

PREPALS MATH WITH BUSINESS APPS



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PrePALS Math with Business Apps

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Glossary

Detailed Licensing

Licensing

A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.

1: WHOLE NUMBERS

1.1: Place Value and Names for Whole Numbers

1.2: Adding and Subtracting Whole Numbers

1.3: Multiplying and Dividing Whole Numbers

1.4: Order of Operations

1.5: Positive and Negative Numbers

1.1: Place Value and Names for Whole Numbers

Learning Objectives

- Identify counting numbers and whole numbers
- Model whole numbers
- Identify the place value of a digit
- Use place value to name whole numbers
- Use place value to write whole numbers
- Round whole numbers

Identify Counting Numbers and Whole Numbers

Learning algebra is similar to learning a language. You start with a basic vocabulary and then add to it as you go along. You need to practice often until the vocabulary becomes easy to you. The more you use the vocabulary, the more familiar it becomes.

Algebra uses numbers and symbols to represent words and ideas. Let's look at the numbers first. The most basic numbers used in algebra are those we use to count objects: 1, 2, 3, 4, 5, ... and so on. These are called the **counting numbers**. The notation "..." is called an ellipsis, which is another way to show "and so on", or that the pattern continues endlessly. Counting numbers are also called natural numbers.

Definition: Counting Numbers

The counting numbers start with 1 and continue.

1, 2, 3, 4, 5 ...

Counting numbers and whole numbers can be visualized on a number line as shown in Figure 1.1.1.



Figure 1.1.1: The numbers on the number line increase from left to right, and decrease from right to left.

The point labeled 0 is called the **origin**. The points are equally spaced to the right of 0 and labeled with the counting numbers. When a number is paired with a point, it is called the **coordinate** of the point.

The discovery of the number zero was a big step in the history of mathematics. Including zero with the counting numbers gives a new set of numbers called the **whole numbers**.

Definition: Whole Numbers

The whole numbers are the counting numbers and zero.

0, 1, 2, 3, 4, 5 ...

We stopped at 5 when listing the first few counting numbers and whole numbers. We could have written more numbers if they were needed to make the patterns clear.

Example 1.1.1: Number Identification

Which of the following are

- a. counting numbers
- b. whole numbers

$$0, \frac{1}{4}, 3, 5.2, 15, 105$$

Solution

- a. The counting numbers start at 1, so 0 is not a counting number. The numbers 3, 15, and 105 are all counting numbers.
- b. Whole numbers are counting numbers and 0. The numbers 0, 3, 15, and 105 are whole numbers. The numbers $\frac{1}{4}$ and 5.2 are neither counting numbers nor whole numbers. We will discuss these numbers later.

Exercise 1.1.1

Which of the following are

- a. whole numbers

$$0, \frac{2}{3}, 2, 9, 11.8, 241, 376$$

Answer a available at the end of this section.

Answer b available at the end of this section.

Exercise 1.1.2

Which of the following are

- a. counting numbers
- b. whole numbers

$$0, \frac{5}{3}, 7, 8.8, 13, 201$$

Answer a available at the end of this section.

Answer b available at the end of this section.

Model Whole Numbers

Our number system is called a **place value system** because the value of a digit depends on its position, or place, in a number. The number 537 has a different value than the number 735. Even though they use the same digits, their value is different because of the different placement of the 3 and the 7 and the 5.

Money gives us a familiar model of place value. Suppose a wallet contains three \$100 bills, seven \$10 bills, and four \$1 bills. The amounts are summarized in Figure 1.1.2. How much money is in the wallet?



Figure 1.1.2

Find the total value of each kind of bill, and then add to find the total. The wallet contains \$374.

$$\begin{array}{c} \$300 + \$70 + \$4 \\ \swarrow \quad \downarrow \quad \searrow \\ \$374 \end{array}$$

Base-10 blocks provide another way to model place value, as shown in Figure 1.1.3. The blocks can be used to represent hundreds, tens, and ones. Notice that the tens rod is made up of 10 ones, and the hundreds square is made of 10 tens, or 100 ones.

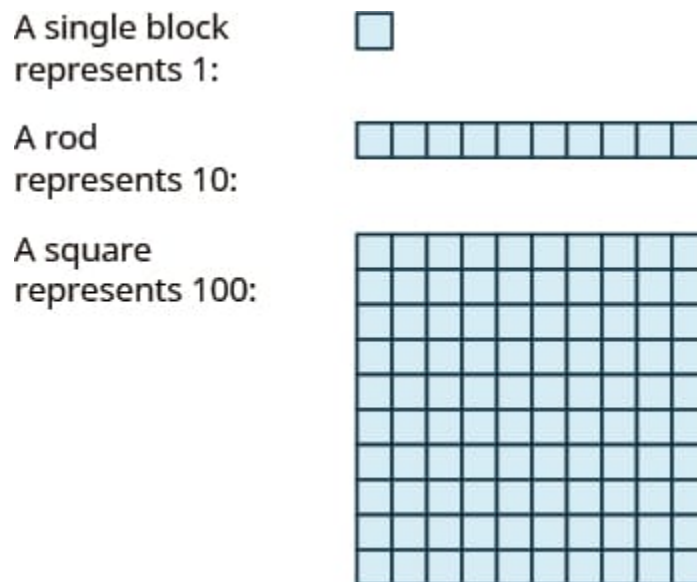


Figure 1.1.3

Figure 1.1.4 shows the number 138 modeled with base-10 blocks.

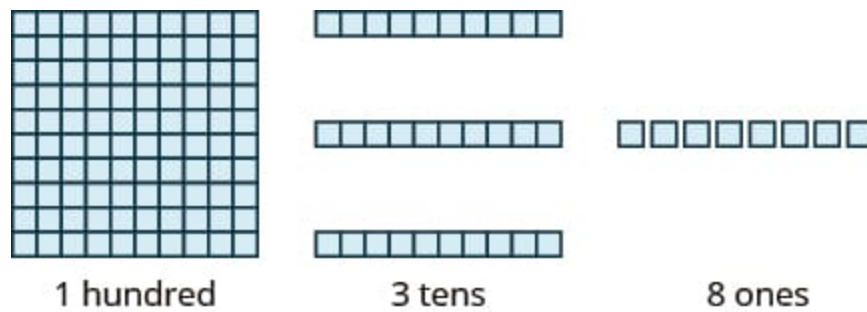


Figure 1.1.4: We use place value notation to show the value of the number 138.

$$100 + 30 + 8$$

138

Digit	Place value	Number	Value	Total value
1	hundreds	1	100	100
3	tens	3	10	30
8	ones	8	1	+8
			Sum =	138

Example 1.1.2: place value notation

Use place value notation to find the value of the number modeled by the base-10 blocks shown.

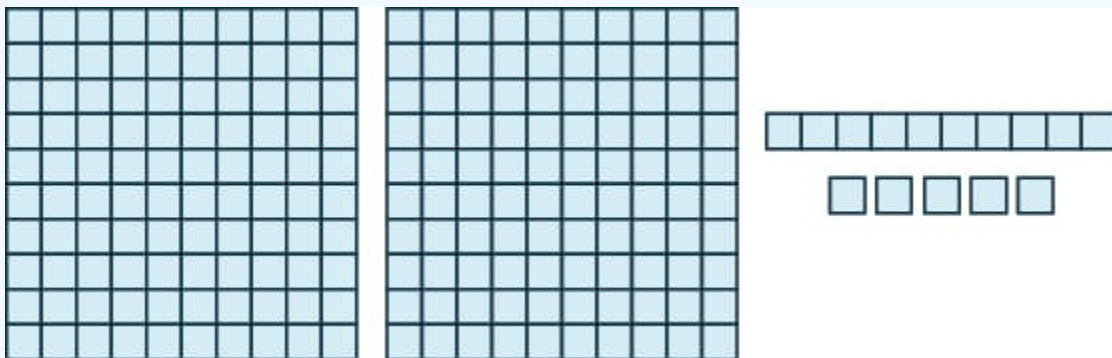


Figure 1.1.5

Solution

There are 2 hundreds squares, which is 200.

There is 1 tens rod, which is 10.

There are 5 ones blocks, which is 5.

$$200 + 10 + 5$$

$$215$$

Digit	Place value	Number	Value	Total value
2	hundreds	2	100	200
1	tens	1	10	10
5	ones	5	1	+5
				215

The base-10 blocks model the number 215.

Identify the Place Value of a Digit

By looking at money and base-10 blocks, we saw that each place in a number has a different value. A place value chart is a useful way to summarize this information. The place values are separated into groups of three, called periods. The periods are *ones*, *thousands*, *millions*, *billions*, *trillions*, and so on. In a written number, commas separate the periods.

Just as with the base-10 blocks, where the value of the tens rod is ten times the value of the ones block and the value of the hundreds square is ten times the tens rod, the value of each place in the place-value chart is ten times the value of the place to the right of it.

Figure 1.1.8 shows how the number 5,278,194 is written in a place value chart.

Place Value														
Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions														
Ten trillions														
Trillions														
Hundred billions														
Ten billions														
Billions														
Hundred millions														
Ten millions														
Millions	5													
Hundred thousands	2													
Ten thousands	7													
Thousands	8													
Hundreds	1													
Tens	9													
Ones	4													

Figure 1.1.8

- The digit 5 is in the millions place. Its value is 5,000,000.
- The digit 2 is in the hundred thousands place. Its value is 200,000.
- The digit 7 is in the ten thousands place. Its value is 70,000.
- The digit 8 is in the thousands place. Its value is 8,000.
- The digit 1 is in the hundreds place. Its value is 100.

- The digit 9 is in the tens place. Its value is 90.
- The digit 4 is in the ones place. Its value is 4.

Example 1.1.3: place value

In the number 63,407,218; find the place value of each of the following digits:

- 7
- 0
- 1
- 6
- 3

Solution

Write the number in a place value chart, starting at the right.

Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
							6	3	4	0	7	2	1	8

Figure 1.1.9

- The 7 is in the thousands place.
- The 0 is in the ten thousands place.
- The 1 is in the tens place.
- The 6 is in the ten millions place.
- The 3 is in the millions place.

Exercise 1.1.6

For each number, find the place value of digits listed: 519,711,641,328

- 9
- 4
- 2
- 6
- 7

Answer a available at the end of this section.

Answer b available at the end of this section.

Answer c available at the end of this section.

Answer d available at the end of this section.

Answer e available at the end of this section.

Use Place Value to Name Whole Numbers

When you write a check, you write out the number in words as well as in digits. To write a number in words, write the number in each period followed by the name of the period without the 's' at the end. Start with the digit at the left, which has the largest place value. The commas separate the periods, so wherever there is a comma in the number, write a comma between the words. The ones period, which has the smallest place value, is not named.



So the number 37,519,248 is written thirty-seven million, five hundred nineteen thousand, two hundred forty-eight. Notice that the word and is not used when naming a whole number.

How to: Name a Whole Number in Words.

Step 1. Starting at the digit on the left, name the number in each period, followed by the period name. Do not include the period name for the ones.

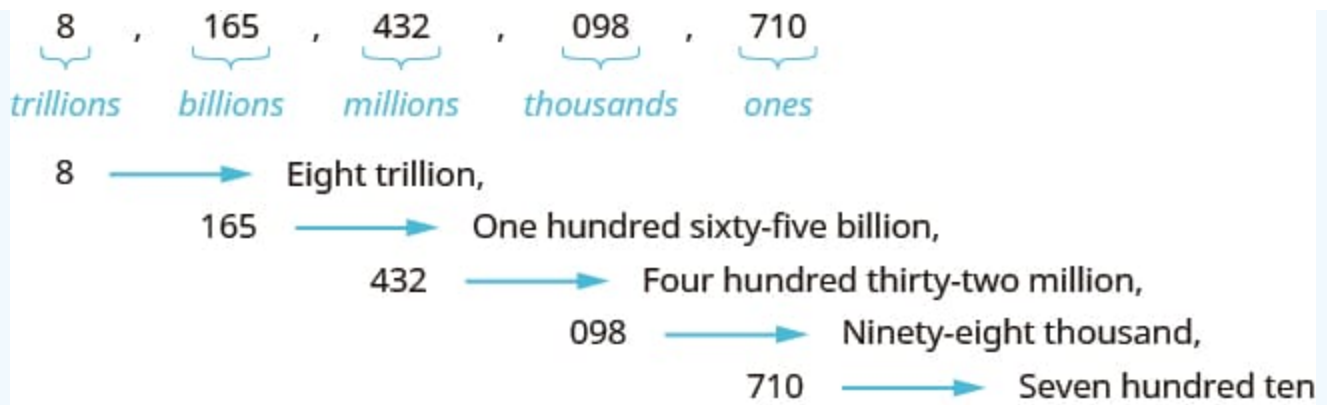
Step 2. Use commas in the number to separate the periods.

Example 1.1.4: name whole numbers

Name the number 8,165,432,098,710 in words.

Solution

Begin with the leftmost digit, which is 8. It is in the trillions place.	eight trillion
The next period to the right is billions.	one hundred sixty-five billion
The next period to the right is millions.	four hundred thirty-two million
The next period to the right is thousands.	ninety-eight thousand
The rightmost period shows the ones.	seven hundred ten



Putting all of the words together, we write 8,165,432,098,710 as eight trillion, one hundred sixty-five billion, four hundred thirty-two million, ninety-eight thousand, seven hundred ten.

Exercise 1.1.7

Name each number in words: 9,258,137,904,061

Answer available at the end of this section.

Example 1.1.5: name whole numbers

A student conducted research and found that the number of mobile phone users in the United States during one month in 2014 was 327,577,529. Name that number in words.

Solution

Identify the periods associated with the number.



Name the number in each period, followed by the period name. Put the commas in to separate the periods.

Millions period: three hundred twenty-seven million

Thousands period: five hundred seventy-seven thousand

Ones period: five hundred twenty-nine

So the number of mobile phone users in the United States during the month of April was three hundred twenty-seven million, five hundred seventy-seven thousand, five hundred twenty-nine.

Exercise 1.1.9

The population in a country is 316,128,839. Name that number

Answer available at the end of this section.

Use Place Value to Write Whole Numbers

We will now reverse the process and write a number given in words as digits.

How to: Use Place Value to Write Whole Numbers

- Step 1. Identify the words that indicate periods. (Remember the ones period is never named.)
- Step 2. Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.
- Step 3. Name the number in each period and place the digits in the correct place value position.

Example 1.1.6: write whole numbers

Write the following numbers using digits.

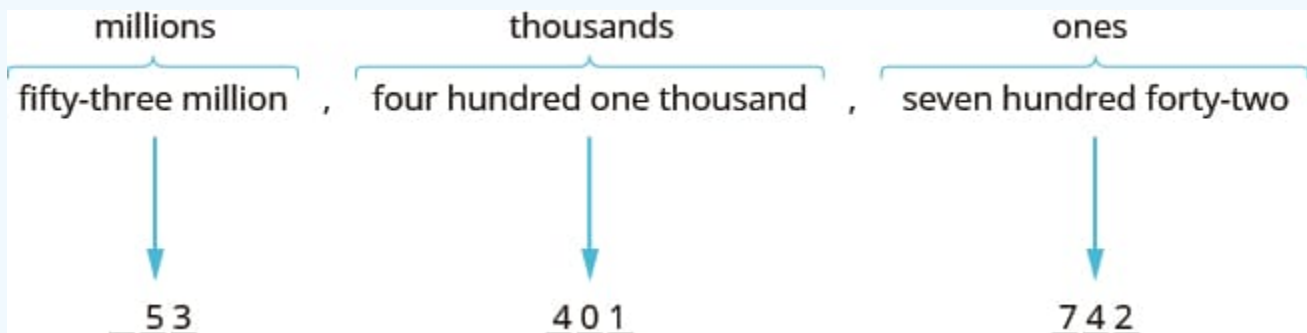
- a. fifty-three million, four hundred one thousand, seven hundred forty-two
- b. nine billion, two hundred forty-six million, seventy-three thousand, one hundred eighty-nine

Solution

- a. Identify the words that indicate periods.

Except for the first period, all other periods must have three places. Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.

Then write the digits in each period.



Put the numbers together, including the commas. The number is 53,401,742.

- b. Identify the words that indicate periods.

Except for the first period, all other periods must have three places. Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.

Then write the digits in each period.

billions	millions	thousands	ones
nine billion	two hundred forty-six million	seventy-three thousand	one hundred eighty-nine
↓	↓	↓	↓
<u> </u> <u>9</u>	<u>2</u> <u>4</u> <u>6</u>	<u>0</u> <u>7</u> <u>3</u>	<u>1</u> <u>8</u> <u>9</u>

The number is 9,246,073,189.

Notice that in part (b), a zero was needed as a place-holder in the hundred thousands place. Be sure to write zeros as needed to make sure that each period, except possibly the first, has three places.

Exercise 1.1.11

Write each number in standard form:

fifty-three million, eight hundred nine thousand, fifty-one

Answer available at the end of this section.

Exercise 1.1.12

Write each number in standard form:

two billion, twenty-two million, seven hundred fourteen thousand, four hundred sixty-six

Answer available at the end of this section.

Example 1.1.7: write standard form

A state budget was about \$77 billion. Write the budget in standard form.

Solution

Identify the periods. In this case, only two digits are given and they are in the billions period. To write the entire number, write zeros for all of the other periods.

billions	millions	thousands	ones
77 billion			
↓	↓	↓	↓
<u> </u> <u>77</u>	<u>0</u> <u>0</u> <u>0</u>	<u>0</u> <u>0</u> <u>0</u>	<u>0</u> <u>0</u> <u>0</u>

So the budget was about \$77,000,000,000.

Exercise 1.1.14

Write each number in standard form:

The total weight of an aircraft carrier is 204 million pounds.

Answer available at the end of this section.

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- Lynn Marecek (Santa Ana College) and MaryAnne Anthony-Smith (formerly of Santa Ana College). This content produced by OpenStax and is licensed under a [Creative Commons Attribution License 4.0](#) license.

Answers

Exercise 1.1.1

2, 9, 241, 376

Exercise 1.1.1

0, 2, 9, 241, 376

Exercise 1.1.2

7, 13, 201

Exercise 1.1.2

0, 7, 13, 201

Exercise 1.1.6

billions

Exercise 1.1.6

ten thousands

Exercise 1.1.6

tens

Exercise 1.1.6

hundred thousands

Exercise 1.1.6

hundred millions

Exercise 1.1.7

nine trillion, two hundred fifty-eight billion, one hundred thirty-seven million, nine hundred four thousand, sixty-one

Exercise 1.1.9

three hundred sixteen million, one hundred twenty-eight thousand, eight hundred thirty nine

Exercise 1.1.11

53, 809, 051

Exercise 1.1.12

2, 022, 714, 466

Exercise 1.1.14

204, 000, 000 *pounds*

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1.2: Adding and Subtracting Whole Numbers

The Commutative Property of Addition

The Commutative Property of Addition

Let a and b represent two whole numbers. Then,

$$a + b = b + a.$$

Grouping Symbols

In mathematics, we use *grouping symbols* to affect the order in which an expression is evaluated. Whether we use parentheses, brackets, or curly braces, the expression inside any pair of grouping symbols must be evaluated first. For example, note how we first evaluate the sum in the parentheses in the following calculation.

$$\begin{aligned}(3 + 4) + 5 &= 7 + 5 \\ &= 12\end{aligned}$$

The rule is simple: Whatever is inside the parentheses is evaluated first.

Writing Mathematics

When writing mathematical statements, follow the mantra:

One equal sign per line.

We can use brackets instead of parentheses.

$$\begin{aligned}5 + [7 + 9] &= 5 + 16 \\ &= 21\end{aligned}$$

Again, note how the expression inside the brackets is evaluated first.

We can also use curly braces instead of parentheses or brackets.

$$\begin{aligned}\{2+3\} + 4 &= 5 + 4 \\ &= 9\end{aligned}$$

Again, note how the expression inside the curly braces is evaluated first.

If grouping symbols are nested, we evaluate the innermost parentheses first.

For example,

$$\begin{aligned}2 + [3 + (4 + 5)] &= 2 + [3 + 9] \\ &= 2 + 12 \\ &= 14.\end{aligned}$$

Grouping Symbols

Use parentheses, brackets, or curly braces to delimit the part of an expression you want evaluated first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.

The Associative Property of Addition

Consider the evaluation of the expression $(2+3)+4$. We evaluate the expression in parentheses first.

$$\begin{aligned}(2 + 3) + 4 &= 5 + 4 \\ &= 9\end{aligned}$$

Now, suppose we change the order of addition to $2 + (3 + 4)$. Then,

$$\begin{aligned}2 + (3 + 4) &= 2 + 7 \\ &= 9.\end{aligned}$$

Although the grouping has changed, the result is the same. That is,

$$(2 + 3) + 4 = 2 + (3 + 4).$$

This property of addition of whole numbers is called the *associate property* of addition.

Associate Property of Addition

Let a , b , and c represent whole numbers. Then,

$$(a + b) + c = a + (b + c).$$

Because of the associate property of addition, when presented with a sum of three numbers, whether you start by adding the first two numbers or the last two numbers, the resulting sum is the same.

The Additive Identity

The Additive Identity Property

The whole number zero is called the additive identity. If a is any whole number, the

$$a + 0 = a.$$

The number zero is called the additive identity because if you add zero to any number, you get the identical number back.

Adding Larger Whole Numbers

For completeness, we include two examples of adding larger whole numbers. Hopefully, the algorithm is familiar from previous coursework.

Example 1

Simplify: $1,234 + 498$.

Solution

Align the numbers vertically, then add, starting at the furthest column to the right. Add the digits in the ones column, $4 + 8 = 12$. Write the 2, then carry a 1 to the tens column. Next, add the digits in the tens column, $3 + 9 = 12$, add the carry to get 13, then write the 3 and carry a 1 to the hundreds column. Continue in this manner, working from right to left

$$\begin{array}{r} 11 \\ 1234 \\ + 498 \\ \hline 1732 \end{array}$$

Therefore, $1,234 + 498 = 1,732$

Exercise

Simplify: $1,286 + 349$.

Answer available at the end of this section.

Add three or more numbers in the same manner.

Example 2

Simplify: $256 + 322 + 418$.

Solution

Align the numbers vertically, then add, starting at the furthest column to the right. Add the digits in the ones column, $6 + 2 + 8 = 16$. Write the 6, then carry a 1 to the tens column. Continue in this manner, working from right to left.

$$\begin{array}{r} 256 \\ 322 \\ +418 \\ \hline 996 \end{array}$$

Therefore, $256 + 322 + 418 = 996$.

Exercise

Simplify: $256 + 342 + 283$

Answer available at the end of this section.

Subtraction of Whole Numbers

The key idea is this: *Subtraction is the opposite of addition.* For example, consider the difference $7 - 4$ depicted on the number line in Figure 1.5.

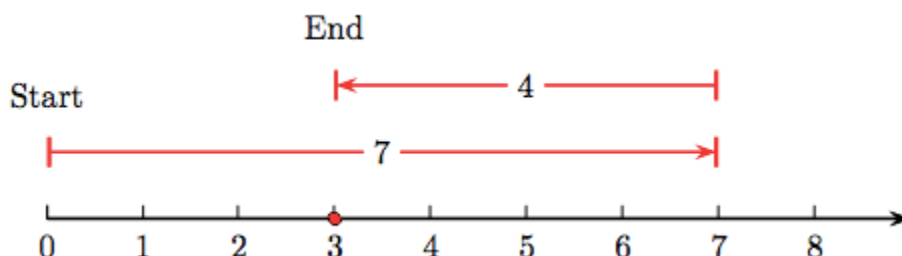


Figure 1.5: Subtraction means *add the opposite*.

If we were adding 7 and 4, we first draw an arrow starting at zero pointing to the right with magnitude (length) seven. Then, to add 4, we would draw a second arrow of magnitude (length) 4, attached to the end of the first arrow and pointing to the right.

However, because subtraction is the *opposite* of addition, in Figure 1.5 we attach an arrow of magnitude (length) four to the end of the first arrow, but *pointing in the opposite direction* (to the left). Note that this last arrow ends at the answer, which is a shaded dot on the number line at 3. That is, $7 - 4 = 3$.

Note that subtraction is **not commutative**; that is, it make no sense to say that $7 - 5$ is the same as $5 - 7$.

Subtraction is **not associative**. It is not the case that $(9 - 5) - 2$ is the same as $9 - (5 - 2)$. On the one hand,

$$\begin{aligned}(9 - 5) - 2 &= 4 - 2 \\ &= 2,\end{aligned}$$

but

$$\begin{aligned}9 - (5 - 2) &= 9 - 3 \\ &= 6.\end{aligned}$$

Subtracting Larger Whole Numbers

Much as we did with adding larger whole numbers, to subtract two large whole numbers, align them vertically then subtract, working from right to left. You may have to “borrow” to complete the subtraction at any step.

Example 3

Simplify: $1,755 - 328$.

Solution

Align the numbers vertically, then subtract, starting at the ones column, then working right to left. At the ones column, we cannot subtract 8 from 5, so we borrow from the previous column. Now, 8 from 15 is 7. Continue in this manner, working from right to left.

$$\begin{array}{r}
 & & 4 & & \\
 & 1 & 7 & \cancel{5} & 15 \\
 - & & 3 & 2 & 8 \\
 \hline
 & 1 & 4 & 2 & 7
 \end{array}$$

Therefore, $1,755 - 328 = 1,427$.

Exercise

Simplify: $5,635 - 288$.

Answer available at the end of this section.

Order of Operations

In the absence of grouping symbols, it is important to understand that addition holds no precedence over subtraction, and vice-versa.

Perform all additions and subtractions in the order presented, moving left to right.

Let's look at an example.

Example 4

Simplify the expression $15 - 8 + 4$.

Solution

This example can be trickier than it seems. However, if we follow the rule (perform all additions and subtractions in the order presented, moving left to right), we should have no trouble. First comes fifteen minus eight, which is seven. Then seven plus four is eleven.

$$\begin{aligned}
 15 - 8 + 4 &= 7 + 4 \\
 &= 11.
 \end{aligned}$$

Exercise

Simplify: $25 - 10 + 8$.

Answer available at the end of this section.

Caution! Incorrect answer ahead!

Note that it is possible to arrive at a different (but incorrect) answer if we favor addition over subtraction in Example 4. If we first add eight and four, then $15 - 8 + 4$ becomes $15 - 12$, which is 3. However, note that **this is incorrect**, because it violates the rule “perform all additions and subtractions in the order presented, moving left to right.”

Exercises

Determine which property of addition is depicted by the given identity.

1. $(51 + 66) + 88 = 51 + (66 + 88)$

2. $64 + 39 = 39 + 64$

3. $79 + 0 = 79$

Simplify the given expression.

4. $16 - 8 + 2$

5. $15 - 5 + 8$

Find the sum.

6. $8583 + 592$

7. $841 + 368 + 919$

8. $(86 + 557) + 80$

Find the difference.

9. $8338 - 7366$

10. $881 - 606$

11. $3838 - (777 - 241)$

12. **Water Subsidies.** Since the drought began in 2007, California farms have received \$79 million in water subsidies. California cotton and rice farmers received an additional \$439 million. How much total water subsidies have farmers received? *Associated Press Times-Standard 4/15/09*

13. **Sun Frost.** Arcata, CA is home to Sun Frost, a manufacturer of highly efficient refrigerators and freezers. The AC model RF12 refrigerator/freezer costs \$2,279 while an R16 model refrigerator/freezer costs \$3,017. How much more does the R16 model cost? Source: www.sunfrost.com/retail/pricelist.html

14. **Earth's Orbit.** Earth orbits the sun in an ellipse. When earth is at its closest to the sun, called *perihelion*, earth is about 147 million kilometers. When earth is at its furthest point from the sun, called *aphelion*, earth is about 152 million kilometers from the sun. What's the difference in millions of kilometers between aphelion and perihelion?

15. **Jobs.** The Times-Standard reports that over the next year, the credit and debit card processing business Humboldt Merchant Services expects to cut 36 of its 80 jobs, but then turn around and hire another 21. How many people will be working for the company then? *Times-Standard 5/6/09*

Answers

1. Associative property of addition
2. Commutative property of addition
3. Additive identity property of addition.
4. 10
5. 18
6. 9175
7. 2128
8. 723
9. 972
10. 275
11. 3302
12. \$518 million
13. \$738
14. 5 million kilometers
15. 65 people

Answers

Exercise

1635

Exercise

881

Exercise

5,347

Exercise

23

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1.3: Multiplying and Dividing Whole Numbers

We begin this section by discussing multiplication of whole numbers. The first order of business is to introduce the various symbols used to indicate multiplication of two whole numbers.

Mathematical symbols that indicate multiplication

Symbol		Example
$\times$	times symbol	3×4
$\cdot$	dot	$3 \cdot 4$
$()$	parentheses	$(3)(4)$ or $3(4)$ or $(3)4$

Products and Factors

In the expression $3 \cdot 4$, the whole numbers 3 and 4 are called the **factors** and $3 \cdot 4$ is called the **product**.

The key to understanding multiplication is held in the following statement.

Multiplication is equivalent to repeated addition.

Suppose, for example, that we would like to evaluate the product $3 \cdot 4$. Because multiplication is equivalent to repeated addition, $3 \cdot 4$ is equivalent to adding three fours. That is,

$$3 \cdot 4 = \underbrace{4 + 4 + 4}_{\text{three fours}}$$

Thus, $3 \cdot 4 = 12$. You can visualize the product $3 \cdot 4$ as the sum of three fours on a number line, as shown in Figure 1.6.

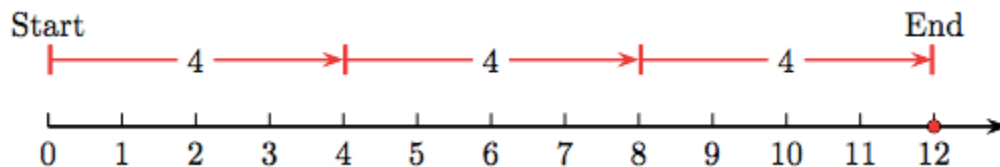


Figure 1.6: Note that $3 \cdot 4 = 4 + 4 + 4$. That is, $3 \cdot 4 = 12$.

Like addition, the order of the factors does not matter.

$$4 \cdot 3 = \underbrace{3 + 3 + 3 + 3}_{\text{four threes}}$$

Thus, $4 \cdot 3 = 12$. Consider the visualization of $4 \cdot 3$ in Figure 1.7.

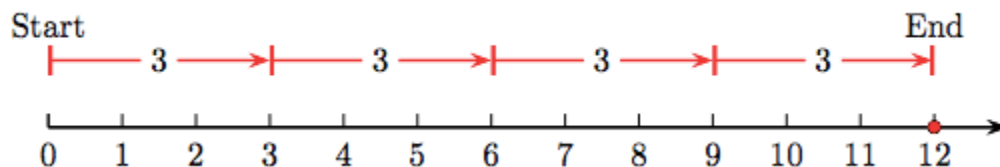


Figure 1.7: Note that $4 \cdot 3 = 3 + 3 + 3 + 3$. That is, $4 \cdot 3 = 12$.

The evidence in Figure 1.6 and Figure 1.7 show us that multiplication is commutative. That is,

$$3 \cdot 4 = 4 \cdot 3$$

Commutative Property of Multiplication

If a and b are any whole numbers, then

$$a \cdot b = b \cdot a.$$

The Multiplicative Identity

In Figure 1.8(a), note that five ones equals 5; that is, $5 \cdot 1 = 5$. On the other hand, in Figure 1.8(b), we see that one five equals five; that is, $1 \cdot 5 = 5$.

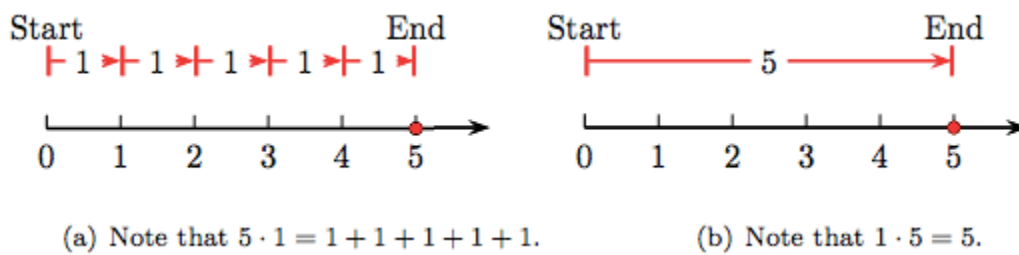


Figure 1.8: Note that $5 \cdot 1 = 5$ and $1 \cdot 5 = 5$.

Because multiplying a whole number by 1 equals that identical number, the whole number 1 is called the multiplicative identity.

The Multiplicative Identity Property

If a is any whole number, then

$$a \cdot 1 = a \text{ and } 1 \cdot a = a.$$

Multiplication by Zero

Because $3 \cdot 4 = 4 + 4 + 4$, we can say that the product $3 \cdot 4$ represents “3 sets of 4,” as depicted in Figure 1.9, where three groups of four boxes are each enveloped in an oval.

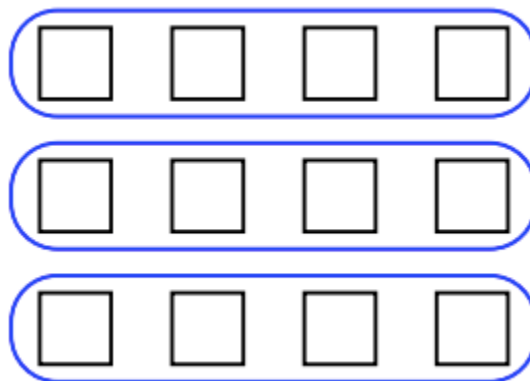


Figure 1.9: Three sets of four: $3 \cdot 4 = 12$.

Therefore, $0 \cdot 4$ would mean zero sets of four. Of course, zero sets of four is zero.

Multiplication by Zero.

If a represents any whole number, then

$$a \cdot 0 = 0 \text{ and } 0 \cdot a = 0.$$

The Associative Property of Multiplication

Like addition, multiplication of whole numbers is associative. Indeed,

$$\begin{aligned} 2 \cdot (3 \cdot 4) &= 2 \cdot 12 \\ &= 24 \end{aligned}$$

and

$$\begin{aligned} (2 \cdot 3) \cdot 4 &= 6 \cdot 4 \\ &= 24. \end{aligned}$$

The Associative Property of Multiplication.

If a , b , and c are any whole numbers, then

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

Multiplying Larger Whole Numbers

Much like addition and subtraction of large whole numbers, we will also need to multiply large whole numbers. Again, we hope the algorithm is familiar from previous coursework.

Example 1

Simplify: $35 \cdot 127$.

Solution

Align the numbers vertically. The order of multiplication does not matter, but we'll put the larger of the two numbers on top of the smaller number. The first step is to multiply 5 times 127. Again, we proceed from right to left. So, 5 times 7 is 35. We write the 5, then carry the 3 to the tens column. Next, 5 times 2 is 10. Add the carry digit 3 to get 13. Write the 3 and carry the 1 to the hundreds column. Finally, 5 times 1 is 5. Add the carry digit to get 6.

$$\begin{array}{r} 1 3 \\ 1 2 7 \\ \times 3 5 \\ \hline 6 3 5 \end{array}$$

The next step is to multiply 3 times 127. However, because 3 is in the tens place, its value is 30, so we actually multiply 30 times 127. This is the same as multiplying 127 by 3 and placing a 0 at the end of the result.

$$\begin{array}{r}
 2 \\
 127 \\
 \times 35 \\
 \hline
 635 \\
 3810 \\
 \hline
 \end{array}$$

After adding the 0, 3 times 7 is 21. We write the 1 and carry the 2 above the 2 in the tens column. Then, 3 times 2 is 6. Add the carry digit 2 to get 8. Finally, 3 times 1 is 3.

All that is left to do is to add the results.

$$\begin{array}{r}
 127 \\
 \times 35 \\
 \hline
 635 \\
 3810 \\
 \hline
 4445
 \end{array}$$

Thus, $35 \cdot 127 = 4,445$.

Alternate Format

It does not hurt to omit the trailing zero in the second step of the multiplication, where we multiply 3 times 127. The result would look like this:

$$\begin{array}{r}
 127 \\
 \times 35 \\
 \hline
 635 \\
 381 \\
 \hline
 4445
 \end{array}$$

In this format, the zero is understood, so it is not necessary to have it physically present. The idea is that with each multiplication by a new digit, we indent the product one space from the right.

Exercise

Simplify: $56 \cdot 335$

Division of Whole Numbers

We now turn to the topic of division of whole numbers. We first introduce the various symbols used to indicate division of whole numbers.

Mathematical Symbols that Indicate Division

Symbol		Example
$\div$	division symbol	$12 \div 4$
$-$	fraction bar	$\frac{12}{4}$
$\longdiv{-}$	division bar	$4 \longdiv{12}$

Note that each of the following say the same thing; that is, “12 divided by 4 is 3.”

$$12 \div 4 = 3 \quad \text{or} \quad \frac{12}{4} = 3 \quad \text{or} \quad 4 \overline{)12}$$

Quotients, Dividends, and Divisors

In the statement

$$\frac{3}{4} \overline{)12}$$

the whole number 12 is called the *dividend*, the whole number 4 is called the *divisor*, and the whole number 3 is called the *quotient*. Note that this division bar notation is equivalent to

$$12 \div 4 = 3 \quad \text{and} \quad \frac{12}{4} = 3.$$

The expression a/b means “ a divided by b ,” but this construct is also called a *fraction*.

Definition: Fraction

The expression

$$\frac{a}{b}$$

is called a **fraction**. The number a on top is called the **numerator** of the fraction; the number b on the bottom is called the **denominator** of the fraction.

The key to understanding division of whole numbers is contained in the following statement.

Division is equivalent to repeated subtraction.

Suppose for example, that we would like to divide the whole number 12 by the whole number 4. This is equivalent to asking the question “how many fours can we subtract from 12?” This can be visualized in a number line diagram, such as the one in Figure 1.10.

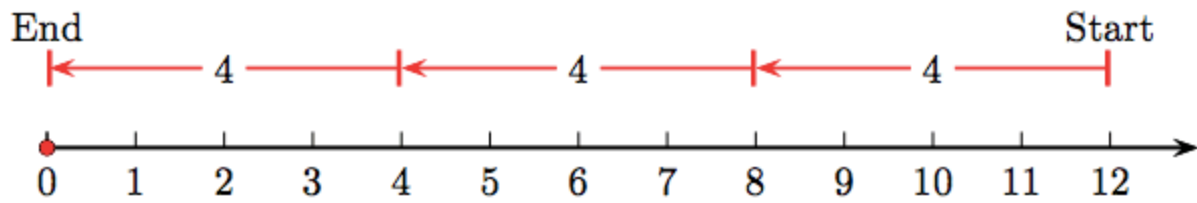


Figure 1.10: Division is repeated subtraction.

In Figure 1.10, note that we if we subtract three fours from twelve, the result is zero. In symbols,

$$12 - \underbrace{4 - 4 - 4}_{\text{three fours}} = 0.$$

Equivalently, we can also ask “How many groups of four are there in 12,” and arrange our work as shown in Figure 1.11, where we can see that in an array of twelve objects, we can circle three groups of four ; i.e., $12 \div 4 = 3$.

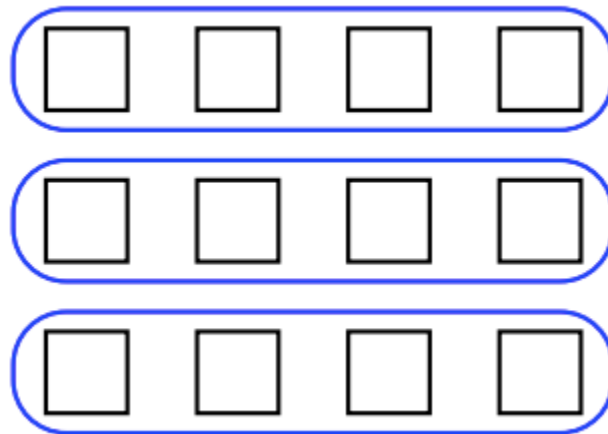


Figure 1.11: There are three groups of four in twelve.

In Figure 1.10 and Figure 1.11, note that the division (repeated subtraction) leaves no remainder. This is not always the case.

Example 2.

Divide 7 by 3.

Solution

In Figure 1.12, we can see that we can subtract two threes from seven, leaving a remainder of one.

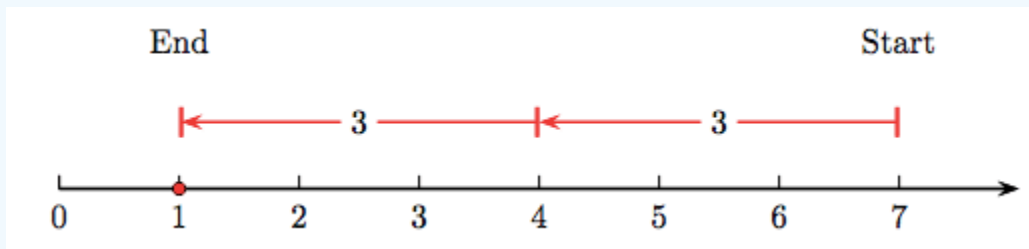


Figure 1.12: Division with a remainder.

Alternatively, in an array of seven objects, we can circle two groups of three, leaving a remainder of one.

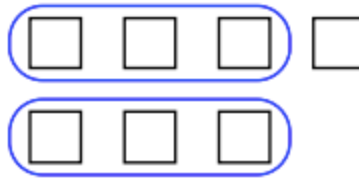


Figure 1.13: Dividing seven by three leaves a remainder of one.

Both Figure 1.12 and Figure 1.13 show that there are two groups of three in seven, with one left over. We say “Seven divided by three is two, with a remainder of one.”

Exercise

Use both the number line approach and the array of boxes approach to divide 12 by 5.

Division is not Commutative

When dividing whole numbers, the order matters. For example

$$12 \div 4 = 3,$$

but $4 \div 12$ is not even a whole number. Thus, if a and b are whole numbers, then $a \div b$ does **not** have to be the same as $b \div a$.

Division is not Associative

When you divide three numbers, the order in which they are grouped will usually affect the answer. For example,

$$\begin{aligned}(48 \div 8) \div 2 &= 6 \div 2 \\ &= 3,\end{aligned}$$

but

$$\begin{aligned}48 \div (8 \div 2) &= 48 \div 4 \\ &= 12.\end{aligned}$$

Thus, if a , b , and c are whole numbers, $(a \div b) \div c$ does not have to be the same as $a \div (b \div c)$.

Division by Zero is Undefined

Suppose that we are asked to divide six by zero; that is, we are asked to calculate $6 \div 0$. In Figure 1.14, we have an array of six objects.

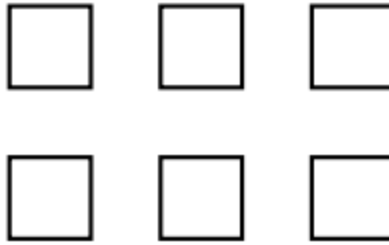


Figure 1.14: How many groups of zero do you see?

Now, to divide six by zero, we must answer the question “How many groups of zero can we circle in Figure 1.14?” Some thought will provide the answer: This is a meaningless request! It makes absolutely no sense to ask how many groups of zero can be circled in the array of six objects in Figure 1.14.

Division by Zero

Division by zero is **undefined**. Each of the expressions

$$6 \div 0 \quad \text{and} \quad \frac{6}{0} \quad \text{and} \quad 0 \overline{)6}$$

is **undefined**.

On the other hand, it make sense to ask “What is zero divided by six?” If we create an array of zero objects, then ask how many groups of six we can circle, the answer is “zero groups of six.” That is, zero divided by six is zero.

$$0 \div 6 = 0 \quad \text{and} \quad \frac{0}{6} = 0 \quad \text{and} \quad 6 \overline{)0}$$

Dividing Larger Whole Numbers

We’ll now provide a quick review of division of larger whole numbers, using an algorithm that is commonly called *long division*. This is not meant to be a thorough discussion, but a cursory one. We’re counting on the fact that our readers have encountered this algorithm in previous courses and are familiar with the process.

Example 3

Simplify: $575/23$.

Solution

We begin by estimating how many times 23 will divide into 57, guessing 1. We put the 1 in the quotient above the 7, multiply 1 times 23, place the answer underneath 57, then subtract.

$$\begin{array}{r} 23 \overline{) 575} \\ \underline{23} \\ 34 \end{array}$$

Because the remainder is larger than the divisor, our estimate is too small. We try again with an estimate of 2.

$$\begin{array}{r} 23 \overline{) 575} \\ \underline{46} \\ 11 \end{array}$$

That’s the algorithm. Divide, multiply, then subtract. You may continue only when the remainder is smaller than the divisor.

To continue, bring down the 5, estimate that 115 divided by 23 is 5, then multiply 5 times the divisor and subtract.

$$\begin{array}{r} 25 \\ 23 \overline{) 575} \\ \underline{46} \\ 115 \\ \underline{115} \\ 0 \end{array}$$

Because the remainder is zero, $575/23 = 25$.

Exercise

Divide $980/35$

Answer available at the end of this section.

Exercises

State the property of multiplication depicted by the given identity.

1. $8 \cdot (5 \cdot 6) = (8 \cdot 5) \cdot 6$

2. $6 \cdot 2 = 2 \cdot 6$

3. $21 \cdot 1 = 21$

Multiply the given numbers.

4. $78 \cdot 3$

5. $128 \cdot 30$

6. $291 \cdot 47$

Simplify the given expression. If the answer doesn't exist or is undefined, write "undefined".

7. $0 \div 5$

8. $24 \div 0$

Divide the given numbers.

9. $\frac{8075}{85}$

10. $\frac{18048}{32}$

Answers

1. Associative property of multiplication

2. Commutative property of multiplication

3. Multiplicative identity property

4. 234

5. 3840

6. 13677

7. 0

8. Undefined

9. 95

10. 564

Answers

Exercise

18,760

Exercise

28

1.4: Order of Operations

The order in which we evaluate expressions matters. Take for example, the expression $4 + 3 \cdot 2$.

If we do the addition first, then

$$\begin{aligned}4 + 3 \cdot 2 &= 7 \cdot 2 \\ &= 14.\end{aligned}$$

On the other hand, if we do the multiplication first, then

$$\begin{aligned}4 + 3 \cdot 2 &= 4 + 6 \\ &= 10.\end{aligned}$$

So, what are we to do? If this expression had grouping symbols, we would start there.

Grouping Symbols

Parentheses, brackets, or curly braces can be used to group parts of an expression. Each of the following are equivalent:

$$(4 + 3) \cdot 2 \text{ or } [4 + 3] \cdot 2 \text{ or } \{4+3\} \cdot 2$$

In each case, the rule is “evaluate the expression inside the grouping symbols first.” If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.

Thus, for example,

$$\begin{aligned}(4 + 3) \cdot 2 &= 7 \cdot 2 \\ &= 14.\end{aligned}$$

Note how the expression contained in the parentheses was evaluated first. Another way to avoid ambiguities in evaluating expressions is to establish an order in which operations should be performed. The following guidelines should always be strictly enforced when evaluating expressions.

Rules Guiding Order of Operations

When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

Example 1

Evaluate $4 + 3 \cdot 2$.

Solution

Because of the established *Rules Guiding Order of Operations*, this expression is no longer ambiguous. There are no grouping symbols or exponents, so we immediately go to rule three, evaluate all multiplications and divisions in the order that they appear, moving left to right. After that we invoke rule four, performing all additions and subtractions in the order that they appear, moving left to right.

$$\begin{aligned}4 + 3 \cdot 2 &= 4 + 6 \\ &= 10\end{aligned}$$

Thus, $4 + 3 \cdot 2 = 10$.

Exercise

Simplify: $8 + 2 \cdot 5$.

Answer available at the end of this section.

Example 2

Evaluate $18 - 2 + 3$.

Solution

Follow the *Rules Guiding Order of Operations*. Addition has no precedence over subtraction, nor does subtraction have precedence over addition. We are to perform additions and subtractions as they occur, moving left to right.

$$\begin{aligned}18 - 2 + 3 &= 16 + 3 \text{ Subtract: } 18 - 2 = 16. \\ &= 19 \text{ Add: } 16 + 3 = 19.\end{aligned}$$

Thus, $18 - 2 + 3 = 19$.

Exercise

Simplify: $17 - 8 + 2$.

Answer available at the end of this section.

Example 3

Evaluate $54 \div 9 \cdot 2$.

Solution

Follow the *Rules Guiding Order of Operations*. Division has no precedence over multiplication, nor does multiplication have precedence over division. We are to perform divisions and multiplications as they occur, moving left to right.

$$\begin{aligned}54 \div 9 \cdot 2 &= 6 \cdot 2 \text{ Divide: } 54 \div 9 = 6. \\ &= 12 \text{ Multiply: } 6 \cdot 2 = 12.\end{aligned}$$

Thus, $54 \div 9 \cdot 2 = 12$.

Exercise

Simplify: $72 \div 9 \cdot 2$.

Answer available at the end of this section.

Example 4

Evaluate $2 \cdot 3^2 - 12$.

Solution

Follow the *Rules Guiding Order of Operations*, exponents first, then multiplication, then subtraction.

'^' allowed only in math mode

Thus, $2 \cdot 3^2 - 12 = 6$.

Exercise

Simplify: $14 + 3 \cdot 4^2$

Answer available at the end of this section.

Example 5

Evaluate $12 + 2(3 + 2 \cdot 5)^2$.

Solution

Follow the *Rules Guiding Order of Operations*, evaluate the expression inside the parentheses first, then exponents, then multiplication, then addition.

$$\begin{aligned} 12 + 2(3 + 2 \cdot 5)^2 &= 12 + 2(3 + 10)^2 && \text{Multiply inside parentheses: } 2 \cdot 5 = 10. \\ &= 12 + 2(13)^2 && \text{Add inside parentheses: } 3 + 10 = 13. \\ &= 12 + 2(169) && \text{Exponents are next: } (13)^2 = 169. \\ &= 12 + 338 && \text{Multiplication is next: } 2(169) = 338. \\ &= 350 && \text{Time to add: } 12 + 338 = 350. \end{aligned}$$

Thus, $12 + 2(3 + 2 \cdot 5)^2 = 350$.

Exercise

Simplify: $3(2 + 3 \cdot 4)^2 - 11$.

Answer available at the end of this section.

Example 6

Evaluate $2\{2 + 2[2 + 2]\}$.

Solution

When grouping symbols are nested, evaluate the expression between the pair of innermost grouping symbols first.

$$\begin{aligned}
 2(2 + 2[2 + 2]) &= 2(2 + 2[4]) && \text{Innermost grouping first: } 2 + 2 = 4. \\
 &= 2(2 + 8) && \text{Multiply next: } 2[4] = 8. \\
 &= 2(10) && \text{Add inside braces: } 2 + 8 = 10. \\
 &= 20 && \text{Multiply: } 2(10) = 20
 \end{aligned}$$

Thus, $2(2 + 2[2 + 2]) = 20$.

Exercise

Simplify: $2\{3 + 2[3 + 2]\}$.

Answer available at the end of this section.

Fraction Bars

Consider the expression

$$\frac{6^2 + 8^2}{(2 + 3)^2}$$

Because a fraction bar means division, the above expression is equivalent to

$$(6^2 + 8^2) \div (2 + 3)^2$$

The position of the grouping symbols signals how we should proceed. We should simplify the numerator, then the denominator, then divide.

Fractional Expressions

If a fractional expression is present, evaluate the numerator and denominator first, then divide.

Example 7

Evaluate the expression

$$\frac{6^2 + 8^2}{(2 + 3)^2}$$

Solution

Simplify the numerator and denominator first, then divide.

$$\begin{aligned}
 \frac{6^2 + 8^2}{(2 + 3)^2} &= \frac{6^2 + 8^2}{(5)^2} && \text{Parentheses in denominator first: } 2 + 3 = 5 \\
 &= \frac{36 + 64}{25} && \text{Exponents are next: } 6^2 = 36, 8^2 = 64, 5^2 = 25. \\
 &= \frac{100}{25} && \text{Add in numerator: } 36 + 64 = 100 \\
 &= 4 && \text{Divide: } 100 \div 25 = 4.
 \end{aligned}$$

Thus, $\frac{6^2 + 8^2}{(2 + 3)^2} = 4$.

Exercise

Simplify: $\frac{12+3 \cdot 2}{6}$

Answer available at the end of this section.

Exercises

Simplify the given expression.

1. $6 \cdot 5 + 4 \cdot 3$

2. $24 - 2 \cdot 5$

3. $27 \div 3 \cdot 3$

4. $22 - 10 + 7$

5. $20 \cdot 10 + 15 \div 5 - 7 \cdot 6$

6. $7 \cdot [8 - 5] - 10$

7. $9 + 6 \cdot (12 + 3)$

8. $9 - 8[6 - (2 + 3)]$

9. $3\{8[6 + 5] - 8[7 + 3]\}$

10. $(5 - 2)^2$

11. $12 \cdot 5^2 + 8 \cdot 9 + 4$

12. $9 - 3 \cdot 2 + 12 \cdot 10^2$

13. $19 + 3[12 - (2^3 + 1)]$

14. $\frac{19 - (4 \cdot 3 - 2)}{6 \cdot 3 - 9}$

Answers

1. 42

2. 14

3. 27

4. 19

5. 161

6. 11

7. 99

8. 1

9. 24

10. 9

11. 376

12. 1203

13. 28

14. 1

¹Later, we'll see that this property applies to all numbers, not just whole numbers

Answers

Exercise

18

Exercise

11

Exercise

16

Exercise

62

Exercise

577

Exercise

26

Exercise

3

1.5: Positive and Negative Numbers

As we saw in the introduction to the chapter, negative numbers have a rich and storied history. One of the earliest applications of negative numbers had to do with credits and debits. For example, if \$5 represents a credit or profit, then -\$5 represents a debit or loss. Of course, the ancients had a different monetary system than ours, but you get the idea. Note that if a vendor experiences a profit of \$5 on a sale, then a loss of -\$5 on a second sale, the vendor breaks even. That is, the sum of \$5 and -\$5 is zero. In much the same way, every whole number has an *opposite* or *negative* counterpart.

The Opposite or Negative of a Whole Number

For every whole number a , there is a unique number $-a$, called the opposite or negative of a , such that $a + (-a) = 0$.

The opposite or negative of any whole number is easily located on the number line.

Number Line Locations

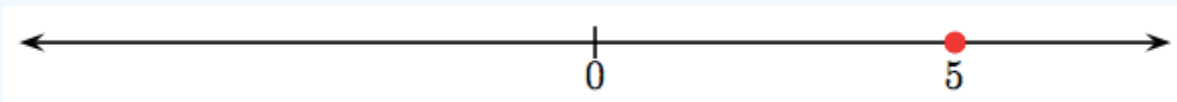
To locate the opposite (or negative) of any whole number, first locate the whole number on the number line. The opposite is the reflection of the whole number through the origin (zero).

Example 1

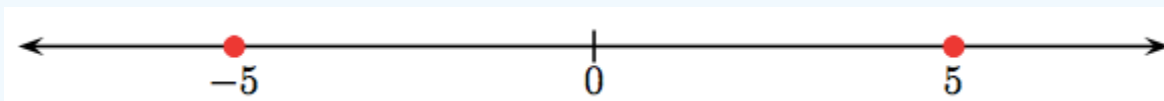
Locate the whole number 5 and its opposite (negative) on the number line.

Solution

Draw a number line, then plot the whole number 5 on the line as a shaded dot.



To find its opposite, reflect the number 5 through the origin. This will be the location of the opposite (negative) of the whole number 5, which we indicate by the symbol -5 .



Note the symmetry. The whole number 5 is located five units to the right of zero. Its negative is located five units to the left of zero.

Exercise

Locate the number -7 and its opposite on the number line.

Important Pronunciation

The symbol -5 is pronounced in one of two ways: (1) “negative five,” or (2) “the opposite of five.”

In similar fashion, we can locate the opposite or negative of any whole number by reflecting the whole number through the origin (zero), which leads to the image shown in Figure 2.1.

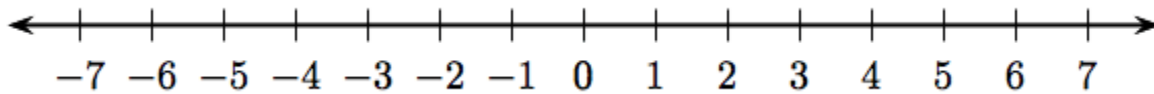


Figure 2.1: The opposite (negative) of any whole number is a reflection of that number through the origin (zero).

The Integers

The collection of numbers arranged on the number line in Figure 2.1 extend indefinitely to the right, and because the numbers on the left are reflections through the origin, the numbers also extend indefinitely to the left. This collection of numbers is called the set of *integers*.

The Integers

The infinite collection of numbers

$$\{..., -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, ...\}$$

is called the **set of integers**.

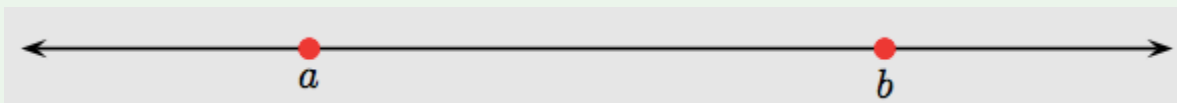
The ellipsis ... at each end of this infinite collection means “etcetera,” as the integers continue indefinitely to the right and left. Thus, for example, both 23,456 and -117, 191 are elements of this set and are therefore integers.

Ordering the Integers

As we saw with the whole numbers, as you move to the right on the number line, the numbers get larger; as you move to the left, the numbers get smaller.

Order on the Number Line

Let a and b be integers located on the number line so that the point representing the integer a lies to the left of the point representing the integer b .



Then the integer a is “less than” the integer b and we write

$$a < b$$

Alternatively, we can also say that the integer b is “larger than” the integer a and write

$$b > a.$$

Example 3

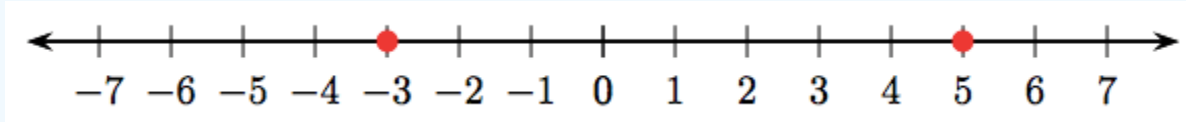
Replace each shaded box with $<$ (less than) or $>$ (greater than) so the resulting inequality is a true statement.

$$-3 \quad \square \quad 5$$

$$-2 \quad \square \quad -4$$

Solution

For the first case, locate -3 and 5 on the number line as shaded dots.

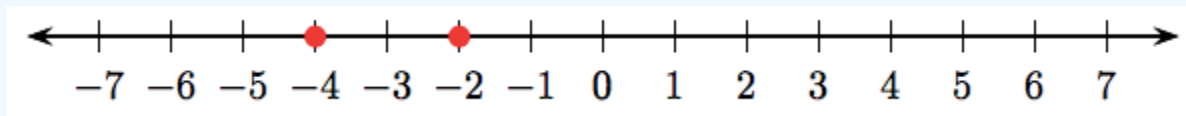


Note that -3 lies to the left of 5 , so:

$$-3 \quad < \quad 5$$

That is, -3 is “less than” 5 .

In the second case, locate -2 and -4 as shaded dots on the number line.



Note that -2 lies to the right of -4 , so:

$$-2 \quad > \quad -4$$

That is, -2 is “greater than” -4 .

Exercise

Compare -12 and -11 .

Answer available at the end of this section.

Important Observation

In Example 3, note that the “pointy end” of the inequality symbol always points towards the smaller number.

We stated earlier that every integer has a unique number called its “opposite” or “negative.” Thus, the integer -5 is the opposite (negative) of the integer 5 . Thus, we can say that the pair -5 and 5 are opposites. Each is the opposite of the other. Logically, this leads us to the conclusion that the opposite of -5 is 5 . In symbols, we would write

$$-(-5) = 5.$$

Opposites of Opposites

Let a be an integer. Then the “opposite of the opposite of a is a .” In symbols, we write

$$-(-a) = a.$$

We can also state that the “negative of a negative a is a .”

Example 4

Simplify $-(-13)$ and $-(-119)$.

Solution

The opposite of the opposite of a number returns the original number. That is,

$$-(-13) = 13 \text{ and } -(-119) = 119.$$

Exercise

Simplify: $-(-50)$.

Answer available at the end of this section.

Positive and Negative

We now define the terms *positive integer* and *negative integer*.

Positive Integer

If a is an integer that lies to the right of zero (the origin) on the number line, then a is a **positive** integer. This means that a is a positive integer if and only if $a > 0$.

Thus, 2, 5, and 117 are positive integers.

Negative Integer

If a is an integer that lies to the left of zero (the origin) on the number line, then a is a **negative** integer. This means that a is a negative integer if and only if $a < 0$.

Thus, -4 , -8 , and -1 , 123 are negative integers.

Zero

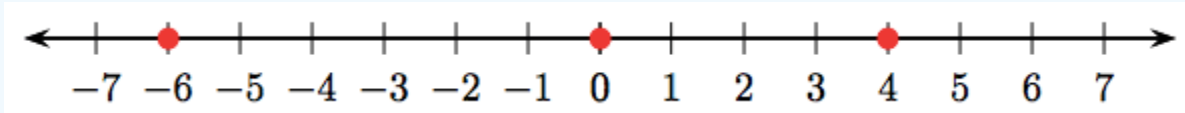
The integer zero is neither positive nor negative.

Example 5

Classify each of the following numbers as negative, positive, or neither: 4, -6, and 0.

Solution

Locate 4, -6, and 0 on the number line.



Thus:

- 4 lies to the right of zero. That is, $4 > 0$, making 4 a positive integer.
- -6 lies to the left of zero. That is, $-6 < 0$, making -6 a negative integer.
- The number 0 is neutral. It is neither negative nor positive.

Exercise

Classify -11 as positive, negative, or neither.

Answer available at the end of this section.

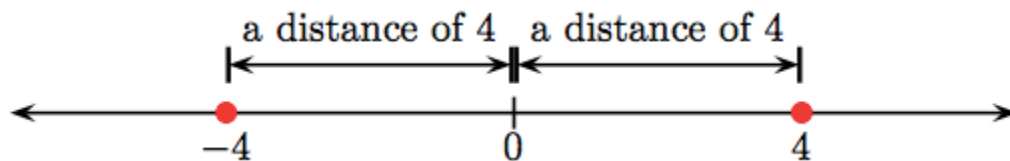
Absolute Value

We define the absolute value of an integer.

Absolute Value

The absolute value of an integer is defined as its distance from the origin (zero).

It is important to note that distance is always a nonnegative quantity (not negative); i.e., distance is either positive or zero. As an example, we've shaded the integers -4 and 4 on a number line.



The number line above shows two cases:

- The integer -4 is 4 units from zero. Because absolute value measures the distance from zero, $|-4| = 4$.
- The integer 4 is also 4 units from zero. Again, absolute value measures the distance from zero, so $|4| = 4$.

Let's look at another example.

Example 6

Determine the value of each expression: a) $|-7|$, b) $|3|$, and c) $|0|$.

Solution

The absolute value of any integer is equal to the distance that number is from the origin (zero) on the number line. Thus:

- a) The integer -7 is 7 units from the origin; hence, $|-7| = 7$.
- b) The integer 3 is 3 units from the origin; hence, $|3| = 3$.
- c) The integer 0 is 0 units from the origin; hence, $|0| = 0$.

Exercise

Simplify: $|-33|$.

Answer available at the end of this section.

Example 7

Determine the value of each expression: a) $-(-8)$ and b) $-|-8|$

Solution

These are distinctly different problems.

- a) The opposite of -8 is 8. That is, $-(-8) = 8$.
- b) However, in this case, we take the absolute value of -8 first, which is 8, then the opposite of that result to get -8 . That is,

$$\begin{aligned} -|-8| &= -(8) \text{ First: } |-8| = 8. \\ &= -8 \text{ Second: The opposite of 8 is } -8. \end{aligned}$$

Exercise

Simplify: $-|-50|$.

Answer available at the end of this section.

Exercises

For #1 and #2, perform the following tasks:

- a) Plot each of the integers on a numberline.
- b) List the numbers in order, from smallest to largest.
 - 1. 5, -6 , 0, and 2
 - 2. -2 , -4 , 3, and -6

Enter the inequality symbol $<$ or the symbol $>$ in the box in order that the resulting inequality is a true statement.

3. $-4 \square 0$

4. $-3 \square -1$

5. $0 \square -3$

6. $1 \square -4$

Simplify each of the following expressions.

7. $-(-4)$

8. $|3|$

9. $|-5|$

10. $-|-20|$

For #11 and #12, provide a number line sketch with your answer.

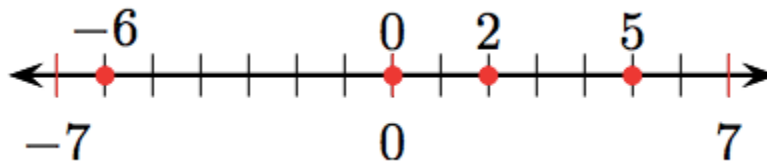
11. Find two integers on the number line that are 4 units away from the integer -1 .

12. Find two integers on the number line that are 3 units away from the integer 0 .

Answers

1.

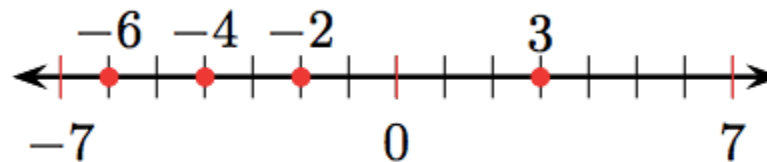
a) Arrange the integers 5, -6 , 0 , and 2 on a number line.



b) -6 , 0 , 2, 5

2.

a) Arrange the integers -2 , -4 , 3, and -6 on a number line.



b) -6 , -4 , -2 , 3

3. $-4 < 0$

4. $-3 < -1$

5. $0 > -3$

6. $1 > -4$

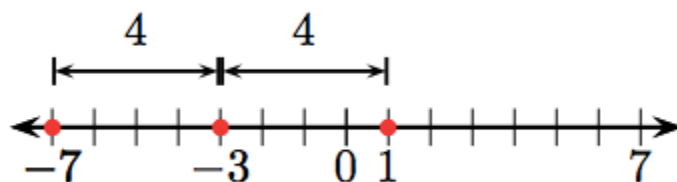
7. 4

8. 3

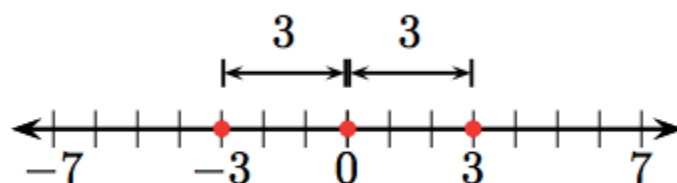
9. 5

10. -20

11. -7 and 1.



12. -3 and 3.



Answers

Exercise

$-12 < -11$

Exercise

50

Exercise

Negative

Exercise

33

Exercise

50

2: FRACTIONS

2.1: Properties of Fractions and Reducing to Lowest Terms

2.2: Multiplying Fractions

2.3: Dividing Fractions

2.4: Adding and Subtracting Fractions

2.5: Multiplying and Dividing Mixed Fractions

2.6: Adding and Subtracting Mixed Fractions

2.1: Properties of Fractions and Reducing to Lowest Terms

In this section we deal with fractions, numbers or expressions of the form a/b .

Definition: Fractions

A number of the form

$$\frac{a}{b}$$

where a and b are numbers is called a *fraction*. The number a is called the *numerator* of the fraction, while the number b is called the *denominator* of the fraction.

Near the end of this section, we'll see that the numerator and denominator of a fraction can also be algebraic expressions, but for the moment we restrict our attention to fractions whose numerators and denominators are integers. We start our study of fractions with the definition of *equivalent fractions*.

Equivalent Fractions

Two fractions are *equivalent* if they represent the same numerical value.

But how can we tell if two fractions represent the same number? Well, one technique involves some simple visualizations. Consider the image shown in Figure 4.1, where the shaded region represents $1/3$ of the total area of the figure (one of three equal regions is shaded).



Figure 4.1: The shaded region is $1/3$ of the whole region.

In Figure 4.2, we've shaded $2/6$ of the entire region (two of six equal regions are shaded).



Figure 4.2: The shaded region is $2/6$ of the whole region.

In Figure 4.3, we've shaded $4/12$ of the entire region (four of twelve equal regions are shaded).



Figure 4.3: The shaded region is $4/12$ of the whole region.

Let's take the diagrams from Figure 4.1, Figure 4.2, and Figure 4.3 and stack them one atop the other, as shown in Figure 4.4.

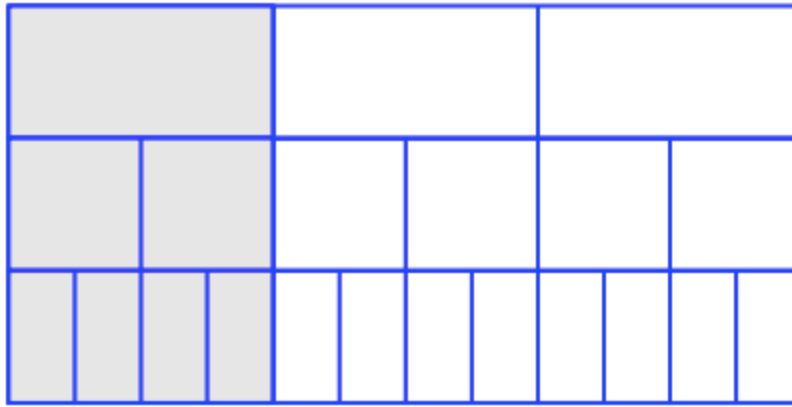


Figure 4.4: One of three equals two of six equals four of twelve.

Figure 4.4 provides solid visual evidence that the following fractions are equivalent.

$$\frac{1}{3} = \frac{2}{6} = \frac{4}{12}$$

Key Observations

1. If we start with the fraction $\frac{1}{3}$, then multiply both numerator and denominator by 2, we get the following result.

$$\begin{aligned} \frac{1}{3} &= \frac{1 \cdot 2}{3 \cdot 2} && \text{Multiply the numerator and denominator by 2.} \\ &= \frac{2}{6} && \text{Simplify numerator and denominator.} \end{aligned}$$

This is precisely the same thing that happens going from Figure 4.1 to 4.2, where we double the number of available boxes (going from 3 available to 6 available) and double the number of shaded boxes (going from 1 shaded to 2 shaded).

2. If we start with the fraction $\frac{1}{3}$, then multiply both numerator and denominator by 4, we get the following result.

$$\begin{aligned} &= \frac{1}{3} = \frac{1 \cdot 4}{3 \cdot 4} && \text{Multiply numerator and denominator by 4.} \\ &= \frac{4}{12} && \text{Simplify numerator and denominator.} \end{aligned}$$

This is precisely the same thing that happens going from Figure 4.1 to 4.3, where we multiply the number of available boxes by 4 (going from 3 available to 12 available) and multiply the number of shaded boxes by 4 (going from 1 shaded to 4 shaded).

The above discussion motivates the following fundamental result.

Creating Equivalent Fractions

If you start with a fraction, then multiply both its numerator and denominator by the same number, the resulting fraction is equivalent (has the same numerical value) to the original fraction. In symbols,

$$\frac{a}{b} = \frac{a \cdot x}{b \cdot x}$$

Arguing in Reverse

Reversing the above argument also holds true.

1. If we start with the fraction $\frac{2}{6}$, then divide both numerator and denominator by 2, we get the following result.

$$\begin{aligned}\frac{2}{6} &= \frac{2 \div 2}{6 \div 2} && \text{Divide numerator and denominator by 2.} \\ &= \frac{1}{3} && \text{Simplify numerator and denominator.}\end{aligned}$$

This is precisely the same thing that happens going backwards from Figure 4.2 to 4.1, where we divide the number of available boxes by 2 (going from 6 available to 3 available) and dividing the number of shaded boxes by 2 (going from 2 shaded to 1 shaded).

2. If we start with the fraction $4/12$, then divide both numerator and denominator by 4, we get the following result.

$$\begin{aligned}\frac{4}{12} &= \frac{4 \div 4}{12 \div 4} && \text{Multiply numerator and denominator by 4.} \\ &= \frac{1}{3} && \text{Simplify numerator and denominator.}\end{aligned}$$

This is precisely the same thing that happens going backwards from Figure 4.3 to 4.1, where we divide the number of available boxes by 4 (going from 12 available to 3 available) and divide the number of shaded boxes by 4 (going from 4 shaded to 1 shaded).

The above discussion motivates the following fundamental result.

Creating Equivalent Fractions

If you start with a fraction, then divide both its numerator and denominator by the same number, the resulting fraction is equivalent (has the same numerical value) to the original fraction. In symbols,

$$\frac{a}{b} = \frac{a \div x}{b \div x}.$$

The Greatest Common Divisor

We need a little more terminology.

Divisor

If d and a are natural numbers, we say that “ d divides a ” if and only if when a is divided by d , the remainder is zero. In this case, we say that “ d is a divisor of a .”

For example, when 36 is divided by 4, the remainder is zero. In this case, we say that “4 is a divisor of 36.” On the other hand, when 25 is divided by 4, the remainder is not zero. In this case, we say that “4 is not a divisor of 25.”

Greatest Common Divisor

Let a and b be natural numbers. The common divisors of a and b are those natural numbers that divide both a and b . The *greatest common divisor* is the largest of these common divisors.

Example 1

Find the greatest common divisor of 18 and 24.

Solution

First list the divisors of each number, the numbers that divide each number with zero remainder.

Divisors of 18 : 1, 2, 3, 6, 9, and 18

Divisors of 24 : 1, 2, 3, 4, 6, 8, 12, and 24

The common divisors are:

Common Divisors : 1, 2, 3, and 6

The greatest common divisor is the largest of the common divisors. That is,

Greatest Common Divisor = 6.

That is, the largest number that divides both 18 and 24 is the number 6.

Exercise

Find the greatest common divisor of 27 and 36.

Answer available at the end of this section.

Reducing a Fraction to Lowest Terms

First, a definition.

Lowest Terms

A fraction is said to be *reduced to lowest terms* if the greatest common divisor of both numerator and denominator is 1.

Thus, for example, $\frac{2}{3}$ is reduced to lowest terms because the greatest common divisor of 2 and 3 is 1. On the other hand, $\frac{4}{6}$ is **not** reduced to lowest terms because the greatest common divisor of 4 and 6 is 2.

Example 2

Reduce the fraction $\frac{18}{24}$ to lowest terms.

Solution

One technique that works well is dividing both numerator and denominator by the greatest common divisor of the numerator and denominator. In Example 1, we saw that the greatest common divisor of 18 and 24 is 6. We divide both numerator and denominator by 6 to get

$$\begin{aligned}\frac{18}{24} &= \frac{18 \div 6}{24 \div 6} && \text{Divide numerator and denominator by 6.} \\ &= \frac{3}{4} && \text{Simplify numerator and denominator.}\end{aligned}$$

Note that the greatest common divisor of 3 and 4 is now 1. Thus, $\frac{3}{4}$ is reduced to lowest terms.

There is a second way we can show division of numerator and denominator by 6. First, factor both numerator and denominator as follows:

$$\frac{18}{24} = \frac{3 \cdot 6}{4 \cdot 6} \quad \text{Factor out a 6.}$$

You can then show “division” of both numerator and denominator by 6 by “crossing out” or “canceling” a 6 in the numerator for a 6 in the denominator, like this:

$$= \frac{3 \cdot \cancel{6}}{4 \cdot \cancel{6}} \text{ Cancel common factor.}$$

$$= \frac{3}{4}$$

Note that we get the same equivalent fraction, reduced to lowest terms, namely $\frac{3}{4}$.

Exercise

Reduce the fraction $\frac{12}{18}$ to lowest terms.

Answer available at the end of this section.

Important Point

In Example 2 we saw that 6 was both a *divisor* and a *factor* of 18. The words *divisor* and *factor* are equivalent.

We used the following technique in our second solution in Example 2.

Cancellation Rule

If you express numerator and denominator as a product, then you may cancel common factors from the numerator and denominator. The result will be an equivalent fraction.

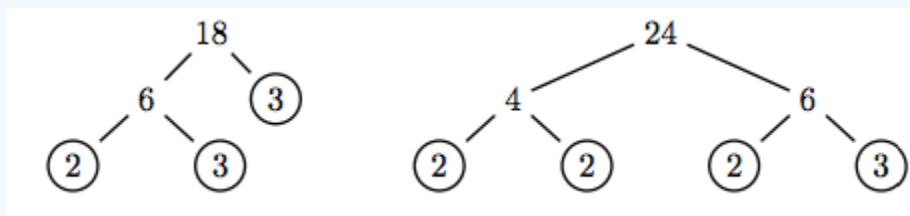
Because of the “Cancellation Rule,” one of the most effective ways to reduce a fraction to lowest terms is to first find prime factorizations for both numerator and denominator, then cancel all common factors.

Example 3

Reduce the fraction $\frac{18}{24}$ to lowest terms.

Solution

Use factor trees to prime factor numerator and denominator.



Once we've factored the numerator and denominator, we cancel common factors.

$$\begin{aligned}
 \frac{18}{24} &= \frac{2 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 3} && \text{Prime factor numerator and denominator.} \\
 &= \frac{\cancel{2} \cdot \cancel{3} \cdot 3}{\cancel{2} \cdot 2 \cdot 2 \cdot \cancel{3}} && \text{Cancel common factors.} \\
 &= \frac{3}{2 \cdot 2} && \text{Remaining factors.} \\
 &= \frac{3}{4} && \text{Simplify denominator.}
 \end{aligned}$$

Thus, $18/24 = 3/4$.

Exercise 2.1.1

Reduce the fraction $28/35$ to lowest terms.

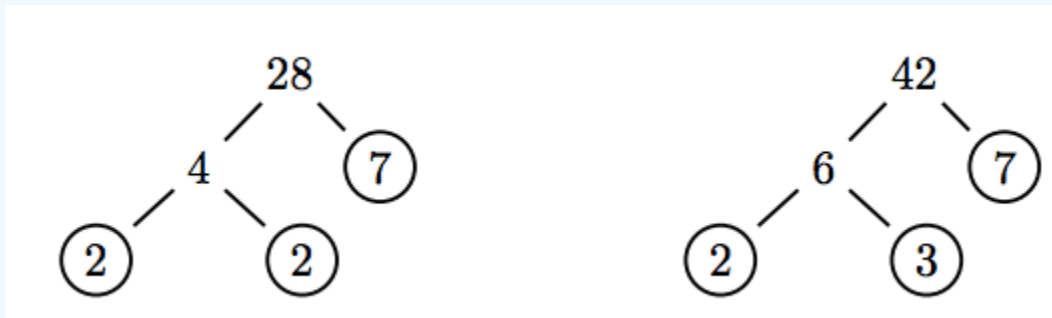
Answer available at the end of this section.

Example 4

Reduce the fraction $28/42$ to lowest terms.

Solution

Use factor trees to prime factor numerator and denominator.



Now we can cancel common factors.

$$\begin{aligned}
 \frac{28}{42} &= \frac{2 \cdot 2 \cdot 7}{2 \cdot 3 \cdot 7} && \text{Prime factor numerator and denominator.} \\
 &= \frac{\cancel{2} \cdot 2 \cdot \cancel{7}}{\cancel{2} \cdot 3 \cdot \cancel{7}} && \text{Cancel common factors.} \\
 &= \frac{2}{3}
 \end{aligned}$$

Thus, $28/42 = 2/3$.

Exercise

Reduce the fraction $36/60$ to lowest terms.

Answer available at the end of this section.

Equivalent Fractions in Higher Terms

Sometimes the need arises to find an equivalent fraction with a different, larger denominator.

Example 6

Express $\frac{3}{5}$ as an equivalent fraction having denominator 20.

Solution

The key here is to remember that multiplying numerator and denominator by the same number produces an equivalent fraction. To get an equivalent fraction with a denominator of 20, we'll have to multiply numerator and denominator of $\frac{3}{5}$ by 4.

$$\begin{aligned}\frac{3}{5} & \quad \text{Multiply numerator and denominator by 4.} \\ = \frac{12}{20} & \quad \text{Simplify numerator and denominator.}\end{aligned}$$

Therefore, $\frac{3}{5}$ equals $\frac{12}{20}$.

Exercise

Express $\frac{2}{3}$ as an equivalent fraction having denominator 21.

Answer available at the end of this section.

Example 7

Express 8 as an equivalent fraction having denominator 5.

Solution

The key here is to note that

$$8 = \frac{8}{1} \quad \text{Understood denominator is 1.}$$

To get an equivalent fraction with a denominator of 5, we'll have to multiply numerator and denominator of $\frac{8}{1}$ by 5.

$$\begin{aligned}& = \frac{8 \cdot 5}{1 \cdot 5} \quad \text{Multiply numerator and denominator by 5.} \\ & = \frac{40}{5} \quad \text{Simplify numerator and denominator.}\end{aligned}$$

Therefore, 8 equals $\frac{40}{5}$.

Exercise

Express 5 as an equivalent fraction having denominator 7.

Answer available at the end of this section.

Negative Fractions

We have to also deal with fractions that are negative. First, let's discuss placement of the negative sign.

- Positive divided by negative is negative, so

$$\frac{3}{-5} = -\frac{3}{5}.$$

- But it is also true that negative divided by positive is negative. Thus,

$$\frac{-3}{5} = -\frac{3}{5}.$$

These two observations imply that all three of the following fractions are equivalent (the same number):

$$\frac{3}{-5} = -\frac{3}{5} = \frac{-3}{5}.$$

Note that there are three possible placements for the negative sign: (1) the denominator, (2) the fraction bar, or (3) the numerator. Any one of these placements produces an equivalent fraction.

Fractions and Negative Signs

Let a and b be any integers. All three of the following fractions are equivalent (same number):

$$\frac{a}{-b} = -\frac{a}{b} = \frac{-a}{b}.$$

Mathematicians prefer to place the negative sign either in the numerator or on the fraction bar. The use of a negative sign in the denominator is discouraged.

Fractions and Negative Signs

Let a and b be any integers. Then,

$$\frac{-a}{-b} = \frac{a}{b}.$$

Exercises

Find the GCD of the given numbers.

1. 36, 60

2. 52, 20

3. 10, 40

Reduce the given fraction to lowest terms.

4. $\frac{28}{56}$

5. $\frac{93}{15}$

6. $\frac{10}{45}$

7. Express 3 as an equivalent fraction having denominator 8.

8. Express $\frac{17}{22}$ as an equivalent fraction having denominator 44.

9. Express $\frac{1}{3}$ as an equivalent fraction having denominator 24.

Reduce the given fraction to lowest terms.

10. $\frac{-48}{14}$

11. $\frac{27}{-75}$

12. $\frac{-60}{-100}$

13. **Hurricanes.** According to the National Atmospheric and Oceanic Administration, in 2008 there were 16 named storms, of which 8 grew into hurricanes and 5 were major.

i) What fraction of named storms grew into hurricanes? Reduce your answer to lowest terms.

ii) What fraction of named storms were major hurricanes? Reduce your answer to lowest terms.

iii) What fraction of hurricanes were major? Reduce your answer to lowest terms.

Answers

1. 12

2. 4

3. 10

4. $\frac{1}{2}$

5. $\frac{31}{5}$

6. $\frac{2}{9}$

7. $\frac{24}{8}$

8. $\frac{34}{44}$

9. $\frac{8}{24}$

10. $\frac{-24}{7}$

11. $\frac{-9}{25}$

12. $\frac{3}{5}$

13.

i) $\frac{1}{2}$

ii) $\frac{5}{16}$

iii) $\frac{5}{8}$

Answers

Exercise

9

Exercise $\frac{2}{3}$ **Exercise 2.1.1** $\frac{4}{5}$ **Exercise** $\frac{3}{5}$ **Exercise** $\frac{14}{21}$ **Exercise** $\frac{35}{7}$

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2.2: Multiplying Fractions

Consider the image in Figure 4.5, where the vertical lines divide the rectangular region into three equal pieces. If we shade one of the three equal pieces, the shaded area represents $\frac{1}{3}$ of the whole rectangular region.



Figure 4.5: The shaded region is $\frac{1}{3}$ of the whole region.

We'd like to visualize taking $\frac{1}{2}$ of $\frac{1}{3}$. To do that, we draw an additional horizontal line which divides the shaded region in half horizontally. This is shown in Figure 4.6. The shaded region that represented $\frac{1}{3}$ is now divided into two smaller rectangular regions, one of which is shaded with a different color. This region represents $\frac{1}{2}$ of $\frac{1}{3}$.



Figure 4.6: Shading $\frac{1}{2}$ of $\frac{1}{3}$.

Next, extend the horizontal line the full width of the rectangular region, as shown in Figure 4.7.



Figure 4.7: Shading $\frac{1}{2}$ of $\frac{1}{3}$.

Note that drawing the horizontal line, coupled with the three original vertical lines, has succeeded in dividing the full rectangular region into six smaller but equal pieces, only one of which (the one representing $\frac{1}{2}$ of $\frac{1}{3}$) is shaded in a new color. Hence, this newly shaded piece represents $\frac{1}{6}$ of the whole region. The conclusion of our visual argument is the fact that $\frac{1}{2}$ of $\frac{1}{3}$ equals $\frac{1}{6}$. In symbols,

$$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

Example 1

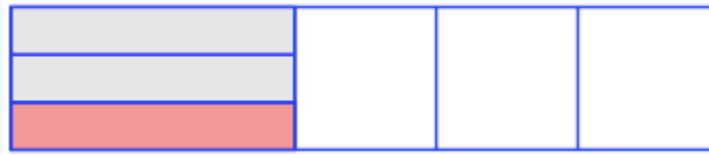
Create a visual argument showing that $\frac{1}{3}$ of $\frac{2}{5}$ is $\frac{2}{15}$.

Solution

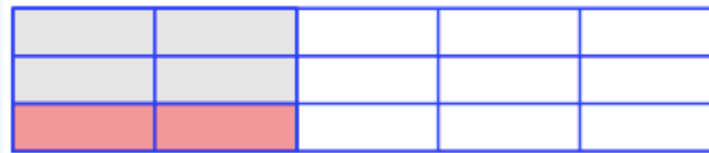
First, divide a rectangular region into five equal pieces and shade two of them. This represents the fraction $\frac{2}{5}$.



Next, draw two horizontal lines that divide the shaded region into three equal pieces and shade 1 of the three equal pieces. This represents taking $\frac{1}{3}$ of $\frac{2}{5}$.



Next, extend the horizontal lines the full width of the region and return the original vertical line from the first image.



Note that the three horizontal lines, coupled with the five original vertical lines, have succeeded in dividing the whole region into 15 smaller but equal pieces, only two of which (the ones representing $\frac{1}{3}$ of $\frac{2}{5}$) are shaded in the new color. Hence, this newly shaded piece represents $\frac{2}{15}$ of the whole region. The conclusion of this visual argument is the fact that $\frac{1}{3}$ of $\frac{2}{5}$ equals $\frac{2}{15}$. In symbols,

$$\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}.$$

Exercise

Create a visual argument showing that $\frac{1}{2}$ of $\frac{1}{4}$ is $\frac{1}{8}$.

Answer available at the end of this section.

Multiplication Rule

In Figure 4.7, we saw that $\frac{1}{2}$ of $\frac{1}{3}$ equals $\frac{1}{6}$. Note what happens when we multiply the numerators and multiply the denominators of the fractions $\frac{1}{2}$ and $\frac{1}{3}$.

$$\begin{aligned} \frac{1}{2} \cdot \frac{1}{3} &= \frac{1 \cdot 1}{2 \cdot 3} && \text{Multiply numerators; multiply denominators.} \\ &= \frac{1}{6} && \text{Simplify numerators and denominators.} \end{aligned}$$

We get $\frac{1}{6}$!

Could this be coincidence or luck? Let's try that again with the fractions from Example 1, where we saw that $\frac{1}{3}$ of $\frac{2}{5}$ equals $\frac{2}{15}$. Again, multiply the numerators and denominators of $\frac{1}{3}$ and $\frac{2}{5}$.

$$\begin{aligned} \frac{1}{3} \cdot \frac{2}{5} &= \frac{1 \cdot 2}{3 \cdot 5} && \text{Multiply numerators; multiply denominators.} \\ &= \frac{2}{15} && \text{Simplify numerators and denominators.} \end{aligned}$$

Again, we get $\frac{2}{15}$!

These two examples motivate the following definition.

Definition: Multiplication Rule

To find the product of the fractions a/b and c/d , multiply their numerators and denominators. In symbols,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Example 2

Multiply $1/5$ and $7/9$.

Solution

Multiply numerators and multiply denominators.

$$\begin{aligned}\frac{1}{5} \cdot \frac{7}{9} &= \frac{1 \cdot 7}{5 \cdot 9} && \text{Multiply numerators; multiply denominators.} \\ &= \frac{7}{45} && \text{Simplify numerators and denominators.}\end{aligned}$$

Exercise

Multiply:

$$\frac{1}{3} \cdot \frac{2}{5}$$

Answer available at the end of this section.

Example 3

Find the product of $-2/3$ and $7/9$.

Solution

The usual rules of signs apply to products. Unlike signs yield a negative result.

$$\begin{aligned}-\frac{2}{3} \cdot \frac{7}{9} &= -\frac{2 \cdot 7}{3 \cdot 9} && \text{Multiply numerators; multiply denominators.} \\ &= -\frac{14}{27} && \text{Simplify numerators and denominators.}\end{aligned}$$

It is not required that you physically show the middle step. If you want to do that mentally, then you can simply write

$$-\frac{2}{3} \cdot \frac{7}{9} = -\frac{14}{27}.$$

Exercise

Multiply:

$$-\frac{3}{5} \cdot \frac{2}{7}$$

Answer available at the end of this section.

Multiply and Reduce

After multiplying two fractions, make sure your answer is reduced to lowest terms (see Section 4.1).

Example 4

Multiply $\frac{3}{4}$ times $\frac{8}{9}$.

Solution

After multiplying, divide numerator and denominator by the greatest common divisor of the numerator and denominator.

$$\begin{aligned}\frac{3}{4} \cdot \frac{8}{9} &= \frac{3 \cdot 8}{4 \cdot 9} && \text{Multiply numerators and denominators.} \\ &= \frac{24}{36} && \text{Simplify numerator and denominator.} \\ &= \frac{24 \div 12}{36 \div 12} && \text{Divide numerator and denominator by GCD.} \\ &= \frac{2}{3} && \text{Simplify numerator and denominator.}\end{aligned}$$

Alternatively, after multiplying, you can prime factor both numerator and denominator, then cancel common factors.

$$\begin{aligned}\frac{3}{4} \cdot \frac{8}{9} &= \frac{24}{36} && \text{Multiply numerators and denominators.} \\ &= \frac{2 \cdot 2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 3 \cdot 3} && \text{Prime factor numerator and denominator.} \\ &= \frac{\cancel{2} \cdot \cancel{2} \cdot 2 \cdot \cancel{3}}{\cancel{2} \cdot \cancel{2} \cdot 3 \cdot \cancel{3}} && \text{Cancel common factors.} \\ &= \frac{2}{3}\end{aligned}$$

Exercise

Multiply:

$$\frac{3}{7} \cdot \frac{14}{9}$$

Answer available at the end of this section.

Multiply and Cancel or Cancel and Multiply

When you are working with larger numbers, it becomes a bit harder to multiply, factor, and cancel. Consider the following argument.

$$\begin{aligned}\frac{18}{30} \cdot \frac{35}{6} &= \frac{630}{180} && \text{Multiply numerators; multiply denominators.} \\ &= \frac{2 \cdot 3 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 3 \cdot 3 \cdot 5} && \text{Prime factor numerators and denominators.} \\ &= \frac{\cancel{2} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{5} \cdot 7}{2 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{5}} && \text{Cancel common factors.} \\ &= \frac{7}{2} && \text{Remaining factors.}\end{aligned}$$

There are a number of difficulties with this approach. First, you have to multiply large numbers, and secondly, you have to prime factor the even larger results.

One possible workaround is to not bother multiplying numerators and denominators, leaving them in factored form.

$$\frac{18}{30} \cdot \frac{35}{6} = \frac{18 \cdot 35}{30 \cdot 6} \quad \text{Multiply numerators; multiply denominators.}$$

Finding the prime factorization of these smaller factors is easier.

$$= \frac{(2 \cdot 3 \cdot 3) \cdot (5 \cdot 7)}{(2 \cdot 3 \cdot 5) \cdot (2 \cdot 3)} \quad \text{Prime factor.}$$

Now we can cancel common factors. Parentheses are no longer needed in the numerator and denominator because both contain a product of prime factors, so order and grouping do not matter.

$$= \frac{\cancel{2} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{5} \cdot \cancel{7}}{\cancel{2} \cdot \cancel{3} \cdot \cancel{5} \cdot 2 \cdot \cancel{3}} \quad \text{Cancel common factors.}$$

$$= \frac{7}{2} \quad \text{Remaining factors.}$$

Another approach is to factor numerators and denominators in place, cancel common factors, then multiply.

$$\frac{18}{30} \cdot \frac{35}{6} = \frac{2 \cdot 3 \cdot 3}{2 \cdot 3 \cdot 5} \cdot \frac{5 \cdot 7}{2 \cdot 3} \quad \text{Factor numerators and denominators.}$$

$$= \frac{\cancel{2} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{5}}{\cancel{2} \cdot \cancel{3} \cdot \cancel{5}} \cdot \frac{\cancel{5} \cdot 7}{2 \cdot \cancel{3}} \quad \text{Cancel common factors.}$$

$$= \frac{7}{2} \quad \text{Remaining factors.}$$

Note that this yields exactly the same result, 7/2.

Cancellation Rule

When multiplying fractions, cancel common factors according to the following rule: "Cancel a factor in a numerator for an identical factor in a denominator."

Example 6

Find the product of 14/15 and 30/140.

Solution

Multiply numerators and multiply denominators. Prime factor, cancel common factors, then multiply.

$$\frac{14}{15} \cdot \frac{30}{140} = \frac{14 \cdot 30}{15 \cdot 140} \quad \text{Multiply numerators; multiply denominators.}$$

$$= \frac{(2 \cdot 7) \cdot (2 \cdot 3 \cdot 5)}{(3 \cdot 5) \cdot (2 \cdot 2 \cdot 5 \cdot 7)} \quad \text{Prime factor numerators and denominators.}$$

$$= \frac{\cancel{2} \cdot \cancel{7} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{5}}{\cancel{3} \cdot 5 \cdot \cancel{2} \cdot \cancel{2} \cdot \cancel{5} \cdot \cancel{7}} \quad \text{Cancel common factors.}$$

$$= \frac{1}{5} \quad \text{Multiply.}$$

Note: Everything in the numerator cancels because you've divided the numerator by itself. Hence, the answer has a 1 in its numerator.

Exercise

Multiply:

$$\frac{6}{35} \cdot \frac{70}{36}$$

Answer available at the end of this section.

When Everything Cancels

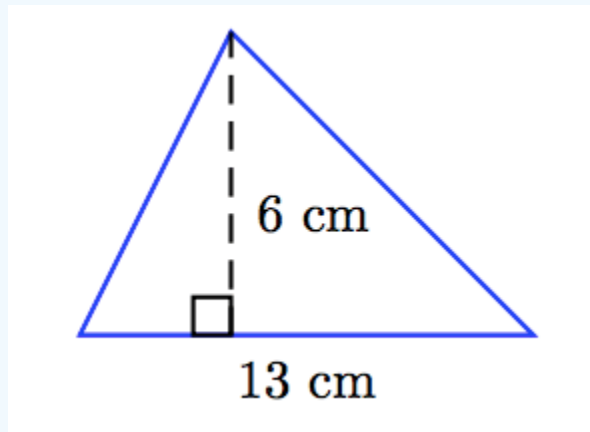
When all the factors in the numerator cancel, this means that you are dividing the numerator by itself. Hence, you are left with a 1 in the numerator. The same rule applies to the denominator. If everything in the denominator cancels, you're left with a 1 in the denominator.

Area of a Triangle

A triangle having base b and height h has area $A = (1/2)bh$. That is, to find the area of a triangle, take one-half the product of the base and height.

Example 9

Find the area of the triangle pictured below.



Solution

To find the area of the triangle, take one-half the product of the base and height.

$$\begin{aligned} A &= \frac{1}{2}bh && \text{Area of a triangle formula.} \\ &= \frac{1}{2}(13 \text{ cm})(6 \text{ cm}) && \text{Substitute: 13 cm for } b, 6 \text{ cm for } h. \\ &= \frac{78 \text{ cm}^2}{2} && \text{Multiply numerators; multiply denominators.} \\ &= 39 \text{ cm}^2. && \text{Simplify.} \end{aligned}$$

Therefore, the area of the triangle is 39 square centimeters.

Exercise

The base of a triangle measures 15 meters. The height is 12 meters. What is the area of the triangle?

Answer available at the end of this section.

Exercises

Multiply the fractions and simplify your result.

1. $\frac{21}{4} \cdot \frac{2}{7}$

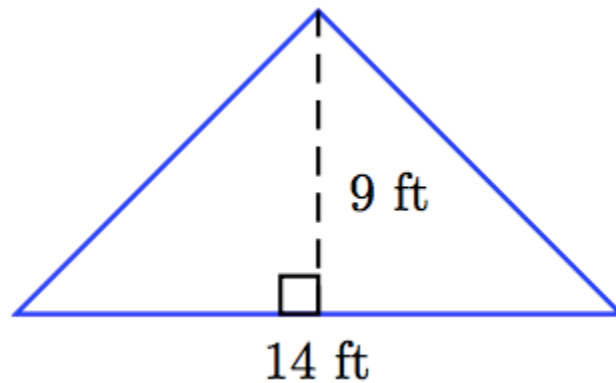
2. $\frac{-9}{2} \cdot \frac{6}{7}$

3. $\frac{2}{15} \cdot \frac{-9}{8}$

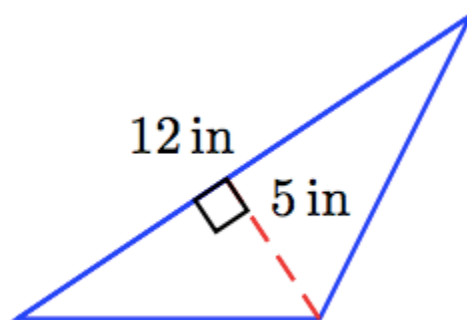
4. $\frac{-21}{20} \cdot \frac{-20}{21}$

Find the area of the triangle shown in the figure. (Note: Figures are not drawn to scale.)

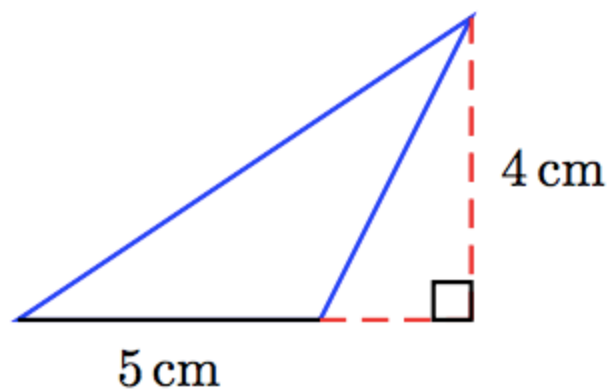
5.



6.



7.



8. **Weight on the Moon.** On the moon, you would only weigh $\frac{1}{6}$ of what you weigh on earth. If you weigh 138 pounds on earth, what would your weight on the moon be?

Answers

1. $\frac{3}{2}$

2. $\frac{-27}{7}$

3. $\frac{-3}{20}$

4. 1

5. 63 ft^2

6. 30 in^2

7. 10 cm^2

8. 23 pounds

Answers

Exercise

Diagram:

Exercise

$\frac{2}{15}$

Exercise

$\frac{6}{35}$

Exercise

$\frac{2}{3}$

Exercise

$\frac{1}{3}$

Exercise

90 square meters

2.3: Dividing Fractions

Suppose that you have four pizzas and each of the pizzas has been sliced into eight equal slices. Therefore, each slice of pizza represents $\frac{1}{8}$ of a whole pizza.

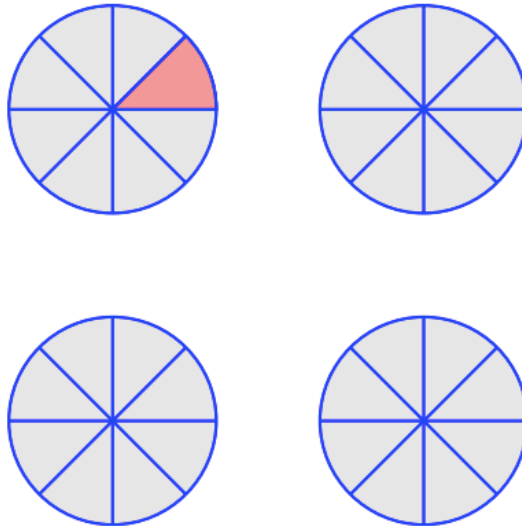


Figure 2.3.1: One slice of pizza is $\frac{1}{8}$ of one whole pizza.

Now for the question: How many one-eighths are there in four? This is a division statement. To find how many one-eighths there are in 4, divide 4 by $\frac{1}{8}$. That is,

$$\text{Number of one-eighths in four} = 4 \div \frac{1}{8}.$$

On the other hand, to find the number of one-eighths in four, Figure 2.3.1 clearly demonstrates that this is equivalent to asking how many slices of pizza are there in four pizzas. Since there are 8 slices per pizza and four pizzas,

$$\text{Number of pizza slices} = 4 \cdot 8.$$

The conclusion is the fact that $4 \div (\frac{1}{8})$ is equivalent to $4 \cdot 8$. That is,

$$\begin{aligned} 4 \div \frac{1}{8} &= 4 \cdot 8 \\ &= 32. \end{aligned}$$

Therefore, we conclude that there are 32 one-eighths in 4.

Reciprocals

The number 1 is still the *multiplicative identity* for fractions.

Multiplicative Identity Property

Let $\frac{a}{b}$ be any fraction. Then,

$$\frac{a}{b} \cdot 1 = \frac{a}{b} \text{ and } 1 \cdot \frac{a}{b} = \frac{a}{b}.$$

The number 1 is called the *multiplicative identity* because the identical number is returned when you multiply by 1.

Next, if we invert $3/4$, that is, if we turn $3/4$ upside down, we get $4/3$. Note what happens when we multiply $3/4$ by $4/3$.

The number $4/3$ is called the *multiplicative inverse* or *reciprocal* of $3/4$. The product of reciprocals is always 1.

Multiplicative Inverse Property

Let a/b be any fraction. The number b/a is called the *multiplicative inverse* or *reciprocal* of a/b . The product of reciprocals is 1.

$$\frac{a}{b} \cdot \frac{b}{a} = 1$$

Note: To find the multiplicative inverse (reciprocal) of a number, simply invert the number (turn it upside down).

For example, the number $1/8$ is the multiplicative inverse (reciprocal) of 8 because

$$8 \cdot \frac{1}{8} = 1.$$

Note that 8 can be thought of as $8/1$. Invert this number (turn it upside down) to find its multiplicative inverse (reciprocal) $1/8$.

Example 2.3.1

Find the multiplicative inverses (reciprocals) of: (a) $2/3$, (b) $-3/5$, and (c) -12 .

Solution

a) Because

$$\frac{2}{3} \cdot \frac{3}{2} = 1,$$

the multiplicative inverse (reciprocal) of $2/3$ is $3/2$.

b) Because

$$-\frac{3}{5} \cdot \left(-\frac{5}{3}\right) = 1,$$

the multiplicative inverse (reciprocal) of $-3/5$ is $-5/3$. Again, note that we simply inverted the number $-3/5$ to get its reciprocal $-5/3$.

c) Because

$$-12 \cdot \left(-\frac{1}{12}\right) = 1,$$

the multiplicative inverse (reciprocal) of -12 is $-1/12$. Again, note that we simply inverted the number -12 (understood to equal $-12/1$) to get its reciprocal $-1/12$.

Exercise 2.3.1

Find the reciprocals of: (a) $-3/7$ and (b) 15

Answer available at the end of this section.

Division

Recall that we computed the number of one-eighths in four by doing this calculation:

$$\begin{aligned}4 \div \frac{1}{8} &= 4 \cdot 8 \\ &= 32.\end{aligned}$$

Note how we inverted the divisor (second number), then changed the division to multiplication. This motivates the following definition of division.

Division Definition

If a/b and c/d are any fractions, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

That is, we invert the divisor (second number) and change the division to multiplication. *Note: We like to use the phrase “invert and multiply” as a memory aid for this definition.*

Example 2.3.2

Divide $1/2$ by $3/5$.

Solution

To divide $1/2$ by $3/5$, invert the divisor (second number), then multiply.

$$\begin{aligned}\frac{1}{2} \div \frac{3}{5} &= \frac{1}{2} \cdot \frac{5}{3} && \text{Invert the divisor (second number).} \\ &= \frac{5}{6} && \text{Multiply.}\end{aligned}$$

Exercise 2.3.2

Divide:

$$\frac{2}{3} \div \frac{10}{3}$$

Answer available at the end of this section.

Example 2.3.3

Simplify the following expressions: (a) $3 \div \frac{2}{3}$ and (b) $\frac{4}{5} \div 5$.

Solution

In each case, invert the divisor (second number), then multiply.

a) Note that 3 is understood to be $3/1$.

$$3 \div \frac{2}{3} = \frac{3}{1} \cdot \frac{3}{2}$$

$$= \frac{9}{2}$$

Invert the divisor (second number).

Multiply numerators; multiply denominators.

b) Note that 5 is understood to be 5/1.

$$\frac{4}{5} \div 5 = \frac{4}{5} \cdot \frac{1}{5}$$

$$= \frac{4}{25}$$

Invert the divisor (second number).

Multiply numerators; multiply denominators.

Exercise 2.3.3

Divide:

$$\frac{15}{7} \div 5$$

Answer

Answer available at the end of this section.

After inverting, you may need to factor and cancel, as we learned to do in Section 4.2.

Example 2.3.4

Divide $-6/35$ by $33/55$.

Solution

Invert, multiply, factor, and cancel common factors.

$$-\frac{6}{35} \div \frac{33}{55} = -\frac{6}{35} \cdot \frac{55}{33}$$

Invert the divisor (second number).

$$= -\frac{6 \cdot 55}{35 \cdot 33}$$

Multiply numerators; multiply denominators.

$$= -\frac{(2 \cdot 3) \cdot (5 \cdot 11)}{(5 \cdot 7) \cdot (3 \cdot 11)}$$

Factor numerators and denominators.

$$= -\frac{2 \cdot \cancel{3} \cdot \cancel{5} \cdot \cancel{11}}{\cancel{5} \cdot 7 \cdot \cancel{3} \cdot \cancel{11}}$$

Cancel common factors.

$$= -\frac{2}{7}$$

Remaining factors.

Note that unlike signs produce a negative answer.

Exercise 2.3.4

Divide:

$$\frac{6}{15} \div \left(-\frac{42}{35}\right)$$

Answer available at the end of this section.

Of course, you can also choose to factor numerators and denominators in place, then cancel common factors.

Exercises

Find the reciprocal of the given number.

1. $16/5$

2. -17

3. $-15/16$

4. 30

Determine which property of multiplication is depicted by the given identity.

5. $\frac{-19}{12} \cdot 1 = \frac{-19}{12}$

6. $-\frac{4}{1} \cdot \left(-\frac{1}{4}\right) = 1$

Divide the fractions and simplify your result.

7. $\frac{8}{23} \div \frac{-6}{11}$

8. $\frac{-10}{21} \div \frac{-6}{5}$

9. $\frac{5}{9} \div \frac{4}{3}$

10. $\frac{-2}{7} \div \frac{-8}{7}$

11. $\frac{20}{17} \div 5$

12. $-6 \div \frac{-21}{8}$

13. $\frac{3}{4} \div (-9)$

Answers

1. $\frac{5}{16}$

2. $-\frac{1}{17}$

3. $-\frac{16}{15}$

4. $\frac{1}{30}$

5. multiplicative identity property

6. multiplicative inverse property

7. $-\frac{44}{69}$

8. $\frac{25}{63}$

9. $\frac{5}{12}$

10. $\frac{1}{4}$

11. $\frac{4}{17}$

12. $\frac{16}{7}$

13. $-\frac{1}{12}$

Answers

Exercise 2.3.1

(a) $-7/3$, (b) $1/15$

Exercise 2.3.2

$1/5$

Exercise 2.3.3

$\frac{3}{7}$

Exercise 2.3.4

$-1/3$

2.4: Adding and Subtracting Fractions

Paul and Tony order a pizza which has been cut into eight equal slices. Thus, each slice is $\frac{1}{8}$ of the whole pizza. Paul eats two slices (shaded in light gray in Figure 2.4.1), or $\frac{2}{8}$ of the whole pizza. Tony eats three slices (shaded in light red (or a darker shade of gray in black-and-white printing) in Figure 2.4.1), or $\frac{3}{8}$ of the whole pizza.

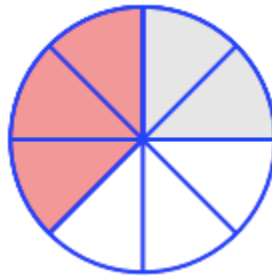


Figure 2.4.1: Paul eats two slices ($\frac{2}{8}$) and Tony eats three slices ($\frac{3}{8}$).

It should be clear that together Paul and Tony eat five slices, or $\frac{5}{8}$ of the whole pizza. This reflects the fact that

$$\frac{2}{8} + \frac{3}{8} = \frac{5}{8}.$$

This demonstrates how to add two fractions with a common (same) denominator. Keep the common denominator and add the numerators. That is,

$$\begin{aligned} \frac{2}{8} + \frac{3}{8} &= \frac{2+3}{8} && \text{Keep denominator; add numerators.} \\ &= \frac{5}{8} && \text{Simplify numerator.} \end{aligned}$$

Adding Fractions with Common Denominators

Let $\frac{a}{c}$ and $\frac{b}{c}$ be two fractions with a common (same) denominator. Their sum is defined as

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

That is, to add two fractions having common denominators, keep the common denominator and add their numerators.

A similar rule holds for subtraction.

Subtracting Fractions with Common Denominators

Let $\frac{a}{c}$ and $\frac{b}{c}$ be two fractions with a common (same) denominator. Their difference is defined as

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

That is, to subtract two fractions having common denominators, keep the common denominator and subtract their numerators.

Example 2.4.1

Find the sum of $\frac{4}{9}$ and $\frac{3}{9}$.

Solution

Keep the common denominator and add the numerators.

$$\begin{aligned}\frac{4}{9} + \frac{3}{9} &= \frac{4+3}{9} && \text{Keep denominator; add numerators.} \\ &= \frac{7}{9} && \text{Simplify numerator.}\end{aligned}$$

Exercise 2.4.1

Add:

$$\frac{1}{8} + \frac{2}{8}$$

Answer available at the end of this section.

Example 2.4.2

Subtract $\frac{5}{16}$ from $\frac{13}{16}$.

Solution

Keep the common denominator and subtract the numerators.

$$\begin{aligned}\frac{13}{16} - \frac{5}{16} &= \frac{13-5}{16} && \text{Keep denominator; subtract numerators.} \\ &= \frac{8}{16} && \text{Simplify numerator.}\end{aligned}$$

Of course, as we learned in Section 4.1, we should always reduce our final answer to lowest terms. One way to accomplish that in this case is to divide numerator and denominator by 8, the greatest common divisor of 8 and 16.

$$\begin{aligned}&= \frac{8 \div 8}{16 \div 8} && \text{Divide numerator and denominator by 8.} \\ &= \frac{1}{2} && \text{Simplify numerator and denominator.}\end{aligned}$$

Exercise 2.4.2

Subtract:

$$\frac{11}{12} - \frac{7}{12}$$

Answer available at the end of this section.

Adding Fractions with Different Denominators

Consider the sum

$$\frac{4}{9} + \frac{1}{6}.$$

We cannot add these fractions because they do not have a common denominator. So, what to do?

Goals

In order to add two fractions with different denominators, we need to:

1. Find a common denominator for the given fractions.
2. Make fractions with the common denominator that are equivalent to the original fractions.

If we accomplish the two items in the “Goal,” we will be able to find the sum of the given fractions.

So, how to start? We need to find a common denominator, but not just any common denominator. Let's agree that we want to keep the numbers as small as possible and find a *least common denominator*.

Definition: Least Common Denominator

The *least common denominator* (LCD) for a set of fractions is the smallest number divisible by each of the denominators of the given fractions.

Consider again the sum we wish to find:

$$\frac{4}{9} + \frac{1}{6}.$$

The denominators are 9 and 6. We wish to find a least common denominator, the smallest number that is divisible by both 9 and 6. A number of candidates come to mind: 36, 54, and 72 are all divisible by 9 and 6, to name a few. But the smallest number that is divisible by both 9 and 6 is 18. This is the least common denominator for 9 and 6.

We now proceed to the second item in “Goal.” We need to make fractions having 18 as a denominator that are equivalent to $4/9$ and $1/6$. In the case of $4/9$, if we multiply both numerator and denominator by 2, we get

$$\begin{aligned} \frac{4}{9} &= \frac{4 \cdot 2}{9 \cdot 2} && \text{Multiply numerator and denominator by 2.} \\ &= \frac{8}{18} && \text{Simplify numerator and denominator.} \end{aligned}$$

In the case of $1/6$, if we multiply both numerator and denominator by 3, we get

$$\begin{aligned} \frac{1}{6} &= \frac{1 \cdot 3}{6 \cdot 3} && \text{Multiply numerator and denominator by 3.} \\ &= \frac{3}{18} && \text{Simplify numerator and denominator.} \end{aligned}$$

Typically, we'll arrange our work as follows.

$$\begin{aligned} \frac{4}{9} + \frac{1}{6} &= \frac{4 \cdot 2}{9 \cdot 2} + \frac{1 \cdot 3}{6 \cdot 3} && \text{Equivalent fractions with LCD = 18.} \\ &= \frac{8}{18} + \frac{3}{18} && \text{Simplify numerators and denominators.} \\ &= \frac{8+3}{18} && \text{Keep common denominator; add numerators.} \\ &= \frac{11}{18} && \text{Simplify numerator.} \end{aligned}$$

Let's summarize the procedure.

Adding or Subtracting Fractions with Different Denominators

1. Find the LCD, the smallest number divisible by all the denominators of the given fractions.
2. Create fractions using the LCD as the denominator that are equivalent to the original fractions.
3. Add or subtract the resulting equivalent fractions. Simplify, including reducing the final answer to lowest terms.

Example 2.4.4

Simplify: $\frac{3}{5} - \frac{2}{3}$.

Solution

The smallest number divisible by both 5 and 3 is 15.

$$\begin{aligned}\frac{3}{5} - \frac{2}{3} &= \frac{3 \cdot 3}{5 \cdot 3} - \frac{2 \cdot 5}{3 \cdot 5} && \text{Equivalent fractions with LCD} = 15. \\ &= \frac{9}{15} - \frac{10}{15} && \text{Simplify numerators and denominators.} \\ &= \frac{9 - 10}{15} && \text{Keep LCD; subtract numerators.} \\ &= \frac{-1}{15} && \text{Simplify numerator.}\end{aligned}$$

Although this answer is perfectly acceptable, negative divided by positive gives us a negative answer, so we could also write

$$= -\frac{1}{15}.$$

Exercise 2.4.4

Subtract:

$$\frac{3}{4} - \frac{7}{5}$$

Answer available at the end of this section.

Example 2.4.5

Simplify: $-\frac{1}{4} - \frac{5}{6}$.

Solution

The smallest number divisible by both 4 and 6 is 12.

$$\begin{aligned}-\frac{1}{4} - \frac{5}{6} &= -\frac{1 \cdot 3}{4 \cdot 3} - \frac{5 \cdot 2}{6 \cdot 2} && \text{Equivalent fractions with LCD} = 12. \\ &= -\frac{3}{12} - \frac{10}{12} && \text{Simplify numerators and denominators.} \\ &= \frac{-3 - 10}{12} && \text{Keep LCD; subtract numerators.} \\ &= \frac{-13}{12} && \text{Simplify numerator.}\end{aligned}$$

Exercise 2.4.5

Subtract: $-\frac{3}{8} - \frac{1}{12}$

Answer available at the end of this section.

Least Common Multiple

First we define the multiple of a number.

Definition: Multiples

The **multiples** of a number d are $1d, 2d, 3d, 4d$, etc. That is, the multiples of d are the numbers nd , where n is a natural number.

For example, the multiples of 8 are $1 \cdot 8, 2 \cdot 8, 3 \cdot 8, 4 \cdot 8$, etc., or equivalently, 8, 16, 24, 32, etc.

Definition: Least Common Multiple

The **least common multiple** (LCM) of a set of numbers is the smallest number that is a multiple of each number of the given set. The procedure for finding an LCM follows:

1. List all of the multiples of each number in the given set of numbers.
2. List the multiples that are in common.
3. Pick the least of the multiples that are in common.

Example 2.4.7

Find the least common multiple (LCM) of 12 and 16.

Solution

List the multiples of 12 and 16.

Multiples of 12 : 12, 24, 36, 48, 60, 72, 84, 96,...

Multiples of 16 : 16, 32, 48, 64, 80, 96, 112,...

Pick the common multiples.

Common Multiples : 48, 96,...

The LCM is the least of the common multiples.

$$\text{LCM}(12,16) = 48$$

Exercise 2.4.7

Find the least common denominator of 6 and 9.

Answer available at the end of this section.

Important Observation

The *least common denominator* is the *least common multiple* of the denominators.

For example, suppose your problem is $5/12 + 5/16$. The LCD is the smallest number divisible by both 12 and 16. That number is 48, which is also the LCM of 12 and 16. Therefore, the procedure for finding the LCM can also be used to find the LCD.

Least Common Multiple Using Prime Factorization

You can also find the LCM using prime factorization.

LCM By Prime Factorization

To find an LCM for a set of numbers, follow this procedure:

1. Write down the prime factorization for each number in compact form using exponents.
2. The LCM is found by writing down every factor that appears in step 1 to the highest power of that factor that appears.

Example 2.4.8

Use prime factorization to find the least common multiple find the least common denominator of 18 and 24. (LCM) of 12 and 16.

Solution

Prime factor 12 and 16.

$$\begin{aligned}12 &= 2 \cdot 2 \cdot 3 \\16 &= 2 \cdot 2 \cdot 2 \cdot 2\end{aligned}$$

Write the prime factorizations in compact form using exponents.

$$\begin{aligned}12 &= 2^2 \cdot 3^1 \\16 &= 2^4\end{aligned}$$

To find the LCM, write down each factor that appears to the highest power of that factor that appears. The factors that appear are 2 and 3. The highest power of 2 that appears is 2^4 . The highest power of 3 that appears is 3^1 .

$$\text{LCM} = 2^4 \cdot 3^1 \text{ Keep highest power of each factor.}$$

Now we expand this last expression to get our LCM.

$$\begin{aligned}&= 16 \cdot 3 \text{ Expand: } 2^4 = 16 \text{ and } 3^1 = 3. \\&= 48. \text{ Multiply.}\end{aligned}$$

Note that this answer is identical to the LCM found in Example 8 that was found by listing multiples and choosing the smallest multiple in common.

Exercise 2.4.8

Use prime factorization to find the least common denominator of 18 and 24.

Answer available at the end of this section.

Example 2.4.10

Simplify: $\frac{5}{28} + \frac{11}{42}$.

Solution

Prime factor the denominators in compact form using exponents.

$$28 = 2 \cdot 2 \cdot 7 = 2^2 \cdot 7$$

$$42 = 2 \cdot 3 \cdot 7 = 2^1 \cdot 3^1 \cdot 7^1$$

To find the LCD, write down each factor that appears to the highest power of that factor that appears. The factors that appear are 2, 3, and 7. The highest power of 2 that appears is 2^2 . The highest power of 3 that appears is 3^1 . The highest power of 7 that appears is 7^1 .

$$\begin{aligned} \text{LCM} &= 2^2 \cdot 3^1 \cdot 7^1 && \text{Keep highest power of each factor.} \\ &= 4 \cdot 3 \cdot 7 && \text{Expand: } 2^2 = 4, 3^1 = 3, 7^1 = 7. \\ &= 84 && \text{Multiply.} \end{aligned}$$

Create equivalent fractions with the new LCD, then add.

$$\begin{aligned} \frac{5}{28} + \frac{11}{42} &= \frac{5 \cdot 3}{28 \cdot 3} + \frac{11 \cdot 2}{42 \cdot 2} && \text{Equivalent fractions with LCD} = 84. \\ &= \frac{15}{84} + \frac{22}{84} && \text{Simplify numerators and denominators.} \\ &= \frac{37}{84} && \text{Keep LCD; add numerators.} \end{aligned}$$

Exercise 2.4.10

Simplify: $\frac{5}{24} + \frac{5}{36}$

Answer available at the end of this section.

Example 2.4.11

Simplify: $-\frac{11}{24} - \frac{1}{18}$.

Solution

Prime factor the denominators in compact form using exponents.

$$24 = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3^1$$

$$18 = 2 \cdot 3 \cdot 3 = 2^1 \cdot 3^2$$

To find the LCD, write down each factor that appears to the highest power of that factor that appears. The factors that appear are 2 and 3. The highest power of 2 that appears is 2^3 . The highest power of 3 that appears is 3^2 .

$$\begin{aligned} \text{LCM} &= 2^3 \cdot 3^2 && \text{Keep highest power of each factor.} \\ &= 8 \cdot 9 && \text{Expand: } 2^3 = 8 \text{ and } 3^2 = 9. \\ &= 72. && \text{Multiply.} \end{aligned}$$

Create equivalent fractions with the new LCD, then subtract.

$$\begin{aligned}
 -\frac{11}{24} - \frac{1}{18} &= -\frac{11 \cdot 3}{24 \cdot 3} - \frac{1 \cdot 4}{18 \cdot 4} && \text{Equivalent fractions with LCD} = 72. \\
 &= -\frac{33}{72} - \frac{4}{72} && \text{Simplify numerators and denominators.} \\
 &= \frac{-33 - 4}{72} && \text{Keep LCD; subtract numerators.} \\
 &= \frac{-37}{72} && \text{Simplify numerator.}
 \end{aligned}$$

Of course, negative divided by positive yields a negative answer, so we can also write our answer in the form

$$-\frac{11}{24} - \frac{1}{18} = -\frac{37}{72}.$$

Exercise 2.4.11

Simplify: $-\frac{5}{24} - \frac{11}{36}$

Answer available at the end of this section.

Comparing Fractions

The simplest way to compare fractions is to create equivalent fractions.

Example 2.4.12

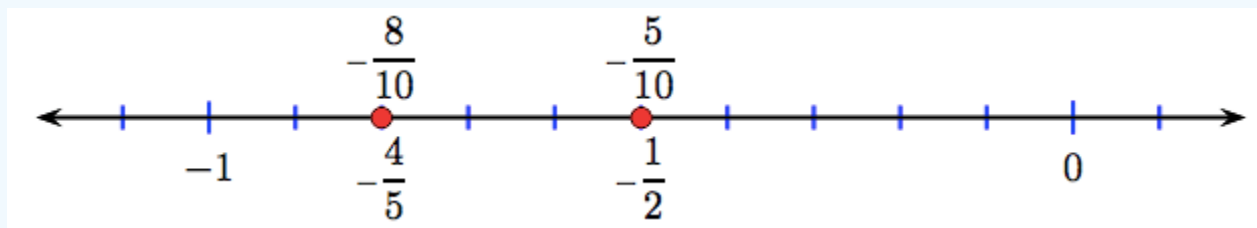
Arrange the fractions $-1/2$ and $-4/5$ on a number line, then compare them by using the appropriate inequality symbol.

Solution

The least common denominator for 2 and 5 is the number 10. First, make equivalent fractions with a LCD equal to 10.

$$\begin{aligned}
 -\frac{1}{2} &= -\frac{1 \cdot 5}{2 \cdot 5} = -\frac{5}{10} \\
 -\frac{4}{5} &= -\frac{4 \cdot 2}{5 \cdot 2} = -\frac{8}{10}
 \end{aligned}$$

To plot tenths, subdivide the interval between -1 and 0 into ten equal increments.



Because $-4/5$ lies to the left of $-1/2$, we have that $-4/5$ is less than $-1/2$, so we write

$$-\frac{4}{5} < -\frac{1}{2}.$$

Exercise 2.4.12

Compare $-3/8$ and $-1/2$.

Answer available at the end of this section.

Exercises

List the multiples of the given numbers, then list the common multiples. Select the LCM from the list of common multiples.

1. 9 and 15

2. 20 and 8

3. 10 and 6

For the given numbers, calculate the LCM using prime factorization.

4. 54 and 12

5. 72 and 108

Add or subtract the fractions, as indicated, and simplify your result.

6. $\frac{7}{12} - \frac{1}{12}$

7. $\frac{3}{7} - \frac{4}{7}$

8. $\frac{4}{11} + \frac{9}{11}$

9. $\frac{1}{6} - \frac{1}{8}$

10. $\frac{1}{5} + \frac{2}{3}$

11. $\frac{2}{3} + \frac{5}{8}$

12. $\frac{1}{6} + \frac{2}{3}$

13. $\frac{5}{6} - \frac{4}{5}$

14. $\frac{1}{6} + \frac{7}{9}$

Answers

1. 45

2. 40

3. 30

4. 108

5. 216

6. $\frac{1}{2}$

7. $-\frac{1}{7}$

8. $\frac{13}{11}$

9. $\frac{1}{24}$

10. $\frac{13}{15}$

11. $\frac{31}{24}$

12. $\frac{5}{6}$

13. $\frac{1}{30}$

14. $\frac{17}{18}$

Answers

Exercise 2.4.1

3/8

Exercise 2.4.2

1/3

Exercise 2.4.4

-13/20

Exercise 2.4.5

-11/24

Exercise 2.4.7

18

Exercise 2.4.8

72

Exercise 2.4.10

25/72

Exercise 2.4.11 $-\frac{37}{72}$ **Exercise 2.4.12**

$$-\frac{1}{2} < -\frac{3}{8}$$

2.5: Multiplying and Dividing Mixed Fractions

We begin with definitions of *proper* and *improper* fractions.

Proper and Improper Fractions

A *proper fraction* is a fraction whose numerator is smaller than its denominator. An *improper fraction* is a fraction whose numerator is larger than or equal to its denominator.

For example,

$$\frac{2}{3}, -\frac{23}{39}, \text{ and } \frac{119}{127}$$

are all examples of proper fractions. On the other hand,

$$\frac{4}{3}, -\frac{317}{123}, \frac{5}{5}, \text{ and } -\frac{233}{101}$$

are all examples of improper fractions.

A mixed fraction¹ is part whole number, part fraction.

Mixed Fractions

The number

$$5\frac{3}{4}$$

is called a *mixed fraction*. It is defined to mean

$$5\frac{3}{4} = 5 + \frac{3}{4}.$$

In the mixed fraction $5\frac{3}{4}$, the 5 is the *whole number part* and the $\frac{3}{4}$ is the *fractional part*.

Changing Mixed Fractions to Improper Fractions

We have all the tools required to change a mixed fraction into an improper fraction. We begin with an example.

Example 1

Change the mixed fraction $4\frac{7}{8}$ into an improper fraction.

Solution

We employ the definition of a mixed fraction, make an equivalent fraction for the whole number part, then add.

$$\begin{aligned}
 4\frac{7}{8} &= 4 + \frac{7}{8} && \text{By definition.} \\
 &= \frac{4 \cdot 8}{8} + \frac{7}{8} && \text{Equivalent fraction with LCD = 8.} \\
 &= \frac{4 \cdot 8 + 7}{8} && \text{Add numerators over common denominator.} \\
 &= \frac{39}{8} && \text{Simplify the numerator.}
 \end{aligned}$$

Thus, $4\frac{7}{8}$ is equal to $39/8$.

Exercise

Change $5\frac{3}{4}$ to an improper fraction.

Answer available at the end of this section.

There is a quick technique you can use to change a mixed fraction into an improper fraction.

Quick Way to Change a Mixed Fraction to an Improper Fraction

To change a mixed fraction to an improper fraction, multiply the whole number part by the denominator, add the numerator, then place the result over the denominator.

Thus, to quickly change $4\frac{7}{8}$ to an improper fraction, multiply the whole number 4 by the denominator 8, add the numerator 7, then place the result over the denominator. In symbols, this would look like this:

$$4\frac{7}{8} = \frac{4 \cdot 8 + 7}{8}.$$

This is precisely what the third step in Example 1 looks like; we're just eliminating a lot of the work.

Example 2

Change $4\frac{2}{3}$ to an improper fraction.

Solution

Take $4\frac{2}{3}$, multiply the whole number part by the denominator, add the numerator, then put the result over the denominator.

$$4\frac{2}{3} = \frac{4 \cdot 3 + 2}{3}$$

Thus, the result is

$$4\frac{2}{3} = \frac{14}{3}.$$

Exercise

Change $7\frac{3}{8}$ to an improper fraction.

Answer available at the end of this section.

You can do the intermediate step in Example 2 mentally, allowing you to skip the intermediate step and go directly from the mixed

fraction to the improper fraction without writing down a single bit of work.

Example 3

Without writing down any work, use mental arithmetic to change $-2\frac{3}{5}$ to an improper fraction.

Solution

To change $-2\frac{3}{5}$ to an improper fraction, ignore the minus sign, proceed as before, then prefix the minus sign to the resulting improper fraction. So, multiply 5 times 2 and add 3. Put the result 13 over the denominator 5, then prefix the resulting improper fraction with a minus sign. That is,

$$-2\frac{3}{5} = -\frac{13}{5}.$$

Exercise

Change $-3\frac{5}{12}$ to an improper fraction.

Answer available at the end of this section.

Changing Improper Fractions to Mixed Fractions

The first step in changing the improper fraction $27/5$ to a mixed fraction is to write the improper fraction as a sum.

$$\frac{27}{5} = \frac{25}{5} + \frac{2}{5}$$

Simplifying equation 4.1, we get

$$\begin{aligned}\frac{27}{5} &= 5 + \frac{2}{5} \\ &= 5\frac{2}{5}\end{aligned}$$

Note: You can't just choose any sum. The sum used in equation 4.1 is constructed so that the first fraction will equal a whole number and the second fraction is proper. Any other sum will fail to produce the correct mixed fraction. For example, the sum

$$\frac{27}{5} = \frac{23}{5} + \frac{4}{5}$$

is useless, because $23/5$ is not a whole number. Likewise, the sum

$$\frac{27}{5} = \frac{20}{5} + \frac{7}{5}$$

is no good. Even though $20/5 = 4$ is a whole number, the second fraction $7/5$ is still improper.

Example 4

Change $25/9$ to a mixed fraction.

Solution

Break $25/9$ into the appropriate sum.

$$\begin{aligned}\frac{25}{9} &= \frac{18}{9} + \frac{7}{9} \\ &= 2 + \frac{7}{9} \\ &= 2\frac{7}{9}\end{aligned}$$

Exercise

Change $25/7$ to a mixed fraction.

Answer available at the end of this section.

Note: A pattern is emerging.

- In the case of $27/5$, note that 27 divided by 5 is equal to 5 with a remainder of 2. Compare this with the mixed fraction result: $27/5 = 5\frac{2}{5}$.
- In the case of Example 4, note that 25 divided by 9 is 2 with a remainder of 7. Compare this with the mixed fraction result: $25/9 = 2\frac{7}{9}$. These observations motivate the following technique.

Quick Way to Change an Improper Fraction to a Mixed Fraction

To change an improper fraction to a mixed fraction, divide the numerator by the denominator. The quotient will be the whole number part of the mixed fraction. If you place the remainder over the denominator, this will be the fractional part of the mixed fraction.

Example 5

Change $37/8$ to a mixed fraction.

Solution

37 divided by 8 is 4, with a remainder of 5. That is:

$$\begin{array}{r} 4 \\ 8 \overline{)37} \\ \underline{32} \\ 5 \end{array}$$

The quotient becomes the whole number part and we put the remainder over the divisor. Thus,

$$\frac{37}{8} = 4\frac{5}{8}.$$

Note: You can check your result with the “Quick Way to Change a Mixed Fraction to an Improper Fraction.” 8 times 4 plus 5 is 37. Put this over 8 to get $37/8$.

Exercise

Change $38/9$ to a mixed fraction.

Answer available at the end of this section.

Example 6

Change $-43/5$ to a mixed fraction.

Solution

Ignore the minus sign and proceed in the same manner as in Example 5. 43 divided by 5 is 8, with a remainder of 3.

$$\begin{array}{r} 8 \\ 5 \overline{)43} \\ \underline{40} \\ 3 \end{array}$$

The quotient is the whole number part, then we put the remainder over the divisor. Finally, prefix the minus sign.

$$-\frac{43}{5} = -8\frac{3}{5}$$

Multiplying and Dividing Mixed Fractions

You have all the tools needed to multiply and divide mixed fractions. First, change the mixed fractions to improper fractions, then multiply or divide as you did in previous sections.

¹A mixed fraction is sometimes called a *mixed number*.

Example 7

Simplify: $-2\frac{1}{12} \cdot 2\frac{4}{5}$.

Solution

Change to improper fractions, factor, cancel, and simplify.

$$\begin{aligned} -2\frac{1}{12} \cdot 2\frac{4}{5} &= -\frac{25}{12} \cdot \frac{14}{5} && \text{Change to improper fractions.} \\ &= -\frac{25 \cdot 14}{12 \cdot 5} && \text{Multiply numerators; multiply denominators.} \\ &= -\frac{(5 \cdot 5) \cdot (2 \cdot 7)}{2 \cdot 2 \cdot 3 \cdot (5)} && \text{Unlike signs; product is negative.} \\ &= -\frac{\cancel{5} \cdot 5 \cdot \cancel{2} \cdot 7}{\cancel{2} \cdot 2 \cdot 3 \cdot \cancel{5}} && \text{Prime factor.} \\ &= -\frac{35}{6} && \text{Cancel common factors.} \\ &= -\frac{35}{6} && \text{Multiply numerators and denominators.} \end{aligned}$$

This is a perfectly good answer, but if you want a mixed fraction answer, 35 divided by 6 is 5, with a remainder of 5. Hence,

$$-2\frac{1}{12} \cdot 2\frac{4}{5} = -5\frac{5}{6}.$$

Exercise 2.5.1

Simplify:

$$-3\frac{3}{4} \cdot 2\frac{2}{5}$$

Answer available at the end of this section.

Example 8

Simplify:

$$-4\frac{4}{5} \div 5\frac{3}{5}.$$

Solution

Change to improper fractions, invert and multiply, factor, cancel, and simplify.

$$\begin{aligned} -4\frac{4}{5} \div 5\frac{3}{5} &= -\frac{24}{5} \div \frac{28}{5} && \text{Change to improper fractions.} \\ &= -\frac{24}{5} \cdot \frac{5}{28} && \text{Invert and multiply.} \\ &= -\frac{2 \cdot 2 \cdot 2 \cdot 3}{5} \cdot \frac{5}{2 \cdot 2 \cdot 7} && \text{Prime factor.} \\ &= -\frac{\cancel{2} \cdot \cancel{2} \cdot 2 \cdot 3}{\cancel{5}} \cdot \frac{\cancel{5}}{\cancel{2} \cdot \cancel{2} \cdot 7} && \text{Cancel common factors.} \\ &= -\frac{6}{7} && \text{Multiply numerators and denominators.} \end{aligned}$$

Exercise

Simplify:

$$-2\frac{4}{9} \cdot 3\frac{2}{3}$$

Answer available at the end of this section.

Exercises

Convert the mixed fraction to an improper fraction.

1. $2\frac{1}{3}$

2. $1\frac{3}{17}$

3. $-1\frac{1}{2}$

Convert the improper fraction to a mixed fraction.

4. $\frac{13}{7}$

5. $\frac{-10}{3}$

6. $\frac{9}{8}$

Multiply the numbers and express your answer as a mixed fraction.

7. $1\frac{1}{7} \cdot 2\frac{1}{2}$

8. $2\frac{1}{4} \cdot 1\frac{5}{11}$

9. $4 \cdot (-2\frac{5}{6})$

10. $(-2\frac{1}{8})(-6)$

Divide the mixed fractions and express your answer as a mixed fraction.

11. $8 \div 2\frac{2}{9}$

12. $(-4\frac{2}{3}) \div 4$

13. $(-5\frac{2}{3}) \div (-2\frac{1}{6})$

14. **Small Lots.** How many quarter-acre lots can be made from $6\frac{1}{2}$ acres of land?

15. **Big Field.** A field was formed from $17\frac{1}{2}$ half-acre lots. How many acres was the resulting field ?

Answers

1. $\frac{7}{3}$

2. $\frac{20}{17}$

3. $-\frac{3}{2}$

4. $1\frac{6}{7}$

5. $-3\frac{1}{3}$

6. $1\frac{1}{8}$

7. $2\frac{6}{7}$

8. $3\frac{3}{11}$

9. $-11\frac{1}{3}$

10. $12\frac{3}{4}$

11. $3\frac{3}{5}$

12. $-1\frac{1}{6}$

13. $2\frac{8}{13}$

14. 26 quarter-acre lots

15. $8\frac{3}{4}$ acres

Answers

Exercise

23/4

Exercise

59/8

Exercise $-41/12$ **Exercise** $3\frac{4}{7}$.**Exercise** $4\frac{2}{9}$ **Exercise 2.5.1** -9 **Exercise** $-2/3$

2.6: Adding and Subtracting Mixed Fractions

In this section, we will learn how to add and subtract mixed fractions.

Adding Mixed Fractions

We can use tools we've already developed to add two or more mixed fractions.

Example 1

Simplify: $2\frac{7}{8} + 1\frac{3}{4}$.

Solution

Change the mixed fractions to improper fractions, make equivalent fractions with a common denominator, then add.

$$\begin{aligned}
 2\frac{7}{8} + 1\frac{3}{4} &= \frac{23}{8} + \frac{7}{4} && \text{Change to equivalent fractions.} \\
 &= \frac{23}{8} + \frac{7 \cdot 2}{4 \cdot 2} && \text{Equivalent fractions with LCD = 8.} \\
 &= \frac{23}{8} + \frac{14}{8} && \text{Simplify numerators and denominators.} \\
 &= \frac{37}{8} && \text{Add numerators over common denominator.}
 \end{aligned}$$

Although this answer is perfectly acceptable, let's change the answer to a mixed fraction: 37 divided by 8 is 4, with a remainder of 5. Thus,

$$2\frac{7}{8} + 1\frac{3}{4} = 4\frac{5}{8}.$$

Exercise

Simplify: $3\frac{2}{3} + 4\frac{1}{8}$

Answer available at the end of this section.

Example 2

Simplify: $3\frac{1}{4} + 2\frac{1}{3}$.

Solution

Change the mixed fractions to improper fractions, make equivalent fractions with a common denominator, then add.

$$\begin{aligned}
 3\frac{1}{4} + 2\frac{1}{3} &= \frac{13}{4} + \frac{7}{3} && \text{Change to improper fractions.} \\
 &= \frac{13 \cdot 3}{4 \cdot 3} + \frac{7 \cdot 4}{3 \cdot 4} && \text{Equivalent fractions with LCD = 12.} \\
 &= \frac{39}{12} + \frac{28}{12} && \text{Simplify numerators and denominators.} \\
 &= \frac{67}{12} && \text{Add numerators and denominators.}
 \end{aligned}$$

Although this answer is perfectly acceptable, let's change the answer to a mixed fraction: 67 divided by 12 is 5, with a remainder of 7. Thus,

$$3\frac{1}{4} + 2\frac{1}{3} = 5\frac{7}{12}.$$

Exercise

Simplify: $8\frac{1}{2} + 2\frac{2}{3}$

Answer available at the end of this section.

Mixed Fraction Approach

There is another possible approach, based on the fact that a mixed fraction is a sum. Let's revisit Example 2.

Example 3

Simplify: $3\frac{1}{4} + 2\frac{1}{3}$.

Solution

Use the commutative and associative properties to change the order of addition, make equivalent fractions with a common denominator, then add.

$$\begin{aligned} 3\frac{1}{4} + 2\frac{1}{3} &= \left(3 + \frac{1}{4}\right) + \left(2 + \frac{1}{3}\right) && \text{Mixed fractions as sums.} \\ &= (3 + 2) + \left(\frac{1}{4} + \frac{1}{3}\right) && \text{Reorder and regroup.} \\ &= 5 + \left(\frac{1 \cdot 3}{4 \cdot 3} + \frac{1 \cdot 4}{3 \cdot 4}\right) && \begin{array}{l} \text{Add whole numbers: } 3 + 2 = 5. \\ \text{Equivalent fractions; LCD} = 12. \end{array} \\ &= 5 + \left(\frac{3}{12} + \frac{4}{12}\right) && \text{Simplify numerators and denominators.} \\ &= 5 + \frac{7}{12} && \text{Add numerators over common denominators.} \end{aligned}$$

This result can be written in mixed fraction form. Thus,

$$3\frac{1}{4} + 2\frac{1}{3} = 5\frac{7}{12}.$$

Note that this solution is identical to the result found in Example 2.

Exercise

Simplify: $7\frac{2}{5} + 3\frac{1}{8}$

Answer available at the end of this section.

Example 3 leads us to the following result.

Adding Mixed Fractions

To add two mixed fractions, add the whole number parts, then add the fractional parts.

Working in Vertical Format

When adding mixed fractions, many prefer to work in a vertical format. For example, here is how we would arrange the solution from Example 2 and Example 3 in vertical format. We create equivalent fractions, then add the whole number parts and fractional parts.

$$\begin{array}{r} 3\frac{1}{4} = 3\frac{1\cdot3}{4\cdot3} = 3\frac{3}{12} \\ + 2\frac{1}{3} = + 2\frac{1\cdot4}{3\cdot4} = + 2\frac{4}{12} \\ \hline 5\frac{7}{12} \end{array}$$

Note that the answer is identical to the answer found in Example 2 and Example 3. That is,

$$3\frac{1}{4} + 2\frac{1}{3} = 5\frac{7}{12}.$$

Example 4

Sarah is making window curtains for two rooms in her house. The kitchen will require $5\frac{2}{3}$ yards of material and the dining room will require $6\frac{5}{8}$ yards of material. How much total material is required?

Solution

To find the total material required for the two rooms, we must add $5\frac{2}{3}$ and $6\frac{5}{8}$. Create equivalent fractions with a common denominator, then add whole number parts and fractional parts.

$$\begin{array}{r} 5\frac{2}{3} = 5\frac{2\cdot8}{3\cdot8} = 5\frac{16}{24} \\ + 6\frac{5}{8} = + 6\frac{5\cdot3}{8\cdot3} = + 6\frac{15}{24} \\ \hline 11\frac{31}{24} \end{array}$$

An answer that is part mixed fraction, part improper fraction, is **not** allowed. To finish, we need to change the improper fractional part to a mixed fraction, then add. 31 divided by 24 is 1, with a remainder of 7. That is, $31/24 = 1\frac{7}{24}$. Now we can add whole number parts and fractional parts.

$$\begin{aligned} 11\frac{31}{24} &= 11 + 1\frac{7}{24} \\ &= 12\frac{7}{24}. \end{aligned}$$

Thus, the total material required is $12\frac{7}{24}$ yards.

Exercise

Jim is working on a project that requires two boards, the first cut to a length of $6\frac{1}{2}$ feet, the second cut to a length of $5\frac{7}{8}$ feet. How many total feet of board is required?

Answer available at the end of this section.

Subtracting Mixed Fractions

Let's look at some examples that subtract two mixed fractions.

Example 5

Simplify: $4\frac{5}{8} - 2\frac{1}{16}$.

Solution

Change the mixed fractions to improper fractions, make equivalent fractions with a common denominator, then subtract.

$$\begin{aligned}
 4\frac{5}{8} - 2\frac{1}{16} &= \frac{37}{8} - \frac{33}{16} && \text{Change to improper fractions.} \\
 &= \frac{37 \cdot 2}{8 \cdot 2} - \frac{33}{16} && \text{Equivalent fractions with LCD = 16.} \\
 &= \frac{74}{16} - \frac{33}{16} && \text{Simplify numerator and denominators.} \\
 &= \frac{41}{16} && \text{Add numerators over common denominator.}
 \end{aligned}$$

Although this answer is perfectly acceptable, let's change the answer to a mixed fraction: 41 divided by 16 is 2, with a remainder of 9. Thus,

$$4\frac{5}{8} - 2\frac{1}{16} = 2\frac{9}{16}.$$

Exercise

Simplify: $5\frac{2}{3} - 3\frac{1}{5}$

Answer available at the end of this section.

Example 6

Simplify: $5\frac{3}{4} - 2\frac{1}{3}$.

Solution

Change the mixed fractions to improper fractions, make equivalent fractions with a common denominator, then subtract.

$$\begin{aligned}
 5\frac{3}{4} - 2\frac{1}{3} &= \frac{23}{4} - \frac{7}{3} && \text{Change to improper fractions.} \\
 &= \frac{23 \cdot 3}{4 \cdot 3} - \frac{7 \cdot 4}{3 \cdot 4} && \text{Equivalent fractions with LCD = 12.} \\
 &= \frac{69}{12} - \frac{28}{12} && \text{Simplify numerators and denominators.} \\
 &= \frac{41}{12} && \text{Add numerators over common denominator.}
 \end{aligned}$$

Although this answer is perfectly acceptable, let's change the answer to a mixed fraction: 41 divided by 12 is 3, with a remainder of 5. Thus,

$$5\frac{3}{4} - 2\frac{1}{3} = 3\frac{5}{12}.$$

Exercise

Simplify: $4\frac{7}{9} - 2\frac{3}{18}$

Answer available at the end of this section.

Mixed Fraction Approach

There is another possible approach, based on the fact that a mixed fraction is a sum. Let's revisit Example 6.

Example 7

Simplify: $5\frac{3}{4} - 2\frac{1}{3}$.

Solution

A mixed fraction is a sum.

$$5\frac{3}{4} - 2\frac{1}{3} = \left(5 + \frac{3}{4}\right) - \left(2 + \frac{1}{3}\right)$$

Distribute the negative sign.

$$= 5 + \frac{3}{4} - 2 - \frac{1}{3}$$

We could change the subtraction to adding the opposite, change the order of addition, then change the adding of opposites back to subtraction. However, it is much easier if we look at this last line as a request to add four numbers, two of which are positive and two of which are negative. Changing the order does not affect the answer.

$$= (5 - 2) + \left(\frac{3}{4} - \frac{1}{3}\right)$$

Note that we did not change the signs of any of the four numbers. We just changed the order. Subtract the whole number parts. Make equivalent fractions with a common denominator, then subtract the fractional parts.

$$\begin{aligned} &= 3 + \left(\frac{3 \cdot 3}{4 \cdot 3} - \frac{1 \cdot 4}{3 \cdot 4}\right) && \text{Create equivalent fractions.} \\ &= 3 + \left(\frac{9}{12} - \frac{4}{12}\right) && \text{Simplify numerators and denominators.} \\ &= 3 + \frac{5}{12} && \text{Subtract fractional parts.} \end{aligned}$$

Thus,

$$5\frac{3}{4} - 2\frac{1}{3} = 3\frac{5}{12}.$$

Note that this is exactly the same answer as that found in Example 6.

Exercise

Simplify: $8\frac{5}{6} - 4\frac{3}{8}$

Answer available at the end of this section.

In Example 6, we see that we handle subtraction of mixed fractions in exactly the same manner that we handle addition of mixed

fractions.

Subtracting Mixed Fractions

To subtract two mixed fractions, subtract their whole number parts, then subtract their fractional parts.

Working in Vertical Format

When subtracting mixed fractions, many prefer to work in a vertical format. For example, here is how we would arrange the solution from Example 6 and Example 7 in vertical format. We create equivalent fractions, then subtract the whole number parts and fractional parts.

$$\begin{array}{r} 5\frac{3}{4} = 5\frac{3\cdot3}{4\cdot3} = 5\frac{9}{12} \\ -2\frac{1}{3} = -3\frac{1\cdot4}{3\cdot4} = -2\frac{4}{12} \\ \hline 3\frac{5}{12} \end{array}$$

Note that the answer is identical to the answer found in Example 6 and Example 7. That is,

$$5\frac{3}{4} - 2\frac{1}{3} = 3\frac{5}{12}.$$

Borrowing in Vertical Format

Consider the following example.

Example 8

Simplify: $8\frac{1}{4} - 5\frac{5}{6}$.

Solution

Create equivalent fractions with a common denominator.

$$\begin{array}{r} 8\frac{1}{4} = 8\frac{1\cdot3}{4\cdot3} = 8\frac{3}{12} \\ -5\frac{5}{6} = -5\frac{5\cdot2}{6\cdot2} = -5\frac{10}{12} \\ \hline \end{array}$$

You can see the difficulty. On the far right, we cannot subtract $10/12$ from $3/12$. The fix is to borrow 1 from 8 in the form of $12/12$ and add it to the $3/12$.

$$\begin{array}{r} 8\frac{3}{12} = 7 + \frac{12}{12} + \frac{3}{12} = 7\frac{15}{12} \\ -5\frac{10}{12} = -5\frac{10}{12} = -5\frac{10}{12} \\ \hline 2\frac{5}{12} \end{array}$$

Now we can subtract. Hence, $8\frac{1}{4} - 5\frac{5}{6} = 2\frac{5}{12}$.

Exercise

Simplify: $7\frac{1}{14} - 2\frac{5}{21}$

Answer available at the end of this section.

Example 9

Jim has a metal rod of length 10 inches. He cuts a length from the metal rod measuring $2\frac{7}{8}$ inches. What is the length of the remaining piece?

Solution

To find the length of the remaining piece, we must subtract $2\frac{7}{8}$ from 10. There is no fractional part on the first number. To remedy this absence, we borrow 1 from 10 in the form of $\frac{8}{8}$. Then we can subtract.

$$\begin{array}{r} 10 = 9 + \frac{8}{8} = 9\frac{8}{8} \\ -2\frac{7}{8} = -2\frac{7}{8} = -2\frac{7}{8} \\ \hline 7\frac{1}{8} \end{array}$$

Hence, the length of the remaining piece of the metal rod is $7\frac{1}{8}$ inches.

Exercise

Sarah has a length of curtain material that measures 12 feet. She cuts a length of $6\frac{2}{3}$ feet from her curtain material. What is the length of the remaining piece?

Answer available at the end of this section.

Exercises

Add or subtract the mixed fractions, as indicated, by first converting each mixed fraction to an improper fraction. Express your answer as a mixed fraction.

1. $9\frac{1}{4} + 9\frac{1}{2}$

2. $9\frac{1}{2} + 7\frac{1}{4}$

3. $8\frac{1}{2} - 1\frac{1}{3}$

4. $4\frac{1}{2} - 1\frac{1}{8}$

5. $4\frac{2}{3} + 1\frac{1}{4}$

Add or subtract the mixed fractions, as indicated, by using vertical format. Express your answer as a mixed fraction.

6. $3\frac{1}{2} + 3\frac{3}{4}$

7. $8\frac{1}{2} - 5\frac{2}{3}$

8. $9\frac{1}{2} - 1\frac{1}{3}$

9. $1\frac{1}{16} + 1\frac{3}{4}$

10. $2\frac{2}{3} + 1\frac{1}{4}$

Answers

1. $18\frac{3}{4}$

2. $16\frac{3}{4}$

3. $7\frac{1}{6}$

4. $3\frac{3}{8}$

5. $5\frac{11}{12}$

6. $7\frac{1}{4}$

7. $2\frac{5}{6}$

8. $8\frac{1}{6}$

9. $2\frac{13}{16}$

10. $3\frac{11}{12}$

Answers

Exercise

$7\frac{19}{24}$

Exercise

$11\frac{1}{6}$

Exercise

$10\frac{21}{40}$

Exercise

$12\frac{3}{8}$ feet.

Exercise

$2\frac{7}{15}$

Exercise

$2\frac{11}{18}$

Exercise

$4\frac{11}{24}$

Exercise

$4\frac{5}{6}$.

Exercise

$5\frac{1}{3}$ feet

3: DECIMALS

3.1: Decimal Notation and Place Value

3.2: Adding and Subtracting Decimals

3.3: Multiplying Decimals

3.4: Dividing Decimals

3.5: Fractions and Decimals

3.1: Decimal Notation and Place Value

Recall that whole numbers are constructed by using *digits*.

The Digits

The set

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

is called the set of *digits*.

As an example, the whole number 55,555 (“fifty-five thousand five hundred fifty-five”) is constructed by using a single digit. However, the position of the digit 5 determines its value in the number 55,555.

5	5	5	5	5
ten thousands	thousands	hundreds	tens	ones
10,000	1,000	100	10	1

Table 3.1: Place value.

The first occurrence of the digit 5 happens in the ten thousands place, so its value is 5 ten thousands, or 50,000. The next occurrence of the digit 5 is in the thousands place, so its value is 5 thousands, or 5,000. Indeed, the whole number 55,555 in expanded form is

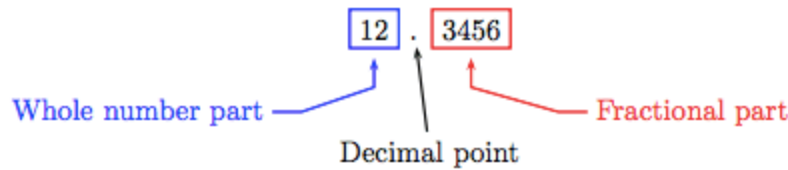
$$50000 + 5000 + 500 + 50 + 5,$$

which reflects the value of the digit 5 in each place.

Decimal Notation

In Table 3.1, each time you move one column to the left, the place value is 10 times larger than the place value of the preceding column. Vice-versa, each time you move one column to the right, the place value is 1/10 of the place value of the preceding column.

Now, consider the *decimal number* 12.3456, which consists of three parts: the whole number part, the decimal point, and the fractional part.



The whole number part of the decimal number is the part that lies strictly to the left of the decimal point, and the place value of each digit in the whole number part is given by the columns shown in Table 3.1.

The fractional part of the decimal number is the part that lies strictly to the right of the decimal point. As we saw in Table 3.1, each column has a value equal to $1/10$ of the value of the column that lies to its immediate left. Thus, it should come as no surprise that:

- The first column to the right of the decimal point has place value $1/10$ (tenths).
- The second column to the right of the decimal point has place value $1/100$ (hundredths).
- The third column to the right of the decimal point has place value $1/1000$ (thousandths).
- The fourth column to the right of the decimal point has place value $1/10000$ (ten-thousandths).

These results are summarized for the decimal number 12.3456 in Table 3.2.

1	2	.	3	4	5	6
tens	ones	decimal point	tenths	hundredths	thousandths	ten-thousandths
10	1	.	$1/10$	$1/100$	$1/1000$	$1/10000$

Table 3.2: Place value.

Pronouncing Decimal Numbers

The decimal number 12.3456 is made up of 1 ten, 2 ones, 3 tenths, 4 hundredths, 5 thousandths, and 6 ten-thousandths (see Table 5.2), and can be written in *expanded form* as

$$12.3456 = 10 + 2 + \frac{3}{10} + \frac{4}{100} + \frac{5}{1000} + \frac{6}{10000}.$$

Note that the whole numbers can be combined and the fractions can be written with a common denominator and summed.

$$\begin{aligned}
 12.3456 &= 12 + \frac{3 \cdot \cancel{1000}}{10 \cdot \cancel{1000}} + \frac{4 \cdot \cancel{100}}{100 \cdot \cancel{100}} + \frac{5 \cdot \cancel{10}}{1000 \cdot \cancel{10}} + \frac{6}{10000} \\
 &= 12 + \frac{3000}{10000} + \frac{400}{10000} + \frac{50}{10000} + \frac{6}{10000} \\
 &= 12 + \frac{3456}{10000}
 \end{aligned}$$

The result tells us how to pronounce the number 12.3456. It is pronounced “twelve and three thousand, four hundred fifty-six ten-thousandths.”

Example 1

Place the decimal number 1,234.56 in expanded form, then combine the whole number part and sum the fractional part over a common denominator. Use the result to help pronounce the decimal number.

Solution

In expanded form,

$$1,234.56 = 1,000 + 200 + 30 + 4 + \frac{5}{10} + \frac{6}{100}$$

Sum the whole number parts. Express the fractional parts as equivalent fractions and combine over one common denominator.

$$\begin{aligned}
 1,234.56 &= 1,234 + \frac{5 \cdot \cancel{10}}{10 \cdot \cancel{10}} + \frac{6}{100} \\
 &= 1,234 + \frac{50}{100} + \frac{6}{100} \\
 &= 1,234 + \frac{56}{100}
 \end{aligned}$$

Hence, 1,234.56 is pronounced “one thousand, two hundred thirty-four and fifty-six hundredths.”

The discussion and example leads to the following result.

How to Read a Decimal Number

1. Pronounce the whole number part to the left of the decimal as you would any whole number.
2. Say the word “and” for the decimal point.
3. State the fractional part to the right of the decimal as you would any whole number, followed by the place value of the digit in the rightmost column.

Example 3

Pronounce the decimal number 34.12.

Solution

The rightmost digit in the fractional part of 34.12 is in the hundredths column. Thus, 34.12 is pronounced “thirty-four and twelve hundredths.”

Exercise

Pronounce 28.73

Answer available at the end of this section.

Important Point

In pronouncing decimal numbers, the decimal point is read as “and.” No other instance of the word “and” should appear in the pronunciation.

Example 4

Explain why “four hundred and thirty-four and two tenths” is an incorrect pronunciation of the decimal number 434.2.

Solution

The decimal point is read as “and.” No other occurrence of the word “and” is allowed in the pronunciation. The correct pronunciation should be “four hundred thirty-four and two tenths.”

Exercise

Pronounce 286.9.

Answer available at the end of this section.

Example 5

Pronounce the decimal number 5,678.123.

Solution

The rightmost digit in the fractional part of 5,678.123 is in the thousandths column. Hence, 5,678.123 is pronounced “five thousand six hundred seventy-eight and one hundred twenty-three thousandths.”

Exercise

Pronounce 7, 002.207

Answer available at the end of this section.

Example 6

Pronounce the decimal number 995.4325.

Solution

The rightmost digit in the fractional part of 995.4325 is in the ten-thousandths column. Hence, 995.4325 is pronounced “nine hundred ninety-five and four thousand three hundred twenty-five ten-thousandths.”

Exercise

Pronounce 500.1205.

Answer available at the end of this section.

Decimals to Fractions

Because we now have the ability to pronounce decimal numbers, it is a simple exercise to change a decimal to a fraction.¹ For example, 134.12 is pronounced “one hundred thirty-four and twelve hundredths,” so it can easily be written as a mixed fraction.

$$134.12 = 134\frac{12}{100}$$

But this mixed fraction can be changed to an improper fraction.

$$\begin{aligned} 134\frac{12}{100} &= \frac{100 \cdot 134 + 12}{100} = \frac{13400 + 12}{100} \\ &= \frac{13412}{100} \end{aligned}$$

Note that the numerator is our original number without the decimal point. There are two decimal places in the original number and the denominator of the final improper fraction contains two zeros.

This discussion leads to the following result.

¹Changing fractions to decimals will be covered in Section 3.5.

Changing Decimals to Improper Fractions

To change a decimal number to an improper fraction, proceed as follows:

1. Create a fraction.
2. Place the decimal number in the numerator without the decimal point.
3. Count the number of decimal places. Place an equal number of zeros in the denominator.

Example 7

Change the following decimal numbers to improper fractions: (a) 1.2345, and (b) 27.198.

Solution

In each case, place the number in the numerator without the decimal point. In the denominator, add a number of zeros equal to the number of decimal places.

a) The decimal number 1.2345 has four decimal places. Hence,

$$1.2345 = \frac{12345}{10000}$$

b) The decimal number 27.198 has three decimal places. Hence,

$$27.198 = \frac{27198}{1000}$$

Exercise

Change 17.205 to an improper fraction.

Answer available at the end of this section.

Example 8

Change each of the following decimals to fractions reduced to lowest terms: (a) 0.35, and (b) 0.125.

Solution

Place each number in the numerator without the decimal point. Place a number of zeros in the denominator equal to the number of decimal places. Reduce to lowest terms.

a) First, place 35 over 100.

$$0.35 = \frac{35}{100}$$

We can divide both numerator and denominator by the greatest common divisor.

$$\begin{aligned} &= \frac{35 \div 5}{100 \div 5} && \text{Divide numerator and denominator by 5.} \\ &= \frac{7}{20} && \text{Simplify numerator and denominator.} \end{aligned}$$

b) First, place 125 over 1000.

$$0.125 = \frac{125}{1000}$$

Prime factor and cancel common factors.

$$\begin{aligned} &= \frac{5 \cdot 5 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5} && \text{Prime factor numerator and denominator.} \\ &= \frac{\cancel{5} \cdot \cancel{5} \cdot \cancel{5}}{2 \cdot 2 \cdot 2 \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5}} && \text{Cancel common factors.} \\ &= \frac{1}{8} && \text{Simplify.} \end{aligned}$$

Exercise

Change 0.375 to a fraction, reduced to lowest terms.

Answer available at the end of this section.

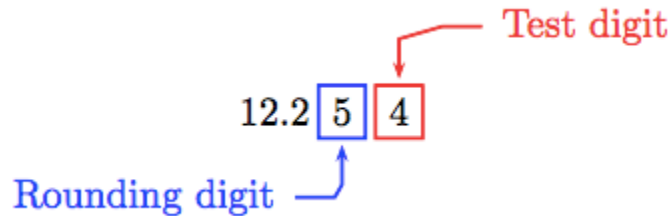
Rounding

The rules for rounding decimal numbers are almost identical to the rules for rounding whole numbers. First, a bit of terminology.

Rounding Digit and Test Digit

The digit in the place to which we wish to round is called the *rounding digit* and the digit that follows on its immediate right is called the *test digit*.

If we want to round the decimal number 12.254 to the nearest hundredth, then the rounding digit is 5 and the test digit is 4.



If we used the rules for rounding whole numbers, because the test digit 4 is less than 5, we would replace all digits to the right of the rounding digit with zeros to obtain the following approximation.

$$12.254 \approx 12.250$$

However, because

$$12.250 = 12\frac{250}{1000} = 12\frac{25}{100},$$

the trailing zero at the end of the fractional part is irrelevant. Hence, we *truncate* every digit after the rounding digit and use the following approximation.

$$12.254 \approx 12.25$$

Important Observation

Deleting trailing zeros from the end of the fractional part of a decimal number does not change its value.

The above discussion motivates the following algorithm for rounding decimal numbers.

Rounding Decimal Numbers

Locate the *rounding digit* and the *test digit*.

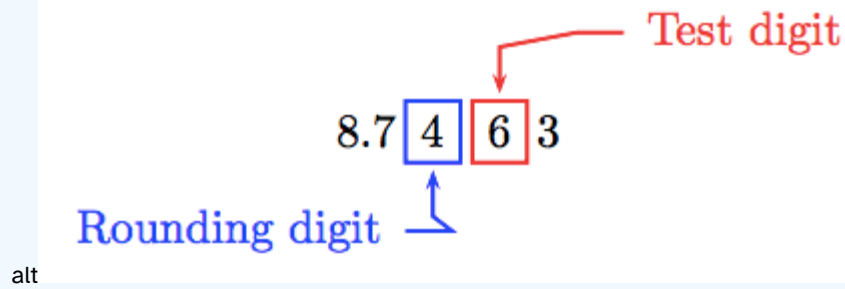
- If the test digit is greater than or equal to 5, add 1 to the rounding digit and truncate all digits to the right of the rounding digit.
- If the test digit is less than 5, simply truncate all digits to the right of the rounding digit.

Example 9

Round 8.7463 to the nearest hundredth.

Solution

Locate the rounding digit in the hundredths place and the test digit to its immediate right.



Because the test digit is greater than 5, add 1 to the rounding digit and truncate all digits to the right of the rounding digit. Hence, to the nearest hundredth:

$$8.7463 \approx 8.75$$

Exercise

Round 9.2768 to the nearest hundredth.

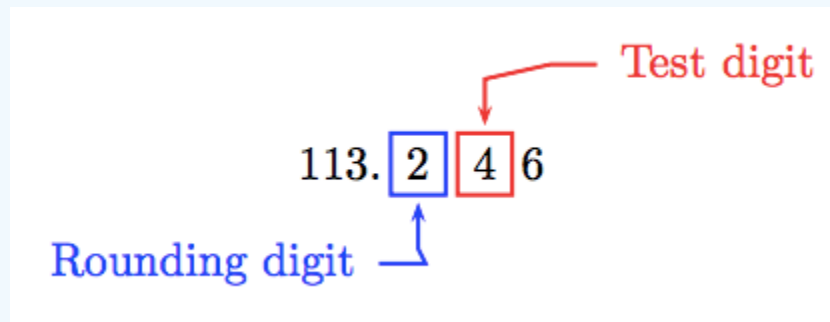
Answer available at the end of this section.

Example 10

Round 113.246 to the nearest tenth.

Solution

Locate the rounding digit in the tenths place and the test digit to its immediate right.



Because the test digit is less than 5, truncate all digits to the right of the rounding digit. Hence, to the nearest tenth:

$$113.246 \approx 113.2$$

Exercise

Round 58.748 to the nearest tenth.

Answer available at the end of this section.

Comparing Decimals

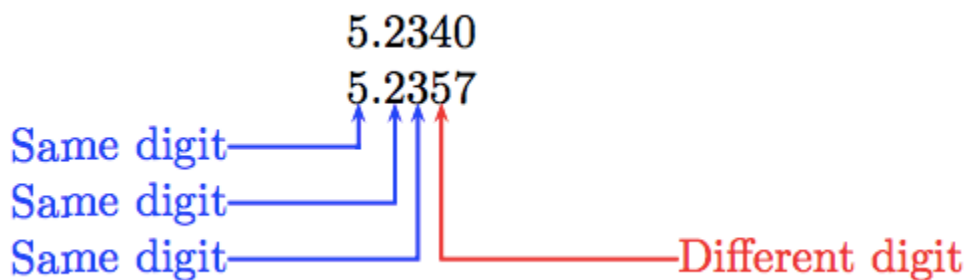
We can compare two positive decimals by comparing digits in each place as we move from left to right, place by place. For example, suppose we wish to compare the decimal numbers 5.234 and 5.2357. First, add enough trailing zeros to the decimal number with the fewer decimal places so that the numbers have the same number of decimal places. In this case, note that

$$5.234 = 5\frac{234}{1000} = 5\frac{2340}{10000} = 5.2340.$$

Important Observation

Adding trailing zeros to the end of the fractional part of a decimal number does not change its value.

Next, align the numbers as follows.



As you scan the columns, moving left to right, the first place that has different digits occurs in the thousandths place, where the digit 5 in the second number is greater than the digit 4 in the first number in the same place. Because 5 is greater than 4, the second number is larger than the first. That is:

$$5.234 < 5.2357$$

This discussion suggests the following algorithm.

Comparing Positive Decimal Numbers

1. Add enough trailing zeros so that both numbers have the same number of decimal places.
2. Compare the digits in each place, moving left to right.
3. As soon as you find two digits in the same place that are different, the decimal number with the greatest digit in this place is the larger number.

Example 11

Compare 4.25 and 4.227.

Solution

Add a trailing zero to the first decimal number and align the numbers as follows.



The first difference is in the hundredths place, where the digit 5 in the first number is greater than the digit 2 in the same place of the second number. Hence, the first number is larger than the second; that is:

$$4.25 > 4.227$$

Exercise

Compare 8.34 and 8.348.

Answer available at the end of this section.

When comparing negative numbers, the number with the larger magnitude is the smaller number. Hence, we have to adjust our algorithm for comparing negative decimal numbers.

Comparing Negative Decimal Numbers

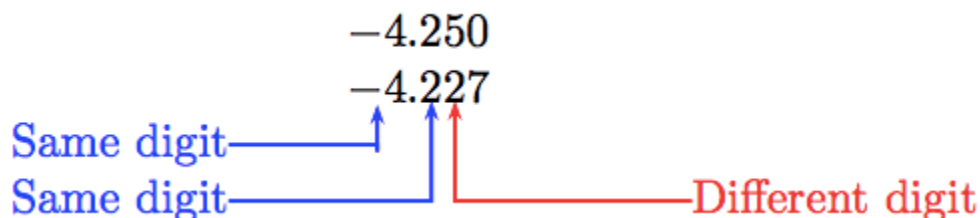
1. Add enough trailing zeros so that both numbers have the same number of decimal places.
2. Compare the digits in each place, moving left to right.
3. As soon as you find two digits in the same place that are different, the decimal number with the greatest digit in this place is the smaller number.

Example 12

Compare -4.25 and -4.227 .

Solution

Add a trailing zero to the first decimal number and align the numbers as follows.



The first difference is in the hundredths place, where the digit 5 in the first number is greater than the digit 2 in the same place of the second number. Hence, the first number is **smaller** than the second; that is:

$$-4.25 < -4.227$$

Exercise

Compare -7.86 and -7.85 .

Answer available at the end of this section.

Exercises

1. Which digit is in the tenths column of the number 4,552.0908?
2. Which digit is in the ten-thousandths column of the number 2,709.50976?
3. Which digit is in the hundredths column of the number 5,705.2193?
4. Which digit is in the tenths column of the number 8,129.3075?
5. Which digit is in the thousandths column of the number 6,971.4289?

Pronounce the given decimal number. Write your answer out in words.

6. 0.9
7. 83.427
8. 1,826.57

Convert the given decimal to a mixed fraction. Do *not* simplify your answer.

9. 98.1
10. 915.239
11. 446.73

Convert the given decimal to an improper fraction. Do *not* simplify your answer.

12. 5.47
13. 2.133
14. 3.9

Convert the given decimal to a fraction. Reduce your answer to lowest terms.

15. 0.35
 16. 0.06
 17. 0.72
-

18. Round 71.2427 to the nearest thousandth.

19. Round 29.379 to the nearest tenth.

20. Round 89.3033 to the nearest thousandth.

21. Round 19.854 to the nearest hundredth.

Determine which of the two given statements is true.

22. $-0.034 < -0.040493$ or $-0.034 > -0.040493$

23. $18.62192 < 18.6293549$ or $18.62192 > 18.6293549$

24. $36.8298 < 36.8296595$ or $36.8298 > 36.8296595$

25. Write the decimal number in words.

i) A recently discovered 7.03-carat blue diamond auctioned at Sotheby's.

ii) The newly launched European Planck telescope will stay in orbit 1.75 years measuring radiation from the Big Bang.

iii) The sun composes 0.9985 of the mass in our solar system.

iv) Clay particles are small - only 0.0001 inch.

Answers

1. 0

2. 7

3. 1

4. 3

5. 8

6. nine tenths

7. eighty-three and four hundred twenty-seven thousandths

8. one thousand, eight hundred twenty-six and fifty-seven hundredths

9. $98\frac{1}{10}$

10. $915\frac{239}{1000}$

11. $446\frac{73}{100}$

12. $\frac{547}{100}$

13. $\frac{2133}{1000}$

14. $\frac{39}{10}$

15. $\frac{7}{20}$

16. $\frac{3}{50}$

17. $\frac{18}{25}$

18. 71.243

19. 29.4

20. 89.303

21. 19.85

22. $-0.034 > -0.040493$

23. $18.62192 < 18.6293549$

24. $36.8298 > 36.8296595$

25.

i) seven and three hundredths

ii) one and seventy-five hundredths

iii) nine thousand nine hundred eighty-five ten-thousandths

iv) one ten-thousandth of an inch

Answers

Exercise

"Twenty-eight and seventy-three hundredths"

Exercise

"Two hundred eighty six and nine tenths"

Exercise

Answer: "Seven thousand two and two hundred seven thousandths."

Exercise

Answer: "Five hundred and one thousand two hundred five ten-thousandths."

Exercise

$$\frac{17205}{100}$$

Exercise

$$\frac{3}{8}$$

Exercise

$$9.28$$

Exercise

58.7

Exercise

$8.34 < 8.348$

Exercise

$-7.86 < -7.85$

3.2: Adding and Subtracting Decimals

Adding Decimals

Addition of decimal numbers is quite similar to addition of whole numbers. For example, suppose that we are asked to add 2.34 and 5.25. We could change these decimal numbers to **mixed fractions** and add.

$$\begin{aligned} 2.34 + 5.25 &= 2\frac{34}{100} + 5\frac{25}{100} \\ &= 7\frac{59}{100} \end{aligned}$$

However, we can also line the decimal numbers on their decimal points and add vertically, as follows.

$$\begin{array}{r} 2.34 \\ +5.25 \\ \hline 7.59 \end{array}$$

Note that this alignment procedure produces the same result, “seven and fifty nine hundredths.” This motivates the following procedure for adding decimal numbers.

Adding Decimals

To add decimal numbers, proceed as follows:

1. Place the numbers to be added in vertical format, aligning the decimal points.
2. Add the numbers as if they were whole numbers.
3. Place the decimal point in the answer in the same column as the decimal points above it.

Example 1

Add 3.125 and 4.814.

Solution

Place the numbers in vertical format, aligning on their decimal points. Add, then place the decimal point in the answer in the same column as the decimal points that appear above the answer.

$$\begin{array}{r} 3.125 \\ +4.814 \\ \hline 7.939 \end{array}$$

Thus, $3.125 + 4.814 = 7.939$.

Exercise

Add: $2.864 + 3.029$

Answer available at the end of this section.

Example 2

Jane has \$4.35 in her purse. Jim has \$5.62 in his wallet. If they sum their money, what is the total?

Solution

Arrange the numbers in vertical format, aligning decimal points, then add.

$$\begin{array}{r} \$4.35 \\ +\$5.62 \\ \hline \$9.97 \end{array}$$

Exercise

Alice has \$8.63 in her purse and Joanna has \$2.29. If they combine sum their money, what is the total?

Answer available at the end of this section.

Before looking at another example, let's recall an important observation.

Important Observation

Adding zeros to the end of the fractional part of a decimal number does not change its value. Similarly, deleting trailing zeros from the end of a decimal number does not change its value.

For example, we could add two zeros to the end of the fractional part of 7.25 to obtain 7.2500. The numbers 7.25 and 7.2500 are identical as the following argument shows:

$$\begin{aligned} 7.2500 &= 7 \frac{2500}{10000} \\ &= 7 \frac{25}{100} \\ &= 7.25 \end{aligned}$$

Example 3

Add 7.5 and 12.23.

Solution

Arrange the numbers in vertical format, aligning their decimal points in a column. Note that we add a trailing zero to improve columnar alignment.

$$\begin{array}{r} 7.50 \\ +12.23 \\ \hline 19.73 \end{array}$$

Hence, $7.5 + 12.23 = 19.73$.

Exercise

Add: $9.7 + 15.86$

Answer available at the end of this section.

Example 4

Find the sum: $12.2 + 8.352 + 22.44$.

Solution

Arrange the numbers in vertical format, aligning their decimal points in a column. Note that we add trailing zeros to improve the columnar alignment.

$$\begin{array}{r} 12.200 \\ 8.352 \\ +22.440 \\ \hline 42.992 \end{array}$$

Hence, $12.2 + 8.352 + 22.44 = 42.992$.

Exercise

Add: $12.9 + 4.286 + 33.97$

Answer available at the end of this section.

Subtracting Decimals

Subtraction of decimal numbers proceeds in much the same way as addition of decimal numbers.

Subtracting Decimals

To subtract decimal numbers, proceed as follows:

1. Place the numbers to be subtracted in vertical format, aligning the decimal points.
2. Subtract the numbers as if they were whole numbers.
3. Place the decimal point in the answer in the same column as the decimal points above it.

Example 5

Subtract 12.23 from 33.57.

Solution

Arrange the numbers in vertical format, aligning their decimal points in a column, then subtract. Note that we subtract 12.23 from 33.57.

$$\begin{array}{r} 33.57 \\ -12.23 \\ \hline 21.34 \end{array}$$

Hence, $33.57 - 12.23 = 21.34$.

Exercise

Subtract: $58.76 - 38.95$

Answer available at the end of this section.

As with addition, we add trailing zeros to the fractional part of the decimal numbers to help columnar alignment.

Example 6

Find the difference: $13.3 - 8.572$.

Solution

Arrange the numbers in vertical format, aligning their decimal points in a column. Note that we add trailing zeros to the fractional part of 13.3 to improve columnar alignment.

$$\begin{array}{r} 13.300 \\ -8.572 \\ \hline 4.728 \end{array}$$

Hence, $13.3 - 8.572 = 4.728$.

Exercise

Subtract: $15.2 - 8.756$

Answer available at the end of this section.

Adding and Subtracting Signed Decimal Numbers

We use the same rules for addition of signed decimal numbers as we did for the addition of integers.

Adding Two Decimals with Like Signs

To add two decimals with like signs, proceed as follows:

1. Add the magnitudes of the decimal numbers.
2. Prefix the common sign.

Example 7

Simplify: $-3.2 + (-18.95)$.

Solution

To add like signs, first add the magnitudes.

$$\begin{array}{r} 3.20 \\ +18.95 \\ \hline 22.15 \end{array}$$

Prefix the common sign. Hence, $-3.2 + (-18.95) = -22.15$

Exercise

Simplify: $-5.7 + (-83.85)$

Answer available at the end of this section.

We use the same rule as we did for integers when adding decimals with unlike signs.

Adding Two Decimals with Unlike Signs

To add two decimals with unlike signs, proceed as follows:

1. Subtract the smaller magnitude from the larger magnitude.
2. Prefix the sign of the decimal number with the larger magnitude.

Example 8

Simplify: $-3 + 2.24$.

Solution

To add unlike signs, first subtract the smaller magnitude from the larger magnitude.

$$\begin{array}{r} 3.00 \\ -2.24 \\ \hline 0.76 \end{array}$$

Prefix the sign of the decimal number with the larger magnitude. Hence, $-3 + 2.24 = -0.76$.

Exercise

Simplify: $-8 + 5.74$

Answer available at the end of this section.

Subtraction still means *add the opposite*.

Example 9

Simplify: $-8.567 - (-12.3)$.

Solution

Subtraction must first be changed to addition by adding the opposite.

$$-8.567 - (-12.3) = -8.567 + 12.3$$

We have unlike signs. First, subtract the smaller magnitude from the larger magnitude.

$$\begin{array}{r} 12.300 \\ -8.567 \\ \hline 3.733 \end{array}$$

Prefix the sign of the decimal number with the larger magnitude. Hence:

$$\begin{aligned} -8.567 - (-12.3) &= -8.567 + 12.3 \\ &= 3.733 \end{aligned}$$

Exercise

Simplify: $-2.384 - (-15.2)$

Answer available at the end of this section.

Order of operations demands that we simplify expressions contained in parentheses first.

Example 10

Simplify: $-11.2 - (-8.45 + 2.7)$.

Solution

We need to add inside the parentheses first. Because we have unlike signs, subtract the smaller magnitude from the larger magnitude.

$$\begin{array}{r} 8.45 \\ -2.70 \\ \hline 5.75 \end{array}$$

Prefix the sign of the number with the larger magnitude. Therefore,

$$-11.2 - (-8.45 + 2.7) = -11.2 - (-5.75)$$

Subtraction means add the opposite.

$$-11.2 - (-5.75) = -11.2 + 5.75$$

Again, we have unlike signs. Subtract the smaller magnitude from the larger magnitude.

$$\begin{array}{r} 11.20 \\ -5.75 \\ \hline 5.45 \end{array}$$

Prefix the sign of the number with the large magnitude.

$$-11.2 + 5.75 = -5.45$$

Exercise

Simplify: $-12.8 - (-7.44 + 3.7)$

Answer available at the end of this section.

Writing Mathematics

The solution to the previous example should be written as follows:

$$\begin{aligned}-11.2 - (-8.45 + 2.7) &= -11.2 - (-5.75) \\ &= -11.2 + 5.75 \\ &= -5.45\end{aligned}$$

Any scratch work, such as the computations in vertical format in the previous example, should be done in the margin or on a scratch pad.

Exercises

Add the decimals.

1. $31.9 + 84.7$

2. $4 + 97.18$

3. $52.671 + 5.97$

Subtract the decimals.

4. $9 - 2.261$

5. $55.672 - 3.3$

6. $60.575 - 6$

7. $8.1 - 2.12$

Add or subtract the decimals, as indicated.

8. $-34.7 + (-56.214)$

9. $8.4 + (-6.757)$

10. $-50.4 + 7.6$

11. $0.19 - 0.7$

12. $2.001 - 4.202$

13. $9.365 + (-5)$

Simplify the given expression.

15. $-19.1 - (1.51 - (-17.35))$

15. $11.55 + (6.3 - (-1.9))$

Answers

1. 116.6
2. 101.18
3. 58.641
4. 6.739
5. 52.372
6. 54.575
7. 5.98
8. -90.914
9. 1.643
10. -42.8
11. -0.51
12. -2.201
13. 4.365
14. -37.96
15. 19.75

Answers

Exercise

5.893

Exercise

\$10.91

Exercise

25.56

Exercise

51.156

Exercise

19.81

Exercise

6.444

Exercise

-89.55

Exercise

-2.26

Exercise

12.816

Exercise

-9.06

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3.3: Multiplying Decimals

Multiplying decimal numbers involves two steps: (1) multiplying the numbers as whole numbers, ignoring the decimal point, and (2) placing the decimal point in the correct position in the product or answer.

For example, consider $(0.7)(0.08)$, which asks us to find the product of “seven tenths” and “eight hundredths.” We could change these decimal numbers to fractions, then multiply.

$$\begin{aligned}(0.7)(0.8) &= \frac{7}{10} \cdot \frac{8}{100} \\ &= \frac{56}{1000} \\ &= 0.056\end{aligned}$$

The product is $56/1000$, or “fifty six thousandths,” which as a decimal is written 0.056 .

Important Observations

There are two very important observations to be made about the example $(0.7)(0.08)$.

1. In fractional form

$$\frac{7}{10} \cdot \frac{8}{100} = \frac{56}{1000},$$

note that the numerator of the product is found by taking the product of the whole numbers 7 and 8. That is, you ignore the decimal points in 0.7 and 0.08 and multiply 7 and 8 as if they were whole numbers.

2. The first factor, 0.7 , has one digit to the right of the decimal point. Its fractional equivalent, $7/10$, has one zero in its denominator. The second factor, 0.08 , has two digits to the right of the decimal point. Its fractional equivalent, $8/100$, has two zeros in its denominator. Therefore, the product $56/1000$ is forced to have three zeros in its denominator and its decimal equivalent, 0.056 , must therefore have three digits to the right of the decimal point.

Multiplying Decimal Numbers

To multiply two decimal numbers, perform the following steps:

1. Ignore the decimal points in the factors and multiply the two factors as if they were whole numbers.
2. Count the number of digits to the right of the decimal point in each factor. Sum these two numbers.
3. Place the decimal point in the product so that the number of digits to the right of the decimal point equals the sum found in step 2.

Example 2

Use the steps outlined in *Multiplying Decimal Numbers* to find the product in Example 1.

Solution

We follow the steps outlined in *Multiplying Decimal Numbers*.

1. The first step is to multiply the factors 2.34 and 1.2 as whole numbers, ignoring the decimal points.

$$\begin{array}{r} 234 \\ \times 12 \\ \hline 468 \\ 234 \\ \hline 2808 \end{array}$$

2. The second step is to find the sum of the number of digits to the right of the decimal points in each factor. Note that 2.34 has two digits to the right of the decimal point, while 1.2 has one digit to the right of the decimal point. Thus, we have a total of three digits to the right of the decimal points in the factors.

3. The third and final step is to place the decimal point in the product or answer so that there are a total of three digits to the right of the decimal point. Thus,

$$(2.34)(1.2) = 2.808.$$

Note that this is precisely the same solution found in Example 1.

What follows is a convenient way to arrange your work in vertical format.

$$\begin{array}{r} 2.34 \\ \times 1.2 \\ \hline 468 \\ 2 34 \\ \hline 2.808 \end{array}$$

Exercise

Multiply: $(5.98)(3.7)$

Answer available at the end of this section.

Example 3

Simplify: $(8.235)(2.3)$.

Solution

We use the convenient vertical format introduced at the end of Example 2.

$$\begin{array}{r} 8.235 \\ \times 2.3 \\ \hline 2 4705 \\ 16 470 \\ \hline 18.9405 \end{array}$$

The factor 8.235 has three digits to the right of the decimal point; the factor 2.3 has one digit to the right of the decimal point. Therefore, there must be a total of four digits to the right of the decimal point in the product or answer.

Exercise

Multiply: $(9.582)(8.6)$

Answer available at the end of this section.

Multiplying Signed Decimal Numbers

The rules governing multiplication of signed decimal numbers are identical to the rules governing multiplication of integers.

Like Signs. The product of two decimal numbers with like signs is positive. That is:

$$(+)(+) = + \text{ and } (-)(-) = +$$

Unlike Signs. The product of two decimal numbers with unlike signs is negative. That is:

$$(+)(-) = - \text{ and } (-)(+) = -$$

Example 4

Simplify: $(-2.22)(-1.23)$.

Solution

Ignore the signs to do the multiplication, then consider the signs in the final answer behind.

As each factor has two digits to the right of the decimal point, there should be a total of 4 decimals to the right of the decimal point in the product.

$$\begin{array}{r} 2.22 \\ \times 1.23 \\ \hline 666 \\ 444 \\ 111 \\ \hline 1.6206 \end{array}$$

Like signs give a positive product. Hence:

$$(-2.22)(-1.23) = 1.6206$$

Exercise

Multiply: $(-3.86)(-5.77)$

Answer available at the end of this section.

Example 5

Simplify: $(5.68)(-0.012)$.

Solution

Ignore the signs to do the multiplication, then consider the signs in the final answer below.

The first factor has two digits to the right of the decimal point, the second factor has three. Therefore, there must be a total of five digits to the right of the decimal point in the product or answer. This necessitates prepending an extra zero in front of our product.

$$\begin{array}{r} 5.68 \\ \times 0.012 \\ \hline 1136 \\ 568 \\ \hline 0.06816 \end{array}$$

Unlike signs give a negative product. Hence:

$$(5.68)(-0.012) = -0.06816$$

Exercise

Multiply: $(9.23)(-0.018)$

Answer available at the end of this section.

Order of Operations

The same *Rules Guiding Order of Operations* also apply to decimal numbers.

Rules Guiding Order of Operations

When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

Powers of Ten

Consider:

$$\begin{aligned} 10^1 &= 10 \\ 10^2 &= 10 \cdot 10 = 100 \\ 10^3 &= 10 \cdot 10 \cdot 10 = 1,000 \\ 10^4 &= 10 \cdot 10 \cdot 10 \cdot 10 = 10,000 \end{aligned}$$

Note the answer for 10^4 , a one followed by four zeros! Do you see the pattern?

Powers of Ten

In the expression 10^n , the exponent matches the number of zeros in the answer. Hence, 10^n will be a 1 followed by n zeros.

Example 7

Simplify: 10^9 .

Solution

10^9 should be a 1 followed by 9 zeros. That is,

$$10^9 = 1,000,000,000,$$

or “one billion.”

Exercise

Simplify: 10^6

Answer available at the end of this section.

Multiplying Decimal Numbers by Powers of Ten

Let's multiply 1.234567 by 10^3 , or equivalently, by 1,000. Ignore the decimal point and multiply the numbers as whole numbers.

$$\begin{array}{r} 1.234567 \\ \times 1000 \\ \hline 1234.567000 \end{array}$$

The sum total of digits to the right of the decimal point in each factor is 6. Therefore, we place the decimal point in the product so that there are six digits to the right of the decimal point.

However, the trailing zeros may be removed without changing the value of the product. That is, 1.234567 times 1000 is 1234.567. Note that the decimal point in the product is three places further to the right than in the original factor. This observation leads to the following result.

Multiplying a Decimal Number by a Power of Ten

Multiplying a decimal number by 10^n will move the decimal point n places to the right.

Example 8

Simplify: $1.234567 \cdot 10^4$

Solution

Multiplying by 104 (or equivalently, by 10,000) moves the decimal 4 places to the right. Thus, $1.234567 \cdot 10,000 = 12345.67$.

Exercise

Simplify: $1.234567 \cdot 10^2$

Answer available at the end of this section.

Exercises

Multiply the decimals.

1. $(6.7)(0.03)$

2. $(4.1)(4.6)$

3. $(75.3)(0.4)$

4. $(3.02)(6.7)$

5. $(-9.41)(0.07)$

6. $(-39.3)(-0.8)$

7. $(1.02)(-0.2)$

Multiply the decimal by the given power of 10.

8. $24.264 \cdot 10$

9. $53.867 \cdot 10^4$

10. $5.096 \cdot 10^3$

11. $61.303 \cdot 100$

Simplify the given expression.

12. $(0.36)(7.4) - (-2.8)^2$

13. $9.4 - (-7.7)(1.2)^2$

14. $6.3 - 4.2(9.3)^2$

15. $(-8.1)(9.4) - 1.8^2$

Answers

1. 0.201
2. 18.86
3. 30.12
4. 20.234
5. -0.6587
6. 31.44
7. -0.204
8. 242.64
9. 538670
10. 5096
11. 6130.3
12. -5.176
13. 20.488
14. -356.958
15. -79.38

Answers

Exercise

22.126

Exercise

82.4052

Exercise

22.2722

Exercise

-0.16614

Exercise

1,000,000

Exercise

123.4567

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3.4: Dividing Decimals

In this and following sections we make use of the terms *divisor*, *dividend*, *quotient*, and *remainder*.

Divisor, Dividend, Quotient, and Remainder

This schematic reminds readers of the position of these terms in the division process.

$$\begin{array}{r} \text{quotient} \\ \text{divisor} \overline{) \text{dividend}} \\ \dots \\ \text{remainder} \end{array}$$

Now that these terms are defined, we begin the discussion of division of decimal numbers. Suppose that we wish to divide 637 by 100. We could do this in fraction form, change the result to a mixed fraction, then the mixed fraction to decimal form.

$$\frac{637}{100} = 6\frac{37}{100} = 6.37$$

We can also arrange the division much as we would the division of two whole numbers.

$$\begin{array}{r} 6.37 \\ 100 \overline{) 637.00} \\ \underline{600} \\ 370 \\ \underline{300} \\ 700 \\ \underline{700} \\ 0 \end{array}$$

Note that adding two zeros after the decimal point in the dividend does not change the value of 637. Further, note that we proceed as if we are dividing two whole numbers, placing the decimal point in the quotient directly above the decimal point in the dividend.

These observations lead to the following algorithm.

Dividing a Decimal by a Whole Number

To divide a decimal number by a whole number, proceed as follows:

1. Set up the long division as you would the division of two whole numbers.

2. Perform the division as if the numbers were both whole numbers, adding zeros to the right of the decimal point in the dividend as necessary to complete the division.
3. Place the decimal point in the quotient immediately above the decimal point in the dividend.

Example 1

Divide 23 by 20.

Solution

Arrange as if using long division to divide whole numbers, adding enough zeros to the right of the decimal point in the dividend to complete the division.

$$\begin{array}{r}
 1.15 \\
 20 \overline{)23.00} \\
 \underline{20} \\
 30 \\
 \underline{20} \\
 100 \\
 \underline{100} \\
 0
 \end{array}$$

Hence, 23 divided by 20 is 1.15.

Adding Zeroes to the Right of the Decimal Point

Usually, one does not immediately see how many zeros to the right of the decimal point in the dividend are needed. These zeros are usually added at each step of the division, until the division is completed or the user is willing to terminate the process and accept only an estimate of the quotient.

Example 2

Divide: $155.2 \div 25$.

Solution

Arrange as if using long division to divide whole numbers, and begin.

$$\begin{array}{r}
 6.2 \\
 25 \overline{)155.2} \\
 \underline{150} \\
 52 \\
 \underline{50} \\
 2
 \end{array}$$

We still have a nonzero remainder. Adding another zero does no good.

$$\begin{array}{r}
 6.20 \\
 25 \overline{)155.20} \\
 \underline{150} \\
 52 \\
 \underline{50} \\
 20
 \end{array}$$

However, if we add one more additional zero, the division completes with a zero remainder.

$$\begin{array}{r}
 6.208 \\
 25 \overline{)155.200} \\
 \underline{150} \\
 52 \\
 \underline{50} \\
 200 \\
 \underline{200} \\
 0
 \end{array}$$

Thus, 155.2 divided by 25 is 6.208.

Exercise

Divide: $42.55 \div 23$

Answer available at the end of this section.

Decimal Divisors When the divisor contains a decimal point, we have a little work to do before we begin the division process. Suppose that we wish to divide 1.25 by 2.5. In fraction form, we could start with

$$\frac{1.25}{2.5},$$

then clear the decimals from the denominator by multiplying both numerator and denominator by 10. *Note: Recall that multiplying by 10 moves the decimal point one place to the right.*

$$\begin{aligned}\frac{1.25}{2.5} &= \frac{1.25 \cdot 10}{2.5 \cdot 10} \\ &= \frac{12.5}{25}\end{aligned}$$

Thus, dividing 1.25 by 2.5 is equivalent to dividing 12.5 by 25. This we know how to do.

$$\begin{array}{r} 0.5 \\ 25 \overline{)12.5} \\ \underline{12\ 5} \\ 0 \end{array}$$

Thus, 1.25 divided by 2.5 is 0.5.

Writing Mathematics

Never write .5. Always add the leading zero in the ones place and write 0.5.

Instead of working in fraction form, we can take care of positioning the decimal point in the long division framework. Start with:

$$2.5 \overline{)1.25}$$

Move the decimal point in the divisor to the end of the divisor, then move the decimal point in the dividend an equal number of places.

$$\begin{array}{r} 2.5 \overline{)1.25} \\ \uparrow \quad \uparrow \\ 25 \overline{)125} \end{array}$$

Thus, the division becomes

$$25 \overline{)12.5}$$

and we proceed as above to find the quotient. This discussion motivates the following algorithm.

Dividing by a Decimal Divisor

If the divisor contains a decimal, proceed as follows:

1. Move the decimal to the end of the divisor.
2. Move the decimal in the dividend an equal number of places.

We can then complete the division using the rules for dividing a decimal by a whole number.

Example 3

Divide: $\overline{)4.392}$

Solution

Move the decimal in the divisor to the end of the divisor. Move the decimal in the dividend an equal number of places (two places) to the right.

$$\begin{array}{r} 0.36 \overline{)4.392} \\ \hline \end{array}$$

Now we can follow the algorithm for dividing a decimal number by a whole number.

$$\begin{array}{r} 12.2 \\ 36 \overline{)439.2} \\ \underline{36} \\ 79 \\ \underline{72} \\ 72 \\ \underline{72} \\ 0 \end{array}$$

Thus, 4.392 divided by 0.36 is 12.2.

Exercise

Divide: $0.45 \overline{)36.99}$

Answer available at the end of this section.

Dividing Signed Decimal Numbers

The rules governing division of signed decimal numbers are identical to the rules governing division of integers.

Like Signs. The quotient of two decimal numbers with like signs is positive. That is:

$$\frac{(+)}{(+)} = + \text{ and } \frac{(-)}{(-)} = +$$

Unlike Signs. The quotient of two decimal numbers with unlike signs is negative. That is:

$$\frac{(+)}{(-)} = - \text{ and } \frac{(-)}{(+)} = -$$

Example 4

Divide: $-0.03 \div 0.024$.

Solution

First, divide the magnitudes. Move the decimal in the divisor to the end of the divisor. Move the decimal in the dividend an equal number of places (three places) to the right. Note that this requires an extra trailing zero in the dividend.

$$0.024 \overline{)0.030}$$

Our problem then becomes:

$$\sqrt[24]{30}$$

We can now follow the algorithm for dividing a decimal number by a whole number. Note that we have to add two trailing zeros in the dividend to complete the division with a zero remainder.

$$\begin{array}{r} 1.25 \\ 24 \overline{)30.00} \\ \underline{24} \\ 60 \\ \underline{48} \\ 120 \\ \underline{120} \\ 0 \end{array}$$

Finally, because the quotient of unlike signs is negative, -0.03 divided by 0.024 is -1.25 . That is,

$$\frac{-0.03}{0.024} = -1.25.$$

Exercise

Divide: $-0.0113 \div 0.05$

Answer available at the end of this section.

Rounding

Sometimes an exact decimal representation of a fraction is not needed and an approximation is more than adequate.

Example 5

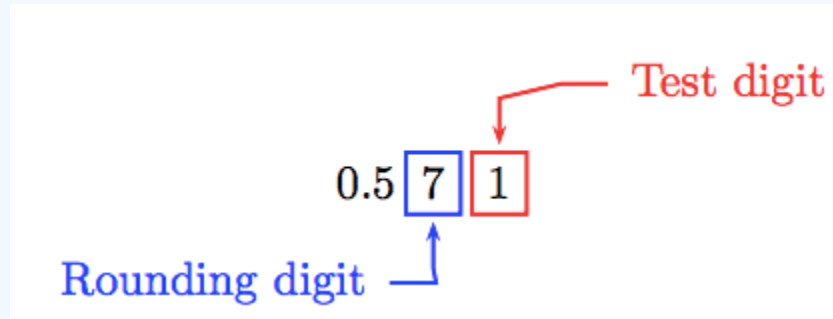
Convert $4/7$ to a decimal. Round your answer to the nearest hundredth.

Solution

We need to carry the division one place beyond the hundredths place.

$$\begin{array}{r} 0.571 \\ 7 \overline{)4.000} \\ \underline{35} \\ 50 \\ \underline{49} \\ 10 \\ \underline{7} \\ 3 \end{array}$$

To round to the nearest hundredth, first identify the rounding and test digits.



Because the “test digit” is less 5, leave the rounding digit alone and truncate. Therefore, correct to the nearest hundredth, $4/7 \approx 0.57$.

Exercise

Convert $5/7$ to a decimal. Round your answer to the nearest hundredth.

Answer available at the end of this section.

Exercises

Divide the numbers.

1. $\frac{39}{52}$

2. $\frac{32.12}{73}$

3. $\frac{17}{25}$

4. $\frac{1.694}{2.2}$

5. $\frac{2.958}{-0.2}$

6. $\frac{1.248}{0.52}$

7. $\frac{-524.35}{100}$

8. Compute the quotient $47/3$, and round your answer to the nearest tenth.

9. Compute the quotient $8/14$, and round your answer to the nearest hundredth.

10. **Media mail.** To promote her business, Theresa mails several packages via Media Mail. One package weighing 2 lbs. costs \$2.77, another package weighing 3 lbs. costs \$3.16, and the third package weighing 5 lbs. costs \$3.94 to mail. What was the average cost per pound to mail the packages? Round your result to the nearest penny. www.usps.com/prices/media-mailprices.htm

Answers

1. 0.75
 2. 0.44
 3. 0.68
 4. 0.77
 5. -14.79
 6. 2.4
 7. -5.2435
 8. 15.7
 9. 0.57
 10. \$0.99 per pound
-

Answers

Exercise

1.85

Exercise

82.2

Exercise

-0.226

Exercise

0.71

3.5: Fractions and Decimals

3.05: Fractions and Decimals

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4: INTRODUCTION TO ALGEBRA

4.1: Mathematical Expressions

4.2: Evaluating Algebraic Expressions

4.3: Simplifying Algebraic Expressions

4.4: Combining Like Terms

4.1: Mathematical Expressions

Recall the definition of a *variable* presented in Section 1.6.

Definition: Variable

A variable is a symbol (usually a letter) that stands for a value that may vary.

Let's add the definition of a *mathematical expression*.

Definition: Mathematical Expression

When we combine numbers and variables in a valid way, using operations such as addition, subtraction, multiplication, division, exponentiation, and other operations and functions as yet unlearned, the resulting combination of mathematical symbols is called a *mathematical expression*.

Thus,

$$2a, x + 5, \text{ and } y^2,$$

being formed by a combination of numbers, variables, and mathematical operators, are valid mathematical expressions. A mathematical expression must be *well-formed*. For example,

$$2 + \div 5x$$

is *not a valid expression* because there is no term following the plus sign (it is not valid to write $+\div$ with nothing between these operators). Similarly,

$$2 + 3(2$$

is not well-formed because parentheses are not balanced.

Translating Words into Mathematical Expressions

In this section we turn our attention to translating word phrases into mathematical expressions. We begin with phrases that translate into *sums*. There is a wide variety of word phrases that translate into sums. Some common examples are given in Table 4.1.1a, though the list is far from complete. In like manner, a number of phrases that translate into differences are shown in Table 4.1.1b.

Table 4.1.1: Translating words into symbols.

Phrase	Translates to:	Phrase	Translates to:
sum of x and 12	$x + 12$	difference of x and 12	$x - 12$
4 greater than b	$b + 4$	4 less than b	$b - 4$
6 more than y	$y + 6$	7 subtracted from y	$y - 7$
44 plus r	$44 + r$	44 minus r	$44 - r$
3 larger than z	$z + 3$	3 smaller than z	$z - 3$
a) Phrases that are sums		b) Phrases that are differences	

Let's look at some examples, some of which translate into expressions involving sums, and some of which translate into expressions involving differences.

Example 1

Translate the following phrases into mathematical expressions:

- a. "12 larger than x ,"
- b. "11 less than y ," and
- c. " r decreased by 9."

Solution

Here are the translations.

- a. "12 larger than x " becomes $x + 12$.
- b. "11 less than y " becomes $y - 11$.
- c. " r decreased by 9" becomes $r - 9$.

Exercise

Translate the following phrases into mathematical expressions:

- a. "13 more than x " and
- b. "12 fewer than y ".

Answer available at the end of this section.

Example 2

Let W represent the width of the rectangle. The length of a rectangle is 4 feet longer than its width. Express the length of the rectangle in terms of its width W .

Solution

We know that the width of the rectangle is W . Because the length of the rectangle is 4 feet longer than the width, we must add 4 to the width to find the length.

$$\begin{array}{ccccccc} \text{Length} & \text{is} & 4 & \text{more than} & \text{the width} \\ \text{Length} & = & 4 & + & W \end{array}$$

Thus, the length of the rectangle in terms of its width, W , is $4 + W$.

Exercise

The width of a rectangle is 5 inches shorter than its length L . Express the width of the rectangle in terms of its length L .

Answer available at the end of this section.

Example 3

A string measures 15 inches is cut into two pieces. Let x represent the length of one of the resulting pieces. Express the length of the second piece in terms of the length x of the first piece.

Solution

The string has original length 15 inches. It is cut into two pieces and the first piece has length x . To find the length of the second piece, we must subtract the length of the first piece from the total length.

$$\begin{array}{ccccccc} \text{Length of the second piece} & \text{is} & \text{Total length} & \text{minus} & \text{the length of the first piece} \\ \text{Length of the second piece} & = & 15 & - & x \end{array}$$

Thus, the length of the second piece, in terms of the length x of the first piece, is $15 - x$.

Exercise

A string is cut into two pieces, the first of which measures 12 inches. Express the total length of the string as a function of x , where x represents the length of the second piece of string.

Answer available at the end of this section.

There is also a wide variety of phrases that translate into products. Some examples are shown in Table 3.2(a), though again the list is far from complete. In like manner, a number of phrases translate into quotients, as shown in Table 3.2(b).

Table 4.1.2: Translating words into symbols.

Phrase	Translates to:	Phrase	Translates to:
product of x and 12	$12x$	quotient of x and 12	$x/12$
4 times b	$4b$	4 divided by b	$4/b$
twice r	$2r$	the ratio of 44 to r	$44/r$
a) Phrases that are products.		b) Phrases that are differences.	

Let's look at some examples, some of which translate into expressions involving products, and some of which translate into expressions involving quotients.

Example 4

Translate the following phrases into mathematical expressions: (a) "11 times x ," (b) "quotient of y and 4," and (c) "twice a ."

Solution

Here are the translations. a) "11 times x " becomes $11x$. b) "quotient of y and 4" becomes $y/4$, or equivalently, $\frac{y}{4}$. c) "twice a " becomes $2a$.

Exercise

Translate into mathematical symbols: (a) "the product of 5 and x " and (b) "12 divided by y ."

Answer available at the end of this section.

Example 5

A plumber has a pipe of unknown length x . He cuts it into 4 equal pieces. Find the length of each piece in terms of the unknown length x .

Solution

The total length is unknown and equal to x . The plumber divides it into 4 equal pieces. To find the length of each piece, we must divide the total length by 4.

$$\begin{array}{ccccccc} \text{Length of each piece} & \text{is} & \text{Total length} & \text{divided by} & 4 \\ \text{Length of each piece} & = & x & \div & 4 \end{array}$$

Thus, the length of each piece, in terms of the unknown length x , is $x/4$, or equivalently, $\frac{x}{4}$.

Exercise

A carpenter cuts a board of unknown length L into three equal pieces. Express the length of each piece in terms of L .

Answer available at the end of this section.

Example 6

Mary invests A dollars in a savings account paying 2% interest per year. She invests five times this amount in a certificate of deposit paying 5% per year. How much does she invest in the certificate of deposit, in terms of the amount A in the savings account?

Solution

The amount in the savings account is A dollars. She invests five times this amount in a certificate of deposit.

$$\begin{array}{ccccccc} \text{Amount in CD} & \text{is} & 5 & \text{times} & \text{Amount in savings} \\ \text{Amount in CD} & = & 5 & \cdot & A \end{array}$$

Thus, the amount invested in the certificate of deposit, in terms of the amount A in the savings account, is $5A$.

Exercise

David invests K dollars in a savings account paying 3% per year. He invests half this amount in a mutual fund paying 4% per year. Express the amount invested in the mutual fund in terms of K , the amount invested in the savings account.

Answer available at the end of this section.

Combinations

Some phrases require combinations of the mathematical operations employed in previous examples.

Example 7

Let the first number equal x . The second number is 3 more than twice the first number. Express the second number in terms of the first number x .

Solution

The first number is x . The second number is 3 more than twice the first number.

$$\begin{array}{lcl} \text{Second number} & \text{is} & 3 \text{ more than } \text{twice the first number} \\ \text{Second number} = & 3+ & 2x \end{array}$$

Therefore, the second number, in terms of the first number x , is $3 + 2x$.

Exercise

A second number is 4 less than 3 times a first number. Express the second number in terms of the first number y .

Answer available at the end of this section.

Example 8

The length of a rectangle is L . The width is 15 feet less than 3 times the length. What is the width of the rectangle in terms of the length L ?

Solution

The length of the rectangle is L . The width is 15 feet less than 3 times the length.

$$\begin{array}{lcl} \text{Width} & \text{is} & 3 \text{ times the length less } 15 \\ \text{Width} = & 3L - & 15 \end{array}$$

Therefore, the width, in terms of the length L , is $3L - 15$.

Exercise

The width of a rectangle is W . The length is 7 inches longer than twice the width. Express the length of the rectangle in terms of its length L .

Answer available at the end of this section.

Exercises

Translate the phrase into a mathematical expression involving the given variable.

1. "8 times the width n "
 2. "6 times the sum of the number n and 3"
 3. "the speed y decreased by 33"
 4. "the supply y doubled"
 5. "13 more than 15 times the number p "
 6. "4 less than 11 times the number x "
7. Representing Numbers. Suppose n represents a whole number.
- i) What does $n + 1$ represent?
 - ii) What does $n + 2$ represent?
 - iii) What does $n - 1$ represent?
-

Answers

1. $8n$
 2. $6(n + 3)$
 3. $y - 33$
 4. $2y$
 5. $15p + 13$
 6. $11x - 4$
 7.
 - i) $n+1$ represents the next whole number after n .
 - ii) $n+2$ represents the next whole number after $n + 1$, or, two whole numbers after n .
 - iii) $n - 1$ represents the whole number before n .
-

Answers

Exercise

- (a) $x + 13$ and
(b) $y - 12$

Exercise

$$L - 5$$

Exercise

$$12 + x$$

Exercise

- (a) $5x$ and (b) $12/y$.

Exercise

$$L/3$$

Exercise

$$\frac{1}{2}K$$

Exercise

$$3y - 4$$

Exercise

$$2W + 7$$

4.2: Evaluating Algebraic Expressions

In this section we will evaluate *algebraic expressions* for given values of the variables contained in the expressions. Here are some simple tips to help you be successful.

Tips for Evaluating Algebraic Expressions

1. Replace all occurrences of variables in the expression with open parentheses. Leave room between the parentheses to substitute the given value of the variable.
2. Substitute the given values of variables in the open parentheses prepared in the first step.
3. Evaluate the resulting expression according to the Rules Guiding Order of Operations.

Let's begin with an example.

Example 1

Evaluate the expression $x^2 - 2xy + y^2$ at $x = -3$ and $y = 2$.

Solution

Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $x^2 - 2xy + y^2$ with open parentheses.

$$x^2 - 2xy + y^2 = ()^2 - 2()() + ()^2$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned}
 & x^2 - 2xy + y^2 && \text{Original expression.} \\
 = & (-3)^2 - 2(-3)(2) + (2)^2 && \text{Substitute } -3 \text{ for } x \text{ and } 2 \text{ for } y. \\
 = & 9 - 2(-3)(2) + 4 && \text{Evaluate exponents first.} \\
 = & 9 - (-6)(2) + 4 && \text{Left to right, multiply } 2(-3) = -6. \\
 = & 9 - (-12) + 4 && \text{Left to right, multiply: } (-6)(2) = -12. \\
 = & 9 + 12 + 4 && \text{Add the opposite.} \\
 = & 25 && \text{Add.}
 \end{aligned}$$

Exercise

If $x = -2$ and $y = -1$, evaluate $x^3 - y^3$.

Answer available at the end of this section.

Example 2

Evaluate the expression $(a - b)^2$ for $a = 3$ and $b = -5$.

Solution

Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $(a - b)^2$ with open parentheses.

$$(a - b)^2 = (() - ())^2$$

Secondly, replace each variable with its given value, and thirdly, follow the “Rules Guiding Order of Operations” to evaluate the resulting expression.

$$\begin{aligned}(a - b)^2 &= ((3) - (-5))^2 && \text{Substitute 3 for } a \text{ and } -5 \text{ for } b. \\ &= (3 + 5)^2 && \text{Add the opposite: } (3) - (-5) = 3 + 5 \\ &= 8^2 && \text{Simplify inside parentheses: } 3 + 5 = 8 \\ &= 64 && \text{Evaluate exponent: } 8^2 = 64\end{aligned}$$

Exercise

If $a = 3$ and $b = -5$, evaluate $a^2 - b^2$.

Answer available at the end of this section.

Example 3

Evaluate the expression $|a| - |b|$ at $a = 5$ and $b = -7$.

Solution

Following “Tips for Evaluating Algebraic Expressions,” first replace all occurrences of variables in the expression $|a| - |b|$ with open parentheses.

$$|a| - |b| = |()| - |()|$$

Secondly, replace each variable with its given value, and thirdly, follow the “Rules Guiding Order of Operations” to evaluate the resulting expression.

$$\begin{aligned}|a| - |b| &= |(5)| = |(-7)| && \text{Substitute 5 for } a \text{ and } -7 \text{ for } b. \\ &= 5 - 7 && \text{Absolute values first: } |(5)| = 5 \text{ and } |(-7)| = 7| \\ &= 5 + (-7) && \text{Add the opposites: } 5 - 7 = 5 + (-7). \\ &= -2 && \text{Add: } 5 + (-7) = -2.\end{aligned}$$

Exercise

If $a = 5$ and $b = -7$, evaluate $2|a| - 3|b|$.

Answer available at the end of this section.

Example 4

Evaluate the expression $|a - b|$ at $a = 5$ and $b = -7$.

Solution

Following “Tips for Evaluating Algebraic Expressions,” first replace all occurrences of variables in the expression $|a - b|$ with open parentheses.

$$|a - b| = |() - ()|$$

Secondly, replace each variable with its given value, and thirdly, follow the “Rules Guiding Order of Operations” to evaluate the resulting expression.

$$\begin{aligned}
 |a - b| &= |(5) - (-7)| && \text{Substitute 5 for } a \text{ and } -7 \text{ for } b. \\
 &= |5 + 7| && \text{Add the opposite: } 5 - (-7) = 5 + 7. \\
 &= |12| && \text{Add: } 5 + 7 = 12. \\
 &= 12 && \text{Take the absolute value: } |12| = 12.
 \end{aligned}$$

Exercise

If $a = 5$ and $b = -7$, evaluate $|2a - 3b|$.

Answer available at the end of this section.

Example 5

Evaluate the expression

$$\frac{ad - bc}{a + b}$$

at $a = 5$, $b = -3$, $c = 2$, and $d = -4$.

Solution

Following “Tips for Evaluating Algebraic Expressions,” first replace all occurrences of variables in the expression with open parentheses.

$$\frac{ad - bc}{a + b} = \frac{()() - ()()}{() + ()}$$

Secondly, replace each variable with its given value, and thirdly, follow the “Rules Guiding Order of Operations” to evaluate the resulting expression.

$$\begin{aligned}
 \frac{ad - bc}{a + b} &= \frac{(5) - (-3)(2)}{(5) + (-3)} && \text{Substitute: 5 for } a, -3 \text{ for } b, 2 \text{ for } c, -4 \text{ for } d. \\
 &= \frac{-20 - (-6)}{2} && \text{Numerator: } (5)(-4) = -20, (-3)(2) = -6. \\
 & && \text{Denominator: } 5 + (-3) = 2. \\
 &= \frac{-20 + 6}{2} && \text{Numerator: Add the opposite.} \\
 &= \frac{-14}{2} && \text{Numerator: } -20 + 6 = -14. \\
 &= -7 && \text{Divide.}
 \end{aligned}$$

Exercise

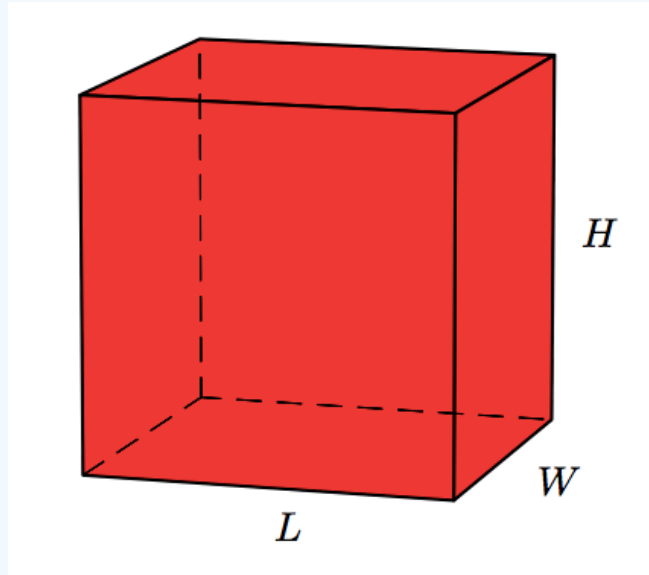
If $a = -7$, $b = -3$, $c = -15$, and $d = -14$, evaluate:

$$\frac{a^2 + b^2}{c + d}$$

Answer available at the end of this section.

Example 6

Pictured below is a rectangular prism.



The volume of the rectangular prism is given by the formula

$$V = LWH,$$

where L is the length, W is the width, and H is the height of the rectangular prism. Find the volume of a rectangular prism having length 12 feet, width 4 feet, and height 6 feet.

Solution

Following “Tips for Evaluating Algebraic Expressions,” first replace all occurrences of L , W , and H in the formula

$$V = LWH$$

with open parentheses.

$$V = () () ()$$

Next, substitute 12 ft for L , 4 ft for W , and 6 ft for H and simplify.

$$\begin{aligned} V &= (12\text{ft})(4\text{ft})(6\text{ft}) \\ &= 288\text{ft}^3 \end{aligned}$$

Hence, the volume of the rectangular prism is 288 cubic feet.

Exercise

The surface area of the prism pictured in this example is given by the following formula:

$$S = 2(WH + LH + LW)$$

If $L = 12$, $W = 4$, and $H = 6$ feet, respectively, calculate the surface area.

Answer available at the end of this section.

Exercises

Evaluate the expression at the given value of x and/or y.

1. $-3x^2 - 6x + 3$ at $x = 7$

2. $-6x - 6$ at $x = 3$

3. $5x^2 + 2x + 4$ at $x = -1$

4. $-9x - 5$ at $x = -2$

5. $12x + 10$ at $x = -12$

6. $|x| - |y|$ at $x = -5$ and $y = 4$

7. $-5x^2 + 2y^2$ at $x = 4$ and $y = 2$

8. $|x - y|$ at $x = 4$ and $y = 5$

9. $5x^2 - 4xy + 3y^2$ at $x = 1$ and $y = -4$

10. $|x - y|$ at $x = 4$ and $y = 4$

11. $5x^2 + 4y^2$ at $x = -2$ and $y = -2$

12. $\frac{-4+9x}{7x}$ at $x = 2$

13. $\frac{-12-7x}{x}$ at $x = -1$

14. The formula

$$d = 16t^2$$

gives the distance (in feet) that an object falls from rest in terms of the time t that has elapsed since its release. Find the distance d (in feet) that an object falls in $t = 4$ seconds.

15. The formula

$$C = \frac{5(F - 32)}{9}$$

gives the Celcius temperature C in terms of the Fahrenheit temperature F . Use the formula to find the Celsius temperature ($^{\circ}\text{C}$) if the Fahrenheit temperature is $F = 230^{\circ}\text{F}$.

Answers

1. -186
 2. -24
 3. 7
 4. 13
 5. -134
 6. 1
 7. -72
 8. 1
 9. 69
 10. 0
 11. 36
 12. 1
 13. 5
 14. 256 feet
 15. 110 degrees
-

Answers

Exercise

-7

Exercise

-16

Exercise

-11

Exercise

31

Exercise

-2

Exercise

288 square feet

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4.3: Simplifying Algebraic Expressions

Recall the commutative and associative properties of multiplication.

The Commutative Property of Multiplication. If a and b are any integers, then

$$a \cdot b = b \cdot a, \text{ or equivalently, } ab = ba.$$

The Associative Property of Multiplication. If a , b , and c are any integers, then

$$(a \cdot b) \cdot c = a \cdot (b \cdot c), \text{ or equivalently, } (ab)c = a(bc).$$

The commutative property allows us to change the order of multiplication without affecting the product or answer. The associative property allows us to regroup without affecting the product or answer.

Example 1

Simplify: $2(3x)$.

Solution

Use the associative property to regroup, then simplify.

$$\begin{aligned} 2(3x) &= (2 \cdot 3)x && \text{Regrouping with the associative property.} \\ &= 6x && \text{Simplify: } 2 \cdot 3 = 6. \end{aligned}$$

Exercise

Simplify: $-5(7y)$

Answer available at the end of this section.

The statement $2(3x)=6x$ is an *identity*. That is, the left-hand side and right-hand side of $2(3x)=6x$ are the same for all values of x . Although the derivation in Example 1 should be the proof of this statement, it helps the intuition to check the validity of the statement for one or two values of x .

If $x = 4$, then

$$\begin{aligned} 2(3x) &= 2(3(4)) && \text{and} && 6x = 6(4) \\ &= 2(12) && && = 24 \\ &= 24 \end{aligned}$$

If $x = -5$, then

$$\begin{aligned} 2(3x) &= 2(3(-5)) && \text{and} && 6x = 6(-5) \\ &= 2(-15) && && = -30 \\ &= -30 \end{aligned}$$

The above calculations show that $2(3x)=6x$ for both $x = 4$ and $x = -5$. Indeed, the statement $2(3x)=6x$ is true, regardless of what is substituted for x .

Example 2

Simplify: $(-3t)(-5)$.

Solution

In essence, we are multiplying three numbers, -3 , t , and -5 , but the grouping symbols ask us to multiply the -3 and the t first. The associative and commutative properties allow us to change the order and regroup.

$$\begin{aligned} (-3t)(-5) &= ((-3)(-5))t && \text{Change the order and regroup.} \\ &= 15t && \text{Multiply: } (-3)(-5) = 15. \end{aligned}$$

Exercise

Simplify: $(-8a)(5)$

Answer available at the end of this section.

Example 3

Simplify: $(-3x)(-2y)$

Solution

In essence, we are multiplying four numbers, -3 , x , -2 , and y , but the grouping symbols specify a particular order. The associative and commutative properties allow us to change the order and regroup.

$$\begin{aligned} (-3x)(-2y) &= ((-3)(-2))(xy) && \text{Change the order and regroup.} \\ &= 6xy && \text{Multiply: } (-3)(-2) = 6. \end{aligned}$$

Exercise

Simplify: $(-4a)(5b)$

Answer available at the end of this section.

Speeding Things Up

The meaning of the expression $2 \cdot 3 \cdot 4$ is clear. Parentheses and order of operations are really not needed, as the commutative and associative properties explain that it doesn't matter which of the three numbers you multiply together first.

- You can multiply 2 and 3 first:

$$\begin{aligned} 2 \cdot 3 \cdot 4 &= (2 \cdot 3) \cdot 4 \\ &= 6 \cdot 4 \\ &= 24. \end{aligned}$$

- Or you can multiply 3 and 4 first:

$$\begin{aligned} 2 \cdot 3 \cdot 4 &= 2 \cdot (3 \cdot 4) \\ &= 2 \cdot 12 \\ &= 24. \end{aligned}$$

- Or you can multiply 2 and 4 first:

$$\begin{aligned}2 \cdot 3 \cdot 4 &= (2 \cdot 4) \cdot 3 \\&= 8 \cdot 3 \\&= 24.\end{aligned}$$

So, it does not matter which two factors you multiply first.

Of course, this would not be the case if there were a mixture of multiplication and other operators (division, addition, subtraction). Then we would have to strictly follow the “Rules Guiding Order of Operations.” But if the only operator is multiplication, the order of multiplication is irrelevant.

Thus, when we see $2(3x)$, as in Example 1, we should think “It’s all multiplication and it doesn’t matter which two numbers I multiply together first, so I’ll multiply the 2 and the 3 and get $2(3x)=6x$.”

Our comments apply equally well to a product of four or more factors. It simply doesn’t matter how you group the multiplication. So, in the case of $(-3x)(-2y)$, as in Example 3, find the product of -2 and -3 and multiply the result by the product of x and y . That is, $(-3x)(-2y)=6xy$.

Example 4

Simplify: $(2a)(3b)(4c)$.

Solution

The only operator is multiplication, so we can order and group as we please. So, we’ll take the product of 2, 3, and 4, and multiply the result by the product of a , b , and c . That is,

$$(2a)(3b)(4c) = 24abc$$

Exercise

Simplify: $(-3x)(-2y)(-4z)$

Answer available at the end of this section.

The Distributive Property

Multiplication is distributive with respect to addition.

The Distributive Property

If a , b , and c are any integers, then

$$a \cdot (b + c) = a \cdot b + a \cdot c, \text{ or equivalently, } a(b + c) = ab + ac.$$

For example, if we follow the “Rules Guiding Order of Operations” and first evaluate the expression inside the parentheses, then

$$\begin{aligned}3(4 + 5) &= 3(9) \quad \text{Parentheses first: } 4 + 5 = 9. \\&= 27. \quad \text{Multiply: } 3(9) = 27.\end{aligned}$$

But if we “distribute” the 3, we get the same answer.

$$\begin{aligned}
 3(4 + 5) &= 3(4 + 5) && \text{Each number in parentheses is multiplied} \\
 &= 3(4) + 3(5) && \text{by the number 3 outside the parentheses.} \\
 &= 12 + 15 && \text{Multiply first: } 3(4) = 12, 3(5) = 15. \\
 &= 27 && \text{Add.}
 \end{aligned}$$

Example 5

Use the distributive property to simplify: $3(4x + 5)$.

Solution

Distribute the 3.

$$\begin{aligned}
 3(4x + 5) &= 3(4x) + 3(5) && \text{Each number in parentheses is multiplied} \\
 &= 12x + 15 && \text{by the number 3 outside the parentheses.} \\
 &&& \text{Multiply first: } 3(4x) = 12x, 3(5) = 15.
 \end{aligned}$$

Exercise

Use the distributive property to simplify: $2(5z+7)$.

Answer available at the end of this section.

Multiplication is also distributive with respect to subtraction.

The Distributive Property

If a , b , and c are any integers, then

$$a \cdot (b - c) = a \cdot b - a \cdot c, \text{ or equivalently, } a(b - c) = ab - ac.$$

The application of this form of the distributive property is identical to the first, the only difference being the subtraction symbol.

Example 6

Use the distributive property to simplify: $5(3x - 2)$.

Solution

Distribute the 5.

$$\begin{aligned}
 5(3x - 2) &= 5(3x) - 5(2) && \text{Each number in parentheses is multiplied} \\
 &= 15x - 10 && \text{by the number 5 outside the parentheses.} \\
 &&& \text{Multiply first: } 5(3x) = 15x, 5(2) = 10.
 \end{aligned}$$

Exercise

Use the distributive property to simplify: $7(4a - 5)$.

Answer available at the end of this section.

Example 7

Remove parentheses: (a) $-9(2t + 7)$, and (b) $-5(4 - 3y)$.

Solution

a) Use the distributive property.

$$\begin{aligned} -9(2t + 7) &= -9(2t) + (-9)(7) && \text{Distribute multiplication by } -9. \\ &= -18t + (-63) && \text{Multiply: } -9(2t) = -18t \text{ and } -9(7) = -63. \\ &= -18t - 63 && \text{Write the answer in simpler form.} \\ &&& \text{Adding } -63 \text{ is the same as subtracting } 63. \end{aligned}$$

b) Use the distributive property.

$$\begin{aligned} -5(4 - 3y) &= -5(4) - (-5)(3y) && \text{Distribute multiplication by } -5. \\ &= -20 - (-15y) && \text{Multiply: } -5(4) = -20 \text{ and } -5(3y) = -15y. \\ &= -20 + 15y && \text{Write the answer in simpler form.} \\ &&& \text{Subtracting } -15y \text{ is the same as adding } 15y. \end{aligned}$$

Exercise

Remove parentheses: $-3(4t - 11)$.

Answer available at the end of this section.

Writing Mathematics

Example 7 stresses the importance of using as few symbols as possible to write your final answer. Hence, $-18t - 63$ is favored over $-18t + (-63)$ and $-20 + 15y$ is favored over $-20 - (-15y)$. You should always make these final simplifications.

Moving a Bit Quicker

Once you've applied the distributive property to a number of problems, showing all the work as in Example 7, you should try to eliminate some of the steps. For example, consider again Example 7(a). It's not difficult to apply the distributive property without writing down a single step, getting:

$$-9(2t + 7) = -18t - 63.$$

Here's the thinking behind this technique:

1. First, multiply -9 times $2t$, getting $-18t$.
2. Second, multiply -9 times $+7$, getting -63 .

Note that this provides exactly the same solution found in Example 7(a).

Let try this same technique on Example 7(b).

$$-5(4 - 3y) = -20 + 15y$$

Here's the thinking behind this technique.

1. First, multiply -5 times 4 , getting -20 .
2. Second, multiply -5 times $-3y$, getting $+15y$.

Note that this provides exactly the same solution found in Example 7(b).

Extending the Distributive Property

Suppose that we add an extra term inside the parentheses.

Distributive Property

If a , b , c , and d are any integers, then

$$a(b + c + d) = ab + ac + ad.$$

Note that we “distributed” the a times each term inside the parentheses. Indeed, if we added still another term inside the parentheses, we would “distribute” a times that term as well.

Example 8

Remove parentheses: $-5(2x - 3y + 8)$.

Solution

We will use the “quicker” technique, “distributing” -5 times each term in the parentheses mentally.

$$-5(2x - 3y + 8) = -10x + 15y - 40$$

Here is our thought process:

1. First, multiply -5 times $2x$, getting $-10x$.
2. Second, multiply -5 times $-3y$, getting $+15y$.
3. Third, multiply -5 times $+8$, getting -40 .

Exercise

Remove parentheses: $-3(4a - 5b + 7)$

Answer available at the end of this section.

Example 9

Remove parentheses: $-4(-3a + 4b - 5c + 12)$.

Solution

We will use the “quicker” technique, “distributing” -4 times each term in the parentheses mentally.

$$-4(-3a + 4b - 5c + 12) = 12a - 16b + 20c - 48$$

Here is our thought process:

1. First, multiply -4 times $-3a$, getting $12a$.
2. Second, multiply -4 times $+4b$, getting $-16b$.
3. Third, multiply -4 times $-5c$, getting $+20c$.
4. Fourth, multiply -4 times $+12$, getting -48 .

Exercise

Remove parentheses: $-2(-2x + 4y - 5z - 11)$.

Answer available at the end of this section.

Distributing a Negative

It is helpful to recall that negating is equivalent to multiplying by -1 .

Multiplying by -1

Let a be any integer, then

$$(-1)a = -a \text{ and } -a = (-1)a.$$

We can use this fact, combined with the distributive property, to negate a sum.

Example 10

Remove parentheses: $-(a + b)$.

Solution

Change the negative symbol into multiplying by -1 , then distribute the -1 .

$$\begin{aligned} -(a + b) &= (-1)(a + b) && \text{Negating is equivalent to multiplying by } -1. \\ &= -a - b && \text{Distribute the } -1. \end{aligned}$$

We chose to use the “quicker” technique of “distributing” the -1 . Here is our thinking:

1. Multiply -1 times a , getting $-a$.
2. Multiply -1 times $+b$, getting $-b$.

Exercise

Remove parentheses: $-(4a - 3c)$

Answer available at the end of this section.

The results in Example 10 and Example 11 show us how to negate a sum: Simply negate each term of the sum. Positive terms change to negative, negative terms turn to positive.

Negating a Sum

To negate a sum, simply negate each term of the sum. For example, if a and b are integers, then

$$-(a + b) = -a - b \text{ and } -(a - b) = -a + b.$$

Example 12

Remove parentheses: $-(5 - 7u + 3t)$.

Solution

Simply negate each term in the parentheses.

$$\mathbb{[-(5 - 7u + 3t) = -5 + 7u - 3t\text{nonumber \}]}$$

Exercise

Remove parentheses: $-(5 - 2x + 4y - 5z)$

Answer available at the end of this section.

Exercises

Use the associative and commutative properties of multiplication to simplify the expression.

1. $10(-4x)$

2. $(-10x)(-3)$

3. $(-4x)10$

4. $(5x)3$

5. $6(2x)$

Simplify the expression.

6. $8(7x + 8)$

7. $9(-2 + 10x)$

8. $2(10 + x)$

9. $3(3 + 4x)$

10. $-(-5 - 7x + 2y)$

11. $4(-6x + 7)$

12. $4(8x - 9)$

13. $-(6x + 2 - 10y)$

Answers

1. $-40x$

2. $30x$

3. $-40x$

4. $15x$

5. $12x$

6. $56x + 64$

7. $-18 + 90x$

8. $20 + 2x$

9. $9 + 12x$

10. $5 + 7x - 2y$

11. $-24x + 28$

12. $32x - 36$

13. $-6x - 2 + 10y$

Answers

Exercise

$-35y$

Exercise

$-40a$

Exercise

$-20ab$

Exercise

$-24xyz.$

Exercise

$10z + 14$

Exercise

$28a - 35$

Exercise

$-12f + 33$

Exercise

$-12a + 15b - 21$

Exercise

$4x - 8y + 10z + 22$

Exercise

$-4a + 3c$

Exercise

$-5 + 2x - 4y + 5z$

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4.4: Combining Like Terms

We begin our discussion with the definition of a *term*.

Definition: Term

A **term** is a single number or variable, or it can be the product of a number (called its coefficient) and one or more variables (called its variable part). The terms in an algebraic expression are separated by addition symbols.

Example 1

Identify the terms in the algebraic expression

$$3x^2 + 5xy + 9y^2 + 12$$

For each term, identify its coefficient and variable part.

Solution

In tabular form, we list each term of the expression $3x^2 + 5xy + 9y^2 + 12$, its coefficient, and its variable part.

Term	Coefficient	Variable Part
$3x^2$	3	x^2
$5xy$	5	xy
$9y^2$	9	y^2
12	12	None

Exercise

How many terms are in the algebraic expression $3x^2 + 2xy - 3y^2$?

Answer available at the end of this section.

Example 2

Identify the terms in the algebraic expression

$$a^3 - 3a^2b + 3ab^2 - b^3$$

For each term, identify its coefficient and variable part.

Solution

The first step is to write each difference as a sum, because the terms of an expression are defined above to be those items separated by addition symbols.

$$a^3 + (-3a^2b) + 3ab^2 + (-b^3)$$

In tabular form, we list each term of the expression $a^3 + (-3a^2b) + 3ab^2 + (-b^3)$, its coefficient, and its variable part.

Term	Coefficient	Variable Part
a^3	1	a^3
$-3a^2b$	-3	a^2b
$3ab^2$	3	ab^2
$-b^3$	-1	b^3

Exercise

How many terms are in the algebraic expression $11 - a^2 - 2ab + 3b^2a$?

Answer available at the end of this section.

Like Terms

We define what is meant by “like terms” and “unlike terms.”

Definition

Like and Unlike Terms. The variable parts of two terms determine whether the terms are like terms or unlike terms.

Like Terms. Two terms are called like terms if they have identical variable parts, which means that the terms must contain the same variables raised to the same exponential powers.

Unlike Terms. Two terms are called unlike terms if their variable parts are different.

Example 3

Classify each of the following pairs as either *like terms* or *unlike terms*: (a) $3x$ and $-7x$, (b) $2y$ and $3y^2$, (c) $-3t$ and $5u$, and (d) $-4a^3$ and $3a^3$.

Solution

Like terms must have *identical* variable parts.

- $3x$ and $-7x$ have identical variable parts. They are “like terms.”
- $2y$ and $3y^2$ do not have identical variable parts (the exponents differ). They are “unlike terms.”
- $-3t$ and $5u$ do not have identical variable parts (different variables). They are “unlike terms.”
- $-4a^3$ and $3a^3$ have identical variable parts. They are “like terms.”

Exercise

Are $-3xy$ and $11xy$ *like* or *unlike* terms?

Answer available at the end of this section.

Combining Like Terms

When using the distributive property, it makes no difference whether the multiplication is on the left or the right, one still distributes the multiplication times each term in the parentheses.

Distributive Property

If a , b , and c are integers, then

$$a(b + c) = ab + ac \text{ and } (b + c)a = ba + ca.$$

In either case, you distribute a times each term of the sum.

“Like terms” can be combined and simplified. The tool used for combining like terms is the distributive property. For example, consider the expression $3y + 7y$, composed of two “like terms” with a common variable part. We can use the distributive property and write

$$3y + 7y = (3 + 7)y$$

Note that we are using the distributive property in reverse, “factoring out” the common variable part of each term. Checking our work, note that if we redistribute the variable part y times each term in the parentheses, we are returned to the original expression $3y + 7y$.

Example 4

Use the distributive property to combine like terms (if possible) in each of the following expressions: (a) $-5x^2 - 9x^2$, (b) $-5ab + 7ab$, (c) $4y^3 - 7y^2$, and (d) $3xy^2 - 7xy^2$.

Solution

If the terms are “like terms,” you can use the distributive property to “factor out” the common variable part.

a) Factor out the common variable part x^2 .

$$\begin{aligned} -5x^2 - 9x^2 &= (-5 - 9)x^2 && \text{Use the distributive property.} \\ &= -14x^2 && \text{Simplify: } -5 - 9 = -5 + (-9) = -14. \end{aligned}$$

b) Factor out the common variable part ab .

$$\begin{aligned} -5ab + 7ab &= (-5 + 7)ab && \text{Use the distributive property.} \\ &= 2ab && \text{Simplify: } -5 + 7 = 2. \end{aligned}$$

c) The terms in the expression $4y^3 - 7y^2$ have different variable parts (the exponents are different). These are “unlike terms” and cannot be combined.

d) Factor out the common variable part xy^2 .

$$\begin{aligned} 3xy^2 - 7xy^2 &= (3 - 7)xy^2 && \text{Use the distributive property.} \\ &= -4xy^2 && \text{Simplify: } 3 - 7 = 3 + (-7) = -4. \end{aligned}$$

Exercise

Simplify: $-8z - 11z$

Speeding Things Up a Bit

Once you've written out all the steps for combining like terms, like those shown in Example 4, you can speed things up a bit by following this rule:

Combining Like Terms

To combine like terms, simply add their coefficients and keep the common variable part.

Thus for example, when presented with the sum of two like terms, such as in $5x + 8x$, simply add the coefficients and repeat the common variable part; that is, $5x + 8x = 13x$.

Example 5

Combine like terms:

- a. $-9y - 8y$,
- b. $-3y^5 + 4y^5$ and
- c. $-3u^2 + 2u^2$.

Solution

a) Add the coefficients and repeat the common variable part. Therefore,

$$-9y - 8y = -17y.$$

b) Add the coefficients and repeat the common variable part. Therefore,

$$-3y^5 + 4y^5 = 1y^5.$$

However, note that $1y^5 = y^5$. Following the rule that the final answer should use as few symbols as possible, a better answer is $-3y^5 + 4y^5 = y^5$.

c) Add the coefficients and repeat the common variable part. Therefore,

$$-3u^2 + 2u^2 = (-1)u^2.$$

However, note that $(-1)u^2 = -u^2$. Following the rule that the final answer should use as few symbols as possible, a better answer is $-3u^2 + 2u^2 = -u^2$.

Simplify

A frequently occurring instruction asks the reader to *simplify* an expression.

Simplify

The instruction simplify is a generic term that means “try to write the expression in its most compact form, using the fewest symbols possible.”

One way you can accomplish this goal is by combining like terms when they are present.

Example 6

Simplify: $2x + 3y - 5x + 8y$.

Solution

Use the commutative property to reorder terms and the associative and distributive properties to regroup and combine like terms.

$$\begin{aligned} 2x + 3y - 5x + 8y &= (2x - 5x) + (3y + 8y) && \text{Reorder and regroup.} \\ &= -3x + 11y && \text{Combine like terms:} \\ &&& 2x - 5x = -3x \text{ and } 3y + 8y = 11y. \end{aligned}$$

Alternate solution

Of course, you do not need to show the regrouping step. If you are more comfortable combining like terms in your head, you are free to present your work as follows:

$$2x + 3y - 5x + 8y = -3x + 11y.$$

Exercise

Simplify: $-3a + 4b - 7a - 9b$

Answer available at the end of this section.

Example 7

Simplify: $-2x - 3 - (3x + 4)$.

Solution

First, distribute the negative sign.

$$-2x - 3 - (3x + 4) = -2x - 3 - 3x - 4 - (3x + 4) = -3x - 4.$$

Next, use the commutative property to reorder, then the associative property to regroup. Then combine like terms.

$$\begin{aligned} &= (-2x - 3x) + (-3 - 4) && \text{Reorder and regroup.} \\ &= -5x + (-7) && \text{Combine like terms:} \\ &&& -2x - 3x = -5x. \\ &= -5x - 7 && \text{Simplify:} \\ &&& -5x + (-7) = -5x - 7. \end{aligned}$$

Alternate solution

You may skip the second step if you wish, simply combining like terms mentally. That is, it is entirely possible to order your work as follows:

$$\begin{aligned} -2x - 3 - (3x + 4) &= -2x - 3 - 3x - 4 && \text{Distribute negative sign.} \\ &= -5x - 7 && \text{Combine like terms.} \end{aligned}$$

Exercise

Simplify: $-9a - 4 - (4a - 8)$

Answer available at the end of this section.

Example 8

Simplify: $2(5 - 3x) - 4(x + 3)$.

Solution

Use the distributive property to expand, then use the commutative and associative properties to group the like terms and combine them.

$$\begin{aligned} 2(5-3x)-4(x+3) &= 10-6x-4x-12 \quad \text{\textcolor{red}{Use the distributive property.}} \\ &= (-6x-4x)+(10-12) \quad \text{\textcolor{red}{Group like terms.}} \\ &= -10x-2 \quad \text{\textcolor{red}{Combine like terms.}} \end{aligned}$$

Alternate solution

You may skip the second step if you wish, simply combining like terms mentally. That is, it is entirely possible to order your work as follows:

$$\begin{aligned} 2(5 - 3x) - 4(x + 3) &= 10 - 6x - 4x - 12 \quad \text{\textcolor{red}{Distribute.}} \\ &= -10x - 2 \quad \text{\textcolor{red}{Combine like terms.}} \end{aligned}$$

Exercise

Simplify: $-2(3a - 4) - 2(5 - a)$

Answer available at the end of this section.

Example 9

Simplify: $-8(3x^2y - 9xy) - 8(-7x^2 - 8xy)$

Solution

We will proceed a bit quicker with this solution, using the distributive property to expand, then combining like terms mentally.

$$\begin{aligned} -8(3x^2y - 9xy) - 8(-7x^2 - 8xy) &= -24x^2y + 72xy + 56x^2y + 64xy \\ &= 32x^2y + 136xy \end{aligned}$$

Exercise

Simplify: $(a^2 - 2ab) - 2(3ab + a^2)$

Answer available at the end of this section.

Exercises

Combine like terms and simplify.

1. $17xy^2 + 18xy^2 + 20xy^2$

2. $12r - 12r$

3. $16s - 5s$

4. $-5q + 7q$

5. $r - 13r - 7r$

6. $19m + m + 15m$

7. $13x^2y + 2x^2y$

8. $-8 + 17n + 10 + 8n$

9. $-2x^3 - 19x^2y - 15x^2y + 11x^3$

10. $-13 + 16m + m + 16$

11. $9n + 10 + 7 + 15n$

12. $3y + 1 + 6y + 3$

13. $3(-4x^2 + 10y^2) + 10(4y^2 - x^2)$

14. $-10q - 10 - (-3q + 5)$

15. $7(8y + 7) - 6(8 - 7y)$

16. $-2(6 + 4n) + 4(-n - 7)$

17. $8 - (4 + 8y)$

18. $7(1 + 7r) + 2(4 - 5r)$

19. $-2(-5 - 8x^2) - 6(6)$

Answers

1. $55xy^2$

2. 0

3. $11s$

4. $2q$

5. $-19r$

6. $35m$

7. $15x^2y$

8. $2 + 25n$

9. $9x^3 - 34x^2y$

10. $3 + 17m$

11. $24n + 17$

12. $9y + 4$

13. $-22x^2 + 70y^2$

14. $-7q - 15$

15. $98y + 1$

16. $-40 - 12n$

17. $4 - 8y$

18. $15 + 39r$

19. $-26 + 16x^2$

Answers

Exercise

3

Exercise

4

Exercise

Like terms

Exercise

$-19z$

Exercise

$-10a - 5b$

Exercise

$-13a + 4$

Exercise

$-4a - 2$

Exercise

$-a^2 - 8ab$

5: SOLVING EQUATIONS

5.1: Solving Equations Part I

5.2: Solving Equations Part II

5.1: Solving Equations Part I

Recall that a variable is a symbol (usually a letter) that stands for a value that varies. If a variable in an equation is replaced by a number and a true statement results, then that number is called a solution of the equation.

Example 1

Is -6 a solution of the equation $2x + 5 = -7$?

Solution

Substitute -6 for x in the equation.

$$\begin{aligned} 2x + 5 &= -7 && \text{Original equation.} \\ 2(-6) + 5 &= -7 && \text{Substitute } -6 \text{ for } x. \\ -12 + 5 &= -7 && \text{On the left, multiply first.} \\ -7 &= -7 && \text{On the left, add.} \end{aligned}$$

Because the last statement is a true statement, -6 is a solution of the equation.

Exercise

Is -4 a solution of $8 - 2x = 5$?

Answer available at the end of this section.

Adding or Subtracting the Same Amount

Two equations having the same set of solutions are *equivalent*. For example, $2x+5 = -7$ and $x = -6$ have the same solutions. Therefore, they are equivalent equations. Certain algebraic operations produce equivalent equations.

Producing Equivalent Equations

Adding the Same Quantity to Both Sides of an Equation. If we start with the equation

$$a = b,$$

then adding c to both sides of the equation produces the equivalent equation

$$a + c = b + c.$$

Subtracting the Same Quantity from Both Sides of an Equation. If we start with the equation

$$a = b,$$

then subtracting c from both sides of the equation produces the equivalent equation

$$a - c = b - c.$$

That is, adding or subtracting the same amount from both sides of an equation will not change the solutions of the equation.

Example 2

Solve for x :

$$x + 3 = -7$$

Solution

To undo the effect of adding 3, subtract 3 from both sides of the equation.

$$\begin{aligned}x + 3 &= -7 && \text{Original equation.}\\x + 3 - 3 &= -7 - 3 && \text{Subtract 3 from both sides.}\\x &= -7 + (-3) && \text{Simplify the left hand side. On the right,}\\&&& \text{express subtractions as adding the opposite.}\\x &= -10\end{aligned}$$

To check the solution, substitute -10 for x in the original equation and simplify.

$$\begin{aligned}x + 3 &= -7 && \text{Original equation.}\\-10 + 3 &= -7 && \text{Substitute } -10 \text{ for } x.\\= 7 &= -7 && \text{Simplify both sides.}\end{aligned}$$

Since the last line of the check is a true statement, this confirms that -10 is a solution.

Exercise

Solve for x :

$$x + 9 = -11$$

Answer available at the end of this section.

Example 3

Solve for x :

$$x - 8 = -11$$

Solution

To undo the effect of subtracting 8, add 8 to both sides of the equation.

$$\begin{aligned}x - 8 &= -11 && \text{Original equation.}\\x - 8 + 8 &= -11 + 8 && \text{Add 8 to both sides.}\\x &= -3 && \text{Simplify both equations.}\end{aligned}$$

To check the solution, substitute -3 for x in the original equation and simplify.

$$\begin{aligned}x - 8 &= -11 && \text{Original equation.}\\-3 - 8 &= -11 && \text{Substitute } -3 \text{ for } x.\\-11 &= -11 && \text{Simplify both sides.}\end{aligned}$$

Since the last line of the check is a true statement, this confirms that -3 is a solution.

Exercise

Solve for x :

$$x - 2 = -7$$

Answer available at the end of this section.

Sometimes a bit of simplification is in order before you start the solution process.

Example 4

Solve for y :

$$-8 + 2 = y - 11(-4)$$

Solution

First, simplify both sides of the equation.

$$\begin{aligned} -8 + 2 &= y - 11(-4) && \text{Original equation.} \\ -6 &= y - (-44) && \text{Simplify. On the left, } -8 + 2 = -6. \\ &&& \text{On the right, } 11(-4) = -44. \\ -6 &= y + 44 - 44 && \text{Subtract 44 from both sides of the equation.} \\ -6 + (-44) &= y && \text{Express subtraction as addition. Simplify on the right.} \\ -50 &= y \end{aligned}$$

To check the solution, substitute -50 for y in the original equation and simplify.

$$\begin{aligned} -8 + 2 &= y - 11(-4) && \text{Original equation.} \\ -8 + 2 &= -50 - 11(-4) && \text{Substitute } -50 \text{ for } y. \\ -6 &= -50 - (-44) && \text{Express subtraction on the right as addition.} \\ -6 &= -6 && \text{On the right, add: } -50 + 44 = -6. \end{aligned}$$

Since the last line of the check is a true statement, this confirms that -50 is a solution.

Exercise

Solve for y :

$$y + 2(-4) = -8 + 6$$

Answer available at the end of this section.

Multiplying or Dividing by the Same Amount

Adding and subtracting is not the only way to produce an equivalent equation.

Producing Equivalent Equations

Multiplying Both Sides of an Equation by the Same Quantity. If we start with the equation

$$a = b,$$

then multiplying both sides of the equation by c produces the equivalent equation

$$a \cdot c = b \cdot c, \text{ or equivalently, } ac = bc,$$

provided $c \neq 0$.

Dividing Both Sides of an Equation by the Same Quantity. If we start with the equation

$$a = b,$$

then dividing both sides of the equation by c produces the equivalent equation

$$\frac{a}{c} = \frac{b}{c},$$

provided $c \neq 0$.

That is, multiplying or dividing both sides of an equation by the same amount will not change the solutions of the equation.

Example 5

Solve for x :

$$-3x = 30$$

Solution

To undo the effect of multiplying by -3 , divide both sides of the equation by -3 .

$$\begin{array}{ll} -3x = 30 & \text{Original equation.} \\ \frac{-3x}{-3} = \frac{30}{-3} & \text{Divide both sides by } -3. \\ x = -10 & \begin{array}{l} \text{On the left, } -3 \text{ times } x, \text{ divided by } -3 \text{ is } x. \\ \text{On the right, } 30/(-3) = -10. \end{array} \end{array}$$

To check the solution, substitute -10 for x in the original equation and simplify.

$$\begin{array}{ll} -3x = 30 & \text{Original equation.} \\ -3(-10) = 30 & \text{Substitute } -10 \text{ for } x. \\ 30 = 30 & \text{Simplify.} \end{array}$$

Because the last line of the check is a true statement, this confirms that -10 is a solution.

Exercise

Solve for z :

$$-4z = -28$$

Answer available at the end of this section.

Example 6

Solve for x :

$$\frac{x}{-2} = -20$$

Solution

To undo the effect of dividing by -2 , multiply both sides of the equation by -2 .

$$\begin{aligned}\frac{x}{-2} &= -20 && \text{Original equation.} \\ -2\left(\frac{x}{-2}\right) &= -2(-20) && \text{Multiply both sides by } -2. \\ x &= 40 && \text{On the left, } x \text{ divided by } -2, \text{ multiplied by } -2, \\ &&& \text{the result is } x. \text{ On the right, } -2(-20) = 40.\end{aligned}$$

To check the solution, substitute 40 for x in the original equation and simplify.

$$\begin{aligned}\frac{x}{-2} &= -20 && \text{Original equation.} \\ \frac{40}{-2} &= -20 && \text{Substitute } 40 \text{ for } x. \\ -20 &= -20 && \text{Simplify both sides.}\end{aligned}$$

Because the last line of the check is a true statement, this confirms that 40 is a solution.

Combining Operations

Consider wrapping and unwrapping a present. To wrap a present we: (1) put the gift paper on, (2) put the tape on, and (3) put the decorative bow on. To unwrap the gift, we must “undo” each of these steps in inverse order. Hence, to unwrap the gift we: (1) take off the decorative bow, (2) take off the tape, and (3) take off the gift paper.

Now, imagine a machine that takes its input, then: (1) multiplies the input by 2 , and (2) adds 3 to the result. This machine is pictured on the left in Figure 2.16.

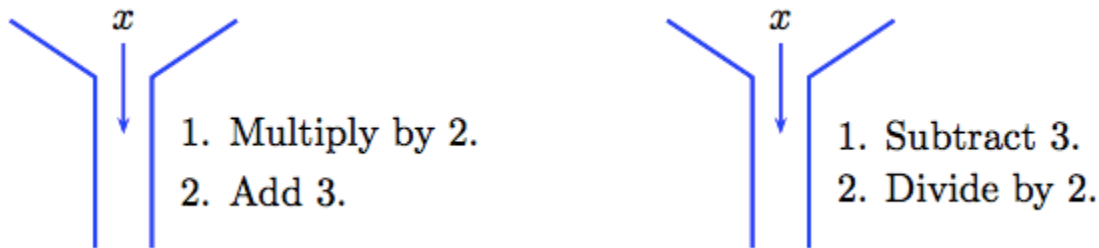


Figure 2.16: The second machine “unwraps” the first machine.

To “unwrap” the effect of the machine on the left, we need a machine that will “undo” each of the steps of the first machine, but in inverse order. The “unwrapping” machine is pictured on the right in Figure 2.16. It will first subtract three from its input, then divide the result by 2 . Note that each of these operations “undoes” the corresponding operation of the first machine, but in inverse order.

For example, drop the integer 7 into the first machine on the left in Figure 2.16. First, we double 7 , then add 3 to the result. The result is $2(7) + 3 = 17$.

Now, to “unwrap” this result, we drop 17 into the second machine. We first subtract 3 , then divide by 2 . The result is $(17 - 3)/2 = 7$, the original integer input into the first machine.

Now, consider the equation

$$2x + 3 = 7.$$

On the left, order of operations demands that we first multiply x by 2 , then add 3 . To solve this equation for x , we must “undo” each of these operations in inverse order. Thus, we will (1) subtract three from both sides of the equation, then (2) divide both sides of the resulting equation by 2 .

$$\begin{aligned}
 2x + 3 - 3 &= 7 - 3 && \text{Subtract 3 from both sides.} \\
 2x &= 4 && \text{Simplify both sides.} \\
 \frac{2x}{2} &= \frac{4}{2} && \text{Divide both sides by 2.} \\
 x &= 2 && \text{Simplify both sides.}
 \end{aligned}$$

Readers should check this solution in the original equation.

Example 7

Solve for x :

$$\frac{x}{4} - 3 = -7$$

Solution

On the left, order of operations demands that we first divide x by 4, then subtract 3. To solve this equation for x , we must “undo” each of these operations in inverse order. Thus, we will (1) add 3 to both sides of the equation, then (2) multiply both sides of the resulting equation by 4.

$$\begin{aligned}
 \frac{x}{4} - 3 &= -7 && \text{Original equation.} \\
 \frac{x}{4} - 3 + 3 &= -7 + 3 && \text{Add 3 to both sides.} \\
 \frac{x}{4} &= -4 && \text{Simplify both sides.} \\
 4\left(\frac{x}{4}\right) &= 4(-4) && \text{Multiply both sides by 4.} \\
 x &= -16 && \text{Simplify both sides.}
 \end{aligned}$$

Check

Substitute -16 for x in the original equation.

$$\begin{aligned}
 \frac{x}{4} - 3 &= -7 && \text{Original equation.} \\
 \frac{-16}{4} - 3 &= -7 && \text{Substitute } -16 \text{ for } x. \\
 -4 - 3 &= -7 && \text{Divide first: } -16/4 = -4. \\
 -7 &= -7 && \text{Subtract: } -4 - 3 = -7.
 \end{aligned}$$

Because the last line of the check is a true statement, -16 is a solution of the original equation.

Exercise

Solve for x :

$$\frac{x}{2} + 6 = 4$$

Answer available at the end of this section.

Example 8

Solve for t :

$$0 = 8 - 2t$$

Solution

On the right, order of operations demands that we first multiply t by -2 , then add 8 . To solve this equation for t , we must “undo” each of these operations in inverse order. Thus, we will (1) subtract 8 from both sides of the equation, then (2) divide both sides of the resulting equation by -2 .

$$\begin{aligned} 0 &= 8 - 2t && \text{Original equation.} \\ 0 - 8 &= 8 - 2t - 8 && \text{Subtract 8 from both sides.} \\ -8 &= -2t && \text{Simplify both sides.} \\ \frac{-8}{-2} &= \frac{-2t}{-2} && \text{Divide both sides by 2.} \\ 4 &= t && \text{Simplify both sides.} \end{aligned}$$

Check

Substitute 4 for t in the original equation.

$$\begin{aligned} 0 &= 8 - 2t && \text{Original equation.} \\ 0 &= 8 - 2(4) && \text{Substitute 4 for } t. \\ 0 &= 8 - 8 && \text{Multiply first: } 2(4) = 8. \\ 0 &= 0 && \text{Subtract: } 8 - 8 = 0. \end{aligned}$$

Because the last line in the check is a true statement, 4 is a solution of the original equation.

Exercise

Solve for r :

$$0 = 9 + 3r$$

Answer available at the end of this section.

Example 9

Solve for p :

$$-12 + 3 = -8 + 4 + \frac{p}{-3}.$$

Solution

Always simplify when possible.

$$\begin{aligned} -12 + 3 &= -8 + 4 + \frac{p}{-3} && \text{Original equation.} \\ -9 &= -4 + \frac{p}{-3} && \text{Simplify both sides.} \end{aligned}$$

On the right, order of operations demands that we first divide p by -3 , then add -4 . To solve this equation for p , we must “undo” each of these operations in inverse order. Thus, we will (1) add a positive 4 to both sides of the equation, then (2) multiply both sides of the resulting equation by -3 .

$$\begin{aligned} -9 + -4 &= -4 + \frac{p}{-3} + 4 && \text{Add 4 to both sides.} \\ -5 &= \frac{p}{-3} && \text{Simplify both sides.} \\ -3(-5) &= -3\left(\frac{p}{-3}\right) && \text{Multiply both sides by } -3. \\ 15 &= p && \text{Simplify both sides.} \end{aligned}$$

Check

Substitute 15 for p in the original equation.

$$-12 + 3 == 8 + 4 + \frac{p}{-3} \quad \text{Original equation.}$$

$$-12 + 3 = -8 + 4 + \frac{15}{-3} \quad \text{Substitute 15 for } p.$$

$$-9 = -8 + 4 + (-5) \quad \begin{array}{l} \text{On the left, add: } -12 + 3 = -9. \text{ On the} \\ \text{right, divide: } 15/(-3) = -5. \end{array}$$

$$-9 = -4 + (-5) \quad \text{On the right, } -8 + 4 = -4.$$

$$-9 = -9 \quad \text{On the right, add: } -4 + (-5) = -9.$$

Because the last line in the check is a true statement, 15 is a solution of the original equation.

Exercise

Solve for q :

$$\frac{q}{-2} - 9 = -8 + 3$$

Answer available at the end of this section.

Applications

Let's look at some applications of equations involving integers. First, we remind readers that a solution of a word problem must incorporate each of the following steps.

Requirements for Word Problem Solutions

1. **Set up a Variable Dictionary.** You must let your readers know what each variable in your problem represents. This can be accomplished in a number of ways:
 1. Statements such as "Let P represent the perimeter of the rectangle."
 2. Labeling unknown values with variables in a table.
 3. Labeling unknown quantities in a sketch or diagram.
2. **Set up an Equation.** Every solution to a word problem must include a carefully crafted equation that accurately describes the constraints in the problem statement.
3. **Solve the Equation.** You must always solve the equation set up in the previous step.
4. **Answer the Question.** This step is easily overlooked. For example, the problem might ask for Jane's age, but your equation's solution gives the age of Jane's sister Liz. Make sure you answer the original question asked in the problem. Your solution should be written in a sentence with appropriate units.
5. **Look Back.** It is important to note that this step does not imply that you should simply check your solution in your equation. After all, it's possible that your equation incorrectly models the problem's situation, so you could have a valid solution to an incorrect equation. The important question is: "Does your answer make sense based on the words in the original problem statement."

Example 10

A student's bank account is overdrawn. After making a his account, Allen finds that he is overdrawn by \$15. What was his account balance before his withdrawal? deposit of \$120, he finds that his account is still overdrawn by an amount of \$75. What was his balance before he made his deposit?

Solution

In our solution, we address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary.* In this case, the unknown is the original balance in the student's account. Let B represent this original balance.

2. *Set up an Equation.* A positive integer represents a healthy balance, while a negative number represents an account that is overdrawn. After the student's deposit, the account is still overdrawn by \$75. We will say that this balance is $-\$75$. Thus,

Original Balance	plus	Student Deposit	equals	Current Balance
B	+	$\$120$	=	$-\$75$

3. *Solve the Equation.* To “undo” the addition, subtract 120 from both sides of the equation.

$$\begin{aligned}
 B + 120 &= -75 && \text{Original equation.} \\
 B + 120 - 120 &= -75 - 120 && \text{Subtract 120 from both sides.} \\
 B &= -195 && \text{Simplify both sides.}
 \end{aligned}$$

4. *Answer the Question.* The original balance was overdrawn to the tune of \$195.

5. *Look Back.* If the original balance was overdrawn by \$195, then we let $-\$195$ represent this balance. The student makes a deposit of \$120. Add this to the original balance to get $-\$195 + \$120 = -\$75$, the correct current balance.

Exercise

After withdrawing \$125 from his account, Allen finds that he is overdrawn by \$15. What was his account balance before his withdrawal?

Answer available at the end of this section.

Example 11

Three more than twice a certain number is -11 . Find the unknown number.

Solution

In our solution, we address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary.* Let x represent the unknown number. 2. Set up an Equation. “Three more than twice a certain number” becomes:

Three	more than	Twice a Certain Number	is	-11
3	+	$2x$	=	11

3. *Solve the Equation.* On the left, order of operations requires that we first multiply x by 2, then add 3. To solve this equation for x , we must “undo” each of these operations in inverse order. Thus, we will (1) subtract 3 from both sides of the equation, then (2) divide both sides of the resulting equation by 2.

$$\begin{aligned}
 3 + 2x &= -11 && \text{Original equation.} \\
 3 + 2x - 3 &= -11 - 3 && \text{Subtract 3 from both sides.} \\
 2x &= -14 && \text{Simplify both sides.} \\
 \frac{2x}{2} &= \frac{-14}{2} && \text{Divide both sides by 2.} \\
 x &= -7 && \text{Simplify both sides.}
 \end{aligned}$$

4. *Answer the Question.* The unknown number is -7 .

5. *Look Back.* Does the answer satisfy the problem constraints? Three more than twice -7 is three more than -14 , or -11 . So the solution is correct.

Exercise 5.1.1

Five less than twice a certain number is -7 . Find the unknown number.

Answer available at the end of this section.

Exercises

1. Is 6 a solution of $3x + 1 = 19$?
2. Is -3 a solution of $4x + 5 = -8$?
3. Is -6 a solution of $-2x + 3 = 15$?

Solve the given equation for x .

4. $x - 13 = 11$
5. $x - 3 = 6$
6. $x + 10 = 17$
7. $x - 6 = 1$
8. $x - 15 = -12$
9. $x + 11 = -19$
10. $x + 2 = 1$
11. $x + 5 = -5$

Solve the given equation for x .

12. $-x = -20$
13. $\frac{x}{-7} = 10$
14. $\frac{x}{9} = -16$
15. $-10x = 20$
16. $-2x = 28$
17. $\frac{x}{-8} = -1$
18. $5x - 14 = -79$
19. $-10x - 16 = 24$

20. $9x + 5 = -85$

21. $-x + 8 = 13$

22. $12x - 15 = -3$

23. $19x + 18 = 113$

24. $-14x + 12 = -2$

25. Two less than eight times an unknown number is -74 . Find the unknown number.

26. Eight more than two times an unknown number is 0 . Find the unknown number.

27. The number -6 is 2 more than an unknown number. Find the unknown number.

28. Three more than eight times an unknown number is -29 . Find the unknown number.

29. The quotient of two integers is 5. One of the integers is -2 . Find the other integer.

30. A student's bank account is overdrawn. After making a deposit of $\$260$, he finds that his account is still overdrawn by an amount of $\$70$. What was his balance before he made his deposit?

31. A student's bank account is overdrawn. After making a deposit of $\$360$, he finds that his account is still overdrawn by an amount of $\$90$. What was his balance before he made his deposit?

32. The number -16 is 4 times larger than an unknown number. Find the unknown number.

33. Two less than nine times an unknown number is 7. Find the unknown number.

Answers

1. Yes

2. No

3. Yes

4. 24

5. 9

6. 7

7. 7

8. 3

9. -30

10. -1

11. -10

12. 20

13. -70

14. -144

15. -2

16. -14

17. 8

18. -13

19. -4

20. -10

21. -5

22. 1

23. 5

24. 1

25. -9

26. -4

27. -8

28. -4

29. -10

30. -\$330

31. -\$450

32. -4

33. 1

Answers

Exercise

No

Exercise

$x = -20$

Exercise

$x = -5$

Exercise

$y = 6$

Exercise

$z = 7$

Exercise

$x = -4$

Exercise

$r = -3$

Exercise

$q = -8$

Exercise

\$110

Exercise 5.1.1

–1

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5.2: Solving Equations Part II

We return to solving equations involving integers, only this time the equations will be a bit more advanced, requiring the use of the distributive property and skill at combining like terms. Let's begin.

Example 1

Solve for x :

$$7x - 11x = 12$$

Solution

Combine like terms.

$$\begin{aligned} 7x - 11x &= 12 && \text{Original equation.} \\ = 4x &= 12 && \text{Combine like terms: } 7x - 11x = -4x. \end{aligned}$$

To undo the effect of multiplying by -4 , divide both sides of the last equation by -4 .

$$\begin{aligned} \frac{-4x}{-4} &= \frac{12}{-4} && \text{Divide both sides by } -4. \\ x &= -3 && \text{Simplify: } 12/(-4) = -3. \end{aligned}$$

Check

Substitute -3 for x in the original equation.

$$\begin{aligned} 7x - 11x &= 12 && \text{Original equation.} \\ 7(-3) - 11(-3) &= 12 && \text{Substitute } -3 \text{ for } x. \\ -21 + 33 &= 12 && \text{On the left, multiply first.} \\ 12 &= 12 && \text{On the left, add.} \end{aligned}$$

Because the last line of the check is a true statement, -3 is a solution of the original equation.

Exercise

Solve for x :

$$-6x - 5x = 22$$

Answer available at the end of this section.

Example 2

Solve for x :

$$12 = 5x - (4 + x)$$

Solution

To take the negative of a sum, negate each term in the sum (change each term to its opposite). Thus, $-(4 + x) = -4 - x$.

$$\begin{aligned}
 12 &= 5x - (4 + x) && \text{Original equation.} \\
 12 &= 5x - 4 - x - (4 + x) = -4 - x. \\
 12 &= 4x - 4 && \text{Combine like terms: } 5x - x = 4x.
 \end{aligned}$$

To undo the effect of subtracting 4, add 4 to both sides of the last equation.

$$\begin{aligned}
 12 + 4 &= 4x - 4 + 4 && \text{Add 4 to both sides.} \\
 16 &= 4x && \text{Simplify both sides.}
 \end{aligned}$$

To undo the effect of multiplying by 4, divide both sides of the last equation by 4.

$$\begin{aligned}
 \frac{16}{4} &= \frac{4x}{4} \sim \text{Divide both sides by 4.} \\
 4 &= x \sim \text{Simplify: } 16/4=4.
 \end{aligned}$$

Check

Substitute 4 for x in the original equation.

$$\begin{aligned}
 12 &= 5x - (4 + x) && \text{Original equation.} \\
 12 &= 5(4) - (4 + 4) && \text{Substitute 4 for } x. \\
 12 &= 20 - 8 && \text{On the right, } 5(4) = 20 \text{ and evaluate} \\
 &&& \text{parentheses: } 4 + 4 = 8. \\
 12 &= 12 && \text{Simplify.}
 \end{aligned}$$

Because the last line of the check is a true statement, 4 is a solution of the original equation.

Exercise

Solve for x :

$$11 = 3x - (1 - x)$$

Answer available at the end of this section.

Variables on Both Sides

Variables can occur on both sides of the equation.

Goal

Isolate the terms containing the variable you are solving for on one side of the equation.

Example 3

Solve for x :

$$5x = 3x - 18$$

Solution

To isolate the variables on one side of the equation, subtract $3x$ from both sides of the equation and simplify.

$$\begin{aligned}
 5x &= 3x - 18 && \text{Original equation.} \\
 5x - 3x &= 3x - 18 - 3x && \text{Subtract } 3x \text{ from both sides.} \\
 2x &= -18 && \text{Combine like terms: } 5x - 3x = 2x \\
 &&& \text{and } 3x - 3x = 0.
 \end{aligned}$$

Note that the variable is now isolated on the left-hand side of the equation. To undo the effect of multiplying by 2, divide both sides of the last equation by 2.

$$\begin{aligned}
 \frac{2x}{2} &= \frac{-18}{2} && \text{Divide both sides by 2.} \\
 x &= -9 && \text{Simplify: } -18/2 = -9.
 \end{aligned}$$

Check

Substitute -9 for x in the original equation.

$$\begin{aligned}
 5x &= 3x - 18 && \text{Original equation.} \\
 5(-9) &= 3(-9) - 18 && \text{Substitute } -9 \text{ for } x. \\
 -45 &= -27 - 18 && \text{Multiply first on both sides.} \\
 -45 &= -45 && \text{Subtract on the right: } -27 - 18 = -45.
 \end{aligned}$$

Because the last line of the check is a true statement, -9 is a solution of the original equation.

Exercise

Solve for x :

$$4x - 3 = x$$

Answer available at the end of this section.

Example 4

Solve for x :

$$5x = 3 + 6x$$

Solution

To isolate the variables on one side of the equation, subtract $6x$ from both sides of the equation and simplify.

$$\begin{aligned}
 5x &= 3 + 6x && \text{Original equation.} \\
 5x - 6x &= 3 + 6x - 6x && \text{Subtract } 6x \text{ from both sides.} \\
 -x &= 3 && \text{Combine like terms: } 5x - 6x = -x \\
 &&& \text{and } 6x - 6x = 0.
 \end{aligned}$$

Note that the variable is now isolated on the left-hand side of the equation.

There are a couple of ways we can finish this solution. Remember, $-x$ is the same as $(-1)x$, so we could undo the effects of multiplying by -1 by dividing both sides of the equation by -1 . Multiplying both sides of the equation by -1 will work equally well. But perhaps the easiest way to proceed is to simply negate both sides of the equation.

$$\begin{aligned}
 -(-x) &= -3 && \text{Negate both sides.} \\
 x &= -3 && \text{Simplify: } -(-x) = x.
 \end{aligned}$$

Check

Substitute -3 for x in the original equation.

$$\begin{aligned} 5x &= 3 + 6x && \text{Original equation.} \\ 5(-3) &= 3 + 6(-3) && \text{Substitute } -3 \text{ for } x. \\ -15 &= 3 - 18 && \text{Multiply first on both sides.} \\ -15 &= -15 && \text{Subtract on the right: } 3 - 18 = -15. \end{aligned}$$

Because the last line of the check is a true statement, -3 is a solution of the original equation.

Exercise

Solve for x :

$$7x = 18 + 9x$$

Answer available at the end of this section.

Dealing with $-x$.

If your equation has the form

$$-x = c,$$

where c is some integer, note that this is equivalent to the equation $(-1)x = c$. Therefore, dividing both sides by -1 will produce a solution for x . Multiplying both sides by -1 works equally well. However, perhaps the easiest thing to do is negate each side, producing

$$-(-x) = -c, \text{ which is equivalent to } x = -c.$$

Example 5

Solve for x :

$$6x - 5 = 12x + 19$$

Solution

To isolate the variables on one side of the equation, subtract $12x$ from both sides of the equation and simplify.

$$\begin{aligned} 6x - 5 &= 12x + 19 && \text{Original equation.} \\ 6x - 5 - 12x &= 12x + 19 - 12x && \text{Subtract } 12x \text{ from both sides.} \\ -6x - 5 &= 19 && \text{Combine like terms: } 6x - 12x = -6x \\ &&& \text{and } 12x - 12x = 0. \end{aligned}$$

Note that the variable is now isolated on the left-hand side of the equation. Next, to “undo” subtracting 5, add 5 to both sides of the equation.

$$\begin{aligned} -6x - 5 + 5 &= 19 + 5 && \text{Add 5 to both sides.} \\ -6x &= 24 && \text{Simplify: } -5 + 5 = 0 \text{ and } 19 + 5 = 24. \end{aligned}$$

Finally, to “undo” multiplying by -6 , divide both sides of the equation by -6 .

$$\begin{aligned} \frac{-6x}{-6} &= \frac{24}{-6} && \text{Divide both sides by } -6. \\ x &= -4 && \text{Simplify: } 24/(-6) = -4. \end{aligned}$$

Check. Substitute -4 for x in the original equation

$$\begin{aligned}
 6x - 5 &= 12x + 19 && \text{Original equation.} \\
 6(-4) - 5 &= 12(-4) + 19 && \text{Substitute } -4 \text{ for } x. \\
 -24 - 5 &= -48 + 19 && \text{Multiply first on both sides.} \\
 -29 &= -29 && \text{Simplify each side: } -24 - 5 = -29, -48 + 19 = -29.
 \end{aligned}$$

Because the last line of the check is a true statement, -4 is a solution of the original equation.

Exercise

Solve for x :

$$2x + 3 = 18 - 3x$$

Answer available at the end of this section.

Example 6

Solve for x :

$$2(3x + 2) - 3(4 - x) = x + 8$$

Solution

Use the distributive property to remove parentheses on the left-hand side of the equation.

$$\begin{aligned}
 2(3x + 2) - 3(4 - x) &= x + 8 && \text{Original equation.} \\
 6x + 4 - 12 + 3x &= x + 8 && \text{Use the distributive property.} \\
 9x - 8 &= x + 8 && \text{Combine like terms: } 6x + 3x = 9x \\
 &&& \text{and } 4 - 12 = -8.
 \end{aligned}$$

Isolate the variables on the left by subtracting x from both sides of the equation

$$\begin{aligned}
 9x - 8 - x &= x + 8 - x && \text{Subtract } x \text{ from both sides.} \\
 8x - 8 &= 8 && \text{Combine like terms: } 9x - x = 8x \\
 &&& \text{and } x - x = 0.
 \end{aligned}$$

Note that the variable is now isolated on the left-hand side of the equation. Next, to “undo” subtracting 8, add 8 to both sides of the equation.

$$\begin{aligned}
 8x - 8 + 8 &= 8 + 8 && \text{Add 8 to both sides.} \\
 8x &= 16 && \text{Simplify: } -8 + 8 = 0 \text{ and } 8 + 8 = 16.
 \end{aligned}$$

Finally, to “undo” multiplying by 8, divide both sides of the equation by 8.

$$\begin{aligned}
 \frac{8x}{8} &= \frac{16}{8} && \text{Divide both sides by 8.} \\
 x &= 2 && \text{Simplify: } 16/8 = 2.
 \end{aligned}$$

Check. Substitute 2 for x in the original equation.

Because the last line of the check is a true statement, 2 is a solution of the original equation.

Exercise

Solve for x :

$$3(2x - 4) - 2(5 - x) = 18$$

Answer available at the end of this section.

Exercises

Solve the equation.

1. $-9x + x = -8$

2. $-4 = 3x - 4x$

3. $27x + 51 = -84$

4. $9 = 5x + 9 - 6x$

5. $41 = 28x + 97$

6. $8x - 8 - 9x = -3$

7. $-85x + 85 = 0$

8. $-6x = -5x - 9$

9. $6x - 7 = 5x$

10. $4x - 3 = 5x - 1$

11. $-3x + 5 = 3x - 1$

12. $6x = 4x - 4$

13. $-8x + 2 = -6x + 6$

14. $1 - (x - 2) = -3$

15. $-7x + 6(x + 8) = -2$

16. $8(-6x - 1) = -8$

17. $2 - 9(x - 5) = -16$

18. $7x + 2(x + 9) = -9$

19. $8 + 2(x - 5) = -4$

20. $9x - 2(x + 5) = -10$

21. $4(-7x + 5) + 8 = 3(-9x - 1) - 2$

22. $-8(-2x - 6) = 7(5x - 1) - 2$

$$23. 2(2x - 9) + 5 = -7(-x - 8)$$

$$24. 2(-2x - 3) = 3(-x + 2)$$

$$25. -5(-9x + 7) + 7 = -(-9x - 8)$$

Answers

1. 1

2. 4

3. -5

4. 0

5. -2

6. -5

7. 1

8. 9

9. 7

10. -2

11. 1

12. -2

13. -2

14. 6

15. 50

16. 0

17. 7

18. -3

19. -1

20. 0

21. 33

22. 3

23. -23

24. -12

25. 1

Answers

Exercise

$$x = -2$$

Exercise

$$x = 3$$

Exercise

$$x = 1$$

Exercise

$$x = -9$$

Exercise

$$x = 3$$

Exercise

$$x = 5$$

6: RATIOS AND PROPORTIONS

6.1: Ratios

6.2: Proportions

6.1: Ratios

We use *ratios* to compare two numeric quantities or quantities with the same units.

Ratio

A *ratio* is the quotient of two numerical quantities or two quantities with the same physical units.

For example, ancient Greek geometers believed that the most pleasing rectangle to the eye had length and width such that the ratio of length to width was a specific number, called the *Golden Ratio*, approximately equal to 1.6180339887.... Architects to this day use this ratio in their designs.

There are a number of equivalent ways of expressing ratios, three of which we will use in this text: fraction notation, “to” notation, and “colon” notation.

- $3/4$ is a ratio, read as “the ratio of 3 to 4.”
- 3 to 4 is a ratio, read as “the ratio of 3 to 4.”
- 3:4 is a ratio, read as “the ratio of 3 to 4.”

Example 1

Express each of the following ratios as a fraction reduced to lowest terms: (a) 36 to 24, and (b) 0.12 : 0.18.

Solution

(a) To express the ratio “36 to 24” as a fraction, place 36 over 24 and reduce.

$$\begin{aligned}\frac{36}{24} &= \frac{3 \cdot \cancel{12}}{2 \cdot \cancel{12}} && \text{Factor.} \\ &= \frac{3 \cdot \cancel{12}}{2 \cdot \cancel{12}} && \text{Cancel common factor.} \\ &= \frac{3}{2}\end{aligned}$$

Thus, the ratio 36 to 24 equals $3/2$.

(b) To express the ratio “0.12:0.18” as a fraction, place 0.12 over 0.18 and reduce.

$$\begin{aligned}\frac{0.12}{0.18} &= \frac{(0.12)(100)}{(0.18)(100)} && \text{Multiply numerator and denominator by 100.} \\ &= \frac{12}{18} && \text{Move each decimal 2 places right.} \\ &= \frac{2 \cdot \cancel{6}}{3 \cdot \cancel{6}} && \text{Factor.} \\ &= \frac{2 \cdot \cancel{6}}{3 \cdot \cancel{6}} && \text{Cancel.} \\ &= \frac{2}{3}\end{aligned}$$

Thus, the ratio 0.12:0.18 equals $2/3$.

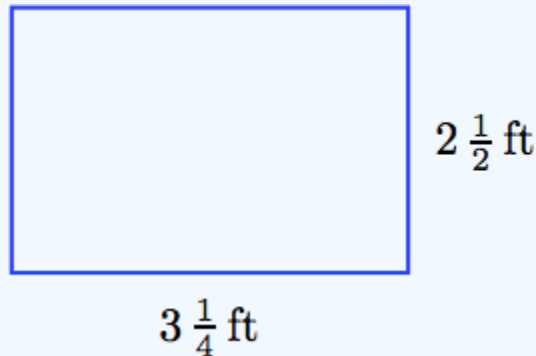
Exercise

Express $0.12 : 0.3$ as a fraction reduced to lowest terms.

Answer available at the end of this section.

Example 2

For the rectangle that follows, express the ratio of length to width as a fraction reduced to lowest terms.



Solution.

The ratio length to width can be expressed as a fraction and reduced as follows.

$$\begin{aligned} \frac{\text{length}}{\text{width}} &= \frac{3 \frac{1}{4} \text{ ft}}{2 \frac{1}{2} \text{ ft}} && \text{Length to width as a fraction.} \\ &= \frac{3 \frac{1}{4} \cancel{\text{ft}}}{2 \frac{1}{2} \cancel{\text{ft}}} && \text{Cancel common units.} \\ &= \frac{\frac{13}{4}}{\frac{5}{2}} && \text{Mixed to improper fractions.} \end{aligned}$$

Invert and multiply, factor, and cancel common factors.

$$\begin{aligned} &= \frac{13}{4} \cdot \frac{2}{5} && \text{Invert and multiply.} \\ &= \frac{26}{20} && \text{Multiply numerators and denominators.} \\ &= \frac{13 \cdot 2}{10 \cdot 2} && \text{Factor numerator and denominator.} \\ &= \frac{13 \cdot \cancel{2}}{10 \cdot \cancel{2}} && \text{Cancel common factors.} \\ &= \frac{13}{10} \end{aligned}$$

Hence, the ratio length to width is $13/10$.

Exercise

A rectangle has length $8 \frac{1}{4}$ inches and width $3 \frac{1}{2}$ inches. Express the ratio of length to width as a fraction reduced to lowest terms.

Answer available at the end of this section.

Rates

We now introduce the concept of *rate*, a special type of ratio.

Rate

A *rate* is a quotient of two measurements with different units.

The physical interpretation of a rate in terms of its units is an important skill.

Example 3

An automobile travels 224 miles on 12 gallons of gasoline. Express the ratio distance traveled to gas consumption as a fraction reduced to lowest terms. Write a short sentence explaining the physical significance of your solution. Include units in your description.

Solution

Place miles traveled over gallons of gasoline consumed and reduce.

$$\begin{aligned}\frac{224 \text{ mi}}{12 \text{ gal}} &= \frac{56 \cdot \cancel{4} \text{ mi}}{3 \cdot \cancel{4} \text{ gal}} && \text{Factor.} \\ &= \frac{56 \cdot \cancel{4} \text{ mi}}{3 \cdot \cancel{4} \text{ gal}} && \text{Cancel common factor.} \\ &= \frac{56 \text{ mi}}{3 \text{ gal}}\end{aligned}$$

Thus, the rate is 56 miles to 3 gallons of gasoline. In plain-speak, this means that the automobile travels 56 miles on 3 gallons of gasoline.

Exercise

Lanny travels 180 kilometers on 14 liters of gasoline. Express the ratio distance traveled to gas consumption as a fraction reduced to lowest terms.

Answer available at the end of this section.

Unit Rates

When making comparisons, it is helpful to have a rate in a form where the denominator is 1. Such rates are given a special name.

Unit Rate

A *unit rate* is a rate whose denominator is 1.

Example 4

Herman drives 120 miles in 4 hours. Find his average rate of speed.

Solution

Place the distance traveled over the time it takes to drive that distance.

$$\frac{120 \text{ miles}}{4 \text{ hours}} = \frac{30 \text{ miles}}{1 \text{ hour}} \quad \text{Divide: } 120/4 = 30.$$

$$= 30 \text{ miles/hour}$$

Hence, Herman's average rate of speed is 30 miles per hour.

Exercise

Jacob drives 120 kilometers in 3 hours. Find his average rate of speed.

Answer available at the end of this section.

Example 5

Aditya works 8.5 hours and receives \$95 for his efforts. What is his hourly salary rate?

Solution

Let's place money earned over hours worked to get the following rate:

$$\frac{95 \text{ dollars}}{8.5 \text{ hours}}$$

We will get a much better idea of Aditya's salary rate if we express the rate with a denominator of 1. To do so, divide. Push the decimal in the divisor to the far right, then move the decimal an equal number of places in the dividend. As we are dealing with dollars and cents, we will round our answer to the nearest hundredth.

$$\begin{array}{r} 11.176 \\ 85 \overline{)950.000} \\ \underline{85} \\ 100 \\ \underline{85} \\ 150 \\ \underline{85} \\ 650 \\ \underline{595} \\ 550 \\ \underline{510} \\ 40 \end{array}$$

Test digit

11.1 7 6

Rounding digit

Because the test digit is greater than or equal to 5, we add 1 to the rounding digit and truncate; i.e., $95/8.5 \approx 11.18$. Hence,

$$\frac{95 \text{ dollars}}{8.5 \text{ hours}} = \frac{11.18 \text{ dollars}}{1 \text{ hour}} \quad \text{Divide: } 95/8.5 \approx 11.18.$$

$$= 11.18 \text{ dollars/hour.}$$

That is, his salary rate is 11.18 dollars per hour.

Exercise

Frannie works 5.5 hours and receives \$120 for her efforts. What is her hourly salary rate? Round your answer to the nearest penny.

Answer available at the end of this section.

Example 6

One automobile travels 422 miles on 15 gallons of gasoline. A second automobile travels 354 miles on 13 gallons of gasoline. Which automobile gets the better gas mileage?

Solution

Decimal division (rounded to the nearest tenth) reveals the better gas mileage.

In the case of the first automobile, we get the following rate:

$$\frac{422 \text{ mi}}{15 \text{ gal}}$$

Divide.

$$\begin{array}{r} 28.13 \\ 15 \overline{)422.00} \\ \underline{30} \\ 122 \\ \underline{120} \\ 20 \\ \underline{15} \\ 50 \\ \underline{45} \\ 5 \end{array}$$

To the nearest tenth, 28.1.

In the case of the first automobile, the mileage rate is 28.1 mi/1 gal, which can be read “28.1 miles per gallon.” In the case of the second automobile, the mileage rate is 27.2 mi/1 gal, which can be read “27.2 miles per gallon.” Therefore, the first automobile gets the better gas mileage.

Exercise

Alicia works 8 hours and makes \$100. Connie works 10 hours and makes \$122. Which woman works at the larger hourly rate?

Answer available at the end of this section.

Exercises

Suggested exercises for PrePALS: 1, 3, 7, 13, 15, 19, 21, 25, 27, 31, 35, 37

In Exercises 1-24, express the given ratio as a fraction reduced to lowest terms.

1. $0.14 : 0.44$

2. $0.74 : 0.2$

3. $0.05 : 0.75$

4. $0.78 : 0.4$

5. $0.1 : 0.95$

6. $0.93 : 0.39$

7. $2\frac{2}{9} : 1\frac{1}{3}$

8. $3\frac{2}{3} : 2\frac{4}{9}$

9. $0.36 : 0.6$

10. $0.58 : 0.42$

11. $15 : 21$

12. $77 : 121$

13. $2\frac{8}{9} : 2\frac{2}{3}$

14. $1\frac{2}{3} : 3\frac{8}{9}$

15. $3\frac{8}{9} : 2\frac{1}{3}$

16. $1\frac{5}{9} : 1\frac{1}{3}$

17. $2\frac{5}{8} : 1\frac{3}{4}$

18. $2\frac{4}{9} : 1\frac{1}{3}$

19. $10 : 35$

20. $132 : 84$

21. 9 : 33

22. 35 : 10

23. 27 : 99

24. 12 : 28

25. One automobile travels 271.8 miles on 10.1 gallons of gasoline. A second automobile travels 257.9 miles on 11.1 gallons of gasoline. Which automobile gets the better gas mileage?

26. One automobile travels 202.9 miles on 13.9 gallons of gasoline. A second automobile travels 221.6 miles on 11.8 gallons of gasoline. Which automobile gets the better gas mileage?

27. Todd is paid 183 dollars for 8.25 hours work. What is his hourly salary rate, rounded to the nearest penny?

28. David is paid 105 dollars for 8.5 hours work. What is his hourly salary rate, rounded to the nearest penny?

29. An automobile travels 140 miles in 4 hours. Find the average rate of speed.

30. An automobile travels 120 miles in 5 hours. Find the average rate of speed.

31. Judah is paid 187 dollars for 8 hours work. What is his hourly salary rate, rounded to the nearest penny?

32. Judah is paid 181 dollars for 8.75 hours work. What is his hourly salary rate, rounded to the nearest penny?

33. One automobile travels 234.2 miles on 10.8 gallons of gasoline. A second automobile travels 270.5 miles on 10.8 gallons of gasoline. Which automobile gets the better gas mileage?

34. One automobile travels 297.6 miles on 10.7 gallons of gasoline. A second automobile travels 298.1 miles on 12.6 gallons of gasoline. Which automobile gets the better gas mileage?

35. An automobile travels 180 miles in 5 hours. Find the average rate of speed.

36. An automobile travels 220 miles in 5 hours. Find the average rate of speed.

37. **Antarctic trek.** Seven women on a 562-mile Antarctic ski trek reached the South Pole 38 days after they began their adventure. What was the ladies' average rate of speed per day? Round your result to the nearest tenth of a mile. *Associated Press-Times-Standard 12/31/09 After 562-mile ski trek, seven women reach the South Pole.*

Answers

1. $\frac{7}{22}$

3. $\frac{1}{15}$

5. $\frac{2}{19}$

7. $\frac{5}{3}$

9. $\frac{3}{5}$

11. $\frac{5}{7}$

13. $\frac{13}{12}$

15. $\frac{5}{3}$

17. $\frac{3}{2}$

19. $\frac{2}{7}$

21. $\frac{3}{11}$

23. $\frac{3}{11}$

25. The first automobile has the better mileage per gallon.

27. 22.18 dollars/hr

29. 35 mi/hr

31. 23.38 dollars/hr

33. The second automobile has the better mileage per gallon.

35. 36 mi/hr

37. 14.8 miles per day

Answers

Exercise

2/5

Exercise

33/14

Exercise

90/7 kilometers per litre

Exercise

40 kilometers per hour

Exercise

\$21.82 per hour

Exercise

Alicia

6.2: Proportions

In Section 6.1, we introduced the concepts of ratio and rate. In this section, we equate these ratios in a construct called a *proportion*.

Proportions

A *proportion* is a statement that equates two ratios or rates.

For example, each of the equations

$$\frac{1}{3} = \frac{2}{6}, \quad \frac{15 \text{ miles}}{2 \text{ hours}} = \frac{30 \text{ miles}}{4 \text{ hours}}, \quad \text{and} \quad \frac{a}{b} = \frac{c}{d}$$

compare two ratios or rates and is a proportion.

The proportion

$$\frac{1}{3} = \frac{2}{6}$$

is read “one is to three as two is to six.” The four numbers that make up this proportion are called the *terms* of the proportion and are ordered in a natural manner.

$$\begin{array}{lcl} \text{First term} & \longrightarrow & 1 \\ \text{Second term} & \longrightarrow & 3 \end{array} = \begin{array}{lcl} 2 & \longleftarrow & \text{Third term} \\ 6 & \longleftarrow & \text{Fourth term} \end{array}$$

Extremes and Means

The first and fourth terms are called the *extremes* of the proportion. The second and third terms are called the *means* of the proportion.

In the proportion

$$\frac{a}{b} = \frac{c}{d},$$

the terms a and d are the extremes; the terms b and c are the means.

$$\frac{\boxed{a}}{b} = \frac{c}{\boxed{d}} \quad \text{extremes}$$

$$a = \frac{\boxed{c}}{\boxed{b}} \quad \text{means}$$

If we multiply both sides of the proportion by the common denominator,

$$bd \left(\frac{a}{b} \right) = bd \left(\frac{c}{d} \right)$$

then cancel,

$$\cancel{d} \left(\frac{a}{\cancel{b}} \right) = b \cancel{d} \left(\frac{c}{\cancel{d}} \right)$$

we get the following result.

$$ad = bc$$

This leads to the following observation.

Product of Extremes and Means

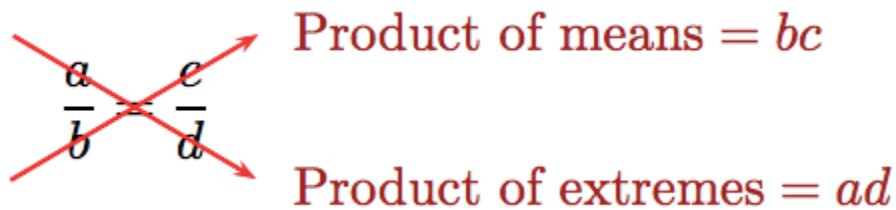
In the proportion

$$\frac{a}{b} = \frac{c}{d}$$

the product of the means equals the product of the extremes. That is,

$$ad = bc.$$

We can get an equivalent result using a technique called *cross multiplication*.



Product of means = bc

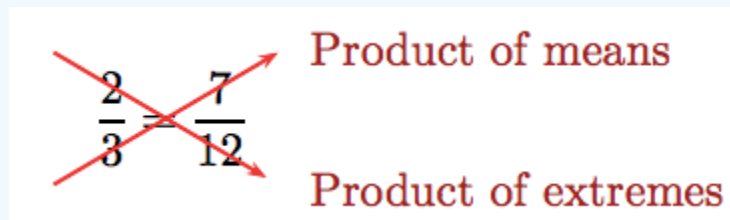
Product of extremes = ad

Example 1

Which of the following is a valid proportion: (a) $\frac{2}{3} = \frac{7}{12}$, or (b) $\frac{4}{9} = \frac{12}{27}$.

Solution

(a) Cross multiply



Product of means

Product of extremes

to get

$$24 = 21.$$

Hence, the product of the extremes does not equal the product of the means, so $2/3=7/12$ is **not** a valid proportion.

(b) Cross multiply

Product of means

Product of extremes

to get

$$108 = 108.$$

Hence, the product of the extremes equals the product of the means, so $4/9 = 12/27$ **is** a valid proportion.

Exercise

Is the following a valid proportion? $\frac{4}{3} = \frac{16}{11}$

Answer available at the end of this section.

Solving Proportions

We already have all the tools needed to solve proportions. Let's begin with the first example.

Example 2

Solve the proportion for x : $\frac{3}{4} = \frac{x}{12}$.

Solution

Cross multiply, then solve the resulting equation.

$$\begin{aligned} \frac{3}{4} &= \frac{x}{12} && \text{Original proportion.} \\ 4 \cdot x &= 3 \cdot 12 && \text{Products of means and extremes are equal.} \\ 4x &= 36 && \text{Simplify.} \\ \frac{4x}{4} &= \frac{36}{4} && \text{Divide both sides by 4.} \\ x &= 9 && \text{Simplify.} \end{aligned}$$

Check

Substitute 9 for x into the original proportion and check.

$$\begin{aligned} \frac{3}{4} &= \frac{x}{12} && \text{Original proportion.} \\ \frac{3}{4} &= \frac{9}{12} && \text{Substitute 9 for } x. \end{aligned}$$

Cross multiply.

$$\frac{3}{4} = \frac{9}{12}$$

Product of means = 36

Product of extremes = 36

Thus, the solution 9 checks.

Exercise

Solve the proportion for n : $\frac{2}{3} = \frac{n}{9}$

Answer available at the end of this section.

Example 3

Solve the proportion for n : $\frac{3}{2} = \frac{24}{n}$.

Solution

Cross multiply, then solve the resulting equation.

$$\begin{aligned} \frac{3}{2} &= \frac{24}{n} && \text{Original proportion.} \\ 3 \cdot n &= 2 \cdot 24 && \text{Products of means and extremes are equal.} \\ 3n &= 48 && \text{Simplify.} \\ \frac{3n}{3} &= \frac{48}{3} && \text{Divide both sides by 3.} \\ n &= 16 && \text{Simplify.} \end{aligned}$$

Check

Substitute 16 for n into the original proportion and check.

$$\begin{aligned} \frac{3}{2} &= \frac{24}{n} && \text{Original proportion.} \\ \frac{3}{2} &= \frac{24}{16} && \text{Substitute 16 for } n. \end{aligned}$$

Cross multiply.

$$\frac{3}{2} = \frac{24}{16}$$

Product of means = 48

Product of extremes = 48

Thus, the solution 16 checks.

Exercise

Solve the proportion for m : $\frac{9}{6} = \frac{m}{4}$

Answer available at the end of this section.

Example 4

Solve the proportion for x : $\frac{2x+1}{15} = \frac{1}{3}$.

Solution

Cross multiply, then solve the resulting equation.

$$\begin{aligned}\frac{2x+1}{15} &= \frac{1}{3} && \text{Original proportion.} \\ 3(2x+1) &= 15(1) && \text{Products of means and extremes are equal.} \\ 6x+3 &= 15 && \text{On the left, distribute.} \\ 6x+3-3 &= 15-3 && \text{On the right, multiply.} \\ 6x &= 12 && \text{Subtract 3 from both sides.} \\ \frac{6x}{6} &= \frac{12}{6} && \text{Simplify.} \\ x &= 2 && \text{Divide both sides by 6.} \\ &&& \text{Simplify both sides.}\end{aligned}$$

Check

We'll leave it to our readers to check this solution.

Exercise

Solve the proportion for y : $\frac{6+2y}{18} = \frac{8}{9}$

Answer available at the end of this section.

Applications

A number of practical applications involve solving a proportion.

Example 5

If 5 oranges cost \$1.15, what will be the cost for 15 oranges (assuming an equal rate)?

Solution

Let x represent the cost for 15 oranges. Assuming the rate for 5 oranges at \$1.15 equals the rate for 15 oranges at an unknown cost x , we set up the following proportion.

$$\frac{5}{1.15} = \frac{15}{x}$$

Cross multiply

$$\frac{5}{1.15} = \frac{15}{x}$$

Product of means = (1.15)(15)

Product of extremes = 5x

to get

$$5x = 17.25.$$

Solve for x.

$$\frac{5x}{5} = \frac{17.25}{5}$$

$$x = 3.45$$

Thus, 15 oranges cost \$3.45.

Exercise

If 7 apples cost \$3.15, how much will 10 apples cost (assuming an equal rate)?

Answer available at the end of this section.

Checking Units is Extremely Important

When setting up a proportion, check to make sure that both numerators have the same units and both denominators have the same units.

For example, in Example 5, both numerators have “oranges” as units and both denominators have “dollars” as units.

$$\begin{array}{lcl} \text{Oranges} \longrightarrow & \frac{5}{1.15} = \frac{15}{x} & \longleftarrow \text{Oranges} \\ \text{Dollars} \longrightarrow & & \longleftarrow \text{Dollars} \end{array}$$

This proportion is set up correctly, because both numerators have the same units and both denominators have the same units. On the other hand, if we had set the proportion up incorrectly as follows,

$$\begin{array}{lcl} \text{Oranges} \longrightarrow & \frac{5}{1.15} = \frac{x}{15} & \longleftarrow \text{Dollars} \\ \text{Dollars} \longrightarrow & & \longleftarrow \text{Oranges} \end{array}$$

a quick check of the units reveals the error; i.e., the numerators have different units and the denominators have different units. Checking units helps us avoid errors!

Example 6

Dylan and David are planning a backpacking trip in Yosemite National Park. On their map, the legend indicates that 1.2 centimeters represents 2 miles. How long is their trip if the route measures 10.6 centimeters on the map? Round your answer to the nearest tenth of a mile.

Solution

Let's set up the proportion with units.

$$\frac{1.2 \text{ cm}}{2 \text{ mi}} = \frac{10.6 \text{ cm}}{x \text{ mi}}$$

Note how including the units aids in the setup of the proportion. Now, let's drop the units and solve for x .

$$\begin{aligned} \frac{1.2}{2} &= \frac{10.6}{x} && \text{Original proportion.} \\ 1.2x &= (2)(10.6) && \text{Cross multiply.} \\ 1.2x &= 21.2 && \text{Simplify right-side hand.} \\ \frac{1.2x}{1.2} &= \frac{21.2}{1.2} && \text{Divide both sides by 1.2.} \\ x &\approx 17.66 && \text{On the right: Divide.} \end{aligned}$$

We carried the division on the right one decimal place past the tenths place. The rounding digit is a 6 and the following test digit is a 6. Add 1 to the rounding digit and truncate.

To the nearest tenth of a mile, the backpacking route is approximately 17.7 miles.

Exercise

Eloise and Susannah are planning a trip to Sequoia National Park. On their map, 3 inches represents 50 miles. How long is their trip if the route measures $4\frac{1}{2}$ inches on the map?

Answer available at the end of this section.

Example 7

A recipe making 2 dozen cookies requires $1\frac{3}{4}$ cups of flour, among other ingredients. If the baker wishes to make twice that number of cookies, how much flour is required?

Solution

Twice 2 dozen is 4 dozen cookies. Let x represent the amount of flour needed for 4 dozen cookies. Assuming an equal rate for 2 dozen cookies (2 dozen requires $1\frac{3}{4}$ cups of flour), we set up the following proportion. Again, using units helps us craft the correct proportion.

$$\frac{2 \text{ dozen}}{1\frac{3}{4} \text{ cups}} = \frac{4 \text{ dozen}}{x \text{ cups}}$$

Note how including the units aids in the setup of the proportion. Now, let's drop the units and solve for x .

$$\frac{2}{1\frac{3}{4}} = \frac{4}{x} \quad \text{Original proportion.}$$

$$2x = 1\frac{3}{4} \cdot 4 \quad \text{Cross multiply.}$$

$$2x = \frac{7}{4} \cdot 4 \quad \text{Change to improper fraction.}$$

$$2x = 7 \quad \text{Multiply.}$$

Divide both sides of the equation by 2 and finish.

$$\frac{2x}{2} = \frac{7}{2} \quad \text{Divide both sides by 2.}$$

$$x = \frac{7}{2}$$

Change the improper fraction to a mixed fraction. Thus, it will take $3\frac{1}{2}$ cups of flour to make 4 dozen cookies.

Exercise

Dough for 3 pizzas requires $8\frac{1}{2}$ cups of flour. If the baker wishes to make 9 pizzas, how many cups of flour are required?

Answer available at the end of this section.

Exercises

Suggested exercises for PrePALS: 1, 3, 13, 15, 21, 33, 37, 39, 47, 49, 51, 53

In Exercises 1-12, which of the following is a true proportion?

1. $\frac{9}{7} = \frac{27}{21}$, $\frac{4}{3} = \frac{9}{7}$, $\frac{7}{2} = \frac{8}{9}$, $\frac{4}{8} = \frac{9}{6}$

2. $\frac{6}{7} = \frac{18}{21}$, $\frac{2}{3} = \frac{8}{6}$, $\frac{4}{3} = \frac{3}{2}$, $\frac{8}{9} = \frac{3}{8}$

3. $\frac{7}{6} = \frac{28}{24}$, $\frac{5}{6} = \frac{5}{4}$, $\frac{9}{5} = \frac{7}{3}$, $\frac{9}{2} = \frac{8}{9}$

4. $\frac{7}{6} = \frac{2}{8}$, $\frac{4}{5} = \frac{5}{7}$, $\frac{3}{4} = \frac{15}{20}$, $\frac{8}{4} = \frac{8}{7}$

5. $\frac{6}{5} = \frac{24}{20}$, $\frac{7}{3} = \frac{2}{4}$, $\frac{2}{4} = \frac{2}{6}$, $\frac{5}{2} = \frac{2}{8}$

6. $\frac{9}{8} = \frac{4}{3}$, $\frac{5}{7} = \frac{10}{14}$, $\frac{8}{6} = \frac{5}{4}$, $\frac{8}{5} = \frac{2}{6}$

7. $\frac{3}{5} = \frac{2}{8}$, $\frac{3}{7} = \frac{6}{14}$, $\frac{5}{6} = \frac{2}{4}$, $\frac{7}{4} = \frac{5}{9}$

8. $\frac{7}{3} = \frac{7}{6}$, $\frac{4}{7} = \frac{8}{14}$, $\frac{5}{3} = \frac{7}{8}$, $\frac{5}{7} = \frac{6}{9}$

9. $\frac{5}{4} = \frac{25}{20}$, $\frac{9}{3} = \frac{9}{6}$, $\frac{7}{4} = \frac{3}{6}$, $\frac{3}{5} = \frac{9}{4}$

10. $\frac{7}{6} = \frac{6}{9}$, $\frac{7}{3} = \frac{2}{5}$, $\frac{6}{7} = \frac{30}{35}$, $\frac{4}{7} = \frac{2}{8}$

11. $\frac{9}{7} = \frac{4}{3}$, $\frac{9}{4} = \frac{7}{9}$, $\frac{3}{5} = \frac{6}{10}$, $\frac{3}{9} = \frac{9}{5}$

$$12. \frac{4}{3} = \frac{8}{7}, \quad \frac{2}{6} = \frac{5}{8}, \quad \frac{7}{2} = \frac{3}{6}, \quad \frac{9}{4} = \frac{36}{16}$$

In Exercises 13-36, solve the given proportion.

$$13. \frac{17}{3} = \frac{x}{18}$$

$$14. \frac{16}{5} = \frac{x}{20}$$

$$15. \frac{6x+10}{6} = \frac{11}{3}$$

$$16. \frac{4x+8}{12} = \frac{5}{3}$$

$$17. \frac{17}{9} = \frac{x}{18}$$

$$18. \frac{8}{9} = \frac{x}{18}$$

$$19. \frac{11}{2} = \frac{x}{8}$$

$$20. \frac{11}{4} = \frac{x}{8}$$

$$21. \frac{7x+15}{15} = \frac{10}{3}$$

$$22. \frac{7x+3}{8} = \frac{5}{4}$$

$$23. \frac{11}{2} = \frac{x}{10}$$

$$24. \frac{19}{6} = \frac{x}{18}$$

$$25. \frac{5x+8}{12} = \frac{2}{3}$$

$$26. \frac{3x+12}{6} = \frac{3}{2}$$

$$27. \frac{2}{15} = \frac{24}{x}$$

$$28. \frac{7}{8} = \frac{14}{x}$$

$$29. \frac{3}{16} = \frac{6}{x}$$

$$30. \frac{4}{21} = \frac{12}{x}$$

$$31. \frac{5}{22} = \frac{20}{x}$$

$$32. \frac{3}{22} = \frac{21}{x}$$

$$33. \frac{2x+10}{6} = \frac{14}{3}$$

$$34. \frac{2x+9}{9} = \frac{13}{3}$$

$$35. \frac{7}{2} = \frac{21}{x}$$

$$36. \frac{2}{15} = \frac{18}{x}$$

37. If 13 dog bones cost \$1.97, what will be the cost for 7 dog bones (assuming an equal rate)? Round your answer to the nearest penny.

38. If 2 watermelons cost \$3.89, what will be the cost for 11 watermelons (assuming an equal rate)? Round your answer to the nearest penny.

39. If 7 bananas cost \$2.55, what will be the cost for 14 bananas (assuming an equal rate)? Round your answer to the nearest penny.
40. If 2 apples cost \$2.05, what will be the cost for 8 apples (assuming an equal rate)? Round your answer to the nearest penny.
41. If 13 oranges cost \$3.61, what will be the cost for 11 oranges (assuming an equal rate)? Round your answer to the nearest penny.
42. If 3 watermelons cost \$1.05, what will be the cost for 9 watermelons (assuming an equal rate)? Round your answer to the nearest penny.
43. If 3 dog bones cost \$1.23, what will be the cost for 13 dog bones (assuming an equal rate)? Round your answer to the nearest penny.
44. If 3 watermelons cost \$4.41, what will be the cost for 7 watermelons (assuming an equal rate)? Round your answer to the nearest penny.
45. If 3 apples cost \$3.24, what will be the cost for 13 apples (assuming an equal rate)? Round your answer to the nearest penny.
46. If 6 apples cost \$3.43, what will be the cost for 7 apples (assuming an equal rate)? Round your answer to the nearest penny.
47. If 4 dog bones cost \$1.03, what will be the cost for 8 dog bones (assuming an equal rate)? Round your answer to the nearest penny.
48. If 4 oranges cost \$4.28, what will be the cost for 3 oranges (assuming an equal rate)? Round your answer to the nearest penny.

49. **Two rolls.** In Haiti, two flat rolls cost 5 gourdes, about 12 cents. How many cents would 20 rolls cost? *Associated Press-Times-Standard 02/18/10 Haiti's earthquake camps turning into shanty towns.*

50. **Turbines.** As proposed, the Shell Wind Energy project consists of 25 ridge-top turbines that can generate up to 50 megawatts, or enough to supply electricity to about 1, 000 homes. Estimate the number of ridge-top turbines that would be needed to supply electricity to 70, 000 homes, the approximate number of properties in Humboldt County, CA. *John Driscoll Times-Standard 12/24/09 Wind power project goes under analysis.*

51. **Dumptrucks.** U.S. Highway 199 had a landslide where as much as 3, 000 cubic yards of material fell on the road, reportedly requiring about 200 large dumptrucks to remove. Only a week earlier, 40, 000 cubic yards of material fell on Highway 96. Estimate the number of dumptrucks needed for that slide rounded to the nearest whole number. *Associated Press-Times-Standard 03/09/10 Another highway closed by slide.*

52. **Timber sales.** Alaska's 26, 000 square mile Tongass National Forest plan allows for timber sales of up to 267 million board-feet per year – enough for nearly 27, 000 two-bedroom homes, but demand for timber is far short of that. Less than 25 million board-feet was logged in the forest in 2009. Forest Service officials have said they hope to increase logging in the Tongass to about 100 million board-feet per year. *Associated Press-Times-Standard 02/18/10 Industry loses lawsuit over logging in Alaska.*

i) Estimate the number of 2-bedroom homes that 25 million board-feet of timber would build.

ii) How many 2-bedroom homes would 100 million board-feet of timber build?

53. **Costly spill.** In Australia, penalties on ships causing oil spills are approximately \$1.75 million Australian dollars, equivalent to about \$1.64 million US dollars. After an oil tanker was grounded onto a coral reef, Australian officials are considering raising the fine to \$10 million Australian dollars. What will the new fine be in US dollars? Round your answer to the nearest hundredth of a million dollars. *Associated Press-Times-Standard 04/13/10 Ship that leaked oil on Great Barrier Reef removed.*

Answers

1. $\frac{9}{7} = \frac{27}{21}$ is a proportion

3. $\frac{7}{6} = \frac{28}{24}$ is a proportion

5. $\frac{6}{5} = \frac{24}{20}$ is a proportion

7. $\frac{3}{7} = \frac{6}{14}$ is a proportion

9. $\frac{5}{4} = \frac{25}{20}$ is a proportion

11. $\frac{3}{5} = \frac{6}{10}$ is a proportion

13. 102

15. 2

17. 34

19. 44

21. 5

23. 55

25. 0

27. 180

29. 32

31. 88

33. 9

35. 6

37. \$1.06

39. \$5.10

41. \$3.05

43. \$5.33

45. \$14.04

47. \$2.06

49. \$0.48

51. 2, 667 loads

53. \$9.37 million US dollars

Answers

Exercise

No

Exercise

6

Exercise

6

Exercise

5

Exercise

\$4.50

Exercise

75 miles

Exercise $25\frac{1}{2}$ cups

7: PERCENT

7.1: Percent, Decimals, Fractions

7.2: Solving Basic Percent Problems

7.3: General Applications of Percent

7.1: Percent, Decimals, Fractions

In the square shown in Figure 7.1.1, a large square has been partitioned into ten rows of ten little squares in each row. In Figure 7.1, we've shaded 20 of 100 possible little squares, or 20% of the total number of little squares.

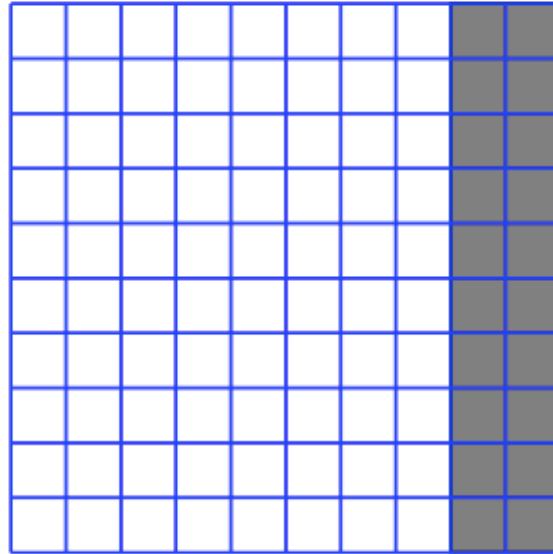


Figure 7.1.1: Shading 20 of 100 little squares, or 20% of the total number of little squares.

The Meaning of Percent

Percent means “parts per hundred.”

In Figure 7.1, 80 out of a possible 100 squares are left unshaded. Thus, 80% of the little squares are unshaded. If instead we shaded 35 out of the 100 squares, then 35% of the little squares would be shaded. If we shaded all of the little squares, then 100% of the little squares would be shaded (100 out of 100).

So, when you hear the word “percent,” think “parts per hundred.”

Changing a Percent to a Fraction

Based on the discussion above, it is fairly straightforward to change a percent to a fraction.

Percent to Fraction

To change a percent to a fraction, drop the percent sign and put the number over 100.

Example 1

Change 24% to a fraction.

Solution

Drop the percent symbol and put 24 over 100.

$$24\% = \frac{24}{100} \quad \text{Percent: Parts per hundred.}$$

$$= \frac{6}{25} \quad \text{Reduce.}$$

Hence, $24\% = 6/25$.

Exercise

Change 36% to a fraction reduced to lowest terms.

Answer available at the end of this section.

Example 2

Change $14\frac{2}{7}\%$ to a fraction.

Solution

Drop the percent symbol and put $14\frac{2}{7}$ over 100 .

$$14\frac{2}{7}\% = \frac{14\frac{2}{7}}{100} \quad \text{Percent: Parts per hundred.}$$

$$= \frac{\frac{100}{7}}{100} \quad \text{Mixed to improper fraction.}$$

$$= \frac{100}{7} \cdot \frac{1}{100} \quad \text{Invert and multiply.}$$

$$= \frac{\cancel{100}}{7} \cdot \frac{1}{\cancel{100}} \quad \text{Cancel.}$$

$$= \frac{1}{7}$$

Hence, $14\frac{2}{7}\% = 1/7$.

Exercise

Change $11\frac{1}{9}\%$ to a fraction reduced to lowest terms.

Answer available at the end of this section.

Example 3

Change 28.4% to a fraction.

Solution

Drop the percent symbol and put 28.4 over 100 .

$$\begin{aligned}
 28.4\% &= \frac{28.4}{100} && \text{Percent: Parts per hundred.} \\
 &= \frac{28.4 \cdot 10}{100 \cdot 10} && \text{Multiply numerator and denominator by 10.} \\
 &= \frac{284}{1000} && \text{Multiplying by 10 moves decimal point one place right.} \\
 &= \frac{71 \cdot 4}{250 \cdot 4} && \text{Factor.} \\
 &= \frac{71}{250} && \text{Cancel common factor.}
 \end{aligned}$$

Exercise

Change 87.5% to a fraction reduced to lowest terms.

Answer available at the end of this section.

Changing a Percent to a Decimal

To change a percent to a decimal, we need only remember that percent means “parts per hundred.”

Example 4

Change 23.25% to a decimal.

Solution

Drop the percent symbol and put 23.25 over 100.

$$\begin{aligned}
 23.25\% &= \frac{23.25}{100} && \text{Percent: Parts per hundred.} \\
 &= 0.2325 && \text{Dividing by 100 moves decimal point 2 places left.}
 \end{aligned}$$

Therefore, $23.25\% = 0.2325$.

Exercise

Change 2.4% to a decimal.

Answer available at the end of this section.

This last example motivates the following simple rule.

Changing a Percent to a Decimal

To change a percent to a decimal, drop the percent symbol and move the decimal point two places to the left.

Example 5

Change $5\frac{1}{2}\%$ to a decimal.

Solution

Note that $1/2=0.5$, then move the decimal 2 places to the left.

$$\begin{aligned}5\frac{1}{2}\% &= 5.5\% \quad 1/2 = 0.5. \\ &= 0.055 \quad \text{Drop} \\ &= 0.055 \quad \text{Move decimal point 2 places left.}\end{aligned}$$

Thus, $5\frac{1}{2}\% = 0.055$.

Exercise 7.1.1

Change $6\frac{3}{4}\%$ to a decimal.

Answer available at the end of this section.

Changing a Decimal to a Percent

Changing a decimal to a percent is the exact opposite of changing a percent to a decimal. In the latter case, we drop the percent symbol and move the decimal point 2 places to the left. The following rule does just the opposite.

Changing a Decimal to a Percent

To change a decimal to a percent, move the decimal point two places to the right and add a percent symbol.

Example 6

Change 0.0725 to a percent.

Solution

Move the decimal point two places to the right and add a percent symbol.

$$\begin{aligned}0.0725 &= 007.25\% \\ &= 7.25\%\end{aligned}$$

Exercise

Change 0.0375 to a percent.

Answer available at the end of this section.

Example 7

Change 1.025 to a percent.

Solution

Move the decimal point two places to the right and add a percent symbol.

$$\begin{aligned} 1.025 &= 102.5\% \\ &= 102.5\% \end{aligned}$$

Exercise

Change 0.525 to a percent.

Answer available at the end of this section.

Changing a Fraction to a Percent

One way to proceed is to first change the fraction to a decimal, then change the resulting decimal to a percent.

Fractions to Percents: Technique #1

To change a fraction to a percent, follow these steps:

1. Divide numerator by the denominator to change the fraction to a decimal.
2. Move the decimal point in the result two places to the right and append a percent symbol.

Example 8

Use Technique #1 to change $5/8$ to a percent.

Solution

Change $5/8$ to a decimal, then change the decimal to a percent.

To change $5/8$ to a decimal, divide 5 by 8. Since the denominator is a product of twos, the decimal should terminate.

$$\begin{array}{r} 0.625 \\ 8 \overline{)5.000} \\ \underline{48} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

To change 0.625 to a percent, move the decimal point 2 places to the right and append a percent symbol.

$$0.625 = 0.625 \times 100\% = 62.5\%$$

Exercise

Change $5/16$ to a percent.

Answer available at the end of this section.

A second technique is to create an equivalent fraction with a denominator of 100.

Fractions to Percents: Technique #2

To change a fraction to a percent, create an equivalent fraction with a denominator of 100.

Example 9

Use Technique #2 to change $5/8$ to a percent.

Solution

Create an equivalent fraction for $5/8$ with a denominator of 100.

$$\frac{5}{8} = \frac{x}{100}$$

Solve this proportion for x .

$$\begin{aligned} 8x &= 500 && \text{Cross multiply.} \\ \frac{8x}{8} &= \frac{500}{8} && \text{Divide both sides by 8.} \\ x &= \frac{125}{2} && \text{Reduce: Divide numerator and denominator by 4.} \\ x &= 62.5 && \text{Divide.} \end{aligned}$$

Thus,

$$\frac{5}{8} = \frac{62.5}{100} = 62.5\%.$$

Alternate Ending

We could also change $125/2$ to a mixed fraction; i.e., $125/2 = 62 \frac{1}{2}$. Then,

$$\frac{5}{8} = \frac{62\frac{1}{2}}{100} = 62\frac{1}{2}\%.$$

Same answer.

Exercise

Change $4/9$ to a percent.

Answer available at the end of this section.

Sometimes we will be content with an approximation.

Example 10

Change $4/13$ to a percent. Round your answer to the nearest tenth of a percent.

Solution

We will use Technique #1.

To change $4/13$ to a decimal, divide 4 by 13. Since the denominator has factors other than 2's and 5's, the decimal will repeat. However, we intend to round to the nearest tenth of a percent, so we will carry the division to four decimal places only. (*Four places are necessary because we will be moving the decimal point two places to the right.*)

$$\begin{array}{r} 0.625 \\ 8 \overline{)5.000} \\ \underline{4 8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

To change the decimal to a percent, move the decimal point two places to the right.

$$0.3076 \approx 0 \ 30.76\% \approx 30.76\%$$

To round to the nearest tenth of a percent, identify the rounding and test digits.

$$30.\boxed{7}\boxed{6}\%$$

Rounding digit Test digit

Because the test digit is greater than or equal to 5, add 1 to the rounding digit and truncate. Thus,

$$0.03076 \approx 30.8\%.$$

Exercise

Change $4/17$ to a percent. Round your answer to the nearest tenth of a percent.

Answer available at the end of this section.

Exercises

Convert the given percent to a fraction, and simplify the result.

1. $4\frac{7}{10}\%$

2. $7\frac{2}{9}\%$

3. 11.76%

4. 13.99%

5. 192%

6. 4.07%

Convert the given percent to a decimal.

7. 124%

8. 0.83%

9. 8%

10. 36.5%

Convert the given decimal to a percent.

11. 8.888

12. 1.681

13. 0.14

14. 0.07

15. 0.044

Convert the given fraction to a percent.

16. $\frac{1}{2}$

17. $\frac{8}{5}$

18. $\frac{9}{4}$

19. Convert $24/29$ to a percent, and round your answer to the nearest hundredth of a percent.

20. Convert $15/7$ to a percent, and round your answer to the nearest tenth of a percent.

21. Convert $7/24$ to a percent, and round your answer to the nearest hundredth of a percent.

22. Convert $8/3$ to a percent, and round your answer to the nearest tenth of a percent.

Answers

1. $\frac{47}{1000}$

2. $\frac{13}{180}$

3. $\frac{147}{1250}$

4. $\frac{1399}{10000}$

5. $\frac{48}{25}$

6. $\frac{407}{10000}$

7. 1.24

8. 0.0083

9. 0.08

10. 0.365

11. 888.8%

12. 168.1%

13. 14%

14. 7%

15. 4.4%

16. 50%

17. 160%

18. 225%

19. 82.76%

20. 214.3%

21. 29.17%

22. 266.7%

Answers

Exercise

9/25

Exercise

1/9

Exercise

7/8

Exercise

0.024

Exercise 7.1.1

0.0675

Exercise

3.75%

Exercise

52.5%

Exercise

31.35%

Exercise

$44\frac{4}{9}\%$

Exercise

23.5%

7.2: Solving Basic Percent Problems

There are three basic types of percent problems:

1. *Find a given percent of a given number.* For example, find 25% of 640.
2. *Find a percent given two numbers.* For example, 15 is what percent of 50?
3. *Find a number that is a given percent of another number.* For example, 10% of what number is 12?

Let's begin with the first of these types.

Find a Given Percent of a Given Number

Let's begin with our first example.

Example 1

What number is 25% of 640?

Solution

Let x represent the unknown number. Translate the words into an equation.

$$\begin{array}{ccccccc} \text{What number} & \text{is} & 25 & \text{of} & 640 \\ x & = & 25\% & \cdot & 640 \end{array}$$

Now, solve the equation for x .

$\cdot$ is only supported in math mode

Thus, 25% of 640 is 160.

Alternate Solution

We could also change 25% to a fraction.

$$\begin{aligned} x &= 25\% \cdot 640 && \text{Original equation.} \\ x &= \frac{1}{4} \cdot 640 && \text{Change 25} \\ x &= \frac{640}{4} && \text{Multiply numerators and denominators.} \\ x &= 160 && \text{Divide: } 640/4 = 160. \end{aligned}$$

Same answer.

Exercise

What number is 36% of 120?

Answer available at the end of this section.

Example 2

What is number $8\frac{1}{3}\%$ of 120?

Solution

Let x represent the unknown number. Translate the words into an equation.

$$\begin{array}{ccccc} \text{What number} & \text{is} & 8\frac{1}{3} & \text{of} & 120 \\ x & = & 8\frac{1}{3}\% & \cdot & 120 \end{array}$$

Now, solve the equation for x . Because

$$8\frac{1}{3}\% = 8.3\% = 0.08\overline{3},$$

working with decimals requires that we work with a repeating decimal. To do so, we would have to truncate the decimal representation of the percent at some place and satisfy ourselves with an approximate answer. Instead, let's change the percent to a fraction and seek an exact answer.

$$\begin{aligned} 8\frac{1}{3}\% &= \frac{8\frac{1}{3}}{100} && \text{Percent: Parts per hundred.} \\ &= \frac{\frac{25}{3}}{100} && \text{Mixed to improper fraction.} \\ &= \frac{25}{3} \cdot \frac{1}{100} && \text{Invert and multiply.} \\ &= \frac{25}{300} && \text{Multiply numerators and denominators.} \\ &= \frac{1}{12} && \text{Reduce: Divide numerator and denominator by 25.} \end{aligned}$$

Now we can solve our equation for x .

$$\begin{aligned} &= 8\frac{1}{3}\% \cdot 120 && \text{Original equation.} \\ x &= \frac{1}{12} \cdot 120 && 8\frac{1}{3}\% = 1/12. \\ x &= \frac{120}{12} && \text{Multiply numerators and denominators.} \\ x &= 10 && \text{Divide: } 120/12 = 10. \end{aligned}$$

Thus, $8\frac{1}{3}\%$ of 120 is 10.

Exercise

What number is $4\frac{1}{6}\%$ of 1,200?

Answer available at the end of this section.

Example 3

What number is $105\frac{1}{4}\%$ of 18.2?

Solution

Let x represent the unknown number. Translate the words into an equation.

$$\begin{array}{ccccccc} \text{What number} & \text{is} & 105 \frac{1}{4} & \text{of} & 18.2 \\ x & = & 105 \frac{1}{4} \% & \cdot & 18.2 \end{array}$$

In this case, the fraction terminates as $1/4=0.25$, so

$$105 \frac{1}{4} \% = 105.25 \quad (7.2.1)$$

Now we can solve our equation for x .

$$\begin{aligned} x &= 105 \frac{1}{4} \% \cdot 18.2 && \text{Original equation.} \\ x &= 1.0525 \cdot 18.2 && 5 \frac{1}{4} \% = 1.0525. \\ x &= 19.1555 && \text{Multiply.} \end{aligned}$$

Thus, $105 \frac{1}{4} \%$ of 18.2 is 19.1555.

Exercise

What number is $105 \frac{3}{4} \%$ of 222?

Answer available at the end of this section.

Find a Percent Given Two Numbers

Now we'll address our second item on the list at the beginning of the section.

Example 4

15 is what percent of 50?

Solution

Let x represent the unknown percent. Translate the words into an equation.

$$\begin{array}{ccccccc} 15 & \text{is} & \text{what percent} & \text{of} & 50 \\ 15 & = & x & \cdot & 50 \end{array}$$

The commutative property of multiplication allows us to change the order of multiplication on the right-hand side of this equation.

$$15 = 50x.$$

Now we can solve our equation for x .

$$\begin{aligned} 15 &= 50x && \text{Original equation.} \\ \frac{15}{50} &= \frac{50x}{50} && \text{Divide both sides by 50.} \\ \frac{15}{50} &= x && \text{Simplify right-hand side.} \\ x &= 0.30 && \text{Divide: } 15/50 = 0.30. \end{aligned}$$

But we must express our answer as a percent. To do this, move the decimal two places to the right and append a percent symbol.

$$0.30 = 0 \text{ } 30.\% = 30\%$$

Thus, 15 is 30% of 50.

Alternative Conversion

At the third step of the equation solution, we had

$$x = \frac{15}{50}.$$

We can convert this to an equivalent fraction with a denominator of 100.

$$x = \frac{15 \cdot 2}{50 \cdot 2} = \frac{30}{100}$$

Thus, $15/50 = 30/100 = 30\%$.

Exercise

14 is what percent of 25?

Answer available at the end of this section.

Example 5

10 is what percent of 80?

Solution

Let x represent the unknown percent. Translate the words into an equation.

$$\begin{array}{ccccccc} 10 & \text{is} & \text{what percent} & \text{of} & 80 \\ 10 & = & x & \cdot & 80 \end{array}$$

The commutative property of multiplication allows us to write the right-hand side as

$$10 = 80x.$$

Now we can solve our equation for x .

$$\begin{array}{ll} 10 = 80x & \text{Original equation.} \\ \frac{10}{80} = \frac{80x}{80} & \text{Divide both sides by 80.} \\ \frac{1}{8} = x & \text{Reduce: } 10/80 = 1/8. \\ 0.125 = x & \text{Divide: } 1/8 = 0.125. \end{array}$$

But we must express our answer as a percent. To do this, move the decimal two places to the right and append a percent symbol.

$$0.125 = 0 \text{ } 12.5\% = 12.5\%$$

Thus, 10 is 12.5% of 80.

Alternative Conversion

At the third step of the equation solution, we had

$$x = \frac{1}{8}.$$

We can convert this to an equivalent fraction with a denominator of 100 by setting up the proportion

$$\frac{1}{8} = \frac{n}{100}$$

Cross multiply and solve for n .

$$\begin{aligned} 8n &= 100 && \text{Cross multiply.} \\ \frac{8n}{8} &= \frac{100}{8} && \text{Divide both sides by 8.} \\ n &= \frac{25}{8} && \text{Reduce: Divide numerator and denominator by 4.} \\ n &= 12\frac{1}{2} && \text{Change } 25/2 \text{ to mixed fraction.} \end{aligned}$$

Hence,

$$\frac{1}{8} = \frac{12\frac{1}{2}}{100} = 12\frac{1}{2}\%.$$

Same answer.

Exercise

10 is what percent of 200?

Answer available at the end of this section.

Find a Number that is a Given Percent of Another Number

Let's address the third item on the list at the beginning of the section.

Example 6

10% of what number is 12?

Solution

Let x represent the unknown number. Translate the words into an equation.

$$\begin{array}{ccccccc} 10 & \text{of} & \text{what number} & \text{is} & 12 \\ 10\% & \cdot & x & = & 12 \end{array}$$

Change 10% to a fraction: $10\% = 10/100 = 1/10$.

$$\frac{1}{10}x = 12$$

Now we can solve our equation for x.

$$10 \left(\frac{1}{10}x \right) = 10(12) \quad \text{Multiply both sides by 10.}$$

$$x = 120 \quad \text{Simplify.}$$

Thus, 10% of 120 is 12.

Alternative Solution

We can also change 10% to a decimal: $10\% = 0.10$. Then our equation becomes

$$0.10x = 12$$

Now we can divide both sides of the equation by 0.10.

$$\frac{0.10x}{0.10} = \frac{12}{0.10} \quad \text{Divide both sides by 0.10.}$$

$$x = 120 \quad \text{Divide: } 12/0.10 = 120.$$

Same answer.

Exercise

20% of what number is 45?

Answer available at the end of this section.

Example 7

$11\frac{1}{9}\%$ of what number is 20?

Solution

Let x represent the unknown number. Translate the words into an equation.

$$11\frac{1}{9}\% \quad \text{of} \quad \text{what number} \quad \text{is} \quad 20$$

$$11\frac{1}{9}\% \cdot x = 20$$

Change $11\frac{1}{9}\%$ to a fraction.

$$11\frac{1}{9}\% \quad \text{Percent: Parts per hundred.}$$

$$= \frac{\frac{100}{9}}{100} \quad \text{Mixed to improper: } 11\frac{1}{9} = 100/9.$$

$$= \frac{100}{9} \cdot \frac{1}{100} \quad \text{Invert and multiply.}$$

$$= \frac{\cancel{100}}{9} \cdot \frac{1}{\cancel{100}} \quad \text{Cancel.}$$

$$= \frac{1}{9} \quad \text{Simplify.}$$

Replace $11\frac{1}{9}\%$ with $1/9$ in the equation and solve for x .

$$\begin{aligned}\frac{1}{9}x &= 20 & 11\frac{1}{9}\% &= 1/9/ \\ 9\left(\frac{1}{9}x\right) &= 9(20) & \text{Multiply both sides by 9.} \\ x &= 180\end{aligned}$$

Thus, $11\frac{1}{9}\%$ of 180 is 20.

Exercise

$12\frac{2}{3}\%$ of what number is 760?

Answer available at the end of this section.

Exercises

1. What number is 22.4% of 125?
2. 60% of what number is 90?
3. 162.5% of what number is 195?
4. 27 is what percent of 45?
5. 37.5% of what number is 57?
6. 50% of what number is 58?
7. 5.6 is what percent of 40?
8. What number is 18.4% of 125?
9. 30.8 is what percent of 40?
10. What number is 89.6% of 125?
11. 27 is what percent of 18?
12. $133\frac{1}{3}\%$ of what number is 80?
13. What number is $54\frac{1}{3}\%$ of 6?
14. $78\frac{1}{2}\%$ of what number is 7.85?

Answers

1. 28
2. 150
3. 120
4. 60

5. 152

6. 116

7. 14

8. 23

9. 77

10. 112

11. 150

12. 60

13. 3.26

14. 10

Answers

Exercise

43.2

Exercise

50

Exercise

234.765

Exercise

56%

Exercise

5%

Exercise

225

Exercise

6,000

7.3: General Applications of Percent

In this section we will look at an assortment of practical problems involving percent.

Example 1

Myrna notes that 20% of her class is absent. If the class has 45 students, how many students are absent?

Solution

Let n represent the number of students that are absent. Then we can translate the problem statement into words and symbols.

$$\begin{array}{ccccccc} \text{Number absent} & \text{is} & 20 & \text{of} & \text{total number of students in the class} \\ n & = & 20\% & \cdot & 45 \end{array}$$

Because $20\% = 0.20$,

$$\begin{aligned} n &= 0.20 \cdot 45 \\ n &= 9 \end{aligned}$$

Multiply: $0.20 \cdot 45 = 9$.

Therefore, 9 students are absent.

Exercise

Aaron notes that 15% of his class is absent. If the class has 80 students, how many students are absent?

Answer available at the end of this section.

Example 2

Misty answered 90% of the questions on her mathematics examination correctly. If Misty had 27 correct answers, how many questions were on the exam?

Solution

Let N represent the number of questions on the examination.

$$\begin{array}{ccccccc} \text{Number of correct answers} & \text{is} & 90 & \text{of} & \text{total number of questions} \\ 27 & = & 90\% & \cdot & N \end{array}$$

Because $90\% = 0.90$, this last equation can be written as

$$27 = 0.90N.$$

Solve for N .

$$\begin{aligned} \frac{27}{0.90} &= \frac{0.90N}{0.90} && \text{Divide both sides by } 0.90. \\ 30 &= N && \text{Divide: } 27/0.90 = 30. \end{aligned}$$

Hence, there were 30 questions on the examination.

Exercise

Erin answered 85% of the questions on her English examination correctly. If she had 34 correct answers, how many questions were on her exam?

Answer available at the end of this section.

Example 3

Misty answered 30 of 40 possible questions on her sociology examination correctly. What percent of the total number of questions did Misty mark correctly?

Solution

Let p represent the percent of the total number of questions marked correctly. Then we can translate the problem statement into words and symbols.

Number of correct answers	is	what percent	of	total number of questions
30	=	p	·	40

Because multiplication is commutative, we can write the last equation in the form

$$30 = 40p.$$

Solve for p .

$$\begin{aligned} \frac{30}{40} &= \frac{40p}{40} && \text{Divide both sides by 40.} \\ \frac{3}{4} &= p && \text{Reduce: } 30/40 = 3/4. \end{aligned}$$

We need to change $p = 3/4$ to a percent. There are two ways to do this:

- We can divide 3 by 4 to get

$$\begin{aligned} p &= \frac{3}{4} \\ &= 0.75 && \text{Divide: } 3/4 = 0.75. \\ &= 75\% && \text{Move decimal point 2 places right.} \end{aligned}$$

- We can create an equivalent fraction with a denominator of 100; i.e.,

$$\begin{aligned} p &= \frac{3}{4} \\ &= \frac{3 \cdot 25}{4 \cdot 25} && \text{Multiply numerator and denominator by 25.} \\ &= \frac{75}{100} && \text{Simplify numerator and denominator.} \\ &= 75\%. && \text{Percent means parts per hundred.} \end{aligned}$$

Either way, Misty got 75% of the questions on her sociology examination correct.

Exercise

Alphonso answered 19 of 25 questions on his biology test correctly. What percent of the questions did he mark correctly?

Answer available at the end of this section.

Example 4

35 millilitres of a 60 millilitre solution is hydrochloric acid. What percent of the solution is hydrochloric acid?

Solution

Let p represent the percent of the percent of the solution that is hydrochloric acid. Then we can translate the problem statement into words and symbols.

$$\begin{array}{ccccccc} \text{Amount of hydrochloric acid} & \text{is} & \text{what percent} & \text{of} & \text{the tota amount of solution} \\ 35 & = & p & \cdot & 60 \end{array}$$

Because multiplication is commutative, we can write the right-hand side of the last equation as follows.

$$35 = 60p$$

Now we can solve for p .

$$\begin{aligned} \frac{35}{60} &= \frac{60p}{60} && \text{Divide both sides by 60.} \\ \frac{7}{12} &= p && \text{Reduce: Divide numerator and denominator by 5.} \end{aligned}$$

Now we must change p to a percent. We can do this exactly by creating an equivalent fraction with a denominator of 100.

$$\frac{7}{12} = \frac{n}{100}$$

Solve for n .

$$\begin{aligned} 12n &= 700 && \text{Cross multiply.} \\ \frac{12n}{12} &= \frac{700}{12} && \text{Divide both sides by 12.} \\ n &= \frac{175}{3} && \text{Reduce: Divide numerator and denominator by 4.} \\ n &= 58\frac{1}{3} && \text{Change improper to mixed fraction.} \end{aligned}$$

Hence,

$$p = \frac{7}{12} = \frac{58\frac{1}{3}}{100} = 58\frac{1}{3}\%.$$

Thus, $58\frac{1}{3}\%$ of the solution is hydrochloric acid.

Approximate Solution

If all that is needed is an approximate answer, say correct to the nearest tenth of a percent, then we would take a different approach starting with the line from above that has

$$\frac{35}{60} = p.$$

We would divide 35 by 60 to get

$$p \approx 0.5833.$$

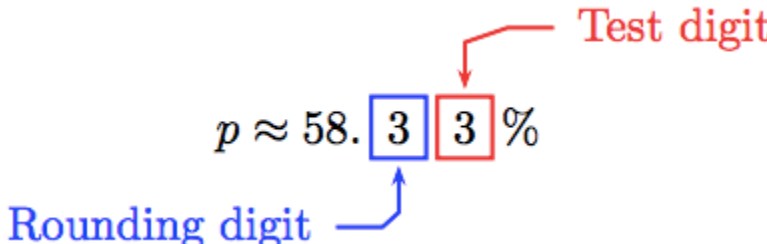
Move the decimal two places to the right and append a percent symbol.

$$p \approx 0.5833 \approx 0 \text{ } 58.33\% \approx 58.33\%.$$


Round to the nearest tenth of a percent.

$$p \approx 58. \boxed{3} \boxed{3} \%$$

Rounding digit Test digit



Because the test digit is less than 5, leave the rounding digit alone and truncate. Thus, correct to the nearest tenth of a percent,

$$p \approx 58.3\%.$$

Note that $p \approx 58.3\%$ is approximate, but $p = 58\frac{1}{3}\%$ is exact.

Exercise

25 millilitres of a 40 millilitre solution is sulfuric acid. What percent of the solution is sulfuric acid?

Answer available at the end of this section.

Exercises

- 31 millilitres of a 250 millilitre solution is sulphuric acid. What percent of the solution is sulphuric acid? Round your answer to the nearest tenth of a percent.
- A family has completed 186 miles of a planned 346 mile trip. Find the percentage of the planned trip already traveled. Round your answer to the nearest percent.
- Erin takes roll in her fifth grade class and finds that 19 out of 34 total students on her roster are present. Find the percentage of the class that is present, correct to the nearest percent.
- A family has traveled 114 miles of a planned trip. This is 37% of the total distance they must travel on the trip. Find, correct to the nearest mile, the total distance they will travel on their trip.
- Tami answered 70% of the questions on the physics examination correctly. If Tami had 98 correct answers, how many questions were on the exam?
- A state charges 8% sales tax on all sales. If the sales tax on a computer is \$20, find the sales price of the computer, correct to the nearest dollar.

7. Kenon earns 6% commission all his sales. If the sale of a computer earns him a \$37 commission, find the sales price of the computer, correct to the nearest dollar.
8. A 23% nitric acid solution contains 59 millilitres of nitric acid. How many total millilitres of solution are present? Round your answer to the nearest millilitre.
9. In a state, a television sold for \$428 is assessed a sales tax of \$45. Find the sales tax rate, rounded to the nearest tenth of a percent.
-
10. **Mars gravity.** The force of gravity on Mars is only 38% of the force of gravity on earth. If you weigh 150 pounds on earth, how much will you weigh on Mars?
11. **Seniors.** 13% of Humboldt County's population is age 65 and older, about 2% more than the state's average. If the population of Humboldt County is approximately 130,000, how many people in Humboldt County are age 65 and older? *Times-Standard 6/10/2009*
12. **Grow faster.** "Approximately 28 million pounds of antibiotics were fed to farm animals in the US during 2008. Thirteen percent of that was fed to healthy animals to make them grow faster." How many pounds of antibiotics were fed to healthy animals? *Associated Press-Times-Standard 12/29/09 Pressure rises to stop antibiotics in agriculture.*
13. **Water supply.** A new water desalination plant, the largest in the Western hemisphere, could come online by 2012 in Carlsbad, California, providing 50 million gallons of drinking water per day, or 10% of the supply for San Diego County. What is the total amount of drinking water supplied to San Diego County daily? *Associated Press-Times-Standard*
14. **Reduce.** Each year, Americans throw out an average of about 1,600 pounds of waste per person. Arcata, CA resident Michael Winkler only uses one trash bag every year – totaling at most 40 pounds. Find the percent of average annual waste per person Mr. Winkler throws out to a tenth of a percent. *Times-Standard Allison White 12/26/09 Waste not...*
-

Answers

1. 12.4
2. 54
3. 56
4. 308 mi
5. 140
6. \$250
7. \$617
8. 257 ml
9. 10.5
10. 57 pounds
11. 16,900
12. 3.84 million pounds
13. 500 million gallons
14. Mr. Winkler throws out 2.5% of the average American's waste.

Answers

Exercise

12

Exercise

40

Exercise

76%

Exercise

62.5%

8: STATISTICS

8.1: Measures of Central Tendency and Dispersion (Ungrouped Data)

8.2: Frequency Distributions (Grouped Data)

8.3: The Normal Distribution

8.1: Measures of Central Tendency and Dispersion (Ungrouped Data)

Learning Objectives

- Recognize, describe, and calculate the measures of the center of data.
- Recognize, describe, and calculate the measures of the spread of data.
- Use the Empirical rule to interpret the mean and standard deviation.

Measures of the Center of the Data

The "center" of a data set is also a way of describing the location. The two most widely used measures of the "center" of the data are the **mean** (average) and the **median**. To calculate the **mean** weight of 50 people, add the 50 weights together and divide by 50. To find the **median** weight of the 50 people, order the data and find the number that splits the data into two equal parts. The median is generally a better measure of the center when there are extreme values or outliers because it is not affected by the precise numerical values of the outliers. The mean is the most common measure of the center.

The letter used to represent the sample mean is an x with a bar over it (pronounced " x bar"): $\bar{x}$. The Greek letter μ (pronounced "mew") represents the **population mean**. One of the requirements for the **sample mean** to be a good estimate of the **population mean** is for the sample taken to be truly random.

You can quickly find the location of the median by using the expression

$$\frac{n + 1}{2} \quad (8.1.1)$$

The letter n is the total number of data values in the sample. If n is an odd number, the median is the middle value of the ordered data (ordered smallest to largest). If n is an even number, the median is equal to the average of the two middle values after the data has been ordered. For example, if the total number of data values is 97, then

$$\frac{n + 1}{2} = \frac{97 + 1}{2} = 49. \quad (8.1.2)$$

The median is the 49th value in the ordered data. If the total number of data values is 100, then

$$\frac{n + 1}{2} = \frac{100 + 1}{2} = 50.5. \quad (8.1.3)$$

The median occurs midway between the 50th and 51st values. The location of the median and the value of the median are **not** the same. The upper case letter M is often used to represent the median. The next example illustrates the location of the median and the value of the median.

Example 8.1.1

The following dataset is in order from smallest to largest:

3; 4; 8; 8; 10; 11; 12; 13; 14; 15

Calculate the mean and the median.

Answer

The calculation for the mean is:

$$\bar{x} = \frac{3 + 4 + 8(2) + 10 + 11 + 12 + 13 + 14 + 15}{10} = 9.8 \quad (8.1.4)$$

To find the median, M , first use the formula for the location. The location is:

$$\frac{n+1}{2} = \frac{10+1}{2} = 5.5 \quad (8.1.5)$$

Starting at the smallest value, the median is located between the 5th and 6th values (the 10 and 11):

3; 4; 8; 8; 10; 11; 12; 13; 14; 15

$$M = \frac{10 + 11}{2} = 10.5 \quad (8.1.6)$$

Exercise 8.1.1

The following dataset is ordered from smallest to largest. Calculate the mean and median.

3; 4; 5; 7; 7; 7; 7; 8; 8; 9; 9; 10; 10; 10; 10; 10; 11; 12; 12; 13; 14; 14; 15; 15; 17; 17; 18; 19; 19; 19; 21; 21; 22; 22; 23; 24; 24; 24; 24

Answer

Mean: $3 + 4 + 5 + 7 + 7 + 7 + 7 + 8 + 8 + 9 + 9 + 10 + 10 + 10 + 10 + 10 + 11 + 12 + 12 + 13 + 14 + 14 + 15 + 15 + 17 + 17 + 18 + 19 + 19 + 19 + 21 + 21 + 22 + 22 + 23 + 24 + 24 + 24 + 24 = 544$

$$\frac{544}{39} = 13.95 \quad (8.1.7)$$

Median: Starting at the smallest value, the median is the 20th term, which is 13.

Interactive Exercise 8.1.1

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Another measure of the center is the mode. The **mode** is the most frequent value. There can be more than one mode in a data set as long as those values have the same frequency and that frequency is the highest. If there are no repeats in a dataset, meaning each value occurs exactly one time, there is no mode.

Example 8.1.2

Statistics exam scores for 20 students are as follows:

50; 53; 59; 59; 63; 63; 72; 72; 72; 72; 72; 76; 78; 81; 83; 84; 84; 84; 90; 93

Find the mode.

Answer

The most frequent score is 72, which occurs five times. Mode = 72.

Exercise 8.1.2

The number of books checked out from the library from 25 students are as follows:

0; 0; 0; 1; 2; 3; 3; 4; 4; 5; 5; 7; 7; 7; 7; 8; 8; 8; 8; 10; 10; 11; 11; 12; 12

Find the mode.

Answer

There is a tie for the most frequent value: 7 and 8 both occur four times. Mode = 7 and 8.

Interactive Exercise 8.1.2

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Interactive Exercise 8.1.3

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Measures of Variation of the Data

One of the differences between the two data sets that any measure of center doesn't capture is the variety of data within the

set. To describe the variation quantitatively, we use **measures of variation** or **measures of spread**. Just as there are several different measures of center, there are also several different measures of variation. In this section, we examine two of the most frequently used measures of variation: the range and standard deviation.

Definition: Range

The **range** of a data set is the difference between the maximum (largest) and minimum (smallest) observations.

Example 8.1.4

Find the range of the data:

8	12	13	11	10	9	14	8	6	14	7	8	13
---	----	----	----	----	---	----	---	---	----	---	---	----

Solution

The range of the data is the difference between the largest and the smallest values in the data set: $14 - 6 = 8$

Interactive Exercise 8.1.4

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Definition: The Standard Deviation

The range only measures the total variation and doesn't capture any variation between the minimum and maximum observed values. In contrast to the range, the **standard deviation** takes into account all the observations. It is the preferred measure of variation when the mean is used as the measure of center. Roughly speaking, the standard deviation measures variation by indicating how far, on average, the observations are from the mean. For a data set with a large amount of variation, the observations will, on average, be far from the mean; so the standard deviation will be large. For a data set with a small amount of variation, the observations will, on average, be close to the mean; so the standard deviation will be small.

Calculating the Standard Deviations

If x is a number, then the difference " $x - \text{mean}$ " is called its **deviation**. In a data set, there are as many deviations as there are items in the data set. The deviations are used to calculate the standard deviation. If the numbers belong to a population, in symbols a deviation is $x - \mu$. For sample data, in symbols a deviation is $x - \bar{x}$.

The procedure to calculate the standard deviation depends on whether the numbers are the entire population or are data from a sample. The calculations are similar, but not identical. Therefore the symbol used to represent the standard deviation depends on whether it is calculated from a population or a sample. The lower case letter s represents the sample standard deviation and the Greek letter σ (sigma, lower case) represents the population standard deviation. If the sample has the same characteristics as the population, then s should be a good estimate of σ .

To calculate the standard deviation, we need to calculate the variance first. The variance is the **average of the squares of the deviations** (the $x - \bar{x}$ values for a sample, or the $x - \mu$ values for a population). The symbol σ^2 represents the population variance; the population standard deviation σ is the square root of the population variance. The symbol s^2 represents the sample variance; the sample standard deviation s is the square root of the sample variance. You can think of the standard deviation as a special average of the deviations.

If the numbers come from a census of the entire **population** and not a sample, when we calculate the average of the squared deviations to find the variance, we divide by N , the number of items in the population. If the data are from a **sample** rather than a population, when we calculate the average of the squared deviations, we divide by $n - 1$, one less than the number of items in the sample.

Formula for the Sample Standard Deviation

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}} \quad (8.1.8)$$

For the sample standard deviation, the denominator is $n - 1$, that is one less than the sample size.

Example 8.1.5

Calculate the sample standard deviation for the following dataset.

5; 6; 10; 10; 14

First we calculate the mean:

$$\bar{x} = \frac{5 + 6 + 10 + 10 + 14}{5} = 9$$

The mean is 9.

The variance may be calculated by using a table. Then the standard deviation is calculated by taking the square root of the variance. We will explain the parts of the table after calculating s .

Data	Deviations	Deviations ²
x	$(x - \bar{x})$	$(x - \bar{x})^2$
5	$5 - 9 = -4$	$(-4)^2 = 16$
6	$6 - 9 = -3$	$(-3)^2 = 9$
10	$10 - 9 = 1$	$(1)^2 = 1$
10	$10 - 9 = 1$	$(1)^2 = 1$
14	$14 - 9 = 5$	$(5)^2 = 25$
		The total is 52

The **sample variance**, s^2 , is equal to the sum of the last column (52) divided by the total number of data values minus one (5 - 1):

$$s^2 = \frac{52}{5 - 1} = 13$$

The **sample standard deviation** s is equal to the square root of the sample variance:

$$s = \sqrt{13} = 3.605551275$$

and this is rounded to two decimal places, $s = 3.61$.

Interactive Exercise 8.1.5.2

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Note

In practice, USE A CALCULATOR OR COMPUTER SOFTWARE TO CALCULATE THE STANDARD DEVIATION such as [this one](#):

[Descriptive Statistics Calculator](#)

Regardless of the tool that you use, you still need to be aware of the context and use the appropriate notation for standard deviation σ or s .

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8.2: Frequency Distributions (Grouped Data)

Learning Objectives

- Organize data for frequency tables.
- Create and interpret time series graphs and scatter plots.

Frequency tables

Frequency tables can be used to summarize a dataset. You will create frequency tables in class. Here, we will practice organizing the data we need to make a frequency table. Begin constructing a frequency table by tallying each observation and then recording the frequency.

Example 8.2.4

In a survey, 20 mothers were asked how many times per week a teenager must be reminded to do his or her chores. Here are the results:

3 5 2 4 4 1 4 3 2 5 2 3 1 2 4 0 4 3 2 5

Number of times teenager is reminded	Tally	Frequency
0	I	1
1	II	2
2	IIII	5
3	IIII	4
4	IIII	5
5	III	3

Exercise 8.2.4

In a survey, 40 people were asked how many times per year they had their car in the shop for repairs. The results are shown in Table. Construct a line graph.

Number of times in shop	Frequency
0	7
1	10
2	14
3	9

Answer

Time Series Graphs

Suppose that we want to study the temperature range of a region for an entire month. Every day at noon we note the temperature

and write this down in a log. A variety of statistical studies could be done with this data. We could find the mean or the median temperature for the month. We could construct a histogram displaying the number of days that temperatures reach a certain range of values. However, all of these methods ignore a portion of the data that we have collected.

One feature of the data that we may want to consider is that of time. Since each date is paired with the temperature reading for the day, we don't have to think of the data as being random. We can instead use the times given to impose a chronological order on the data. A graph that recognizes this ordering and displays the changing temperature as the month progresses is called a time series graph.

To construct a time series graph, we must look at both pieces of our **paired data set**. We start with a standard Cartesian coordinate system. The horizontal axis is used to plot the date or time increments, and the vertical axis is used to plot the values of the variable that we are measuring. By doing this, we make each point on the graph correspond to a date and a measured quantity. The points on the graph are typically connected by straight lines in the order in which they occur.

Exercise 8.2.6

The following table is a portion of a data set from www.worldbank.org. Use the table to construct a time series graph for CO₂ emissions for the United States.

CO₂ Emissions

	Ukraine	United Kingdom	United States
2003	352,259	540,640	5,681,664
2004	343,121	540,409	5,790,761
2005	339,029	541,990	5,826,394
2006	327,797	542,045	5,737,615
2007	328,357	528,631	5,828,697
2008	323,657	522,247	5,656,839
2009	272,176	474,579	5,299,563

This is a times series graph that matches the supplied data. The x-axis shows years from 2003 to 2012, and the y-axis shows the annual CPI.

Figure 8.2.8.

Time series graphs are important tools in various applications of statistics. When recording values of the same variable over an extended period of time, sometimes it is difficult to discern any trend or pattern. However, once the same data points are displayed graphically, some features jump out. Time series graphs make trends easy to spot.

Scatter Plots

When both variables are quantitative, we can construct a scatter plot to organize the data. To construct a **scatterplot**, we use a horizontal axis for the observations of one variable and a vertical axis for the observations of the other. When picking which axis to use for each variable consider whether you suspect that one variable depends on the other. The independent variable will be on the x-axis and dependent will be on the y axis. Each pair of observations is then plotted as a point.

Example 8.2.7

The summary of the ages and the mileages of a sample of 5 cars is shown below:

Table 8.2.A: Table showing the ages and the mileages of a sample of 5 cars.

Car	x (age in years)	y (mileage in miles)
1	7	78524
2	2	12574
3	1	24914
4	5	65813
5	3	39824

To construct the **scatterplot**, we use a horizontal axis for the ages of cars and a vertical axis for the mileage.

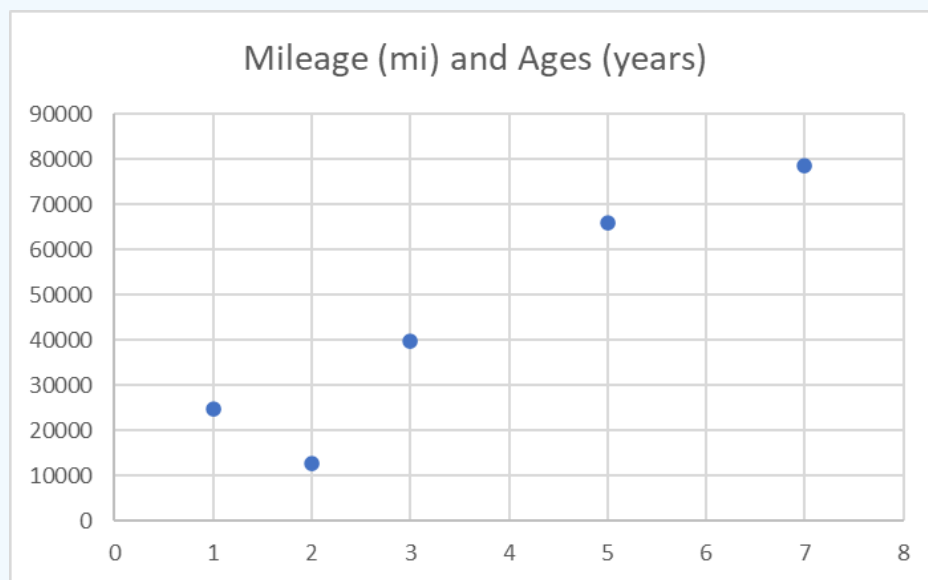


Figure 8.2.B: Scatter plot showing the ages and the mileages of a sample of 5 cars.

Exercise 8.2.7

Amelia plays basketball for her high school. She wants to improve to play at the college level. She notices that the number of points she scores in a game goes up in response to the number of hours she practices her jump shot each week. She records the following data:

X (hours practicing jump shot)	Y (points scored in a game)
5	15
7	22

X (hours practicing jump shot)	Y (points scored in a game)
9	28
10	31
11	33
12	36

Construct a scatter plot and state if what Amelia thinks appears to be true.

Answer available at the end of this section.

Answers

Exercise 8.2.7

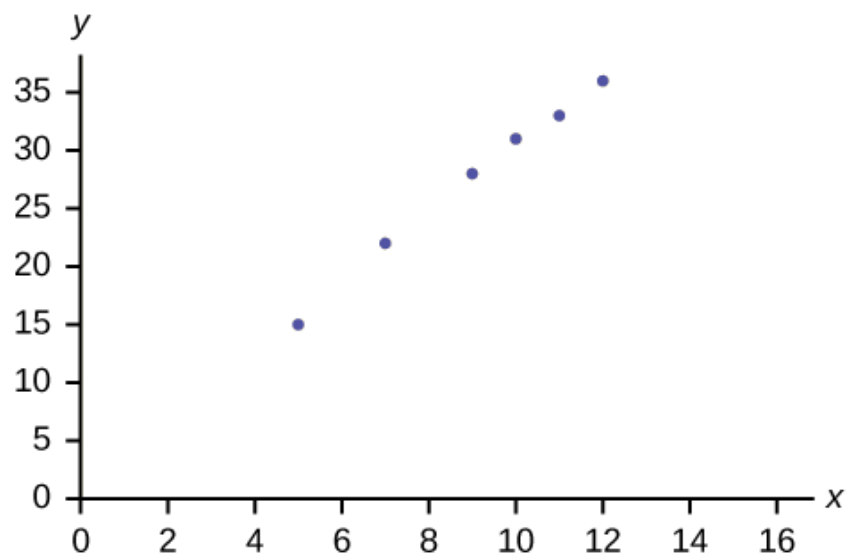


Figure 8.2.2

Yes, Amelia's assumption appears to be correct. The number of points Amelia scores per game goes up when she practices her jump shot more.

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8.3: The Normal Distribution

Learning Objectives

- Identify the characteristics of a normal distribution
- Apply the Empirical Rule (68-95-99.7 Rule) for normal distributions
- Calculate and interpret z-scores

Introduction

Most high schools have a set amount of time in between classes during which students must get to their next class. If you were to stand at the door of your math class and watch the students coming in, think about how the students would enter. Usually, one or two students enter early, then more students come in, then a large group of students enter, and finally, the number of students entering decreases again, with one or two students barely making it on time, or perhaps even coming in late!

Now consider this. Have you ever popped popcorn in a microwave? Think about what happens in terms of the rate at which the kernels pop. For the first few minutes, nothing happens, and then, after a while, a few kernels start popping. This rate increases to the point at which you hear most of the kernels popping, and then it gradually decreases again until just a kernel or two pop.

Here's something else to think about. Try measuring the height, shoe size, or the width of the hands of the students in your class. In most situations, you will probably find that there are a couple of students with very low measurements and a couple with very high measurements, with the majority of students centered on a particular value.



All of these examples show a typical pattern that seems to be a part of many real-life phenomena. In statistics, because this pattern is so pervasive, it seems to fit to call it normal, or more formally, the **normal distribution**. The normal distribution is an extremely important concept, because it occurs so often in the data we collect from the natural world, as well as in many of the more theoretical ideas that are the foundation of statistics. This chapter explores the details of the normal distribution.

The Characteristics of a Normal Distribution

Normal Distribution

The **normal distribution** is a commonly occurring statistical distribution whose bell shape is determined by the mean and standard deviation of the distribution.

The normal distribution is a continuous probability distribution. The distribution is represented by a smoothed-out histogram. The total area under the normal distribution curve, which represents the probabilities of all outcomes in the sample space, is always 1.

The Empirical Rule for Normal Distributions

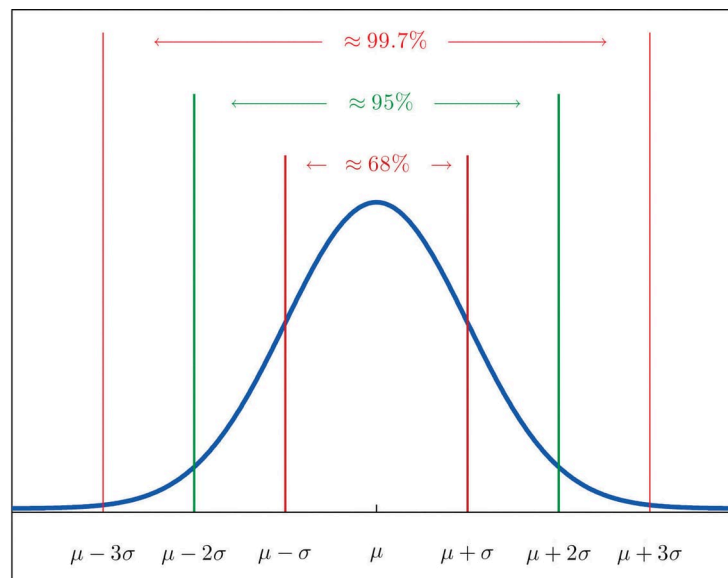
Since all normal distributions have the same shape and the total area under the curve is 1 or 100%, it turns out that the amount of data between the mean and the standard deviations for any approximately normal distribution is consistent.

Empirical Rule

For data sets that are approximately bell-shaped:

- approximately 68% of the data lie within one standard deviation of the mean, that is, in the interval with endpoints $\mu \pm \sigma$
- approximately 95% of the data lie within two standard deviations of the mean, that is, in the interval with endpoints $\mu \pm 2\sigma$
- approximately 99.7% of the data lie within three standard deviations of the mean, that is, in the interval with endpoints $\mu \pm 3\sigma$

The Empirical Rule is also referred to as the 68-95-99.7 Rule. The figure below illustrates the Empirical Rule.



Two key points in regard to the Empirical Rule are that the data distribution must be approximately bell-shaped and that the percentages are only approximately true. The Empirical Rule does not apply to data sets that are not bell-shaped, especially severely asymmetric distributions, and the actual percentage of observations in any of the intervals specified by the rule could be either greater or less than those given in the rule.

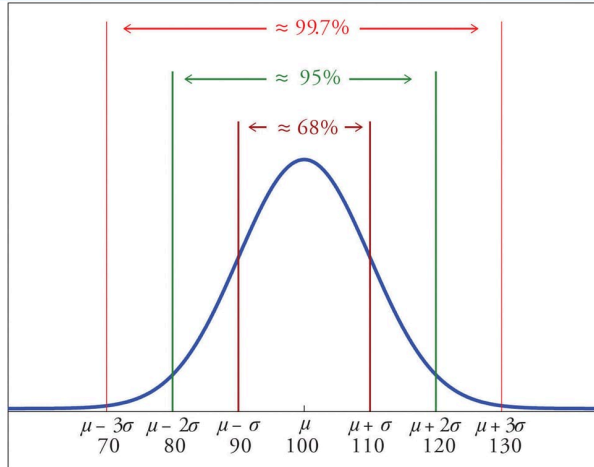
Example 8.3.1

Scores on IQ tests have a bell-shaped distribution with mean $\mu = 100$ and standard deviation $\sigma = 10$. Discuss what the Empirical Rule implies concerning individuals with IQ scores of 110, 120, and 130.

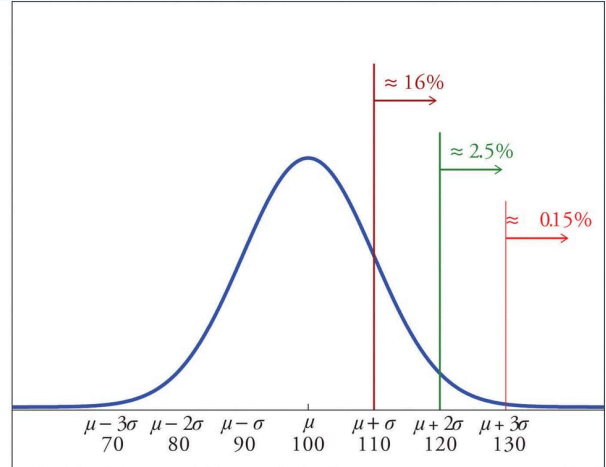
Solution:

A sketch of the IQ distribution is shown below. The Empirical Rule states that

- approximately 68% of the IQ scores in the population lie between 90 and 110
- approximately 95% of the IQ scores in the population lie between 80 and 120
- approximately 99.7% of the IQ scores in the population lie between 70 and 130.



(a) Whole Spectrum



(b) Higher End

1. Since 68% of the IQ scores lie *within* the interval from 90 to 110, it must be the case that $100\% - 68\% = 32\%$ lie *outside* that interval. By symmetry approximately half of that 32%, or 16% of all IQ scores, will lie above 110. If 16% lie above 110, then $100\% - 16\% = 84\%$ lie below. We conclude that the IQ score 110 is the 84th percentile.
2. The same analysis applies to the score 120. Since approximately 95% of all IQ scores lie within the interval from 80 to 120, only 5% lie outside it, and half of them, or 2.5% of all scores, are above 120. The IQ score 120 is thus higher than 97.5% of all IQ scores, and is quite a high score.
3. By a similar argument, only $\frac{15}{100}$ of 1% of all adults, or about one or two in every thousand, would have an IQ score above 130. This fact makes the score 130 extremely high.

Try It 8.3.1

Heights of 18-year-old males have a bell-shaped distribution with mean 69.6 inches and standard deviation 1.4 inches.

- a. About what proportion of all such men are between 68.2 and 71 inches tall?
- b. What interval centered on the mean should contain about 95% of all such men?

Answer available at the end of this section.

Calculating and Interpreting z-Scores

z-score

A **z-score** is a measure of the number of standard deviations a particular data point is away from the mean.

For example, let's say the mean score on a test for your math class was an 82, with a standard deviation of 7 points. If your score was an 89, it is exactly one standard deviation to the right of the mean; therefore, your z-score would be 1. If, on the other hand, you scored a 75, your score would be exactly one standard deviation below the mean, and your z-score would be -1 . All values that are below the mean have negative z-scores, while all values that are above the mean have positive z-scores. A z-score of -2 would represent a value that is exactly 2 standard deviations below the mean, so in this case, the value would be $82 - 2(7) = 82 - 14 = 68$.

To calculate a z-score for which the numbers are not so obvious, you take the deviation and divide it by the standard deviation.

$$z = \frac{\text{deviation}}{\text{standard deviation}}$$

You may recall that deviation is the mean value of the variable, μ , subtracted from the observed value, x , so in symbolic terms, the z-score would be:

$$z = \frac{x - \mu}{\sigma}$$

As previously stated, since σ is always positive, z will be positive when x is greater than μ and negative when x is less than μ . A z-score of zero means that the term has the same value as the mean. The value of z represents the number of standard deviations the given value of x is above or below the mean and is usually rounded to 2 decimal places.

Example 8.3.2

What is the z-score for an A on the test described above, which has a mean score of 82? (Assume that an A is a 93.)

Solution

For this problem, $x = 93$, $\mu = 82$ and $\sigma = 7$. The z-score can be calculated as follows:

$$\begin{aligned} z &= \frac{x - \mu}{\sigma} \\ z &= \frac{93 - 82}{7} \\ z &\approx 1.57 \end{aligned}$$

A score of 93 on the test is about 1.57 standard deviations above the average test score.

If we know that the test scores from the last example are distributed normally, then a z-score can tell us something about how our test score relates to the rest of the class. From the Empirical Rule, we know that about 68% of the students would have scored between a z-score of -1 and 1 , or between 75 and 89, on the test. 95% of the students scored between a z-score of -2 and 2 , or between 68 and 96.

Try It 8.3.2

On a nationwide math test, the mean was 65 and the standard deviation was 10. If Robert scored 81, what was his z-score?

Answer available at the end of this section.

Notes:

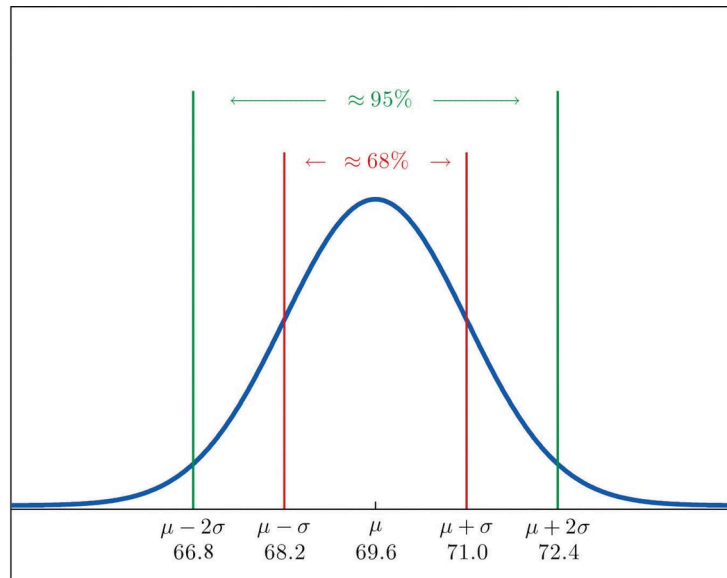
1. Data available on the College Board Website: <http://professionals.collegeboard.com/data-reportsresearch/ap/archived/2007>

Answers

Try It 8.3.1

A sketch of the distribution of heights is shown below.

- Since the interval from 68.2 to 71.0 has endpoints $\mu - \sigma$ and $\mu + \sigma$, by the Empirical Rule about 68% of all 18-year-old males should have heights in this range.
- By the Empirical Rule, the shortest such interval has endpoints $\mu - 2\sigma$ and $\mu + 2\sigma$. Since $\mu - 2\sigma = 69.6 - 2(1.4) = 66.8$ and $\mu + 2\sigma = 69.6 + 2(1.4) = 72.4$ the interval in question is the interval from 66.8 inches to 72.4 inches.



Try It 8.3.2

For this problem, $x = 81$, $\mu = 65$ and $\sigma = 10$. The z-score can be calculated as follows:

$$z = \frac{x - \mu}{\sigma}$$

$$z = \frac{81 - 65}{10}$$

$$z = 1.60$$

Robert scored 1.6 standard deviations above the mean.

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