than 4. You can write your solution in interval notation (review in the domain section) as $(-\infty, -2)$.

2. Again, undo the multiplication by 9 and subsequent subtraction by 3 by adding 3 and dividing by 9.

$$9x - 3 \geq 0 \quad \longrightarrow \quad 9x \geq 0 + 3 \quad \longrightarrow \quad \frac{9x}{9} \geq \frac{3}{9} \quad \longrightarrow \quad x \geq \frac{1}{3}$$

Because x can be greater than or equal to $\frac{1}{3}$, plugging in $\frac{1}{3}$ itself should work. Test: $9 \cdot \frac{1}{3} - 3$ does indeed equal 0. In interval notation, the answer is $[\frac{1}{3}, \infty)$.

3. Here as we go to undo the operations, we will have to divide by a -1. At that point, we have to flip the inequality sign!

$$6 - x \leq 1 \quad \longrightarrow \quad -x \leq 1 - 6 \quad \longrightarrow \quad \frac{-x}{-1} \geq \frac{-5}{-1} \quad \longrightarrow \quad x \geq 5$$

Plug in something larger than 5 for x to test. In interval notation, we say $[5, \infty)$.

4. As we isolate x, we will need to multiply by a negative number, so again watch out for flipping the sign!

$$-\frac{x}{2} + 1 > 10 \quad \longrightarrow \quad -\frac{x}{2} > 10 - 1 \quad \longrightarrow \quad -\frac{x}{2} \cdot (-2) < 9 \cdot (-2) \quad \longrightarrow \quad x < -18$$

Plug in something strictly greater than -18 to check. In interval notation, we say $(-\infty, -18)$.

? Exercise 1

Solve the inequalities.

1. $12x - 144 \geq 0$
2. $4 - \frac{3}{2}x < -5$

Answer

1. $x \geq 12$ or $[12, \infty)$
2. $x > 6$ or $(6, \infty)$

When more powers of x or factors are involved, we need to use logic and cases to cover all our bases. The easiest method is using a number line, so let me show you that technique.

✓ Quadratic Inequality

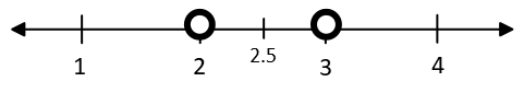
Solve the inequality $x^2 \geq 5x - 6$.

Solution

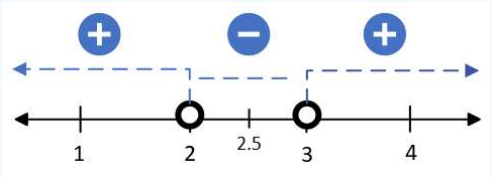
If this were an equation, we would move everything to one side and factor, so let's start with that.

$$x^2 - 5x + 6 \geq 0 \quad \longrightarrow \quad (x - 3)(x - 2) \geq 0$$

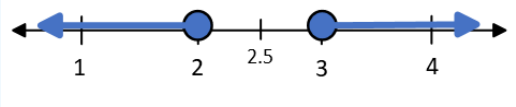
This is saying we want the product $(x - 3)(x - 2)$ to turn out to be 0 or positive. For a product to be positive, we could have multiplied two positive quantities, **or** two negative quantities. If you follow this number line analysis method, you will cut to the chase on covering those cases. Start by marking 2 and 3 on a number line, the values that would have been solutions to this quadratic if it had been an equation instead.



These two points divide the number line into three zones: $(-\infty, 2)$, $(2, 3)$, and $(3, \infty)$. In each zone, pick an easy test point to plug into the factors to see if the result is positive or negative. In the first zone, I pick 0 and plug in: $(0 - 3)(0 - 2) = (-3)(-2) = 6$ is positive. In the middle, I pick 2.5 and plug in: $(2.5 - 3)(2.5 - 2) = (-0.5)(0.5)$ results in a negative number. In the last zone, I pick 4: $(4 - 3)(4 - 2) = (1)(2)$ is positive. Mark these.



Since we want the product to be 0 or positive, we report the positive zones and include their endpoints, giving the answer $(-\infty, 2] \cup [3, \infty)$. We would show this answer on a number line with closed dots, like so:

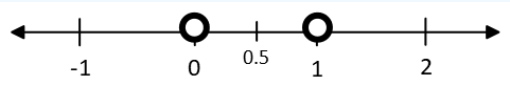


Repeated Factors

Solve the inequality $x(x-1)^2 > 0$.

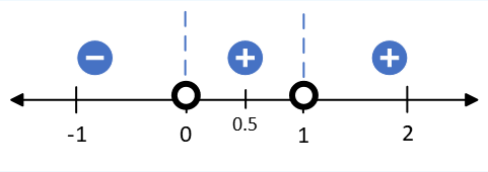
Solution

Here we're already factored nicely, and we note that if this were an equation, the solutions would be $x = 0, 1$. Mark them on a number line.

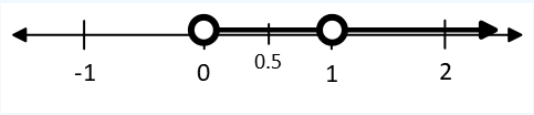


We are divided into three zones again: $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$. Pick test points again and determine the positive and negative zones. I'm going to test...

- $x = -1$ gives $(-1)(-1-1)^2 = (-1)(-2)^2 =$ negative.
- $x = 0.5$ gives $(0.5)(0.5-1)^2 = (0.5)(-0.5)^2 =$ positive.
- $x = 2$ gives $(2)(2-1)^2 = (2)(1)^2 =$ positive.



I want to report positive zones in my answer, but since the inequality is strict, I cannot include the endpoints! I report $(0, 1) \cup (1, \infty)$, and I could draw this as:



Fractional Inequality

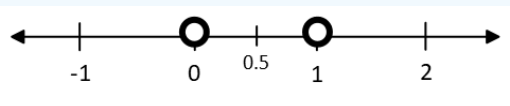
Solve the inequality $\dfrac{1+x}{1-x} \leq 1$

Solution

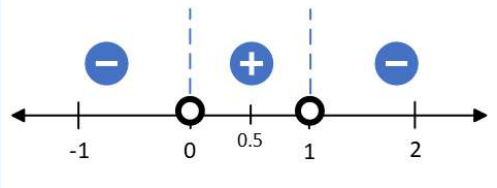
The first step is to get 0 on one side and a single rational expression on the other, so we can use positive/negative analysis as we have been doing. We subtract over the 1 and find a common denominator so we can create one fraction.

$$\frac{1+x}{1-x} - 1 \leq 0 \quad \longrightarrow \quad \frac{1+x}{1-x} - \frac{1-x}{1-x} \leq 0 \quad \longrightarrow \quad \frac{1+x-(1-x)}{1-x} \leq 0$$

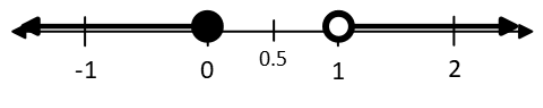
Simplifying gives us the inequality $\dfrac{2x}{1-x} \leq 0$. Once again, look at the factors involved, $2x$ and $(1-x)$. These are zero if x is 0 or 1. Mark those points of interest.



Now we analyze zones by testing easy points to plug in (you try!) and get the following result.



We want to report the negative zones, but we still need to think about the endpoints. We plug in $x = 0$ to the original inequality to test: yes, $\frac{1+0}{1-0}$ is indeed less than or equal to 1. But at $x = 1$, the left side of the inequality is undefined due to dividing by zero!! We leave him out. The answer is $(-\infty, 0] \cup (1, \infty)$.



Exercise 2

Solve the inequalities.

1. $x^2 - 4x > 0$
2. $\dfrac{x}{x+1} + \dfrac{1}{x} \leq 0$ (Remember to check your answers against the **original** inequality.)

Answer

1. $x < 0$ or $x > 4$, or in interval notation, $(-\infty, 0) \cup (4, \infty)$.
2. $-1 < x < 0$, or in interval notation, $(-1, 0)$.

Bookends and uhhhhh bookCases

Sometimes you might see inequalities that contain **two signs**, meaning that the quantity is supposed to satisfy two conditions. For example,

$$0 < x + 4 \leq 10$$

means that $(x + 4)$ is supposed to be a value greater than 0, but not more than 10. I think of these as bookends. You can still do the algebraic operations to this inequality, you just have to apply them across the entire thing.

Example 2

Solve the simultaneous inequality $0 < x + 4 \leq 10$.

Solution

My goal is always to get x by himself. Currently there is an addition by 4 that I don't want, so to get rid of that, I subtract 4 from each part of the inequality and simplify.

$$0 - 4 < x + 4 - 4 \leq 10 - 4 \quad \longrightarrow \quad -4 < x \leq 6$$

This tells me that the solution is for x to be bigger than -4 but not more than 6. You can leave it like that, or you can report in interval notation, $(-4, 6]$.

Remember the absolute value bars? We described the effect they have using **cases.** What matters is whether the inside is positive or negative. For example,

$$|x + 4| = \begin{cases} x + 4 & \text{if } (x + 4) \geq 0 \\ -(x + 4) & \text{if } (x + 4) < 0 \end{cases}$$

Caution

A mistake I see ALL THE TIME is when people think the effect of absolute value bars is to individually change the sign of each term inside. "Absolute"ly not! (lol) You must take the opposite of the whole entire inside. Split into the two cases: "|inside|" will...

- **spit out "inside" when the inside is nonnegative**
- **spit out "−(inside)" when the inside is negative**

If you see absolute value bars appear in an inequality, all you gotta do is isolate that part, and then break it down into these two cases!

- For $<$ and $\leq$, split into cases using AND:

$$|\text{inside}| < a \quad \Longrightarrow \quad (\text{inside}) < a \quad \text{AND} \quad -(\text{inside}) < a$$

- For $>$ and $\geq$, split into cases using OR:

$$|\text{inside}| > a \quad \Longrightarrow \quad (\text{inside}) > a \quad \text{OR} \quad -(\text{inside}) > a$$

Example 3

Solve the inequalities.

1. $|3x + 2| \leq 8$
2. $|12x - 30| - 6 > 0$

Solution

1. The absolute value is by itself on one side, so we go ahead and look at the inequality sign. Since it's "$\leq$," we split into the two options with an AND:

$$(3x + 2) \leq 8 \quad \text{AND} \quad -(3x + 2) \leq 8$$

Now solve each inequality separately. We get as our answer, "$x \leq 2$ and $x \geq -\frac{10}{3}$," which could also be written "$-\frac{10}{3} \leq x \leq 2$." In interval notation, this is $\left[-\frac{10}{3}, 2\right]$.

2. First isolate the absolute value part! Add 6 on both sides to get $|12x - 30| > 6$. Now split into cases with an OR:

$$(12x - 30) > 6 \quad \text{OR} \quad -(12x - 30) > 6$$

Solving each inequality separately, we get "$x > 3$ or $x < 2$," **which cannot be written as one statement do not do it I will mail you a dead fish and you'll regret it.** But you can write it in interval notation, as $(-\infty, 2) \cup (3, \infty)$.

Now you try.

Exercise 3

Solve the inequalities.

1. $-2 \leq 2x - 1 \leq 5$
2. $|x - 10| - 5 < 0$
3. $|2x + 4| \geq 6$

Answer

1. $-\frac{1}{2} \leq x \leq 3$, or $\left[-\frac{1}{2}, 3\right]$
2. $5 < x < 15$, or $(5, 15)$
3. "$x \geq 1$ or $x \leq -5$," or $(-\infty, 5] \cup [1, \infty)$

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3.2E Exercises

Linear Inequalities

Solve the inequalities.

1. $14x + 32 < 4$
2. $\frac{3}{2}x + 6 > 0$
3. $6q + 2 \geq 4q - 3$
4. $2x - 1 \leq 12$
5. $5 - 3x > -4$
6. $-4x + \frac{1}{2} \leq 0$
7. $ax + b \leq c$, where $a, b,$ and c are constants, and $a > 0$
8. $ax + b \leq c$, where $a, b,$ and c are constants, and $a < 0$

Answer

1. $x < -2$ or $(-\infty, 2)$
2. $x > -4$ or $(-4, \infty)$
3. $q \geq -\frac{5}{2}$ or $\left[-\frac{5}{2}, \infty\right)$
4. $x \leq \frac{13}{2}$ or $\left(-\infty, \frac{13}{2}\right]$
5. $x < 3$
6. $x \geq \frac{1}{8}$
7. $x \leq \frac{c-b}{a}$
8. $x \geq \frac{c-b}{a}$

Inequalities With Factors

Solve the inequalities.

1. $x^2 \geq 4$
2. $x^4 \geq 16$
3. $x^2 + 3x < -2$
4. $x^2 - 5x + 6 < 0$
5. $\dfrac{x}{x+2} \leq 0$
6. $\dfrac{2}{x+3} \leq 0$

Answer

1. $x \geq 2$ or $x \leq -2$, or $(-\infty, -2] \cup [2, \infty)$
2. $x \geq 2$ or $x \leq -2$, or $(-\infty, -2] \cup [2, \infty)$
3. $-2 < x < -1$, or $(-2, -1)$
4. $2 < x < 3$, or $(2, 3)$
5. $-2 < x \leq 0$, or $(-2, 0]$
6. $x < -3$, or $(-\infty, -3)$

Simultaneous and Absolute Value Inequalities

Solve the inequalities.

1. $-1 \leq 3x + 4 \leq 1$
2. $|2x + 8| < 2$
3. $\left|\dfrac{x+1}{2}\right| \geq 3$

4. $|3 - x| > 9$

Answer

1. $-\frac{5}{3} \leq x \leq -1$, or $\left[-\frac{5}{3}, -1\right]$
2. $-5 < x < -3$, or $(-5, -3)$
3. $x \leq -7$ or $x \geq 5$, or $(-\infty, -7] \cup [5, \infty)$
4. $x < -6$ or $x > 12$, or $(-\infty, -6) \cup (12, \infty)$

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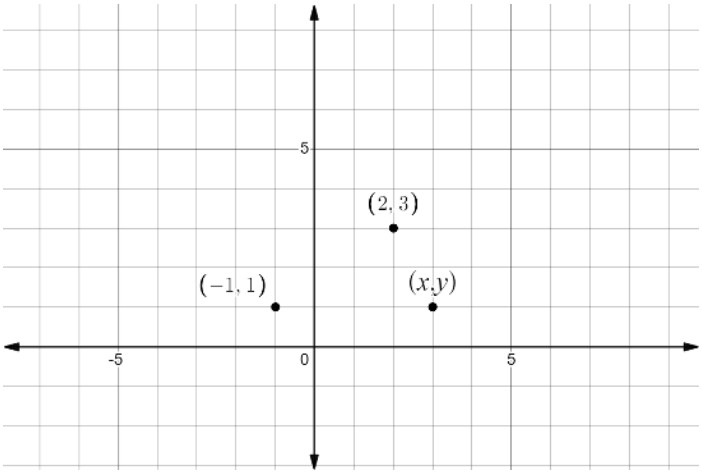
3.3 Lines and Circles

Learning Objectives

By the end of this section, you will be able to:

- give and use the point-slope and slope-intercept forms for the equation of a line
- find the intercepts and slope of a line
- connect the idea of slope to parallel and perpendicular lines
- find where two lines intersect
- sketch the graph of a line
- know and use the essential geometric formulas for circles
- recognize the standard equation of a circle and what it tells you
- sketch the graph of a circle

In this section, we're going to find out how to describe straight lines and circles using equations. We first need a place for our lines to live. For this, we use a **coordinate plane.**



You might hear this called the **Cartesian plane**, too. Anyway, it's a two-dimensional plane defined by two perpendicular axes, the **(horizontal)** x**-axis** and the **(vertical)** y**-axis.** Points on the plane are defined by a **coordinate pair** of numbers, (x, y), where the **first coordinate** is always the x**-value** and the **second coordinate** is always the y**-value.** (In the olden days these coordinates were called the abscissa and the ordinate, but now if you say that people will be like "Oh yeah, weren't those the two monsters they had to sail between in the Odyssey?" No. But I digress.) You can see I've labeled a few points as examples. If you look at the point $(2, 3)$ and track down toward the x-axis, you'll see his x-value is 2. Track horizontally from the point over to the y-axis, and you'll see his y-value is 3. Check the same for $(-1, 1)$. You can now take any point on the plane and determine its coordinates "address," and if I gave you a set of coordinates, you could locate that point on the plane. The point $(0, 0)$ is the location where the axes intersect, and it's called the **origin.**

? Exercise 1

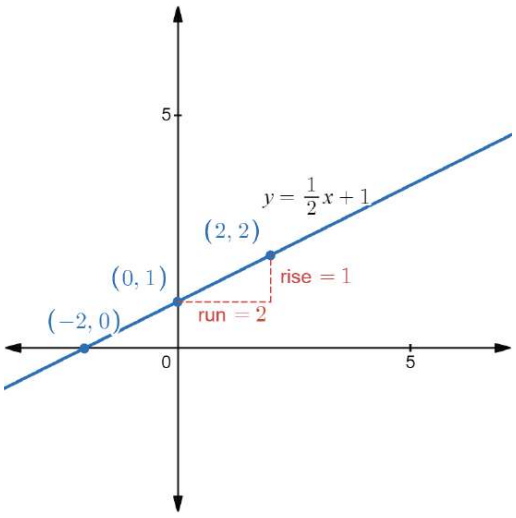
What are the x- and y-coordinates of that point I labeled (x, y) up there?

Answer

$3, 1$

Lines

Now, here's a picture of a straight line **graphed** (drawn) in the coordinate plane. Take a gander and then we'll discuss.



This line is straight as heck, we agree. I labeled the line with an equation, which we'll get to momentarily. For now, look at the points I deigned to label. The $(-2, 0)$ is sitting on the x-axis, so it's called an x-**intercept**. Likewise, the point $(0, 1)$ is sitting on the y-axis, so it's called a y-**intercept.** This line also has a quality of **steepness,** which we need to describe somehow. Notice that if I travel from the point $(0, 1)$ to the point $(2, 2)$, I need to go a distance of 2 units to the right horizontally, as well as 1 unit up vertically. We say this is 2 units in the positive x direction and 1 unit in the positive y direction. Horizontal travel is nicknamed **run,** while vertical travel is nicknamed **rise.** Steepness essentially describes how much altitude you gain while traveling a certain amount of horizontal distance. For every two steps to the right, my line goes up one step. We call this ratio the **slope** of the line, and usually denote it m. How do we calculate it?

The run of 2 can be computed as the difference of the x-coordinates of the two points $(0, 1)$ and $(2, 2)$. Similarly, the rise is computed as the difference of the y-coordinates. So in this case,

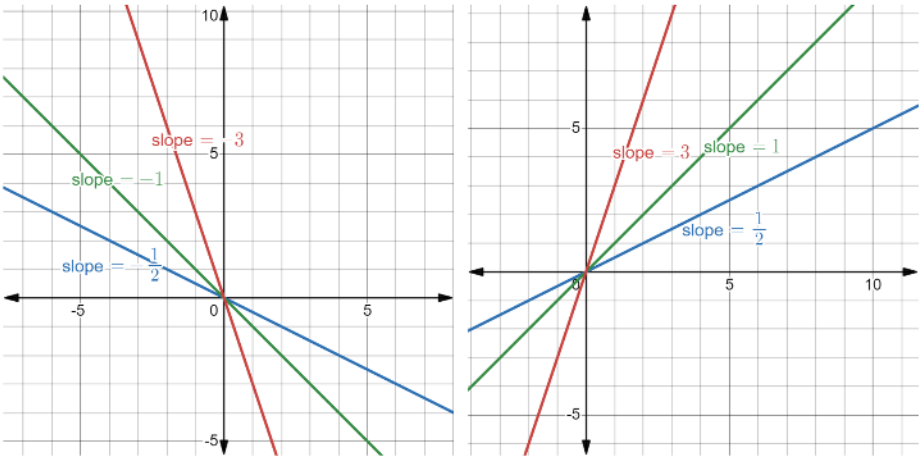
$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{2-1}{2-0} = \frac{1}{2}$$

- An x-**intercept** is a point where the line crosses the x-axis, and it will always have a y-coordinate of 0.
- A y-**intercept** is a point where the line cross the y-axis, and it will always have an x-coordinate of 0.
- The **slope** m of a line can be calculated using two distinct points on the line, (x_1, y_1) and (x_2, y_2), by computing rise over run:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

When calculating slope, it doesn't matter which order you subtract the coordinates in, as long as you are consistent on top and bottom! Also, the result will be the same for **any** two points you choose on the line. If we computed the slope using the points $(-2, 0)$ and $(0, 1)$ instead, we would get the same thing! In other words, a straight line has **constant slope,** that is, it never changes steepness.

Before we talk about equations, I want to give you a few more general ideas about slope. Compare this picture to the block of important info that comes right after it:

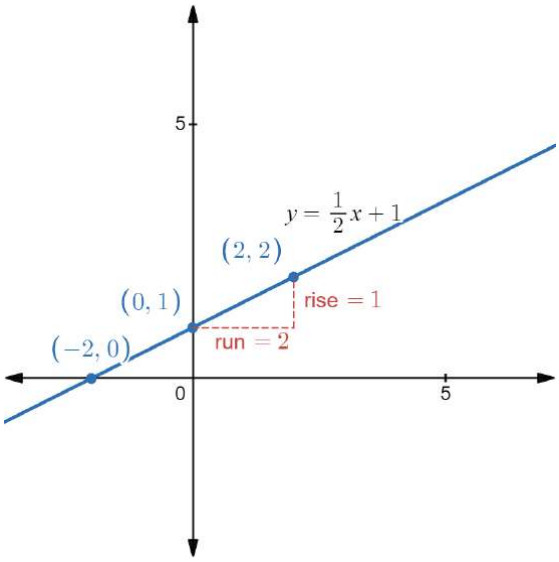


- **Uphill / Increasing lines** have **positive** slopes. Lines with whole number slopes bigger than 1 look **steeper** and lines with little fraction slopes smaller than 1 look **less steep.** The general idea is the bigger the number, the steeper the line.
- **Downhill / Decreasing lines** have **negative** slopes. Again, whole numbers more negative than -1 make **steeper** lines and little fractional slopes like $-\frac{1}{2}$ make **less steep** lines.
- Fun fact: if the slope of a line is **zero,** it's actually a flat **horizontal line.** The rise must be zero for this to happen, so the line can't go uphill or downhill.
- The slope of a **vertical line** is **undefined**, because if you're vertical, you are not changing horizontally at all, so the run would be zero and we can't divide by zero!

Okay, let's get to equations. Here's the bottom line (no pun intended, but I'm not taking it out).

All of the points (x, y) lying on a straight line will **satisfy** a certain equation of the form $y = mx + b$.

Look back at this picture:



The equation that goes with this line is $y = \frac{1}{2}x + 1$. Take the point on the line $(0, 1)$ and plug in 0 for x and 1 for y:

$$y = \frac{1}{2}x + 1 \quad \longrightarrow \quad 1 \stackrel{?}{=} \frac{1}{2}(0) + 1$$

Yep, $1 = 0 + 1$ any day. Plug in the x- and y-values of any other point on that line, and the equation will still be **true.** Now, there are some very special features of this equation form.

Slope-Intercept Equation of a Line

The equation of a line with slope m and y-intercept $(0, b)$ is

$$y = mx + b$$

If you look back at the example, you can see the y-intercept was $(0, 1)$ and yes indeed, we see a 1 in the b position of his equation.

Exercise 2

What is the slope-intercept equation of a line with slope $m = -3$ and y-intercept $(0, 4)$?

Answer

We identify that $b = 4$ and plug in: $y = -3x + 4$.

Everybody looooves good old em-ex-plus-bee, but the equation form you'll use most for the rest of your calculus life will be the **point-slope form**, so prioritize memorizing that. It needs two ingredients: a value m for the slope, and a point you want the line to go through, (x_0, y_0).

Point-Slope Equation of a Line

The equation of a line with slope m passing through the point (x_0, y_0) is

$$y - y_0 = m(x - x_0)$$

Example 1

Use point-slope form to write the equation of a line with slope -2 that passed through the point $(1, 3)$. Then use algebra to simplify the equation to slope-intercept form.

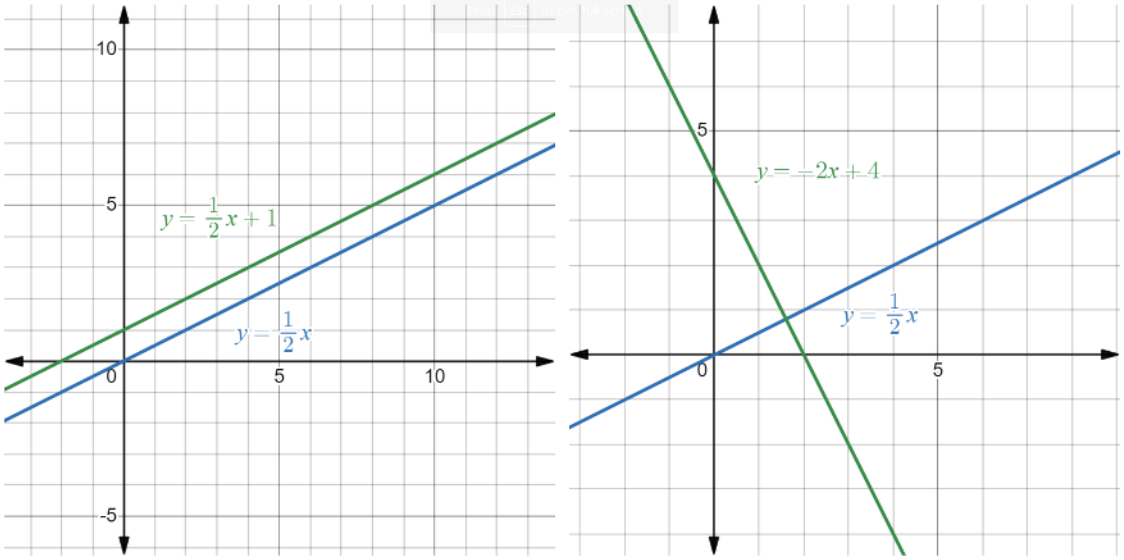
Solution

We have $m = -2$ and $(x_0, y_0) = (1, 3)$, so we just plug in: $y - 3 = -2(x - 1)$. Now we simplify and isolate y until we match the form $y = mx + b$.

$$y - 3 = -2(x - 1) \quad \longrightarrow \quad y = -2x + 2 + 3 \quad \longrightarrow \quad y = -2x + 5$$

From this form, we can tell that the y-intercept of this line would be $(0, 5)$.

Let's look at two interesting situations real quick. On the left, the lines go on forever without intersecting, which is called being **parallel lines.** On the right, they meet at **right angles** (90 degrees), which called being **perpendicular lines.**



- **Parallel lines** have **the same slope.**
- If two lines are **perpendicular** to each other then their slopes are **negative reciprocals.** (The negative reciprocal of a number $\frac{a}{b}$ is $-\frac{b}{a}$.)

Example 2

1. Find the equation of the line that is parallel to the line $y = 2x - 5$ and passes through the point $(-1, 8)$.

2. Find the equation of the line that is perpendicular to the line $y = 2x - 5$ and passes through the point $(3, 2)$.

Solution

1. We have the ingredient $(x_0, y_0) = (-1, 8)$, so we just need to find a slope to use. If we want to be parallel to $y = 2x - 5$, we better have the same slope. Since that equation is in slope-intercept form, we can just read off that the slope is 2. So we plug in $m = 2$ and the point we have to get $y - 8 = 2(x + 1)$.

2. This time we want to pass through $(x_0, y_0) = (3, 2)$, and we want to be perpendicular to $y = 2x - 5$. We know that guy's slope is 2, which we can think of as $\frac{2}{1}$. Then the negative reciprocal is $-\frac{1}{2}$. We'll use that for m and plug in to get $y - 2 = -\frac{1}{2}(x - 3)$.

Exercise 3

Find the equation of a line that is:

1. parallel to $y + 2 = -3x$ and passes through $(1, 1)$.
2. perpendicular to $y + \frac{4}{5}x + 1 = 0$ and passes through $(10, 0)$.

Hint: You need to simplify the given equations until they are in the form $y = mx + b$ so that you can see what their slopes are!

Answer

1. $y - 1 = -3(x - 1)$
2. $y = \frac{5}{4}(x - 10)$

If two lines **aren't** parallel, then they are **guaranteed to intersect** somewhere, eventually. At the location of intersection, they are at the exact same y-value for the exact same x-value. In slope-intercept form, $y = mx + b$, we can see that y "depends" on x. So in a sense, we could "hypothesize" that the y-values are equal, and solve for the x-value that would make that true.

To find where two lines $y = m_1x + b_1$ and $y = m_2x + b_2$ intersect, set $m_1x + b_1 = m_2x + b_2$ and solve for x. Then to find the corresponding y-coordinate, plug that x solution into either of the original equations.

Sometimes, after simplification, you will find the two equations are **equivalent**, in which case the lines are not actually distinct. They have ALL points in common.

Example 3

If the two lines intersect, find the point of intersection.

1. $y + 2x = -1$ and $y = -2x + 4$
2. $y + 2 = x$ and $2y = 6x + 8$

Solution

1. The second line is already in slope-intercept form, but the other one needs rearranging to get $y = -2x - 1$. We see at a glance that their slopes are the same! These are parallel lines and will never intersect. If you didn't notice this and set them equal anyway, you would be trying to solve

$$-2x - 1 \stackrel{?}{=} -2x + 4 \quad \longrightarrow \quad -2x - 1 + 2x \stackrel{?}{=} -2x + 4 + 2x \quad \longrightarrow \quad -1 \stackrel{?}{=} 4$$

and that is what we in the biz call a dirty rotten lie.

2. Both of these need a little cleaning to be in slope-intercept form, so do the algebra and report back. Did you get $y = x - 2$ and $y = 3x + 4$? Me too. Now set the right hand sides equal to each other and solve for x:

$$x - 2 = 3x + 4 \quad \longrightarrow \quad 2x = -6 \quad \longrightarrow \quad x = -3$$

This means the lines are at the exact same point when $x = -3$. To find the y-coordinate, we plug in to one of the equations:

$$y = (-3) - 2 \quad \longrightarrow \quad y = -5$$

The point of intersection is $(-3, -5)$.

Let's end this part with some tips for drawing lines quickly and easily based on their equations.

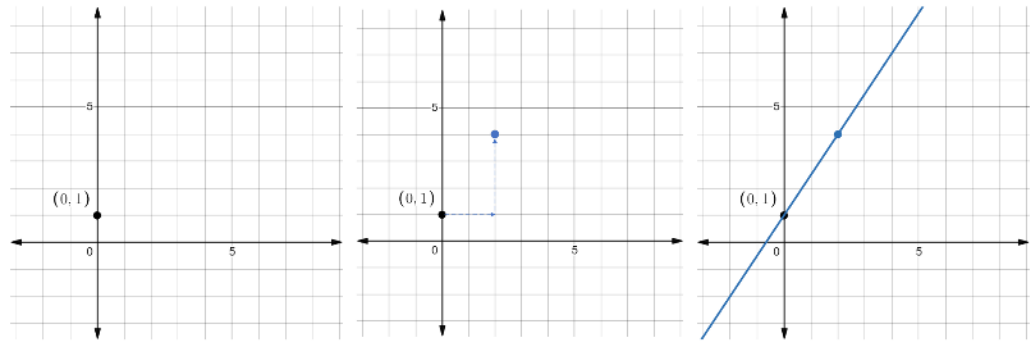
- If you know one point and the slope $m = \frac{a}{b}$ is positive, plot the point and then find another point just by tracking b units to the **right**, and then a units **up**. Drop a point in right there, connect them with a straight line, and you're done.
- If the slope $m = -\frac{a}{b}$ is negative, then track b units to the right, and then a units **down** instead.
- If a slope-intercept form has no b term, like this $y = 3x$, that means b is secretly 0. Thus you automatically know that line passes through the origin since its y-intercept is $(0, 0)$.
- An equation $y = b$ with no x term actually means $y = 0x + b$, so slope is zero and the line is horizontal, passing through b on the y-axis.
- An equation $x = a$ with no y term is a vertical line with undefined slope, passing through a on the x-axis.

Exercise 4

Sketch the graph of the line $y = \frac{3}{2}x + 1$

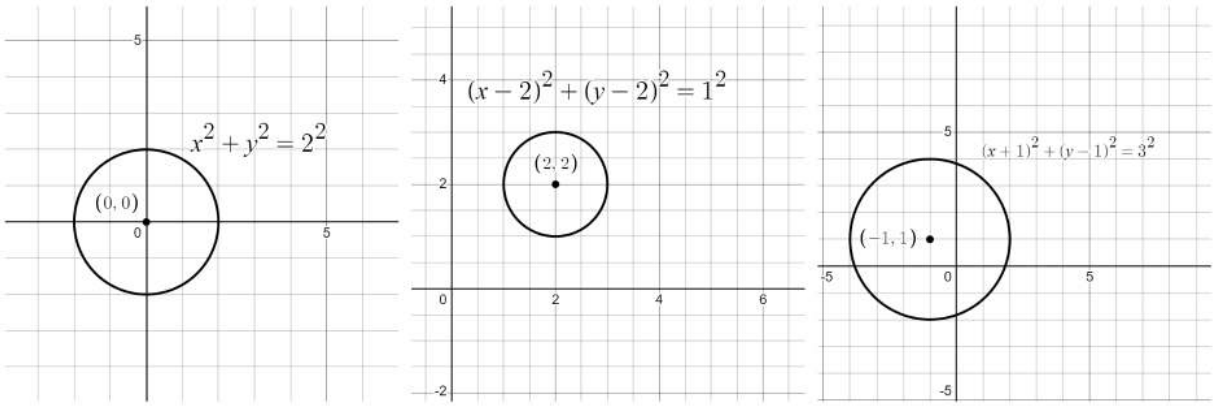
Answer

The slope-intercept form tells me the y-intercept $(0, 1)$ so I plot that. Then I count over 2 steps to the right and count up 3 steps, and drop another point in right there. Then I connect the dots like it's kindergarten!



Circles

Here are some circles drawn on coordinate planes. Each one is labeled with an equation. **All** of the points lying on the circle **satisfy** the equation given.



What do you deduce about the relationship between the equation and how the circle looks? The **radius** of the circle (the uniform distance from the center to any point on the circle) seems to be the number getting squared on the right hand side. And the coordinates of the **center?** They seem to be the numbers subtracted inside the parentheses. Interesting...

Standard Equation of a Circle

The equation of a circle centered at the point (h, k) with radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

Exercise 5

1. Describe the circle whose equation is $(x+2)^2 + (y-1)^2 = 16$.
2. Find the equation of a circle centered at $(1, 2)$ with radius 5.

Answer

1. We should manipulate this equation a little bit so we can see exactly what lines up with the standard form. In standard form, I see subtractions inside the parentheses, so we should make that match by changing the $+$ to minus-a-minus. And we should have somebody getting squared on the right hand side...

$$(x-(-2))^2 + (y-1)^2 = 4^2$$

Now we can see that the center of the circle must be $(-2, 1)$ and the radius must be 4.

2. For this, we identify that the center we want is $(h, k) = (1, 2)$ and the radius we want is $r = 5$. Then we just plug in to get $(x-1)^2 + (y-2)^2 = 5^2$.

Here are the essential features of a circle that I expect you to know:

- The **radius** is the uniform distance from the center to any point on the circle.
- The **diameter** is the distance from one side of the circle across to the other, passing through the center, so it's double the radius. That is, diameter $d = 2r$.
- The **area** of a circle with radius r is πr^2.
- The **circumference** of a circle is the distance all the way around it, as in how many steps your Apple watch would say you covered if you walked the whole perimeter, and it's calculated $2\pi r$.

If you're given an equation containing x^2 and y^2 terms in it, and asked whether it's the equation of a circle or not, the only thing to do is try to get it to *look* like the standard form of a circle's equation. This means completing the square...twice!

Example 4

Is $x^2 + 2x + y^2 - 2y = 7$ the equation of some circle? If so, tell the center and the radius.

Solution

We are aiming for something looking like $(x-h)^2 + (y-k)^2 = r^2$, and there are two perfect square products in there, so we will complete the square on the x terms AND on the y terms to try to make some perfect squares appear. First the x's... We identify the coefficient $b = 2$, so we need to add $\left(\frac{b}{2}\right)^2 = 1$ to both sides of the equation.

$$(x^2 + 2x \quad) + y^2 - 2y = 7 \quad \longrightarrow \quad (x^2 + 2x + 1) + y^2 - 2y = 7 + 1$$

Then factor and simplify to get $(x+1)^2 + y^2 - 2y = 8$. So far so good. Now the y's... We identify the coefficient $b = -2$ this time, so we will be adding $\left(\frac{b}{2}\right)^2 = 1$ again.

$$(x+1)^2 + (y^2 - 2y \quad) = 8 \quad \longrightarrow \quad (x+1)^2 + (y^2 - 2y + 1) = 8 + 1$$

Finally, factor and simplify, and we have achieved our goal! We have $(x+1)^2 + (y-1)^2 = 9$, which looks exactly like the standard form. We read off the center to be $(-1, 1)$ and the radius must be 3.

Exercise 6

Prove that $x^2 + 4x + y^2 + 2y + 1 = 0$ is the equation of a circle by completing the square to find standard form. Then tell the center and radius of the circle.

Answer

The equation is $(x+2)^2 + (y+1)^2 = 2^2$ so the center is $(-2, -1)$ and the radius is 2.

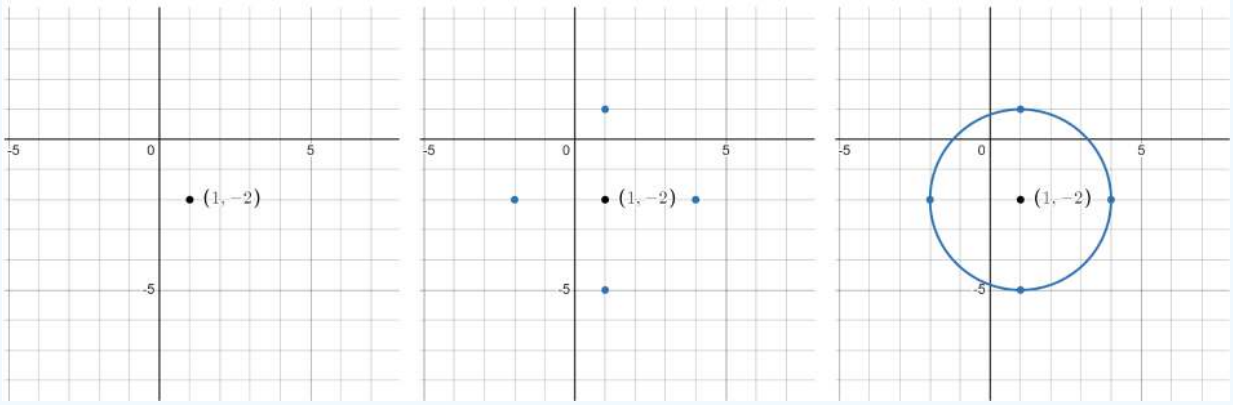
Sketching a circle based on an equation is pretty straightforward in theory, if uhhhhhhhhhh aesthetically challenging in practice (well, unless you're Giotto). Really, you just identify and plot the center, and then draw four cardinal points at the appropriate distance away for the radius. And then do your darnedest to connect the dots in a nice circle.

Exercise 7

Plot the circle with equation $(x-1)^2 + (y+2)^2 = 3^2$ on a coordinate plane.

Answer

Identify that the center is $(1, -2)$ and that the radius is 3, and then channel your inner artiste.

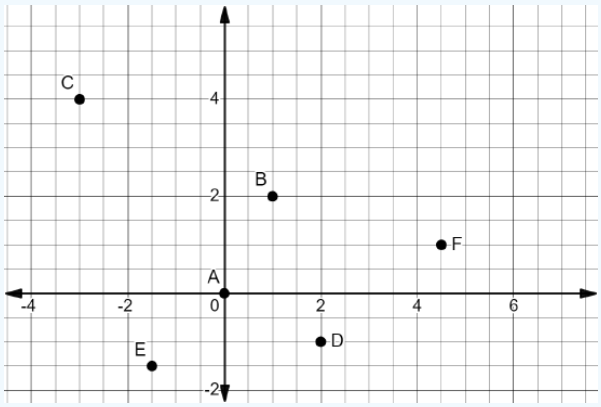


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3.3E Exercises

Understanding the Coordinate Plane

Give the coordinates of the points labeled below.



Answer

- $A = (0, 0)$
- $B = (1, 2)$
- $C = (-3, 4)$
- $D = (2, -1)$
- $E = (-1.5, -1.5)$
- $F = (4.5, 1)$

Identifying Equations

Identify whether the equation is that of a line, a circle, or neither. Then, if it's a line, tell the slope and y-intercept. If it's a circle, tell the center and radius.

1. $y = 2x - 18$
2. $3x + 3y = 3$
3. $x^2 + y^2 = 25$
4. $(x + 2)^2 = (y - 1)^2 + 3$
5. $y^2 - 3 = -x^2 - 2x - 1$
6. $-2x + 2y = x + y + 4$
7. (Complete the square twice) $x^2 + 4x + y^2 - 6y = -8$
8. $x^2 + y^2 - 2x - 8y + 17 = 9$

Answer

Hey don't look down here without trying! Hint: you need to use algebra to manipulate the equations to see if they look like $y = mx + b$ or $(x - h)^2 + (y - k)^2 = r^2$. Remember that it's okay to see a $(x + 3)^2$, for example, because that can be written $(x - (-3))^2$ to match the form. Now go back and try!

1. Line. The slope is 2 and y-intercept is $(0, -18)$.
2. Line, with rearrangement $y = -x + 1$. The slope is -1 and the intercept is $(0, 1)$.
3. Circle. The center is $(0, 0)$ since like x^2 is like $(x - 0)^2$, and the radius is 5.
4. With rearrangement, this is $(x + 2)^2 - (y - 1)^2 = 3$, which does NOT match a circle's equation because of the subtraction of the squared terms. In fact, this is the standard equation of a hyperbola.
5. With factoring and rearrangement, this is $(x + 1)^2 + y^2 = 3$, which is a circle. The center is $(-1, 0)$ and the radius is $\sqrt{3}$.
6. With simplification, this is $y = 3x + 4$, a line. The slope is 3 and intercept is $(0, 4)$.

7. After completing the square on x terms and y terms separately, we have $(x+2)^2+(y-3)^2=5$, a circle. The center is $(-2,3)$ and the radius is $\sqrt{5}$.
8. This becomes $(x-1)^2+(y-4)^2=9$, a circle. The center is $(1,4)$ and the radius is 3.

Writing Equations

1. Find the equation of the line with slope 4 and y-intercept $(0,45)$.
2. Find the equation of the line with slope $-\frac{1}{3}$ passing through the point $(2,3)$.
3. Find the equation of the line parallel to $y+x=0$ and passing through the point $(0,1)$.
4. Find the equation of the line perpendicular to $y=\frac{7}{8}x+9$ and passing through the point $(7,6)$.
5. Find the equation of the circle centered at the origin with radius 1.
6. Find the equation of the circle centered at $(1,2)$ with radius 3.
7. Find the equation of the circle centered at $(-6,-2)$ with radius $\sqrt{47}$.

Answer

1. $y=4x+45$
2. $y-3=-\frac{1}{3}(x-2)$ or $y=-\frac{1}{3}x+\frac{11}{3}$
3. $y=-x+1$
4. $y-6=-\frac{8}{7}(x-7)$ or $y=-\frac{8}{7}x+14$
5. $x^2+y^2=1$
6. $(x-1)^2+(y-2)^2=9$
7. $(x+6)^2+(y+2)^2=47$

Finding Intersections

Identify whether the two lines intersect, and if so, give the coordinates of the point of intersection.

1. $y=2x+4$ and $y=7x-1$
2. $y+2x=1$ and $2y=-4x$
3. $y-x=3$ and $8y=8x+24$

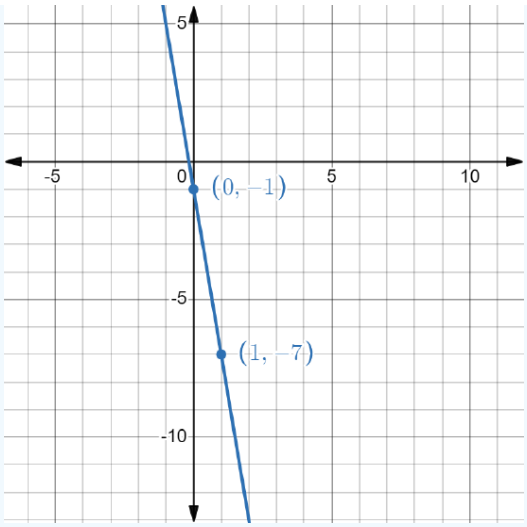
Answer

1. These lines intersect at $(1,6)$.
2. These lines are parallel, so they don't intersect.
3. These lines are actually the same line.

Graphing Lines

Sketch the line with equation $3y+4x+1=2(y-x)$ on a coordinate plane.

Answer



Geometry of Circles

Given the equation of a circle $(x-1)^2+(y+3)^2=49$, identify the center and radius. Then compute:

1. the diameter
2. the area
3. the circumference

Answer

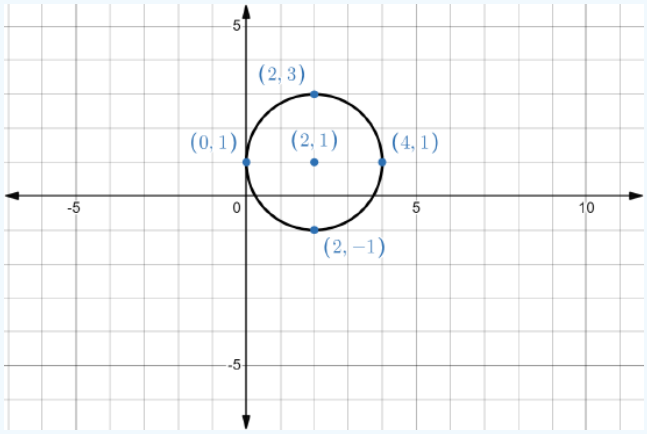
The center is $(1,-3)$ and the radius is 7.

1. The diameter is 14 units.
2. The area is 49π square units.
3. The circumference is 14π units.

Sketching a Circle

Sketch the circle whose equation is $(x-2)^2+(y-1)^2=4$ on a coordinate plane.

Answer



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3.4 Modeling With Equations

Learning Objectives

By the end of this section, you will be able to:

- recognize and describe direct and inverse variation using equations containing variables
- combine different types of variation into a single equation of multiple variables
- set up equations describing the relationships between real life quantities
- solve an equation for a variable in terms of others

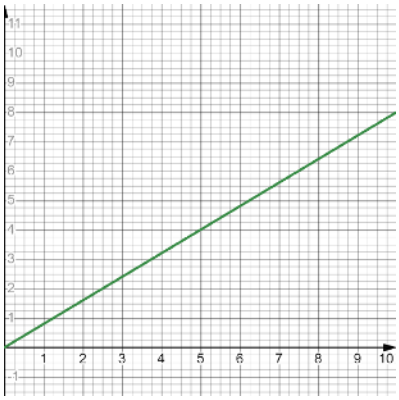
A **mathematical model** is a way of describing a real-world situation or phenomenon using math symbols in an equation or formula. This involves representing quantities with variables and then writing down how they relate to each other. Since they depend upon each other, changing one variable induces a change in the other(s). Sometimes the relationship between them is very simple.

Direct Variation

Direct Variation

If the quantities x and y are related by an equation of the form $y = kx$**,** for some constant $k \neq 0$, then we say y **varies directly as** x, or y **is directly proportional to** x. The constant (plain ol' number) k is the **constant of proportionality.**

Translation: one variable is a constant multiple of the other. This is also just called being **proportional**, like the default type of proportionality is "direct." Notice that the equation $y = kx$ looks like the equation of a straight line with slope k and y-intercept $(0, 0)$. Looking at the graph below, we see that when k is positive, increasing x causes an increase in y (as I go to the right, I also go up). This shows how y is directly proportional to x.



Lots of real-world physical situations can be modeled by direct variation.

Example 1

1. As I produce more and more washing machines, my cost goes up, because I'm paying for more materials and more labor. Each washing machine costs me $225.37 to make. Find a model relating cost, denoted C, and the number of machines I make, denoted x. Then use it to figure out what it will cost to make 100 washing machines.

2. I'm driving down I-70 in my 2005 Scion xA at a constant speed of 100mph. (Well, I don't have cruise control because it's a manual, but I put a brick on the gas pedal. (Do not do this.)) As time in hours, denoted t, goes on, I cover more distance in miles, denoted d. Find a model relating distance and time, and use it to figure out how long it will take to travel 150 miles.

Solution

1. Every washing machine costs $225.37 to make, so the total cost can be calculated with multiplication: $C = 225.37x$. This model shows that C is directly proportional to the number of machines I make, x, with the constant of proportionality k

$= 225.37$. If I want to make 100 machines, I can calculate the associated cost by plugging in for x: $C = 225.37(100) = \$22,537$.

2. Distance covered and time are directly proportional, with equation $d = 100t$. If I want to figure out how long it will take to cover 150 miles, I just plug that quantity in for d and solve for t. I get

$$150 = 100t \quad \rightarrow \quad t = \frac{150}{100} = 1.5 \text{ hrs}$$

("But De Wolf," you say, "even with a brick on the gas pedal, you won't have a constant speed due to other road factors!" Shush, yes I will, this is I-70 driving across Kansas, it's as straight and flat as they come.)

Exercise 1

The cost of printing 10 pages is \$5.50, and we know the cost-per-page is a constant rate, so cost is directly proportional to the number of pages needed. How much will it cost to print 100 pages?

Answer

Name cost C and number of pages n. "Cost is directly proportional to number of pages" is code for $C = kn$. We know that $n = 10$ goes with cost $C = 5.5$, so we can determine k.

$$5.5 = k(10) \quad \longrightarrow \quad k = \frac{5.5}{10} = 0.55$$

Now I want to know what C is if $n = 100$. Simply plug in:

$$C = (0.55)(100) = \$55.00$$

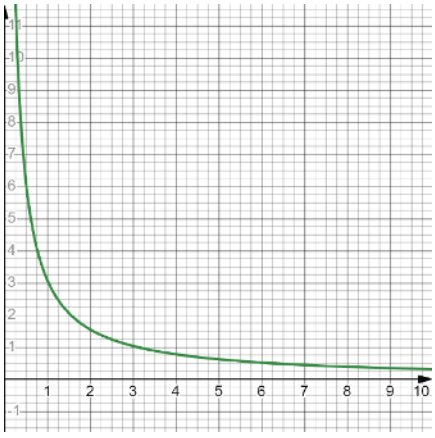
Inverse Variation

Sometimes two quantities are related in such a way that if I raise one, the other gets smaller.

Inverse Variation

If the quantities x and y are related by an equation of the form $y = \frac{k}{x}$, for some constant $k \neq 0$, then we say y **varies inversely as** x, or y **is inversely proportional to** x. The constant (plain ol' number) k is still the **constant of proportionality.**

This equation is NOT the equation of a straight line. Think about what happens when x gets bigger. Since k is constant, staying the same, when the denominator of the fraction gets bigger, the fraction as a whole gets smaller. Then y, who is defined by this fraction, will be getting smaller. On a graph, this relationship looks like this when k is positive:



We see as we go to the right, the curve falls downhill, showing that as x increases, the y-value decreases.

Example 2

1. A perpetuity is a bond that pays the same amount every year forever. For example, you could pay a price, denoted P, today and in return, receive \$1000 every year. The current price of that perpetuity depends on the market interest rate, written as a decimal and denoted i, according to the equation $P = \frac{1000}{i}$. If interest rates go up, what will happen to the prices of such bonds? If the price is currently sitting at \$10,000, what must the market interest rate be?

2. Boyle's Law states that, if the temperature and amount of an ideal gas are held constant, the pressure P exerted by the gas is inversely proportional to the volume it occupies. If I compress a gas into a smaller volume without letting any escape, the pressure will increase. Boyle's Law is often stated in math symbols as $PV = nRT$, where n and R are constants specific to the situation. Find a way of writing that equation so that it matches the form in the definition of inverse variation.

Solution

1. Price and interest rate are inversely proportional, so if interest rates go up, the prices will go down. We can plug in the price \$10,000 and solve for i to determine the associated rate.

$$10000 = \frac{1000}{i} \quad \longrightarrow \quad i = \frac{1000}{10000} = 0.1$$

The market interest rate is currently 10%.

2. Since temperature T is being held constant, the right hand side of the equation all boils down to some plain ol' number, and we could nickname it k. Then with a little rearrangement, we can make the equation look like the formula for inverse variation between two variables $y = \frac{k}{x}$, seeing the model as

$$PV = k \quad \longrightarrow \quad P = \frac{k}{V}$$

Exercise 2

If I move house all by myself, it takes me 12 hours to pack everything. If I ask my mother to help, it takes only 6 hours. The more people I ask to help, the less time it takes, so the time t is inversely proportional to the number of people working, n. If I ask my whole family of 6 people to help, how long will it take to pack?

Answer

We write the model with inverse variation, $t = \frac{k}{n}$. I said if I work alone, meaning $n = 1$, it takes me 12 hours. Plugging these values in, we get $12 = \frac{k}{1}$, which means $k = 12$. We can confirm because if my mom helps, so that $n = 2$, we should get 6 hours, and yep, $t = \frac{12}{2} = 6$. Great. Now if I have $n = 6$, the time will be $t = \frac{12}{6} = 2$ hours!

More Than Two Variables

In the Boyle's Law example, we held temperature constant, but in real life, of course it could vary, in which case T would be seen as a variable in the equation $PV = (nR)T$. There may be more than two quantities involved in a model that can all vary, giving us multiple variables in our equation. In the info box below, k always represents a constant number.

- If x, y, and z are related by the equation $z = kxy$, we say z is **proportional to the product** of x and y, or z is **jointly proportional** to x and y, or z **varies jointly** as x and y.
- Looking at an equation solved for one variable, like the above or like $z = k\frac{x}{y}$, we know that z is **directly proportional** to any variables in the **numerator,** and **inversely proportional** to any variables in the **denominator.**
- You can also be proportional to expressions other than a variable alone! For example, if $z = k(x + y)$, we say z is directly proportional **to the sum** of x and y. Or if $z = \frac{k}{x^2}$, we say that z is inversely proportional **to the square** of x.

Example 3

1. Newton's Law of Gravity tells us the gravitational force F between two objects with masses m_1 and m_2 that are a distance r apart, computed using the model $F = G\frac{m_1 m_2}{r^2}$, where G is the universal gravitational constant ($6.67430 \times 10^{-11}\ \frac{N \cdot m^2}{kg^2}$). Describe the relationships between F and the other variables m_1, m_2, r. If the distance between the masses goes up, what happens to the gravitational force they exert on each other?

2. If I have a spherical balloon with radius r, the volume inside is given by $V = \frac{4}{3}\pi r^3$. Describe the variation relationship between V and r modeled by this equation, and identify the constant of proportionality. If the radius increases by a factor of 2, what happens to the volume of the balloon?

Solution

1. F is directly proportional to the product of m_1 and m_2. F is inversely proportional to the square of the distance r. If the distance r increases, then certainly the the square of the distance, r^2, increases, which makes the fraction as a whole get smaller. Thus the force will decrease.

2. V is directly proportional to the cube of r. Remember that π is just a plain ol' number even though he looks like a letter, so he gets lumped in with the $\frac{4}{3}$ as part of the constant of proportionality. If the radius increases by a factor of 2, the new radius is $2r$. Let's plug that in to the model to see what happens.

$$V_{\text{new}} = \frac{4}{3}\pi(2r)^3 = \frac{4}{3}\pi(8r^3) = 8\left(\frac{4}{3}\pi r^3\right) = 8V_{\text{original}}$$

The volume also increases (it'd better) but by a factor of 8.

Exercise 3

1. Think back to that perpetuity price vs. market interest rates example... I picked a specific payment of \$1000 per year for that example, but of course you have a lot of different payment options to choose from. Let's nickname the payment amount with a variable, $\$X$. Rewrite the relationship between price, payment, and interest rate, and describe the relationships. If the payment you want increases, what happens to the price of the bond?

2. The **arithmetic mean (average)** of a set of k sample values $\{x_1, x_2, ..., x_k\}$ is calculated

$$\overline{x} = \frac{x_1 + x_2 + ... + x_k}{k}$$

If the number of sample values is kept the same, but the values themselves change, describe the variation. Also, give the constant of proportionality.

Answer

1. The price appeared in the numerator of the fraction, so the equation is now $P = \frac{X}{i}$, and we're seeing all the letters as variables now. P is directly proportional to X and inversely proportional to i. If the interest rates stay the same and you want to buy a perpetuity that pays a higher amount each year, then the price will also increase.

2. The average $\overline{x}$ is directly proportional to the sum of the values, $x_1 + x_2 + ... + x_k$. Even though this doesn't currently look quite right, remember that we can write the division like this

$$\overline{x} = \frac{1}{k}(x_1 + x_2 + ... + x_k)$$

to see that the constant of proportionality is $\frac{1}{k}$, still just a plain ol' number!

Solving for a variable in terms of other variables

If you have an equation relating multiple variable quantities, and it's solved for the guy you're curious about, fantastic. It was very easy to calculate the price P given an interest rate i in one go, because we had P isolated on one side of the equation. All we had to do was plug in a number for i and calculate! On the other hand, solving for an interest rate when given a price was a little more

involved, because the variable was NOT isolated, but rather stuck in a denominator. If we wanted to find interest rates associated with a whole bunch of different price options, this constant repeated solving would become tedious.

Good news, life doesn't have to be that way! If we had solved the equation for the variable i in terms of the other variables, we could easily calculate results with one step of a calculator.

To **solve an equation for a desired variable in terms of the other variables,** simply use algebra to isolate the desired variable on one side, doing inverse operations as necessary to move other variables over. This can be done on any equation, not just models using variation.

Example 4

1. Suppose I know the volume that I want inside my spherical balloon, and I want to compute the necessary radius. Solve $V = \frac{4}{3}\pi r^3$ for r. If I want the volume to be 36 cubic inches, what should the radius be?

2. Solve the equation $F = G\frac{m_1 m_2}{r^2}$ for the variable m_1. Also, solve for r in terms of the other variables.

3. The monthly cost of running De Wolf's De Washing Machines depends on several factors: my facility's rent, C_r; the cost of my utilities which depends on temperature outside, T, and the number of machines I make, x; and the materials and labor costs, which also depend on x. The total cost is given by $C = C_r + (k_1 T + k_2 x) + 250x$, where k_1 and k_2 are just constants. Solve this equation for x.

Solution

1. To start, I use whatever inverse operations are needed to get all the clutter away from the r^3. I need to multiply both sides of the equation by $\frac{3}{4\pi}$:

$$\frac{3}{4\pi}\cdot V = \frac{4}{3}\pi r^3 \cdot \frac{3}{4\pi} \quad \longrightarrow \quad r^3 = \frac{3V}{4\pi}$$

Then to finish, we need to undo the cubing by taking a cube root on both sides, resulting in $r = \sqrt[3]{\frac{3V}{4\pi}}$. If we want $V = 36$, we just plug that in and simplify to find r:

$$r = \sqrt[3]{\frac{3(36)}{4\pi}} = \sqrt[3]{\frac{3(9)}{\pi}} = \sqrt[3]{\frac{27}{\pi}} = \frac{3}{\sqrt[3]{\pi}}$$

2. To isolate m_1, I need to divide away the things he's multiplied by, and multiply over the things he's divided by.

$$\frac{r^2}{Gm_2}\cdot F = G\frac{m_1 m_2}{r^2}\cdot\frac{r^2}{Gm_2} \quad \longrightarrow \quad m_1 = \frac{Fr^2}{Gm_2}$$

To solve instead for r^2, use your favorite trick for getting a variable out of a denominator, maybe cross-multiplication? Or "multiplying over" the r^2. However you want to think about it.

$$F = G\frac{m_1 m_2}{r^2} \quad \longrightarrow \quad Fr^2 = Gm_1 m_2$$

Then to isolate r, we need to divide both sides by F and then take a square root.

$$r = \sqrt{\frac{Gm_1 m_2}{F}}$$

3. To start isolating x, I need to bring together my like terms to make sure I only have one x term. I also need to subtract all the non-x terms over to the same side.

$$C = C_r + k_1 T + k_2 x + 250x \quad \longrightarrow \quad k_2 x + 250x = C - C_r - k_1 T$$
$$\longrightarrow \quad (k_2 + 250)x = C - C_r - k_1 T$$

Having factored x out, I am ready to isolate him by dividing over the expression in parentheses. I get $x = \frac{C - C_r - k_1 T}{k_2 + 250}$.

Exercise 4

1. The surface area of a cylinder-shaped tin can, with radius r and height h, is given by $A = 2\pi r^2 + 2\pi rh$. Solve this equation for h.
2. The volume of a cone with base radius r and height h is given by $V = \pi r^2 \frac{h}{3}$. Solve this equation for r. Also solve it for h.

Answer

1. $h = \frac{A - 2\pi r^2}{2\pi r}$
2. $r = \sqrt{\frac{3V}{\pi h}}$ and $h = \frac{3V}{\pi r^2}$

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3.4E Exercises

Identifying Variation With Two Variables

For each equation,

- Check whether y varies directly or inversely as x, or neither. To do this, algebraically manipulate the equation to see if a variable can be written either as a constant multiple of the other ($y = kx$) or as a constant multiple of the reciprocal of the other $\left(y = \frac{k}{x}\right)$.
- Describe in words the nature of the relationship between the variables.

1. $y = \frac{1}{3}x$
2. $y = \frac{12}{x}$
3. $y = 4x + 2$
4. $y - x = 0$
5. $\frac{x}{2y} = 1$
6. $\frac{y+1}{3x} = 2$
7. $\frac{1}{y} = \frac{8}{x}$
8. $xy = 9$

Answer

1. y is directly proportional to x with constant of proportionality $\frac{1}{3}$.
2. y is inversely proportional to x with constant 12.
3. This is not a proportional relationship, as it cannot be written in either form, due to the $+2$ constant term.
4. y is directly proportional to x with constant 1.
5. y is directly proportional to x with constant $\frac{1}{2}$.
6. This is not a proportional relationship.
7. y is directly proportional to x with constant $\frac{1}{8}$.
8. y is inversely proportional to x with constant 9.

Writing Equations of Proportionality

Write an equation that expresses the relationship described. Denote your constant of proportionality by k.

1. y is directly proportional to x.
2. v is inversely proportional to u.
3. P varies directly as t.
4. F varies inversely as a.
5. w is directly proportional to the product of u and v.
6. A is inversely proportional to q and directly proportional to p.
7. V is directly proportional to h and inversely proportional to the square of x.

Answer

1. $y = kx$
2. $v = \frac{k}{u}$
3. $P = kt$
4. $F = \frac{k}{a}$
5. $w = kuv$
6. $A = k\frac{p}{q}$
7. $V = k\frac{h}{x^2}$

Finding Constants of Proportionality.

Use the equations of proportionality you found above as well as the given information to determine the constants of proportionality.

1. If $x = 10$, then $y = 20$.
2. If $u = 2$, then $v = 4$.
3. If $t = 25$, then $P = 100$.
4. If $a = 9$, then $F = 12$.
5. If $u = 6$ and $v = 8$, then $w = 1$.
6. If $p = 3$ and $q = 6$, then $A = 12$.
7. If $h = 1$ and $x = 2$, then $V = 2$.

Answer

1. $k = 2$
2. $k = 8$
3. $k = 4$
4. $k = 108$
5. $k = \frac{1}{48}$
6. $k = 24$
7. $k = 8$

Applications

1. Hooke's Law states that the force F needed to keep a spring stretched x units beyond its natural length is directly proportional to that displacement x. The constant of proportionality k is called the **spring constant**. Write the equation of proportionality. If a spring's natural length is 6cm and a force of 20 Newtons is needed to keep it stretched to a length of 10cm, find the spring constant. What force will be needed to keep the same spring stretched to 12cm?

2. The power P generated by a windmill turning is directly proportional to the cube of the wind speed v. Write the equation of proportionality. If a wind speed of 10mph generates 50 watts of power, how much power will the windmill produce if the wind speed drops to 5mph?

3. I'm racing go karts around a circular track with radius r. The force F needed to keep the kart from drifting is jointly proportional to the weight w of the kart and the square of my speed v, and inversely proportional to the radius of the curve. Write the equation of proportionality and solve it for v.

Answer

1. $F = kx$, $k = 5$, $F = 30\text{N}$
2. $P = kv^3$, $k = \frac{1}{20}$, $P = \frac{125}{20} = \frac{25}{4} = 6.25$ watts
3. $F = k\frac{wv^2}{r}$, $v = \sqrt{\frac{Fr}{kw}}$

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3.5 Solving Systems of Linear Equations

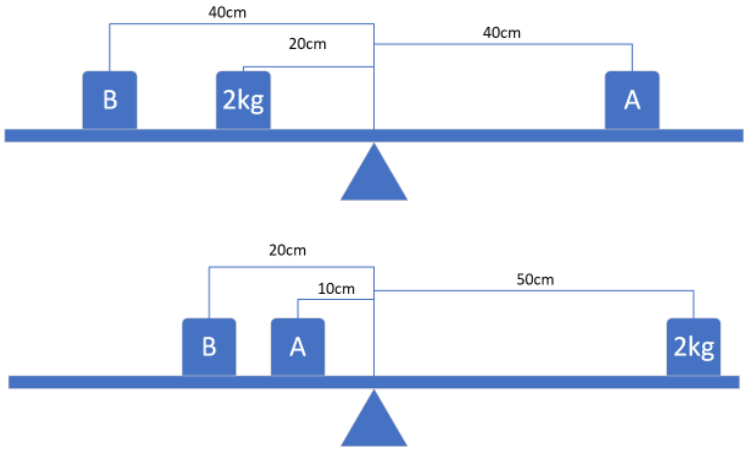
Learning Objectives

By the end of this section, you will:

- know the definition of a system of linear equations and how they relate to intersections of lines
- be able to solve 2×2 systems of linear equations with the substitution and elimination methods
- be able to solve 3×3 systems of linear equations with the elimination method

The other day, I was exploring an ancient tomb on a trip to Egypt. As I came around a corner, a wall of stone suddenly slid into place behind me, and I was confronted by an enchanted sphinx angry at the invasion of its privacy (fair). It showed me a flat plank balanced on a fulcrum point, and three lumps of solid gold. "This first lump is exactly 2kg," it intoned. "If you can determine the weights of the other two lumps, I will release you—and what's more, you can keep the gold." Obviously, I figured it out and survived to write this textbook. How did I do it?

I gingerly placed the blocks on the plank such that they perfectly balanced each other, and noted the distances from the fulcrum. Then I did it again in a different configuration.



I nicknamed the weights of the unknown blocks with the variables A and B. I know that in order to be perfectly balanced, the sum of the moments (turning effect of the weight of each block on a lever, aka the weight of the block multiplied by the distance from the fulcrum) must balance out on either side of the plank. This fact gave me two equations that must be true:

$$\begin{cases} 40B + 20(2) = 40A \\ 10A + 20B = 50(2) \end{cases}$$

All I had to do is figure out how to find values for A and B that would **satisfy both equations simultaneously.** This is called a **system of two linear equations in two unknowns (2×2),** because in each equation the variables only appear to the first power. How did I do this? First, I organized my terms, rewriting the system as

$$\begin{cases} 40A - 40B = 40 \\ 10A + 20B = 100 \end{cases}$$

Then I noticed that every term in the first equation has a factor of 40 in common, so I divided both sides by 40 just to clean things up. I also divided the second equation by 10. At this point, I had two choices. The first is called **substitution**, and the second is called **elimination.** Both are equally easy in this particular example, so let me show you.

Example 1

Solve the system of equations $\begin{cases} A - B = 1 \\ A + 2B = 10 \end{cases}$ two ways.

Solution

Substitution Method: This method takes advantage of the fact that the top equation can easily be solved for one of the variables; I choose A. Algebra results in $A = B + 1$. Now if A is equal to $B + 1$, I can substitute $(B + 1)$ wherever A appears in the second equation without changing anything, technically. This gives me an equation with only one unknown, easily solved!

$$A + 2B = 10 \quad \longrightarrow \quad (B + 1) + 2B = 10 \quad \longrightarrow \quad \cdots \quad \longrightarrow \quad B = 3$$

So gold lump B is 3kg! To figure out A, I just sub that value in for B in either original equation. The first one is easy.

$$A - 3 = 1 \quad \longrightarrow \quad A = 4$$

Elimination Method: This method will always work, even if equations are too ugly to solve nicely for one variable. The idea is that if two things are equal, $X = Y$, and two other things are also equal, $W = Z$, then adding the equations will give another true equation, $X + W = Y + Z$. To make this fact useful, I first multiply through one equation by whatever factor is needed to make the A or B terms match, but have opposite signs. For example, multiplying through the first equation by 2 gives me

$$\begin{cases} 2 \cdot (A - B) = 1 \cdot 2 \\ A + 2B = 10 \end{cases} \quad \longrightarrow \quad \begin{cases} 2A - 2B = 2 \\ A + 2B = 10 \end{cases}$$

To add the equations, all you have to do is make sure all the A and B and constant terms are lined up vertically, and then add down each "column." I get

$$\begin{cases} 2A - 2B = 2 \\ A + 2B = 10 \end{cases} \quad \xrightarrow{\text{add}} \quad 3A + 0B = 12$$

The $0B$ is of course nothing, I'm just trying to show you what happened there. Now I have a single equation with one unknown, $3A = 12$, which solves to get $A = 4$kg. To find B, just plug into an original equation again, say, $4 - B = 1$, telling me that $B = 3$kg. Confirmed!

It seems many students default to the substitution method, but I myself have always prefered the elimination method because it's more useful when systems get more complicated, and like Post Malone, *when I started ballin' I was young.* You can pick what you like, and I won't judge, but you do have to get the right answer in the end... Let's hit a quick formal definition and summarize.

2×2 Linear Systems

Systems of Equations

- A **system of equations** is a set of equations that have the exact same variables, for which a **solution** is a set of values chosen for each variable that will satisfy **all** of the equations simultaneously.
- A **linear equation** is an equation in which all variables appear only to the first power, such as $ax + by = c$ or $ax + by + cz = d$, where a, b, c, d are constant coefficients. In general and formally, a linear equation in n variables can be written in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = c,$$

  where $a_1, a_2, \ldots, a_n$ are constant coefficients and $x_1, x_2, \ldots, x_n$ are different variables.
- A **system of linear equations** is, shocker, a system in which all the equations are linear equations.

To **solve a system of two linear equations in two unknowns**, you can use...

- **Substitution:** Solve one equation for a particular variable, then sub in for that variable in the second equation, yielding an equation with one unknown, and solve that. Then plug that result into an original equation to find the second unknown.
- **Elimination:** Multiply one or both equations by whatever is needed such that, for a particular variable, the equation's terms have opposite coefficients. Then line up all the variable terms and constant terms in columns correctly and add down the columns, yielding an equation with one unknown, and solve that. Then plug the result into an original equation to get the other unknown.

? Exercise 1

Solve the linear systems using both methods.

1. $\begin{cases} 5A - B = 10 \\ 10A + B = 20 \end{cases}$
2. $\begin{cases} A + B = 6 \\ 2A - 4B = 24 \end{cases}$
3. $\begin{cases} 2A + 3B = 4 \\ 3A + 2B = -4 \end{cases}$

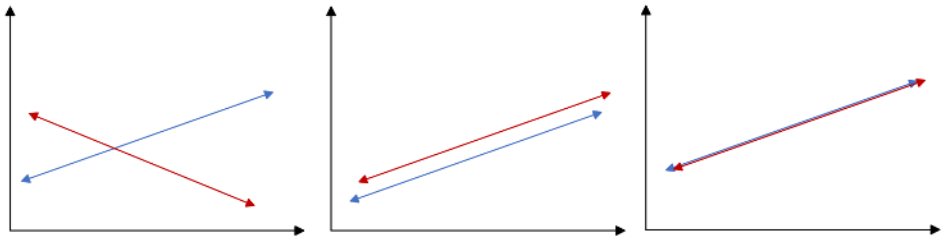
Answer

1. $A = 2, B = 0$
2. $A = 8, B = -2$
3. $A = -4, B = 4$ (Hint: multiply the first equation by 3 and the second equation by -2 and use elimination.)

Okay now, remember in last section how, given two equations of lines, we wanted to find their intersection point? It all came down to finding values for x and y that would satisfy **both** equations. Aha! You already solved a 2×2 system of equations without knowing. Which method did we use? Technically it was substitution, because by setting the right hand sides equal to each other, we were subbing in one expression for y into the other equation.

$$\begin{cases} y = 2x + 1 \\ y = 3x - 4 \end{cases} \quad \longrightarrow \quad 2x + 1 = 3x - 4$$

You can always think of a 2×2 linear system as a question of two *lines* intersecting or not (it literally has LINE in the name). Thus there are three options for the outcome of solving a 2×2:



A system of two linear equations in two unknowns can have:

- **one unique solution** (the equations' lines intersect once)
- **no solutions** (the equations' lines are parallel and never intersect; in this case the system is called **inconsistent**)
- **infinitely many solutions** (the equations' lines are smack on top of each other, the same line really, and have all points in common)

We've seen the first case, but what will it look like in practice when the other two cases appear? Investigate the systems below.

? Exercise 2

1. $\begin{cases} x + y = 3 \\ 3x + 3y = 9 \end{cases}$
2. $\begin{cases} 2x + 3y = 4 \\ -4x - 6y = -7 \end{cases}$

Answer

1. Say you start trying to do substitution, so you solve the first equation for y, $y = -x + 3$, and sub into the second equation. You get

$$3x + 3(-x + 3) = 9 \quad \longrightarrow \quad 3x - 3x + 9 = 9 \quad \longrightarrow \quad 9 = 9$$

Well, uh, that's certainly a fact, but I didn't get any information about x??? Now what? Look back at the equations...did you notice that the second one is just a constant multiple of the first one? I can make the equations match if I multiply the first one by 3. Both equations describe the same line. I have infinitely many solutions to this system.

2. I don't love substitution for this one because I'll have fractions, so let's try elimination. If I am trying to make the x terms matching opposites, I could multiply the first equation through by 2, to get

$$\begin{cases} 4x + 6y = 8 \\ -4x - 6y = -7 \end{cases}$$

But now when I go to add down the columns, **both** the x's and the y's kill each other off! I end up with the statement $0 = 1$, which in this universe is NOT TRUE. There is NO solution to this system.

So that's what those look like. Let's ramp it up.

3×3 Linear Systems

There's no reason you can only solve systems with two equations and two unknowns. As long as I have enough equations giving me enough information, I can solve systems involving as many unknowns as I want!

Example 2

In addition to running De Wolf's De Washing Machines manufacturing company, I also sell illegal exotic ant farms on the black market. I'm smuggling in three new varieties of ant: East African siafu army ants, Costa Rican Ecitons, and Brazilian leaf-cutters. All in all I should acquire 12 colonies, and I have $600 to spend on this restock. Army ants cost $60 per colony, Ecitons cost $40, and leaf-cutters cost $50. I definitely want to have twice as many leaf-cutters as army ants, because they're fun to watch. Denoting the number of each type of colony as $A, E,$ and L, respectively, write a system of equations using all of these requirements.

Solution

If I want 12 new colonies in all, the total of $A, E,$ and L should be 12. This gives me one equation, $A + E + L = 12$. If I have $600 to spend, I use the individual cost information to set up another equation about the money involved,

$$60A + 40E + 50L = 600$$

Finally, if I want twice as many leaf-cutters as army ants, I want L to be $2 \cdot A$, giving the equation $L = 2A$. This is now a system of three equations with three unknowns. I write them together with a brace, organizing each equation so the variables are in the same order. Since there is no E term in the third equation, I put in a placeholder $0E$.

$$\begin{cases} A + E + L = 12 \\ 60A + 40E + 50L = 600 \\ 2A + 0E - L = 0 \end{cases}$$

Solving a 3×3 comes down to reducing it to a 2×2 system, solving that, and then going back for the third variable. You can use a variety of strategies to achieve this. Since this one has an equation that already contains only two variables, substitution is a great idea.

Example 3

Solve the system $\begin{cases} A + E + L = 12 \\ 60A + 40E + 50L = 600 \\ 2A + 0E - L = 0 \end{cases}$

Solution

Using the third equation, $L = 2A$, we sub $2A$ for L in the other two equations and simplify everything we can. This will result in a nice 2×2 system.

$$\begin{cases} A + E + 2A = 12 \\ 60A + 40E + 50(2A) = 600 \end{cases} \longrightarrow \begin{cases} 3A + E = 12 \\ 16A + 4E = 60 \end{cases}$$

Solve this smaller system with whatever method you prefer, to find $A = 3$ and $E = 3$. Then plug 3 in for A in the third equation to get $L = 2A = 6$. I should order 3 colonies of army ants, 3 of Ecitons, and 6 of leaf-cutters.

If you don't have that nice freebie of one equation only having two variables in it, you have to make it from scratch. Use elimination on two of the equations to do this.

Example 4

Solve the system $\begin{cases} 3x + y - z = 1 \\ 2x + y + z = 4 \\ -x - y + 2z = 0 \end{cases}$.

Solution

I notice that the first equation has a y term while the third equation has a $-y$ term. Extracting those two, I add them to eliminate the y terms.

$$\begin{cases} 3x + y - z = 1 \\ -x - y + 2z = 0 \end{cases} \xrightarrow{\text{add}} 2x + z = 1$$

I can now solve that equation for $z = 1 - 2x$ and substitute it into two equations from the system. I must be sure to use the middle equation that I had ignored so far, but other than that I choose the first one.

$$\begin{cases} 3x + y - (1 - 2x) = 1 \\ 2x + y + (1 - 2x) = 4 \end{cases} \longrightarrow \begin{cases} 5x + y = 2 \\ y = 3 \end{cases}$$

Ope, in that second one it actually turned out to simplify all the way to part of the answer! We now know $y = 3$. (If this doesn't happen, just solve the 2×2 like usual.) Now, I can plug that in to find x:

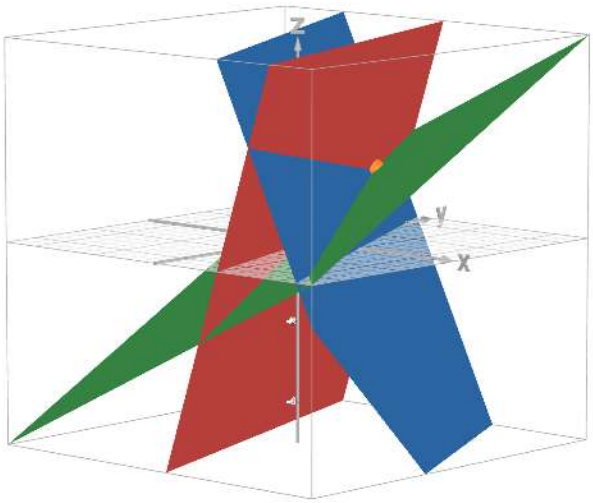
$$5x + 3 = 2 \quad \longrightarrow \quad x = -\frac{1}{5}$$

and plug THAT into $z = 1 - 2x$ to find z:

$$z = 1 - 2\left(-\frac{1}{5}\right) = \frac{7}{5} \tag{1}$$

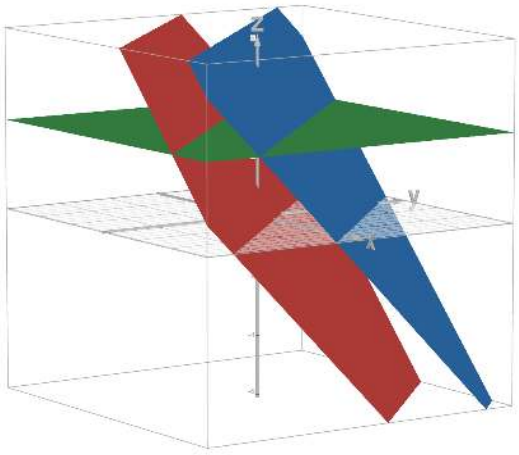
The solution to the system is $x = -\frac{1}{5}, y = 3, z = \frac{7}{5}$.

You might be wondering, is there a way to see these systems as questions about intersections? Yes, but when there are three variables involved, we're living in 3D space rather than on the 2D coordinate plane. Linear equations like the above in $x, y,$ and z are the equations of flat planes through 3D space. This is what the equations from the last system look like in space:

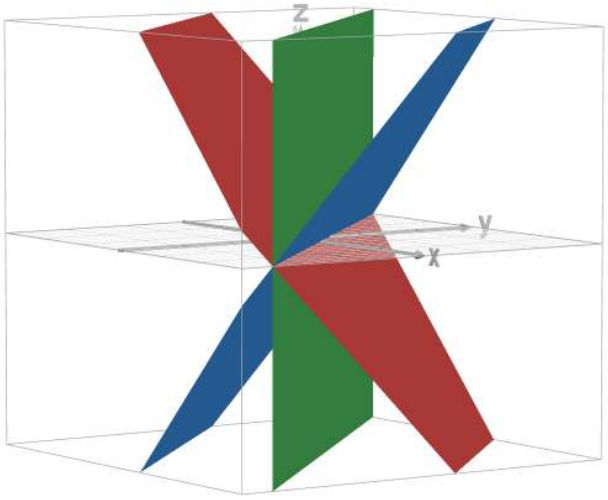


The solution we found is the point in 3D space that I marked with the bright orange ball there--the point where all three planes intersect simultaneously! Here is a picture of a no-solutions situation: two of the planes are parallel, so there is no place where **all**

three planes intersect at once.



And here is a picture of an infinitely-many-solutions situation: the planes all intersect each other on a single line that goes on forever, so that's infinitely many points that they have in common.



Now you try!

? Exercise 3

Solve the systems, or state if a system has no solutions:

1. $\begin{cases} x+2y+z=3 \\ y-z=-4 \\ -x-2y+3z=9 \end{cases}$
2. $\begin{cases} x+y+z=4 \\ x+3y+3z=10 \\ 2x+y-z=3 \end{cases}$
3. $\begin{cases} x+y-z=1 \\ 2x+2y-2z=3 \\ 2z=2 \end{cases}$

Answer

1. $x=2, y=-1, z=3$
2. $x=1, y=2, z=1$

3. No solution.

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3.5E Exercises

Identifying Linear Equations

Tell whether the equation is linear in multiple unknowns. All letters below represent variables, not constants!

1. $x + 2y = 1$
2. $x + y - z = 0$
3. $\sqrt{x} + 2y = 1$
4. $xy + yz + xz = 2$
5. $x^2 = 17$
6. $v_1 + 2v_2 + 3v_3 + 4v_4 = 5$
7. $\sqrt{3}x + \pi y = e^2$

Answer

1. Linear in two variables.
2. Linear in three variables.
3. Not linear, the $\sqrt{x}$ is $x^{\frac{1}{2}}$ and linear equations can only have variables to the first power.
4. Not linear, the variables are being multiplied by each other rather than constant coefficients!
5. Not linear, the variable is raised to the second power.
6. Linear in four variables.
7. Linear! Even though the coefficients look suspicious, $\sqrt{3}, \pi,$ and e^2 are all just plain ol' numbers, and the variables only appear to the first power.

Solving 2×2 Linear Systems

Solve the systems using any method you like, or tell if the system is inconsistent.

1. $\begin{cases} 2x + 3y = -2 \\ x - y = 4 \end{cases}$
2. $\begin{cases} 3x - 2y = 4 \\ 4x + y = 9 \end{cases}$
3. $\begin{cases} 3x - 2y = 4 \\ -6x + 4y = 7 \end{cases}$
4. $\begin{cases} x + 2y = 5 \\ 3x - y = 8 \end{cases}$
5. $\begin{cases} 2x - 3y = -2 \\ x + 4y = 10 \end{cases}$
6. $\begin{cases} x - y = \frac{5}{2} \\ 3x - 3y = 9 \end{cases}$
7. $\begin{cases} 3x + y = 18 \\ 2x - 2y = 4 \end{cases}$

Answer

1. $x = 2, y = -2$
2. $x = 2, y = 1$
3. No solution.
4. $x = 3, y = 1$
5. $x = 2, y = 2$
6. No solution.
7. $x = 5, y = 3$

Solving 3×3 Linear Systems

Solve the systems using any method you like, or tell if the system is inconsistent.

1. $\begin{cases} x+2y+z=7 \\ 2x-y+3z=-7 \\ 3x+y-2z=6 \end{cases}$
2. $\begin{cases} x+2y+z=6 \\ 2x-y+4z=-5 \\ 2x+2y=1 \end{cases}$
3. $\begin{cases} x-y-z=-1 \\ -8x+8y+8z=2 \\ x-y=1 \end{cases}$
4. $\begin{cases} x+y=4 \\ y+z=9 \\ x+z=10 \end{cases}$
5. $\begin{cases} x+y+z=7 \\ y+z=4 \\ x+z=10 \end{cases}$
6. $\begin{cases} 2x-y-z=-1 \\ x+z=3 \\ x-y-2z=8 \end{cases}$

Answer

1. $x=0, y=4, z=-1$
2. $x=-3.5, y=4, z=1.5$
3. No solutions.
4. $x=2.5, y=1.5, z=7.5$.
5. $x=3, y=-3, z=7$
6. No solutions.

Setting Up Linear Systems

1. The sum of two numbers is twice their difference. The larger number is 10 more than the smaller. Write a 2×2 system and solve to find the numbers.

2. I am nine years older than my youngest sister. My middle sister is five years older than the youngest. At some point in time, my age was twice my middle sister's age. Write a 3×3 system describing our ages at that point in time, and interpret its solution as to our ages.

3. Tickets to come tour my illicit ant farm emporium cost $5 per adult and $2 per child under twelve. Last weekend, I sold 36 tickets total and made $117 total. Write a 2×2 system and solve to find out how many adults attended, and how many children attended.

4. A gas station sells three types of gas: Regular at $3.49 per gallon, Midgrade at $3.59 per gallon, and Premium at $3.71 per gallon. On a certain day, 4750 gallons of gas were sold for a total of $15,702. The gas station sold twice as much Regular as Midgrade. Write a 3×3 system to find how many gallons of each type were sold, but do not solve (or solve with the aid of a calculator).

Answer

1. The system starts out as $\begin{cases} x+y=2(x-y) \\ x=10+y \end{cases}$. The solution is $x=15, y=5$.

2. Say my age at the time of interest is x, my middle sister is y, and my youngest sister is z. At the time that I was twice as old as my middle sister, $x=2y$. We also have $x=z+9$ and $y=z+5$. This makes a 3×3 system. The solution is $x=8, y=4, z=-1$. The interpretation is that when my middle sister was 4, I was 8, but it was actually one year BEFORE the youngest was born.

3. Calling the number of adults A and the number of children C, the system is $\begin{cases} A + C = 36 \\ 5A + 2C = 117 \end{cases}$. The solution is $A = 15, C = 21$.

4. The system is $\begin{cases} R + M + P = 4750 \\ 3.49R + 3.59M + 3.71P = 15702 \\ R = 2M \end{cases}$.

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3.6 Vectors and Matrices

Learning Objectives

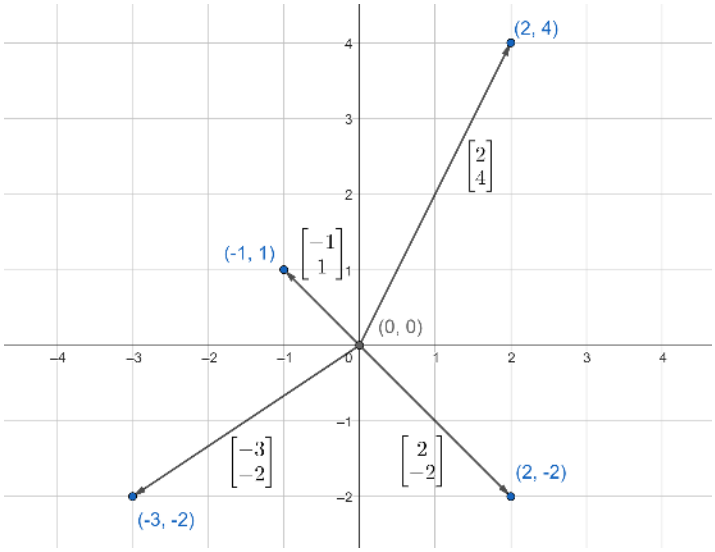
By the end of this section, you will be able to:

- add and subtract vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$
- add and subtract 2×2 matrices
- perform a matrix-vector multiplication with a 2×2 matrix and a 2×1 vector
- use the inverse of a 2×2 matrix to solve a linear system of equations

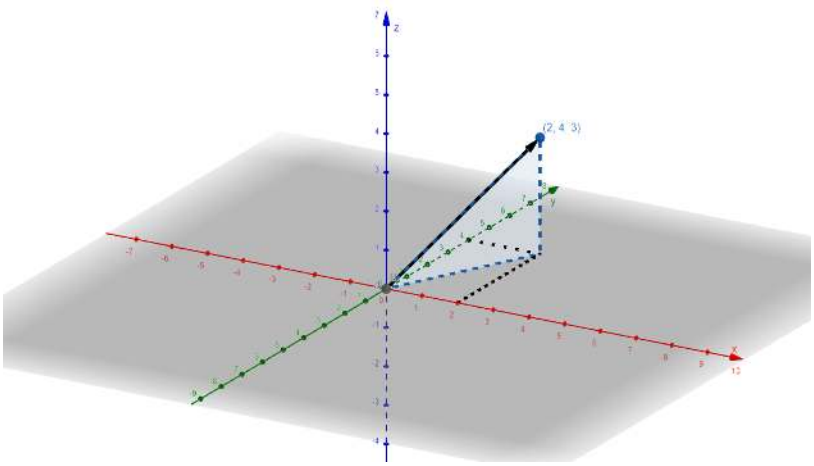
Maybe you just read those objectives and said, "Add and subtract what in where????" Keep your shirt on, Mabel, I'm gonna tell you.

We've been identifying points in the 2D plane using pairs of coordinates, (x, y). Really this is just a very short list of coordinates, and if we consider the set of all possible lists-of-two-coordinates, we call that set $\mathbb{R}^2$. In other words, $\mathbb{R}^2$ means "the set of all different pairs of numbers, where both of the numbers in each pair are real numbers specifically." All I want you to think right now is "$\mathbb{R}^2$ = code for 2D space, xy-plane."

We write members of $\mathbb{R}^2$ as column vectors of length 2, like this: $\begin{bmatrix} x \\ y \end{bmatrix}$. A **vector** is visualized on the plane as an arrow whose tail starts at the origin, and whose arrowhead is at the point with coordinates (x, y).



Similarly, $\mathbb{R}^3$ is code for "all the different lists of three real numbers," which are written as column vectors of length 3, like this: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. You can think of $\mathbb{R}^3$ as 3D space, if you see these vectors as an arrow pointing from the origin to the coordinates of a point (x, y, z) in 3D space. The picture below shows the vector $\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$.



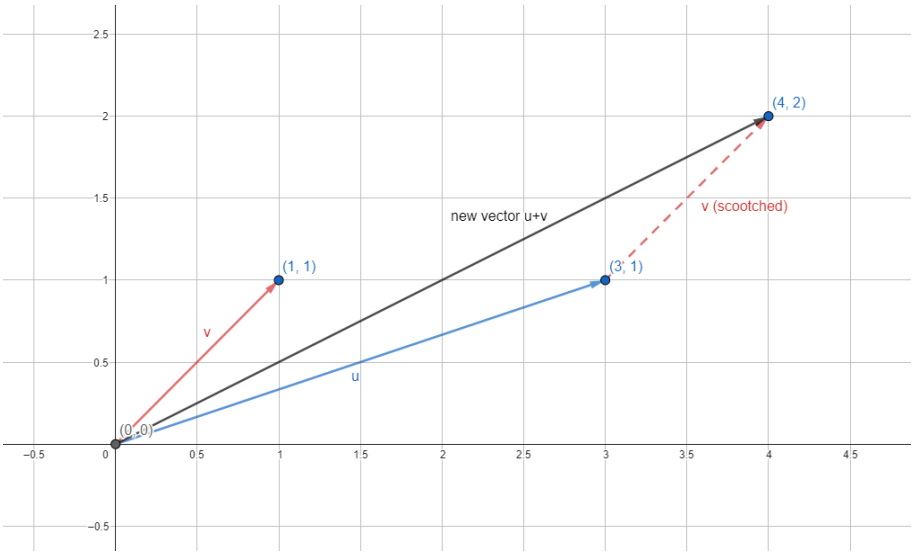
Arithmetic With Vectors

Adding and subtracting vectors is extremely simple algebraically, but interesting geometrically.

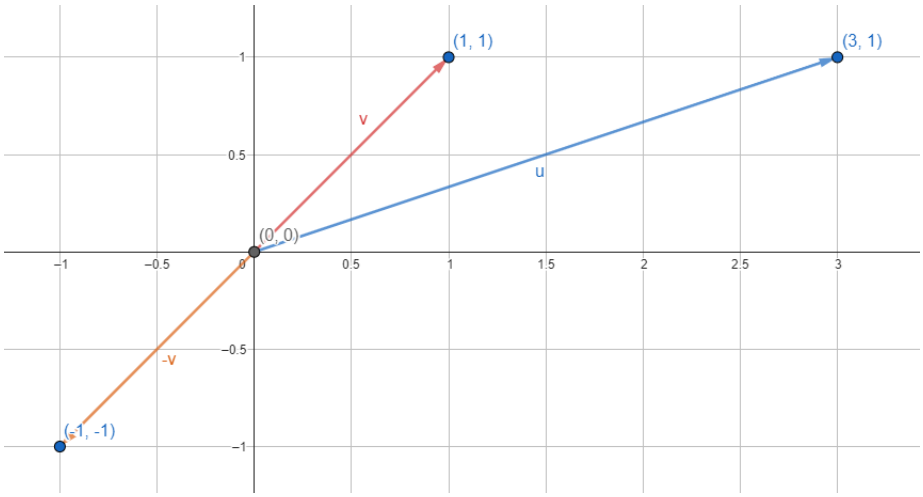
To **add** or **subtract vectors,** just add or subtract in each component separately and write the answer as a vector of the same size.

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}, \qquad \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a-c \\ b-d \end{bmatrix}$$

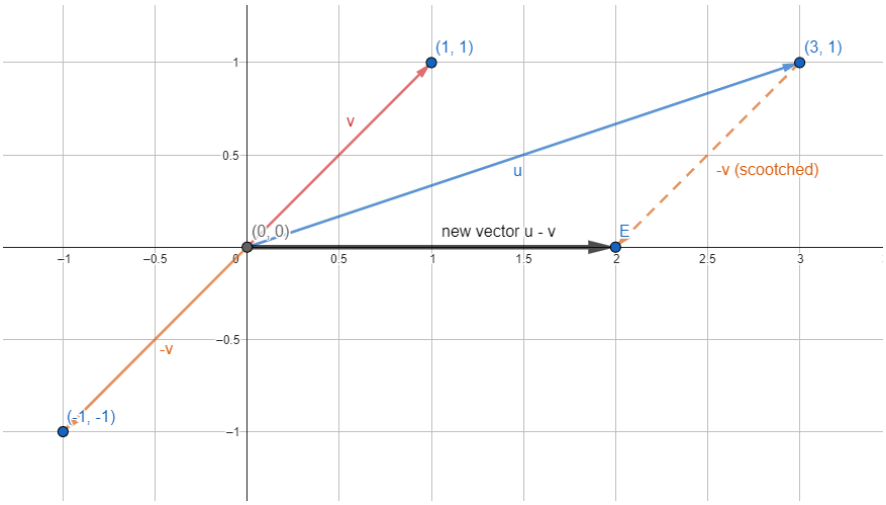
When seeing vectors as arrows, what is happening here? Addition can be thought of as taking the tail end of one vector and scooting it up to start at the arrowhead of the other. Then you just connect from the origin to the head of the scootched vector. That new vector is the sum. Look at the picture below and notice how $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ according to the rule above.



What about subtraction, you say? Well, if you want to subtract two vectors nicknamed u and v, you're computing $u - v$, aka $u + (-v)$. And $-v$ is just the **opposite** of v, meaning the vector of same length pointing directly backwards! See the picture below.



Since the red vector is $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the orange vector is $-v = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. If you want to do subtraction, just use the negative vector and do the same addition process.

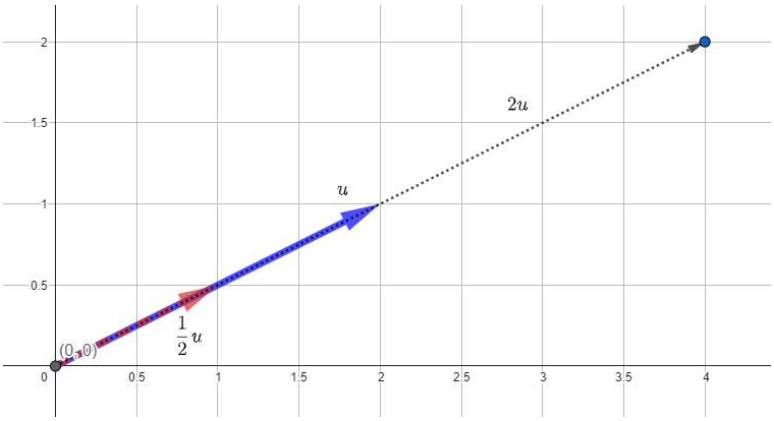


The other thing you can do with vectors is stretch or shrink them without changing their direction, which is handled mathematically by **scalar multiplication** ("scalar" is another word for "single plain ol' constant number").

To **multiply a vector by a constant** k**,** just multiply each entry in the vector by k.

$$k \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ kc \end{bmatrix}$$

Geometrically, the effects are thus:



We've already seen that multiplying by a (-1) has the effect of flipping the arrow to the exact opposite direction, because $(-1)v = -v$.

Exercise 1

If $u = \begin{bmatrix}1\\3\end{bmatrix}$, $v = \begin{bmatrix}2\\-2\end{bmatrix}$, $w = \begin{bmatrix}0\\5\end{bmatrix}$, compute:

1. $u + v$
2. $v + w$
3. $u + v + w$
4. $u - w$
5. $5u$
6. $3u + 2w$
7. $-4v$

Answer

1. $\begin{bmatrix}3\\1\end{bmatrix}$
2. $\begin{bmatrix}2\\3\end{bmatrix}$
3. $\begin{bmatrix}3\\6\end{bmatrix}$
4. $\begin{bmatrix}1\\-2\end{bmatrix}$
5. $\begin{bmatrix}5\\15\end{bmatrix}$
6. $\begin{bmatrix}3\\19\end{bmatrix}$
7. $\begin{bmatrix}-8\\8\end{bmatrix}$

Exercise 2

If $u = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $v = \begin{bmatrix}2\\-1\\1\end{bmatrix}$, and $w = \begin{bmatrix}0\\0\\3\end{bmatrix}$, compute:

1. $u - v + w$
2. $-\frac{1}{3}w$
3. $2u + v$

Answer

1. $\begin{bmatrix}-1\\1\\3\end{bmatrix}$

2. $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$
3. $\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$

Working With Matrices

A vector in $\mathbb{R}^2$ was a single column with two rows in it, which is also called a 2×1 "array." A 2×2 array is called a 2×2 **matrix** and looks like this:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

You can also have **matrices** (plural of "matrix" (thanks Latin)) of other sizes, such as the 2×3 and 3×3 examples below.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

(2 rows, 3 columns), (3 rows, 3 columns)

We can add and subtract matrices entry-by-entry in the same way as vectors, AS LONG AS they have the exact same size.

To **add or subtract matrices of the same size,** just add or subtract the corresponding components:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

To **perform scalar multiplication by a constant k on a matrix,** just multiply each entry by k:

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

? Exercise 3

If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$, compute:

1. $A + B$
2. $A - B$
3. $2A$
4. $-4B$

Answer

1. $A + B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$
2. $A - B = \begin{bmatrix} -2 & -1 \\ -3 & -2 \end{bmatrix}$
3. $2A = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$
4. $-4B = \begin{bmatrix} -12 & -12 \\ -12 & -12 \end{bmatrix}$ or $-\begin{bmatrix} 12 & 12 \\ 12 & 12 \end{bmatrix}$

Now, you can't multiply vectors together, but there is such a thing as **matrix-vector multiplication.** To do it, the **matrix must have the same number of columns** as the **vector has entries**. This means we can multiply a 2×2 matrix A onto a 2×1 vector v.

The **matrix-vector multiplication** of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ results in a new 2×1 vector

$$Av = \begin{bmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{bmatrix}$$

This might look intimidating to memorize, but if you notice the highlighted relationships below, it should get easier to remember.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{bmatrix}$$

Example 1

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $v = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, compute Av.

Solution

I just set up the matrix-vector multiplication and follow the formula.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1(5) + 2(6) \\ 3(5) + 4(6) \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

For the first entry in the new vector, I looked at the first row of the matrix, multiplied corresponding entries with the column vector, and added them up. Then I did the same for the second entry, but with the second row of the matrix.

Make sure to do the exercise below before moving on! It's got somebody super important in it.

Exercise 4

1. Compute $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.
2. Compute $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

Answer

1. $\begin{bmatrix} 17 \\ 6 \end{bmatrix}$
2. $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ — Notice that this matrix-vector multiplication had no effect on the vector! This matrix is a special type called the 2×2 **identity matrix.** He has 1's on the diagonal and 0's off the diagonal. He's called the identity matrix because multiplying a vector by him doesn't result in an identity crisis. The vector stays the exact same. An identity matrix is often denoted I, so we have $Iv = v$ for any vector v.

Solving Systems With Matrices

I'm telling you all of this for two reasons: first, some physics professors said they wished freshmen would learn about vectors and matrices sooner, and second, because there's a cool way to use them on the systems of equations we were just solving in the last section. To start, there is a way to translate from a system of equations to an equation with a matrix in it, and vice versa. This is called the **matrix equation** for the system. Study the example below carefully and look for patterns.

$$\begin{cases} 3x + 2y - z = 0 \\ x - y + z = 3 \\ x + z = 2 \end{cases} \iff \underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}}_{b}$$

All of the coefficients on the variables in the system were placed into the **coefficient matrix** A, in careful order (notice how there was no y term in the third equation—there is a placeholder 0 in the y spot of the coefficient matrix in that row), and all of the

constants from the other sides of the equals signs appeared in the b vector on the far right of the matrix equation. Meanwhile, all of the variable names got slapped in a vector v being multiplied by A. If you really investigate the result of the matrix-vector multiplication, you will rediscover the same equations from the system...

$$\begin{bmatrix}3 & 2 & -1\\ 1 & -1 & 1\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 3\\ 2\end{bmatrix} \quad\rightarrow\quad \begin{bmatrix}3x+2y-z\\ x-y+z\\ x+0y+z\end{bmatrix}=\begin{bmatrix}0\\ 3\\ 2\end{bmatrix}$$

See it? Try translating back and forth in the 2×2 case.

? Exercise 5

1. Translate the system $\begin{cases}2x+y=0\\ x-y=3\end{cases}$ into a matrix equation.
2. Translate the matrix equation $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}5\\ 6\end{bmatrix}$ into a system of equations.

Answer

1. $\begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 3\end{bmatrix}$
2. $\begin{cases}x+2y=5\\ 3x+4y=6\end{cases}$

Solving systems, especially when they get bigger, is wayyyyyy more fun using the language of matrices. If you want to learn about that, take a course in linear algebra. It's my favorite subject, so there's that endorsement.

Often in real-world applications, we will have a bunch of systems we need to solve, all with the same left hand sides, but different choices of right hand side constants. Like

$$\begin{cases}2x+y=0\\ x-y=3\end{cases},\qquad \begin{cases}2x+y=1\\ x-y=5\end{cases},\qquad \begin{cases}2x+y=9\\ x-y=10\end{cases},\ldots$$

Doing a substitution or elimination process by hand every single time is super inefficient and unfun. But if we translated each of these to a 2×2 matrix system, we would notice that the coefficient matrix **never changes**, it's only the vectors of constants!

$$\begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 3\end{bmatrix},\qquad \begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1\\ 5\end{bmatrix},\qquad \begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}9\\ 10\end{bmatrix},\ldots$$

Some square matrices have a special partner matrix called an **inverse.** The inverse of a matrix A is denoted A^{-1}, and it has the effect of undoing multiplication by A. There are a few ways I can write this in math: one way is $AA^{-1}=A^{-1}A=I$, where I is the **identity matrix** you saw in Exercise 4 above. Or I can say $A^{-1}Av=v$. The idea there is that if I take v and multiply it by A using matrix-vector multiplication, and THEN do matrix-vector multiplication on the result using A^{-1}, I will get back to the original v. This is the important feature, because watch what happens if I apply A^{-1} to both sides of a matrix equation...

$$Av=b \quad\longrightarrow\quad A^{-1}Av=A^{-1}b \quad\longrightarrow\quad Iv=A^{-1}b \quad\longrightarrow\quad v=A^{-1}b$$

Since v was the vector holding all my variable names, this would give me a vector of solution numbers for them!

✓ Example 2

If the matrix equation $Av=b$ is given below,

$$\begin{bmatrix}-2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix}$$

and $A^{-1}=\begin{bmatrix}-1 & -1\\ -1 & -2\end{bmatrix}$, find $v=\begin{bmatrix}x\\ y\end{bmatrix}$.

Solution

We saw above that we can compute $v=A^{-1}b$, so all we need to do is perform matrix-vector multiplication with A^{-1} and b:

$$v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$$

This tells us that $x = -3$ and $y = -5$.

If you're trying to solve a system for lots of b vector options, it's much faster to just figure out the inverse of the coefficient matrix, and then apply it to all the b vectors with quick matrix-vector multiplications. So how do you find the inverse?? With 2×2s, there's a handy formula.

The **inverse** of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The quantity $ad - bc$ is called the **determinant** of A, and **must not equal zero for this to work!** If a matrix's determinant turns out to be zero, it is **not invertible.**

To help you memorize, translate the formula into English: to make a matrix's inverse, swap the diagonal entries a and d, flip the signs on the off-diagonal entries b and c, and then compute $ad - bc$ and divide everybody by it.

Exercise 6

Give the inverse of the matrix, if it is invertible.

1. $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$
2. $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
3. $C = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$

Answer

1. $A^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$
2. Uh oh, in this one $ad - bc = 1(1) - 1(1) = 0$, and we can't divide by zero, so this matrix is not invertible.
3. $C^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -0.5 & 2.5 \\ 0.5 & -1.5 \end{bmatrix}$

Now bring it all together by using the matrix inverse to solve the following systems, which all have the same coefficient matrix:

Exercise 7

Solve the systems.

1. $\begin{cases} 2x + 3y = 1 \\ x + 2y = 1 \end{cases}$
2. $\begin{cases} 2x + 3y = 2 \\ x + 2y = 4 \end{cases}$
3. $\begin{cases} 2x + 3y = -3 \\ x + 2y = 2 \end{cases}$

Answer

First find the coefficient matrix: $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. Then use the formula to find its inverse, $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. Then apply it to the constant vector of each system to find the solutions.

1. $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ so $x = -1, y = 1$.
2. $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \end{bmatrix}$ so $x = -8, y = 6$.
3. $x = -12, y = 7$.

In real life, systems are often much bigger than 3×3 or even 4×4 (which I wouldn't ask you to solve by hand) and matrices are basically the only way to deal with them! We won't worry about matrix inverses for matrices larger than 2×2 in this course, but I do want you to practice *setting up* the kind of system that could arise from a "real-world" application. Check out a classic example below, and then try your hand in the Exercises section!

Example 3

A pharmaceutical company designs a pill that contains 10 units of Vitamin A, 8 units of Vitamin B, and 5 units of Vitamin C. These pills are produced by combining three powders that are available:

- Powder X contains 30% Vitamin A, 20% Vitamin B, and 15% Vitamin C.
- Powder Y contains 25% Vitamin A, 10% Vitamin B, and 35% Vitamin C.
- Powder Z contains 20% Vitamin A, 25% Vitamin B, and no Vitamin C.

Set up a system (but do not solve) to figure out how many units of each powder should be mixed together to produce a pill with the desired properties.

Solution

Okay, the idea here is that for each vitamin, the amounts coming from the ingredient powders must total up to the desired quantity. First, let's pick some variable names. We're trying to figure out the amounts of each powder to use, so let $X =$ "units of Powder X used," $Y =$ "units of Powder Y," and $Z =$ "units of Powder Z."

Consider the Vitamin A we want... However much we use of Powder X, aka "X," we know that 30% of that quantity will be Vitamin A. Similarly, we know the proportions of the Y and Z amounts that will consist of Vitamin A. All together, we want to end up with 10 units of Vitamin A. We distill this information into an equation based on Vitamin A:

$$0.3X + 0.25Y + 0.2Z = 10$$

Similarly, we set up an equation based on Vitamin B:

$$0.2X + 0.1Y + 0.25Z = 8$$

Since Powder Z contains no Vitamin C, our third equation looks like this:

$$0.15X + 0.35Y = 5$$

We have three equations and three variables, and we've run out of vitamin types to analyze, so we are finished constructing our system! We just put the equations together.

$$\begin{cases} 0.3X + 0.25Y + 0.2Z = 10 \\ 0.2X + 0.1Y + 0.25Z = 8 \\ 0.15X + 0.35Y = 5 \end{cases}$$

Translated into a matrix equation, this would become

$$\begin{bmatrix} 0.3 & 0.25 & 0.2 \\ 0.2 & 0.1 & 0.25 \\ 0.15 & 0.35 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 5 \end{bmatrix}$$

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3.6E Exercises

Computing Linear Combinations

A **linear combination of vectors** is when you take a bunch of vectors, multiply them by some constants, and add them all up. If v_1, v_2, v_3 are vectors and k_1, k_2, k_3 are constants, then the expression $k_1v_1 + k_2v_2 + k_3v_3$ is a linear combination. All you need to know to evaluate them is scalar multiplication and addition of vectors. Given the vectors

$$v_1 = \begin{bmatrix}1\\0\\2\end{bmatrix}, \quad v_2 = \begin{bmatrix}0\\3\\3\end{bmatrix}, \quad v_3 = \begin{bmatrix}7\\6\\5\end{bmatrix}$$

compute the following:

1. $2v_1 + 3v_2$
2. $v_1 + v_2 + v_3$
3. $5v_1 - v_3$
4. $\frac{1}{3}v_2 + v_3$

Answer

1. $\begin{bmatrix}2\\9\\13\end{bmatrix}$
2. $\begin{bmatrix}8\\9\\10\end{bmatrix}$
3. $\begin{bmatrix}-2\\-6\\5\end{bmatrix}$
4. $\begin{bmatrix}7\\7\\6\end{bmatrix}$

Arithmetic With Matrices

In order to add matrices, they must be **exactly the same** size and shape. Given the matrices below, evaluate the expressions, **if** they are valid.

$$A = \begin{bmatrix}1 & 3\\2 & 6\\7 & 7\end{bmatrix}, \quad B = \begin{bmatrix}2 & 4\\0 & 8\end{bmatrix}, \quad C = \begin{bmatrix}3 & 2\\1 & 0\end{bmatrix}, \quad D = \begin{bmatrix}1 & 0\\0 & 1\\2 & 5\end{bmatrix}$$

1. $A + B$
2. $B + C$
3. $A + D$
4. $4B + D$
5. $C - 3B$
6. $A - 2D$

Answer

1. These are not compatible sizes. A is 3×2 and B is 2×2.
2. $B + C = \begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$
3. $A + D = \begin{bmatrix}2 & 3\\2 & 7\\9 & 12\end{bmatrix}$

4. These are not compatible sizes.

5. $C - 3B = \begin{bmatrix} -3 & -10 \\ 1 & -24 \end{bmatrix}$

6. $A - 2D = \begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 3 & -3 \end{bmatrix}$

? Matrix-Vector Multiplication

Perform the 2×2 matrix-vector multiplications.

1. $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix}$
3. $\begin{bmatrix} 2 & -1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
4. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ (This is a trick question—see if you can figure out the answer by noticing something...)

Answer

1. $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$

2. This is actually the identity matrix, which will have no effect on the vector when multiplied.

3. $\begin{bmatrix} 1 \\ -10 \end{bmatrix}$

4. Aha! Even though we're not worrying about 3×3 matrix-vector multiplication in this course, we can recognize this is the 3×3 version of the identity matrix, with 1s on the diagonal and 0s elsewhere. Multiplying by it will not change the vector, so the answer is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

? Translating Between Systems and Matrix Equations

No need to solve the systems below, this is a translation problem.

1. Translate the system $\begin{cases} 12u + 11v + w = 18 \\ 7u - 3w = 6 \\ 8v = 9 \end{cases}$ into a matrix equation.

2. Translate the system $\begin{cases} 0.5x + y - 0.75z = 9 \\ 4x + 2y + z = 20 \\ x - y = 3 \end{cases}$ into a matrix equation.

3. Translate the matrix equation into a system

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 7 & 7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

4. Translate the matrix equation into a system

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 5 \end{bmatrix}$$

Answer

1. $\begin{bmatrix} 12 & 11 & 1 \\ 7 & 0 & -3 \\ 0 & 8 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \\ 9 \end{bmatrix}$
2. $\begin{bmatrix} 0.5 & 1 & -0.75 \\ 4 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 20 \\ 3 \end{bmatrix}$
3. $\begin{cases} a_1 + 3a_2 = 1 \\ 2a_1 + 6a_2 = 2 \\ 7a_1 + 7a_2 = 3 \end{cases}$
4. $\begin{cases} x_2 + 2x_3 = 9 \\ x_1 - x_2 = 0 \\ 3x_1 + 3x_2 + 4x_3 = 5 \end{cases}$

Setting Up Applications

1. De Wolf's De Washing Machines sells two high-end washer models, the HydroLuxumator 2.0 and the SaniSpinner III. We sell the Hydro for $1449.99 each and the Spinner for $1879.99 each. Last quarter, we sold a total of 123 washing machines between the two, and the total revenue combined was $157,942. Set up a system that could be used to find the number of each model sold. (Do not solve, I didn't cook these numbers.)

2. I need a certain concentration of salt brine for my large seawater aquarium, but I only have access to three concentration options: Solution A has 5 grams of salt per liter of liquid, Solution B has 8g/L, and Solution C has 10g/L. I need 100L of a final mixture with a concentration of 7g/L. Solution A is the cheapest, so I'd like to use twice as much of that as the other two combined. Write a system that could allow me to solve for the amount I should use of each solution type. (Do not solve.)

3. I have $10,000 that I want to split between a high-return but slightly high-risk investment option A and a low-risk but lower-return investment option B. Option A has an average annual return of 12%, and Option B has an average annual return of 7%. I'm aiming to balance out around an annual return of 10%. Write a system to determine how much should I put into each option. (Do not solve.)

Answer

1. $\begin{cases} H + S = 123 \\ 1449.99H + 1879.99S = 157924 \end{cases}$

2. The equation you might not see how to set up is an equation giving info about the amount of salt involved. Solution A has 5 grams per liter, so if I use a number of liters A, I will have $5A$ grams of salt, and so on for B and C. All of the salt coming from the solutions must combine to give me a total amount of salt that is 7g/L in 100L.

$$\begin{cases} A + B + C = 100 \\ 5A + 8B + 10C = 7(100) \\ A = 2(B + C) \end{cases}$$

3. I want to receive a return of 10% of $10,000 in a year, and this must come from making 12% of the amount I put in Option A, and 7% of the amount I put in Option B. Can you see how this is described by the second equation below?

$$\begin{cases} A + B = 10000 \\ 0.12A + 0.07B = (0.10)(10000) \end{cases}$$

Solving Systems With Matrix Inverse

Translate the systems to matrix equations, then use the inverse of the coefficient matrix to solve.

1. $\begin{cases} 3x + y = 5 \\ 2x - y = 25 \end{cases}$
2. $\begin{cases} 7x + 5y = 1 \\ 4x + 3y = 2 \end{cases}$

Answer

1. $x = 6, y = -13$

2. $x = -7, y = 10$

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3.7 Chapter 3 Study Guide

Definitions

- **equation:** a mathematical statement that two expressions are equal.
- **solution:** a set of values that, when plugged in for the variables, will make an equation's statement true.
- **equivalent equations:** equations that have exactly the same solutions. When this is the case, you can algebraically manipulate one equation until it matches the other.
- **linear equation in n unknowns:** an equation with n variables that appear only to the first power. Examples: $3x + 7 = 20$, $4x_1 + 2x_2 - x_3 = 0$. Non-examples: $\sqrt{x} + y = 1$, $x^2 + xy = 2$.
- **inverse operation:** an operation that undoes another operation. Examples: addition/subtraction, multiplication/division, raising to powers/taking corresponding roots.
- **opposite:** the opposite of a number or expression is the same number or expression, but with the opposite sign. Examples: opposite of -3 is 3, opposite of $x + 1$ is $-(x + 1)$.
- **absolute value:** the absolute value function $|\text{inside}|$ will return inside when the inside is nonnegative, and returns the opposite, $-(\text{inside})$, if it's negative.
- **quadratic equation:** a polynomial equation where the highest power on the variable is 2. **Standard form** is $ax^2 + bx + c = 0$.
- **inequality:** a mathematical statement that two expressions are related by one of the inequality signs, $<, \leq, >, \geq$.
- **simultaneous inequality:** an inequality with two bookends, like $-4 < x + 2 < 15$.
- **coordinate/Cartesian plane:** the 2D plane defined by x-axis and y-axis. A point on the plane is labeled with a **coordinate pair** (x, y).
- **origin:** the point of intersection of the axes, aka coordinates $(0, 0)$.
- **intercepts:** an x-intercept is a point where a graph crosses the x-axis; a y-intercept is a point where a graph crosses the y-axis.
- **slope:** describes the steepness of a line or the rate of vertical change relative to the horizontal change. Denoted $m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$. If slope is positive, the line runs uphill. If negative, the line runs downhill.
- **slope-intercept form:** an equation of a line in the form $y = mx + b$, where m gives the slope, and the y-intercept is $(0, b)$.
- **point-slope form:** an equation of a line in the form $y - y_0 = m(x - x_0)$, where m is the slope and the line passes through point (x_0, y_0).
- **standard equation of a circle:** an equation of the form $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center of the circle and r is the radius.
- **mathematical model:** a way of describing a real-world situation with an equation or formula. Variables are used to represent quantities.
- **direct variation:** if $y = kx$, where k is some **constant of proportionality**, then y **varies directly as** or **is directly proportional to** x.
- **inverse variation:** if $y = \frac{k}{x}$, where k is some **constant of proportionality**, then y **varies inversely as** or **is inversely proportional to** x.
- **system of linear equations:** a set of linear equations using the same exact variables, for which we look for a **solution**, a set of values for the variables that will satisfy all of the equations **simultaneously.**
- **substitution:** the method of solving a system that involves solving an equation for a single variable, then substituting that expression into the other equation(s).
- **elimination:** the method that involves multiplying equation(s) by constants to achieve the cancelling out of a variable entirely when the equations are added.
- **inconsistent:** if a system has no solutions, it is called inconsistent.
- $\mathbb{R}^2$ **(**$\mathbb{R}^3$**):** the set of all pairs (or triples, respectively) of real numbers, which are written as...
- **vectors:** written like $\begin{bmatrix} a \\ b \end{bmatrix}$ or $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. These can be thought of as arrows pointing from the origin to a point with coordinates (a, b) (in 2D space) or (a, b, c) (in 3D space).
- $n \times m$ **matrix:** an array of entries with n rows and m columns. A 2×2 matrix looks like $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- $n \times n$ **identity matrix:** a square matrix with 1s on the diagonal entries and 0s elsewhere, such as $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Multiplying a vector by an identity matrix will not change the vector: $Iv = v$.
- **inverse of a matrix:** the inverse of a square matrix A is another square matrix A^{-1} such that $A^{-1}(Av) = v$, or in other words, $AA^{-1} = I$.

Solving a Quadratic Equation

To solve a quadratic equation in the **standard form** $ax^2 + bx + c = 0$, there are three main techniques. Ranked from most convenient to least convenient, on average, they are:

1. Basic factoring when $a = 1$. This is the "find two numbers that multiply to c and add to b" method. If $a \neq 1$, try the ac method to factor by grouping. After factoring, you have an expression containing factors in the form $(x - c)$ multiplied together, with zero on the other side. Each such factor has the potential to wipe out the entire expression were it to equal 0, and $x = c$ would cause that! Then each factor gives a solution to the equation.
2. Completing the square on the first two terms, while moving the constant term c to the other side, gives an equation of the form $(x + m)^2 = n$. Then by taking a square root of both sides and subtracting the m, we can isolate x.
3. Solutions can always be found using **the quadratic formula:**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Possible Outcomes When Solving Quadratic Equations

- Distinct real roots: either something like "$(x - 3)(x - 2) = 0 \implies x = 2, 3$," or a result like $x = 2 \pm \sqrt{3}$.
- Repeated real root: two copies of the same number, coming from something like "$(x + 1)^2 = 0$."
- No real roots, but a **complex conjugate pair**: when the discriminant $b^2 - 4ac$ turns out to be negative, these will look like $x = a \pm bi$.

Caution

Get in the habit of checking your answers by plugging them back into the original equation or inequality. Sometimes when solving radical equations or absolute value equations, extraneous invalid solutions appear that must be thrown out!

Algebra Rules for Inequalities

- You can **add** or **subtract** the same thing on both sides of an inequality, and it won't change: $A < B \iff A + C < B + C$.
- You can **multiply** or **divide** something **positive** on both sides of an inequality, and it won't change: if $C \geq 0$, $A < B \iff AC < BC$.
- **ALERT!!!** If you multiply or divide both sides by a **negative** number, you must **flip the direction of the inequality sign:** if $C < 0$, $A < B \iff AC > BC$ or $\frac{A}{C} > \frac{B}{C}$.

Inequalities Involving Absolute Value

- For $<$ and $\leq$, split into cases using AND: $|\text{inside}| < a \implies (\text{inside}) < a$ AND $-(\text{inside}) < a$.
- For $>$ and $\geq$, split into cases using OR: $|\text{inside}| > a \implies (\text{inside}) > a$ OR $-(\text{inside}) > a$

Caution

AVOID THIS COMMON MISTAKE: $|3 - x| \neq 3 + x$!!!!!! It splits into two situations: $(3 - x)$ if $(3 - x) \geq 0$, and $-(3 - x) = x - 3$ if $(3 - x) < 0$.

Fun Facts About Lines

- An x-intercept is a point where the line crosses the x-axis, and it will always have a y-coordinate of 0.
- A y-intercept is a point where the line cross the y-axis, and it will always have an x-coordinate of 0.
- The slope m of a line can be calculated using two distinct points on the line, (x_1, y_1) and (x_2, y_2), by computing rise over run:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in y}}{\text{change in x}} = \frac{y_2 - y_1}{x_2 - x_1}$$

- **Uphill / Increasing lines** have **positive** slopes. The bigger the number, the steeper the line.
- **Downhill / Decreasing lines** have **negative** slopes. Again, whole numbers more negative than -1 make **steeper** lines and little fractional slopes like $-\frac{1}{2}$ make **less steep** lines.
- If the slope of a line is **zero,** it's actually a flat **horizontal line.**
- The slope of a **vertical line** is **undefined.**
- **All** of the points (x, y) lying on a straight line will **satisfy** the equation of the line, meaning they can be plugged in and the equation will be true, and meaning that the line is made up of all the points (x, y) that do satisfy the equation.
- The equation of a line with slope m and y-intercept $(0, b)$ is $y = mx + b$.
- The equation of a line with slope m passing through the point (x_0, y_0) is $y - y_0 = m(x - x_0)$.
- **Parallel lines** have **the same slope.**
- If two lines are **perpendicular** to each other then their slopes are **negative reciprocals.** (The negative reciprocal of a number $\frac{a}{b}$ is $-\frac{b}{a}$.)
- To find where two lines $y = m_1x + b_1$ and $y = m_2x + b_2$ **intersect**, set $m_1x + b_1 = m_2x + b_2$ and solve for x. Then to find the corresponding y-coordinate, plug that x solution into either of the original equations.
- To sketch the graph of a line:
  - If you know one point and the slope $m = \frac{a}{b}$ is positive, plot the point and then find another point just by tracking b units to the **right**, and then a units **up**. Drop a point in right there, connect them with a straight line, and you're done.
  - If the slope $m = \frac{a}{b}$ is negative, then track b units to the right, and then a units **down** instead.
  - If a slope-intercept form has no b term, like this $y = 3x$, you automatically know that line passes through the origin since its y-intercept is $(0, 0)$.
  - The equation of a horizontal line passing through b on the y-axis is $y = b$.
  - The equation of a vertical line passing through a on the x-axis is $x = a$.

Fun Facts About Circles

- The equation of a circle centered at the point (h, k) with radius r is $(x - h)^2 + (y - k)^2 = r^2$.
- The **radius** is the uniform distance from the center to any point on the circle.
- The **diameter** is the distance from one side of the circle across to the other, passing through the center, so it's double the radius. That is, diameter $d = 2r$.
- The **area** of a circle with radius r is πr^2.
- The **circumference** of a circle is the distance all the way around it, as in how many steps your Apple watch would say you covered if you walked the whole perimeter, and it's calculated $2\pi r$.
- To sketch the graph of a circle, identify and plot the center, and then draw four cardinal points at the appropriate distance away for the radius. Connect the dots with curves as best you can to create a circle shape.
- To see if an equation with x^2 and y^2 terms is the equation of a circle, **complete the square on x terms and y terms separately** to see if it can be made to match the standard form.

Variation

- **Direct variation:** if $y = kx$, where k is some **constant of proportionality**, then y **varies directly as** or **is directly proportional to** x.
- **Inverse variation:** if $y = \frac{k}{x}$, where k is some **constant of proportionality**, then y **varies inversely as** or **is inversely proportional to** x.

- If x, y, and z are related by the equation $z = kxy$, we say z is **proportional to the product** of x and y, or z is **jointly proportional** to x and y, or z **varies jointly** as x and y.
- Looking at an equation solved for one variable, like the above or like $z = k\frac{x}{y}$, we know that z is **directly proportional** to any variables in the **numerator,** and **inversely proportional** to any variables in the **denominator.**
- You can also be proportional to expressions other than a variable alone! For example, if $z = k(x + y)$, we say z is directly proportional **to the sum** of x and y. Or if $z = \frac{k}{x^2}$, we say that z is inversely proportional **to the square** of x.

Solving Systems

- **Substitution:** Solve one equation for a particular variable, then sub in for that variable in the second equation, yielding an equation with one unknown, and solve that. Then plug that result into an original equation to find the second unknown.
- **Elimination:** Multiply one or both equations by whatever is needed so that, on a particular variable, the equations have opposite coefficients. Then line up all the variable terms and constant terms in columns correctly and add down the columns, yielding an equation with one unknown, and solve that. Then plug the result into an original equation to get the other unknown.
- A system of two linear equations in two unknowns can have:
  - **one unique solution** (the equations' lines intersect once)
  - **no solutions** (the equations' lines are parallel and never intersect; in this case the system is called **inconsistent**) This will happen if as you're solving, you end up with an equation that is **a lie,** like $0 = 5$.
  - **infinitely many solutions** (the equations' lines are smack on top of each other, the same line really, and have all points in common) This will happen if as you're solving, you end up with all the variables killing each other off, resulting in a true equation, like $0 = 0$. This always happens if one equation is a constant multiple of another.

Vector Arithmetic

- Vector addition/subtraction:

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}, \qquad \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a-c \\ b-d \end{bmatrix}$$

  Geometrically, you are "chaining" up the vectors nose to tail and drawing a new vector from the origin to the final arrowhead.
- Scalar multiplication: $k \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ kc \end{bmatrix}$. Geometrically, this stretches or shrinks a vector's length without changing its direction. A negative scalar will cause a 180° flip in direction.

Matrix Arithmetic

- Matrix addition/subtraction (requires the same exact size): $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & f+b \\ c+g & d+h \end{bmatrix}$
- Scalar multiplication: $k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$
- 2×2 **matrix-vector multiplication** of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ results in a new 2×1 vector

$$Av = \begin{bmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{bmatrix}$$

Inverse of a Matrix

The **inverse** of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. The quantity $ad - bc$ is called the **determinant** of A, and **must not equal zero for this to work!** If a matrix's determinant turns out to be zero, it is **not invertible.**

Solving 2×2 Systems With Matrix Inverses

To solve a system $\begin{cases} ax + by = v_1 \\ cx + dy = v_2 \end{cases}$,

1. Translate to a matrix equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

2. Find the inverse of the coefficient matrix using the formula $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ as long as $ad - bc \neq 0$. If it is 0, the system is **inconsistent** and has no solutions.

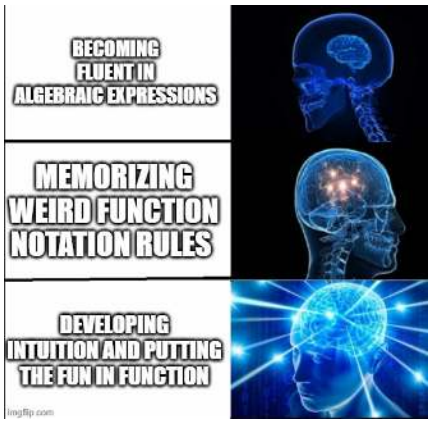
3. Solve for $\begin{bmatrix} x \\ y \end{bmatrix}$ by multiplying $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

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CHAPTER OVERVIEW

Chapter 4: Functions

In this chapter, all the groundwork we've laid in the first three finally starts to come to fruition. Functions are everywhere in math, sciences, business, technology, etc! We rely on algebraic expressions to describe many functions succinctly, but if you're super visual like me, we can also gain powerful understanding from the ability to graph functions on a coordinate plane. Once we have the foundational toolbox of this chapter, we will be able to go on to specific types of functions for the rest of the book. At first, this may feel like learning just another weird language you have to memorize the rules of, but I want you to really try to absorb the essence of functions as machines that take inputs and outputs. The more you lift your gaze from the trees of notation and get your head around the forest of what's really happening, the better off you're going to be in Calc I and science courses. So really try to dig in!



4.1 Functions, Domain, and Notation
4.1E Exercises
4.2 Graphing Functions With Transformations
4.2E Exercises
4.3 Compositions of Functions
4.3E Exercises
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4.1 Functions, Domain, and Notation

Learning Objectives

By the end of this section, you will:

- know the definition of a function
- think of functions defined in four ways
- be able to evaluate functions
- be able to analyze the domain and range of a function

There are many ways to describe the concept of a **function.** I'm going to give you a few different ones so you can find your favorite one that makes it "click," but each of them is important in its own way for future applications of functions in different subjects, so make sure to read through each version. The first time I learned about functions, personally, I remember just thinking, "code for y lol," but I hope we'll get a little more rigorous in this section. Here is a formal definition to get us started, and then we'll proceed to elaboration.

Function

Given two sets of elements (like numbers) A and B, a **function** f is a rule that takes each element x of A and assigns to it **exactly one** element, nicknamed "$f(x)$," in the set B.

Caution

I emphasize the *nicknameness* of the **function notation** $f(x)$ because a common mistake is to treat those parentheses as multiplication. It's not!!! It is telling you that x is a variable representing an input, and the result that f assigns to it is denoted by $f(x)$.

Functions as Machines

A function named f is like a machine into which you feed an **input** x from a set of inputs called the **domain** of f. The variable representing the inputs is called the **independent** variable. Inside the machine, some activity happens, and then some **output** (also called the **image** of x under f) named $f(x)$ gets spit out the other end. When a variable (such at y) represents the output values, it is called the **dependent** variable. The activity inside the machine is really the essence of "f," like f is the **instructions** about what to do to the input to produce the output.



(hopper image from Minecraft Wiki)

You could say this in words. For example, there is a function called the **sign function** (or signum function), which takes real numbers as its inputs, and when you feed it a number, it checks whether it's positive or negative. If it's a positive number, the sign function spits out "1," and if it's negative, it returns "-1," and if it's zero, it gives zero right back. We can translate these instructions from words to math using the following notation,

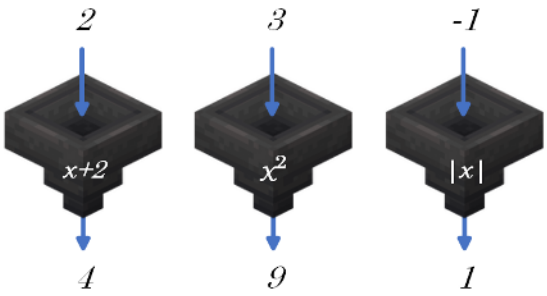
$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

This type of function, where the instructions are split up into cases for different types of inputs, is called a **piecewise-defined** (or **piecewise**) **function.**

This process of translating from instructions to math notation is done in a consistent way. It goes like this:

$$\text{output named } f(x) = \text{ algebraic instructions about what to do to input to get output}$$

For example, "$f(x) = x + 2$" is the machine that adds 2 to any input to produce the output. So if I feed it 2 as the input, it will spit out 4. See examples below.



Feeding f a specific input like 3 is called **evaluating** f **at** 3, or informally "plugging 3 into f," and written $f(3)$. To evaluate at an input, you just read through the algebraic expression defining the function, plugging in the input everywhere you see the variable. Take that middle example above. It's defined by $f(x) = x^2$, with instructions "square that guy," so specifically $f(3) = 3^2 = 9$. Whatever appears in the box, $f(\square)$, you slap in wherever the variable appears (often using parentheses!).

$$f(4) = 4^2, \qquad f(a) = a^2, \qquad f(x-1) = (x-1)^2$$

Example 1

For the following functions, describe in words what the instructions are, and then evaluate at the given inputs.

1. $f(x) = x^2 + 1$, at $x = 0, 1, -2, 3x$
2. $g(x) = \sqrt{x}$, at $x = 1, 4, 9, x^2$
3. $h(t) = 14 - 3t$, at $t = 0, 4, 5, a - 1$

Solution

1. The instructions are, "square the input and then add 1 to the result." Evaluations:

$$f(0) = 0^2 + 1 = 1, \quad f(1) = 1^2 + 1 = 2, \quad f(-2) = (-2)^2 + 1 = 5,$$
$$f(3x) = (3x)^2 + 1 = 9x^2 + 1$$

2. The instructions are, "take the square root of the input." Evaluations:

$$g(1) = \sqrt{1} = 1, \quad g(4) = \sqrt{4} = 2, \quad g(9) = \sqrt{9} = 3, \quad f(x^2) = \sqrt{x^2} = |x|$$

3. The instructions are "multiply the input by 3, and then subtract the result from 14." Evaluations:

$$h(0) = 14 - 3(0) = 14, \quad h(4) = 14 - 3(4) = 2, \quad h(5) = 14 - 3(5) = -1,$$
$$h(a-1) = 14 - 3(a-1) = 17 - 3a$$

Exercise 1

For the following functions, describe in words what the instructions are, and then evaluate at the given inputs.

1. $f(x) = (x-3)^3$, at $x = 1, 2, \star$
2. $g(u) = \dfrac{u+2}{3}$, at $u = -2, 0, 4, 2u$
3. $h(t) = |t|$, at $t = -3, 0, 3, t + 1$

Answer

1. "Subtract 3 from the input, then cube the result." Evaluations: $f(1) = -8, f(2) = -1, f(\star) = (\star - 3)^3$
2. "Add 2 to the input, then divide the result by 3." Evaluations: $g(-2) = 0, g(0) = \frac{2}{3}, g(4) = 2, g(2u) = \frac{2u+2}{3}$
3. "Take the absolute value of the input." Evaluations: $h(-3) = 3, h(0) = 0, h(3) = 0, h(t+1) = |t+1|$

Functions From Tables

One way to give information about assigning outputs to inputs is with a table of values.

x	$f(x)$
0	0
1	1
2	4
3	9
4	16
5	25

Can you tell what definition of f would make sense from this table? Yeah, $f(x) = x^2$ would be consistent.

Exercise 2

Guess the function's definition if the following table shows the outputs assigned to various inputs.

x	$f(x)$
0	1
1	$\frac{1}{2}$
2	$\frac{1}{3}$
5	$\frac{1}{6}$
10	$\frac{1}{11}$
100	$\frac{1}{101}$

Answer

Looks like the function should be $f(x) = \frac{1}{x+1}$. That would be a way to consistently get the desired outputs to come from the indicated inputs.

Caution

A **key feature** of the definition of a function is that it assigns **exactly one** output to each input! There can be no ambiguity about what will happen to an input. There's no way to give a function a single input and have two different options of output!

Exercise 3

Based on what I just yelled about, why can't the table below define a function?

inputs	**outputs**
1	2

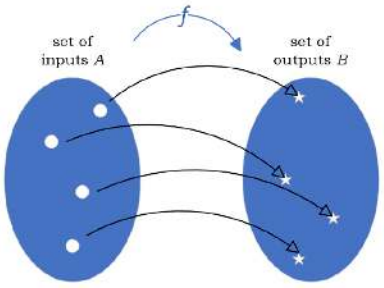
2	3
7	12
2	11
9	15

Answer

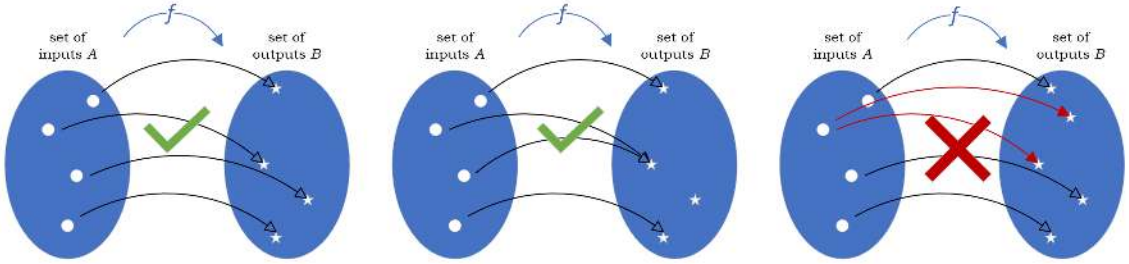
Look at that sneaky row saying that 2 goes with 11 even though earlier it said 2 goes with 3! This can't be a function, because it's assigned two different outputs to the same input. Illegal!

Functions as Maps

You could also imagine a function as a map that directs you from Input Land to Output Land by some rules. In this sense, the function is a bunch of arrows telling you where to send each input.



This sense of a function lets us visualize the "exactly one output for each input" rule mentioned above.



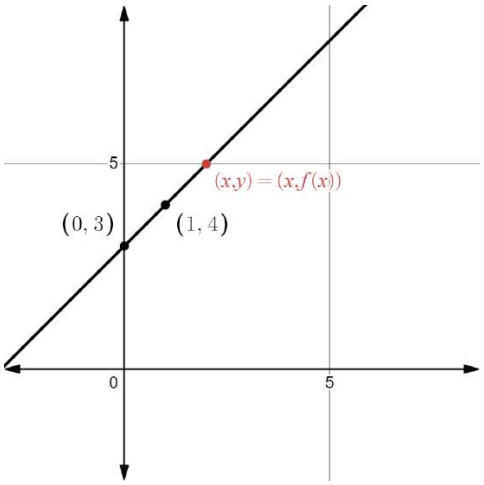
It's A-OK to assign two inputs to the same output, but it's illegal to send a single input to two different outputs. That would give f an existential crisis anytime you asked him to do anything. Can't have that.

This type of visualization of a function as a map will serve us well when we talk about compositions, so that's why I wanted you to see it here.

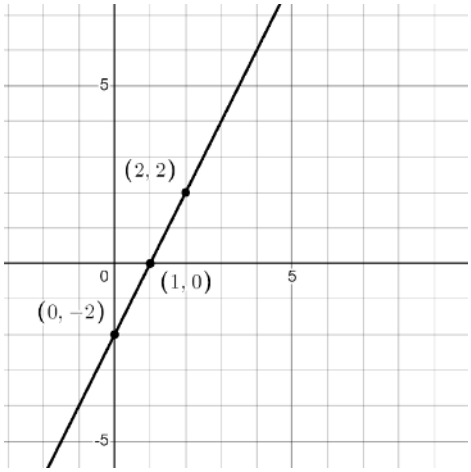
Functions From Graphs

Since a function assigns an output to each input, we could name the input x and name the output y, and write them in a little pair together, (x, y). Ope, that looks familiar?? It looks like the coordinates of a point in the 2D plane! In the next section, we will learn how to draw **graphs of functions,** but for now let's connect the ideas basically.

Consider the function $f(x) = x + 3$. If we nickname the output $f(x)$ as "y," suddenly we have the equation of a line in the coordinate plane, $y = x + 3$. For any input x we feed to f, the output will give us a y-value that goes with that x-value on the line! That is, we have $f(0) = 3$, and the point $(0, 3)$ is on the line. Similarly, $f(1) = 4$, and the point $(1, 4)$ is on the line. For any input x, the y-coordinate will be defined using f to be $x + 3$. In this scenario, the function value depends on the input x, so x is the independent variable measured on the horizontal axis, and the function value output is telling us the value of the dependent variable measured on the vertical axis.



Going the other direction, let's try to express a function who describes the graph below.



I'm looking for a function to define the y-values, $y = f(x) =$???, to match all the points on this line. I could write these and similar points in a table like this:

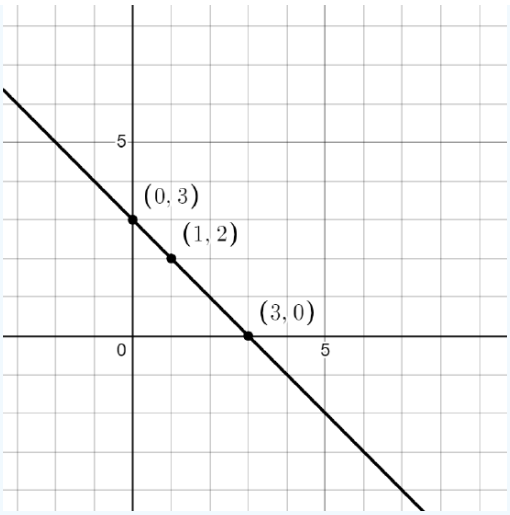
x	$y = f(x)$
0	-2
1	0
2	2
3	4

Nice. I'd like to find the algebraic expression that defines the rule. It will not surprise me if it turns out to look like a linear equation, something in the form $ax + b$, because my graph is a straight line. I see a y-intercept of -2, and for every step to the right I take, I take two steps up. Rise-over-run-wise, that's a slope of positive 2. Maybe $f(x) = 2x - 2$ will work? Confirm using the table and the picture that every point (x, y) on the line would satisfy this function definition...

$$x = 0 \longrightarrow f(0) = -2\checkmark, \quad x = 2 \longrightarrow f(2) = 2\checkmark, \quad \ldots$$

? Exercise 4

Try to identify the algebraic expression definition of the function whose graph is shown below.



Answer

$f(x) = -x + 3$ will be satisfied by all points on the line, $(x, y) = (x, f(x))$, and matches the slope of -1 and y-intercept of 3.

Domain Again

Domain of a Function

The **domain** of a function is the set of all inputs for the function. This may be explicitly given in a situation, such as

$$f(x) = x + 4, \quad 0 \leq x \leq 20$$

or it may be implicitly assumed to be the domain of whatever algebraic expression is defining the function. In that case, the domain is whatever real numbers are **legal** to be plugged in for x. Finally, in a real-world application problem, you should assess the domain by only including values that make sense in context.

It's often easier to talk about what **isn't in the domain** than what is...like it's usually a shorter list. When I ask you about domain, you will always say to yourself, "Self? What problem children will cause legality issues if I evaluate the function there?" This is exactly the same as analyzing the domain of an algebraic expression. We report our answers the same way: with interval notation, words, set notation, whatever as long as it's clear, communicative, and true.

Example 2

Find the domains of the functions given.

1. $p(x) = x^2 + 2$
2. $g(x) = \dfrac{1}{x}$
3. $h(t) = 12 - \sqrt{t}$
4. $f(n) = \dfrac{2}{\sqrt{n+2}}$
5. $V(x) = x^2(30 - x)$, which gives the volume of a certain box with square base, dependent on the side length of the base, x.

Solution

1. This function is defined using a polynomial, and everything is legal to plug into polynomials. I'm always looking out for two things: dividing by zero, and taking square roots of negative numbers. Neither is an issue here. The domain is thus all real numbers, written $\mathbb{R}$ or in interval notation, $(-\infty, \infty)$. **Note:** this is not a coordinate point, it's an open interval!
2. This function has division involved, so we cannot allow x to be 0. We say "all real numbers, $x \neq 0$," or maybe $(-\infty, 0) \cup (0, \infty)$, or maybe $\{x \mid x \neq 0\}$.

3. This function has a square root, so I can't plug in any negative numbers for t. We say, "all nonnegative real numbers," or $[0, \infty)$, etc.
4. This function has division and square roots! First of all, I can't let n be anything that would cause the inside of the square root to be negative. Aka, I must require that $n + 2 \geq 0$. By solving the inequality, I see this is guaranteed if $n \geq -2$. BUT I also can't divide by zero, and if $n = -2$, that happens. :(So I can't allow -2 in my domain either. My final answer is $n > -2$, or $(-2, \infty)$.
5. This describes a physical quantity, so we think about what answers would be silly. It won't be much of a box if its base has dimensions 0×0, so we could restrict $x > 0$. It also doesn't make sense to have a negative dimension or negative volume, which would happen if $x > 30$ because of the $(30 - x)$ factor. Since $x = 30$ would also result in a volume of 0, I'd probably throw that out and say my domain of interest for this function is just $0 < x < 30$.

? Exercise 5

Find the domains of the functions given.

1. $q(x) = (x + 1)(x - 1)(x + 2)$
2. $f(x) = \sqrt{2x - 1} + 4$
3. $g(y) = \frac{y-1}{y+3}$
4. $A(r) = \pi r^2$, a function giving the area of a circle I plan to cut out of a 12in $\times$ 20in rectangle of cardboard.

Answer

1. All real numbers.
2. $\{x \mid x \geq \frac{1}{2}\}$
3. $y \neq -3$
4. It wouldn't make any sense to cut out a circle with negative radius, and I don't want a radius of zero (would that be just stabbing a pin through the cardboard??) so I'm going to restrict $r > 0$. But also, the smallest dimension of my cardboard is 12in, so even if I position the circle in the middle of the rectangle, I can't have a diameter larger than 12in. That restricts my radius to 6in max. The domain I'm interested in is $0 < r \leq 6$.

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4.1E Exercises

Describing Functions

Describe in words what instructions are being given in the algebraic definitions of functions below.

1. $g(x) = x^2 - 1$
2. $K(m) = \frac{8}{5}m$
3. $A(r) = \pi r^2$
4. $h(x) = \dfrac{3x+1}{5}$
5. $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
6. $s(t) = 6 + 3t - 4.9t^2$
7. $P(u) = 1300u - (1.5u^2 + 30u)$
8. $F(C) = \frac{9}{5}C + 32$

Answer

This is all in your own words, however makes sense to you, AS LONG AS it is correct in the order of operations happening!

1. Square the input, then subtract 1.
2. Multiply the input by $\frac{8}{5}$. This function approximates a conversion from speed in miles per hour to speed in kilometers per hour.
3. Square the input, then multiply by π. This function computes the area of a circle depending on its radius.
4. Multiply the input by 3, then add 1, then divide the result by 5.
5. Take the absolute value of the input; that is, if the input is nonnegative, return the exact input, but if it is negative, return its opposite.
6. Multiply the input by 3 and add it to 6, then subtract from that the result of "squaring the input and then multiplying by 4.9".
7. Square the input, multiply by 1.5, and add the result of "multiplying 30 by the input." Subtract all of that from the input multiplied by 1300. This could be a profit function depending on number of units sold, u, which subtracts a formula for costs from a formula calculating revenue.
8. Multiply the input by $\frac{9}{5}$ and then add 32. This function takes a Celsius temperature as input and converts it to the Fahrenheit equivalent.

Evaluating Functions

Evaluate the functions at the indicated inputs. Report your answers with appropriate function notation!

1. $g(x) = x^2 - 1$ at $x = -2, 0, 5$, and $x + h$.
2. $K(m) = \frac{8}{5}m$ at $m = 0, 25, 45$, and 75.
3. $H(x) = \dfrac{3x+1}{5}$ at $x = 0, \frac{1}{3}, 3$, and $x + h$.
4. $s(t) = 6 + 3t - 4.9t^2$ at $t = 0, 1$, and t_f.
5. (Calculator allowed) $P(u) = 1300u - (1.5u^2 + 30u)$ at $u = 2, 10$, and $\blacksquare$.
6. $F(C) = \frac{9}{5}C + 32$ at $C = 0, 20$, and 100.

Answer

1. $g(-2) = 3$, $g(0) = -1$, $g(5) = 24$, $g(x+h) = (x+h)^2 - 1$
2. $K(0) = 0, K(25) = 40, K(45) = 72, K(75) = 120$
3. $H(0) = \frac{1}{5}, H(1/3) = \frac{2}{5}, H(3) = 2, H(x+h) = \dfrac{3(x+h)+1}{5}$
4. $s(0) = 6, s(1) = 4.1, s(t_f) = 6 + 3t_f - 4.9t_f^2$
5. $P(2) = 2534, P(10) = 12550, P(\blacksquare) = 1300\blacksquare - (1.5\blacksquare^2 + 30\blacksquare)$
6. $F(0) = 32, F(20) = 68, F(100) = 212$

Difference Quotients

A **difference quotient** is an expression you will see roughly 2130329028 times in Calc I of the form $\frac{f(x+h)-f(x)}{h}$. For each function in particular, write the difference quotient and simplify if reasonable. Be careful when plugging in, when subtracting, and when cancelling! For example, $f(x)=x^2+1$ would yield

$$\frac{(x+h)^2+1-(x^2+1)}{h}=\frac{2hx+h^2}{h}=2x+h$$

1. $f(x)=x^3$
2. $f(x)=2x+3$
3. $f(x)=4$

Answer

1. $\frac{(x+h)^3-x^3}{h}=\cdots=3hx+3x^2+h^2$
2. $\frac{2(x+h)+3-(2x+3)}{h}=\cdots=2$
3. $\frac{4-4}{h}=0$ (Watch out! There is no variable in the function's defining expression. This is called a **constant function**, and we'll see it later again.)

Functions From Tables

For each table of values, tell whether it can represent a function, and if so, guess the function that the value pairs would satisfy.

1.

inputs	outputs
-2	2
-1	1
0	0
4	4

2.

inputs	outputs
5	11
6	13
7	15
8	17

3.

inputs	outputs
-1	1
17	4
2	0
-1	9

Answer

1. Yes, it's fine for multiple inputs to be assigned the same output. The function looks like $f(x) = |x|$.
2. Yes, looks like $f(x) = 2x + 1$.
3. No, there is an input sent to two distinct outputs, so this is not a function.

? Domain of a Function

Give the domains for the following functions.

1. $C(x) = x^2 + 40x, \quad 0 \leq x \leq 100$
2. $p(x) = 3x^3 + 2x^2 + x$
3. (Calculator allowed) $h(t) = 100 + 3t - 4.9t^2$, which gives the height above the ground, in feet, at time t seconds, for a rock dropped off a 100ft platform.
4. $r(x) = \dfrac{3}{x^2 - 5x - 6}$
5. $s(x) = \sqrt{x^2 - x}$ (Secretly also an inequality solving problem.)
6. $f(n) = \dfrac{n+1}{n^2+1}$, where $n = 1, 2, 3, ...$
7. $q(x) = x^7 - x^2 + 3$
8. $C(x) = 0.25x^2 - 15x + 2500$, which gives the total cost of producing x washing machines.
9. $g(x) = \dfrac{1}{x^2 - 16}$
10. $F(x) = \dfrac{\sqrt{x}}{(x-2)\sqrt{x-1}}$

Answer

1. The domain is given to us, $0 \leq x \leq 100$. This could be a cost function for producing x objects where it doesn't make sense to make a negative number of objects and it isn't feasible to make more than 100.
2. All real numbers, $(-\infty, \infty)$.
3. I start the clock at time 0, as negative time doesn't make sense, outside of Doctor Who I suppose, so $t \geq 0$. But it also doesn't make sense to continue looking at time values after the rock hits the ground, aka when $h(t) = 0$ft. I'm going to stop my domain at $t = 4.8$ seconds. Figure out why (you can use a calculator and review how to solve a quadratic equation)... Final answer is $0 \leq t \leq 4.8$.
4. $\{x \mid x \neq -1, 6\}$
5. The inside must be nonnegative, and it factors as $x(x-1)$. There are two ways for this to be 0, when $x = 0, 1$, and that's fine. There are also two ways this expression can be positive: a) if both $x > 0$ AND $x - 1 > 0$, which will be true if $x > 1$, or b) if both $x < 0$ AND $x - 1 < 0$, which will happen if $x < 0$. So my domain includes all real numbers up to and including 0, and then all real numbers greater than or equal to 1. We write this as $(-\infty, 0] \cup [1, \infty)$.
6. Domain: all natural numbers (positive integers) n.
7. $(-\infty, \infty)$
8. $\{x \mid x \geq 0\}$ (I can't make negative washing machines... Would that be taking washing machines out back and destroying them with a sledgehammer?)
9. $\{x \mid x \neq \pm 4\}$
10. $\{x \mid x > 1, x \neq 2\}$

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4.2 Graphing Functions With Transformations

Learning Objectives

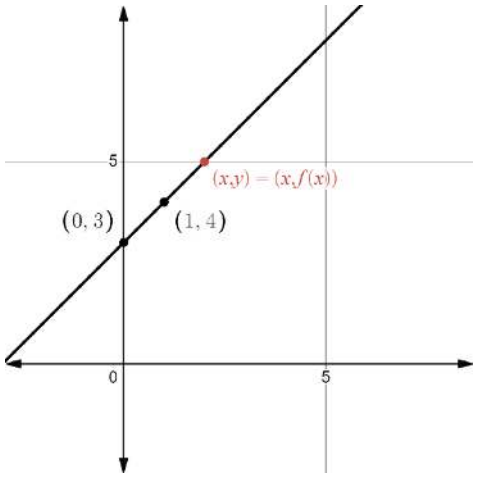
By the end of this section, you will be able to:

- grasp the relationship between a function and its graph
- assess whether a graph is the graph of a function using the Vertical Line Test
- sketch graphs of functions
- identify the effect of transformations of functions on their graphs
- identify even and odd functions

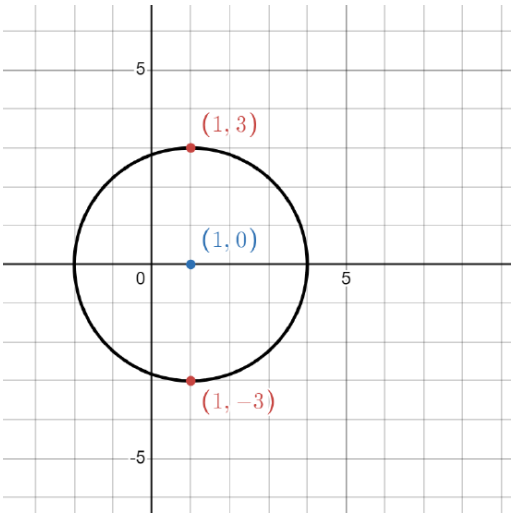
A **graph** is a diagram drawn on a coordinate plane showing the relationship between two variables, each of which is measured along an axis. We've already seen the graphs of lines and circles in detail when learning about equations. Since functions also describe relationships between two variables, an **independent** input variable and a **dependent** output variable, we have been able to sketch the graph of a function, essentially drawing on a plane all of the points whose coordinates are related via that function. All functions can be graphed (in some way, although maybe not prettily), but not all graphs represent functions! Let's talk about that.

Vertical Line Test

We saw this graph in the last section:



Here we have the set of infinitely many points that lie on that straight line, whose equation we would recognize as $y = x + 3$. This the graph of the function $f(x) = x + 3$, because at each input x-value, that function f will spit out the correct output value for the y-coordinate. Now consider this graph:

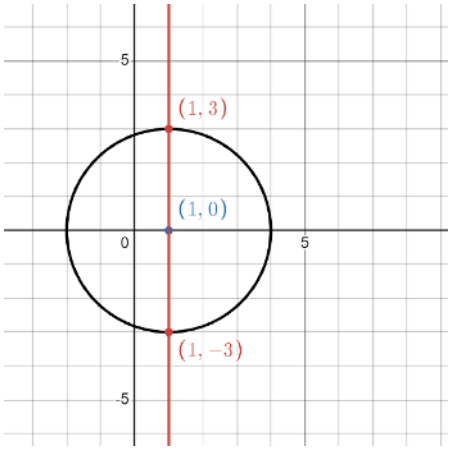


Could this graph be representing a function? Aka, could the y-coordinates of points on this circle be defined by function values given the x-coordinates as inputs? I highlighted two points on the circle there. Notice that they both have the x-coordinate 1, and yet they have two distinct y-coordinates... A function is only allowed to assign **one** output to each input, so this **cannot** be the graph of a function! There's a handy trick for assessing this at a glance.

Vertical Line Test (VLT)

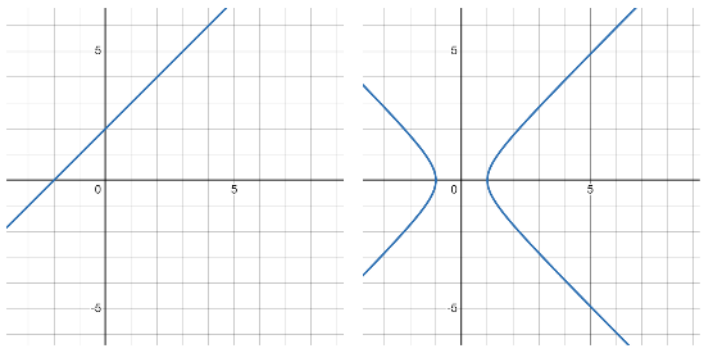
If there is any place on a given graph where you could draw a **vertical line** that **intersects** the graph at **multiple distinct points**, then the graph **cannot** be the graph of a function.

A vertical line has an equation of the form $x = a$ and is passing through all the points with x-coordinate a. If it crosses a graph more than once, that means the graph has multiple y-values assigned to the same x-value. Illegal for functions!



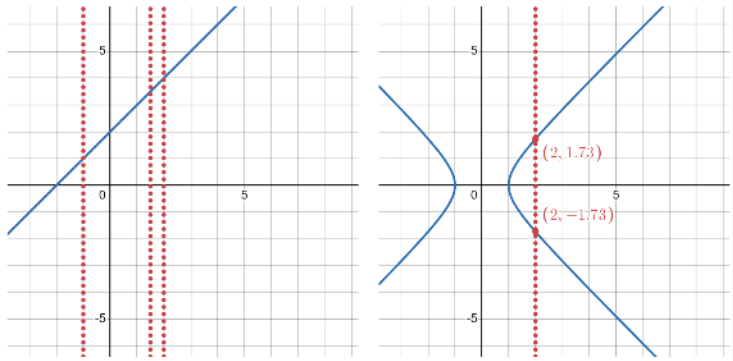
Example 1

Tell whether the following graphs could represent functions.



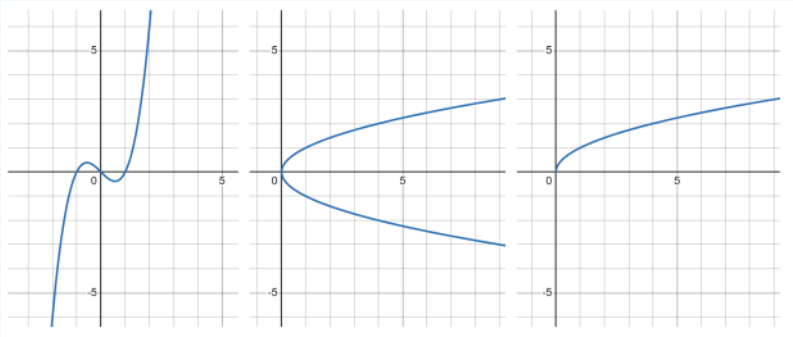
Solution

The first passes the VLT, as you can draw a vertical line anywhere and have only one intersection, so it represents a linear function. The second is the graph of a hyperbola and fails the VLT, so no.



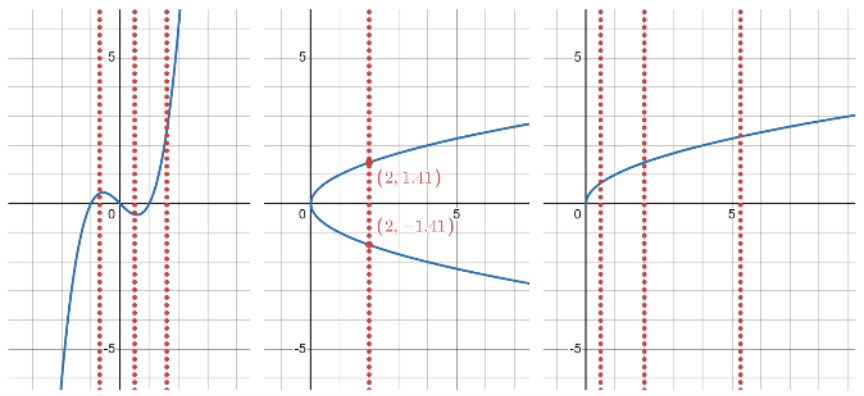
Exercise 1

Tell whether the graphs could represent functions.



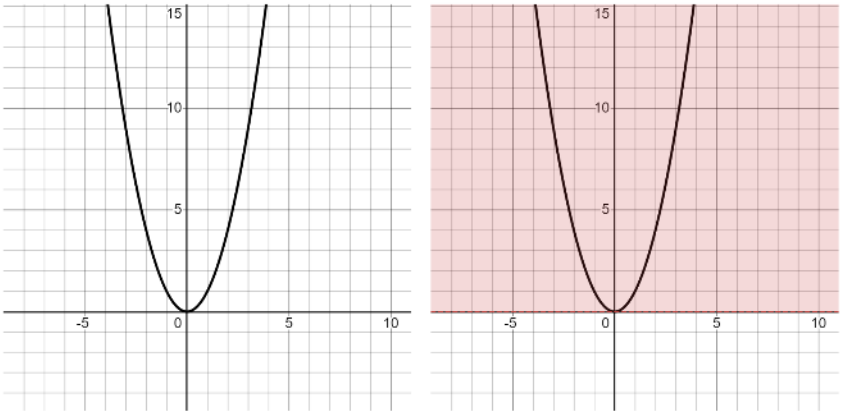
Answer

Yes, no, and yes. VLT demonstrated below.



Domain and Range via Graphs

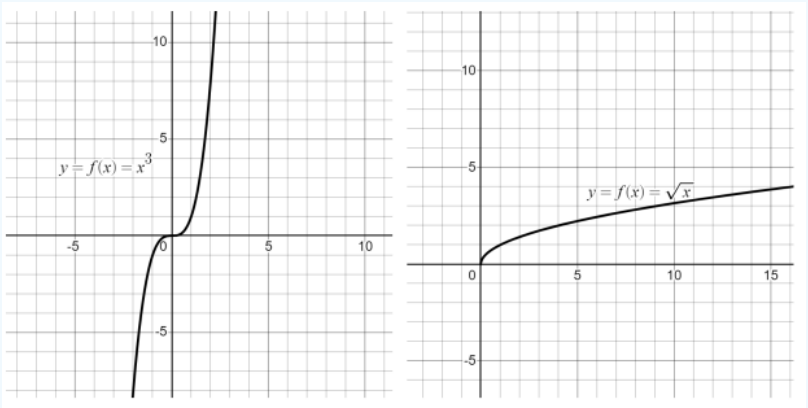
Recall that the **domain** of a function is its set of inputs (most often, all real numbers that are legal to use as inputs). The **range** of a function is the set of real numbers that it actually produces as outputs. So like, I could think of the function $f(x) = x^2$ as taking real numbers and inputs and giving real numbers as outputs, which is certainly true, but more specifically, f will only give **nonnegative** real numbers as outputs. Let's look at the graph of this function.



The arms of this parabola continue forever upward and outward. In regard to the x-axis, all input x-values are allowed in the domain. But looking at the y-axis, we see that the graph never passes below the height $y = 0$. This graph never attains any negative y-values, but it does attain all positive y-values as the arms go up and up forever. For this function, the domain is all real numbers, and the range is real numbers $y \geq 0$.

Example 2

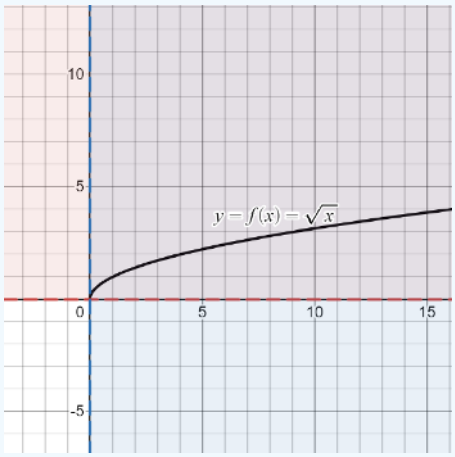
Assess the domain and range of the following graphed functions.



Solution

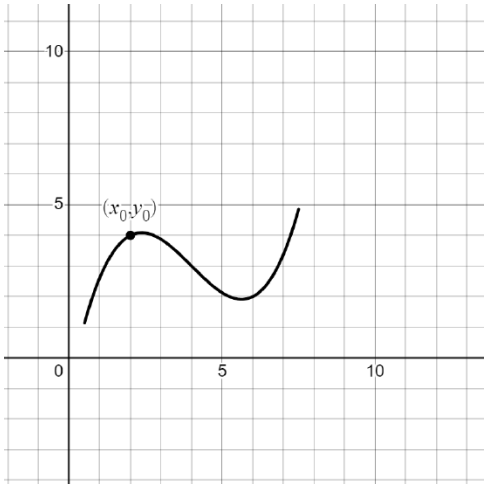
On the left, we see that the function's graph heads down and to the left forever, which means all negative x- and y-values can be attained, and up and to the right forever, which means all positive x- and y-values can be attained. We know that $f(x) = x^3$ can take any real number as input, and will give as outputs both positive and negative real numbers. Both the domain and range for this function are $\mathbb{R}$, all real numbers.

On the right, we see that the graph doesn't appear in the left half of the 2D plane, so it is not defined for any negative x-values. The graph also stays above the x-axis forever, meaning it never attains negative y-values. The domain here is $\{x \mid x \geq 0\}$ and the range is $\{y \mid y \geq 0\}$.

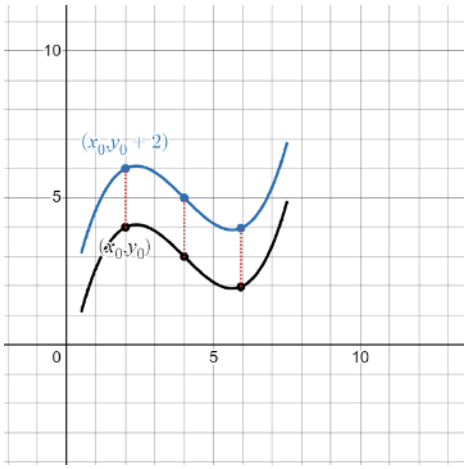


Transformations of Graphs

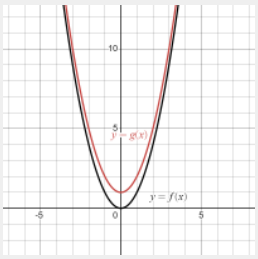
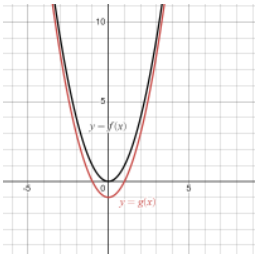
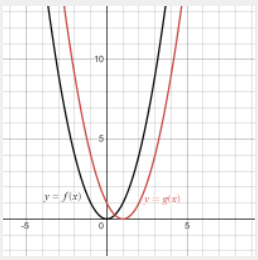
Suppose I have a graph of some function like the one given below.

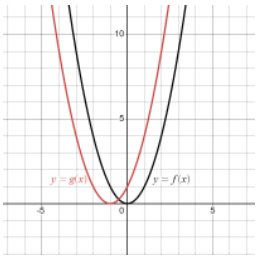
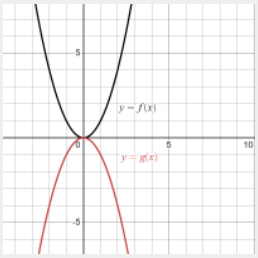
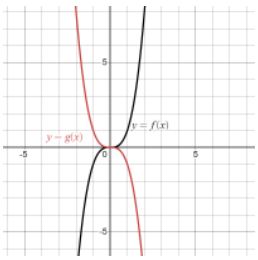
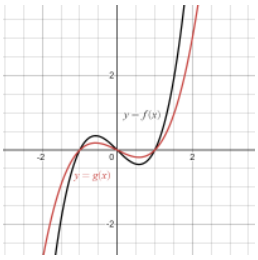


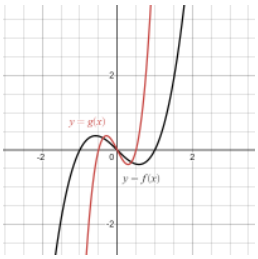
To every single point (x, y) on the graph, I perform a little operation by adding 2 to the y-value. This has the effect of scootching each point exactly two units upward. The entire graph appears to have **shifted vertically upward by 2,** without otherwise changing the shape of the graph.



If this was the graph of a function, $y = f(x)$, then the new blue line is the graph $y = f(x) + 2$. There are several different little transformations you can apply to an original function whose graphical consequences are easy to predict, just like this. I'm going to give you a big table here, but it's important that you read through line by line and understand as much as possible why each operation causes the indicated graph change. Each transformation starts with an original function $f(x)$ and makes a change to it like $f(x) + 2$ to produce a new function, which you'll find in the second column.

Transformation Description	$f(x) \rightarrow$ **New Function**	**English Translation**	**Example**	**Graph**
Vertical shift **up** by k	$f(x) + k$	*An original function gets a positive constant **added** on to the very end, which causes a vertical change.*	$f(x) = x^2, \rightarrow g(x) = x^2 + 1$	
Vertical shift **down** by k	$f(x) - k$	*An original function gets a constant **subtracted** on the very end, which causes a vertical change.*	$f(x) = x^2, \rightarrow g(x) = x^2 - 1$	
Horizontal shift **right** by h	$f(x - h)$	*In an original function, each instance of the variable was replaced by the variable **minus** a constant, causing a horizontal change.*	$f(x) = x^2, \rightarrow g(x) = (x - 1)^2$	

Horizontal shift **left** by h	$f(x+h)$	*In an original function, each instance of the variable was replaced by the variable **plus** a constant, causing a horizontal change.*	$f(x)=x^2, \rightarrow g(x)$ $=(x+1)^2$	
Reflection across the x-axis	$-f(x)$	*Flip the sign on the whole original function, which has the effect of flipping the sign on all y-values.*	$f(x)=x^2, \rightarrow g(x)=-x^2$	
Reflection across the y-axis	$f(-x)$	*In an original function, each instance of the variable x was replaced by $(-x)$.*	$f(x)=x^3, \rightarrow g(x)$ $=(-x)^3$	
Vertical **stretch** by a factor of k	$k \cdot f(x)$, where $k>1$	*An entire original function is multiplied by a constant **larger** than 1, causing a vertical change.*	$f(x)=x^3-x, \rightarrow g(x)$ $=3(x^3-x)$	
Vertical **shrink** by a factor of k	$k \cdot f(x)$, where $0<k<1$	*An entire original function is multiplied by a positive fraction **smaller** than 1, causing a vertical change.*	$f(x)=x^3-x, \rightarrow g(x)$ $=\frac{1}{2}(x^3-x)$	
Horizontal **stretch** by a factor of $\frac{1}{h}$	$f(hx)$, where $0<h<1$	*In an original function, each instance of x is replaced by x times a small positive fraction, causing a horizontal change.*	$f(x)=x^3-x, \rightarrow g(x)$ $=(\frac{1}{2}x)^3-\frac{1}{2}x$	

Horizontal **shrink** by a factor of $\frac{1}{h}$	$f(hx)$, where $h > 1$	*In an original function, each instance of x is replaced by x times a constant larger than 1, causing a horizontal change.*	$f(x) = x^3 - x, \rightarrow g(x) = (2x)^3 - 2x$	

All of these transformation types can be combined one after another. When describing this process, sometimes the order in which you perform transformations really matters! See the example below.

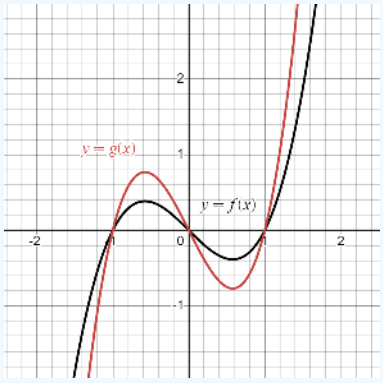
Example 3

Describe the transformation(s) done to $f(x) = x^3 - x$ to produce the following functions.

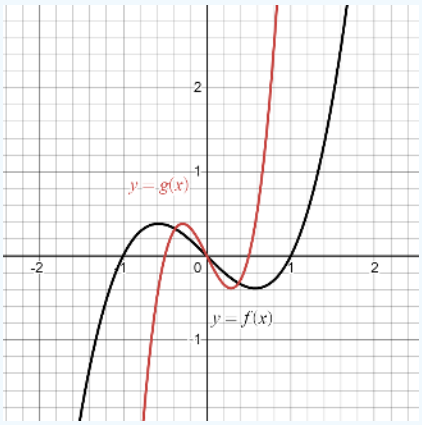
1. $g(x) = 2x^3 - 2x$
2. $g(x) = 8x^3 - 2x$
3. $g(x) = (x-2)^3 - x + 2$
4. $g(x) = (x+1)^3 - x - 3$
5. $g(x) = 4(x^3 - x) + 1$
6. $g(x) = -2((x-1)^3 - (x-1))$

Solution

1. We can see that this g was made by taking $f(x)$ and multiplying the whole thing by a 2, so $g(x) = 2f(x)$. According to the table, this causes a vertical stretch by a factor of 2. On the graph, this looks like...

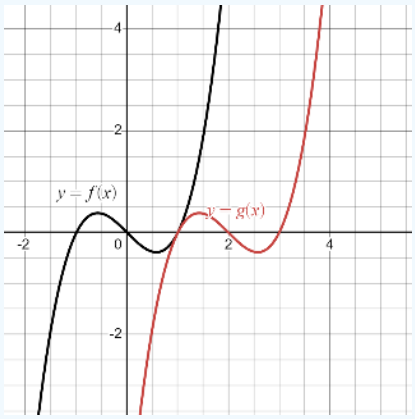


2. This one is slightly different, but if we consider that $8 = 2^3$, we can see that $g(x) = f(2x)$. If you go through f and replace each x with $(2x)$, you get g. This causes a horizontal change, and since $2 > 1$, we are looking at a horizontal shrink by a factor of $\frac{1}{2}$.



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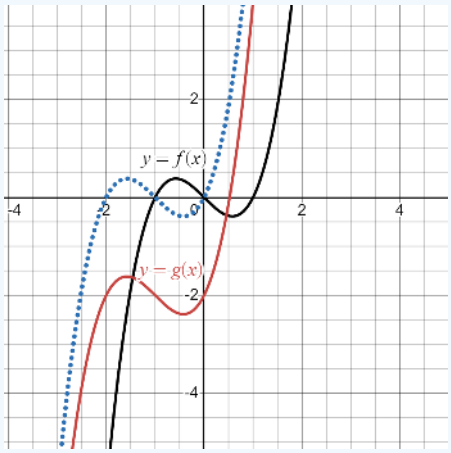
3. In this one, we notice that $(x-h)$ type expression and ask ourselves if this could be a horizontal shift. With a little algebra, we can see that $g(x) = (x-2)^3 - (x-2) = f(x-2)$. This gives us a horizontal shift to the right by 2.



4. We see that $(x+h)$ expression and try to do the same algebra trick to see a horizontal shift situation, but we get $g(x) = (x+1)^3 - (x+1) - 2$. Well that's okay, we just have a two transformations going on here. First, a horizontal shift left by 1, and then a vertical shift down by 2.

$$f(x) = x^3 - x \quad \longrightarrow \quad f(x+1) = (x+1)^3 - (x+1)$$
$$\longrightarrow f(x+1) - 2 = (x+1)^3 - (x+1) - 2 = g(x)$$

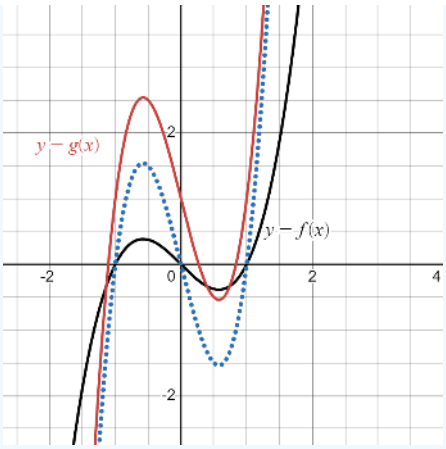
In the graph below, the "intermediate" step after just the horizontal shift is shown in blue.



5. This one looks like multiple steps again. We get g by taking f, multiplying it as a whole by 4, and then adding a 1. This means a vertical stretch by a factor of 4, followed by a vertical shift up by 1.

$$f(x) = x^3 - x \quad \longrightarrow \quad 4f(x) = 4(x^3 - x) \quad \longrightarrow \quad 4f(x) + 1 = 4(x^3 - x) + 1 = g(x)$$

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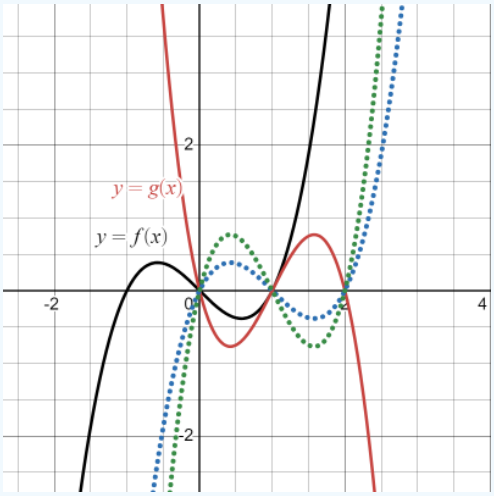


6. We pick this apart from the inside out as well. The $(x-h)$ expressions are clues for a horizontal shift to the right. Then the entire result was multiplied by 2, causing a vertical stretch.

$$f(x)=x^3-x \quad \longrightarrow \quad f(x-1)=(x-1)^3-(x-1) \quad \longrightarrow \quad 2f(x-1)=2((x-1)^3-(x-1))$$

Finally, the entire result was made negative, which causes a reflection over the x-axis.

$$\cdots \longrightarrow \quad -2f(x-1)=-2((x-1)^3-(x-1))=g(x)$$



Exercise 2

Describe the transformation(s) performed on $f(x)=\sqrt{x}$ to produce the following functions. If desired, graph both functions using Desmos to reinforce your answers like I did above.

1. $g(x)=\sqrt{x+2}$
2. $g(x)=4\sqrt{x}$
3. $g(x)=-2\sqrt{x-1}$
4. $g(x)=\sqrt{-x}+3$
5. $g(x)=\sqrt{\frac{1}{2}x-1}$

Answer

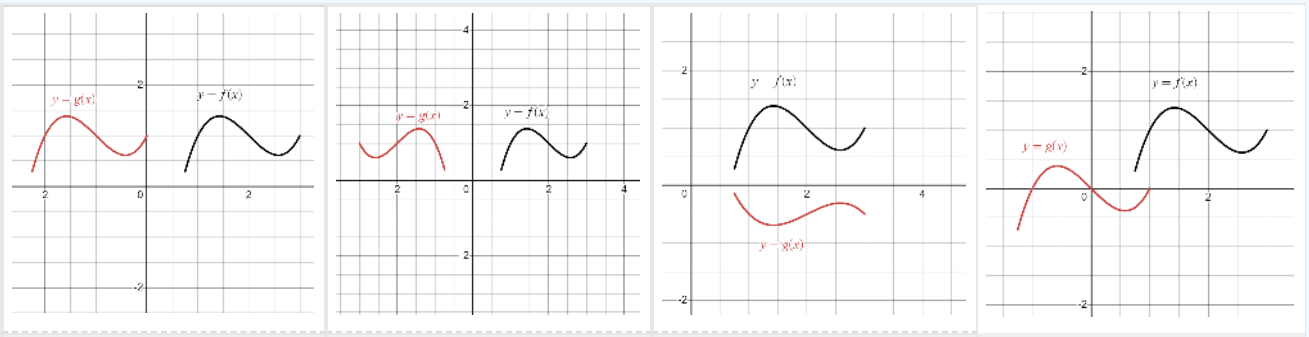
1. Horizontal shift to the left by 2.
2. Vertical stretch by a factor of 4.
3. Horizontal shift to the right by 1, then vertical stretch by 2, then reflection over the x-axis.
4. Reflection over the y-axis, then vertical shift up by 3.

5. Horizontal shift right by 1, THEN horizontal stretch by a factor of 2. Watch out, if you mix up the order, you will get a different function!

$$\sqrt{x} \longrightarrow \sqrt{\frac{1}{2}x} \longrightarrow \sqrt{\frac{1}{2}(x-1)} \quad \neq \quad \sqrt{x} \longrightarrow \sqrt{x-1} \longrightarrow \sqrt{\frac{1}{2}x-1}$$

Exercise 3

Describe the transformation(s) applied to the black graph to produce the red graph. Hint: pick a point or two and try to see how they in particular moved and changed. Write the formula for $g(x)$ depending on $f(x)$.



1. 2. 3. 4.

Answer

1. Horizontal shift to the left by 3 (look at the right endpoint, who was at an x-value of 3 but is now at an x-value of 0. We have $g(x) = f(x+3)$.
2. Reflection across the y-axis. We have $g(x) = f(-x)$.
3. Vertical shrink by a factor of $\frac{1}{2}$ and reflection across the x-axis. We have $g(x) = -\frac{1}{2}f(x)$.
4. Horizontal shift to the left by 2 and vertical shift down by 1. We have $g(x) = f(x+2) - 1$.

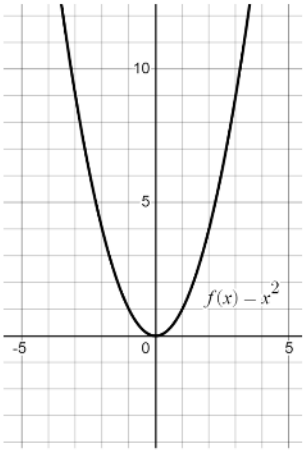
Even and Odd Functions

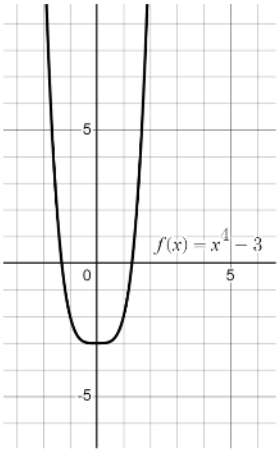
Now that we know about reflections of graphs across various axes, we can discuss some special types of functions.

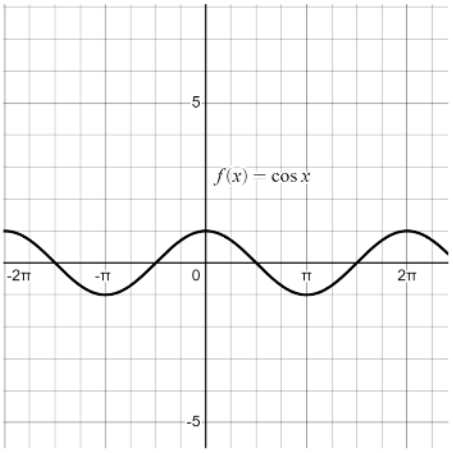
Even and Odd Functions

- A function f is called an **even function** if $f(-x) = f(x)$. Aka, reflecting its graph across the y-axis changes nothing. Aka aka, its graph is symmetric about the y-axis.
- A function f is called an **odd function** if $f(-x) = -f(x)$. Aka, reflecting its graph across the y-axis has the same effect as reflecting it across the x-axis. Aka aka, the graph is symmetric about the origin. Aka aka aka, the graph has 180° rotational symmetry.
- A function need not be one or the other! A function can be **neither** even nor odd.

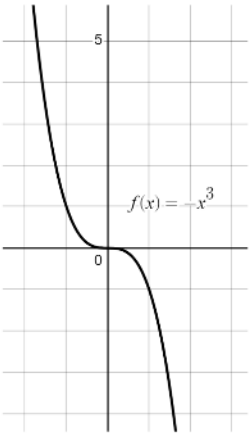
Some examples of even functions and their graphs:

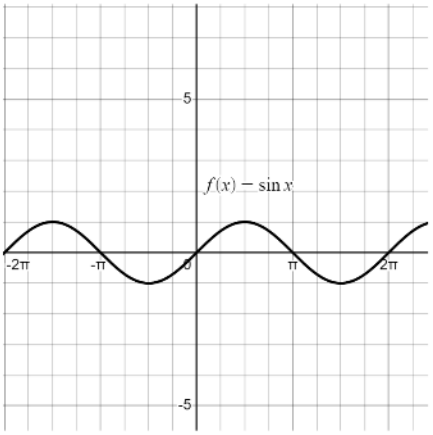


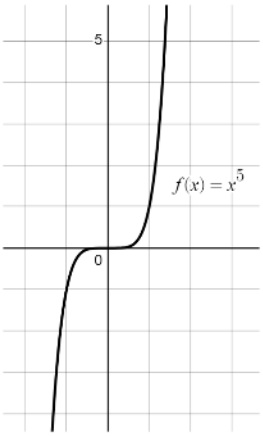




Some examples of odd functions and their graphs:







? Exercise 4

Confirm the examples by checking the defining conditions algebraically.

1. $f(x) = x^2$ is an even function.
2. $f(x) = x^4 - 3$ is an even function.
3. $f(x) = -x^3$ is an odd function.
4. $f(x) = x^5$ is an odd function.

Answer

1. We plug in $-x$ as the input and see if the result simplifies to the original function again.

$$f(-x) = (-x)^2 = x^2 = f(x) \quad \checkmark$$

2. Same thing.

$$f(-x) = (-x)^4 - 3 = x^4 - 3 = f(x) \quad \checkmark$$

3. This time we're looking for the result to simplify to the opposite of the whole original function.

$$f(-x) = -(-x)^3 = -(-x^3) = -f(x) \quad \checkmark$$

4. Same thing.

$$f(-x) = (-x)^5 = -x^5 = -f(x) \quad \checkmark$$

The other two examples, $\sin x$ and $\cos x$, will be dealt with in detail in Chapter 8. Right now you can just look at the symmetry in their graphs. :)

Sketching Graphs of Functions

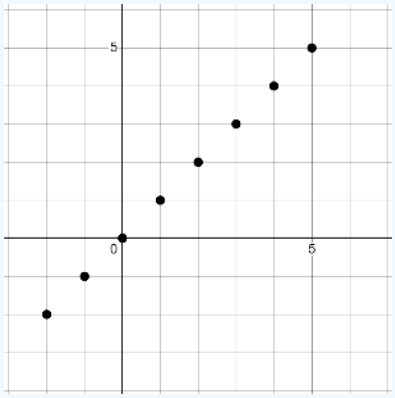
Later on as we study specific types of functions in detail, you will learn about the shapes of different polynomial functions, exponential and logarithmic functions, etc. etc. The power of the transformations above (which I use all the dang time when teaching Calc I, actually) is the ability to immediately sketch a function of interest from my own head and get some visual intuition. For now, we will sketch graphs just based on checking a few points and thinking about transformations when convenient. I'm going to demonstrate a challenge problem in the next example, but don't panic—the exercise afterward is much chiller.

Example 4

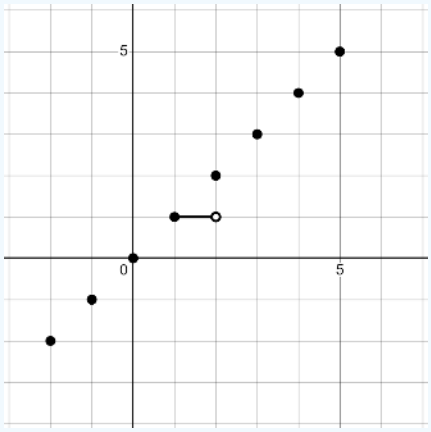
The **floor function,** denoted $\lfloor x \rfloor$, takes a real number x as input and spits out the largest integer less than or equal to it. Aka, it takes a real number and, if it's an integer, leaves it the same, but if it's between integers, rounds it down always to the lower integer. Sketch a graph of $f(x) = \lfloor x \rfloor$ and then use it to sketch a graph of $g(x) = \lfloor x \rfloor + 1$.

Solution

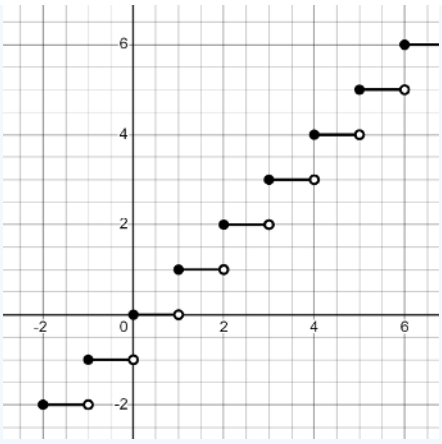
I start by considering some points to get a sense of what this function does. For example, if $x = 0$, that's already an integer. The function will spit out $y = 0$. Likewise for other integers. I could plot some of those...



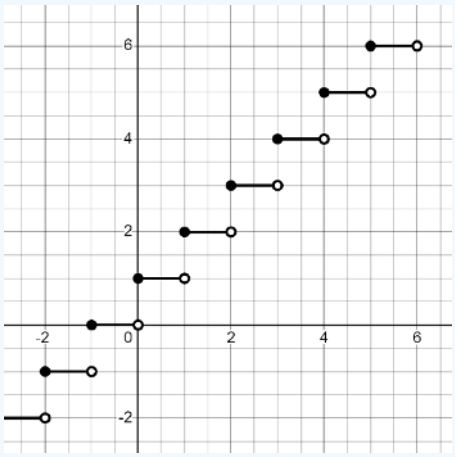
Now I ask myself what's happening in between? For $x = 1.5$, the floor function will round down to 1 and spit that out. Same for $x = 1.25, 1.75, 1.9999, \ldots$ and everything in between 1 and 2, BUT as soon as $x = 2$, the function jumps up to the point $(2, 2)$ already plotted. I draw this on the graph like so:



The same phenomenon will happen between the other integers, hanging out as a flat line until it jumps to the next level. So the graph looks like this:



I recognize that going from $f(x) = \lfloor x \rfloor$ to $g(x) = \lfloor x \rfloor + 1$ just took a vertical shift up by 1. So I just take every point on my graph and imagine sliding it directly upward by 1 unit. This allows me to sketch the graph of g below.

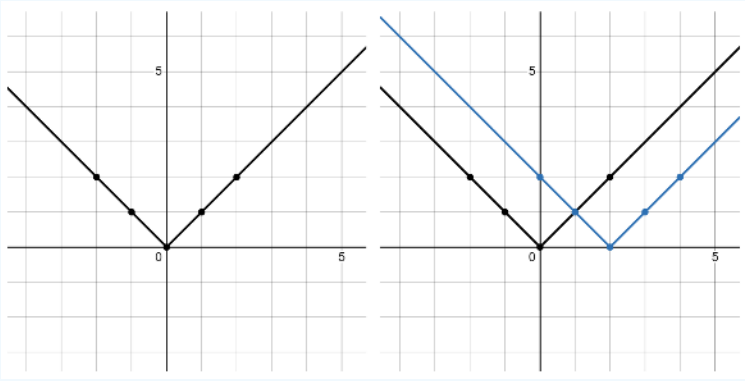


Exercise 5

Sketch a graph of the absolute value function $f(x) = |x|$, and use it to sketch the graph of $g(x) = |x - 2|$.

Answer

Plot a few points and recognize the shape will use straight lines. The g is obtained from f by a horizontal shift to the right by 2. Your graphs should look like this (I don't care if you plot them both on the same plane like I did in the second image):



In studying transformations, we have technically been making functions out of other functions by performing additions, subtractions, and multiplications by scalars. Next, we're going to study how to make new functions from other functions by

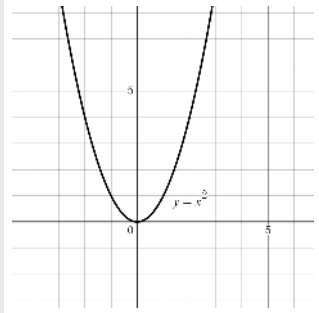
combining them in various ways!

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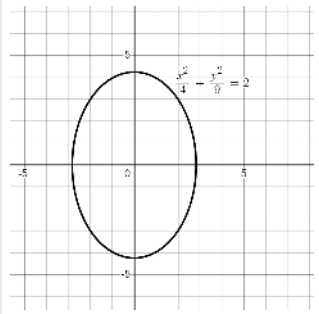
4.2E Exercises

Vertical Line Test

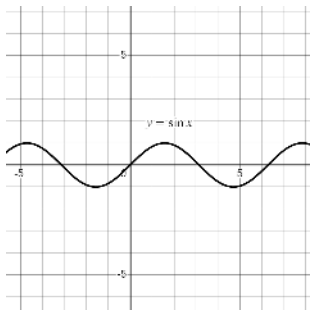
Tell whether the following graphs could represent functions.



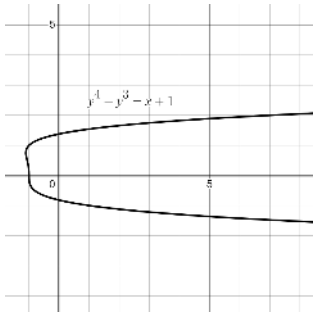
1.



2.



3.



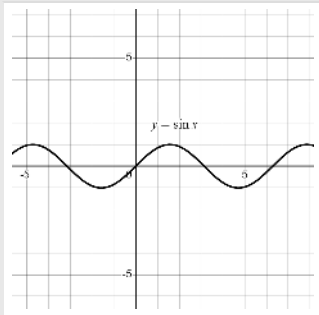
4.

Answer

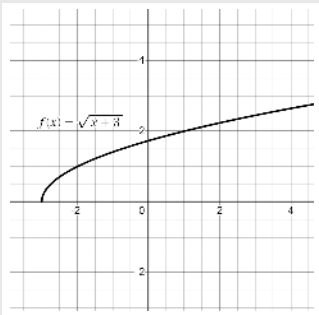
1. Yes
2. No
3. Yes
4. No

Domain and Range From Graphs

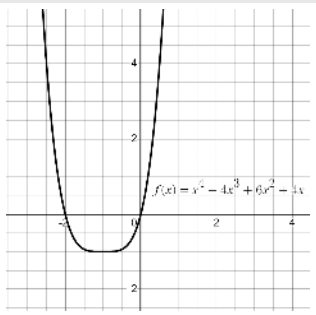
Give the domain and range of the following graphed functions.



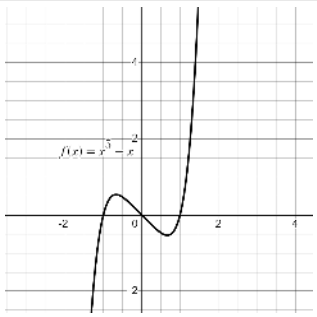
1.



2.



3.



4.

Answer

1. Dom: $(-\infty, \infty)$, Range: $[-1, 1]$
2. Dom: $[-3, \infty)$, Range: $[0, \infty)$
3. Dom: $(-\infty, \infty)$, Range: $[-1, \infty)$
4. Dom: $(-\infty, \infty)$, Range: $(-\infty, \infty)$

Describing Transformations

Describe the transformation(s) done to $f(x) = x^2$ to produce the following functions.

1. $g(x) = -2x^2$
2. $g(x) = x^2 + 3$
3. $g(x) = \frac{1}{4}(x-1)^2$

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4. $g(x) = (x+2)^2 - 4$
5. $g(x) = -(2x+3)^2$

Answer

1. Vertical stretch by a factor of 2 and reflection across the x-axis.
2. Vertical shift up by 3.
3. Horizontal shift right by 1, then vertical shrink by a factor of $\frac{1}{4}$. (Or, horizontal stretch by a factor of 2, and then horizontal shift right by 1.)
4. Horizontal shift left by 2 and vertical shift down by 4.
5. Horizontal shift left by 3, then horizontal shrink by a factor of $\frac{1}{2}$, then reflect over the x-axis. (Or, horizontal shrink by a factor of $\frac{1}{2}$, then a horizontal shift left by $\frac{3}{2}$, then reflect over x-axis.)

Producing New Functions

Write the function $g(x)$ obtained from $f(x) = x^2$ by performing the transformations in order.

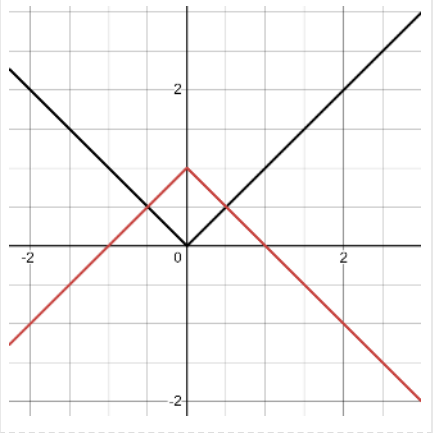
1. Horizontal shift right by 1 and vertical shift down by 3.
2. Vertical stretch by a factor of 3, then reflect across the x-axis, then shift vertically up by 2.
3. Reflect across the y-axis, then horizontal shift left by 2, then horizontal stretch by a factor of 3.

Answer

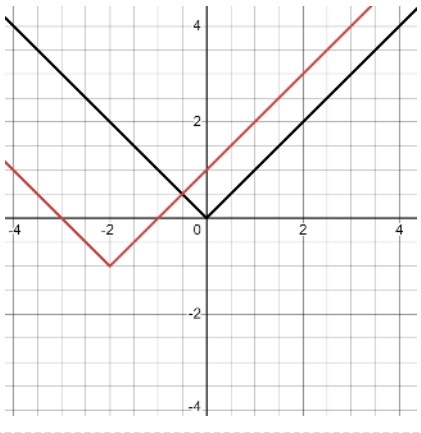
1. $g(x) = (x-1)^2 - 3$
2. $g(x) = -3x^2 + 2$
3. $g(x) = \left(\frac{1}{3}x + 2\right)^2$ (Note that the first step had no effect, because $(-x)^2 = x^2$. This function is symmetric about the y-axis.)

Describing Transformations From Graphs

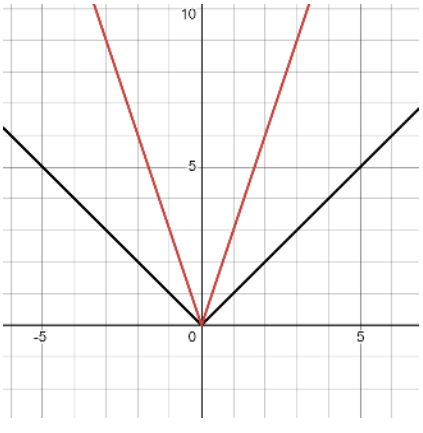
Describe the transformations that changed the black graph into the red graph.



1.



2.



3.

Answer

1. Reflect across the y-axis, then shift up by 1.
2. Shift left by 2 and down by 1.
3. Vertical stretch by a factor of 3. (Or horizontal shrink by a factor of $\frac{1}{3}$. This can be seen by comparing the points $(1,1)$ on the black graph and $(1,3)$ on the red graph. This is a picture of $f(x) = |x|$ and $g(x) = |3x| = 3|x|$, so it can be seen as vertical stretch or a horizontal shrink, equivalently.

Even and Odd Functions

Determine whether the function is even, odd, or neither.

1. $f(x) = x^5 - x$
2. $p(x) = x^3 + 1$
3. $g(x) = 2x + x^2$
4. $f(x) = x^7$
5. $h(x) = x^4 - 4x^2$
6. $f(x) = |x|$

Answer

1. Odd
2. Neither
3. Neither
4. Odd
5. Even
6. Even

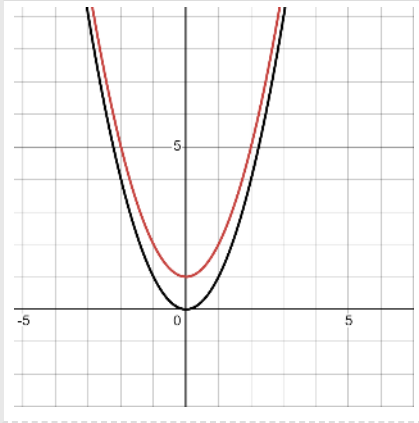
Sketching Graphs of Functions

Sketch the graphs of the functions by plotting points and connecting the dots, and then use them to sketch the graphs of the transformed functions.

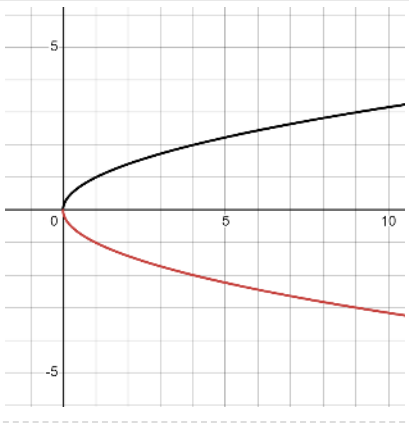
1. $f(x) = x^2$ becoming $g(x) = x^2 + 1$
2. $f(x) = \sqrt{x}$ becoming $g(x) = -\sqrt{x}$
3. $f(x) = 3x + 1$ becoming $g(x) = -(3x + 1)$

Answer

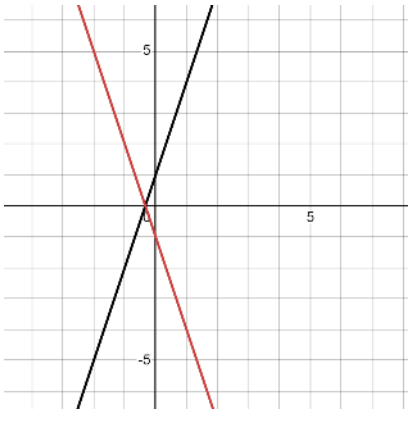
1. Plot the points $(0, 0), (1, 1), (-1, 1), (2, 4), (-2, 4), (3, 9)$, etc. Connect the dots smoothly. Then use that to sketch $g(x)$.
2. Plot the points $(0, 0), (1, 1), (4, 2), (9, 3)$, etc. Connect the dots smoothly. Use that to sketch $g(x)$.
3. This is a straight line, so sketch as usual by plotting y-intercept and following the slope. Connect the dots with a straight line. Use that to sketch $g(x)$.



1.



2.



3.

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4.3 Compositions of Functions

Learning Objectives

By the end of this section, you will be able to:

- produce new functions from other functions via addition, subtraction, multiplication, and division
- produce new functions by composing other functions
- decompose functions into constituent parts

They say there's no such thing as one cockroach, and the same is true of functions (ew). What I'm saying is, as soon as you have a couple o' functions, they can produce more just by combining them in different ways. These are the basic ones, real quick...

Given two functions, f and g, you can make:

- their **sum,** a new function named "$f+g$," defined by $(f+g)(x)=f(x)+g(x)$.
- their **difference,** a new function named "$f-g$," defined by $(f-g)(x)=f(x)-g(x)$. (ALERT: subtract the WHOLE function g, using parentheses!)
- their **product,** a new function named "fg," defined by $(fg)(x)=f(x)g(x)$. (ALERT: if the functions have multiple terms, this would require expanding! Again, use parentheses!)
- their **quotient,** a new function named "$\frac{f}{g}$," defined by $\left(\frac{f}{g}\right)(x)=\dfrac{f(x)}{g(x)}$. (ALERT: any x such that $g(x)=0$ is not allowed in the domain of this new function!)

These functions are defined on domains where both f and g are defined, so we say the domain is the intersection of the domains of f and g. The domain of the quotient of the functions is further restricted because you can't divide by zero.

Example 1

For $f(x)=x^2$ and $g(x)=\frac{1}{x-1}$, find the functions and tell their domains.

1. $(f+g)(x)$
2. $(fg)(x)$
3. $\left(\frac{g}{f}\right)(x)$

Solution

1. $(f+g)(x)=f(x)+g(x)=(x^2)+\left(\frac{1}{x-1}\right)=x^2+\frac{1}{x-1}$. This function's domain is $\{x\mid x\neq 1\}$.
2. $(fg)(x)=f(x)g(x)=(x^2)\left(\frac{1}{x-1}\right)=\frac{x^2}{x-1}$. This function's domain is $\{x\mid x\neq 1\}$.
3. $\left(\frac{g}{f}\right)(x)=\dfrac{g(x)}{f(x)}=\dfrac{\frac{1}{x-1}}{x^2}=\dfrac{1}{x^2(x-1)}$. This function's domain is $\{x\mid x\neq 0,1\}$.

Exercise 1

For $f(x)=\sqrt{x}$ and $g(x)=x+2$, find the functions and tell their domains.

1. $(f+g)(x)$
2. $(f-g)(x)$
3. $(fg)(x)$
4. $\left(\frac{f}{g}\right)(x)$

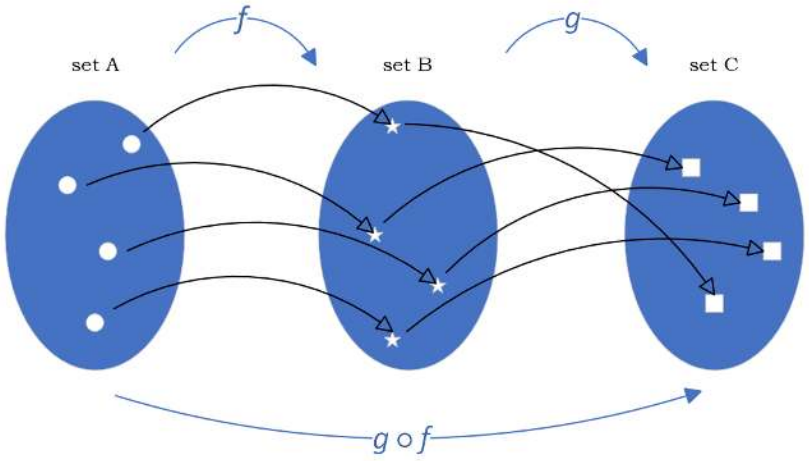
Answer

1. $(f+g)(x)=\sqrt{x}+x+2$. Domain: $x\geq 0$.
2. $(f-g)(x)=\sqrt{x}-x-2$. Domain: $x\geq 0$.
3. $(fg)(x)=\sqrt{x}(x+2)=x\sqrt{x}+2\sqrt{x}$. Domain: $x\geq 0$.

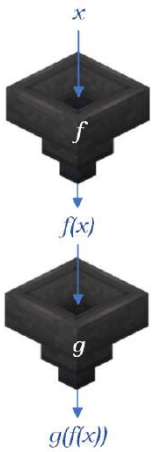
4. $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{x+2}$. Domain: $x \geq 0$. (The problematic $x = -2$ is a non-issue because we're already not allowing negative numbers.)

Compositions of Functions

Recall that way of thinking of functions as maps from one set of elements to another. Let's say I have a function f with domain A mapping to a set B where his outputs live. Then say that the set B is in fact the set of *inputs* for another function, g, who maps to yet another set C. Then I could chain up these two functions, first following f and then following g, to find a way of assigning an output from C to each input in A. Aka, defining a new function. This daisychain function is called the **composition** of g and f, and denoted $g \circ f$.



If you prefer the machine analogy, it's this process:



Composing a function g with another function f results in the **composite function,** $g \circ f$**,** defined by $(g \circ f)(x) = g(f(x))$. This function is defined for x values where both $f(x)$ and $g(f(x))$ are defined.

You can compose two functions in either order to get new functions, and you can even compose a function with himself...and then compose that with himself again... The sky is the limit.

Example 2

Given $f(x) = x^2$ and $g(x) = \frac{1}{x-1}$, find the functions and their domains.

1. $(g \circ f)(x)$
2. $(f \circ g)(x)$
3. $(f \circ f)(x)$
4. $(g \circ f \circ f)(x)$

Solution

1. We know that $(g \circ f)(x) = g(f(x))$. We are going to **work from the inside outward** just applying the definitions of these functions. We replace $f(x)$ with his definition first. Then we treat him as the input that we plug into the definition of g.

$$(g \circ f)(x) = g(f(x)) = g(x^2) = \frac{1}{(x^2)-1} = \frac{1}{x^2-1}$$

The domain of this function is all real numbers but $x \neq \pm 1$.

2. We repeat the same process, just being careful with who goes where. You will see that the final answer is NOT the same as the previous one! With composition, order matters!

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x-1}\right) = \left(\frac{1}{x-1}\right)^2$$

The domain of this function is all real numbers but $x \neq 1$.

3. This time, we are composing f with another copy of itself.

$$(f \circ f)(x) = f(f(x)) = f(x^2) = (x^2)^2 = x^4$$

The domain is $\mathbb{R}$.

4. You can compose more than just two things. First, convince yourself that $(g \circ f \circ f)(x) = g(f(f(x)))$. Then work from the inside outward.

$$g(f(f(x))) = g(f(x^2)) = g((x^2)^2) = g(x^4) = \frac{1}{x^4-1}$$

The domain is all real numbers but $x \neq \pm 1$.

Exercise 2

For $f(x) = \sqrt{x}$ and $g(x) = x + 2$, find the functions and tell their domains.

1. $(g \circ f)(x)$
2. $(f \circ g)(x)$
3. $(f \circ f)(x)$
4. $(g \circ f \circ f)(x)$

Answer

1. $(g \circ f)(x) = \sqrt{x} + 2$. Domain: $x \geq 0$.
2. $(f \circ g)(x) = \sqrt{x+2}$. Domain: $x \geq -2$.
3. $(f \circ f)(x) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$. (Recall that you can write these roots as $(x^{1/2})^{1/2}$ and simplify with power rules.) Domain: $x \geq 0$.
4. $(g \circ f \circ f)(x) = \sqrt[4]{x} + 2$. Domain: $x \geq 0$.

Decomposing Functions

It's also important for our critical thinking and function intuition skills to be able to identify what constituents could have been composed to produce a complicated function. As always, I try to only make you learn things that are going to actually serve you later in math. This decomposing skill, like the basic skill of understanding the order of operations in a function, will come in handy

when you do something called chain rule and u-substitution in various Calculuses. Tbh, though, this is no biggie if you just remember how to describe operations in English.

Example 3

Identify two functions f and g such that $h = f \circ g$.

1. $h(x) = \sqrt{x+1}$
2. $h(x) = (x-2)^2 + x - 2$
3. $h(x) = (x^2 + 3x + 1)^3$
4. $h(x) = |x-1|$
5. $h(x) = \sqrt{1+\sqrt{x}}$

Solution

1. A good place to start is to ask myself what happened over the course of this function? Looks like an input was added to 1, and then the square root was taken. Hone in on the "and then" in that sentence. That's a great place to split up the action. If I take $g(x) = x + 1$, the function "add 1 to something," and $f(x) = \sqrt{x}$, the function "take the square root of something," then the composition is

$$(f \circ g)(x) = f(g(x)) = f(x+1) = \sqrt{x+1} \quad = h(x) \quad \checkmark$$

2. In this one, I notice something interesting going on: $h(x) = (x-2)^2 + (x-2)$. It kinda looks like the action "subtract 2 from something" has been inserted into another action, "square something and add it to itself." If I take $g(x) = x - 2$ and $f(x) = x^2 + x$, then the composition is

$$(f \circ g)(x) = f(g(x)) = f(x-2) = (x-2)^2 + (x-2) \quad = h(x) \quad \checkmark$$

3. Here, the process is "plug an input into a certain quadratic polynomial, and then cube the whole thing." I take $g(x)$ to be the part before the "and then," so $g(x) = x^2 + 3x + 1$. Then I take the second step for f, getting $f(x) = x^3$. Then the composition is

$$(f \circ g)(x) = f(g(x)) = f(x^2 + 3x + 1) = (x^2 + 3x + 1)^3 \quad = h(x) \quad \checkmark$$

4. The process is, "subtract 1 from the input, and then take the absolute value of the result." We use $g(x) = x - 1$ and $f(x) = |x|$. Then the composition is

$$(f \circ g)(x) = f(g(x)) = f(x-1) = |x-1| \quad = h(x) \quad \checkmark$$

5. Sometimes you actually have a few choices on which pairs of functions to choose. I describe this process, "Take the square root, then add 1, then take the square root again." I could break up my functions at either "then" in the sentence. One option is to have $g(x) = 1 + \sqrt{x}$ and then $f(x) = \sqrt{x}$. Check this composition to make sure it matches $h(x)$. Another option is to have $g(x) = \sqrt{x}$ and $f(x) = \sqrt{1+x}$. Check this one too to confirm!

Exercise 3

Identify two functions f and g such that $h = f \circ g$.

1. $h(x) = 2x + 1$
2. $h(x) = (x+2)^2 - 3$
3. $h(x) = \dfrac{1}{1+\sqrt{x}}$
4. $h(x) = \sqrt{1-x^2}$

Answer

YMMV, but these are some options:

1. $g(x) = 2x, f(x) = x + 1$
2. $g(x) = x + 2, f(x) = x^2 - 3$
3. $g(x) = \sqrt{x}, f(x) = \dfrac{1}{1+x}$
4. $g(x) = 1 - x^2, f(x) = \sqrt{x}$

Now be sure to go over the exercises section carefully, because we'll see some real-life applications of the composition of functions in there!

* *(Minecraft hopper image from Minecraft Wiki)*

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4.3E Exercises

Algebra With Functions

Given each pair of functions, find $f+g, f-g, fg$, and $\frac{f}{g}$, and give their domains.

1. $f(x)=x^4+x^2,\ \ g(x)=x^3+1$
2. $f(x)=\sqrt{25-x^2},\ \ g(x)=x+1$
3. $f(x)=\dfrac{x}{x+1},\ \ g(x)=x^2+2$

Answer

1.

- $(f+g)(x)=x^4+x^3+x^2+1$, dom: $(-\infty,\infty)$
- $(f-g)(x)=x^4-x^3+x^2-1$, dom: $(-\infty,\infty)$
- $(fg)(x)=x^7+x^5+x^4+x^2$, dom: $(-\infty,\infty)$
- $\left(\frac{f}{g}\right)(x)=\dfrac{x^4+x^2}{x^3+1}$, dom: $x\neq -1$

2.

- $(f+g)(x)=\sqrt{25-x^2}+x+1$, dom: $[-5,5]$
- $(f-g)(x)=\sqrt{25-x^2}-x-1$, dom: $[-5,5]$
- $(fg)(x)=(x+1)\sqrt{25-x^2}$, dom: $[-5,5]$
- $\left(\frac{f}{g}\right)(x)=\dfrac{\sqrt{25-x^2}}{x+1}$, dom: $\{x\mid -5\leq x\leq 5, x\neq -1\}$ or $[-5,-1)\cup(-1,5]$

3.

- $(f+g)(x)=\dfrac{x}{x+1}+x^2+2$, dom: $x\neq -1$
- $(f-g)(x)=\dfrac{x}{x+1}-x^2-2$, dom: $x\neq -1$
- $(fg)(x)=\dfrac{x(x^2+2)}{x+1}$, dom: $x\neq -1$
- $\left(\frac{f}{g}\right)(x)=\dfrac{x}{(x+1)(x^2+2)}$, dom: $x\neq -1$

Composing Functions

Given each pair of functions, find $f\circ g, g\circ f, f\circ f$, and $g\circ g$. Simplify if reasonable.

1. $f(x)=x^4+x^2,\ \ g(x)=x^2+1$
2. $f(x)=\sqrt{25-x^2},\ \ g(x)=x+1$
3. $f(x)=\dfrac{x}{x+1},\ \ g(x)=x^2-1$

Answer

1.

- $(f\circ g)(x)=(x^2+1)^4+(x^2+1)^2$
- $(g\circ f)(x)=(x^4+x^2)^2+1$
- $(f\circ f)(x)=(x^4+x^2)^4+(x^4+x^2)^2$
- $(g\circ g)(x)=(x^2+1)^2+1=x^4+2x^2+2$

2.

- $(f\circ g)(x)=\sqrt{25-(x+1)^2}$
- $(g\circ f)(x)=\sqrt{25-x^2}+1$
- $(f\circ f)(x)=\sqrt{25-(\sqrt{25-x^2})^2}=\sqrt{25-25+x^2}=\sqrt{x^2}=|x|$
- $(g\circ g)(x)=x+2$

3.

- $(f \circ g)(x) = \dfrac{x^2-1}{x^2} = 1 - \dfrac{1}{x^2}$
- $(g \circ f)(x) = \left(\dfrac{x}{x+1}\right)^2 - 1$
- $(f \circ f)(x) = \dfrac{\frac{x}{x+1}}{\frac{x}{x+1}+1} = \ldots = \dfrac{x}{2x+1}$ (Hint: get a common denominator in the big denominator.)
- $(g \circ g)(x) = (x^2-1)^2 - 1 = x^4 - 2x^2$

Composing Three Functions

Given three functions f, g, and h, find $h \circ g \circ f$.

1. $f(x) = x^2,\ \ g(x) = x+4,\ \ h(x) = \sqrt{x}$
2. $f(x) = x-1,\ \ g(x) = x^3,\ \ h(x) = \dfrac{1}{x}$
3. $f(x) = 2,\ \ g(x) = \sqrt{x+1},\ \ h(x) = x^2$

Answer

1. $(h \circ g \circ f)(x) = \sqrt{x^2+4}$
2. $(h \circ g \circ f)(x) = \dfrac{1}{(x-1)^3}$
3. $(h \circ g \circ f)(x) = (\sqrt{2+1})^2 = 3$

Decomposing Functions

Find two functions f and g such that $h = f \circ g$.

1. $h(x) = (x-3)^5$
2. $h(x) = \dfrac{x^2}{x^2+1}$
3. $h(x) = (x-1)^3 + (x-1)^2 + 3$

Answer

1. $g(x) = x-3, f(x) = x^5$
2. $g(x) = x^2, f(x) = \dfrac{x}{x+1}$
3. $g(x) = x-1, f(x) = x^3 + x^2 + 3$

Volume of a Balloon

If I'm blowing up a spherical balloon, then the radius of the balloon is growing as time goes on. Aka, I can see the radius r **as a function of** time t. Let's say that the radius is growing at a rate of 3 inches per second, so the function giving the radius at time t is $r(t) = 3t$. Meanwhile, I see volume V as a function of radius r with the formula $V(r) = \frac{4}{3}\pi r^3$. Use composition to write volume as a function of time, $V(t)$. (This type of idea will come up when you learn about **related rates** in Calc I.)

Answer

Volume V depends on radius r, which itself depends on time t. I can daisychain $V(r) = V(r(t)) = (V \circ r)(t)$ to have a function that will take a time t as input and give a volume $V(t)$ as output. With these formulas, we have

$$V(r(t)) = V(3t) = \frac{4}{3}\pi(3t)^3 = 36\pi t^3$$

Water Filling a Tank

Say water is pouring into a cylindrical tank such that the height of the water level is increasing by 5 inches per minute. At time $t = 0$, the tank started out empty, and the cylinder's base has radius 15 inches. Write a function that expresses the volume of water in the tank as a function of time. (Recall the volume of a cylinder can be computed using the radius and the height, $V = \pi r^2 h$.)

Answer

We have $h(t) = 5t$ and $V(h) = \pi(15)^2 h = 225\pi h$. Then $V(t) = V(h(t)) = 225\pi(5t) = 1125\pi t$.

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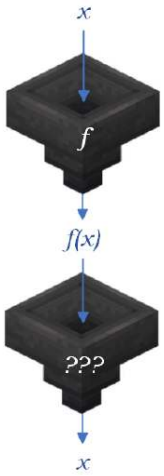
4.4 Inverse Functions

Learning Objectives

By the end of this section, you will be able to:

- identify one-to-one functions using the Horizontal Line Test
- find the inverse of a one-to-one function and check your answer with composition
- assess the domain and range of inverse functions

Hearken back to when we talked about inverse operations that undo each others' effects, such as addition/subtraction, or multiplication/division. Well, we ponder the same idea when it comes to functions. We ask ourselves, "Self? How could I feed an input x into f, and then take the output and throw that directly into another machine, such that the final result is the same x that I started with?"



That new machine, "???", if we can find it, will be called the **inverse function** for f, and we denote it f^{-1}. Note that this does NOT mean the reciprocal $\frac{1}{f(x)}$!!! The superscript is just a decoration that means "inverse function." But the unfortunate reality is that not all functions have inverses. Only a certain type, which is called **one-to-one**.

One-to-One Functions

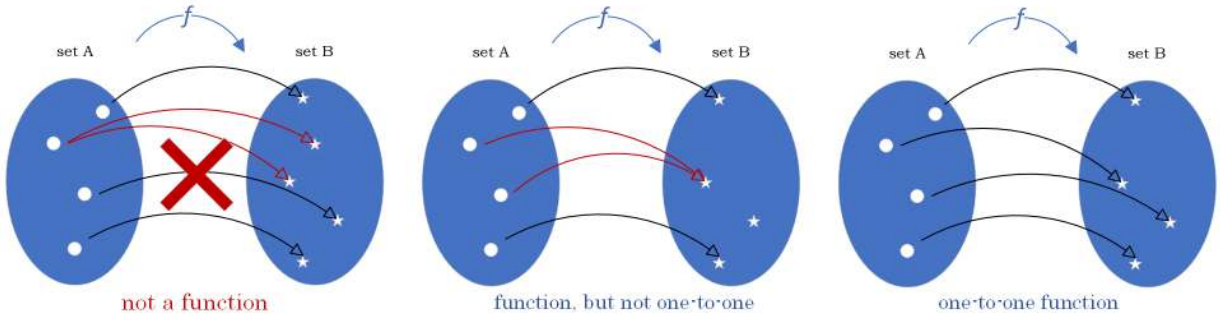
One-to-One Functions

A function f is called **one-to-one** (or **injective**, if you go on to upper level math courses later) if no two elements in its domain have the same image. Aka, **no two distinct inputs get sent to the exact same output.** In math symbols,

$$\text{if } x_1 \neq x_2, \quad \text{then } f(x_1) \neq f(x_2)$$

In general, it's okay for a function to send two inputs to the same output. For example, $f(x) = x^2$ sends both 2 and -2 to 4. The issue is, with those functions, we don't know how to reliably undo them. If I wanted an inverse function to $f(x) = x^2$, I would be looking for a way to take x^2 as an input and track it back to x. There's no way for me to know whether to assign the positive or negative version of x as the result. There's ambiguity there!

The solution is to look only at one-to-one functions. When a function is one-to-one, there is no ambiguity about which original input resulted in the output $f(x)$. With the mapping visualization of functions, the picture below shows these cases.

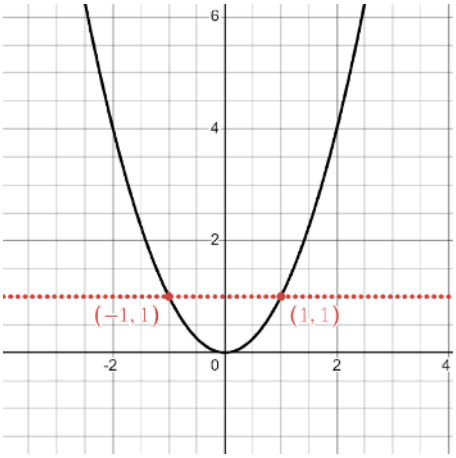


There is a handy dandy trick for assessing at a glance whether a function is one-to-one, using its graph.

Horizontal Line Test (HLT)

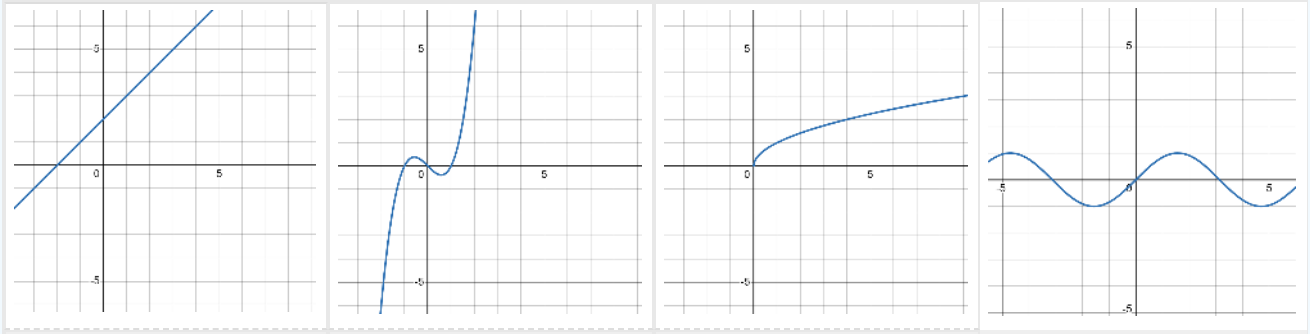
If there is **any place** on the graph of a function where you can draw a **horizontal line intersecting the graph multiple times,** then it is **not** one-to-one.

A horizontal line has the equation $y = a$ passing through some point a on the y-axis. If a graph intersects with this line, it means there is some point (x, a) where input x is sent to $f(x) = a$. If there are multiple such points, (x_1, a) and (x_2, a), that means there are two inputs sent to the same output. For example, we already realized that $f(x) = x^2$ is a problematic fave. We can see from his graph that he fails the Horizontal Line Test.



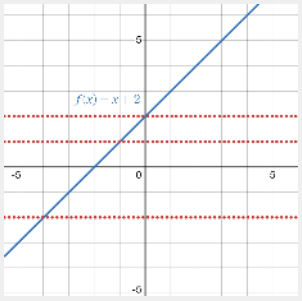
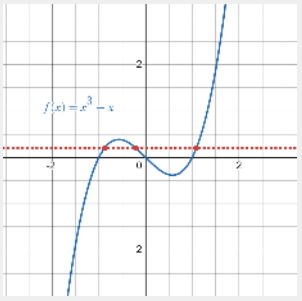
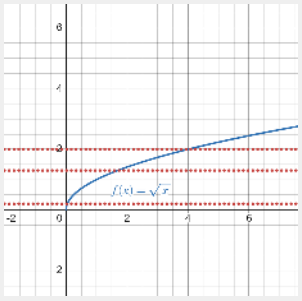
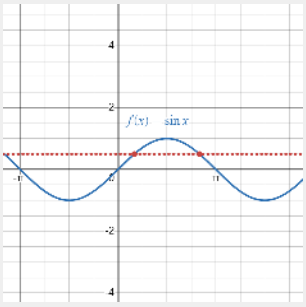
Exercise 1

Tell whether each graphed function is one-to-one or not.



1. 2. 3. 4.

Answer

1. Yes, passes HLT	2. No, fails HLT	3. Yes, passes HLT	4. No, fails HLT
			

Inverse Functions

The Inverse of a Function

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A, and its values are defined by

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y$$

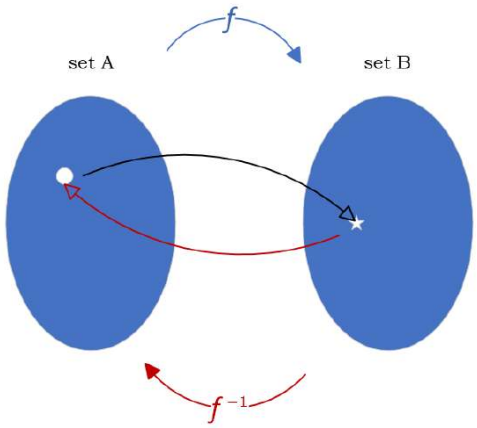
where y is in B. This means that

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x \quad \text{and} \quad (f \circ f^{-1})(y) = f(f^{-1}(y)) = y$$

for any x in A, and any y in B.

We also say that two functions with this relationship are **inverses of each other.** That is, the inverse of f^{-1} is f. In math notation, this is the statement $(f^{-1})^{-1} = f$.

In English, the inverse function takes an element y from B as input, and the function value it spits out will be the element from A that f had mapped to y. Since $f(x) = y$ is sometimes called the **image of** x **under** f**,** you might hear x called a **preimage** of y.



Example 2

Given the following table representation of a function,

x	$f(x)$
0	0
1	2

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2	4
3	6

find $f^{-1}(0), f^{-1}(2), f^{-1}(4)$, and $f^{-1}(6)$.

Solution

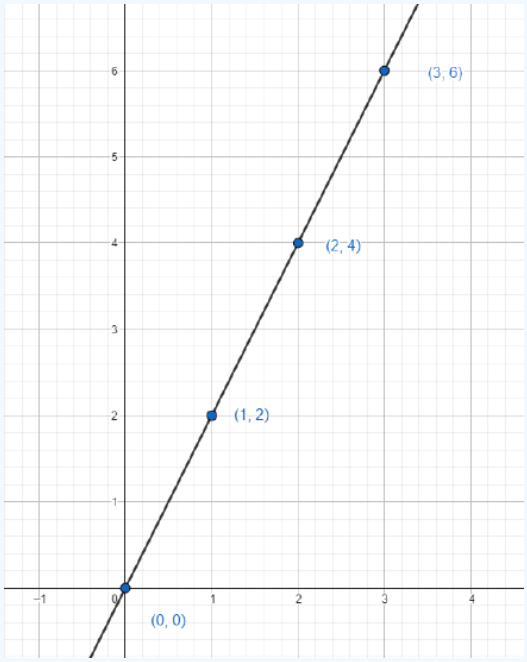
To find inverse function values, we just look *backwards* on each row of the table. So

- $f^{-1}(0) = 0$
- $f^{-1}(2) = 1$
- $f^{-1}(4) = 2$
- $f^{-1}(6) = 3$

The function from the table looks like $f(x) = 2x$, "multiply something by 2," and we notice that the inverse function looks like the rule "divide something by 2," aka $f^{-1}(x) = \frac{x}{2}$. This checks out, because the inverse operation of multiplication is division.

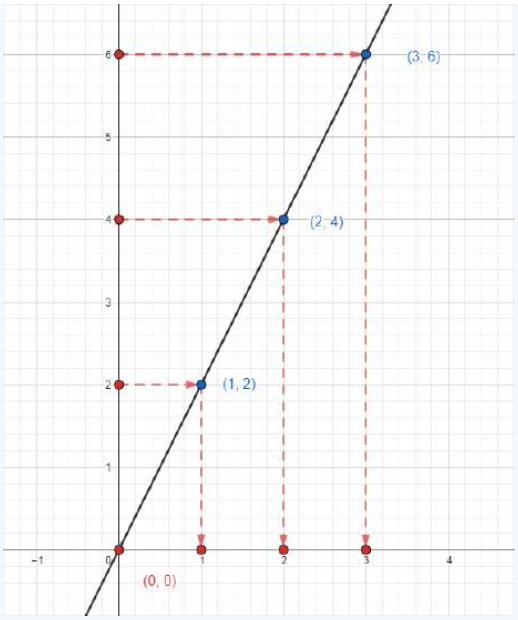
Example 2

Given the graphed function $f(x)$, find the values $f^{-1}(0), f^{-1}(2), f^{-1}(4)$, and $f^{-1}(6)$.



Solution

To read the values of the inverse function off of a graph of f, just start with the y-value of interest, track it over to the graph, and then track downward to the x-axis to see which input value went with him. This is the same function as the previous example, so use the graph to confirm the answers.



Example 3

Confirm algebraically that $f(x) = 2x + 1$ and $f^{-1}(x) = \dfrac{x-1}{2}$ are inverses of each other.

Solution

The domain and range of both of these functions is $\mathbb{R}$. We want to confirm that these functions undo each other by checking that $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(x) = x$, for all real numbers x. We compute:

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(2x+1) = \frac{(2x+1)-1}{2} = \frac{2x}{2} = x \quad \checkmark$$

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right) + 1 = x - 1 + 1 = x \quad \checkmark$$

Exercise 2

Confirm algebraically that $f(x) = x^3$ and $f^{-1}(x) = x^{\frac{1}{3}}$ are inverses of each other.

Answer

For all x in $\mathbb{R}$,

$$(f^{-1} \circ f)(x) = f^{-1}(x^3) = (x^3)^{\frac{1}{3}} = x \quad \checkmark$$

$$(f \circ f^{-1})(x) = f(x^{\frac{1}{3}}) = (x^{\frac{1}{3}})^3 = x \quad \checkmark$$

Given a particular function, how do you systematically figure out its inverse function? Follow this process:

To **find the inverse of a function** $f(x)$,

1. Write $y = f(x)$, aka replace the left hand side of the function definition with "y."
2. Solve this equation for x in terms of y, if possible.
3. Swap the variable names: replace all the x's with y's and vice versa.
4. Replace y with "$f^{-1}(x)$."

Example 4

Find the inverse function, if possible.

1. $f(x) = 4x - 3$
2. $f(x) = \dfrac{x+1}{x-1}$
3. $f(x) = (x+4)^2$

Solution

1. Write $y = 4x - 3$. We solve this equation for x in terms of y, getting $x = \dfrac{y+3}{4}$. Now we swap all the x's and y's to get $y = \dfrac{x+3}{4}$. We report the final answer as $f^{-1}(x) = \dfrac{x+3}{4}$.

2. Write $y = \dfrac{x+1}{x-1}$. We start solving this equation for x...

$$y = \frac{x+1}{x-1} \quad \rightarrow \quad y(x-1) = x+1 \quad \rightarrow \quad xy - y = x+1$$

At this point, bring all the x terms to one side and everything else to the other.

$$xy - x = y + 1 \quad \rightarrow \quad x(y-1) = y+1 \quad \rightarrow \quad x = \frac{y+1}{y-1}$$

Swapping the variable names, we have $y = \dfrac{x+1}{x-1}$, so the final answer is $f^{-1}(x) = \dfrac{x+1}{x-1}$.

3. We write $y = (x+4)^2$ and start trying to solve for x. The first thing we try to do is get rid of the power of 2...but when we do that, we get some ambiguity!

$$x + 4 = \pm\sqrt{y}$$

We can't get an inverse to this function! If we graphed this function, it would fail the HLT.

Exercise 3

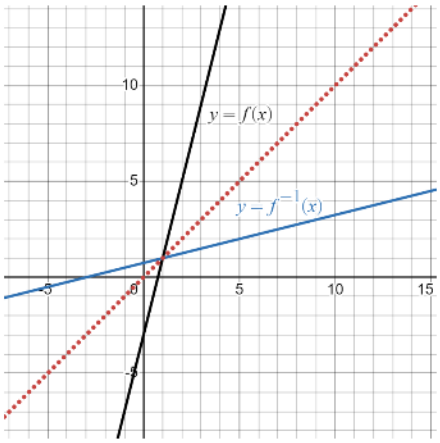
Find the inverse function, if possible.

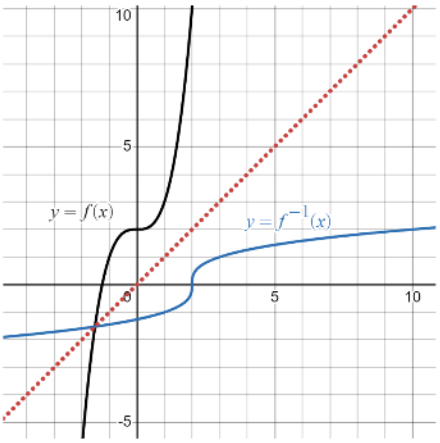
1. $f(x) = x^3 + 2$
2. $f(x) = \dfrac{2x+1}{x-3}$
3. $f(x) = -2x^2$

Answer

1. $f^{-1}(x) = \sqrt[3]{x-2}$
2. $f^{-1}(x) = \dfrac{3x+1}{x-2}$
3. This function has no inverse. I run into an issue when I try to take a square root to undo the square on the x.

Let's wrap up by noticing something interesting about the graphs of inverse functions. Below, I've graphed some functions together with their inverse functions on the same coordinate planes, as well as a supplementary dotted line $y = x$. What do you observe?





The graphs of inverse functions look like mirror images of each other when reflected over the line $y = x$. Niftyyyyy. Look at the first graph and find the point $(2, 5)$ on the black line. That set of coordinates is telling us that $f(2) = 5$. We know that the inverse function must then satisfy $f^{-1}(5) = 2$. Oh but wait, that means that the point $(5, 2)$ should appear on the graph of f^{-1}! Look at the blue line. It does indeed pass through the point $(5, 2)$. Math! Still undefeated.

Okay, that's the scoop on inverses. Check out the exercises section to see some real-world applications like the one below!

Example 5

Back in Section 4.1, we saw a function $A(r) = \pi r^2$, which calculates the area of a circle with radius r that I was cutting out of a piece of cardboard. In that physical context, we determined that the sensible domain for this function should be $0 \leq r \leq 6$. Find the range for this function. Then find the inverse function giving the radius needed for a particular area value. Then determine the domain and range of the inverse function.

Solution

The area will increase as the radius increases, so the smallest area is found using the smallest possible radius, $r = 0$ giving $A = 0$, and the largest area is found using the largest possible radius, $r = 6$ giving $A = 36\pi$. Thus the function values that are actually attained are the range $0 \leq A \leq 36\pi$. The inverse function will have that set as its domain, and we think of it as taking area values as inputs and returning the radius that would be needed to achieve that area. We can go about this using the method above, with a few notational changes that just make sense.

Write $A = \pi r^2$ and solve this equation for r. As we go along, we reach a point where we take a square root, which normally is a red flag.

$$A = \pi r^2 \quad \rightarrow \quad r^2 = \frac{A}{\pi} \quad \rightarrow \quad r = \pm\sqrt{\frac{A}{\pi}}$$

Except!!! In this example, the domain of $A(r)$ did NOT include negative numbers! So we actually have no ambiguity here, we are only taking the positive root. We have $r = \sqrt{\frac{A}{\pi}}$.

Now, here I'm going to deviate a little bit from the process described above. I don't want to swap the variable names, because these letters were chosen to describe actual real-life quantities. I want to keep calling area values A and radius values r. But I can indicate that I'm now seeing radius as a function of area by giving the answer $r(A) = \sqrt{\frac{A}{\pi}}$. The range of this function will match the domain of the original, $0 \leq r \leq 6$.

These notational changes make it easy to know how to use this function later. Say I need a circle with area 4π square inches. I just plug that area value into my new function to get $r(4\pi) = \sqrt{\frac{4\pi}{\pi}} = 2$ inches.

** (Minecraft hopper image from Minecraft Wiki)*

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4.4E Exercises

Identifying One-to-One Functions

Investigate the functions algebraically and intuitively and determine if you think they are one-to-one.

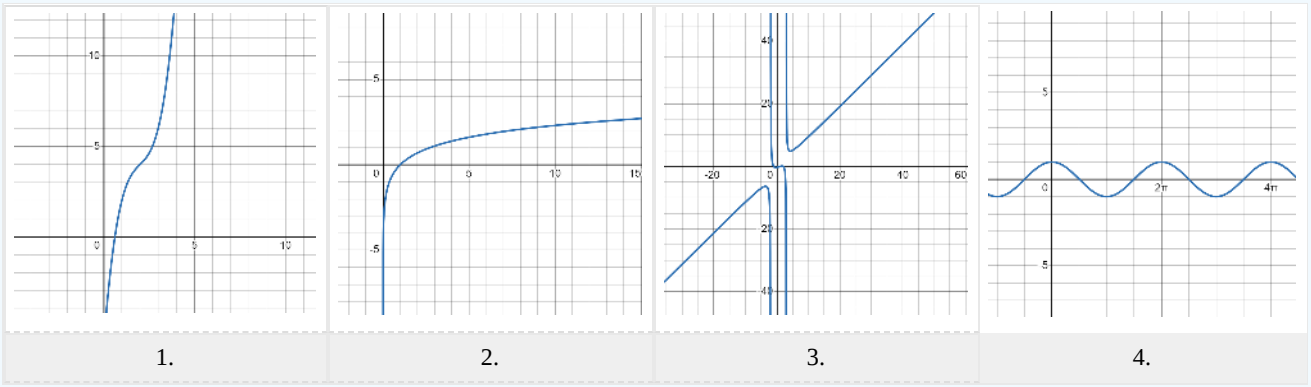
1. The floor function. (see Section 4.2)
2. The absolute value function.
3. Any linear function $f(x) = mx + b$.
4. The square root function.

Answer

1. No. For example, the floor function sends *all* real numbers $1 \leq x < 2$ to the same value, 1.
2. No. The absolute value function's graph looks like a V, which would fail the HLT.
3. Yes, any straight line graph will pass the HLT.
4. Yes, the square root function by convention only returns the positive root.

Horizontal Line Test

Determine whether the graphed functions are one-to-one.



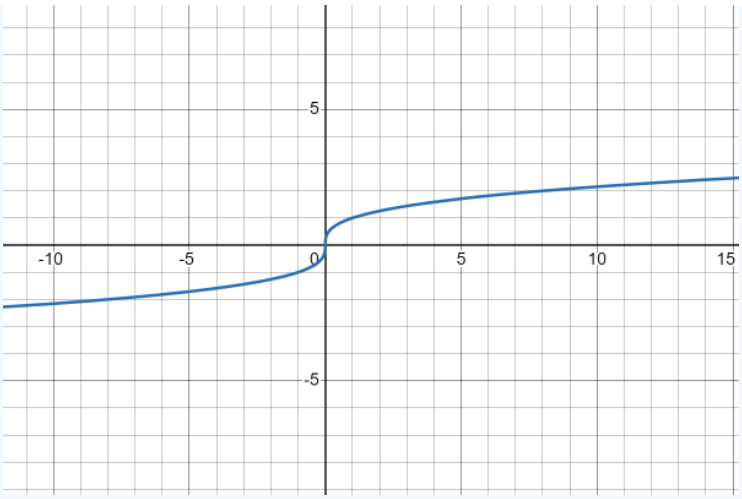
1. 2. 3. 4.

Answer

1. Yes
2. Yes
3. No
4. No

Reading Inverse Function Values From a Graph

Using the graph of $f(x)$, find $f^{-1}(-2), f^{-1}(-1), f^{-1}(0), f^{-1}(1)$, and $f^{-1}(2)$. Try to identify the function that was graphed, as well as its inverse function.



Answer

- $f^{-1}(-2) = -8$
- $f^{-1}(-1) = -1$
- $f^{-1}(0) = 0$
- $f^{-1}(1) = 1$
- $f^{-1}(2) = 8$.

The graphed function is $f(x) = \sqrt[3]{x}$. The inverse function is $f^{-1}(x) = x^3$.

Testing Inverses by Composing

Test whether the functions are inverses of each other by checking the conditions:

$$(g \circ f)(x) = x, \text{ for all } x \text{ in domain of } f, \quad \text{and} \quad (f \circ g)(x) = x, \text{ for all } x \text{ in domain of } g$$

1. $f(x) = 14 - 3x$ and $g(x) = \dfrac{14 - x}{3}$
2. $f(x) = (x - 5)^3$ and $g(x) = \sqrt[3]{x} + 5$
3. $f(x) = x$ and $g(x) = \frac{1}{x}$
4. $f(x) = \dfrac{x + 1}{x + 2}$ and $g(x) = \dfrac{2x - 1}{x - 1}$

Answer

1. Yes.
2. Yes.
3. No.
4. No.

Finding Inverses

Find the inverse of the function, if possible. Tell the domain and range of both functions.

1. $f(x) = x^2 + 1$
2. $f(x) = \sqrt{x}$
3. $f(x) = \frac{x+2}{3}$
4. $f(x) = \sqrt{2x - 1} + 4$
5. $f(x) = \frac{1}{x}$
6. $f(x) = |x|$

Answer

1. The domain of f is $(-\infty, \infty)$ and the range is $[1, \infty)$, but this function has no inverse.
2. The domain and range of f are both $[0, \infty)$. The inverse function is $f^{-1}(x) = x^2$, defined on domain $x \geq 0$.
3. The domain and range of f are both $(-\infty, \infty)$. The inverse function is $f^{-1}(x) = 3x - 2$, with same domain and range.
4. The domain of f is $x \geq \frac{1}{2}$ and the range is $[4, \infty)$. The inverse function is $f^{-1}(x) = \dfrac{(x-4)^2 + 1}{2}$, defined only on $x \geq 4$, with range $\left[\frac{1}{2}, \infty\right)$.
5. The domain and range of f are both $\{x \mid x \neq 0\}$. The inverse function is $f^{-1}(x) = \frac{1}{x}$, with same domain and range. Aka, this function is its own inverse!
6. The domain of f is $(-\infty, \infty)$ and the range is $[0, \infty)$, but this function has no inverse.

? Applications

1. The function $k(m) = \frac{8}{5}m$ computes the approximate equivalent in kilometers per hour, given as input a speed in miles per hour. Find a function that converts (approximately) from kilometers per hour to miles per hour instead. Then convert 80 kph into mph.

2. The function $F(C) = \frac{9}{5}C + 32$ takes a Celsius temperature as input and computes the Fahrenheit equivalent as output. Find a function that converts from Fahrenheit to Celsius instead. Then convert 95 degrees Fahrenheit into Celsius.

3. The function $V(h) = \pi r^2 h$ calculates the volume of a cylinder with (constant) radius r, as a function of the height h. Find a function that could take as input a desired volume and report as output the necessary height. Then, if a cylinder has radius 2 feet and a volume of 16π cubic feet is desired, find the necessary height.

Answer

1. The function is $m(k) = \frac{5}{8}k$, and $m(80) = 50$ mph.
2. The function is $C(F) = \frac{5}{9}(F - 32)$, and $C(95) = 35$ degrees Celsius.
3. The function is $h(V) = \frac{V}{\pi r^2}$, and $h(16\pi) = \frac{16\pi}{\pi 2^2} = 4$ feet.

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4.5 Chapter 4 Study Guide

Definitions

- **function:** a rule that assigns to each element x of a set A exactly one element, $f(x)$, in a set B. Aka, a function takes each input $(x-)$value and gives a single output $(y-)$value.
- **domain:** the set A of possible (legal) inputs for a particular function.
- **range:** the set of output values that a function actually attains.
- **piecewise-defined function:** a function whose definition is split into cases for certain inputs or ranges of inputs, written with a large brace on the left.
- **evaluating:** evaluating a function at an input value or expression is plugging in that value or expression everywhere the variable appeared in the function definition.
- **graph:** the graph of a function is the set of all points $(x, f(x))$ plotted on the coordinate plane. Aka, the x-value tells you the input, and the function output $f(x)$ tells you the y-value that goes with it.
- **even function:** a function such that plugging in $(-x)$ for x causes no change to the function after simplification, so $f(-x) = f(x)$. This looks like being symmetric across the y-axis (like a butterfly).
- **odd function:** a function such that plugging in $(-x)$ for x makes the same function, but negative, after simplification, so $f(-x) = -f(x)$. This looks like being symmetric about the origin (180-degree rotation has no effect).
- **floor function:** a function that rounds down any input to the nearest lower integer. So $\lfloor 3 \rfloor = 3$ and $\lfloor 3.7 \rfloor = 3$, etc.
- **composition of functions:** plugging a whole function in as the input to another function. Written $(f \circ g)(x) = f(g(x))$.
- **one-to-one function:** a function that sends distinct inputs to distinct outputs. Aka, it never sends multiple inputs to the same output. Example: $f(x) = x^2$ is not one-to-one because both $-2, 2$ are sent to 4.
- **inverse functions:** functions that undo each other, or functions such that if you do one and then the other, you get back the original input. The inverse of f is f^{-1}, with opposite domain and range, such that $(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$ and $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$. Example: $f(x) = x^3$ and $f^{-1}(x) = \sqrt[3]{x}$.

Evaluating a Function Example

Evaluate $f(x) = x^2 - x + 1$ at $x = 3, \pi$, and $x + h$.

- $f(3) = 3^2 - 3 + 1 = 7$
- $f(\pi) = \pi^2 - \pi + 1$
- $f(x + h) = (x + h)^2 - (x + h) + 1$

ALERT: use parentheses and be careful with subtraction and powers of negative inputs.

Vertical Line Test (VLT)

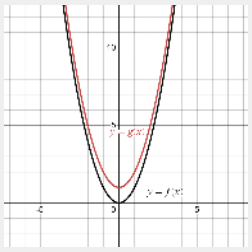
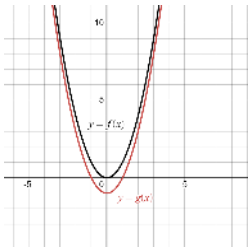
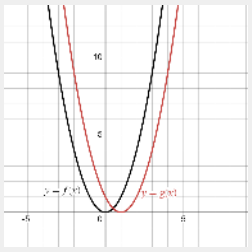
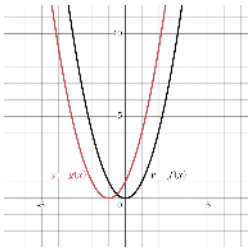
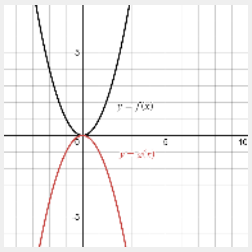
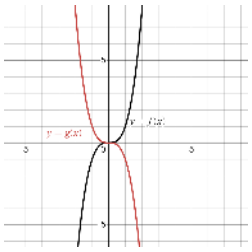
If there is any place on a given graph where you could draw a **vertical line** that **intersects** the graph at **multiple distinct points**, then the graph **cannot** be the graph of a function.

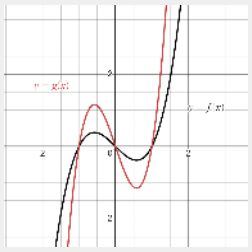
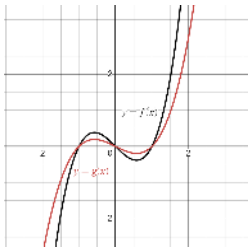
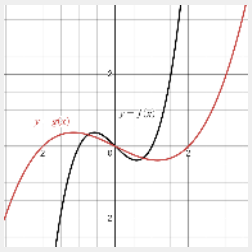
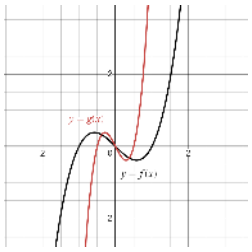
Assessing Domain & Range

- Determine domain of a function by figuring out what numbers are legal to plug in for the variable. Look for inputs that cause dividing by zero or taking an even root of negative numbers. Don't allow those inputs into the domain.
- Determine range of a function by figuring out what kind of numbers it can produce. Look for things like even powers or absolute value bars, which can never produce negative numbers, or radical signs, which by convention give the positive roots.

Transformations Table

Transformation Description	$f(x) \to$ **New Function**	**English Translation**	**Example**	**Graph**

Vertical shift **up** by k	$f(x)+k$	*An original function gets a constant **added** on to the very end, which causes a vertical change.*	$f(x) = x^2, \rightarrow\ g(x)\ = x^2 + 1$	
Vertical shift **down** by k	$f(x)-k$	*An original function gets a constant **subtracted** on the very end, which causes a vertical change.*	$f(x) = x^2, \rightarrow\ g(x) = x^2 - 1$	
Horizontal shift **right** by h	$f(x-h)$	*In an original function, each instance of the variable was replaced by the variable **minus** a constant, causing a horizontal change.*	$f(x) = x^2, \rightarrow\ g(x) = (x-1)^2$	
Horizontal shift **left** by h	$f(x+h)$	*In an original function, each instance of the variable was replaced by the variable **plus** a constant, causing a horizontal change.*	$f(x) = x^2, \rightarrow\ g(x) = (x+1)^2$	
Reflection across the x-axis	$-f(x)$	*Flip the sign on the whole original function, which has the effect of flipping the sign on all y-values.*	$f(x) = x^2, \rightarrow\ g(x) = -x^2$	
Reflection across the y-axis	$f(-x)$	*In an original function, each instance of the variable x was replaced by $(-x)$.*	$f(x) = x^3, \rightarrow\ g(x) = (-x)^3$	

Vertical **stretch** by a factor of k	$k \cdot f(x)$, where $k > 1$	*An entire original function is multiplied by a constant **larger** than 1, causing a vertical change.*	$f(x) = x^3 - x, \to g(x) = 3(x^3 - x)$	
Vertical **shrink** by a factor of k	$k \cdot f(x)$, where $0 < k < 1$	*An entire original function is multiplied by a positive fraction **smaller** than 1, causing a vertical change.*	$f(x) = x^3 - x, \to g(x) = \frac{1}{2}(x^3 - x)$	
Horizontal **stretch** by a factor of $\frac{1}{h}$	$f(hx)$, where $0 < h < 1$	*In an original function, each instance of x is replaced by x times a small positive fraction, causing a horizontal change.*	$f(x) = x^3 - x, \to g(x) = (\frac{1}{2}x)^3 - \frac{1}{2}x$	
Horizontal **shrink** by a factor of $\frac{1}{h}$	$f(hx)$, where $h > 1$	*In an original function, each instance of x is replaced by x times a constant larger than 1, causing a horizontal change.*	$f(x) = x^3 - x, \to g(x) = (2x)^3 - 2x$	

Identifying Transformations

1. Look at the operations that were done to the input variable, **in order.**
2. For each operation type, note the transformation effect from the table above.
3. Write down in words the effects, in order.

Example: $f(x) = (x-1)^2 + 3$ could have started out as x^2, then replaced the input with $(x-1)$, which causes a shift right according to the table. Then a $+3$ was tacked onto the end, which causes a shift up according to the table. So the transformations were: "Shift right by 1 and shift up by 3."

Even and Odd Functions

- A function f is called an **even function** if $f(-x) = f(x)$. Aka, reflecting its graph across the y-axis changes nothing. Aka aka, its graph is symmetric about the y-axis.
- A function f is called an **odd function** if $f(-x) = -f(x)$. Aka, reflecting its graph across the y-axis has the same effect as reflecting it across the x-axis. Aka aka, the graph is symmetric about the origin.
- A function need not be one or the other! A function can be **neither** even nor odd.

Algebra With Functions

Given two functions, f and g, you can make:

- their **sum,** a new function named "$f+g$," defined by $(f+g)(x)=f(x)+g(x)$.
- their **difference,** a new function named "$f-g$," defined by $(f-g)(x)=f(x)-g(x)$. (ALERT: subtract the WHOLE function g, using parentheses!)
- their **product,** a new function named "fg," defined by $(fg)(x)=f(x)g(x)$. (ALERT: if the functions have multiple terms, this would require expanding! Again, use parentheses!)
- their **quotient,** a new function named "$\frac{f}{g}$," defined by $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$. (ALERT: any x such that $g(x)=0$ is not allowed in the domain of this new function!)

These functions are defined on domains where both f and g are defined, so we say the domain is the intersection of the domains of f and g. The domain of the quotient of the functions is further restricted because you can't divide by zero.

Composing and Decomposing

Composing functions: to compose $(f\circ g)(x)$,

1. Translate to nested parentheses $f(g(x))$.
2. Replace $g(x)$ with its definition.
3. Go through f and everywhere you see an x, replace it with the whole input expression in parentheses.

Example: for $f(x)=x^2+3$, $g(x)=x-1$, we have $(f\circ g)(x)=f(g(x))=f(x-1)=(x-1)^2+3$.

Decomposing: to find f and g such that a function $h=f\circ g$,

1. Identify what happened to the input and in what order. Write in words if needed.
2. Split the operations at an "and then" phrase. Use the first activities as the inside function g and the later activities as f.
3. Check answer by composing to see if it matches h.

Example: $h(x)=\sqrt{x+2}$ is "add 2 to the input, and then take the square root." Then $g(x)=x+2$ and $f(x)=\sqrt{x}$.

Horizontal Line Test (HLT)

If there is **any place** on the graph of a function where you can draw a **horizontal line intersecting the graph multiple times,** then it is **not** one-to-one.

Finding the Inverse of a Function

To find the inverse of a one-to-one function $f(x)$,

1. Replace "$f(x)$" with "y."
2. Solve this equation for x in terms of y, if possible.
3. Swap the variable names: replace all the x's with y's and vice versa.
4. Replace y with "$f^{-1}(x)$."

Note: in an application problem, the variable names are usually descriptive in some way, so skip Step 3 in those.

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CHAPTER OVERVIEW

Chapter 5: Polynomial Functions

In this chapter, we study functions whose algebraic definitions consist of polynomial expressions. Much of this will feel familiar if you've come through Chapters 2 and 3, since working with polynomial functions necessarily involves a lot of simplifying of polynomial expressions and solving of polynomial equations! We'll build up through the degrees, starting with linear functions (officially), proceeding to quadratic functions, and then generalizing to higher-degree polynomials. There's a detour through polynomial long division, a big ol' roundup of visualization and graphing information, and a return to inequalities at the end. In calculus, polynomials are hugely important and everybody's favorite because it's very easy to take their derivatives, and a whole entire section of a Calc I class is usually devoted to finding their maximum and minimum values, so hopefully this chapter will make you extremely comfortable with functions of this type.

5.1 Linear Functions and Rate of Change

5.1E Exercises

5.2 Quadratic Functions

5.2E Exercises

5.3 Polynomial Functions

5.3E Exercises

5.4 Roots of Polynomial Functions

5.4E Exercises

5.5 Chapter 5 Study Guide

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5.1 Linear Functions and Rate of Change

Learning Objectives

By the end of this section, you will be able to:

- recognize a linear function and identify its important characteristics
- connect linear functions to everything we know about lines and linear equations
- understand the concept of a secant line and relate it to average rate of change
- recognize a line tangent to a graphed curve at a point
- construct and use mathematical models using linear functions

Omg, you say, enough about lines already???? I know, I know, I hear you. What I want to do in this section is just round up what we know about lines and linear equations, write it in the language of functions real quick, and actually then move on to some concepts that are probably going to show up in the first week of your Calc I courses. When your professor is like, "Blah blah secant line blah blah tangent line," you can nod emphatically and make a good first impression. You're welcome.

Linear Functions

A **linear function** is a first-degree polynomial function, i.e. it can be written in the form

$$f(x) = a_1 x + a_0$$

where $a_1 (\neq 0)$ and a_0 are constant coefficients. Fun facts about linear functions:

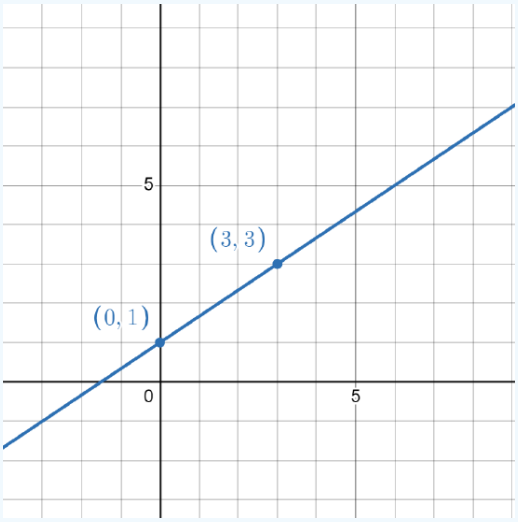
- Both the **domain** and the **range** of a linear function (unless specified due to an application) are **all real numbers, $\mathbb{R}$.**
- The graphs of linear functions are straight lines. If you rename $a_1 = m$ and $a_0 = b$, you can recognize the slope-intercept equation of a line.
- Consequently, the leading coefficient a_1 gives the slope of the line, and the constant term a_0 gives the y-intercept $(0, a_0)$ of the graph.
- Because of what we know about positive and negative slopes, if $a_1 < 0$, the function is **decreasing** (going downhill from left to right) and if $a_1 > 0$, the function is **increasing** (going uphill from left to right).

Linear functions are then extremely easy to graph by hand, and hopefully by now that process is pretty second nature.

Example 1

1. Describe what you know about the function $f(x) = 5x$ just based on looking at him.

2. Write the linear function whose graph is shown below.



3. Graph the linear function $f(x) = -\frac{1}{2}x + 3$.

Solution

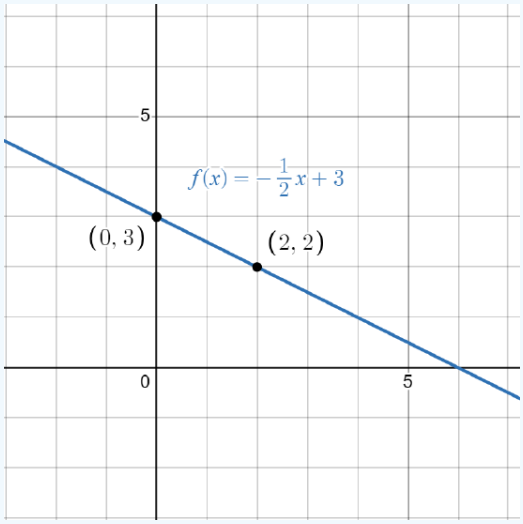
1. We see that this is certainly a linear function, with $a_1 = 5$ and $a_0 = 0$. The former tells us that the slope is 5, which is a) uphill and b) pretty steep, being greater than 1... The latter tells us that the y-intercept is the origin $(0, 0)$, which is also the x-intercept.

2. This line passes through the y-intercept $(0, 1)$, which tells us that $a_0 = 1$. If we travel from $(0, 1)$ to $(3, 3)$, we see it requires three steps in the x direction and two steps in the y direction, so it's pretty clear that the slope is $\frac{\text{rise}}{\text{run}} = \frac{2}{3}$, but you could also calculate it with the slope formula using these two points:

$$a_1 = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{3 - 0} = \frac{2}{3}$$

Thus the function must be $f(x) = \frac{2}{3}x + 1$.

3. We immediately identify the y-intercept either by knowing the $y = mx + b$ formula, or by plugging in $x = 0$ (remember, in order to be intercepting the y-axis, you darn well better have an x-value of 0, so you can always find y-intercepts this way). It's $(0, 3)$, so we plot that point. We then use the slope $\frac{-1}{2}$ by taking two steps to the right (run) and one step down (negative rise) and plot another point, $(2, 2)$. Connect the dots with a straight line.



Something we definitely need to be able to do is identify what functions are linear and what aren't! It's all about being able to match the defining form.

? Exercise 1

Are the functions linear?

1. $f(x) = 2 - 3x$
2. $f(x) = x + 1 + x^2$
3. $f(x) = 4^x$
4. $f(x) = 4x$
5. $f(x) = 6\sqrt{x} + 1$

Answer

1. Yes, this function has an x term to the first power and a constant term and nothing else.
2. No, this function has an x^2 term!
3. No, this function has the x stuck in an exponent.
4. Yes, this function has no constant term but that's okay. In the form $f(x) = a_1 x + a_0$, we would just say that $a_0 = 0$.
5. No, this function has an $x^{\frac{1}{2}}$ term.

Rate of Change

The discussion below will feel reminiscent of our discussion of variation, but the term "rate of change" is going to come up a lot in your future, so we're going to connect ideas here.

When two variables depend on each other in some way, making a change in the independent variable will induce a change in the dependent variable. Exactly how and how much the dependent variable will react to a change in the independent variable depends on the nature of the relationship. We work with this idea intuitively all the time in daily life. For example,

- The more washing machines I sell, the more money I make. The quantity "money I have" depends on the quantity "number of machines sold." The rate at which my money is increasing is directly related to the rate at which my sales are increasing. If each washing machine costs \$999, then the money I make can be written as a function of the number of machines sold, via $M(x) = 999x$. Every time I sell just one more machine, I make \$999 more.
- The longer I drive on the highway, the more ground I cover. The quantity "distance covered" depends on the quantity "time I've been driving." In fact, I have a word for this dependence, because I know about the concept of speed. Speed is the rate at which my distance changes, relative to how time is changing. A speed of 70mph means that when time changes from 0hr to 1hr, my distance covered will change from 0mi to 70mi.
- If I pump water into a pool using a hose, then the volume of water in the pool will increase as time goes on. The quantity "volume" is related to the quantity "time," and the rate at which the volume increases will depend directly on how long I let the hose flow into the pool. If the hose is adding 10 gallons per minute, then whenever a minute of time passes, my dependent variable volume will increase by 10 gallons.

In math language, we write the dependent variable as a function of the independent variable, and then talk about the **rate of change of [dependent quantity] with respect to [independent quantity].**

- We have the function $M(x) = 999x$. The rate of change of money, M, with respect to number of machines sold, x, is \$999 per machine.
- Remembering "distance equals speed times time," we write a function $d(t) = 70t$. Speed is the rate of change of distance with respect to time. (The rate at which my distance is changing as time goes on.)
- Say I started with 100 gallons in the pool, and I'm adding 10 more every minute. We could write volume V in gallons as a function of time t in minutes, $V(t) = 100 + 10t$. The rate of change of volume with respect to time is 10 gallons per minute.

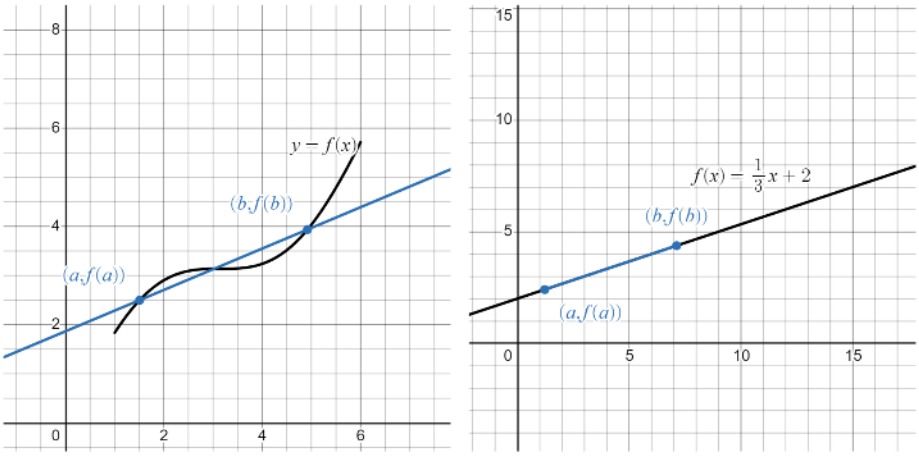
Notice that the word "per" shows up when talking about these changing quantities. "Miles per hour," "gallons per minute." This is a clue about how rate of change is calculated, because "per" can indicate division.

> The **average rate of change** of a function $y = f(x)$ between $x = a$ and $x = b$ is
>
> $$\frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

In the examples I gave above, the rates of change are actually all **constant.** A constant \$999 per machine, constant 70 miles per hour, constant 10 gallons per minute. Notice what kind of functions we were using... In each one, the independent variable appears only to the first power, with a constant coefficient. That's right, those are linear functions!

> **Linear functions have constant rate of change.**

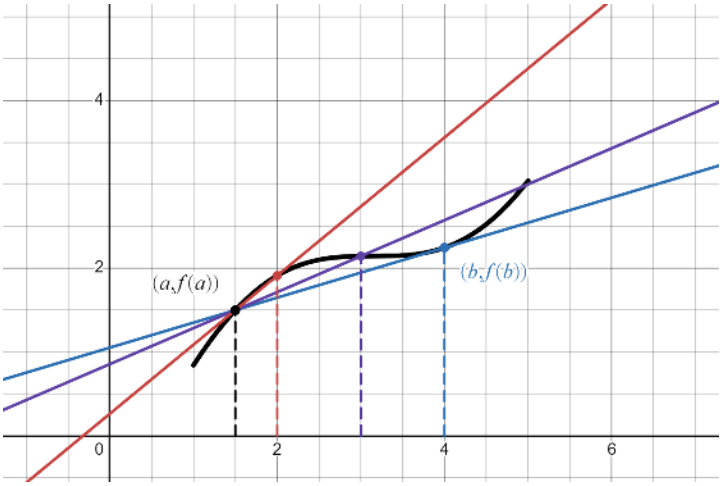
Now, maybe you've already noticed something interesting about the way we calculate rate of change. We've seen the language "change in y vs change in x" before, when calculating slope! Study the graphs below.



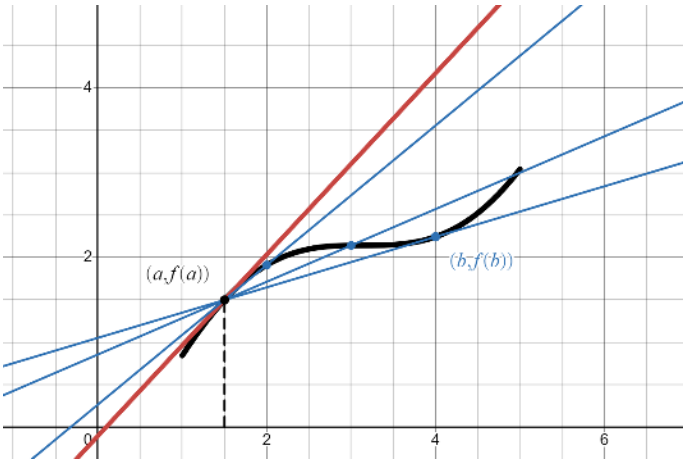
The formula $\frac{\text{change in } y}{\text{change in } x} = \frac{f(b)-f(a)}{b-a}$ is exactly the rise-over-run calculation of the **slope** of the blue line through $(a, f(a))$ and $(b, f(b))$! And on the right, when the function is linear, the rate of change calculation will give the exact same answer every single time, because the slope of the line never changes.

When you draw a straight line through two points on the graph of a function, like on the left above, it's called a **secant line.** The average rate of change of f between two points $x = a$ and $x = b$ is the **slope of the secant line** between those points $(a, f(a))$ and $(b, f(b))$.

We're going to do practice problems shortly that are inspired by real-life linear functions, but before that I want to give you a spoiler to help you out in Calc I later. It starts with a secant line between two points on a curve. I'm going to hold the first point stable, and gradually scootch the second point back closer and closer to the first point. Observe.



As I move the $(b, f(b))$ point, the slopes of my secant lines change. The closer I come to the $(a, f(a))$ point, the more they approach a special kind of line, shown in red below.



This red line is called **the line tangent to the curve** $y = f(x)$ **at the point** $(a, f(a))$**.** You agree with me that while you're driving your car, you have a certain "instantaneous" speed at any split second in time? In a similar way, at the exact point $(a, f(a))$, this red line has exactly the same instantaneous slope as the black graphed curve. This idea of instantaneous rate of change is huge in a Calc I class. So get hype.

Constructing and Using Linear Functions

Let's practice setting up and using linear functions in application problems, since like, all the algebra of working with linear functions is pretty much covered by everything we learned in lines and equations... This will help us practice our reasoning and interpretation skills as well. Double whammy, as my father would say.

Example 2

I am going on a trip and I need to pay a petsitter to take care of my cat. Petsitter A charges a flat fee of $20 and then $20 per day of the trip. Petsitter B charges a flat fee of $80 and then $10 per day. For each option, write the total cost of the petsitting as a function of n, the number of days I am gone. If I go on a 4-day trip, which petsitter should I choose? What about a two-week trip? For which trip length will the cost for each petsitter be the same? Graph both functions.

Solution

To calculate the cost of using Petsitter A, I start with the flat fee and add on $20 for every day of my trip. I can write the function performing this calculation as $A(n) = 20 + 20n$. Similarly, the function giving the cost of Petsitter B is $B(n) = 80 + 10n$. These are linear functions, because the variable n appears to the first power.

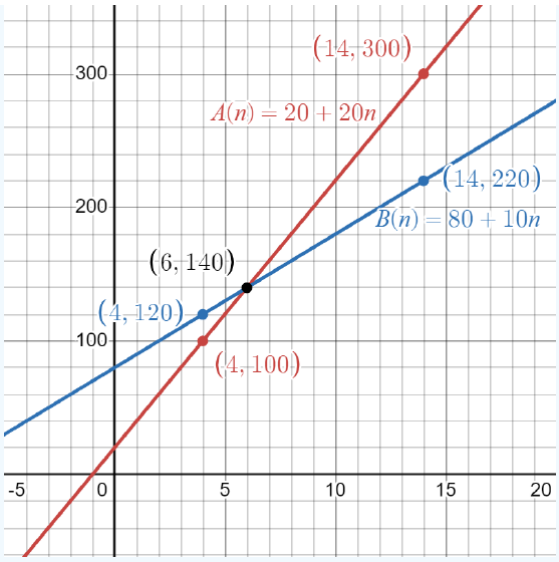
If I go on a 4-day trip, that means $n = 4$. Plugging that value into my functions will tell me the cost of each option. We have $A(4) = 100$ and $B(4) = 120$. Since Petsitter A costs less, I should use them for this trip.

If I go on a two-week trip, that's $n = 14$. We plug in again to find that $A(14) = 300$ and $B(14) = 220$. In this case, I should use Petsitter B.

Now, how could I go about finding the n value for which the costs turn out the same? I just notice that if the two costs are $A(n)$ and $B(n)$, then my goal is to have

$$A(n) = B(n) \quad \longrightarrow \quad 20 + 20n = 80 + 10n$$

All we have to do is solve for n, finding that a $n = 6$ day trip will result in equal costs. Wait...did that process seem familiar somehow? Let's graph these functions and see if we notice anything.



Oh duh, finding that n-value that yields equal costs is just finding the point of intersection of these two lines!

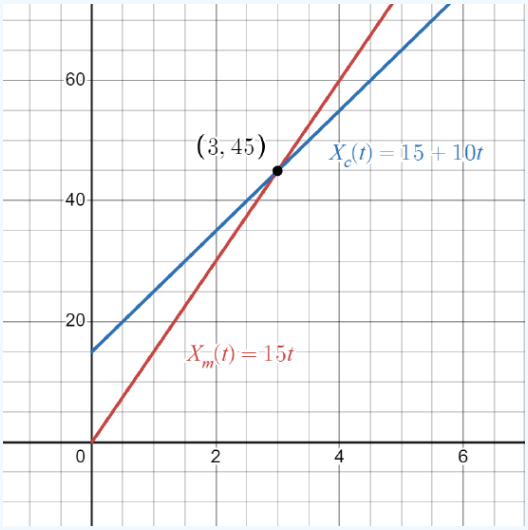
Exercise 2

My younger cousin and I are having a foot race. To be fair, I let her starting line be 15 feet ahead of my starting line. We start sprinting at the same time, and my speed is 15 feet per second, while hers is 10 ft/s. Write a function for each of us, giving our positions as a function of time after the clock starts. At what time will I catch up to her? Graph both functions.

Answer

Let's say that my starting line is position zero. Then my cousin is starting at a position value of 15. It doesn't matter what you name your functions...I guess I'll use X_m for my position and X_c for my cousin's. Then $X_m(t) = 0 + 15t$ and $X_c(t) = 15 + 10t$.

The question of when I will catch up is asking, "At what time value will our positions be equal?" Whoop, just an intersection problem again. You should get $t = 3$ seconds. Finally, your graph should look like this:



Example 3

At time $t = 0$, an object is dropped from the top of a skyscraper and falls freely under the influence of gravity. The table below gives the velocity and height values at different times. Determine in this scenario whether velocity is a linear function of time, and whether position is a linear function of time. If one or both is a linear function, write the function in standard form.

t **in seconds**	$v(t)$ **in meters/second**	$h(t)$ **in meters**
0	0	50
0.5	-4.9	48.775
1	-9.8	45.1
1.5	-14.7	38.975
2	-19.6	30.4
2.5	-24.5	19.375
3	-29.4	5.9

Solution

One thing we could do here is plot these data points on a coordinate plane and try to visually identify whether they lie on a straight line. Let's do that at the end to confirm our answer, but not rely on that strategy, because in real life like this, the numbers are kinda gross. Instead, let's use the fun fact that linear functions have constant rates of change. Let's try computing rate of change for $v(t)$ and $h(t)$ using different pairs of numbers, and see if either of them have a single consistent answer. For these calculations, feel free to use a calculator since the numbers aren't cooked.

First, let's check out $v(t)$... Choosing the data points $(0, 0)$ and $(0.5, -4.9)$, we do a quick slope calculation

$$\frac{v(0.5) - v(0)}{0.5 - 0} = \frac{-4.9 - 0}{0.5} = \frac{-4.9}{0.5} = -9.8$$

of the rate of change of $v(t)$ between those points. Picking another pair, say, $(1.5, -14.7)$ and $(3, -29.4)$, we get

$$\frac{v(3) - v(1.5)}{3 - 1.5} = \frac{-29.4 - (-14.7)}{1.5} = \frac{-14.7}{1.5} = -9.8$$

Oh interesting, that's the same... You can check some more pairs of points, but any time you do this slope-esque rate of change calculation, you will get a constant -9.8! That indicates that $v(t)$ is a linear function here. It must have a form something like $v(t) = a_1 t + a_0$. We know that the constant slope IS the rate of change, so that value can go in for a_1. How can we find a_0? Notice that when $t = 0$, the table says $v(0) = 0$. So we should be able to plug in 0 to the function and get the initial velocity we want.

$$v(0) = -9.8(0) + a_0 = a_0 \quad \rightarrow \quad \text{since } v(0) = 0, \text{ we must have } a_0 = 0.$$

We update our velocity function to be $v(t) = -9.8t$.

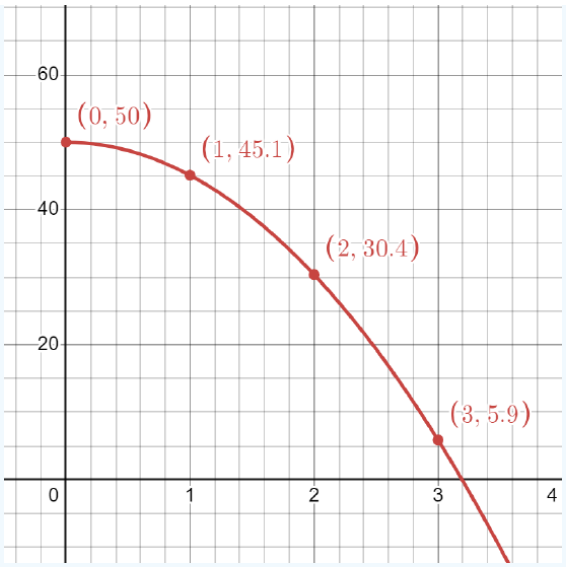
Okay, now let's check $h(t)$... With the data points $(0, 50)$ and $(1, 45.1)$, we calculate a rate of change of

$$\frac{h(1) - h(0)}{1 - 0} = \frac{45.1 - 50}{1 - 0} = -4.9$$

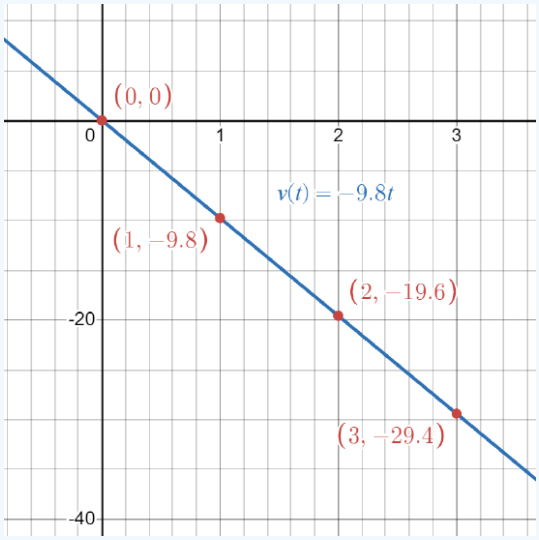
but if we use another pair of points, say $(0, 50)$ and $(3, 5.9)$, we get

$$\frac{h(3) - h(0)}{3 - 0} = \frac{5.9 - 50}{3} = \frac{-44.1}{3} = -14.7$$

That's a different number! This position function $h(t)$ does not have constant rate of change, so it must not be a linear function. The graph below shows some of the data points for $h(t)$, and you can see that they lie on a curve rather than a straight line. In fact, the function for height is $h(t) = 50 - 4.9t^2$, as you may know from studying physics. Since the independent variable t appears squared, this is indeed NOT a linear function.



On the other hand, the data points for velocity certainly lie on the straight line whose function we found.



Exercise 3

The table below gives the perimeters and areas of squares with various side lengths. Determine whether perimeter is a linear function of side length, and whether area is a linear function of side length. If one or both is a linear function, write the function in standard form.

x **in inches**	$P(x)$ **in inches**	$A(x)$ **square inches**
1	4	1
2	8	4
3	12	9
4	16	16
5	20	25

Answer

With any pair of data points chosen, the rate of change of $P(x)$ should turn out to be 4. With a constant rate of change, we have a linear function, $P(x) = 4x$. On the other hand, calculating rate of change of $A(x)$ with different choices of points yields a variety of answers. In fact, you already know how area depends on side length for a square! $A(x) = x^2$, which is clearly not a linear function at all, because the x appears to the second power.

Check out the exercises section for more practice with identifying, setting up, and using linear functions!

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5.1E Exercises

Identifying Linear Functions

Tell whether the function is linear and write it in standard form if so, identifying the leading coefficient and the constant term.

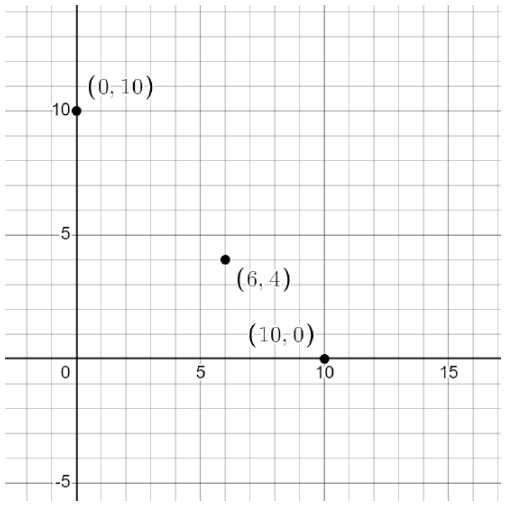
1. $f(x) = 5 + 3x$
2. $g(x) = 2x^2 + 1$
3. $h(x) = 9x + 1 - \sqrt{x}$
4. $p(x) = c_1 x + c_0$, where c_0, c_1 are real numbers.
5. $q(x) = 2(x + 3)$
6. $r(x) = \dfrac{2x + 3}{x + 1}$
7. $C(x) = 30x + 300$
8. $R(x) = 150x$
9. $P(x) = 200x - (1200 + 4x^2)$
10. $T(t) = t + \dfrac{1}{t}$

Answer

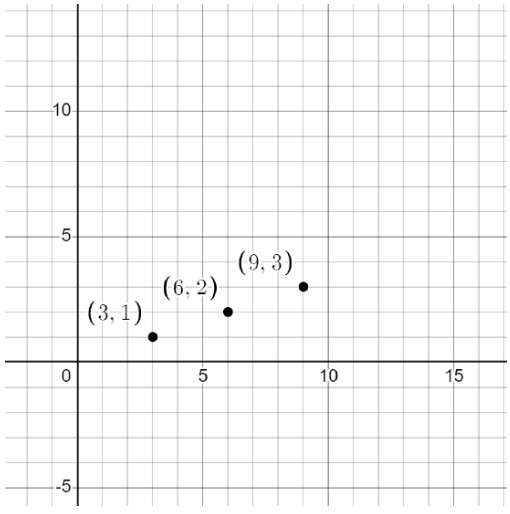
1. Yes, $a_1 = 3$, $a_0 = 5$.
2. No.
3. No.
4. Yes, c_1 is leading coefficient (gives slope) and c_0 is the constant term.
5. Yes, $q(x) = 2x + 6$, so $a_1 = 2, a_0 = 6$.
6. No.
7. Yes, $a_1 = 30, a_0 = 300$.
8. Yes, $a_1 = 150, a_0 = 0$.
9. No.
10. No.

Identifying Linear Functions' Graphs

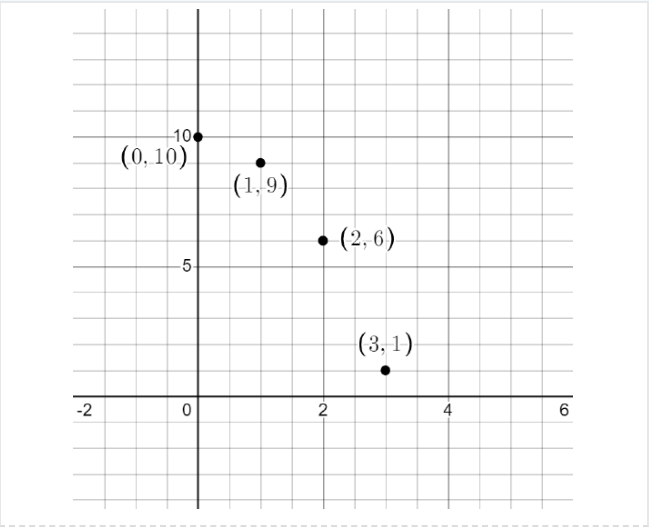
If a function passes through the graphed points, tell whether you believe it is a linear function. If so, write the function in standard form.



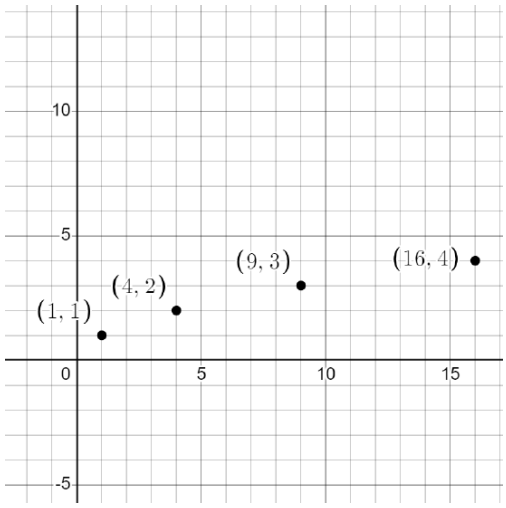
1.



2.



3.



4.

Answer

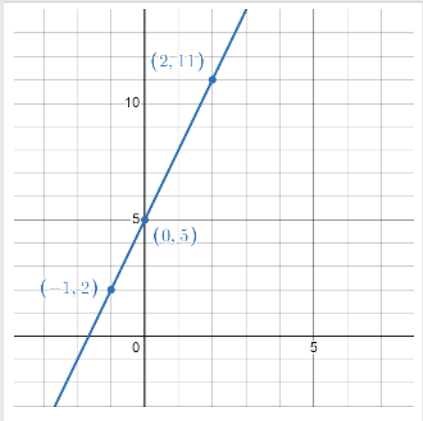
1. Yes, this is $f(x) = -x + 10$.
2. Yes, this is $f(x) = \frac{1}{3}x$.
3. No.
4. No.

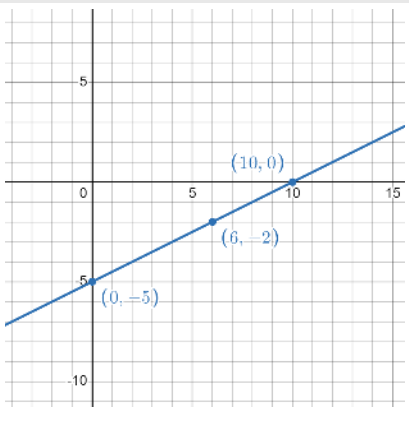
Graphing Linear Functions

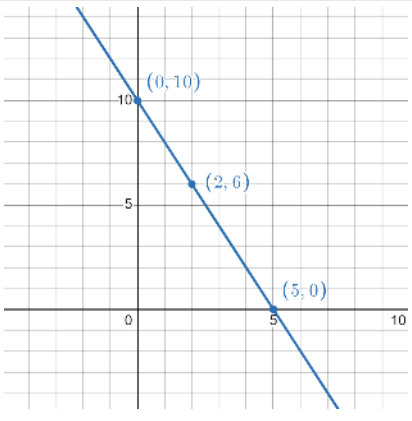
Sketch the graph of the linear function (without using technology).

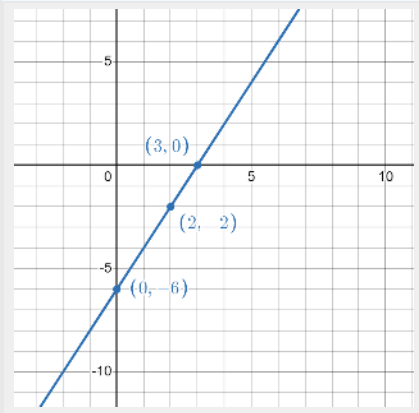
1. $f(x) = 5 + 3x$
2. $g(x) = \frac{1}{2}x - 5$
3. $h(x) = -2x + 10$
4. $p(x) = 2(x - 3)$
5. $C(x) = 30x + 300$
6. $R(x) = 150x$

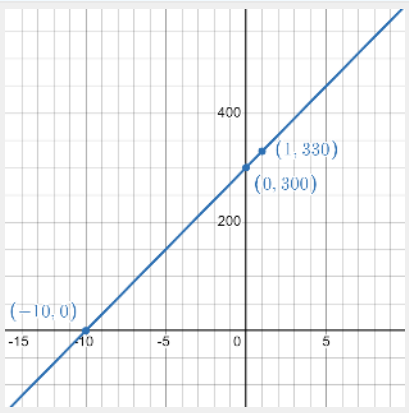
Answer

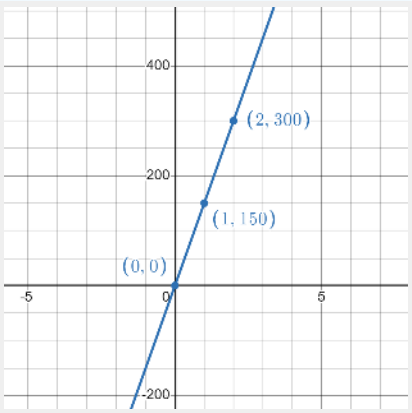












Average Rate of Change Using Tables

Using the table of values, find the average rate of change of the function for the indicated intervals.

t	$f(t)$
0	0
1	2
2	2
3	5
4	6

1. $t = 0$ to $t = 1$
2. $t = 1$ to $t = 2$
3. $t = 0$ to $t = 3$
4. $t = 1$ to $t = 3$
5. $t = 3$ to $t = 4$
6. $t = 2$ to $t = 4$

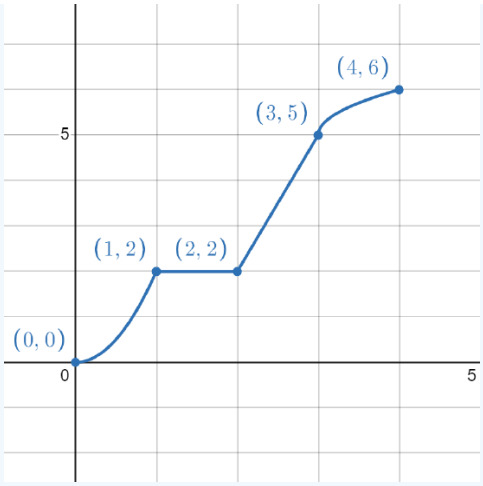
Answer

Identify the two points of interest on the table, $(t_1, f(t_1))$ and $(t_2, f(t_2))$, and then compute the average rate of change (slope calculation) between them.

1. 2
2. 0
3. $\frac{5}{3}$
4. $\frac{3}{2}$
5. 1
6. 2

Average Rate of Change Using Secant Lines

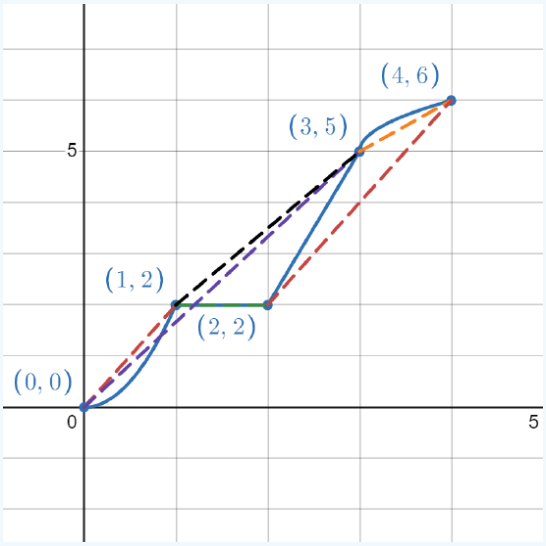
Recall that the average rate of change of a function between two points is the slope of the secant line drawn through those same points. Using the graph below, sketch the secant lines and find the average rate of change for the indicated intervals.



1. $t = 0$ to $t = 1$
2. $t = 1$ to $t = 2$
3. $t = 0$ to $t = 3$
4. $t = 1$ to $t = 3$
5. $t = 3$ to $t = 4$
6. $t = 2$ to $t = 4$

Answer

You should get the same answers as the last problem in the end. The secant lines should look like this:



? Constructing Linear Models

Note: the numbers in the problems below have been thoroughly cooked, so even though they're "applications," you should not be using a calculator. This kind of thing is totally reasonable on a test without calculators.

1. I used terrarium-grade silicone caulk rather than aquarium-grade in constructing my axolotl habitat, like a n00b. Now water is seeping out the seams at a constant rate of 0.25 gallon per hour. If the tank originally had 100 gallons in it, write the volume of water left in the tank as a function of time. If the tank leaks all night (for 8 hours), how much water will be in the tank in the morning? How long would it take to completely drain if I don't fix it asap?

2. Swimming at the surface of the ocean, the pressure is the same as the air pressure at sea level, 1 atmosphere (14.6 psi). Diving down, the pressure increases by 1 atmosphere every 10 meters deeper you go. Find a linear model that gives the

pressure as a function of depth below the surface (in meters). At a depth of 50 meters, what will the pressure be? At what depth will the pressure be 101 atmospheres?

3. Back in the day before unlimited phone plans, my monthly phone bill charged a $15 flat fee for up to 100 text messages. For every text message over 100, it charged an extra $0.10. Denoting the number of texts sent in a month by x, write a linear function to calculate the cost of my bill, assuming I always send more than 100 messages. If I send 120 texts in a month, what will the bill be?

Answer

1. $V(t) = 100 - 0.25t$. $V(8) = 98$gal. It will take $t = 400$hrs for all the water to leak out, so I have time to fix the tank!
2. $P(d) = 1 + \frac{1}{10}d$. $P(50) = 6$atm. $P = 101$atm if $d = 1000$m.
3. $C(x) = 15 + 0.1(x - 100)$. $C(120) = \$17$.

? City Population Growth

From 2000 to 2020, the population growth rates of Podunk Town, Kansas and One-Horse Town, Kansas were both roughly linear. If Podunk had a population of 4100 in the year 2000, and the population grew by about 80 people each year, write a linear model for population as a function of time in years *after 2000*. Do the same for One-Horse, which started with 6500 people but only had net population growth of about 40 people each year. What was each town's population in 2010? Was there any year between 2000 and 2020 when Podunk's population overtook that of One-Horse?

Answer

Podunk's population is $P(t) = 4100 + 80t$ where t is number of years after 2000. Likewise, One-Horse's population is $O(t) = 6500 + 40t$. 2010 is 10 years past 2000, so we find $P(10) = 4900$ and $O(10) = 6900$.

Rather than checking the populations for each year to see if Podunk ever caught up with One-Horse, we could note that such a crossing-over would happen at an intersection of the two linear functions. We try setting the functions equal to each other to see if there is a reasonable value for t where they intersect.

$$4100 + 80t = 6500 + 40t \quad \longrightarrow \quad t = 60$$

Hmm, no, it would take 60 years after 2000 for Podunk to win out, but this model is only valid for 2000-2020. One-Horse always had the larger population in this time period.

? Identifying Linear Relationships From Tables

The table below gives the monthly cost and revenue values for the production and sale of x washing machines. Identify whether cost and/or revenue are linear functions of x. If so, find the function's algebraic definition.

x	$C(x)$	$R(x)$
0	$2,000	$0
20	$2,582	$22,400
50	$3,429	$56,000
100	$4,800	$112,000
200	$7,434	$224,000

Answer

Cost does not demonstrate a linear relationship, but revenue does. The rate of change is a constant 1120, so the function is $R(x) = 1120x$.

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5.2 Quadratic Functions

Learning Objectives

By the end of this section, you will be able to:

- identify quadratic functions and their important features: vertex, roots, end behavior
- write quadratic functions in standard and vertex forms
- assess the domain and range of a quadratic function
- find real roots of quadratic functions using factoring techniques
- sketch graphs of quadratic functions

Next up in our tour of polynomial functions, you will see the degree-two polys coming up here on my left, your right. These are known as **quadratic functions.** For this section, you may want to review Section 2.2 Factoring Techniques if you haven't already, because I am going to assume that you're generally proficient with those methods. You will have to solve quadratics *ad nauseum* in Calc I optimization problems, so make sure you come out of this section with solid mastery!

Quadratic Functions

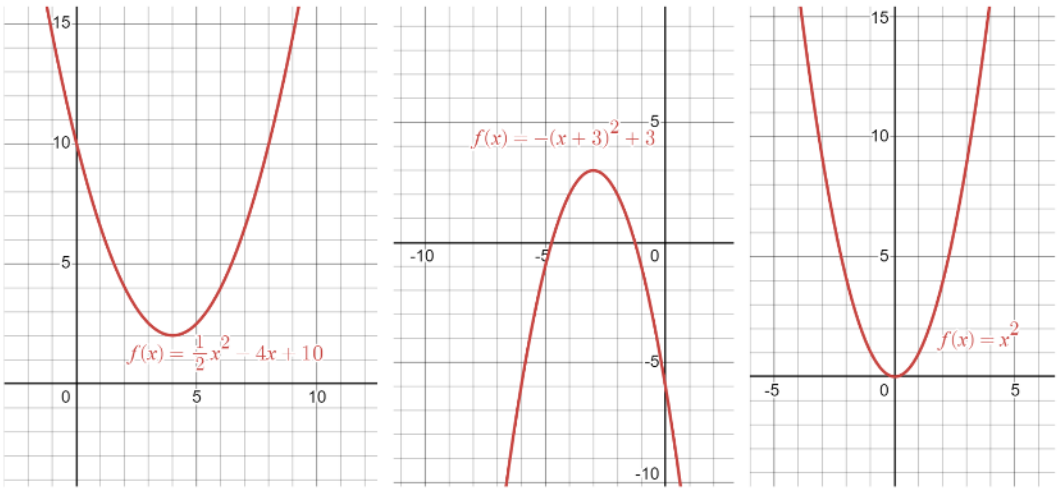
A **quadratic function** is a degree-two polynomial function, i.e. it can be written in the **general or standard form**

$$f(x) = a_2x^2 + a_1x + a_0 \quad \text{or more often} \quad f(x) = ax^2 + bx + c$$

where $a(\neq 0), b$, and c are constants. Fun facts about quadratic functions:

- The graphs of these functions are **parabolas**, which can open upward or downward. If the leading coefficient $a > 0$, it opens upward. If $a < 0$, it opens downward. There is a point called the **vertex**, which is the peak of the hill if the parabola opens downward, or the bottom of the trough if the parabola opens upward. The parabola is **symmetric** about the vertical line passing through the vertex.
- Quadratics are often written in **vertex form,** $f(x) = a(x-h)^2 + k$, highlighting the location of the parabola's vertex, (h, k). ALERT: be careful with the signs because of that subtraction!
- The **domain** of a quadratic function (unless specified due to an application) is **all real numbers,** $\mathbb{R}$.
- The **range** of a quadratic function is limited due to its parabola shape. Identifying the vertex will help you determine the range, since it will give the maximum or minimum possible function value.
- The x-intercept(s) of quadratic functions are called **roots** or **zeros.** As always, to find these, set the whole function equal to 0 and solve for x.

Let's look at some graphs while we're here talking about parabolas. Quadratic functions look something like this:



Depending on whether the parabola opens upward or downward, the quadratic function will either have a (**global** or **absolute**) **maximum** (a y-value that it never goes above) or a (**global** or **absolute**) **minimum** (a y-value that it never goes below). That max or min is the **vertex** of the parabola. On the flip side, in whichever direction the parabola opens, it will go on forever and ever that way, which brings us to the subject of **end behavior.** End behavior means the behavior of a function as x goes off to the right forever to infinity, and as x goes off to the left forever to negative infinity. We can use symbols and words to describe end behavior. For example, the first graph above goes up forever as we send x off to positive infinity, and also up forever as we send x to negative infinity. We write:

- as $x \to \infty$, $f(x) \to \infty$
- as $x \to -\infty$, $f(x) \to \infty$

Similarly, for the downward-opening second graph,

- as $x \to \infty$, $f(x) \to -\infty$
- as $x \to -\infty$, $f(x) \to -\infty$

This means that the **range** of a quadratic function will have one infinity symbol in it. For example, the first function hits all y-values from 2 onward, so the range is $[2, \infty)$. The second function has range $(-\infty, 3]$. What is the range of the third function?

While we're here, notice also that there are three possibilities when it comes to x-intercepts. The first example never touches the x-axis, so it has **no real roots.** The second example crosses the x-axis in two distinct locations, so it has **distinct real roots.** The third example bounces off the x-axis without crossing over, so it has what is called a **repeated root.** We'll see why it's called repeated shortly, when we learn how to find these roots algebraically.

Example 1

Tell whether the functions are quadratic functions, and if necessary, FOIL them into standard form.

1. $f(x) = 11x^2 + 12x + 13$
2. $g(x) = \dfrac{x^2 + x + 1}{x - 1}$
3. $h(x) = x^2 - x^3$
4. $p(x) = \frac{1}{4} + \frac{1}{2}x + x^2$
5. $q(x) = (x + 3)^2$
6. $A(r) = \pi r^2$
7. $C(x) = 3000 + 30x - 0.5x^2$

Solution

1. Yes.
2. No.
3. No, this is degree-three.
4. Yes, you could order the terms from largest power of x to smallest, if you want.
5. Yes, $q(x) = x^2 + 6x + 9$.
6. Yes, the variable r is squared. You could view this in standard form as $A(r) = \pi r^2 + 0r + 0$ meaning $a = \pi, b = 0, c = 0$.
7. Yes.

Exercise 1

Tell whether the functions are quadratic functions, and if necessary, rewrite them in standard form.

1. $f(x) = \sqrt{x^2 + x + 1}$
2. $g(x) = 3x^2 - 1$
3. $h(t) = 100 - 4.9t^2$
4. $p(x) = 2x^2 + 4(x - 1)^2$
5. $q(x) = (x + 1)(x - 2)$
6. $r(x) = \dfrac{x^2 + 6x + 1}{x - 2}$
7. $C(x) = 200 + 40x - x^{\frac{3}{2}}$

Answer

1. No.
2. Yes.
3. Yes.
4. Yes, $p(x) = 6x^2 - 8x + 4$.
5. Yes, $q(x) = x^2 - x - 2$.
6. No.
7. No.

Vertex Form

The vertex form of a quadratic function is very useful, because knowing the location of the vertex helps you sketch a graph, tells you where the function's max or min is, and in combination with knowledge of which way the parabola opens, tells you everything you need to assess range and narrow down root possibilities. A quadratic function is written in **vertex form** if it looks like this:

$$f(x) = a(x-h)^2 + k$$

where $a, h,$ and k are constants. We know here that if $a > 0$, the parabola opens up, and if $a < 0$, it opens down. We also know that (h, k) is the location of the vertex.

(Why?? Remember transformations? To go from a "basic" function $g(x) = x^2$ to the function $f(x) = a(x-h)^2 + k$, we see that several transformations happened: a vertical stretch or shrink, then a shift to the right by h, and a shift up by k. If the vertex started at the origin, $(0,0)$ and was shifted right by h and up by k, it's now living at (h,k)!)

To identify the vertex, simply make sure the function matches the vertex form above, and then read off the values of h and k.

? Exercise 2

Identify the vertex.

1. $f(x) = 3(x-1)^2 + 2$
2. $f(x) = (x+2)^2 - 16$
3. $f(x) = x^2 + 1$
4. $f(x) = \frac{1}{2}(x+10)^2$

Answer

1. Here $h = 1$ and $k = 2$, so the vertex is at $(1, 2)$.
2. We could rewrite this slightly as $f(x) = (x-(-2))^2 + (-16)$ so that it's easy to identify $(-2, -16)$.
3. We could rewrite this as $f(x) = (x-0)^2 + 1$ to identify $(0, 1)$.
4. Similarly, we notice $f(x) = \frac{1}{2}(x-(-10))^2 + 0$ to identify $(-10, 0)$.

That's great and all, but what if the vertex form isn't obvious or already given? Then we have to find it ourselves. There's a process for this.

To convert from standard form $f(x) = ax^2 + bx + c$ into vertex form $f(x) = a(x-h)^2 + k$,

1. If $a \neq 1$, factor it out like so: $f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$.
2. Working **inside** the parentheses, **complete the square** on the first two terms, $x^2 + \frac{b}{a}x$, making sure to both add **and** subtract the necessary term inside the parentheses.
3. If necessary, distribute the factored-out a back onto the constant term to finally reach the form $f(x) = a(x-h)^2 + k$ for some h and k.

Once you know the location of the vertex and whether the parabola opens up or down, you can easily determine the range. Let's include that information in our examples.

Example 2

1. Convert $f(x) = x^2 - 2x + 3$ into vertex form. Then identify the vertex and the range of the function.

2. Convert $f(x) = -3x^2 - 6x + 3$ into vertex form. Then identify the vertex and the range of the function.

Solution

1. Here, $a = 1$ so we don't have to worry about that. We can go straight to completing the square. We take $b = -2$, divide by 2, and square the result to get 1. Then we sneakily add zero to the function by adding and subtracting 1.

$$f(x) = x^2 - 2x \quad +3 \quad \longrightarrow \quad f(x) = x^2 - 2x + 1 + 3 - 1$$

Now the $x^2 - 2x + 1$ has been engineered to factor as $(x-1)^2$, so we finish simplifying:

$$f(x) = (x-1)^2 + 2$$

We can see the vertex is $(1, 2)$, and since the leading coefficient is positive, the parabola must open upward. That means $y = 2$ is the **lowest** my function ever goes. The range must be $[2, \infty)$.

2. In this case, $a \neq 1$, so we will have to carefully push him around throughout. First, we get him out of the way for now by writing $f(x) = -3(x^2 + 2x - 1)$, and we focus on the inside of the parentheses. If I want to complete the square on $x^2 + 2x$, I just take the 2, divide by 2, and square the result to get 1. Still inside the parentheses, I add and subtract that result.

$$f(x) = -3(x^2 + 2x + 1 - 1 - 1) \quad \longrightarrow \quad f(x) = -3((x+1)^2 - 2)$$

Now, I just distribute that 3 we had factored out onto the perfect square term and onto the constant -2 so that I fully match the vertex form:

$$f(x) = -3(x+1)^2 + 6.$$

The vertex is $(-1, 6)$. Since the leading coefficient -3 was negative, the parabola opens downward. That means the $y = 6$ height is the **maximum** function value, so my range is $(-\infty, 6]$.

Exercise 3

Convert to vertex form. Then identify the vertex and range of the function.

1. $f(x) = x^2 - 4x + 8$
2. $f(x) = 2x^2 + 12x + 17$
3. $f(x) = -2x^2 - 20x - 50$

Answer

1. $f(x) = (x-2)^2 + 4$. The vertex is $(2, 4)$ and the range is $[4, \infty)$.
2. $f(x) = 2(x+3)^2 - 1$. The vertex is $(-3, -1)$ and the range is $[-1, \infty)$.
3. $f(x) = -2(x+5)^2 + 0$. The vertex is $(-5, 0)$ and the range is $(-\infty, 0]$.

Finding Roots of Quadratic Functions

The roots of a quadratic function are just the x-values of the x-intercepts. To be on the x-axis, I must have a y-value of 0, so I am looking for the locations at which

$$f(x) = ax^2 + bx + c = 0.$$

Whoop, that's just giving me a quadratic equation, $ax^2 + bx + c = 0$. We've solved the heck out of those already, using various techniques.

Now, when you go about this process, there are three scenarios that could arise. Let me demonstrate them with an example:

Example 3

Find all real roots of the quadratic functions:

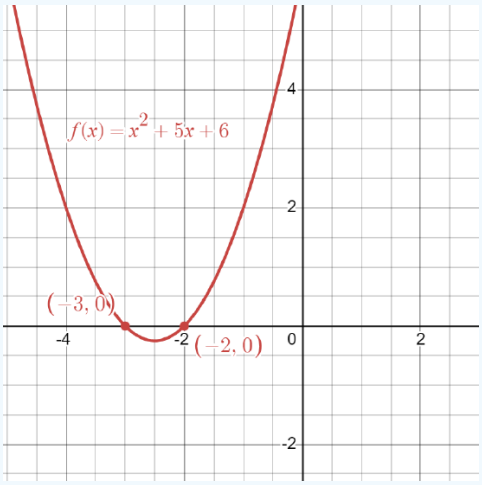
1. $f(x) = x^2 + 5x + 6$
2. $f(x) = 2x^2 + 4x + 2$
3. $f(x) = x^2 - 4x + 10$

Solution

1. Always just **set the function equal to 0** as the first step. I solve the resulting quadratic equation for x:

$$x^2 + 5x + 6 = 0 \quad \longrightarrow \quad (x+2)(x+3) = 0 \quad \longrightarrow \quad x = -2, -3$$

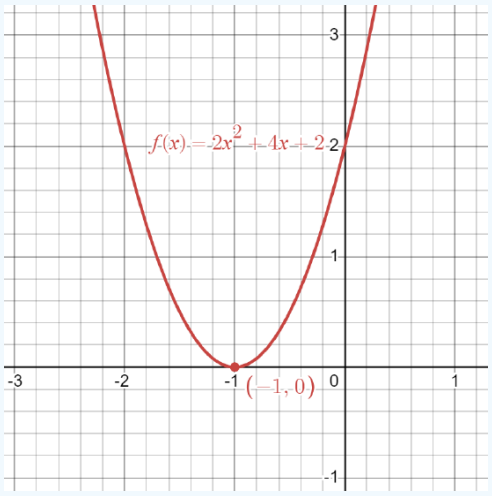
Here, my quadratic factored nicely into two distinct factors, resulting in two different real number roots. This guy looks like this:



2. Notice that each term has a 2 in it, so I try a factoring method again:

$$2x^2 + 4x + 2 = 0 \quad \longrightarrow \quad 2(x^2 + 2x + 1) = 0 \quad \longrightarrow \quad 2(x+1)^2 = 0$$

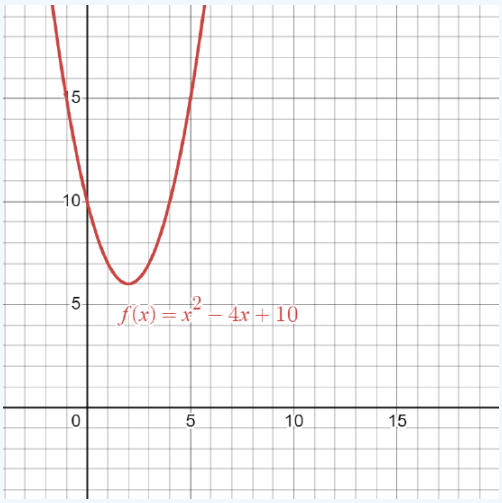
Notice that this one turned out to have two copies of the same factor, $(x+1)$. Aka, the factor was *repeated*. In this case, solving for x yields only one value, $x = -1$, but we call this a repeated root, or a **root with multiplicity 2**, because the factor that it came from appeared twice. Sometimes I write this answer as "$x = -1, -1$" so I remember what's going on. This graph looks like this, bouncing off the x-axis:



3. I set this guy equal to 0 as usual, $x^2 - 4x + 10 = 0$, but I don't see an easy way to factor him at the moment. I don't waste too much time thinking about it before giving the quadratic formula a shot. I indentify $a = 1, b = -4, c = 10$ and plug in...

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 4(1)(10)}}{2} = \frac{4 \pm \sqrt{-24}}{2}$$

Ope, companyyyyyy halt! The discriminant, $b^2 - 4ac$, turned out to be negative here. There are no real roots! If we pushed on into the complex numbers, the solution to this equation would be a complex conjugate pair. Look what happens on the graph... the parabola doesn't hit the x-axis at all!



Now you practice finding roots! You can use any method you like for this, including the ac method, completing the square, grouping, whatever.

? Exercise 4

Find any and all real roots of the quadratic functions.

1. $f(x) = x^2 - 2x - 8$
2. $f(x) = x^2 - 6x + 9$
3. $f(x) = 3x^2 - x - 2$
4. $f(x) = 4x^2 - 12x + 8$
5. $f(x) = x^2 - x + 2$

Answer

1. $x = -2, 4$
2. $x = 3, 3$
3. $x = -\frac{2}{3}, 1$
4. $x = 1, 2$
5. No real roots.

Sketching Graphs of Quadratic Functions

To **sketch the graph of a quadratic function**,

1. Find the vertex form and plot the vertex. Note whether the parabola should open up or down.
2. Find a few more points by plugging in different x-values and computing the function values. You can also choose to find the roots instead of random points.
3. Using knowledge of symmetry and how parabolas look, fit a parabola to the the plotted points smoothly.

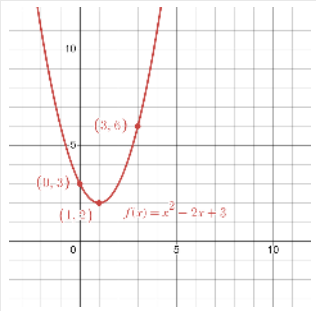
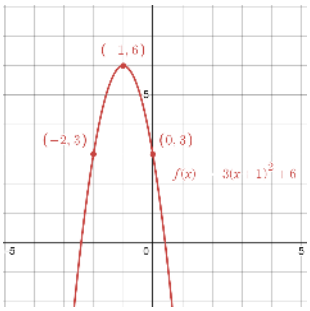
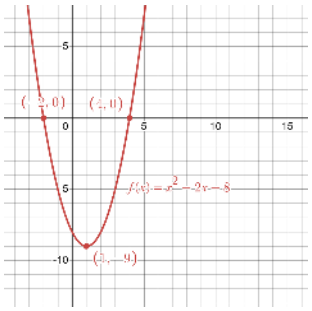
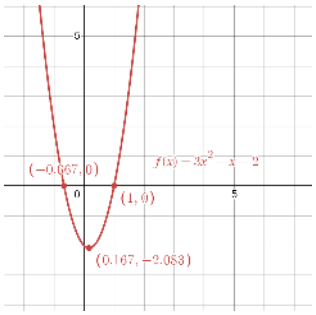
Or, use knowledge of transformations to sketch the graph, starting with the most basic $f(x) = x^2$ and performing the appropriate transformations.

Example 4

Sketch the graphs of the quadratic functions.

1. $f(x) = x^2 - 2x + 3$
2. $f(x) = -3x^2 - 6x + 3$
3. $f(x) = x^2 - 2x - 8$
4. $f(x) = 3x^2 - x - 2$

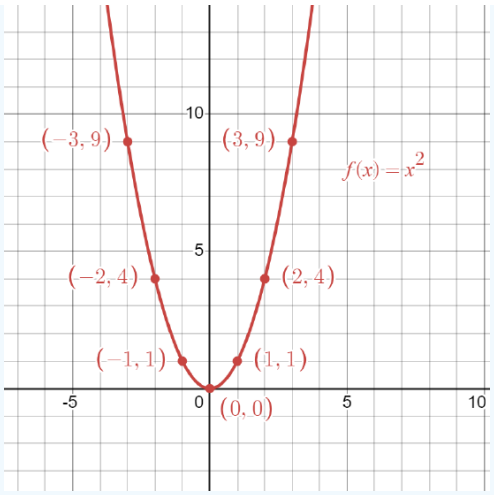
Solution

1.	2.	3.	4.
We've already found the vertex form in a previous example: $f(x) = (x-1)^2 + 2$. Plot the vertex $(1, 2)$. I compute a point on either side of the vertex: $f(0) = 3 \longrightarrow (0, 3)$ $f(3) = 6 \longrightarrow (3, 6)$ Connect the dots knowing that the parabola opens upward forever.	We have the vertex form: $f(x) = -3(x+1)^2 + 6$. Plot the vertex $(-1, 6)$. Note that this parabola needs to open downward. I compute an easy point: $f(0) = 3 \longrightarrow (0, 3)$ and use symmetry to deduce another point, $(-2, 3)$. Connect the dots.	I see the roots of this function are $x = -2, 4$, which gives me the two x-intercept points, $(-2, 0)$ and $(4, 0)$. I still need the vertex to draw a decent graph. Completing the square gets me to the vertex form $f(x) = (x-1)^2 - 9$ so I see the vertex is $(1, -9)$ and the parabola should open upwards. Connect the dots.	We found the roots of this function to be $x = -\frac{2}{3}, 1$ so I have two points, $(-\frac{2}{3}, 0)$ and $(1, 0)$. I complete the square to get the vertex form $f(x) = 3\left(x - \frac{1}{6}\right)^2 - \frac{25}{12}$ so I see the vertex is $(\frac{1}{6}, -\frac{25}{12})$. Noting that $\frac{1}{6}$ is less than 0.2, and $\frac{25}{12}$ is just a hair more than 2, I sketch the point and connect the dots.
			

Often, though, all you really need is to know exactly what $f(x) = x^2$ looks like, and then be able to recognize transformations.

Example 5

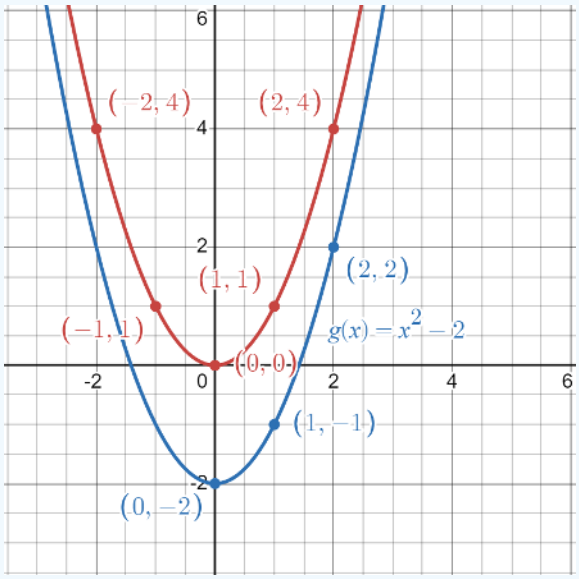
Without performing any real calculations, use knowledge of transformations to sketch the graphs of the quadratic functions, starting with the graph of $f(x) = x^2$:



1. $g(x) = x^2 - 2$
2. $g(x) = (x-1)^2 + 3$
3. $g(x) = -x^2 + 4$

Solution

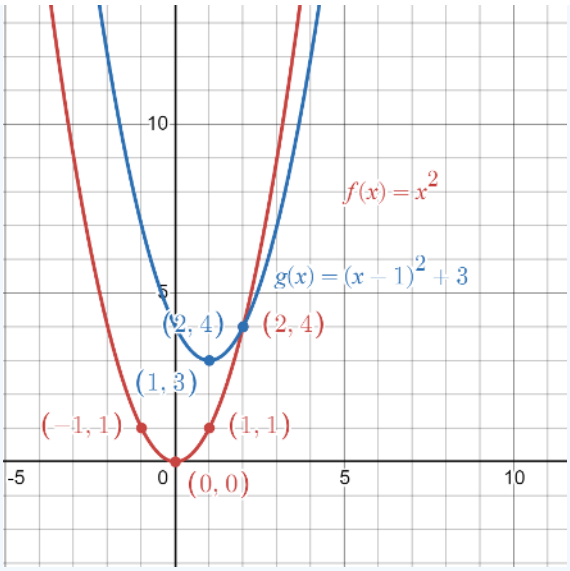
1. We recognize that the transformation "$x^2 \longrightarrow x^2 - 2$" is a shift down by 2 units. We take some signpost points on the original graph, like $(0,0)$, $(1,1)$, $(2,4)$, and imagine sliding them down by 2 units to help us sketch.



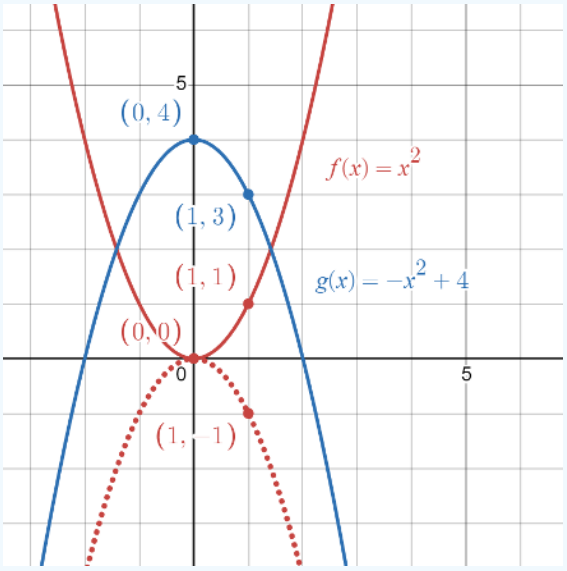
2. We recognize the transformations

$$x^2 \quad \longrightarrow \quad (x-1)^2 \quad \longrightarrow \quad (x-1)^2 + 3$$

are a shift right by 1 and shift up by 3. We take a few signpost points and perform those shifts on them to help us sketch. For example, $(0,0)$ becomes $(1,3)$.



3. We recognize that the transformations here were first a reflection over the x-axis, and then a shift up by 4. We can imagine an intermediate step if that helps us sketch the graph, shown in dotted red below.



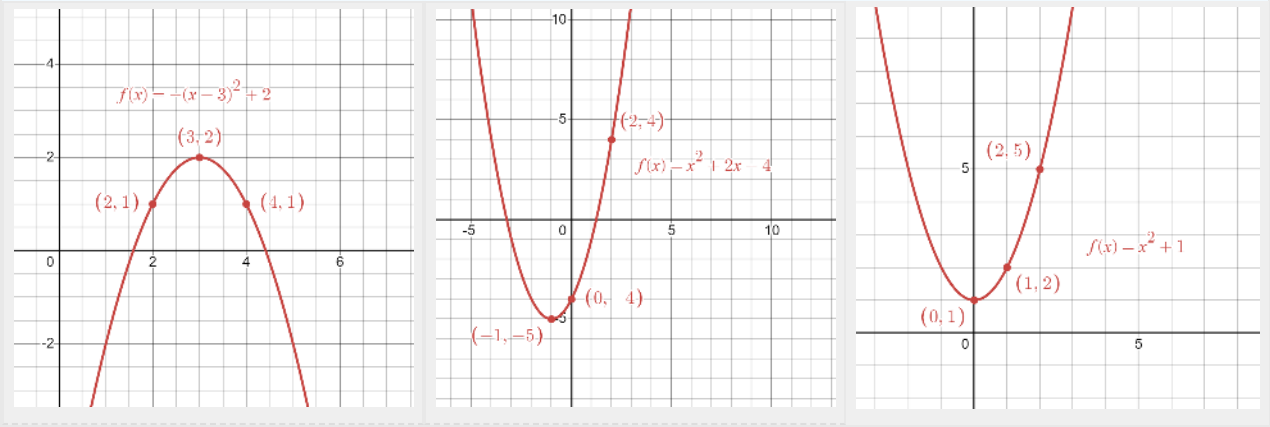
Exercise 5

Using either of the above methods, sketch the graphs of the quadratic functions.

1. $f(x) = -(x-3)^2 + 2$
2. $f(x) = x^2 + 2x - 4$
3. $f(x) = x^2 + 1$

Answer

1.	2.	3.



Now, be sure to do the exercises section, because we're going to see some real-life application problems over there!

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5.2E Exercises

Identifying Quadratic Functions

Tell whether the functions are quadratic functions, and if necessary, write them in standard form.

1. $f(x) = 3x^2$
2. $g(x) = 2x^{\frac{3}{2}} - x$
3. $h(x) = \frac{1}{x^2} + 3$
4. $p(x) = (x+1)(x-2)$
5. $q(x) = (x+1)^2(x-2)$
6. $r(x) = 4 - (x+1)^2$

Answer

1. Yes.
2. No.
3. No.
4. Yes, $p(x) = x^2 - x - 2$.
5. No, when expanded, this will have an x^3 term.
6. Yes, $r(x) = -x^2 - 2x + 3$.

Standard vs. Vertex Form

If the quadratic function is in standard form, convert it to vertex form and identify the vertex (h, k). If it is already in vertex form, convert it to standard form and identify the constants a, b, and c.

1. $f(x) = -2(x+1)^2 + 5$
2. $f(x) = (x+2)^2 - 13$
3. $g(x) = x^2 + 14x + 39$
4. $g(x) = -4x^2 - 16x - 21$

Answer

1. $f(x) = -2x^2 - 4x + 3$, $a = -2, b = -4, c = 3$.
2. $f(x) = x^2 + 4x - 9$, $a = 1, b = 4, c = -9$.
3. $g(x) = (x+7)^2 - 10$, $(h, k) = (-7, -10)$.
4. $g(x) = -4(x+2)^2 - 5$, $(h, k) = (-2, -5)$.

Domain and Range

Give the domain and range of each quadratic function.

1. $q(x) = x^2$
2. $f(x) = 4 - (x+1)^2$
3. $g(x) = -2(x+1)^2 + 5$
4. $p(x) = 3x^2$
5. $h(x) = (x+2)^2 - 13$
6. $r(x) = -x^2 + 4$

Answer

1. Domain: $\mathbb{R}$. Range: $[0, \infty)$.
2. Domain: $\mathbb{R}$. Range: $(-\infty, 4]$.
3. Domain: $\mathbb{R}$. Range: $(-\infty, 5]$.
4. Domain: $\mathbb{R}$. Range: $[0, \infty)$.
5. Domain: $\mathbb{R}$. Range: $[-13, \infty)$.

6. Domain: $\mathbb{R}$. Range: $(-\infty, 4]$.

Finding Real Roots

Find any and all real roots of the quadratic functions. Classify the scenario: (I) one repeated real root, (II) two distinct real roots, or (III) no real roots.

1. $p(x) = x^2 - x - 2$
2. $q(x) = x^2$
3. $r(x) = x^2 - 6x + 9$
4. $f(x) = 2x^2 + x + 4$
5. $h(x) = 3x^2 + x + 12$
6. $g(x) = 2x^2 - x - 1$

Answer

1. $x = -1, 2$ (II)
2. $x = 0, 0$ (I)
3. $x = 3, 3$ (I)
4. No real roots (III)
5. No real roots (III)
6. $x = -\frac{1}{2}, 1$ (II)

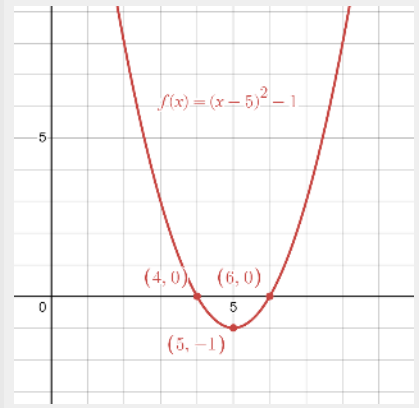
Sketching Graphs of Quadratic Functions

Sketch the graphs of the functions (without needing technology).

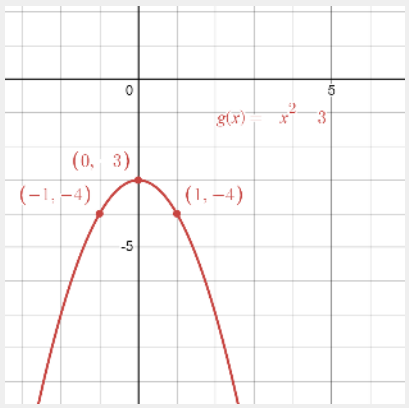
1. $f(x) = (x-5)^2 - 1$
2. $g(x) = -x^2 - 3$
3. $h(x) = x^2 + 2x - 15$

Answer

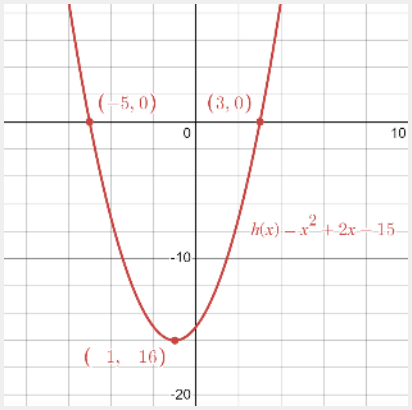
1.



2.



3.



Parabolic Motion

When a ball is launched straight upward from the ground with an initial velocity v_0 ft/s, its height in feet after t seconds is given by $h(t) = v_0 t - 16t^2$. Let v_0 be 32 ft/s and convert this function into vertex form. What is the maximum height achieved by the ball, and when does it begin to fall down? When will it hit the ground again?

Repeat the process for $v_0 = 160$ ft/s.

Answer

For $v_0 = 32$ ft/s: $h(t) = -16(t-1)^2 + 16$. Max height: 16ft, reached at time $t = 1$s. It hits the ground after 2s.

For $v_0 = 160$ ft/s: $h(t) = -16(t-5)^2 + 400$. Max height: 400ft, reached at time $t = 5$s. It hits the ground after 10s.

? Maximizing Profits

If my monthly revenue function (money coming in) at De Wolf's De Washing Machines is given by $R(x) = 1200x$, where x is the number of machines I sell, and my monthly cost function (money going out) is given by $C(x) = x^2 + 400x$, find the quadratic function that calculates my profits. What will the profit be if 100 machines sell? How many machines sold would result in a profit of $120,000? Find the maximum possible profit and the number of machines sold to achieve it.

Answer

$P(x) = R(x) - C(x) = 800x - x^2$. If 100 machines sell, profit is $70,000. Profit will be $120,000 if $x = 200$ or 600.

Converted to vertex form: $P(x) = -(x-400)^2 + 160000$. Max profit: $160,000, when $x = 400$.

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5.3 Polynomial Functions

Learning Objectives

By the end of this section, you will be able to:

- identify polynomial functions and their important features: degree, leading coefficient, roots, end behavior
- write polynomial functions in standard form
- find real roots in easy factoring situations
- identify the degree of a polynomial function based on its graph and understand how features of the graph relate to the function

In the book *Watership Down*, the rabbits can't really count. In their rabbit language, they have the counting words, "one, two, three, four.....*many*." In our survey of polynomial functions, we've done degree-one (linear) and degree-two (quadratic) polynomials, and now we're going to open it up to general degree-many. Oh, I mean **degree-**n.

Polynomial Functions

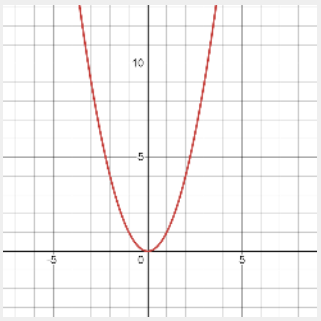
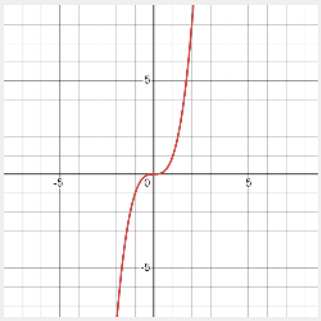
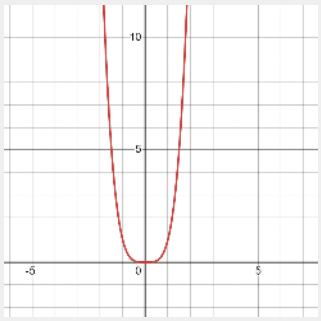
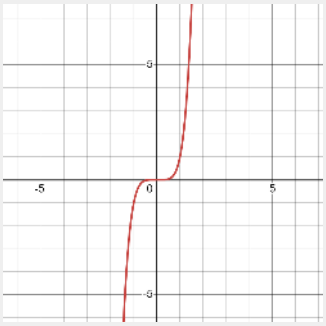
A **polynomial function** is a function that can be written in the form

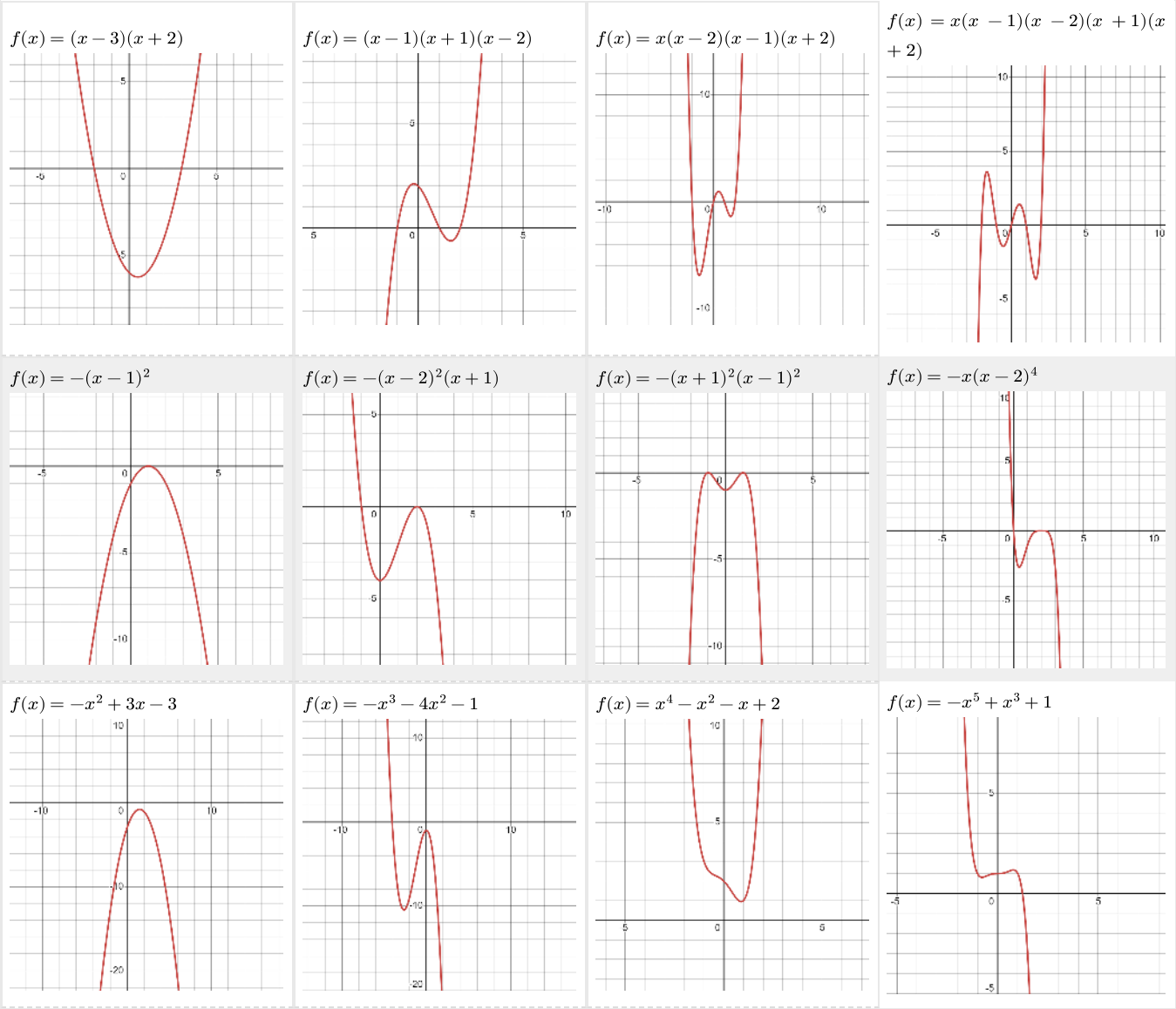
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer, $a_n \neq 0$, and $a_0, a_1, ..., a_n$ are constants. Fun facts about polynomial functions:

- The highest appearing power n is called the **degree.**
- These functions always have **all real numbers** as their **domain.**
- The **range** of a polynomial function depends on whether its degree is even or odd and the sign of its leading coefficient. We'll talk about this and end behavior later.
- The x-intercepts are called the **roots** or **zeros** of the polynomial function. The maximum number of real roots for a given degree-n polynomial, counting with multiplicity, is n.
- The graphs of polynomials are always nicely smoothly curved, with no skips or jumps or corners.

Let's be structured about how we look at example graphs. A lot can be told about the shape of a function's graph by knowing the degree and sign of the leading coefficient of the polynomial, and vice versa. Carefully study the following table column-by-column and try to notice any patterns.

degree 2 (quadratic)	degree 3 (cubic)	degree 4 (quartic)	degree 5
$f(x) = x^2$ 	$f(x) = x^3$ 	$f(x) = x^4$ 	$f(x) = x^5$ 

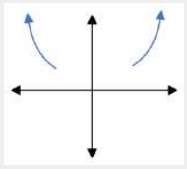
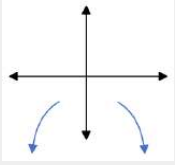
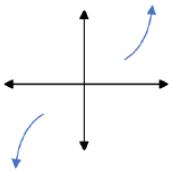
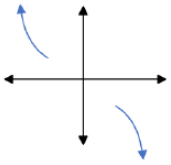


Here are the takeaway observations:

- Monomials like $f(x) = x^n$ for even power n kind of look like parabolas but flattening out more at the origin. If n is odd, they look like the basic cubic, but flattening out more at the origin.
- Even-degree polynomials always come in and go out in the same direction. Look at degree-two and degree-four. There may be some squiggles in the middle, but in the end they will either go up forever or down forever in both x directions. If the leading coefficient is positive, they will open upward. If the leading coefficient is negative, they will open downward.
- Odd-degree polynomials always come in from one direction and leave in the other. Look at degree-three and degree-five. There may be squiggles in the middle, but eventually they go down forever to the left and up forever to the right, or vice versa. If the leading coefficient is positive, the function generally travels uphill, going down forever as $x \to -\infty$ and up forever as $x \to \infty$. If the leading coefficient is negative, the function generally travels downhill, going up forever as $x \to -\infty$ and down forever as $x \to \infty$.
- As for the squiggles, a degree-n polynomial function can "turn around" (have a peak or a trough) up to $n-1$ times. For example, the degree-four polynomials can't have more than three turnarounds.
- A degree-n polynomial function can have up to n distinct real roots. For example, look at the cubics column. The first one has a single root at 0 (and it has multiplicity 3, which we'll talk about in five seconds, just hang on). The second has three distinct roots. The third has two distinct roots, and the one at $x = 2$ has multiplicity 2, causing that bouncing-off behavior. The fourth

has only one real root again (and secretly a complex conjugate pair of roots, otherwise). A cubic can't cross the x-axis more than three times, so it can only have up to 3 distinct real roots.

Here's the end behavior observations organized into a table.

degree vs. sign of leading coefficient	$a_n > 0$	$a_n < 0$
n **even**	• as $x \to \infty$, $f(x) \to \infty$ • as $x \to -\infty$, $f(x) \to \infty$ 	• as $x \to \infty$, $f(x) \to -\infty$ • as $x \to -\infty$, $f(x) \to -\infty$ 
n **odd**	• as $x \to \infty$, $f(x) \to \infty$ • as $x \to -\infty$, $f(x) \to -\infty$ 	• as $x \to \infty$, $f(x) \to -\infty$ • as $x \to -\infty$, $f(x) \to \infty$ 

Example 1

Tell whether the functions are polynomial functions, and if necessary, write them in standard form. If they are, identify the degree and the leading coefficient.

1. $f(x) = x^{16} + x^{11} - x^3 + 1$
2. $g(x) = x^{-2} + x^{-1} + x + x^2$
3. $h(x) = -2(x+1)(x-2)^2$
4. $p(x) = \sqrt{x^3 + x}$
5. $q(x) = 3x^5 + (x-1)^2$
6. $r(x) = \dfrac{x^3 - 1}{x^2 + 2}$

Solution

1. Yes, the degree is 16 and the leading coefficient is a secret 1.
2. No, the negative powers of x are not allowed in a polynomial.
3. Yes, when expanded this becomes $h(x) = -2x^3 + 6x^2 - 8$, so the degree is 3 and the leading coefficient is -2.
4. No, the square root is not allowed.
5. Yes, when expanded and simplified this becomes $q(x) = 3x^5 + x^2 - 2x + 1$, so the degree is 5 and the leading coefficient is 3.
6. No, this is two polynomials being divided.

Exercise 1

Tell whether the functions are polynomial functions, and if necessary, write them in standard form.

1. $f(x) = x^3 + x + \dfrac{1}{x}$
2. $g(x) = 1 - x - x^2 - x^3 - x^4$
3. $h(x) = (x^2 + 1)^{\frac{3}{2}}$
4. $p(x) = x^2 - 7x^3 + x$
5. $q(x) = x^7(x-1)(x-2)$
6. $r(x) = \dfrac{x^5 + x + 2}{x^7 + 9}$

Answer

1. No.
2. Yes, degree 4, leading coefficient -1.
3. No.
4. Yes, in order, $p(x) = -7x^3 + x^2 + x$, so degree 3 and leading coefficient -7.
5. Yes, $q(x) = x^9 - 3x^8 + 2x^7$, so degree 9 and leading coefficient 1.
6. No.

Example 2

For the functions in the previous example that turned out to be polynomials, describe the end behavior.

1. $f(x) = x^{16} + x^{11} - x^3 + 1$
2. $h(x) = -2(x+1)(x-2)^2$
3. $q(x) = 3x^5 + (x-1)^2$

Solution

1. This has even degree and positive leading coefficient, so as $x \to \infty$, $f(x) \to \infty$, and as $x \to -\infty$, $f(x) \to \infty$.
2. This has odd degree and negative leading coefficient (try to see why, or look back at the expanded standard form we found), so as $x \to \infty$, $h(x) \to -\infty$, and as $x \to -\infty$, $h(x) \to \infty$.
3. This has odd degree and positive leading coefficient, so so as $x \to \infty$, $q(x) \to \infty$, and as $x \to -\infty$, $q(x) \to -\infty$.

Exercise 2

Do the same for the polynomials in the previous exercise.

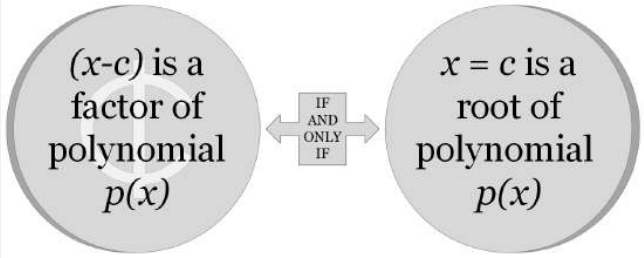
1. $g(x) = 1 - x - x^2 - x^3 - x^4$
2. $p(x) = x^2 - 7x^3 + x$
3. $q(x) = x^7(x-1)(x-2)$

Answer

1. As $x \to \infty$, $g(x) \to -\infty$, and as $x \to -\infty$, $g(x) \to -\infty$.
2. As $x \to \infty$, $p(x) \to \infty$, and as $x \to -\infty$, $p(x) \to \infty$.
3. As $x \to \infty$, $q(x) \to \infty$, and as $x \to -\infty$, $q(x) \to -\infty$.

Roots and Multiplicity

Here's a big ol' key to relating the equation of a polynomial to its roots. Remember how we found roots of quadratic functions? We said, "If you want to be a root, you gotta be on the x-axis, which means you gotta have a y-coordinate of 0, which means the function value spit out needs to be 0." So we set the function equal to 0 and solved for x, and generally the easiest way to do that was to factor the quadratic. Then if any factor is zero, the entire expression will be wiped out, so we were able to see immediately what values of x would result in a root. When we factor a polynomial, the factors that appear and the roots of the function are two sides of the same coin.



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When factoring a polynomial, you will break it down into a set of factors that are either **linear factors** of the form $(x-c)$, or **irreducible quadratic factors** of the form (ax^2+bx+c), where that quadratic does **not** have real roots; i.e. the discriminant b^2-4ac turned out to be negative. So finding all real roots of a polynomial is as simple as looking at the linear factors $(x-c)$ and writing down a list of the c's. Note: you may have quadratic factors that you **can** find ugly real roots for using the quadratic formula, which are not considered irreducible. Technically you would still be able to write linear factors for them, it's just gross like

$$p(x)=\left(x+\frac{\sqrt{5}}{2}\right)\left(x-\frac{\sqrt{5}}{2}\right)(x-1).$$

Now, I mentioned this in the quadratic functions section, but there is a concept called **multiplicity,** which we can be precise about now that we know the factor-root association fact above.

The real number c is a **root (zero)** of a polynomial function $p(x)$ if and only **(iff)** the factor $(x-c)$ appears in the factorization of $p(x)$. The **multiplicity** of the root c is the number of times that factor is repeated in the factorization. So if

$$p(x)=(\cdots)\cdots(x-c)^n\cdots(\cdots)$$

then the multiplicity of root c is n.

We'll learn some tactics for factoring higher-degree polynomials in the next section, but for now I'm hard at work in the kitchen cooking up these examples so that the factoring is manageable. Don't say I never gave you nothin'. And don't forget about the factoring by grouping technique!

Example 3

Find all real roots of the polynomial functions, with multiplicities.

1. $f(x)=x^3-x^2-2x$
2. $g(x)=(x-3)(x^2-x-6)$
3. $p(x)=2x^3-2x+x^2-1$
4. $q(x)=x^3+x$
5. $h(x)=x^5$

Solution

1. First, factor completely like so: $f(x)=x(x^2-x-2)=x(x-2)(x+1)$. Notice that the factor of x could be seen as $(x-0)$, and the factor $(x+1)$ should be seen as $(x-(-1))$. Then we can see that the c's that appear in those factors are $-1, 0,$ and 2. Each of those factors appears only once, so the multiplicity of each root is 1.
2. This is already partially factored, so finish the job to get $g(x)=(x-3)(x-3)(x+2)=(x-3)^2(x+2)$. We read the roots off as $x=-2, 3$, but we notice that there are two copies of the factor $(x-3)$. The root $x=3$ has multiplicity 2, and $x=-2$ has multiplicity 1.
3. Factoring by grouping, we have $p(x)=2x(x^2-1)+(x^2-1)=(2x+1)(x^2-1)=(2x+1)(x+1)(x-1)$. If you're unsure what to do with that $(2x+1)$, remember that what's going on behind the scenes of solving for roots is setting each factor equal to zero and solving for x. So from $2x+1=0$ we get a root $x=-\frac{1}{2}$. The other roots are $x=-1, 1$. All three have multiplicity 1 again.
4. Factoring out the x in common, we have $q(x)=x(x^2+1)$. That x factor out front tells us a real root $x=0$, but the (x^2+1) factor is an irreducible quadratic with no real roots (try setting it equal to zero—you will end up trying to square root a negative number). So the only real root we report is $x=0$, with multiplicity 1.
5. Remember that this can be thought of as $h(x)=(x-0)^5$, so we have a real root of $x=0$, with multiplicity 5.

Exercise 3

Find all real roots of the polynomial functions, with multiplicities.

1. $f(x)=(x-17)^4$
2. $g(x)=x^3+9x$
3. $h(x)=x(x+1+2x^2+2x)$

4. $p(x) = (x+8)(x^2+9x+8)$

Answer

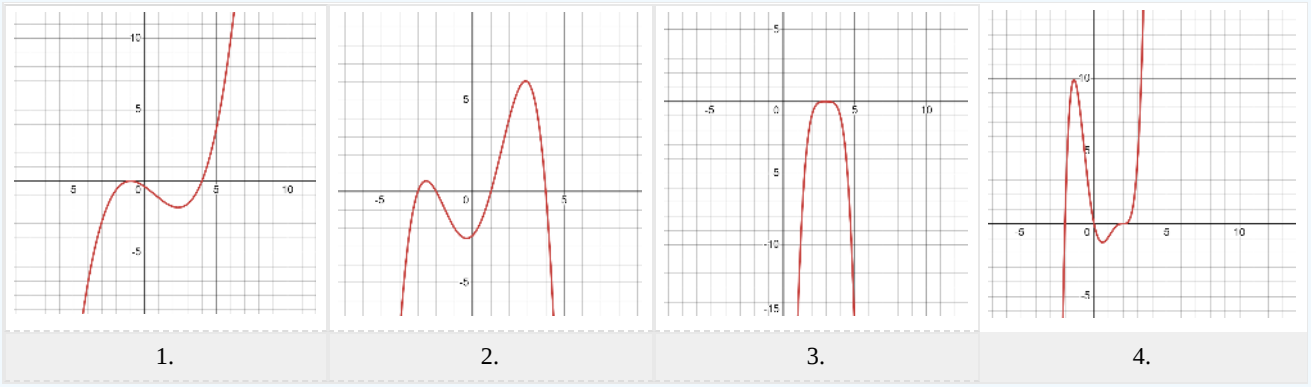
1. $x = 17$, mult 4.
2. $x = 0$, mult 1 (the other factor is an irreducible quadratic).
3. $x = -1, -\frac{1}{2}, 0$, all mult 1 (try grouping).
4. $x = -8$ with mult 2, $x = -1$ with mult 1.

The last thing I want to talk about in this section is how to recognize multiplicities from graphs. If you look back at the table of polynomial examples, you will notice a couple things.

- When a root has multiplicity 1, the graph passes straight through the x-axis at that point.
- When a root has multiplicity 2, the graph bounces off the x-axis at that point.
- When a root has multiplicity 3, the graph passes through the x-axis, but it hangs out for a second, flattening out a bit.
- When a root has multiplicity 4, the graph bounces off the x-axis, but it hangs out for a second, flattening out a bit.
- The pattern continues: pass through the x-axis at odd multiplicity roots, but don't cross over at even multiplicity roots, and the higher the multiplicity, the more flattening behavior.

Example 4

Based on the graph given, determine a possible set of real roots with their multiplicities, write the factors that must appear in the factorization of the polynomial function, and use end behavior to determine the sign of the leading coefficient.



1.	2.	3.	4.

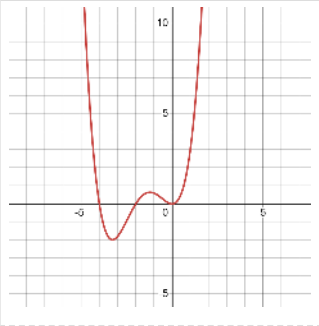
Solution

1. The graph bounces off at $x = -1$, so that's a root with multiplicity 2. It passes straight through $x = 4$, so that's a root with multiplicity 1. The factors $(x+1)^2$ and $(x-4)$ must appear. The graph comes in from below and leaves going up, so it is indeed a degree-3 polynomial with positive leading coefficient.
2. The graph passes straight through $x = -3, -2, 1, 4$, so those are all roots with multiplicity 1. The factors that must appear are $(x+3)(x+2)(x-1)(x-4)$. This graph comes in from below and leaves going down, so it is indeed a degree-4 polynomial with negative leading coefficient.
3. The graph hangs out a bit at $x = 3$ but ultimately doesn't pass through, so this looks like a root with multiplicity 4, so the factors must be $(x-3)^4$. Again, coming in from below and leaving going down tells us it's a degree-4 polynomial with negative leading coefficient.
4. The graph passes straight through $x = -2$ and $x = 0$, roots with multiplicity 1. It hangs out a bit and then passes through at $x = 2$, so that's a root with multiplicity 3. The factors are $x(x+2)(x-2)^3$, and this is a degree-5 polynomial with positive leading coefficient.

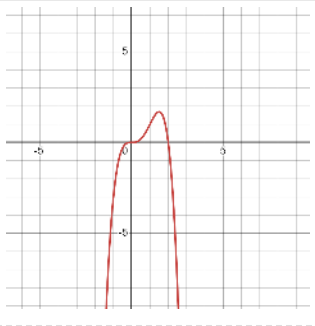
Now you try, and then hit the exercises!

Exercise 4

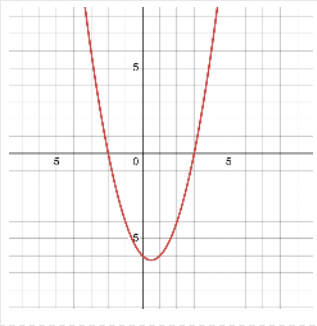
Based on the graph given, determine a possible set of real roots with their multiplicities, write the factors that must appear in the factorization of the polynomial function, and use end behavior to determine the sign of the leading coefficient.



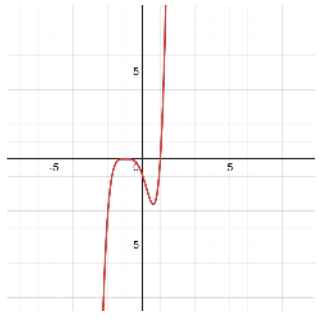
1.



2.



3.



4.

Answer

1. Roots: -4 (mult 1), -2 (mult 1), 0 (mult 2). Factors: $x^2(x+4)(x+2)$. Degree-4 with positive leading coefficient.
2. Roots: 0 (mult 3), 2 (mult 1). Factors: $x^3(x-2)$. Degree-4 with negative leading coefficient.
3. Roots: -2 and 3, both mult 1. Factors: $(x+2)(x-3)$. Quadratic with positive leading coefficient.
4. Roots: -1 (mult 4), 1 (mult 1). Factors: $(x+1)^4(x-1)$. Degree-5 with positive leading coefficient.

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5.3E Exercises

Identifying Polynomial Functions

Tell whether the functions are polynomial functions, and if necessary, write them in standard form. If they are polynomial functions, identify the degree and the leading coefficient.

1. $f(x) = 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5$
2. $g(x) = \sqrt{x^2 - 3x + 5}$
3. $h(x) = x^{-2} + x + x^7$
4. $p(x) = x(x + 1) + x^2(1 - x)$
5. $q(x) = (x - 3)(x + 1)(x - 4)^2$
6. $r(x) = \dfrac{x}{x^2 + 2}$

Answer

1. Yes, degree 5, leading coefficient 32.
2. No.
3. No.
4. Yes, degree 3, leading coefficient -1. Standard form: $p(x) = -x^3 + 2x^2 + x$.
5. Yes, degree 4, leading coefficient 1. Standard form: $q(x) = x^4 - 10x^3 + 29x^2 - 8x - 48$.
6. No.

Describing End Behavior

Describe the end behavior of the polynomial functions.

1. $f(x) = -x^3 + x$
2. $f(x) = x^2 + 2x + 1$
3. $f(x) = 2(x + 1)^5$
4. $f(x) = -x(x + 7)$
5. $f(x) = -2x^8 + x^4 + 16$
6. $f(x) = 12x + 12$

Answer

1. As $x \to -\infty$, $f(x) \to \infty$. As $x \to \infty$, $f(x) \to -\infty$.
2. As $x \to -\infty$, $f(x) \to \infty$. As $x \to \infty$, $f(x) \to \infty$.
3. As $x \to -\infty$, $f(x) \to -\infty$. As $x \to \infty$, $f(x) \to \infty$.
4. As $x \to -\infty$, $f(x) \to -\infty$. As $x \to \infty$, $f(x) \to -\infty$.
5. As $x \to -\infty$, $f(x) \to -\infty$. As $x \to \infty$, $f(x) \to -\infty$.
6. As $x \to -\infty$, $f(x) \to -\infty$. As $x \to \infty$, $f(x) \to \infty$.

Finding Real Roots With Easy Factoring

Find all real roots of the polynomial functions, with multiplicities.

1. $p(x) = x^3 - x$
2. $p(x) = (x + 1)(x^2 + 2x + 1)$
3. $p(x) = 3x^3 - 12x - x^2 + 4$
4. $p(x) = x^6 - 3x^5$
5. $p(x) = (x + 2)(x^2 - 4)$
6. $p(x) = (x - 4)(x - 3)(x^2 - x - 2)$

Answer

1. $x = -1, 0, 1$ (all mult 1).

2. $x = -1$ (mult 3).
3. $x = \frac{1}{3}, -2, 2$ (all mult 1).
4. $x = 0$ (mult 5), $x = 3$ (mult 1).
5. $x = -2$ (mult 2), $x = 2$ (mult 1).
6. $x = 4, 3, 2, -1$ (all mult 1).

? Quadratic Factors

The following polynomial functions are partially factored. Determine if the quadratic factors are irreducible. List all real roots.

1. $f(x) = (x-2)(x^2+7x+6)$
2. $g(x) = (x^2-1)(x^2+1)$
3. $h(x) = x(x^2-x+12)$
4. $p(x) = (x+2)(x-1)(x^2+9)$
5. $q(x) = (x^3-1)(x^2+2x+10)$

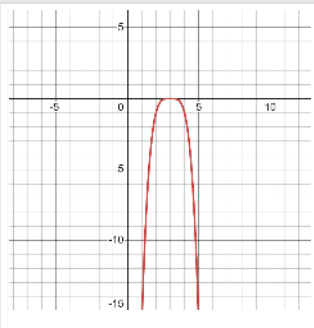
Answer

1. The quadratic factors further as $(x+1)(x+6)$. Roots: $x = -6, -1, 2$.
2. (x^2+1) is irreducible, but $(x^2-1) = (x+1)(x-1)$ gives roots $x = \pm 1$.
3. The discriminant $b^2 - 4ac = 1 - 48$ is negative, so the quadratic is irreducible. Only real root is 0.
4. (x^2+9) is irreducible, so roots are $x = -2, 1$.
5. The discriminant $b^2 - 4ac = 4 - 400$ is negative, so it's irreducible. The other factor does give a real root of $x = 1$.

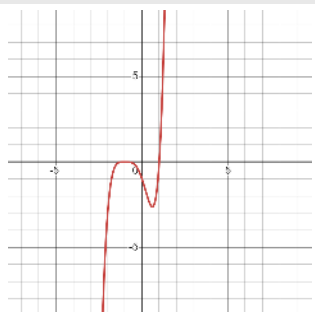
? Matching Polynomial Functions to Graphs

Match the following polynomial functions to the graphs given below.

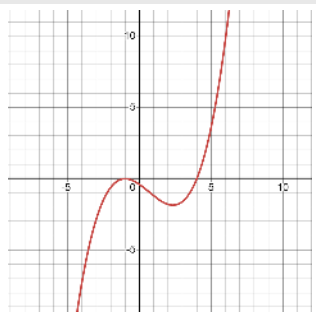
1. $p(x) = (x+1)^4(x-1)$
2. $p(x) = 0.1(x+1)^2(x-4)$
3. $p(x) = -(x-3)^4$
4. $p(x) = 0.3x(x+2)(x-2)^3$



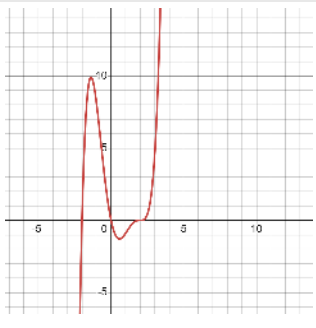
A.



B.



C.



D.

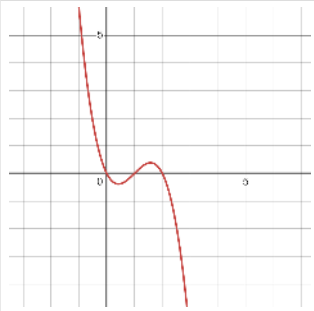
Answer

The funky leading coefficients are necessary to make the graphs fit nicely, lol.

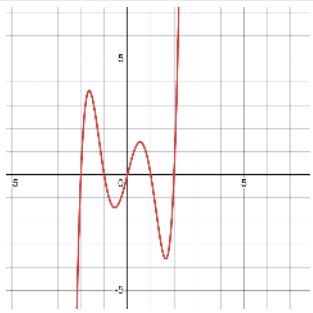
1. B
2. C
3. A
4. D

Deducing Information From Graphs

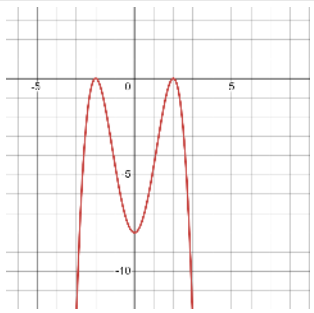
Based on the graph given, determine a possible set of real roots with their multiplicities, write the factors that must appear in the factorization of the polynomial function, and use end behavior to determine the sign of the leading coefficient.



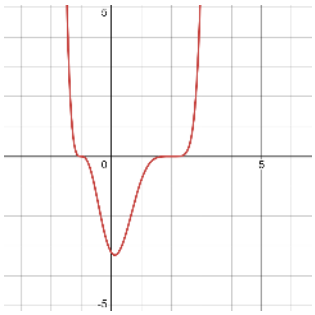
1.



2.



3.



4.

Answer

1. Roots: $x = 0, 1, 2$ (mult 1). Factors: $x(x-1)(x-2)$. Degree-3, negative leading coeff.
2. Roots: $x = -2, -1, 0, 1, 2$ (mult 1). Factors: $x(x-1)(x+1)(x-2)(x+2)$. Degree-5, positive leading coeff.
3. Roots: $x = \pm 2$ (both mult 2). Factors: $(x-2)^2(x+2)^2$. Degree-4, negative leading coeff.
4. Roots: $x = -1$ (mult 3), $x = 2$ (mult 5). Factors: $(x+1)^3(x-2)^5$. Degree-8, positive leading coeff.

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5.4 Roots of Polynomial Functions

Page ID 155141

Learning Objectives

By the end of this section, you will be able to:

- perform polynomial long division
- factor polynomials using inspection and polynomial division
- apply the Rational Root Theorem to find roots
- factor higher-degree polynomials and find roots using a combination of techniques

When we're at a loss to factor a quadratic function, we can always fall back on the quadratic formula, but we're not so lucky in higher degrees. There's a closed formula for cubics but it sucks to use, quartics get truly demented, and everything is terrible after that. So really, we just want to figure out how to factor as best we can to break higher-degree polynomials down into linear and quadratic factors. A tool we need in our toolbox for this (and other math courses) is called **polynomial long division.**

Polynomial Long Division

We want to divide a **(dividend)** polynomial $p(x)$ by another polynomial **divisor** $d(x)$, resulting in some kind of **quotient** polynomial $q(x)$ and possibly a **remainder** polynomial $r(x)$. Forget polys for now and think about numbers. If I divide 37 by 5, I can write two (true) equations

$$\frac{p}{d} = q + \frac{r}{d} \quad \rightarrow \quad \frac{37}{5} = 7 + \frac{2}{5}$$

$$p = qd + r \quad \rightarrow \quad 37 = 7(5) + 2$$

Make sure you agree with me about those statements. The same statements are true when I divide polynomials instead of numbers.

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)} \quad \text{and} \quad p(x) = q(x) \cdot d(x) + r(x)$$

Each of the ingredients in the above equations may be polynomials of some sort, whether constants, linear functions, quadratics, etc. The only rule is that the degree of $r(x)$ must be smaller than the degree of $d(x)$.

So given a division of $p(x)$ by $d(x)$, how do we figure out $q(x)$ and $r(x)$? Here's a summary in English:

1. Set up a long division bar with $p(x)$ underneath and the divisor $d(x)$ out front.
2. Compare the first terms of $p(x)$ and $d(x)$ and ask yourself, "What do I need to multiply $d(x)$ by in order to make these first terms match exactly?" Write what is needed above the long division bar.
3. Actually perform the multiplication through $d(x)$, writing the result underneath the $p(x)$. Subtract down the relevant columns, resulting in a remainder. Bring down the next term in the dividend $p(x)$.
4. Compare the first terms of the remainder and the $d(x)$, and repeat Steps 2-3.
5. Continue until you run out of terms to bring down.
6. The terms you wrote above the long division bar give you the quotient $q(x)$, and the final remainder gives you $r(x)$.

It's much easier to learn this via example, though.

Example 1

Divide $p(x) = 2x^2 + 12x - 4$ by $d(x) = x - 2$. Write the result in the form $p(x) = q(x) \cdot d(x) + r(x)$, and in the form $\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$.

Solution

1. Set up the problem:

$$x-2\overline{)\,2x^2+12x-4}$$

2. Compare the first terms: $p(x)$ has $2x^2$ and $d(x)$ has x. If I multiply x by $2x$, it will match the $2x^2$. Write $2x$ above the bar, lined up properly.

$$\begin{array}{r} 2x \\ x-2\overline{)\,2x^2+12x-4} \end{array}$$

3. Perform the multiplication $(2x)(x-2)$ and write it lined up below $p(x)$, and subtract down the appropriate columns. Bring down the next term (shown in red).

$$\begin{array}{r} 2x \\ x-2\overline{)\,2x^2+12x-4} \\ -\ (2x^2-4x) \\ \hline 0+16x-4 \end{array}$$

4. Compare the first terms x and $16x$. To match, we must multiply the x by 16. Write that above, multiply it through $d(x)$, and write the result underneath. Subtract down the columns.

$$\begin{array}{r} 2x+16 \\ x-2\overline{)\,2x^2+12x-4} \\ -\ (2x^2-4x) \\ \hline 0+16x-4 \\ -\ (16x-32) \\ \hline 0+28 \end{array}$$

5. There is no other term to bring down, and our remainder is just a constant, so we stop here. The quotient is $q(x)=2x+16$ and the remainder is $r(x)=28$. Our final answer is

$$2x^2+12x-4=(2x+16)(x-2)+28 \quad \text{or} \quad \frac{2x^2+12x-4}{x-2}=2x+16+\frac{28}{x-2}$$

Exercise 1

Divide $p(x)=3x^2+9x+12$ by $d(x)=x+1$. Write the result in the form $p(x)=q(x)\cdot d(x)+r(x)$, and in the form $\frac{p(x)}{d(x)}=q(x)+\frac{r(x)}{d(x)}$.

Answer

$$\begin{array}{r} 3x+6 \\ x+1\overline{)\,3x^2+9x+12} \\ -\ (3x^2+3x) \\ \hline 0+6x+12 \\ -\ (6x+6) \\ \hline 0+6 \end{array}$$

We have

$$3x^2+9x+12=(3x+6)(x+1)+6 \quad \text{or} \quad \frac{3x^2+9x+12}{x+1}=3x+6+\frac{6}{x+1}$$

ALERT: One important thing to watch out for is when dividends are missing certain powers of x terms! **Before** performing polynomial long division, **insert placeholders with zero coefficients** as needed into your $p(x)$. For example:

$$x^3-1 \quad \rightarrow \quad x^3+0x^2+0x-1$$

Otherwise you will run into trouble and nonsense when you try to line your columns up and subtract correctly.

There's no reason you can only divide by using linear polynomials as $d(x)$. Let's step it up with a more complex example and exercise.

Example 2

Divide $p(x) = x^4 - 1$ by $d(x) = x^2 - 1$. Write the result in the form $p(x) = q(x) \cdot d(x) + r(x)$, and in the form $\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$.

Solution

First, I insert placeholders: $p(x) = x^4 + 0x^3 + 0x^2 + 0x - 1$. I write this below the long division sign and put $d(x)$ out front.

$$x^2 - 1 \overline{)\, x^4 + 0x^3 + 0x^2 + 0x - 1}$$

Comparing the first terms, I need to multiply x^2 by x^2 to make it match the x^4 so that they will kill each other off when I subtract. Place that up above, and perform the multiplication, writing it underneath in the appropriate columns. Subtract down the columns and bring down the next term(s).

$$\begin{array}{r} x^2 \\ x^2 - 1 \overline{)\, x^4 + 0x^3 + 0x^2 + 0x - 1} \\ -\ (x^4 \qquad - x^2) \\ \hline 0 + 0 + x^2 + 0x - 1 \end{array}$$

Now we compare the first terms and see that x^2 already matches x^2. When this happens, you don't need to multiply by anything but 1, so put a 1 above the division bar. We perform the multiplication and write it underneath, lined up correctly. Subtract again.

$$\begin{array}{r} x^2 \quad + 1 \\ x^2 - 1 \overline{)\, x^4 + 0x^3 + 0x^2 + 0x - 1} \\ -\ (x^4 \qquad - x^2) \\ \hline 0 + 0 + x^2 + 0x - 1 \\ -\ (x^2 \qquad - 1) \\ \hline 0 \end{array}$$

Notice that on our last subtraction, everything wiped out, leaving a remainder of $r(x) = 0$! This means that the divisor $d(x)$ divides evenly into $p(x)$. We write our final answer:

$$x^4 - 1 = (x^2 + 1)(x^2 - 1) \quad \text{or} \quad \frac{x^4 - 1}{x^2 - 1} = x^2 + 1$$

Exercise 2

Divide $p(x) = x^3 - 27$ by $d(x) = x^2 + 3x + 9$. Write the result in the form $p(x) = q(x) \cdot d(x) + r(x)$, and in the form $\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$.

Answer

$$\begin{array}{r} x - 3 \\ x^2 + 3x + 9 \overline{)\, x^3 + 0x^2 + 0x - 27} \\ -\ (x^3 + 3x^2 + 9x) \\ \hline 0 - 3x^2 - 9x - 27 \\ -\ (-3x^2 - 9x - 27) \\ \hline 0 \end{array}$$

$$x^3 - 27 = (x - 3)(x^2 + 3x + 9) \quad \text{or} \quad \frac{x^3 - 27}{x^2 + 3x + 9} = x - 3$$

Say I'm dividing $p(x)$ by a certain $d(x) = x - c$. Then I can write the result as

$$p(x) = q(x) \cdot (x - c) + r$$

like we've been doing. Here, r is a constant number, because I'm dividing by a degree-one linear factor $x - c$. What would happen if I evaluate $p(x)$ *at* $x = c$? Let's plug it in.

$$p(c) = q(c) \cdot (c - c) + r = 0 + r = r$$

This gives me an interesting fun fact: the function value $p(c)$ is exactly the remainder r that you get when dividing $p(x)$ by $x - c$! Now, on the flip side, say I'm in the situation of the above exercise, where the remainder turned out to be zero. If that ever happens, you know for a fact that $p(c) = 0$, telling you that c is indeed a root.

Rational Root Theorem

We're going to pick up another tool to use with polynomial long division when factoring to find roots. Here's the official statement:

Rational Root Theorem

If a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ has **integer** coefficients a_n and a_0 (neither being zero), then every **rational root** of f is of the form $\frac{p}{q}$, where p is a factor of a_0, and q is a factor of a_n.

In other words, to find the list of possible rational number roots of f, make two lists,

- all the integer factors $\pm p_1, \pm p_2, \pm p_3 \ldots$ of the constant term a_0,
- all the integer factors $\pm q_1, \pm q_2, \pm q_3 \ldots$ of the leading coefficient a_n,

and then list all possible combinations $\pm \frac{p_i}{q_j}$ where the numerator is from the first list and the denominator is from the second list. These are the only rational numbers that could be roots of f, and you can check which are actually roots by evaluating f to see if you get zero there.

Example 3

1. List all rational numbers that could be roots of $f(x) = 6x^3 + 23x^2 - 6x - 8$. Test the options until you find two that are in fact roots. You can use a calculator this time to speed things up.

2. Which of the following polynomials might have a root at $x = \frac{5}{4}$?

- $f(x) = 12x^4 + x^2 + 10$
- $g(x) = 6x^3 - 2x^2 - 5$
- $h(x) = 20x^5 + 15x^3 - 17x^2 + x - 20$

Solution

1. We first make the two lists of factors:

- $a_0 = -8$ has factors $\pm 1, \pm 2, \pm 4, \pm 8$
- $a_n = 6$ has factors $\pm 1, \pm 2, \pm 3, \pm 6$

Now the possible combinations of numerators and denominators are, after simplification and without writing duplicates,

$$\pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{8}{3}, \pm \frac{1}{6}$$

Generally, pick the easiest ones to test out first, especially if you're not using a calculator. Plug the options into the polynomial and see which ones turn out to give 0. The ones that do work are $x = -4, -\frac{1}{2}, \frac{2}{3}$.

2. We are looking for polynomials that have 5 as a factor of their constant term and 4 as a factor of their leading coefficient. This is true of f because 5 divides 10 and 4 divides 12. Since 4 doesn't divide 6, g is disqualified, but h works.

? Exercise 3

1. List all rational numbers that could be roots of $f(x) = 2x^3 + 5x^2 - 4x - 3$. Test the options until you find two that are in fact roots.

2. Which of the following polynomials might have a root at $x = \frac{2}{3}$?

- $f(x) = 12x^4 + x^2 + 10$
- $g(x) = 9x^3 - x - 12$
- $h(x) = 20x^5 + 15x^3 - 17x^2 + x - 20$

Answer

1. Options: $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$. The ones that are actually roots are $x = 1, -3, -\frac{1}{2}$.

2. The ones with compatible a_0 and a_n factors are f and g.

Factoring Higher-Degree Polynomial Functions

Armed with these tools, we are now equipped to factor polynomials, which will also tell us their roots. It's a simple process:

To **factor a polynomial function** f and **find its real roots:**

Step 1: Find a root c by **inspection** (look at the polynomial and making an educated guess at a number easy to plug in, like ± 1 or ± 2, or use the **Rational Root Theorem**, and test options). One you find c, you know $(x - c)$ **must** appear in the polynomial's factorization, like $f(x) = (x - c)(\text{something else})$.

Step 2: Divide f by $(x - c)$ using polynomial division. The quotient $q(x)$ is the (something else) factor.

Step 3: If the $q(x)$ is not a linear factor or an irreducible quadratic, repeat the process on $q(x)$ to further factor it. So on and so forth, until every factor is linear of the form $(x - c)$ or an irreducible quadratic.

Step 4: Determine the real roots using the linear factors.

✓ Example 4

Factor $f(x) = 2x^3 - x^2 - 2x + 1$ and find all real roots.

Solution

First solution: We first check if something easy to plug in will result in f giving 0. The easiest thing to plug in first is 1: $f(1) = 2 - 1 - 2 + 1 = 0$. Yep! So $x = 1$ is a root, and we know that f must factor as $f(x) = (x - 1)(\text{something})$. To determine the other factor, we divide by $(x - 1)$:

$$
\begin{array}{r}
2x^2 + x - 1 \\
x - 1 \overline{)\, 2x^3 - x^2 - 2x + 1} \\
\underline{-\ (2x^3 - 2x^2)} \\
0 + x^2 - 2x \\
\underline{-\ \ (x^2 - x)} \\
0 + -x + 1 \\
\underline{-\ \ (-x + 1)} \\
0
\end{array}
$$

So now we have $f(x) = (x - 1)(2x^2 + x - 1)$, and we can factor the quadratic further using any preferred method to get a final answer of $f(x) = (x - 1)(2x - 1)(x + 1)$. The real roots are $x = -1, \frac{1}{2}, 1$.

Second solution: Since this function's a_0 and a_n are very small numbers with limited factor options, it would also be quick to just make the rational roots list and test all the options to see if we get three roots. Remember that since f is degree-3, it can't possibly have more than three distinct real roots! If we find them all we'll be done. Using the Rational Root Theorem, we have

a short list of possibilities: $\pm 1, \pm \frac{1}{2}$. We already saw $f(1) = 0$, and $f(-1) = -2 - 1 + 2 + 1 = 0$, so that's good too. Then $f\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{4} - 1 + 1 = 0$ is also easy to compute. We stop there, having found three real roots!

Third solution: This function is a great candidate for factoring by grouping. We start with $f(x) = x^2(2x - 1) - (2x - 1)$ to get $f(x) = (x^2 - 1)(2x - 1)$. Then we recognize the difference of squares and factor completely as $f(x) = (x + 1)(x - 1)(2x - 1)$.

Try it yourself below, and then get more practice in the exercises section!

? Exercise 4

Factor $f(x) = 3x^4 + 4x^3 - 5x^2 - 2x$ and find all real roots.

Answer

First factor $x(3x^3 + 4x^2 - 5x - 2)$ and focus on the cubic factor. Notice $x = 1$ is a root, so divide by $(x - 1)$ to get to $x(x - 1)(3x^2 + 7x + 2) = x(x - 1)(3x + 1)(x + 2)$. The roots are $x = 0, 1, -\frac{1}{3}, -2$.

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5.4E Exercises

Polynomial Long Division

Divide $p(x)$ by $d(x)$ and write the result in both forms: $p(x) = q(x) \cdot d(x) + r(x)$ and $\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$.

1. $p(x) = 2x^3 + 5x^2 + x - 2$, $d(x) = x + 2$
2. $p(x) = x^4 - x^3 - 9x + 9$, $d(x) = x - 1$
3. $p(x) = 4x^3 + 4x^2 + 11x + 8$, $d(x) = x^2 + 2$
4. $p(x) = x^5 + x^3 + x^2 + 5$, $d(x) = x^2 + 1$

Answer

1. $q(x) = 2x^2 + x - 1$, $r(x) = 0$.
2. $q(x) = x^3 - 9$, $r(x) = 0$.
3. $q(x) = 4x + 4 = 4(x + 1)$, $r(x) = 3x$.
4. $q(x) = x^3 + 1$, $r(x) = 4$.

Rational Root Theorem (RRT)

A.) List all rational numbers that could be roots of f. Test options until you find one that is indeed a root.

1. $f(x) = 5x^4 - 4x^3 + 3x^2 - 2x - 2$
2. $f(x) = 6x^3 + 23x^2 - 6x - 8$
3. $f(x) = 3x^3 - 5x^2 - 11x + 18$

B.) Which of the following polynomial functions might have $x = \frac{3}{4}$ as a root?

1. $f(x) = 7x^3 + 2x^2 - x + 3$
2. $g(x) = 8x^4 - 6x + 6$
3. $h(x) = 32x^5 + 2x^4 - x^2 - x + 15$
4. $p(x) = 4x^6 + 7x^3 + 5$

Answer

A.)

1. $\pm 1, \pm 2, \pm\frac{1}{5}, \pm\frac{2}{5}$, $x = 1$.
2. $\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{2}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}, \pm\frac{1}{6}$, $x = -\frac{1}{2}$.
3. $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18, \pm\frac{1}{3}, \pm\frac{2}{3}$, $x = 2$.

B.) $g(x)$ and $h(x)$

Factoring Polynomials

Factor the polynomial function (using any method you prefer, there may be shortcuts) and find all real roots.

1. $f(x) = 3x^3 - 6x^2 - 27x + 54$
2. $g(x) = 4x^4 - 1$
3. $h(x) = x^4 - 5x^2 + 4$
4. $p(x) = x^3 + 7x^2 + 16x + 12$
5. $q(x) = x^4 - 6x^3 + 10x^2 - 6x + 9$

Answer

1. (Grouping or RRT + division) $f(x) = 3(x - 3)(x + 3)(x - 2)$, roots $x = 2, \pm 3$.

2. (Difference of squares) $g(x) = (2x^2 + 1)(2x^2 - 1)$, first factor does not give real roots, second factor gives real roots $x = \pm\sqrt{\frac{1}{2}}$.
3. (Factoring trick or RRT + Division) $h(x) = (x + 1)(x - 1)(x + 2)(x - 2)$, roots $x = \pm 1, \pm 2$.
4. (RRT + Division) $p(x) = (x + 2)^2(x + 3)$, roots $x = -2$ (mult 2), $x = -3$.
5. (RRT + Division) $q(x) = (x - 3)^2(x^2 + 1)$, roots $x = 3$ (mult 2), second factor does not give real roots.

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5.5 Chapter 5 Study Guide

Definitions

- (Recall) **linear function:** a degree-1 polynomial function of the form $f(x) = a_1x + a_0$, aka $f(x) = mx + b$.
- **average rate of change:** the rate of change of $y = f(x)$ with respect to x between $x = a$ and $x = b$ is $\frac{f(b) - f(a)}{b - a}$, aka like slope of a straight line between two data points $(a, f(a)), (b, f(b))$.
- **secant line:** a straight line drawn between two points on the graph of a function, $(a, f(a))$ and $(b, f(b))$.
- **quadratic function:** a degree-2 polynomial function of the form $f(x) = a_2x^2 + a_1x + a_0$, aka $f(x) = ax^2 + bx + c$.
- **vertex form:** the equation of a parabola in the form $f(x) = a(x - h)^2 + k$, highlighting the location of the vertex (h, k).
- **polynomial function:** a function that can be written in the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$.
- **roots / zeros:** the roots $x = c$ of a polynomial function are the locations of the x-intercepts, found by setting the function equal to 0 and solving for x.
- **multiplicity:** the multiplicity of a root $x = c$ is the number of copies of $(x - c)$ that appear in the polynomial's factorization.
- **linear factors:** factors of the form $(x - c)$.
- **irreducible quadratic factors:** factors of the form $(ax^2 + bx + c)$ that do **not** yield real roots, aka the discriminant $b^2 - 4ac < 0$.
- **polynomial long division:** the process of dividing a polynomial $p(x)$ by a divisor polynomial $d(x)$, resulting in a quotient polynomial $q(x)$ and potentially a remainder $r(x)$. Results are written in the forms

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)} \quad \text{and} \quad p(x) = q(x) \cdot d(x) + r(x)$$

- **end behavior:** how a function behaves as you look off in the positive or negative x-direction forever, written like "as $x \to \infty$, $f(x) \to \infty$" for example.

Facts About Linear Functions

- Both the **domain** and the **range** of a linear function (unless specified due to an application) are **all real numbers, $\mathbb{R}$.**
- The graphs of linear functions are straight lines. If you rename $a_1 = m$ and $a_0 = b$, you can recognize the slope-intercept equation of a line.
- Consequently, the leading coefficient a_1 gives the slope of the line, and the constant term a_0 gives the y-intercept $(0, a_0)$ of the graph.
- Because of what we know about positive and negative slopes, if $a_1 < 0$, the function is **decreasing** (going downhill from left to right) and if $a_1 > 0$, the function is **increasing** (going uphill from left to right).
- Linear functions have **constant rate of change** (slope is the same no matter what two points are used to calculate it). You can check if a table of values defines a linear function by checking if all slope calculations between data points give the same result.

Facts About Quadratic Functions

- The graphs of these functions are **parabolas**, which can open upward or downward. If the leading coefficient $a > 0$, it opens upward. If $a < 0$, it opens downward. There is a point called the **vertex**, which is the peak of the hill if the parabola opens downward, or the bottom of the trough if the parabola opens upward. The parabola is **symmetric** about the vertical line passing through the vertex.
- Quadratics are often written in **vertex form,** $f(x) = a(x - h)^2 + k$, highlighting the location of the parabola's vertex, (h, k). ALERT: be careful with the signs because of that subtraction!
- The **domain** of a quadratic function (unless specified due to an application) is **all real numbers, $\mathbb{R}$.**
- The **range** of a quadratic function is limited due to its parabola shape. Identifying the vertex will help you determine the range, since it will give the **maximum** or **minimum** possible function value.
- The x-intercept(s) of quadratic functions are called **roots** or **zeros.** As always, to find these, set the whole function equal to 0 and solve for x.

- A quadratic can have (a) **one repeated real root**, (b) **two distinct real roots**, or (c) **no real roots** (but rather a complex conjugate pair of roots).
- In case (a), the parabola bounces off the x-axis. In case (b), the parabola intersects the x-axis at two locations. In case (c), the parabola is too low or too high and never intersects the x-axis.

Converting From Standard Form to Vertex Form

To convert from standard form $f(x) = ax^2 + bx + c$ into vertex form $f(x) = a(x-h)^2 + k$,

1. If $a \neq 1$, factor it out like so: $f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$.
2. Working **inside** the parentheses, **complete the square** on the first two terms, $x^2 + \frac{b}{a}x$, making sure to both add **and** subtract the necessary term inside the parentheses.
3. If necessary, distribute the factored-out a back onto the constant term to finally reach the form $f(x) = a(x-h)^2 + k$ for some h and k.

Sketching Quadratic Functions

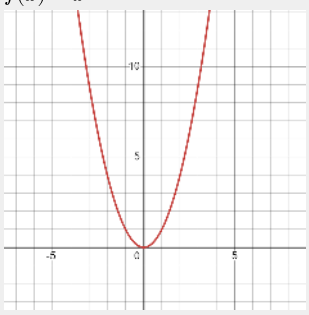
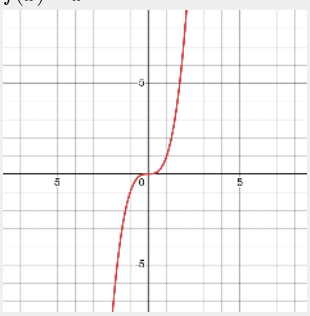
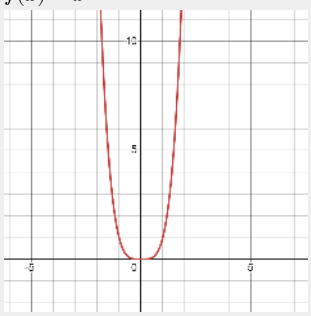
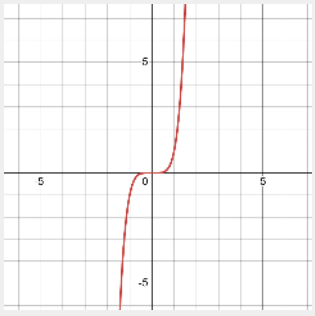
1. Find the vertex form and plot the vertex. Note whether the parabola should open up or down.
2. Find a few more points by plugging in different x-values and computing the function values. You can also choose to find the roots instead of random points.
3. Using knowledge of symmetry and how parabolas look, fit a parabola to the the plotted points smoothly.

Or, use knowledge of transformations to sketch the graph, starting with the most basic $f(x) = x^2$ and performing the appropriate transformations.

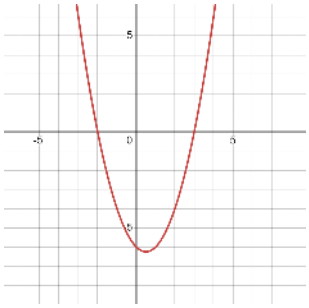
Facts About Polynomial Functions

- The highest appearing power n is called the **degree.**
- These functions always have **all real numbers** as their **domain.**
- The **range** of a polynomial function depends on whether its degree is even or odd and the sign of its leading coefficient. We'll talk about this and end behavior later.
- The x-intercepts are called the **roots** or **zeros** of the polynomial function. The maximum number of real roots for a given degree-n polynomial, counting with multiplicity, is n.
- The graphs of polynomials are always nicely smoothly curved, with no skips or jumps or corners.

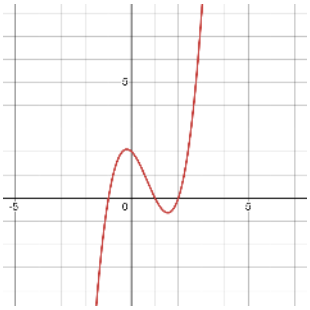
Table of Polynomial Examples

degree 2 (quadratic)	degree 3 (cubic)	degree 4 (quartic)	degree 5
$f(x) = x^2$ 	$f(x) = x^3$ 	$f(x) = x^4$ 	$f(x) = x^5$ 

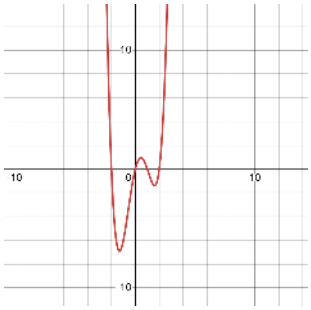
$f(x) = (x-3)(x+2)$



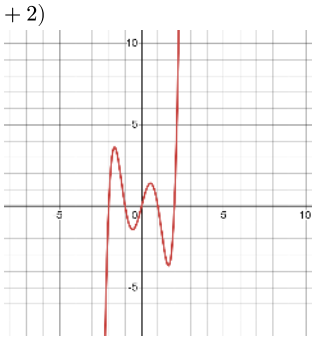
$f(x) = (x-1)(x+1)(x-2)$



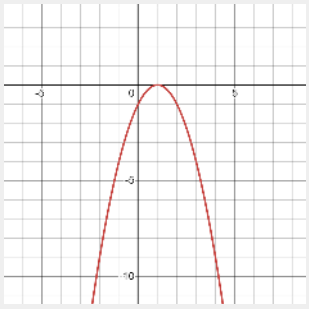
$f(x) = x(x-2)(x-1)(x+2)$



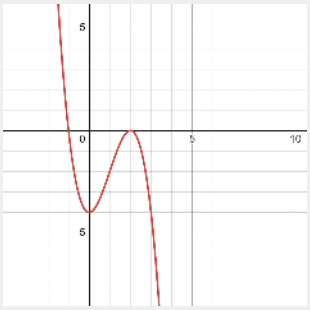
$f(x) = x(x-2)(x-2)(x+1)(x+2)$



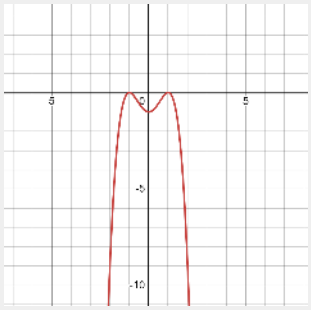
$f(x) = -(x-1)^2$



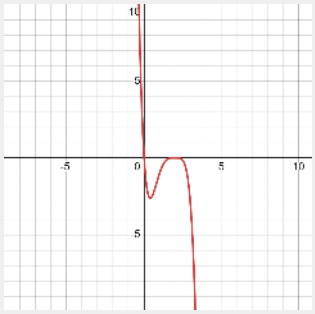
$f(x) = -(x-2)^2(x+1)$



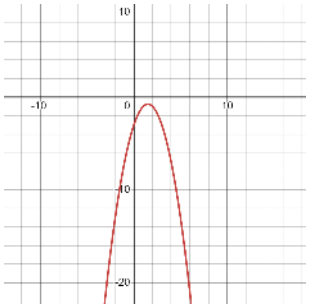
$f(x) = -(x+1)^2(x-1)^2$



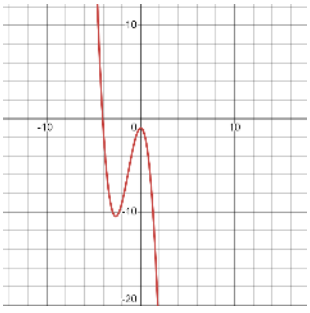
$f(x) = -x(x-2)^4$



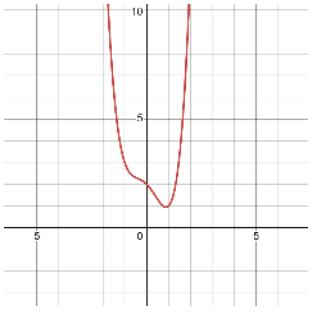
$f(x) = -x^2 + 3x - 3$



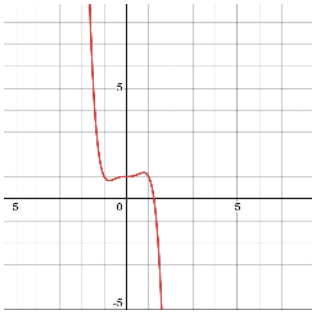
$f(x) = -x^3 - 4x^2 - 1$



$f(x) = x^4 - x^2 - x + 2$



$f(x) = -x^5 + x^3 + 1$

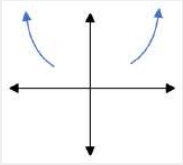
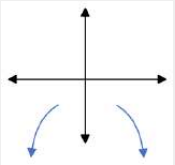
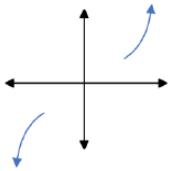
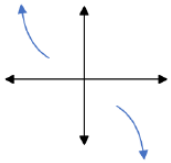


Notes:

- Monomials like $f(x) = x^n$ for even power n kind of look like parabolas but flattening out more at the origin. If n is odd, they look like the basic cubic, but flattening out more at the origin.
- A degree-n polynomial function can "turn around" (have a peak or a trough) up to $n-1$ times. For example, the degree-four polynomials can't have more than three turnarounds.
- A degree-n polynomial function can have up to n distinct real roots.
- End behavior: see table summarizing below.

End Behavior

degree vs. sign of leading coefficient	$a_n > 0$	$a_n < 0$

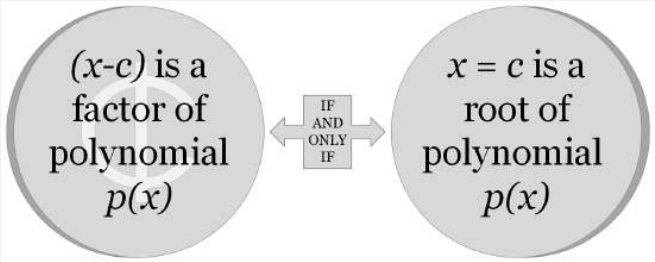
n **even**	• as $x \to \infty$, $f(x) \to \infty$ • as $x \to -\infty$, $f(x) \to \infty$ 	• as $x \to \infty$, $f(x) \to -\infty$ • as $x \to -\infty$, $f(x) \to -\infty$ 
n **odd**	• as $x \to \infty$, $f(x) \to \infty$ • as $x \to -\infty$, $f(x) \to -\infty$ 	• as $x \to \infty$, $f(x) \to -\infty$ • as $x \to -\infty$, $f(x) \to \infty$ 

Roots and Multiplicity

The real number c is a root (zero) of a polynomial function $p(x)$ if and only (iff) the factor $(x - c)$ appears in the factorization of $p(x)$. The multiplicity of the root c is the number of times that factor is repeated in the factorization. So if

$$p(x) = (\cdots)\cdots(x-c)^n\cdots(\cdots)$$

then the multiplicity of root c is n.



Multiplicity and graphs:

- When a root has multiplicity 1, the graph passes straight through the x-axis at that point.
- When a root has multiplicity 2, the graph bounces off the x-axis at that point.
- When a root has multiplicity 3, the graph passes through the x-axis, but it hangs out for a second, flattening out a bit.
- When a root has multiplicity 4, the graph bounces off the x-axis, but it hangs out for a second, flattening out a bit.
- The pattern continues: pass through the x-axis at odd multiplicity roots, but don't cross over at even multiplicity roots, and the higher the multiplicity, the more flattening behavior.

Polynomial Long Division

1. Insert placeholders with zero coefficients for any missing powers of x in the dividend $p(x)$. Set up long division bar with $p(x)$ underneath and $d(x)$ out front.
2. Compare first terms to see what is missing. Write whatever factors are needed to make them match up above the division bar in the appropriate column.
3. Perform the multiplication through $d(x)$, writing it underneath $p(x)$ in appropriate columns. Subtract down the columns. Bring down next term.
4. Compare the first terms, repeat Steps 2-3 until no more terms to bring down.
5. Terms written above the bar are $q(x)$, and remainder is $r(x)$.

Example

Divide $p(x) = x^4 - 1$ by $d(x) = x^2 - 1$. Write the result in the form $p(x) = q(x) \cdot d(x) + r(x)$, and in the form $\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$.

Solution

First, I insert placeholders: $p(x) = x^4 + 0x^3 + 0x^2 + 0x - 1$. I write this below the long division sign and put $d(x)$ out front.

$$x^2 - 1 \overline{)\, x^4 + 0x^3 + 0x^2 + 0x - 1}$$

Comparing the first terms, I need to multiply x^2 by x^2 to make it match the x^4 so that they will kill each other off when I subtract. Place that up above, and perform the multiplication, writing it underneath in the appropriate columns. Subtract down the columns and bring down the next term(s).

$$\begin{array}{r} x^2 \\ x^2 - 1 \overline{)\, x^4 + 0x^3 + 0x^2 + 0x - 1} \\ \underline{-\ (x^4 \qquad - x^2)} \\ 0 + 0 + x^2 + 0x - 1 \end{array}$$

Now we compare the first terms and see that x^2 already matches x^2. When this happens, you don't need to multiply by anything but 1, so put a 1 above the division bar. We perform the multiplication and write it underneath, lined up correctly. Subtract again.

$$\begin{array}{r} x^2 \quad + 1 \\ x^2 - 1 \overline{)\, x^4 + 0x^3 + 0x^2 + 0x - 1} \\ \underline{-\ (x^4 \qquad - x^2)} \\ 0 + 0 + x^2 + 0x - 1 \\ \underline{-\ (x^2 \qquad - 1)} \\ 0 \end{array}$$

Notice that on our last subtraction, everything wiped out, leaving a remainder of $r(x) = 0$! This means that the divisor $d(x)$ divides evenly into $p(x)$. We write our final answer:

$$x^4 - 1 = (x^2 + 1)(x^2 - 1) \quad \text{or} \quad \frac{x^4 - 1}{x^2 - 1} = x^2 + 1$$

Rational Root Theorem

If a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ has **integer** coefficients $a_n \neq 0$ and $a_0 \neq 0$, then every **rational root** of f is of the form $\frac{p}{q}$, where p is a factor of a_0, and q is a factor of a_n.

In other words, to find the list of possible rational number roots of f, make two lists,

- all the integer factors $\pm p_1, \pm p_2, \pm p_3 \ldots$ of the constant term a_0,
- all the integer factors $\pm q_1, \pm q_2, \pm q_3 \ldots$ of the leading coefficient a_n,

and then list all possible combinations $\pm \frac{p_i}{q_j}$ where the numerator is from the first list and the denominator is from the second list. These are the only rational numbers that could be roots of f, and you can check which are actually roots by evaluating f to see if you get zero there.

Factoring Higher-Degree Polynomial Functions

To factor a polynomial function f and find its real roots:

Step 1: Find a root c by **inspection**. One you find c, you know $(x-c)$ **must** appear in the polynomial's factorization, like $f(x) = (x-c)(\text{something else})$.

Step 2: Divide f by $(x-c)$ using polynomial division. The quotient $q(x)$ is the (something else) factor.

Step 3: If the $q(x)$ is not a linear factor or an irreducible quadratic, repeat the process on $q(x)$ to further factor it. So on and so forth, until every factor is linear of the form $(x-c)$ or an irreducible quadratic.

Step 4: Determine the real roots using the linear factors.

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CHAPTER OVERVIEW

Chapter 6: Intro to More Functions

This chapter includes rational functions and a basics crash course on exponential and logarithmic functions. In my opinion, rational functions are really boring until you learn about limits in Calc I, so I'm gonna go ahead and give you a sneak peek at the intuition of limits while we're at it. For a more in-depth look at exponentials and logarithms and how they relate to each other, as well as their applications, check out Chapter 7.

6.1 Rational Functions

6.1E Exercises

6.2 Asymptotes and Limits

6.2E Exercises

6.3 Basics of Exponential Functions

6.3E Exercises

6.4 Basics of Logarithmic Functions

6.4E Exercises

6.5 Chapter 6 Study Guide

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6.1 Rational Functions

Learning Objectives

By the end of this section, you will be able to:

- identify rational functions and their important features: domain, roots, asymptotes, holes
- manipulate rational functions using polynomial long division and the important tool Partial Fraction Decomposition
- solve rational and polynomial inequalities

Just as a rational number is a ratio of two integers, a **rational function** is a **ratio of two polynomials.** That's all there is to it! Stuff like this:

$$f(x) = \frac{1}{x}, \qquad g(x) = \frac{x^2 + x + 1}{x - 3}, \qquad h(x) = \frac{x + 2}{x^5 + 4x^2 + 1}$$

Rational Functions

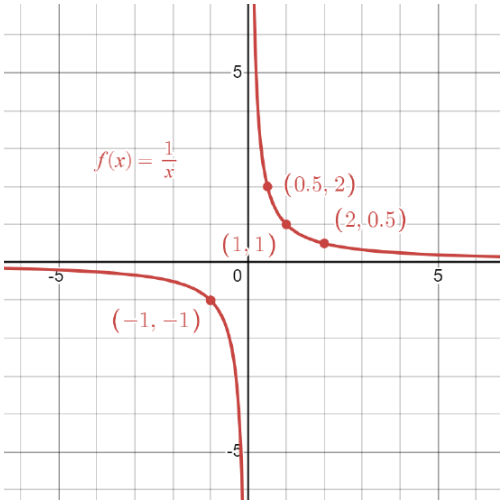
A rational function is a function of the form

$$r(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials. (Assume that the fraction is fully reduced; i.e. p and q have no factors in common.) Fun facts about rational functions:

- Their **domain** is all real numbers **except** any values of x that cause $q(x)$ to be zero. You can write this in set notation as $\{x \mid q(x) \neq 0\}$.
- Their graphs tend to be kinda complicated, with vertical lines they aren't allowed to cross **(vertical asymptotes)**, horizontal lines they hug as they go off to the right or left **(horizontal asymptotes)**, and sometimes **slant asymptotes,** diagonal straight lines that they hug instead (see the next section for more). See a note at the end of this section to learn about **holes**.

The most basic example is the function $f(x) = \frac{1}{x}$. It has a very simple constant function $p(x) = 1$ as numerator and very simple linear function $q(x) = x$ as denominator. You should know what this guy in particular looks like:

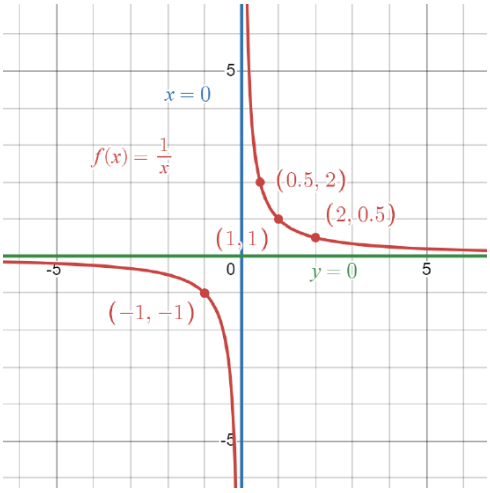


If you forget, you can always plot a handful of points like I did above to reminder yourself of his shape. Notice that $x = 0$ is **not** allowed in the domain of f, because that would cause me to divide by zero. Looking at the graph, you can see that the function is defined for x-values getting **close** to 0, but not **at** $x = 0$. This creates an invisible vertical line with equation $x = 0$, which f hugs closely but can never cross. This is called a **vertical asymptote (V.A.).**

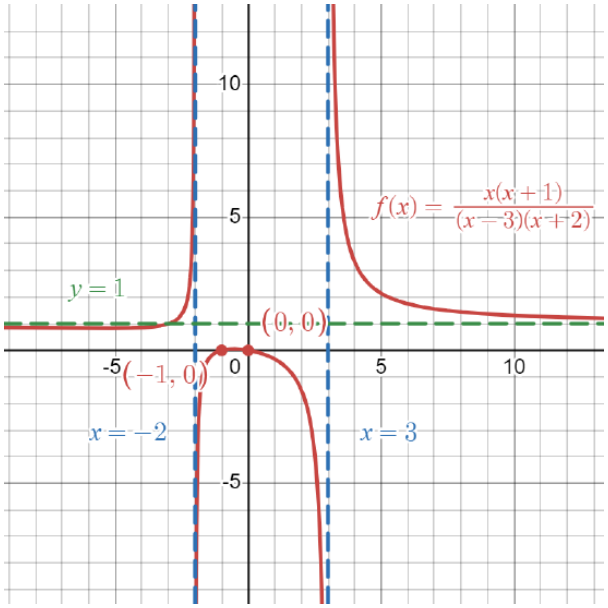
Notice also that as we travel off to the right in the positive x direction, we are plugging in larger and larger numbers as x-values. What happens to a fraction whose denominator becomes very large? Let's think about it...

$$\frac{1}{2} = 0.5, \qquad \frac{1}{10} = 0.1, \qquad \frac{1}{100} = 0.01, \qquad \frac{1}{1,000,000} = 0.000001, \qquad \cdots$$

The function values are getting tiny! So we see the graph hugs the x-axis, with smaller and smaller heights as $x \to \infty$. Looking off to the left forever, the function will have negative y-values like $-\frac{1}{10}, -\frac{1}{1000}$, etc. so it will hug the x-axis from below as $x \to -\infty$. It's as if we have an invisible horizontal line with equation $y = 0$ (the x-axis) that the function hugs with its end behavior, which is called a **horizontal asymptote (H.A.).** Finally, ask yourself, is it possible to plug in a value for x that will make $f(x) = \frac{1}{x} = 0$? No, it's not possible. So that's why we never see the function intersect with the x-axis! This particular rational function has no roots.



Let's look at a more complicated example briefly. The rational function $f(x) = \dfrac{x(x+1)}{(x-3)(x+2)}$ is graphed below.



Now, I've intentionally written the top and bottom of $f(x)$ in factored forms, because I want you to see if you notice any patterns here. Take a gander before reading on and visually identify roots, vertical or horizontal asymptotes, and any connections to the function's definition...

Observations:

- There are roots at $x = -1$ and $x = 0$, which are exactly the x-values that cause the numerator to be 0, which of course causes the entire fraction to be 0.

- There are vertical asymptotes at $x = -2$ and $x = 3$, which are exactly the values that cause the denominator to be 0, making the function undefined there.
- There is a horizontal asymptote, $y = 1$, which I can see visually on the graph. Going off as $x \to \infty$, the function hugs it from above. Going off as $x \to -\infty$, the function crosses over the line and then hugs it from below. So it's legal to cross over a horizontal asymptote!

These are the big important features of rational functions that you want to be able to identify algebraically and graphically, so let's summarize the information.

It's often best to first fully factor the top and bottom of the rational function. Then, to identify...

- the **domain** of a rational function, just look at the denominator alone, set it equal to 0, and solve for x. Those values cause division by zero, so they cannot be allowed in the domain. Those values also give the **vertical asymptotes,** which will have equations of the form $x = c$.
- the **roots (x-intercepts)** of a rational function, just look at the numerator alone, set it equal to 0, and solve for x. Aka, the roots of the numerator give the roots of the rational function.
- any **horizontal asymptotes** of a rational function using the graph, look for an invisible horizontal line that the function hugs as $x \to \infty$ or $x \to -\infty$. If the line passes through c on the y-axis, then the equation of the horizontal asymptote is $y = c$.

Example 1

Algebraically identify any roots and vertical asymptotes of the rational functions.

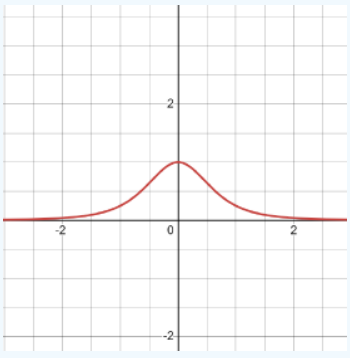
1. $f(x) = \dfrac{x^2 + 3x + 2}{x - 4}$
2. $g(x) = \dfrac{x^2 - 4}{x^2 - 9}$
3. $r(x) = \dfrac{1}{x^4 + 2x^2 + 1}$

Solution

1. First, factor everybody to get $f(x) = \dfrac{(x+2)(x+1)}{x-4}$. To identify any roots, look at what x-values make the numerator zero... Those would be $x = -1, -2$. To identify any V.A.s, check for x-values that make the denominator zero... That would be $x = 4$, so that's a vertical asymptote.

2. First, factor to get $g(x) = \dfrac{(x+2)(x-2)}{(x-3)(x+3)}$. Looking at the numerator, we identify roots at $x = \pm 2$. Looking at the denominator, we identify V.A.s with equations $x = -3, x = 3$.

3. First, notice that the denominator can be seen as $(x^2)^2 + 2(x^2) + 1$, which can be factored as a perfect square to get $r(x) = \dfrac{1}{(x^2+1)^2}$. Looking at the numerator, we see it can never be 0 because it's always 1, so there are no roots! Looking at the denominator, we ask ourselves if any x-value would cause division by zero... Nope! Everything going on here is strictly positive. So there's also no vertical asymptote. If you're curious, this guy looks like this:



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Exercise 1

Algebraically identify any roots and vertical asymptotes of the rational functions.

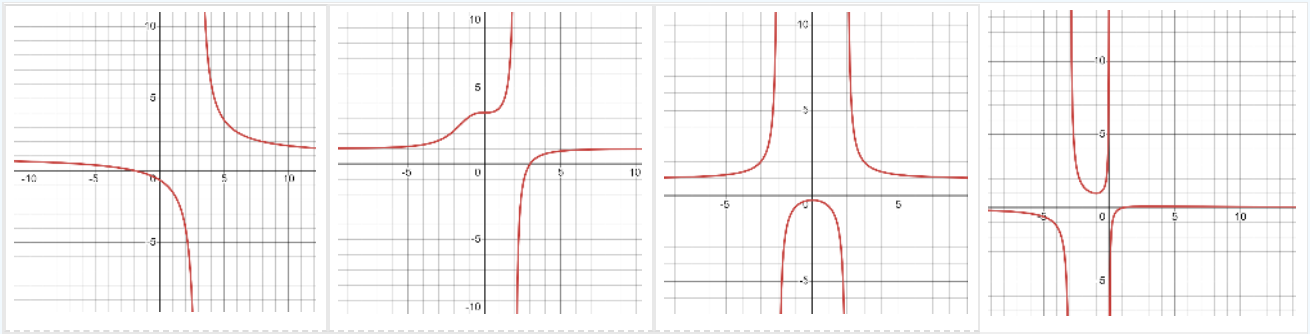
1. $f(x) = \dfrac{x^2 + 6x + 8}{x^3 + 3x^2 - x - 3}$
2. $g(x) = \dfrac{x^2 + x + 2}{x^2 - 2x + 1}$
3. $h(x) = \dfrac{x^3 + x^2}{x - 2}$

Answer

1. (Factor by grouping in denominator.) Roots: $x = -2, -4$. V.A.s: $x = -3, x = -1, x = 1$.
2. Roots: no real roots. V.A.s: $x = 1$.
3. Roots: $x = 0, -1$. V.A.s: $x = 2$.

Exercise 2

Graphically, identify any roots, vertical asymptotes, and horizontal asymptotes.



1. | 2. | 3. | 4.

Answer

1. Roots: $x = -2$. V.A.: $x = 3$. H.A.: $y = 1$.
2. Roots: $x = 3$. V.A.: $x = 2$. H.A.: $y = 1$.
3. Roots: none. V.A.s: $x = -2$, $x = 2$. H.A.: $y = 1$.
4. Roots: $x = 1$. V.A.s: $x = 0, x = -3$. H.A.: $y = 0$.

We're going to get serious about algebraically finding horizontal asymptotes in the next section, when we talk about limits. For now, I just want to be sure you can deal with these functions if they come up using a couple of tools. One thing you might need to do at some point is translate from a rational expression to the equivalent quotient-and-remainder version. You'll use polynomial long division to achieve this. This is productive if (a) the degree of the top is **larger** than the degree of the bottom, or (b) the degrees are the same. If the degree of the bottom is larger, there's nothing to do with the division.

Example 2

Write the rational function $f(x) = \dfrac{p(x)}{d(x)}$ as $f(x) = q(x) + \dfrac{r(x)}{d(x)}$.

1. $f(x) = \dfrac{x^2 + 3x + 2}{x - 4}$
2. $f(x) = \dfrac{x^2 - 4}{x^2 - 9}$
3. $f(x) = \dfrac{1}{x^4 + 2x^2 + 1}$

Solution

1. Simply perform the division:

$$\begin{array}{r} x+7 \\ x-4\overline{)\,x^2+3x+2} \\ -\ (x^2-4x) \\ \hline 0+7x+2 \\ -\quad (7x-28) \\ \hline 0+30 \end{array}$$

We get $f(x)=x+7+\dfrac{30}{x-4}$

2. You can either do long division, which is a little funky:

$$\begin{array}{r} 1 \\ x^2-9\overline{)\,x^2+0x-4} \\ -\ (x^2\quad -9) \\ \hline 0+5 \end{array}$$

or just notice that $-4=-9+5$, do a sneaky replacement, and split the fraction:

$$f(x)=\frac{x^2-4}{x^2-9}=\frac{x^2-9+5}{x^2-9}=\frac{x^2-9}{x^2-9}+\frac{5}{x^2-9}=1+\frac{5}{x^2-9}$$

Either way, you get the same thing.

3. This is already in the desired form actually, because if you try to do long division, you will find $q(x)=0$ and $r(x)=f(x)$.

Go back and try it yourself on the rational functions from Exercise 1! (Check your answers: 1. already in the form; 2. $1+\frac{3x+1}{x^2-2x+1}$; 3. $x^2+3x+6+\frac{12}{x-2}$.)

Partial Fraction Decomposition

Another way to rewrite rational functions in a useful way (this comes up in Calc II and Differential Equations, for example) uses a special technique called **partial fractions.** Polynomial long division above allowed us to split up a big rational expression as long as the degree of the top wasn't smaller than the degree of the bottom. The partial fractions method allows to deal with that left-out situation! I want to introduce you to it here at a simplified level. The idea is to kinda undo the process of getting a common denominator. In numbers, it would be like writing:

$$\frac{41}{30}=\frac{1}{2}+\frac{2}{3}+\frac{1}{5}$$

What we want is to split a rational expression into several fractions whose denominators are **factors** of the original denominator, like this:

$$\frac{x+1}{x(x-1)(x+2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2}$$

Our mission is to figure out what the numerators A, B, and C need to be to make this equation true. The basic process is simply to clear the denominators and consider like terms separately. See the example below.

Example 3

Find A, B, and C to write $f(x)=\dfrac{x+1}{x(x-1)(x+2)}=\dfrac{A}{x}+\dfrac{B}{x-1}+\dfrac{C}{x+2}$.

Solution

First, multiply through the entire equation carefully by the left hand side's whole denominator, which will have the effect of **clearing all denominators** when you distribute through the right hand side.

$$x(x-1)(x+2)\cdot\frac{x+1}{x(x-1)(x+2)}=\left(\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2}\right)\cdot x(x-1)(x+2)$$

$$x(x-1)(x+2) \quad \backslash x \quad x-1 \quad x+2 \,/$$

This boils down to $x+1 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$. Next, expand everything as much as possible by FOILing and distributing.

$$x+1 = Ax^2 + Ax - 2A + Bx^2 + 2Bx + Cx^2 - Cx$$

Now, we **compare the coefficients** of like terms on both sides of the equation. For example, on the left we have NO x^2 terms, but on the right there are $Ax^2 + Bx^2 + Cx^2$. In order for these to turn out the same, I must have $0 = A + B + C$. Next, compare the x terms. On the left, there is a single x, and on the right we have $Ax + 2Bx - Cx$. This means we must have $1 = A + 2B - C$. Finally, compare the constant terms. On the left, we have 1, and on the right, $-2A$. Our last equation is $1 = -2A$.

Everything has boiled down to a system of equations!

$$\begin{cases} A + B + C & = 0 \\ A + 2B - C & = 1 \\ -2A & = 1 \end{cases}$$

Solving this system using the methods of Section 3.5, we find $A = -\frac{1}{2}, B = \frac{2}{3}, C = -\frac{1}{6}$. Thus, we can write

$$\frac{x+1}{x(x-1)(x+2)} = -\frac{1}{2} \cdot \frac{1}{x} + \frac{2}{3} \cdot \frac{1}{(x-1)} - \frac{1}{6} \cdot \frac{1}{(x+2)}$$

? Exercise 3

1. Use partial fractions to write $f(x) = \dfrac{2x}{x^2+5x+6}$ as a sum of terms whose denominators are linear factors of the denominator of f. Meaning, factor the denominator first, and then find A and B such that $f(x) = \dfrac{2x}{x^2+5x+6} = \dfrac{A}{x+3} + \dfrac{B}{x+2}$.

2. Use partial fractions to write $f(x) = \dfrac{x-3}{x^3-x}$ as a sum of terms whose denominators are linear factors of the denominator of f.

Answer

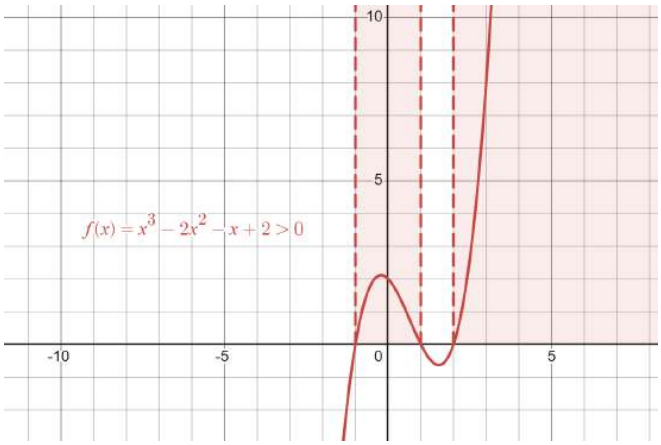
1. $f(x) = \dfrac{2x}{x^2+5x+6} = \dfrac{6}{x+3} - \dfrac{4}{x+2}$
2. $f(x) = \dfrac{x-3}{x^3-x} = \dfrac{3}{x} - \dfrac{2}{x+1} - \dfrac{1}{x-1}$

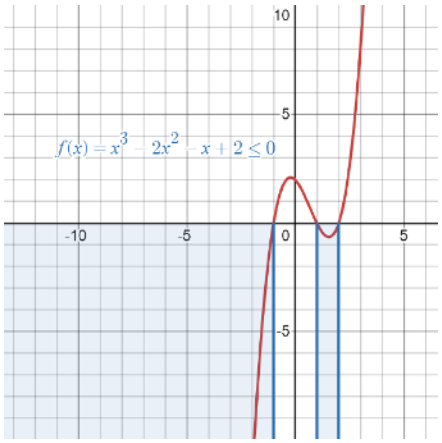
The above method works whenever all the factors in the denominator are linear and distinct. In the case of repeated or irreducible quadratic factors, things get a little more complicated. I leave that for when it becomes necessary in later courses, but if you're really curious you can check out the table at this link now!

Polynomial and Rational Inequalities

The methods in this section are going to be very similar to the number line process from Section 3.2, if you want to review that real quick. If you're all set, read on.

By looking at a graph of $y = f(x)$, it's pretty easy to assess an inequality of the form, say, $f(x) > 0$, or $f(x) \leq 0$. This is simply a matter of the function values, aka y-values, being positive versus negative. Well, if you have positive y-coordinate, you're above the x-axis, and if you have negative y-coordinate, you're below it. So we can easily read off the zones that provide the solution to such an inequality. See the graphical example below.





Notice that the only way for a polynomial function to change from negative to positive values or vice versa is for it to pass through the x-axis, at the location of a root! So the roots of the function are the signpost points around which we will base our analysis. Here's the process:

To solve a **polynomial inequality:**

1. Move all the terms to one side and get 0 on the other.
2. **Factor** the polynomial into linear factors and irreducible quadratic factors, and find the real roots.
3. Make a number line, divided into zones (intervals) by the **roots**.
4. In each zone, determine whether the function is positive or negative by using logic or testing points and comparing the signs of the factors.
5. Report the desired zones as intervals, based on the inequality sign.

Example 4

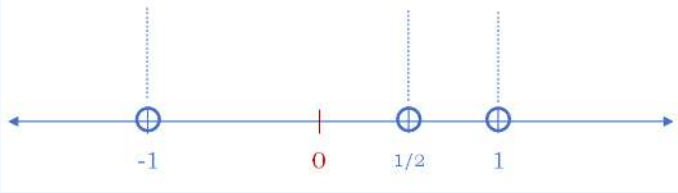
Solve the polynomial inequality $2x^3 - x^2 - 2x > -1$.

Solution

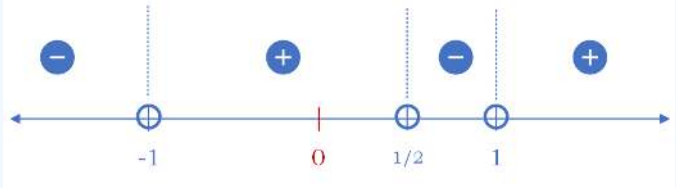
1. We move everything to the left side: $2x^3 - x^2 - 2x + 1 > 0$.

2. This polynomial was factored in Example 4 of Section 5.4, so refer back to that if you get stuck. We get $(2x - 1)(x - 1)(x + 1) > 0$. The roots are $x = -1, \frac{1}{2}, 1$.

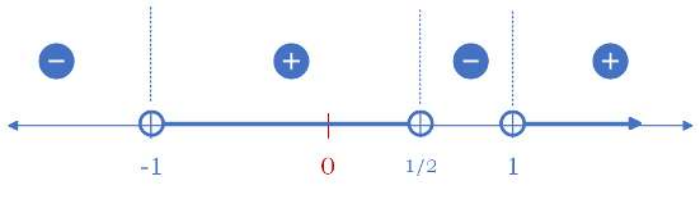
3. We set up the number line, split at the roots. (The red 0 is just for reference.)



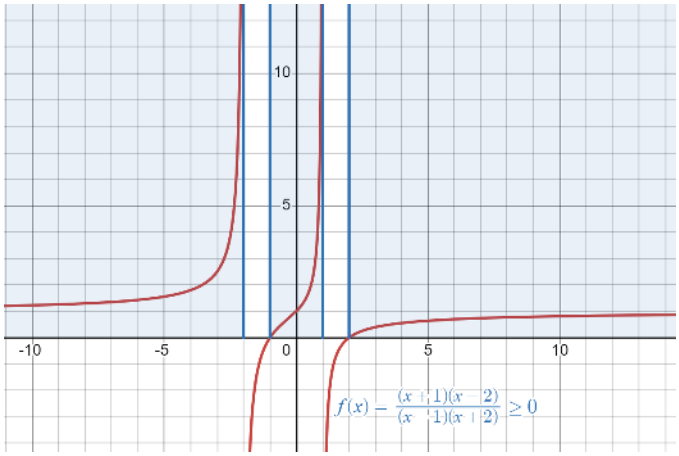
4. In each zone, we determine the sign of the polynomial. For the first zone, you could try plugging in -2 to see that the factors $(\text{neg})(\text{neg})(\text{neg})$ will result in a negative number. For the second zone, plug in 0, which would give the factors $(\text{neg})(\text{neg})(\text{pos}) = \text{pos}$. For the third zone, we find $(\text{pos})(\text{neg})(\text{pos}) = \text{neg}$. For the last zone, we find $(\text{pos})(\text{pos})(\text{pos}) = \text{pos}$. Label the zones.



5. We are looking for the positive zones because we want to be greater than 0. Since the inequality is strict, we won't include the endpoints of the intervals. Our final answer is the list of intervals: $\left(-1, \frac{1}{2}\right) \cup (1, \infty)$.



The process is extremely similar for solving rational inequalities. The only difference to account for is that it's possible for a rational function to jump over the x-axis without passing through it, due to the vertical asymptotes. See below, where the graph is positive on one side of the V.A. $x = -2$ and negative when you hop to the other side, etc.



To solve a **rational inequality:**

1. Move all the terms to one side into a **single** giant fraction and get 0 on the other.
2. Factor the numerator **and** the denominator. Identify the **points of interest** where **either** the numerator or the denominator are zero. (Basically, roots of the top poly and roots of the bottom poly.)
3. Make a number line, divided into zones (intervals) by the points of interest.
4. In each zone, determine whether the function is positive or negative by using logic or testing points and comparing the signs of the factors.
5. Report the desired zones as intervals, based on the inequality sign. **Test** the endpoints carefully to see if they should be included, because the function may not be defined at some of them!

Example 5

Solve the rational inequality $\dfrac{4}{x+2} + 1 \leq x$.

Solution

1. We move all terms over, get a common denominator, and combine into a single rational expression.

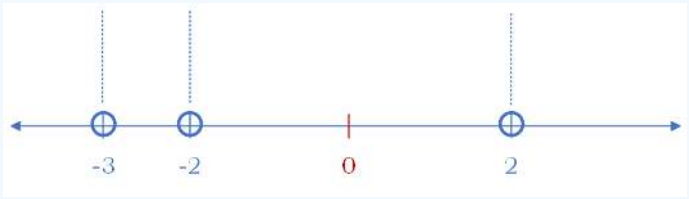
$$\frac{4}{x+2} + 1 - x \leq 0 \quad \rightarrow \quad \frac{4}{x+2} + \frac{(1-x)(x+2)}{x+2} \leq 0 \quad \rightarrow \quad \frac{4 + (1-x)(x+2)}{x+2} \leq 0$$

2. We expand and simplify the numerator so that we can factor it. The denominator is already a linear factor.

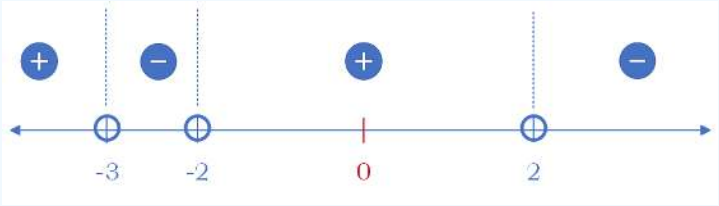
$$\frac{4 - x^2 - x + 2}{x+2} = \frac{-(x^2 + x - 6)}{x+2} = \frac{-(x-2)(x+3)}{x+2}$$

The points of interest are $x = -3, -2, 2$.

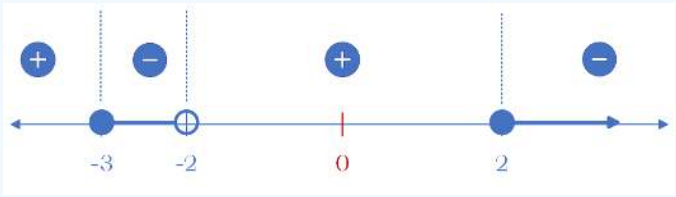
3. We make the number line.



4. In the first zone, we imagine plugging in -4 and deduce that the rational expression will be $\frac{-(\text{neg})(\text{neg})}{\text{neg}} = \text{pos}$. In the second zone, $\frac{-(\text{neg})(\text{pos})}{\text{neg}} = \text{neg}$. In the third zone, $\frac{-(\text{neg})(\text{pos})}{\text{pos}} = \text{pos}$. In the last zone, $\frac{-(\text{pos})(\text{pos})}{\text{pos}} = \text{neg}$.



5. We want to be less than or equal to 0, so we take the negative zones and **test** the endpoints. At $x = -3$, the expression equals 0, so that's fine. At $x = -2$, the expression is not defined, so we leave that out! At $x = 2$, the expression is 0 again. Our final answer is $[-3, -2) \cup [2, \infty)$.



Now you try!

? Exercise 4

Solve the inequalities.

1. $x^3 - 4x \geq 0$
2. $x^2 - 4 < \frac{1}{x^2 - 1}$
3. $\frac{x^2 + 2x}{x^2 - 3x + 2} \leq 0$
4. $\frac{2x}{x + 1} > x^2$

Answer

1. $[-2, 0] \cup [2, \infty)$
2. $(-2, -1) \cup (1, 2)$
3. $[-2, 0] \cup (1, 2)$
4. $(-2, -1) \cup (0, 1)$

Throughout this section, we have assumed that our rational functions are fully reduced. Of course, if you construct or come across rational functions in the course of your math career, this may not be the case yet! When a function is defined using a rational expression that is not fully reduced, it produces a **hole** or **removable discontinuity** in the graph. You will investigate this in the exercises section!

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6.1E Exercises

Rational Functions

Algebraically, identify the domain and any roots and vertical asymptotes of the rational functions.

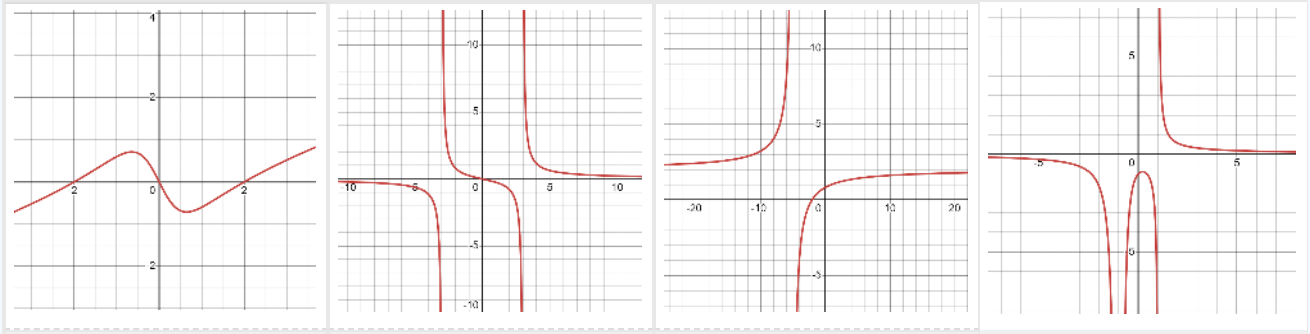
1. $f(x) = \dfrac{x}{x+1}$
2. $g(x) = \dfrac{2x^2 - 3x}{x-4}$
3. $h(x) = \dfrac{5}{x^2+1}$
4. $p(x) = \dfrac{3x^2 - 5x + 2}{x^2 - 4x + 4}$
5. $q(x) = \dfrac{x^3 - x^2}{2x^2 + x - 6}$
6. $r(x) = \dfrac{2x}{x^2 - 9}$
7. $C(t) = \dfrac{t^2 - 2t + 1}{t^3 - 8}$

Answer

1. Domain: all real numbers $x \neq -1$. Roots: $x = 0$. V.A. $x = -1$.
2. Domain: all real numbers $x \neq 4$. Roots: $x = 0, \frac{3}{2}$. V.A. $x = 4$.
3. Domain: all real numbers. Roots: none. No V.A.s.
4. Domain: all real numbers $x \neq 2$. Roots: $x = \frac{2}{3}, 1$. V.A. $x = 2$.
5. Domain: all real numbers $x \neq \frac{3}{2}, -2$. Roots: $x = 0, 1$. V.A.s $x = \frac{3}{2}$ and $x = -2$.
6. Domain: all real numbers $x \neq \pm 3$. Roots: $x = 0$. V.A.s $x = -3$ and $x = 3$.
7. Domain: all real numbers $t \neq 2$. Roots: $t = 1$. V.A. $t = 2$.

Graphs of Rational Functions

By looking at the graph, identify any roots, vertical asymptotes, and horizontal asymptotes of the rational function. Then match each graph to its potential function.



1. | 2. | 3. | 4.

A. $f(x) = \dfrac{x^2 + 1}{(x+1)^2(x-1)}$

B. $f(x) = \dfrac{x^3 - 4x}{3x^2 + 2}$

C. $f(x) = \dfrac{2x + 4}{x + 5}$

D. $f(x) = \dfrac{2x}{x^2 - 9}$

Answer

1. No V.A. or H.A., roots at $x = -2, 0, 2$. Goes with B.
2. Root at $x = 0$. V.A.s $x = -3$ and $x = 3$. H.A. $y = 0$. Goes with D.
3. Root at $x = -2$. V.A. $x = -5$. H.A. $y = 2$. Goes with C.
4. No roots. V.A.s $x = -1$ and $x = 1$. H.A. $y = 0$. Goes with A.

Manipulating Rational Functions

If the degree of the numerator is greater than or equal to the degree of the denominator, write $\frac{p(x)}{d(x)}$ in the form $q(x) + \frac{r(x)}{d(x)}$. If not, use partial fractions to write as a sum of terms with linear factors in the denominators.

1. $g(x) = \dfrac{2x^2 - 3x}{x - 4}$
2. $p(x) = \dfrac{3x^2 - 5x + 2}{x^2 - 4x + 4}$
3. $r(x) = \dfrac{2x}{x^2 - 9}$
4. $q(x) = \dfrac{2x^3 - 2x^2 + 18}{2x^2 + x - 6}$
5. $f(x) = \dfrac{6x^2}{(x - 2)(x^2 - 1)}$
6. $h(x) = \dfrac{2x + 1}{x^2 - 5x + 6}$

Answer

1. $2x + 5 + \dfrac{20}{x - 4}$
2. $3 + \dfrac{7x - 10}{x^2 - 4x + 4}$
3. $\dfrac{1}{x + 3} + \dfrac{1}{x - 3}$
4. $x - \frac{3}{2} + \dfrac{\frac{15}{2}x + 9}{2x^2 + x - 6}$
5. $-\dfrac{3}{x - 1} + \dfrac{1}{x + 1} + \dfrac{8}{x - 2}$
6. $\dfrac{7}{x - 3} - \dfrac{5}{x - 2}$

Polynomial and Rational Inequalities

Solve the inequalities.

1. $x^3 - 2x^2 - 3x \geq 0$
2. $x^3 < 9x$
3. $\dfrac{x}{x^2 - 1} + \dfrac{1}{x + 1} \leq 0$
4. $\dfrac{1}{x} > \dfrac{2}{x - 1}$

Answer

1. $[-1, 0] \cup [3, \infty)$
2. ALERT: you can't just divide both sides by x because x can take negative and positive values, so you won't know whether to flip the inequality sign! $(-\infty, -3) \cup (0, 3)$
3. $(-\infty, -1) \cup \left[\frac{1}{2}, 1\right)$

4. $(-\infty, -1) \cup (0, 1)$

Holes

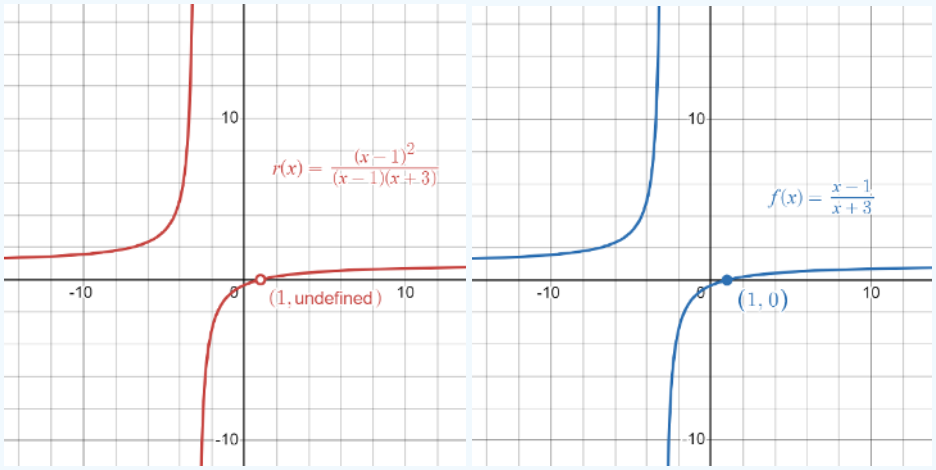
If a function is defined by a rational expression that is **not** fully reduced, for example,

$$r(x) = \frac{x^2 - 2x + 1}{x^2 + 2x - 3} = \frac{(x-1)(x-1)}{(x-1)(x+3)}$$

then its domain does **not** include the problem value $x = 1$ that causes the denominator to be zero (and of course, we can't allow $x = -3$ either). However, this function doesn't have a vertical asymptote at $x = 1$ like it does at $x = -3$! For all purposes **besides** the actual problem value, the function will **behave** exactly like the function defined by the simplified rational expression,

$$f(x) = \frac{x-1}{x+3}$$

Both are graphed below. The graph of $r(x)$ has a **hole** (or **removable discontinuity**) at $x = 1$, but otherwise it looks just like the graph of $f(x)$.



Now, for the functions given, identify their domain issues, and indicate whether each problem value yields a hole or a vertical asymptote. Find the fully reduced rational function f whose graph will match (apart from holes).

1. $r(x) = \dfrac{x^2 + 6x + 5}{2x^2 + 3x + 1}$
2. $r(x) = \dfrac{x^3 - x}{2x^2 + 4x}$
3. $r(x) = \dfrac{(x+1)(x-2)(x+3)}{(x-1)(x+1)(x-2)(x+2)}$

Answer

1. Domain: $x \neq -\frac{1}{2}, -1$. Vertical asymptote $x = -\frac{1}{2}$ and hole at $x = -1$. $f(x) = \dfrac{x+5}{2x+1}$.
2. Domain: $x \neq 0, -2$. Vertical asymptote $x = -2$, hole at $x = 0$. $f(x) = \dfrac{x^2 - 1}{2(x+2)}$.
3. Domain: $x \neq \pm 1, \pm 2$. Vertical asymptotes $x = 1$ and $x = -2$. Holes at $x = -1, 2$. $f(x) = \dfrac{x+3}{(x-1)(x+2)}$.

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6.2 Asymptotes and Limits

Learning Objectives

By the end of this section, you will be able to:

- intuitively understand the concept of the limit of a function
- understand limit notation
- describe asymptotic behavior using limits
- intuitively evaluate limits of rational functions to assess vertical, horizontal, and slant asymptotes

It's highly possible that you will spend the first chunk of your Calculus I course talking about **limits**, as they're a vital tool building up to the definition of a derivative, which is an elusive mythical Pokemon that perhaps you've heard of and are excited to meet. I want to give you a sneak peek at the intuitive and graphical conception of a limit, and along the way be a little more precise about **vertical, horizontal,** and **slant asymptotes** as pertains to rational functions.

Limit of a Function (by ~vibes~)

When looking at a graph of $y = f(x)$, the **limit of** f **at** $x = a$ is the y-value that the function appears to be heading toward as you move towards a particular x-value of interest, a, from both the left and the right. The **existence** of a limit does depend on the function actually approaching a value from **both** directions.

In other words, nicknaming the limit "L" if it exists, we're saying that as $x \to a$, $f(x) \to L$, in the arrow notation we've used before for end behavior. **NOTE:** the limit **does not care** what the **actual** function value $f(a)$ is, or even whether it's defined! It's all about where the function ACTS like it's going.

We use the **limit notation**

$$\lim_{x \to a} f(x) = L$$

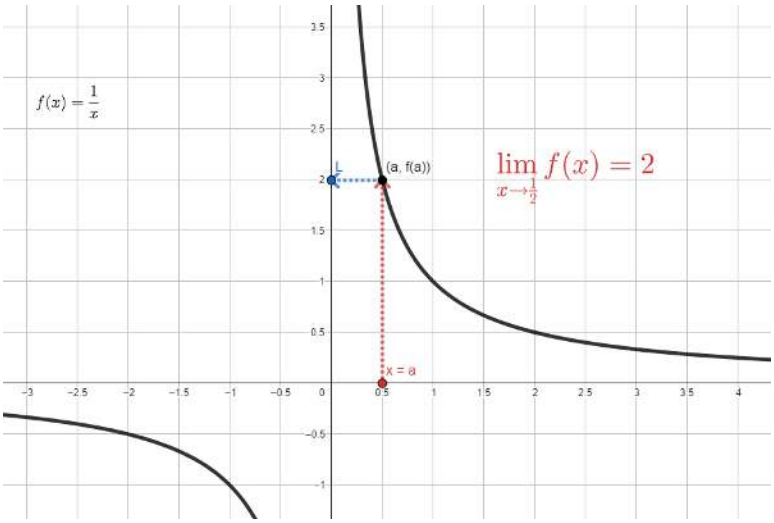
if the limit exists properly. If a function tends toward $\pm\infty$ when approaching $x = a$ from both directions, the limit is infinite, and although ∞ is not a number, we still use the notation

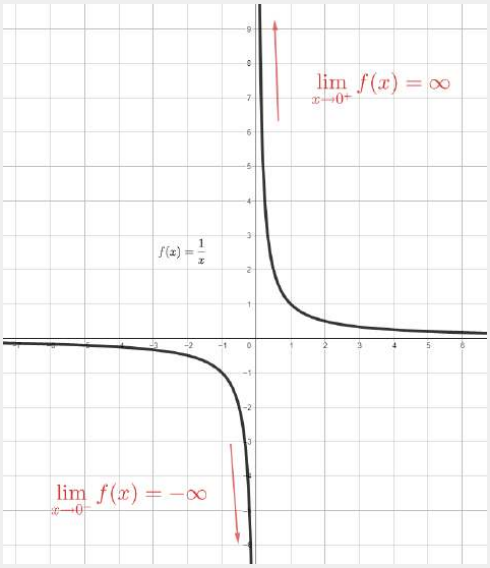
$$\lim_{x \to a} f(x) = (\pm)\infty$$

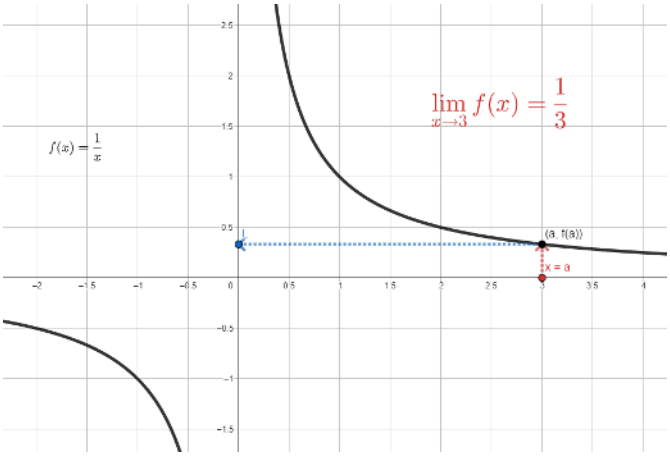
Sometimes we are interested in **one-sided limits,** looking at the behavior of the function as we come in from only one side. In this case, we use a "$^+$" decoration to indicate approaching **from the right,** and a "$^-$" to indicate approaching **from the left.**

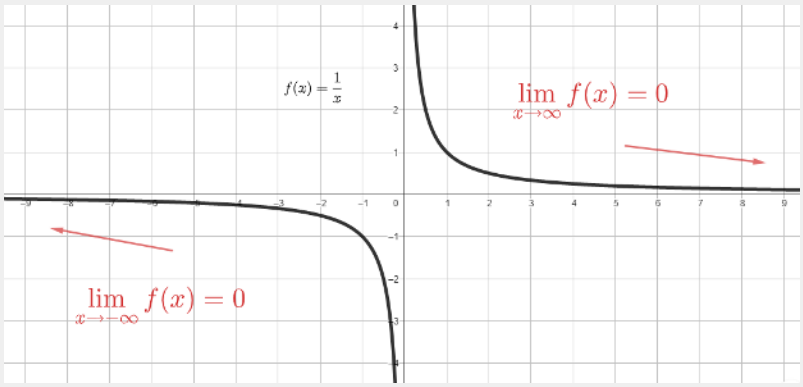
$$\text{approaching } x = a \text{ from the right only: } \lim_{x \to a^+} f(x) = L,$$
$$\text{approaching from the left only: } \lim_{x \to a^-} f(x) = L$$

The information above will make a lot more sense after looking at some pictures. Study the examples below carefully to interpret the definition.

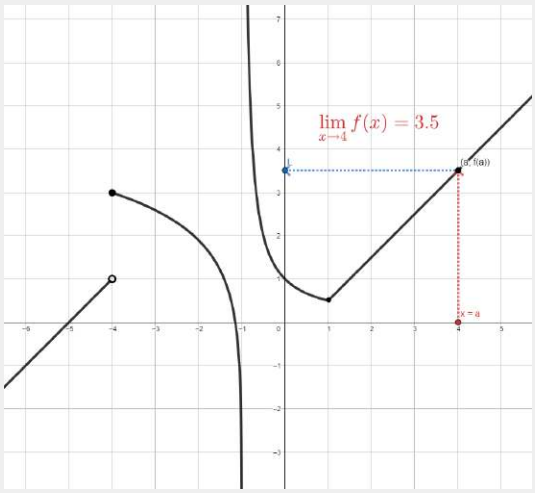
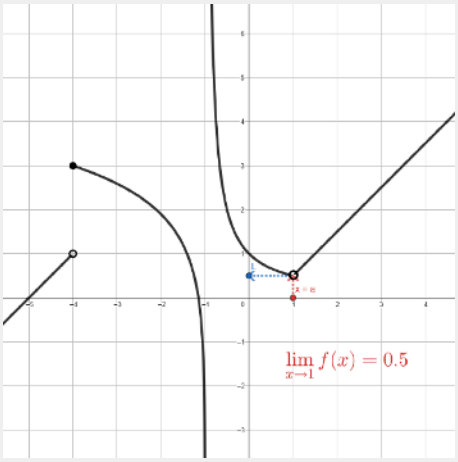
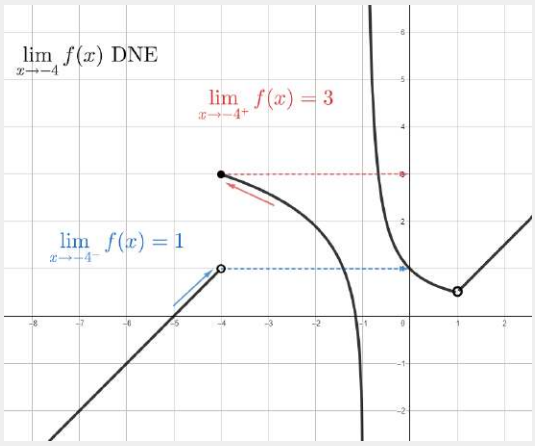
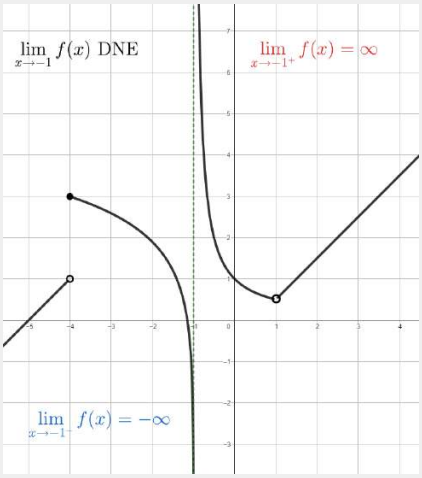

$f(x) = \frac{1}{x}$
$\lim_{x \to \frac{1}{2}} f(x) = 2$
(a, f(a))
L
x = a


$\lim_{x \to 0^+} f(x) = \infty$
$f(x) = \frac{1}{x}$
$\lim_{x \to 0^-} f(x) = -\infty$

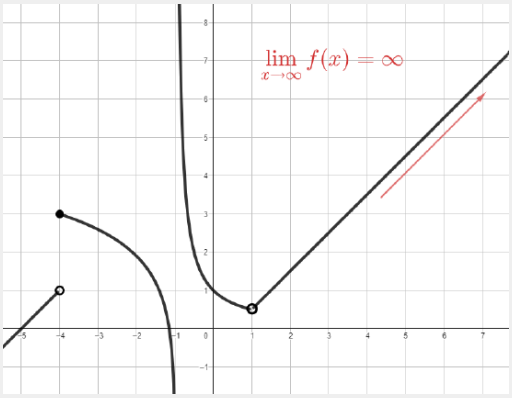

$\lim_{x \to 3} f(x) = \frac{1}{3}$
$f(x) = \frac{1}{x}$
(a, f(a))
L
x = a



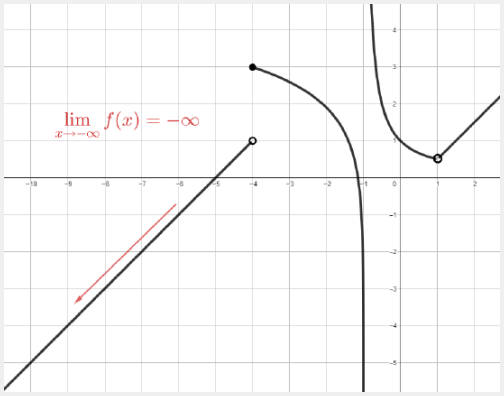
You can see that to find the limit as $x \to a$, just find a on the x-axis, track it up to the function, and then track the height over to find the y-value at that point. This works great with the functions above because they're not very chaotic, there are no holes or jumps, etc. We need to make sure we grasp the fringe cases when looking for limits graphically. Study the table below.

The limit exists and agrees with the function value at that location.	The limit exists even though the function is not defined at that location.
	
The one-sided limits exist, but do not agree, so the limit as a whole does not exist.	**The one-sided limits are infinite, but do not agree, so the limit as a whole does not exist.**
	

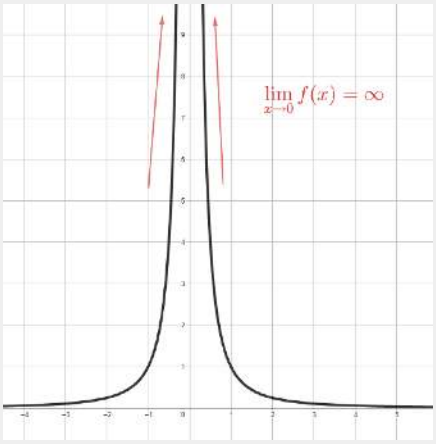
The function goes up forever, so the "limit at infinity" is infinite.



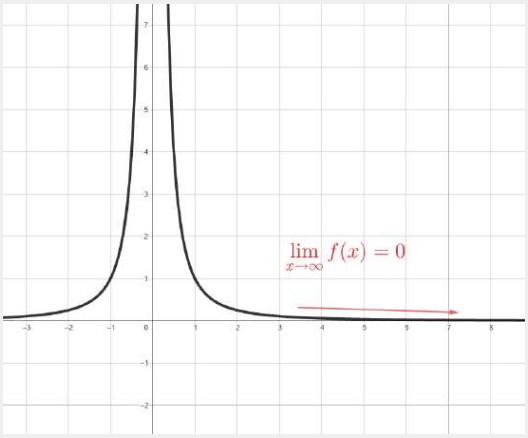
The function goes down forever, so the "limit at negative infinity" is negative infinity.



The one-sided limits are infinite, but they do agree, so we say the limit is infinity.



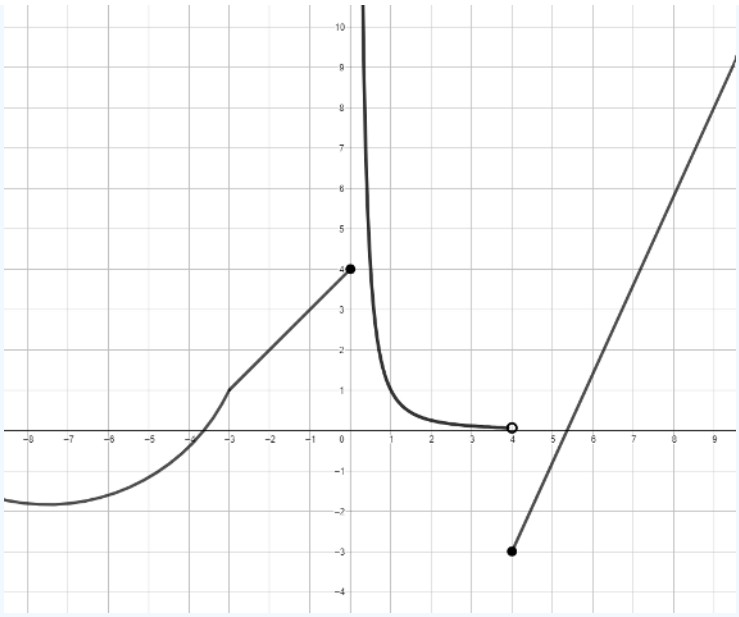
The "limit at infinity" is leveling out, hugging a certain horizontal line, so it is finite.



You can practice reading limits off of a graph below.

? Exercise 1

Using the graph of $y = f(x)$ below, find the limits or tell if they do not exist.



$$\lim_{x\to -3} f(x), \quad \lim_{x\to 0^-} f(x), \quad \lim_{x\to 0^+} f(x), \quad \lim_{x\to 0} f(x),$$

$$\lim_{x\to 1} f(x), \quad \lim_{x\to 4^+} f(x), \quad \lim_{x\to 4} f(x), \quad \lim_{x\to \infty} f(x)$$

Answer

1. $\lim_{x\to -3} f(x) = 1$
2. $\lim_{x\to 0^-} f(x) = 4$
3. $\lim_{x\to 0^+} f(x) = \infty$
4. $\lim_{x\to 0} f(x)$ DNE.
5. $\lim_{x\to 1} f(x) = 1$
6. $\lim_{x\to 4^+} f(x) = -3$
7. $\lim_{x\to 4} f(x)$ DNE.
8. $\lim_{x\to \infty} f(x) = \infty$

Now let's connect these ideas to the asymptotes we were learning about.

Vertical Asymptotes

The vertical line $x = a$ is a vertical asymptote of a function $f(x)$ if ANY of the following are true:

$\lim_{x\to a} f(x) = \infty$	$\lim_{x\to a^+} f(x) = \infty$	$\lim_{x\to a^-} f(x) = \infty$
$\lim_{x\to a} f(x) = -\infty$	$\lim_{x\to a^+} f(x) = -\infty$	$\lim_{x\to a^-} f(x) = -\infty$

Horizontal Asymptotes

The horizontal line $y = L$ is a horizontal asymptote of a function $f(x)$ if

$$\lim_{x\to \infty} f(x) = L \quad \text{and/or} \quad \lim_{x\to -\infty} f(x) = L$$

I want to do a little bit of intuitive limit-finding using algebra, which will be made more rigorous in Calc I.

Example 1

Find the horizontal asymptotes of the functions by using logic and algebra to analyze $\lim_{x\to\pm\infty} f(x)$.

1. $f(x) = \dfrac{1}{x}$
2. $f(x) = \dfrac{x}{x+1}$
3. $f(x) = \dfrac{x^2+1}{2x^2+x+3}$

Solution

1. To find H.A.s, we are interested in what happens as $x \to \infty$ and $x \to -\infty$. Let's think about this... if x is getting bigger and bigger, what happens to $\dfrac{1}{x}$? Think about things like $\frac{1}{2}, \frac{1}{4}, \frac{1}{10}, \frac{1}{10000}$... what's happening? It's getting smaller and smaller, getting close to zero but never exactly zero. It doesn't matter that it never quite hits zero. It's always *heading towards* zero, so that's what we report as the *limit*.

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

The same thing happens as $x \to -\infty$. Think about $-\frac{1}{2}, -\frac{1}{10}, -\frac{1}{10000}$... Still getting closer and closer to zero forever! We have a horizontal asymptote $y = 0$.

2. This is a little trickier. If $x \to \infty$, the numerator x will blow up forever, but so will the denominator $x+1$! Let's imagine some possibilities as x blows up...

$$\frac{2}{3} \quad \cdots \quad \frac{5}{6} \quad \cdots \quad \frac{20}{21} \quad \cdots \quad \frac{1,000}{1,001} \quad \cdots \quad \frac{1,000,000}{1,000,001}$$

Do you agree that as x gets bigger, the $+1$ happening in the denominator becomes basically irrelevant? These fractions are all getting closer and closer to the number 1. We have a horizontal asymptote $y = 1$. Here's a trick to see this easily... Multiply top and bottom by $\frac{1}{x}$:

$$\frac{x}{(x+1)} \frac{\cdot\frac{1}{x}}{\cdot\frac{1}{x}} = \frac{1}{1+\frac{1}{x}}$$

Now think about what happens when $x \to \infty$. The $\frac{1}{x}$ term will head towards 0, making the whole fraction head towards $\dfrac{1}{1+0} = 1$.

3. Let's try the same trick on this one. The **highest power** of x in the denominator is x^2, so **multiply top and bottom by the reciprocal** of that.

$$\frac{(x^2+1)}{(2x^2+x+3)} \frac{\cdot\frac{1}{x^2}}{\cdot\frac{1}{x^2}} = \frac{1+\frac{1}{x^2}}{2+\frac{1}{x}+\frac{3}{x^2}}$$

As $x \to \infty$, the $\frac{1}{x^2}$ heads to zero real quick. So does $\frac{1}{x}$ as we saw before. And convince yourself that even though there's a 3 in $\frac{3}{x^2}$, it will quickly be dominated by the denominator blowing up. So our whole function is heading towards

$$\frac{1+0}{2+0+0} = \frac{1}{2}$$

We have a horizontal asymptote $y = \frac{1}{2}$.

Exercise 2

Find the horizontal asymptotes of the rational functions by analyzing $\lim_{x\to\pm\infty} f(x)$.

1. $f(x) = \dfrac{2}{x^2}$
2. $f(x) = \dfrac{4x^2-2}{x^2-9}$

3. $f(x) = \dfrac{1}{(x^2+1)^2}$

Answer

1. H.A. $y = 0$
2. H.A. $y = 4$
3. H.A. $y = 0$

Example 2

Tell whether $x = 3$ is a vertical asymptote of the functions by analyzing $\lim_{x\to 3} f(x)$.

1. $f(x) = \dfrac{1}{x-3}$
2. $f(x) = \dfrac{x+2}{x+3}$
3. $f(x) = \dfrac{x-3}{x^2-9}$

Solution

1. Let's think about what happens to $f(x)$ if x gets close to 3. Consider the denominator and imagine plugging in options for x coming in from the right, numbers bigger than 3...

x-value near 3	4	3.5	3.1	3.01
denominator $x-3$	$4-3=1$	$3.5-3=0.5$	$3.1-3=0.1$	$3.01-3=0.01$

Okay, we can see that the denominator is getting super tiny, heading towards 0 as x heads towards 3. If the denominator of a fraction gets smaller, the fraction as a whole blows up! So we have $x = 3$ as a V.A. because

$$\lim_{x\to 3^+} f(x) = \infty$$

2. Thinking about x becoming 3, we see the numerator will head towards $3+2=5$ and the denominator will head towards $3+3=6$. Nothing special happening here, the fraction as a whole is heading towards $\frac{5}{6}$. That's a finite number, not $\pm\infty$, so we do not have a vertical asymptote here.

3. If $x \to 3$, we see that both our numerator and denominator are heading towards 0, so we don't know what the fraction is doing as a whole... Let's try simplifying this expression to see if anything becomes clearer.

$$\frac{x-3}{x^2-9} = \frac{x-3}{(x+3)(x-3)} = \frac{1}{x+3}$$

Aha! This function f isn't defined at $x = 3$, but it *behaves* exactly like $\dfrac{1}{x+3}$. When $x \to 3$, that expression heads toward $\frac{1}{9}$. In this case, $x = 3$ is a hole in the graph, but not a vertical asymptote!

Exercise 3

Tell whether $x = 0$ is a vertical asymptote of the functions by analyzing $\lim_{x\to 0} f(x)$.

1. $f(x) = \dfrac{1}{x^2}$
2. $f(x) = \dfrac{x}{x+1}$
3. $f(x) = \dfrac{x^2+x}{x}$
4. $f(x) = \dfrac{x+1}{x^2}$

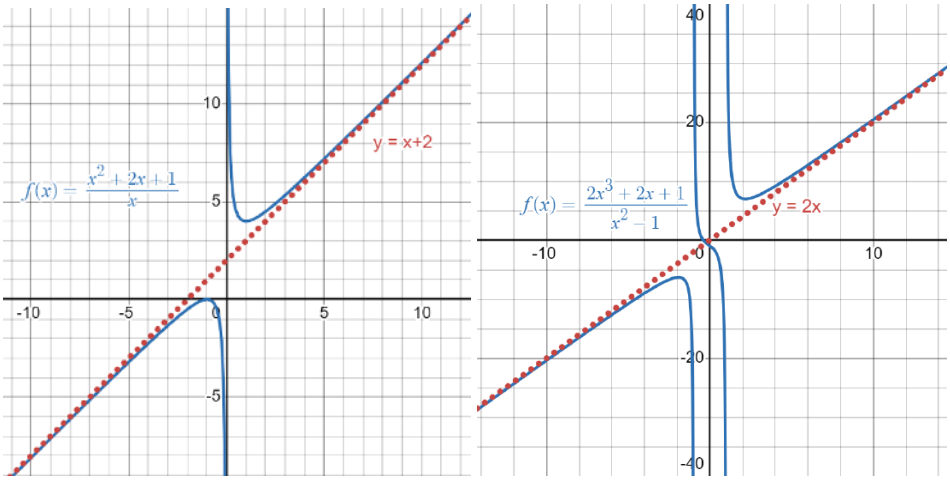
Answer

1. Yes, because $\lim_{x\to 0} f(x) = \infty$.

2. No, $\lim_{x\to 0} \dfrac{x}{x+1} = \dfrac{0}{1} = 0$.
3. Simplifying the expression, f behaves like $x+1$, and $\lim_{x\to 0}(x+1) = 1$, so no.
4. Yes, as $x \to 0$, the numerator heads to 1, but the denominator gets very tiny very quickly, making the whole fraction blow up to ∞.

Slant Asymptotes

I want to use limits to talk about one final concept relating to rational functions. Sometimes a rational function appears to do some crazy stuff, but then hug a straight line going off forever to infinity, like the pictures below:



What's going on here is that, as $x \to \pm\infty$, the function ends up **behaving** like a linear function. Look at the first example. Notice what happens if I start trying to analyze the limit as $x \to \infty$.

$$\lim_{x\to\infty}\left(\frac{x^2+2x+1}{x}\right) = \lim_{x\to\infty}\left(\frac{x^2}{x}+\frac{2x}{x}+\frac{1}{x}\right) = \lim_{x\to\infty}\left(x+2+\frac{1}{x}\right)$$

Hmm... As x blows up, that $\frac{1}{x}$ term is going to contribute less and less to the behavior of the function. My graph will end up acting just like the first piece, a straight line, $y = x+2$.

Slant Asymptote

A rational function $f(x) = \dfrac{p(x)}{d(x)}$ has a slant asymptote when the **degree of the numerator is exactly one more than the degree of the denominator.** To find the equation of the slant asymptote line, perform polynomial long division on $\dfrac{p(x)}{d(x)}$, writing $\dfrac{p(x)}{d(x)} = q(x) + \dfrac{r(x)}{d(x)}$. The quotient $q(x)$ will be a linear function, and the $\dfrac{r(x)}{d(x)}$ will become irrelevant as $x \to \infty$. Then the slant asymptote is $y = q(x)$.

Example 3

Find the slant asymptote of the second example above, $f(x) = \dfrac{2x^3+2x+1}{x^2-1}$.

Solution

Polynomial long division yields $f(x) = (2x) + \dfrac{4x-1}{x^2-1}$. That second term will eventually become tiny as $x \to \infty$, which we can see by doing that trick for analyzing infinite limits:

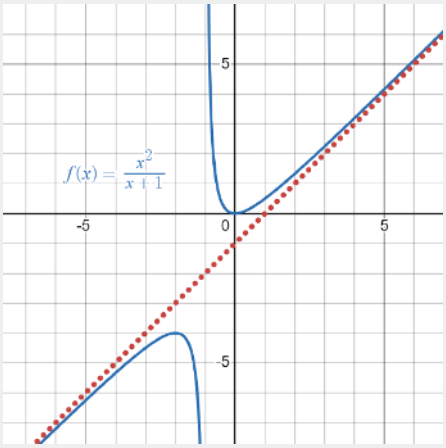
$$\frac{(4x-1)}{(x^2-1)}\frac{\cdot\frac{1}{x^2}}{\cdot\frac{1}{x^2}} = \frac{\frac{4}{x}-\frac{1}{x^2}}{1-\frac{1}{x^2}}$$

The numerator heads to 0, while the denominator heads to 1. Well, $\frac{0}{1} = 0$, so we see that term will become so tiny it's irrelevant! Thus f will act like $q(x) = 2x$, a line.

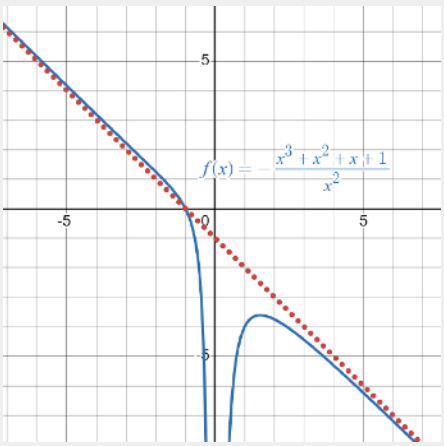
Exercise 4

Find the slant asymptotes of the rational functions below. Confirm afterwards by looking at the graph.

$$f(x) = \frac{x^2}{x+1}$$



$$f(x) = -\frac{x^3+x^2+x+1}{x^2}$$



Answer

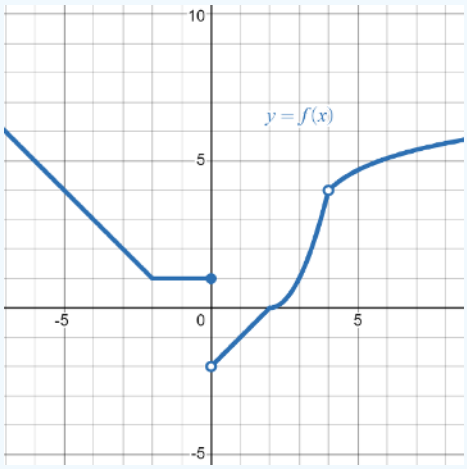
1. Long division yields $(x-1) + \frac{1}{x+1}$. The slant asymptote is $y = x - 1$.
2. Long division (or just splitting the fraction and cancelling) yields $-\left(x + 1 + \frac{1}{x} + \frac{1}{x^2}\right)$. The last two terms approach zero as $x \to \infty$, so the function hugs the slant asymptote $y = -x - 1$.

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6.2E Exercises

Finding Limits From Graphs

By looking at the graph of $f(x)$ below, find the indicated limits.



$\lim_{x\to -5} f(x)$, $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$,

$\lim_{x\to 4^-} f(x)$, $\lim_{x\to 4^+} f(x)$, $\lim_{x\to 4} f(x)$, $\lim_{x\to -\infty} f(x)$

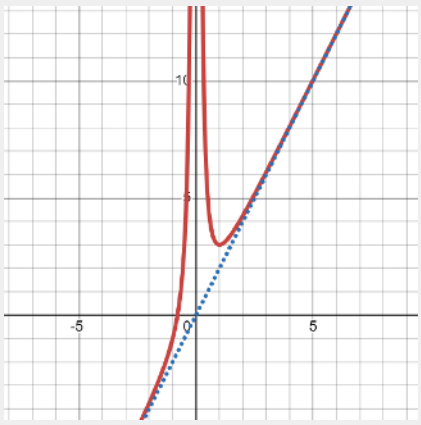
Answer

- $\lim_{x\to -5} f(x) = 4$
- $\lim_{x\to 0^-} f(x) = 1$
- $\lim_{x\to 0^+} f(x) = -2$
- $\lim_{x\to 0} f(x)$ DNE.
- $\lim_{x\to 4^-} f(x) = 4$
- $\lim_{x\to 4^+} f(x) = 4$
- $\lim_{x\to 4} f(x) = 4$
- $\lim_{x\to -\infty} f(x) = \infty$

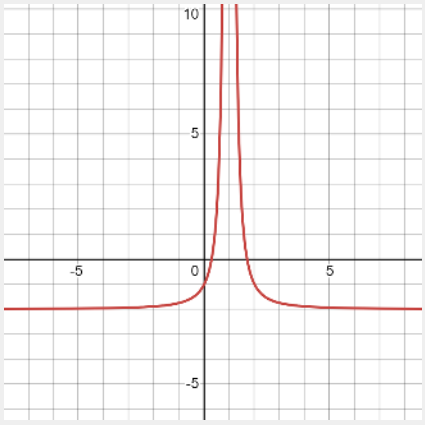
Identifying Asymptotes From Graphs

By looking at the graphs of rational functions below, identify any vertical, horizontal, or slant asymptotes.

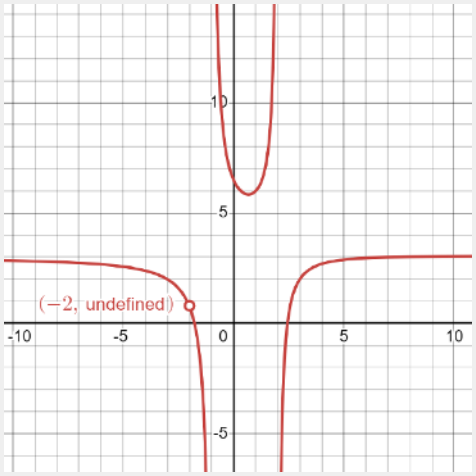
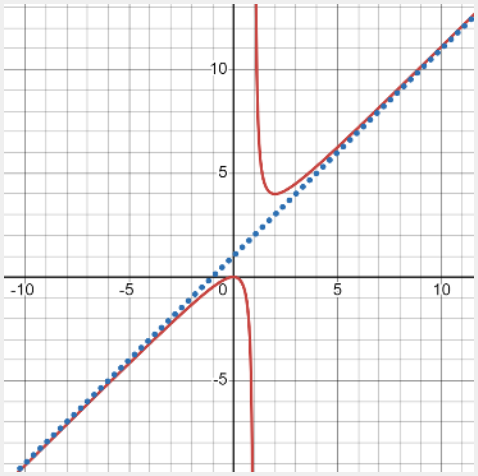
1.



2.



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3.	4.
	

Answer

1. V.A. $x = 0$, Slant $y = 2x$.
2. V.A. $x = 1$, H.A. $y = -2$.
3. V.A.s $x = -1, x = 2$, H.A. $y = 3$.
4. V.A. $x = 1$, Slant $y = x + 1$.

Algebraically Finding Vertical Asymptotes

Find any vertical asymptotes of the rational functions by analyzing appropriate limits.

1. $f(x) = \dfrac{1}{x^2 - 5x + 6}$
2. $f(x) = \dfrac{x}{x + 2}$
3. $f(x) = \dfrac{-2x^2 + 4x - 1}{(x - 1)^2}$
4. $f(x) = \dfrac{(x + 2)(x + 3)(x - 3)}{(x + 2)(x - 2)(x + 1)}$
5. $f(x) = \dfrac{2x^5 + x^2}{x^4}$

Answer

HINT: If you can't get started, the steps are 1) identify domain issues and 2) analyze the limits as x approaches those locations to see if any of them have a limit of $\pm\infty$. Go do them and come back to check your answers.

1. $x = 2$ and $x = 3$ are V.A.s.
2. $x = -2$ is a V.A.
3. $x = 1$ is a V.A. (As $x \to 1$, the numerator heads toward 1 while the denominator heads toward 0, making the fraction blow up.)
4. $x = -1$ and $x = 2$ are V.A.s. (When you consider $x \to -2$, notice that by cancelling in the top and bottom as possible, the fraction's value heads toward $-\frac{5}{4}$, not $\pm\infty$.)
5. $x = 0$ is a V.A. (Write $\dfrac{2x^5 + x^2}{x^4}$ as $2x + \frac{1}{x^2}$. As $x \to 0$, the first term goes away but the second term blows up forever.)

Algebraically Finding Horizontal Asymptotes

Find any horizontal asymptotes of the rational functions by analyzing appropriate limits.

1. $f(x) = \dfrac{1}{x^2 - 5x + 6}$
2. $f(x) = \dfrac{x}{x + 2}$
3. $f(x) = \dfrac{-2x^2 + 4x - 1}{(x - 1)^2}$
4. $f(x) = \dfrac{(x + 2)(x + 3)(x - 3)}{(x + 2)(x - 2)(x + 1)}$
5. $f(x) = \dfrac{2x^5 + x^2}{x^4}$

Answer

HINT: If you can't get started, the limits you need to analyze are as $x \to \infty$ and/or $x \to -\infty$. Remember the trick of dividing top and bottom by the highest power of x from the denominator.

1. H.A. $y = 0$.
2. H.A. $y = 1$.
3. H.A. $y = -2$.
4. H.A. $y = 1$.
5. No H.A.

Algebraically Finding Slant Asymptotes

Find any slant asymptotes of the rational functions by analyzing appropriate limits.

1. $f(x) = \dfrac{x^2 + 1}{x}$
2. $f(x) = \dfrac{2x^5 + x^2}{x^4}$
3. $f(x) = \dfrac{3x^3 + 2x^2 - x + 1}{x^2 + 1}$

Answer

HINT: The question is, does the function behave like a linear function as $x \to \pm\infty$? Use polynomial long division (or splitting the fraction and cancelling) to see.

1. $y = x$
2. $y = 2x$
3. $y = 3x + 2$

Limit Application

The zombie apocalypse has begun (womp womp). Currently, the spread of the outbreak is modeled by the function $P(d) = \frac{1}{2}d^3 - d + 1000$, giving the number of zombies on day d after the outbreak started. If nothing changes and time goes on with this growth function unchecked, will the number of zombies (a) decrease and go extinct, (b) level out at a certain number, or (c) grow until they take over the Earth?

Answer

If time is going on forever, we're looking at sending $d \to \infty$. As d grows, the term $\frac{1}{2}d^3$ will grow very quickly. Even though the $-d$ term is subtracting off some zombies, it can't offset the quick growth of the cubic term! Think about it...

$$\frac{1}{2}2^3 - 2 = 2, \quad \ldots \quad \frac{1}{2}10^3 - 10 = 490, \quad \ldots \quad \frac{1}{2}100^3 - 100 = 499,900, \quad \ldots$$

> Believe me? So as time goes on, the zombies take over the Earth. (rip)

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6.3 Basics of Exponential Functions

Learning Objectives

By the end of this section, you will be able to:

- recognize exponential functions, their important features, and their graphs
- evaluate exponential functions and use technology to solve problems

We can raise numbers to powers in our sleep at this point, but mostly we've done integer and rational powers. That is, you see stuff like $4^2, 9^{\frac{1}{2}}, 100^{-2}$ without having an allergic reaction (hopefully). But what would you do if I jump out the bushes and throw a 3^π across your path?? We've basically ignored a whole set of real numbers as power options—the irrational numbers. Stuff like

$$2^\pi, \quad 5^{\sqrt{2}}, \quad 8^e.$$

Until today! The way we deal with irrational powers is to approximate them with rational powers till you're "close enough" that any error is irrelevant for your purposes. We use calculators to do this, don't worry! Remember that we can approximate $\pi \approx 3.14159...$, $\sqrt{2} \approx 1.414213...$, and $e \approx 2.718281...$. A calculator can then evaluate

$$2^\pi \approx 8.824978..., \quad 5^{\sqrt{2}} \approx 9.7385177..., \quad 8^e \approx 285.005408...$$

All of the power rules that we know already still apply!

Technology Note: when you need to type exponents into various calculators/websites, you will generally use the carat symbol ^ (shift+6 on a keyboard). If your power is very complicated, you'll want to also use parentheses. For example, to enter $2^{-\sqrt{5}}$ into WolframAlpha, you can type "2^(-sqrt(5))" and it will understand.

Okay, the whole point of this is that now we can make sense of the **exponential expression** a^x for **any** real number x. This allows us to define **exponential functions.**

Exponential Functions

The **exponential function with base** a is defined by $f(x) = a^x$, for all real numbers x, where $a > 0$ and $a \neq 1$.

Fun facts about exponential functions:

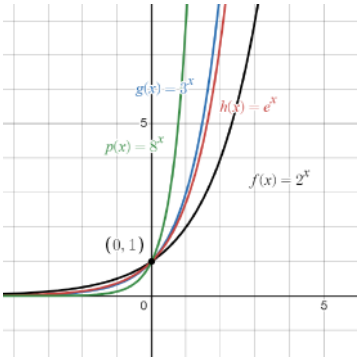
- In an exponential function, the **base** is a plain ol' number and the **variable** is in the **exponent.** Not to be confused with polynomial expressions where the variable is the base and a number is the power, like x^3! Whole different animal.
- We don't want $a = 1$ because 1^x is always 1, giving a super boring constant function $f(x) = 1$.
- We don't want to consider negative bases because raising negative numbers to even (or certain fractional) powers is a mess in the real numbers.
- The **domain** of an exponential function is all real numbers.
- If $a > 1$, then the function will be **increasing,** but if $0 < a < 1$, the function will be **decreasing** (we'll explore this).
- No matter what a is, $f(0) = 1$. Aka, the y-intercept of an exponential function like this is always $(0, 1)$.

Exponential functions are very sensitive. A small change in the variable can cause significant change in the function value. For example, if $f(x) = 8^x$, consider...

x-value	0	1	2	3	4
$f(x)$	$8^0 = 1$	$8^1 = 8$	$8^2 = 64$	$8^3 = 512$	$8^4 = 4096$

Going from $x = 2$ to $x = 3$ doesn't feel like much, but the function value shoots from 64 all the way to 512.

When you raise a base $a > 1$ to bigger and bigger powers, the results **increase**, just like the example above with $a = 8$. Let's see some graphs.



Observations:

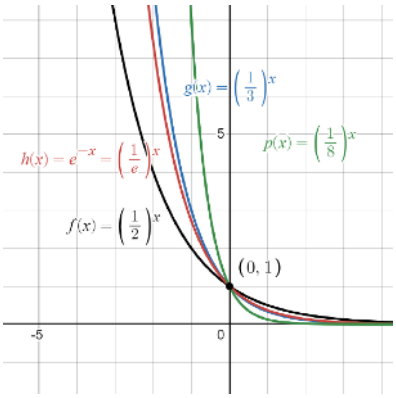
- All $f(x) = a^x$ pass through the point $(0, 1)$, as mentioned in the fun facts.
- There's no reason a base has to be a whole number! Recall the special number $e \approx 2.718$. You can raise him to the x no problem, and you can see that his (red) graph lies between $f(x) = 2^x$ and $g(x) = 3^x$, just as $2 < 2.718 < 3$.
- On the left side of the graph, as $x \to -\infty$, the graphs hug a horizontal asymptote $y = 0$. This makes sense because plugging in negative numbers as powers produces smaller and smaller numbers. For example,

$$2^{-1} = \frac{1}{2}, \quad 2^{-2} = \frac{1}{4}, \quad 2^{-5} = \frac{1}{32}, \quad ...$$

- These functions are always positive! (They're always above the x-axis.)
- Since $a^1 = a$ for any a, you can see that they each pass through the point $(1, a)$ matching their base. For example, $f(x) = 2^x$ passes through $(1, 2)$.
- Looking out at end behavior as $x \to \infty$, the bigger the base, the steeper the function (the faster it grows).

On the other hand, if $0 < a < 1$, raising to higher and higher powers causes the function value to **decrease.** Consider $a = \frac{1}{2}$...

$$\left(\frac{1}{2}\right)^{-1} = 2, \quad \left(\frac{1}{2}\right)^0 = 1, \quad \left(\frac{1}{2}\right)^1 = \frac{1}{2}, \quad \left(\frac{1}{2}\right)^2 = \frac{1}{4}, \quad ...$$



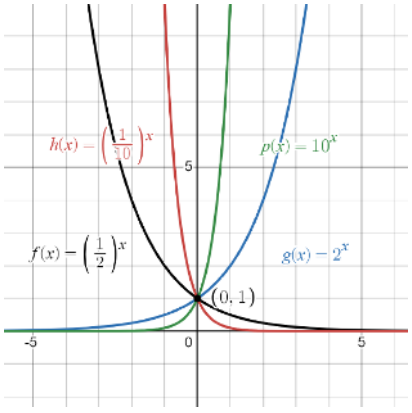
Observations (reason through them as above):

- The graphs all still pass through $(0, 1)$.
- The graphs hug a horizontal asymptote $y = 0$ as $x \to \infty$.
- The functions are still always positive (above the x-axis).
- The functions still pass through the point $(1, a)$ where a is their base.
- The functions also pass through the point $(-1, \frac{1}{a})$, which is sometimes easier to pinpoint.

There's one more thing I want to call out, which I've hinted at in the graph above, labeling $h(x) = e^{-x} = \frac{1}{e}^x$. Remember your power rules! There is a relationship between the functions $f(x) = a^x$ and $g(x) = (\frac{1}{a})^x$.

$$g(x) = \left(\frac{1}{a}\right)^x = \frac{1}{a^x} = a^{-x}$$

You can transform f into g by replacing the input x with $-x$, which we recall from function transformations causes a **reflection** about the y-axis. Observe below.



Test your understanding with the exercises below, and then we'll get into working with these functions.

? Exercise 1

Answer True/False:

1. The function $f(x) = x^7$ is exponential with base 7.
2. The function $f(x) = 7^x$ is exponential with base 7.
3. The function $f(x) = 4^x$ is a horizontal reflection across the y-axis of the function $g(x) = 4^{-x}$.
4. The function $f(x) = 4^x$ is a horizontal reflection across the y-axis of the function $g(x) = (\frac{1}{4})^x$.
5. The graph of $f(x) = (1.5)^x$ is increasing.
6. The graph of $f(x) = (0.25)^x$ is decreasing.
7. The graph of $f(x) = 3^x$ is decreasing.
8. As $x \to \infty$, the graph of $f(x) = 2^x$ is steeper (faster growing) than the graph of $g(x) = 3^x$.
9. As $x \to \infty$, the graph of $f(x) = 5^x$ is steeper (faster growing) than the graph of $g(x) = e^x$.
10. The graph of $f(x) = 6^x$ passes through $(1, 6)$.
11. The graph of $f(x) = 3^x$ passes through $(3, 1)$.
12. The graph of $f(x) = a^x$ passes through $(0, 1)$ for any base a.
13. The graph of $f(x) = a^x$ passes through $(1, 0)$ for any base a.
14. There exists a value of x such that $a^x < 0$ for any base a.

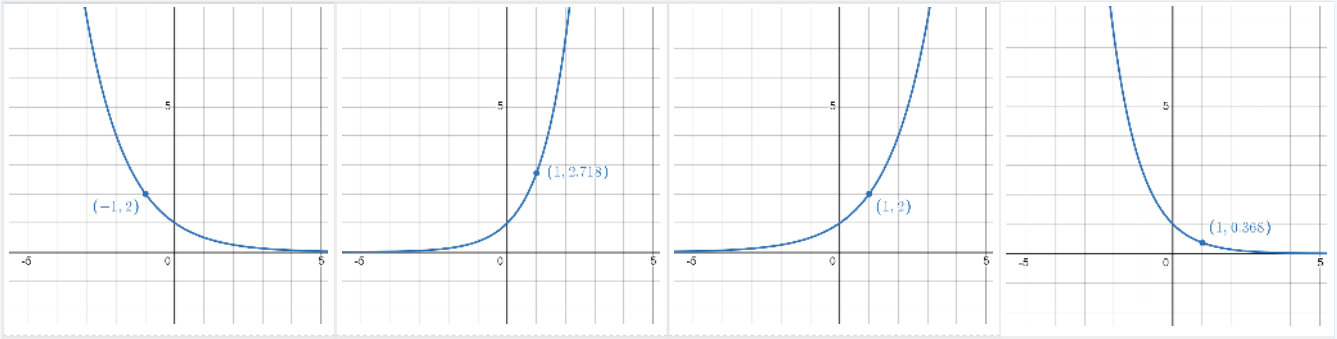
Answer

1. False. That's a polynomial (monomial) with degree 7.
2. True.
3. True.
4. True, this is equivalent to the previous statement.
5. True, the base is bigger than 1.
6. True, the base is smaller than 1.
7. False.
8. False, the bigger the base, the faster the growth.

9. True, 5 is a bigger base than $e \approx 2.71...$.
10. True, $f(1) = 6$.
11. False, it passes through $(1, 3)$ though. At $x = 3$, $f(3) = 9$.
12. True, $a^0 = 1$ for any a.
13. False, $a^1 \neq 0$.
14. False, the function $f(x) = a^x$ is always positive.

Exercise 2

Match the exponential functions with their graphs.



1.	2.	3.	4.
$f(x) = e^x$	$g(x) = 2^x$	$h(x) = \left(\frac{1}{2}\right)^x$	$p(x) = e^{-x}$

Answer

1. h
2. f
3. g
4. p

Using Exponential Functions

These functions appear all the time in real-world problems. Check out the examples below, then try the exercises section!

Example 1

Billybob Joe borrows $100 from Marybeth Sue and she charges him 10% interest, compounded annually. That is, after one year, he owes her $100(1 + 0.1) = 110$. (Review working with percents if you need to.) After two years, he's charged another 10% of interest on *that* new amount, so he owes $110(1 + 0.1) = 121$. But wait... Notice that we could have computed $100(1 + 0.1)(1 + 0.1) = 100(1 + 0.1)^2$ to find the amount owed after two years! Do you see the pattern?

The amount owed after t years is given by an exponential function $A(t) = A_0(1 + i)^t$, where $A_0 = 100$ is the initial amount, and $i = 0.1$ is the interest rate written as a decimal. Use this function (and a calculator) to find the amount Billybob Joe owes after

1. 3 years
2. 5 years
3. 10 years

If he can talk Marybeth Sue down to 7% interest, what will the new function be? Use it to find the amount he would owe at the same times, using the new interest rate.

Solution

Simply plug in the values for t, rounding to two decimal places because this is money:

1. $A(3) = 100(1.1)^3 = 133.10$
2. $A(5) = 100(1.1)^5 = 161.05$
3. $A(10) = 100(1.1)^{10} = 259.37$

If the interest rate changes to 7%, we just change i to 0.07 and keep everything else the same. The new function is $A(t) = 100(1.07)^t$, and the new amounts owed would be

1. $A(3) = 100(1.07)^3 = 122.50$
2. $A(5) = 140.26$
3. $A(10) = 196.72$

Example 2

I have a Petri dish containing 200 bacteria at the moment, but the count is doubling every half hour. Construct an exponential function $N(t)$ that will give the number of bacteria after t hours. Use it and a calculator to find the number of bacteria after

1. 1 hour
2. 2 hours
3. 5 hours

Solution

Let's reason through this like we did for interest... If we're doubling every half hour, then after 0.5 hr, I have $200(2) = 400$. After another half hour, I have $400(2) = 200(2)(2) = 200(2)^2 = 800$. If h is the number of half-hours that have passed, I have $200(2)^h$ bacteria. How can I translate this to a function that takes an input of t in hours? Well, if 1 hour has passed, that's 2 half-hours, so two doublings. If 2 hours have passed, then 4 half-hours have passed, that's four doublings. I just need to raise 2 to the power $2t$ to have this effect! My function is $N(t) = 200(2)^{2t}$. I plug in the desired t values:

1. $N(1) = 200(2)^2 = 800$

2. $N(2) = 200(2)^4 = 3200$
3. $N(5) = 200(2)^{10} = 204,800$

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6.3E Exercises

Definition True/False

Answer True/False. If False, explain why.

1. The function $f(x) = x^{\frac{1}{2}}$ is exponential with base $\frac{1}{2}$.
2. The function $f(x) = \left(\frac{1}{2}\right)^x$ is exponential with base $\frac{1}{2}$.
3. The expression $\left(\frac{1}{2}\right)^x$ is equivalent to the expression $\frac{1}{2^x}$.
4. The function $f(x) = 5^x$ is exponential with base 5.
5. The function $f(x) = 5^{-x}$ is exponential with base 5.
6. The function $f(x) = 5^{-x}$ is exponential with base $\frac{1}{5}$.
7. The function $f(x) = 3^x$ is increasing.
8. The function $f(x) = \left(\frac{1}{3}\right)^x$ is increasing.
9. The function $f(x) = \left(\frac{1}{3}\right)^x$ is decreasing.
10. For $f(x) = 3^x$, we have $f(0) = 3$.
11. For $f(x) = 3^x$, we have $f(0) = 1$.
12. For $f(x) = 3^x$, we have $f(1) = 3$.
13. The function $f(x) = e^x$ can take negative values. That is, there exists some x such that $f(x) < 0$.
14. The function $f(x) = 3^x$ can take negative values.

Answer

1. F, the variable isn't in the exponent.
2. T
3. T
4. T
5. F, rewrite 5^{-x} as $\left(\frac{1}{5}\right)^x$ to see the base.
6. T
7. T
8. F, when the base is less than 1, the function is decreasing (downhill behavior).
9. T
10. F, $f(0) = 3^0 = 1$.
11. T
12. T
13. F, exponential functions like this never take negative values (stay above the x-axis).
14. F, same reason.

Graphs True/False

Answer True/False. If False, justify or find the error.

1. The graph of $f(x) = a^x$ passes through $(0, 1)$ for any base with $a > 0$ and $a \neq 1$.
2. The graph of $f(x) = 2^x$ is increasing.
3. The graph of $f(x) = \left(\frac{4}{5}\right)^x$ is decreasing.
4. The graph of $f(x) = \left(\frac{5}{4}\right)^x$ is decreasing.
5. As $x \to \infty$, the graph of $f(x) = 5^x$ is steeper (faster growing) than the graph of $g(x) = 10^x$.
6. As $x \to \infty$, the graph of $g(x) = 10^x$ is steeper (faster growing) than the graph of $f(x) = 5^x$.
7. The graph of $f(x) = e^x$ is increasing.
8. As $x \to \infty$, the graph of $f(x) = e^x$ is steeper than the graph of $g(x) = 3^x$, for $x > 0$.
9. The graph of $f(x) = 6^x$ passes through $(1, 6)$.
10. The graph of $f(x) = 6^x$ passes through $\left(-1, \frac{1}{6}\right)$.
11. The graph of $f(x) = 6^x$ passes through $(6, 1)$.
12. The graph of $f(x) = e^x$ passes below the x-axis.

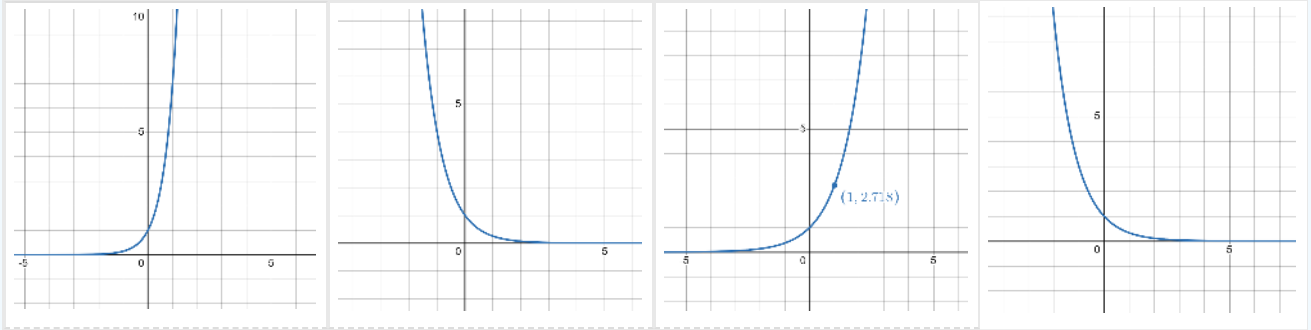
13. The graph of $f(x) = a^x$ passes through $(1, a)$ for any base a.
14. The graph of $f(x) = 5^{-x}$ is the reflection across the y-axis of the graph of $g(x) = 5^x$.
15. The graph of $f(x) = \left(\frac{1}{7}\right)^x$ is the reflection across the y-axis of the graph of $g(x) = 7^x$.

Answer

1. T
2. T
3. T
4. F, the base $\frac{5}{4} > 1$ so it's increasing.
5. F, bigger base gives faster growth.
6. T
7. T
8. F, $e \approx 2.71...$ is less than 3, so g has the larger base and thus faster growth.
9. T
10. T
11. F, $f(6) = 6^6$.
12. F, as mentioned, these exponential functions are always positive.
13. T
14. T
15. T

Matching Graphs

By analyzing signal points like $(1, a)$ and $\left(-1, \frac{1}{a}\right)$ and using your knowledge of exponential function graphs, match the graphs to their functions.



1.	2.	3.	4.
$f(x) = e^x$	$g(x) = \left(\frac{1}{3}\right)^x$	$h(x) = 7^x$	$p(x) = 4^{-x}$

Answer

1. h
2. p
3. f
4. g

Working With Exponential Functions

1. (No calculator) Fill in the table of values for the function $f(x) = 2^x$.

x	$f(x)$	**x**	$f(x)$
1		**0**	

2		**-1**	
3		**-2**	

2. (No calculator) Fill in the table of values for the function $f(x) = \left(\frac{2}{3}\right)^x$.

x	$f(x)$	**x**	$f(x)$
1		**0**	
2		**-1**	
3		**-2**	

3. (No calculator) Fill in the table of values for the function $f(x) = 5^{-x}$.

x	$f(x)$	**x**	$f(x)$
1		**0**	
2		**-1**	
3		**-2**	

4. For $f(x) = 5^x$, what should x be to get $f(x) = 5$? What should x be to get $f(x) = 25$?
5. For $f(x) = 5^x$, what should x be to get $f(x) = \frac{1}{5}$? What should x be to get $f(x) = 1$?
6. For $f(x) = 4^x$, what should x be to get $f(x) = 2$?
7. For $f(x) = 8^x$, what should x be to get $f(x) = 2$?
8. If $f(x) = 2^x$ and $g(x) = 4^x$, is there an x-value for which $f(x) = g(x)$?
9. Is there an x-value such that $e^x = -2$?
10. Using technology (calculator or WolframAlpha, for example), fill in the table of values for the function $f(x) = (1.2)^x$.

x	$f(x)$	**x**	$f(x)$
-2		**0**	
-1		**3.25**	
-0.5		**10**	

Answer

1.

x	$f(x)$	**x**	$f(x)$
1	2	**0**	1
2	4	**-1**	$\frac{1}{2}$
3	8	**-2**	$\frac{1}{4}$

2.

x	$f(x)$	**x**	$f(x)$
1	$\frac{2}{3}$	**0**	1
2	$\frac{4}{9}$	**-1**	$\frac{3}{2}$
3	$\frac{8}{27}$	**-2**	$\frac{9}{4}$

3.

x	$f(x)$	**x**	$f(x)$
1	$\frac{1}{5}$	**0**	1

2	$\frac{1}{25}$	**-1**	5
3	$\frac{1}{125}$	**-2**	25

4. For $f(x) = 5$, x should be 1. For $f(x) = 25$, x should be 2.
5. For $f(x) = \frac{1}{5}$, x should be -1. For $f(x) = 1$, x should be 0.
6. We want x such that $f(x) = 4^x = 2$. What kind of power should 4 be raised to to get out...its square root? That's right, $x = \frac{1}{2}$. Check: $f(1/2) = 4^{\frac{1}{2}} = \sqrt{4} = 2$.
7. Similar reasoning, $x = \frac{1}{3}$.
8. Yes, both pass through $(0, 1)$. Aka, if $x = 0$, then $f(x) = g(x) = 1$.
9. No, there is no possible input that will make $f(x) = e^x$ pass below the x-axis into negative function values.
10. *Let me teach you a technology trick for this kind of thing (for when you're allowed to use technology, of course, not for cheating). Go to wolframalpha.com and type in "table of values (1.2)^x for x = -2, -1, -0.5, 0, 3.25, 10" and hit Enter. You're welcome.*

x	$f(x)$	x	$f(x)$
-2	0.694444	**0**	1
-1	0.833333	**3.25**	1.80859
-0.5	0.912871	**10**	6.19174

Solving for x

Determine what x must be in the following equations.

1. $2^x = 8$
2. $5^{2x} = 25$
3. $e^{x-1} = e^7$
4. $10^x = 100$
5. $4^x = 2$

Answer

1. $x = 3$
2. $x = 1$
3. $x = 8$
4. $x = 2$
5. $x = \frac{1}{2}$

Applications

1. Unless you receive a raise every year equivalent to the rate of inflation, you are essentially taking a wage cut as the purchasing power of your money goes down! My dad's current salary is $100,000 per year. If inflation is currently steady at a rate of 4% per year, and his employer promises to give him a fair cost-of-living raise to match that each year, what does he expect his annual salary to be after 2 years? After 10 years? Write a function that gives his salary, S, as a function of how many years, y, have passed. You can use a calculator to perform the computations.

2. My video is going marginally viral on social media right now. Right before it took off, it was sitting at only 40 views, but now the number of views is doubling every minute! Assume the algorithm blesses me for ten minutes with this growth. After 2 minutes, how many views does the video have? After 5 minutes? After the 10 minutes? Write a function giving the total number of views on my video as a function of how many minutes have passed.

Answer

1. After one year, his salary will be raised to $100000(1.04) = 104000$. After two years, it would be $100000(1.04)^2 = 108160$. Following the pattern, after ten years it will be $100000(1.04)^{10} = 148024$. The function is $S(y) = 100000(1.04)^y$.

2. After one minute, I have $40(2) = 80$ views. Every time a minute passes, I must multiply by 2 to effectively double the count. The function is thus $N(t) = 40 \cdot 2^t$. At $t = 2$, $N = 160$ views. At $t = 5$, $N = 1280$. At $t = 10$, $N \approx 41$k views.

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6.4 Basics of Logarithmic Functions

Learning Objectives

By the end of this section, you will be able to:

- recognize logarithmic functions, their important features, and their graphs
- relate logarithms to exponentials
- use logarithm laws to expand and condense logarithmic expressions
- evaluate logarithmic functions and use technology to solve problems

In the exercises section you just finished, I snuck in a certain type of question. I asked something like, "If $f(x) = 9^x$, what does x need to be in order to make $f(x) = 3$?" In other words, I want to have $9^x = 3$, and I wonder what x will make that true? Wait... 3 is the square root of 9, and we write roots with fractional exponents. So $x = \frac{1}{2}$ will work! We can check: $f\left(\frac{1}{2}\right) = 9^{\frac{1}{2}} = \sqrt{9} = 3$.

Logarithms, at the end of the day, are just the way we deal with that question-and-answer process. You can also think of them as a way to **undo** the "raising to a power" process that happens inside an exponential equation. We've already learned that functions that undo other functions are called **inverse functions.** If you forgot that stuff, review Section 4.4.

Every exponential function $f(x) = a^x$ (with $a > 0$ and $a \neq 1$) passes the Horizontal Line Test and is thus one-to-one. Therefore, it has an inverse function f^{-1}. Recall that being an inverse function means that the values $f^{-1}(x)$ are defined by

$$f^{-1}(x) = y \quad \iff \quad f(y) = x.$$

Aka, if you want to know what $f^{-1}(x)$ spits out, it's the number y such that $f(y) = x$.

Logarithmic Functions

For $a > 0$ and $a \neq 1$, the **logarithmic function with base a,** which we denote $\log_a$, is the inverse function of the exponential function with the same base a. Thus, it is defined by the relationship

$$\log_a x = y \quad \iff \quad a^y = x. \qquad (\star)$$

We say, "log base a of x is y." Basically, "$\log_a(x) =$?" is asking the **question**, "What do I need to raise the base a to in order to get out this x?" The **answer** to that question is the output of the function $\log_a$.

When in particular the base is the special number e, we write $\log_e x$ as $\ln x$, and call it the **natural log of x.**

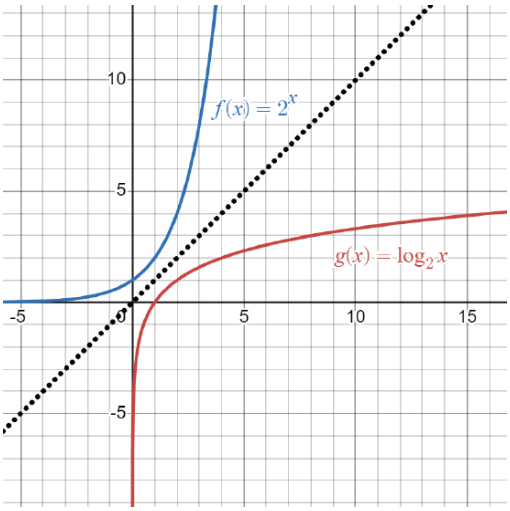
Fun facts about logarithmic functions:

- Their **domain** is **only** positive real numbers! You are **not** allowed to put 0 or a negative number into a log, because that would be the same as saying $a^y = 0$, and there's no possible y that will satisfy that!
- Since $a^0 = 1$ for any a, we have $\log_a(1) = 0$ for any base a. This is something you should know without using a calculator.
- Similarly, since $a^1 = a$, we have $\log_a(a) = 1$ for any a. Aka, if you take the log of its own base, you will always get 1. This is the other evaluation you should know without a calculator.

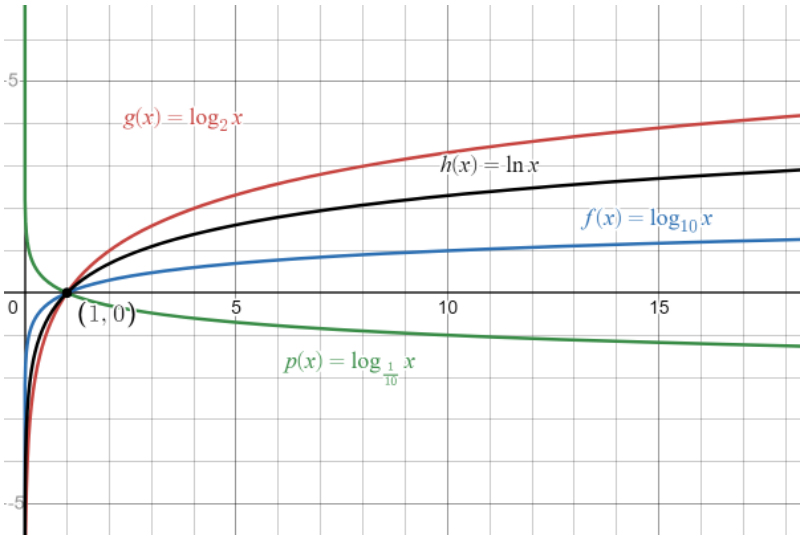
The fact that a is called the **base** in both equations where it appears should help you remember that $\log_a$ is related to a^x. Whenever the input for a log function might be ambiguous, you can use parentheses; for example, $\log_a(x + 1)$ tells you $(x + 1)$ is the whole input, whereas $\log_a x + 1$ would be interpreted as "compute $\log_a$ of x and then add 1 to the result."

I just want to emphasize again that the relationship $(\star)$ in the definition above is your bread and butter here. You will generally always be translating back and forth between these two forms whenever working with logarithms in functions or equations. Memorize that guy!!!

Now let's see what these functions look like. Recall that the graphs of inverse functions look like mirror images, reflected across the diagonal line $y = x$. This means that to remember what the graphs of logarithmic functions look like, you just have to remember exponential graphs and then reflect them. The beautiful thing about math is that you don't have to memorize a bunch of random stuff. You can just memorize a few things and then deduce the rest from them!



Study the graphs below and see if you can detect any patterns and observations.



Observations:

- While exponential functions exhibited **fast** growth (or decay), logarithmic functions exhibit **slow growth** (or decay).
- They all pass through the point $(1, 0)$, which makes sense because $\log_a(1) = 0$, as mentioned in the fun facts.
- Looking closely, you can see each one pass through $(a, 1)$ for its base. Example: $f(x) = \log_{10}(x)$ passes through $(10, 1)$.
- They all have a vertical asymptote $x = 0$. Aka, they never touch or cross over the y-axis, because their domain is only defined for $x > 0$.
- If $a > 1$, the function is increasing (going uphill). If $0 < a < 1$, the function is decreasing.
- Notice that $p(x)$ is the reflection of $f(x)$ over the x-axis, and their bases are reciprocals. This will always happen.
- I've included on this graph the Big Guys that your science professors asked me to cover. Mostly people use base 10 and base e (the natural log) and you'll find these buttons on a calculator. Computer science uses a log of base 2, in a shock to no one.
- Looking out at end behavior, so for $x \to \infty$, we see that the bigger the base, the slower the growth. Thus you see since $2 < e < 10$, the graph of $\ln x$ lies between the graphs of $\log_{10} x$ and $\log_2 x$.

We should also make sure we understand what it means for exponential and logarithmic functions to be inverses. Let's discuss and then summarize. First, consider what happens if we plug an input x into an exponential function, and then apply the corresponding logarithm, by definition:

$$\log_a(a^x) = ? \quad \Longleftrightarrow \quad a^? = a^x$$

Oh, duh. We must have $? = x$ to make the translated equation true. So doing an exponential and then a log gives you back the original input. Next, consider the other direction of composition, by definition:

$$a^{\log_a x} = ? \quad \Longleftrightarrow \quad \log_a(?) = \log_a(x)$$

Aha, again we must have $? = x$ to make the translation true. So doing a log and then the corresponding exponential gives you back the original input. Inverse relationship confirmed!

$$\log_a(a^x) = x, \text{ for all real numbers } x \quad \text{and} \quad a^{\log_a x} = x, \text{ for all } x > 0$$

Test your understanding with the exercises below.

Exercise 1

Answer True/False.

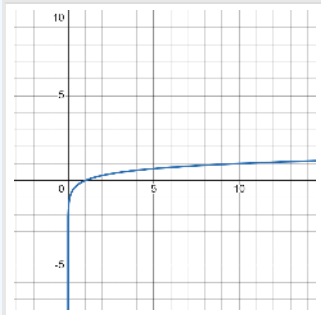
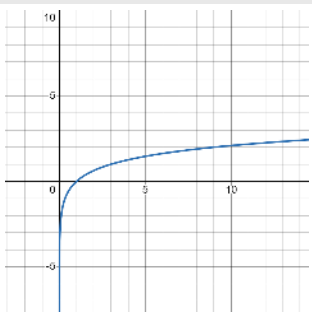
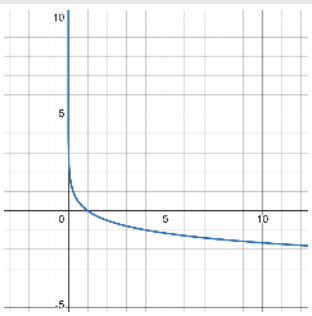
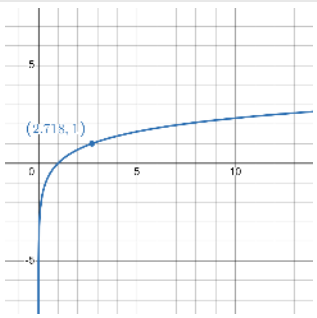
1. For $f(x) = \log_2(x)$, we have $f(4) = 2$.
2. For $f(x) = \log_2(x)$, we have $f\left(\frac{1}{2}\right) = -1$.
3. For $f(x) = \log_{10}(x)$, we have $f(1) = 100$.
4. I am allowed to plug in $x = 0$ to the function $f(x) = \ln x$.
5. I am allowed to plug in $x = 1$ to the function $f(x) = \ln x$.
6. I am allowed to plug in $x = 1$ to the function $f(x) = \ln(x - 1)$.
7. The graph of the function $f(x) = \log_2 x$ is the reflection of $g(x) = 2^x$ across the y-axis.
8. The graph of the function $f(x) = \log_2 x$ is the reflection of $g(x) = 2^x$ across the line $y = x$.
9. The graph of the function $f(x) = \log_{1/2} x$ is the reflection of $g(x) = \log_2 x$ across the x-axis.
10. The graph of $f(x) = \log_3 x$ passes through $(1, 0)$.
11. The graph of $f(x) = \log_3 x$ passes through $(0, 1)$.
12. The graph of $f(x) = \log_3 x$ passes through $(3, 1)$.
13. The graph of $f(x) = \log_a x$ passes through $(a, 1)$.
14. For $f(x) = e^x$ and $g(x) = \ln x$, we know $(f \circ g)(x) = x$, for all $x > 0$.
15. As $x \to \infty$, the graph of $f(x) = \log_5 x$ exhibits slower growth (is less steep) than $g(x) = \ln x$, for $x > 1$.
16. $\ln(x) > \log_5 x$ for $x > 1$.
17. As $x \to \infty$, the graph of $f(x) = \log_5 x$ exhibits faster growth (is steeper) than $g(x) = \log_{10} x$, for $x > 1$.

Answer

1. True, we translate $f(4) = \log_2(4) = ?$ to $2^? = 4$ and see that it must give the answer 2.
2. True, we translate $f\left(\frac{1}{2}\right) = \log_2\left(\frac{1}{2}\right) = ?$ to $2^? = \frac{1}{2}$ and see that the power must be -1.
3. False, we translate $\log_{10}(1) = ?$ to $10^? = 1$ and see that the power must be 0, not 100. Or just remember that for any base, the log of 1 is 0.
4. False, we can't plug 0 into any logs.
5. True, $x = 1$ is in the domain.
6. False, plugging in $x = 1$ results in plugging in $1 - 1 = 0$ into the log, which is illegal.
7. False, these are inverse functions, so the graphs are reflections across the line $y = x$.
8. True.
9. True, we observed that above.
10. True, all such logs pass through $(1, 0)$.
11. False, we can't even plug in the x-value 0.
12. True, if we plug in $x = 3$, we are taking the log of its own base, which gives 1.
13. True, this always works.
14. True, since $\ln$ means log base e, these are inverse functions.
15. True, the bigger the base, the slower the growth, and $5 > e$.
16. True, slower growth means that at some x-value like $x = 2$, the graph of $\log_5 x$ will be lower than the graph of $\ln x$, so has a smaller function value.
17. True, the bigger the base, the slower the growth, so base 5 is faster growing than base 10.

Exercise 2

Match the graphs to their functions.

			
1.	2.	3.	4.
$f(x) = \ln x$	$g(x) = \log_{10} x$	$h(x) = \log_3 x$	$p(x) = \log_{0.25} x$

Answer

1. This graph passes through $(10, 1)$, so the base is 10 and goes with g.
2. This graph passes through $(3, 1)$, so it goes with h.
3. This graph must have a fractional base because it's descending, so it must go with p. We can confirm by checking the point $(4, -1)$ that the graph passes through. Indeed, $\log_{0.25}(4) =?$ translates to $\left(\frac{1}{4}\right)^{?} = 4$ and the function value must be -1. Goes with p.
4. Remember that $e \approx 2.71...$ so that's the point being labeled. Must go with f.

Log Laws

I want to briefly introduce you to the **Logarithm Laws.** These laws all follow from power rules (and if you're interested in going deeper into that, check out Chapter 7). These laws are used to **expand** and **condense** expressions containing logarithms. So they let us work with stuff like this:

$$\log_2(x) + \log_2(x+1), \qquad \ln\left(\frac{x}{y}\right)^3, \qquad 3\log_{10} x + 2\log_{10} x - \log_{10}(10x)$$

These types of expressions must be dealt with in courses like Calc I and Calc II for sure. Let's get right into them.

Log Law	English Translation	Caveman Translation
$\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of expressions (numbers, variables, whatever as long as it is being multiplied) is the sum of the logarithms of the expressions.	**Multiplication** inside the log can be written as **addition** outside the log. Used backwards, addition outside the log translates to multiplication inside the log.
$\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of expressions is the difference of the logarithms of the expressions.	Division inside the log can be written as subtraction outside the log. Used backwards, subtraction outside the log translates to division inside the log. The **denominator** must be the input whose log term is subtracted!
$\log_a(A^B) = B\log_a A$	The logarithm of an expression raised to a power is the power times the logarithm of the expression.	If the whole input is raised to a power, you can **bring powers down** in front as coefficients.

Note: The **bases must match** in order to use these laws backwards! Using them **forwards** is called **expanding**, and using them **backwards** is called **condensing.** These laws are remixed to deal with big complicated combo expressions as well.

Example 1

Expand fully or condense fully. Simplify if reasonable.

1. $\log_2(xy)$
2. $\log_2(2xyz)$
3. $\ln\left(\frac{1}{x}\right)$
4. $\ln(x^3y^5)$
5. $2\log_5 x - \log_5 x$
6. $\ln A - \ln B + \ln C - \ln D$
7. $3\log_3 x + 2\log_3 y$

Solution

1. We expand $\log_2(xy) = \log_2 x + \log_2 y$.

2. We split all the multiplications using additions outside the logs: $\log_2(2xyz) = \log_2(2) + \log_2 x + \log_2 y + \log_2 z$. We also simplify the $\log_2(2)$ term while we're at it, to get $1 + \log_2 x + \log_2 y + \log_2 z$.

3. We expand $\ln\left(\frac{1}{x}\right) = \ln 1 - \ln x$ and simplify since any log with 1 plugged in will give 0, to get $-\ln x$. By the way, you could get straight here by remembering that $\frac{1}{x} = x^{-1}$ and using the bring powers down rule...

4. We expand and bring the powers down to get $\ln(x^3y^5) = \ln x^3 + \ln y^5 = 3\ln x + 5\ln y$. Note that we **cannot** write $(3)(5)(\ln xy)$ because the **entire** input must be raised to a single power in order to bring that power down in front!

5. We're condensing now. **First** move any coefficients up as powers, **then** condense any additions or subtractions. We get $\log_5 x^2 - \log_5 x = \log_5\left(\frac{x^2}{x}\right) = \log_5(x)$. You can also get straight here by recognizing that $2A - A = A$ no matter how ugly A is.

6. Any additions get multiplied, and any subtractions get divided, so just sort them between the numerator and denominator of a fraction in the input! We get $\ln\left(\frac{AC}{BD}\right)$.

7. Bring powers up first and then condense! We get $\log_3 x^3 + \log_3 y^2 = \log_3(x^3y^2)$.

Exercise 3

Expand fully or condense fully. Simplify if reasonable.

1. $\ln[(x+3)(x-1)]$
2. $\log_{10}\left(\frac{10x}{y}\right)$
3. $\log_2(2x^2y^3)$
4. $\ln x + \ln y - 3\ln z$
5. $4\log_3(A+1) - 2\log_3(B)$

Answer

1. $\ln(x+3) + \ln(x-1)$
2. $1 + \log_{10} x - \log_{10} y$
3. $1 + 2\log_2 x + 3\log_2 y$
4. $\ln\left(\frac{xy}{z^3}\right)$
5. $\log_3\left(\frac{(A+1)^4}{B^2}\right)$

Using Logarithms

Generally, you will need to use a calculator of some sort to deal with logarithm application problems. Here are some basic examples.

Example 2

The age of an ancient object can be determined by seeing how much radioactive carbon-14 is left in it, because we know how fast carbon-14 dissipates as a substance ages. If we know the original amount of carbon-14 was C_0, and we measure only C remaining, then we can calculate the age in years using the formula

$$A = -8267 \ln \frac{C}{C_0}$$

To avoid dealing with annoying numbers, let's say that whatever C_0 was, I know that C is only 50% of it. (Using a calculator) find the age of the object. What if we're all the way down to 25% of the original amount?

Solution

If we're down to 50%, then in math terms we know $C = 0.5C_0$. We can plug this into the formula and simplify:

$$A = -8267 \ln \frac{0.5C_0}{C_0} = -8267 \ln(0.5)$$

Using a calculator, we get about $A = 5730$ years old. If $C = 0.25C_0$ instead, we get

$$A = -8267 \ln \frac{0.25C_0}{C_0} = -8267 \ln(0.25) = 11,460 \text{ years}$$

Example 3

You will notice that since many populations exhibit exponential growth, many formulas for computations *about* populations will involve their inverse functions, logarithms. I had that Petri dish of bacteria, remember? It doubles in population every half hour, meaning that every half hour, each bacterium divides into two copies. I started with 200 bacteria, and it turns out that I can compute the amount of time (in hours) needed to reach N bacteria using the formula

$$t = \frac{\ln \frac{N}{200}}{2 \ln 2}$$

Find how long it will take for me to have 800 bacteria, without using a calculator actually. Then using a calculator, find how long it will take to reach 1000 bacteria.

Solution

If you want to learn how to find that formula, check out Chapter 7. For now, let's just use it to compute:

$$t = \frac{\ln \frac{800}{200}}{2 \ln 2} = \frac{\ln 4}{2 \ln 2}$$

Now you say, "I need my calculator!" No you don't! Notice that $\ln 4 = \ln(2^2)$, right? Then use the bringing-down-powers rule...

$$t = \frac{2 \ln 2}{2 \ln 2} = 1 \text{ hour.}$$

You can also just use common sense here... if I'm doubling every half hour and I notice that going from 200 to 800 is just doubling twice, then of course it will only take one hour!

To deal with $N = 1000$, you can calculate:

$$t = \frac{\ln \frac{1000}{200}}{2 \ln 2} = \frac{\ln 5}{2 \ln 2} \approx 1.161 \text{ hours.}$$

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6.4E Exercises

Definition True/False

Answer True/False. If False, explain why.

1. If $\log_a x = y$, then $a^x = y$.
2. If $\log_b a = c$, then $b^c = a$.
3. There is a logarithmic function with base 1, $f(x) = \log_1 x$.
4. The natural log function $\ln x$ is the logarithmic function with base π.
5. Let $f(x) = \log_a x$ be a logarithmic function ($a > 0, a \neq 1$). The domain of f is $\{x \mid x \geq 0\}$.
6. Let $f(x) = \log_a x$ be a logarithmic function ($a > 0, a \neq 1$). The domain of f is $\{x \mid x > 0\}$.
7. I am allowed to plug in $x = -2$ to the function $f(x) = \ln(x + 3)$.
8. I am allowed to plug in $x = -2$ to the function $f(x) = \ln(x + 1)$.
9. For any base a, $\log_a(a) = 1$.
10. For any base a, $\log_a(0) = 1$.
11. For any base a, $\log_a(1) = 0$.
12. For $f(x) = \log_{10} x$ and $g(x) = 10^x$, we know $(g \circ f)(x) = x$ for all $x > 0$.

Answer

1. F, this does not match the definition. It must be $a^y = x$.
2. T
3. F, we do not allow the base $a = 1$. (Why doesn't it make sense?)
4. F, it's base e.
5. F, we can't have $\geq$ because 0 is not allowed.
6. T
7. T
8. F, if $x = -2$ then the input to the log is $-2 + 1 = -1$, a negative number.
9. T
10. F, 0 isn't even in the domain.
11. T
12. T

Graphs True/False

Answer True/False. If False, justify or find the error.

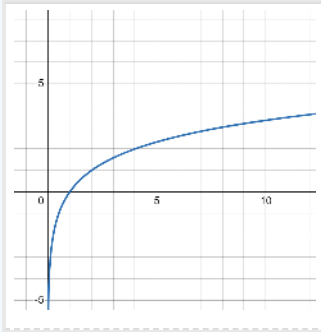
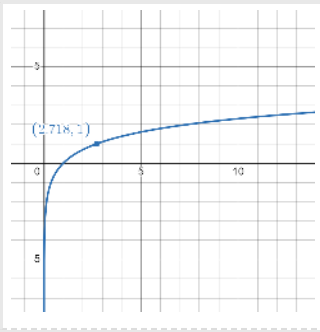
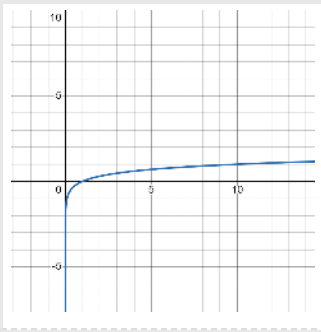
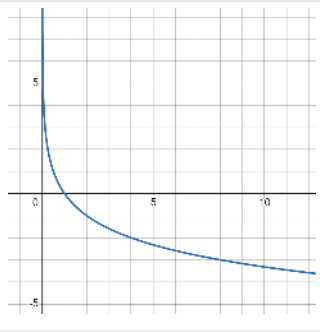
1. The graph of $f(x) = \log_3 x$ passes through $(1, 0)$.
2. The graph of $f(x) = \log_a x$ passes through $(1, a)$ for any base a.
3. The graph of $f(x) = \log_a x$ passes through $(a, 1)$ for any base a.
4. The graph of the function $f(x) = \log_3 x$ is the reflection of the graph of $g(x) = \log_{\frac{1}{3}} x$ across the x-axis.
5. The graph of the function $f(x) = \ln x$ is the reflection of the graph of $g(x) = e^x$ across the x-axis.
6. The graph of $f(x) = \log_2 x$ exhibits faster growth than the graph of $g(x) = \ln x$, for $x > 1$.
7. The graph of $f(x) = \log_{10} x$ exhibits faster growth than the graph of $g(x) = \ln x$, for $x > 1$.

Answer

1. T
2. F, it passes through $(1, 0)$.
3. T
4. T
5. F, it's the reflection across the line $y = x$ because they're inverse functions.
6. T
7. F, the base $10 > e$.

Matching Graphs

By analyzing signal points like $(a, 1)$ and using your knowledge of logarithmic function graphs, match the graphs to their functions.

			
1.	2.	3.	4.
$f(x) = \log_{10} x$	$g(x) = \ln x$	$h(x) = \log_{\frac{1}{2}} x$	$p(x) = \log_2 x$

Answer

1. p
2. g
3. f
4. h

Working With Logarithms

Without using a calculator, compute the following using the definition and Log Laws:

1. $\log_3 9$
2. $\log_{17}(1)$
3. $\log_2\left(\frac{1}{4}\right)$
4. $\ln e$
5. $\ln e^4$
6. $\log_{10} 1000$
7. $\log_2 4 + 2\log_2 8$
8. $\log_3(6) - \log_3(2)$

Answer

1. 2
2. 0
3. -2
4. 1
5. 4
6. 3
7. 8
8. 1 (Hint: consider $\log_3(6) = \log_3(2 \cdot 3)$.)

Condensing/Expanding

Fully expand or fully condense. Simplify if reasonable.

1. $\log_2\left(\frac{x}{2}\right)$
2. $\log_{10}(100x^2y^3)$

3. $\ln(A^2 + 2AB + B^2)$
4. $\log_3 11 + \log_3 7$
5. $\ln x + \ln y - 2\ln z - 3\ln(w+1)$
6. $3\log_a(A+B) - 4\log_a(A-B)$

Answer

1. $\log_2 x - 1$
2. $2 + 2\log_{10} x + 3\log_{10} y$
3. $2\ln(A+B)$
4. $\log_3(77)$
5. $\ln\left(\dfrac{xy}{z^2(w+1)^3}\right)$
6. $\log_a\left(\dfrac{(A+B)^3}{(A-B)^4}\right)$

? Logarithms Vs. Exponentials True/False

Answer True/False. Justify your answer.

1. $\log_2(2^x) = x$
2. $\log_2(8) = 3$
3. For any base a, $\log_a(x^a) = x$.
4. $e^{\ln x} = x$ (for $x > 0$)
5. $e^{\log_{10} x} = x$ (for $x > 0$)
6. $10^{\log_{10} x} = x$ (for $x > 0$)

Answer

1. T, inverse functions.
2. T, write as $\log_2(2^3) = 3\log_2 2$ or translate with definition.
3. F, it should be $\log_a(a^x) = x$.
4. T, inverse functions.
5. F, the bases must match.
6. T, inverse functions.

? Solving for x

By translating back and forth from logarithm form to exponential form, solve for x in the equations below.

1. $\log_{10}(x) = 3$
2. $\log_2(x) = 1$
3. $\ln(x+1) = 4$
4. $\log_5(5x) = 2$
5. $\ln(x) - 3 = 0$
6. $2\ln x = 4$

Answer

1. $x = 1000$
2. $x = 2$
3. $x = e^4 - 1$
4. $x = 5$
5. $x = e^3$
6. $x = e^2$

Applications

1. We saw how exponential functions can be used to calculate interest compounding annually. In fact, you can choose to compound multiple times a year, or even continuously. For **continuously compounding interest,** the amount of money you have at time t years after investing an initial amount A_0 is given by $A = A_0 e^{rt}$, where r is the interest rate written as a decimal. Say I want to double my money, so that A is $2A_0$. Then my equation becomes $2 = e^{rt}$. Give the equivalent logarithmic equation. Solve it for t. If the interest rate is 12%, find the time t it will take to double my money. (Use a calculator if needed.)

2. If I take a turkey out of the oven with internal temperature 165°F and place it in an ambient room temperature of 65°F, then the amount of time (in minutes) it will take to cool to an internal temperature of T is given by

$$t = -40 \ln\left(\frac{T-65}{100}\right)$$

Use a calculator to find how long it takes to cool down to 150°F, 125°F, and 100°F.

Answer

1. The equivalent equation is $\ln(2) = rt$. Solved for t, we have $t = \frac{\ln 2}{r}$. If $r = .12$, then $t \approx 5.78$ years.
2. For $T = 150^\circ$F, $t \approx 6.5$ minutes. For $T = 125^\circ$F, $t \approx 20.4$ minutes. For $T = 100^\circ$F, $t \approx 42$ minutes.

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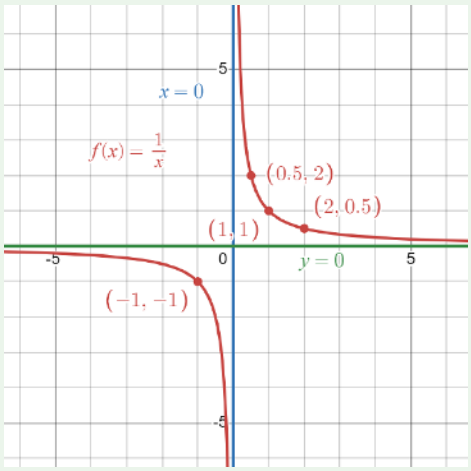
6.5 Chapter 6 Study Guide

Definitions

- **rational function:** a function that is a quotient of two polynomials, $f(x) = \dfrac{p(x)}{q(x)}$.
- **vertical asymptote:** a vertical line with equation $x = a$ that a function approaches infinitely but never touches. As x approaches a from one side or the other, the function shoots off to positive or negative infinity, hugging the vertical line.
- **horizontal asymptote:** a horizontal line with equation $y = a$ that a function approaches as $x \to \infty$ and/or $x \to -\infty$.
- **hole:** a location where a rational expression is not defined due to a domain issue, but doesn't have a V.A.
- **slant asymptote:** a line (linear function) that a rational function "behaves like" as $x \to \infty$. Only happens if the degree of the numerator is exactly one more than the degree of the denominator.
- **limit of f at $x = a$:** the y-value that the function's graph is approaching as x approaches a from both the left and the right.
- **one-sided limit:** a limit where you only approach $x = a$ from the left **or** the right.
- **exponential function:** a function of the form $f(x) = a^x$, with $a > 0, a \neq 1$. The number a is the **base.**
- **logarithmic function:** the logarithmic function with base a, $\log_a x$, is defined by $\log_a x = y \iff a^y = x$.
- **natural log:** the natural log function $\ln x$ is the logarithmic function with base e.

Facts About Rational Functions

- Their **domain** is all real numbers **except** any values of x that cause $q(x)$ to be zero. You can write this in set notation as $\{x \mid q(x) \neq 0\}$.
- Often, your first move should be to fully factor the top and bottom. Then to identify...
  - the **domain** of a rational function, just look at the denominator alone, set it equal to 0, and solve for x. Those values cause division by zero, so they cannot be allowed in the domain. Those values also give the vertical asymptotes, which will have equations of the form $x = c$.
  - the **roots** (x-intercepts) of a rational function, just look at the numerator alone, set it equal to 0, and solve for x. Aka, the roots of the numerator give the roots of the rational function.
  - any **horizontal asymptotes** of a rational function using the graph, look for an invisible horizontal line that the function hugs as $x \to \infty$ or $x \to -\infty$. If the height of that line is c, then the equation of the horizontal asymptote is $y = c$.



Partial Fraction Decomposition

To determine $A, B,$ and C such that $f(x) = \dfrac{x+1}{x(x-1)(x+2)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+2}$,

1. Multiply through the entire equation by the left hand side's whole denominator to clear all denominators.
2. Compare coefficients of like terms on both sides of the equation. Each power of x will give an equation involving $A, B,$ and C.
3. Solve the resulting system of equations to find values for $A, B,$ and C.

Solving a Polynomial Inequality

1. Move all the terms to one side and get 0 on the other.
2. Factor the polynomial into linear factors and irreducible quadratic factors, and find the real roots.
3. Make a number line, divided into zones (intervals) by the roots.
4. In each zone, determine whether the function is positive or negative by using logic or testing points and comparing the signs of the factors.
5. Report the desired zones as intervals, based on the inequality sign.

Solving a Rational Inequality

1. Move all the terms to one side into a **single** giant fraction and get 0 on the other.
2. Factor the numerator **and** the denominator. Identify the **points of interest** where **either** the numerator or the denominator are zero. (Basically, roots of the top poly and roots of the bottom poly.)
3. Make a number line, divided into zones (intervals) by the points of interest.
4. In each zone, determine whether the function is positive or negative by using logic or testing points and comparing the signs of the factors.
5. Report the desired zones as intervals, based on the inequality sign. **Test** the endpoints carefully to see if they should be included, because the function may not be defined at some of them!

Limit of a Function

When looking at a graph of $y = f(x)$, the **limit of** f **at** $x = a$ is the y-value that the function appears to be heading toward as you move towards a particular x-value of interest, a, from both the left and the right. The **existence** of a limit does depend on the function actually approaching a value from both directions.

In other words, nicknaming the limit "L" if it exists, we're saying that as $x \to a$, $f(x) \to L$, in the arrow notation we've used before. **NOTE:** the limit **does not care** what the **actual** function value $f(a)$ is, or even whether it's defined! It's all about where the function ACTS like it's going.

We use the **limit notation**

$$\lim_{x \to a} f(x) = L$$

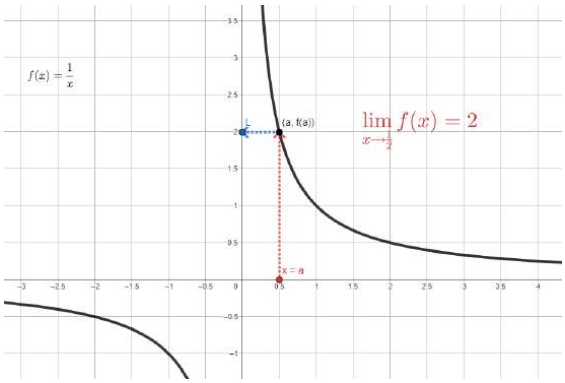
if the limit exists properly. If a function tends toward $\pm\infty$ when approaching $x = a$ from both directions, the limit is infinite, and although ∞ is not a number, we still use the notation

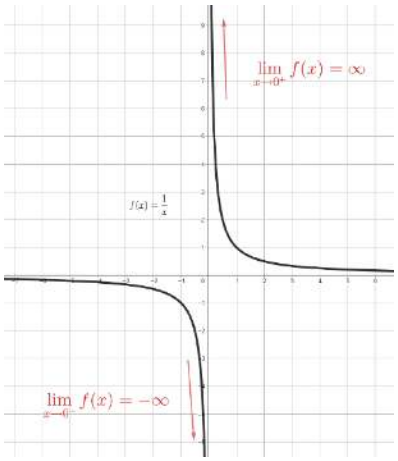
$$\lim_{x \to a} f(x) = (\pm)\infty$$

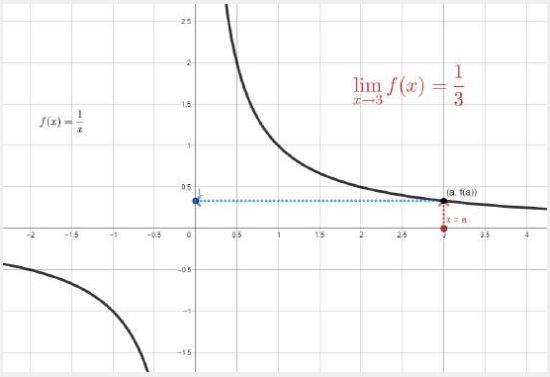
Sometimes we are interested in **one-sided limits,** looking at the behavior of the function as we come in from only one side. In this case, we use a "$+$" decoration to indicate approaching **from the right,** and a "$-$" to indicate approaching **from the left.**

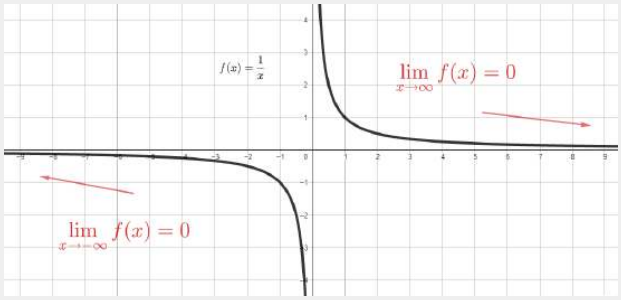
$$\text{approaching } x = a \text{ from the right only: } \lim_{x \to a^+} f(x) = L,$$
$$\text{approaching from the left only: } \lim_{x \to a^-} f(x) = L$$

Limit Examples



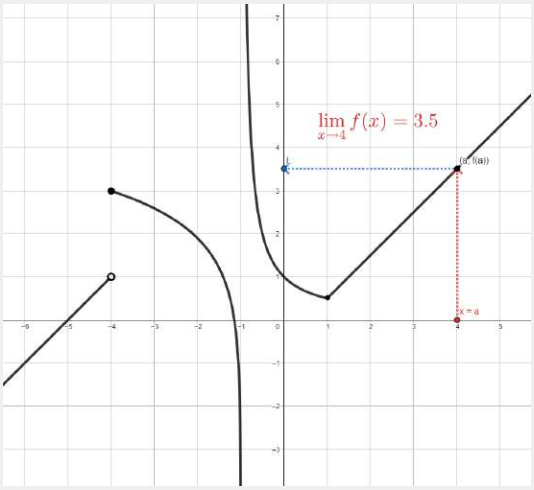


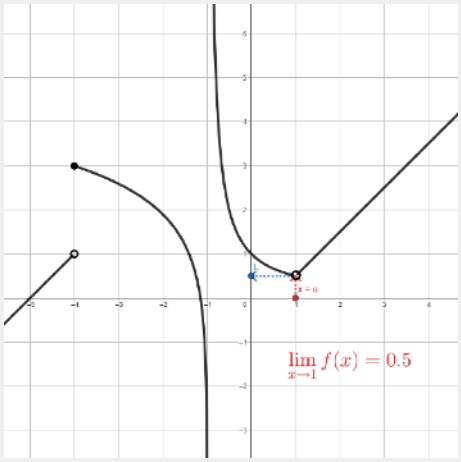




The limit exists and agrees with the function value at that location.

The limit exists even though the function is not defined at that location.

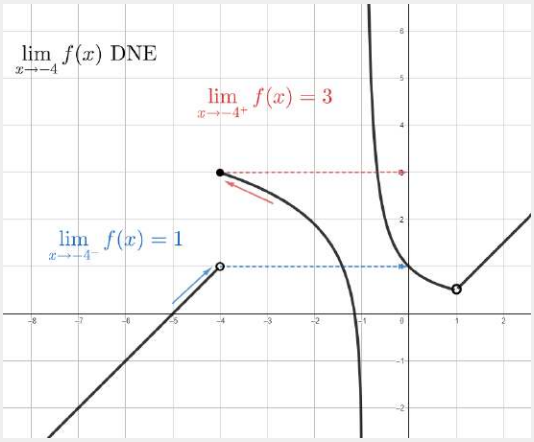




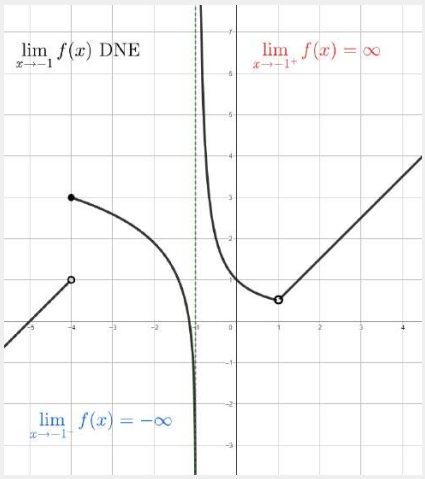
The one-sided limits exist, but do not agree, so the limit as a whole does not exist.

The one-sided limits are infinite, but do not agree, so the limit as a whole does not exist.

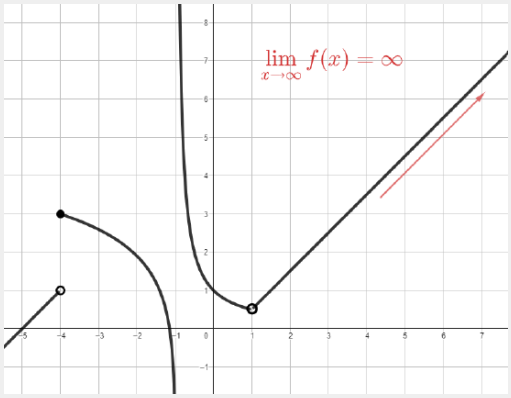
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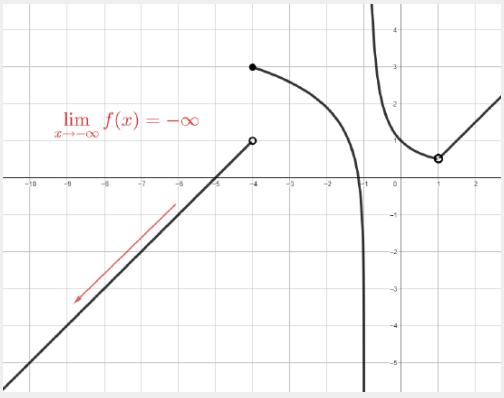
The function goes up forever, so the "limit at infinity" is infinite.



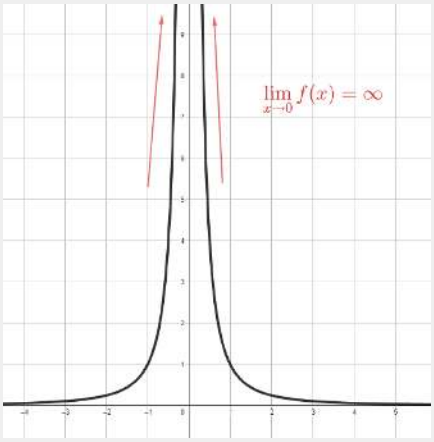
The function goes down forever, so the "limit at negative infinity" is negative infinity.

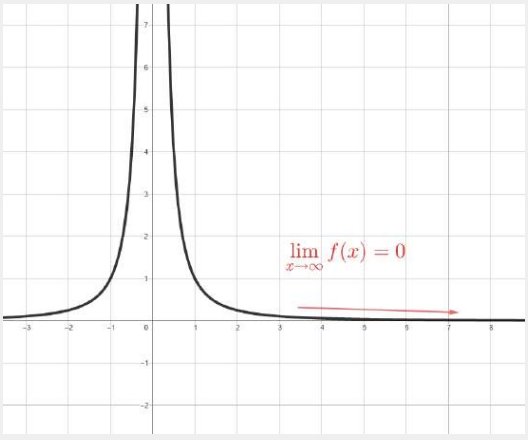


The one-sided limits are infinite, but they do agree, so we say the limit is infinity.



The "limit at infinity" is leveling out, hugging a certain horizontal line, so it is finite.





Vertical Asymptotes via Limits

The vertical line $x = a$ is a vertical asymptote of a function $f(x)$ if ANY of the following are true:

$\lim_{x\to a} f(x) = \infty$	$\lim_{x\to a^+} f(x) = \infty$	$\lim_{x\to a^-} f(x) = \infty$
$\lim_{x\to a} f(x) = -\infty$	$\lim_{x\to a^+} f(x) = -\infty$	$\lim_{x\to a^-} f(x) = -\infty$

Horizontal Asymptotes via Limits

The horizontal line $y = L$ is a horizontal asymptote of a function $f(x)$ if

$$\lim_{x\to\infty} f(x) = L \quad \text{and/or} \quad \lim_{x\to-\infty} f(x) = L$$

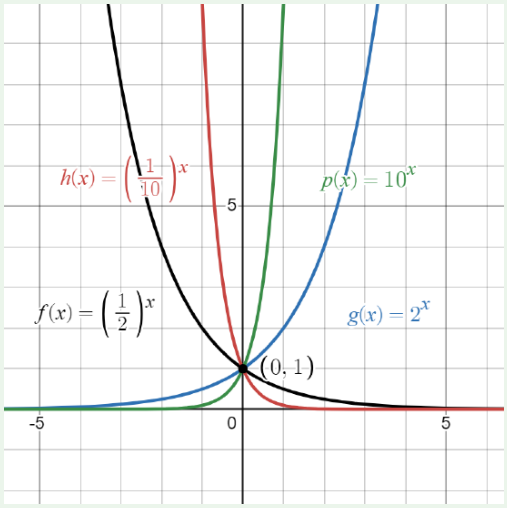
Finding a Slant Asymptote

1. Make sure you're looking at a rational function $\frac{p(x)}{d(x)}$ where the degree of $p(x)$ is exactly one more than the degree of $d(x)$.
2. Perform polynomial long division, writing $\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$.
3. The slant asymptote is $y = q(x)$.

Facts About Exponential Functions

Exponential functions are functions of the form $f(x) = a^x$ for some **base** $a > 0, a \neq 1$.

- In an exponential function, the **base** is a plain ol' number and the **variable** is in the **exponent.** Not to be confused with polynomial expressions where the variable is the base and a number is the power, like x^3! Whole different animal.
- The **domain** of an exponential function is all real numbers.
- If $a > 1$, the function is **increasing** (graph going uphill from left to right). If $0 < a < 1$, the function is **decreasing** (graph going downhill from left to right).
- **End behavior for** $a > 1$**:** As $x \to -\infty$, the graph hugs a horizontal asymptote at the x-axis. As $x \to \infty$, the graph shoots up rapidly, and the bigger the base, the faster the growth.
- **End behavior for** $0 < a < 1$**:** As $x \to -\infty$, the graph is blowing up to the left. As $x \to \infty$, the graph hugs the horizontal asymptote at the x-axis forever.
- All exponential functions pass through $(0, 1)$ as well as $(1, a)$, for whatever their base a is.
- The function values are always positive, aka above the x-axis, aka the **range** is $(0, \infty)$.
- For $f(x) = a^x$ and $g(x) = \left(\frac{1}{a}\right)^x$, note that $g(x) = \left(\frac{1}{a}\right)^x = \frac{1}{a^x} = a^{-x} = f(-x)$.



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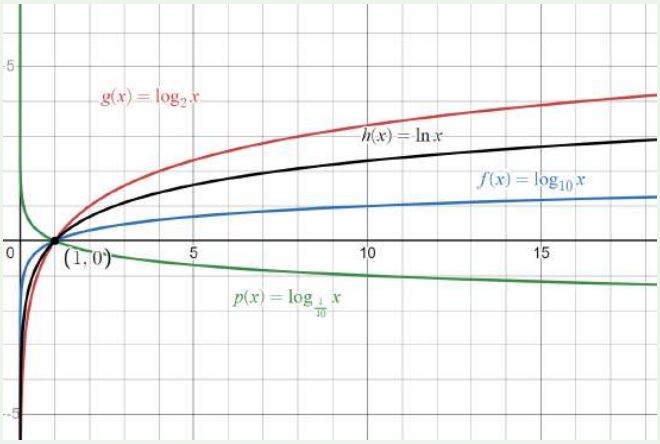
Facts About Logarithmic Functions

The logarithmic function with base a ($a > 0, a \neq 1$), $\log_a x$, is defined by

$$\log_a x = y \quad \iff \quad a^y = x.$$

- Basically, "$\log_a(x) =$?" is asking the **question**, "What do I need to raise the base a to in order to get out this x?" The **answer** to that question is the output of the function $\log_a$.
- When in particular the base is the special number e, we write $\log_e x$ as $\ln x$, and call it the **natural log of x.**
- Their **domain** is **only** positive real numbers! You are **not** allowed to put 0 or a negative number into a log, because that would be the same as saying $a^y = 0$, and there's no possible y that will satisfy that!
- Since $a^0 = 1$ for any a, we have $\log_a(1) = 0$ for any base a. Aka, they pass through the point $(1, 0)$.
- Similarly, since $a^1 = a$, we have $\log_a(a) = 1$ for any a. Aka, if you take the log of its own base, you will always get 1. Also aka, they pass through the point $(a, 1)$.
- The functions $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions, so their graphs are mirror images across the line $y = x$.
- While exponential functions exhibited **fast** growth (or decay), logarithmic functions exhibit **slow growth** (or decay).
- They have a vertical asymptote $x = 0$ due to the domain issue.
- If $a > 1$, the function is increasing (going uphill). If $0 < a < 1$, the function is decreasing.
- The graph $y = \log_a x$ is the reflection of the graph $y = \log_{1/a} x$ over the x-axis.
- A function with larger base will be growing slower (after $x > 1$) than a function with a smaller base. See picture below.
- Inverse Functions: $f(x) = a^x$ and $g(x) = \log_a x$ undo each other, i.e.

$$\log_a(a^x) = x, \text{ for all real numbers } x \quad \text{and} \quad a^{\log_a x} = x, \text{ for all } x > 0$$



Log Laws

Log Law	English Translation	Caveman Translation
$\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of expressions (numbers, variables, whatever as long as it is being multiplied) is the sum of the logarithms of the expressions.	**Multiplication** inside the log can be written as **addition** outside the log. Used backwards, addition outside the log translates to multiplication inside the log.
$\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of expressions is the difference of the logarithms of the expressions.	Division inside the log can be written as subtraction outside the log. Used backwards, subtraction outside the log translates to division inside the log. The **denominator** must be the input whose log term is subtracted!

$\log_a(A^B) = B\log_a A$	The logarithm of an expression raised to a power is the power times the logarithm of the expression.	If the whole input is raised to a power, you can **bring powers down** in front as coefficients.

Note: The **bases must match** in order to use these laws backwards! Using them **forwards** is called **expanding**, and using them **backwards** is called **condensing.**

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CHAPTER OVERVIEW

Chapter 7: Exponential and Logarithmic Functions

In this chapter, we get seriously into working with exponential and logarithmic functions. In the first two sections, we reproduce or summarize the basics from Chapter 6, if you've already read that, with the addition of transformed functions, more advanced graphing, and some extra topics. Then we'll get good at using these functions and their inverse relationship to solve equations that we haven't been able to deal with before. Finally, we check out many real-world applications involving these functions, because boy howdy are they all over the place in science and money!

7.1 Exponential Functions

7.1E Exercises

7.2 Logarithmic Functions

7.2E Exercises

7.3 Solving Exponential and Logarithmic Equations

7.3E Exercises

7.4 Applications

7.4E Exercises

7.5 Chapter 7 Study Guide

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7.1 Exponential Functions

Learning Objectives

By the end of this section, you will be able to:

- recognize exponential functions, their important features, and their graphs
- recognize transformations of exponential functions and their graphs
- sketch the graph of an exponential function
- evaluate exponential functions and use technology to solve problems

As you've probably deduced from the name, **exponential functions** involve raising something to some kind of power. We've seen functions where a variable is raised to a plain ol' number power already; those are polynomials like $p(x) = x^3$. Well, this is the other way around. This time, we're taking a plain ol' number as the **base** and putting the *variable* upstairs in the exponent. Like the functions $f(x) = 3^x$ or $g(x) = 17^x$, or even $\exp(t) = e^t$, which uses the number $e \approx 2.718...$ as the base!

We're secretly making a claim here that we know how to raise a number to ANY real number power, because we want to say that x can be anything. We know how to do integer powers and rational powers, but how do you deal with irrational numbers? A calculator can handle it, by simply rounding off the decimal approximation of an irrational number somewhere, and computing that rational number power instead. In practice, you can get "close enough" for your various purposes using technology to do this approximation technique.

Here's some good news: All of the power rules that we know already still apply!

Technology Note: when you need to type exponents into various calculators/websites, you will generally use the carat symbol ^ (shift+6 on a keyboard). If your power is very complicated, you'll want to also use parentheses. For example, to enter $2^{-\sqrt{5}}$ into WolframAlpha, you can type "2^(-sqrt(5))" and it will understand.

Exponential Functions

The **exponential function with base** a is defined by $f(x) = a^x$, for all real numbers x, where $a > 0$ and $a \neq 1$.

Fun facts about exponential functions:

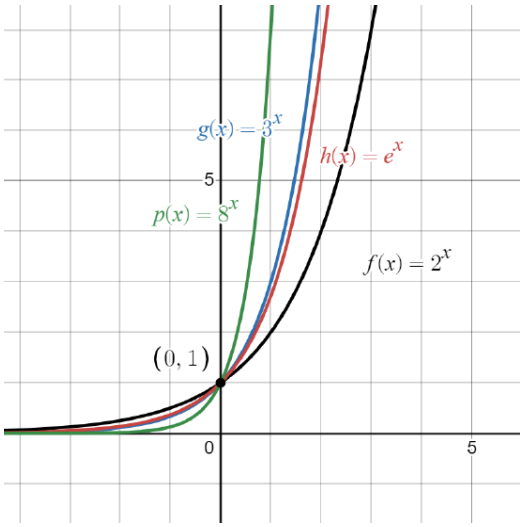
- We don't allow $a = 1$ because 1^x is always 1, giving a super boring constant function $f(x) = 1$.
- We don't allow negative bases because raising negative numbers to even (and some fractional) powers is a mess for the real numbers.
- The **domain** of an exponential function is all real numbers.

Exponential functions are very sensitive. A small change in the variable can cause significant change in the function value. For example, if $f(x) = 8^x$, consider

x-value	0	1	2	3	4
$f(x)$	$8^0 = 1$	$8^1 = 8$	$8^2 = 64$	$8^3 = 512$	$8^4 = 4096$

Going from $x = 2$ to $x = 3$ doesn't feel like much, but the function value shoots from 64 all the way to 512.

When you raise a base $a > 1$ to bigger and bigger powers, the results **increase**, just like the example above with $a = 8$. Let's see some graphs.



Observations:

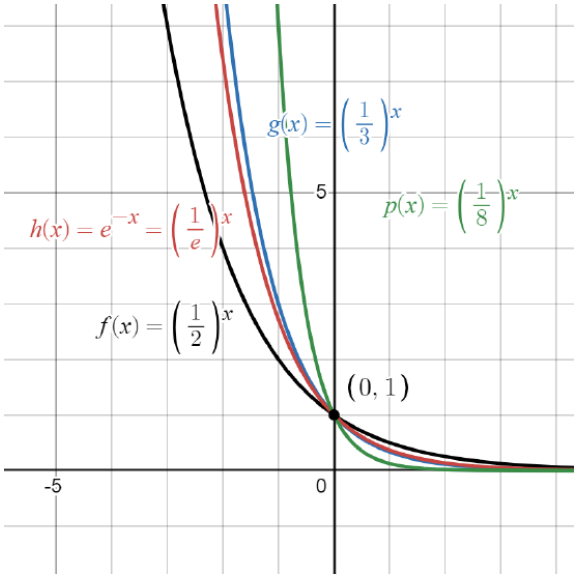
- All $f(x) = a^x$ pass through the point $(0, 1)$. This makes sense because anything raised to the 0 is 1, so $f(0) = a^0 = 1$, regardless of what a is.
- There's no reason a base has to be a whole number! Recall the special number $e \approx 2.718$. You can use him as a base number, and you can see that his graph lies between $f(x) = 2^x$ and $g(x) = 3^x$, just as $2 < 2.718 < 3$.
- On the left side of the graph, as $x \to -\infty$, the graphs hug a horizontal asymptote $y = 0$. This makes sense because plugging in negative numbers as powers produces smaller and smaller numbers. For example,

$$2^{-1} = \frac{1}{2}, \qquad 2^{-2} = \frac{1}{4}, \qquad 2^{-5} = \frac{1}{32}, \quad \ldots$$

- These functions are always positive! The graph never crosses the x-axis. Why? Is there any option for x such that $a^x = 0$?
- Since $a^1 = a$ for any a, you can see that they each pass through the point $(1, a)$ matching their base. For example, $f(x) = 2^x$ passes through $(1, 2)$.
- Looking out at end behavior as $x \to \infty$, the bigger the base, the steeper the function (the faster it grows).

On the other hand, if $0 < a < 1$, raising to higher and higher powers causes the function value to **decrease.** For example, consider raising $a = \frac{1}{2}$ to powers...

$$\left(\frac{1}{2}\right)^{-1} = 2, \qquad \left(\frac{1}{2}\right)^{0} = 1, \qquad \left(\frac{1}{2}\right)^{1} = \frac{1}{2}, \qquad \left(\frac{1}{2}\right)^{2} = \frac{1}{4}, \quad \ldots$$



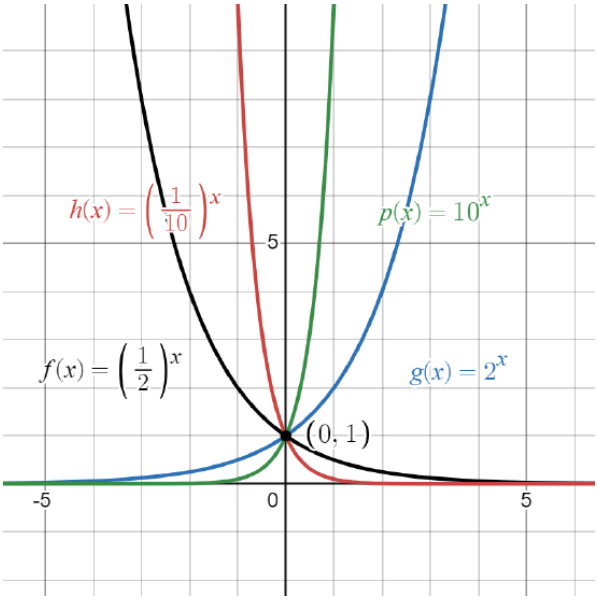
Observations (reason through them as above):

- The graphs all still pass through $(0,1)$.
- The graphs hug a horizontal asymptote $y=0$, this time as $x \to \infty$ to the right.
- The functions are still always positive (above the x-axis).
- The functions still pass through the point $(1,a)$ where a is their base. They also pass through the point $(-1, \frac{1}{a})$, which is often easier to see on the graph. For example, $f(x)$ above passes through $(-1,2)$.

There's one more thing I want to call out, which I've hinted at in the graph above, labeling $h(x)=e^{-x}=\frac{1}{e}^{x}$. Remember your power rules! There is a relationship between the functions $f(x)=a^x$ and $g(x)=\left(\frac{1}{a}\right)^x$.

$$g(x)=\left(\frac{1}{a}\right)^{x}=\frac{1}{a^x}=a^{-x}=f(-x)$$

You can transform f into g by replacing the input x with $-x$, which we recall from function transformations causes a **reflection** about the y-axis. Observe below.



To sum up,

- If $a>1$, the function is **increasing** (graph going uphill from left to right). If $0<a<1$, the function is **decreasing** (graph going downhill from left to right). These trends are called exponential growth or exponential decay, respectively.
- **End behavior for $a>1$:** As $x \to -\infty$, the graph hugs a horizontal asymptote at the x-axis. As $x \to \infty$, the graph shoots up rapidly, and the bigger the base, the faster the growth.
- **End behavior for $0<a<1$:** As $x \to -\infty$, the graph is blowing up to the left. As $x \to \infty$, the graph hugs the horizontal asymptote at the x-axis forever.
- All exponential functions pass through $(0,1)$ as well as $(1,a)$, for whatever their base a is.
- The function values are always positive, aka above the x-axis, aka the **range** is $(0,\infty)$.
- For $f(x)=a^x$ and $g(x)=\left(\frac{1}{a}\right)^x$, note that $g(x)=\left(\frac{1}{a}\right)^x=\frac{1}{a^x}=a^{-x}=f(-x)$.

Test your understanding with the exercises below.

? Exercise 1

Answer True/False:

1. The function $f(x)=x^7$ is exponential with base 7.
2. The function $f(x)=7^x$ is exponential with base 7.
3. The function $f(x)=4^x$ is a horizontal reflection across the y-axis of the function $g(x)=4^{-x}$.

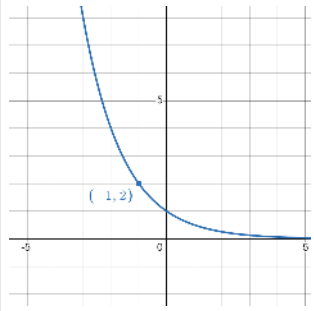
4. The function $f(x) = 4^x$ is a horizontal reflection across the y-axis of the function $g(x) = \left(\frac{1}{4}\right)^x$.
5. The graph of $f(x) = (1.5)^x$ is increasing.
6. The graph of $f(x) = (0.25)^x$ is decreasing.
7. The graph of $f(x) = 3^x$ is decreasing.
8. As $x \to \infty$, the graph of $f(x) = 2^x$ is steeper (faster growing) than the graph of $g(x) = 3^x$.
9. As $x \to \infty$, the graph of $f(x) = 5^x$ is steeper (faster growing) than the graph of $g(x) = e^x$.
10. The graph of $f(x) = 6^x$ passes through $(1, 6)$.
11. The graph of $f(x) = 3^x$ passes through $(3, 1)$.
12. The graph of $f(x) = a^x$ passes through $(0, 1)$ for any base a.
13. The graph of $f(x) = a^x$ passes through $(1, 0)$ for any base a.
14. There exists a value of x such that $a^x < 0$ for any base a.

Answer

1. False. That's a polynomial (monomial) with degree 7.
2. True.
3. True.
4. True, this is equivalent to the previous statement.
5. True, the base is bigger than 1.
6. True, the base is smaller than 1.
7. False.
8. False, the bigger the base, the faster the growth.
9. True, 5 is a bigger base than $e \approx 2.71\ldots$.
10. True, $f(1) = 6$.
11. False, it passes through $(1, 3)$ though. At $x = 3$, $f(3) = 9$.
12. True, $a^0 = 1$ for any a.
13. False, $a^1 \neq 0$.
14. False, the function $f(x) = a^x$ is always positive.

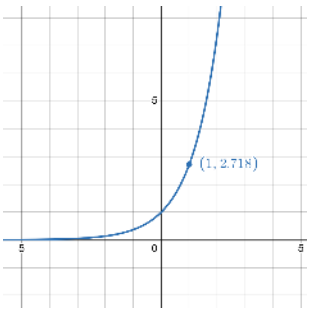
Exercise 2

Match the exponential functions with their graphs.



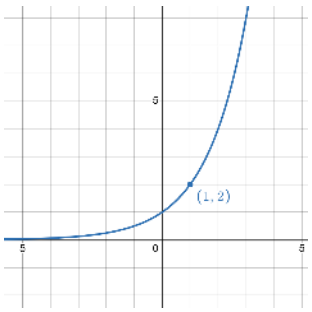
1.

$f(x) = e^x$



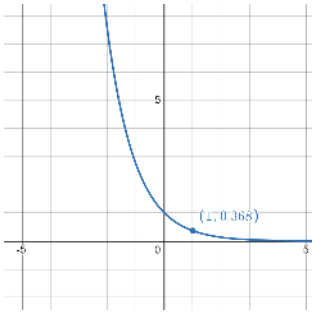
2.

$g(x) = 2^x$



3.

$h(x) = \left(\frac{1}{2}\right)^x$



4.

$p(x) = e^{-x}$

Answer

1. h
2. f
3. g
4. p

https://math.libretexts.org/@go/page/155683

Transformations of Exponential Functions

Recall from our study of function transformations that you can...

- Shift a function vertically by changing from $f(x)$ to $f(x) \pm k$.
- Shift a function horizontally by changing from $f(x)$ to $f(x \pm k)$.
- Stretch or shrink a function vertically by changing from $f(x)$ to $kf(x)$.
- Stretch or shrink a function horizontally by changing from $f(x)$ to $f(kx)$.
- Reflect a function across the y-axis by changing from $f(x)$ to $f(-x)$.
- Reflect a function across the x-axis by changing from $f(x)$ to $-f(x)$.

Although the definition of "the" exponential functions above is a simple $f(x) = a^x$, you will hear people use the term "exponential function" for anything that looks like the guy below, where $c_0, c_1, c_2,$ and c_3 are all constants that have shown up due to transformations.

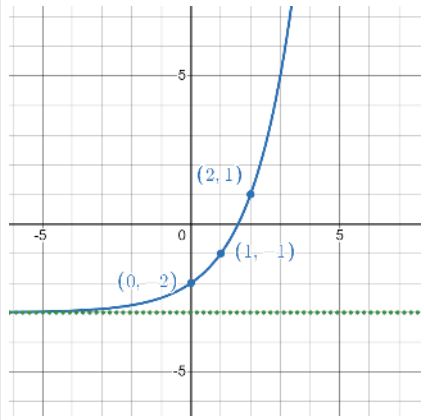
$$f(x) = c_1 a^{c_2 x + c_3} + c_0$$

(This would be like taking a^x and doing a horizontal shift then stretch, then a vertical stretch then shift... if c_1 or c_2 are negative, maybe there was a reflection in there, yikes...)

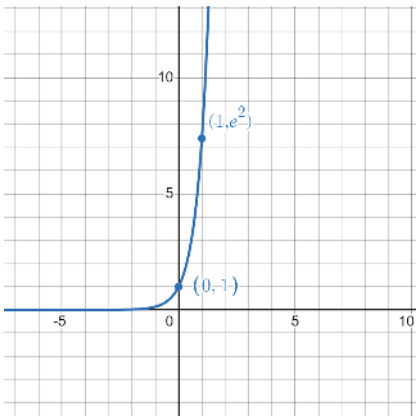
We will see a whole bunch of functions like this when we practice application problems. For now, I want you to apply your previous knowledge to match graphs to their functions.

Example 1

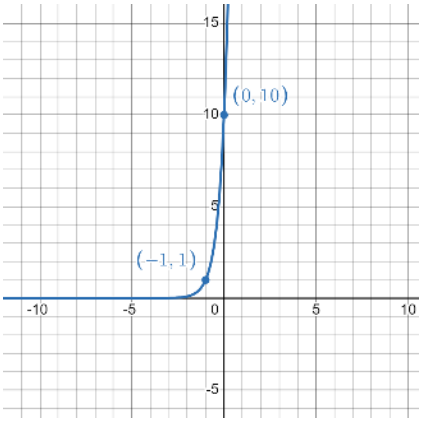
Give the function whose graph is shown. I've marked some signal points to use in the reasoning. I will give you matching in your own exercise, but this will help you understand the process.



1.



2.



3.

Solution

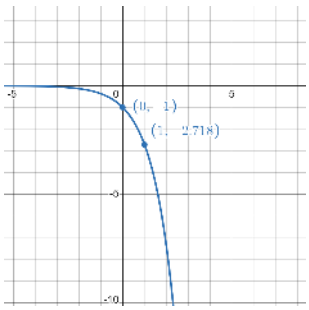
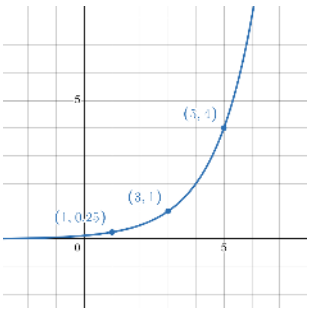
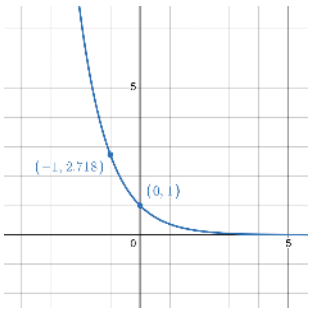
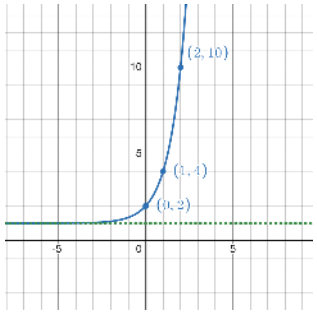
1. The easiest thing to look out for first is where the horizontal asymptote has moved to. It used to be at $y = 0$ but now it's at $y = -3$, which is a vertical shift down by 3, so I start out thinking about "$a^x - 3$." Normally, the graph passes through the point $(0, 1)$, and I see now it passes through $(0, -2)$, so that's consistent with the vertical shift. Before the shift, the point $(1, -1)$ would have been at $(1, 2)$, and that looks like $(1, a)$ for $a = 2$. So I think I'm looking at $f(x) = 2^x - 3$. Let's check the other point, $(2, 1)$ by plugging in $x = 2$. We have $f(2) = 2^2 - 3 = 1$, yep! **Note:** if a vertical shift is involved, it's totally possible for a transformed function like this to take negative values. It's just the a^x guys that are guaranteed to be always positive.

2. First, the horizontal asymptote is still at 0, so I don't have a vertical shift. That point $(1, e^2)$ is an interesting choice of call-out... What kind of function of the form a^{kx} could produce e^2 when $x = 1$? Well, the a better be base e, and $k = 2$ would work! This function is $f(x) = e^{2x}$. Visually, this looks like e^x got a horizontal squish.

3. Again, no vertical shift happening. Now, usually the function should pass through $(0, 1)$, but it looks like that point has shifted horizontally over to $(-1, 1)$. That's a horizontal shift to the left by 1. That means the point $(0, 10)$ was originally $(1, 10)$, and who passes through that? We must have started with 10^x and then performed the shift left. That gives us $f(x) = 10^{x+1}$.

Exercise 3

Match the graphs to their functions.

1.	2.	3.	4.
			
$f(x) = e^{-x}$	$g(x) = -e^{x}$	$h(x) = 3^{x} + 1$	$p(x) = 2^{x-3}$

Answer

1. Remember that $e \approx 2.718...$ so that point is $(1, -e)$. This graph looks exactly like $y = e^{x}$ reflected over the x-axis, which would become $-e^{x}$. Must be g.
2. No vertical shift here. Usually we pass through $(0, 1)$ but this time, we pass through $(3, 1)$. That looks like a horizontal shift by 3 to the right. Let's hypothesize "a^{x-3}" and see what would satisfy the point $(5, 4)$. We want to get 4 when plugging in $x = 5$, so what would make

$$a^{5-3} = a^{2} = 4$$

   be true? The base $a = 2$ will do that! If you want to double check, try plugging in $x = 1$ to confirm the point $(1, 0.25)$. This goes with p.
3. Aha, e again! This time, we've had a reflection over the y-axis. That happens if you replace x with $(-x)$ in the function e^{x}. So this goes with f.
4. The asymptote has moved from the x-axis up by 1, so we have a vertical shift. We expect something like "a^x + 1" and there's only one function that looks like that! Well, also only one function left to choose, lol. You can confirm that h will pass through the given points.

If you ever find yourself in a pinch (marooned on a deserted island? jumped in an alleyway by a mathematician?) needing to graph an exponential function by hand, here's what I recommend.

To **sketch the graph of an exponential function,**

1. Find the y-intercept by plugging in $x = 0$ and drop that point in.
2. Compute a couple more easy-to-evaluate points, like plugging in $x = \pm 1, \pm 2$.
3. Check if there's a "$+k$" of some sort on the end of the whole function. That'll be a vertical shift. You could draw in your horizontal asymptote.
4. Check if there's a negative somewhere causing a reflection.
5. Use all that knowledge to connect the dots in the general shape you expect for an exponential, so...a big swoop.

Working With Exponential Functions

If you need to simplify an exponential expression, simply apply your power rules! For very simple (or very well-cooked) problems, you can evaluate exponential functions by hand, but for more complicated functions or ugly numbers, you will need a calculator or technology of some sort. Let's practice.

Example 2

Without using technology, simplify the functions (first), and (then) evaluate them at the given x-values.

1. $f(x) = 3^x 9^x + 1$, at $x = -1, 0, 1$.
2. $g(x) = 2(2^{x+1})$, at $x = -1, 0, 1, 2$.
3. $h(x) = \dfrac{3^x}{6} - 2$, at $x = -1, 0, 1, 2$.

Solution

1. First, simplify by using power rules (I'm going to break this down this time, but if this is not second nature to you, review power rules asap)...

$$f(x) = 3^x(3^2)^x + 1 = 3^x 3^{2x} + 1 = 3^{x+2x} + 1 = 3^{3x} + 1 \quad \text{or} \quad = 27^x + 1$$

Then just plug in to find $f(-1) = \frac{1}{27} + 1 = \frac{28}{27}$, $f(0) = 27^0 + 1 = 2$, and $f(1) = 28$.

2. First, simplify as $g(x) = 2^1 2^{x+1} = 2^{x+2}$ and then evaluate $g(-1) = 2$, $g(0) = 4$, $g(1) = 8$, and $g(2) = 16$.

3. First, simplify as

$$h(x) = \frac{3^x}{2 \cdot 3} - 2 = \frac{1}{2} \cdot 3^x 3^{-1} - 2 = \frac{1}{2} \cdot 3^{x-1} - 2$$

Then evaluate $h(-1) = \frac{1}{2}(3^{-2}) - 2 = \frac{1}{18} - 2 = \frac{1}{18} - \frac{36}{18} = -\frac{35}{18}$, $h(0) = -\frac{11}{6}$, $h(1) = -\frac{3}{2}$, and $h(2) = -\frac{1}{2}$.

Exercise 4

Without using technology, simplify and evaluate the functions at the given x-values.

1. $f(x) = 2^x + 2^x - 2$, at $x = -1, 0, 1, 2$.
2. $g(x) = \dfrac{9^x}{81^x}$, at $x = -1, 0, 1, 2$.
3. $h(x) = 5(1 + 0.5)^x$, at $x = -1, 0, 1, 2$.

Answer

1. $f(x) = 2(2^x) - 2 = 2^{x+1} - 2$ and $f(-1) = -1$, $f(0) = 0$, $f(1) = 2$, $f(2) = 6$.
2. $g(x) = \dfrac{1}{9^x}$ or 9^{-x}, and $g(-1) = 9$, $g(0) = 1$, $g(1) = \frac{1}{9}$, $g(2) = \frac{1}{81}$.
3. $h(x) = 5\left(\frac{3}{2}\right)^x$ and $h(-1) = \frac{10}{3}$, $h(0) = 5$, $h(1) = \frac{15}{2}$, $h(2) = \frac{45}{4}$.

Exercise 5

Using technology, fill in the table of values for the function $f(t) = 100\left(1 + \frac{0.12}{2}\right)^{2t}$. (I recommend simplifying first.) Round to two decimal places if needed.

t	$f(t)$	t	$f(t)$
0		**5**	
1		**10**	
2		**20**	
3		**100**	

Answer

Simplified, $f(t) = 100(1.06)^{2t}$.

Let me teach you a technology trick for this kind of thing (for when you're allowed to use technology, of course, not for cheating). Go to wolframalpha.com and type in "table of values 100(1.06)^(2t) for t = 0, 1, 2, 3, 5, 10, 20, 100" and hit Enter. You're welcome.

t	$f(t)$	t	$f(t)$
0	100	**5**	179.09
1	112.36	**10**	320.71
2	126.25	**20**	1028.57
3	141.85	**100**	11,512,600

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7.1E Exercises

(Note: Most of these problems come from Section 6.3E, except for the transformation and sketching material.)

Definition True/False

Answer True/False. If False, justify or find the error.

1. The function $f(x) = x^{\frac{1}{2}}$ is exponential with base $\frac{1}{2}$.
2. The function $f(x) = \left(\frac{1}{2}\right)^x$ is exponential with base $\frac{1}{2}$.
3. The expression $\left(\frac{1}{2}\right)^x$ is equivalent to the expression $\frac{1}{2^x}$.
4. The function $f(x) = 5^x$ is exponential with base 5.
5. The function $f(x) = 5^{-x}$ is exponential with base 5.
6. The function $f(x) = 5^{-x}$ is exponential with base $\frac{1}{5}$.
7. The function $f(x) = 3^x$ is increasing.
8. The function $f(x) = \left(\frac{1}{3}\right)^x$ is increasing.
9. The function $f(x) = \left(\frac{1}{3}\right)^x$ is decreasing.
10. For $f(x) = 3^x$, we have $f(0) = 3$.
11. For $f(x) = 3^x$, we have $f(0) = 1$.
12. For $f(x) = 3^x$, we have $f(1) = 3$.
13. The function $f(x) = e^x$ can take negative values. That is, there exists some x such that $f(x) < 0$.
14. The function $f(x) = 3^x$ can take negative values.

Answer

1. F, the variable isn't in the exponent.
2. T
3. T
4. T
5. F, rewrite 5^{-x} as $\left(\frac{1}{5}\right)^x$ to see the base.
6. T
7. T
8. F, when the base is less than 1, the function is decreasing (downhill behavior).
9. T
10. F, $f(0) = 3^0 = 1$.
11. T
12. T
13. F, exponential functions like this never take negative values (stay above the x-axis).
14. F, same reason.

Graphs True/False

Answer True/False. If False, justify or find the error.

1. The graph of $f(x) = a^x$ passes through $(0, 1)$ for any base with $a > 0$ and $a \neq 1$.
2. The graph of $f(x) = 2^x$ is increasing.
3. The graph of $f(x) = \left(\frac{4}{5}\right)^x$ is decreasing.
4. The graph of $f(x) = \left(\frac{5}{4}\right)^x$ is decreasing.
5. As $x \to \infty$, the graph of $f(x) = 5^x$ is steeper (faster growing) than the graph of $g(x) = 10^x$.
6. As $x \to \infty$, the graph of $g(x) = 10^x$ is steeper (faster growing) than the graph of $f(x) = 5^x$.
7. The graph of $f(x) = e^x$ is increasing.
8. The graph of $f(x) = e^x$ is steeper than the graph of $g(x) = 3^x$, for $x > 0$.
9. The graph of $f(x) = 6^x$ passes through $(1, 6)$.
10. The graph of $f(x) = 6^x$ passes through $(-1, \frac{1}{6})$.
11. The graph of $f(x) = 6^x$ passes through $(6, 1)$.

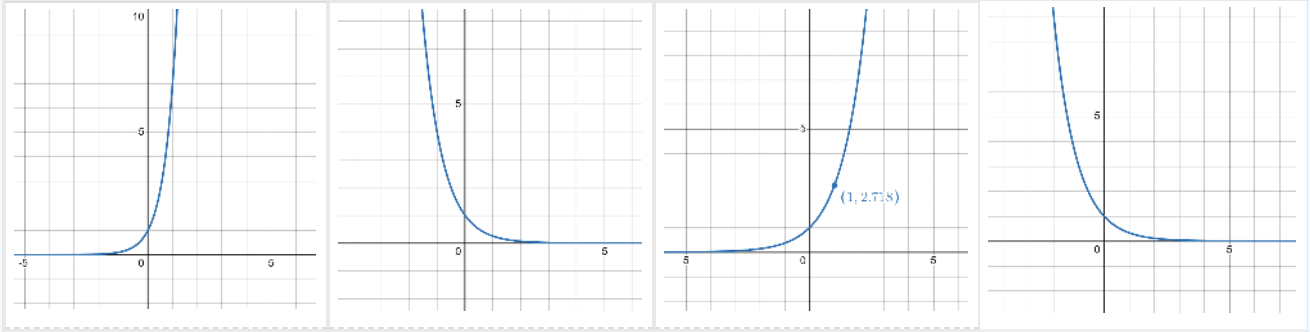
12. The graph of $f(x) = e^x$ passes below the x-axis.
13. The graph of $f(x) = a^x$ passes through $(1, a)$ for any base a.
14. The graph of $f(x) = 5^{-x}$ is the reflection across the y-axis of the graph of $g(x) = 5^x$.
15. The graph of $f(x) = \left(\frac{1}{7}\right)^x$ is the reflection across the y-axis of the graph of $g(x) = 7^x$.

Answer

1. T
2. T
3. T
4. F, the base $\frac{5}{4} > 1$ so it's increasing.
5. F, bigger base gives faster growth.
6. T
7. T
8. F, $e \approx 2.71...$ is less than 3, so g has the larger base and thus faster growth.
9. T
10. T
11. F, $f(6) = 6^6$.
12. F, as mentioned, these exponential functions are always positive.
13. T
14. T
15. T

Matching Graphs

By analyzing signal points like $(1, a)$ and $\left(-1, \frac{1}{a}\right)$ and using your knowledge of exponential function graphs, match the graphs to their functions.



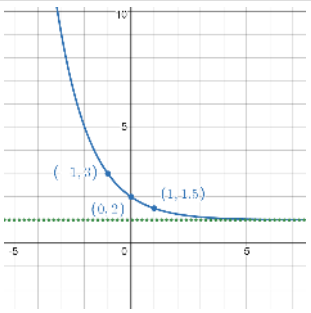
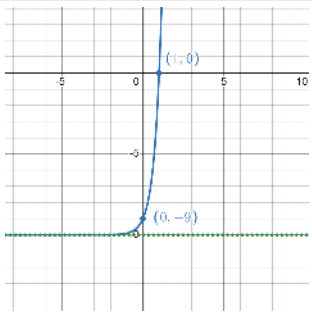
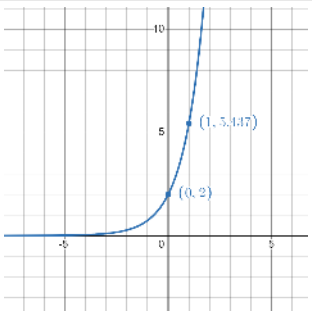
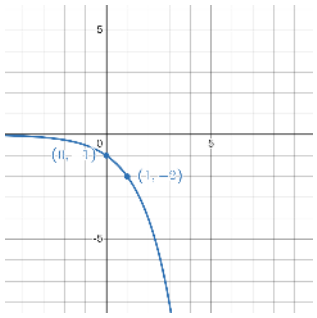
1.	2.	3.	4.
$f(x) = e^x$	$g(x) = \left(\frac{1}{3}\right)^x$	$h(x) = 7^x$	$p(x) = 4^{-x}$

Answer

1. h
2. p
3. f
4. g

Matching Graphs With Transformations

Match the graphs to their functions.

1.	2.	3.	4.
			
$f(x) = 2e^x$	$g(x) = 10^x - 10$	$h(x) = \left(\frac{1}{2}\right)^x + 1$	$p(x) = -2^x$

Answer

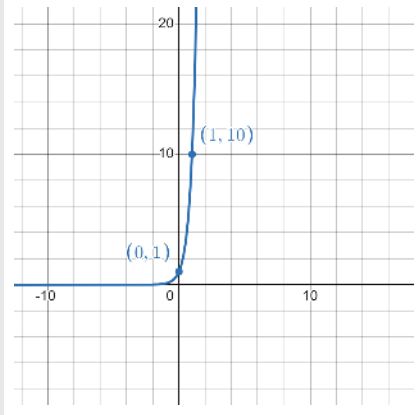
1. h
2. g
3. f
4. p

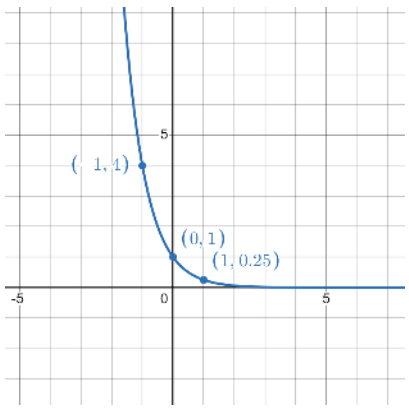
Sketching Graphs

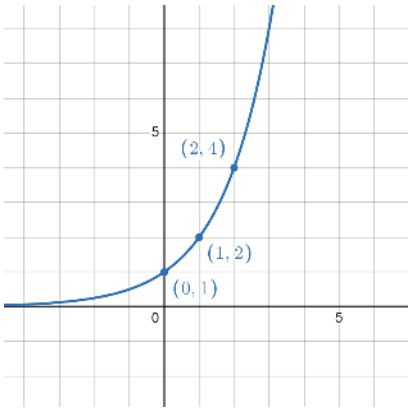
Sketch the graphs of the exponential functions.

1. $f(x) = 10^x$
2. $f(x) = (0.25)^x$
3. $f(x) = 2^x$
4. $f(x) = 2^{x+1}$
5. $f(x) = -3^x$
6. $f(x) = -3^x + 1$

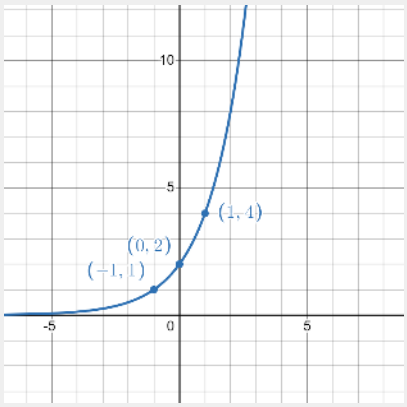
Answer

1. 

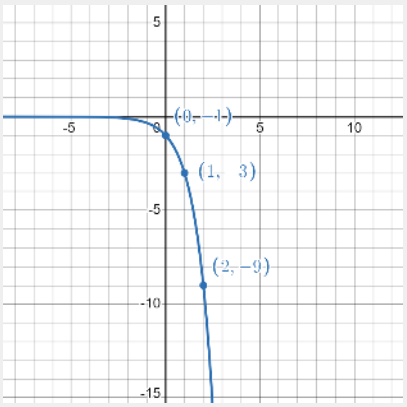
2. 

3. 

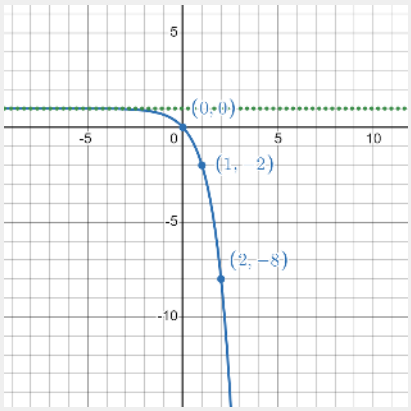
4.



5.



6.



Working With Exponential Functions

1. (Without calculator) Fill in the table of values for the function $f(x) = 2^x$.

x	$f(x)$	x	$f(x)$
1		**0**	
2		**-1**	
3		**-2**	

2. (Without calculator) Fill in the table of values for the function $f(x) = \left(\frac{2}{3}\right)^x$.

x	$f(x)$	x	$f(x)$
1		**0**	
2		**-1**	
3		**-2**	

3. (Without calculator) Fill in the table of values for the function $f(x) = 5^{-x}$.

x	$f(x)$	x	$f(x)$
1		**0**	
2		**-1**	
3		**-2**	

4. For $f(x) = 5^x$, what should x be to get $f(x) = 5$? What should x be to get $f(x) = 25$?
5. For $f(x) = 5^x$, what should x be to get $f(x) = \frac{1}{5}$? What should x be to get $f(x) = 1$?
6. For $f(x) = 4^x$, what should x be to get $f(x) = 2$?
7. For $f(x) = 8^x$, what should x be to get $f(x) = 2$?
8. If $f(x) = 2^x$ and $g(x) = 4^x$, is there an x-value for which $f(x) = g(x)$?
9. Is there an x-value such that $e^x = -2$?
10. Using technology (calculator or WolframAlpha, for example), fill in the table of values for the function $f(x) = (1.2)^x$.

x	$f(x)$	x	$f(x)$
-2		**0**	
-1		**3.25**	

https://math.libretexts.org/@go/page/155685

-0.5		**10**	

Answer

1.

x	$f(x)$	x	$f(x)$
1	2	**0**	1
2	4	**-1**	$\frac{1}{2}$
3	8	**-2**	$\frac{1}{4}$

2.

x	$f(x)$	x	$f(x)$
1	$\frac{2}{3}$	**0**	1
2	$\frac{4}{9}$	**-1**	$\frac{3}{2}$
3	$\frac{8}{27}$	**-2**	$\frac{9}{4}$

3.

x	$f(x)$	x	$f(x)$
1	$\frac{1}{5}$	**0**	1
2	$\frac{1}{25}$	**-1**	5
3	$\frac{1}{125}$	**-2**	25

4. For $f(x) = 5$, x should be 2. For $f(x) = 25$, x should be 2.
5. For $f(x) = \frac{1}{5}$, x should be -1. For $f(x) = 1$, x should be 0.
6. We want x such that $f(x) = 4^x = 2$. What kind of power should 4 be raised to to get out...its square root? That's right, $x = \frac{1}{2}$. Check: $f(1/2) = 4^{\frac{1}{2}} = \sqrt{4} = 2$.
7. Similar reasoning, $x = \frac{1}{3}$.
8. Yes, both pass through $(0, 1)$. Aka, if $x = 0$, then $f(x) = g(x) = 1$.
9. No, there is no possible input that will make $f(x) = e^x$ pass below the x-axis into negative function values.
10.

x	$f(x)$	x	$f(x)$
-2	0.694444	**0**	1
-1	0.833333	**3.25**	1.80859
-0.5	0.912871	**10**	6.19174

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7.2 Logarithmic Functions

Learning Objectives

By the end of this section, you will be able to:

- recognize logarithmic functions, their important features, and their graphs
- relate logarithms to exponentials
- sketch the graphs of logarithmic functions and simple transformations
- use Logarithm Laws, including the change of base formula, to condense and expand logarithmic expressions
- evaluate logarithms and use technology to solve problems

In the exercises section you just finished, I snuck in a certain type of question. I asked something like, "If $f(x) = 9^x$, what does x need to be in order to make $f(x) = 3$?" In other words, I want to have $9^x = 3$, and I wonder what x will make that true? Wait... 3 is the square root of 9, and we write roots with fractional exponents. So $x = \frac{1}{2}$ will work! We can check: $f\left(\frac{1}{2}\right) = 9^{\frac{1}{2}} = \sqrt{9} = 3$.

Logarithms, at the end of the day, are just the way we deal with that question-and-answer process. You can also think of them as a way to **undo** the "raising to a power" process that happens in an exponential function. We've already learned that functions that undo other functions are called **inverse functions.** If you forgot that stuff, review Section 4.4.

Every exponential function $f(x) = a^x$ (with $a > 0$ and $a \neq 1$) passes the Horizontal Line Test and is thus one-to-one. Therefore, it has an inverse function f^{-1}. Recall that being an inverse function means that the values $f^{-1}(x)$ are defined by

$$f^{-1}(x) = y \quad \iff \quad f(y) = x.$$

Aka, if you want to know what $f^{-1}(x)$ spits out, it's the number y such that $f(y) = x$.

Logarithmic Functions

For $a > 0$ and $a \neq 1$, the **logarithmic function with base a,** which we denote $\log_a$, is the inverse function of the exponential function with the same base a. Thus, it is defined by the relationship

$$\log_a x = y \quad \iff \quad a^y = x. \qquad (\star)$$

We say, "log base a of x is y." Basically, "$\log_a(x) =$?" is asking the **question**, "What do I need to raise the base a to, in order to get out this x?" The **answer** to that question is the output of the function $\log_a$.

When in particular the base is the special number e, we write $\log_e x$ as $\ln x$, and call it the **natural log of x.**

Fun facts about logarithmic functions:

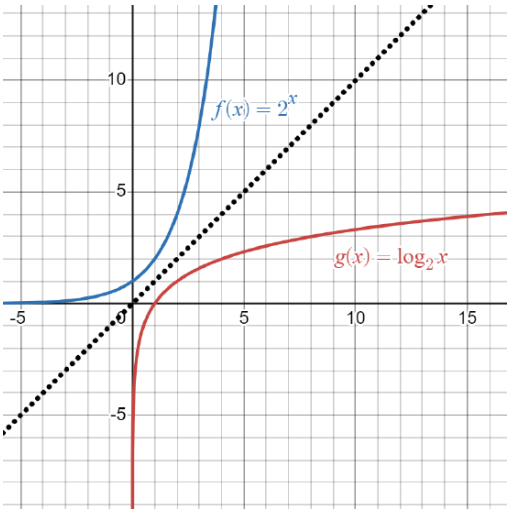
- Their **domain** is **only** positive real numbers! You are **not** allowed to put 0 or a negative number into a log, because, for example, that would be the same as saying $a^y = 0$, and there's no possible y that will satisfy that!
- Since $a^0 = 1$ for any a, we have $\log_a(1) = 0$ for any base a. This is something you should know without using a calculator.
- Similarly, since $a^1 = a$, we have $\log_a(a) = 1$ for any a. Aka, if you take the log of its own base, you will always get 1. This is the other value you should know without a calculator.

The fact that a is called the **base** in both equations where it appears should help you remember that $\log_a$ is related to a^x. Whenever the input for a log function might be ambiguous, you can use parentheses; for example, $\log_a(x + 1)$ tells you $(x + 1)$ is the whole input, whereas $\log_a x + 1$ would be interpreted as "compute $\log_a$ of x and then add 1 to the result."

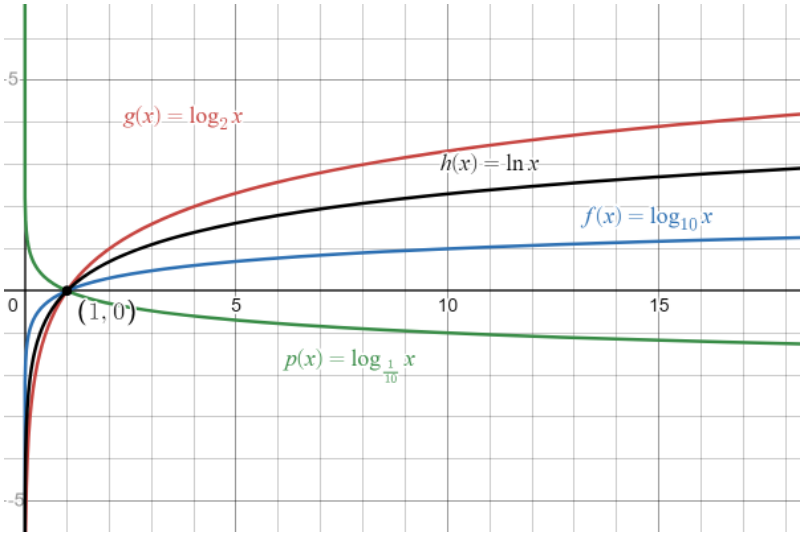
I just want to emphasize again that the relationship $(\star)$ in the definition above is your bread and butter here. You will generally always be translating back and forth between these two forms whenever working with logarithms in functions or equations. Memorize that guy!!!

Now let's see what these functions look like. Recall that the graphs of inverse functions look like mirror images, reflected across the diagonal line $y = x$. This means that to remember what the graphs of logarithmic functions look like, you just have to remember

exponential graphs and then reflect them. The beautiful thing about math is that you don't have to memorize a bunch of random stuff. You can just memorize a few things and then deduce the rest from them!



Study the graphs below and see if you can detect any patterns and observations.



Observations:

- While exponential functions exhibited **fast** growth (or decay), logarithmic functions exhibit **slow growth** (or decay) as $x \to \infty$.
- They all pass through the point $(1, 0)$, which makes sense because $\log_a(1) = 0$ translates to the fact $a^0 = 1$ for any a.
- Looking closely, you can see each one pass through $(a, 1)$, for base a. Example: $f(x) = \log_{10}(x)$ passes through $(10, 1)$.
- They all have a vertical asymptote $x = 0$. Aka, they never touch or cross over the y-axis, because their domain is only defined for $x > 0$.
- If $a > 1$, the function is increasing (going uphill). If $0 < a < 1$, the function is decreasing.
- Notice that $p(x)$ is the reflection of $f(x)$ over the x-axis, and their bases are reciprocals. This follows from power rules applied to their translations to exponential equations.
- I've included on this graph the Big Guys that your science professors asked me to cover. Mostly people use base 10 and base e (the natural log) and you'll find these buttons on a calculator. Computer science uses log base 2 a lot, in a shock to no one.
- Looking out at end behavior, as $x \to \infty$, we see that the bigger the base, the slower the growth. For example, since $2 < e < 10$, the graph of $\ln x$ lies above the graph of $\log_{10} x$ and below that of $\log_2 x$.

We should also make sure we understand what it means for exponential and logarithmic functions to be inverses. Let's discuss and then summarize. First, consider what happens if we plug an input x into an exponential function, and then apply the corresponding logarithm, by definition:

$$\log_a(a^x) = ? \quad \iff \quad a^? = a^x$$

Oh, duh. We must have $? = x$ to make the translated equation true. So doing an exponential and then a log gives you back the original input. Next, consider the other direction of composition, by definition:

$$a^{\log_a x} = ? \quad \iff \quad \log_a(?) = \log_a(x)$$

Aha, again we must have $? = x$ to make the translation true. So doing a log and then the corresponding exponential gives you back the original input. Inverse relationship confirmed!

$$\log_a(a^x) = x, \text{ for all real numbers } x \quad \text{and} \quad a^{\log_a x} = x, \text{ for all } x > 0$$

Test your understanding with the exercises below.

? Exercise 1

Answer True/False.

1. For $f(x) = \log_2(x)$, we have $f(4) = 2$.
2. For $f(x) = \log_2(x)$, we have $f\left(\frac{1}{2}\right) = -1$.
3. For $f(x) = \log_{10}(x)$, we have $f(1) = 100$.
4. I am allowed to plug in $x = 0$ to the function $f(x) = \ln x$.
5. I am allowed to plug in $x = 1$ to the function $f(x) = \ln x$.
6. I am allowed to plug in $x = 1$ to the function $f(x) = \ln(x - 1)$.
7. The graph of the function $f(x) = \log_2 x$ is the reflection of $g(x) = 2^x$ across the y-axis.
8. The graph of the function $f(x) = \log_2 x$ is the reflection of $g(x) = 2^x$ across the line $y = x$.
9. The graph of the function $f(x) = \log_{1/2} x$ is the reflection of $g(x) = \log_2 x$ across the x-axis.
10. The graph of $f(x) = \log_3 x$ passes through $(1, 0)$.
11. The graph of $f(x) = \log_3 x$ passes through $(0, 1)$.
12. The graph of $f(x) = \log_3 x$ passes through $(3, 1)$.
13. The graph of $f(x) = \log_a x$ passes through $(a, 1)$.
14. For $f(x) = e^x$ and $g(x) = \ln x$, we know $(f \circ g)(x) = x$.
15. The graph of $f(x) = \log_5 x$ exhibits slower growth (is less steep) than $g(x) = \ln x$, for $x > 1$.
16. $\ln(x) > \log_5 x$ for $x > 1$.
17. The graph of $f(x) = \log_5 x$ exhibits faster growth (is steeper) than $g(x) = \log_{10} x$, for $x > 1$.

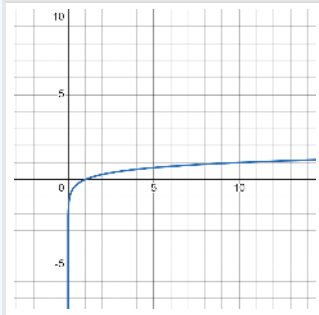
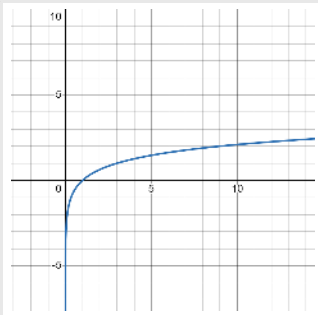
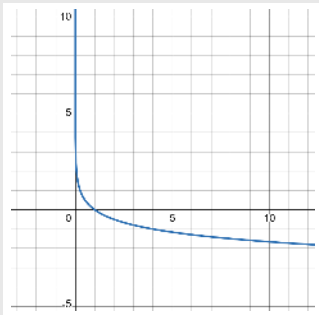
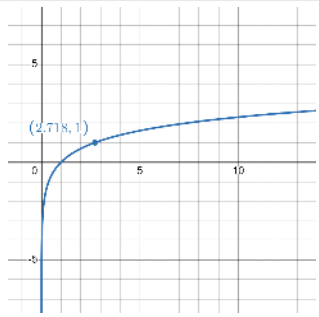
Answer

1. True, we translate $f(4) = \log_2(4) = ?$ to $2^? = 4$ and see that it must give the answer 2.
2. True, we translate $f\left(\frac{1}{2}\right) = \log_2\left(\frac{1}{2}\right) = ?$ to $2^? = \frac{1}{2}$ and see that the power must be -1.
3. False, we translate $\log_{10}(1) = ?$ to $10^? = 1$ and see that the power must be 0, not 100. Or just remember that for any base, the log of 1 is 0.
4. False, we can't plug 0 into any logs.
5. True, $x = 1$ is in the domain.
6. False, plugging in $x = 1$ results in plugging in $1 - 1 = 0$ into the log, which is illegal.
7. False, these are inverse functions, so the graphs are reflections across the line $y = x$.
8. True.
9. True, we observed that above.
10. True, all such logs pass through $(1, 0)$.
11. False, we can't even plug in the x-value 0.
12. True, if we plug in $x = 3$, we are taking the log of its own base, which is also 1.
13. True, this always works.
14. True, since $\ln$ means log base e, these are inverse functions.
15. True, the bigger the base, the slower the growth, and $5 > e$.

16. True, slower growth means that at some x-value like $x = 2$, the graph of $\log_5 x$ will be lower than the graph of $\ln x$, having a smaller function value.
17. True, the bigger the base, the slower the growth, so base 5 is faster growing than base 10.

Exercise 2

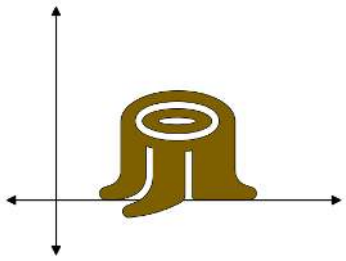
Match the graphs to their functions.

			
1.	2.	3.	4.
$f(x) = \ln x$	$g(x) = \log_{10} x$	$h(x) = \log_3 x$	$p(x) = \log_{0.25} x$

Answer

1. This graph passes through $(10, 1)$, so the base is 10, and it goes with g.
2. This graph passes through $(3, 1)$, so it goes with h.
3. This graph must have a fractional base because it's descending, so it must go with p. We can confirm by checking the point $(4, -1)$ that the graph passes through. Indeed, $\log_{0.25}(4) =?$ translates to $\left(\frac{1}{4}\right)^{?} = 4$ and the function value must be -1.
4. Remember that $e \approx 2.718...$ so that's the point being labeled. Must go with f.

Of course, logarithmic functions can be transformed just like any other, but in this course, I'm going to keep it really simple. Let's kill two birds with one stone and deal with them while learning to sketch the graph of a log. Here you go:



lol jk i'm hilarious

To **sketch the graph of a logarithmic function,**

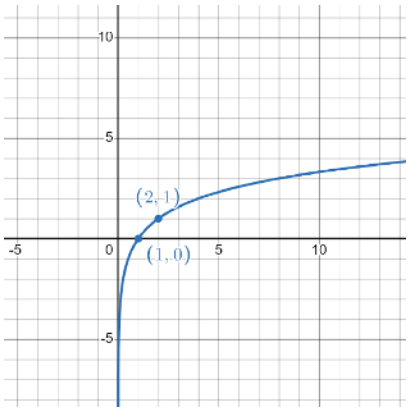
1. First identify the OG log function (the "lOG"? if you will?) if you're looking at a transformation. We'll sketch the OG first.
   1. Remember you have a vertical asymptote, the y-axis.
   2. Plot the points $(1, 0)$ and $(a, 1)$, where a is your base.
   3. Draw a swoosh through the two points, hugging the y-axis forever going down, and slowly growing as $x \to \infty$.
2. Identify any transformations that happened to make your desired function. Adjust the signal points you plotted accordingly, draw the new vertical asymptote if any horizontal shifts happened, and connect the dots nicely.

Example 1

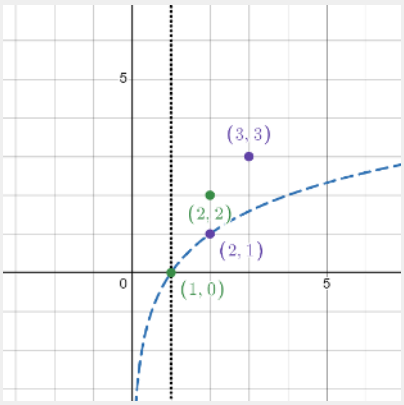
Sketch the graph of $f(x) = \log_2(x-1) + 2$.

Solution

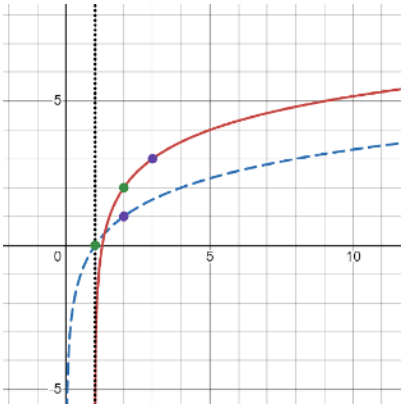
First, let's sketch a basic $y = \log_2 x$. We will pass through $(1, 0)$ and $(2, 1)$.



Next, we identify that there was a horizontal shift to the right by 1, and a vertical shift up by 2. The horizontal shift will move our vertical asymptote to $x = 1$. If we perform both shifts on our signal points, we will have $(2, 2)$ and $(3, 3)$. In the picture, the green points show $(1, 0) \longrightarrow (2, 2)$ and the purple points show $(2, 1) \longrightarrow (3, 3)$.



Finally, using the V.A. and the new signal points, we connect the dots in a nice logarithmic shape.



Exercise 3

Sketch the graphs of the functions.

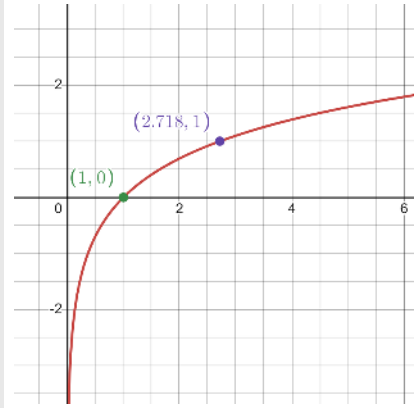
1. $f(x) = \ln x$
2. $f(x) = \log_{10} x - 2$
3. $f(x) = \log_3(x + 1)$

Answer

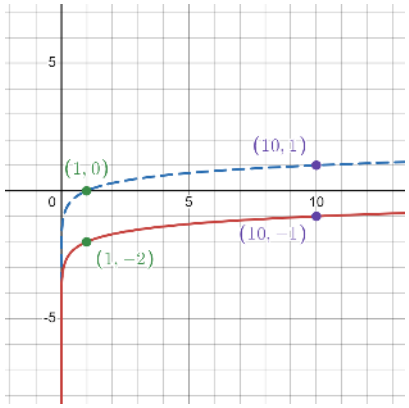
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If applicable, the "OG log" used as a sketching aid is shown in dotted blue. Corresponding transformed signal points are color-coded as well.

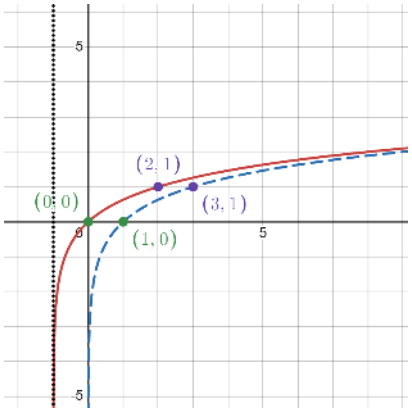
1.



2.



3.



Log Laws

Because of the relationship between logarithms and powers, we have a set of **Logarithm Laws** that follow from power rules. For example, let's investigate the relationship between the logarithm of a product, $\log_a(AB)$, and the logarithms of the factors separately, $\log_a(A)$ and $\log_a(B)$. We don't know what their output values are yet, so let's nickname them with letters:

- $\log_a(AB) = ?$
- $\log_a A = x$
- $\log_a B = y$

Each of those little log equations translate into an equivalent exponential equation.

- $a^? = AB$
- $a^x = A$
- $a^y = B$

I'm going to take the last two equations and substitute into the first equation for A and B. Ope, I can use power rules on that result!

$$a^? = AB \quad \longrightarrow \quad a^? = a^x a^y \quad \longrightarrow \quad a^? = a^{x+y}$$

So now I know that $?$ must be $x + y$. We have $\log_a(AB) = x + y$. But wait! Those x and y were just nicknames we made up! Replacing them with their secret identities, we get our first Logarithm Law,

$$\log_a(AB) = \log_a A + \log_a B$$

Similar goofing around with power rules will get you the rest of the Log Laws given in the table below.

Log Law	English Translation	Caveman Translation
$\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of expressions (numbers, variables, whatever as long as it is being multiplied) is the sum of the logarithms of the expressions.	**Multiplication** inside the log can be written as **addition** outside the log. Used backwards, addition outside the log translates to multiplication inside the log.
$\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of expressions is the difference of the logarithms of the expressions.	**Division** inside the log can be written as **subtraction** outside the log. Used backwards, subtraction outside the log translates to division inside the log. The **denominator** must be the input whose log term is subtracted!

$\log_a(A^B) = B\log_a A$	The logarithm of an expression raised to a power is the power times the logarithm of the expression.	If the whole input is raised to a power, you can **bring powers down** in front as coefficients.
$\log_b x = \dfrac{\log_a x}{\log_a b}$	**Change of Base Formula:** to express $\log_b x$ in terms of $\log_a x$, divide $\log_a x$ by the constant $\log_a b$.	Maybe you want to replace $\log_b x$ with an equivalent expression that uses base a instead. You can switch from $\log_b$ to $\log_a$, but you have to divide by the number $\log_a b$.

Note: The **bases must match** in order to use the first two laws backwards! Using them **forwards** is called **expanding**, and using them **backwards** is called **condensing.** These laws are remixed to deal with big complicated combo expressions as well.

✔ Example 2

Expand fully or condense fully. Simplify if reasonable.

1. $\log_2(xy)$
2. $\log_2(2xyz)$
3. $\ln\left(\frac{1}{x}\right)$
4. $\ln(x^3y^5)$
5. $2\log_5 x - \log_5 x$
6. $\ln A - \ln B + \ln C - \ln D$
7. $3\log_3 x + 2\log_3 y$

Solution

1. We expand $\log_2(xy) = \log_2 x + \log_2 y$.

2. We split all the multiplications using additions outside the logs: $\log_2(2xyz) = \log_2(2) + \log_2 x + \log_2 y + \log_2 z$. We also simplify the $\log_2(2)$ term while we're at it, to get $1 + \log_2 x + \log_2 y + \log_2 z$.

3. We expand $\ln\left(\frac{1}{x}\right) = \ln 1 - \ln x$ and simplify, since any log with 1 plugged in will give 0, to get $-\ln x$. By the way, you could get straight here by remembering that $\frac{1}{x} = x^{-1}$ and using the bring powers down rule...

4. We expand and bring the powers down to get $\ln x^3 + \ln y^5 = 3\ln x + 5\ln y$. Note that we **cannot** write $(3)(5)(\ln xy)$ because the **entire** input must be raised to a single power in order to bring that power down in front!

5. We're condensing now. **First** move any coefficients up as powers, **then** condense any additions or subtractions. We get $\log_5 x^2 - \log_5 x = \log_5\left(\dfrac{x^2}{x}\right) = \log_5(x)$. You can also get straight here by recognizing that $2A - A = A$ no matter how ugly A is.

6. Any additions get multiplied, and any subtractions get divided, so just sort them between the numerator and denominator of a fraction! We get $\ln\left(\frac{AC}{BD}\right)$.

7. Bring powers up first and then condense! We get $\log_3 x^3 + \log_3 y^2 = \log_3(x^3y^2)$.

? Exercise 4

Expand fully or condense fully. Simplify if relevant.

1. $\ln[(x+3)(x-1)]$
2. $\log_{10}\left(\frac{10x}{y}\right)$
3. $\log_2(2x^2y^3)$
4. $\ln x + \ln y - 3\ln z$
5. $4\log_3(A+1) - 2\log_3(B)$

Answer

1. $\ln(x+3) + \ln(x-1)$
2. $1 + \log_{10} x - \log_{10} y$

3. $1 + 2\log_2 x + 3\log_2 y$
4. $\ln\left(\frac{xy}{z^3}\right)$
5. $\log_3\left(\frac{(A+1)^4}{B^2}\right)$

Now let's try bringing in the change of base formula.

Example 3

Mathematicians loooove the natural log, while computer scientists looooove log base 2. Help us get along by expressing each logarithm in terms of the other. On the last one, use the $\ln$ button on a calculator to evaluate the final answer, rounded to 4 decimal places.

1. $\ln x$
2. $\ln 5$
3. $\log_2 n$
4. $\log_2 1000$

Solution

1. We want to write $\ln x =$ something with $\log_2$. We identify $b = e$ and $a = 2$. Plug into the change of base formula:

$$\ln x = \frac{\log_2 x}{\log_2 e}$$

2. In this case, $x = 5$. Using the formula we just found, $\ln 5 = \frac{\log_2 5}{\log_2 e}$.

3. In this case, $b = 2$ and $a = e$. We get the formula $\log_2 n = \frac{\ln n}{\ln 2}$.

4. Here $n = 1000$ in the formula we just found, so we have $\log_2 1000 = \frac{\ln 1000}{\ln 2}$. Using a calculator to evaluate the fraction, we get 9.9658. This process allows us to compute non-tradish logs using a calculator that only has $\ln$, for example. Let's use common sense and estimation to check our answer. We can translate $\log_2 1000 =?$ to $2^? = 1000$. Do you know your powers of 2 for binary? $2^{10} = 1024$, which is pretty close to 1000. So yes, I do expect my answer to be a bit less than 10!

Exercise 5

Write the logarithms in terms of natural logs ($\ln$) and common logs ($\log_{10}$) so that you can use a standard calculator to compute the answer to four decimal places.

1. $\log_2 x$, give formulas and then evaluate at $x = 60$. Check your answer with common sense.
2. $\log_5 x$, give formulas and then evaluate at $x = 10$.
3. $\log_b x$, give formulas and then plug $x = e$ into the natural log version and simplify.

Answer

1. $\log_2 x = \frac{\ln x}{\ln 2}$ and $\log_2 x = \frac{\log_{10} x}{\log_{10} 2}$. For $x = 60$, we get 5.9069. This makes sense, because $\log_2 60 =?$ means $2^? = 60$, and we know $2^6 = 64$. We expect the answer to be close to 6, yes.

2. $\log_5 x = \frac{\ln x}{\ln 5}$ and $\log_5 x = \frac{\log_{10} x}{\log_{10} 5}$. For $x = 10$, we plug into the second formula and see

$$\log_5 10 = \frac{\log_{10} 10}{\log_{10} 5} = \frac{1}{\log_{10} 5} = 1.4307$$

3. $\log_b x = \frac{\ln x}{\ln b}$ and $\log_b x = \frac{\log_{10} x}{\log_{10} b}$. Plugging in $x = e$,

$$\log_b e = \frac{\ln e}{\ln b} = \frac{1}{\ln b}$$

Btw, in general, $\log_b(a) = \frac{1}{\log_a(b)}$.

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7.2E Exercises

(Note: Most of these problems come from Section 6.4E, except for the sketching, change of base, and justification of Log Laws material.)

? Definition True/False

Answer True/False. If False, explain why.

1. If $\log_a x = y$, then $a^x = y$.
2. If $\log_b a = c$, then $b^c = a$.
3. There is a logarithmic function with base 1, $f(x) = \log_1 x$.
4. The natural log function $\ln x$ is the logarithmic function with base π.
5. Let $f(x) = \log_a x$ be a logarithmic function ($a > 0, a \neq 1$). The domain of f is $\{x \mid x \geq 0\}$.
6. Let $f(x) = \log_a x$ be a logarithmic function ($a > 0, a \neq 1$). The domain of f is $\{x \mid x > 0\}$.
7. I am allowed to plug in $x = -2$ to the function $f(x) = \ln(x + 3)$.
8. I am allowed to plug in $x = -2$ to the function $f(x) = \ln(x + 1)$.
9. For any base a, $\log_a(a) = 1$.
10. For any base a, $\log_a(0) = 1$.
11. For any base a, $\log_a(1) = 0$.
12. For $f(x) = \log_{10} x$ and $g(x) = 10^x$, we know $(g \circ f)(x) = x$ for all $x > 0$.

Answer

1. F, this does not match the definition. It must be $a^y = x$.
2. T
3. F, we do not allow the base $a = 1$. (Why?)
4. F, it's base e.
5. F, we can't have $\geq$ because 0 is not allowed.
6. T
7. T
8. F, if $x = -2$ then the input to the log is $-2 + 1 = -1$, a negative number.
9. T
10. F, 0 isn't even in the domain.
11. T
12. T

? Graphs True/False

Answer True/False. If False, explain why.

1. The graph of $f(x) = \log_3 x$ passes through $(1, 0)$.
2. The graph of $f(x) = \log_a x$ passes through $(1, a)$ for any base a.
3. The graph of $f(x) = \log_a x$ passes through $(a, 1)$ for any base a.
4. The graph of the function $f(x) = \log_3 x$ is the reflection of the graph of $g(x) = \log_{\frac{1}{3}} x$ across the x-axis.
5. The graph of the function $f(x) = \ln x$ is the reflection of the graph of $g(x) = e^x$ across the x-axis.
6. The graph of $f(x) = \log_2 x$ exhibits faster growth than the graph of $g(x) = \ln x$, for $x > 1$.
7. The graph of $f(x) = \log_{10} x$ exhibits faster growth than the graph of $g(x) = \ln x$, for $x > 1$.

Answer

1. T
2. F, it passes through $(1, 0)$.
3. T
4. T
5. F, it's the reflection across the line $y = x$ because they're inverse functions.

6. T
7. F, the base $10 > e$.

Logarithms Vs. Exponentials True/False

Answer True/False. Justify your answer.

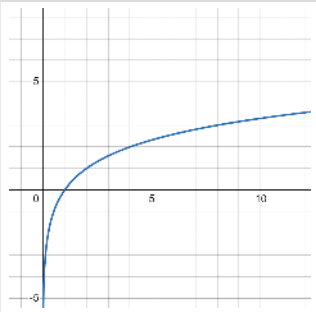
1. $\log_2(2^x) = x$
2. $\log_2(8) = 3$
3. For any base a, $\log_a(x^a) = x$.
4. $e^{\ln x} = x$, for $x > 0$
5. $e^{\log_{10} x} = x$, for $x > 0$
6. $10^{\log_{10} x} = x$, for $x > 0$

Answer

1. T, inverse functions.
2. T, write as $\log_2(2^3) = 3\log_2 2$ or translate with definition.
3. F, it should be $\log_a(a^x) = x$.
4. T, inverse functions.
5. F, the bases must match.
6. T, inverse functions.

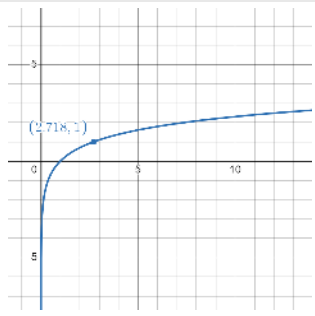
Matching Graphs

By analyzing signal points like $(a, 1)$ and using your knowledge of logarithmic function graphs, match the graphs to their functions.



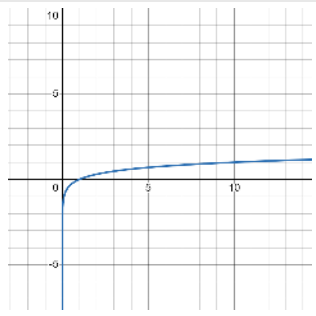
1.

$f(x) = \log_{10} x$



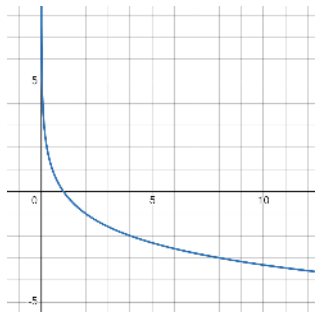
2.

$g(x) = \ln x$



3.

$h(x) = \log_{\frac{1}{2}} x$



4.

$p(x) = \log_2 x$

Answer

1. p
2. g
3. f
4. h

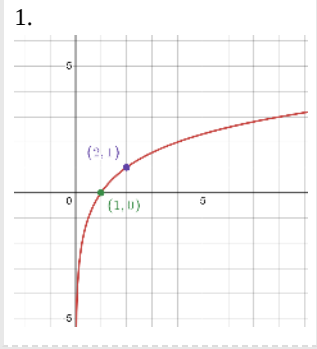
Sketching Graphs

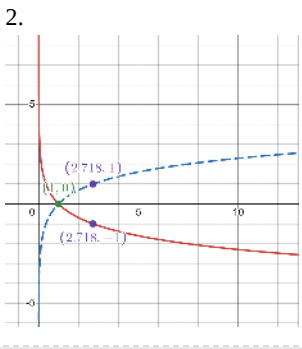
Sketch the graphs of the functions.

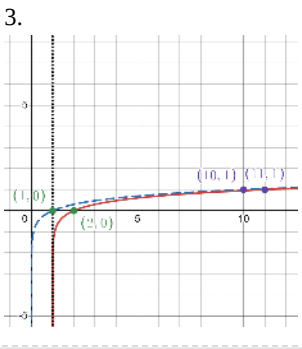
1. $f(x) = \log_2 x$
2. $f(x) = -\ln x$
3. $f(x) = \log_{10}(x - 1)$

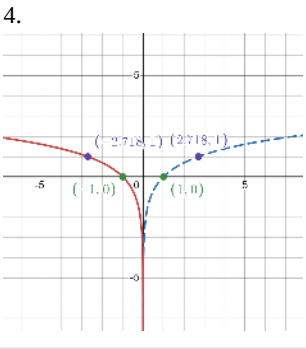
4. $f(x) = \ln(-x)$

Answer

1. 

2. 

3. 

4. 

Working With Logarithms

Without using a calculator, compute the following using the definition and Log Laws:

1. $\log_3 9$
2. $\log_{17}(1)$
3. $\log_2\left(\frac{1}{4}\right)$
4. $\ln e$
5. $\ln e^4$
6. $\log_{10} 1000$
7. $\log_2 4 + 2\log_2 8$
8. $\log_3(6) - \log_3(2)$

Answer

1. 2
2. 0
3. -2
4. 1
5. 4
6. 3
7. 8
8. 1

Condensing/Expanding

Fully expand or fully condense. Simplify if reasonable.

1. $\log_2\left(\frac{x}{2}\right)$
2. $\log_{10}(100x^2y^3)$
3. $\ln(A^2 + 2AB + B^2)$
4. $\log_3 11 + \log_3 7$
5. $\ln x + \ln y - 2\ln z - 3\ln(w+1)$
6. $3\log_a(A+B) - 4\log_a(A-B)$

Answer

1. $\log_2 x - 1$
2. $2 + 2\log_{10} x + 3\log_{10} y$
3. $2\ln(A+B)$
4. $\log_3(77)$

5. $\ln\left(\dfrac{xy}{z^2(w+1)^3}\right)$
6. $\log_a\left(\dfrac{(A+B)^3}{(A-B)^4}\right)$

Change of Base

Write the logarithms in terms of natural ($\ln$) or common ($\log_{10}$) logarithms, and then evaluate, rounding to 4 decimal places. (Use a calculator if needed.)

1. $\log_3 7$
2. $\log_2(0.25)$
3. $\log_{17} 15$
4. $\log_\pi 3$

Answer

1. 1.7712
2. -2
3. 0.9558
4. 0.9597

Justifying Log Laws

1. By noticing that $A^2 = AA$, use the Log Law "$\log_a(AB) = \log_a A + \log_a B$" to prove the fun fact "$\log_a(A^2) = 2\log_a A$."

2. By noticing that $A^3 = AAA$, prove the fact "$\log_a(A^3) = 3\log_a A$." Now do you believe the bringing-down-powers rule? Use it in the next problem.

3. By noticing that $\frac{A}{B} = AB^{-1}$, prove that $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$. ("Prove" meaning start with "$\log_a\left(\frac{A}{B}\right)$" and manipulate it until you get "$\log_a A - \log_a B$" — you can't just use the law from the section! You're showing WHY it's true.)

Answer

1. Substitute AA in for A^2. Then apply the multiplication-inside-translates-to-addition-outside rule:

$$\log_a(A^2) = \log_a(AA) = \log_a A + \log_a A = 2(\log_a A)$$

Yup, it works!

2. Similarly, using the same law,

$$\log_a(A^3) = \log_a(AAA) = \log_a A + \log_a A + \log_a A = 3\log_a A \quad \checkmark$$

3. We start with the same strategy:

$$\log_a\left(\frac{A}{B}\right) = \log_a(AB^{-1}) = \log_a A + \log_a B^{-1}$$

and now bring down the -1 power to get

$$\log_a\left(\frac{A}{B}\right) = \ldots = \log_a A + (-1)\log_a B = \log_a A - \log_a B \quad \checkmark$$

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7.3 Solving Exponential and Logarithmic Equations

Learning Objectives

By the end of this section, you will be able to:

- solve equations where the variable is in an exponent
- solve equations where the variable is in a logarithm

This section is going to feel like a lot of mechanics, and hopefully go by pretty quickly. We're going to do a whole section on the various applications of exponential and logarithmic functions, so what I want to do here is make sure you have the ability to work with the equations that arise. Thus, we'll be doing a lot of practice problems and throwing it back to algebra, as in, what are *legal* moves you can do to equations where the variable is stuck in an exponent or a log. Here we go.

The "why" behind our actions throughout this section is this: Recall that exponential functions are **one-to-one** (that's why they had an inverse function, the log with the same base). That means that the **only** way to have the same output, i.e. to have $a^A = a^B$, you **must have actually had the same input,** i.e. $A = B$. Logarithmic equations have the same feature!

(Because exponential and logarithmic functions are one-to-one,)

$$a^A = a^B \quad \iff \quad A = B$$

and similarly,

$$\log_a A = \log_a B \quad \iff \quad A = B$$

What's the point??

1. We can take an equation and apply "$\log_a(\ )$" to both sides and the equality will still be true.
2. We can (oh god I can never figure out how to put this in English) uhh put both sides of an equation into the exponents of the same base and it will still give an equation? Like this, for example: $A = B \quad \longrightarrow \quad e^A = e^B$.

These two techniques are all you need to solve for x whenever he is trapped in an exponent or log. In special simple circumstances, it feels like there are "shortcuts" that you can use, and those are great, but behind the scenes this is all that's happening. Let's just do a bunch of problems to get comfortable with these techniques.

Solving Exponential Equations

Example 1

The simplest type of exponential equation is just one *of that form* we saw above. To solve these equations, just apply that one-to-one feature.

1. $3^x = 3^2$
2. $e^x = e^{2x-1}$
3. $10^{x^2} = 10^{2x-1}$
4. $2^x = 4^{x+1}$

Solution

1. That looks exactly like $a^A = a^B$, if $a = 3$ and the expressions are $A = x$ and $B = 2$. Therefore, $A = B$ means $x = 2$. Whoops, done already. NEXT!

2. This time, $a = e$, $A = x$, and $B = 2x - 1$. Again, $x = 2x - 1$. Solve for x normally to get $x = 1$.

3. This time, $a = 10$, $A = x^2$, and $B = 2x - 1$. Set $x^2 = 2x - 1$. This is now a quadratic equation, so solve that by moving everything to the left side, factoring, and concluding that $x = 1$.

4. Wait, these bases don't match! ...Yet, my friend, *yet.* Replace 4 with 2^2 and simplify using power rules...

$$2^x = (2^2)^{x+1} = 2^{2x+2} \quad \rightarrow \quad x = 2x + 2 \quad \rightarrow \quad x = -2$$

Exercise 1

Solve the equations.

1. $2^n = 2^{2n}$
2. $e^{3x} = e^{x+8}$
3. $5^{x^2} = 25^{2x}$

Answer

1. The only way that $n = 2n$ can be true is if $n = 0$.
2. $x = 4$.
3. $x = 0, x = 4$.

For more complicated problems, we can always default to the step-by-step process below.

How to Solve Exponential Equations

1. Boil everything down to a **single** base-to-a-power term on one side of the equals.
2. Take the (appropriate) log of both sides and use the bring-powers-down Log Law. At this point, no variables should be stuck upstairs anymore.
3. Solve for the variable as usual. **Check** your answer(s) by plugging back in. There may or may not be a valid solution!

Set the scene: x is trapped in a burning building, screaming from the upstairs window "Aaaah help meeee" and you look around for something to help. You spot a fallen log lying nearby, lean it up against the wall, and the variable sliiiides down the log safely to the ground. Applause. You're a hero.

Example 2

Solve (if possible) and simplify, but leave your answer exact (meaning, no need to put into a calculator).

1. $e^x = e^{2x-1}$
2. $5^{x+1} = 2$
3. $4e^{3x} = 1$
4. $3^x + 3^{2x} = 0$
5. $3xe^x + x^2e^x = 0$

Solution

1. This one...we did above, very quickly. But I just want to prove to you real quick that the technique above will work on anything. It's secretly the same 3-step process. Using power rules,

$$e^x = e^{2x-1} \quad \rightarrow \quad \frac{e^x}{e^{2x-1}} = 1 \quad \rightarrow \quad e^x e^{-2x+1} = e^{x-2x+1} = e^{-x+1} = 1$$

So we apply $\ln$ to both sides of the equation $e^{-x+1} = 1$ to get

$$\ln(e^{-x+1}) = \ln 1 \quad \rightarrow \quad (-x+1)\ln e = \ln 1$$

But we know that $\ln e = 1$ and $\ln 1 = 0$, so we really have $-x + 1 = 0$, aka $x = 1$ just like before. Do it the short way like in Example 1, of course. But math is allllll connected to and confirmed by itself. It's a good way to convince yourself of *why* you can use *different* methods and still get the same answer! Okay, now let's buckle down.

2. Step 1 is handled for free. Because the base is 5, we take $\log_5$ of both sides.

$$\log_5(5^{x+1}) = \log_5(2) \quad \rightarrow \quad (x+1)\log_5(5) = \log_5(2) \quad \rightarrow \quad x + 1 = \log_5(2)$$

Finish solving for x to get $x = \log_5(2) - 1$. Once you understand what's happening, you can go straight from the original equation to the last step in the above line. You don't have to show all the work in between unless you want to.

3. Step 1 is not quite done: there is a 4 on the front of our exponential. Divide him over to get $e^{3x} = \frac{1}{4}$. Now take $\ln$ of both sides.

$$\ln(e^{3x}) = \ln(0.25) \quad \rightarrow \quad 3x \ln e = \ln(0.25) \quad \rightarrow \quad 3x = \ln(0.25) \tag{1}$$

Finish isolating x to get $x = \dfrac{\ln(0.25)}{3}$. This is also equivalent to $-\frac{1}{3}\ln(4)$, if you think that's prettier. (For log practice, make sure you see why those answers are equivalent.)

4. Hmm, how to get those exponential terms together somehow? Well, they have a lot in common... If we think of it as $3^x + (3^x)^2$, we could factor to get $3^x(1 + 3^x) = 0$. Now recall that exponential functions like a^x never turn out to equal zero, so we can divide both sides by 3^x with no risk.

$$\frac{1}{3^x}3^x(1 + 3^x) = \frac{0}{3^x} \quad \rightarrow \quad 1 + 3^x = 0 \quad \rightarrow \quad 3^x = -1$$

Hold up... we can't apply $\log_3$ on both sides, because we can't take the log of that negative 1! There is **no** x-value that will make 3^x equal -1. This equation has **no solution**.

5. First, factor out everything the two terms have in common to get $xe^x(3 + x) = 0$. Again, you can divide by e^x because it never equals 0, but you **cannot** do the same with the x because he might!! We now have $x(3 + x) = 0$. This gives two solutions: $x = 0, -3$. You can check both by plugging them in.

Exercise 2

Solve (if possible) and simplify, but leave your answer exact.

1. $2^{x+1} = 7$
2. $1 + 2e^{x-1} = 5$
3. $3^x - 3^{2x} = 0$
4. $e^{2x} - e^x - 2 = 0$

Answer

1. $x = \log_2(7) - 1$
2. $x = 1 + \ln 2$
3. This one is slightly different from #4 up there, and it does have a solution, $x = 0$.
4. Factoring this one as if it's a quadratic $z^2 - z - 2$ will give you two options: $e^x = 2$ or $e^x = -1$. The first one gives you a solution, $x = \ln 2$, but the second option is invalid because e^x is never negative.

Solving Logarithmic Equations

Again, these can be very simple if they're already in the form $\log_a A = \log_a B$.

Example 3

Solve the equations.

1. $\log_2(x + 2) = \log_2(14)$
2. $\ln(2x) = \ln(x + 2)$
3. $\log_{10}(x^2) = \log_{10}(x)$
4. $\log_{10}(10^x) = \log_{10}(100)$

Solution

1. We must have $x + 2 = 14$, so $x = 12$. And yes, it's legal to plug in 12, no issues with taking logs of negatives or anything.

2. We must have $2x = x + 2$, so $x = 2$.

3. We must have $x^2 = x$, which gives two possibilities: $x = 0, 1$. When we test, we see that $x = 1$ does indeed work, but we're not allowed to plug in $x = 0$ to a log! So we just have the one valid solution.

4. This is a little funky. We have that $10^x = 100$, so we can eyeball that $x = 2$. Or we can simplify the logs first, to see that

$$\log_{10}(10^x) = \log_{10}(10^2) \quad \rightarrow \quad x\log_{10}(10) = 2\log_{10}(10) \quad \rightarrow \quad x = 2$$

Exercise 3

Solve the equations.

1. $\ln(2x) = \ln(3)$
2. $\log_5(x+2) = \log_5(2x)$
3. $\log_{10}(x^2 - 3x) = \log_{10}(4)$

Answer

1. $x = \frac{3}{2}$.
2. $x = 2$.
3. $x = 4, -1$ (both valid, check them).

If we are confronted with something more complicated...

How to Solve Logarithmic Equations

1. Condense completely (using Log Laws) until you get one single logarithm term on one side by itself.
2. Raise the base to each side of the equation (or translate to exponential form of the logarithmic equation).
3. Solve for the variable.

Example 4

Solve the equations (and check your solutions).

1. $\ln(2x) = 8$
2. $4 + \log_2(4 - x) = 2$
3. $\log_2 x + \log_2(x+1) - \log_2(x^2) = 1$
4. $\log_2 x + \log_4 x = 9$

Solution

1. Step 1 is handled, so we move to Step 2. You can translate this $\log_a b = c$ straight to $a^c = b$ form, or else go through the process. The base of $\ln$ is e, so we have $e^{\ln(2x)} = e^8$. By the inverse relationship $a^{\log_a x} = x$, we can simplify to

$$2x = e^8 \quad \rightarrow \quad x = \frac{e^8}{2}$$

2. First we need to isolate the log term. We get $\log_2(4-x) = -2$. Now we can translate to $2^{-2} = 4 - x$ and solve for $x = \frac{15}{4}$. We check, and this is valid to plug in.

3. Here we have a bunch of log terms, but we just condense them down since all the bases match. Using Log Laws, we have

$$\log_2\left(\frac{x(x+1)}{x^2}\right) = \log_2\left(\frac{x+1}{x}\right) = 1$$

Next, recall that $\log_2(2) = 1$, so that whole input must be equal to 2... Or go through the process,

$$2^{\log_2\left(\frac{x+1}{x}\right)} = 2^1 \quad \rightarrow \quad \frac{x+1}{x} = 2$$

To solve this, I'll use cross multiplication to get $x + 1 = 2x$ and conclude $x = 1$. And that's legal to plug back in, so we're good.

4. You can't just condense these logs as is, because the bases don't match! We first use the change-of-base formula to replace the $\log_4$ term with an equivalent term involving only $\log_2$, and then simplify:

$$\log_2 x + \frac{\log_2 x}{\log_2 4} \quad \rightarrow \quad \log_2 x + \frac{\log_2 x}{\log_2(2^2)} \quad \rightarrow \quad \log_2 x + \frac{\log_2 x}{2}$$

Now we have $\log_2 x + \frac{1}{2}\log_2 x = 9$ with all the bases matching, and we can proceed.

$$\log_2 x + \frac{1}{2}\log_2 x = \frac{3}{2}\log_2 x = 9 \quad \rightarrow \quad \log_2 x = 9 \cdot \frac{2}{3} \quad \rightarrow \log_2 x = 6$$

We translate to $2^6 = x$ and see that $x = 64$.

Exercise 4

Solve and check.

1. $\log_3(x+1) = 3$
2. $4 + 4\log_{10}(2x) = 16$
3. $\ln(x+1) + \ln(x-1) = 0$
4. $\log_3(2x) + \log_{1/3}(x) = x$ (Hint: your first thought should be change-of-base formula just to see if anything good happens.)

Answer

1. $x = 26$
2. $x = 500$
3. $x = \sqrt{2}$ (meaning only the positive version)
4. $x = \log_3(2)$

I know this section was mostly problems, but heh there are even more in the exercises section for you to practice. Then in the last section of this chapter, you will apply them to real-world scenarios!

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7.3E Exercises

Solving Simple Exponential Equations

Solve (leave answers exact, do not use a calculator and round).

1. $e^{x+1} = e^3$
2. $2^{3x} = 2^{12}$
3. $4^x = 2^{x+3}$
4. $10^{x^2+3x} = 10^{-2}$
5. $a^{6x} = (\sqrt{a})^x$
6. $a^2 = \sqrt{a^x}$

Answer

1. $x = 2$
2. $x = 4$
3. $x = 3$
4. $x = -1, -2$
5. $x = 0$
6. $x = 4$

Solving Exponential Equations

Solve (leave answers exact, do not use a calculator and round).

1. $2^{x+1} = 16$
2. $3e^x = 45$
3. $3^{2x} = 10$
4. $10^x + 10^{x+1} = 22$
5. $2^{2x} - 2(2^x) + 1 = 0$
6. $10^{3x} = \left(\frac{1}{10}\right)^{x-1}$

Answer

1. $x = 3$
2. $x = \ln(15)$
3. $x = \dfrac{\log_3(10)}{2}$
4. $x = \log_{10} 2$ (Hint: $10^{x+1} = 10 \cdot 10^x$)
5. $x = 0$
6. $x = \frac{1}{4}$

Solving Simple Logarithmic Equations

Solve and check (leave answers exact, do not use a calculator and round).

1. $\ln(x+1) = \ln 12$
2. $\log_{10}(x) = \log_{10}(2 - 3x)$
3. $\log_2(n^2) = \log_2(4n - 4)$
4. $2 \ln x = \ln 16$
5. $\log_2 x = 1$

Answer

1. $x = 11$
2. $x = \frac{1}{2}$

3. $n = 2$
4. $x = 4$ ($x = -4$ is not a valid solution)
5. $x = 2$ (just translate)

? Solving Logarithmic Equations

Solve and check (leave answers exact, do not use a calculator and round).

1. $\log_2 x = 3$
2. $\log_{10}(x + 2) = 4$
3. $\ln(x) + 2\ln(x) = 0$
4. $\log_5(x^2 - 1) - \log_5(x + 1) = 2$
5. $\ln x + \log_2 x = 0$

Answer

1. $x = 8$
2. $x = 9998$
3. $x = 1$
4. $x = 26$
5. $x = 1$ (Remember that $\frac{1}{\ln 2}$ is just a constant number!)

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7.4 Applications

Learning Objectives

By the end of this section, you will be able to:

- implement exponential and logarithmic functions and equations to solve common problems in science and business

In this section, I want to demonstrate common real-world application problems one-by-one, doing an example with you and then giving you a related exercise. There are just a bunch of different configurations these problems can take, and I think a learn-by-example approach will be the most efficient. In these problems, we will generally need to use calculators/technology. Let's start in the area of money...

Say you make an initial investment of $\$P$ at an annual interest rate of 4%. That means that after one year, 4% of P will be calculated and added on to the amount in your account. Thus you will have $P(1+0.04)$ dollars in the account. After another year, the same thing happens, but with your new total, so that you have $(P(1.04))(1+0.04) = P(1.04)^2$. Each year, you pick up another factor of (1.04), so if you want to compute the amount of money after t years at interest rate $i = 0.04$, you can use an exponential function

$$A(t) = P(1+i)^t.$$

In reality, interest can be calculated and paid to you *more* than once a year, so let's start with that concept.

Example: Compound Interest

When interest on an investment is **compounded** multiple times a year (quarterly, monthly, weekly, etc.), an exponential function is used to compute the amount of money in the account after a certain number of years. In the formula below, P (the **principal**) is the initial amount, r is the (**nominal**) interest rate **per year** (even though it's compounded n times per year) expressed as a decimal, and $A(t)$ is amount after t **years.**

$$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$$

1. Explain why the formula makes sense for this scenario.
2. If \$1000 is invested at an interest rate of 8% per year, compounded semiannually, find the amount in the account after 5 years.
3. For the same situation, find the interest rate that, compounded annually, would result in the same amount of money after 5 years.
4. Now find the amount in the account after 5 years if interest is compounded quarterly. What about monthly? What do you notice about the amounts the more times you compound?
5. How long will it take for the account to double in the situation of part (2.)?

Solution

1. Every time the interest is compounded, the percentage used to calculate the interest money is $\frac{r}{n}$. To find the total amount including the principal *and* this extra interest money added on after one compounding period, we would multiply $P\left(1+\frac{r}{n}\right)$. In a year, there are n periods, and every single time, we are multiplying by another copy of $\left(1+\frac{r}{n}\right)$. Thus, after t years, aka nt periods, we will have $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$.

2. We identify all the information given to us: $P = 1000$, $r = 0.08$, $n = 2$ (twice a year), and we want to plug in $t = 5$ years. Putting all of this into the formula, we are computing $A(5) = 1000\left(1+\frac{0.08}{2}\right)^{2(5)}$. Simplifying and then plugging into a calculator, we have

$$A(5) = 1000\,(1.04)^{10} = \$1480.24$$

3. If interest is only being calculated once per year, we would have some rate i and calculate the amount after t years using $P(1+i)^t$. What we want is to end up with the exact same amount of money after 5 years as our previous answer, \$1480.24. We can set that amount equal to our hypothetical annual situation and solve for that new rate i.

$$\$1480.24 = 1000(1+i)^5 \quad \rightarrow \quad (1+i)^5 = 1.48024 \quad \rightarrow \quad 1+i = \sqrt[5]{1.48024} \approx 1.081599...$$

This tells us that i must be 8.16% to achieve the same amount, if interest is only compounded once per year.

4. All we need to do here is go back to our compound interest formula and adjust the n accordingly. First, for quarterly (there are four quarters in a whole year) we use $n = 4$. We compute $A(5) = 1000\left(1 + \frac{0.08}{4}\right)^{4(5)} = \$1485.95$ in this case. Next, for monthly, there are twelve months in a year so we use $n = 12$. We compute $A(5) = 1000\left(1 + \frac{0.08}{12}\right)^{12(5)} = \$1489.85$ in this case. We notice that every time we increase n, the amount of money goes up. The more times you compound, the more interest you make in the end.

5. If we want the account to double and we started with $P = 1000$, we want to have $A(t)$ turn out to be 2000. We just plug that in where we see $A(t)$ and solve for the amount of time t needed to make the equation true. I first plug in all my numbers and simplify a bit.

$$2000 = 1000\left(1 + \frac{0.08}{2}\right)^{2t} \quad \rightarrow \quad (1.04)^{2t} = 2$$

Now I see my variable t trapped upstairs. I apply $\ln$ to both sides so I can bring him down outta there. Then finish isolating t.

$$\ln((1.04)^{2t}) = \ln 2 \quad \rightarrow \quad 2t\ln(1.04) = \ln 2 \quad \rightarrow \quad t = \frac{\ln 2}{2\ln(1.04)} \approx 8.84 \text{ years}$$

? Exercise 1

Ian invests \$1,000,000 at a (nominal) rate of 15% per year, compounded quarterly.

1. Write the function that will compute the amount of money in Ian's account after t years.
2. Fill in the table of values for different t values below.

t	$A(t)$	t	$A(t)$
1		**5**	
2		**10**	

3. Find the interest rate that, compounded annually, would result in the same amount of money after one year as in the compounded quarterly situation. This is called the annual percentage yield.
4. How long will it take for the account to reach \$3,000,000?

Answer

1. $A(t) = 1000000\left(1 + \frac{0.15}{4}\right)^{4t} = 1000000(1.0375)^{4t}$.
2.

t	$A(t)$	t	$A(t)$
1	\$1,158,650	**5**	\$2,088,150
2	\$1,342,470	**10**	\$4,360,380

3. We want the factor $\left(1 + \frac{.15}{4}\right)^4$ to have the same effect as a factor $(1 + i)^1$. Setting them equal and solving for i gives 0.15865, so about 15.87%.
4. $t \approx 7.46$ years.

If we increase the number n of compounding periods to every day...twice a day...every hour...every minute...every second...(!) until $n \to \infty$, we are talking about **continuously compounding interest.**

✓ Example: Continuously Compounding Interest

The formula that calculates the amount of money $A(t)$ in an account after t years, with principal P and interest rate r compounded continuously, is

$$A(t) = Pe^{rt}$$

1. If $5000 is invested at an interest rate of 5% per year, compounded continuously, write the function $A(t)$.
2. How much money will be in the account after 10 years?
3. For any initial investment P, how long will it take for the money to double?
4. What is the equivalent percentage annual yield? (That is, the rate i for which $(1+i)^1$ is equal to $e^{r(1)}$.)

Solution

1. We identify $P = 5000$ and $r = 0.05$, and write our function $A(t) = 5000e^{0.05t}$.

2. We are looking for $A(10)$, so we just plug in $t = 10$ and compute $A(10) = 5000e^{0.05(10)} = \$8243.61$.

3. If we put in P and we want double that, we simply want $A(t)$ to turn out to be $2P$. Plug that in and solve for t. It's a good idea to use $\ln$ here because it has base e.

$$2P = Pe^{0.05t} \quad \rightarrow \quad e^{0.05t} = 2 \quad \rightarrow \quad 0.05t = \ln 2 \quad \rightarrow \quad t \approx 13.86 \text{ years}$$

4. We want to solve for i in the equation $1 + i = e^{0.05}$. Plugging in $e^{0.05}$ to a calculator, we get 1.051271... so we know $i \approx 0.05127$ or about 5.13%.

? Exercise 2

A sum of $45,000 is invested at a rate of 13% compounded continuously.

1. Write the function $A(t)$ for this scenario.
2. Fill in the table of values for different times t.

t	$A(t)$	t	$A(t)$
1		**5**	
2		**10**	

3. How long will it take for the investment to double in value?

Answer

1. $A(t) = 45000e^{0.13t}$
2.

t	$A(t)$	t	$A(t)$
1	$51,247	**5**	$86,199
2	$58,362	**10**	$165,118

3. $t \approx 5.33$ years.

Switching gears to biology: exponential functions show up all the time in population growth modeling...

✓ Example: Population Doubling

Bacteria reproduce by splitting themselves in two every certain amount of time. Thus, after a length of time a (minutes, hours, days, years, whatever), their population will double. If you start with a population of size n_0, then to find the size of the population $n(t)$ at time t (minutes, hours, days, years, whatever, consistent with a's units), we can use the formula

$$n(t) = n_0 2^{t/a}$$

1. Explain why this formula makes sense.
2. If we start with a Petri dish containing 200 bacteria, and they are doubling every 3 hours, how many bacteria will we have after 24 hours?
3. How long will it take to reach 1000 bacteria?

Solution

1. Say the population is doubling every half hour, so that $a = 0.5$hr. Then after t hours, the population will have doubled (multiplied itself by 2) a total of $2t$ times. So if we started with n_0 and multiplied it by 2 over and over, we will end up with $n(t) = n_0 2^{t/0.5} = n_0 2^{2t}$ after t hours.

2. We identify $n_0 = 200$ and $a = 3$hrs. At $t = 24$, we can just calculate $n(24) = 200 \cdot 2^{24/3} = 200 \cdot 2^8 = 51,200$ bacteria.

3. All we are saying here is that we want to have $n(t)$ turn out to be 1000 for some t. We just plug in what we want and solve for t. I'll rescue the t trapped upstairs using $\ln$ since he's my favorite log...

$$1000 = 200 \cdot 2^{t/3} \quad \rightarrow \quad 5 = 2^{t/3} \quad \rightarrow \quad \ln 5 = \frac{t}{3} \ln 2$$

Then just finish solving to get $t = 3\frac{\ln 5}{\ln 2} \approx 6.97$ hours.

? Exercise 3

Rabbits have discovered the strawberry patch :(and on this nutritious banquet provided free of charge, their population is booming. At this point, the rabbit population in the garden is about 10 and doubling every 28 days.

1. Write the formula that gives the rabbit population after t weeks.
2. There are 15 more weeks of summer. If this keeps up, what size will the population be at the end of the summer?
3. How long will it take to reach my lucky number of 51 rabbits?

Answer

1. $n(t) = 10 \cdot 2^{t/4}$ (watch out for the differences in units)
2. $n(15) = 135$ rabbits (round to a whole number, you can't have a half a rabbit (alive at least))
3. $t \approx 9.4$ weeks.

In the population-doubling scenario, the base used in the exponential function is 2, but this is not required. You could talk about "tripling time" and use base 3... If we choose to use e as the base in a model, giving for example $n(t) = n_0 e^{rt}$, then t's coefficient r is called the **relative growth rate,** which is the rate of growth expressed as a **proportion** of the population at any time. This scenario is basically the same as compound interest, just with different letters, lol.

✓ Example: Exponential Growth

The civilian noninstitutionalized population (people 16yo and up, not living in institutions, not on active military duty) in the United States is projected to increase at a relative growth rate around 0.41% between 2023 and 2053 (down from a rate of 1.1% over the past 40 years). The mathematical model for this exponential growth is

$$n(t) = n_0 e^{rt}$$

where $n(t)$ gives population in millions, t years after 2023, with initial population size n_0, and relative growth rate r.

1. In 2023, the civilian noninstitutionalized population was 266 million people. Find the formula that will estimate the projected population t years after 2023 in this scenario.
2. Estimate the population at the end of this projection period, in 2053.
3. In what year do you expect the population to cross 275 million?

Solution

1. We identify $n_0 = 266$ and $r = 0.0041$. Then we just plug into the formula to get $n(t) = 266e^{0.0041t}$.

2. The year 2053 is 30 years after 2023, so $t = 30$. We plug this into the formula we just found to get $n(30) = 266e^{0.0041(30)} \approx 301$ million people.

3. If we want to know when $n(t) = 275$, we just set the formula equal to that and solve for the necessary t, using $\ln$ to take care of the e situation.

$$275 = 266e^{0.0041t} \quad \rightarrow \quad 1.033835 = e^{0.0041t} \quad \rightarrow \quad 0.0041t = \frac{\ln 1.033835}{0.0041} \approx 8.12$$

So we expect it to happen sometime in 2031.

Exercise 4

A few weeks have passed, the strawberry patch has been picked clean, and the sudden lack of food is having a *negative* effect on the rabbit population in our yard. I've observed that the population is declining at a relative rate of 10% per week. (This means you should use a *negative* number as your relative growth rate.) Currently, I count about 17 rabbits in the yard.

1. Write a model for the population $n(t)$ we expect t weeks from now.
2. Assuming trends stay the same, approximately when will we be down to just two rabbits in the yard?

Answer

1. $n(t) = 17e^{-0.1t}$
2. $t \approx 21$ weeks from now.

Radioactive substances decay as they emit radiation over time. The rate of decay is proportional to the mass of the substance, which is similar to the population growth situation, but **decreasing**, like the rabbit population we just saw. Physicists talk about this process in terms of **half-life**: the time it takes for a sample to decay to half its original mass. However, it's still the general move to express the mathematical model for a radioactive decay situation in terms of base e.

Example: Radioactive Decay

A substance has initial mass m_0 and half-life h. We want to express an exponential model for its radioactive decay in the form

$$m(t) = m_0 e^{-rt}$$

where r is the **relative decay rate** (it'll be positive, but the effect will be decay because of the negative sign in the exponent).

1. Find a way to express r in terms of the half-life time h.
2. If I have 10 grams of radium-226, which has a half-life of 1,600 years, then how long (in years) will it take to decay to 6 grams?

Solution

1. If the amount of time h is the half-life, that means that when $t = h$, we have $m(t) = \frac{m_0}{2}$. We can just plug this data point into the model and solve for r.

$$\frac{m_0}{2} = m_0 e^{-rh} \quad \rightarrow \quad \frac{1}{2} = e^{-rh} \quad \rightarrow \quad \ln(0.5) = -rh$$

To finish isolating r, we divide both sides by $-h$ and find that, given h, you can compute $r = -\frac{\ln(0.5)}{h}$, or equivalently and prettier, $r = \frac{\ln 2}{h}$.

2. We identify that $m_0 = 10$ and $h = 1600$, and that I want $m(t) = 6$. We just figured out that we can compute $r = \frac{\ln 2}{1600} \approx 0.0004332$. Now we can just plug everything into the general formula and solve for t.

$$6 = 10e^{-0.0004332t} \quad \rightarrow \quad 0.6 = e^{-0.0004332t} \quad \rightarrow \quad t = \frac{\ln 0.6}{-0.0004332} \approx 1179.19 \text{ years}$$

Does this answer make sense? Always stop and check. Going from 10g to 6g is decaying by less than half, so I expect my answer to be less than the half-life, 1600 years. So yeah, 1180ish years sounds pretty reasonable.

Exercise 5

The substance used for carbon dating of ancient objects, carbon-14, has half-life 5,730 years. If I have a wooden artifact containing 3mg of carbon-14 **now,** and I know that **originally** it should have contained 12mg of carbon-14 (based on the amount that is present in non-aged wood), about how old is it?

1. First find r and the formula for $m(t)$.
2. Of the two numbers given, 3mg and 12mg, which one is m_0 and which one is a value for $m(t)$?
3. Plug in the information appropriately and solve for t.

Answer

1. $r = 0.00012097$ so $m(t) = m_0 e^{-.00012197t}$
2. $m_0 = 12$ and $m(t) = 3$ now.
3. $t \approx 11,366$ years old.

As evidenced by the rabbit population situation, maybe plain old exponential growth isn't always the best way to model population, because in real life, a given environment can't actually support an unlimited population. Limited resources result in a set maximum population possible, called the **carrying capacity.** Populations start off growing exponentially, but eventually the limiting factors cause them to level off around the carrying capacity. The way to model these situations is using a **logistic function.**

Example: Logistic Growth

A population that experiences **logistic growth** increases according to the model

$$n(t) = \frac{M}{1 + \frac{M - n_0}{n_0} e^{-rt}}$$

where $n(t)$ is the population at time t years, M is the carrying capacity (a plain ol' number), r is the initial relative growth rate, and n_0 is the initial population.

1. My koi pond can support a maximum population of 40 fish. I introduce 4 koi fish as my initial population, and the (highly technical) pet store told me their initial relative growth rate is 0.15. Find the logistic growth function that will model the population in my pond.
2. When will I have 20 fish?
3. Graph the function using technology to see what will happen "eventually" as time goes on forever.

Solution

1. We identify $M = 40$, $r = 0.15$, and $n_0 = 4$. The model is thus

$$n(t) = \frac{40}{1 + \frac{40-4}{4} e^{-0.15t}} = \frac{40}{1 + 9e^{-0.15t}}$$

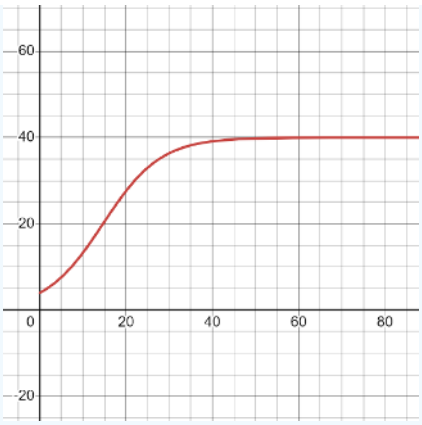
2. I want $n(t) = 20$, so I plug that in and start solving for t.

$$20 = \frac{40}{1 + 9^{-0.15t}} \quad \rightarrow \quad (1 + 9e^{-0.15t})(20) = 40 \quad \rightarrow \quad 1 + 9e^{-0.15t} = 2$$

Now we just isolate the exponential term and solve as usual.

$$\rightarrow \quad e^{-0.15t} = \frac{1}{9} \quad \rightarrow \quad -0.15t = \ln\left(\frac{1}{9}\right) \quad \rightarrow \quad t \approx 14.6 \text{ years}$$

3. This is the graph of the function:



We can see that the population first grows quickly and then slows down, leveling out around the carrying capacity as time goes on.

Exercise 6

An infectious disease begins turning people into zombies in a small city in Florida (obviously). The city has a population of 12,000, and initially 4 downhome country boys are infected after swimming in a particular swamp hole. The initial relative growth rate of the epidemic is 0.67 per day, and the number of people infected after t days can be predicted by the model

$$n(t) = \frac{8000}{1 + 1999e^{-.67t}}$$

1. Will the entire city's population become zombies?
2. After 1 week, how many zombies will there be?

Answer

1. This model seems to fit the formula for logistic growth, so we can identify the carrying capacity as $M = 8000$ by looking at the numerator. We confirm that $\frac{8000-4}{4} = 1999$ accounts for the correct coefficient in the denominator as well. So we expect the infection to spread initially but level out around 8000 zombies in the long run.
2. At $t = 7$ days, we compute $n(7) \approx 413$ zombies.

Okay, last little topic here is **Newton's Law of Cooling,** which you'll learn a lot more about if you take Differential Equations.

Example: Newton's Law of Cooling

If an object with internal temperature T_0 is placed into an environment with surrounding temperature T_s, then it will cool down to match the surrounding temperature as time goes on. The temperature $T(t)$ of the object at time t is given by the formula

$$T(t) = T_s + (T_0 - T_s)e^{-rt}$$

where r is the coefficient of heat transfer, depending on the type of object.

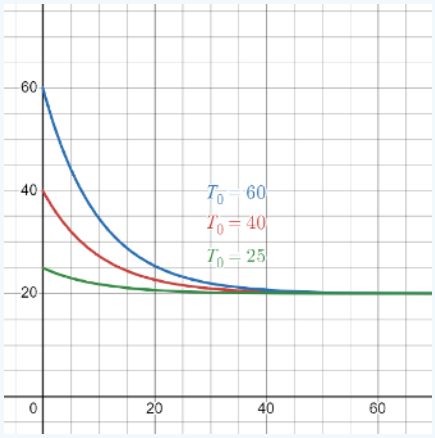
1. Hold T_s constant and look at graphs (using technology) trying various options for T_0. What do you notice about the rate of cooling relative to the **difference** between T_0 and T_s?
2. I take a turkey out of the oven with an internal temperature of 165°F and place it in an ambient room temperature of 65°F. After about 20 minutes, it's cooled to a temperature of 125°F. Find r.
3. After 30 minutes, what will be the internal temperature of the turkey?
4. I want to know how long it will take for the turkey to cool to various temperatures. First find a function that expresses time t in terms of a desired temperature T (aka the inverse function to $T(t)$, which you could name $t(T)$ to be clear). Then use that function and technology to fill in the table of values.

Temperature T	**time to reach** T	**Temperature** T	**time to reach** T
150		100	

130		70	

Solution

1. Using desmos, I generated these graphs by choosing $T_s = 20$ and $r = 0.1$ and then picking a few different options for T_0.



We observe that when the difference between the object's temperature and the surrounding temperature is **large**, the object's temperature drops rapidly, initially. You can see that when $T_0 = 60$, much higher than $T_s = 20$, the blue graph falls steeply at the beginning of time. However, as the gap between the temperatures gets **smaller**, the rate of cooling slows down; i.e. each graph begins to level out, getting less steep, as time goes on.

2. We are given essentially two **data points** here: at $t = 0$, we have $T(0) = T_0 = 165$, and at $t = 20$, we have $T(20) = 125$. We want a function of the form $T(t) = T_s + (T_0 - T_s)e^{-rt}$ that passes through both of these points. First off, we recognize that $T_s = 65$ and $T_0 = 165$. Let's plug in the values of the second data point:

$$125 = 65 + (165 - 65)e^{-r(20)} \quad \rightarrow \quad 125 = 65 + 100e^{-20r}$$

This gives us an equation with only one unknown: the r that we are trying to figure out. Solve for r.

$$60 = 100e^{-20r} \quad \rightarrow \quad 0.6 = e^{-20r} \quad \rightarrow \quad \ln(0.6) = -20r \quad \rightarrow \quad r \approx 0.02554$$

We can update our model for this situation to be $T(t) = 65 + 100e^{-0.02554t}$.

3. At $t = 30$, we have $T(30) = 65 + 100e^{-0.02554(30)} \approx 111°$F.

4. We have done this type of problem a few times thus far, but generally by taking specific numbers that we plug in for $T(t)$ and then solving for t. This time, we're going to solve for t in general **first,** and then use the result to do computations with technology. Aka, we want to see t as depending on some input temperature T, so we'll write our function as $t(T)$ in the end.

$$T = 65 + 100e^{-0.02554t} \quad \rightarrow \quad \frac{T-65}{100} = e^{-0.02554t} \quad \rightarrow \quad -0.02554t = \ln\left(\frac{T-65}{100}\right)$$
$$\rightarrow \quad t(T) = -\frac{1}{0.02554}\ln\left(\frac{T-65}{100}\right)$$

So for any T we are interested in, we can just plug that value into the right hand side of the equation above and use a calculator to compute the time it would take to cool to that temperature. I'm going to use WolframAlpha to help me fill in the table efficiently by typing in "table of values -(1/0.02554) ln((T-65)/100) for T = 150, 130, 100, 70" and hitting Enter. I'll round my answers to one decimal place.

Temperature T°F	**time to reach T**	**Temperature T°F**	**time to reach T**
150	6.4 min	100	41.1 min
130	16.9 min	70	117.3 min (almost 2 hours)

Exercise 7

The melting point of a certain metal is 800°C. I melt a small amount of metal in a crucible and pour it into a mold, leaving it in an ambient temperature of 20°C to cool off. After 60 minutes, it has cooled to 49°C. Find the appropriate r for the model

$$T(t) = T_s + (T_0 - T_s)e^{-rt}$$

and write a function that takes as input a particular temperature and gives as output the time it takes to cool to that temperature. Fill in the table below using this formula.

Temperature T°C	**time to reach T**	**Temperature T°C**	**time to reach T**
90		40	
60		30	

Answer

We find $r = 0.055$ and $T(t) = 20 + (800 - 20)e^{-0.055t}$. Then $t(T) = -\frac{1}{0.055}\ln\left(\frac{T-20}{780}\right)$. Using technology,

Temperature T°C	**time to reach T**	**Temperature T°C**	**time to reach T**
90	43.8 min	40	66.6 min
60	54.0 min	30	79.2 min

Okay, that's your survey of application problems where exponential and logarithmic functions and equations show up! We're done with this chapter!

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7.4E Exercises

(Some of these exercises are based on material from Chapter 6. They may look slightly different from the ones in this section, but they should be simpler and you should try to do them on your own as a test of your mastery. Those are labeled "Level 1," and more advanced problems like in this section are labeled "Level 2." The goal is not to memorize the exact configuration of problem covered in this section! The goal is to be able to try things and figure out how to solve problems you've never seen before.)

? Interest Compounded Annually (Level 1)

1. Billybob Joe borrows $100 from Marybeth Sue and she charges him 10% interest, compounded yearly. Write an exponential function that gives the amount Billybob Joe owes after t years. Use this function (and a calculator) to find the amount Billybob Joe owes after 3, 5, and 10 years. If he can talk Marybeth Sue down to 7% interest, what will the new function be? Use it to find the amount he would owe at the same times, using the new interest rate.

2. Unless you receive a raise every year equivalent to the rate of inflation, you are essentially taking a wage cut as the purchasing power of your money goes down! My dad's current salary is $100,000 per year. If inflation is currently steady at a rate of 4% per year, and his employer promises to give him a fair cost-of-living raise to match that each year, what does he expect his annual salary to be after 2 years? After 10 years? Write a function that gives his salary, S, as a function of how many years, y, have passed. You can use a calculator to perform the computations.

Answer

Hint: if you're ever stuck on how to make the function, just use common sense to compute the first few years and see if you find a pattern.

1.

- $A(3) = 100(1.1)^3 = 133.10$
- $A(5) = 100(1.1)^5 = 161.05$
- $A(10) = 100(1.1)^{10} = 259.37$
- $A_7(3) = 100(1.07)^3 = 122.50$
- $A_7(5) = 140.26$
- $A_7(10) = 196.72$

2.

- $S(1) = 100000(1.04) = 104000$
- $S(2) = 100000(1.04)^2 = 108160$
- $S(10) = 100000(1.04)^{10} = 148024$
- $S(y) = 100000(1.04)^y$.

? Compounding Interest (Level 2)

I invest $20,000 in a long-term slow-growth mutual fund, giving a nominal annual interest rate of 9%, compounded quarterly.

1. Write the function that will compute the amount of money in my account after t years.
2. Fill in the table of values for different t values below.

t	$A(t)$	t	$A(t)$
1		**5**	
2		**10**	

3. Find the annual percentage yield: the rate that, compounded annually, would result in the same amount of money after one year as the compounded quarterly situation.
4. How long will it take for the account to reach $30,000?

Answer

1. $A(t) = 20000(1.0225)^{4t}$.
2.

t	$A(t)$	t	$A(t)$
1	$21,862	**5**	$31,210
2	$23,897	**10**	$48,704

3. 9.31%
4. $t \approx 4.56$ years.

Continuously Compounding Interest (Level 1)

For continuously compounding interest, the amount of money you have at time t years after investing an initial amount A_0 is given by $A = A_0 e^{rt}$, where r is the interest rate written as a decimal. If the interest rate is 12%, find the time t it will take to double my money.

Answer

$t \approx 5.78$ years

Continuously Compounding Interest (Level 2)

A sum of $120,000 is invested at a rate of 11% compounded continuously.

1. Write the function $A(t)$ for this scenario.
2. Fill in the table of values for different times t.

t	$A(t)$	t	$A(t)$
1		**5**	
2		**10**	

3. How long will it take for the investment to double in value?

Answer

1. $A(t) = 120000e^{0.11t}$
2.

t	$A(t)$	t	$A(t)$
1	$133,953	**5**	$207,990
2	$149,529	**10**	$360,500

3. $t \approx 6.3$ years.

Doubling (Level 1)

1. I have a Petri dish containing 200 bacteria at the moment, but the count is doubling every half hour. Construct an exponential function $N(t)$ that will give the number of bacteria after t hours. Use it and a calculator to find the number of bacteria after 1, 2, and 5 hours.

2. My video is going marginally viral on social media right now. Right before it took off, it was sitting at only 40 views, but now the number of views is doubling every minute! Assume the algorithm blesses me for ten minutes with this growth. After 2 minutes, how many views does the video have? After 5 minutes? After the 10 minutes? Write a function giving the total number of views on my video as a function of how many minutes have passed.

Answer

1. $N(t) = 200 \cdot 2^{2t}$
   - $N(1) = 200(2)^2 = 800$
   - $N(2) = 200(2)^4 = 3200$
   - $N(5) = 200(2)^{10} = 204,800$
2. $N(t) = 40 \cdot 2^t$
   - $N(2) = 160$ views.
   - $N(5) = 1280$.
   - $N(10) \approx 41$k views.

? Working With Logs (Level 1)

I had that Petri dish of bacteria, remember? It doubles in population every half hour, meaning that every half hour, each bacterium divides into two copies. I started with 200 bacteria, and it turns out that I can compute the amount of time (in hours) needed to reach N bacteria using the formula

$$t = \frac{\ln \frac{N}{200}}{2 \ln 2}$$

Find how long it will take for me to have 800 bacteria, without using a calculator actually. Then using a calculator, find how long it will take to reach 1000 bacteria.

Answer

For 800, compute or use common sense to get $t = 1$hr. For 1000, ≈ 1.161hr.

? Negative Relative Growth Rate (Level 2)

Mites have infiltrated my ant farm and are decimating my ant population. ("Decimating" comes from the Latin "decimus," which means "tenth," and means that 10% of a population is wiped out.) My ant population is declining at a relative rate of 10% per day. There are currently 150 ants left alive.

1. Write a model for the population $n(t)$ we expect t days from now.
2. Assuming trends stay the same, approximately when will I have only 50 ants in the farm?

Answer

1. $n(t) = 150e^{-0.1t}$
2. $t \approx 11$ days from now.

? Logistic Population Growth (Level 2)

Iguanas went locally extinct on the Galapagos island of Santiago in the early 1900s, but the neighboring North Seymour Island was at max capacity of 5000 iguanas a hundred years later. In 2019, about 1500 iguanas were relocated from North Seymour Island to Santiago to re-seed its population. The North Seymour population was thus depleted, but assuming the iguanas there

exhibit logistic growth with relative growth rate 0.2, they should be repopulating toward the carrying capacity of 5000. Write a function that models the logistic growth of the iguana population on North Seymour Island. Based on that model, estimate the number of iguanas that were present in 2023.

Answer

Giving number of iguanas at time t years after 2019, $n(t) = \dfrac{5000}{1 + \frac{1500}{3500}e^{-0.2t}}$. At $t = 4$, we expect ≈ 4200 iguanas.

? Radioactive Decay (Level 1)

The age of an ancient object can be determined by seeing how much radioactive carbon-14 is left in it, because we know how fast carbon-14 dissipates as a substance ages. If we know the original amount of carbon-14 was C_0, and we measure only C remaining, then we can calculate the age in years using the formula

$$A = -8267 \ln \frac{C}{C_0}$$

To avoid dealing with annoying numbers, let's say that whatever C_0 was, I know that C is only 50% of it. (Using a calculator) find the age of the object. What if we're all the way down to 25% of the original amount?

Answer

At 50%, $C = 0.5C_0$, so $A = 5730$ years old. At 25%, $11,460$ years.

? Radioactive Decay (Level 2)

Cobalt-60 has a half-life of 5.2714 years.

1. Find the relative decay rate r and write the exponential model for its radioactive decay.
2. I start with a 100mg sample. How long will it take to decay to 75mg?
3. If I have a sample containing 40mg that I expect to have started out at 100mg, how old is it?

Answer

1. $r = 0.1315$ so $m(t) = m_0 e^{-0.1315t}$
2. 2.19 years.
3. about 7 years old.

? Newton's Law of Cooling (Level 2)

I bonked my noggin and need to ice it, so I take a bag of frozen peas out of the freezer with an internal temperature of 0°F. The ambient temperature in the room is 75°F. After 5 minutes, the bag has warmed up to 15°F. Write the function $T(t)$ that gives the temperature of the frozen peas after t minutes. Also, find the inverse function, which gives as output the time that it takes to reach a certain temperature input. When the bag reaches the temperature 35°F, I need to put it back in the freezer, so when will that happen?

Answer

Start with $T(t) = 75 + (0 - 75)e^{-rt}$ and use the data point to find $r = .0446$. The function is $T(t) = 75 - 75e^{-.0446t}$. The inverse function is $t(T) = -\frac{1}{0.0446}\ln\left(\frac{75-T}{75}\right)$. It will take about 14 minutes. Hopefully my goose egg will go down by then.

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7.5 Chapter 7 Study Guide

Definitions

- **exponential function:** a function of the form $f(x) = a^x$, where $a > 0$ and $a \neq 1$.
- **base (of an exponential function):** the number a being raised to the power x in an exponential function.
- **end behavior:** how a function behaves "eventually," as $x \to \pm\infty$.
- **logarithmic function:** a function of the form $f(x) = \log_a(x)$, where $a > 0$ and $a \neq 1$, whose values are defined by the relationship

$$\log_a x = y \quad \iff \quad a^y = x.$$

- **natural log:** the logarithmic function with base e.
- **expanding:** the process of taking a single logarithm and writing it as the sum/difference of multiple simpler logarithms, including bringing powers down as coefficients if possible.
- **condensing:** the process of taking a sum/difference of multiple logarithms and combining them into a single log using the Log Laws.
- **compound interest:** interest that is calculated and paid on an investment every certain amount of time. Starting with initial investment amount P, with nominal annual rate r written as a decimal, but compounded n times a year, after t years, you will have

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

- **population doubling:** population growth modeled by an exponential function with base 2. Starting with population initial size n_0, doubling every a seconds/minutes/days/etc., the population at time t will be

$$n(t) = n_0 2^{t/a}$$

- **relative growth rate:** the rate r of growth expressed as a proportion of the population at any time, which appears when modeling growth using base e instead of 2 like above. In this case, the function takes the form

$$n(t) = n_0 e^{rt}$$

- **logistic growth:** a population exhibits logistic growth if it can be modeled using a logistic function:

$$n(t) = \frac{M}{1 + \frac{M - n_0}{n_0} e^{-rt}}$$

- **carrying capacity:** the maximum population an environment can support due to limiting factors. This is the M in the logistic function above.
- **half-life:** the amount of time it takes for a substance to decay to half its original amount.
- **radioactive decay:** the decay of substances as they emit radiation over time. Starting with a mass m_0 and half-life h, the mass remaining at time t is modeled with the function

$$m(t) = m_0 e^{-rt}$$

  where r is the **relative decay rate,** which can be computed from h using $r = \frac{\ln 2}{h}$.
- **Newton's Law of Cooling:** the fact that an object with initial temperature T_0 placed in an environment with surrounding temperature T_s will approach the environment's temperature, with temp at time t given by

$$T(t) = T_s + (T_0 - T_s)e^{-rt}$$

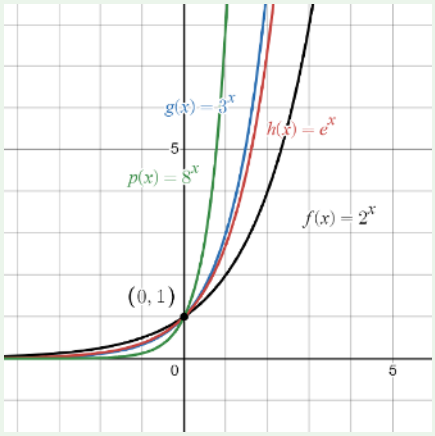
  where r is the **coefficient of heat transfer.**

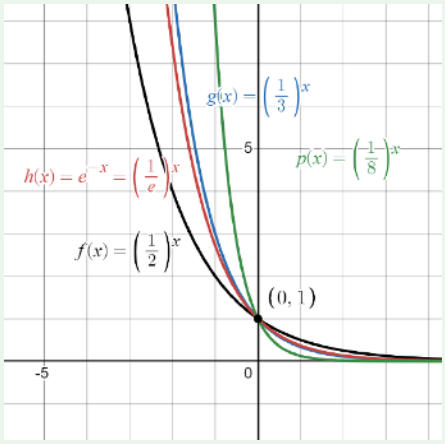
Exponential Functions

The **exponential function with base** a is defined by $f(x) = a^x$, for all real numbers x, where $a > 0$ and $a \neq 1$.

Fun facts about exponential functions:

- The **domain** of an exponential function is all real numbers.
- If $a > 1$, the function is **increasing** (graph going uphill from left to right). If $0 < a < 1$, the function is **decreasing** (graph going downhill from left to right). See graphs below.
- **End behavior for** $a > 1$**:** As $x \to -\infty$, the graph hugs a horizontal asymptote at the x-axis. As $x \to \infty$, the graph shoots up rapidly, and the bigger the base, the faster the growth.
- **End behavior for** $0 < a < 1$**:** As $x \to -\infty$, the graph is blowing up to the left. As $x \to \infty$, the graph hugs the horizontal asymptote at the x-axis forever.
- All exponential functions pass through $(0, 1)$ as well as $(1, a)$, for whatever their base a is.
- The function values are always positive, aka above the x-axis, aka their **range** is $(0, \infty)$.
- For $f(x) = a^x$ and $g(x) = \left(\frac{1}{a}\right)^x$, note that $g(x) = \left(\frac{1}{a}\right)^x = \frac{1}{a^x} = a^{-x} = f(-x)$.

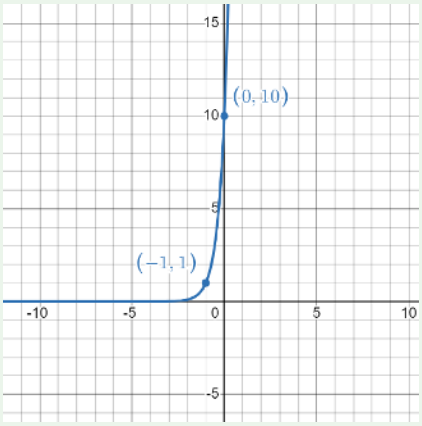




Sketching the Graph of an Exponential Function

1. Find the y-intercept by plugging in $x = 0$ and drop that point in.
2. Compute a couple more easy-to-evaluate points, like plugging in $x = \pm 1, \pm 2$.
3. Check if there's a "$+k$" of some sort on the end of the whole function. That'll be a vertical shift. You could draw in your horizontal asymptote.
4. Check if there's a negative somewhere causing a reflection.
5. Use all that knowledge to connect the dots in the general shape you expect for an exponential, so...a big swoop.

Example: this is $f(x) = 10^{x+1}$. No vertical shift, so H.A. stays at x-axis.



Logarithmic Functions

For $a > 0$ and $a \neq 1$, the **logarithmic function with base a,** which we denote $\log_a$, is the inverse function of the exponential function with the same base a. Thus, it is defined by the relationship

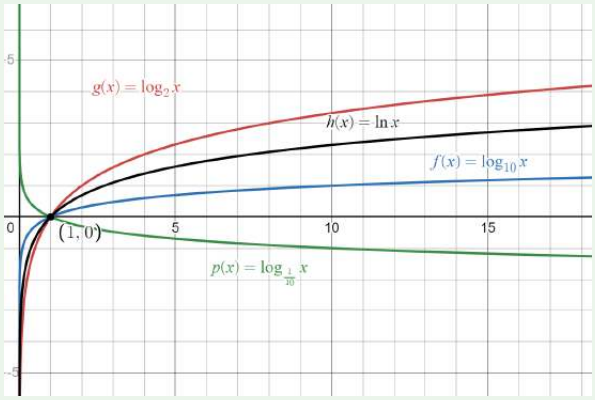
$$\log_a x = y \quad \iff \quad a^y = x.$$

We say, "log base a of x is y." Basically, "$\log_a(x) =$?" is asking the **question**, "What do I need to raise the base a to in order to get out this x?" The **answer** to that question is the output of the function $\log_a$.

When in particular the base is the special number e, we write $\log_e x$ as $\ln x$, and call it the **natural log of x.**

Fun facts about logarithmic functions:

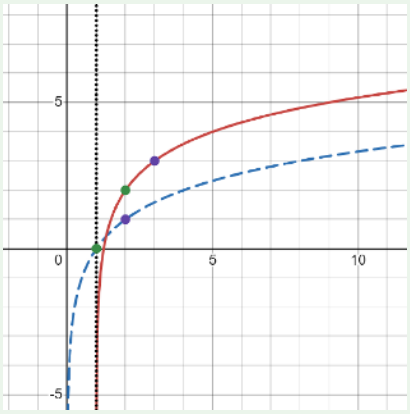
- Their **domain** is **only** positive real numbers! You are **not** allowed to put 0 or a negative number into a log, because that would be the same as saying $a^y = 0$, and there's no possible y that will satisfy that!
- Since $a^0 = 1$ for any a, we have $\log_a(1) = 0$ for any base a. This is something you should know without using a calculator.
- Similarly, since $a^1 = a$, we have $\log_a(a) = 1$ for any a. Aka, if you take the log of its own base, you will always get 1. This is the other evaluation you should know.
- Inverse functions relationship: $\log_a(a^x) = x$, for all real numbers x, and $a^{\log_a x} = x$, for all $x > 0$



Sketching the Graph of a Logarithmic Function

1. First identify the OG log function if you're looking at a transformation. We'll sketch that first.
    1. Remember you have a vertical asymptote, the y-axis.
    2. Plot the points $(1, 0)$ and $(a, 1)$, where a is your base.
    3. Draw a swoosh through the two points, hugging the y-axis forever going down, and slowly growing off as $x \to \infty$.
2. Identify any transformations that happened to make your desired function. Adjust the signal points you plotted accordingly, draw the new vertical asymptote if any horizontal shifts happened, and connect the dots nicely.

Example: this is $f(x) = \log_2(x - 1) + 2$. The dotted blue graph is the OG, $\log_2 x$



Logarithm Laws

Log Law	English Translation	Caveman Translation
$\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of expressions (numbers, variables, whatever as long as it is being multiplied) is the sum of the logarithms of the expressions.	**Multiplication** inside the log can be written as **addition** outside the log. Used backwards, addition outside the log translates to multiplication inside the log.
$\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of expressions is the difference of the logarithms of the expressions.	**Division** inside the log can be written as **subtraction** outside the log. Used backwards, subtraction outside the log translates to division inside the log. The **denominator** must be the input whose log term is subtracted!
$\log_a(A^B) = B\log_a A$	The logarithm of an expression raised to a power is the power times the logarithm of the expression.	If the whole input is raised to a power, you can **bring powers down** in front as coefficients.
$\log_b x = \frac{\log_a x}{\log_a b}$	**Change of Base Formula:** to express $\log_b x$ in terms of $\log_a x$, divide (\log_a x\) by the constant $\log_a b$.	Maybe you want to replace $\log_b x$ with an equivalent expression that uses base a instead. You can switch from $\log_b$ to $\log_a$, but you have to divide by the number $\log_a b$.

Note: The **bases must match** in order to use the first two laws backwards! Using them **forwards** is called **expanding**, and using them **backwards** is called **condensing.**

How to Solve Exponential Equations

1. Get everything boiled down to a single base-to-a-power term on one side of the equals.
2. Take the (appropriate) log of both sides and use the bring-powers-down Log Law.
3. Solve for the variable as usual. Check your answer by plugging back in. There may or may not be a valid solution!

How to Solve Logarithmic Equations

1. Condense completely until you get one single logarithm term on one side by itself.
2. Raise the base to each side of the equation (or translate to exponential form of the logarithmic equation).
3. Solve for the variable.

The best way to study application problems is to review the examples and exercises carefully, since there's such a variety!

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CHAPTER OVERVIEW

Chapter 8: Geometry and Trigonometry

In this final chapter, we cover some basic geometry that everybody should have memorized in order to be ready for common application problems in Calculus courses, and practice setting up problems that assume this kind of knowledge. Next, we cover trigonometric functions, first via triangles and the classic "sohcahtoa" approach, but quickly moving on to your new best friend: the Unit Circle. After learning how to find values for trigonometric functions and graph the most important ones, we will solve trig equations and learn the most commonly used trig identities as well. Note that many Trig & Analytic Geometry courses would be way more involved than this chapter. We're not doing side-angle-side and parallelogram proofs. I'm only putting in here what I personally remember using often throughout my undergraduate math career, and trusting that if you need more later, you can pick it up along the way. On the flip side of that, the stuff that *does* make it into this chapter is stuff that professors can entirely reasonably expect you to know off the top of your head! So let's buckle down and finish strong!

8.1 Geometry Essentials

8.1E Exercises

8.2 Solving Problems With Geometry

8.2E Exercises

8.3 Trigonometry via Triangles

8.3E Exercises

8.4 Trigonometric Functions

8.4E Exercises

8.5 More (and Inverse) Trig Functions

8.5E Exercises

8.6 Trigonometric Identities

8.6E Exercises

8.7 Solving Trigonometric Equations

8.7E Exercises

8.8 Chapter 8 Study Guide

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8.1 Geometry Essentials

Learning Objectives

By the end of this section, you will be able to:

- identify essential features of, as well as compute area and perimeter of, basic geometric shapes: circles, triangles, polygons
- identify and use right triangles and related facts such as the Pythagorean Theorem
- identify essential features of, as well as compute volume and surface area of, basic 3D solids: cylinders, prisms, spheres

This section is going to cover the essentials of geometry that everybody should have on instant recall before going into Calculus I. Many application problems use physical situations involving cylindrical or conical tanks, or spherical balloons, or spreading circles of oil spills... Calc I is basically devoted to the concept of a derivative, which is code for "rate of change" physically, so **related rates** problems involve things like water leaking from a container or the volume of a balloon expanding. In **optimization** problems, you need to know basic geometry of boxes or cylinders, so that you can find the dimensions that will result in the maximum volume or minimum surface area under certain constraints, for example. A professor might provide you with more obscure formulas, like the surface area of a cone, say, but they will certainly expect you to know how to find the area of a circle or triangle right off the top of your head. In other words, this section is again a big survey of "fun facts" about shapes that you need to memorize if you haven't already!

Area and Perimeter

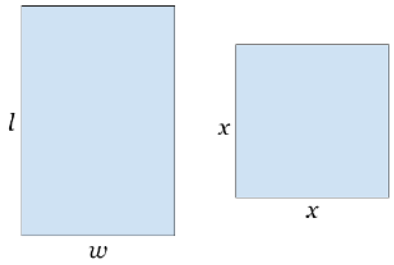
My brother just bought a house. In taking up the rattling old tiles, he discovered that there is terrazzo flooring underneath. Instead of covering up that nice surface, he is going to have it restored/refinished. Such services are priced depending on the size of the job. If the refinishers will charge $2 per square foot, then what he needs to know is the **area** of his house's floors! Area comes with units that are "square" such as square feet (ft^2) or square miles (mi^2).

Perimeter is like measuring how many yards of fencing material you would need to enclose a whole entire yard. It's the sum of the lengths of all the sides of a shape (or in a circle, the length of a string if you laid it all the way around the edge of the circle, and then stretched it out and measured it). As such, the units of a perimeter are distance measures like centimeters, feet, miles, etc.

Here are the basic shapes for which you must know how to find area and perimeter.

Rectangles

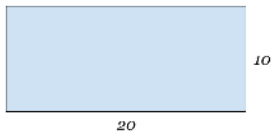
Rectangles have four sides, with opposite pairs having equal length, and all their internal angles are right angles (90°). **Squares are just special rectangles** where **all** the sides are the same length.



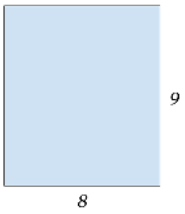
The area of a rectangle is "length times width," $A_{\text{rect}} = lw$ square units. In the case of a square, length and width are the same, so if the side length is x, then $A_{\text{sq}} = x^2$ square units. The perimeter of a rectangle adds up the two sides of length l and the two sides of length w, so $P_{\text{rect}} = 2l + 2w$ units. What is the perimeter of a square with side length x?

Exercise 1

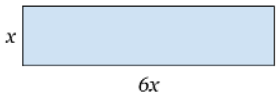
Find the area and perimeter for each rectangle. Figures may not be drawn to scale!



1.



2.



3.

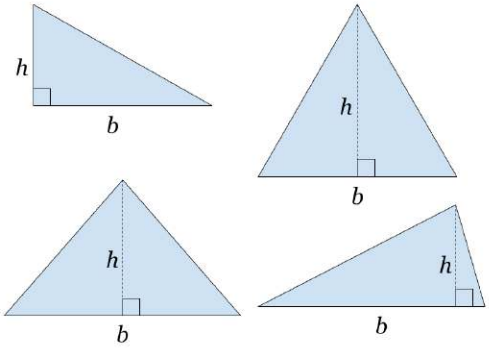
Answer

1. $A = 200$ square units. $P = 60$ units.
2. $A = 72$ square units. $P = 34$ units.
3. $A = 6x^2$ square units. $P = 2x + 2(6x) = 14x$ units.

Triangles

Triangles are three-sided shapes. There are a few types of triangles to know about:

- **Right triangles** have a right (90°) internal angle. The sides on either side of the right angle are the **legs** and the third (longest) side is the **hypotenuse.** For these triangles, the **height** is the same as the length of a vertical leg.
- **Equilateral triangles** have all three sides of the same length. All three of their internal angles are also the same, 60°. The height of these triangles is NOT the same as their side length.
- **Isosceles triangles** have two sides of equal length. The angles opposite those sides will also be equal. The height of these triangles is NOT the same as their side length. (Technically, an equilateral triangle is a special isosceles triangle, just like a square is a special rectangle...)
- **Scalene triangles** have all different length sides. The height of these triangles is NOT the same as their side length.



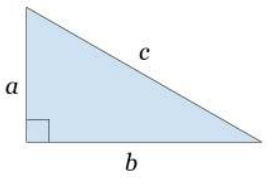
The area of a triangle is always "one-half base times height," $A_{\text{tri}} = \frac{1}{2}bh$. The perimeter of a triangle is the sum of its side lengths. If you know those lengths, you just add them up. If you don't know them, it can be slightly tricky to find them... But if it's a right triangle, there is a super important math fact that makes life easier.

Pythagorean Theorem

In a **right triangle** with **leg** lengths a and b, and **hypotenuse** length c, the following is true:

$$a^2 + b^2 = c^2$$

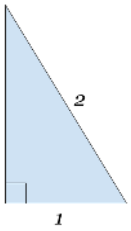
This can be used to find the length of any side of a right triangle if you know the other two. Simply plug in the known values and solve for the third.



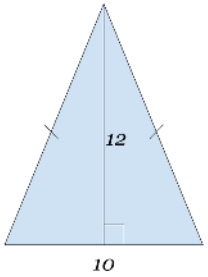
$a^2+b^2 = c^2$

Example 1

Find the area and perimeter of the triangles below. Note that diagrams may not be to scale, so you must use math.



1.



2.

Solution

1. This is a right triangle and we're given two side lengths: one leg and the hypotenuse. To find the area, we need to find the height, the length of that other leg. We use the Pythagorean Theorem, plugging in the sides we know and using a variable name for the other leg:

$$1^2 + b^2 = 2^2 \quad \rightarrow \quad b^2 = 4 - 1 = 3, \quad \rightarrow \quad b = \sqrt{3}$$

Note that we only use the positive square root because a negative side length doesn't make sense. Now that we know all three sides (and thus the height), we compute:

$$A = \frac{1}{2}bh = \frac{1}{2}(1)(\sqrt{3}) = \frac{\sqrt{3}}{2} \qquad P = 1 + 2 + \sqrt{3} = 3 + \sqrt{3}$$

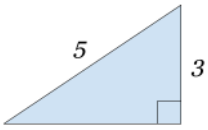
2. (Tick marks in sides of shapes mean that the lengths are equal, if you didn't know.) This is an isoceles triangle, and we're given the base and the height. We can get the area straightaway: $A = \frac{1}{2}bh = \frac{1}{2}(12)(10) = 60$. To find the perimeter, we need to know length of the other sides (which are the same). To find this, notice that cutting the triangle in half down the given height line creates two right triangles! The base of each right triangle is just half of the whole triangle's base, so it must be 5. The unknown side we want is the hypotenuse of the right triangle. We can find it using the Pythagorean Theorem.

$$5^2 + 12^2 = c^2 \quad \rightarrow \quad 25 + 144 = c^2 \quad \rightarrow \quad 169 = c^2 \quad \rightarrow \quad c = 13$$

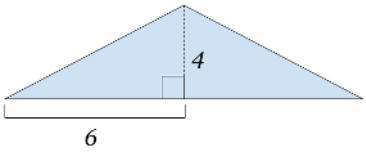
Now we can compute the perimeter, knowing the base is 10 and each of the other sides is 13: $P = 10 + 13 + 13 = 36$.

Exercise 2

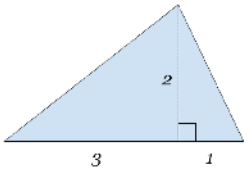
Find the area and perimeter of the triangles. Leave answers exact (don't use a calculator, just write square roots).



1.



2.



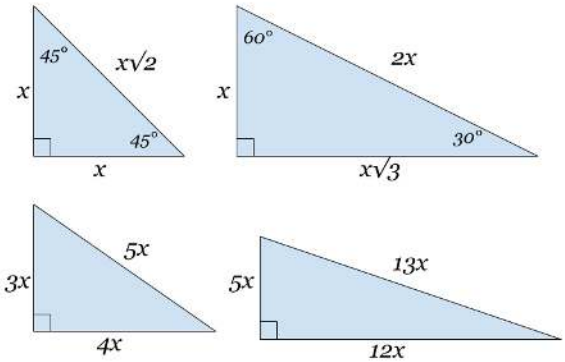
3.

Answer

1. $A = 12$ square units, $P = 12$ units.
2. $A = 24$ square units. $P = 2\sqrt{52} + 12$ units.
3. $A = 8$ square units. $P = \sqrt{5} + \sqrt{13} + 4$ units.

Special Triangles

Here are some special triangle types that, if you learn to recognize them, will make your life wayyyy easier. I explain them below the graphic.



45-45-90 Right Triangles: These triangles have a 90° angle so they're right triangles, and the other two angles are both 45°. This means that the legs have equal length. It also turns out that the hypotenuse is $\sqrt{2}$ times the length of the legs (use the Pythagorean Theorem to check this).

30-60-90 Right Triangles: These triangles have a 90° angle so they're right triangles, and the other two angles are 30° and 60°. Whatever the shortest leg (across from the 30°) is, the hypotenuse is twice that value, and the other leg is $\sqrt{3}$ times that value (again, comes from Pythagorus).

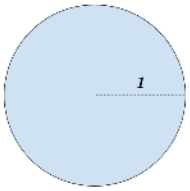
Pythagorean Triples: These are right triangles whose side lengths are all integers. For example, a triangle with side lengths $(3, 4, 5)$ is always a right triangle. The other easiest one is $(5, 12, 13)$, and you might see $(7, 24, 25)$. Note that any MULTIPLE of these side lengths is also guaranteed to be a right triangle. Stuff like $(6, 8, 10)$.

Circles

Circles have one essentially defining dimension: the **radius, r.** This is the uniform distance from any point on the edge of the circle to the center. The total maximum distance across a circle passing through the center is called the **diameter,** and it is simply double the radius. The **area of a circle** is $A_c = \pi r^2$. The perimeter of a circle, or the distance all the way around its edge, is called the **circumference,** and computed $C = 2\pi r$.

? Exercise 3

Give the radius, diameter, circumference, and area of the circles.

	
1.	2.

Answer

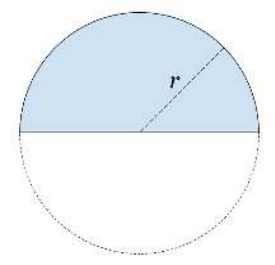
1. $r = 1$ units, $d = 2$ units, $C = 2\pi$ units, $A = \pi$ sq. units
2. $d = 8$ units, $r = 4$ units, $C = 8\pi$ units, $A = 16\pi$ sq. units

Pieces of Pizzas

You can also use your common sense about the area of portions of circles. This is very easy when you see semicircles or quarter-circles: just take the appropriate fraction of the area of an imaginary full circle!

Example 2

This shape is a semicircle, which is half of a full circle that I've shown with a dashed line. Find the area and perimeter of the semicircle.



Solution

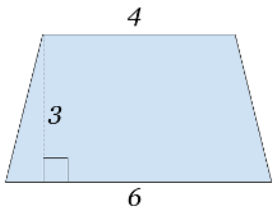
The radius of the full circle would be r, so the area of the full circle would be $A = \pi r^2$. The area of the semicircle is half of that, so we report $\frac{1}{2}\pi r^2$. Now, be careful with the perimeter... Part of the perimeter is that round boundary of the semicircle, which would be half of the full circle's circumference. That piece must be $\frac{1}{2}(2\pi r) = \pi r$. But there is also that bottom boundary, which is a straight line. It would actually be the diameter of the full circle, so it has length $2r$. The total perimeter of the shape is then $2r + \pi r$ or $(2 + \pi)r$, if you prefer.

Decomposing Shapes

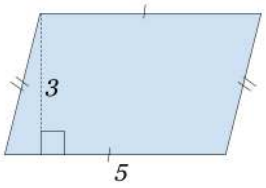
For other 2D shapes, the key to finding area boils down to just cutting them into pieces whose area you can find easily. I personally don't usually have the area formula of a trapezoid on rote recall, because it's easy for me to re-derive it if I really need it, using slice-and-dice reasoning. Let's do some examples.

Example 3

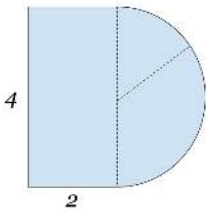
Find the area. (In 1. you may assume that this is an **isosceles** trapezoid, meaning the slanty sides are equal.)



1.



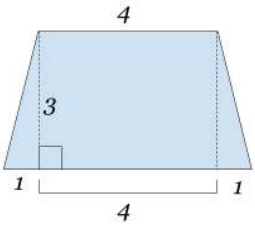
2.



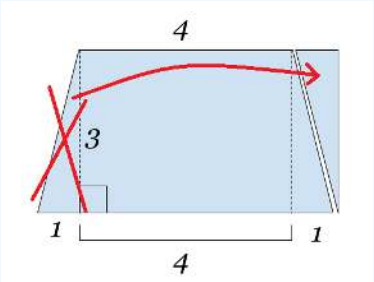
3.

Solution

1. This is a symmetrical trapezoid, and rather than memorizing the formulas for those, I just imagine slicing it into pieces. One option is to cut two congruent right triangles off the ends, like this:

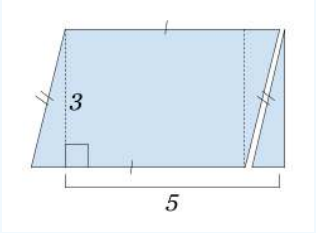


This breaks the shape down into three simple shapes: two identical right triangles with base 1 and height 3, and a rectangle with length 4 and width 3. You can get those areas and add them up, or another shortcut is to imagine cutting off one triangle, rotating it, and sliding it over to the other side... This shows that the area of the trapezoid is exactly the same as the area of a rectangle with length 5 and width 3!



Either way, the area of the shape must be $A = 15$ square units. By the way, if you do want the formula for the area of a trapezoid with base b and top side a and height h, it would be $A = \frac{a+b}{2}h$.

2. This is a parallelogram, and we've been given the base and the height. To find the area, once again break down the shape into triangles and rectangles, or imagine cutting off the triangular portion on the left and sliding it over to fit against the right side of the shape... The area of a parallelogram is the same as the area of a rectangle with the same base and height!



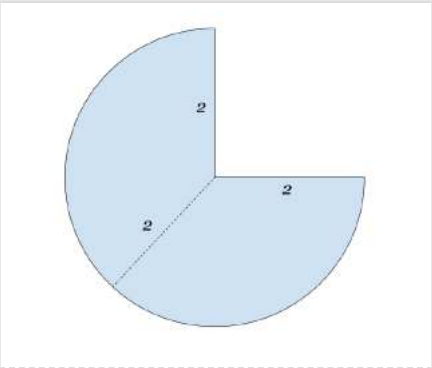
Thus the area is $A = 15$.

3. This shape can be divided into two pieces: a rectangle with length 4 and width 2, and a semicircle of some size. We observe that the diameter of the circle is 4, so the radius must be 2. The area of a circle with radius 2 is $\pi r^2 = \pi(2^2) = 4\pi$, so the area of a **semicircle** is half of that. Thus the total area of the shape is $A = 2(4) + \frac{1}{2}4\pi = 8 + 2\pi$ square units.

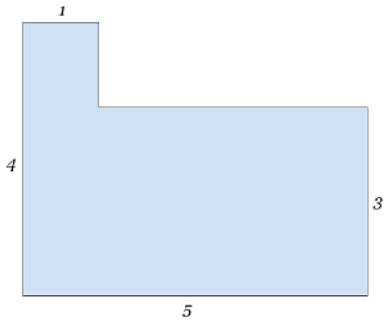
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Exercise 4

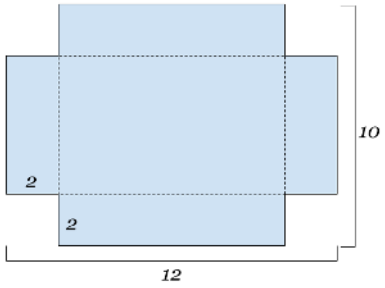
Find the area. Figures may not be drawn to scale!



1.



2.



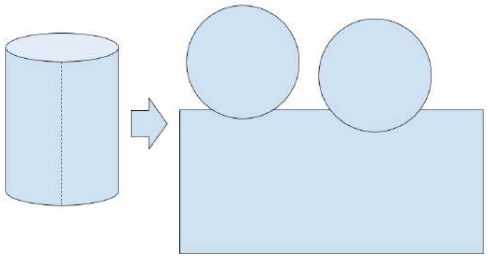
3.

Answer

1. $A = \frac{3}{4}(4\pi) = 3\pi$ square units. (This is three-fourths of a circle with radius 2.)
2. $A = 16$ square units.
3. $A = 104$ square units.

Volume and Surface Area

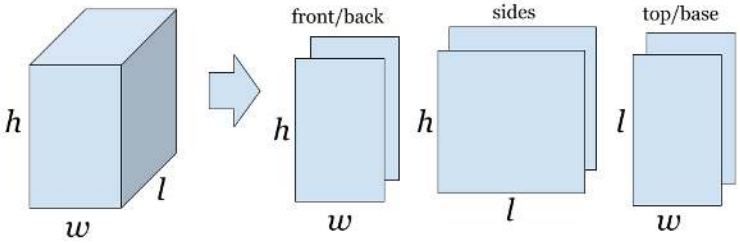
Let's move from 2-dimensional shapes in the plane on to 3-dimensional shapes like cubes, spheres, cylinders, rectangular prisms (boxes), etc. Imagine filling a cylindrical tin can with a liquid. The **volume** of the can is how much liquid it can hold, reported in "cubic" units like cubic centimeters (cm^3) or cubic feet (ft^3), or other volume units like liters (1 L = 0.001 m^3). Now imagine that you take the tin can, cut off the circular top and bottom, and then slice down the side and flatten out the tin sheet it was made from:



The total area of those three 2D shapes is called the **surface area** of the cylinder. There are handful of important standard 3D shapes you should know.

Boxes

A box is a rectangular prism (this includes cubes, as they're just special boxes where all sides have the same length). You can think of them as having a top, bottom, and four sides. There are three defining dimensions for a box: the length, width, and height.

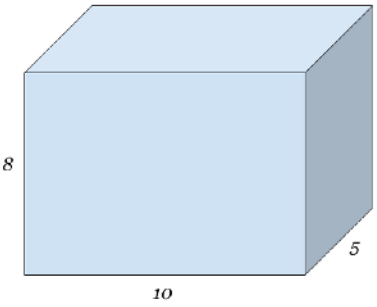


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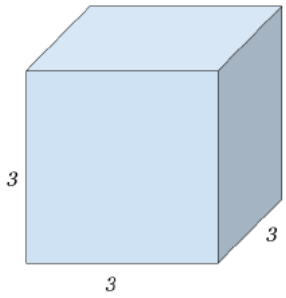
The volume of a box is "length times width times height," $V = lwh$, or in the case of a cube where all sides have length x, $V = x^3$. The surface area can be found by imagining cutting apart a cardboard box like in the figure above. There are pairs of sides with different areas, hw or hl or lw. The total surface area of the box is $S = 2hw + 2hl + 2lw$. What is the surface area of a cube then?

? Exercise 5

Find the volume and surface area.



1.



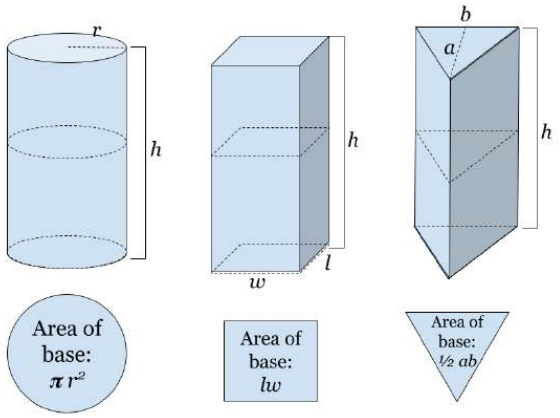
2.

Answer

1. $V = 400$ cubic units. $S = 340$ square units.
2. $V = 27$ cubic units. $S = 54$ square units.

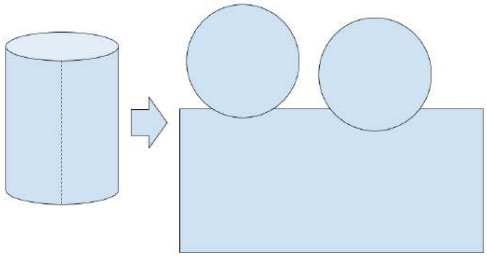
Cylinders (Extruded Solids)

As a child, you might have played with Play-Doh and certain accoutrements called extruders, which have some kind of shaped opening, through which you force a blob of dough. The result is a long snake with consistent cross section, such as a long cylinder, or a long triangular prism. It's easy to calculate the volume of any such shape with consistent cross section area and height h. You just multiply the area of the base times the height. Here are some examples:



Thus the volume of the cylinder is $V_{\text{cyl}} = \pi r^2 h$. The volume of the box is consistent with the formula we just learned: $V_{\text{box}} = (lw)(h)$. The volume of the triangular prism shape is $V_{tr} = \frac{1}{2}abh$.

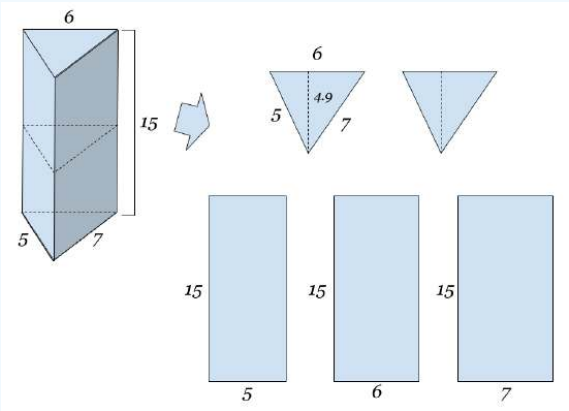
Finding surface area of such shapes is all about imagining cutting apart the shape as if it were made out of cardboard. Just like the tin can example we used before:



The surface area of a cylinder consists of two circles, both with area πr^2, and a rectangular sheet when flattened out. The height of that rectangle is the height of the cylinder, but what is the length? Well, it *was* wrapped all the way around the circular base, so the length is exactly the circumference of the circle! Thus the area of the rectangle is $2\pi rh$. The surface area of a cylinder as a whole is thus $S_{\text{cyl}} = 2\pi r^2 + 2\pi rh$.

? Exercise 6

Find the volume (you can use a calculator for the numbers) and the surface area (break down into the pieces shown).

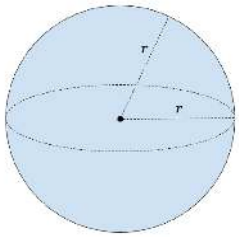


Answer

$V = \frac{1}{2}(6)(4.9)(15) = 220.5$ cubic units. $S = 2\left(\frac{1}{2}(6)(4.9)\right) + 5(15) + 6(15) + 7(15) = 299.4$ square units.

Spheres

A sphere, just like a circle, has just one defining dimension: its radius r. A sphere is all the points in 3D space that are exactly a distance of r away from the center.



Once you know r, you can compute the volume and surface area:

$$V_{\text{sphere}} = \frac{4}{3}\pi r^3, \qquad S_{\text{sphere}} = 4\pi r^2$$

Unfortunately, I don't have a super quick and easy trick for you to see *why* these formulas are true, but if you want to do a bit of a deep dive, this is a cool video! For now, you'll probably just want to memorize these formulas. At least, you can remember that since we're doing something circlish, there better be a π in there somewhere! And then since volume comes in cubic units, it makes

sense that the r should be *cubed* in that formula, while the surface area has r *squared* and area comes in square units. Maybe those tricks will get you started. Meanwhile, practice using the formulas in the exercise below.

? Exercise 7

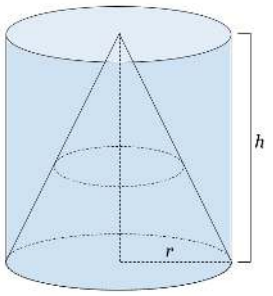
Find the volume and surface area of the sphere

1. with radius 1 in.
2. with radius 4 cm.
3. with radius x units.

Answer

1. $V = \frac{4}{3}\pi$ cubic inches. $S = 4\pi$ square inches.
2. $V = \frac{256}{3}\pi$ cm^3. $S = 64\pi$ cm^2.
3. $V = \frac{4}{3}\pi x^3$ cubic units. $S = 4\pi x^2$ square units.

Okay, those are the basic formulas that you will need going forward. There are, of course, other shapes that professors may or may not expect you to know off the top of your head. For example, the volume of a right circular cone is one-third the volume of a cylinder with the same base and height:



$$V_{\text{cyl}} = \pi r^2 h \quad \rightarrow \quad V_{\text{cone}} = \frac{1}{3}\pi r^2 h$$

When in doubt, just ask your professor what kind of geometric shapes and solids they want you to have on instant recall.

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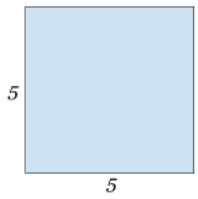
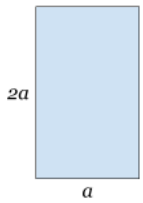
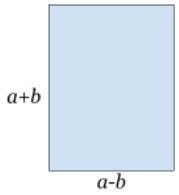
8.1E Exercises

Note: Figures may not be drawn to scale! Yes, I'm a talented artiste, but I am also lazyyyy.

Also note: When numbers are "well cooked" (small and whole numbers) don't use a calculator. If dimensions have fractions or decimals or you need to use the Pythagorean Theorem and it gets ugly, you can use a calculator.

Rectangles

Find the area and perimeter.

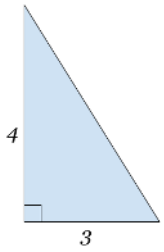
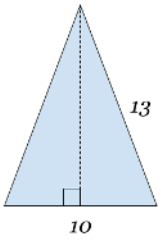
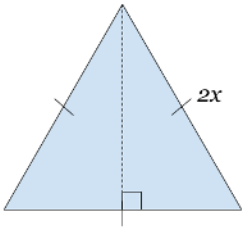
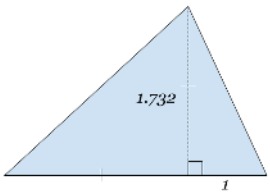
			
1.	2.	3.	4.

Answer

1. $A = 25$ sq units. $P = 20$ units.
2. $A = 51$ sq units. $P = 40$ units.
3. $A = 2a^2$ sq units. $P = 6a$ units.
4. $A = (a+b)(a-b) = a^2 - b^2$ sq units. $P = 2(a+b) + 2(a-b) = 4a$ units.

Triangles

Find the area and perimeter.

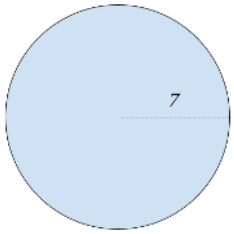
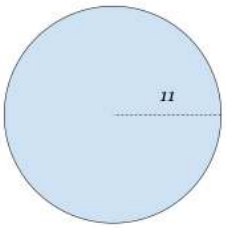
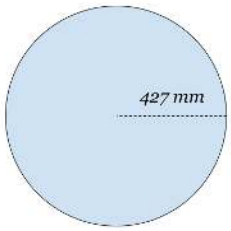
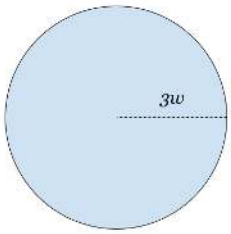
			
1.	2.	3.	4.

Answer

1. $A = 6$ sq units. $P = 12$ units.
2. $A = 60$ sq units. $P = 36$ units.
3. $A = \sqrt{3}x^2$ sq units. $P = 6x$ units.
4. $A = 2.365$ sq units. $P = 7.18$ units.

Circles

Give the diameter, circumference, and area.

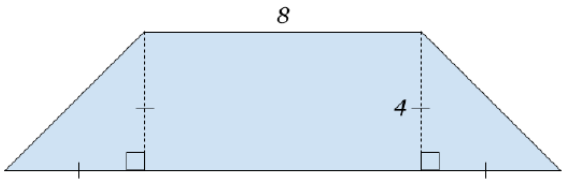
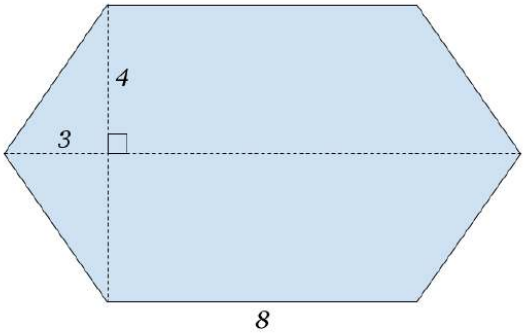
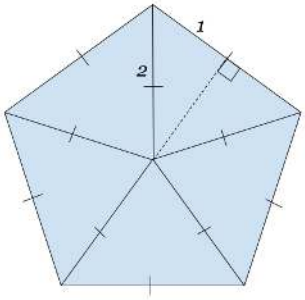
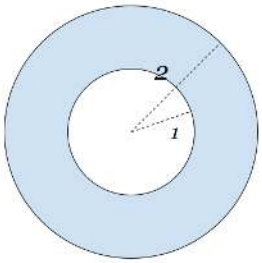
1.	2.	3.	4.
			

Answer

1. $d = 14$ units, $C = 14\pi$ units, $A = 49\pi$ sq units
2. $d = 22$ units, $C = 22\pi$ units, $A = 121\pi$ sq units
3. $d = 854$ mm, $C = 854\pi$ mm, $A = 182329\pi$ sq mm.
4. $d = 6w$ units, $C = 6\pi w$ units, $A = 9\pi w^2$ sq units.

? Composite Shapes

Find the area and perimeter.

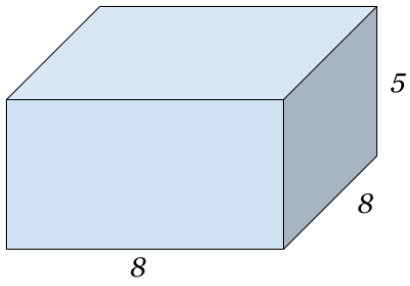
1.	2.
	
3.	**4.**
	

Answer

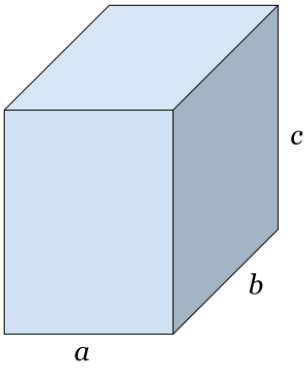
1. $A = 48$ sq units. $P = 8 + 16 + 2(4\sqrt{2}) = 24 + 8\sqrt{2}$ units.
2. $A = 88$ sq units. $P = 36$ units.
3. $A = 5\sqrt{3}$ sq units. $P = 10$ units.
4. $A = 4\pi - \pi = 3\pi$ sq units. Perimeter is the inner circumference plus the outer circumference, $P = 4\pi + 2\pi = 6\pi$ units.

Boxes

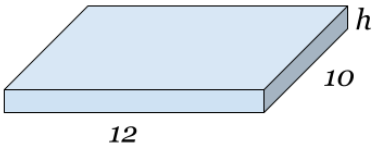
Find the volume and surface area.



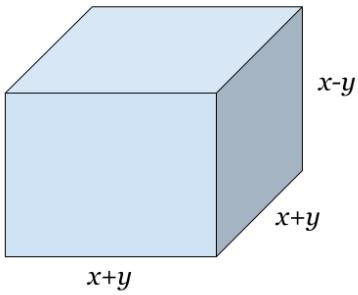
1.



2.



3.



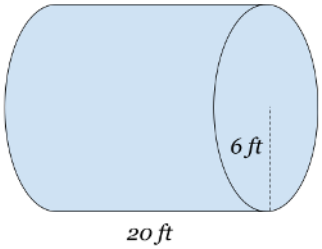
4.

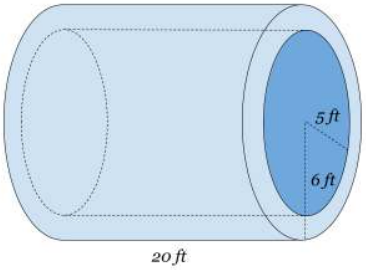
Answer

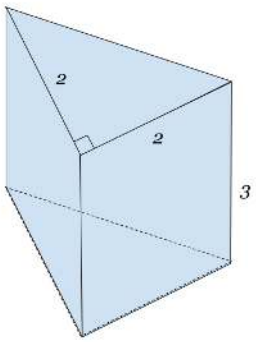
1. $V = 320$ cu units. $S = 288$ sq units.
2. $V = abc$ cu units. $S = 2ab + 2bc + 2ac$ sq units.
3. $V = 120h$ cu units. $S = 240 + 44h$ sq units.
4. $V = (x+y)^2(x-y) = x^3 + x^2y - xy^2 - y^3$ cu units. $S = 2(x+y)^2 + 4(x+y)(x-y) = 6x^2 + 4xy - 2y^2$ sq units.

Extruded Solids

Find the volume and surface area.







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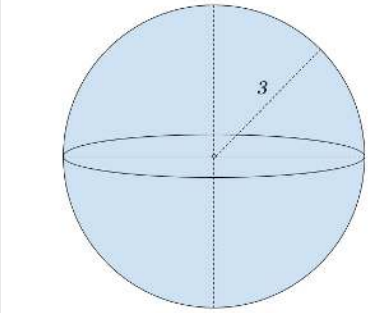
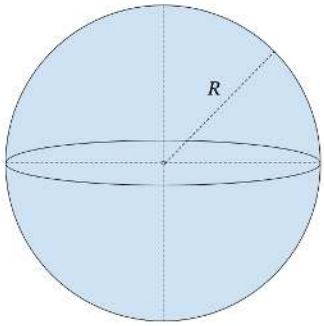
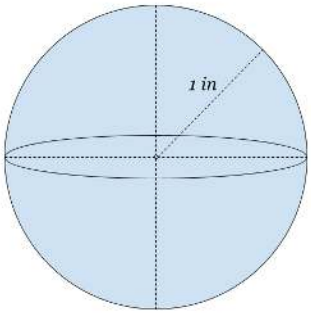
1.	2.	3.

Answer

1. $V = 720\pi$ ft^3. $S = 2(36\pi) + 20(12\pi) = 312\pi$ ft^2.
2. This is a hollow shape, the same size as the previous cylinder, but subtracting an inner cylinder with radius 5ft and length 20ft. The volume of the interior cylinder is 500π, so the volume of the shell shape is $V = 720\pi - 500\pi = 220\pi$ ft^3. The surface area is tricky. There is the outer wall, $20(12\pi)$, and the inner wall, $20(10\pi)$, and the front and back surfaces both have area $30\pi - 25\pi$. All together, it's $S = 450\pi$ ft^2.
3. $V = 6$ cubic units. $S = 16 + 6\sqrt{2}$ square units.

? Spheres

Find the volume and surface area.

		
1.	2.	3.

Answer

1. $V = 36\pi$ cubic units. $S = 36\pi$ square units.
2. $V = \frac{4}{3}R^3\pi$ cubic units. $S = 4\pi R^2$ square units.
3. $V = \frac{4}{3}\pi$ cubic inches. $S = 4\pi$ square inches.

? Lassoing the Earth

I have 25,000 miles worth of rope lying around so I decide to wrap it all the way around the Earth, right on the equator. Then I decide I want to raise the rope 1 meter above the ground everywhere, all the way around. How much more rope do I need? Pause first and intuitively make a guess...the answer might surprise you!

Answer

Hint: The rope forms a circle which has some radius. How does the radius change?

? Square Pies?

My sister and her husband are coming to visit me for Thanksgiving and I plan to make two pumpkin pies (the correct choice of pie). The recipe on the can of pumpkin says it makes two 9-in pies. However, I only have one traditional round 9-inch-diameter pie pan. I do have several other sizes of rectangular glass baking dishes that I could frankenstein a pie crust into, though...

- An 8-inch by 8-inch square
- A 7-inch by 11-inch rectangle
- A 9-inch by 13-inch rectangle

Compare the areas of all the shapes to see which of the three I should use in addition to my standard pie pan. (You can use a calculator.)

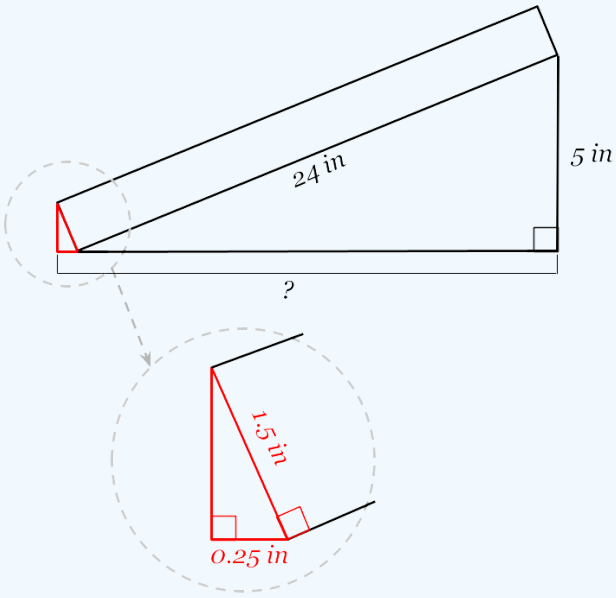
Answer

Find the area of the pie pan by finding its radius first. The 8x8 pan is the best approximation of the circular pan's area.

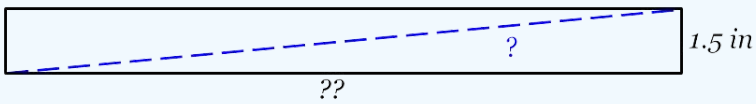
? My Brother's Pantry

As mentioned in the section, my brother bought a house. He actually really truly literally called to have me help solve the following problem while installing pantry shelving, since the house was built in the 60s and has wonky dimensions. You can use a calculator on this problem, and although figures are not drawn to scale, you can assume angles that look like right angles are right angles.

Step 1: Here are the dimensions he was able to measure, since it was difficult physically to get accurate measurements inside the pantry. Use the given dimensions on the triangles to find the length "?" below. Round your answer to four decimal places.



Step 2: Plug the number you got in Step 1 into the blue dimension "?" below, and find the length labeled "??" rounded to four decimal places.



Step 3: My brother uses a standard tape measure while cutting his boards, on which the smallest tick marks are $\frac{1}{16}$ of an inch. Find a way to write your decimal approximation for "??" as a number he can see easily on the tape measure.

Answer

Step 1: Using the Pythagorean Theorem on the black triangle and adding appropriately, you should get ? = 23.7234 inches.

Step 2: You should get ?? = 23.6759 inches.

Step 3: Using my calculator to find decimal value conversions of fractions with denominator 16, I find that 0.6759 is just a hair under $\frac{11}{16} = 0.6875$. This tick mark would be the small one right after a $\frac{5}{8}$-ths tick. He should measure the board to be

$23\frac{11}{16}$ inches.

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8.2 Solving Problems With Geometry

Learning Objectives

By the end of this section, you will be able to:

- set up and solve real-world problems that involve geometric shapes
- use the distance formula to find the distance between two points
- use similar triangles to set up equations
- view geometric quantities as functions

In this section, we'll do a survey of problems that require you to know some geometry in order to set up or solve them. As my old professor used to say, **"You might want to draw a pitchur."** (I went to college in the South.)

To solve a problem involving geometry, try...

1. Sketching a picture of the situation.
2. Labeling any known quantities, dimensions, etc.
3. Identifying what exactly you are trying to find for your answer.
4. Using geometric relationships between the different quantities to solve for your answer.

Here are some examples.

Example 1

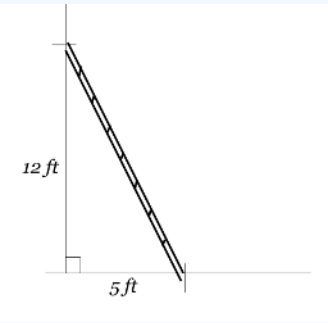
1. I need to lean a ladder up against my house's exterior wall to reach a second-story window. Because of a bush planted close to the house, I can place the base of the ladder at a distance of 5ft away from the wall. I need the ladder to reach up the wall to a height of 12ft. How long of a ladder do I need?

2. In Minecraft, I am making a rectangular pasture for my rainbow sheep flock. I will use a cliff face as one side of the rectangle, and make the rest of the sides with fencing. If I have 600 blocks of fencing and I divide them evenly amongst the sides, what is the area of my pasture?

3. I am going to make an open box by cutting the corners out of a square piece of cardboard and folding up the sides. If I want the box to have dimensions $3 \times 3 \times 3$ inches, what size cardboard sheet must I start with?

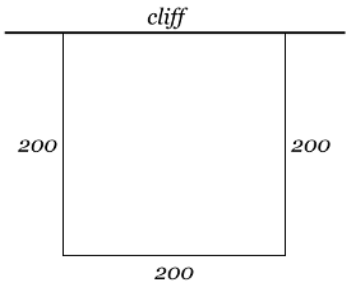
Solution

1. I sketch my house wall, the ground, and the ladder leaning against the wall. Then, I label the given dimensions on the picture. Since my house stands straight up (hopefully) I know that this is a right triangle.



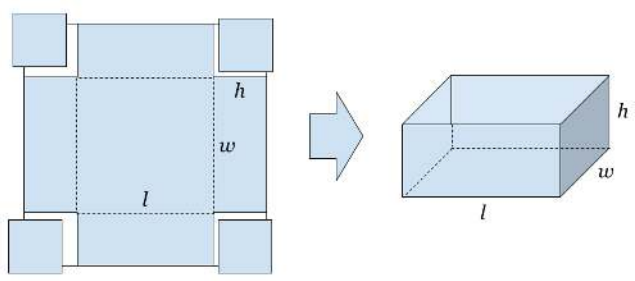
Now, I identify that what I want to know is the length of the ladder, which is the hypotenuse of my right triangle. You can either use the Pythagorean Theorem to find that length, or notice that this is a special (5, 12, 13) right triangle, so the ladder length must be 13 ft.

2. I sketch the rectangular pasture, with three sides of fencing and one side made by the cliff. If I am splitting 600 blocks equally amongst three sides, each side will get 200 blocks. The picture looks like this:

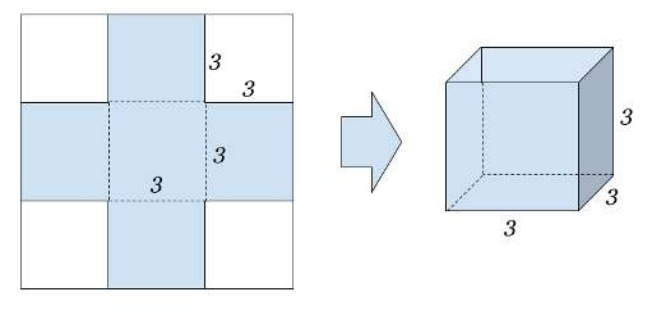


Ope, the area of the interior of the pasture is just the area of a square with side length 200. So the pasture has area $A = 200^2 = 40000$ blocks.

3. First I draw a square and sketch what it looks like to cut squares out of the corners so I can imagine folding up the sides into a box. I can also sketch the finished box, because I need to connect the dimensions of the finished box to some kind of dimensions on the cardboard diagram...



Now that I have a sense, I think about the actual givens in this problem. The original cardboard is square, so its side lengths must be the same, and the squares I'm cutting out of the corners must all be the same size of course, and I want the final box to be $3 \times 3 \times 3$. I adjust my picture.



Now I identify that I'm supposed to say what the original dimensions of the cardboard square were. Looking at the left side of the picture, I see the big outer square is 9×9. Thus that is the size needed to make this box.

Exercise 1

1. Two cities are planning to build a lightning rail speed train along the absolute shortest path between them. City A is 300 miles north and 400 miles east of City B. How long will the lightning rail route be?

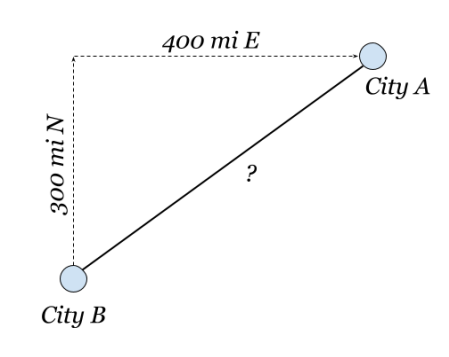
2. I am building a large rectangular frame for a huge oil painting. I want the outer dimensions of the frame to be 10ft by 12ft, with a rectangular inner opening such that the width of the frame is consistent on all sides, say some value x ft. I will purchase fancy trim for both the inner and outer edges of the frame. Depending on the width x chosen, how many feet of trim will I need?

3. My camping tent is a triangular prism: the front and back surfaces are equilateral triangles, with the point of the triangle at the top of course, and the sides are flat rectangles, as is the floor. Over the years, it's gotten a bit saggy, so I want to construct

some wooden supports to place straight down the middle of the tent to hold it up, without distorting its shape. If the sides and floor of the tent are all 2 meters wide, what length should I cut my wooden supports to?

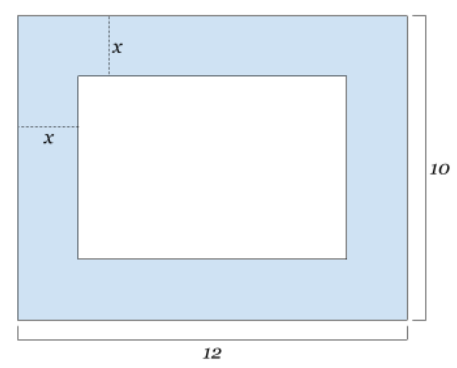
Answer

1. The shortest route between two points is the straight line between them. I sketch the locations of the cities and realize that the straight line I'm looking for is the hypotenuse of a right triangle...



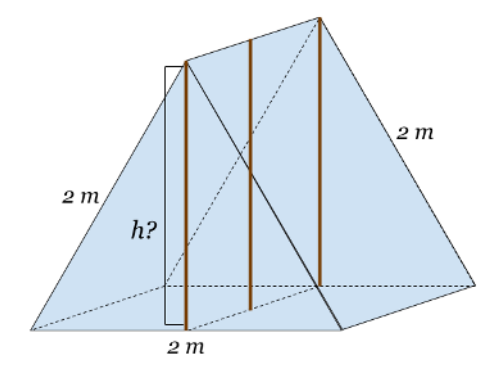
Either use the Pythagorean Theorem to find the length of the hypotenuse, or recognize that this is a multiple of the (3, 4, 5) special right triangle. Thus the hypotenuse must be 500mi long.

2. I have a rectangular frame with a rectangular inner opening. The outer dimensions are 10ft by 12ft, and the width of the frame is consistent. I would sketch the situation like this:



If I'm going to add trim on all the inner and outer edges, I need to find the total perimeter of this shape. That is $P = 2(12) + 2(10) + 2(10 - 2x) + 2(12 - 2x) = 88 - 8x$ ft.

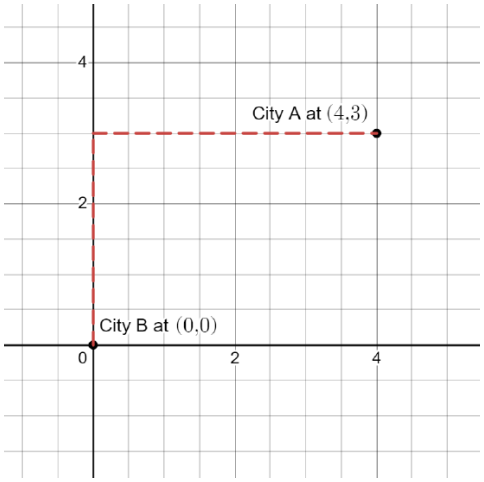
3. I imagine a triangular camping tent and sketch a picture of the situation. I want to place supports in the tent, and the question is what height to make them.



Finding h is just finding the height of the equilateral triangle with all sides 2m. Using the methods of the last section, we find $h = \sqrt{3} \approx 1.732$m.

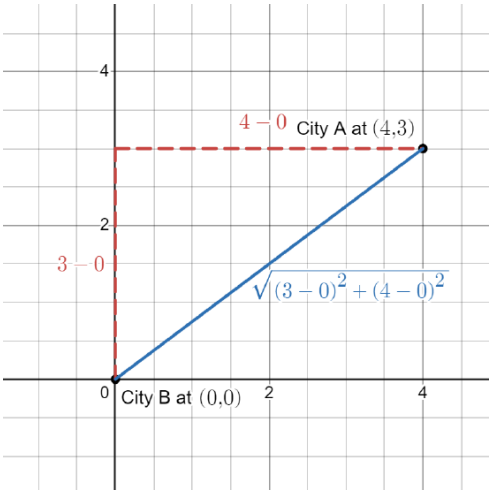
The Distance Formula

In the exercise above, we had to find the shortest possible distance between two cities, having been given an amount to travel in the vertical direction and an amount to travel in the horizontal direction. We could have imagined these two cities as being on a 2D plane, with City B at $(0, 0)$ and City A at $(4, 3)$, if we decide that the x- and y-values represent hundreds of miles.



We can find the straight line hypotenuse distance between the two points, which is the shortest possible path to travel between them, using the Pythagorean Theorem:

$$a^2 + b^2 = c^2 \quad \rightarrow \quad (3-0)^2 + (4-0)^2 = c^2 \quad \rightarrow \quad c = \sqrt{(3-0)^2 + (4-0)^2} = 5$$



Let's generalize this process to any two points, (x_1, y_1) and (x_2, y_2).

The Distance Formula

The distance between two points $A : (x_1, y_1)$ and $B : (x_2, y_2)$ is given by

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

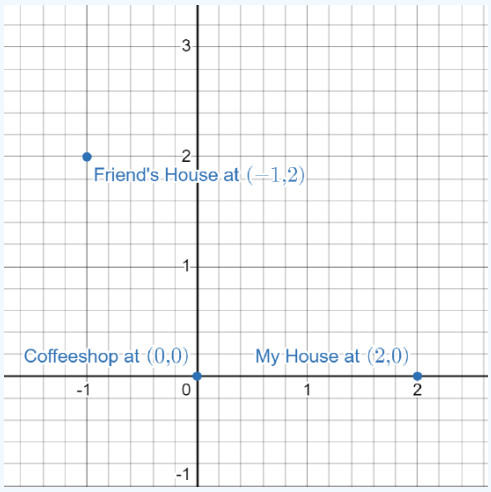
Example 2

My Midwestern hometown is laid out on a grid, as is typical. The streets run north-south and east-west, so you can see my city from a bird's-eye-view as a coordinate plane. I live two blocks east of a coffeeshop, and my friend lives one block west and two blocks north of the same coffeeshop. Find the distance as the crow flies:

1. between my house and the coffeeshop
2. between my friend's house and the coffeeshop
3. between my house and my friend's house

Solution

It's a good idea to sketch a picture. I will place the coffeeshop at the origin $(0,0)$ because I'm given "the least" information about it in the problem. In general, if locations are given *relative* to another location, you should put the "reference location" at the origin. My house is two blocks (two steps) to the east (to the right) of the origin. My friend's house is one step west (to the *left*) of the origin and two steps north (up).



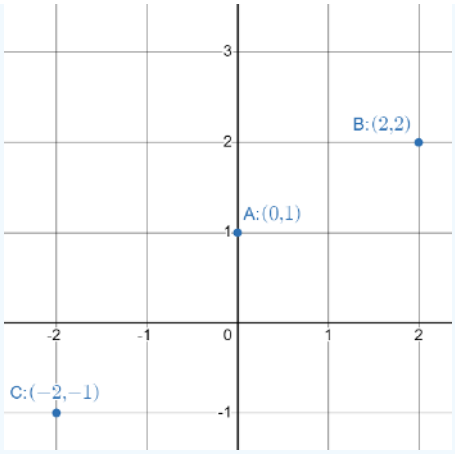
Having identified each location's coordinates, we can just plug into the distance formula. My units are understood to be "one block's length."

1. $d((2,0),(0,0)) = \sqrt{(2-0)^2+(0-0)^2} = \sqrt{2^2+0} = 2$
2. $d((-1,2),(0,0)) = \sqrt{(-1-0)^2+(2-0)^2} = \sqrt{(-1)^2+2^2} = \sqrt{1+4} = \sqrt{5}$
3. $d((2,0),(-1,2)) = \sqrt{(-1-2)^2+(2-0)^2} = \sqrt{(-3)^2+2^2} = \sqrt{9+4} = \sqrt{13}$

Exercise 2

Given the graphed points, find the distance

1. $d(A,B)$
2. $d(B,A)$
3. $d(A,C)$
4. $d(B,C)$
5. $d((0,0),B)$
6. $d(A,A)$

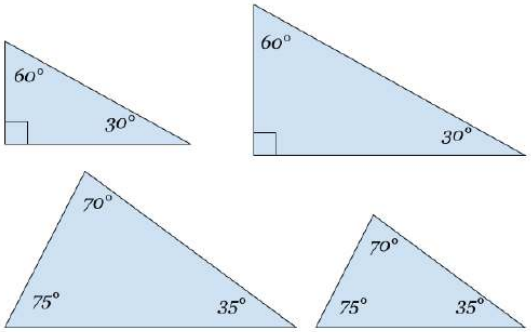


Answer

1. $\sqrt{5}$
2. $\sqrt{5}$ (The distance between two points is the same, in any order, bc it's a positive measured quantity.)
3. $\sqrt{8}$
4. 5
5. $\sqrt{8}$
6. 0 (The distance from a point to itself is of course always 0.)

Similar Triangles

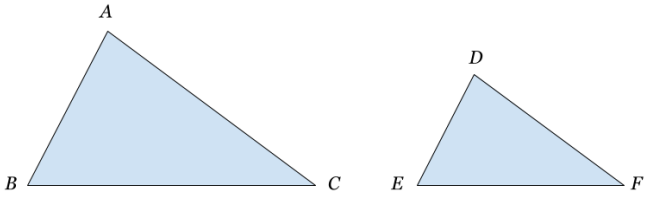
A geometric concept that comes up a lotttt in setting up Calc I problems, and which unfortunately many students forget about, is the proportionality of similar triangles. **Similar triangles** are triangles that have exactly all the same interior angles. You will notice that similar triangles look "like the same shape" but scaled up or down.



In math symbols, you represent a triangle with vertices at points A, B, and C as "ΔABC" and if two triangles are similar, we write

$$\Delta ABC \sim \Delta DEF$$

making sure to "line up" the corresponding parts when we write the names. See the diagram below.



Similar Triangles

The **corresponding sides** of similar triangles are **proportional**. So if $\Delta ABC \sim \Delta DEF$, we have

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

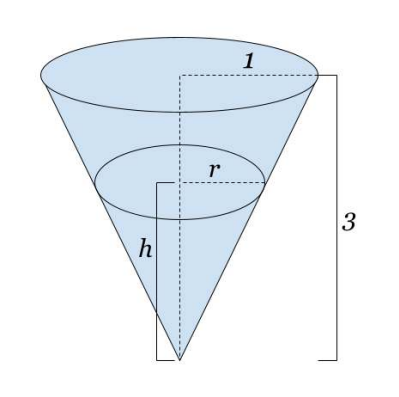
Any legal manipulation of these proportions is also true, of course. For example, "$\frac{AB}{DE} = \frac{BC}{EF}$" is saying, "The ratio of the (length of) big triangle's side AB to the small triangle's corresponding side DE is the same as the ratio of the big triangle's side BC to the small triangle's corresponding side EF." But of course, by algebraic manipulation, we have

$$\frac{AB}{DE} = \frac{BC}{EF} \quad \rightarrow \quad DE \cdot \frac{1}{BC} \cdot \frac{AB}{DE} = \frac{BC}{EF} \cdot \frac{1}{BC} \cdot DE \quad \rightarrow \quad \frac{AB}{BC} = \frac{DE}{EF}$$

The English translation of *this* proportion is, "The ratio of two sides of the big triangle is the same as the ratio of the corresponding two sides of the small triangle." It all boils down to the same math fact, but different forms might be more natural to use in different scenarios. The example below is like the number one way I use similar triangles in Calc I.

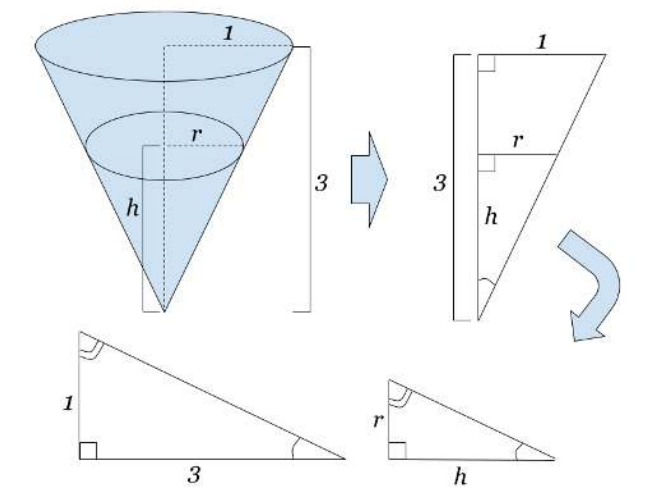
Example 3

I have a conical tank holding a certain amount of water, shown below with dimensions. Find r in terms of h.



Solution

The area of the water's surface will be different depending on the depth of the water in the tank, right? If h is smaller (lower water level), then r is smaller, and if h is bigger (higher water level), then r will be bigger. We just need to find a way to write down that relationship. The key is to recognize that we can slice out a pair of triangles from this cone:



https://math.libretexts.org/@go/page/156921

The large and small triangles are similar because they have all the same angles. How can you tell? They share their smallest angle, and they both have a right angle. Since the interior angles of triangles must sum to 180° total, their third angles must also be equal. Since they are similar triangles, the corresponding sides can be written in proportion(a variety of ways):

$$\frac{1}{3} = \frac{r}{h} \quad \text{or} \quad \frac{r}{1} = \frac{h}{3} \quad \text{etc.}$$

The second option there is a good one to use, because I want to express r in terms of h, and it's basically there. We write $r = \frac{h}{3}$ or $r = \frac{1}{3}h$.

I'll let you practice doing the same type of problem in the exercises section. For now, I want to just connect these problems back to the idea of **functions.** In the scenario above, we thought of h as if it were an independent variable. Like the water level can vary, maybe by draining or pumping water into the tank. Then we think of h as dictating what r will be. So in a sense, we are thinking of r as depending on h, an output determined by an input h, or radius "$r(h)$" being a **function of** h.

In the same way, you will often think of quantities like area and volume as functions of some dimension. The area of a circle depends on the size of its radius, so we can think of area, $A(r) = \pi r^2$, as a function taking input r. In the scenario above, the volume of the cone of water inside the tank can be calculated using the formula $V = \frac{1}{3}\pi r^2 h$, as we mentioned in the last section. But the example gave us the relationship $r = \frac{h}{3}$ as well! We can replace r with what he's equal to, getting

$$V = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h = \frac{1}{27}\pi h^3 \quad \rightarrow \quad V(h) = \frac{1}{27}\pi h^3$$

This allows us to think of the volume of the water as a function of h, the height of the water. This concept comes up all the time in optimization problems down the line in calculus, so I wanted to familiarize you with it now.

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8.2E Exercises

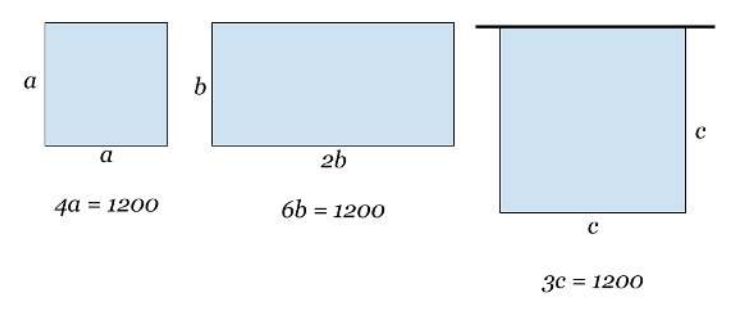
Fences

Cretaceous Park has 1200m of fencing material to build a paddock for their velociraptors (velociraptors are from the late Cretaceous period, not the Jurassic).

1. If they want to make a square paddock, what should its dimensions be? What would be the resulting area inside the fencing?
2. If they want to make a rectangular paddock whose long side is twice the short side, what should the dimensions be? What would be the area in this case?
3. Velociraptors are scared of fast water (I made this up, don't fact-check me), so the park can use a straight-running river on the property as the "fourth wall" of the paddock, thus only using fencing for three sides. If they want a square paddock again, still using up all of the fencing material, what should the dimensions be? What would be the area?

Answer

1. 300m by 300m. Area will be 90,000 square meters.
2. 200m by 400m. Area will be 80,000 square meters.
3. 400m by 400m. Area will be 160,000 square meters.



Note: You can use a calculator on the rest of these problems.

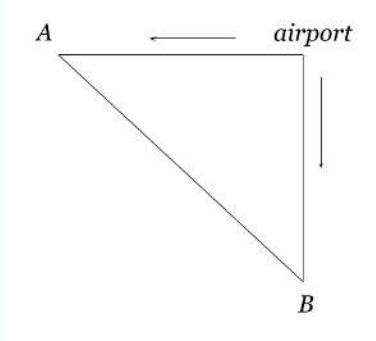
Airplanes

Plane A takes off from an airport and flies due west at a speed of 550mph. At the same time, Plane B leaves the same airport and flies due south at a speed of 570mph.

1. After 2 hours, how far has Plane A traveled from the airport? How far as Plane B traveled?
2. After 2 hours, how far apart (as the crow flies) are Plane A and Plane B?
3. After a certain amount of time, Plane A and Plane B are 990mi apart, and Plane A is 687.5mi west of the airport. How far is Plane B from the airport?

Answer

1. Plane A has gone 1100mi. Plane B has gone 1140mi.
2. The distance between them is about 1584mi.
3. This time the hypotenuse and one leg of the triangle are given. The length of the other leg is about 712mi.



Expanding Railroads

Steel rails expand when heated, so railroad tracks have "expansion gaps" built in to prevent buckling as the rails lengthen. Suppose this was not the case... Say a continuous length of railroad track is 1 mile long, fixed in place at both ends. On a hot day, the metal expands and the rails lengthen by 2 feet. The track buckles upward, forming an isosceles triangle. What is the height of the middle point above the ground? Take a moment to think intuitively and make a guess...you may be surprised by the final answer!

Answer

Hints: A mile is 5280ft. Draw a sketch of the situation. Eventually, you should use the Pythagorean Theorem in some way...

Crossing a River

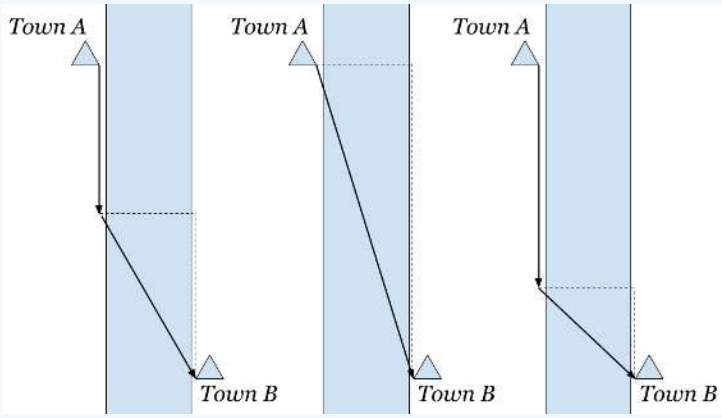
Town A is on the bank of a 40-meter-wide river, and Town B is on the other bank, 1 kilometer downstream. I start in Town A and run 500 meters down the bank. Then I jump into the river and swim directly towards Town B, diagonally across the water.

1. How far do I have to swim? How much distance do I cover total, both running and swimming?
2. If I were to jump directly in the river at Town A and swim diagonally through the river to Town B, how far would I swim?
3. If I were to run 700m first and then start swimming, how far would I swim? What is the total distance traveling from Town A to Town B?

Answer

1. The swim is about 501.6 m. The total distance is then 1001.6 m.
2. The swim is about 1000.8 m, and that's the full distance.
3. The swim is about 302.7 m. The total distance covered is 1002.7 m.

Figure is not drawn to scale!



Building an Aquarium

I'm designing a tank for my axolotl and I will have to order pieces of plate glass to assemble it out of. The plate glass will cost me $25 per square foot. I will use a different material to make the lid of my aquarium, so I am going to build an open glass box.

1. If I want the tank to be 2ft wide, 3ft long, and 2ft tall, how much will the glass cost me?
2. If I want to make a cubical tank and can't afford more than $600 worth of glass, what's the biggest size tank I could make (find the max side length)?

Answer

1. It will cost me $650.
2. The side length should be about 2.2 ft.

Planning a Pool

I have a rectangular space in my yard that is 20ft by 30ft. I hire Drew and Matt to build me a circular pool in this space.

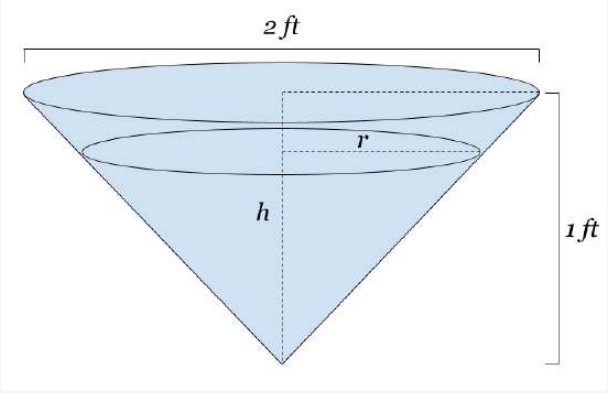
1. What is the largest radius we can use for the circular pool without going outside the space?
2. What would be the area of the resulting pool?
3. We're going to put fancy mosaic tile all the way around the rim of the pool. How many feet of tile border will we need?
4. If I fill the pool to a depth of 5ft, what will be the volume of water in the pool?
5. How much tile will we need to tile the whole wall around the 5ft-deep pool, as well as the floor?

Answer

1. 10ft
2. 100π sq.ft.
3. 20π ft
4. 500π cu.ft.
5. 200π sq.ft.

Conical Tank

Express the radius of the circular water surface, r, in terms of the height of the water level, h. If the height increases by 0.5ft, how will the radius r change? Express the volume of water inside the tank as a function of h.



Answer

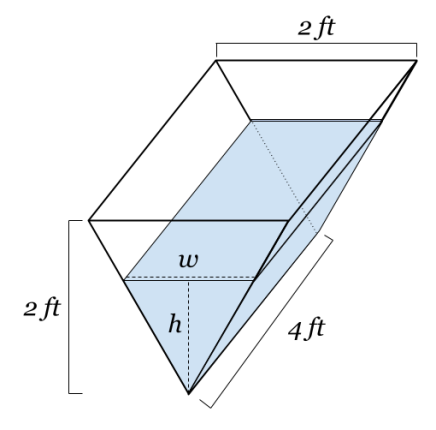
Using similar triangles, we have $\frac{r}{1}=\frac{h}{1}$... oh wait, $r=h$ here. Did you recognize that the triangles are 45-45-90? The leg lengths are equal. If height changes by any amount, then since $r=h$, the radius will change by the same amount!

The volume of a cone is $V=\frac{1}{3}\pi r^2 h$, and we use the fact that $r=h$ in this case to express $V(h)=\frac{1}{3}\pi h^3$ as a function of h alone.

Water Trough

A trough, shown below, is filled partway with water.

1. Find the maximum volume capacity of the trough.
2. Express the dimension w in terms of the water level height h.
3. Write the volume of the water as a function of h alone.



Answer

1. 8 cubic feet.
2. Using similar triangles, $\frac{w}{2} = \frac{h}{2}$ so $w = h$.
3. $V(h) = 2wh = 2h^2$.

Distance Formula

Find the distance between the points.

1. $(1, 2)$ and $(3, 4)$
2. $(-1, -1)$ and $(0, 5)$
3. $(2, 7)$ and $(0, 0)$
4. $(10, 2)$ and $(-4, 5)$

Answer

1. $\sqrt{8} \approx 2.83$
2. $\sqrt{37} \approx 6.08$
3. $\sqrt{53} \approx 7.28$
4. $\sqrt{205} \approx 14.32$

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8.3 Trigonometry via Triangles

Learning Objectives

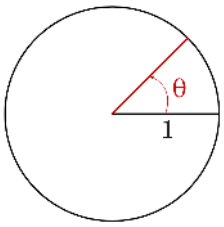
By the end of this section, you will be able to:

- understand angle measure in degrees and radians
- identify and find the trigonometric ratios sine, cosine, tangent, secant, cosecant, and cotangent
- use a given trigonometric ratio to find the other ratios at the same angle

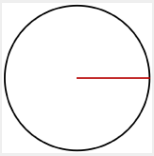
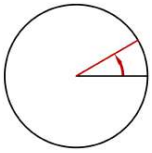
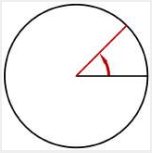
The main way that I want you to think about trig functions is going to be using the **unit circle** in the next section, spoiler alert. However, rather than doggedly memorizing the unit circle, I personally use the special triangles we discussed in Section 8.1, together with the trig ratios we will discuss in this section, to deduce all the information I need about the unit circle from scratch on the fly. Plus, in applications, you will definitely need to remember all this triangle stuff for geometric-style calculus problems, so we'll start here! First, a quick recap of how to measure angles.

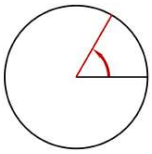
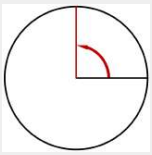
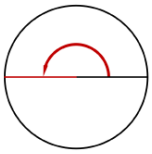
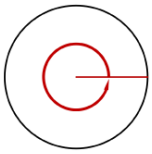
Geometry of Angle Measure

"Did a full one-eighty...Crazy..." You already know that turning around to face the exact opposite direction is a 180° turn. Spinning in a full circle to face the exact same way you started is a 360°. What we're describing with these numbers is the **measure of an angle.** We can portray angles using a circle with radius one, by starting with a standard horizontal "starting line" and swinging the angle in a **counterclockwise** direction:

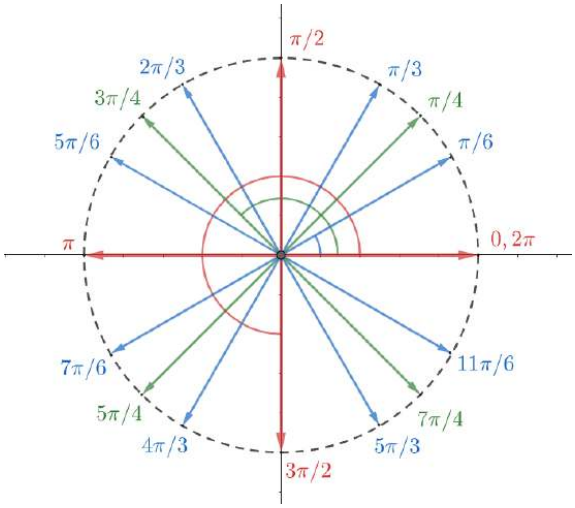


Angles have two options for their units: **degrees (°)** and **radians (rad).** You might find degrees more natural because high school loves them (God knows why, they're terrible???) but you will pretty much always use radians in future calculus courses (I promise it's better for you, like eating broccoli). These are the important angles you should have on instant recall:

Angle in Degrees	Angle in Radians	Angle on Circle
0°	0	
30°	$\frac{\pi}{6}$	
45°	$\frac{\pi}{4}$	

60°	$\frac{\pi}{3}$	
90°	$\frac{\pi}{2}$	
180°	π	
270°	$\frac{3\pi}{2}$	
360°	2π	

Here are all the most commonly used angles on a single unit circle (remember we measure counterclockwise from zero).

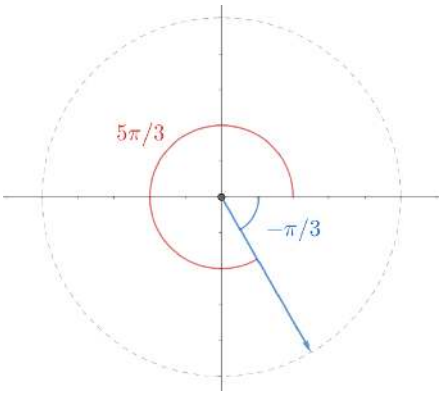


It's really best to just memorize these angles, by imagining you're slicing this circle into pizza pieces. You can see that if you cut the top half of the pizza into thirds, the cuts line up with the angles "a third of π" and "two thirds of π," right? For other, weirder angles, if you ever find yourself needing to convert back and forth between degrees and radians, the key is to recognize that $180^\circ = \pi$ radians. That means you can do a unit conversion with the factors $\frac{180}{\pi}$ or $\frac{\pi}{180}$.

To **convert from degrees to radians,** multiply the angle by $\frac{\pi}{180}$ and simplify.

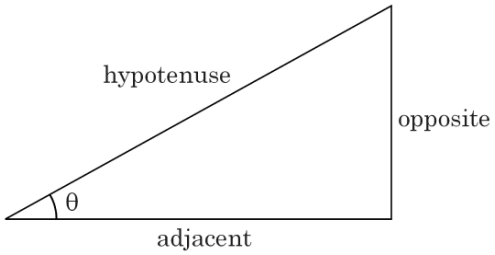
To **convert from radians to degrees,** multiply the angle by $\frac{180}{\pi}$ and simplify.

One last thing: You can also talk about **negative angles**, which are measured in a **clockwise direction.** For example, the angle $-\frac{\pi}{3}$ takes you to the same location as $\frac{5\pi}{3}$. You will practice working with negative angles (and angles bigger than 360°) in the exercises section.



Trigonometric Ratios

Consider a right triangle with a certain acute angle θ. The **trigonometric ratios** are certain ratios of the sides of the triangle relative to the chosen angle θ.



The Trigonometric Ratios

The trigonometric ratios **sine, cosine, tangent, cosecant, secant,** and **cotangent** are defined as

$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp} \qquad \tan\theta = \frac{opp}{adj}$$

$$\csc\theta = \frac{hyp}{opp} \qquad \sec\theta = \frac{hyp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$

Note: the ratios in the second row are simply the **reciprocals** of the ratios in the first row:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Also Note: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$! This is very important. Confirm by dividing the side ratios of sine and cosine to see that they turn out to be $\frac{opp}{adj}$, etc.

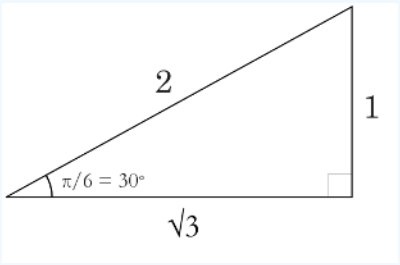
You can memorize these ratios with the "SOH-CAH-TOA" mnemonic, which stands for "Sine:Opposite/Hypotenuse, Cosine:Adjacent/Hypotenuse, Tangent:Opposite/Adjacent." The following example demonstrates how to find the values of all these ratios for a particular angle.

Example 1

Find the trigonometric ratios for $\theta = \frac{\pi}{6}$.

Solution

First, let's sketch the special right triangle that features the angle $\frac{\pi}{6} = 30^\circ$, a 30-60-90 triangle. We already know the standard side lengths for this triangle, too.



We identify the opposite side to be 1, the adjacent side to be $\sqrt{3}$, and the hypotenuse to be 2. We can now compute the main trig ratios:

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

We can use these ratios to find $\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, and $\cot\theta = \frac{1}{\tan\theta}$:

$$\csc\left(\frac{\pi}{6}\right) = 2 \qquad \sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}} \qquad \cot\left(\frac{\pi}{6}\right) = \sqrt{3}$$

? Exercise 1

Find the trigonometric ratios for $\theta = \frac{\pi}{4}$

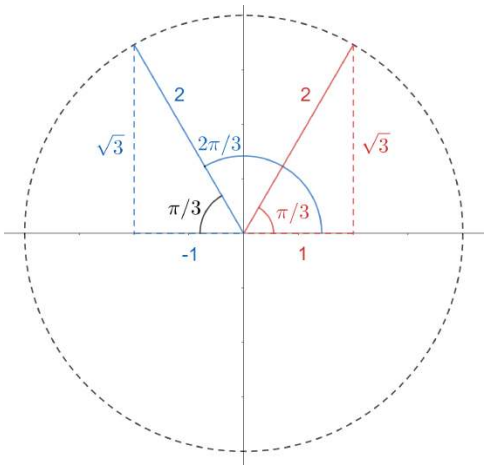
Answer

Hint: $\frac{\pi}{4} = 45^\circ$ so you can use a special triangle.

$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \qquad \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \qquad \tan\left(\frac{\pi}{4}\right) = 1$$

$$\csc\left(\frac{\pi}{4}\right) = \sqrt{2} \qquad \sec\left(\frac{\pi}{4}\right) = \sqrt{2} \qquad \cot\left(\frac{\pi}{4}\right) = 1$$

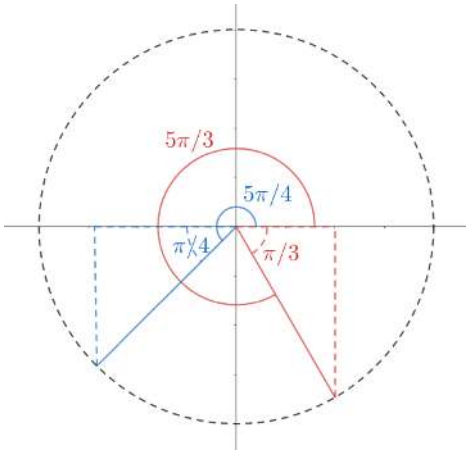
This is great and all for angles less than $\frac{\pi}{2}$ (90°), but what about the rest of the unit circle? Let's try sketching our triangles on the plane instead...



I've drawn radii on that circle showing two different angles. Consider the red angle $\frac{\pi}{3}$. That's 60°, so I drop in some dotted lines to create a 30-60-90 right triangle. I also label the side lengths that I know because it's a special triangle. Now I'm ready to compute, say, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. What about that blue angle, $\frac{2\pi}{3}$? Here's the key: drop a vertical line from the endpoint of your radius to the x-axis. In this case, that's straight downward. Sketch in your triangle and determine the **reference angle** *inside* that triangle. The reference

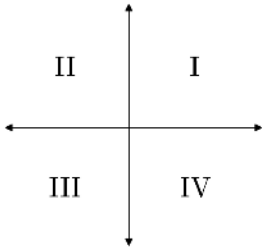
angle for $\frac{2\pi}{3}$ is $\frac{\pi}{3}$. Now, I label the sides of my special triangle again, **but** because my adjacent leg is on the **negative** side of the x-axis, I label that -1.

At this point, you can use the blue triangle to find any trig ratios you need. For example, $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos\frac{2\pi}{3} = \frac{-1}{2}$. Here are some more examples of reference angles and triangles for different locations on a circle:



Extrapolating Trig Ratios

The last thing I want to do in this section is make sure you can reverse-engineer trig ratios from a single given one. You will be given a value like $\sin\theta = \frac{a}{b}$ and information about which **quadrant** the angle lies in. The quadrants of the plane are shown below:



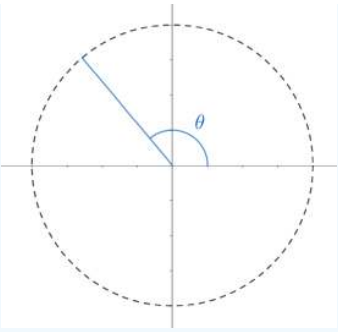
1. Sketch a right triangle in the plane in the appropriate quadrant and label θ. Label two of the sides a and b, according to the trig ratio given.
2. Use the Pythagorean Theorem to find the third side of the triangle.
3. Use the triangle to find the other trig ratios as usual.

Example 2

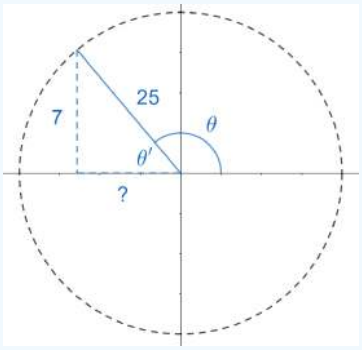
Find $\cos\theta, \tan\theta, \csc\theta, \sec\theta,$ and $\cot\theta$, given that $\sin\theta = \frac{7}{25}$ and θ is in Quadrant II.

Solution

It doesn't matter if our drawing is to scale or correctly proportioned, we just gotta get in the ballpark. I drop a potential angle θ into Quadrant II:



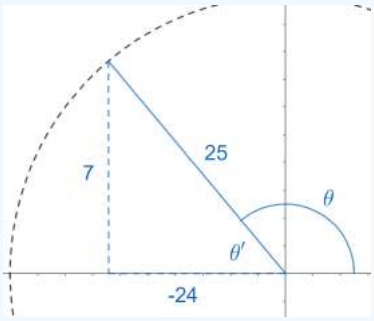
I sketch a right triangle and label the known sides, given that $\sin\theta = \frac{opp}{hyp} = \frac{7}{25}$. The angle with respect to which you determine "opposite" vs "adjacent" in this case is the reference angle, which I've labeled θ'. I don't know anything about the adjacent side yet, because sine doesn't use him!



To figure out that ? adjacent side, we can use the Pythagorean Theorem.

$$a^2 + b^2 = c^2 \quad \rightarrow \quad ?^2 + 7^2 = 25^2 \quad \rightarrow \quad ? = \sqrt{576} = 24$$

Since this triangle is on the negative side of the x-axis, I label the unknown side with -24. Now I know all three sides of this triangle and I just hone in on that guy to find all the trig ratios.



$$\sin\theta = \frac{7}{25} \qquad \cos\theta = -\frac{24}{25} \qquad \tan\theta = -\frac{7}{24}$$

$$\csc\theta = \frac{25}{7} \qquad \sec\theta = -\frac{25}{24} \qquad \cot\theta = -\frac{24}{7}$$

My picture is SUPER not to scale, but it doesn't matter, because the numbers themselves will never steer you wrong!

? Exercise 2

Find all trig ratios for θ, using the given and the quadrant.

1. $\tan\theta = \sqrt{3}$ in Quadrant I. (Hint: think of $\sqrt{3}$ as $\frac{\sqrt{3}}{1}$.)
2. $\cos\theta = -\frac{1}{\sqrt{2}}$ in Quadrant III.

Answer

1. $\sin\theta = \frac{\sqrt{3}}{2}, \cos\theta = \frac{1}{2}, \tan\theta = \sqrt{3}, \csc\theta = \frac{2}{\sqrt{3}}, \sec\theta = 2, \cot\theta = \frac{1}{\sqrt{3}}$
2. $\sin\theta = -\frac{1}{\sqrt{2}}, \cos\theta = -\frac{1}{\sqrt{2}}, \tan\theta = 1, \csc\theta = -\sqrt{2}, \sec\theta = -\sqrt{2}, \cot\theta = 1$

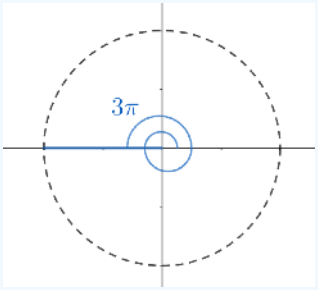
Be sure to check out the exercises section, because you will further explore things like **coterminal angles** and reference angles in there!

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8.3E Exercises

Angles Greater Than 2π

We mostly saw angles between 0 and 2π, a full circle, in this section. However, it's possible to make sense of an angle greater than 2π, by continuing to loop around the origin to achieve more than 360° of rotation. For example, an angle of 3π will make a full circle (2π's worth) and then continue another π radians to arrive at the location shown below.



Notice that this location also matches the angle of just π... We say that 3π and π are **coterminal angles.** Given an angle greater than 2π, you can find the coterminal angle between 0 and 2π by subtracting off copies of 2π as many times as possible without going negative. In this example, $3\pi - 2\pi = \pi$ gives me my coterminal angle. The same can be done for angles greater than 360°, by subtracting off copies of 360° until you get a coterminal angle smaller than 360°. Find the coterminal angles in this preferred range for the angles below.

1. 5π
2. 720°
3. $\frac{7\pi}{6}$
4. 370°
5. $\frac{9\pi}{4}$
6. $\frac{13\pi}{3}$

Answer

1. π
2. 0°
3. $\frac{7\pi}{6}$ (this angle is already between 0 and 2π)
4. 10°
5. $\frac{\pi}{4}$
6. $\frac{\pi}{3}$

Negative Angles

Find the positive angle between 0 and 2π (or between 0 and 360°) that is coterminal with the negative angles below. It can often be helpful to draw a picture! Or, you can convert, for degrees for example, by subtracting: $360-$ (size of the negative angle) $=$ (positive coterminal angle).

1. -45°
2. -180°
3. $-\frac{\pi}{6}$
4. $-\frac{3\pi}{2}$
5. -90°
6. -20°
7. $-\frac{5\pi}{4}$
8. $-\frac{11\pi}{6}$

Answer

1. 315°
2. 180°
3. $\frac{11\pi}{6}$
4. $\frac{\pi}{2}$
5. 270°
6. 340°
7. $\frac{3\pi}{4}$
8. $\frac{\pi}{6}$

Converting Between Degrees and Radians

Convert from degrees to radians or vice versa. You can use a calculator, but most of these you should be able to eyeball. Try sketching them on a unit circle!

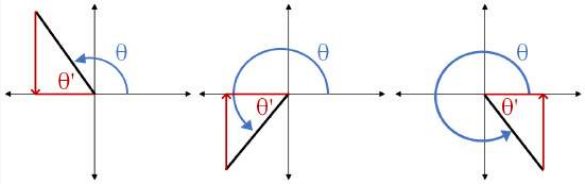
1. 180°
2. 0 rad
3. 30°
4. 45°
5. $\frac{\pi}{3}$
6. $\frac{3\pi}{2}$
7. $\frac{4\pi}{3}$
8. 225°

Answer

1. π
2. 0°
3. $\frac{\pi}{6}$
4. $\frac{\pi}{4}$
5. 60°
6. 270°
7. 240°
8. $\frac{5\pi}{4}$

Reference Angles

Recall that to find the reference angle of θ, you take the most direct route from your angle's location straight to the x-axis. The reference angle θ' is the interior angle of the resulting right triangle. For example:



Find the reference angle for each angle below.

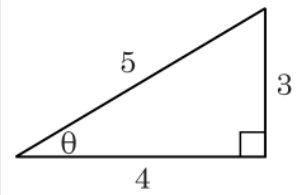
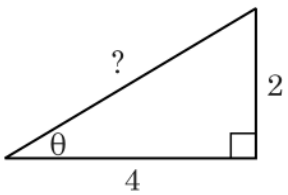
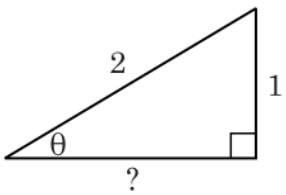
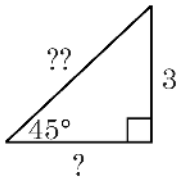
1. $\frac{5\pi}{4}$
2. 225°
3. $\frac{11\pi}{6}$
4. $\frac{\pi}{3}$
5. 45°
6. $\frac{5\pi}{3}$
7. 120°
8. $\frac{7\pi}{6}$

Answer

1. $\frac{\pi}{4}$
2. 45°
3. $\frac{\pi}{6}$
4. $\frac{\pi}{3}$ (in the first quadrant, the reference angle just *is* the angle)
5. 45°
6. $\frac{\pi}{3}$
7. 60°
8. $\frac{\pi}{6}$

? Finding Trig Ratios From Triangles

Find $\sin\theta, \cos\theta$, and $\tan\theta$ using the triangles (determine any missing side lengths first). Note: figures are not drawn to scale!

			
1.	2.	3.	4.

Answer

1. $\sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}, \tan\theta = \frac{3}{4}$
2. $\sin\theta = \frac{1}{\sqrt{5}}, \cos\theta = \frac{2}{\sqrt{5}}, \tan\theta = \frac{1}{2}$
3. $\sin\theta = \frac{1}{2}, \cos\theta = \frac{\sqrt{3}}{2}, \tan\theta = \frac{1}{\sqrt{3}}$
4. $\sin(45^\circ) = \frac{1}{\sqrt{2}}, \cos(45^\circ) = \frac{1}{\sqrt{2}}, \tan(45^\circ) = 1$

? Finding Trig Ratios of Larger Angles

Find $\sin\theta, \cos\theta$, and $\tan\theta$ for the given angles. You will probably need to sketch them on a plane to find the reference angles. Watch out for when triangles' sides should be labeled as negative!

1. $\frac{2\pi}{3}$
2. $\frac{5\pi}{4}$
3. $-\frac{\pi}{6}$
4. $-\frac{5\pi}{6}$
5. $\frac{3\pi}{4}$
6. $\frac{5\pi}{3}$

Answer

1. $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}, \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}, \tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$
2. $\sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}, \cos\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}, \tan\left(\frac{5\pi}{4}\right) = 1$
3. $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}, \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$
4. $\sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{2}, \cos\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}, \tan\left(-\frac{5\pi}{6}\right) = \frac{1}{\sqrt{3}}$
5. $\sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}, \cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}, \tan\left(\frac{3\pi}{4}\right) = -1$
6. $\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}, \cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}, \tan\left(\frac{5\pi}{3}\right) = -\sqrt{3}$

? Finding Trig Ratios From Each Other

From the given trig ratio and quadrant, find all six trig ratios for that angle.

1. $\sin\theta = \frac{1}{\sqrt{2}}$, Quadrant I
2. $\cos\theta = \frac{\sqrt{3}}{2}$, Quadrant IV
3. $\tan\theta = 1$, Quadrant III
4. $\sin\theta = \frac{1}{2}$, Quadrant II

Answer

1. $\sin\theta = \frac{1}{\sqrt{2}}, \cos\theta = \frac{1}{\sqrt{2}}, \tan\theta = 1, \csc\theta = \sqrt{2}, \sec\theta = \sqrt{2}, \cot\theta = 1$
2. $\sin\theta = -\frac{1}{2}, \cos\theta = \frac{\sqrt{3}}{2}, \tan\theta = -\frac{1}{\sqrt{3}}, \csc\theta = -2, \sec\theta = \frac{2}{\sqrt{3}}, \cot\theta = -\sqrt{3}$
3. $\sin\theta = -\frac{1}{\sqrt{2}}, \cos\theta = -\frac{1}{\sqrt{2}}, \tan\theta = 1, \csc\theta = -\sqrt{2}, \sec\theta = -\sqrt{2}, \cot\theta = 1$
4. $\sin\theta = \frac{1}{2}, \cos\theta = -\frac{\sqrt{3}}{2}, \tan\theta = -\frac{1}{\sqrt{3}}, \csc\theta = 2, \sec\theta = -\frac{2}{\sqrt{3}}, \cot\theta = -\sqrt{3}$

? Quadrants

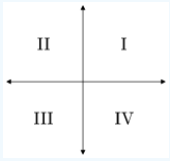
1. Sketch and label the quadrants of the plane.

2. Identify which quadrant each angle lies in: $\frac{\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{4}, \frac{5\pi}{3}$

3. Determine the range of angle values contained in each quadrant. For example, Quadrant I comprises angles between 0 and $\pi/2$ radians.

4. In each quadrant, certain trig ratios always have negative or positive values. For example, for θ in Quadrant III, $\sin\theta$ is always negative, $\cos\theta$ is always negative, and $\tan\theta$ is always positive. Give such facts for the other three quadrants.

Answer

1. 

2. I, II, III, and IV, respectively.

3. Quadrant I: 0 to $\pi/2$. Quadrant II: $\pi/2$ to π. Quadrant III: π to $3\pi/2$. Quadrant IV: $3\pi/2$ to 2π.

4. Quadrant I: sine, cosine, and tangent all positive. Quadrant II: sine positive, cosine and tangent negative. Quadrant III: sine and cosine negative, tangent positive. Quadrant IV: sine and tangent negative, cosine positive. You can remember this fact with the mnemonic "**A**ll **S**tudents **T**ake **C**alculus," which tells you in order, "All positive, Sine positive, Tangent positive, Cosine positive."

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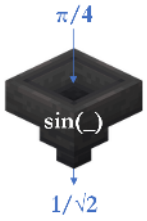
8.4 Trigonometric Functions

Learning Objectives

By the end of this section, you will be able to:

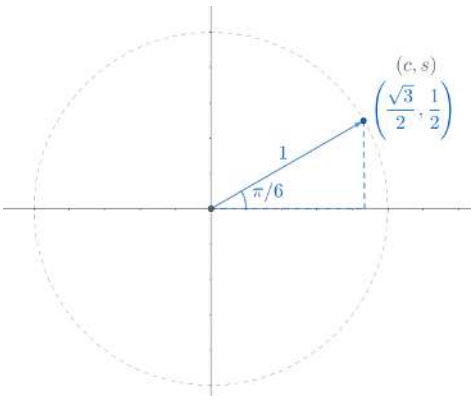
- view sine, cosine, tangent, cosecant, secant, and cotangent as functions of an input angle
- use the unit circle to find values for trigonometric functions at common angles
- recognize the graphs of sine, cosine, and tangent on sight, and sketch them yourself

In the previous section, we learned the trigonometric "ratios" defined using the sides of a right triangle, relative to a chosen acute angle. We also saw that trig ratios still make sense even when considering an angle that isn't acute, by drawing a two-dimensional plane and sketching reference triangles. At this point, it'll hopefully be one short step for man to transition to thinking of sine, cosine, and tangent (etc.) as **functions.** Because side ratios are proportional for similar triangles, it didn't actually matter what exactly the side lengths were in the triangles we were looking at; everything really boiled down to depending on the angle θ. If we consider θ to be the independent input variable, feed it into sine, for example, and see what "$\sin\theta$" spits out as an output, we're seeing sine as a function of θ.

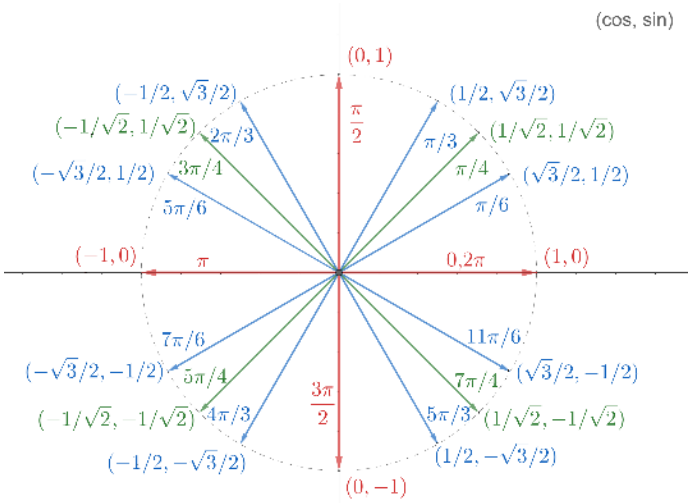


The Unit Circle

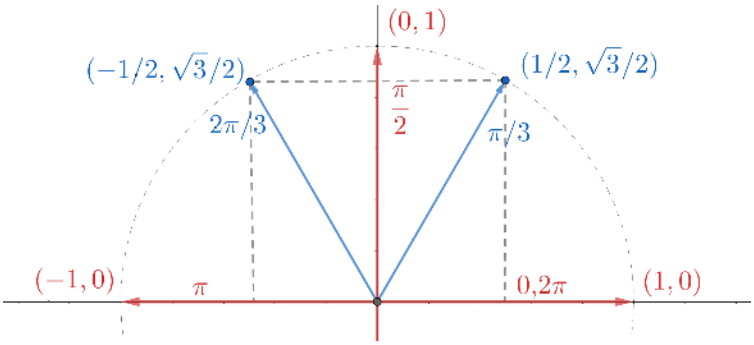
An excellent tool to have on instant recall is what's called "the" **unit circle.** The name "unit" indicates that the radius of the circle is 1 (thanks Latin! still undefeated) and the circle is drawn on a 2D plane, centered at the origin. If you draw a ray at a particular angle θ on that circle just like we did in the last section, you will pick out a particular point on the circle. The coordinates of that point **give you the values of cosine and sine for that angle!** Just repeat to yourself "cosine, sine" until you remember the order, because it matters! Here's the angle $\pi/6$ as an example: $\cos(\pi/6) = \frac{\sqrt{3}}{2}$ and $\sin(\pi/6) = \frac{1}{2}$, and as long as the radius is 1, those are the exact coordinates of that point on the circle.



If you know your unit circle, you are going to have a MUCH easier time in calculus courses (and beyond). Now, here's the whole thing, but I'm going to give you a trick so you don't have to like rote memorize every tiny detail separately.



When I myself took Calc I, approximately 187 years ago, my professor handed out a blank version of this to us every single day (yes we had class five days a week at 7:45am (uphill both ways my guy)) and made us fill it out from memory. Now, the trick is that all you really need to know is the first quadrant, and you can figure everything else out from there! Observe:



If I know the values at $\pi/3$, then I can extrapolate that information to find the coordinates for $2\pi/3$. The two points are the exact same height, so they have the same y-value. They're also symmetric across the y-axis, so their x-values are the same number but opposite sign! If you study the full circle carefully, you will see that using symmetry and common sense, as well as paying attention to which coordinates should be negative or positive according to the x- and y-axes, it's a piece of cake to figure out any location with logic! Work smarter, not harder. You only need to find the values that you need in the moment on a problem.

✔ Example 1

Use the unit circle to evaluate the trig functions:

1. $\sin\left(\frac{5\pi}{4}\right)$
2. $\cos(\pi)$
3. $\tan\left(\frac{11\pi}{6}\right)$
4. $\csc\left(\frac{5\pi}{4}\right)$

Solution

1. Look at the unit circle (or eventually, sketch it yourself) to see that the coordinates at $5\pi/4$ are $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. This makes sense because if I sketch in a reference triangle, it will be 45-45-90, and being in Quadrant III, I have both x- and y-coordinates negative. The y-coordinate is who tells us about sine, so my answer is $-\frac{1}{\sqrt{2}}$.
2. Looking at π, the coordinates are $(-1, 0)$, and the x-coordinate is who tells us about cosine, so the answer is -1.
3. "Wait a second," you say, "we've been scammed! There's nothing on the unit circle mentioning tangent!" It's okay, we just remember the fact that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and use the sine and cosine values to compute! Looking at $11\pi/6$, the sine value is $-1/2$ and the cosine value is $\sqrt{3}/2$. Dividing those (order matters!), we get the answer $-1/\sqrt{3}$.
4. Recall that $\csc\theta = \frac{1}{\sin\theta}$, so we can just take our answer from part 1 and flip it over: $-\sqrt{2}$.

? Exercise 1

Use the unit circle to evaluate the trig functions:

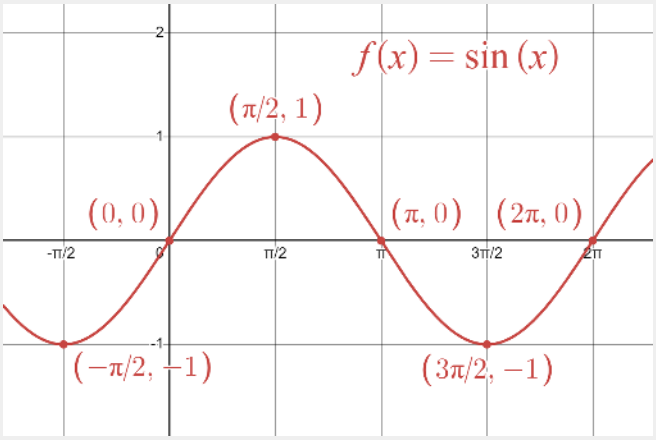
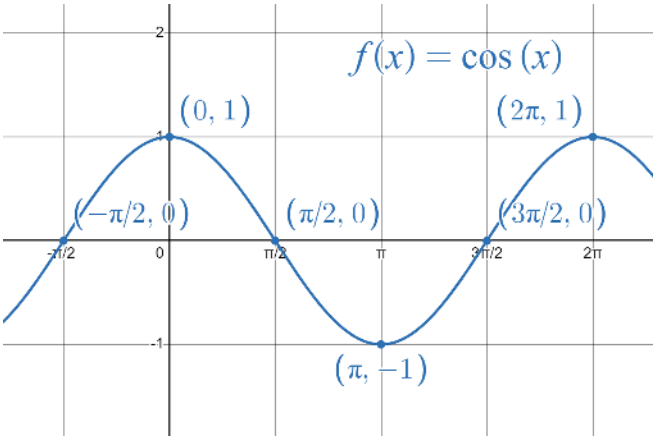
1. $\sin(0)$
2. $\cos\left(\frac{\pi}{4}\right)$
3. $\tan\left(\frac{5\pi}{4}\right)$
4. $\sec\left(\frac{\pi}{4}\right)$

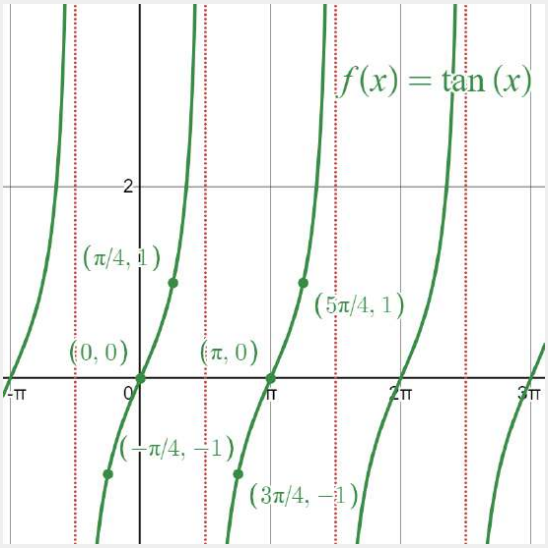
Answer

1. 0
2. $\frac{1}{\sqrt{2}}$
3. 1
4. $\sqrt{2}$

The Graphs of Trig Functions

I want to emphasize that the unit circle is a picture tool that we use to figure out the values of sine and cosine, but it is NOT somehow a graph of sine or cosine! Trig functions have graphs like any other function, where the independent variable (on the horizontal axis) is the input θ, and the output of the function gives you a height or y-value on the vertical axis. Here are the graphs of the Big Three.

Graph	Observations
	• **Periodic behavior:** The function starts at $(0, 0)$, does a hill, does a valley, and returns to the height of 0 by $x = 2\pi$...and then it repeats the same pattern. This is called being **periodic** with **period** 2π, because it takes from 0 to 2π to do one full cycle. • The points called out on the graph are the easiest ones to find on the unit circle, the cardinal points. • Sine passes through $(0, 0)$. That is, $\sin(0) = 0$. He is **not** symmetric about the y-axis. He **is** symmetric about the origin. • The function values are **bounded** between -1 and 1. Aka, the function never goes higher than 1 or lower than -1. • Sine is nice and smooth and continuous, meaning you can draw him without ever having to pick up your pencil. • The function is defined for negative values of x: these are just negative angles, which were discussed in the last section and its exercises. In fact, his **domain** is all real numbers.
	• **Periodic behavior:** Cosine also repeats his behavior, starting the cycle over every time you travel a distance of 2π. You can see he starts at a height of 1 at $x = 0$, performs a full down-and-up cycle, and comes back to a height of 1 at $x = 2\pi$. • The points labeled are again the easiest ones to find from the unit circle. You can compare them to confirm. • Cosine passes through $(0, 1)$. That is, $\cos(0) = 1$. He **is** symmetric about the y-axis. • His y-value heights are also trapped between -1 and 1. • Cosine is also nice and smooth and continuous, with **domain** all real numbers again.



- **Periodic behavior:** Tangent repeats himself, but his period is only π.
- **Vertical asymptotes:** Tangent is **undefined** at any x-value that is an odd multiple of $\pi/2$. This is because at those locations, $\cos x$ is zero, and you can't compute $\tan x = \frac{\sin x}{\cos x}$! You see vertical asymptotes at those locations, and tangent hugs them but never touches them. This means tangent is not continuous because you have to keep picking up your pencil to hop over those asymptotes!
- The locations where tangent is 1 are $\pi/4$ and $5\pi/4$. These are the "quarters" locations in Quadrants I and III, and these are the locations where sine and cosine have the same value. Thus when you divide them to find tangent, you get 1.
- Tangent passes through $(0, 0)$, and passes through the x-axis at every multiple of π. In math symbols, you can write "$\tan x = 0$ whenever $x = nx$ for some integer n."
- Due to those undefined values, the **domain** of tangent is all real numbers **except** odd multiples of $\pi/2$. You can write this in math as "$x \neq (2k-1)\frac{\pi}{2}$ for k any integer." Try some options for k to see that this expression will give you odd multiples like $-\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, etc.

To sketch these graphs yourself, you really only need a few factoids and a sprinkling of artistic talent, as long as you stare at those pictures until they're burned into your retinas so you have an idea of what trig functions *should* look like...

To **sketch the graph of sine or cosine,**

1. Sketch a baby unit circle in the margin and label the cardinal points $(1, 0), (0, 1), (-1, 0)$, and $(0, -1)$. These give you the values of sine and cosine at the inputs $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, respectively.
2. Draw the axes for your function's graph and mark those same angles on the horizontal axis. Then use your unit circle to plot the function's points at those values.
3. Use your knowledge of symmetry and the hilly shape of sine/cosine to smoothly connect the dots and follow the pattern to repeat periodically.

To **sketch the graph of tangent,**

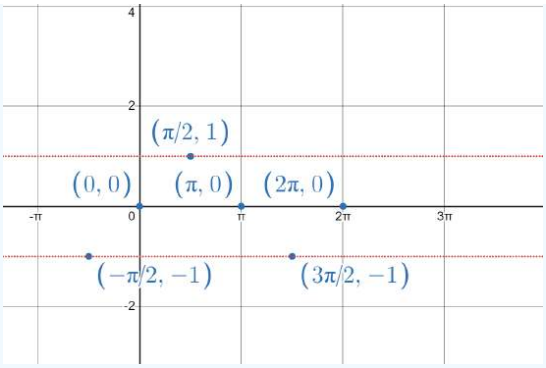
1. Sketch the x- and y-axes and mark the x-values $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \ldots$ etc.
2. Recall the locations where tangent is 0 (anywhere that sine is 0): at $\ldots, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$ etc. At those locations, the function crosses the x-axis, so plot those roots.
3. Recall the locations where tangent is 1: at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$, as discussed above. Plot those points, $(\pi/4, 1)$ and $5\pi/4$.
4. Recall the locations where tangent is -1: at $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$, etc. Plot those points too.
5. Sketch in the vertical asymptotes: dotted vertical lines passing through $\frac{\pi}{2}, \frac{3\pi}{2}$, etc. on the x-axis.
6. Use your knowledge of the shape of tangent to smoothly connect the dots and follow the asymptotes up and down. Repeat the behavior periodically.

✔ Example 2

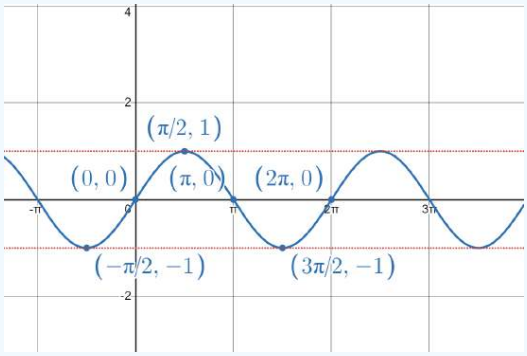
Sketch the graphs of $\sin(x)$ and $\tan(x)$.

Solution

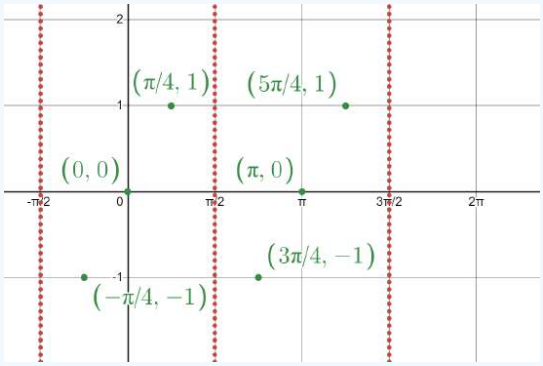
To graph $\sin x$, I use the unit circle to pick out the points $(x, \sin x)$ that are easiest: $(0, 0), (\pi/2, 1), (\pi, 0), (3\pi/2, -1), (2\pi, 0)$. I drop those in, and also note to myself that I'm not going to be going above $y = 1$ or below $y = -1$:



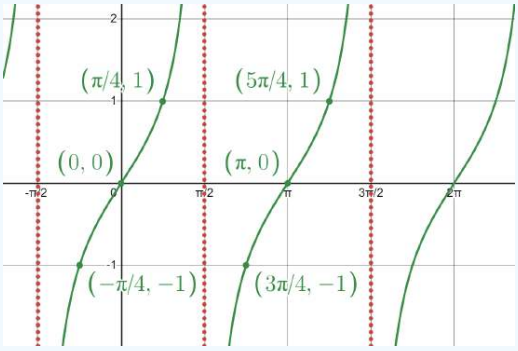
Now that I can see how my repeating wave pattern is going to look, I smoothly sketch it in:



To graph $\tan x$, I plot some signpost points $(0, 0)$, $(-\pi/4, -1)$ and $(\pi/4, 1)$, and maybe one more round of that pattern. I also drop in the vertical asymptotes between which I will draw my repeating lines.



Then, knowing that $\tan$ will go smoothly through these points and hug the vertical asymptotes, and repeat itself periodically, I sketch in the graph.



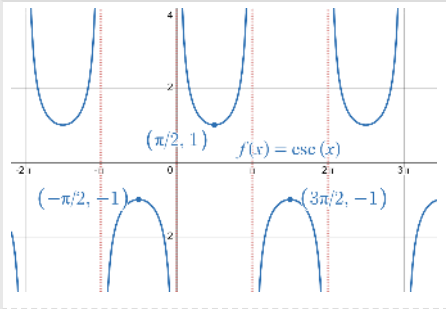
Exercise 2

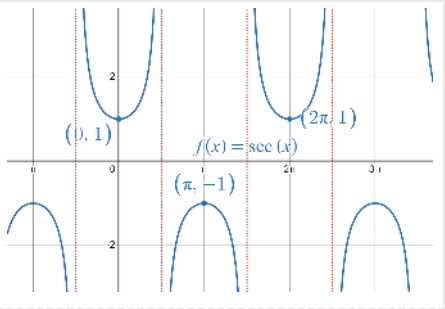
Sketch the graph of $\cos x$.

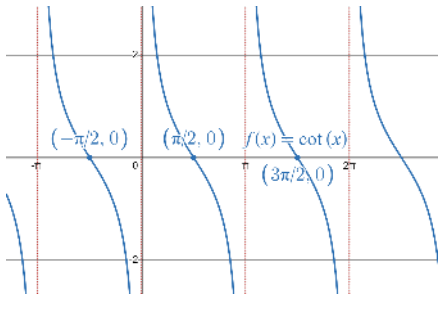
Answer

You can check your answer by looking back at the graph of $\cos x$ given in the table earlier.

There are of course graphs for the reciprocal trig functions cosecant, secant, and cotangent as well. They are...less intuitive and in my experience don't seem to come up all that much later on? Ask your professor if you need to have these memorized. For now, here's what they look like, just so you're aware. I'll guide you through some exploration of these in the exercises section!





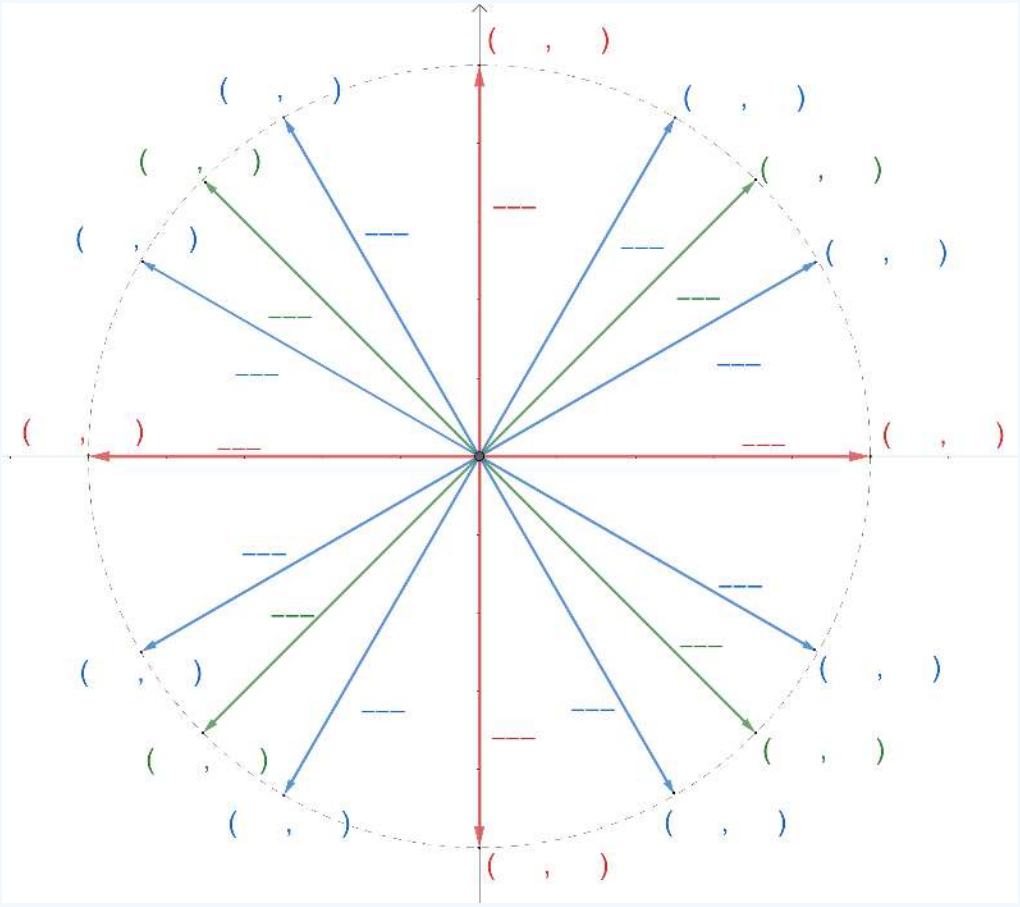


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8.4E Exercises

The Unit Circle

Fill in the blanks. (Try to do this from memory as much as possible.)



Answer

AFTER you try as much as possible, you can look back to the previous section for hints. Practice using Quadrant I to deduce the rest of the blanks!

Using the Unit Circle

Use the unit circle to evaluate the trig functions.

1. $\sin(0)$
2. $\cos\left(\frac{3\pi}{2}\right)$
3. $\tan(0)$
4. $\sin\left(\frac{\pi}{6}\right)$
5. $\cos\left(\frac{2\pi}{3}\right)$
6. $\tan\left(\frac{\pi}{4}\right)$
7. $\sec(\pi)$
8. $\csc\left(\frac{\pi}{4}\right)$
9. $\cot\left(\frac{5\pi}{6}\right)$

Answer

1. 0
2. 0
3. 0
4. $\frac{1}{2}$
5. $-\frac{1}{2}$
6. 1
7. -1
8. $\sqrt{2}$
9. $-\sqrt{3}$

Sketching Basic Trig Functions

From memory and using your knowledge of the unit circle, sketch the graphs of $\sin x, \cos x,$ and $\tan x$, labeling intercepts and any asymptotes.

Answer

You can check by comparing to the previous section.

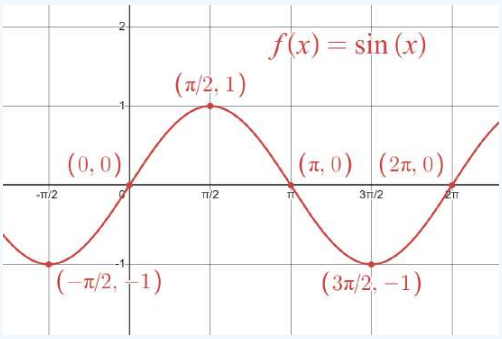
Exploring Cosecant

1. How is cosecant related to sine?

2. At which x values is $\sin x$ equal to 0? What can you say about cosecant at those locations?

3. At which x values is $\sin x$ equal to $\csc x$?

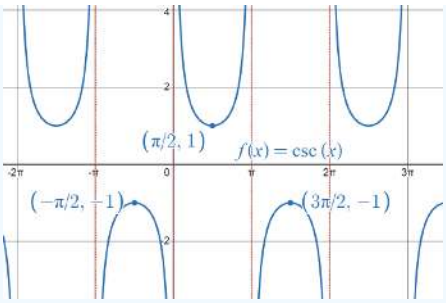
4. Look at the graph of sine below and imagine approaching $x = 0$ from the **right**. (Aka, your angle value x is positive but getting smaller.) What are the function values (y-values) doing as you approach? What do you expect the function values of cosecant to do as x approaches 0 from the right, as a result? Check your hypothesis by looking at the graph of $\csc x$ in the Answer.



5. Go to desmos.com and graph **both** sine and cosecant together on the same graph.

Answer

1. $\csc x = \frac{1}{\sin x}$
2. $\sin x = 0$ if $x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$ etc. Since we can't divide by 0, cosecant is **undefined** at these locations.
3. Whenever $\sin x = \pm 1$, $\csc x = \frac{1}{\pm 1} = \pm 1$. That happens at $x = -\pi/2, \pi/2, 3\pi/2, 5\pi/2, \ldots$ etc.
4. Coming into $x = 0$ from the right, the graph of sine is decreasing toward the origin. Aka the function values are positive but getting super tiny. If $\sin x = \frac{1}{1000}$, for example, $\csc x = 1000$, so when sine gets tiny, cosecant blows up. Indeed, we see that approaching $x = 0$ from the right, the graph below blows upward along the vertical asymptote! Similar reasoning explains the behavior near the other asymptotes as well.



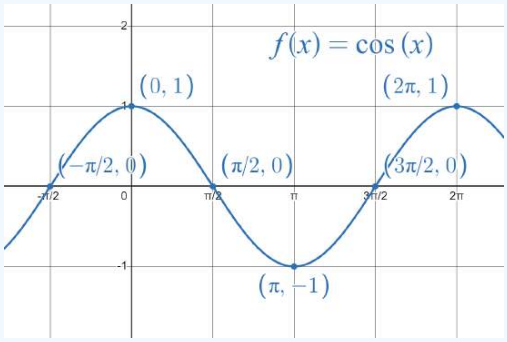
Exploring Secant

1. How is secant related to cosine?

2. At which x values is $\cos x$ equal to 0? What can you say about secant at those locations?

3. At which x values is $\cos x$ equal to $\sec x$?

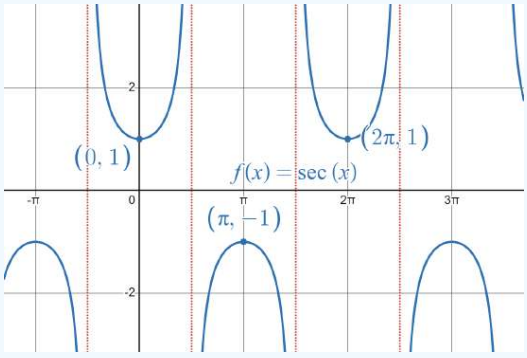
4. Look at the graph of cosine below and imagine approaching $x = \frac{\pi}{2}$ from the **left**. What are the function values (y-values) doing as you approach? What do you expect the function values of secant to do as x approaches $\frac{\pi}{2}$ from the left, as a result? Check your hypothesis by looking at the graph of $\sec x$ in the Answer.



5. Go to desmos.com and graph **both** cosine and secant together on the same graph.

Answer

1. $\sec x = \frac{1}{\cos x}$
2. $\cos x = 0$ when $x = \ldots -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ etc. Then secant is undefined at those inputs.
3. Anytime $\cos x = \pm 1$. This happens at $x = \ldots -\pi, 0, \pi, 2\pi, \ldots$ etc.
4. Coming in to $x = \pi/2$ from the left, the y-values cosine gives us are positive but getting smaller. We expect the values of secant to get larger. Indeed, in the graph we see secant blow up along the left side of the vertical asymptote $x = \pi/2$.



Exploring Cotangent

1. What is the relationship between cotangent and tangent? What is the relationship between cotangent and sine and cosine?

2. At what x-values is $\tan x$ equal to zero? What can you say about $\cot x$ at those locations?

3. Where is $\cot x$ equal to zero?

4. If $\tan x$ is a large positive number at some input value x, what do you expect about the value of $\cot x$ there? If $\cot x$ crosses the x-axis at an x-value, what do you know about tangent at that location? Hint: look back at your relationship between tan and cotan.

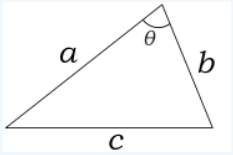
5. Go to desmos.com and graph **both** tangent and cotangent together on the same graph.

Answer

1. $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$
2. $\tan x = 0$ anytime $\sin x = 0$, which we've already seen is at $x = ..., -2\pi, -\pi, 0, \pi, 2\pi, ...$ etc. At those x-values, cotan is undefined.
3. $\cot x = 0$ anytime $\cos x = 0$, which we've already seen is at $x = ... -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...$ etc.
4. We expect $\cot x$ to be a very small (but positive) number like a teeny fraction. If $\cot x$ has a root at x, then $\tan x = \frac{1}{\cot x}$ will be undefined!

Law of Cosines

We're all familiar by now with the Pythagorean Theorem, but it only applies to right triangles. Turns out, there is a **generalization** of the Pythagorean Theorem called the Law of Cosines. For the triangle below,



the Law of Cosines says that

$$c^2 = a^2 + b^2 - 2ab\cos\theta,$$

where θ is the angle opposite side c. Being a "generalization" means that this is a broader overarching rule, of which the Pythagorean Theorem is supposed to be a **special case.** Explain why the Pythagorean Theorem can be derived as a special case of the Law of Cosines.

Answer

Hint: What happens if θ is a right angle? What do you know about the function cosine?

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8.5 More (and Inverse) Trig Functions

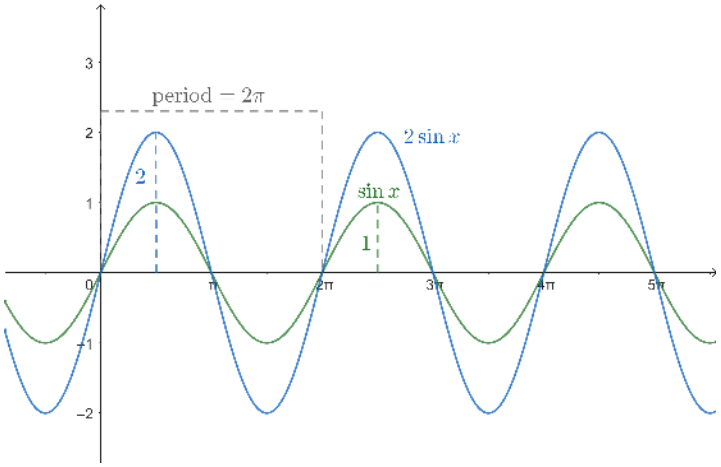
Learning Objectives

By the end of this section, you will be able to:

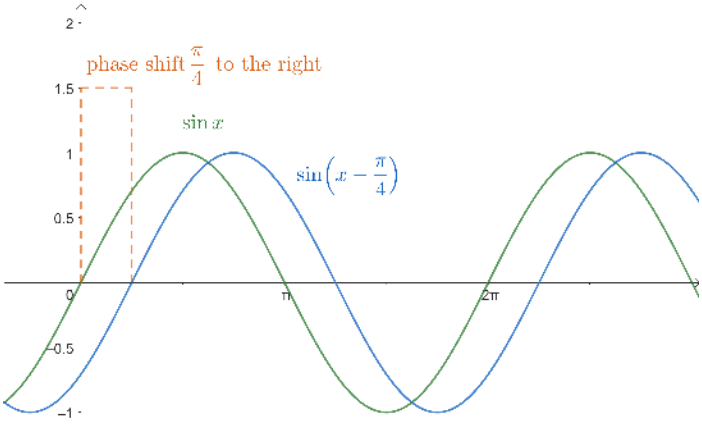
- recognize and graph transformations of sine and cosine
- understand and recognize the inverse trigonometric functions and their domains
- evaluate inverse trig functions

Transformations of Trig Functions

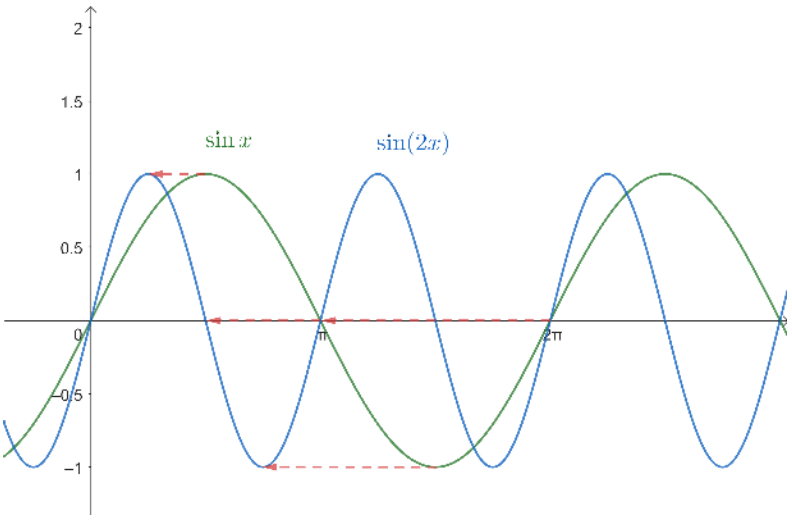
Vocab-wise, we mentioned the **period** of a trig function last section: the amount of time it takes for it to complete one full cycle of behavior and start over. To help you out in future science classes, I want to make sure we cover a couple more terms. The **amplitude** of a sine or cosine wave is the maximum distance between the graph of the function and the horizontal axis; i.e. the max height (and max dip). If you perform a horizontal shift on a sine or cosine wave, you might hear the term **phase shift.** We'll start with amplitude:



The amplitude of $f(x) = \sin x$ is just 1, since sine only goes as high as 1 and as low as -1. The blue graph is $g(x) = 2\sin x$, and we recall that multiplying a whole function by a constant results in a vertical stretch, which has the effect of increasing the amplitude to 2. Both of these functions complete a full cycle between $x = 0$ and $x = 2\pi$, and then repeat themselves forever, so their period is 2π. Now consider the next example:



This time, $f(x) = \sin x$ has had its input replaced by $(x - \frac{\pi}{4})$, which we recall causes a horizontal shift to the right. Last transformation type:



This last picture shows the difference between $f(x) = \sin x$ and $g(x) = \sin(2x)$. Multiplying by 2 inside the input causes a horizontal shrink, and we see that all of the "milestones" on the green graph (max values, min values, hitting the x-axis...) are happening "sooner" on the blue graph. Instead of peaking at $x = \frac{\pi}{2}$, the squished version peaks sooner at $x = \frac{\pi}{4}$, etc. This horizontal squish also affects the period, because the blue graph finishes a cycle twice as fast, making his period just π.

For a function of the form $f(x) = a\sin(x + c)$ or $a\cos(x + c)$,

- $|a|$ is the **amplitude**
- c gives the **phase shift** (to the left if $+$, to the right if $-$)
- the **period** is 2π

For a function of the form $f(x) = a\sin(bx)$ or $a\cos(bx)$,

- $|a|$ is the **amplitude**
- b causes a horizontal shrink if $b > 1$, and a horizontal stretch if $0 < b < 1$
- the **period** is $\frac{2\pi}{b}$

To **sketch the graph of a transformed trig function,**

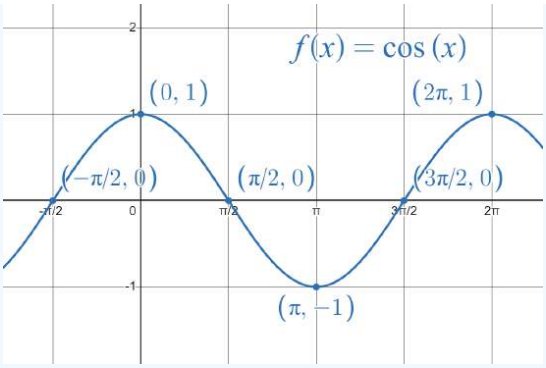
1. Sketch the OG trig function first with its important signpost points (max values, min values, roots, basically the info you get from the cardinal points on the unit circle). This will help you imagine what the transformed trig function should look like based on the transformations that happened.
2. Determine the transformations that have occurred based on the function's definition. Analyze how they change the signpost points. This can be done using common sense or with a from-the-ground-up approach, so read the following examples carefully to see strategies.
3. Plot the new signpost points and smoothly connect the dots.

Example 1

Sketch the graph of $f(x) = 2\cos(x + \pi)$.

Solution

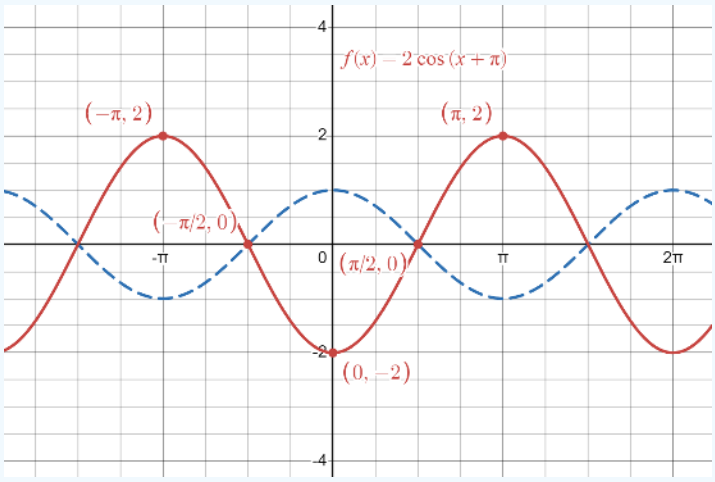
First, we sketch plain ol' $\cos x$ with his important points:



Next, we identify that the multiplication by 2 should cause a vertical shift, stretching the amplitude to 2. The $(x + \pi)$ will cause a horizontal shift to the left by π. Here is how that will affect the signpost points, performing both transformations one after another, moving from left column to right:

Original	**After Vertical Stretch** (multiply y-coordinate only)	**After Horizontal Shift** (subtract π to slide x-coordinates to the left)
$(0, 1)$	$(0, 2)$	$(-\pi, 2)$
$(\pi/2, 0)$	$(\pi/2, 0)$	$(-\pi/2, 0)$
$(\pi, -1)$	$(\pi, -2)$	$(0, -2)$
$(3\pi/2, 0)$	$(3\pi/2, 0)$	$(\pi/2, 0)$
$(2\pi, 1)$	$(2\pi, 2)$	$(\pi, 2)$

Now, we plot the new (third column) points and connect the dots. The original $\cos x$ is shown in dotted blue if you want to compare.



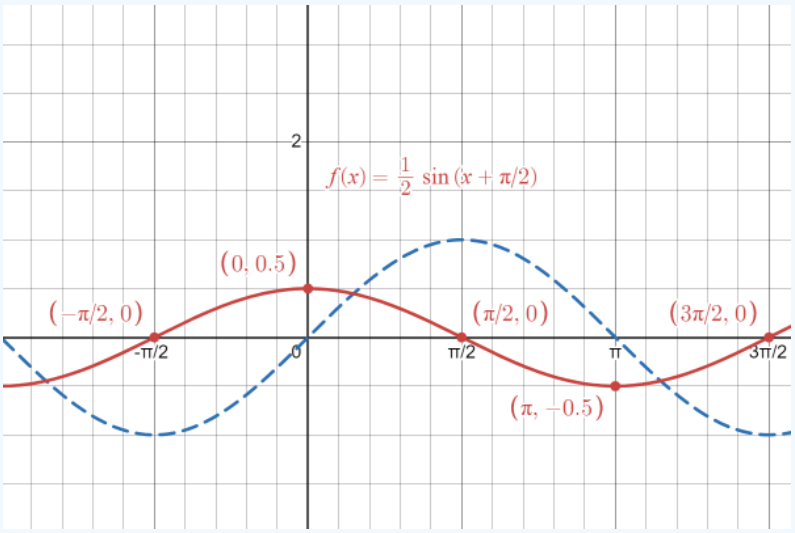
? Exercise 1

Sketch the graph of $f(x) = \frac{1}{2}\sin\left(x + \frac{\pi}{2}\right)$.

Answer

Original	**After Vertical Shrink**	**After Phase Shift**
$(0, 0)$	$(0, 0)$	$(-\pi/2, 0)$
$(\pi/2, 1)$	$(\pi/2, 1/2)$	$(0, 1/2)$

$(\pi, 0)$	$(\pi, 0)$	$(\pi/2, 0)$
$(3\pi/2, -1)$	$(3\pi/2, -1/2)$	$(\pi, -1/2)$
$(2\pi, 0)$	$(2\pi, 0)$	$(3\pi/2, 0)$



Example 2

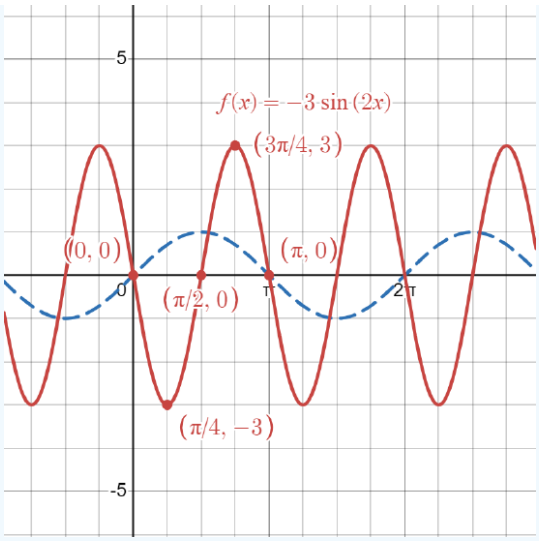
Sketch the graph of $f(x) = -3\sin(2x)$.

Solution

We see that -3 multiplication out front and recognize two things: we will have a stretched amplitude of 3, and our graph will have been reflected across the x-axis. We also see $2x$ in the input. Basically, $\sin(2x)$ does everything $\sin x$ does, but faster, at smaller x-values. If $\sin x$ is 1 at $\pi/2$, then $\sin(2x)$ will reach 1 at $\pi/4$ instead. I'm going to break this down into a few steps over some columns: the first column will be the x-coordinates for some sample points. The second column will be $2x$, which then is the input for sine in the third column. Finally, we multiply by -3 to get our final y-values.

x-value	$2x$	$\sin(2x)$	$-3\sin(2x)$	**Points**
0	0	0	0	$(0, 0)$
$\pi/4$	$\pi/2$	1	-3	$(\pi/4, -3)$
$\pi/2$	π	0	0	$(\pi/2, 0)$
$3\pi/4$	$3\pi/2$	-1	3	$(3\pi/4, 3)$
π	2π	0	0	$(\pi, 0)$

One more note: I chose my sample points to be small increments in fourths, knowing that when they're multiplied by 2 in the second column, they will *become* the easy values $0, \pi/2, \pi,$ etc. Now I plot the points and connect the dots, extending the periodic behavior.



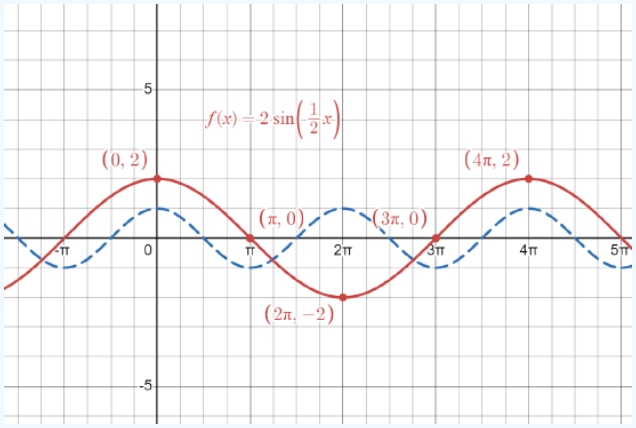
? Exercise 2

Sketch the graph of $f(x) = 2\cos\left(\frac{1}{2}x\right)$.

Answer

Hint: choose sample x-values such that they can easily be divided by 2 and stay nice!

x-value	$\frac{1}{2}x$	$\cos\left(\frac{1}{2}x\right)$	$2\cos\left(\frac{1}{2}x\right)$	Points
0	0	1	2	$(0, 2)$
π	$\pi/2$	0	0	$(\pi, 0)$
2π	π	-1	-2	$(2\pi, -2)$
3π	$3\pi/2$	0	0	$(3\pi, 0)$
4π	2π	1	2	$(4\pi, 2)$

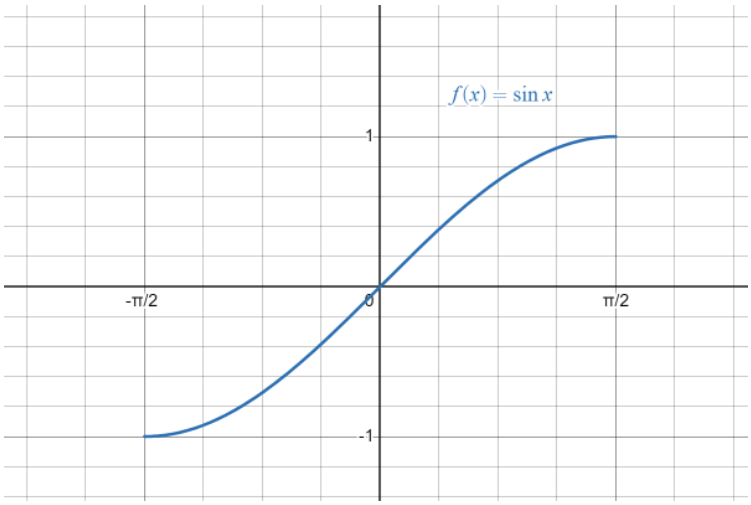


Inverse Trig Functions

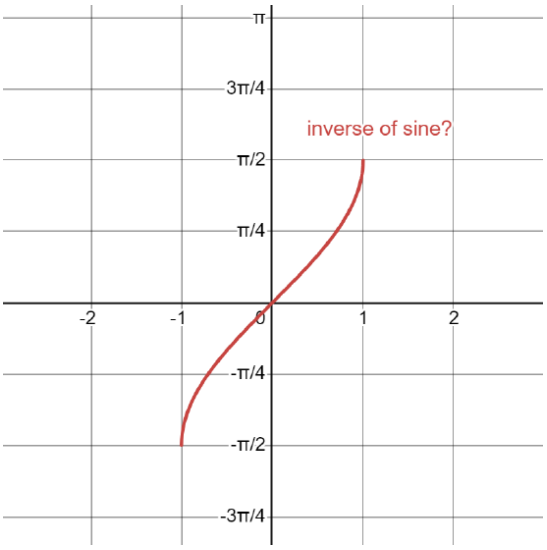
Inverse functions "undo" each other, as we recall. Betcha can't guess what inverse trig functions do.

"Hold the phone!" you say? "Not so fast... Only one-to-one functions can have inverses!" You're completely correct. And yes, sine, cosine, and tangent all fail the Horizontal Line Test, so I guess we can all close our books and go home!

Wait, come back, I have a solution. We're going to zoom in on our trig functions and only consider them on a **particular domain.** For example, if I only consider $\sin x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ (the "right half" of the unit circle), then he passes the HLT with flying colors.



It's perfectly reasonable to define an inverse function for this little piece. Recalling that inverse functions look like reflections across the diagonal line $y = x$, we expect it to look like this:



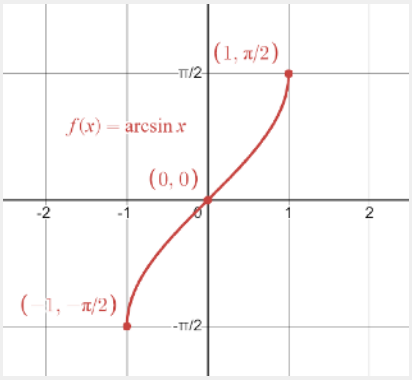
This new red function is called **arcsine.** Whereas we feed sine an angle and it returns a number between -1 and 1, we turn around and feed arcsine a number between -1 and 1, and it returns an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. As always with inverse functions, their domains and ranges are swapped!

For cosine and tangent, we can also restrict their domains to a particular piece (called a "branch") so that they pass the HLT, and define their inverse functions, which are called **arccosine** and **arctangent.** I will always use this notation:

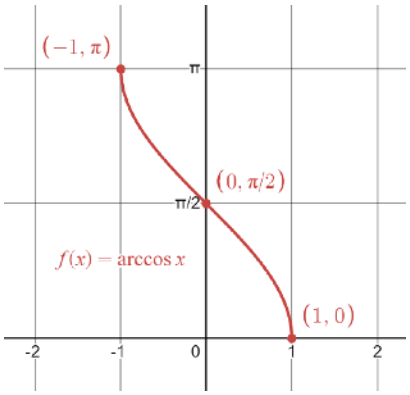
$$\arcsin x \qquad \arccos x \qquad \arctan x$$

but you will also come across the notation style $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$, because of how we've written f^{-1} to mean the inverse of f. In particular, you'll see this on calculators because they want the buttons to be small and they can't fit "arc" on them... However, I find this notation morally and pragmatically reprehensible, because students will often interpret the $^{-1}$ to mean reciprocal, like $4^{-1} = \frac{1}{4}$. So I'm a big "arc" apologist. Here's the gist on the Big Three.

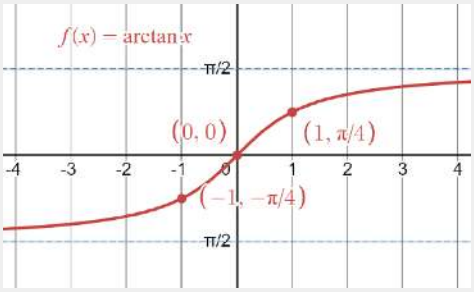
Function	Observations



- **Domain:** arcsine only takes inputs $-1 \leq x \leq 1$, because those are the only outputs of sine. In interval notation, dom: $[-1, 1]$.
- **Range:** arcsine only outputs values from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$, because that was the restricted domain we chose for sine. You can think of those angles as the "right half" of the unit circle. In interval notation, range: $[-\pi/2, \pi/2]$.
- Arcsine passes through $(0, 0)$ because $\sin(0) = 0$, so we must have $\arcsin(0) = 0$. Similarly, it has signpost endpoints $(-1, -\pi/2)$ and $(1, \pi/2)$.



- **Domain:** arccosine takes inputs $-1 \leq x \leq 1$ for the same reason as arcsine. In interval notation, dom: $[-1, 1]$.
- **Range:** arccosine only spits out values from 0 to π, because that's the restricted domain we use for cosine (by tradition). You can remember those angles as the "top half" of the unit circle. In interval notation, range: $[0, \pi]$.
- Arccosine's signpost points are the endpoints $(-1, \pi)$ and $(1, 0)$, and the y-intercept $(0, \pi/2)$.



- **Domain:** Arctangent takes all real numbers as inputs, because that's the range of tangent. In interval notation, dom: $(-\infty, \infty)$.
- **Range:** Arctangent spits out values **strictly** between $-\pi/2$ and $\pi/2$, because that's the conventional restricted domain of tangent. In interval notation, range: $(-\pi/2, \pi/2)$. Going off to the right forever, arctangent approaches the horizontal asymptote $y = \pi/2$ from below. Going off to the left, it approaches $y = -\pi/2$ from above.
- When sketching arctangent, you should plot the easy points $(0, 0)$, $(-1, -\pi/4)$, and $(1, \pi/4)$, drop in your horizontal asymptotes, and then smoothly swoop in through the points.

There are, of course, inverse functions for the reciprocal trig functions; these are **arccosecant** ($\operatorname{arccsc} x$), **arcsecant** ($\operatorname{arcsec} x$), and **arccotangent** ($\operatorname{arccot} x$), but for the life of me I can't remember needing to have them on instant recall in my other courses. I'll guide you through exploring those a bit in the exercises, and assume that if you need them later in life, you can Google them along the way.

Evaluating inverse trig functions is a total scam, because all you do is translate back into regular trig world, sketch a unit circle, and use your brain.

Example 3

Evaluate.

1. $\arcsin(-1)$
2. $\arccos\left(\frac{\sqrt{2}}{2}\right)$
3. $\arctan(\sqrt{3})$

Solution

1. We want to find $\arcsin(-1) = ?$, and he spits out angles, so I want to find the θ such that $\sin\theta = -1$. Well, where is sine -1? Remember your unit circle... At $\frac{3\pi}{2}$, with coordinates $(0, -1)$, sine is -1. BUT WAIT, that's not in our allowed range of angles for arcsine. Simply recall that $\frac{3\pi}{2}$ is coterminal with the negative angle $-\frac{\pi}{2}$. That's our final answer.

2. Again, translate to regular trig. We are looking for the angle θ where $\cos\theta = \frac{\sqrt{2}}{2}$.That's just $\frac{1}{\sqrt{2}}$ after rationalizing the denominator. You will see people write it this way OFTEN. So anyways, on the unit circle, cosine has this value at two angles: $\frac{\pi}{4}$ and $\frac{7\pi}{4}$. We only report the first one as our answer, because the other option is not in our allowable range!

3. We are looking for θ such that $\tan\theta = \sqrt{3}$. In other words, we need $\frac{\sin\theta}{\cos\theta} = \sqrt{3}$. With the unit circle, we notice that at $\frac{\pi}{3}$, we have coordinates $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Remember that those are (cosine, sine) values, so if you divide them in the correct order, the fraction simplifies to $\sqrt{3}$. This angle is in our allowed range, so we report back $\frac{\pi}{3}$.

Exercise 3

Evaluate.

1. $\arccos(1)$
2. $\arcsin\left(\frac{1}{2}\right)$
3. $\arctan(-1)$

Answer

1. 0
2. $\pi/6$
3. $-\pi/4$

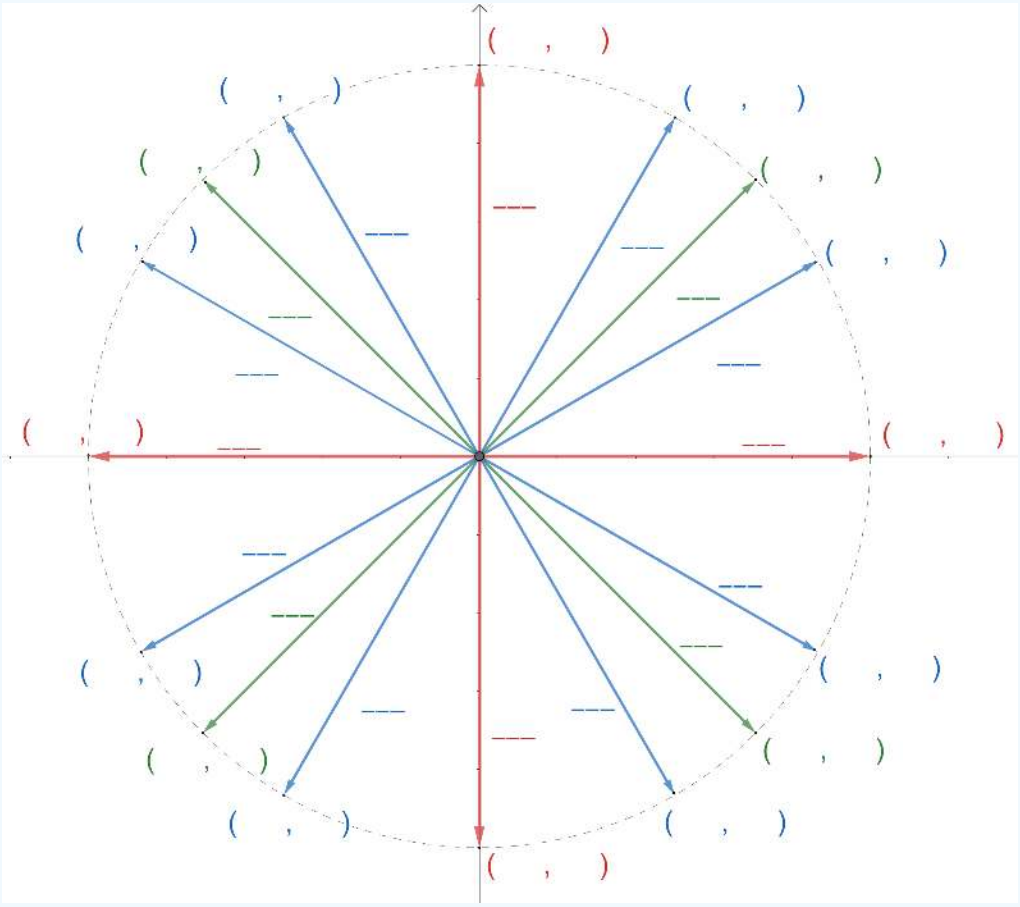
Okay, you can go do a lot more practice with those in the exercises!

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8.5E Exercises

The Unit Circle Again

Fill in the blanks. (Try to do this from memory as much as possible.)



Answer

If needed, refer back to Section 8.4.

Vocabulary

For the given functions, tell the amplitude, phase shift (if any, and give the direction), and period.

1. $f(x) = \sin(x - \pi)$
2. $f(x) = 2\cos x$
3. $f(x) = \tan(2x)$
4. $f(x) = -3\sin\left(x + \frac{\pi}{2}\right)$
5. $f(x) = -\frac{1}{3}\cos(4x)$
6. $f(x) = 4\tan\left(\frac{1}{2}x\right)$

Answer

Go to desmos.com and graph both the OG trig function and the transformed one to visually confirm your answers.

Sketching Transformed Trig Functions

Sketch the functions.

1. $f(x) = 2\sin x$
2. $f(x) = \frac{1}{2}\cos x$
3. $f(x) = -3\sin x$
4. $f(x) = \cos(x - \pi)$
5. $f(x) = \sin(2x)$
6. $f(x) = 2\cos\left(x + \frac{\pi}{2}\right)$
7. $f(x) = \sin\left(x - \frac{\pi}{4}\right)$
8. $f(x) = \frac{1}{3}\cos\left(\frac{1}{2}x\right)$

Answer

Go to desmos.com and graph the function to visually confirm your answers.

Inverse Trig Function Facts

1. As we saw, the restricted domain we use for sine is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Give the domain and range of arcsine. Go to desmos.com and type in "$y = \sin x\{\pi/2 \leq x \leq \pi/2\}$" to the first line. Type in "$y = \arcsin x$" to the second line. Describe the relationship you see between the graphs.

2. The restricted domain used for cosine is $[0, \pi]$. Give the domain and range of arccosine. Graph both on desmos and describe the relationship.

3. The restricted domain used for tangent is $(-\frac{\pi}{2}, \frac{\pi}{2})$. Give the domain and range for arctangent. Graph both on desmos and describe the relationship.

Answer

For all three, the graphs should look like reflections across the line $y = x$.

1. dom: $[-1, 1]$. range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
2. dom: $[-1, 1]$. range: $[0, \pi]$.
3. dom: $(-\infty, \infty)$. range: $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Exploring Inverse Reciprocal Trig

1. Go to desmos.com and graph $f(x) = \operatorname{arccsc} x$. Give the domain and range. By reverse engineering, determine the restricted domain used for $\csc x$. Graph $g(x) = \csc x$ to see the branch used and compare the graphs.

2. Do the same for $f(x) = \operatorname{arcsec} x$.

3. Do the same for $f(x) = \operatorname{arccot} x$.

Answer

1. dom: $(-\infty, -1] \cup [1, \infty)$. range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (this interval is also the restricted domain for cosecant).
2. dom: $(-\infty, -1] \cup [1, \infty)$. range: $[0, \pi]$ (this is also the restricted domain for secant).
3. dom: $(-\infty, \infty)$. range: $(0, \pi)$ (this is also the restricted domain for cotangent).

Evaluating Inverse Trig Functions

Evaluate.

1. $\arcsin(0)$
2. $\arcsin(1)$
3. $\arcsin(-1)$

4. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$
5. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$
6. $\arcsin\left(\frac{1}{\sqrt{2}}\right)$
7. $\arccos(0)$
8. $\arccos(1)$
9. $\arccos(-1)$
10. $\arccos\left(\frac{1}{2}\right)$
11. $\arccos\left(\frac{1}{\sqrt{2}}\right)$
12. $\arccos\left(-\frac{1}{\sqrt{2}}\right)$
13. $\arctan(0)$
14. $\arctan(1)$
15. $\arctan(-1)$
16. $\arctan\left(\frac{1}{\sqrt{3}}\right)$

Answer

1. 0
2. $\pi/2$
3. $-\pi/2$
4. $\pi/3$
5. $-\pi/3$
6. $\pi/4$
7. $\pi/2$
8. 0
9. π
10. $\pi/3$
11. $\pi/4$
12. $3\pi/4$
13. 0
14. $\pi/4$
15. $-\pi/4$
16. $\pi/6$

? Evaluating Inverse Reciprocal Trig Functions

Evaluate $\operatorname{arcsec}(2)$ by following these steps:

1. Translate the equation $\operatorname{arcsec}(2) = \theta$ into an equation involving only secant.
2. Translate the new equation into an equivalent equation involving only cosine.
3. Solve the equation for the cosine term.
4. Find the angle θ using the unit circle and your knowledge of cosine. Refer back to your exploration of arcsecant to determine which range of angles to report your answer in.

Do the same process to evaluate $\operatorname{arccsc}\left(\frac{2}{\sqrt{3}}\right)$.

Answer

1. $\sec\theta = 2$
2. $\frac{1}{\cos\theta} = 2$
3. $\cos\theta = \frac{1}{2}$
4. Cosine is $\frac{1}{2}$ at $\frac{\pi}{3}$. This is within the allowable range $[0, \pi]$.

   And $\operatorname{arccsc}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{3}$.

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8.6 Trigonometric Identities

Learning Objectives

By the end of this section, you will be able to:

- simplify trigonometric expressions
- know and use the fundamental Pythagorean identity, and derive the others from it
- know and use the even/odd, sum, difference, and double angle identities
- legally manipulate expressions involving trig functions

In this section, we're going to learn ways to legally manipulate expressions involving trig functions. Beyond using definitions and fraction simplifying techniques, we're also going to learn some essential **trig identities.** These are more like..... secret identities. They tell you what trig expressions are equivalent—secretly the same guy, like Clark Kent and Superman (spoiler alert). Applying these true statements to rearrange expressions into more useful forms is really just like a puzzle! Gear up for the idea that sometimes you just need to throw a lot of spaghetti at the wall and see what sticks, and that's okay. Just a) write down *anything* you know to be *true*, and b) don't be afraid to try something and see what happens. Okay, let's go.

Technique #1 when simplifying or manipulating expressions featuring trig function is simple to **write everything in terms of sines and cosines** and see if anything magical happens. Technically, in doing this, we are using facts we already learned, which I'll now tell you are known as the **Reciprocal Identities,**

$$\sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}, \quad \cot x = \frac{1}{\tan x},$$

and the **Quotient Identities,**

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}.$$

Example 1

Simplify the expression $\csc x \tan x + \sec x$.

Solution

Let's translate everybody to sines and cosines only:

$$\csc x \tan x + \sec x = \frac{1}{\sin x} \cdot \frac{\sin x}{\cos x} + \frac{1}{\cos x}$$

Ope, nice, we have matching terms on top and bottom that we can cancel! NOTE: The quantity we're cancelling is $(\sin x)$ AS A WHOLE. You **cannot** cancel like this: $\frac{\sin x}{x} \to \sin$. That's a crime—a sin, if you will. Anyways, let's simplify:

$$\frac{1}{\sin x} \cdot \frac{\sin x}{\cos x} + \frac{1}{\cos x} = \frac{1}{\cos x} + \frac{1}{\cos x} = \frac{2}{\cos x} \quad \text{or} \quad 2\sec x$$

Exercise 1

Simplify the expression $\sin x \cot x - \frac{1}{\sec x}$.

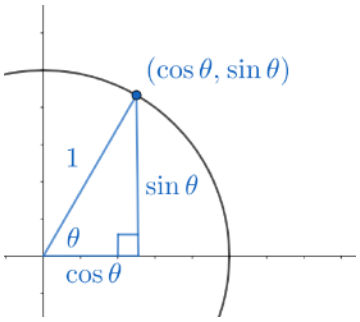
Answer

0.

Pythagorean Identities

When we look at triangles on the unit circle to determine values of $\sin\theta$ and $\cos\theta$, we always have the hypotenuse of length 1 (since it's a radius of the circle). Then the vertical leg of the right triangle is of size "$\sin\theta$" (whatever that number turns out to be) and the

horizontal leg is of size "$\cos\theta$."



Writing down the Pythagorean Theorem for this triangle gives us our first important identity. Several more follow as a direct consequence of it as well.

The Pythagorean Identities

For any consistent expression A, we have

$$\sin^2 A + \cos^2 A = 1.$$

(Notation note: $\sin^2 A$ just means $(\sin A)^2$.)

We also have

$$\tan^2 A + 1 = \sec^2 A, \qquad 1 + \cot^2 A = \csc^2 A$$

Example 2

You don't need to memorize the other two, as long as you know the first one. We can derive them from the OG using a strategic division across both sides of the equation. For example, divide by $\cos^2 A$ and see what happens.

Solution

We divide through the whole equation by $\cos^2 A$.

$$\frac{1}{\cos^2 A} \cdot (\sin^2 A + \cos^2 A) = 1 \cdot \frac{1}{\cos^2 A} \quad \rightarrow \quad \frac{\sin^2 A}{\cos^2 A} + 1 = \frac{1}{\cos^2 A}$$

Now we notice that $\frac{\sin^2 A}{\cos^2 A} = \left(\frac{\sin A}{\cos A}\right)^2 = \tan^2 A$, and note that $\frac{1}{\cos^2 A} = \sec^2 A$, by definition of secant. All together, we get

$$\tan^2 A + 1 = \sec^2 A$$

Exercise 2

Use a strategic division and simplification to derive $1 + \cot^2 A = \csc^2 A$ from the original $\sin^2 A + \cos^2 A = 1$.

Answer

Hint: what do you need to divide everyone by in order to make the first term into a 1?

Note that A can be anything in these identities, as long as it's consistent in the terms. You can also rearrange the equation in any way you need, *legally*. For example,

$$\sin^2 x + \cos^2 x = 1, \qquad \cos^2(a+b) + \sin^2(a+b) = 1, \qquad 1 - \sin^2(2\theta) = \cos^2(2\theta)$$

are all true, but

$$\sin^2\theta - \cos^2\theta \neq 1, \qquad \sin^2 x + \cos^2 3x \neq 1, \qquad \sin x + \cos x \neq 1$$

and so on. Make sure you look at each example there and understand *why*.

Even/Odd Identities

Since $\sin x$ is symmetric about the origin, it's an **odd function** (review Section 4.2 if needed). Since $\cos x$ is symmetric across the y-axis, it's an **even function.** These facts give us the following identities:

$$\sin(-x) = -\sin x \quad \text{and} \quad \cos(-x) = \cos x$$

Exercise 3

Find an even/odd identity for tangent by considering what $\tan(-x) = \ldots$ turns out to be.

1. Write tangent in terms of sines and cosines.
2. Apply the two facts above and simplify the result.

Answer

$\tan(-x) = -\tan x$.

Sum/Difference Identities

I actually saw these come up the last time I taught Calc I, so I want you to be familiar with the identities involving sine and cosine below, but I don't expect you to memorize the ones involving tangent (ask your professor if they want you to!). These allow you to rewrite trig functions with complicated inputs as expressions involving simpler inputs.

Sum/Difference Formulas

For **sine,** the formulas for addition or subtraction in the **input** have mixed terms and matching operation:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B, \qquad \sin(A-B) = \sin A\cos B - \cos A\sin B$$

For **cosine,** we have matching terms and opposite operation:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B, \qquad \cos(A-B) = \cos A\cos B + \sin A\sin B$$

Also, for **tangent,** if needed, we have:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}, \qquad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

I've done some strategic color-coding above to help you notice important features while memorizing. Once again, if you find yourself forgetting the subtraction formulas above, you can actually just deduce them from the addition versions, using the even/odd identities!

Example 3

Prove the difference formula for sine using the addition formula $\sin(A+B) = \sin A\cos B + \cos A\sin B$ and the even/odd identities.

Solution

We want to figure out how to rewrite $\sin(A-B)$, so we start with that and try manipulating it. The first thing to observe is that $A - B = A + (-B)$. Now we can apply the sum formula given above, treating A and $(-B)$ as the added components.

$$\sin(A + (-B)) = \sin A\cos(-B) + \cos A\sin(-B)$$

Next, we apply the facts that $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$, to get

$$\sin(A-B) = \sin A\cos B + (\cos A)(-\sin B) = \sin A\cos B - \cos A\sin B$$

Now, that was with a lot of explanation, but let's say I asked you to "prove" something like this on a test. It would be sufficient (in my opinion, I'm a chill grader) to just write down a **true** chain of equations showing all the intermediate steps, like this:

$$\sin(A-B) = \sin(A+(-B)) = \sin A\cos(-B) + \cos A\sin(-B) = \sin A\cos B - \cos A\sin B$$

But of course, since the idea is to show **why** something is true, you cannot leave out any of those intermediate steps!

Exercise 4

Prove the difference formula for cosine using the addition formula $\cos(A+B) = \cos A \cos B - \sin A \sin B$ and the even/odd identities.

Answer

$$\cos(A-B) = \cos(A+(-B)) = \cos A \cos(-B) - \sin A \sin(-B) = \cos A \cos B + \sin A \sin B$$

Double Angle Identities

The name **double angle** means that these formulas are used when the input for a trig function is "2 times something" specifically.

Double Angle Formulas

For **sine,** we have

$$\sin 2A = 2 \sin A \cos A$$

For **cosine,** we have several equivalent expressions

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$$

Also, if needed, we have for **tangent**

$$\tan 2A = \frac{2\tan A}{1-\tan^2 x}$$

Let's prove some of these double angle formulas using the sum/difference identities we already learned. (Are you sensing a theme here?)

Example 4

Prove the double angle identities for $\cos 2A$.

Solution

The key here is to recognize that $2A = A + A$. Using the sum formula with A plugged in for **both** A and B, we have

$$\cos 2A = \cos(A+A) = \cos A \cos A - \sin A \sin A = (\cos A)^2 - (\sin A)^2 = \cos^2 A - \sin^2 A$$

Great, but how do we show that the other expressions are equivalent to the one above? For this, we're going to use the Pythagorean Identity $\sin^2 A + \cos^2 A = 1$. First, we rearrange it to solve for $\cos^2 A$:

$$\cos^2 A = 1 - \sin^2 A$$

Then we go back to our double-angle identity expression and replace $\cos^2 A$ with what he's equal to, and simplify:

$$\cos^2 A - \sin^2 A = (1 - \sin^2 A) - \sin^2 A = 1 - \sin^2 A - \sin^2 A = 1 - 2\sin^2 A$$

To get the final one, we instead solve the Pythagorean Identity for the $\sin^2 A$ term:

$$\sin^2 A = 1 - \cos^2 A$$

And then do the same substitution trick (use parentheses!):

$$\cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A) = \cos^2 A - 1 + \cos^2 A = 2\cos^2 A - 1$$

Since everybody is equal to $\cos^2 A - \sin^2 A$ and that guy is equal to $\cos 2A$, we're done!

? Exercise 5

Prove the double-angle formula for $\sin 2A$ using the sum formula we learned above (this is less work than the example we just did).

Answer

$$\sin 2A = \sin(A + A) = \sin A \cos A + \sin A \cos A = 2 \sin A \cos A$$

I want to emphasize that you should know these identities forwards and backwards! Here's an example of using a double-angle formula backwards...

✔ Example 5

Evaluate $\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$ even though $\pi/8$ isn't a "typical" angle.

Solution

We notice that the expression is of the form $\sin A \cos A$, which we know appears on the right hand side of the double angle identity

$$\sin 2A = 2 \sin A \cos A.$$

We can rearrange this identity to become

$$\sin A \cos A = \frac{1}{2}\sin(2A).$$

Now, letting $A = \frac{\pi}{8}$, we have

$$\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) = \frac{1}{2}\sin\left(\frac{2\pi}{8}\right) = \frac{1}{2}\sin\left(\frac{\pi}{4}\right),$$

which evaluates to $\frac{1}{2\sqrt{2}}$.

These aren't the only trig identities that people study in something like a dedicated Analytical Trigonometry course, and you may see more crop up from time to time in specific contexts. These include things called **cofunction identities, half-angle formulas, product-sum formulas**, and whatever other fun little equivalences people prove and write down. The ones I've chosen to include in this class are the ones that I feel were most helpful to have on instant recall throughout my undergraduate math career. If you're interested in other identities, we'll explore some of them in the exercises, and there are a lot of nice "cheat sheet" lists out there, such as this one.

I hope by this point in the course you're starting to notice that studying math is not like other subjects where you have to memorize a ton of seemingly random factoids, like dates and dead people, or Latin names for things... Math builds on itself, and we use math facts to deduce or prove other math facts. Sure, you need to memorize a few facts to start out, but once you have a basic toolbox, you can derive everything you need using logic. The best thing you can take away from this class is the ability to puzzle out the consequences of facts, or know when something has been logically justified from start to finish. Like how we used the even/odd identities to figure out the difference formulas, or used the sum formulas to figure out the double-angle formulas. Math is not black magic and chaos that you have no hope of retaining. Really try to engage with the WHY and HOW behind anything you're learning; that synthesis and integration is what *learning* is all about!

Anyways, all this to say, go do the exercises to explore utilizing all these facts.

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8.6E Exercises

Basic Simplification

Simplify each expression. Your answer should be a number or a single trig function.

1. $\cos x \csc x$
2. $\cot x \sin x$
3. $\cot x \tan x$
4. $\frac{\sin x}{\tan x}$
5. $\dfrac{\csc x \sin x}{\cot x}$
6. $\cos^2\theta + \frac{1}{\csc^2\theta}$
7. $\sin x \tan x + \cos x$
8. $\frac{\sin\theta}{\cot\theta} + \cos\theta$

Answer

1. $\cot x$
2. $\cos x$
3. 1
4. $\cos x$
5. $\tan x$
6. 1
7. $\frac{1}{\cos x}$ or $\sec x$
8. $\frac{1}{\cos\theta}$ or $\sec\theta$

Manipulating and Simplifying Trig Expressions

1. FOIL and simplify: $(1-\sin x)(1+\sin x)$
2. FOIL and simplify: $(\cos\theta + \sin\theta)^2$
3. Show the steps simplifying $\frac{\tan\theta}{\csc\theta}\cos\theta$ to $\sin^2\theta$.
4. Write the expression $\dfrac{\sin x + x}{\cos x}$ as two terms involving $\tan x$ and/or $\sec x$.
5. Write the expression $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1+\sin\theta}$ as a single fraction term and then simplify as far as possible.
6. Write the expression $\dfrac{1}{1-\sin x} - \dfrac{1}{1+\sin x}$ as a single simplified term involving $\tan x$ and/or $\sec x$.
7. Write the expression $\csc\theta + \cot\theta$ as a single fraction term involving $\sin\theta$ and/or $\cos\theta$
8. Substitute $\cos\theta$ for x in the expression $\sqrt{1-x^2}$ and simplify using the Pythagorean Identity (assume θ is an angle such that $\sin\theta \geq 0$).
9. Use factoring and a Pythagorean Identity to simplify $\cos x \sin^2 + \cos^3 x$.
10. Use a Pythagorean Identity to simplify $(1+\tan^2 x)\sin^2 x$.

Answer

1. $1-\sin^2 x$ aka $\cos^2 x$.
2. $1 + 2\cos\theta\sin\theta$
3. $\frac{\tan\theta}{\csc\theta}\cos\theta = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\sin\theta}}\cos\theta = \frac{\sin^2\theta}{\cos\theta}\cos\theta = \sin^2\theta$
4. $\tan x + x\sec x$
5. $\sec\theta$
6. $2\tan x \sec x$

7. $\dfrac{1+\cos\theta}{\sin\theta}$
8. $\sqrt{1-\cos^2\theta} = \sqrt{\sin^2\theta} = \sin\theta$
9. $\cos x$.
10. $\tan^2 x$.

Using Sum/Difference Formulas

(Make sure to do all of these.)

1. Simplify $\sin(x+\pi)$.

2. Simplify $\cos\left(\frac{\pi}{2}-\theta\right)$. (This is called a **cofunction identity**.)

3. Find a **triple-angle formula** for $\sin(3A)$ by writing $3A = 2A + A$ and using a sum formula and double angle formulas.

4. Use a sum formula backwards to evaluate $\sin 15^\circ \cos 30^\circ + \cos 15^\circ \sin 30^\circ$ even though 15° isn't a "typical" angle we have memorized.

5. (This is an expression that could show up in Calc I.) Plug in $f(x) = \sin x$ to the expression

$$\frac{f(x+h)-f(x)}{h}$$

and simplify until it looks like this: $\sin x\left(\dfrac{\cos h - 1}{h}\right) + \cos x\left(\dfrac{\sin h}{h}\right)$.

Answer

1. $-\sin x$
2. $\sin\theta$
3. $4\sin A\cos^2 A - \sin A$
4. This is equivalent to $\sin(15^\circ + 30^\circ) = \sin(45^\circ) = \frac{1}{\sqrt{2}}$.
5. Plugged in, you should have $\dfrac{\sin(x+h)-\sin x}{h}$, and you use the sum formula on the $\sin(x+h)$ term. Then algebraically manipulate by factoring, splitting fractions, etc...

Formulas for Lowering Powers

The results of these problems provide a formula you can use to get rid of a squared trig term and replace it with some trig term to the first power only.

1. Use a double-angle formula to find an expression equivalent to $\sin^2 x$ that involves $\cos 2x$.
2. Use a double-angle formula to find an expression equivalent to $\cos^2 x$ that involves $\cos 2x$.
3. Use the formulas you derived to evaluate $\sin^2\left(\frac{\pi}{12}\right)$ even though it's not a "typical" angle.

Answer

1. Using $\cos 2x = 1 - 2\sin^2 x$, we get $\sin^2 x = \frac{1-\cos 2x}{2}$.
2. $\cos^2 x = \frac{1+\cos 2x}{2}$
3. $\frac{1}{2} - \frac{\sqrt{3}}{4}$ or $\frac{2-\sqrt{3}}{4}$ if you prefer.

Product-Sum Formulas

The results of these problems provide a way to rewrite a product of trig functions as a sum.

1. Recall that two equations can be added to form a new, also true, equation. That is,

$$A = B \text{ and } C = D \quad \Longrightarrow \quad A + C = B + D.$$

Use this fact to add the sum formula for sine to the difference formula for sine.

2. Simplify the result so that one side has a single term involving $\sin A \cos B$. Fully isolate this term to find a formula

$$\sin A \cos B = \ldots$$

3. Use this formula to evaluate $\sin\left(\frac{3\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$.

Answer

1. $\sin(A+B)+\sin(A-B) = \sin A\cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B$
2. $\sin A \cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B))$
3. $\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)$.

Fill-in-the-Blanks Practice for Memorization

Fill in the blanks in the identities below.

1. $\sin^2 x + [\quad] = 1$.
2. $\tan^2 x + 1 = [\quad]$.
3. $[\quad] + \cot^2 x = \csc^2 x$.
4. $\sin(A+B) = \sin A \cos B\, [\quad] \cos A \sin B$.
5. $\sin(A-B) = \sin A \cos B\, [\quad] \cos A \sin B$.
6. $[\quad] = \cos A \cos B - \sin A \sin B$.
7. $\cos(A-B) = [\quad] + [\quad]$.
8. $[\quad] = \cos^2 A - \sin^2 A$.
9. $\sin 2A = [\quad]$.

Answer

After trying from scratch, check your answers against the previous section. Rinse and repeat.

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8.7 Solving Trigonometric Equations

Learning Objectives

By the end of this section, you will be able to:

- find all solutions to equations involving trig functions
- use trig identities and other tricks to solve equations involving multiple trig functions

Sometimes students escape high school under the impression that trig can be entirely handled with the trusty $\sin^{-1}$ button on a calculator, but in calculus courses this is **not** the case. Any time you try to tackle a trig equation by slapping it with an inverse trig function, you are actually losing a lot of information! This is because, in order to define the inverse functions, we had to pretend most of sine, cosine, and tangent didn't exist. We restricted our domain to a tiny chunk of the x-axis. In some applications, that may be fine, but in others, all that lost information was important! So now we're going to learn how to find solutions to trig equations as will be needed in future courses. For example, how do you figure out all possible angles θ such that "$\sin\theta - \frac{1}{2} = 0$" will be a true statement?

To **find all solutions to a trigonometric equation,**

1. Use algebra to manipulate the equation until it consists of one single trig function isolated on one side of the equals.
2. Find all the solutions in the usual unit circle (angles between 0 and 2π).
3. List all solutions by adding integer multiples of the period to the solutions.

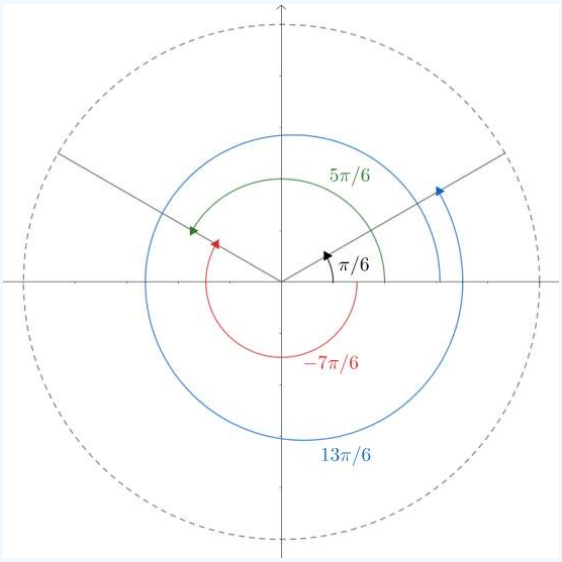
Example 1

Give at least four solutions to the equation, and write a formula that will describe all solutions.

1. $\sin\theta - \frac{1}{2} = 0$
2. $\cos\theta = \sin\theta$
3. $2\cos\theta = \cos\theta$

Solution

1. First, we get the trig term by himself: $\sin\theta = \frac{1}{2}$. Now we ask ourselves, at what angles does sine turn out to be $\frac{1}{2}$? Recalling our unit circle, that will happen if θ is $\frac{\pi}{6}$ or $\frac{5\pi}{6}$. BUT ALSO, it will happen at any other angle values that are **coterminal** with those! See four options below:



So $\frac{\pi}{6}+2\pi=\frac{13\pi}{6}$ is a location where $\sin\theta=\frac{1}{2}$, and so is $\frac{\pi}{6}+4\pi=\frac{25\pi}{6}$, and any such angle created by adding a multiple of 2π to $\frac{\pi}{6}$. The same is true for $\frac{5\pi}{6}$ and his coterminal angles $\frac{5\pi}{6}+2\pi=\frac{17\pi}{6}$, or $\frac{5\pi}{6}+4\pi=\frac{29\pi}{6}$, etc. etc. Also, coterminal **negative** angles are solutions too! For example, $-\frac{7\pi}{6}$ is coterminal with $\frac{5\pi}{6}$ and $-\frac{11\pi}{6}$ is coterminal with $\frac{\pi}{6}$. The way to summarize all of these solutions is with formulas like this:

$$\theta=\frac{\pi}{6}+2k\pi,\quad \theta=\frac{5\pi}{6}+2k\pi,\quad \text{where } k \text{ is any integer}$$

Since k can be a positive or negative integer, this will cover all our bases.

2. Our first step is to boil everything down to a single trig function isolated on one side of the equation. We might like to divide both sides by $\cos\theta$ to get $1=\tan\theta$. By doing so, however, we are assuming that $\cos\theta\neq 0$! Is that okay? Well, any angle θ where $\cos\theta=0$, such at $\theta=\frac{\pi}{2},\frac{3\pi}{2}$, etc. will not be a solution to this equation, because at those angles, $\sin\theta\neq 0$. So it's okay to make this assumption! Now, when does tangent turn out to be 1? Anywhere on the unit circle where $\sin\theta$ and $\cos\theta$ are equal. This happens at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$. Our final answer is then $\theta=\frac{\pi}{4}+2k\pi, \theta=\frac{5\pi}{4}+2k\pi$, where k is any integer. You can also just use common sense on this equation instead of doing the isolation step to convert to tangent. Just look at the equation and ask yourself, "Hmm, where are cosine and sine the same?" and ponder your unit circle. Specific solutions we could list out by choosing k options could be: $\theta=-\frac{7\pi}{4},\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4},\frac{13\pi}{4}$, etc.

3. We bring together the like terms and isolate the trig function to get $\cos\theta=0$. Now, where is cosine zero? At $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Any angle given by $\theta=\frac{\pi}{2}+2k\pi$ or $\theta=\frac{3\pi}{2}+2k\pi$ will be a solution. Some specific examples are: $-\frac{3\pi}{2},-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2}$, etc. This can also be described as "odd multiples of $\frac{\pi}{2}$," if you like.

? Exercise 1

Give at least three specific solutions to the equation, and describe all possible solutions.

1. $\sin\theta=0$
2. $2\cos\theta=-\sqrt{2}$
3. $\tan^2\theta=1$

Answer

1. $\sin\theta$ is zero if $\theta=-\pi,0,\pi,2\pi$, etc. Aka, $\theta=0+2k\pi$ and $\theta=\pi+2k\pi$, or simplified, any integer multiple of π.
2. Remember that $-\frac{\sqrt{2}}{2}$ is another way of writing $-\frac{1}{\sqrt{2}}$. So $\theta=\frac{3\pi}{4}+2k\pi$, $\theta=\frac{5\pi}{4}+2k\pi$, where k is any integer.
3. Hint: if $\tan\theta=\pm 1$, then the equation $\tan^2\theta=1$ will be true, so you're looking for angles where tangent is 1 or -1. So $\theta=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$ etc. all work! This can be described as all odd multiples of $\frac{\pi}{4}$. One way you can write this in math code as $\theta=(2k+1)\frac{\pi}{4}$ (can you see why?) if you want.

Sometimes an equation has multiple trig function terms in it and we need to use identities to manipulate it into an easily solvable form. Here are some more complicated examples. Each one will show you a special technique.

✓ Example 2

Solve the equations.

1. $1+\sin\theta=1-\sin^2\theta$ (Factoring technique.)
2. $\sin\theta=2\cos^2\theta-1$ (Using an identity and factoring.)
3. $\cos 3\theta+1=0$ (Dealing with horizontal stretch/shrink in the input.)
4. $\tan\left(\theta+\frac{\pi}{4}\right)+\sin^2\theta=1-\cos^2\theta$ (Using an identity and dealing with a phase shift.)

Solution

1. First, I bring all the trig stuff together and move everything else over to the other side to clean things up.

$$\sin^2\theta+\sin\theta=1-1=0$$

Now, I notice that the two terms on the left have a common factor, so I pull it out.

$$(\sin\theta)(\sin\theta+1)=0$$

Now, there are two ways that this equation could turn out to be true: either $\sin\theta = 0$ or $\sin\theta + 1 = 0$. Either of those cases would cause the whole left side to wipe out. In the first case, $\sin\theta = 0$ when $\theta = 0, \pi, 2\pi, \ldots$ etc. On the other hand, $\sin\theta = -1$ at $\theta = \frac{3\pi}{2}, \frac{7\pi}{2}, -\frac{\pi}{2}$, etc. So this equation has solutions $\theta = k\pi$ and $\theta = \frac{3\pi}{2} + 2k\pi$, where k can be any integer.

2. The thing to notice here is the $\cos^2\theta$ term, which we could rewrite in terms of $\sin^2\theta$ using the Pythagorean Identity. Since $\cos^2\theta = 1 - \sin^2\theta$, we sub in and simplify:

$$\sin\theta = 2(1 - \sin^2\theta) - 1 \quad \rightarrow \quad \sin\theta = -2\sin^2\theta + 1$$

If we're going to use that factoring trick above, we need to get zero on one side. We manipulate...

$$2\sin^2\theta + \sin\theta - 1 = 0 \quad \rightarrow \quad (2\sin\theta - 1)(\sin\theta + 1) = 0$$

Again, we have two cases. We have $2\sin\theta - 1 = 0$ if $\sin\theta = \frac{1}{2}$, which happens at $\theta = \frac{\pi}{6} + 2k\pi, \theta = \frac{5\pi}{6} + 2k\pi$, as we've seen. In the other case, we have $\sin\theta + 1 = 0$ if $\sin\theta = -1$, which happens at $\theta = -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$, etc. So we have solutions

$$\theta = \frac{\pi}{6} + 2k\pi, \quad \theta = \frac{5\pi}{6} + 2k\pi, \quad \theta = \frac{3\pi}{2} + 2k\pi$$

3. The weirdo here is the 3θ input for cosine, but all we do is ignore that for now. Pretend it's some other nickname. Right now, we're just looking for angles x where $\cos x + 1 = 0$. Isolating the trig function term, that is $\cos x = -1$. We know that happens if $x = \pi$ on the unit circle, and again for every angle $\pi + 2k\pi$, where k is any integer, but that's not our final answer! Time to un-nickname and realize this means we want $3\theta = \pi + 2k\pi$. Solving for θ, this tells us that our solution list is $\theta = \frac{\pi + 2k\pi}{3}$, where k is any integer. You could write that as $(2k+1)\frac{\pi}{3}$, if you want.

4. It looks like there's a lot going on here, but the $\sin^2$ and $\cos^2$ appearing makes me think I should use the Pythagorean Identity to clean things up first. Simplifying, I get

$$\tan\left(\theta + \frac{\pi}{4}\right) + \sin^2\theta + \cos^2\theta = 1 \quad \rightarrow \quad \tan\left(\theta + \frac{\pi}{4}\right) + 1 = 1 \quad \rightarrow \quad \tan\left(\theta + \frac{\pi}{4}\right) = 0$$

I start by ignoring the garbage in tangent's input. Just look for angles x where $\tan x = 0$. This will happen anytime $\sin x = 0$, which we've already seen is when $x = k\pi$, where k is any integer. That means we want $\theta + \frac{\pi}{4} = k\pi$. Solving for θ, we get the answer $\theta = k\pi - \frac{\pi}{4}$, where k is any integer. These are values such as $\theta = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$, etc.

? Exercise 2

Solve the equations.

1. $\sin(2\theta) = \frac{\sqrt{3}}{2}$
2. $\cos(\theta - \pi) + 1 = 0$
3. $\sin 2\theta - \cos\theta = 0$
4. $\cos\theta + 1 = \sin\theta$ (Hints: First square both sides of the equation, then use an identity to write everything in terms of cosine. Then you can use the factoring technique. At the end, you MUST check your answers to make sure they're valid in the original equation.)

Answer

1. $\theta = \frac{\pi}{6} + k\pi, \theta = \frac{\pi}{3} + k\pi$, where k is any integer.

2. $\theta = 2k\pi$, where k is any integer.

3. Hint: use a double-angle formula on the first term before anything else. Use factoring on this and find solutions for two cases. All together, θ can be any odd multiple of $\frac{\pi}{2}$, as well as $\theta = \frac{\pi}{6} + 2k\pi, \theta = \frac{5\pi}{6} + 2k\pi$, for any integer k.

4. After squaring both sides, which requires FOILing, you should have $\cos^2\theta + 2\cos\theta + 1 = \sin^2\theta$. The next hint is to make everything in terms of cosine by using the Pythagorean Identity to replace $\sin^2\theta$... After factoring and solving your cases, you should get hypothetical solutions $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ and $\theta = \pi$. Checking these answers by plugging them into the original equation, we see that only $\frac{\pi}{2}$ and π actually work. So we have $\theta = \frac{\pi}{2} + 2k\pi, \theta = (2k+1)\pi$ (which is odd multiples of π).

More practice can be found in the exercises section! I'll leave you with an example straight outta Calc I where you need these skills.

Example 3

[SPOILER ALERT] The derivative of a function $f(x)$ is another function who describes the instantaneous slope of the curve $y = f(x)$ at any particular location x (the slope of the tangent line at that point, if you remember that discussion from Section 5.1). It's denoted $f'(x)$, and you will learn allll about this concept in Calc I. It turns out that for $f(x) = \sin x$, the derivative is $f'(x) = \cos x$. Now, when doing optimization problems (problems where you are looking for an "optimal" situation, like maximizing profits, or minimizing costs) you must find certain x-values known as critical numbers. These are locations at which $f'(x) = 0$.

So to find the critical numbers for the function $f(x) = \sin x$, you are looking for x where $f'(x) = \cos x = 0$. What are all such critical numbers of f?

Solution

All I want to do is figure out what x should be to cause $\cos x = 0$. I know from the unit circle that cosine is zero if $x = \frac{\pi}{2}, \frac{3\pi}{2}$. Any $x = \frac{\pi}{2} + 2k\pi$ or $x = \frac{3\pi}{2} + 2k\pi$ will therefore be a critical number. Writing out some specific examples, we have

$$x = ..., -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, ...$$

from which we can see the critical numbers are all odd multiples of $\frac{\pi}{2}$.

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8.7E Exercises

Solving Trig Equations

Give at least three specific solutions to the equation, and describe all possible solutions.

1. $-\sin\theta + 1 = 0$
2. $\sqrt{2}\cos x = 1$
3. $\tan\theta = -1$
4. $\sin\left(x + \frac{\pi}{2}\right) = \frac{1}{2}$
5. $\cos^2\theta = 1$
6. $\cot(4x) = \frac{1}{\sqrt{3}}$ (Hint: what must $\tan 4x$ be for this equation to be true?)

Answer

1. $\theta = -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$ aka $\theta = \frac{\pi}{2} + 2k\pi$ where k is any integer.
2. $x = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}, \ldots$ aka $\theta = \frac{\pi}{4} + 2k\pi, \theta = \frac{7\pi}{4} + 2k\pi$ where k is any integer.
3. $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \ldots$ aka $\theta = \frac{3\pi}{4} + k\pi$ where k is any integer.
4. $\theta = -\frac{\pi}{3}, \frac{5\pi}{3}, \ldots$ and $\theta = \frac{\pi}{3}, \frac{7\pi}{3}, \ldots$ aka $\theta = \frac{\pi}{3} + 2k\pi, \theta = \frac{5\pi}{3} + 2k\pi$, where k is any integer.
5. Hint: there are two ways that $(\cos\theta)^2$ can turn out to be positive 1. It turns out $\theta = 0, \pi, 2\pi, 3\pi, \ldots$ aka $\theta = k\pi$, where k is any integer, or "multiples of π."
6. $4x = \frac{\pi}{3} + \pi k$, so $x = \frac{1}{4}\left(\frac{\pi}{3} + \pi k\right)$, of which some examples would be $\frac{\pi}{12}$ (if $k = 0$), $\frac{\pi}{3}$ (if $k = 1$), and $\frac{7\pi}{12}$ (if $k = 2$), etc.

Solving Trig Equations With Factoring

Find all solutions.

1. $\sin^2\theta - 1 = 0$
2. $\cos^2\theta + 2\cos\theta = -1$
3. $\tan\theta\sin\theta + \sin\theta = 0$
4. $2\sin\theta\cos\theta - \cos\theta = 0$

Answer

1. $\theta = \frac{\pi}{2} + 2k\pi, \theta = \frac{3\pi}{2} + 2k\pi$ where k is any integer. These can be combined as $\theta = \frac{\pi}{2} + k\pi$. To convince yourself of this, write out a list of options for different choices of k and compare. (For example, if $k = 0$, we get $\frac{\pi}{2}, \frac{3\pi}{2}$...if $k = 1$, we get $\frac{5\pi}{2}, \frac{7\pi}{2}$... Keep going. You will get all "odd multiples" of $\frac{\pi}{2}$, which are also covered by the formula $\frac{\pi}{2} + k\pi$, if you write out a list of HIS examples.)
2. $\theta = \pi + 2k\pi$ where k is any integer
3. $\theta = k\pi$ and $\theta = \frac{3\pi}{4} + k\pi$, where k is any integer.
4. $\theta = \frac{\pi}{2} + 2k\pi, \theta = \frac{3\pi}{2} + 2k\pi$ and $\theta = \frac{\pi}{6} + 2k\pi, \theta = \frac{5\pi}{6} + 2k\pi$ where k is any integer.

Solving Trig Equations Using Identities

Give at least three specific solutions to the equation, and describe all possible solutions.

1. $2\sin\theta\cos\theta = 0$
2. $\frac{\sin^2\theta}{\cos\theta} + \cos\theta = 1$
3. $\tan x = \sec^2 x - 1$

Answer

1. Hint: this is one half of a certain identity, so it's equivalent to something simpler... Turns out to be $\theta = k\frac{\pi}{2}$ where k is any integer.
2. Hint: common denominator... $\theta = 2k\pi$ where k is any integer.
3. $\theta = \frac{\pi}{4} + k\pi$, $\theta = k\pi$ where k is any integer.

Solving a Trig Equation Involving Absolute Value

Follow the steps to find all solutions to the equation

$$\left|\tan\theta - \frac{2}{\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$$

1. Recall that the effect of absolute value bars is split into two cases:

$$|\text{inside}| = \begin{cases} \text{inside} & \text{if inside } \geq 0 \\ -(\text{inside}) & \text{if inside } < 0 \end{cases}$$

   Use this fact to split this equation into two cases.
2. Find the solutions to both case equations with the usual techniques.

Answer

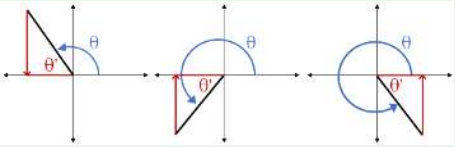
1. The cases are $\tan\theta - \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$ and $-(\tan\theta - \frac{2}{\sqrt{3}}) = \frac{1}{\sqrt{3}}$. Simplify these equations so that you can solve them.
2. Hint: you can write $\frac{3}{\sqrt{3}}$ as $\sqrt{3}$. $\theta = \frac{\pi}{6} + k\pi, \theta = \frac{\pi}{3} + k\pi$, where k is any integer.

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8.8 Chapter 8 Study Guide

Definitions

- **area:** the measure of a two-dimensional region's interior space. Area quantities come with squared units like square feet, square miles, etc.
- **perimeter:** the sum of the lengths of all sides of a two-dimensional region. This comes with length units like inches, centimeters, etc.
- **right triangles:** triangles with one right interior angle. The sides on either side of the right angle are the **legs** and the other (longest) side is the **hypotenuse.**
- **equilateral triangles:** triangles with all sides equal. All interior angles are also equal, and must be 60°.
- **isosceles triangles:** triangles with two equal sides. The angles opposite those two sides will also be equal.
- **scalene triangles:** triangles with sides of all different lengths.
- **45-45-90 triangles:** special triangles whose angles are $45^\circ, 45^\circ$, and 90°. The legs will be the same length, say x, and the hypotenuse will be $x\sqrt{2}$.
- **30-60-90 triangles:** special triangles whose angles are $30^\circ, 60^\circ$, and 90°. The shortest leg is opposite the smallest angle, and if that leg has length x, then the other leg is $x\sqrt{3}$ and the hypotenuse is $2x$.
- **Pythagorean triples:** special right triangles whose side lengths are integers. The nicest ones are (3, 4, 5) and (5, 12, 13).
- **rectangular prism:** another word for box.
- **similar triangles:** triangles with all the same angles. Their corresponding side lengths are proportional!
- **unit circle:** a circle centered at the origin with radius 1, which we use to describe angles measured as counterclockwise rotation. Sweeping all the way around the circle travels from 0 to 2π (0° to 360°).
- **degrees:** units for the measure of an angle in which a full circle is 360°.
- **radians:** units for the measure of an angle in which a full circle is 2π radians.
- **negative angles:** angles measured in a clockwise direction instead of counterclockwise.
- **coterminal angles:** angles that "end up" in the exact same location on the unit circle. For example, 90° is $\frac{\pi}{2}$, and so is $\frac{5\pi}{2}$, it just loops all the way around the circle once before arriving there.
- **quadrants:** the four quarters of the 2D plane, divided up by the axes. Starting in the top right and moving counterclockwise, these are numbered I, II, III, and IV.
- **reference angle:** when you draw an angle in Quadrants II, III, or IV, the reference angle is the acute angle found by traveling directly to the horizontal axis.



- **period:** how long it takes for a trig function to complete a full cycle of its behavior. (How far you travel along the horizontal axis before it starts to repeat itself.)
- **amplitude:** the maximum distance sine or cosine travel away from the horizontal axis before turning around. For basic sine and cosine, this is 1.
- **phase shift:** a horizontal shift of the graph of sine or cosine.

Geometry Part

Essential 2D Formulas

- **Rectangle with side lengths x and y:** area $A = xy$, perimeter $P = 2x + 2y$.
- **Square with side length x:** $A = x^2$, $P = 4x$.
- **Triangle with base length b and height h:** $A = \frac{1}{2}bh$. If a right triangle, the base and height are just the legs. To find perimeter, find all three side lengths and add.
- **Circle with radius r:** $A = \pi r^2$, perimeter is **circumference** $C = 2\pi r$. To find the area of only part of a circle, just multiply by the proportion; for example, the area of a semicircle would be $\frac{1}{2}\pi r^2$.

Essential 3D Formulas

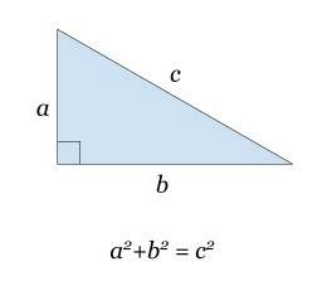
- **Rectangular Box with length x, width y, and height z:** volume $V = xyz$, surface area $S = 2xy + 2xz + 2yz$.
- **Cube with side length x:** $V = x^3$, $S = 6x^2$.
- **Cylinder with radius of base r and height h:** $V = (\text{area of base})(\text{height}) = \pi r^2 h$, $S = 2\pi r^2 + 2\pi rh$.
- **Any extruded solid with consistent cross section, with area of the base A_b and height h:** $V = A_b h$. To find surface area, decompose into the various sides as if you're cutting apart a cardboard shape, find their areas, and add.
- **Sphere with radius r:** $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$.
- **(If needed) Right Circular Cone with radius of base r and height h:** $V = \frac{1}{3}\pi r^2 h$.

Pythagorean Theorem

In a **right triangle** with **leg** lengths a and b, and **hypotenuse** length c, the following is true:

$$a^2 + b^2 = c^2$$

This can be used to find the length of any side of a right triangle if you know the other two. Simply plug in the known values and solve for the third.



Attacking Problems Involving Geometry

Try...

1. Sketching a picture of the situation.
2. Labeling any known quantities, dimensions, etc.
3. Identifying what exactly you are trying to find for your answer.
4. Using geometric relationships between the different quantities to solve for your answer.

The Distance Formula

The distance between two points $A : (x_1, y_1)$ and $B : (x_2, y_2)$ is given by

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

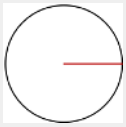
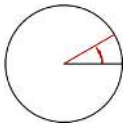
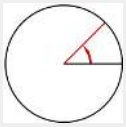
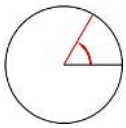
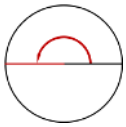
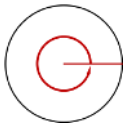
Note: It doesn't matter which point you use as the "starting point," aka in what order you subtract the x-values and y-values, but you must be consistent in the choice. So you **can't** do $(x_2 - x_1)^2$ with $(y_1 - y_2)^2$, you gotta keep the same order both times.

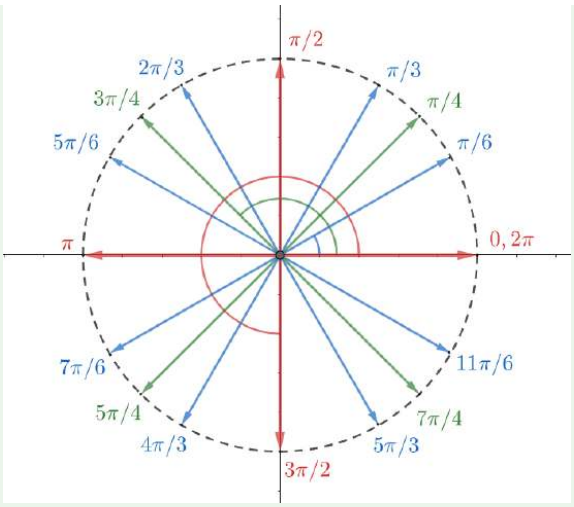
Similar Triangles

The **corresponding sides** of similar triangles are **proportional**. So if $\Delta ABC \sim \Delta DEF$, we have

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Angles on the Unit Circle

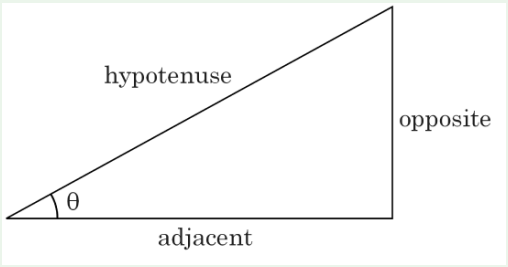
Angle in Degrees	Angle in Radians	Angle on Circle
0°	0	
30°	$\frac{\pi}{6}$	
45°	$\frac{\pi}{4}$	
60°	$\frac{\pi}{3}$	
90°	$\frac{\pi}{2}$	
180°	π	
270°	$\frac{3\pi}{2}$	
360°	2π	



To **convert from degrees to radians,** multiply the angle by $\frac{\pi}{180}$ and simplify.

To **convert from radians to degrees,** multiply the angle by $\frac{180}{\pi}$ and simplify.

The Trigonometric Ratios



The trigonometric ratios **sine, cosine, tangent, cosecant, secant,** and **cotangent** are defined as

$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp} \qquad \tan\theta = \frac{opp}{adj}$$

$$\csc\theta = \frac{hyp}{opp} \qquad \sec\theta = \frac{hyp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$

Note: the ratios in the second row is simply the **reciprocals** of the ratios in the first row.

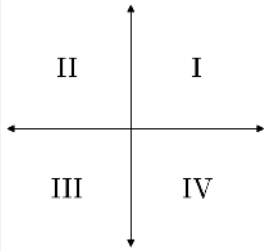
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Also Note: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$! This is very important. Confirm by dividing the side ratios of sine and cosine to see that they turn out to be $\frac{opp}{adj}$, etc.

Memory Trick: You can memorize these ratios with the "SOH-CAH-TOA" mnemonic, which stands for "Sine:Opposite/Hypotenuse, Cosine:Adjacent/Hypotenuse, Tangent:Opposite/Adjacent."

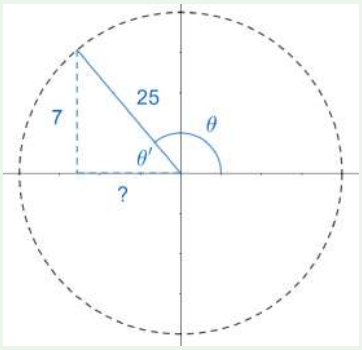
Using One Trig Ratio to Find the Others

You will be given a value like $\sin\theta = \frac{a}{b}$ and information about which **quadrant** the angle lies in. The quadrants of the plane are shown below:

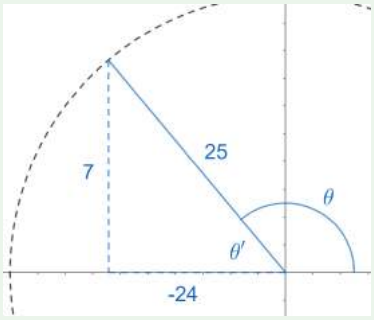


1. Sketch a right triangle in the plane in the appropriate quadrant and label θ. Label two of the sides a and b, according to the trig ratio given.
2. Use the Pythagorean Theorem to find the third side of the triangle.
3. Use the triangle to find the other trig ratios as usual.

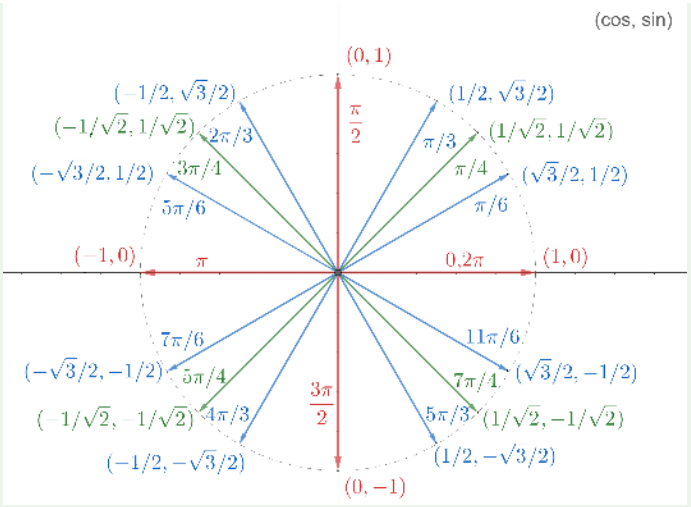
Example: for the given "$\sin\theta = \frac{7}{25}$ and θ is in Quadrant II," you would sketch this triangle:



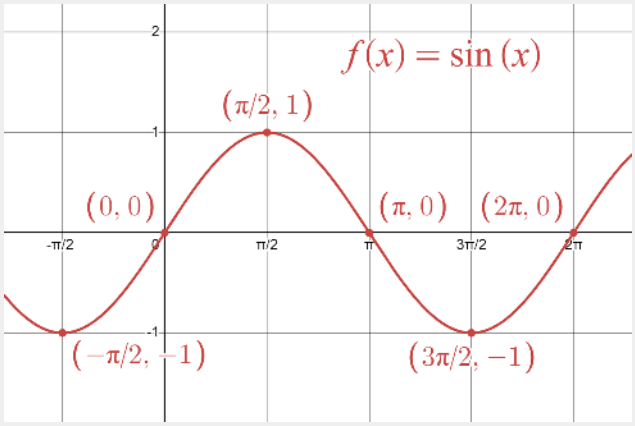
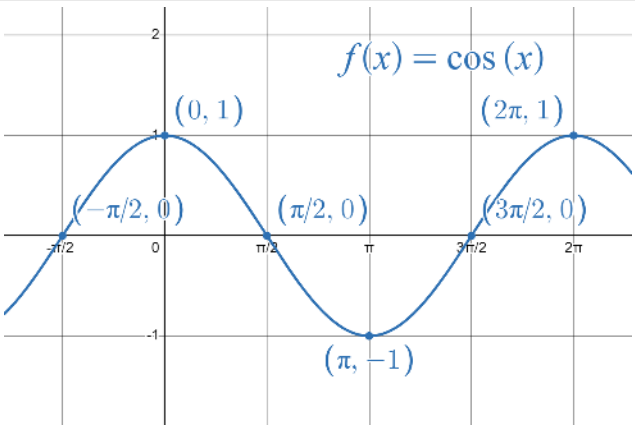
Then use the Pythagorean Theorem to complete the triangle and find all the trig ratios by definition.

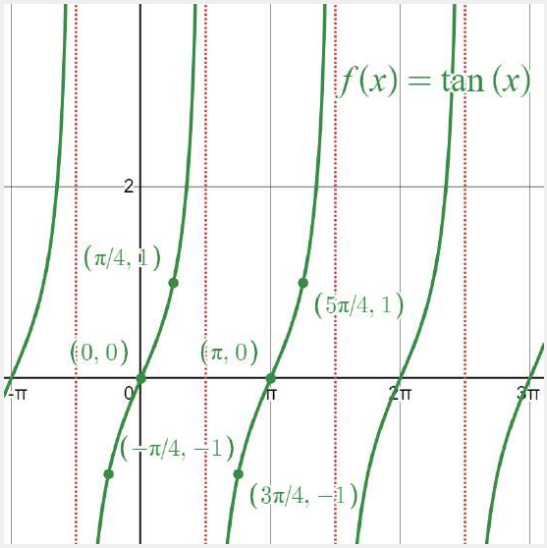


Values of Sine and Cosine From Unit Circle



Graphs of Trig Functions

Graph	Observations
	• **Periodic behavior:** The function starts at $(0,0)$, does a hill, does a valley, and returns to the height of 0 by $x = 2\pi$...and then it repeats the same pattern. This is called being **periodic** with **period** 2π, because it takes from 0 to 2π to do one full cycle. • The points called out on the graph are the easiest ones to find on the unit circle, the cardinal points. • Sine passes through $(0,0)$. That is, $\sin(0) = 0$. He is **not** symmetric about the y-axis. • The function values are **bounded** between -1 and 1. Aka, the function never goes higher than 1 or lower than -1. • Sine is nice and smooth and continuous, meaning you can draw him without ever having to pick up your pencil. • The function is defined for negative values of x: these are just negative angles, which were discussed in the last section and its exercises. In fact, his **domain** is all real numbers.
	• **Periodic behavior:** Cosine also repeats his behavior, starting the cycle over every time you travel a distance of 2π. You can see he starts at a height of 1 at $x = 0$, performs a full down-and-up cycle, and comes back to a height of 1 at $x = 2\pi$. • The points labeled are again the easiest ones to find from the unit circle. You can compare them to confirm. • Cosine passes through $(0,1)$. That is, $\cos(0) = 1$. He **is** symmetric about the y-axis. • His y-value heights are also trapped between -1 and 1. • Cosine is also nice and smooth and continuous, with **domain** all real numbers again.



- **Periodic behavior:** Tangent repeats himself, but his period is only π.
- **Vertical asymptotes:** Tangent is **undefined** at any x-value that is an odd multiple of $\pi/2$. This is because at those locations, $\cos x$ is zero, and you can't compute $\tan x = \frac{\sin x}{\cos x}$! You see vertical axymptotes at those locations, and tangent hugs them but never touches them. This means tangent is not continuous because you have to keep picking up your pencil to hop over those asymptotes!
- The locations where tangent is 1 are $\pi/4$ and $5\pi/4$. These are the "quarters" locations in Quadrants I and III, and these are the locations where sine and cosine have the same value. Thus when you divide them to find tangent, you get 1.
- Tangent passes through $(0, 0)$, and passes through the x-axis at every multiple of π. In math symbols, you can write "$\tan x = 0$ whenever $x = nx$ for some integer n."
- Due to those undefined values, the **domain** of tangent is all real numbers **except** odd multiples of $\pi/2$. You can write this in math as "$x \neq (2k-1)\frac{\pi}{2}$ for k any integer." Try some options for k to see that this expression will give you odd multiples like $-\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, etc.

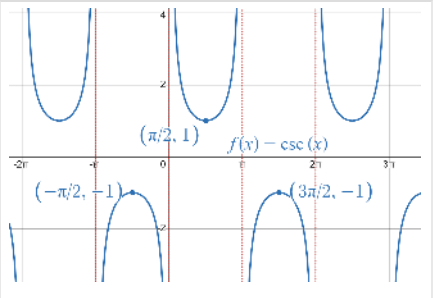
To **sketch the graph of sine or cosine,**

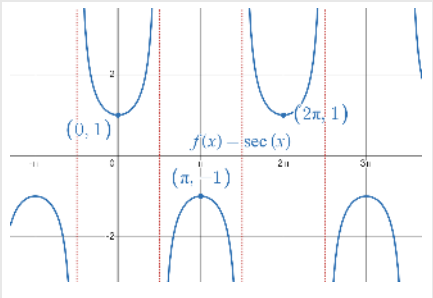
1. Sketch a baby unit circle in the margin and label the cardinal points $(1, 0), (0, 1), (-1, 0)$, and $(0, -1)$. These give you the values of sine and cosine at the inputs $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, respectively.
2. Draw the axes for your function's graph and mark those same angles on the horizontal axis. Then use your unit circle to plot the function's points at those values.
3. Use your knowledge of symmetry and the hilly shape of sine/cosine to smoothly connect the dots and follow the pattern to repeat periodically.

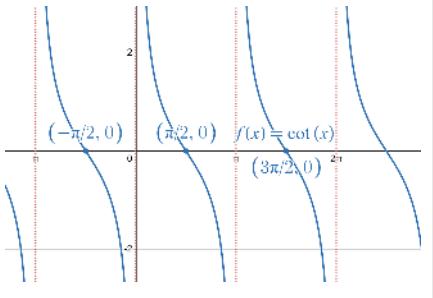
To **sketch the graph of tangent,**

1. Sketch the x- and y-axes and mark the x-values $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \ldots$ etc.
2. Recall the locations where tangent is 0 (anywhere that sine is 0): at $\ldots, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$ etc. At those locations, the function crosses the x-axis, so plot those roots.
3. Recall the locations where tangent is 1: at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$, as discussed above. Plot those points, $(\pi/4, 1)$ and $5\pi/4$.
4. Recall the locations where tangent is -1: at $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$, etc. Plot those points too.
5. Sketch in the vertical asymptotes: dotted vertical lines passing through $\frac{\pi}{2}, \frac{3\pi}{2}$, etc. on the x-axis.
6. Use your knowledge of the shape of tangent to smoothly connect the dots and follow the asymptotes up and down. Repeat the behavior periodically.

Graphs of the three reciprocal trig functions:







Transformed Trig Functions

For a function of the form $f(x) = a\sin(x + c)$ or $a\cos(x + c)$,

- $|a|$ is the **amplitude**
- c gives the **phase shift** (to the left if $+$, to the right if $-$)
- the period is 2π

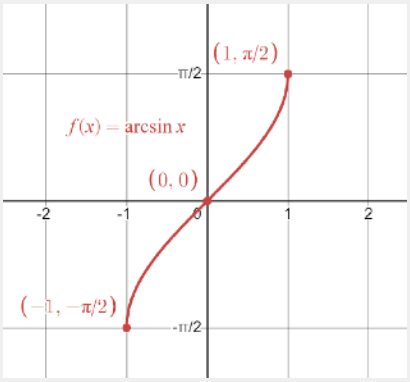
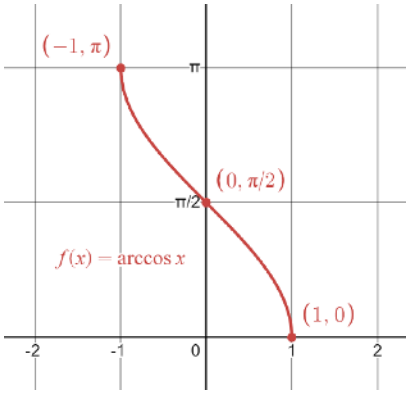
For a function of the form $f(x) = a\sin(bx)$ or $a\cos(bx)$,

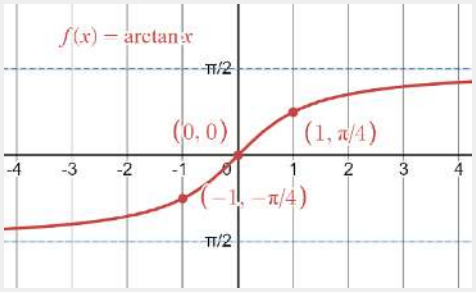
- $|a|$ is the **amplitude**
- b causes a horizontal shrink if $b > 1$, and a horizontal stretch if $0 < b < 1$
- the period is $\frac{2\pi}{b}$

To **sketch the graph of a transformed trig function,**

1. sketch the OG trig function first with its important signpost points
2. determine the transformations that have occurred and how they change the signpost points
3. plot the new signpost points and smoothly connect the dots

Inverse Trig Functions

Function	Observations
	• **Domain:** arcsine only takes inputs $-1 \le x \le 1$, because those are the only outputs of sine. In interval notation, $[-1, 1]$. • **Range:** arcsine only outputs values from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$, because that was the restricted domain we chose for sine. You can think of those angles as the "right half" of the unit circle. In interval notation, $[-\pi/2, \pi/2]$. • Arcsine passes through $(0, 0)$ because $\sin(0) = 0$, so we must have $\arcsin(0) = 0$. Similarly, it has signpost endpoints $(-1, -\pi/2)$ and $(1, \pi/2)$.
	• **Domain:** arccosine takes inputs $-1 \le x \le 1$ for the same reasons as arcsine. In interval notation, $[-1, 1]$. • **Range:** arccosine only spits out values from 0 to π, because that's the restricted domain we use for cosine (by tradition). You can remember those angles as the "top half" of the unit circle. In interval notation, $[0, \pi]$. • Arccosine's signpost points are the endpoints $(-1, \pi)$ and $(1, 0)$, and the y-intercept $(0, \pi/2)$.



- **Domain:** Arctangent takes all real numbers as inputs, because that's the range of tangent (he goes up forever and down forever). In interval notation, $[-\infty, \infty]$.
- **Range:** Arctangent spits out values **strictly** between $-\pi/2$ and $\pi/2$, because that's the conventional restricted domain of tangent. In interval notation, $(-\pi/2, \pi/2)$. Going off to the right forever, arctangent approaches the horizontal asymptote $y = \pi/2$ from below. Going off to the left, it approaches $y = -\pi/2$ from above.
- When sketching arctangent, you should plot the easy points $(0, 0), (-1, -\pi/4)$, and $(1, \pi/4)$, drop in your horizontal asymptotes, and then smoothly swoop in through the points.

There are inverse functions for the reciprocal trig functions:

- **arccosecant** ($\operatorname{arccsc} x$): domain $(-\infty, -1] \cup [1, \infty)$ and range $[-\pi/2, \pi/2]$ (this is also the restricted domain for cosecant)
- **arcsecant** ($\operatorname{arcsec} x$): domain $(-\infty, -1] \cup [1, \infty)$ and range $[0, \pi]$ (this is also the restricted domain for secant)
- **arccotangent** ($\operatorname{arccot} x$): domain $(-\infty, \infty)$ and range $(0, \pi)$ (this is also the restricted domain for cotangent)

To **evaluate an inverse trig function,**

1. Translate it to an equation involving the regular trig function. For example, "$\arcsin(1) =$?" translates to "$\sin(?) = 1$."
2. Recall your unit circle to figure out which angles give you the desired outcome.
3. Report the angle that falls in the allowable range of the inverse trig function.

The Pythagorean Identities

For any consistent expression A, we have

$$\sin^2 A + \cos^2 A = 1.$$

(Notation note: recall that $\sin^2 A$ just means $(\sin A)^2$.)

We also have

$$\tan^2 A + 1 = \sec^2 A, \qquad 1 + \cot^2 A = \csc^2 A$$

Even/Odd Identities

Since **sine** is an **odd function** and **cosine** is an **even function,**

$$\sin(-x) = -\sin x \quad \text{and} \quad \cos(-x) = \cos x$$

Sum/Difference Formulas

For **sine,** the formulas for addition or subtraction in the **input** have mixed terms:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \qquad \sin(A - B) = \sin A \cos B - \cos A \sin B$$

For **cosine,** we have matching terms:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \qquad \cos(A - B) = \cos A \cos B + \sin A \sin B$$

Also, for **tangent,** if needed, we have:

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \qquad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Double Angle Formulas

For **sine,** we have

$$\sin 2A = 2\sin A\cos A$$

For **cosine,** we have several equivalent expressions

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$$

Also, if needed, we have for **tangent**

$$\tan 2A = \frac{2\tan A}{1-\tan^2 x}$$

Solving Trig Equations

To **find all solutions to a trigonometric equation,**

1. Use algebra to manipulate the equation until it consists of one single trig function isolated on one side of the equals.
2. Find all the solutions in the usual unit circle (angles between 0 and 2π).
3. List all solutions by adding integer multiples of the period to the solutions.

Algebra tricks and techniques to try:

- Translate everything into sines and cosines and simplify.
- If there are multiple fraction terms, find a common denominator.
- On the other hand, if there is a fraction with multiple terms added/subtracted in the numerator, try splitting it into multiple fractions in case anything will simplify.
- Bring everything to one side and then try factoring and splitting into cases.
- If there are absolute value bars, split into two cases.
- Try squaring both sides to see if it makes it easier, but you must check your answers at the end to make sure they work in the original equation!
- Look for expressions that look like double-angle or sum/difference formulas in case you can replace them with something simpler.
- Try getting all the trig to be in terms of a single type (like using the Pythagorean Identity to replace $\sin^2 x$ with $1-\cos^2 x$).
- If there is a complicated input, first ignore it (you could replace it with a nickname variable) to find the angles you want, and then adjust as necessary.
- When in doubt, just rewrite the equation in many different ways using identities to replace/condense/expand/simplify/manipulate the various parts, until something works! "Write down true things."

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CHAPTER OVERVIEW

Appendices

- 4-Week Study Guide before going from MATH040 into MATH110
- Drill Sheets
  - Basic Solving Practice
  - Combo Simplification Practice
  - Exponents and Radicals Drills
  - Fill-in-the-Blanks Unit Circle
  - Fractions Drills
  - Multiplication Drills

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4-Week Study Guide before going from MATH040 into MATH110

	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7
Week 1	Review 6.1 and read 6.2 (do examples and exercises)	Do 6.2E	Read 7.1	Do 7.1E	Read 7.2	Do 7.2E	Day off / Catch-up
Week 2	Read 7.3	Do 7.3E	Read 7.4	Do 7.4E	Day off / Catch-up	Write Ch. 7 study guide and compare to 7.5	Read 8.3
Week 3	Do 8.3E	Read 8.4	Do 8.4E	Day off / Catch-up	Read 8.5	Do 8.5E	Read 8.6
Week 4	Do 8.6E	Day off / Catch-up	Read 8.7	Do 8.7E	Write Ch. 8 study guide and compare to 8.8	Finish any undone Exercises	Make a practice test from examples and do it without notes

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SECTION OVERVIEW

Drill Sheets

Basic Solving Practice

Combo Simplification Practice

Exponents and Radicals Drills

Fill-in-the-Blanks Unit Circle

Fractions Drills

Multiplication Drills

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Basic Solving Practice

Solve for x.

1. $2x + 3 = 7$
2. $4x - 5 = 15$
3. $\frac{3x}{2} + 4 = 10$
4. $5x - 2 = 3x + 6$
5. $7 - x = 2x + 8$
6. $\frac{x-1}{3} = 2$
7. $6x + 5 = 2x + 17$
8. $3(x - 4) = 2x + 1$
9. $2(x + 3) - 4 = 10$
10. $\frac{5x+2}{4} = 3$
11. $8 - \frac{x}{2} = 3$
12. $4x + 7 = 3(x - 2)$
13. $2(x + 5) = 3x - 4$
14. $\frac{2x-3}{5} = \frac{x+1}{2}$
15. $6x - 8 = 4x + 10$
16. $\frac{3x+4}{2} - 5 = 7$
17. $9 - \frac{2x}{3} = 1$
18. $4(x - 2) + 3 = 5x - 1$
19. $\frac{5x+3}{3} = 2x + 1$
20. $7x - 4 = 3x + 12$

Solve the quadratic equations.

1. $x^2 - 5x + 6 = 0$
2. $x^2 + 4x - 12 = 0$
3. $2x^2 - 8x + 6 = 0$
4. $x^2 - 9 = 0$
5. $3x^2 + 2x - 8 = 0$
6. $x^2 + 6x + 9 = 0$
7. $4x^2 - 4x - 8 = 0$
8. $x^2 - 4x - 5 = 0$
9. $2x^2 + 3x - 2 = 0$
10. $x^2 - 2x - 15 = 0$
11. $x^2 + 2x - 35 = 0$
12. $3x^2 - 14x + 8 = 0$
13. $5x^2 - 20 = 0$
14. $x^2 + 8x + 16 = 0$
15. $2x^2 - 5x + 3 = 0$

16. $x^2 - 6x + 8 = 0$
17. $4x^2 + 7x - 2 = 0$
18. $x^2 - 3x - 10 = 0$
19. $3x^2 + x - 4 = 0$
20. $x^2 + 5x + 6 = 0$

Solve the absolute value equations.

1. $|x - 3| = 7$
2. $|2x + 1| = 5$
3. $|x + 4| = 9$
4. $|3x - 2| = 4$
5. $|x - 5| = 3x - 7$
6. $|2x - 1| = 6$
7. $|x + 2| = |x - 4|$
8. $|x - 1| = 2x + 3$
9. $|x + 5| = 2x - 1$
10. $|3x + 4| = |x - 2|$

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Combo Simplification Practice

Simplify.

$3x+5x$		$\frac{2}{3}(3x+6)-4x$	
$2a-4a+7$		$\frac{5}{2}x-\frac{3}{2}x$	
$\frac{1}{2}x-\frac{3}{4}x+1$		$4(a-b)+2b$	
$4y^2+2y-6y^2+5$		$3x+\frac{1}{2}(4x-6)$	
$5x-3x+2$		$6-2(2x+1)$	
$\sqrt{16}+x^2$		$\sqrt{25}+x^2-3$	
$3(x+2)$		$2a+3b-a+2b$	
$\frac{1}{3}(3x+6)$		$6x+\frac{1}{2}(4x-2)$	
$4x^2-2x+6-3x^2$		x^2+2x-x^2+3	
$2(a+b)-3a$		$\frac{1}{4}x+\frac{3}{4}x$	
$5-3(2x-1)$		$4(a+b)-3(a-b)$	

FOIL or Factor:

$(x+2)(x+3)$		x^2+5x+6	
$(2x-5)(x+4)$		$2x^2-8x+6$	
$(3a+4)(2a-1)$		$a^2-9a+14$	
$(x-6)(x+7)$		$x^2-7x+10$	
$(5y+2)(3y-1)$		$3y^2+5y-2$	
$(x+2)(x-1)(x+3)$		$x^2+4x-12$	
$(2x-3)(x+4)(x-1)$		$2a^2-3a-5$	
$(a+1)(a-2)(a+3)$		$x^2-10x+21$	
$(y+2)(y-3)(y+4)$		a^2-4a+3	
$(2x+1)(x-2)(x+5)$		$x^2-11x+24$	

Simplify.

$\frac{2x^2+4x}{2x}$	
$\frac{3x^2-12}{3x}$	
$\frac{x^2-4}{x^2-2x}$	
$\frac{x^2-9}{x^2-6x+9}$	
$\frac{5x^2-10x}{x(x-2)}$	
$\frac{2x^2-8}{4x}$	

$\frac{x^2-16}{x^2-4x+4}$	
$\frac{6x^2-3x}{3x}$	
$\frac{x^2+5x+6}{x^2+2x}$	

$\frac{2}{x}+\frac{3}{x+1}$	
$\frac{4}{x+2}-\frac{2}{x+3}$	
$\frac{3}{x}+\frac{5}{x+2}$	
$\frac{1}{x+1}-\frac{1}{x+3}$	
$\frac{2x}{x+3}-\frac{3}{x+2}$	
$\frac{5}{x-1}+\frac{7}{x+1}$	
$\frac{2x}{x+4}+\frac{3}{x+2}$	
$\cdot\ \frac{x+3}{x+1}-\frac{x-1}{x+2}$	
$\frac{3x+5}{x+2}-\frac{x+2}{x}$	

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Exponents and Radicals Drills

1. $2^3 \cdot 2^5 =$
2. $\frac{3^4}{3^2} =$
3. $5^6 \cdot \frac{1}{5^3} =$
4. $(2^2)^3 =$
5. $(4^3)^{-2} =$
6. $(6^4)^{\frac{1}{2}} =$
7. $\frac{2^8}{2^4} =$
8. $3^5 \cdot 3^0 =$
9. $(5^2)^{-3} =$
10. $(2^3)^2 =$
11. $(4^2)^{-2} =$
12. $(6^3)^{\frac{1}{3}} =$
13. $\frac{2^7}{2^2} =$
14. $\frac{3^3}{3^3} =$
15. $5^{-4} \cdot 5^4 =$
16. $(2^4)^{-3} =$
17. $(4^5)^{-1} =$
18. $(6^2)^{\frac{1}{2}} =$
19. $\frac{2^5}{2^3} =$
20. $3^2 \cdot 3^{-2} =$
21. $(5^3)^{-2} =$
22. $(2^6)^{-1} =$
23. $(4^3)^{\frac{1}{3}} =$
24. $6^0 =$
25. $2^4 \cdot 2^6 =$
26. $\frac{3^0}{3^6} =$
27. $5^{-2} \cdot 5^3 =$
28. $(2^2)^{-4} =$
29. $(4^4)^{-2} =$
30. $(6^5)^{\frac{1}{5}} =$
31. $\frac{2^9}{2^7} =$
32. $3^3 \cdot 3^2 =$
33. $5^0 \cdot 5^5 =$
34. $(2^5)^{-3} =$
35. $(4^1)^{-2} =$
36. $(6^4)^{\frac{1}{4}} =$
37. $\frac{2^6}{2^6} =$
38. $\frac{1}{3^3} \cdot 3^3 =$
39. $\frac{5^4}{5^2} =$
40. $\frac{1}{2^7} =$

https://math.libretexts.org/@go/page/153749

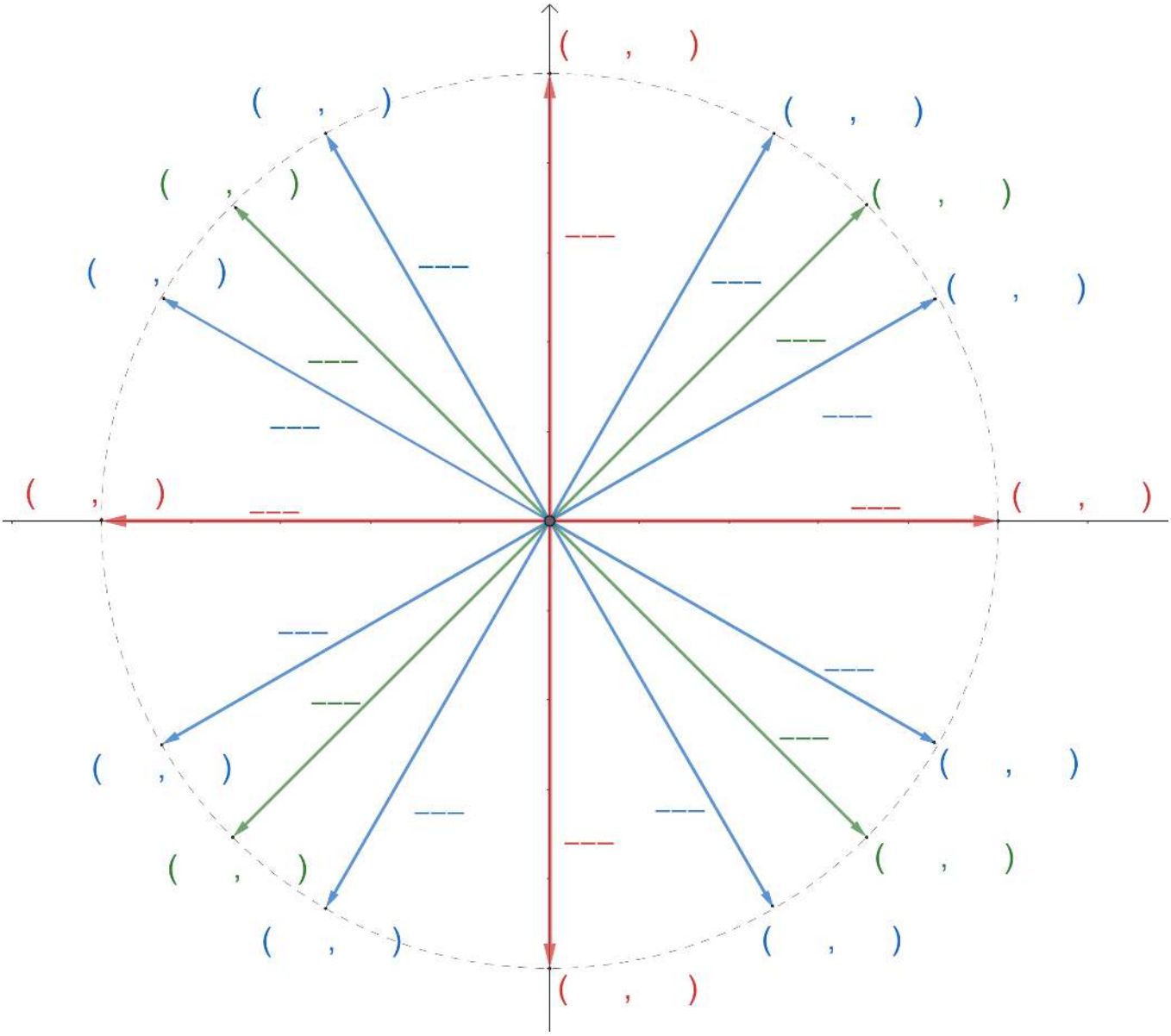
1. $a^3 \cdot a^5 =$
2. $b^4 \cdot b^6 =$
3. $c^2 \cdot c^{-3} =$
4. $d^3 \cdot \frac{1}{d^2} =$
5. $(e^2)^3 =$
6. $(f^{-2})^{-3} =$
7. $\frac{g^4}{g^{-2}} =$
8. $h^5 \cdot h^{-3} =$
9. $(i^3)^{\frac{1}{2}} =$
10. $\frac{j^6}{j^4} =$
11. $k^7 \cdot k^{-4} =$
12. $\frac{l^8}{l^{-3}} =$
13. $(m^2)^{-3} =$
14. $(n^{-1})^4 =$
15. $\frac{o^5}{o^2} =$
16. $p^3 \cdot p^{-2} =$
17. $(q^4)^{\frac{1}{3}} =$
18. $\frac{r^7}{r^5} =$
19. $s^{-3} \cdot s^2 =$
20. $(t^3)^{-2} =$

1. $2^3 \cdot 2^5$
2. $3^4 \cdot 3^6$
3. $x^2 \cdot x^5$
4. $y^3 \cdot y^{-2}$
5. $(2x)^3 \cdot (2x)^2$
6. $(3y)^{-2} \cdot (3y)^3$
7. $(a^2b)^3 \cdot (a^2b)^{-1}$
8. $(4x^2y)^2 \cdot (4x^2y)^{-3}$
9. $(5xy^2)^{-3} \cdot (5xy^2)^{-1}$
10. $(ab^{-2}c^3)^2 \cdot (ab^{-2}c^3)^{-1}$
11. $\frac{2^7}{2^4}$
12. $\frac{3^8}{3^3}$
13. $\frac{x^6}{x^3}$
14. $\frac{y^5}{y^{-2}}$
15. $\frac{(2x)^4}{(2x)^2}$
16. $\frac{(3y)^{-3}}{(3y)^2}$
17. $\frac{(a^3b)^4}{(a^3b)^2}$
18. $\frac{(4x^3y)^5}{(4x^3y)^{-2}}$
19. $\frac{(5xy^{-2}z)^6}{(5xy^{-2}z)^3}$
20. $\frac{(ab^{-3}c^2)^7}{(ab^{-3}c^2)^{-2}}$

1. $\sqrt{16} =$
2. $\sqrt{25} =$
3. $\sqrt{36} =$
4. $\sqrt{49} =$
5. $\sqrt{64} =$
6. $\sqrt{81} =$
7. $\sqrt{100} =$
8. $\sqrt{121} =$
9. $\sqrt{144} =$
10. $\sqrt{169} =$
11. $\sqrt{x^2} =$
12. $\sqrt{y^2} =$
13. $\sqrt{a^2} =$
14. $\sqrt{b^2} =$
15. $\sqrt{x^4} =$
16. $\sqrt{y^4} =$
17. $\sqrt{a^6} =$
18. $\sqrt{b^6} =$
19. $\sqrt{x^6} =$
20. $\sqrt{y^6} =$
21. $\sqrt[3]{8x^3} =$
22. $\sqrt[3]{27y^6} =$
23. $\sqrt[3]{64a^6} =$
24. $\sqrt[3]{125b^9} =$
25. $\sqrt[4]{16x^4} =$
26. $\sqrt[4]{81y^8} =$
27. $\sqrt[4]{256a^8} =$
28. $\sqrt[4]{625b^8} =$
29. $\sqrt[5]{32x^{10}} =$
30. $\sqrt[5]{64y^{15}} =$
31. $\sqrt[4]{(\sqrt{y})^{12}} =$
32. $\sqrt{(\sqrt{y})^4} =$
33. $\sqrt[4]{\left(\sqrt{y^4}\right)^2} =$
34. $\left(\sqrt{y^3}\right)^2 =$
35. $\sqrt[3]{\left(\sqrt{x^2}\right)^3} =$
36. $\sqrt[4]{(\sqrt{y})^3} =$
37. $\left(\sqrt{(\sqrt{x})^2}\right)^3 =$
38. $\sqrt[3]{(\sqrt{x})^6} =$
39. $\sqrt{(\sqrt{x})^2} =$
40. $(\sqrt[4]{y})^3 =$

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Fill-in-the-Blanks Unit Circle



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Fractions Drills

1. $\frac{1}{2} + \frac{1}{3} =$ _____
2. $\frac{2}{3} + \frac{3}{4} =$ _____
3. $\frac{3}{5} + \frac{2}{5} =$ _____
4. $\frac{2}{7} + \frac{5}{7} =$ _____
5. $\frac{1}{4} + \frac{3}{8} =$ _____
6. $\frac{3}{6} + \frac{2}{3} =$ _____
7. $\frac{2}{9} + \frac{4}{9} =$ _____
8. $\frac{4}{6} + \frac{3}{4} =$ _____
9. $\frac{5}{8} + \frac{1}{8} =$ _____
10. $\frac{1}{2} + \frac{1}{4} =$ _____
11. $\frac{3}{4} + \frac{2}{5} =$ _____
12. $\frac{5}{6} + \frac{1}{2} =$ _____
13. $\frac{4}{7} + \frac{3}{7} =$ _____
14. $\frac{3}{8} + \frac{1}{8} =$ _____
15. $\frac{5}{9} + \frac{4}{9} =$ _____
16. $\frac{1}{5} + \frac{2}{10} =$ _____
17. $\frac{3}{10} + \frac{1}{4} =$ _____
18. $\frac{2}{5} + \frac{3}{10} =$ _____
19. $\frac{3}{6} + \frac{1}{3} =$ _____
20. $\frac{2}{3} + \frac{2}{9} =$ _____

Time: _____

21. $\frac{3}{4} - \frac{1}{3} =$ _____
22. $\frac{2}{3} - \frac{3}{4} =$ _____
23. $\frac{3}{5} - \frac{2}{5} =$ _____
24. $\frac{2}{7} - \frac{5}{7} =$ _____
25. $\frac{1}{4} - \frac{3}{8} =$ _____
26. $\frac{3}{6} - \frac{2}{3} =$ _____
27. $\frac{2}{9} - \frac{4}{9} =$ _____
28. $\frac{4}{6} - \frac{3}{4} =$ _____
29. $\frac{5}{8} - \frac{1}{8} =$ _____
30. $\frac{1}{2} - \frac{1}{4} =$ _____
31. $\frac{5}{6} - \frac{1}{2} =$ _____
32. $\frac{5}{8} - \frac{1}{8} =$ _____
33. $\frac{2}{7} - \frac{5}{7} =$ _____
34. $\frac{1}{4} - \frac{3}{8} =$ _____
35. $\frac{5}{9} - \frac{4}{9} =$ _____
36. $\frac{1}{5} - \frac{2}{10} =$ _____
37. $\frac{3}{10} - \frac{1}{4} =$ _____
38. $\frac{2}{5} - \frac{3}{10} =$ _____
39. $\frac{3}{6} - \frac{1}{3} =$ _____
40. $\frac{2}{3} - \frac{2}{9} =$ _____

Time: _____

1. $\frac{1}{5}+\frac{1}{9}=$ _____
2. $\frac{1}{6}+\frac{1}{4}=$ _____
3. $\frac{1}{3}+\frac{1}{5}=$ _____
4. $\frac{1}{2}+\frac{1}{8}=$ _____
5. $\frac{1}{21}+\frac{1}{16}=$ _____
6. $\frac{1}{32}+\frac{1}{8}=$ _____
7. $\frac{1}{3}+\frac{1}{32}=$ _____
8. $\frac{1}{28}+\frac{1}{4}=$ _____
9. $\frac{1}{15}+\frac{1}{6}=$ _____
10. $\frac{1}{35}+\frac{1}{7}=$ _____
11. $\frac{1}{7}+\frac{1}{8}=$ _____
12. $\frac{1}{2}+\frac{1}{10}=$ _____
13. $\frac{1}{4}+\frac{1}{6}=$ _____
14. $\frac{1}{12}+\frac{1}{9}=$ _____
15. $\frac{1}{21}+\frac{1}{7}=$ _____
16. $\frac{1}{24}+\frac{1}{6}=$ _____
17. $\frac{1}{20}+\frac{1}{12}=$ _____
18. $\frac{1}{8}+\frac{1}{3}=$ _____
19. $\frac{1}{16}+\frac{1}{32}=$ _____
20. $\frac{1}{18}+\frac{1}{27}=$ _____

Time: _____

1. $\frac{1}{20}-\frac{1}{16}=$ _____
2. $\frac{1}{24}-\frac{1}{8}=$ _____
3. $\frac{1}{3}-\frac{1}{36}=$ _____
4. $\frac{1}{32}-\frac{1}{8}=$ _____
5. $\frac{1}{28}-\frac{1}{14}=$ _____
6. $\frac{1}{8}-\frac{1}{16}=$ _____
7. $\frac{1}{10}-\frac{1}{7}=$ _____
8. $\frac{1}{35}-\frac{1}{4}=$ _____
9. $\frac{1}{6}-\frac{1}{4}=$ _____
10. $\frac{1}{3}-\frac{1}{33}=$ _____
11. $\frac{1}{3}-\frac{1}{5}=$ _____
12. $\frac{1}{12}-\frac{1}{9}=$ _____
13. $\frac{1}{5}-\frac{1}{9}=$ _____
14. $\frac{1}{7}-\frac{1}{8}=$ _____
15. $\frac{1}{35}-\frac{1}{5}=$ _____
16. $\frac{1}{2}-\frac{1}{10}=$ _____
17. $\frac{1}{20}-\frac{1}{7}=$ _____
18. $\frac{1}{15}-\frac{1}{30}=$ _____
19. $\frac{1}{28}-\frac{1}{10}=$ _____
20. $\frac{1}{10}-\frac{1}{6}=$ _____

Time: _____

1. $\frac{1}{2} \times \frac{2}{3} =$ _____
2. $\frac{2}{5} \times \frac{3}{4} =$ _____
3. $\frac{4}{7} \times \frac{2}{5} =$ _____
4. $\frac{3}{5} \times \frac{1}{2} =$ _____
5. $\frac{2}{3} \times \frac{3}{4} =$ _____
6. $\frac{5}{6} \times \frac{2}{5} =$ _____
7. $\frac{4}{9} \times \frac{1}{3} =$ _____
8. $\frac{3}{8} \times \frac{2}{7} =$ _____
9. $\frac{1}{2} \times \frac{3}{5} =$ _____
10. $\frac{3}{7} \times \frac{2}{3} =$ _____
11. $\frac{1}{3} \times \frac{4}{7} =$ _____
12. $\frac{2}{5} \times \frac{1}{2} =$ _____
13. $\frac{3}{4} \times \frac{1}{3} =$ _____
14. $\frac{1}{2} \times \frac{3}{4} =$ _____
15. $\frac{3}{5} \times \frac{2}{3} =$ _____
16. $\frac{2}{3} \times \frac{1}{4} =$ _____
17. $\frac{3}{8} \times \frac{1}{5} =$ _____
18. $\frac{4}{9} \times \frac{3}{4} =$ _____
19. $\frac{2}{5} \times \frac{4}{7} =$ _____
20. $\frac{1}{3} \times \frac{5}{8} =$ _____
21. $\frac{3}{4} \times \frac{1}{5} =$ _____
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36. $\frac{2}{5} \times \frac{1}{3} =$ _____
37. $\frac{1}{2} \times \frac{3}{7} =$ _____
38. $\frac{2}{3} \times \frac{2}{5} =$ _____
39. $\frac{1}{4} \times \frac{2}{9} =$ _____
40. $\frac{3}{7} \times \frac{1}{4} =$ _____

Time: _____

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×	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										

(1)

1. $2 \times 7 =$ __
2. $8 \times 3 =$ __
3. $1 \times 10 =$ __
4. $5 \times 8 =$ __
5. $9 \times 2 =$ __
6. $4 \times 3 =$ __
7. $6 \times 7 =$ __
8. $10 \times 1 =$ __
9. $3 \times 9 =$ __
10. $7 \times 2 =$ __
11. $6 \times 1 =$ __
12. $4 \times 6 =$ __
13. $9 \times 7 =$ __
14. $5 \times 2 =$ __
15. $10 \times 4 =$ __
16. $1 \times 2 =$ __
17. $3 \times 7 =$ __
18. $8 \times 6 =$ __
19. $4 \times 5 =$ __
20. $7 \times 10 =$ __
21. $4 \times 9 =$ __
22. $10 \times 5 =$ __
23. $6 \times 4 =$ __
24. $3 \times 6 =$ __
25. $7 \times 5 =$ __
26. $2 \times 10 =$ __
27. $1 \times 8 =$ __
28. $9 \times 4 =$ __
29. $5 \times 6 =$ __
30. $8 \times 10 =$ __
31. $2 \times 5 =$ __
32. $3 \times 10 =$ __
33. $1 \times 6 =$ __
34. $8 \times 9 =$ __
35. $7 \times 1 =$ __
36. $6 \times 8 =$ __
37. $9 \times 10 =$ __
38. $2 \times 3 =$ __
39. $5 \times 9 =$ __
40. $1 \times 4 =$ __

1. $2 \times 7 =$ __
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3. $3 \times 9 =$ __
4. $5 \times 8 =$ __
5. $9 \times 2 =$ __
6. $4 \times 3 =$ __
7. $6 \times 7 =$ __
8. $7 \times 2 =$ __
9. $6 \times 1 =$ __
10. $4 \times 6 =$ __
11. $10 \times 4 =$ __
12. $2 \times 3 =$ __
13. $8 \times 9 =$ __
14. $5 \times 2 =$ __
15. $1 \times 4 =$ __
16. $3 \times 8 =$ __
17. $4 \times 7 =$ __
18. $7 \times 3 =$ __
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27. $4 \times 9 =$ __
28. $8 \times 6 =$ __
29. $9 \times 10 =$ __
30. $1 \times 7 =$ __
31. $3 \times 7 =$ __
32. $5 \times 9 =$ __
33. $7 \times 6 =$ __
34. $4 \times 5 =$ __
35. $6 \times 8 =$ __
36. $9 \times 5 =$ __
37. $1 \times 10 =$ __
38. $10 \times 6 =$ __
39. $8 \times 5 =$ __
40. $2 \times 8 =$ __

1. $10 \times 11 =$ __
2. $20 \times 30 =$ __
3. $70 \times 12 =$ __
4. $1000 \times 40 =$ __
5. $60 \times 50 =$ __
6. $400 \times 10 =$ __
7. $30 \times 90 =$ __
8. $60 \times 80 =$ __
9. $20 \times 70 =$ __
10. $300 \times 70 =$ __
11. $25 \times 40 =$ __
12. $900 \times 10 =$ __
13. $60 \times 70 =$ __
14. $100 \times 50 =$ __
15. $80 \times 100 =$ __
16. $1000 \times 20 =$ __
17. $800 \times 10 =$ __
18. $70 \times 500 =$ __
19. $30 \times 600 =$ __
20. $500 \times 700 =$ __
21. $300 \times 70 =$ __
22. $90 \times 70 =$ __
23. $12 \times 100 =$ __
24. $80 \times 50 =$ __
25. $25 \times 20 =$ __
26. $700 \times 80 =$ __
27. $500 \times 12 =$ __
28. $700 \times 40 =$ __
29. $80 \times 100 =$ __
30. $90 \times 200 =$ __
31. $1000 \times 700 =$ __
32. $600 \times 20 =$ __
33. $50 \times 60 =$ __
34. $40 \times 80 =$ __
35. $30 \times 90 =$ __
36. $200 \times 700 =$ __
37. $900 \times 100 =$ __
38. $600 \times 400 =$ __
39. $40 \times 800 =$ __
40. $100 \times 900 =$ __

1. $1^2 =$ __
2. $2^2 =$ __
3. $3^2 =$ __
4. $4^2 =$ __
5. $5^2 =$ __
6. $6^2 =$ __
7. $7^2 =$ __
8. $8^2 =$ __
9. $9^2 =$ __
10. $10^2 =$ __
11. $11^2 =$ __
12. $12^2 =$ __

1. $10^2 =$ __
2. $20^2 =$ __
3. $30^2 =$ __
4. $40^2 =$ __
5. $50^2 =$ __
6. $60^2 =$ __
7. $70^2 =$ __
8. $80^2 =$ __
9. $90^2 =$ __
10. $100^2 =$ __

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