

DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA



Anup Lamichhane
Oho Northern University

Differential Equations and Linear Algebra

Anup Lamichhane

Oho Northern University

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CHAPTER OVERVIEW

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1.1: Introduction

A **differential equation** is an equation that contains one or more derivatives of an unknown function. The **order** of a differential equation is the order of the highest derivative that it contains. A differential equation is an **ordinary differential equation** if it involves an unknown function of only one variable, or a **partial differential equation** if it involves partial derivatives of a function of more than one variable. For now we'll consider only ordinary differential equations, and we'll just call them **differential equations**.

Throughout this text, all variables and constants are real unless it is stated otherwise. We'll usually use x for the independent variable unless the independent variable is time; then we'll use t .

The simplest differential equations are first order equations of the form

$$\frac{dy}{dx} = f(x)$$

or equivalently

$$y' = f(x),$$

where f is a known function of x .

We'll usually consider differential equations that can be written as

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \quad (1.1.1)$$

where at least one of the functions $y, y', \dots, y^{(n-1)}$ actually appears on the right. Here are some examples:

$$\frac{dy}{dx} - x^2 = 0 \quad (\text{first order}),$$

$$\frac{dy}{dx} + 2xy^2 = -2 \quad (\text{first order}),$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2x \quad (\text{second order}),$$

$$xy''' + y^2 = \sin x \quad (\text{third order}),$$

$$y^{(n)} + xy' + 3y = x \quad (\text{n-th order}).$$

Although none of these equations is written as in Equation 1.1.1, all of them **can** be written in this form:

$$y' = x^2,$$

$$y' = -2 - 2xy^2,$$

$$y'' = 2x - 2y' - y,$$

$$y''' = \frac{\sin x - y^2}{x},$$

$$y^{(n)} = x - xy' - 3y.$$

Solutions of Differential Equations

A **solution** of a differential equation is a function that satisfies the differential equation on some open interval; thus, y is a solution of Equation 1.1.1 if y is n times differentiable and

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x))$$

for all x in some open interval (a, b) . In this case, we also say that y is a **solution of Equation 1.1.1 on (a, b)** . Functions that satisfy a differential equation at isolated points are not interesting. For example, $y = x^2$ satisfies

$$xy' + x^2 = 3x$$

if and only if $x = 0$ or $x = 1$, but it is not a solution of this differential equation because it does not satisfy the equation on an open interval.

The graph of a solution of a differential equation is a **solution curve**. More generally, a curve C is said to be an **integral curve** of a differential equation if every function $y = y(x)$ whose graph is a segment of C is a solution of the differential equation. Thus, any solution curve of a differential equation is an integral curve, but an integral curve need not be a solution curve.

✓ Example 1.1.1

If a is any positive constant, the circle

$$x^2 + y^2 = a^2 \quad (1.1.2)$$

is an integral curve of

$$y' = -\frac{x}{y}. \quad (1.1.3)$$

Solution

To see this, note that the only functions whose graphs are segments of Equation 1.1.2 are

$$y_1 = \sqrt{a^2 - x^2} \quad \text{and} \quad y_2 = -\sqrt{a^2 - x^2}.$$

We leave it to you to verify that these functions both satisfy Equation 1.1.3 on the open interval $(-a, a)$. However, Equation 1.1.2 is not a solution curve of Equation 1.1.3, since it is not the graph of a function.

✓ Example 1.1.2

Verify that

$$y = \frac{x^2}{3} + \frac{1}{x} \quad (1.1.4)$$

is a solution of

$$xy' + y = x^2 \quad (1.1.5)$$

on $(0, \infty)$ and on $(-\infty, 0)$.

Solution

Substituting Equation 1.1.4 and

$$y' = \frac{2x}{3} - \frac{1}{x^2}$$

into Equation 1.1.5 yields

$$xy'(x) + y(x) = x \left(\frac{2x}{3} - \frac{1}{x^2} \right) + \left(\frac{x^2}{3} + \frac{1}{x} \right) = x^2$$

for all $x \neq 0$. Therefore y is a solution of Equation 1.1.5 on $(-\infty, 0)$ and $(0, \infty)$. However, y isn't a solution of the differential equation on any open interval that contains $x = 0$, since y is not defined at $x = 0$.

Figure 1.1.1 shows the graph of Equation 1.1.4. The part of the graph of Equation 1.1.4 on $(0, \infty)$ is a solution curve of Equation 1.1.5, as is the part of the graph on $(-\infty, 0)$.

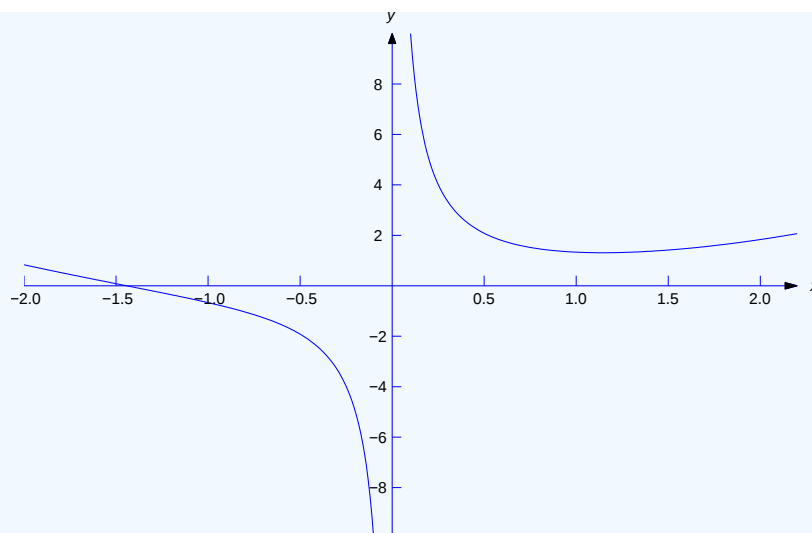


Figure 1.1.1 : $y = \frac{x^2}{3} + \frac{1}{x}$

✓ Example 1.1.3

Show that if c_1 and c_2 are constants then

$$y = (c_1 + c_2 x)e^{-x} + 2x - 4 \quad (1.1.6)$$

is a solution of

$$y'' + 2y' + y = 2x \quad (1.1.7)$$

on $(-\infty, \infty)$.

Solution

Differentiating Equation 1.1.6 twice yields

$$y' = -(c_1 + c_2 x)e^{-x} + c_2 e^{-x} + 2$$

and

$$y'' = (c_1 + c_2 x)e^{-x} - 2c_2 e^{-x},$$

so

$$\begin{aligned} y'' + 2y' + y &= (c_1 + c_2 x)e^{-x} - 2c_2 e^{-x} + 2[-(c_1 + c_2 x)e^{-x} + c_2 e^{-x} + 2] + (c_1 + c_2 x)e^{-x} + 2x - 4 \\ &= (1 - 2 + 1)(c_1 + c_2 x)e^{-x} + (-2 + 2)c_2 e^{-x} + 4 + 2x - 4 \\ &= 2x \end{aligned}$$

for all values of x . Therefore y is a solution of Equation 1.1.7 on $(-\infty, \infty)$.

Initial Value Problems

In Example 1.1.3 we saw that the differential equation $y^{(n)} = e^{2x}$ has an infinite family of solutions that depend upon the n arbitrary constants $c_1, c_2, \dots, c_n$. In the absence of additional conditions, there's no reason to prefer one solution of a differential equation over another. However, we'll often be interested in finding a solution of a differential equation that satisfies one or more specific conditions. The next example illustrates this.

✓ Example 1.1.5

Find a solution of

$$y' = x^3$$

such that $y(1) = 2$.

Solution

We can verify that the solutions of $y' = x^3$ are

$$y = \frac{x^4}{4} + c.$$

To determine a value of c such that $y(1) = 2$, we set $x = 1$ and $y = 2$ here to obtain

$$2 = y(1) = \frac{1}{4} + c$$

so

$$c = \frac{7}{4}.$$

Therefore the required solution is

$$y = \frac{x^4 + 7}{4}.$$

Figure 1.1.2 shows the graph of this solution. Note that imposing the condition $y(1) = 2$ is equivalent to requiring the graph of y to pass through the point $(1, 2)$.

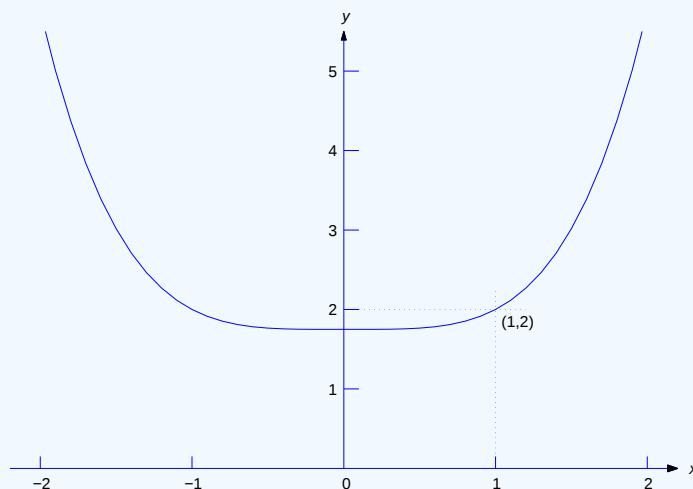


Figure 1.1.2 : $y = \frac{x^4 + 7}{4}$

We can rewrite the problem considered in Example 1.1.5 more briefly as

$$y' = x^3, \quad y(1) = 2.$$

We call this an **initial value problem**. The requirement $y(1) = 2$ is an **initial condition**. Initial value problems can also be posed for higher order differential equations. For example,

$$y'' - 2y' + 3y = e^x, \quad y(0) = 1, \quad y'(0) = 2 \quad (1.1.8)$$

is an initial value problem for a second order differential equation where y and y' are required to have specified values at $x = 0$. In general, an initial value problem for an n -th order differential equation requires y and its first $n - 1$ derivatives to have specified values at some point x_0 . These requirements are the **initial conditions**. We'll denote an initial value problem for a differential equation by writing the initial conditions after the equation, as in Equation 1.1.8. For example, we would write an initial value problem for Equation 1.1.1 as

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \quad y(x_0) = k_0, \quad y'(x_0) = k_1, \quad \dots, \quad y^{(n-1)}(x_0) = k_{n-1}. \quad (1.1.9)$$

Consistent with our earlier definition of a solution of the differential equation in Equation 1.1.9, we say that y is a solution of the initial value problem Equation 1.1.9 if y is n times differentiable and

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x))$$

for all x in some open interval (a, b) that contains x_0 , and y satisfies the initial conditions in Equation 1.1.9. The largest open interval that contains x_0 on which y is defined and satisfies the differential equation is the **interval of validity** of y .

✓ Example 1.1.6

In Example 1.1.5 we saw that

$$y = \frac{x^4 + 7}{4} \quad (1.1.10)$$

is a solution of the initial value problem

$$y' = x^3, \quad y(1) = 2.$$

Since the function in Equation 1.1.10 is defined for all x , the interval of validity of this solution is $(-\infty, \infty)$.

✓ Example 1.1.7

In Example 1.1.2 we verified that

$$y = \frac{x^2}{3} + \frac{1}{x} \quad (1.1.11)$$

is a solution of

$$xy' + y = x^2$$

on $(0, \infty)$ and on $(-\infty, 0)$. By evaluating Equation 1.1.11 at $x = \pm 1$, you can see that Equation 1.1.11 is a solution of the initial value problems

$$xy' + y = x^2, \quad y(1) = \frac{4}{3} \quad (1.1.12)$$

and

$$xy' + y = x^2, \quad y(-1) = -\frac{2}{3}. \quad (1.1.13)$$

The interval of validity of Equation 1.1.11 as a solution of Equation 1.1.12 is $(0, \infty)$, since this is the largest interval that contains $x_0 = 1$ on which Equation 1.1.11 is defined. Similarly, the interval of validity of Equation 1.1.11 as a solution of Equation 1.1.13 is $(-\infty, 0)$, since this is the largest interval that contains $x_0 = -1$ on which Equation 1.1.11 is defined.

Free Fall Under Constant Gravity

The term **initial value problem** originated in problems of motion where the independent variable is t (representing elapsed time), and the initial conditions are the position and velocity of an object at the initial (starting) time of an experiment.

✓ Example 1.1.8

An object falls under the influence of gravity near Earth's surface, where it can be assumed that the magnitude of the acceleration due to gravity is a constant g .

- Construct a mathematical model for the motion of the object in the form of an initial value problem for a second order differential equation, assuming that the altitude and velocity of the object at time $t = 0$ are known. Assume that gravity is the only force acting on the object.
- Solve the initial value problem derived above to obtain the altitude as a function of time.

Solution a

Let $y(t)$ be the altitude of the object at time t . Since the acceleration of the object has constant magnitude g and is in the downward (negative) direction, y satisfies the second order equation

$$y'' = -g,$$

where the prime now indicates differentiation with respect to t . If y_0 and v_0 denote the altitude and velocity when $t = 0$, then y is a solution of the initial value problem

$$y'' = -g, \quad y(0) = y_0, \quad y'(0) = v_0. \quad (1.1.14)$$

Solution b

Integrating Equation 1.1.14 twice yields

$$\begin{aligned} y' &= -gt + c_1, \\ y &= -\frac{gt^2}{2} + c_1t + c_2. \end{aligned}$$

Imposing the initial conditions $y(0) = y_0$ and $y'(0) = v_0$ in these two equations shows that $c_1 = v_0$ and $c_2 = y_0$. Therefore the solution of the initial value problem Equation 1.1.14 is

$$y = -\frac{gt^2}{2} + v_0t + y_0.$$

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1.2: Integrals as solutions

A first order ODE is an equation of the form

$$\frac{dy}{dx} = f(x, y)$$

or just

$$y' = f(x, y)$$

In general, there is no simple formula or procedure one can follow to find solutions. In the next few sections we will look at special cases where solutions are not difficult to obtain. In this section, let us assume that f is a function of x alone, that is, the equation is

$$y' = f(x) \tag{1.2.1}$$

We could just integrate (antidifferentiate) both sides with respect to x .

$$\int y'(x) dx = \int f(x) dx + C$$

that is

$$y(x) = \int f(x) dx + C$$

This $y(x)$ is actually the general solution. So to solve Equation 1.2.1, we find some antiderivative of $f(x)$ and then we add an arbitrary constant to get the general solution.

✓ Example 1.2.1

Solve the initial value problem $y' = 3x + 2$, $y(0) = 2$.

Solution



✓ Example 1.2.4

Find the general solution of $y' = y^2$.

Solution

First we note that $y = 0$ is a solution. We can now assume that $y \neq 0$. Write

$$\frac{dx}{dy} = \frac{1}{y^2}$$

We integrate to get

$$x = \frac{-1}{y} + C$$

We solve for $y = \frac{1}{C-x}$. So the general solution is

$$y = \frac{1}{C-x} \text{ or } y = 0$$

Note the singularities of the solution. If for example $C = 1$, then the solution as we approach $x = 1$. See Figure 1.2.1. Generally, it is hard to tell from just looking at the equation itself how the solution is going to behave. The equation $y' = y^2$ is very nice and defined everywhere, but the solution is only defined on some interval $(-\infty, C)$ or (C, ∞) . Usually when this happens we only consider one of these the solution. For example if we impose a condition $y(0) = 1$, then the solution is $y = \frac{1}{1-x}$, and we would consider this solution only for x on the interval $(-\infty, 1)$. In the figure, it is the left side of the graph.

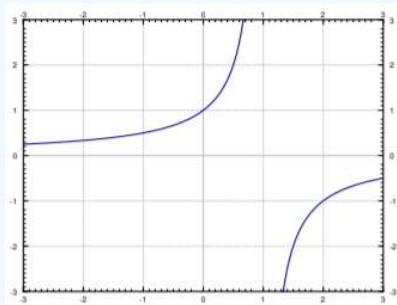


Figure 1.2.1: Plot of $y = \frac{1}{1-x}$.

Classical problems leading to differential equations solvable by integration are problems dealing with velocity, acceleration and distance. You have surely seen these problems before in your calculus class.

✓ Example 1.2.5

Suppose a car drives at a speed $e^{t/2}$ meters per second, where t is time in seconds. How far did the car get in 2 seconds (starting at $t = 0$)? How far in 10 seconds?

Solution

Let x denote the distance the car traveled. The equation is

$$x' = e^{t/2}$$

We can just integrate this equation to get that

$$x(t) = 2e^{t/2} + C$$

We still need to figure out C . We know that when $t = 0$, then $x = 0$. That is, $x(0) = 0$. So

$$0 = x(0) = 2e^{0/2} + C = 2 + C$$

Thus $C = -2$ and

$$x(t) = 2e^{t/2} - 2$$

Now we just plug in to get where the car is at 2 and at 10 seconds. We obtain

$$x(2) = 2e^{2/2} - 2 \approx 3.44 \text{ meters}, \quad x(10) = 2e^{10/2} - 2 \approx 294 \text{ meters}$$

✓ Example 1.2.6

Suppose that the car accelerates at a rate of $t^2 \frac{m}{s^2}$. At time $t = 0$ the car is at the 1 meter mark and is traveling at $10 \frac{m}{s}$. Where is the car at time $t = 10$.

Solution

Well this is actually a second order problem. If x is the distance traveled, then x' is the velocity, and x'' is the acceleration. The equation with initial conditions is

$$x'' = t^2, \quad x(0) = 1, \quad x'(0) = 10$$

What if we say $x' = v$. Then we have the problem

$$v' = t^2, \quad v(0) = 10$$

Once we solve for v , we can integrate and find x .

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1.3: Slope fields

The general first order equation we are studying looks like

$$y' = f(x, y)$$

In general, we cannot simply solve these kinds of equations explicitly. It would be nice if we could at least figure out the shape and behavior of the solutions, or if we could find approximate solutions. At this point it may be good to first try the Lab I and/or Project I from the [IODE website](#).

1.3.1: Slope fields

The equation $y' = f(x, y)$ gives you a slope at each point in the (x, y) -plane. And this is the slope a solution $y(x)$ would have at x if its value was y . In other words, $f(x, y)$ is the slope of a solution whose graph runs through the point (x, y) . At a point (x, y) , we plot a short line with the slope $f(x, y)$. For example, if $f(x, y) = xy$, then at point $(2, 1.5)$ we draw a short line of slope $xy = 2 \times 1.5 = 3$. So, if $y(x)$ is a solution and $y(2) = 1.5$, then the equation mandates that $y'(2) = 3$. See Figure 1.3.1.

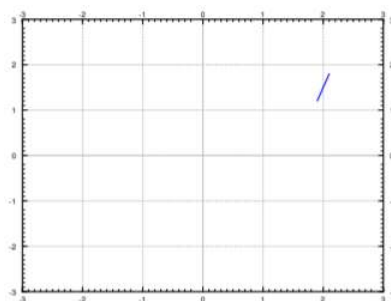


Figure 1.3.1: The slope $y' = xy$ at $(2, 1.5)$.

To get an idea of how solutions behave, we draw such lines at lots of points in the plane, not just the point $(2, 1.5)$. We would ideally want to see the slope at every point, but that is just not possible. Usually we pick a grid of points fine enough so that it shows the behavior, but not too fine so that we can still recognize the individual lines. We call this picture the of the equation. See Figure 1.3.2 for the slope field of the equation $y' = xy$. Usually in practice, one does not do this by hand, but has a computer do the drawing.

Below is a video on slope fields.



Suppose we are given a specific initial condition $y(x_0) = y_0$. A solution, that is, the graph of the solution, would be a curve that follows the slopes we drew. For a few sample solutions, see Figure 1.3.3. It is easy to roughly sketch (or at least imagine) possible solutions in the slope field, just from looking at the slope field itself. You simply sketch a line that roughly fits the little line segments and goes through your initial condition.

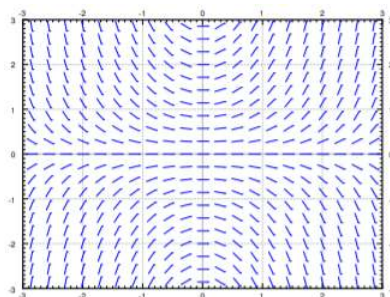


Figure 1.3.3: Slope field of $y' = xy$.

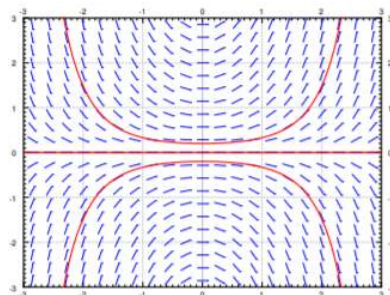


Figure 1.3.3: Slope field of $y' = xy$ with a graph of solutions satisfying $y(0) = 0.2$, $y(0) = 0$, and $y(0) = -0.2$.

By looking at the slope field we get a lot of information about the behavior of solutions without having to solve the equation. For example, in Figure 1.3.3 we see what the solutions do when the initial conditions are $y(0) > 0$, $y(0) = 0$ and $y(0) < 0$. A small change in the initial condition causes quite different behavior. We see this behavior just from the slope field and imagining what solutions ought to do.

We see a different behavior for the equation $y' = -y$. The slope field and a few solutions is in see Figure 1.3.4. If we think of moving from left to right (perhaps x is time and time is usually increasing), then we see that no matter what $y(0)$ is, all solutions tend to zero as x tends to infinity. Again that behavior is clear from simply looking at the slope field itself.

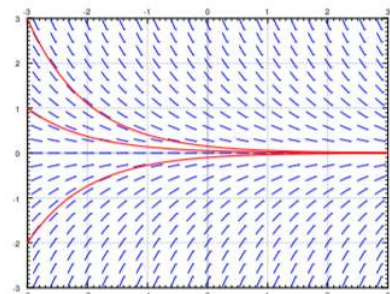


Figure 1.3.4: Slope field of $y' = -y$ with a graph of a few solutions.

Below is a video on choosing which differential equation corresponds to the given slope field.



1.3.2: Existence and uniqueness

We wish to ask two fundamental questions about the problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

- i. Does a solution exist?
- ii. Is the solution unique (if it exists)?

What do you think is the answer? The answer seems to be yes to both does it not? Well, pretty much. But there are cases when the answer to either question can be no.

Since generally the equations we encounter in applications come from real life situations, it seems logical that a solution always exists. It also has to be unique if we believe our universe is deterministic. If the solution does not exist, or if it is not unique, we have probably not devised the correct model. Hence, it is good to know when things go wrong and why.

✓ Example 1.3.1

Attempt to solve

$$y' = \frac{1}{x}, \quad y(0) = 0.$$

Solution

Integrate to find the general solution $y = \ln|x| + C$. Note that the solution does not exist at $x = 0$. See Figure 1.3.5 on the following page. The equation may have been written as the seemingly harmless $xy' = 1$.

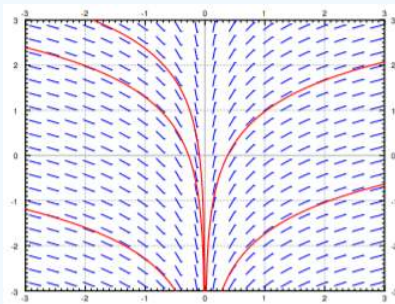


Figure 1.3.5: Slope field of $y' = \frac{1}{x}$.

✓ Example 1.3.2

Solve:

$$y' = 2\sqrt{|y|}, \quad y(0) = 0.$$

Solution

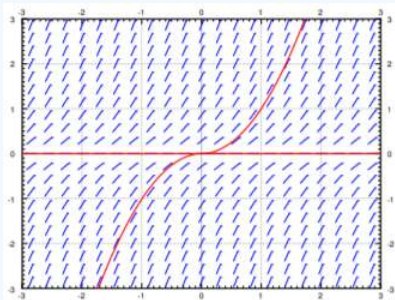


Figure 1.3.6: Slope field of $y' = 2\sqrt{|y|}$ with two solutions satisfying $y(0) = 0$.

Note that $y = 0$ is a solution. But another solution is the function

$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

It is hard to tell by staring at the slope field that the solution is not unique. Is there any hope? Of course there is. We have the following theorem, known as **Picard's theorem**.¹

🔗 Theorem 1.3.1

Picard's theorem on existence and uniqueness

If $f(x, y)$ is continuous (as a function of two variables) and $\frac{\partial f}{\partial y}$ exists and is continuous near some (x_0, y_0) , then a solution to

$$y' = f(x, y), \quad y(x_0) = y_0$$

exists (at least for x in some small interval) and is unique.

Note that the problems $y' = \frac{1}{x}$, $y(0) = 0$ and $y' = 2\sqrt{|y|}$, $y(0) = 0$ do not satisfy the hypothesis of the theorem. Even if we can use the theorem, we ought to be careful about this existence business. It is quite possible that the solution only exists for a short while.

✓ Example 1.3.3

For some constant A , solve:

$$y' = y^2 \quad y(0) = A$$

Solution

We know how to solve this equation. First assume that $A \neq 0$, so y is not equal to zero at least for some x near 0. So $x' = \frac{1}{y^2}$, so $x = -\frac{1}{y} + C$, so $y = \frac{1}{C-x}$. If $y(0) = A$, then $C = \frac{1}{A}$ so

$$y = \frac{1}{\frac{1}{A} - x}$$

If $A = 0$, then $y = 0$ is a solution.

For example, when $A = 1$ the solution “blows up” at $x = 1$. Hence, the solution does not exist for all x even if the equation is nice everywhere. The equation $y' = y^2$ certainly looks nice.

For most of this course we will be interested in equations where existence and uniqueness holds, and in fact holds “globally” unlike for the equation $y' = y^2$.

1.3.3: Footnotes

[1] Named after the French mathematician Charles Émile Picard (1856 – 1941)

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1.4: Separable Equations

When a differential equation is of the form $y' = f(x)$, we can just integrate: $y = \int f(x)dx + C$. Unfortunately this method no longer works for the general form of the equation $y' = f(x, y)$. Integrating both sides yields

$$y = \int f(x, y)dx + C$$

Notice the dependence on y in the integral.

1.4.0.1: Separable equations

Let us suppose that the equation is *separable*. That is, let us consider

$$y' = f(x)g(y),$$

for some functions $f(x)$ and $g(y)$. Let us write the equation in the Leibniz notation

$$\frac{dy}{dx} = f(x)g(y)$$

Then we rewrite the equation as

$$\frac{dy}{g(y)} = f(x)dx$$

Now both sides look like something we can integrate. We obtain

$$\int \frac{dy}{g(y)} = \int f(x)dx + C$$

If we can find closed form expressions for these two integrals, we can, perhaps, solve for y .

✓ Example 1.4.1

Take the equation

$$y' = xy$$

First note that $y = 0$ is a solution, so assume $y \neq 0$ from now on, so that we can divide by y . Write the equation as $\frac{dy}{dx} = xy$, then

$$\int \frac{dy}{y} = \int xdx + C.$$

We compute the antiderivatives to get

$$\ln|y| = \frac{x^2}{2} + C$$

Or

$$|y| = e^{\frac{x^2}{2} + C} = e^{\frac{x^2}{2}} e^C = De^{\frac{x^2}{2}}$$

where $D > 0$ is some constant. Because $y = 0$ is a solution and because of the absolute value we actually can write:

$$y = De^{\frac{x^2}{2}}$$

for any number D (including zero or negative).

We check:

$$y' = Dxe^{\frac{x^2}{2}} = x \left(De^{\frac{x^2}{2}} \right) = xy$$

Yay!

We should be a little bit more careful with this method. You may be worried that we were integrating in two different variables. We seemingly did a different operation to each side. Let us work through this method more rigorously. Take

$$\frac{dy}{dx} = f(x)g(y)$$

We rewrite the equation as follows. Note that $y = y(x)$ is a function of x and so is $\frac{dy}{dx}$!

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x)$$

We integrate both sides with respect to x .

$$\int \frac{1}{g(y)} \frac{dy}{dx} dx = \int f(x) dx + C$$

We use the change of variables formula (substitution) on the left hand side:

$$\int \frac{1}{g(y)} dy = \int f(x) dx + C$$

And we are done.

1.4.1: Implicit solutions

It is clear that we might sometimes get stuck even if we can do the integration. For example, take the separable equation

$$y' = \frac{xy}{y^2 + 1}$$

We separate variables,

$$\frac{y^2 + 1}{y} dy = \left(y + \frac{1}{y} \right) dy = x dx$$

We integrate to get

$$\frac{y^2}{2} + \ln |y| = \frac{x^2}{2} + C$$

or perhaps the easier looking expression (where $D = 2C$)

$$y^2 + 2 \ln |y| = x^2 + D$$

It is not easy to find the solution explicitly as it is hard to solve for y . We, therefore, leave the solution in this form and call it an *implicit solution*. It is still easy to check that an implicit solution satisfies the differential equation. In this case, we differentiate with respect to x , and remember that y is a function of x , to get

$$y' \left(2y + \frac{2}{y} \right) = 2x$$

Multiply both sides by y and divide by $2(y^2 + 1)$ and you will get exactly the differential equation. We leave this computation to the reader.

If you have an implicit solution, and you want to compute values for y , you might have to be tricky. You might get multiple solutions y for each x , so you have to pick one. Sometimes you can graph x as a function of y , and then flip your paper. Sometimes you have to do more.

Computers are also good at some of these tricks. More advanced mathematical software usually has some way of plotting solutions to implicit equations. For example, for $C = 0$ if you plot all the points (x, y) that are solutions to $y^2 + 2 \ln |y| = x^2$, you find the two curves in Figure 1.4.1. This is not quite a graph of a function. For each x there are two choices of y . To find a function you would have to pick one of these two curves. You pick the one that satisfies your initial condition if you have one. For example, the top curve satisfies the condition $y(1) = 1$. So for each C we really got two solutions. As you can see, computing values from an implicit solution can be somewhat tricky. But sometimes, an implicit solution is the best we can do.

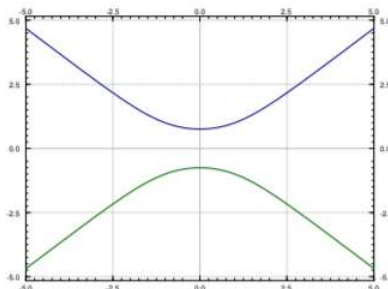


Figure 1.4.1: The implicit solution $y^2 + 2 \ln |y| = x^2$ to $y' = \frac{xy}{y^2 + 1}$.

The equation above also has the solution $y = 0$. So the general solution is

$$y^2 + 2 \ln |y| = x^2 + C, \quad \text{and} \quad y = 0.$$

These outlying solutions such as $y = 0$ are sometimes called *singular solutions*.

Below is a video on solving a separable differential equation.



✓ Example 1.4.2

Solve $x^2 y' = 1 - x^2 + y^2 - x^2 y^2$, $y(1) = 0$.

Solution

First factor the right hand side to obtain

$$x^2 y' = (1 - x^2) (1 + y^2)$$

Separate variables, integrate, and solve for y

$$\begin{aligned}\frac{y'}{1+y^2} &= \frac{1-x^2}{x^2}, \\ \frac{y'}{1+y^2} &= \frac{1}{x^2} - 1, \\ \arctan(y) &= -\frac{1}{x^2} - x + C, \\ y &= \tan\left(-\frac{1}{x} - x + C\right)\end{aligned}\tag{1.4.1}$$

Solve for the initial condition, $0 = \tan(-2 + C)$ to get $C = 2$ (or $C = 2 + \pi$, or $C = 2 + 2\pi$, etc.). The particular solution we seek is, therefore,

$$y = \tan\left(\frac{-1}{x} - x + 2\right).$$

✓ Example 1.4.3

Juan made a cup of coffee, and Juan likes to drink coffee only once reaches 60 degrees Celsius and will not burn him. Initially at time $t = 0$ minutes, Juan measured the temperature and the coffee was 89 degrees Celsius. One minute later, Juan measured the coffee again and it had 85 degrees. The temperature of the room (the ambient temperature) is 22 degrees. When should Juan start drinking?

Solution

Let T be the temperature of the coffee in degrees Celsius, and let A be the ambient (room) temperature, also in degrees Celsius. states that the rate at which the temperature of the coffee is changing is proportional to the difference between the ambient temperature and the temperature of the coffee. That is,

$$\frac{dT}{dt} = k(A - T),$$

for some constant k . For our setup $A = 22$, $T(0) = 89$, $T(1) = 85$. We separate variables and integrate (let C and D denote arbitrary constants)

$$\begin{aligned}\frac{1}{T-A} \frac{dT}{dt} &= -k, \\ \ln(T-A) &= -kt + C, \quad (\text{note that } T-A > 0) \\ T-A &= De^{-kt}, \\ T &= A + De^{-kt}\end{aligned}\tag{1.4.2}$$

That is, $T = 22 + De^{-kt}$. We plug in the first condition: $89 = T(0) = 22 + D$, and hence $D = 67$. So $T = 22 + 67e^{-kt}$. The second condition says $85 = T(1) = 22 + 67e^{-k}$. Solving for k we get $k = -\ln \frac{85-22}{67} \approx 0.0616$. Now we solve for the time t that gives us a temperature of 60 degrees. Namely, we solve

$$60 = 22 + 67e^{-0.0616t}$$

to get $t = -\frac{\ln \frac{60-22}{67}}{0.0616} \approx 9.21$ minutes. So Juan can begin to drink the coffee at just over 9 minutes from the time Juan made it. That is probably about the amount of time it took us to calculate how long it would take. See Figure 1.4.2.

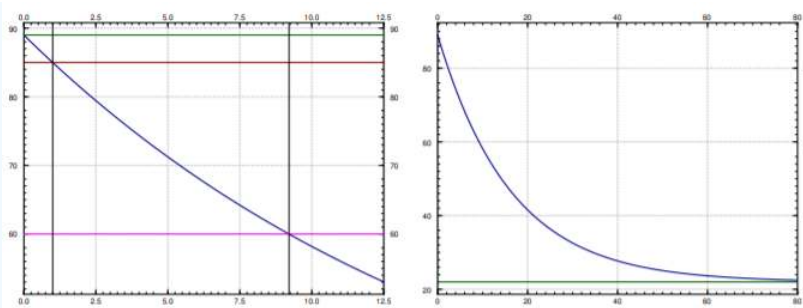


Figure 1.4.2: Graphs of the coffee temperature function $T(t)$. On the left, horizontal lines are drawn at temperatures 60, 85, and 89. Vertical lines are drawn at $t = 1$ and $t = 9.21$. Notice that the temperature of the coffee hits 85 at $t = 1$, and 60 at $t \approx 9.21$. On the right, the graph is over a longer period of time, with a horizontal line at the ambient temperature 22.

✓ Example 1.4.4

Find the general solution to $y' = \frac{-xy^2}{3}$ (including singular solutions).

Solution

First note that $y = 0$ is a solution (a singular solution). Now assume that $y \neq 0$.

$$\begin{aligned} -\frac{3}{y^2}y' &= x, \\ \frac{3}{y} &= \frac{x^2}{2} + C, \\ y &= \frac{3}{\frac{x^2}{2} + C} = \frac{6}{x^2 + 2C}. \end{aligned} \tag{1.4.3}$$

So the general solution is,

$$y = \frac{6}{x^2 + 2C}, \quad \text{and} \quad y = 0.$$

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1.5: Linear equations and the integrating factor

One of the most important types of equations we will learn how to solve are the so-called linear equations. In fact, the majority of the course is about linear equations. In this lecture we focus on the first order linear equation. A first order equation is linear if we can put it into the form:

$$y' + p(x)y = f(x). \quad (1.5.1)$$

Here the word “linear” means linear in y and y' ; no higher powers nor functions of y or y' appear. The dependence on x can be more complicated.

Solutions of linear equations have nice properties. For example, the solution exists wherever $p(x)$ and $f(x)$ are defined, and has the same regularity (read: it is just as nice). But most importantly for us right now, there is a method for solving linear first order equations. The trick is to rewrite the left hand side of (1.5.1) as a derivative of a product of y with another function. To this end we find a function $r(x)$ such that

$$r(x)y' + r(x)p(x)y = \frac{d}{dx}[r(x)y]$$

This is the left hand side of (1.5.1) multiplied by $r(x)$. So if we multiply (1.5.1) by $r(x)$, we obtain

$$\frac{d}{dx}[r(x)y] = r(x)f(x)$$

Now we integrate both sides. The right hand side does not depend on y and the left hand side is written as a derivative of a function. Afterwards, we solve for y . The function $r(x)$ is called the integrating factor and the method is called the integrating factor method.

We are looking for a function $r(x)$, such that if we differentiate it, we get the same function back multiplied by $p(x)$. That seems like a job for the exponential function! Let

$$r(x) = e^{\int p(x)dx}$$

We compute:

$$\begin{aligned} y' + p(x)y &= f(x), \\ e^{\int p(x)dx} y' + e^{\int p(x)dx} p(x)y &= e^{\int p(x)dx} f(x), \\ \frac{d}{dx} [e^{\int p(x)dx} y] &= e^{\int p(x)dx} f(x), \\ e^{\int p(x)dx} y &= \int e^{\int p(x)dx} f(x) dx + C, \\ y &= e^{-\int p(x)dx} \left(\int e^{\int p(x)dx} f(x) dx + C \right). \end{aligned} \quad (1.5.2)$$

Of course, to get a closed form formula for y , we need to be able to find a closed form formula for the integrals appearing above.

Below is a video on solving a differential equation using an integrating factor.



✓ Example 1.5.1

Solve

$$y' + 2xy = e^{x-x^2}, \quad y(0) = -1$$

Solution

First note that $p(x) = 2x$ and $f(x) = e^{x-x^2}$. The integrating factor is $r(x) = e^{\int p(x)dx} = e^{x^2}$. We multiply both sides of the equation by $r(x)$ to get

$$\begin{aligned} e^{x^2}y' + 2xe^{x^2}y &= e^{x-x^2}e^{x^2}, \\ \frac{d}{dx} [e^{x^2}y] &= e^x. \end{aligned} \tag{1.5.3}$$

We integrate

$$\begin{aligned} e^{x^2}y &= e^x + C, \\ y &= e^{x-x^2} + Ce^{-x^2}. \end{aligned} \tag{1.5.4}$$

Next, we solve for the initial condition $-1 = y(0) = 1 + C$, so $C = -2$. The solution is

$$y = e^{x-x^2} - 2e^{-x^2}.$$

Note that we do not care which antiderivative we take when computing $e^{\int p(x)dx}$. You can always add a constant of integration, but those constants will not matter in the end.

? Exercise 1.5.1

Try it! Add a constant of integration to the integral in the integrating factor and show that the solution you get in the end is the same as what we got above. An advice: Do not try to remember the formula itself, that is way too hard. It is easier to remember the process and repeat it.

Since we cannot always evaluate the integrals in closed form, it is useful to know how to write the solution in definite integral form. A definite integral is something that you can plug into a computer or a calculator. Suppose we are given

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

. Look at the solution and write the integrals as definite integrals.

$$y(x) = e^{\int_{x_0}^x p(s)ds} \left(\int_{x_0}^x e^{\int_{x_0}^t p(s)ds} f(t)dt + y_0 \right) \tag{1.5.5}$$

You should be careful to properly use dummy variables here. If you now plug such a formula into a computer or a calculator, it will be happy to give you numerical answers.

? Exercise 1.5.2

Check that $y(x_0) = y_0$ in formula (1.5.5).

? Exercise 1.5.3

Write the solution of the following problem as a definite integral, but try to simplify as far as you can. You will not be able to find the solution in closed form.

$$y' + y = e^{x^2 - x}, \quad y(0) = 10$$

📌 Note

Before we move on, we should note some interesting properties of linear equations. First, for the linear initial value problem $y' + p(x)y = f(x)$, $y(x_0) = y_0$, there is always an explicit formula (1.5.5) for the solution. Second, it follows from the formula (1.5.5) that if $p(x)$ and $f(x)$ are continuous on some interval (a, b) , then the solution $y(x)$ exists and is differentiable on (a, b) . Compare with the simple nonlinear example we have seen previously, $y' = y^2$, and compare to Theorem 1.2.1.

✓ Example 1.5.2

Let us discuss a common simple application of linear equations. This type of problem is used often in real life. For example, linear equations are used in figuring out the concentration of chemicals in bodies of water (rivers and lakes).

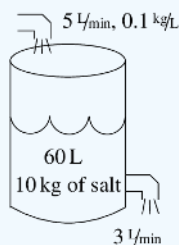


Figure 1.5.1

A 100 liter tank contains 10 kilograms of salt dissolved in 60 liters of water. Solution of water and salt (brine) with concentration of 0.1 kilograms per liter is flowing in at the rate of 5 liters a minute. The solution in the tank is well stirred and flows out at a rate of 3 liters a minute. How much salt is in the tank when the tank is full?

Solution

Let us come up with the equation. Let x denote the kilograms of salt in the tank, let t denote the time in minutes. For a small change Δt in time, the change in x (denoted Δx) is approximately

$$\Delta x \approx (\text{rate in} \times \text{concentration in})\Delta t - (\text{rate out} \times \text{concentration out})\Delta t.$$

Dividing through by Δt and taking the limit $\Delta t \rightarrow 0$ we see that

$$\frac{dx}{dt} = (\text{rate in} \times \text{concentration in}) - (\text{rate out} \times \text{concentration out})$$

In our example, we have

$$\begin{aligned} \text{rate in} &= 5, \\ \text{concentration in} &= 0.1, \\ \text{rate out} &= 3, \\ \text{concentration out} &= \frac{x}{\text{volume}} = \frac{x}{60 + (5 - 3)t}. \end{aligned} \tag{1.5.6}$$

Our equation is, therefore,

$$\frac{dx}{dt} = (5 \times 0.1) - \left(3 \frac{x}{60+2t}\right)$$

Or in the form (1.5.1)

$$\frac{dx}{dt} + \frac{3}{60+2t}x = 0.5$$

Let us solve. The integrating factor is

$$r(t) = \exp\left(\int \frac{3}{60+2t} dt\right) = \exp\left(\frac{3}{2} \ln(60+2t)\right) = (60+2t)^{3/2}$$

We multiply both sides of the equation to get

$$\begin{aligned} (60+2t)^{3/2} \frac{dx}{dt} + (60+2t)^{3/2} \frac{3}{60+2t} x &= 0.5(60+2t)^{3/2}, \\ \frac{d}{dt} \left[(60+2t)^{3/2} x \right] &= 0.5(60+2t)^{3/2}, \\ (60+2t)^{3/2} x &= \int 0.5(60+2t)^{3/2} dt + C, \\ x &= (60+2t)^{-3/2} \int \frac{(60+2t)^{3/2}}{2} dt + C(60+2t)^{-3/2}, \\ x &= (60+2t)^{-3/2} \frac{1}{10} (60+2t)^{5/2} + C(60+2t)^{-3/2}, \\ x &= \frac{(60+2t)}{10} + C(60+2t)^{-3/2}. \end{aligned} \tag{1.5.7}$$

We need to find C . We know that at $t = 0$, $x = 10$. So

$$10 = x(0) = \frac{60}{10} + C(60)^{-3/2} = 6 + C(60)^{-3/2}$$

or

$$C = 4(60^{3/2}) \approx 1859.03.$$

We are interested in x when the tank is full. So we note that the tank is full when $60+2t = 100$, or when $t = 20$. So

$$\begin{aligned} x(20) &= \frac{60+40}{10} + C(60+40)^{-3/2} \\ &\approx 10 + 1859.03(100)^{-3/2} \approx 11.86. \end{aligned} \tag{1.5.8}$$

See Figure 1.5.2 for the graph of x over t .

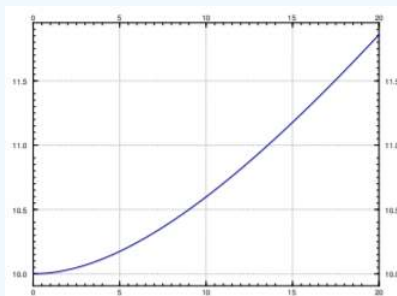


Figure 1.5.2: Graph of the solution x kilograms of salt in the tank at time t .

The concentration at the end is approximately 0.1186 kg/liter and we started with $\frac{1}{6}$ or 0.167 kg/liter .

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1.6: Existence and Uniqueness of Solutions of Nonlinear Equations

Although there are methods for solving some nonlinear equations, it is impossible to find useful formulas for the solutions of most. Whether we are looking for exact solutions or numerical approximations, it is useful to know conditions that imply the existence and uniqueness of solutions of initial value problems for nonlinear equations. In this section we state such a condition and illustrate it with examples.

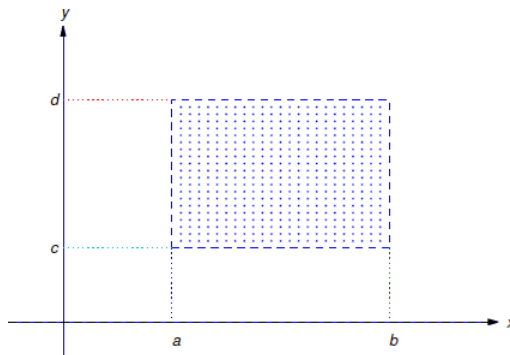


Figure 2.5.1 : An open rectangle

Some terminology: an *open rectangle* R is a set of points (x, y) such that

$$a < x < b \quad \text{and} \quad c < y < d$$

(Figure 2.5.1). We'll denote this set by $R : \{a < x < b, c < y < d\}$. “Open” means that the boundary rectangle (indicated by the dashed lines in Figure 2.5.1) is not included in R .

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for first order nonlinear differential equations. We omit the proof, which is beyond the scope of this book.

Theorem 2.5.1 : existence and uniqueness

a. If f is continuous on an open rectangle

$$R : \{a < x < b, c < y < d\}$$

that contains (x_0, y_0) then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{1.6.1}$$

has at least one solution on some open subinterval of (a, b) that contains x_0 .

b. If both f and f_y are continuous on R then Equation 1.6.1 has a unique solution on some open subinterval of (a, b) that contains x_0

It's important to understand exactly what Theorem 2.5.1 says.

- (a) is an *existence theorem*. It guarantees that a solution exists on some open interval that contains x_0 , but provides no information on how to find the solution, or to determine the open interval on which it exists. Moreover, (a) provides no information on the number of solutions that Equation 1.6.1 may have. It leaves open the possibility that Equation 1.6.1 may have two or more solutions that differ for values of x arbitrarily close to x_0 . We will see in Example 2.5.6 that this can happen.
- (b) is a *uniqueness theorem*. It guarantees that Equation 1.6.1 has a unique solution on some open interval (a, b) that contains x_0 . However, if $(a, b) \neq (-\infty, \infty)$, Equation 1.6.1 may have more than one solution on a larger interval that contains (a, b) . For example, it may happen that $b < \infty$ and all solutions have the same values on (a, b) , but two solutions y_1 and y_2 are defined on some interval (a, b_1) with $b_1 > b$, and have different values for $b < x < b_1$; thus, the graphs of the y_1 and y_2 “branch off” in different directions at $x = b$. (See Example 2.5.7 and Figure 2.5.3). In this case, continuity implies that $y_1(b) = y_2(b)$ (call their common value $\bar{y}$), and y_1 and y_2 are both solutions of the initial value problem

$$y' = f(x, y), \quad y(b) = \bar{y} \tag{1.6.2}$$

that differ on every open interval that contains b . Therefore f or f_y must have a discontinuity at some point in each open rectangle that contains (b, y) , since if this were not so, 1.6.2 would have a unique solution on some open interval that contains b . We leave it to you to give a similar analysis of the case where $a > -\infty$.

✓ Example 2.5.1

Consider the initial value problem

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad y(x_0) = y_0. \quad (1.6.3)$$

Since

$$f(x, y) = \frac{x^2 - y^2}{1 + x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{2y(1 + 2x^2)}{(1 + x^2 + y^2)^2}$$

are continuous for all (x, y) , Theorem 2.5.1 implies that if (x_0, y_0) is arbitrary, then Equation 1.6.3 has a unique solution on some open interval that contains x_0 .

✓ Example 2.5.2

Consider the initial value problem

$$y' = \frac{x^2 - y^2}{x^2 + y^2}, \quad y(x_0) = y_0. \quad (1.6.4)$$

Here

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{4x^2y}{(x^2 + y^2)^2}$$

are continuous everywhere except at $(0, 0)$. If $(x_0, y_0) \neq (0, 0)$, there's an open rectangle R that contains (x_0, y_0) that does not contain $(0, 0)$. Since f and f_y are continuous on R , Theorem 2.5.1 implies that if $(x_0, y_0) \neq (0, 0)$ then Equation 1.6.4 has a unique solution on some open interval that contains x_0 .

✓ Example 2.5.3

Consider the initial value problem

$$y' = \frac{x + y}{x - y}, \quad y(x_0) = y_0. \quad (1.6.5)$$

Here

$$f(x, y) = \frac{x + y}{x - y} \quad \text{and} \quad f_y(x, y) = \frac{2x}{(x - y)^2}$$

are continuous everywhere except on the line $y = x$. If $y_0 \neq x_0$, there's an open rectangle R that contains (x_0, y_0) that does not intersect the line $y = x$. Since f and f_y are continuous on R , Theorem 2.5.1 implies that if $y_0 \neq x_0$, Equation 1.6.5 has a unique solution on some open interval that contains x_0 .

✓ Example 2.5.4

In Example 2.2.4, we saw that the solutions of

$$y' = 2xy^2 \quad (1.6.6)$$

are

$$y \equiv 0 \quad \text{and} \quad y = -\frac{1}{x^2 + c},$$

where c is an arbitrary constant. In particular, this implies that no solution of Equation 1.6.6 other than $y \equiv 0$ can equal zero for any value of x . Show that Theorem 1.6.1b implies this.

We'll obtain a contradiction by assuming that Equation 1.6.6 has a solution y_1 that equals zero for some value of x , but is not identically zero. If y_1 has this property, there's a point x_0 such that $y_1(x_0) = 0$, but $y_1(x) \neq 0$ for some value of x in every open interval that contains x_0 . This means that the initial value problem

$$y' = 2xy^2, \quad y(x_0) = 0 \tag{1.6.7}$$

has two solutions $y \equiv 0$ and $y = y_1$ that differ for some value of x on every open interval that contains x_0 . This contradicts Theorem 2.5.1 (b), since in Equation 1.6.6 the functions

$$f(x, y) = 2xy^2 \quad \text{and} \quad f_y(x, y) = 4xy.$$

are both continuous for all (x, y) , which implies that Equation 1.6.7 has a unique solution on some open interval that contains x_0 .

Below is a video on finding values where there is no guarantee of existence and uniqueness of a solution to a differential equation.



✓ Example 2.5.5

Consider the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(x_0) = y_0. \tag{1.6.8}$$

- For what points (x_0, y_0) does Theorem 1.6.1a imply that Equation 1.6.8 has a solution?
- For what points (x_0, y_0) does Theorem 1.6.1b imply that Equation 1.6.8 has a unique solution on some open interval that contains x_0 ?

Solution a

Since

$$f(x, y) = \frac{10}{3}xy^{2/5}$$

is continuous for all (x, y) , Theorem 2.5.1 implies that Equation 1.6.8 has a solution for every (x_0, y_0) .

Solution b

Here

$$f_y(x, y) = \frac{4}{3}xy^{-3/5}$$

is continuous for all (x, y) with $y \neq 0$. Therefore, if $y_0 \neq 0$ there's an open rectangle on which both f and f_y are continuous, and Theorem 2.5.1 implies that Equation 1.6.8 has a unique solution on some open interval that contains x_0 .

If $y = 0$ then $f_y(x, y)$ is undefined, and therefore discontinuous; hence, Theorem 2.5.1 does not apply to Equation 1.6.8 if $y_0 = 0$.

✓ Example 2.5.6

Example 2.5.5 leaves open the possibility that the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = 0 \quad (1.6.9)$$

has more than one solution on every open interval that contains $x_0 = 0$. Show that this is true.

Solution

By inspection, $y \equiv 0$ is a solution of the differential equation

$$y' = \frac{10}{3}xy^{2/5}. \quad (1.6.10)$$

Since $y \equiv 0$ satisfies the initial condition $y(0) = 0$, it is a solution of Equation 1.6.9.

Now suppose y is a solution of Equation 1.6.10 that is not identically zero. Separating variables in Equation 1.6.10 yields

$$y^{-2/5}y' = \frac{10}{3}x$$

on any open interval where y has no zeros. Integrating this and rewriting the arbitrary constant as $5c/3$ yields

$$\frac{5}{3}y^{3/5} = \frac{5}{3}(x^2 + c).$$

Therefore

$$y = (x^2 + c)^{5/3}. \quad (1.6.11)$$

Since we divided by y to separate variables in Equation 1.6.10, our derivation of Equation 1.6.11 is legitimate only on open intervals where y has no zeros. However, Equation 1.6.11 actually defines y for all x , and differentiating Equation 1.6.11 shows that

$$y' = \frac{10}{3}x(x^2 + c)^{2/3} = \frac{10}{3}xy^{2/5}, \quad -\infty < x < \infty$$

Therefore Equation 1.6.11 satisfies Equation 1.6.10 on $(-\infty, \infty)$ even if $c \leq 0$, so that $y(\sqrt{|c|}) = y(-\sqrt{|c|}) = 0$. In particular, taking $c = 0$ in Equation 1.6.11 yields

$$y = x^{10/3}$$

as a second solution of Equation 1.6.9. Both solutions are defined on $(-\infty, \infty)$, and they differ on every open interval that contains $x_0 = 0$ (Figure 2.5.2). In fact, there are *four* distinct solutions of Equation 1.6.9 defined on $(-\infty, \infty)$ that differ from each other on every open interval that contains $x_0 = 0$. Can you identify the other two?

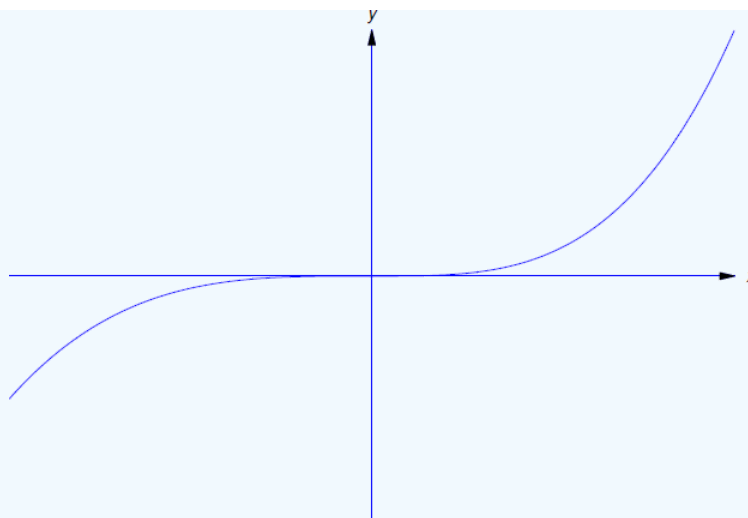


Figure 2.5.2 : Two solutions ($y = 0$ and $y = x^{1/2}$) of Equation 1.6.9 that differ on every interval containing $x_0 = 0$

✓ Example 2.5.7

From Example 2.5.5, the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = -1 \quad (1.6.12)$$

has a unique solution on some open interval that contains $x_0 = 0$. Find a solution and determine the largest open interval (a, b) on which it is unique.

Solution

Let y be any solution of Equation 1.6.12. Because of the initial condition $y(0) = -1$ and the continuity of y , there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form Equation 1.6.11. Setting $x = 0$ and $y = -1$ in Equation 1.6.11 yields $c = -1$, so

$$y = (x^2 - 1)^{5/3} \quad (1.6.13)$$

for x in I . Therefore every solution of Equation 1.6.12 differs from zero and is given by Equation 1.6.13 on $(-1, 1)$; that is, Equation 1.6.13 is the unique solution of Equation 1.6.12 on $(-1, 1)$. This is the largest open interval on which Equation 1.6.12 has a unique solution. To see this, note that Equation 1.6.13 is a solution of Equation 1.6.12 on $(-\infty, \infty)$. From Exercise 2.2.15, there are infinitely many other solutions of Equation 1.6.12 that differ from Equation 1.6.13 on every open interval larger than $(-1, 1)$. One such solution is

$$y = \begin{cases} (x^2 - 1)^{5/3}, & -1 \leq x \leq 1, \\ 0, & |x| > 1. \end{cases}$$



Figure 2.5.3 : Two solutions of Equation 1.6.12 on $(-1, 1)$ that coincide on $(-1, 1)$, but on no larger open interval. (right)

✓ Example 2.5.8

From Example 2.5.5), the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = 1 \quad (1.6.14)$$

has a unique solution on some open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it is unique.

Solution

Let y be any solution of Equation 1.6.14. Because of the initial condition $y(0) = 1$ and the continuity of y , there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form Equation 1.6.11. Setting $x = 0$ and $y = 1$ in Equation 1.6.11 yields $c = 1$, so

$$y = (x^2 + 1)^{5/3} \quad (1.6.15)$$

for x in I . Therefore every solution of Equation 1.6.14 differs from zero and is given by Equation 1.6.15 on $(-\infty, \infty)$; that is, Equation 1.6.15 is the unique solution of Equation 1.6.14 on $(-\infty, \infty)$. Figure 2.5.4) shows the graph of this solution.

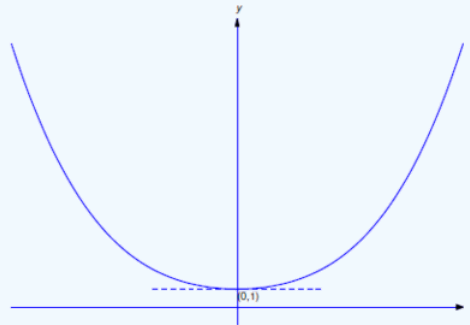


Figure 2.5.4 : The unique solution of Equation 1.6.14.

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1.6E: Existence and Uniqueness of Solutions of Nonlinear Equations (Exercises)

Q2.3.1

In **Exercises 2.3.1-2.3.13**, find all (x_0, y_0) for which Theorem 2.3.1 implies that the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ has (a) a solution and (b) a unique solution on some open interval that contains x_0 .

1. $y' = \frac{x^2 + y^2}{\sin x}$

2. $y' = \frac{e^x + y}{x^2 + y^2}$

3. $y' = \tan xy$

4. $y' = \frac{x^2 + y^2}{\ln xy}$

5. $y' = (x^2 + y^2)y^{1/3}$

6. $y' = 2xy$

7. $y' = \ln(1 + x^2 + y^2)$

8. $y' = \frac{2x + 3y}{x - 4y}$

9. $y' = (x^2 + y^2)^{1/2}$

10. $y' = x(y^2 - 1)^{2/3}$

11. $y' = (x^2 + y^2)^2$

12. $y' = (x + y)^{1/2}$

13. $y' = \frac{\tan y}{x - 1}$

Q2.3.2

14. Apply Theorem 2.3.1 to the initial value problem

$$y' + p(x)y = q(x), \quad y(x_0) = y_0$$

for a linear equation, and compare the conclusions that can be drawn from it to those that follow from Theorem 2.1.2.

15.

a. Verify that the function

$$y = \begin{cases} (x^2 - 1)^{5/3}, & -1 < x < 1, \\ 0, & |x| \geq 1, \end{cases}$$

is a solution of the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = -1$$

on $(-\infty, \infty)$. HINT: You'll need the definition

$$y'(\bar{x}) = \lim_{x \rightarrow \bar{x}} \frac{y(x) - y(\bar{x})}{x - \bar{x}}$$

to verify that y satisfies the differential equation at $\bar{x} = \pm 1$.

b. Verify that if $\epsilon_i = 0$ or 1 for $i = 1, 2$ and $a, b > 1$, then the function

$$y = \begin{cases} \epsilon_1(x^2 - a^2)^{5/3}, & -\infty < x < -a, \\ 0, & -a \leq x \leq -1, \\ (x^2 - 1)^{5/3}, & -1 < x < 1, \\ 0, & 1 \leq x \leq b, \\ \epsilon_2(x^2 - b^2)^{5/3}, & b < x < \infty, \end{cases}$$

is a solution of the initial value problem of a on $(-\infty, \infty)$.

16. Use the ideas developed in **Exercise 2.3.15** to find infinitely many solutions of the initial value problem

$$y' = y^{2/5}, \quad y(0) = 1$$

on $(-\infty, \infty)$.

17. Consider the initial value problem

$$y' = 3x(y - 1)^{1/3}, \quad y(x_0) = y_0. \quad (\text{A})$$

- For what points (x_0, y_0) does Theorem 2.3.1 imply that (A) has a solution?
- For what points (x_0, y_0) does Theorem 2.3.1 imply that (A) has a unique solution on some open interval that contains x_0 ?

18. Find nine solutions of the initial value problem

$$y' = 3x(y - 1)^{1/3}, \quad y(0) = 1$$

that are all defined on $(-\infty, \infty)$ and differ from each other for values of x in every open interval that contains $x_0 = 0$.

19. From Theorem 2.3.1, the initial value problem

$$y' = 3x(y - 1)^{1/3}, \quad y(0) = 9$$

has a unique solution on an open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it is unique.

20.

- From Theorem 2.3.1, the initial value problem

$$y' = 3x(y - 1)^{1/3}, \quad y(3) = -7 \quad (\text{A})$$

has a unique solution on some open interval that contains $x_0 = 3$. Determine the largest such open interval, and find the solution on this interval.

- Find infinitely many solutions of (A), all defined on $(-\infty, \infty)$.

21. Prove:

- If

$$f(x, y_0) = 0, \quad a < x < b, \quad (\text{A})$$

and x_0 is in (a, b) , then $y \equiv y_0$ is a solution of

$$y' = f(x, y), \quad y(x_0) = y_0$$

on (a, b) .

- If f and f_y are continuous on an open rectangle that contains (x_0, y_0) and (A) holds, no solution of $y' = f(x, y)$ other than $y \equiv y_0$ can equal y_0 at any point in (a, b) .

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1.7: Substitution

Just as when solving integrals, one method to try is to change variables to end up with a simpler equation to solve.

1.7.1: Substitution

The equation

$$y' = (x - y + 1)^2 \quad (1.7.1)$$

is neither separable nor linear. What can we do? How about trying to change variables, so that in the new variables the equation is simpler. We use another variable v , which we treat as a function of x . Let us try

$$v = x - y + 1. \quad (1.7.2)$$

We need to figure out y' in terms of v' , v and x . We differentiate (in x) to obtain $v' = 1 - y'$. So $y' = 1 - v'$. We plug this into the equation to get

$$1 - v' = v^2 \quad (1.7.3)$$

In other words, $v' = 1 - v^2$. Such an equation we know how to solve by separating variables:

$$\frac{1}{1 - v^2} dv = dx \quad (1.7.4)$$

So

$$\frac{1}{2} \ln \left| \frac{v+1}{v-1} \right| = x + C, \quad \text{or} \quad \left| \frac{v+1}{v-1} \right| = e^{2x+2C}, \quad \text{or} \quad \frac{v+1}{v-1} = De^{2x},$$

for some constant D . Note that $v = 1$ and $v = -1$ are also solutions.

Now we need to “unsubstitute” to obtain

$$\frac{x - y + 2}{x - y} = De^{2x} \quad (1.7.5)$$

and also the two solutions $x - y + 1 = 1$ or $y = x$, and $x - y + 1 = -1$ or $y = x + 2$. We solve the first equation for y .

$$\begin{aligned} x - y + 2 &= (x - y) De^{2x}, \\ x - y + 2 &= Dxe^{2x} - yDe^{2x}, \\ -y + yDe^{2x} &= Dxe^{2x} - x - 2, \\ y(-1 + De^{2x}) &= Dxe^{2x} - x - 2, \\ y &= \frac{Dxe^{2x} - x - 2}{De^{2x} - 1}. \end{aligned} \quad (1.7.6)$$

Note that $D = 0$ gives $y = x + 2$, but no value of D gives the solution $y = x$.

Substitution in differential equations is applied in much the same way that it is applied in calculus. You guess. Several different substitutions might work. There are some general things to look for. We summarize a few of these in a table.

When you see	Try substituting
yy'	$v = y^2$
y^2y'	$v = y^3$
$(\cos y)y'$	$v = \sin y$
$(\sin y)y'$	$v = \cos y$
$y'e^y$	$v = e^y$

Usually you try to substitute in the “most complicated” part of the equation with the hopes of simplifying it. The above table is just a rule of thumb. You might have to modify your guesses. If a substitution does not work (it does not make the equation any simpler), try a different one.

1.7.2: Bernoulli Equations

There are some forms of equations where there is a general rule for substitution that always works. One such example is the so-called Bernoulli equation.¹

$$y' + p(x)y = q(x)y^n \quad (1.7.7)$$

This equation looks a lot like a linear equation except for the y^n . If $n = 0$ or $n = 1$, then the equation is linear and we can solve it. Otherwise, the substitution $v = y^{1-n}$ transforms the Bernoulli equation into a linear equation. Note that n need not be an integer.

✓ Example 1.5.1: Bernoulli Equation

Solve

$$xy' + y(x+1) + xy^5 = 0, \quad y(1) = 1$$

Solution

First, the equation is Bernoulli $p(x) = \frac{x+1}{x}$ (and $q(x) = -1$). We substitute

$$v = y^{1-5} = y^{-4}, \quad v' = -4y^{-5}y'$$

In other words, $\left(\frac{-1}{4}\right)y^5v' = y'$. So

$$\begin{aligned} xy' + y(x+1) + xy^5 &= 0, \\ \frac{-xy^5}{4}v' + y(x+1) + xy^5 &= 0, \\ \frac{-x}{4}v' + y^{-4}(x+1) + x &= 0, \\ \frac{-x}{4}v' + v(x+1) + x &= 0, \end{aligned} \quad (1.7.8)$$

and finally

$$v' - \frac{4(x+1)}{x}v = 4$$

Now the equation is linear. We can use the integrating factor method. In particular, we use formula (1.4.17). Let us assume that $x > 0$ so $|x| = x$. This assumption is OK, as our initial condition is $x = 1$. Let us compute the integrating factor. Here $p(s)$ from formula (1.4.17) is $\frac{-4(s+1)}{s}$.

$$\begin{aligned} e^{\int_1^x p(s)ds} &= \exp\left(\int_1^x \frac{-4(s+1)}{s}ds\right) = e^{-4x-4\ln(x)+4} = e^{-4x+4}x^{-4} = \frac{e^{-4x+4}}{x^4}, \\ e^{-\int_1^x p(s)ds} &= e^{4x+4\ln(x)-4} = e^{4x-4}x^4 \end{aligned} \quad (1.7.9)$$

We now plug in to (1.4.17)

$$\begin{aligned} v(x) &= e^{-\int_1^x p(s)ds} \left(\int_1^x e^{\int_1^t p(s)ds} 4dt + 1 \right), \\ &= e^{4x-4}x^4 \left(\int_1^x 4 \frac{e^{-4t+4}}{t^4} dt + 1 \right) \end{aligned} \quad (1.7.10)$$

Note that the integral in this expression is not possible to find in closed form. As we said before, it is perfectly fine to have a definite integral in our solution. Now “unsubstitute”

$$y^{-4} = e^{4x-4} x^4 \left(4 \int_1^x \frac{e^{-4t+4}}{t^4} dt + 1 \right),$$

$$y = \frac{e^{-x+1}}{x \left(4 \int_1^x \frac{e^{-4t+4}}{t^4} dt + 1 \right)^{1/4}} \quad (1.7.11)$$

1.7.3: Homogeneous Equations

Another type of equations we can solve by substitution are the so-called homogeneous equations. Suppose that we can write the differential equation as

$$y' = F\left(\frac{y}{x}\right) \quad (1.7.12)$$

Here we try the substitutions

$$v = \frac{y}{x} \quad \text{and therefore} \quad y' = v + xv' \quad (1.7.13)$$

We note that the equation is transformed into

$$v + xv' = F(v) \quad \text{or} \quad xv' = F(v) - v \quad \text{or} \quad \frac{v'}{F(v) - v} = \frac{1}{x} \quad (1.7.14)$$

Hence an implicit solution is

$$\int \frac{1}{F(v) - v} dv = \ln|x| + C \quad (1.7.15)$$

✓ Example 1.5.2

Solve

$$x^2 y' = y^2 + xy, \quad y(1) = 1$$

Solution

We put the equation into the form $y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$. We substitute $v = \frac{y}{x}$ to get the separable equation

$$xv' = v^2 + v - v = v^2$$

$$v^{-2} dv = \frac{1}{x} dx$$

which has a solution

$$\int \frac{1}{v^2} dv = \ln|x| + C,$$

$$\frac{-1}{v} = \ln|x| + C, \quad (1.7.16)$$

$$v = \frac{-1}{\ln|x| + C}.$$

We unsubstute

$$\frac{y}{x} = \frac{-1}{\ln|x| + C},$$

$$y = \frac{-x}{\ln|x| + C} \quad (1.7.17)$$

We want $y(1) = 1$, so

$$1 = y(1) = \frac{-1}{\ln|1| + C} = \frac{-1}{C}$$

Thus $C = -1$ and the solution we are looking for is

$$y = \frac{-x}{\ln|x| - 1}$$

1.7.4: Footnotes

[1] There are several things called Bernoulli equations, this is just one of them. The Bernoullis were a prominent Swiss family of mathematicians. These particular equations are named for Jacob Bernoulli (1654–1705).

- - [Jiří Lebl \(Oklahoma State University\)](#). These pages were supported by NSF grants DMS-0900885 and DMS-1362337.

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1.8: Autonomous equations

Let us consider general differential equation problems of the form

$$\frac{dx}{dt} = f(x)$$

where the derivative of solutions depends only on x (the dependent variable). Such equations are called **autonomous** equations. If we think of t as time, the naming comes from the fact that the equation is independent of time.

Let us come back to the cooling coffee problem (see Example 1.3.3). Newton's law of cooling says that

$$\frac{dx}{dt} = -k(x - A)$$

where x is the temperature, t is time, k is some constant, and A is the ambient temperature. See Figure 1.8.1 for an example with $k = 0.3$ and $A = 5$.

Note the solution $x = A$ (in the figure $x = 5$). We call these constant solutions the equilibrium solutions. The points on the x axis where $f(x) = 0$ are called critical points. The point $x = A$ is a critical point. In fact, each critical point corresponds to an equilibrium solution. Note also, by looking at the graph, that the solution $x = A$ is "stable" in that small perturbations in x do not lead to substantially different solutions as t grows. If we change the initial condition a little bit, then as $t \rightarrow \infty$ we get $x \rightarrow A$. We call such critical points stable. In this simple example it turns out that all solutions in fact go to A as $t \rightarrow \infty$. If a critical point is not stable we would say it is unstable.

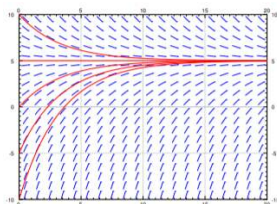


Figure 1.8.1: The slope field and some solutions of $x' = 0.3(5 - x)$.

Let us consider the logistic equation

$$\frac{dx}{dt} = kx(M - x)$$

for some positive k and M . This equation is commonly used to model population if we know the limiting population M , that is the maximum sustainable population. The logistic equation leads to less catastrophic predictions on world population than $x' = kx$. In the real world there is no such thing as negative population, but we will still consider negative x for the purposes of the math (see Figure 1.8.2 for an example).

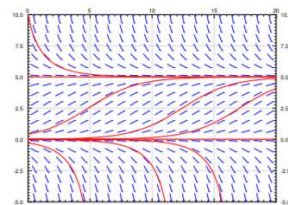


Figure 1.8.2: The slope field and some solutions of $x' = 0.1x(5 - x)$.

Note two critical points, $x = 0$ and $x = 5$. The critical point at $x = 5$ is stable. On the other hand the critical point at $x = 0$ is unstable.

It is not really necessary to find the exact solutions to talk about the long term behavior of the solutions. For example, from the above slope field plot, we can easily see that

$$\lim_{t \rightarrow \infty} x(t) = \begin{cases} 5 & \text{if } x(0) > 0, \\ 0 & \text{if } x(0) = 0, \\ \text{DNE or } -\infty & \text{if } x(0) < 0. \end{cases}$$

Where DNE means "does not exist." From just looking at the slope field we cannot quite decide what happens if $x(0) < 0$. It could be that the solution does not exist for t all the way to ∞ . Think of the equation $x' = x^2$, we have seen that it only exists for some finite period of time. Same can happen here. In our example equation above it will actually turn out that the solution does not exist for all time, but to see that we would have to solve the equation. In any case, the solution does go to $-\infty$, but it may get there rather quickly.

If we are interested only in the long term behavior of the solution, we would be doing unnecessary work if we solved the equation exactly. We could draw the slope field, but it is easier to just look at the arrows, which is a simple way to visualize the behavior of autonomous equations. In this case there is one dependent variable x . We draw the x -axis, we mark all the critical points, and then we draw arrows in between. Since x is the dependent variable we draw the axis vertically, as it appears in the slope field diagrams above. If $f(x) > 0$, we draw an up arrow. If $f(x) < 0$, we draw a down arrow. To figure this out, we could just plug in some x between the critical points, $f(x)$ will have the same sign at all x between two critical points as long as $f(x)$ is continuous. For example, $f(6) = -0.6 < 0$, so $f(x) < 0$ for $x > 5$, and the arrow above $x = 5$ is a down arrow. Next, $f(1) = 0.4 > 0$, so $f(x) > 0$ whenever $0 < x < 5$, and the arrow points up. Finally, $f(-1) = -0.6 < 0$ so $f(x) < 0$ when $x < 0$, and the arrow points down.

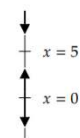


Figure 1.8.3

Armed with the phase diagram, it is easy to sketch the solutions approximately: As time t moves from left to right, the graph of a solution goes up if the arrow is up, and it goes down if the arrow is down.

Below is a video on solving an autonomous differential equation that describes logistic growth.



? Exercise 1.8.1

Try sketching a few solutions simply from looking at the phase diagram. Check with the preceding graphs if you are getting the type of curves.

Once we draw the phase diagram, we can easily classify critical points as stable or unstable.¹



Figure 1.8.4

Since any mathematical model we cook up will only be an approximation to the real world, unstable points are generally bad news.

Let us think about the logistic equation with harvesting. Suppose an alien race really likes to eat humans. They keep a planet with humans on it and harvest the humans at a rate of h million humans per year. Suppose x is the number of humans in millions on the planet and t is time in years. Let M be the limiting population when no harvesting is done and $k > 0$ is some constant depending on how fast humans multiply. Our equation becomes

$$\frac{dx}{dt} = kx(M - x) - h$$

We expand the right hand side and solve for critical points

$$\frac{dx}{dt} = -kx^2 + kMx - h$$

Solving for the critical points A and B from the quadratic equations:

$$A = \frac{kM + \sqrt{(kM)^2 - 4hk}}{2k}, \quad B = \frac{kM - \sqrt{(kM)^2 - 4hk}}{2k}$$

Below is a video on solving an autonomous initial value problem.



? Exercise 1.8.2

Sketch a phase diagram for different possibilities. Note that these possibilities are $A > B$, or $A = B$, or A and B both complex (i.e. no real solutions). Hint: Fix some simple k and M and then vary h .

For example, let $M = 8$ and $k = 0.1$. When $h = 1$, then A and B are distinct and positive. The slope field we get is in Figure 1.8.5. As long as the population starts above B , which is approximately 1.55 million, then the population will not die out. It will in fact tend towards $A \approx 6.45$ million. If ever some catastrophe happens and the population drops below B , humans will die out, and the fast food restaurant serving them will go out of business.

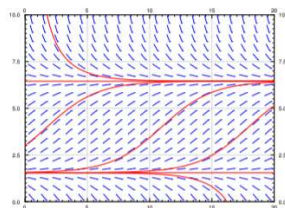


Figure 1.8.5: Slope field and some solutions of $x' = 0.1x(8 - x) - 1$.

When $h = 1.6$, then $A = B = 4$ and there is only **one** critical point and it is **unstable**. When the population starts above 4 million it will tend towards 4 million. If it ever drops below 4 million, humans will die out on the planet. This scenario is not one that we (as the human fast food proprietor) want to be in. A small perturbation of the equilibrium state and we are out of business; there is no room for error (see Figure 1.8.6).

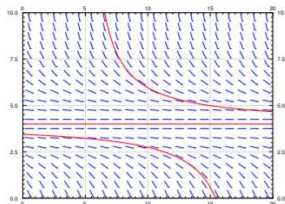


Figure 1.8.6: The slope field and some solutions of $x' = 0.1x(8 - x) - 1.6$.

Finally if we are harvesting at 2 million humans per year, there are no critical points. The population will always plummet towards zero, no matter how well stocked the planet starts (see Figure 1.8.7).

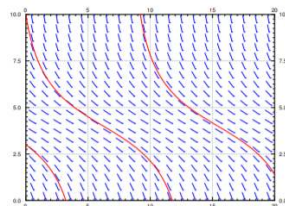


Figure 1.8.7: Slope field and some solutions of $x' = 0.1x(8 - x) - 2$.

1.8.1: Footnotes

[1] Unstable points with one of the arrows pointing towards the critical point are sometimes called semistable.

1.8.2: References

1. Paul W. Berg and James L. McGregor, [Elementary Partial Differential Equations](#), Holden-Day, San Francisco, CA, 1966.
2. William E. Boyce, Richard C. DiPrima, [Elementary Differential Equations and Boundary Value Problems](#), 9th edition, John Wiley & Sons Inc., New York, NY, 2008.
3. C.H. Edwards and D.E. Penney, [Differential Equations and Boundary Value Problems: Computing and Modeling](#), 4th edition, Prentice Hall, 2008.
4. Stanley J. Farlow, [An Introduction to Differential Equations and Their Applications](#), McGraw-Hill, Inc., Princeton, NJ, 1994. (Published also by Dover Publications, 2006.)
5. E.L. Ince, [Ordinary Differential Equations](#), Dover Publications, Inc., New York, NY, 1956.

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1.9: Numerical methods- Euler's method

At this point it may be good to first try the Lab II and/or Project II from the IODE website: www.math.uiuc.edu/iode/. As we said before, unless $f(x, y)$ is of a special form, it is generally very hard if not impossible to get a nice formula for the solution of the problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

If the equation can be solved in closed form, we should do that. But what if we have an equation that cannot be solved in closed form? What if we want to find the value of the solution at some particular x ? Or perhaps we want to produce a graph of the solution to inspect the behavior. In this section we will learn about the basics of numerical approximation of solutions.

The simplest method for approximating a solution is Euler's Method.¹ It works as follows: Take x_0 and compute the slope $k = f(x_0, y_0)$. The slope is the change in y per unit change in x . Follow the line for an interval of length h on the x -axis. Hence if $y = y_0$ at x_0 , then we say that y_1 (the approximate value of y at $x_1 = x_0 + h$) is $y_1 = y_0 + hk$. Rinse, repeat! Let $k = f(x_1, y_1)$, and then compute $x_2 = x_1 + h$, and $y_2 = y_1 + hk$. Now compute x_3 and y_3 using x_2 and y_2 , etc. Consider the equation $y' = \frac{y^2}{3}$, $y(0) = 1$, and $h = 1$. Then $x_0 = 0$ and $y_0 = 1$. We compute

$$\begin{aligned} x_1 &= x_0 + h = 0 + 1 = 1, & y_1 &= y_0 + h f(x_0, y_0) = 1 + 1 \cdot \frac{1}{3} = \frac{4}{3} \approx 1.333, \\ x_2 &= x_1 + h = 1 + 1 = 2, & y_2 &= y_1 + h f(x_1, y_1) = \frac{4}{3} + 1 \cdot \left(\frac{4}{3}\right)^2 = \frac{52}{27} \approx 1.926. \end{aligned} \quad (1.9.1)$$

We then draw an approximate graph of the solution by connecting the points (x_0, y_0) , (x_1, y_1) , (x_2, y_2) , ... For the first two steps of the method see Figure 1.9.1.

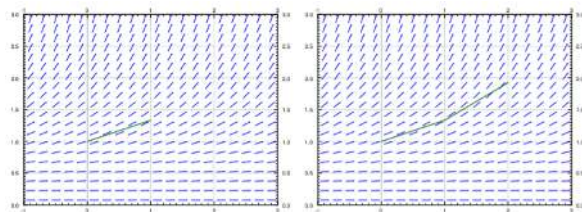


Figure 1.9.1: First two steps of Euler's method with $h = 1$ for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$.

More abstractly, for any $i = 0, 1, 2, 3, \dots$, we compute

$$x_{i+1} = x_i + h, \quad y_{i+1} = y_i + h f(x_i, y_i).$$

The line segments we get are an approximate graph of the solution. Generally it is not exactly the solution. See Figure 1.9.2 for the plot of the real solution and the approximation.

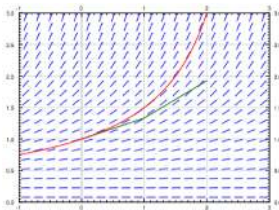


Figure 1.9.2: Two steps of Euler's method (step size 1) and the exact solution for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$.

We continue with the equation $y' = \frac{y^2}{3}$, $y(0) = 1$. Let us try to approximate $y(2)$ using Euler's method. In Figures 1.9.1 and 1.9.2 we have graphically approximated $y(2)$ with step size 1. With step size 1, we have $y(2) \approx 1.926$. The real answer is 3. We are approximately 1.074 off. Let us halve the step size. Computing y_4 with $h = 0.5$, we find that $y(2) \approx 2.209$, so an error of about 0.791. Table 1.9.1 gives the values computed for various parameters.

? Exercise 1.9.1

Solve this equation exactly and show that $y(2) = 3$.

The difference between the actual solution and the approximate solution we will call the error. We will usually talk about just the size of the error and we do not care much about its sign. The main point is, that we usually do not know the real solution, so we only have a vague understanding of the error. If we knew the error exactly ... what is the point of doing the approximation?

h	Approximate $y(2)$	Error	$\frac{\text{Error}}{\text{Previous error}}$
1	1.92593	1.07407	
0.5	2.20861	0.79139	0.73681
0.25	2.47250	0.52751	0.66656
0.125	2.68034	0.31966	0.60599
0.0625	2.82040	0.17960	0.56184
0.03125	2.90412	0.09588	0.53385
0.015625	2.95035	0.04965	0.51779
0.0078125	2.97472	0.02528	0.50913

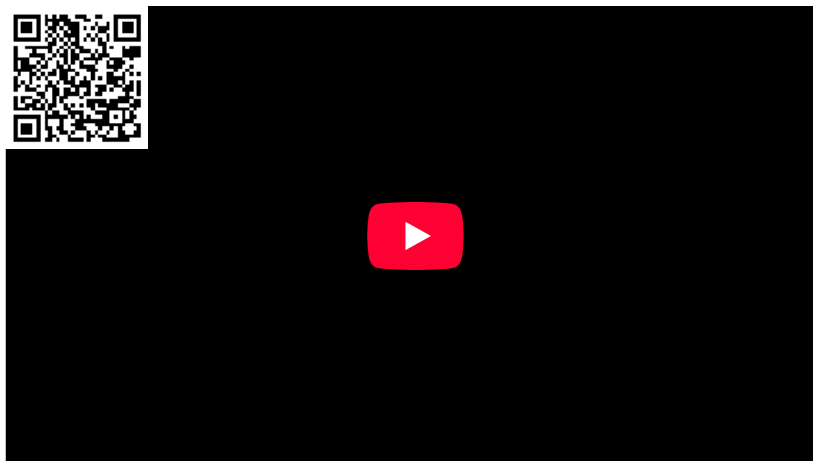
Table 1.9.1: Euler's method approximation of $y(2)$ where $y' = \frac{y^2}{3}$, $y(0) = 1$.

We notice that except for the first few times, every time we halved the interval the error approximately halved. This halving of the error is a general feature of Euler's method as it is a first order method. In the IODE Project II you are asked to implement a second order method. A second order method reduces the error to approximately one quarter every time we

halve the interval (second order as $\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$).

To get the error to be within 0.1 of the answer we had to already do 64 steps. To get it to within 0.01 we would have to halve another three or four times, meaning doing 512 to 1024 steps. That is quite a bit to do by hand. The improved Euler method from IODE Project II should quarter the error every time we halve the interval, so we would have to approximately do half as many “halvings” to get the same error. This reduction can be a big deal. With 10 halvings (starting at $h = 1$) we have 1024 steps, whereas with 5 halvings we only have to do 32 steps, assuming that the error was comparable to start with. A computer may not care about this difference for a problem this simple, but suppose each step would take a second to compute (the function may be substantially more difficult to compute than $\frac{y^2}{3}$). Then the difference is 32 seconds versus about 17 minutes. Note: We are not being altogether fair, a second order method would probably double the time to do each step. Even so, it is 1 minute versus 17 minutes. Next, suppose that we have to repeat such a calculation for different parameters a thousand times. You get the idea.

Below is a video on using Euler's Method to approximate the solution to a differential equation.



Note that in practice we do not know how large the error is! How do we know what is the right step size? Well, essentially we keep halving the interval, and if we are lucky, we can estimate the error from a few of these calculations and the assumption that the error goes down by a factor of one half each time (if we are using standard Euler).

? Exercise 1.9.2

In the table above, suppose you do not know the error. Take the approximate values of the function in the last two lines, assume that the error goes down by a factor of 2. Can you estimate the error in the last time from this? Does it (approximately) agree with the table? Now do it for the first two rows. Does this agree with the table?

Let us talk a little bit more about the example $y' = \frac{y^2}{3}$, $y(0) = 1$. Suppose that instead of the value $y(2)$ we wish to find $y(3)$. The results of this effort are listed in Table 1.9.2 for successive halvings of h . What is going on here? Well, you should solve the equation exactly and you will notice that the solution does not exist at $x = 3$. In fact, the solution goes to infinity when you approach $x = 3$.

h	Approximate $y(3)$
1	3.16232
0.5	4.54329
0.25	6.86079
0.125	10.80321
0.0625	17.59893
0.03125	29.46004
0.015625	50.40121
0.0078125	87.75769

Table 1.9.2: Attempts to use Euler's to approximate $y(3)$ where $y' = \frac{y^2}{3}$, $y(0) = 1$.

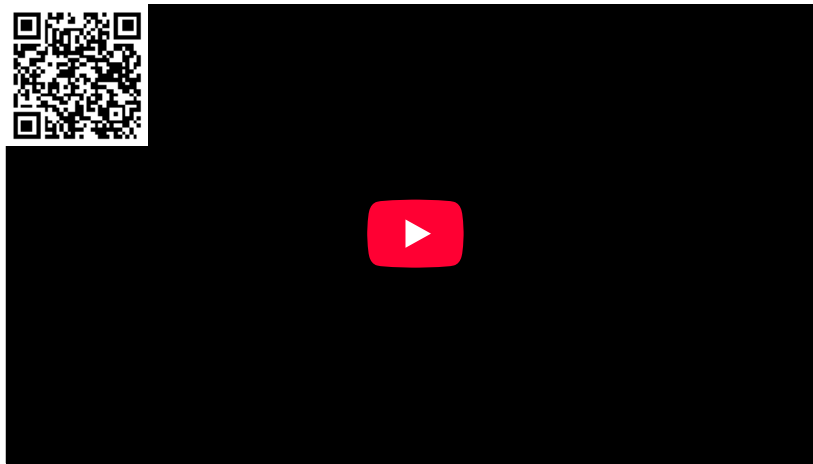
Another case when things can go bad is if the solution oscillates wildly near some point. Such an example is given in IODE Project II. The solution may exist at all points, but even a much better numerical method than Euler would need an insanely small step size to approximate the solution with reasonable precision. And computers might not be able to easily handle such a small step size.

In real applications we would not use a simple method such as Euler's. The simplest method that would probably be used in a real application is the standard Runge-Kutta method (see exercises). That is a fourth order method, meaning that if we halve the interval, the error generally goes down by a factor of 16 (it is fourth order as $\frac{1}{16} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$).

Choosing the right method to use and the right step size can be very tricky. There are several competing factors to consider.

- Computational time: Each step takes computer time. Even if the function f is simple to compute, we do it many times over. Large step size means faster computation, but perhaps not the right precision.
- Roundoff errors: Computers only compute with a certain number of significant digits. Errors introduced by rounding numbers off during our computations become noticeable when the step size becomes too small relative to the quantities we are working with. So reducing step size may in fact make errors worse.
- Stability: Certain equations may be numerically unstable. What may happen is that the numbers never seem to stabilize no matter how many times we halve the interval. We may need a ridiculously small interval size, which may not be practical due to roundoff errors or computational time considerations. Such problems are sometimes called stiff. In the worst case, the numerical computations might be giving us bogus numbers that look like a correct answer. Just because the numbers have stabilized after successive halving, does not mean that we must have the right answer.

Below is a video on using Euler's Method to approximation a solution to a differential equation.



We have seen just the beginnings of the challenges that appear in real applications. Numerical approximation of solutions to differential equations is an active research area for engineers and mathematicians. For example, the general purpose method used for the ODE solver in Matlab and Octave (as of this writing) is a method that appeared in the literature only in the 1980s.

1.9.1: Footnotes

[1] Named after the Swiss mathematician Leonhard Paul Euler (1707–1783). The correct pronunciation of the name sounds more like “oiler.”

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1.10: Exact Equations

Another type of equation that comes up quite often in physics and engineering is an *exact equation*. Suppose $F(x, y)$ is a function of two variables, which we call the *potential function*. The naming should suggest potential energy, or electric potential. Exact equations and potential functions appear when there is a conservation law at play, such as conservation of energy. Let us make up a simple example. Let

$$F(x, y) = x^2 + y^2.$$

We are interested in the lines of constant energy, that is lines where the energy is conserved; we want curves where $F(x, y) = C$, for some constant C . In our example, the curves $x^2 + y^2 = C$ are circles. See Figure 1.10.1.

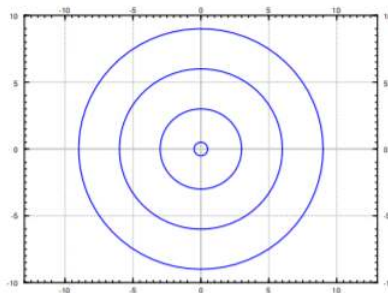


Figure 1.10.1: Solutions to $F(x, y) = x^2 + y^2 = C$ for various C .

We take the *total derivative* of F :

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy.$$

For convenience, we will make use of the notation of $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$. In our example,

$$dF = 2x dx + 2y dy.$$

We apply the total derivative to $F(x, y) = C$, to find the differential equation $dF = 0$. The differential equation we obtain in such a way has the form

$$M dx + N dy = 0, \quad \text{or} \quad M + N \frac{dy}{dx} = 0.$$

An equation of this form is called *exact* if it was obtained as $dF = 0$ for some potential function F . In our simple example, we obtain the equation

$$2x dx + 2y dy = 0, \quad \text{or} \quad 2x + 2y \frac{dy}{dx} = 0.$$

Since we obtained this equation by differentiating $x^2 + y^2 = C$, the equation is exact. We often wish to solve for y in terms of x . In our example,

$$y = \pm \sqrt{C^2 - x^2}.$$

An interpretation of the setup is that at each point $\vec{v} = (M, N)$ is a vector in the plane, that is, a direction and a magnitude. As M and N are functions of (x, y) , we have a *vector field*. The particular vector field $\vec{v}$ that comes from an exact equation is a so-called *conservative vector field*, that is, a vector field that comes with a potential function $F(x, y)$, such that

$$\vec{v} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right).$$

Let γ be a path in the plane starting at (x_1, y_1) and ending at (x_2, y_2) . If we think of $\vec{v}$ as force, then the work required to move along γ is

$$\int_{\gamma} \vec{v}(\vec{r}) \cdot d\vec{r} = \int_{\gamma} M dx + N dy = F(x_2, y_2) - F(x_1, y_1).$$

That is, the work done only depends on endpoints, that is where we start and where we end. For example, suppose F is gravitational potential. The derivative of F given by $\vec{v}$ is the gravitational force. What we are saying is that the work required to move a heavy box from the ground floor to the roof, only depends on the change in potential energy. That is, the work done is the same no matter what path we took; if we took the stairs or the elevator. Although if we took the elevator, the elevator is doing the work for us. The curves $F(x, y) = C$ are those where no work need be done, such as the heavy box sliding along without accelerating or breaking on a perfectly flat roof, on a cart with incredibly well oiled wheels.

An exact equation is a conservative vector field, and the implicit solution of this equation is the potential function.

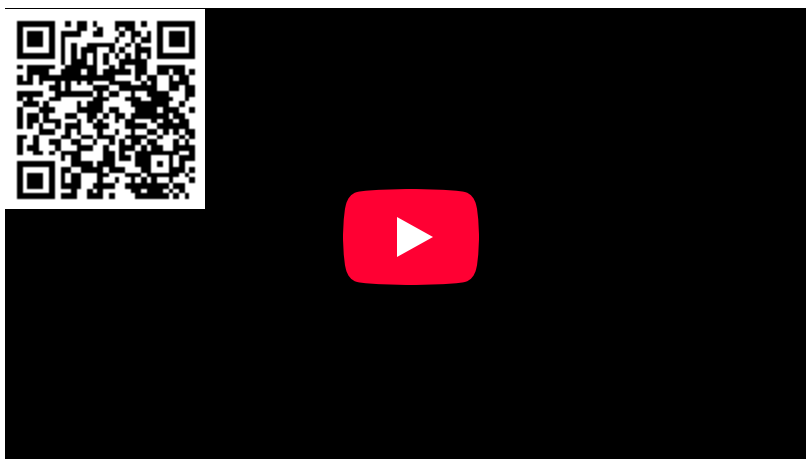
1.10.1: Solving exact equations

Now you, the reader, should ask: Where did we solve a differential equation? Well, in applications we generally know M and N , but we do not know F . That is, we may have just started with $2x + 2y \frac{dy}{dx} = 0$, or perhaps even

$$x + y \frac{dy}{dx} = 0.$$

It is up to us to find some potential F that works. Many different F will work; adding a constant to F does not change the equation. Once we have a potential function F , the equation $F(x, y(x)) = C$ gives an implicit solution of the ODE.

Below is a video on solving an exact first order differential equation.



Below is another video on solving an exact first order differential equation.



✓ Example 1.10.1

Let us find the general solution to $2x + 2y \frac{dy}{dx} = 0$. Forget we knew what F was.

Solution

If we know that this is an exact equation, we start looking for a potential function F . We have $M = 2x$ and $N = 2y$. If F exists, it must be such that $F_x(x, y) = 2x$. Integrate in the x variable to find

$$F(x, y) = x^2 + A(y), \quad (1.10.1)$$

for some function $A(y)$. The function A is the , though it is only constant as far as x is concerned, and may still depend on y . Now differentiate (1.10.1) in y and set it equal to N , which is what F_y is supposed to be:

$$2y = F_y(x, y) = A'(y).$$

Integrating, we find $A(y) = y^2$. We could add a constant of integration if we wanted to, but there is no need. We found $F(x, y) = x^2 + y^2$. Next for a constant C , we solve

$$F(x, y(x)) = C.$$

for y in terms of x . In this case, we obtain $y = \pm\sqrt{C^2 - x^2}$ as we did before.

? Exercise 1.10.1

Why did we not need to add a constant of integration when integrating $A'(y) = 2y$? Add a constant of integration, say 3, and see what F you get. What is the difference from what we got above, and why does it not matter?

The procedure, once we know that the equation is exact, is:

- Integrate $F_x = M$ in x resulting in $F(x, y) = \text{something} + A(y)$.
- Differentiate this F in y , and set that equal to N , so that we may find $A(y)$ by integration.

The procedure can also be done by first integrating in y and then differentiating in x . Pretty easy huh? Let's try this again.

✓ Example 1.10.2

Consider now $2x + y + xy \frac{dy}{dx} = 0$.

OK, so $M = 2x + y$ and $N = xy$. We try to proceed as before. Suppose F exists. Then $F_x(x, y) = 2x + y$. We integrate:

$$F(x, y) = x^2 + xy + A(y)$$

for some function $A(y)$. Differentiate in y and set equal to N :

$$N = xy = F_y(x, y) = x + A'(y).$$

But there is no way to satisfy this requirement! The function xy cannot be written as x plus a function of y . The equation is not exact; no potential function F exists.

But there is no way to satisfy this requirement! The function xy cannot be written as x plus a function of y . The equation is not exact; no potential function F exists

Below is a video on solving an exact first order differential equation.



Below is another video on solving an exact first order differential equation.



Below is another video on solving an exact first order differential equation.



Below is another video on solving an exact first order differential equation.



Is there an easier way to check for the existence of F , other than failing in trying to find it? Turns out there is. Suppose $M = F_x$ and $N = F_y$. Then as long as the second derivatives are continuous,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

Let us state it as a theorem. Usually this is called the Poincaré Lemma.¹

Theorem 1.10.1

Poincaré

If M and N are continuously differentiable functions of (x, y) , and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then near any point there is a function $F(x, y)$ such that $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$.

The theorem doesn't give us a global F defined everywhere. In general, we can only find the potential locally, near some initial point. By this time, we have come to expect this from differential equations.

Let us return to Example 1.10.2 where $M = 2x + y$ and $N = x - 1$. Notice $M_y = 1$ and $N_x = 1$, which are clearly not equal. The equation is not exact.

Example 1.10.3

Solve

$$\frac{dy}{dx} = \frac{-2x - y}{x - 1}, \quad y(0) = 1.$$

Solution

We write the equation as

$$(2x + y) + (x - 1) \frac{dy}{dx} = 0,$$

so $M = 2x + y$ and $N = x - 1$. Then

$$M_y = 1 = N_x.$$

The equation is exact. Integrating M in x , we find

$$F(x, y) = x^2 + xy + A(y).$$

Differentiating in y and setting to N , we find

$$x - 1 = x + A'(y).$$

So $A'(y) = -1$, and $A(y) = -y$ will work. Take $F(x, y) = x^2 + xy - y$. We wish to solve $x^2 + xy - y = C$. First let us find C . As $y(0) = 1$ then $F(0, 1) = C$. Therefore $0^2 + 0 \times 1 - 1 = C$, so $C = -1$. Now we solve $x^2 + xy - y = -1$ for y to get

$$y = \frac{-x^2 - 1}{x - 1}.$$

✓ Example 1.10.4

Solve

$$-\frac{y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy = 0, \quad y(1) = 2.$$

Solution

We leave to the reader to check that $M_y = N_x$.

This vector field (M, N) is not conservative if considered as a vector field of the entire plane minus the origin. The problem is that if the curve γ is a circle around the origin, say starting at $(1, 0)$ and ending at $(1, 0)$ going counterclockwise, then if F existed we would expect

$$0 = F(1, 0) - F(1, 0) = \int_{\gamma} F_x dx + F_y dy = \int_{\gamma} \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy = 2\pi.$$

That is nonsense! We leave the computation of the path integral to the interested reader, or you can consult your multivariable calculus textbook. So there is no potential function F defined everywhere outside the origin $(0, 0)$.

If we think back to the theorem, it does not guarantee such a function anyway. It only guarantees a potential function locally, that is only in some region near the initial point. As $y(1) = 2$ we start at the point $(1, 2)$. Considering $x > 0$ and integrating M in x or N in y , we find

$$F(x, y) = \arctan\left(\frac{y}{x}\right).$$

The implicit solution is $\arctan\left(\frac{y}{x}\right) = C$. Solving, $y = \tan(C)x$. That is, the solution is a straight line. Solving $y(1) = 2$ gives us that $\tan(C) = 2$, and so $y = 2x$ is the desired solution. See Figure 1.10.1, and note that the solution only exists for $x > 0$.

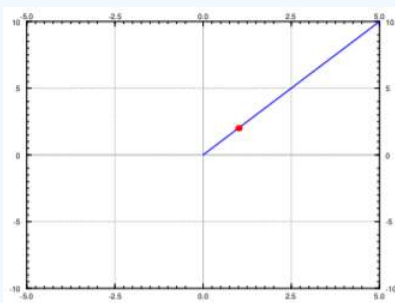


Figure 1.10.1: Solution to $-\frac{y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy = 0$, $y(1) = 2$, with initial point marked.

✓ Example 1.10.5

Solve

$$x^2 + y^2 + 2y(x + 1)\frac{dy}{dx} = 0.$$

Solution

The reader should check that this equation is exact. Let $M = x^2 + y^2$ and $N = 2y(x + 1)$. We follow the procedure for exact equations

$$F(x, y) = \frac{1}{3}x^3 + xy^2 + A(y),$$

and

$$2y(x+1) = 2xy + A'(y).$$

Therefore $A'(y) = 2y$ or $A(y) = y^2$ and $F(x, y) = \frac{1}{3}x^3 + xy^2 + y^2$. We try to solve $F(x, y) = C$. We easily solve for y^2 and then just take the square root:

$$y^2 = \frac{C - (\frac{1}{3})x^3}{x+1}, \quad \text{so} \quad y = \pm \sqrt{\frac{C - (\frac{1}{3})x^3}{x+1}}.$$

When $x = -1$, the term in front of $\frac{dy}{dx}$ vanishes. You can also see that our solution is not valid in that case. However, one could in that case try to solve for x in terms of y starting from the implicit solution $\frac{1}{3}x^3 + xy^2 + y^2 = C$. The solution is somewhat messy and we leave it as implicit.

1.10.2: Integrating factors

Sometimes an equation $M dx + N dy = 0$ is not exact, but it can be made exact by multiplying with a function $u(x, y)$. That is, perhaps for some nonzero function $u(x, y)$,

$$u(x, y)M(x, y) dx + u(x, y)N(x, y) dy = 0$$

is exact. Any solution to this new equation is also a solution to $M dx + N dy = 0$.

In fact, a linear equation

$$\frac{dy}{dx} + p(x)y = f(x), \quad \text{or} \quad (p(x)y - f(x)) dx + dy = 0$$

is always such an equation. Let $r(x) = e^{\int p(x) dx}$ be the integrating factor for a linear equation. Multiply the equation by $r(x)$ and write it in the form of $M + N \frac{dy}{dx} = 0$.

$$r(x)p(x)y - r(x)f(x) + r(x)\frac{dy}{dx} = 0.$$

Then $M = r(x)p(x)y - r(x)f(x)$, so $M_y = r(x)p(x)$, while $N = r(x)$, so $N_x = r'(x) = r(x)p(x)$. In other words, we have an exact equation. Integrating factors for linear functions are just a special case of integrating factors for exact equations.

But how do we find the integrating factor u ? Well, given an equation

$$M dx + N dy = 0,$$

u should be a function such that

$$\frac{\partial}{\partial y} [uM] = u_y M + u M_y = \frac{\partial}{\partial x} [uN] = u_x N + u N_x.$$

Therefore,

$$(M_y - N_x)u = u_x N - u_y M.$$

At first it may seem we replaced one differential equation by another. True, but all hope is not lost.

A strategy that often works is to look for a u that is a function of x alone, or a function of y alone. If u is a function of x alone, that is $u(x)$, then we write $u'(x)$ instead of u_x , and u_y is just zero. Then

$$\frac{M_y - N_x}{N} u = u'.$$

In particular, $\frac{M_y - N_x}{N}$ ought to be a function of x alone (not depend on y). If so, then we have a linear equation

$$u' - \frac{M_y - N_x}{N} u = 0.$$

Letting $P(x) = \frac{M_y - N_x}{N}$, we solve using the standard integrating factor method, to find $u(x) = Ce^{\int P(x) dx}$. The constant in the solution is not relevant, we need any nonzero solution, so we take $C = 1$. Then $u(x) = e^{\int P(x) dx}$ is the integrating factor.

Similarly we could try a function of the form $u(y)$. Then

$$\frac{M_y - N_x}{M} u = -u'.$$

In particular, $\frac{M_y - N_x}{M}$ ought to be a function of y alone. If so, then we have a linear equation

$$u' + \frac{M_y - N_x}{M} u = 0.$$

Letting $Q(y) = \frac{M_y - N_x}{M}$, we find $u(y) = Ce^{-\int Q(y) dy}$. We take $C = 1$. So $u(y) = e^{-\int Q(y) dy}$ is the integrating factor.

✓ Example 1.10.6

Solve

$$\frac{x^2 + y^2}{x + 1} + 2y \frac{dy}{dx} = 0.$$

Solution

Let $M = \frac{x^2 + y^2}{x + 1}$ and $N = 2y$. Compute

$$M_y - N_x = \frac{2y}{x + 1} - 0 = \frac{2y}{x + 1}.$$

As this is not zero, the equation is not exact. We notice

$$P(x) = \frac{M_y - N_x}{N} = \frac{2y}{x + 1} \frac{1}{2y} = \frac{1}{x + 1}$$

is a function of x alone. We compute the integrating factor

$$e^{\int P(x) dx} = e^{\ln(x+1)} = x + 1.$$

We multiply our given equation by $(x + 1)$ to obtain

$$x^2 + y^2 + 2y(x + 1) \frac{dy}{dx} = 0,$$

which is an exact equation that we solved in Example 1.10.5. The solution was

$$y = \pm \sqrt{\frac{C - (\frac{1}{3})x^3}{x + 1}}.$$

✓ Example 1.10.7

Solve

$$y^2 + (xy + 1) \frac{dy}{dx} = 0.$$

Solution

First compute

$$M_y - N_x = 2y - y = y.$$

As this is not zero, the equation is not exact. We observe

$$Q(y) = \frac{M_y - N_x}{M} = \frac{y}{y^2} = \frac{1}{y}$$

is a function of y alone. We compute the integrating factor

$$e^{-\int Q(y) dy} = e^{-\ln y} = \frac{1}{y}.$$

Therefore we look at the exact equation

$$y + \frac{xy+1}{y} \frac{dy}{dx} = 0.$$

The reader should double check that this equation is exact. We follow the procedure for exact equations

$$F(x, y) = xy + A(y),$$

and

$$\frac{xy+1}{y} = x + \frac{1}{y} = x + A'(y).$$

Consequently $A'(y) = \frac{1}{y}$ or $A(y) = \ln y$. Thus $F(x, y) = xy + \ln y$. It is not possible to solve $F(x, y) = C$ for y in terms of elementary functions, so let us be content with the implicit solution:

$$xy + \ln y = C.$$

We are looking for the general solution and we divided by y above. We should check what happens when $y = 0$, as the equation itself makes perfect sense in that case. We plug in $y = 0$ to find the equation is satisfied. So $y = 0$ is also a solution.

1.10.3: Footnotes

[1] Named for the French polymath [Jules Henri Poincaré](#) (1854–1912).

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1.11: First Order Linear PDE

We only considered ODE so far, so let us solve a linear first order PDE. Consider the equation

$$a(x, t) u_x + b(x, t) u_t + c(x, t) u = g(x, t), \quad u(x, 0) = f(x), \quad -\infty < x < \infty, \quad t > 0,$$

where $u(x, t)$ is a function of x and t . The *initial condition* $u(x, 0) = f(x)$ is now a function of x rather than just a number. In these problems, it is useful to think of x as position and t as time. The equation describes the evolution of a function of x as time goes on. Below, the coefficients a , b , c , and the function g are mostly going to be constant or zero. The method we describe works with nonconstant coefficients, although the computations may get difficult quickly.

This method we use is the . The idea is that we find lines along which the equation is an ODE that we solve. We will see this technique again for second order PDE when we encounter the wave equation in [Section 4.8](#).

✓ Example 1.11.1

Consider the equation

$$u_t + \alpha u_x = 0, \quad u(x, 0) = f(x).$$

This particular equation, $u_t + \alpha u_x = 0$, is called the *transport equation*.

The data will propagate along curves called characteristics. The idea is to change to the so-called *characteristic coordinates*. If we change to these coordinates, the equation simplifies. The change of variables for this equation is

$$\xi = x - \alpha t, \quad s = t.$$

Let's see what the equation becomes. Remember the chain rule in several variables.

$$\begin{aligned} u_t &= u_\xi \xi_t + u_s s_t = -\alpha u_\xi + u_s, \\ u_x &= u_\xi \xi_x + u_s s_x = u_\xi. \end{aligned} \tag{1.11.1}$$

The equation in the coordinates ξ and s becomes

$$\underbrace{(-\alpha u_\xi + u_s)}_{u_t} + \alpha \underbrace{(u_\xi)}_{u_x} = 0,$$

or in other words

$$u_s = 0.$$

That is trivial to solve. Treating ξ as simply a parameter, we have obtained the ODE $\frac{du}{ds} = 0$.

The solution is a function that does not depend on s (but it does depend on ξ). That is, there is some function A such that

$$u = A(\xi) = A(x - \alpha t).$$

The initial condition says that:

$$f(x) = u(x, 0) = A(x - \alpha 0) = A(x),$$

so $A = f$. In other words,

$$u(x, t) = f(x - \alpha t).$$

Everything is simply moving right at speed α as t increases. The curve given by the equation

$$\xi = \text{constant}$$

is called the characteristic. See Figure 1.11.1. In this case, the solution does not change along the characteristic.

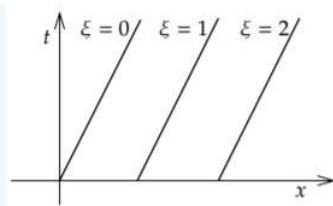


Figure 1.11.1: Characteristic curves.

In the (x, t) coordinates, the characteristic curves satisfy $t = \frac{1}{\alpha}(x - \xi)$, and are in fact lines. The slope of characteristic lines is $\frac{1}{\alpha}$, and for each different ξ we get a different characteristic line.

We see why $u_t + \alpha u_x = 0$ is called the transport equation: everything travels at some constant speed. Sometimes this is called . An example application is material being moved by a river where the material does not diffuse and is simply carried along. In this setup, x is the position along the river, t is the time, and $u(x, t)$ the concentration the material at position x and time t . See Figure 1.11.2 for an example.

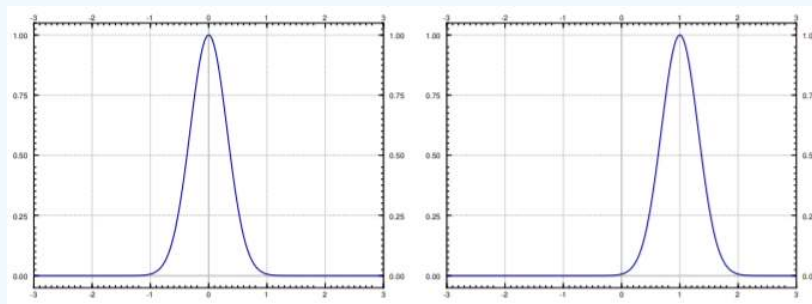


Figure 1.11.2: Example of “transport” in $u_t - u_x = 0$ (that is, $\alpha = 1$) where the initial condition $f(x)$ is a peak at the origin. On the left is a graph of the initial condition $u(x, 0)$. On the right is a graph of the function $u(x, 1)$, that is at time $t = 1$. Notice it is the same graph shifted one unit to the right.

We use similar idea in the more general case:

$$au_x + bu_t + cu = g, \quad u(x, 0) = f(x).$$

We change coordinates to the characteristic coordinates. Let us call these coordinates (ξ, s) . These are coordinates where $au_x + bu_t$ becomes differentiation in the s variable.

Along the characteristic curves (where ξ is constant), we get a new ODE in the s variable. In the transport equation, we got the simple $\frac{du}{ds} = 0$. In general, we get the linear equation

$$\frac{du}{ds} + cu = g. \quad (1.11.2)$$

We think of everything as a function of ξ and s , although we are thinking of ξ as a parameter rather than an independent variable. So the equation is an ODE. It is a linear ODE that we can solve using the integrating factor.

To find the characteristics, think of a curve given parametrically $(x(s), t(s))$. We try to have the curve satisfy

$$\frac{dx}{ds} = a, \quad \frac{dt}{ds} = b.$$

Why? Because when we think of x and t as functions of s we find, using the chain rule,

$$\frac{du}{ds} + cu = \underbrace{\left(u_x \frac{dx}{ds} + u_t \frac{dt}{ds} \right)}_{\frac{du}{ds}} + cu = au_x + bu_t + cu = g.$$

So we get the ODE (1.11.2), which then describes the value of the solution u of the PDE along this characteristic curve. It is also convenient to make sure that $s = 0$ corresponds to $t = 0$, that is $t(0) = 0$. It will be convenient also for $x(0) = \xi$. See Figure 1.11.3

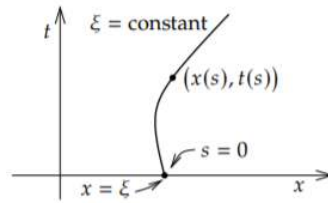


Figure 1.11.3: General characteristic curve.

✓ Example 1.11.2

Consider

$$u_x + u_t + u = x, \quad u(x, 0) = e^{-x^2}.$$

We find the characteristics, that is, the curves given by

$$\frac{dx}{ds} = 1, \quad \frac{dt}{ds} = 1.$$

So

$$x = s + c_1, \quad t = s + c_2,$$

for some c_1 and c_2 . At $s = 0$ we want $t = 0$, and x should be ξ . So we let $c_1 = \xi$ and $c_2 = 0$:

$$x = s + \xi, \quad t = s.$$

The ODE is $\frac{du}{ds} + u = x$, and $x = s + \xi$. So, the ODE to solve along the characteristic is

$$\frac{du}{ds} + u = s + \xi.$$

The general solution of this equation, treating ξ as a parameter, is $u = Ce^{-s} + s + \xi - 1$, for some constant C . At $s = 0$, our initial condition is that u is $e^{-\xi^2}$, since at $s = 0$ we have $x = \xi$. Given this initial condition, we find $C = e^{-\xi^2} - \xi + 1$. So,

$$\begin{aligned} u &= (e^{-\xi^2} - \xi + 1)e^{-s} + s + \xi - 1 \\ &= e^{-\xi^2-s} + (1-\xi)e^{-s} + s + \xi - 1. \end{aligned} \tag{1.11.3}$$

Substitute $\xi = x - t$ and $s = t$ to find u in terms of x and t :

$$\begin{aligned} u &= e^{-\xi^2-s} + (1-\xi)e^{-s} + s + \xi - 1 \\ &= e^{-(x-t)^2-t} + (1-x+t)e^{-t} + x - 1. \end{aligned} \tag{1.11.4}$$

See Figure 1.11.4 for a plot of $u(x, t)$ as a function of two variables.

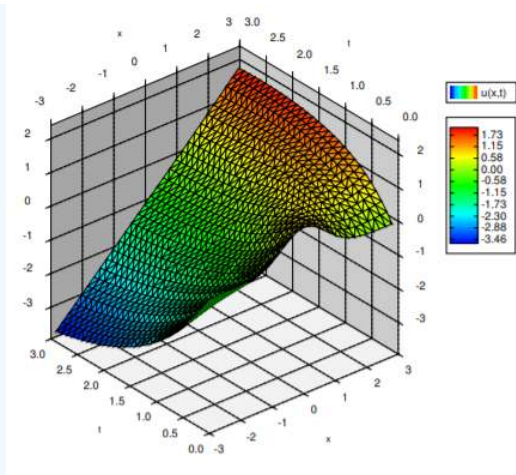


Figure 1.11.4: Plot of the solution $u(x, t)$ to $u_x + u_t + u = x$, $u(x, 0) = e^{-x^2}$.

When the coefficients are not constants, the characteristic curves are not going to be straight lines anymore.

✓ Example 1.11.3

Consider the following variable coefficient equation:

$$xu_x + u_t + 2u = 0, \quad u(x, 0) = \cos(x). \quad (1.11.5)$$

We find the characteristics, that is, the curves given by

$$\frac{dx}{ds} = x, \quad \frac{dt}{ds} = 1.$$

So

$$x = c_1 e^s, \quad t = s + c_2.$$

At $s = 0$, we wish to get the line $t = 0$, and x should be ξ . So

$$x = \xi e^s, \quad t = s.$$

OK, the ODE we need to solve is

$$\frac{du}{ds} + 2u = 0.$$

This is for a fixed ξ . At $s = 0$, we should get that u is $\cos(\xi)$, so that is our initial condition. Consequently,

$$u = e^{-2s} \cos(\xi) = e^{-2t} \cos(xe^{-t}).$$

We make a few closing remarks. One thing to keep in mind is that we would get into trouble if the coefficient in front of u_t , that is the b , is ever zero. Let us consider a quick example of what can go wrong:

$$u_x + u = 0, \quad u(x, 0) = \sin(x).$$

This problem has no solution. If we had a solution, it would imply that $u_x(x, 0) = \cos(x)$, but $u_x(x, 0) + u(x, 0) = \cos(x) + \sin(x) \neq 0$. The problem is that the characteristic curve is now the line $t = 0$, and the solution is already provided on that line!

As long as b is nonzero, it is convenient to ensure that b is positive by multiplying by -1 if necessary, so that positive s means positive t .

Another remark is that if a or b in the equation are variable, the computations can quickly get out of hand, as the expressions for the characteristic coordinates become messy and then solving the ODE becomes even messier. In the examples above, b was

always 1, meaning we got $s = t$ in the characteristic coordinates. If b is not constant, your expression for s will be more complicated.

Finding the characteristic coordinates is really a system of ODE in general if a depends on t or if b depends on x . In that case, we would need techniques of systems of ODE to solve, see [Chapter 3](#) or [Chapter 8](#). In general, if a and b are not linear functions or constants, finding closed form expressions for the characteristic coordinates may be impossible.

Finally, the method of characteristics applies to nonlinear first order PDE as well. In the nonlinear case, the characteristics depend not only on the differential equation, but also on the initial data. This leads to not only more difficult computations, but also the formation of singularities where the solution breaks down at a certain point in time. An example application where first order nonlinear PDE come up is traffic flow theory, and you have probably experienced the formation of singularities: traffic jams. But we digress.

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1.E: First order ODEs (Exercises)

This is a textbook targeted for a one semester first course on differential equations, aimed at engineering students. Prerequisite for the course is the basic calculus sequence.

1.E.1: 1.1: Introduction

1. Find the order of the equation.

a. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} \frac{d^3y}{dx^3} + x = 0$

b. $y'' - 3y' + 2y = x^7$

c. $y' - y^7 = 0$

d. $y''y - (y')^2 = 2$

2. Verify that the function is a solution of the differential equation on some interval, for any choice of the arbitrary constants appearing in the function.

a. $y = ce^{2x}$; $y' = 2y$

b. $y = \frac{x^2}{3} + \frac{c}{x}$; $xy' + y = x^2$

c. $y = \frac{1}{2} + ce^{-x^2}$; $y' + 2xy = x$

d. $y = (1 + ce^{-x^2/2})(1 - ce^{-x^2/2})^{-1}$; $2y' + x(y^2 - 1) = 0$

e. $y = \tan\left(\frac{x^3}{3} + c\right)$; $y' = x^2(1 + y^2)$

f. $y = (c_1 + c_2x)e^x + \sin x + x^2$; $y'' - 2y' + y = -2\cos x + x^2 - 4x + 2$

g. $y = c_1e^x + c_2x + \frac{2}{x}$; $(1 - x)y'' + xy' - y = 4(1 - x - x^2)x^{-3}$

h. $y = x^{-1/2}(c_1 \sin x + c_2 \cos x) + 4x + 8$; $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 4x^3 + 8x^2 + 3x - 2$

3. Verify that the function is a solution of the initial value problem.

a. $y = x \cos x$; $y' = \cos x - y \tan x$, $y(\pi/4) = \frac{\pi}{4\sqrt{2}}$

b. $y = \frac{1+2\ln x}{x^2} + \frac{1}{2}$; $y' = \frac{x^2 - 2x^2y + 2}{x^3}$, $y(1) = \frac{3}{2}$

c. $y = \tan\left(\frac{x^2}{2}\right)$; $y' = x(1 + y^2)$, $y(0) = 0$

d. $y = \frac{2}{x-2}$; $y' = \frac{-y(y+1)}{x}$, $y(1) = -2$

4. Verify that the function is a solution of the initial value problem.

a. $y = x^2(1 + \ln x)$; $y'' = \frac{3xy' - 4y}{x^2}$, $y(e) = 2e^2$, $y'(e) = 5e$

b. $y = \frac{x^2}{3} + x - 1$; $y'' = \frac{x^2 - xy' + y + 1}{x^2}$, $y(1) = \frac{1}{3}$, $y'(1) = \frac{5}{3}$

c. $y = (1 + x^2)^{-1/2}$; $y'' = \frac{(x^2 - 1)y - x(x^2 + 1)y'}{(x^2 + 1)^2}$, $y(0) = 1$, $y'(0) = 0$

d. $y = \frac{x^2}{1-x}$; $y'' = \frac{2(x+y)(xy' - y)}{x^3}$, $y(1/2) = 1/2$, $y'(1/2) = 3$

5. Suppose an object is launched from a point 320 feet above the earth with an initial velocity of 128 ft/sec upward, and the only force acting on it thereafter is gravity. Take $g = 32$ ft/sec².

- Find the highest altitude attained by the object.
- Determine how long it takes for the object to fall to the ground.

8. Let a be a nonzero real number.

- Verify that if c is an arbitrary constant then

$$y = (x - c)^a \quad (A)$$

is a solution of

$$y' = ay^{(a-1)/a} \quad (B)$$

on (c, ∞) .

b. Suppose $a < 0$ or $a > 1$. Can you think of a solution of (B) that isn't of the form (A)?

6. Verify that

$$y = \begin{cases} e^x - 1, & x \geq 0, \\ 1 - e^{-x}, & x < 0, \end{cases}$$

is a solution of

$$y' = |y| + 1$$

on $(-\infty, \infty)$.

7.

(a) Verify that if c is any real number then

$$y = c^2 + cx + 2c + 1 \quad (A)$$

satisfies

$$y' = \frac{-(x+2) + \sqrt{x^2 + 4x + 4}}{2} \quad (B)$$

on some open interval. Identify the open interval.

(b) Verify that

$$y_1 = \frac{-x(x+4)}{4}$$

also satisfies (B) on some open interval, and identify the open interval. (Note that y_1 can't be obtained by selecting a value of c in (A).)

1.E.2: 1.2: Integrals as solutions

? Exercise 1.E. 1.1.1

Solve for v , and then solve for x . Find $x(10)$ to answer the question.

? Exercise 1.E. 1.1.2

Solve $\frac{dy}{dx} = x^2 + x$ for $y(1) = 3$.

? Exercise 1.E. 1.1.3

Solve $\frac{dy}{dx} = \sin(5x)$ for $y(0) = 2$.

? Exercise 1.E. 1.1.4

Solve $\frac{dy}{dx} = \frac{1}{x^2 - 1}$ for $y(0) = 0$.

? Exercise 1.E. 1.1.5

Solve $y' = y^3$ for $y(0) = 1$.

? Exercise 1.E. 1.1.6: (little harder)

Solve $y' = (y - 1)(y + 1)$ for $y(0) = 3$.

? Exercise 1.E. 1.1.7

Solve $\frac{dy}{dx} = \frac{1}{y+1}$ for $y(0) = 0$.

? Exercise 1.E. 1.1.8: (harder)

Solve $y'' = \sin x$ for $y(0) = 0, y'(0) = 2$.

? Exercise 1.E. 1.1.9

A spaceship is traveling at the speed $2t^2 + 1$ km/s (t is time in seconds). It is pointing directly away from earth and at time $t = 0$ it is 1000 kilometers from earth. How far from earth is it at one minute from time $t = 0$?

? Exercise 1.E. 1.1.10

Solve $\frac{dx}{dt} = \sin(t^2) + t$, $x(0) = 20$. It is OK to leave your answer as a definite integral.

? Exercise 1.E. 1.1.11

A dropped ball accelerates downwards at a constant rate 9.8 meters per second squared. Set up the differential equation for the height above ground h in meters. Then supposing $h(0) = 100$ meters, how long does it take for the ball to hit the ground.

? Exercise 1.E. 1.1.12

Find the general solution of $y' = e^x$, and then $y' = e^y$.

? Exercise 1.E. 1.1.13

Solve $\frac{dy}{dx} = e^x + x$ and $y(0) = 10$.

Answer

$$y = e^x + \frac{x^2}{2} + 9$$

? Exercise 1.E. 1.1.14

Solve $x' = \frac{1}{x^2}$, $x(1) = 1$.

Answer

$$x = (3t - 2)^{1/3}$$

? Exercise 1.E. 1.1.15

Solve $x' = \frac{1}{\cos(x)}$, $x(0) = \frac{\pi}{2}$.

Answer

$$Ax = \sin^{-1}(t+1)$$

? Exercise 1.E. 1.1.16

Sid is in a car traveling at speed $10t + 70$ miles per hour away from Las Vegas, where t is in hours. At $t = 0$ the Sid is 10 miles away from Vegas. How far from Vegas is Sid 2 hours later?

Answer

170

? Exercise 1.E. 1.1.17

Solve $y' = y''$, $y(0) = 1$, where n is a positive integer. Hint: You have to consider different cases.

Answer

If $n \neq 1$, then $y = ((1-n)x + 1)^{1/(1-n)}$. If $n = 1$, then $y = e^x$.

? Exercise 1.E. 1.1.18

The rate of change of the volume of a snowball that is melting is proportional to the surface area of the snowball. Suppose the snowball is perfectly spherical. The volume (in centimeters cubed) of a ball of radius r centimeters is $\frac{4}{3}\pi r^3$. The surface area is $4\pi r^2$. Set up the differential equation for how the radius r is changing. Then, suppose that at time $t = 0$ minutes, the radius is 10 centimeters. After 5 minutes, the radius is 8 centimeters. At what time t will the snowball be completely melted?

Answer

The equation is $r' = -C$ for some constant C . The snowball will be completely melted in 25 minutes from time $t = 0$.

? Exercise 1.E. 1.1.19

Find the general solution to $y'''' = 0$. How many distinct constants do you need?

Answer

$$y = Ax^3 + Bx^2 + Cx + D, \text{ so 4 constants.}$$

1.E.3: 1.2: Slope fields

? Exercise 1.E. 1.2.1

Sketch slope field for $y' = e^{x-y}$. How do the solutions behave as x grows? Can you guess a particular solution by looking at the slope field?

? Exercise 1.E. 1.2.2

Sketch slope field for $y' = x^2$.

? Exercise 1.E. 1.2.3

Sketch slope field for $y' = y^2$.

? Exercise 1.E. 1.2.4

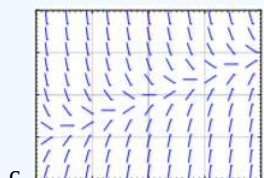
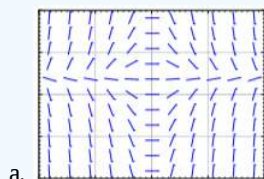
Is it possible to solve the equation $y' = \frac{xy}{\cos x}$ for $y(0) = 1$? Justify.

? Exercise 1.E. 1.2.5

Is it possible to solve the equation $y' = y\sqrt{|x|}$ for $y(0) = 0$? Is the solution unique? Justify.

? Exercise 1.E. 1.2.6

Match equations $y' = 1 - x$, $y' = x - 2y$, $y' = x(1 - y)$ to slope fields. Justify.



? Exercise 1.E. 1.2.7: (challenging)

Take $y' = f(x, y)$, $y(0) = 0$, where $f(x, y) > 1$ for all x and y . If the solution exists for all x , can you say what happens to $y(x)$ as x goes to positive infinity? Explain.

? Exercise 1.E. 1.2.8: (challenging)

Take $(y - x)y' = 0$, $x(0) = 0$.

- Find two distinct solutions.
- Explain why this does not violate Picard's theorem.

? Exercise 1.E. 1.2.9

Suppose $y' = f(x, y)$. What will the slope field look like, explain and sketch an example, if you know the following about $f(x, y)$:

- f does not depend on y .
- f does not depend on x .
- $f, (t, t) = 0$ for any number t .
- $f(x, 0) = 0$ and $f(x, 1) = 1$ for all x .

? Exercise 1.E. 1.2.10

Find a solution to $y' = |y|$, $y(0) = 0$. Does Picard's theorem apply?

? Exercise 1.E. 1.2.11

Take an equation $y' = (y - 2x)g(x, y) + 2$ for some function $g(x, y)$. Can you solve the problem for the initial condition $y(0) = 0$, and if so what is the solution?

? Exercise 1.E. 1.2.12: (challenging)

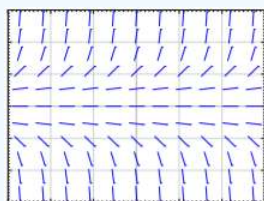
Suppose $y' = f(x, y)$ is such that $f(x, 1) = 0$ for every x , f is continuous and $\frac{\partial f}{\partial y}$ exists and is continuous for every x and y .

- Guess a solution given the initial condition $y(0) = 1$.
- Can graphs of two solutions of the equation for different initial conditions ever intersect?
- Given $y(0) = 0$, what can you say about the solution. In particular, can $y(x) > 1$ for any x ? Can $y(x) = 1$ for any x ? Why or why not?

? Exercise 1.E. 1.2.13

Sketch the slope field of $y' = y^3$. Can you visually find the solution that satisfies $y(0) = 0$?

Answer



$y = 0$ is a solution such that $y(0) = 0$

? Exercise 1.E. 1.2.14

Is it possible to solve $y' = xy$ for $y(0) = 0$? Is the solution unique?

Answer

Yes a solution exists. The equation is $y' = f(x, y)$ where $f(x, y) = xy$. The function $f(x, y)$ is continuous and $\frac{\partial f}{\partial y} = x$, which is also continuous near $(0, 0)$. So a solution exists and is unique. (In fact, $y = 0$ is the solution.)

? Exercise 1.E. 1.2.15

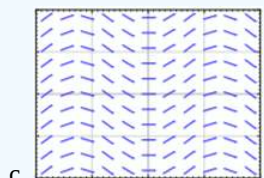
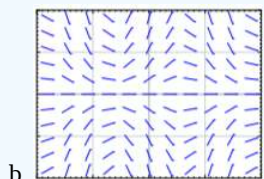
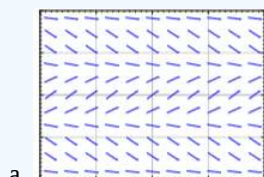
Is it possible to solve $y' = \frac{x}{x^2-1}$ for $y(1) = 0$?

Answer

No, the equation is not defined at $(x, y) = (1, 0)$.

? Exercise 1.E. 1.2.16

Match equations $y' = \sin x$, $y' = \cos y$, $y' = y \cos(x)$ to slope fields. Justify.



Answer

- a. $y' = \cos y$
- b. $y' = y \cos(x)$
- c. $y' = \sin x$

Justification left to reader.

? Exercise 1.E. 1.2.17: (tricky)

Suppose

$$f(y) = \begin{cases} 0 & \text{if } y > 0, \\ 1 & \text{if } y \leq 0. \end{cases}$$

Does $y' = f(y)$, $y(0) = 0$ have a continuously differentiable solution? Does Picard apply? Why, or why not?

Answer

Picard does not apply as f is not continuous at $y = 0$. The equation does not have a continuously differentiable solution. Suppose it did. Notice that $y'(0) = 1$. By the first derivative test, $y(x) > 0$ for small positive x . But then for those x we would have $y'(x) = 0$, so clearly the derivative cannot be continuous.

? Exercise 1.E. 1.2.18

Consider an equation of the form $y' = f(x)$ for some continuous function f , and an initial condition $y(x_0) = y_0$. Does a solution exist for all x ? Why or why not?

Answer

The solution is $y(x) = \int_{x_0}^x f(s)ds + y_0$, and this does indeed exist for every x .

1.E.4: 1.3: Separable Equations

? Exercise 1.E. 1.3.1

Solve $y' = \frac{x}{y}$.

? Exercise 1.E. 1.3.2

Solve $y' = x^2 y$.

? Exercise 1.E. 1.3.3

Solve $\frac{dx}{dt} = (x^2 - 1)$, for $x(0) = 0$.

? Exercise 1.E. 1.3.4

Solve $\frac{dx}{dt} = x \sin(t)$, for $x(0) = 1$.

? Exercise 1.E. 1.3.5

Solve $\frac{dy}{dx} = xy + x + y + 1$. Hint: Factor the right hand side.

? Exercise 1.E. 1.3.6

Solve $xy' = y + 2x^2 y$, where $y(1) = 1$.

? Exercise 1.E. 1.3.7

Solve $\frac{dy}{dx} = \frac{y^2+1}{x^2+1}$, for $y(0) = 1$.

? Exercise 1.E. 1.3.8

Find an implicit solution for $\frac{dy}{dx} = \frac{x^2+1}{y^2+1}$, for $y(0) = 1$.

? Exercise 1.E. 1.3.9

Find an explicit solution for $y' = xe^{-y}$, $y(0) = 1$.

? Exercise 1.E. 1.3.10

Find an explicit solution for $xy' = e^{-y}$, for $y(1) = 1$.

? Exercise 1.E. 1.3.11

Find an explicit solution for $y' = ye^{-x^2}$, $y(0) = 1$. It is alright to leave a definite integral in your answer.

? Exercise 1.E. 1.3.12

Suppose a cup of coffee is at 100 degrees Celsius at time $t = 0$, it is at 70 degrees at $t = 10$ minutes, and it is at 50 degrees at $t = 20$ minutes. Compute the ambient temperature.

? Exercise 1.E. 1.3.13

Solve $y' = 2xy$.

Answer

$$y = Ce^{x^2}$$

? Exercise 1.E. 1.3.14

Solve $x' = 3xt^2 - 3t^2$, $x(0) = 2$.

Answer

$$y = e^{t^3} + 1$$

? Exercise 1.E. 1.3.15

Find an implicit solution for $x' = \frac{1}{3x^2+1}$ $x(0) = 1$.

Answer

$$x^3 + x = t + 2$$

? Exercise 1.E. 1.3.16

Find an explicit solution to $xy' = y^2$, $y(1) = 1$.

Answer

$$y = \frac{1}{1 - \ln x}$$

? Exercise 1.E. 1.3.17

Find an implicit solution to $y' = \frac{\sin(x)}{\cos(y)}$.

Answer

$$\sin(y) = -\cos(x) + C$$

? Exercise 1.E. 1.3.18

Take Example 1.3.3 with the same numbers: 89 degrees at $t = 0$, 85 degrees at $t = 1$, and ambient temperature of 22 degrees. Suppose these temperatures were measured with precision of ± 0.5 degrees. Given this imprecision, the time it takes the coffee to cool to (exactly) 60 degrees is also only known in a certain range. Find this range. Hint: Think about what kind of error makes the cooling time longer and what shorter.

Answer

The range is approximately 7.45 to 12.15 minutes.

? Exercise 1.E. 1.3.19

A population x of rabbits on an island is modeled by $x' = x - (\frac{1}{1000})x^2$, where the independent variable is time in months. At time $t = 0$, there are 40 rabbits on the island.

- Find the solution to the equation with the initial condition.
- How many rabbits are on the island in 1 month, 5 months, 10 months, 15 months (round to the nearest integer)

Answer

- $x = \frac{1000e^t}{e^t + 24}$.
- 102 rabbits after one month, 861 after 5 months, 999 after 10 months, 1000 after 15 months.

1.E.5: 1.4: Linear equations and the integrating factor

In the exercises, feel free to leave answer as a definite integral if a closed form solution cannot be found. If you can find a closed form solution, you should give that.

? Exercise 1.E. 1.4.1

Solve $y' + xy = x$.

? Exercise 1.E. 1.4.2

Solve $y' + 6y = e^x$.

? Exercise 1.E. 1.4.3

Solve $y' + 3x^2y = \sin(x)e^{-x^3}$ with $y(0) = 1$.

? Exercise 1.E. 1.4.4

Solve $y' + \cos(x)y = \cos(x)$.

? Exercise 1.E. 1.4.5

Solve $\frac{1}{x^2+1}y' + xy = 3$ with $y(0) = 0$.

? Exercise 1.E. 1.4.6

Suppose there are two lakes located on a stream. Clean water flows into the first lake, then the water from the first lake flows into the second lake, and then water from the second lake flows further downstream. The in and out flow from each lake is 500 liters per hour. The first lake contains 100 thousand liters of water and the second lake contains 200 thousand liters of water. A truck with 500 kg of toxic substance crashes into the first lake. Assume that the water is being continually mixed perfectly by the stream.

- Find the concentration of toxic substance as a function of time in both lakes.
- When will the concentration in the first lake be below 0.001 kg per liter?
- When will the concentration in the second lake be maximal?

? Exercise 1.E. 1.4.7

Newton's law of cooling states that $\frac{dx}{dt} = -k(x - A)$ where x is the temperature, t is time, A is the ambient temperature, and $k > 0$ is a constant. Suppose that $A = A_0 \cos(\omega t)$ for some constants A_0 and ω . That is, the ambient temperature oscillates (for example night and day temperatures).

- Find the general solution.
- In the long term, will the initial conditions make much of a difference? Why or why not?

? Exercise 1.E. 1.4.8

Initially 5 grams of salt are dissolved in 20 liters of water. Brine with concentration of salt 2 grams of salt per liter is added at a rate of 3 liters a minute. The tank is mixed well and is drained at 3 liters a minute. How long does the process have to continue until there are 20 grams of salt in the tank?

? Exercise 1.E. 1.4.9

Initially a tank contains 10 liters of pure water. Brine of unknown (but constant) concentration of salt is flowing in at 1 liter per minute. The water is mixed well and drained at 1 liter per minute. In 20 minutes there are 15 grams of salt in the tank. What is the concentration of salt in the incoming brine?

? Exercise 1.E. 1.4.10

Solve $y' + 3x^2y + x^2 = 0$.

Answer

$$y = Ce^{-x^3} + \frac{1}{3}$$

? Exercise 1.E. 1.4.11

Solve $y' + 2 \sin(2x)y = 2 \sin(2x)$ with $y(\pi/2) = 3$.

Answer

$$y = 2e^{\cos(2x)+1} + 1$$

? Exercise 1.E. 1.4.12

Suppose a water tank is being pumped out at $3 \frac{\text{L}}{\text{min}}$. The water tank starts at 10 L of clean water. Water with toxic substance is flowing into the tank at $2 \frac{\text{L}}{\text{min}}$, with concentration $20t \frac{\text{g}}{\text{L}}$ at time t . When the tank is half empty, how many grams of toxic substance are in the tank (assuming perfect mixing)?

Answer

250 grams

? Exercise 1.E. 1.4.13

Suppose we have bacteria on a plate and suppose that we are slowly adding a toxic substance such that the rate of growth is slowing down. That is, suppose that $\frac{dP}{dt} = (2 - 0.1t)P$. If $P(0) = 1000$, find the population at $t = 5$.

Answer

$$P(5) = 1000e^{2 \times 5 - 0.05 \times 5^2} = 1000e^{8.75} \approx 6.31 \times 10^6$$

? Exercise 1.E. 1.4.14

A cylindrical water tank has water flowing in at I cubic meters per second. Let A be the area of the cross section of the tank in meters. Suppose water is flowing from the bottom of the tank at a rate proportional to the height of the water level. Set up the differential equation for h , the height of the water, introducing and naming constants that you need. You should also give the units for your constants.

Answer

$Ah' = I - kh$, where k is a constant with units $\frac{m^2}{s}$.

1.E.6: 1.5: Substitution

Hint: Answers need not always be in closed form.

? Exercise 1.E. 1.5.1

Solve $y' + y(x^2 - 1) + xy^6 = 0$, with $y(1) = 1$.

? Exercise 1.E. 1.5.2

Solve $2yy' + 1 = y^2 + x$, with $y(0) = 1$.

? Exercise 1.E. 1.5.3

Solve $y' + xy = y^4$, with $y(0) = 1$.

? Exercise 1.E. 1.5.4

Solve $yy' + x = \sqrt{x^2 + y^2}$.

? Exercise 1.E. 1.5.5

Solve $y' = (x + y - 1)^2$.

? Exercise 1.E. 1.5.6

Solve $y' = \frac{x^2 - y^2}{xy}$, with $y(1) = 2$.

? Exercise 1.E. 1.5.7

Solve $xy' + y + y^2 = 0$, $y(1) = 2$.

Answer

$$y = \frac{2}{3x-2}$$

? Exercise 1.E. 1.5.8

Solve $xy' + y + x = 0$, $y(1) = 1$.

Answer

$$y = \frac{3-x^2}{2x}$$

? Exercise 1.E. 1.5.9

Solve $y^2y' = y^3 - 3x$, $y(0) = 2$.

Answer

$$y = (7e^{3x} + 3x + 1)^{1/3}$$

? Exercise 1.E. 1.5.10

Solve $2yy' = e^{y^2-x^2} + 2x$.

Answer

$$y = \sqrt{x^2 - \ln(C - x)}$$

1.E.7: 1.6: Autonomous equations

? Exercise 1.E. 1.6.1

Consider $x' = x^2$.

- Draw the phase diagram, find the critical points and mark them stable or unstable.
- Sketch typical solutions of the equation.
- Find $\lim_{t \rightarrow \infty} x(t)$ for the solution with the initial condition $x(0) = -1$.

? Exercise 1.E. 1.6.2

Let $x' = \sin x$.

- Draw the phase diagram for $-4\pi \leq x \leq 4\pi$. On this interval mark the critical points stable or unstable.
- Sketch typical solutions of the equation.
- Find $\lim_{t \rightarrow \infty} x(t)$ for the solution with the initial condition $x(0) = 1$.

? Exercise 1.E. 1.6.3

Suppose $f(x)$ is positive for $0 < x < 1$, it is zero when $x = 0$ and $x = 1$, and it is negative for all other x .

- Draw the phase diagram for $x' = f(x)$, find the critical points and mark them stable or unstable.
- Sketch typical solutions of the equation.
- Find $\lim_{t \rightarrow \infty} x(t)$ for the solution with the initial condition $x(0) = 0.5$.

? Exercise 1.E. 1.6.4

Start with the logistic equation $\frac{dx}{dt} = kx(M - x)$. Suppose that we modify our harvesting. That is we will only harvest an amount proportional to current population. In other words we harvest hx per unit of time for some $h > 0$ (Similar to earlier example with h replaced with hx).

- Construct the differential equation.
- Show that if $kM > h$, then the equation is still logistic.
- What happens when $kM < h$?

? Exercise 1.E. 1.6.5

A disease is spreading through the country. Let x be the number of people infected. Let the constant S be the number of people susceptible to infection. The infection rate $\frac{dx}{dt}$ is proportional to the product of already infected people, x , and the number of susceptible but uninfected people, $S - x$.

- Write down the differential equation.
- Supposing $x(0) > 0$, that is, some people are infected at time $t = 0$, what is $\lim_{t \rightarrow \infty} x(t)$.
- Does the solution to part b) agree with your intuition? Why or why not?

? Exercise 1.E. 1.6.6

Let $x' = (x - 1)(x - 2)x^2$.

- Sketch the phase diagram and find critical points.
- Classify the critical points.
- If $x(0) = 0.5$ then find $\lim_{t \rightarrow \infty} x(t)$.

Answer

- 0, 1, 2 are critical points.
- $x = 0$ is unstable (semistable), $x = 1$ is stable, and $x = 2$ is unstable.
- 1

? Exercise 1.E. 1.6.7

Let $x' = e^{-x}$.

- Find and classify all critical points.
- Find $\lim_{t \rightarrow \infty} x(t)$ given any initial condition.

Answer

- There are no critical points.
- ∞

? Exercise 1.E. 1.6.8

Assume that a population of fish in a lake satisfies $\frac{dx}{dt} = kx(M - x)$. Now suppose that fish are continually added at A fish per unit of time.

- Find the differential equation for x .
- What is the new limiting population?

Answer

- $\frac{dx}{dt} = kx(M - x) + A$
- $\frac{kM + \sqrt{(kM)^2 + 4Ak}}{2k}$

? Exercise 1.E. 1.6.9

Suppose $\frac{dx}{dt} = (x - \alpha)(x - \beta)$ for two numbers $\alpha < \beta$.

- Find the critical points, and classify them.

For b), c), d), find $\lim_{t \rightarrow \infty} x(t)$ based on the phase diagram.

- $x(0) < \alpha$,
- $\alpha < x(0) < \beta$,
- $\beta < x(0)$.

Answer

- α is a stable critical point, β is an unstable one.
- α
- α
- ∞ or DNE.

1.E.8: 1.7: Numerical methods: Euler's method

? Exercise 1.E. 1.7.1

Consider $\frac{dx}{dt} = (2t - x)^2$, $x(0) = 2$. Use Euler's method with step size $h = 0.5$ to approximate $x(1)$.

? Exercise 1.E. 1.7.2

Consider $\frac{dx}{dt} = t - x$, $x(0) = 1$.

- Use Euler's method with step sizes $h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ to approximate $x(1)$.
- Solve the equation exactly.
- Describe what happens to the errors for each h you used. That is, find the factor by which the error changed each time you halved the interval.

? Exercise 1.E. 1.7.3

Approximate the value of e by looking at the initial value problem $y' = y$ with $y(0) = 1$ and approximating $y(1)$ using Euler's method with a step size of 0.2.

? Exercise 1.E. 1.7.4

Example of numerical instability: Take $y' = -5y$, $y(0) = 1$. We know that the solution should decay to zero as x grows. Using Euler's method, start with $h = 1$ and compute y_1, y_2, y_3, y_4 to try to approximate $y(4)$. What happened? Now halve the interval. Keep halving the interval and approximating $y(4)$ until the numbers you are getting start to stabilize (that is, until they start going towards zero). Note: You might want to use a calculator.

The simplest method used in practice is the Runge-Kutta method. Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ and a step size h . Everything is the same as in Euler's method, except the computation of y_{i+1} and x_{i+1} .

$$\begin{aligned} k_1 &= f(x_i, y_i), \\ k_2 &= f\left(x_i + \frac{h}{2}, y_i + k_1 \frac{h}{2}\right) & x_{i+1} &= x_i + h, \\ k_3 &= f\left(x_i + \frac{h}{2}, y_i + k_2 \frac{h}{2}\right) & y_{i+1} &= y_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}h, \\ k_4 &= f(x_i + h, y_i + k_3 h). \end{aligned} \tag{1.E.1}$$

? Exercise 1.E. 1.7.5

Consider $\frac{dy}{dx} = yx^2$, $y(0) = 1$.

- Use Runge-Kutta (see above) with step sizes $h = 1$ and $h = \frac{1}{2}$ to approximate $y(1)$.
- Use Euler's method with $h = 1$ and $h = \frac{1}{2}$.
- Solve exactly, find the exact value of $y(1)$, and compare.

? Exercise 1.E. 1.7.6

Let $x' = \sin(xt)$, and $x(0) = 1$. Approximate $x(1)$ using Euler's method with step sizes 1, 0.5, 0.25. Use a calculator and compute up to 4 decimal digits.

Answer

Approximately: 1.0000, 1.2397, 1.382

? Exercise 1.E. 1.7.7

Let $x' = 2t$, and $x(0) = 0$.

- Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1.
- Solve exactly, and compute the errors.
- Compute the factor by which the errors changed.

Answer

- 0, 8, 12
- $x(4) = 16$, so errors are: 16, 8, 4
- Factors are 0.5, 0.5, 0.5

? Exercise 1.E. 1.7.8

Let $x' = xe^{xt+1}$, and $x(0) = 0$.

- Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1.
- Guess an exact solution based on part a) and compute the errors.

Answer

- 0, 0, 0
- $x = 0$ is a solution so errors are: 0, 0, 0

There is a simple way to improve Euler's method to make it a second order method by doing just one extra step. Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, and a step size h . What we do is to pretend we compute the next step as in Euler, that is, we start with (x_i, y_i) , we compute a slope $k_1 = f(x_i, y_i)$, and then look at the point $(x_i + h, y_i + k_1 h)$. Instead of letting our new point be $(x_i + h, y_i + k_1 h)$, we compute the slope at that point, call it k_2 , and then take the average of k_1 and k_2 , hoping that the average is going to be closer to the actual slope on the interval from x_i to $x_i + h$. And we are correct, if we halve the step, the error should go down by a factor of $2^2 = 4$. To summarize, the setup is the same as for regular Euler, except the computation of y_{i+1} and x_{i+1} .

$$\begin{aligned} k_1 &= f(x_i, y_i), & x_{i+1} &= x_i + h, \\ k_2 &= f(x_i + h, y_i + k_1 h), & y_{i+1} &= y_i + \frac{k_1 + k_2}{2} h. \end{aligned} \tag{1.E.2}$$

? Exercise 1.E. 1.7.9

Consider $\frac{dy}{dx} = x + y$, $y(0) = 1$.

- Use the improved Euler's method (see above) with step sizes $h = \frac{1}{4}$ and $h = \frac{1}{8}$ to approximate $y(1)$.
- Use Euler's method with $h = \frac{1}{4}$ and $h = \frac{1}{8}$.
- Solve exactly, find the exact value of $y(1)$.
- Compute the errors, and the factors by which the errors changed.

Answer

- Improved Euler: $y(1) \approx 3.3897$ for $h = 1/4$, $y(1) \approx 3.4237$ for $h = 1/8$,
- Standard Euler: $y(1) \approx 2.8828$ for $h = 1/4$, $y(1) \approx 3.1316$ for $h = 1/8$,
- $y = 2e^x - x - 1$, so $y(2)$ is approximately 3.4366
- Approximate errors for improved Euler: 0.046852 for $h = 1/4$, and 0.012881 for $h = 1/8$. For standard Euler: 0.55375 for $h = 1/4$, and 0.30499 for $h = 1/8$. Factor is approximately 0.27 for improved Euler, and 0.55 for standard Euler.

1.E.9: 1.8 Exact Equations

? Exercise 1.E. 1.8.1

Solve the following exact equations, implicit general solutions will suffice:

- $(2xy + x^2) dx + (x^2 + y^2 + 1) dy = 0$
- $x^5 + y^5 \frac{dy}{dx} = 0$
- $e^x + y^3 + 3xy^2 \frac{dy}{dx} = 0$
- $(x + y) \cos(x) + \sin(x) + \sin(x)y' = 0$

? Exercise 1.E. 1.8.2

Find the integrating factor for the following equations making them into exact equations:

- $e^{xy} dx + \frac{y}{x} e^{xy} dy = 0$
- $\frac{e^x + y^3}{y^2} dx + 3x dy = 0$
- $4(y^2 + x) dx + \frac{2x + 2y^2}{y} dy = 0$
- $2 \sin(y) dx + x \cos(y) dy = 0$

? Exercise 1.E. 1.8.3

Suppose you have an equation of the form: $f(x) + g(y) \frac{dy}{dx} = 0$.

- Show it is exact.
- Find the form of the potential function in terms of f and g .

? Exercise 1.E. 1.8.4

Suppose that we have the equation $f(x) dx - dy = 0$.

- Is this equation exact?
- Find the general solution using a definite integral.

? Exercise 1.E. 1.8.5

Find the potential function $F(x, y)$ of the exact equation $\frac{1+xy}{x} dx + (\frac{1}{y} + x) dy = 0$ in two different ways.

- Integrate M in terms of x and then differentiate in y and set to N .
- Integrate N in terms of y and then differentiate in x and set to M .

? Exercise 1.E. 1.8.6

A function $u(x, y)$ is said to be a if $u_{xx} + u_{yy} = 0$.

- Show if u is harmonic, $-u_y dx + u_x dy = 0$ is an exact equation. So there exists (at least locally) the so-called function $v(x, y)$ such that $v_x = -u_y$ and $v_y = u_x$.

Verify that the following u are harmonic and find the corresponding harmonic conjugates v :

- $u = 2xy$
- $u = e^x \cos y$
- $u = x^3 - 3xy^2$

? Exercise 1.E. 1.8.7

Solve the following exact equations, implicit general solutions will suffice:

- $\cos(x) + ye^{xy} + xe^{xy}y' = 0$
- $(2x + y)dx + (x - 4y)dy = 0$
- $e^x + e^y \frac{dy}{dx} = 0$
- $(3x^2 + 3y)dx + (3y^2 + 3x)dy = 0$

Answer

- $e^{xy} + \sin(x) = C$
- $x^2 + xy - 2y^2 = C$
- $e^x + e^y = C$
- $x^3 + 3xy + y^3 = C$

? Exercise 1.E. 1.8.8

Find the integrating factor for the following equations making them into exact equations:

- $\frac{1}{y}dx + 3ydy = 0$
- $dx - e^{-x-y}dy = 0$
- $\left(\frac{\cos(x)}{y^2} + \frac{1}{y}\right)dx + \frac{x}{y^2}dy = 0$
- $\left(2y + \frac{y^2}{x}\right)dx + (2y + x)dy = 0$

Answer

- Integrating factor is y , equation becomes $dx + 3y^2dy = 0$.
- Integrating factor is e^x , equation becomes $e^x dx - e^{-y}dy = 0$.
- Integrating factor is y^2 , equation becomes $(\cos(x) + y)dx + xdy = 0$.
- Integrating factor is x , equation becomes $(2xy + y^2)dx + (x^2 + 2xy)dy = 0$.

? Exercise 1.E. 1.8.9

- Show that every separable equation $y' = f(x)g(y)$ can be written as an exact equation, and verify that it is indeed exact.
- Using this rewrite $y' = xy$ as an exact equation, solve it and verify that the solution is the same as it was in Example 1.3.1.

Answer

- The equation is $-f(x)dx + \frac{1}{g(y)}dy$, and this is exact because $M = -f(x)$, $N = \frac{1}{g(y)}$, so $M_y = 0 = N_x$.
- $-xdx + \frac{1}{y}dy = 0$, leads to potential function $F(x, y) = -\frac{x^2}{2} + \ln|y|$, solving $F(x, y) = C$ leads to the same solution as the example.

1.E.10: 1.9: First Order Linear PDE

? Exercise 1.E. 1.9.1

Solve

- $u_t + 9u_x = 0$, $u(x, 0) = \sin(x)$,
- $u_t - 8u_x = 0$, $u(x, 0) = \sin(x)$,
- $u_t + \pi u_x = 0$, $u(x, 0) = \sin(x)$,
- $u_t + \pi u_x + u = 0$, $u(x, 0) = \sin(x)$.

? Exercise 1.E. 1.9.2

Solve $u_t + 3u_x = 1$, $u(x, 0) = x^2$.

? Exercise 1.E. 1.9.3

Solve $u_t + 3u_x = x$, $u(x, 0) = e^x$.

? Exercise 1.E. 1.9.4

Solve $u_x + u_t + xu = 0$, $u(x, 0) = \cos(x)$.

? Exercise 1.E. 1.9.5

a. Find the characteristic coordinates for the following equations:

$$u_x + u_t + u = 1, u(x, 0) = \cos(x), \quad 2u_x + 2u_t + 2u = 2, u(x, 0) = \cos(x).$$

b. Solve the two equations using the coordinates.

c. Explain why you got the same solution, although the characteristic coordinates you found were different.

? Exercise 1.E. 1.9.6

Solve $(1 + x^2)u_t + x^2u_x + e^xu = 0$, $u(x, 0) = 0$. Hint: Think a little out of the box.

? Exercise 1.E. 1.9.7

Solve

a. $u_t - 5u_x = 0$, $u(x, 0) = \frac{1}{1+x^2}$,

b. $u_t + 2u_x = 0$, $u(x, 0) = \cos(x)$.

Answer

a. $u = \frac{1}{1+(x+5t)^2}$

b. $u = \cos(x - 2t)$

? Exercise 1.E. 1.9.8

Solve $u_x + u_t + tu = 0$, $u(x, 0) = \cos(x)$.

Answer

$$u = \cos(x - t)e^{-t^2/2}$$

? Exercise 1.E. 1.9.9

Solve $u_x + u_t = 5$, $u(x, 0) = x$.

Answer

$$u = x + 4t$$

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CHAPTER OVERVIEW

2: Applications of First Order Equations

In this chapter, we consider applications of first order differential equations.

2.1: Growth and Decay

2.1E: Growth and Decay (Exercises)

2.2: Cooling and Mixing

2.2E: Cooling and Mixing (Exercises)

2.3: Elementary Mechanics

2.3E: Elementary Mechanics (Exercises)

2.4: Autonomous Second Order Equations

2.4E: Autonomous Second Order Equations (Exercises)

2.5: Applications to Curves

2.5E: Applications to Curves (Exercises)

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2.1: Growth and Decay

This section begins with a discussion of exponential growth and decay, which you have probably already seen in calculus. We consider applications to radioactive decay, carbon dating, and compound interest. We also consider more complicated problems where the rate of change of a quantity is in part proportional to the magnitude of the quantity, but is also influenced by other factors for example, a radioactive substance is manufactured at a certain rate, but decays at a rate proportional to its mass, or a saver makes regular deposits in a savings account that draws compound interest.

Since the applications in this section deal with functions of time, we'll denote the independent variable by t . If Q is a function of t , Q' will denote the derivative of Q with respect to t ; thus,

$$Q' = \frac{dQ}{dt}.$$

Exponential Growth and Decay

One of the most common mathematical models for a physical process is the **exponential model**, where it is assumed that the rate of change of a quantity Q is proportional to Q ; thus

$$Q' = aQ, \quad (2.1.1)$$

where a is the constant of proportionality.

From Example 2.1.3, the general solution of Equation 2.1.1 is

$$Q = ce^{at}$$

and the solution of the initial value problem

$$Q' = aQ, \quad Q(t_0) = Q_0$$

is

$$Q = Q_0 e^{a(t-t_0)}. \quad (2.1.2)$$

Since the solutions of $Q' = aQ$ are exponential functions, we say that a quantity Q that satisfies this equation **grows exponentially** if $a > 0$, or **decays exponentially** if $a < 0$ (Figure 4.1.1).

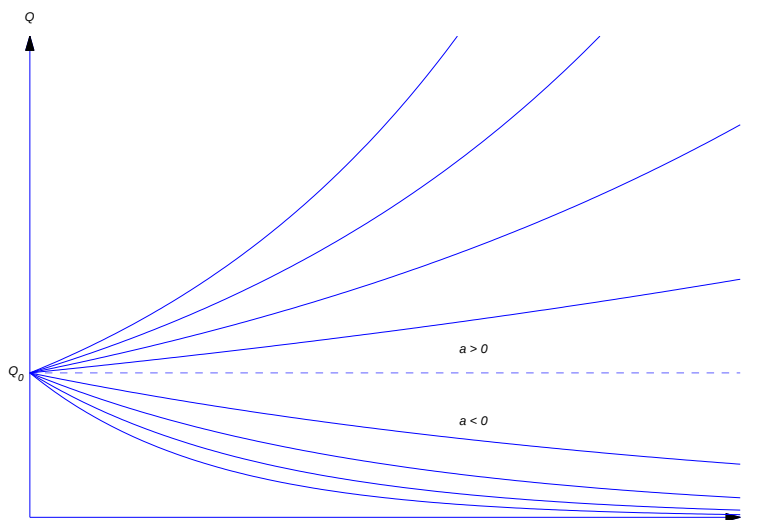


Figure 4.1.1 : Exponential growth and decay

Radioactive Decay

Experimental evidence shows that radioactive material decays at a rate proportional to the mass of the material present. According to this model the mass $Q(t)$ of a radioactive material present at time t satisfies Equation 2.1.1, where a is a negative constant

whose value for any given material must be determined by experimental observation. For simplicity, we'll replace the negative constant a by $-k$, where k is a positive number that we'll call the **decay constant** of the material. Thus, Equation 2.1.1 becomes

$$Q' = -kQ.$$

If the mass of the material present at $t = t_0$ is Q_0 , the mass present at time t is the solution of

$$Q' = -kQ, \quad Q(t_0) = Q_0.$$

From Equation 2.1.2 with $a = -k$, the solution of this initial value problem is

$$Q = Q_0 e^{-k(t-t_0)}. \quad (2.1.3)$$

The **half-life** τ of a radioactive material is defined to be the time required for half of its mass to decay; that is, if $Q(t_0) = Q_0$, then

$$Q(\tau + t_0) = \frac{Q_0}{2}. \quad (2.1.4)$$

From Equation 2.1.3 with $t = \tau + t_0$, Equation 2.1.4 is equivalent to

$$Q_0 e^{-k\tau} = \frac{Q_0}{2},$$

so

$$e^{-k\tau} = \frac{1}{2}.$$

Taking logarithms yields

$$-k\tau = \ln \frac{1}{2} = -\ln 2,$$

so the half-life is

$$\tau = \frac{1}{k} \ln 2. \quad (2.1.5)$$

(Figure 4.1.2). The half-life is independent of t_0 and Q_0 , since it is determined by the properties of material, not by the amount of the material present at any particular time.

✓ Example 4.1.1

A radioactive substance has a half-life of 1620 years.

- If its mass is now 4 g (grams), how much will be left 810 years from now?
- Find the time t_1 when 1.5 g of the substance remain.

Solution a

From Equation 2.1.3 with $t_0 = 0$ and $Q_0 = 4$,

$$Q = 4e^{-kt}, \quad (2.1.6)$$

where we determine k from Equation 2.1.5, with $\tau = 1620$ years:

$$k = \frac{\ln 2}{\tau} = \frac{\ln 2}{1620}.$$

Substituting this in Equation 2.1.6 yields

$$Q = 4e^{-(t \ln 2)/1620}. \quad (2.1.7)$$

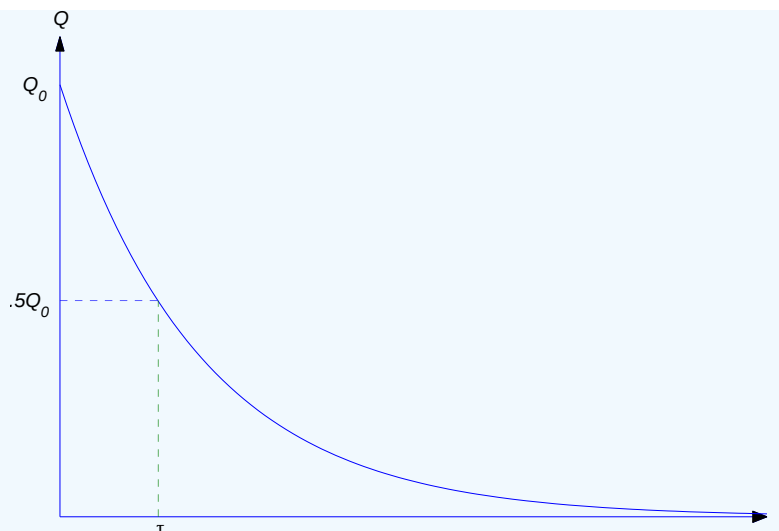


Figure 4.1.2 : Half-life of a radioactive substance

Therefore the mass left after 810 years will be

$$\begin{aligned} Q(810) &= 4e^{-(810 \ln 2)/1620} = 4e^{-(\ln 2)/2} \\ &= 2\sqrt{2} \text{ g.} \end{aligned}$$

Solution b

Setting $t = t_1$ in Equation 2.1.7 and requiring that $Q(t_1) = 1.5$ yields

$$\frac{3}{2} = 4e^{(-t_1 \ln 2)/1620}.$$

Dividing by 4 and taking logarithms yields

$$\ln \frac{3}{8} = -\frac{t_1 \ln 2}{1620}.$$

Since $\ln 3/8 = -\ln 8/3$,

$$t_1 = 1620 \frac{\ln 8/3}{\ln 2} \approx 2292.4 \text{ years.}$$

Interest Compounded Continuously

Suppose we deposit an amount of money Q_0 in an interest-bearing account and make no further deposits or withdrawals for t years, during which the account bears interest at a constant annual rate r . To calculate the value of the account at the end of t years, we need one more piece of information: how the interest is added to the account, or—as the bankers say—how it is **compounded**. If the interest is compounded annually, the value of the account is multiplied by $1 + r$ at the end of each year. This means that after t years the value of the account is

$$Q(t) = Q_0(1 + r)^t.$$

If interest is compounded semiannually, the value of the account is multiplied by $(1 + r/2)$ every 6 months. Since this occurs twice annually, the value of the account after t years is

$$Q(t) = Q_0 \left(1 + \frac{r}{2}\right)^{2t}.$$

In general, if interest is compounded n times per year, the value of the account is multiplied n times per year by $(1 + r/n)$; therefore, the value of the account after t years is

$$Q(t) = Q_0 \left(1 + \frac{r}{n}\right)^{nt}. \quad (2.1.8)$$

Thus, increasing the frequency of compounding increases the value of the account after a fixed period of time. Table 4.1.1 shows the effect of increasing the number of compoundings over $t = 5$ years on an initial deposit of $Q_0 = 100$ (dollars), at an annual interest rate of 6%.

Table 4.1.1 : The effect of compound interest

n (number of compoundings per year)	$\$100 \left(1 + \frac{.06}{n}\right)^{5n}$ (value in dollars after 5 years)
1	\$133.82
2	\$134.39
4	\$134.68
8	\$134.83
364	\$134.98

You can see from Table 4.1.1 that the value of the account after 5 years is an increasing function of n . Now suppose the maximum allowable rate of interest on savings accounts is restricted by law, but the time intervals between successive compoundings isn't; then competing banks can attract savers by compounding often. The ultimate step in this direction is to **compound continuously**, by which we mean that $n \rightarrow \infty$ in Equation 2.1.8. Since we know from calculus that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = e^r,$$

this yields

$$\begin{aligned} Q(t) &= \lim_{n \rightarrow \infty} Q_0 \left(1 + \frac{r}{n}\right)^{nt} = Q_0 \left[\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n\right]^t \\ &= Q_0 e^{rt}. \end{aligned}$$

Observe that $Q = Q_0 e^{rt}$ is the solution of the initial value problem

$$Q' = rQ, \quad Q(0) = Q_0;$$

that is, with continuous compounding the value of the account grows exponentially.

✓ Example 4.1.2

If \$150 is deposited in a bank that pays $5\frac{1}{2}\%$ annual interest compounded continuously, the value of the account after t years is

$$Q(t) = 150e^{.055t}$$

dollars. (Note that it is necessary to write the interest rate as a decimal; thus, $r = .055$.) Therefore, after $t = 10$ years the value of the account is

$$Q(10) = 150e^{.55} \approx \$259.99.$$

✓ Example 4.1.3

We wish to accumulate \$10,000 in 10 years by making a single deposit in a savings account bearing $5\frac{1}{2}\%$ annual interest compounded continuously. How much must we deposit in the account?

Solution

The value of the account at time t is

$$Q(t) = Q_0 e^{.055t}. \quad (2.1.9)$$

Since we want $Q(10)$ to be \$10,000, the initial deposit Q_0 must satisfy the equation

$$10000 = Q_0 e^{.55}, \quad (2.1.10)$$

obtained by setting $t = 10$ and $Q(10) = 10000$ in Equation 2.1.9. Solving Equation 2.1.10 for Q_0 yields

$$Q_0 = 10000e^{-.55} \approx \$5769.50.$$

Mixed Growth and Decay

✓ Example 4.1.4

A radioactive substance with decay constant k is produced at a constant rate of a units of mass per unit time.

- Assuming that $Q(0) = Q_0$, find the mass $Q(t)$ of the substance present at time t .
- Find $\lim_{t \rightarrow \infty} Q(t)$.

Solution a:

Here

$$Q' = \text{rate of increase of } Q - \text{rate of decrease of } Q.$$

The rate of increase is the constant a . Since Q is radioactive with decay constant k , the rate of decrease is kQ . Therefore

$$Q' = a - kQ.$$

This is a linear first order differential equation. Rewriting it and imposing the initial condition shows that Q is the solution of the initial value problem

$$Q' + kQ = a, \quad Q(0) = Q_0. \quad (2.1.11)$$

Since e^{-kt} is a solution of the complementary equation, the solutions of Equation 2.1.11 are of the form $Q = ue^{-kt}$, where $u'e^{-kt} = a$, so $u' = ae^{kt}$. Hence,

$$u = \frac{a}{k}e^{kt} + c$$

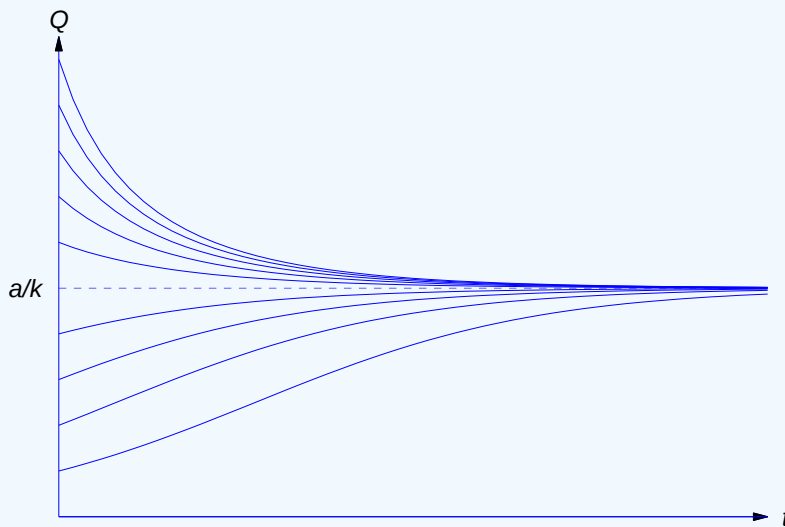


Figure 4.1.3 : $Q(t)$ approaches the steady state value $\frac{a}{k}$ as $t \rightarrow \infty$

and

$$Q = ue^{-kt} = \frac{a}{k} + ce^{-kt}.$$

Since $Q(0) = Q_0$, setting $t = 0$ here yields

$$Q_0 = \frac{a}{k} + c \quad \text{or} \quad c = Q_0 - \frac{a}{k}.$$

Therefore

$$Q = \frac{a}{k} + \left(Q_0 - \frac{a}{k}\right) e^{-kt}. \quad (2.1.12)$$

b. Since $k > 0$, $\lim_{t \rightarrow \infty} e^{-kt} = 0$, so from Equation 2.1.12

$$\lim_{t \rightarrow \infty} Q(t) = \frac{a}{k}.$$

This limit depends only on a and k , and not on Q_0 . We say that a/k is the **steady state** value of Q . From Equation 2.1.12 we also see that Q approaches its steady state value from above if $Q_0 > a/k$, or from below if $Q_0 < a/k$. If $Q_0 = a/k$, then Q remains constant (Figure 4.1.3).

Carbon Dating

The fact that Q approaches a steady state value in the situation discussed in Example 4 underlies the method of **carbon dating**, devised by the American chemist and Nobel Prize Winner **W.S. Libby**.

Carbon 12 is stable, but carbon-14, which is produced by cosmic bombardment of nitrogen in the upper atmosphere, is radioactive with a half-life of about 5570 years. Libby assumed that the quantity of carbon-12 in the atmosphere has been constant throughout time, and that the quantity of radioactive carbon-14 achieved its steady state value long ago as a result of its creation and decomposition over millions of years. These assumptions led Libby to conclude that the ratio of carbon-14 to carbon-12 has been nearly constant for a long time. This constant, which we denote by R , has been determined experimentally.

Living cells absorb both carbon-12 and carbon-14 in the proportion in which they are present in the environment. Therefore the ratio of carbon-14 to carbon-12 in a living cell is always R . However, when the cell dies it ceases to absorb carbon, and the ratio of carbon-14 to carbon-12 decreases exponentially as the radioactive carbon-14 decays. This is the basis for the method of carbon dating, as illustrated in the next example.

✓ Example 4.1.5

An archaeologist investigating the site of an ancient village finds a burial ground where the amount of carbon-14 present in individual remains is between 42 and 44% of the amount present in live individuals. Estimate the age of the village and the length of time for which it survived.

Solution

Let $Q = Q(t)$ be the quantity of carbon-14 in an individual set of remains t years after death, and let Q_0 be the quantity that would be present in live individuals. Since carbon-14 decays exponentially with half-life 5570 years, its decay constant is

$$k = \frac{\ln 2}{5570}.$$

Therefore

$$Q = Q_0 e^{-t(\ln 2)/5570}$$

if we choose our time scale so that $t_0 = 0$ is the time of death. If we know the present value of Q we can solve this equation for t , the number of years since death occurred. This yields

$$t = -5570 \frac{\ln Q/Q_0}{\ln 2}.$$

It is given that $Q = 0.42Q_0$ in the remains of individuals who died first. Therefore these deaths occurred about

$$t_1 = -5570 \frac{\ln .42}{\ln 2} \approx 6971$$

years ago. For the most recent deaths, $Q = .44Q_0$; hence, these deaths occurred about

$$t_2 = -5570 \frac{\ln .44}{\ln 2} \approx 6597$$

years ago. Therefore it is reasonable to conclude that the village was founded about 7000 years ago, and lasted for about 400 years.

A Savings Program

✓ Example 4.1.6

A person opens a savings account with an initial deposit of \$1000 and subsequently deposits \$50 per week. Find the value $Q(t)$ of the account at time $t > 0$, assuming that the bank pays 6% interest compounded continuously.

Solution

Observe that Q isn't continuous, since there are 52 discrete deposits per year of \$50 each. To construct a mathematical model for this problem in the form of a differential equation, we make the simplifying assumption that the deposits are made continuously at a rate of \$2600 per year. This is essential, since solutions of differential equations are continuous functions. With this assumption, Q increases continuously at the rate

$$Q' = 2600 + 0.06Q$$

and therefore Q satisfies the differential equation

$$Q' - .06Q = 2600. \quad (2.1.13)$$

(Of course, we must recognize that the solution of this equation is an approximation to the true value of Q at any given time. We'll discuss this further below.) Since $e^{.06t}$ is a solution of the complementary equation, the solutions of Equation 2.1.13 are of the form $Q = ue^{.06t}$, where $u'e^{.06t} = 2600$. Hence, $u' = 2600e^{-.06t}$,

$$u = -\frac{2600}{.06}e^{-.06t} + c$$

and

$$Q = ue^{.06t} = -\frac{2600}{.06} + ce^{.06t}. \quad (2.1.14)$$

Setting $t = 0$ and $Q = 1000$ here yields

$$c = 1000 + \frac{2600}{0.06},$$

and substituting this into Equation 2.1.14 yields

$$Q = 1000e^{.06t} + \frac{2600}{.06}(e^{.06t} - 1) \quad (2.1.15)$$

where the first term is the value due to the initial deposit and the second is due to the subsequent weekly deposits.

Mathematical models must be tested for validity by comparing predictions based on them with the actual outcome of experiments. Example 6 is unusual in that we can compute the exact value of the account at any specified time and compare it with the approximate value predicted by Equation 2.1.15 (See **Exercise 4.1.21**). The following table gives a comparison for a ten year period. Each exact answer corresponds to the time of the year-end deposit, and each year is assumed to have exactly 52 weeks.

Table 4.1.3

Year	Approximate Value of Q (Example 4.1.6)	Exact Value of P (Exercise 4.1.21)	Error $Q - P$	Percentage Error $(Q - P)/P$
1	\$3741.42	\$3739.87	\$1.55	.0413
2	6652.36	6649.17	3.19	.0479
3	9743.30	9738.37	4.93	.0506
4	13,025.38	13,018.60	6.78	.0521
5	16,510.41	16,501.66	8.75	.0530

Year	Approximate Value of Q (Example 4.1.6)	Exact Value of P (Exercise 4.1.21)	Error $Q - P$	Percentage Error $(Q - P)/P$
6	20, 210.94	20, 200.11	10.83	.0536
7	24, 140.30	24, 127.25	13.05	.0541
8	28, 312.63	28, 297.23	15.40	.0544
9	32, 742.97	32, 725.07	17.90	.0547
10	37, 447.27	37, 426.72	20.55	.0549

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2.1E: Growth and Decay (Exercises)

Q4.1.1

1. The half-life of a radioactive substance is 3200 years. Find the quantity $Q(t)$ of the substance left at time $t > 0$ if $Q(0) = 20$ g.
2. The half-life of a radioactive substance is 2 days. Find the time required for a given amount of the material to decay to $1/10$ of its original mass.
3. A radioactive material loses 25% of its mass in 10 minutes. What is its half-life?
4. A tree contains a known percentage p_0 of a radioactive substance with half-life τ . When the tree dies the substance decays and isn't replaced. If the percentage of the substance in the fossilized remains of such a tree is found to be p_1 , how long has the tree been dead?
5. If t_p and t_q are the times required for a radioactive material to decay to $1/p$ and $1/q$ times its original mass (respectively), how are t_p and t_q related?
6. Find the decay constant k for a radioactive substance, given that the mass of the substance is Q_1 at time t_1 and Q_2 at time t_2 .
7. A process creates a radioactive substance at the rate of 2 g/hr and the substance decays at a rate proportional to its mass, with constant of proportionality $k = .1(\text{hr})^{-1}$. If $Q(t)$ is the mass of the substance at time t , find $\lim_{t \rightarrow \infty} Q(t)$.
8. A bank pays interest continuously at the rate of 6%. How long does it take for a deposit of Q_0 to grow in value to $2Q_0$?
9. At what rate of interest, compounded continuously, will a bank deposit double in value in 8 years?
10. A savings account pays 5% per annum interest compounded continuously. The initial deposit is Q_0 dollars. Assume that there are no subsequent withdrawals or deposits.
 - a. How long will it take for the value of the account to triple?
 - b. What is Q_0 if the value of the account after 10 years is \$100,000 dollars?
11. A candymaker makes 500 pounds of candy per week, while his large family eats the candy at a rate equal to $Q(t)/10$ pounds per week, where $Q(t)$ is the amount of candy present at time t .
 - a. Find $Q(t)$ for $t > 0$ if the candymaker has 250 pounds of candy at $t = 0$.
 - b. Find $\lim_{t \rightarrow \infty} Q(t)$.
12. Suppose a substance decays at a yearly rate equal to half the square of the mass of the substance present. If we start with 50 g of the substance, how long will it be until only 25 g remain?
13. A super bread dough increases in volume at a rate proportional to the volume V present. If V increases by a factor of 10 in 2 hours and $V(0) = V_0$, find V at any time t . How long will it take for V to increase to $100V_0$?
14. A radioactive substance decays at a rate proportional to the amount present, and half the original quantity Q_0 is left after 1500 years. In how many years would the original amount be reduced to $3Q_0/4$? How much will be left after 2000 years?
15. A wizard creates gold continuously at the rate of 1 ounce per hour, but an assistant steals it continuously at the rate of 5% of however much is there per hour. Let $W(t)$ be the number of ounces that the wizard has at time t . Find $W(t)$ and $\lim_{t \rightarrow \infty} W(t)$ if $W(0) = 1$.
16. A process creates a radioactive substance at the rate of 1 g/hr, and the substance decays at an hourly rate equal to $1/10$ of the mass present (expressed in grams). Assuming that there are initially 20 g, find the mass $S(t)$ of the substance present at time t , and find $\lim_{t \rightarrow \infty} S(t)$.
17. A tank is empty at $t = 0$. Water is added to the tank at the rate of 10 gal/min, but it leaks out at a rate (in gallons per minute) equal to the number of gallons in the tank. What is the smallest capacity the tank can have if this process is to continue forever?
18. A person deposits \$25,000 in a bank that pays 5% per year interest, compounded continuously. The person continuously withdraws from the account at the rate of \$750 per year. Find $V(t)$, the value of the account at time t after the initial deposit.
19. A person has a fortune that grows at rate proportional to the square root of its worth. Find the worth W of the fortune as a function of t if it was \$1 million 6 months ago and is \$4 million today.

20. Let $p = p(t)$ be the quantity of a product present at time t . The product is manufactured continuously at a rate proportional to p , with proportionality constant $1/2$, and it is consumed continuously at a rate proportional to p^2 , with proportionality constant $1/8$. Find $p(t)$ if $p(0) = 100$.

21.

a. In the situation of Example 4.1.6 find the exact value $P(t)$ of the person's account after t years, where t is an integer. Assume that each year has exactly 52 weeks, and include the year-end deposit in the computation.

HINT: At time t the initial \$1000 has been on deposit for t years. There have been $52t$ deposits of \$50 each. The first \$50 has been on deposit for $t - 1/52$ years, the second for $t - 2/52$ years ... in general, the j th \$50 has been on deposit for $t - j/52$ years ($1 \leq j \leq 52t$). Find the present value of each \$50 deposit assuming 6% interest compounded continuously, and use the formula

$$1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad (x \neq 1)$$

to find their total value.

b. Let

$$p(t) = \frac{Q(t) - P(t)}{P(t)}$$

be the relative error after t years. Find

$$p(\infty) = \lim_{t \rightarrow \infty} p(t).$$

22. A homebuyer borrows P_0 dollars at an annual interest rate r , agreeing to repay the loan with equal monthly payments of M dollars per month over N years.

a. Derive a differential equation for the loan principal (amount that the homebuyer owes) $P(t)$ at time $t > 0$, making the simplifying assumption that the homebuyer repays the loan continuously rather than in discrete steps. (See Example 4.1.6.)

b. Solve the equation derived in (a).

c. Use the result of (b) to determine an approximate value for M assuming that each year has exactly 12 months of equal length.

d. It can be shown that the exact value of M is given by

$$M = \frac{rP_0}{12} (1 - (1 + r/12)^{-12N})^{-1}.$$

Compare the value of M obtained from the answer in (c) to the exact value if (i) $P_0 = \$50,000$, $r = 7\frac{1}{2}\%$, $N = 20$ (ii) $P_0 = \$150,000$, $r = 9.0\%$, $N = 30$.

23. Assume that the homebuyer of **Exercise 4.1.22** elects to repay the loan continuously at the rate of αM dollars per month, where α is a constant greater than 1. (This is called **accelerated payment**.)

a. Determine the time $T(\alpha)$ when the loan will be paid off and the amount $S(\alpha)$ that the homebuyer will save.

b. Suppose $P_0 = \$50,000$, $r = 8\%$, and $N = 15$. Compute the savings realized by accelerated payments with $\alpha = 1.05, 1.10$, and 1.15 .

24. A benefactor wishes to establish a trust fund to pay a researcher's salary for T years. The salary is to start at S_0 dollars per year and increase at a fractional rate of a per year. Find the amount of money P_0 that the benefactor must deposit in a trust fund paying interest at a rate r per year. Assume that the researcher's salary is paid continuously, the interest is compounded continuously, and the salary increases are granted continuously.

25. A radioactive substance with decay constant k is produced at the rate of

$$\frac{at}{1 + btQ(t)}$$

units of mass per unit time, where a and b are positive constants and $Q(t)$ is the mass of the substance present at time t ; thus, the rate of production is small at the start and tends to slow when Q is large.

a. Set up a differential equation for Q .

- b. Choose your own positive values for a , b , k , and $Q_0 = Q(0)$. Use a numerical method to discover what happens to $Q(t)$ as $t \rightarrow \infty$. (Be precise, expressing your conclusions in terms of a , b , k . However, no proof is required.)
26. Follow the instructions of **Exercise 4.1.25**, assuming that the substance is produced at the rate of $at/(1 + bt(Q(t))^2)$ units of mass per unit of time.
27. Follow the instructions of **Exercise 4.1.25**, assuming that the substance is produced at the rate of $at/(1 + bt)$ units of mass per unit of time.

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2.2: Cooling and Mixing

Newton's Law of Cooling

Newton's law of cooling states that if an object with temperature $T(t)$ at time t is in a medium with temperature $T_m(t)$, the rate of change of T at time t is proportional to $T(t) - T_m(t)$; thus, T satisfies a differential equation of the form

$$T' = -k(T - T_m). \quad (2.2.1)$$

Here $k > 0$, since the temperature of the object must decrease if $T > T_m$, or increase if $T < T_m$. We'll call k the **temperature decay constant of the medium**.

For simplicity, in this section we'll assume that the medium is maintained at a constant temperature T_m . This is another example of building a simple mathematical model for a physical phenomenon. Like most mathematical models it has its limitations. For example, it is reasonable to assume that the temperature of a room remains approximately constant if the cooling object is a cup of coffee, but perhaps not if it is a huge cauldron of molten metal. (For more on this see **Exercise 4.2.17**.)

To solve Equation 2.2.1, we rewrite it as

$$T' + kT = kT_m.$$

Since e^{-kt} is a solution of the complementary equation, the solutions of this equation are of the form $T = ue^{-kt}$, where $u'e^{-kt} = kT_m$, so $u' = kT_me^{kt}$. Hence,

$$u = T_me^{kt} + c,$$

so

$$T = ue^{-kt} = T_m + ce^{-kt}.$$

If $T(0) = T_0$, setting $t = 0$ here yields $c = T_0 - T_m$, so

$$T = T_m + (T_0 - T_m)e^{-kt}. \quad (2.2.2)$$

Note that $T - T_m$ decays exponentially, with decay constant k .

✓ Example 4.2.1

A ceramic insulator is baked at 400°C and cooled in a room in which the temperature is 25°C . After 4 minutes the temperature of the insulator is 200°C . What is its temperature after 8 minutes?

Solution

Here $T_0 = 400$ and $T_m = 25$, so Equation 2.2.2 becomes

$$T = 25 + 375e^{-kt}. \quad (2.2.3)$$

We determine k from the stated condition that $T(4) = 200$; that is,

$$200 = 25 + 375e^{-4k};$$

hence,

$$e^{-4k} = \frac{175}{375} = \frac{7}{15}.$$

Taking logarithms and solving for k yields

$$k = -\frac{1}{4} \ln \frac{7}{15} = \frac{1}{4} \ln \frac{15}{7}.$$

Substituting this into Equation 2.2.3 yields

$$T = 25 + 375e^{-\frac{t}{4} \ln \frac{15}{7}}$$

(Figure 4.2.1). Therefore the temperature of the insulator after 8 minutes is

$$\begin{aligned} T(8) &= 25 + 375e^{-2 \ln \frac{15}{7}} \\ &= 25 + 375\left(\frac{7}{15}\right)^2 \approx 107^\circ \text{C}. \end{aligned}$$

✓ Example 4.2.2

An object with temperature 72°F is placed outside, where the temperature is -20°F . At 11:05 the temperature of the object is 60°F and at 11:07 its temperature is 50°F . At what time was the object placed outside?

Solution

Let $T(t)$ be the temperature of the object at time t . For convenience, we choose the origin $t_0 = 0$ of the time scale to be 11:05 so that $T_0 = 60$. We must determine the time τ when $T(\tau) = 72$. Substituting $T_0 = 60$ and $T_m = -20$ into Equation 2.2.2 yields

$$T = -20 + (60 - (-20))e^{-kt}$$

or

$$T = -20 + 80e^{-kt}. \quad (2.2.4)$$

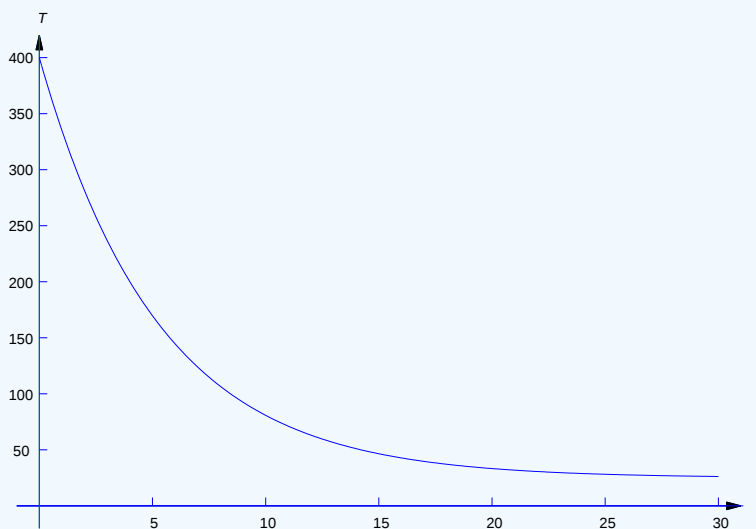


Figure 4.2.1 : $T = 25 + 375e^{-(t/4) \ln 15/7}$

We obtain k from the stated condition that the temperature of the object is 50°F at 11:07. Since 11:07 is $t = 2$ on our time scale, we can determine k by substituting $T = 50$ and $t = 2$ into Equation 2.2.4 to obtain

$$50 = -20 + 80e^{-2k}$$

This is shown in Figure 4.2.2 .

Hence,

$$e^{-2k} = \frac{70}{80} = \frac{7}{8}.$$

Taking logarithms and solving for k yields

$$k = -\frac{1}{2} \ln \frac{7}{8} = \frac{1}{2} \ln \frac{8}{7}.$$

Substituting this into Equation 2.2.4 yields

$$T = -20 + 80e^{-\frac{t}{2} \ln \frac{8}{7}},$$

and the condition $T(\tau) = 72$ implies that

$$72 = -20 + 80e^{-\frac{\tau}{2} \ln \frac{8}{7}};$$

hence,

$$e^{-\frac{\tau}{2} \ln \frac{8}{7}} = \frac{92}{80} = \frac{23}{20}.$$

Taking logarithms and solving for τ yields

$$\tau = -\frac{2 \ln \frac{23}{20}}{\ln \frac{8}{7}} \approx -2.09 \text{ min.}$$

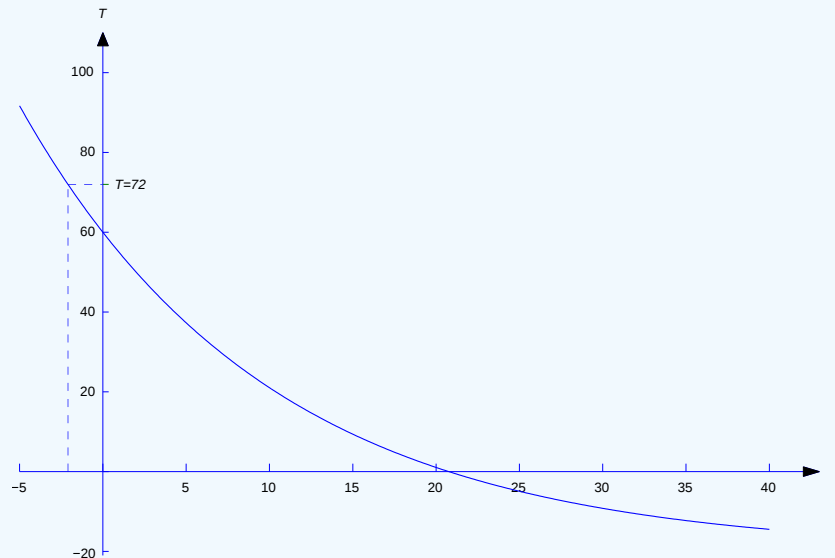


Figure 4.2.2 : $-20 + 80e^{-\frac{t}{2} \ln \frac{8}{7}}$

Therefore the object was placed outside about 2 minutes and 5 seconds before 11:05; that is, at 11:02:55.

Mixing Problems

In the next two examples a saltwater solution with a given concentration (weight of salt per unit volume of solution) is added at a specified rate to a tank that initially contains saltwater with a different concentration. The problem is to determine the quantity of salt in the tank as a function of time. This is an example of a **mixing problem**. To construct a tractable mathematical model for mixing problems we assume in our examples (and most exercises) that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Exercises 4.2.22 and 4.2.23 deal with situations where this isn't so, but the distribution of salt becomes approximately uniform as $t \rightarrow \infty$.

✓ Example 4.2.3

A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains 1/2 pound of salt per gallon is poured into the tank at the rate of 4 gal/min and the mixture is drained from the tank at the same rate (Figure 4.2.3).

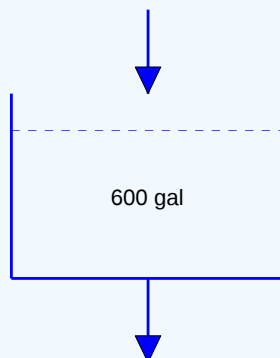
- Find a differential equation for the quantity $Q(t)$ of salt in the tank at time $t > 0$, and solve the equation to determine $Q(t)$.
- Find $\lim_{t \rightarrow \infty} Q(t)$.

Solution a

To find a differential equation for Q , we must use the given information to derive an expression for Q' . But Q' is the rate of change of the quantity of salt in the tank changes with respect to time; thus, if **rate in** denotes the rate at which salt enters the tank and **rate out** denotes the rate by which it leaves, then

$$Q' = \text{rate in} - \text{rate out.} \quad (2.2.5)$$

4 gal/min; .5 lb/gal



4 gal/min

Figure 4.2.3 : A mixing problem

The rate in is

$$\left(\frac{1}{2} \text{ lb/gal}\right) \times (4 \text{ gal/min}) = 2 \text{ lb/min.}$$

Determining the rate out requires a little more thought. We're removing 4 gallons of the mixture per minute, and there are always 600 gallons in the tank; that is, we are removing $1/150$ of the mixture per minute. Since the salt is evenly distributed in the mixture, we are also removing $1/150$ of the salt per minute. Therefore, if there are $Q(t)$ pounds of salt in the tank at time t , the rate out at any time t is $Q(t)/150$. Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned} \text{rate out} &= (\text{concentration}) \times (\text{rate of flow out}) \\ &= (\text{lb/gal}) \times (\text{gal/min}) \\ &= \frac{Q(t)}{600} \times 4 \\ &= \frac{Q(t)}{150}. \end{aligned}$$

We can now write Equation 2.2.5 as

$$Q' = 2 - \frac{Q}{150}.$$

This first order equation can be rewritten as

$$Q' + \frac{Q}{150} = 2.$$

Since $e^{-t/150}$ is a solution of the complementary equation, the solutions of this equation are of the form $Q = ue^{-t/150}$, where $u'e^{-t/150} = 2$, so $u' = 2e^{t/150}$. Hence,

$$u = 300e^{t/150} + c,$$

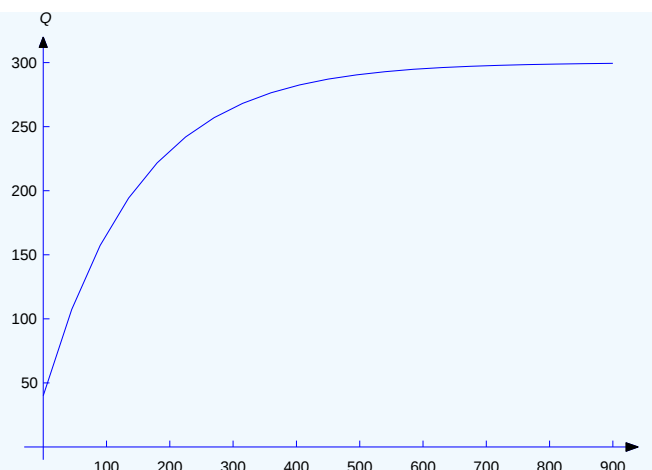


Figure 4.2.4 : $Q = 300 - 260e^{-t/150}$

so

$$Q = ue^{-t/150} = 300 + ce^{-t/150} \quad (2.2.6)$$

(Figure 4.2.4). Since $Q(0) = 40$, $c = -260$; therefore,

$$Q = 300 - 260e^{-t/150}.$$

Solution b

From Equation 2.2.6, we see that that $\lim_{t \rightarrow \infty} Q(t) = 300$ for any value of $Q(0)$. This is intuitively reasonable, since the incoming solution contains 1/2 pound of salt per gallon and there are always 600 gallons of water in the tank.

✓ Example 4.2.4

A 500-liter tank initially contains 10 g of salt dissolved in 200 liters of water. Starting at $t_0 = 0$, water that contains 1/4 g of salt per liter is poured into the tank at the rate of 4 liters/min and the mixture is drained from the tank at the rate of 2 liters/min (Figure 4.2.5). Find a differential equation for the quantity $Q(t)$ of salt in the tank at time t prior to the time when the tank overflows and find the concentration $K(t)$ (g/liter) of salt in the tank at any such time.

Solution

We first determine the amount $W(t)$ of solution in the tank at any time t prior to overflow. Since $W(0) = 200$ and we are adding 4 liters/min while removing only 2 liters/min, there's a net gain of 2 liters/min in the tank; therefore,

$$W(t) = 2t + 200.$$

Since $W(150) = 500$ liters (capacity of the tank), this formula is valid for $0 \leq t \leq 150$.

Now let $Q(t)$ be the number of grams of salt in the tank at time t , where $0 \leq t \leq 150$. As in Example 4.2.3

$$Q' = \text{rate in} - \text{rate out}. \quad (2.2.7)$$

4 liters/min; .25 g/liter

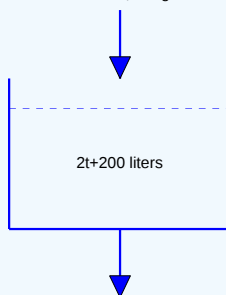


Figure 4.2.5 : Another mixing problem

The rate in is

$$\left(\frac{1}{4} \text{ g/liter}\right) \times (4 \text{ liters/min}) = 1 \text{ g/min.} \quad (2.2.8)$$

To determine the rate out, we observe that since the mixture is being removed from the tank at the constant rate of 2 liters/min and there are $2t + 200$ liters in the tank at time t , the fraction of the mixture being removed per minute at time t is

$$\frac{2}{2t + 200} = \frac{1}{t + 100}.$$

We're removing this same fraction of the salt per minute. Therefore, since there are $Q(t)$ grams of salt in the tank at time t ,

$$\text{rate out} = \frac{Q(t)}{t + 100}. \quad (2.2.9)$$

Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned} \text{rate out} &= (\text{concentration}) \times (\text{rate of flow out}) \\ &= (\text{g/liter}) \times (\text{liters/min}) \\ &= \frac{Q(t)}{2t + 200} \times 2 = \frac{Q(t)}{t + 100}. \end{aligned}$$

Substituting Equation 2.2.8 and Equation 2.2.9 into Equation 2.2.7 yields

$$Q' = 1 - \frac{Q}{t + 100}, \quad \text{so} \quad Q' + \frac{1}{t + 100} Q = 1. \quad (2.2.10)$$

By separation of variables, $1/(t + 100)$ is a solution of the complementary equation, so the solutions of Equation 2.2.10 are of the form

$$Q = \frac{u}{t + 100}, \quad \text{where} \quad \frac{u'}{t + 100} = 1, \quad \text{so} \quad u' = t + 100.$$

Hence,

$$u = \frac{(t + 100)^2}{2} + c. \quad (2.2.11)$$

Since $Q(0) = 10$ and $u = (t + 100)Q$, Equation 2.2.11 implies that

$$(100)(10) = \frac{(100)^2}{2} + c,$$

so

$$c = 100(10) - \frac{(100)^2}{2} = -4000$$

and therefore

$$u = \frac{(t + 100)^2}{2} - 4000.$$

Hence,

$$Q = \frac{u}{t + 200} = \frac{t + 100}{2} - \frac{4000}{t + 100}.$$

Now let $K(t)$ be the concentration of salt at time t . Then

$$K(t) = \frac{1}{4} - \frac{2000}{(t + 100)^2}$$

This is shown in Figure 4.2.6 .

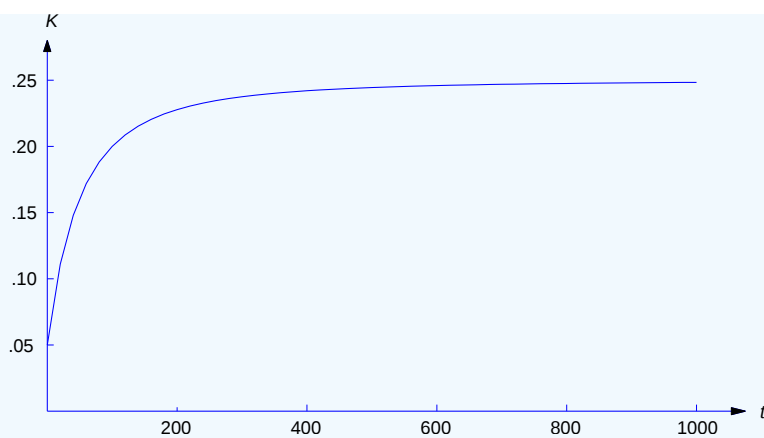


Figure 4.2.6 : $K(t) = \frac{1}{4} - \frac{2000}{(t+100)^2}$

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2.2E: Cooling and Mixing (Exercises)

Q4.2.1

1. A thermometer is moved from a room where the temperature is 70°F to a freezer where the temperature is 12°F . After 30 seconds the thermometer reads 40°F . What does it read after 2 minutes?
2. A fluid initially at 100°C is placed outside on a day when the temperature is -10°C , and the temperature of the fluid drops 20°C in one minute. Find the temperature $T(t)$ of the fluid for $t > 0$.
3. At 12:00 pm a thermometer reading 10°F is placed in a room where the temperature is 70°F . It reads 56° when it is placed outside, where the temperature is 5°F , at 12:03. What does it read at 12:05 pm?
4. A thermometer initially reading 212°F is placed in a room where the temperature is 70°F . After 2 minutes the thermometer reads 125°F .
 - a. What does the thermometer read after 4 minutes?
 - b. When will the thermometer read 72°F ?
 - c. When will the thermometer read 69°F ?
5. An object with initial temperature 150°C is placed outside, where the temperature is 35°C . Its temperatures at 12:15 and 12:20 are 120°C and 90°C , respectively.
 - a. At what time was the object placed outside?
 - b. When will its temperature be 40°C ?
6. An object is placed in a room where the temperature is 20°C . The temperature of the object drops by 5°C in 4 minutes and by 7°C in 8 minutes. What was the temperature of the object when it was initially placed in the room?
7. A cup of boiling water is placed outside at 1:00 pm. One minute later the temperature of the water is 152°F . After another minute its temperature is 112°F . What is the outside temperature?
8. A tank initially contains 40 gallons of pure water. A solution with 1 gram of salt per gallon of water is added to the tank at 3 gal/min, and the resulting solution drains out at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t > 0$.
9. A tank initially contains a solution of 10 pounds of salt in 60 gallons of water. Water with $1/2$ pound of salt per gallon is added to the tank at 6 gal/min, and the resulting solution leaves at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t > 0$.
10. A tank initially contains 100 liters of a salt solution with a concentration of .1 g/liter. A solution with a salt concentration of .3 g/liter is added to the tank at 5 liters/min, and the resulting mixture is drained out at the same rate. Find the concentration $K(t)$ of salt in the tank as a function of t .
11. A 200 gallon tank initially contains 100 gallons of water with 20 pounds of salt. A salt solution with $1/4$ pound of salt per gallon is added to the tank at 4 gal/min, and the resulting mixture is drained out at 2 gal/min. Find the quantity of salt in the tank as it is about to overflow.
12. Suppose water is added to a tank at 10 gal/min, but leaks out at the rate of $1/5$ gal/min for each gallon in the tank. What is the smallest capacity the tank can have if the process is to continue indefinitely?
13. A chemical reaction in a laboratory with volume V (in ft^3) produces q_1 ft^3/min of a noxious gas as a byproduct. The gas is dangerous at concentrations greater than $\bar{c}$, but harmless at concentrations $\leq \bar{c}$. Intake fans at one end of the laboratory pull in fresh air at the rate of q_2 ft^3/min and exhaust fans at the other end exhaust the mixture of gas and air from the laboratory at the same rate. Assuming that the gas is always uniformly distributed in the room and its initial concentration c_0 is at a safe level, find the smallest value of q_2 required to maintain safe conditions in the laboratory for all time.
14. A 1200-gallon tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains $1/2$ pound of salt per gallon is added to the tank at the rate of 6 gal/min and the resulting mixture is drained from the tank at 4 gal/min. Find the quantity $Q(t)$ of salt in the tank at any time $t > 0$ prior to overflow.
15. Tank T_1 initially contain 50 gallons of pure water. Starting at $t_0 = 0$, water that contains 1 pound of salt per gallon is poured into T_1 at the rate of 2 gal/min. The mixture is drained from T_1 at the same rate into a second tank T_2 , which initially contains 50

gallons of pure water. Also starting at $t_0 = 0$, a mixture from another source that contains 2 pounds of salt per gallon is poured into T_2 at the rate of 2 gal/min. The mixture is drained from T_2 at the rate of 4 gal/min.

- Find a differential equation for the quantity $Q(t)$ of salt in tank T_2 at time $t > 0$.
- Solve the equation derived in (a) to determine $Q(t)$.
- Find $\lim_{t \rightarrow \infty} Q(t)$.

16. Suppose an object with initial temperature T_0 is placed in a sealed container, which is in turn placed in a medium with temperature T_m . Let the initial temperature of the container be S_0 . Assume that the temperature of the object does not affect the temperature of the container, which in turn does not affect the temperature of the medium. (These assumptions are reasonable, for example, if the object is a cup of coffee, the container is a house, and the medium is the atmosphere.)

- Assuming that the container and the medium have distinct temperature decay constants k and k_m respectively, use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time t .
- Assuming that the container and the medium have the same temperature decay constant k , use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time t .
- Find $\lim_{t \rightarrow \infty} S(t)$ and $\lim_{t \rightarrow \infty} T(t)$.

17. In our previous examples and exercises concerning Newton's law of cooling we assumed that the temperature of the medium remains constant. This model is adequate if the heat lost or gained by the object is insignificant compared to the heat required to cause an appreciable change in the temperature of the medium. If this isn't so, we must use a model that accounts for the heat exchanged between the object and the medium. Let $T = T(t)$ and $T_m = T_m(t)$ be the temperatures of the object and the medium, respectively, and let T_0 and T_{m0} be their initial values. Again, we assume that T and T_m are related by Newton's law of cooling,

$$T' = -k(T - T_m). \quad (\text{A})$$

We also assume that the change in heat of the object as its temperature changes from T_0 to T is $a(T - T_0)$ and that the change in heat of the medium as its temperature changes from T_{m0} to T_m is $a_m(T_m - T_{m0})$, where a and a_m are positive constants depending upon the masses and thermal properties of the object and medium, respectively. If we assume that the total heat of the system consisting of the object and the medium remains constant (that is, energy is conserved), then

$$a(T - T_0) + a_m(T_m - T_{m0}) = 0. \quad (\text{B})$$

- Equation (A) involves two unknown functions T and T_m . Use (A) and (B) to derive a differential equation involving only T .
- Find $T(t)$ and $T_m(t)$ for $t > 0$.
- Find $\lim_{t \rightarrow \infty} T(t)$ and $\lim_{t \rightarrow \infty} T_m(t)$.

18. Control mechanisms allow fluid to flow into a tank at a rate proportional to the volume V of fluid in the tank, and to flow out at a rate proportional to V^2 . Suppose $V(0) = V_0$ and the constants of proportionality are a and b , respectively. Find $V(t)$ for $t > 0$ and find $\lim_{t \rightarrow \infty} V(t)$.

19. Identical tanks T_1 and T_2 initially contain W gallons each of pure water. Starting at $t_0 = 0$, a salt solution with constant concentration c is pumped into T_1 at r gal/min and drained from T_1 into T_2 at the same rate. The resulting mixture in T_2 is also drained at the same rate. Find the concentrations $c_1(t)$ and $c_2(t)$ in tanks T_1 and T_2 for $t > 0$.

20. An infinite sequence of identical tanks $T_1, T_2, \dots, T_n, \dots$, initially contain W gallons each of pure water. They are hooked together so that fluid drains from T_n into T_{n+1} ($n = 1, 2, \dots$). A salt solution is circulated through the tanks so that it enters and leaves each tank at the constant rate of r gal/min. The solution has a concentration of c pounds of salt per gallon when it enters T_1 .

- Find the concentration $c_n(t)$ in tank T_n for $t > 0$.
- Find $\lim_{t \rightarrow \infty} c_n(t)$ for each n .

21. Tanks T_1 and T_2 have capacities W_1 and W_2 liters, respectively. Initially they are both full of dye solutions with concentrations c_1 and c_2 grams per liter. Starting at $t_0 = 0$, the solution from T_1 is pumped into T_2 at a rate of r liters per minute, and the solution from T_2 is pumped into T_1 at the same rate.

- Find the concentrations $c_1(t)$ and $c_2(t)$ of the dye in T_1 and T_2 for $t > 0$.
- Find $\lim_{t \rightarrow \infty} c_1(t)$ and $\lim_{t \rightarrow \infty} c_2(t)$.

22. Consider the mixing problem of Example 4.2.3, but without the assumption that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Assume instead that the distribution approaches uniformity as $t \rightarrow \infty$. In this

case the differential equation for Q is of the form

$$Q' + \frac{a(t)}{150}Q = 2$$

where $\lim_{t \rightarrow \infty} a(t) = 1$.

- Assuming that $Q(0) = Q_0$, can you guess the value of $\lim_{t \rightarrow \infty} Q(t)$?
- Use numerical methods to confirm your guess in the these cases:

$$(i) a(t) = t/(1+t) \quad (ii) a(t) = 1 - e^{-t^2} \quad (iii) a(t) = 1 - \sin(e^{-t}).$$

23. Consider the mixing problem of Example 4.2.4 in a tank with infinite capacity, but without the assumption that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Assume instead that the distribution approaches uniformity as $t \rightarrow \infty$. In this case the differential equation for Q is of the form

$$Q' + \frac{a(t)}{t+100}Q = 1$$

where $\lim_{t \rightarrow \infty} a(t) = 1$.

- Let $K(t)$ be the concentration of salt at time t . Assuming that $Q(0) = Q_0$, can you guess the value of $\lim_{t \rightarrow \infty} K(t)$?
- Use numerical methods to confirm your guess in the these cases:

$$(i) a(t) = t/(1+t) \quad (ii) a(t) = 1 - e^{-t^2} \quad (iii) a(t) = 1 + \sin(e^{-t}).$$

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2.3: Elementary Mechanics

Newton's Second Law of Motion

In this section we consider an object with constant mass m moving along a line under a force F . Let $y = y(t)$ be the displacement of the object from a reference point on the line at time t , and let $v = v(t)$ and $a = a(t)$ be the velocity and acceleration of the object at time t . Thus, $v = y'$ and $a = v' = y''$, where the prime denotes differentiation with respect to t . Newton's second law of motion asserts that the force F and the acceleration a are related by the equation

$$F = ma. \quad (2.3.1)$$

Note: Units

In applications there are three main sets of units in use for length, mass, force, and time: the cgs, mks, and British systems. All three use the second as the unit of time. Table 4.3.1 shows the other units. Consistent with Equation 2.3.1, the unit of force in each system is defined to be the force required to impart an acceleration of (one unit of length)/s² to one unit of mass.

Table 4.3.1

Set	Length	Force	Mass
cgs	centimeter (cm)	dyne (d)	gram (g)
mks	meter (m)	newton (N)	kilogram (kg)
British	foot (ft)	pound (lb)	slug (sl)

If we assume that Earth is a perfect sphere with constant mass density, Newton's law of gravitation (discussed later in this section) asserts that the force exerted on an object by Earth's gravitational field is proportional to the mass of the object and inversely proportional to the square of its distance from the center of Earth. However, if the object remains sufficiently close to Earth's surface, we may assume that the gravitational force is constant and equal to its value at the surface. The magnitude of this force is mg , where g is called the **acceleration due to gravity**. (To be completely accurate, g should be called the **magnitude of the acceleration due to gravity at Earth's surface**.) This quantity has been determined experimentally. Approximate values of g are

$$\begin{aligned} g &= 980 \text{ cm/s}^2 && (\text{cgs}) \\ g &= 9.8 \text{ m/s}^2 && (\text{mks}) \\ g &= 32 \text{ ft/s}^2 && (\text{British}). \end{aligned}$$

In general, the force F in Equation 2.3.1 may depend upon t , y , and y' . Since $a = y''$, Equation 2.3.1 can be written in the form

$$my'' = F(t, y, y'), \quad (2.3.2)$$

which is a second order equation. We'll consider this equation with restrictions on F later; however, since Chapter 2 dealt only with first order equations, we consider here only problems in which Equation 2.3.2 can be recast as a first order equation. This is possible if F does not depend on y , so Equation 2.3.2 is of the form

$$my'' = F(t, y').$$

Letting $v = y'$ and $v' = y''$ yields a first order equation for v :

$$mv' = F(t, v). \quad (2.3.3)$$

Solving this equation yields v as a function of t . If we know $y(t_0)$ for some time t_0 , we can integrate v to obtain y as a function of t .

Equations of the form Equation 2.3.3 occur in problems involving motion through a resisting medium.

Motion Through a Resisting Medium Under Constant Gravitational Force

Now we consider an object moving vertically in some medium. We assume that the only forces acting on the object are gravity and resistance from the medium. We also assume that the motion takes place close to Earth's surface and take the upward direction to be positive, so the gravitational force can be assumed to have the constant value $-mg$. We'll see that, under reasonable assumptions on the resisting force, the velocity approaches a limit as $t \rightarrow \infty$. We call this limit the **terminal velocity**.

✓ Example 4.3.1

An object with mass m moves under constant gravitational force through a medium that exerts a resistance with magnitude proportional to the speed of the object. (Recall that the speed of an object is $|v|$, the absolute value of its velocity v .) Find the velocity of the object as a function of t , and find the terminal velocity. Assume that the initial velocity is v_0 .

Solution

The total force acting on the object is

$$F = -mg + F_1, \quad (2.3.4)$$

where $-mg$ is the force due to gravity and F_1 is the resisting force of the medium, which has magnitude $k|v|$, where k is a positive constant. If the object is moving downward ($v \leq 0$), the resisting force is upward (Figure 4.3.1a), so

$$F_1 = k|v| = k(-v) = -kv.$$

On the other hand, if the object is moving upward ($v \geq 0$), the resisting force is downward (Figure 4.3.1b), so

$$F_1 = -k|v| = -kv.$$

Thus, Equation 2.3.4 can be written as

$$F = -mg - kv, \quad (2.3.5)$$

regardless of the sign of the velocity.

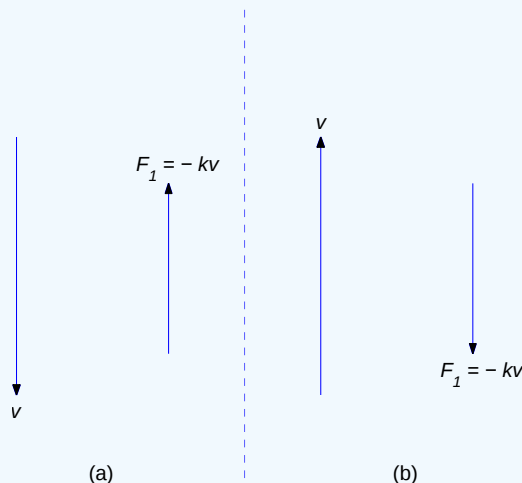


Figure 4.3.1 : Resistive forces

From Newton's second law of motion,

$$F = ma = mv',$$

so Equation 2.3.5 yields

$$mv' = -mg - kv,$$

or

$$v' + \frac{k}{m}v = -g. \quad (2.3.6)$$

Since $e^{-kt/m}$ is a solution of the complementary equation, the solutions of Equation 2.3.6 are of the form $v = ue^{-kt/m}$, where $u'e^{-kt/m} = -g$, so $u' = -ge^{kt/m}$. Hence,

$$u = -\frac{mg}{k}e^{kt/m} + c,$$

so

$$v = ue^{-kt/m} = -\frac{mg}{k} + ce^{-kt/m}. \quad (2.3.7)$$

Since $v(0) = v_0$,

$$v_0 = -\frac{mg}{k} + c,$$

so

$$c = v_0 + \frac{mg}{k}$$

and Equation 2.3.7 becomes

$$v = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-kt/m}.$$

Letting $t \rightarrow \infty$ here shows that the terminal velocity is

$$\lim_{t \rightarrow \infty} v(t) = -\frac{mg}{k},$$

which is independent of the initial velocity v_0 (Figure 4.3.2).

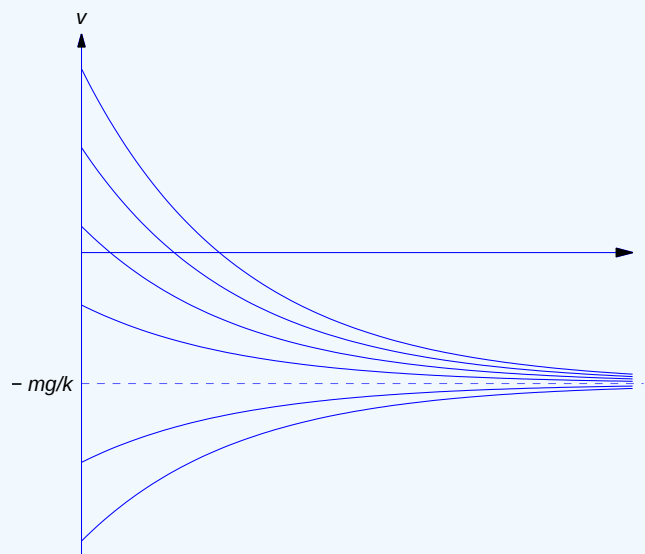


Figure 4.3.2 : Solutions of $mv' = -mg - kv$

✓ Example 4.3.2 : Terminal Velocity

A 960-lb object is given an initial upward velocity of 60 ft/s near the surface of Earth. The atmosphere resists the motion with a force of 3 lb for each ft/s of speed. Assuming that the only other force acting on the object is constant gravity, find its velocity v as a function of t , and find its terminal velocity.

Solution

Since $mg = 960$ and $g = 32$, $m = 960/32 = 30$. The atmospheric resistance is $-3v$ lb if v is expressed in feet per second. Therefore

$$30v' = -960 - 3v,$$

which we rewrite as

$$v' + \frac{1}{10}v = -32.$$

Since $e^{-t/10}$ is a solution of the complementary equation, the solutions of this equation are of the form $v = ue^{-t/10}$, where $u'e^{-t/10} = -32$, so $u' = -32e^{t/10}$. Hence,

$$u = -320e^{t/10} + c,$$

so

$$v = ue^{-t/10} = -320 + ce^{-t/10}. \quad (2.3.8)$$

The initial velocity is 60 ft/s in the upward (positive) direction; hence, $v_0 = 60$. Substituting $t = 0$ and $v = 60$ in Equation 2.3.8 yields

$$60 = -320 + c,$$

so $c = 380$, and Equation 2.3.8 becomes

$$v = -320 + 380e^{-t/10} \text{ ft/s}$$

The terminal velocity is

$$\lim_{t \rightarrow \infty} v(t) = -320 \text{ ft/s}.$$

✓ Example 4.3.3

A 10 kg mass is given an initial velocity $v_0 \leq 0$ near Earth's surface. The only forces acting on it are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the resistance is 8 N if the speed is 2 m/s, find the velocity of the object as a function of t , and find the terminal velocity.

Solution

Since the object is falling, the resistance is in the upward (positive) direction. Hence,

$$mv' = -mg + kv^2, \quad (2.3.9)$$

where k is a constant. Since the magnitude of the resistance is 8 N when $v = 2$ m/s,

$$k(2^2) = 8,$$

so $k = 2 \text{ N}\cdot\text{s}^2/\text{m}^2$. Since $m = 10$ and $g = 9.8$, Equation 2.3.9 becomes

$$10v' = -98 + 2v^2 = 2(v^2 - 49). \quad (2.3.10)$$

If $v_0 = -7$, then $v \equiv -7$ for all $t \geq 0$. If $v_0 \neq -7$, we separate variables to obtain

$$\frac{1}{v^2 - 49}v' = \frac{1}{5}, \quad (2.3.11)$$

which is convenient for the required partial fraction expansion

$$\frac{1}{v^2 - 49} = \frac{1}{(v-7)(v+7)} = \frac{1}{14} \left[\frac{1}{v-7} - \frac{1}{v+7} \right]. \quad (2.3.12)$$

Substituting Equation 2.3.12 into Equation 2.3.11 yields

$$\frac{1}{14} \left[\frac{1}{v-7} - \frac{1}{v+7} \right] v' = \frac{1}{5},$$

so

$$\left[\frac{1}{v-7} - \frac{1}{v+7} \right] v' = \frac{14}{5}.$$

Integrating this yields

$$\ln|v-7| - \ln|v+7| = 14t/5 + k.$$

Therefore

$$\left| \frac{v-7}{v+7} \right| = e^k e^{14t/5}.$$

Since Theorem 2.3.1 implies that $(v-7)/(v+7)$ cannot change sign (why?), we can rewrite the last equation as

$$\frac{v-7}{v+7} = ce^{14t/5}, \quad (2.3.13)$$

which is an implicit solution of Equation 2.3.10. Solving this for v yields

$$v = -7 \frac{c + e^{-14t/5}}{c - e^{-14t/5}}. \quad (2.3.14)$$

Since $v(0) = v_0$, it Equation 2.3.13 implies that

$$c = \frac{v_0 - 7}{v_0 + 7}.$$

Substituting this into Equation 2.3.14 and simplifying yields

$$v = -7 \frac{v_0(1 + e^{-14t/5}) - 7(1 - e^{-14t/5})}{v_0(1 - e^{-14t/5}) - 7(1 + e^{-14t/5})}.$$

Since $v_0 \leq 0$, v is defined and negative for all $t > 0$. The terminal velocity is

$$\lim_{t \rightarrow \infty} v(t) = -7 \text{ m/s},$$

independent of v_0 . More generally, it can be shown (**Exercise 4.3.11**) that if v is any solution of Equation 2.3.9 such that $v_0 \leq 0$ then

$$\lim_{t \rightarrow \infty} v(t) = -\sqrt{\frac{mg}{k}}.$$

This is demonstrated in Figure 4.3.3.

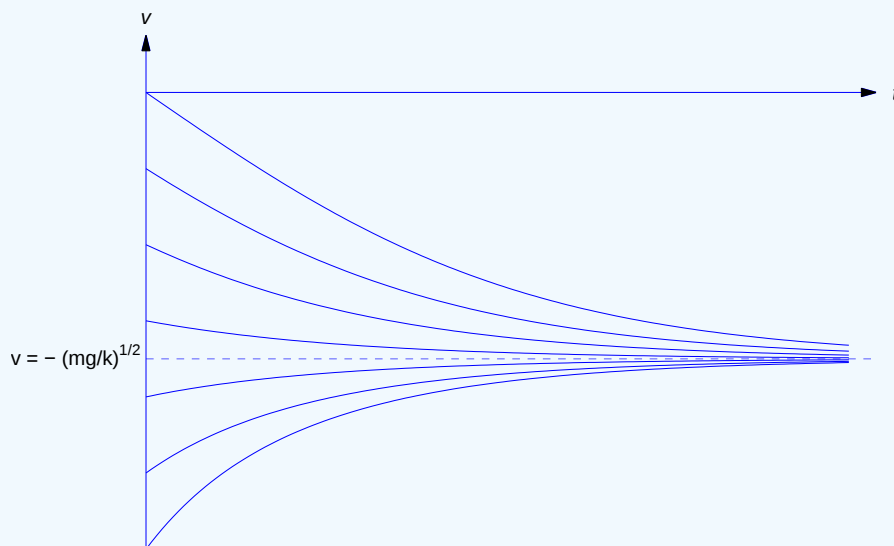


Figure 4.3.3 : Solutions of $mv' = -mg + kv^2$, $v(0) = v_0 \leq 0$

✓ Example 4.3.4

A 10-kg mass is launched vertically upward from Earth's surface with an initial velocity of v_0 m/s. The only forces acting on the mass are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the atmospheric resistance is 8 N if the speed is 2 m/s, find the time T required for the mass to reach maximum altitude.

Solution

The mass will climb while $v > 0$ and reach its maximum altitude when $v = 0$. Therefore $v > 0$ for $0 \leq t < T$ and $v(T) = 0$; therefore, we replace Equation 2.3.10 by

$$10v' = -98 - 2v^2. \quad (2.3.15)$$

Separating variables yields

$$\frac{5}{v^2 + 49} v' = -1,$$

and integrating this yields

$$\frac{5}{7} \tan^{-1} \frac{v}{7} = -t + c.$$

(Recall that $\tan^{-1} u$ is the number θ such that $-\pi/2 < \theta < \pi/2$ and $\tan \theta = u$.) Since $v(0) = v_0$,

$$c = \frac{5}{7} \tan^{-1} \frac{v_0}{7},$$

so v is defined implicitly by

$$\frac{5}{7} \tan^{-1} \frac{v}{7} = -t + \frac{5}{7} \tan^{-1} \frac{v_0}{7}, \quad 0 \leq t \leq T. \quad (2.3.16)$$

Solving this for v yields

$$v = 7 \tan \left(-\frac{7t}{5} + \tan^{-1} \frac{v_0}{7} \right). \quad (2.3.17)$$

Using the identity

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

with $A = \tan^{-1}(v_0/7)$ and $B = 7t/5$, and noting that $\tan(\tan^{-1} \theta) = \theta$, we can simplify Equation 2.3.17 to

$$v = 7 \frac{v_0 - 7 \tan(7t/5)}{7 + v_0 \tan(7t/5)}.$$

Since $v(T) = 0$ and $\tan^{-1}(0) = 0$, Equation 2.3.16 implies that

$$-T + \frac{5}{7} \tan^{-1} \frac{v_0}{7} = 0.$$

Therefore

$$T = \frac{5}{7} \tan^{-1} \frac{v_0}{7}.$$

Since $\tan^{-1}(v_0/7) < \pi/2$ for all v_0 , the time required for the mass to reach its maximum altitude is less than

$$\frac{5\pi}{14} \approx 1.122 \text{ s}$$

regardless of the initial velocity. Figure 4.3.4 shows graphs of v over $[0, T]$ for various values of v_0 .

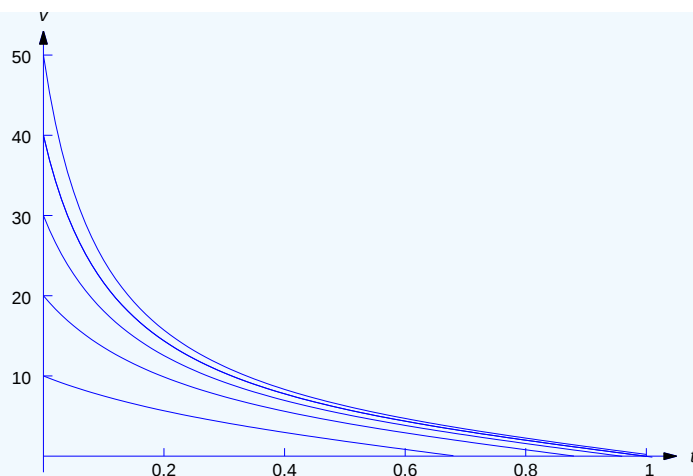


Figure 4.3.4 : Solutions of 4.3.15 for various $v_0 > 0$

Escape Velocity

Suppose a space vehicle is launched vertically and its fuel is exhausted when the vehicle reaches an altitude h above Earth, where h is sufficiently large so that resistance due to Earth's atmosphere can be neglected. Let $t = 0$ be the time when burnout occurs. Assuming that the gravitational forces of all other celestial bodies can be neglected, the motion of the vehicle for $t > 0$ is that of an object with constant mass m under the influence of Earth's gravitational force, which we now assume to vary inversely with the square of the distance from Earth's center; thus, if we take the upward direction to be positive then gravitational force on the vehicle at an altitude y above Earth is

$$F = -\frac{K}{(y + R)^2}, \quad (2.3.18)$$

where R is Earth's radius (Figure 4.3.5).

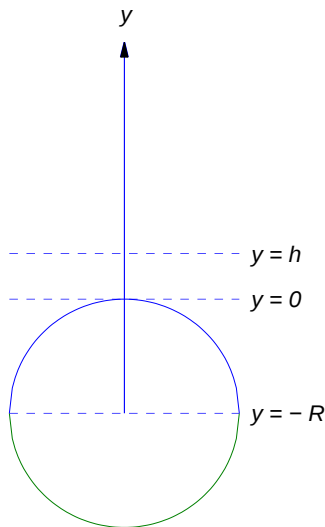


Figure 4.3.5 : Escape velocity

Since $F = -mg$ when $y = 0$, setting $y = 0$ in Equation 2.3.18 yields

$$-mg = -\frac{K}{R^2};$$

therefore $K = mgR^2$ and Equation 2.3.18 can be written more specifically as

$$F = -\frac{mgR^2}{(y + R)^2}. \quad (2.3.19)$$

From Newton's second law of motion,

$$F = m \frac{d^2 y}{dt^2},$$

so Equation 2.3.19 implies that

$$\frac{d^2 y}{dt^2} = -\frac{gR^2}{(y+R)^2}. \quad (2.3.20)$$

We'll show that there's a number v_e , called the **escape velocity**, with these properties:

1. If $v_0 \geq v_e$ then $v(t) > 0$ for all $t > 0$, and the vehicle continues to climb for all $t > 0$; that is, it "escapes" Earth. (Is it really so obvious that $\lim_{t \rightarrow \infty} y(t) = \infty$ in this case? For a proof, see **Exercise 4.3.20**.)
2. If $v_0 < v_e$ then $v(t)$ decreases to zero and becomes negative. Therefore the vehicle attains a maximum altitude y_m and falls back to Earth.

Since Equation 2.3.20 is second order, we cannot solve it by methods discussed so far. However, we are concerned with v rather than y , and v is easier to find. Since $v = y'$ the chain rule implies that

$$\frac{d^2 y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = v \frac{dv}{dy}.$$

Substituting this into Equation 2.3.20 yields the first order separable equation

$$v \frac{dv}{dy} = -\frac{gR^2}{(y+R)^2}. \quad (2.3.21)$$

When $t = 0$, the velocity is v_0 and the altitude is h . Therefore we can obtain v as a function of y by solving the initial value problem

$$v \frac{dv}{dy} = -\frac{gR^2}{(y+R)^2}, \quad v(h) = v_0.$$

Integrating Equation 2.3.21 with respect to y yields

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + c. \quad (2.3.22)$$

Since $v(h) = v_0$,

$$c = \frac{v_0^2}{2} - \frac{gR^2}{h+R},$$

so Equation 2.3.22 becomes

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h+R} \right). \quad (2.3.23)$$

If

$$v_0 \geq \left(\frac{2gR^2}{h+R} \right)^{1/2},$$

the parenthetical expression in Equation 2.3.23 is nonnegative, so $v(y) > 0$ for $y > h$. This proves that there's an escape velocity v_e . We'll now prove that

$$v_e = \left(\frac{2gR^2}{h+R} \right)^{1/2}$$

by showing that the vehicle falls back to Earth if

$$v_0 < \left(\frac{2gR^2}{h+R} \right)^{1/2}. \quad (2.3.24)$$

If Equation 2.3.24 holds then the parenthetical expression in Equation 2.3.23 is negative and the vehicle will attain a maximum altitude $y_m > h$ that satisfies the equation

$$0 = \frac{gR^2}{y_m + R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h + R} \right).$$

The velocity will be zero at the maximum altitude, and the object will then fall to Earth under the influence of gravity.

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2.3E: Elementary Mechanics (Exercises)

Except where directed otherwise, assume that the magnitude of the gravitational force on an object with mass m is constant and equal to mg . In exercises involving vertical motion take the upward direction to be positive.

Q4.3.1

1. A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to his speed, with $k = 2.5$ lb-s/ft. Assuming that he starts from rest, find his velocity as a function of time and find his terminal velocity.
2. A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to her speed, with constant of proportionality k . Find k , given that her terminal velocity is -16 ft/s, and then find her velocity v as a function of t . Assume that she starts from rest.
3. A boat weighs 64,000 lb. Its propellor produces a constant thrust of 50,000 lb and the water exerts a resistive force with magnitude proportional to the speed, with $k = 2000$ lb-s/ft. Assuming that the boat starts from rest, find its velocity as a function of time, and find its terminal velocity.
4. A constant horizontal force of 10 N pushes a 20 kg-mass through a medium that resists its motion with .5 N for every m/s of speed. The initial velocity of the mass is 7 m/s in the direction opposite to the direction of the applied force. Find the velocity of the mass for $t > 0$.
5. A stone weighing $1/2$ lb is thrown upward from an initial height of 5 ft with an initial speed of 32 ft/s. Air resistance is proportional to speed, with $k = 1/128$ lb-s/ft. Find the maximum height attained by the stone.
6. A 3200-lb car is moving at 64 ft/s down a 30-degree grade when it runs out of fuel. Find its velocity after that if friction exerts a resistive force with magnitude proportional to the square of the speed, with $k = 1$ lb-s²/ft². Also find its terminal velocity.
7. A 96 lb weight is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the speed. Find its velocity as a function of time if its terminal velocity is -128 ft/s.
8. An object with mass m moves vertically through a medium that exerts a resistive force with magnitude proportional to the speed. Let $y = y(t)$ be the altitude of the object at time t , with $y(0) = y_0$. Use the results of Example 4.3.1 to show that

$$y(t) = y_0 + \frac{m}{k}(v_0 - v - gt).$$

9. An object with mass m is launched vertically upward with initial velocity v_0 from Earth's surface ($y_0 = 0$) in a medium that exerts a resistive force with magnitude proportional to the speed. Find the time T when the object attains its maximum altitude y_m . Then use the result of **Exercise 4.3.8** to find y_m .
10. An object weighing 256 lb is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the square of the speed. The magnitude of the resisting force is 1 lb when $|v| = 4$ ft/s. Find v for $t > 0$, and find its terminal velocity.
11. An object with mass m is given an initial velocity $v_0 \leq 0$ in a medium that exerts a resistive force with magnitude proportional to the square of the speed. Find the velocity of the object for $t > 0$, and find its terminal velocity.
12. An object with mass m is launched vertically upward with initial velocity v_0 in a medium that exerts a resistive force with magnitude proportional to the square of the speed.
 - a. Find the time T when the object reaches its maximum altitude.
 - b. Use the result of **Exercise 4.3.11** to find the velocity of the object for $t > T$.
13. An object with mass m is given an initial velocity $v_0 \leq 0$ in a medium that exerts a resistive force of the form $a|v|/(1 + |v|)$, where a is positive constant.
 - a. Set up a differential equation for the speed of the object.
 - b. Use your favorite numerical method to solve the equation you found in (a), to convince yourself that there's a unique number a_0 such that $\lim_{t \rightarrow \infty} s(t) = \infty$ if $a \leq a_0$ and $\lim_{t \rightarrow \infty} s(t)$ exists (finite) if $a > a_0$. (We say that a_0 is the **bifurcation value** of a .) Try to find a_0 and $\lim_{t \rightarrow \infty} s(t)$ in the case where $a > a_0$.

14. An object of mass m falls in a medium that exerts a resistive force $f = f(s)$, where $s = |v|$ is the speed of the object. Assume that $f(0) = 0$ and f is strictly increasing and differentiable on $(0, \infty)$.

- Write a differential equation for the speed $s = s(t)$ of the object. Take it as given that all solutions of this equation with $s(0) \geq 0$ are defined for all $t > 0$ (which makes good sense on physical grounds).
- Show that if $\lim_{s \rightarrow \infty} f(s) \leq mg$ then $\lim_{t \rightarrow \infty} s(t) = \infty$.
- Show that if $\lim_{s \rightarrow \infty} f(s) > mg$ then $\lim_{t \rightarrow \infty} s(t) = s_T$ (terminal speed), where $f(s_T) = mg$.

15. A 100-g mass with initial velocity $v_0 \leq 0$ falls in a medium that exerts a resistive force proportional to the fourth power of the speed. The resistance is .1 N if the speed is 3 m/s.

- Set up the initial value problem for the velocity v of the mass for $t > 0$.
- Use **Exercise 4.3.14** (c) to determine the terminal velocity of the object.
- To confirm your answer to (b), use one of the numerical methods studied in Chapter 3 to compute approximate solutions on $[0, 1]$ (seconds) of the initial value problem of (a), with initial values $v_0 = 0, -2, -4, \dots, -12$. Present your results in graphical form similar to Figure 4.3.3.

16. A 64-lb object with initial velocity $v_0 \leq 0$ falls through a dense fluid that exerts a resistive force proportional to the square root of the speed. The resistance is 64 lb if the speed is 16 ft/s.

- Set up the initial value problem for the velocity v of the mass for $t > 0$.
- Use **Exercise 4.3.14** (c) to determine the terminal velocity of the object.
- To confirm your answer to (b), use one of the numerical methods studied in Chapter 3 to compute approximate solutions on $[0, 4]$ (seconds) of the initial value problem of (a), with initial values $v_0 = 0, -5, -10, \dots, -30$. Present your results in graphical form similar to Figure 4.3.3.

Q4.3.2

In **Exercises 4.3.17-4.3.20**, assume that the force due to gravity is given by Newton's law of gravitation. Take the upward direction to be positive.

17. A space probe is to be launched from a space station 200 miles above Earth. Determine its escape velocity in miles/s. Take Earth's radius to be 3960 miles.

18. A space vehicle is to be launched from the moon, which has a radius of about 1080 miles. The acceleration due to gravity at the surface of the moon is about 5.31 ft/s^2 . Find the escape velocity in miles/s.

19.

- Show that (Equation 4.3.27) can be rewritten as

$$v^2 = \frac{h-y}{y+R} v_e^2 + v_0^2.$$

- Show that if $v_0 = \rho v_e$ with $0 \leq \rho < 1$, then the maximum altitude y_m attained by the space vehicle is

$$y_m = \frac{h + R\rho^2}{1 - \rho^2}.$$

- By requiring that $v(y_m) = 0$, use (Equation 4.3.26) to deduce that if $v_0 < v_e$ then

$$|v| = v_e \left[\frac{(1 - \rho^2)(y_m - y)}{y + R} \right]^{1/2},$$

where y_m and ρ are as defined in (b) and $y \geq h$.

- Deduce from (c) that if $v < v_e$, the vehicle takes equal times to climb from $y = h$ to $y = y_m$ and to fall back from $y = y_m$ to $y = h$.

20. In the situation considered in the discussion of escape velocity, show that $\lim_{t \rightarrow \infty} y(t) = \infty$ if $v(t) > 0$ for all $t > 0$.

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2.4: Autonomous Second Order Equations

A second order differential equation that can be written as

$$y'' = F(y, y') \quad (2.4.1)$$

where F is independent of t , is said to be **autonomous**. An autonomous second order equation can be converted into a first order equation relating $v = y'$ and y . If we let $v = y'$, Equation 2.4.1 becomes

$$v' = F(y, v). \quad (2.4.2)$$

Since

$$v' = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = v \frac{dv}{dy}, \quad (2.4.3)$$

Equation 2.4.2 can be rewritten as

$$v \frac{dv}{dy} = F(y, v). \quad (2.4.4)$$

The integral curves of Equation 2.4.4 can be plotted in the (y, v) plane, which is called the **Poincaré phase plane** of Equation 2.4.1. If y is a solution of Equation 2.4.1 then $y = y(t)$, $v = y'(t)$ is a parametric equation for an integral curve of Equation 2.4.4. we will call these integral curves **trajectories** of Equation 2.4.1, and we will call Equation 2.4.4 the **phase plane equivalent** of Equation 2.4.1.

In this section we will consider autonomous equations that can be written as

$$y'' + q(y, y')y' + p(y) = 0. \quad (2.4.5)$$

Equations of this form often arise in applications of Newton's second law of motion. For example, suppose y is the displacement of a moving object with mass m . It's reasonable to think of two types of time-independent forces acting on the object. One type - such as gravity - depends only on position. We could write such a force as $-mp(y)$. The second type - such as atmospheric resistance or friction - may depend on position and velocity. (Forces that depend on velocity are called **damping** forces.) We write this force as $-mq(y, y')y'$, where $q(y, y')$ is usually a positive function and we've put the factor y' outside to make it explicit that the force is in the direction opposing the motion. In this case Newton's, second law of motion leads to Equation 2.4.5.

The phase plane equivalent of Equation 2.4.5 is

$$v \frac{dv}{dy} + q(y, v)v + p(y) = 0. \quad (2.4.6)$$

Some statements that we will be making about the properties of Equation 2.4.5 and Equation 2.4.6 are intuitively reasonable, but difficult to prove. Therefore our presentation in this section will be informal: we will just say things without proof, all of which are true if we assume that $p = p(y)$ is continuously differentiable for all y and $q = q(y, v)$ is continuously differentiable for all (y, v) . We begin with the following statements:

Properties

- **Statement 1:** If y_0 and v_0 are arbitrary real numbers then Equation 2.4.5 has a unique solution on $(-\infty, \infty)$ such that $y(0) = y_0$ and $y'(0) = v_0$.
- **Statement 2:** If $y = y(t)$ is a solution of Equation 2.4.5 and τ is any constant then $y_1 = y(t - \tau)$ is also a solution of Equation 2.4.5, and y and y_1 have the same trajectory.
- **Statement 3:** If two solutions y and y_1 of Equation 2.4.5 have the same trajectory then $y_1(t) = y(t - \tau)$ for some constant τ .
- **Statement 4:** Distinct trajectories of Equation 2.4.5 cannot intersect; that is, if two trajectories of Equation 2.4.5 intersect, they are identical.
- **Statement 5:** If the trajectory of a solution of Equation 2.4.5 is a closed curve then $(y(t), v(t))$ traverses the trajectory in a finite time T , and the solution is periodic with period T ; that is, $y(t + T) = y(t)$ for all t in $(-\infty, \infty)$.

If $\bar{y}$ is a constant such that $p(\bar{y}) = 0$ then $y \equiv \bar{y}$ is a constant solution of Equation 2.4.5. We say that $\bar{y}$ is an **equilibrium** of Equation 2.4.5 and $(\bar{y}, 0)$ is a **critical point** of the phase plane equivalent equation Equation 2.4.6. We say that the equilibrium and the critical point are **stable** if, for any given $\epsilon > 0$ **no matter how small**, there's a $\delta > 0$, **sufficiently small**, such that if

$$\sqrt{(y_0 - \bar{y})^2 + v_0^2} < \delta$$

then the solution of the initial value problem

$$y'' + q(y, y')y' + p(y) = 0, \quad y(0) = y_0, \quad y'(0) = v_0$$

satisfies the inequality

$$\sqrt{(y(t) - \bar{y})^2 + (v(t))^2} < \epsilon$$

for all $t > 0$. Figure 4.4.1 illustrates the geometrical interpretation of this definition in the Poincaré phase plane: if (y_0, v_0) is in the smaller shaded circle (with radius δ), then $(y(t), v(t))$ must be in the larger circle (with radius ϵ) for all $t > 0$.

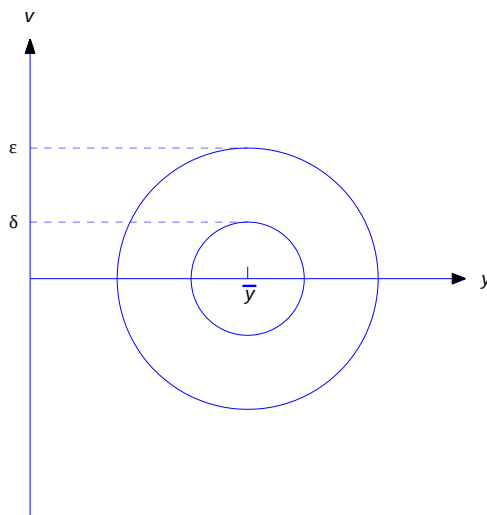


Figure 4.4.1 : Stability: if (y_0, v_0) is in the smaller circle then $(y(t), v(t))$ is in the larger circle for all $t > 0$

If an equilibrium and the associated critical point are not stable, we say they are **unstable**. To see if you really understand what **stable** means, try to give a direct definition of **unstable** (Exercise 4.4.22). we will illustrate both definitions in the following examples.

The Undamped Case

We will begin with the case where $q \equiv 0$, so Equation 2.4.5 reduces to

$$y'' + p(y) = 0. \tag{2.4.7}$$

We say that this equation - as well as any physical situation that it may model - is **undamped**. The phase plane equivalent of Equation 2.4.7 is the separable equation

$$v \frac{dv}{dy} + p(y) = 0.$$

Integrating this yields

$$\frac{v^2}{2} + P(y) = c, \tag{2.4.8}$$

where c is a constant of integration and $P(y) = \int p(y) dy$ is an antiderivative of p .

If Equation 2.4.7 is the equation of motion of an object of mass m , then $mv^2/2$ is the kinetic energy and $mP(y)$ is the potential energy of the object; thus, Equation 2.4.8 says that the total energy of the object remains constant, or is **conserved**. In particular, if

a trajectory passes through a given point (y_0, v_0) then

$$c = \frac{v_0^2}{2} + P(y_0).$$

✓ Example 4.4.1 : The Undamped Spring - Mass System

Consider an object with mass m suspended from a spring and moving vertically. Let y be the displacement of the object from the position it occupies when suspended at rest from the spring (Figure 4.4.2).

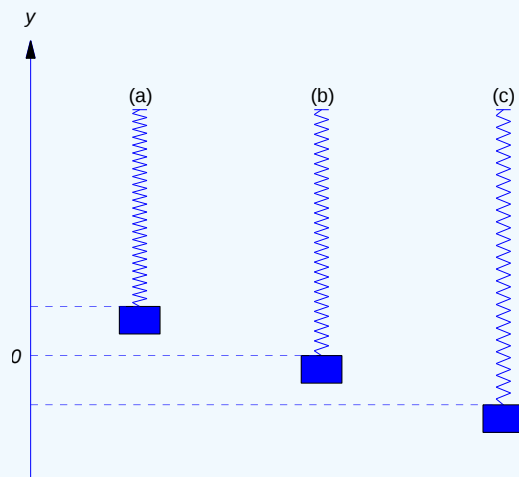


Figure 4.4.2 : (a) $y > 0$ (b) $y = 0$ (c) $y < 0$.

Assume that if the length of the spring is changed by an amount ΔL (positive or negative), then the spring exerts an opposing force with magnitude $k|\Delta L|$, where k is a positive constant. In Section 6.1, it will be shown that if the mass of the spring is negligible compared to m and no other forces act on the object then Newton's second law of motion implies that

$$my'' = -ky, \quad (2.4.9)$$

which can be written in the form Equation 2.4.7 with $p(y) = ky/m$. This equation can be solved easily by a method that we will study in Section 5.2, but that method isn't available here. Instead, we will consider the phase plane equivalent of Equation 2.4.9.

From Equation 2.4.3, we can rewrite Equation 2.4.9 as the separable equation

$$mv \frac{dv}{dy} = -ky.$$

Integrating this yields

$$\frac{mv^2}{2} = -\frac{ky^2}{2} + c,$$

which implies that

$$mv^2 + ky^2 = \rho \quad (2.4.10)$$

($\rho = 2c$). This defines an ellipse in the Poincaré phase plane (Figure 4.4.3).

We can identify ρ by setting $t = 0$ in Equation 2.4.10; thus, $\rho = mv_0^2 + ky_0^2$, where $y_0 = y(0)$ and $v_0 = v(0)$. To determine the maximum and minimum values of y we set $v = 0$ in Equation 2.4.10; thus,

$$y_{\max} = R \quad \text{and} \quad y_{\min} = -R, \quad \text{with } R = \sqrt{\frac{\rho}{k}}. \quad (2.4.11)$$

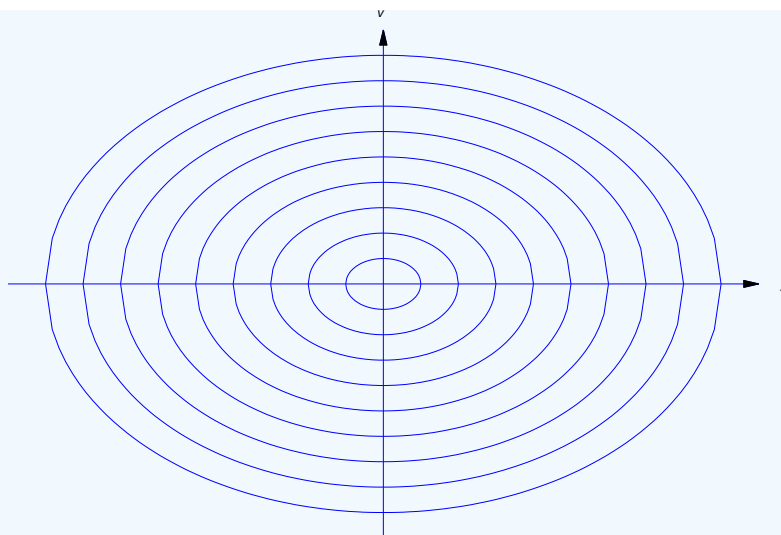


Figure 4.4.3 : Trajectories of $my'' + ky = 0$

Equation 2.4.9 has exactly one equilibrium, $\bar{y} = 0$, and it is stable. You can see intuitively why this is so: if the object is displaced in either direction from equilibrium, the spring tries to bring it back.

In this case we can find y explicitly as a function of t . (Don't expect this to happen in more complicated problems!) If $v > 0$ on an interval I , Equation 2.4.10 implies that

$$\frac{dy}{dt} = v = \sqrt{\frac{\rho - ky^2}{m}}$$

on I . This is equivalent to

$$\frac{\sqrt{k}}{\sqrt{\rho - ky^2}} \frac{dy}{dt} = \omega_0, \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}. \quad (2.4.12)$$

Since

$$\begin{aligned} \int \frac{\sqrt{k} dy}{\sqrt{\rho - ky^2}} &= \sin^{-1} \left(\sqrt{\frac{k}{\rho}} y \right) + c \\ &= \sin^{-1} \left(\frac{y}{R} \right) + c \end{aligned}$$

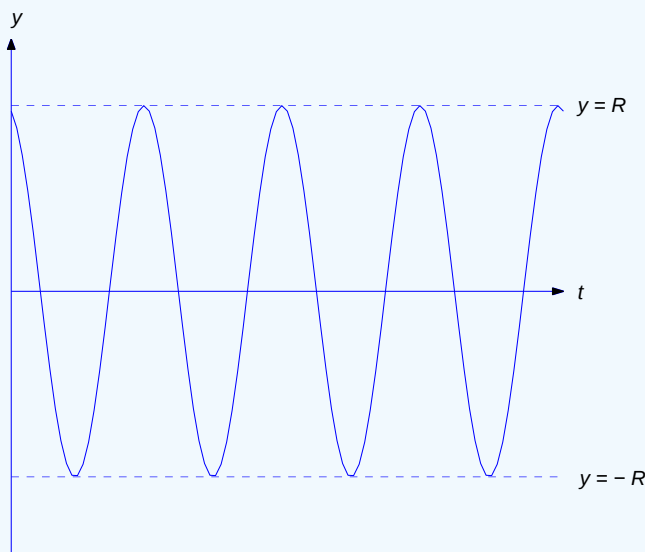


Figure 4.4.4 : $y = R \sin(\omega_0 t + \phi)$

(see Equation 2.4.11), Equation 2.4.12 implies that there's a constant ϕ such that

$$\sin^{-1}\left(\frac{y}{R}\right) = \omega_0 t + \phi$$

or

$$y = R \sin(\omega_0 t + \phi)$$

for all t in I . Although we obtained this function by assuming that $v > 0$, you can easily verify that y satisfies Equation 2.4.9 for all values of t . Thus, the displacement varies periodically between $-R$ and R , with period $T = 2\pi/\omega_0$ (Figure 4.4.4). (If you've taken a course in elementary mechanics you may recognize this as **simple harmonic motion**.)

✓ Example 4.4.2 : The Undamped Pendulum

Now we consider the motion of a pendulum with mass m , attached to the end of a weightless rod with length L that rotates on a frictionless axle (Figure 4.4.5). We assume that there's no air resistance.

Let y be the angle measured from the rest position (vertically downward) of the pendulum, as shown in Figure 4.4.5 .

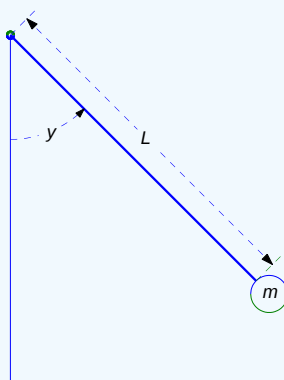


Figure 4.4.5 : The undamped pendulum

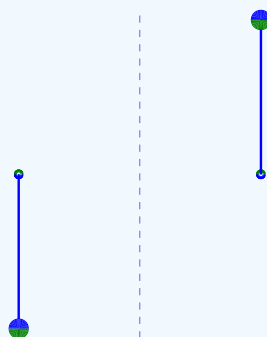
Newton's second law of motion says that the product of m and the tangential acceleration equals the tangential component of the gravitational force; therefore, from Figure 4.4.5 ,

$$mLy'' = -mg \sin y,$$

or

$$y'' = -\frac{g}{L} \sin y. \quad (2.4.13)$$

Since $\sin n\pi = 0$ if n is any integer, Equation 2.4.13 has infinitely many equilibria $\bar{y}_n = n\pi$. If n is even, the mass is directly below the axle (Figure 2.4.6a) and gravity opposes any deviation from the equilibrium.



(a) Stable equilibrium

(b) Unstable equilibrium

Figure 4.4.6 : (a) Stable equilibrium (b) Unstable equilibrium

However, if n is odd, the mass is directly above the axle (Figure 2.4.6b) and gravity increases any deviation from the equilibrium. Therefore we conclude on physical grounds that $\bar{y}_{2m} = 2m\pi$ is stable and $\bar{y}_{2m+1} = (2m+1)\pi$ is unstable.

The phase plane equivalent of Equation 2.4.13 is

$$v \frac{dv}{dy} = -\frac{g}{L} \sin y,$$

where $v = y'$ is the angular velocity of the pendulum. Integrating this yields

$$\frac{v^2}{2} = \frac{g}{L} \cos y + c. \quad (2.4.14)$$

If $v = v_0$ when $y = 0$, then

$$c = \frac{v_0^2}{2} - \frac{g}{L},$$

so Equation 2.4.14 becomes

$$\begin{aligned} \frac{v^2}{2} &= \frac{v_0^2}{2} - \frac{g}{L} (1 - \cos y) \\ &= \frac{v_0^2}{2} - \frac{2g}{L} \sin^2 \frac{y}{2}, \end{aligned}$$

which is equivalent to

$$v^2 = v_0^2 - v_c^2 \sin^2 \frac{y}{2}, \quad (2.4.15)$$

where

$$v_c = 2\sqrt{\frac{g}{L}}.$$

The curves defined by Equation 2.4.15 are the trajectories of Equation 2.4.13. They are periodic with period 2π in y , which isn't surprising, since if $y = y(t)$ is a solution of Equation 2.4.13 then so is $y_n = y(t) + 2n\pi$ for any integer n . Figure 4.4.7 shows trajectories over the interval $[-\pi, \pi]$.

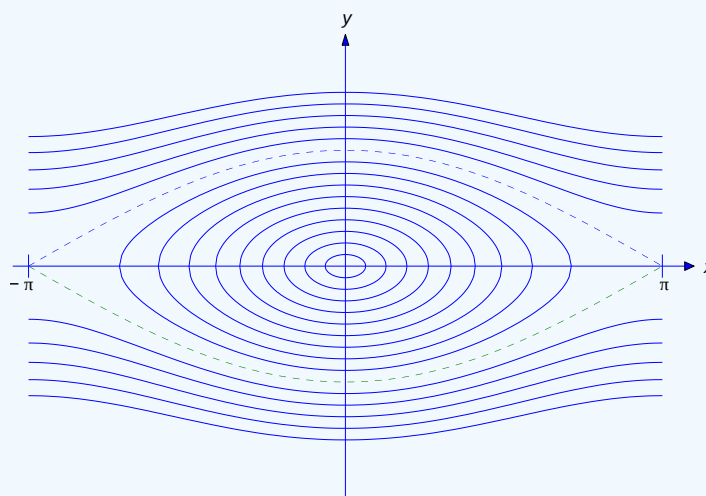


Figure 4.4.7 : Trajectories of the undamped pendulum

From Equation 2.4.15, you can see that if $|v_0| > v_c$ then v is nonzero for all t , which means that the object whirls in the same direction forever, as in Figure 4.4.8. The trajectories associated with this whirling motion are above the upper dashed curve and below the lower dashed curve in Figure 4.4.7. You can also see from Equation 2.4.15 that if $0 < |v_0| < v_c$, then $v = 0$ when $y = \pm y_{\max}$, where

$$y_{\max} = 2 \sin^{-1}(|v_0|/v_c).$$

In this case the pendulum oscillates periodically between $-y_{\max}$ and $y_{\max}$, as shown in Figure 4.4.9 . The trajectories associated with this kind of motion are the ovals between the dashed curves in Figure 4.4.7 .

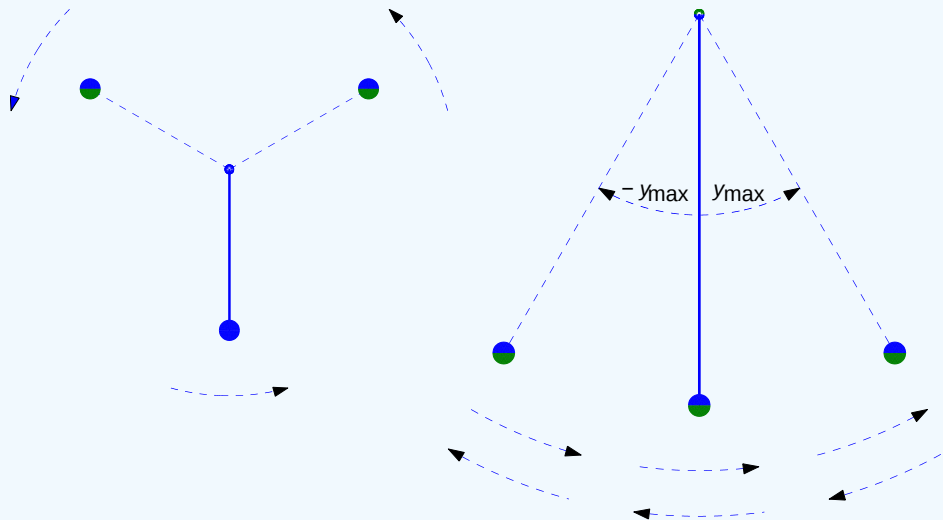


Figure 4.4.8 (left) and 4.4.9 (right): (left) The whirling undamped pendulum and (b) The oscillating undamped pendulum

It can be shown (see **Exercise 4.4.21** for a partial proof) that the period of the oscillation is

$$T = 8 \int_0^{\pi/2} \frac{d\theta}{\sqrt{v_c^2 - v_0^2 \sin^2 \theta}}. \quad (2.4.16)$$

Although this integral cannot be evaluated in terms of familiar elementary functions, you can see that it is finite if $|v_0| < v_c$.

The dashed curves in Figure 4.4.7 contain four trajectories. The critical points $(\pi, 0)$ and $(-\pi, 0)$ are the trajectories of the unstable equilibrium solutions $\bar{y} = \pm\pi$. The upper dashed curve connecting (but not including) them is obtained from initial conditions of the form $y(t_0) = 0, v(t_0) = v_c$. If y is any solution with this trajectory then

$$\lim_{t \rightarrow \infty} y(t) = \pi \quad \text{and} \quad \lim_{t \rightarrow -\infty} y(t) = -\pi.$$

The lower dashed curve connecting (but not including) them is obtained from initial conditions of the form $y(t_0) = 0, v(t_0) = -v_c$. If y is any solution with this trajectory then

$$\lim_{t \rightarrow \infty} y(t) = -\pi \quad \text{and} \quad \lim_{t \rightarrow -\infty} y(t) = \pi.$$

Consistent with this, the integral Equation 2.4.16 diverges to ∞ if $v_0 = \pm v_c$. (**Exercise 4.4.21**).

Since the dashed curves separate trajectories of whirling solutions from trajectories of oscillating solutions, each of these curves is called a **separatrix**.

In general, if Equation 2.4.7 has both stable and unstable equilibria then the separatrices are the curves given by Equation 2.4.8 that pass through unstable critical points. Thus, if $(\bar{y}, 0)$ is an unstable critical point, then

$$\frac{v^2}{2} + P(y) = P(\bar{y}) \quad (2.4.17)$$

defines a separatrix passing through $(\bar{y}, 0)$.

Stability and Instability Conditions for $y'' + p(y) = 0$

It can be shown (**Exercise 4.4.23**) that an equilibrium $\bar{y}$ of an undamped equation

$$y'' + p(y) = 0 \quad (2.4.18)$$

is stable if there's an open interval (a, b) containing $\bar{y}$ such that

$$p(y) < 0 \text{ if } a < y < \bar{y} \text{ and } p(y) > 0 \text{ if } \bar{y} < y < b \quad (2.4.19)$$

If we regard $p(y)$ as a force acting on a unit mass, Equation 2.4.19 means that the force resists all sufficiently small displacements from $\bar{y}$.

We've already seen examples illustrating this principle. The equation Equation 2.4.9 for the undamped spring-mass system is of the form Equation 2.4.18 with $p(y) = ky/m$, which has only the stable equilibrium $\bar{y} = 0$. In this case Equation 2.4.19 holds with $a = -\infty$ and $b = \infty$. The equation Equation 2.4.13 for the undamped pendulum is of the form Equation 2.4.18 with $p(y) = (g/L) \sin y$. We've seen that $\bar{y} = 2m\pi$ is a stable equilibrium if m is an integer. In this case

$$p(y) = \sin y < 0 \text{ if } (2m-1)\pi < y < 2m\pi$$

and

$$p(y) > 0 \text{ if } 2m\pi < y < (2m+1)\pi$$

It can also be shown (Exercise 4.4.24) that $\bar{y}$ is unstable if there's a $b > \bar{y}$ such that

$$p(y) < 0 \text{ if } \bar{y} < y < b \quad (2.4.20)$$

or an $a < \bar{y}$ such that

$$p(y) > 0 \text{ if } a < y < \bar{y} \quad (2.4.21)$$

If we regard $p(y)$ as a force acting on a unit mass, Equation 2.4.20 means that the force tends to increase all sufficiently small positive displacements from $\bar{y}$, while Equation 2.4.21 means that the force tends to increase the magnitude of all sufficiently small negative displacements from $\bar{y}$.

The undamped pendulum also illustrates this principle. We've seen that $\bar{y} = (2m+1)\pi$ is an unstable equilibrium if m is an integer. In this case

$$\sin y < 0 \text{ if } (2m+1)\pi < y < (2m+2)\pi$$

so Equation 2.4.20 holds with $b = (2m+2)\pi$, and

$$\sin y > 0 \text{ if } 2m\pi < y < (2m+1)\pi$$

so Equation 2.4.21 holds with $a = 2m\pi$.

✓ Example 4.4.3

The equation

$$y'' + y(y-1) = 0 \quad (2.4.22)$$

is of the form Equation 2.4.18 with $p(y) = y(y-1)$. Therefore $\bar{y} = 0$ and $\bar{y} = 1$ are the equilibria of Equation 2.4.22. Since

$$\begin{aligned} y(y-1) &> 0 && \text{if } y < 0 \text{ or } y > 1 \\ &< 0 && \text{if } 0 < y < 1 \end{aligned}$$

$\bar{y} = 0$ is unstable and $\bar{y} = 1$ is stable.

The phase plane equivalent of Equation 2.4.22 is the separable equation

$$v \frac{dv}{dy} + y(y-1) = 0.$$

Integrating yields

$$\frac{v^2}{2} + \frac{y^3}{3} - \frac{y^2}{2} = C,$$

which we rewrite as

$$v^2 + \frac{1}{3}y^2(2y-3) = c \quad (2.4.23)$$

after renaming the constant of integration. These are the trajectories of Equation 2.4.22. If y is any solution of Equation 2.4.22, the point $(y(t), v(t))$ moves along the trajectory of y in the direction of increasing y in the upper half plane ($v = y' > 0$), or in the direction of decreasing y in the lower half plane ($v = y' < 0$).

Figure 4.4.10 shows typical trajectories. The dashed curve through the critical point $(0, 0)$, obtained by setting $c = 0$ in Equation 2.4.23, separates the $y-v$ plane into regions that contain different kinds of trajectories; again, we call this curve a **separatrix**. Trajectories in the region bounded by the closed loop (b) are closed curves, so solutions associated with them are periodic. Solutions associated with other trajectories are not periodic. If y is any such solution with trajectory not on the separatrix, then

$$\begin{aligned}\lim_{t \rightarrow \infty} y(t) &= -\infty, & \lim_{t \rightarrow -\infty} y(t) &= -\infty, \\ \lim_{t \rightarrow \infty} v(t) &= -\infty, & \lim_{t \rightarrow -\infty} v(t) &= \infty.\end{aligned}$$

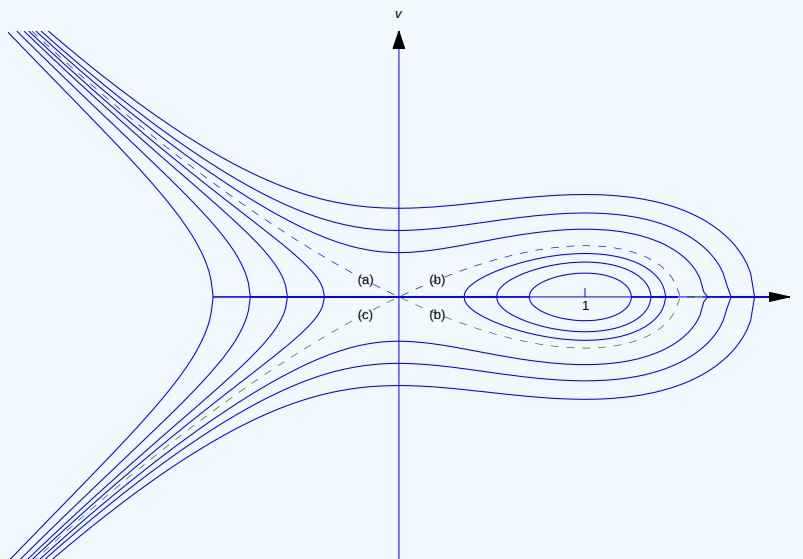


Figure 4.4.10 : Trajectories of $y'' + y(y - 1) = 0$

The separatrix contains four trajectories of Equation 2.4.22. One is the point $(0, 0)$, the trajectory of the equilibrium $\bar{y} = 0$. Since distinct trajectories cannot intersect, the segments of the separatrix marked (a), b, and (c) – which don't include $(0, 0)$ – are distinct trajectories, none of which can be traversed in finite time. Solutions with these trajectories have the following asymptotic behavior:

$$\begin{aligned}\lim_{t \rightarrow \infty} y(t) &= 0, & \lim_{t \rightarrow -\infty} y(t) &= -\infty, \\ \lim_{t \rightarrow \infty} v(t) &= 0, & \lim_{t \rightarrow -\infty} v(t) &= \infty & \text{(on (a))} \\ \lim_{t \rightarrow \infty} y(t) &= 0, & \lim_{t \rightarrow -\infty} y(t) &= 0, \\ \lim_{t \rightarrow \infty} v(t) &= 0, & \lim_{t \rightarrow -\infty} v(t) &= 0 & \text{(on (b))} \\ \lim_{t \rightarrow \infty} y(t) &= -\infty, & \lim_{t \rightarrow -\infty} y(t) &= 0, \\ \lim_{t \rightarrow \infty} v(t) &= -\infty, & \lim_{t \rightarrow -\infty} v(t) &= 0 & \text{(on (c)).}\end{aligned}$$

The Damped Case

The phase plane equivalent of the damped autonomous equation

$$y'' + q(y, y')y' + p(y) = 0 \tag{2.4.24}$$

is

$$v \frac{dv}{dy} + q(y, v)v + p(y) = 0.$$

This equation isn't separable, so we cannot solve it for v in terms of y , as we did in the undamped case, and conservation of energy doesn't hold. (For example, energy expended in overcoming friction is lost.) However, we can study the qualitative behavior of its solutions by rewriting it as

$$\frac{dv}{dy} = -q(y, v) - \frac{p(y)}{v} \quad (2.4.25)$$

and considering the direction fields for this equation.

In the following examples we will also be showing computer generated trajectories of this equation, obtained by numerical methods. The exercises call for similar computations. The methods discussed in Chapter 3 are not suitable for this task, since $p(y)/v$ in Equation 2.4.25 is undefined on the y axis of the Poincaré phase plane. Therefore we are forced to apply numerical methods briefly discussed in Section 10.1 to the system

$$\begin{aligned} y' &= v \\ v' &= -q(y, v)v - p(y), \end{aligned}$$

which is equivalent to Equation 2.4.24 in the sense defined in Section 10.1. Fortunately, most differential equation software packages enable you to do this painlessly.

In the text we will confine ourselves to the case where q is constant, so Equation 2.4.24 and Equation 2.4.25 reduce to

$$y'' + cy' + p(y) = 0 \quad (2.4.26)$$

and

$$\frac{dv}{dy} = -c - \frac{p(y)}{v}.$$

(we will consider more general equations in the exercises.) The constant c is called the **damping constant**. In situations where Equation 2.4.26 is the equation of motion of an object, c is positive; however, there are situations where c may be negative.

The Damped Spring-Mass System

Earlier we considered the spring - mass system under the assumption that the only forces acting on the object were gravity and the spring's resistance to changes in its length. Now we will assume that some mechanism (for example, friction in the spring or atmospheric resistance) opposes the motion of the object with a force proportional to its velocity. In Section 6.1 it will be shown that in this case Newton's second law of motion implies that

$$my'' + cy' + ky = 0, \quad (2.4.27)$$

where $c > 0$ is the **damping constant**. Again, this equation can be solved easily by a method that we will study in Section 5.2, but that method isn't available here. Instead, we will consider its phase plane equivalent, which can be written in the form Equation 2.4.25 as

$$\frac{dv}{dy} = -\frac{c}{m} - \frac{ky}{mv}. \quad (2.4.28)$$

(A minor note: the c in Equation 2.4.26 actually corresponds to c/m in this equation.) Figure 4.4.11 shows a typical direction field for an equation of this form. Recalling that motion along a trajectory must be in the direction of increasing y in the upper half plane ($v > 0$) and in the direction of decreasing y in the lower half plane ($v < 0$), you can infer that all trajectories approach the origin in clockwise fashion.

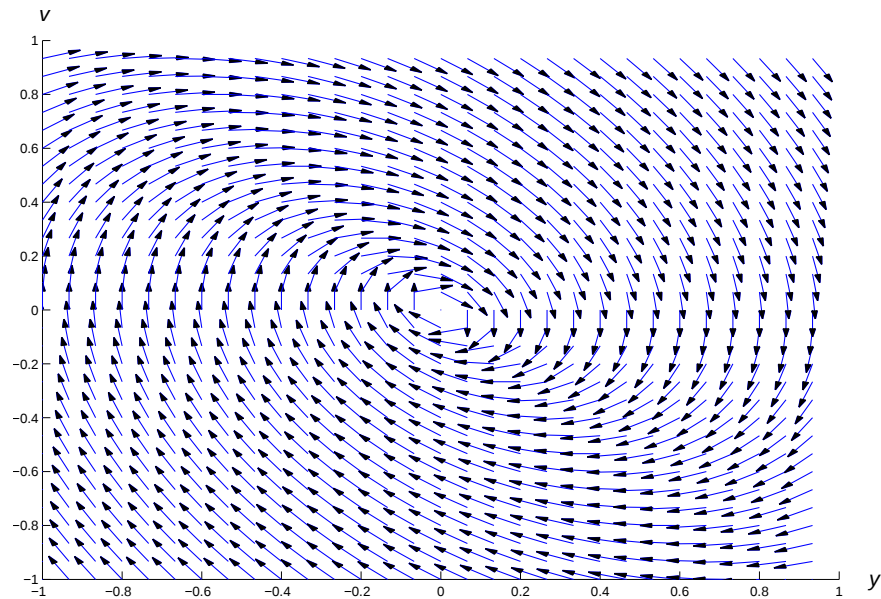


Figure 4.4.11 : A typical direction field for $my'' + cy' + ky = 0$ with $0 < c < c_1$

To confirm this, Figure 4.4.12 shows the same direction field with some trajectories filled in. All the trajectories shown there correspond to solutions of the initial value problem

$$my'' + cy' + ky = 0, \quad y(0) = y_0, \quad y'(0) = v_0,$$

where

$$mv_0^2 + ky_0^2 = \rho \quad (\text{a positive constant});$$

thus, if there were no damping ($c = 0$), all the solutions would have the same dashed elliptic trajectory, shown in Figure 4.4.14 .

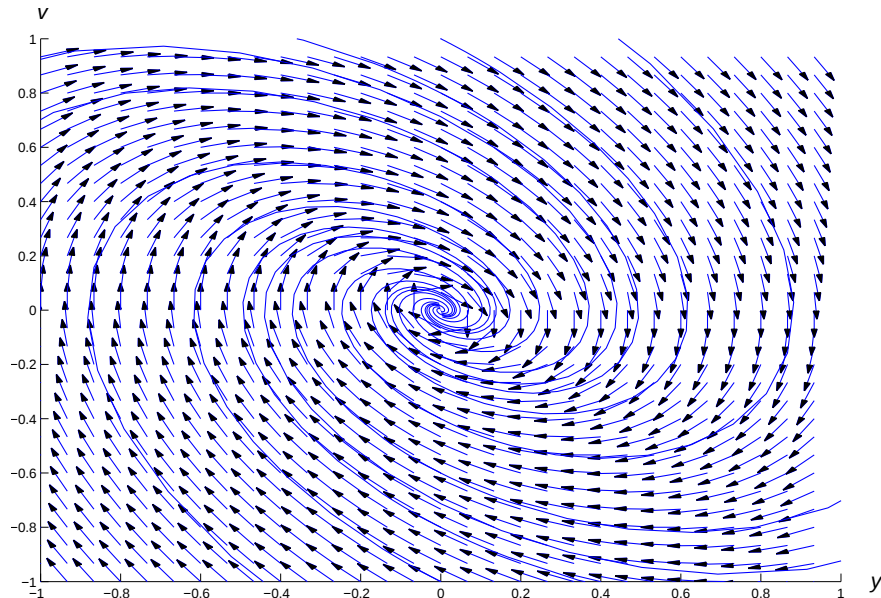


Figure 4.4.11 with some trajectories added

Solutions corresponding to the trajectories in Figure 4.4.12 cross the y -axis infinitely many times. The corresponding solutions are said to be **oscillatory** (Figure 4.4.13)

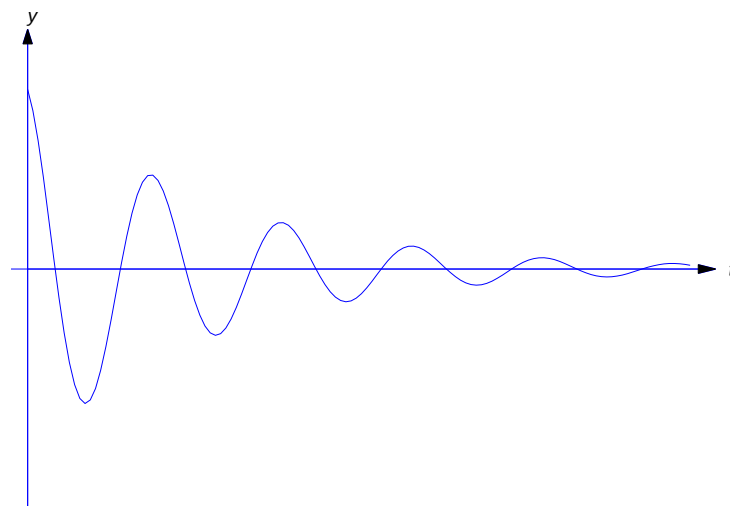


Figure 4.4.13 : An oscillatory solution of $my'' + cy' + ky = 0$

It is shown in Section 6.2 that there's a number c_1 such that if $0 \leq c < c_1$ then all solutions of (4.4.27) are oscillatory, while if $c \geq c_1$, no solutions of (4.4.27) have this property. (In fact, no solution not identically zero can have more than two zeros in this case.) Figure 4.4.14 shows a direction field and some integral curves for (4.4.28) in this case.

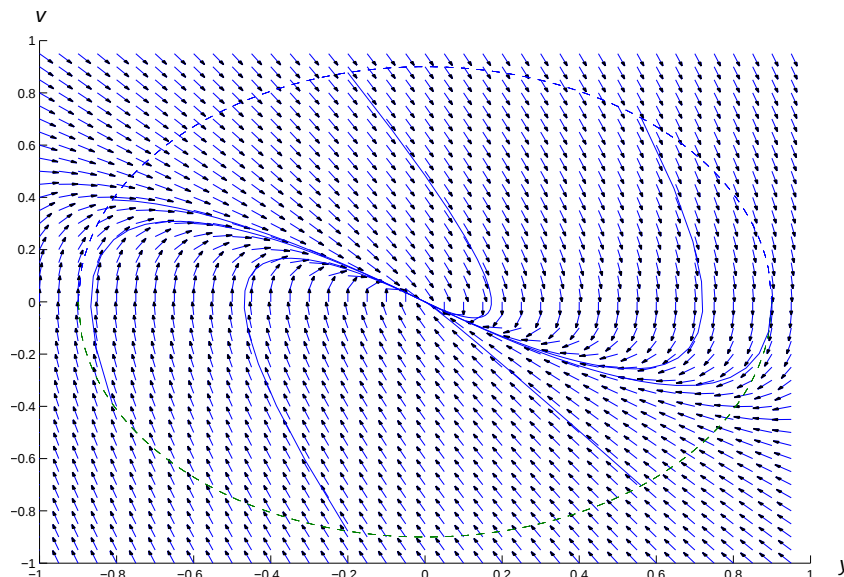


Figure 4.4.14 : A typical direction field for $my'' + cy' + ky = 0$ with $c > c_1$

✓ Example 4.4.4

Now we return to the pendulum. If we assume that some mechanism (for example, friction in the axle or atmospheric resistance) opposes the motion of the pendulum with a force proportional to its angular velocity, Newton's second law of motion implies that

$$mLy'' = -cy' - mg\sin y, \quad (2.4.29)$$

where $c > 0$ is the damping constant. (Again, a minor note: the c in Equation 2.4.26 actually corresponds to c/mL in this equation.) To plot a direction field for Equation 2.4.29 we write its phase plane equivalent as

$$\frac{dv}{dy} = -\frac{c}{mL} - \frac{g}{Lv} \sin y.$$

Figure 4.4.15 shows trajectories of four solutions of Equation 2.4.29, all satisfying $y(0) = 0$. For each $m = 0, 1, 2, 3$, imparting the initial velocity $v(0) = v_m$ causes the pendulum to make m complete revolutions and then settle into decaying oscillation about the stable equilibrium $\bar{y} = 2m\pi$.

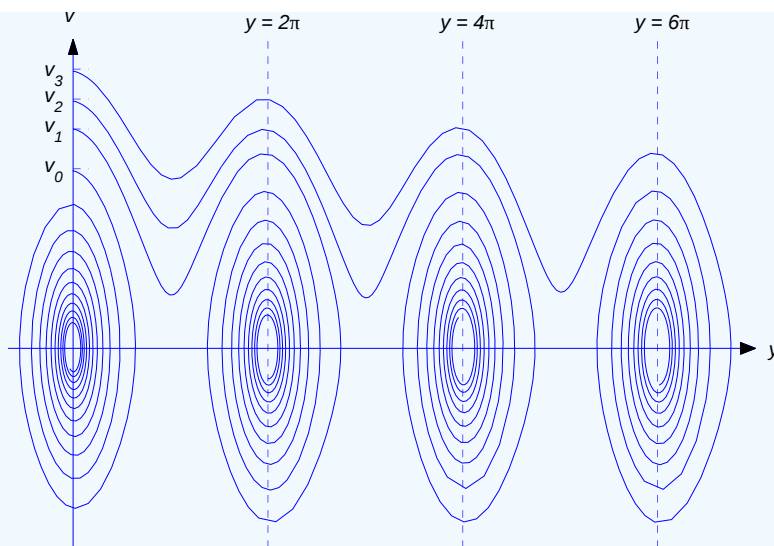


Figure 4.4.15 : Four trajectories of the damped pendulum.

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2.4E: Autonomous Second Order Equations (Exercises)

Q4.4.1

In **Exercises 4.4.1–4.4.4** find the equations of the trajectories of the given undamped equation. Identify the equilibrium solutions, determine whether they are stable or unstable, and plot some trajectories.

1. $y'' + y^3 = 0$

2. $y'' + y^2 = 0$

3. $y'' + y|y| = 0$

4. $y'' + ye^{-y} = 0$

Q4.4.2

In **Exercises 4.4.5–4.4.8** find the equations of the trajectories of the given undamped equation. Identify the equilibrium solutions, determine whether they are stable or unstable, and find the equations of the separatrices (that is, the curves through the unstable equilibria). Plot the separatrices and some trajectories in each of the regions of Poincaré plane determined by them.

5. $y'' - y^3 + 4y = 0$

6. $y'' + y^3 - 4y = 0$

7. $y'' + y(y^2 - 1)(y^2 - 4) = 0$

8. $y'' + y(y - 2)(y - 1)(y + 2) = 0$

Q4.4.3

In **Exercises 4.4.9–4.4.12** plot some trajectories of the given equation for various values (positive, negative, zero) of the parameter

a. Find the equilibria of the equation and classify them as stable or unstable. Explain why the phase plane plots corresponding to positive and negative values of a differ so markedly. Can you think of a reason why zero deserves to be called the **critical value** of a ?

9. $y'' + y^2 - a = 0$

10. $y'' + y^3 - ay = 0$

11. $y'' - y^3 + ay = 0$

12. $y'' + y - ay^3 = 0$

Q4.4.4

In **Exercises 4.4.13–4.4.18** plot trajectories of the given equation for $c = 0$ and small nonzero (positive and negative) values of c to observe the effects of damping.

13. $y'' + cy' + y^3 = 0$

14. $y'' + cy' - y = 0$

15. $y'' + cy' + y^3 = 0$

16. $y'' + cy' + y^2 = 0$

17. $y'' + cy' + y|y| = 0$

18. $y'' + y(y - 1) + cy = 0$

Q4.4.5

19. The **van der Pol** equation

$$y'' - \mu(1 - y^2)y' + y = 0, \quad (\text{A})$$

where μ is a positive constant and y is electrical current (Section 6.3), arises in the study of an electrical circuit whose resistive properties depend upon the current. The damping term $-\mu(1 - y^2)y'$ works to reduce $|y|$ if $|y| < 1$ or to increase $|y|$ if $|y| > 1$. It can be shown that van der Pol's equation has exactly one closed trajectory, which is called a **limit cycle**. Trajectories inside the limit cycle spiral outward to it, while trajectories outside the limit cycle spiral inward to it (Figure [figure:4.4.16]). Use your favorite differential equations software to verify this for $\mu = .5, 1, 1.5, 2$. Use a grid with $-4 < y < 4$ and $-4 < v < 4$.

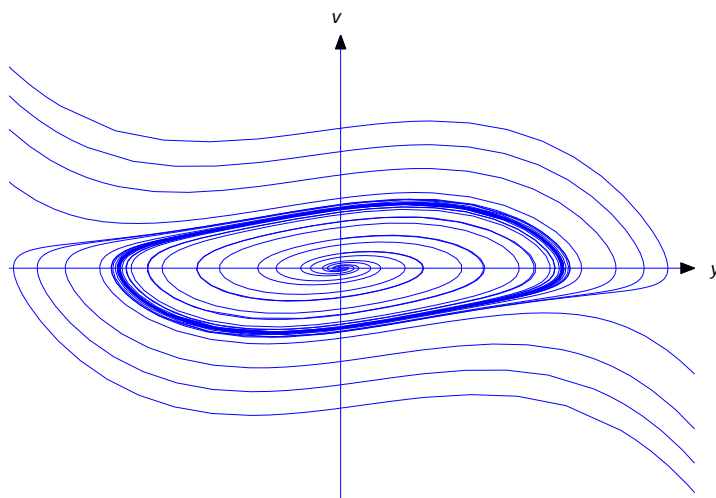


Figure 4.4E.16 : Trajectories of van der Pol's equation

20. Rayleigh's equation,

$$y'' - \mu(1 - (y')^2/3)y' + y = 0$$

also has a limit cycle. Follow the directions of **Exercise 4.4.19** for this equation.

21. In connection with Equation 4.4.16, suppose $y(0) = 0$ and $y'(0) = v_0$, where $0 < v_0 < v_c$.

a. Let T_1 be the time required for y to increase from zero to $y_{\max} = 2 \sin^{-1}(v_0/v_c)$. Show that

$$\frac{dy}{dt} = \sqrt{v_0^2 - v_c^2 \sin^2 y/2}, \quad 0 \leq t < T_1. \quad (\text{A})$$

b. Separate variables in (A) and show that

$$T_1 = \int_0^{y_{\max}} \frac{du}{\sqrt{v_0^2 - v_c^2 \sin^2 u/2}} \quad (\text{B})$$

c. Substitute $\sin u/2 = (v_0/v_c) \sin \theta$ in (B) to obtain

$$T_1 = 2 \int_0^{\pi/2} \frac{d\theta}{\sqrt{v_c^2 - v_0^2 \sin^2 \theta}}. \quad (\text{C})$$

d. Conclude from symmetry that the time required for $(y(t), v(t))$ to traverse the trajectory

$$v^2 = v_0^2 - v_c^2 \sin^2 y/2$$

is $T = 4T_1$, and that consequently $y(t+T) = y(t)$ and $v(t+T) = v(t)$; that is, the oscillation is periodic with period T .

e. Show that if $v_0 = v_c$, the integral in (C) is improper and diverges to ∞ . Conclude from this that $y(t) < \pi$ for all t and $\lim_{t \rightarrow \infty} y(t) = \pi$.

22. Give a direct definition of an unstable equilibrium of $y'' + p(y) = 0$.

23. Let p be continuous for all y and $p(0) = 0$. Suppose there's a positive number ρ such that $p(y) > 0$ if $0 < y \leq \rho$ and $p(y) < 0$ if $-\rho \leq y < 0$. For $0 < r \leq \rho$ let

$$\alpha(r) = \min \left\{ \int_0^r p(x) dx, \int_{-r}^0 |p(x)| dx \right\} \quad \text{and} \quad \beta(r) = \max \left\{ \int_0^r p(x) dx, \int_{-r}^0 |p(x)| dx \right\}.$$

Let y be the solution of the initial value problem

$$y'' + p(y) = 0, \quad y(0) = v_0, \quad y'(0) = v_0,$$

and define $c(y_0, v_0) = v_0^2 + 2 \int_0^{y_0} p(x) dx$.

a. Show that

$$0 < c(y_0, v_0) < v_0^2 + 2\beta(|y_0|) \quad \text{if} \quad 0 < |y_0| \leq \rho.$$

b. Show that

$$v^2 + 2 \int_0^y p(x) dx = c(y_0, v_0), \quad t > 0.$$

c. Conclude from (b) that if $c(y_0, v_0) < 2\alpha(r)$ then $|y| < r$, $t > 0$.

d. Given $\epsilon > 0$, let $\delta > 0$ be chosen so that

$$\delta^2 + 2\beta(\delta) < \max \{ \epsilon^2/2, 2\alpha(\epsilon/\sqrt{2}) \}.$$

Show that if $\sqrt{y_0^2 + v_0^2} < \delta$ then $\sqrt{y^2 + v^2} < \epsilon$ for $t > 0$, which implies that $\bar{y} = 0$ is a stable equilibrium of $y'' + p(y) = 0$.

e. Now let p be continuous for all y and $p(\bar{y}) = 0$, where $\bar{y}$ is not necessarily zero. Suppose there's a positive number ρ such that $p(y) > 0$ if $\bar{y} < y \leq \bar{y} + \rho$ and $p(y) < 0$ if $\bar{y} - \rho \leq y < \bar{y}$. Show that $\bar{y}$ is a stable equilibrium of $y'' + p(y) = 0$.

24. Let p be continuous for all y .

a. Suppose $p(0) = 0$ and there's a positive number ρ such that $p(y) < 0$ if $0 < y \leq \rho$. Let ϵ be any number such that $0 < \epsilon < \rho$. Show that if y is the solution of the initial value problem

$$y'' + p(y) = 0, \quad y(0) = y_0, \quad y'(0) = 0$$

with $0 < y_0 < \epsilon$, then $y(t) \geq \epsilon$ for some $t > 0$. Conclude that $\bar{y} = 0$ is an unstable equilibrium of $y'' + p(y) = 0$.

b. Now let $p(\bar{y}) = 0$, where $\bar{y}$ isn't necessarily zero. Suppose there's a positive number ρ such that $p(y) < 0$ if $\bar{y} < y \leq \bar{y} + \rho$. Show that $\bar{y}$ is an unstable equilibrium of $y'' + p(y) = 0$.

c. Modify your proofs of (a) and (b) to show that if there's a positive number ρ such that $p(y) > 0$ if $\bar{y} - \rho \leq y < \bar{y}$, then $\bar{y}$ is an unstable equilibrium of $y'' + p(y) = 0$.

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2.5: Applications to Curves

We begin with two examples of families of curves generated by varying a parameter over a set of real numbers.

✓ Example 4.5.1

For each value of the parameter c , the equation

$$y - cx^2 = 0 \quad (2.5.1)$$

defines a curve in the xy -plane. If $c \neq 0$, the curve is a parabola through the origin, opening upward if $c > 0$ or downward if $c < 0$. If $c = 0$, the curve is the x axis (Figure 4.5.1).

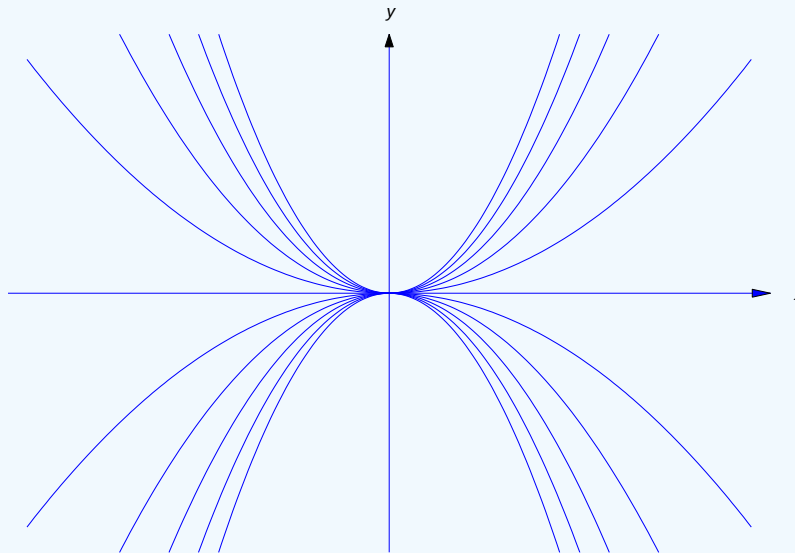


Figure 4.5.1 : A family of curves defined by $y - cx^2 = 0$

✓ Example 4.5.2

For each value of the parameter c the equation

$$y = x + c \quad (2.5.2)$$

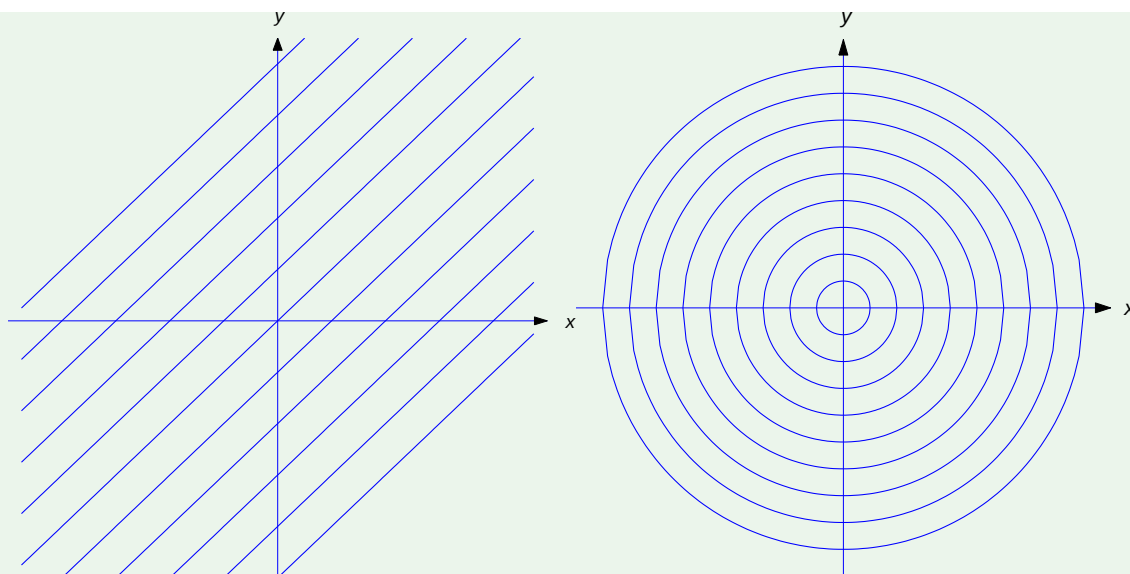
defines a line with slope 1(Figure 4.5.2).

Definition 4.5.1

An equation that can be written in the form

$$H(x, y, c) = 0 \quad (2.5.3)$$

is said to define a **one-parameter family of curves** if, for each value of c in in some nonempty set of real numbers, the set of points (x, y) that satisfy Equation 2.5.3 forms a curve in the xy -plane.



Figures 4.5.2 and 4.5.3 : (left) A family of lines defined by $y = x + c$. (right) A family of circles defined by $x^2 + y^2 - c^2 = 0$

Equations Equation 2.5.1 and Equation 2.5.2 define one-parameter families of curves. (Although Equation 2.5.2 isn't in the form Equation 2.5.3, it can be written in this form as $y - x - c = 0$.)

✓ Example 4.5.3

If $c > 0$, the graph of the equation

$$x^2 + y^2 - c = 0 \quad (2.5.4)$$

is a circle with center at $(0, 0)$ and radius $\sqrt{c}$. If $c = 0$, the graph is the single point $(0, 0)$. (We don't regard a single point as a curve.) If $c < 0$, the equation has no graph. Hence, Equation 2.5.4 defines a one-parameter family of curves for positive values of c . This family consists of all circles centered at $(0, 0)$ (Figure 4.5.3).

✓ Example 4.5.4

The equation

$$x^2 + y^2 + c^2 = 0$$

does not define a one-parameter family of curves, since no (x, y) satisfies the equation if $c \neq 0$, and only the single point $(0, 0)$ satisfies it if $c = 0$.

Recall from Section 1.2 that the graph of a solution of a differential equation is called an **integral curve** of the equation. Solving a first order differential equation usually produces a one-parameter family of integral curves of the equation. Here we are interested in the converse problem: given a one-parameter family of curves, is there a first order differential equation for which every member of the family is an integral curve. This suggests the next definition.

✎ Definition 4.5.2

4.5.2 If every curve in a one-parameter family defined by the equation

$$H(x, y, c) = 0 \quad (2.5.5)$$

is an integral curve of the first order differential equation

$$F(x, y, y') = 0, \quad (2.5.6)$$

then Equation 2.5.6 is said to be a **differential equation for the family**.

To find a differential equation for a one-parameter family we differentiate its defining equation Equation 2.5.5 implicitly with respect to x , to obtain

$$H_x(x, y, c) + H_y(x, y, c)y' = 0. \quad (2.5.7)$$

If this equation doesn't, then it is a differential equation for the family. If it does contain c , it may be possible to obtain a differential equation for the family by eliminating c between Equation 2.5.5 and Equation 2.5.7.

✓ Example 4.5.5

Find a differential equation for the family of curves defined by

$$y = cx^2. \quad (2.5.8)$$

Solution

Differentiating Equation 2.5.8 with respect to x yields

$$y' = 2cx.$$

Therefore $c = y'/2x$, and substituting this into Equation 2.5.8 yields

$$y = \frac{xy'}{2}$$

as a differential equation for the family of curves defined by Equation 2.5.8. The graph of any function of the form $y = cx^2$ is an integral curve of this equation.

The next example shows that members of a given family of curves may be obtained by joining integral curves for more than one differential equation.

✓ Example 4.5.6

- Try to find a differential equation for the family of lines tangent to the parabola $y = x^2$.
- Find two tangent lines to the parabola $y = x^2$ that pass through $(2, 3)$, and find the points of tangency.

Solution a

The equation of the line through a given point (x_0, y_0) with slope m is

$$y = y_0 + m(x - x_0). \quad (2.5.9)$$

If (x_0, y_0) is on the parabola, then $y_0 = x_0^2$ and the slope of the tangent line through (x_0, x_0^2) is $m = 2x_0$; hence, Equation 2.5.9 becomes

$$y = x_0^2 + 2x_0(x - x_0),$$

or, equivalently,

$$y = -x_0^2 + 2x_0x. \quad (2.5.10)$$

Here x_0 plays the role of the constant c in Definition 4.5.1; that is, varying x_0 over $(-\infty, \infty)$ produces the family of tangent lines to the parabola $y = x^2$.

Differentiating Equation 2.5.10 with respect to x yields $y' = 2x_0$. We can express x_0 in terms of x and y by rewriting Equation 2.5.10 as

$$x_0^2 - 2x_0x + y = 0$$

and using the quadratic formula to obtain

$$x_0 = x \pm \sqrt{x^2 - y}. \quad (2.5.11)$$

We must choose the plus sign in Equation 2.5.11 if $x < x_0$ and the minus sign if $x > x_0$; thus,

$$x_0 = \left(x + \sqrt{x^2 - y}\right) \text{ if } x < x_0.$$

and

$$x_0 = \left(x - \sqrt{x^2 - y}\right) \text{ if } x > x_0.$$

Since $y' = 2x_0$, this implies that

$$y' = 2 \left(x + \sqrt{x^2 - y}\right), \quad \text{if } x < x_0 \quad (2.5.12)$$

and

$$y' = 2 \left(x - \sqrt{x^2 - y}\right), \quad \text{if } x > x_0. \quad (2.5.13)$$

Neither Equation 2.5.12 nor Equation 2.5.13 is a differential equation for the family of tangent lines to the parabola $y = x^2$. However, if each tangent line is regarded as consisting of two **tangent half lines** joined at the point of tangency, Equation 2.5.12 is a differential equation for the family of tangent half lines on which x is less than the abscissa of the point of tangency (Figure 4.5.4 a), while Equation 2.5.13 is a differential equation for the family of tangent half lines on which x is greater than this abscissa (Figure 4.5.4 (b)). The parabola $y = x^2$ is also an integral curve of both Equation 2.5.12 and Equation 2.5.13.

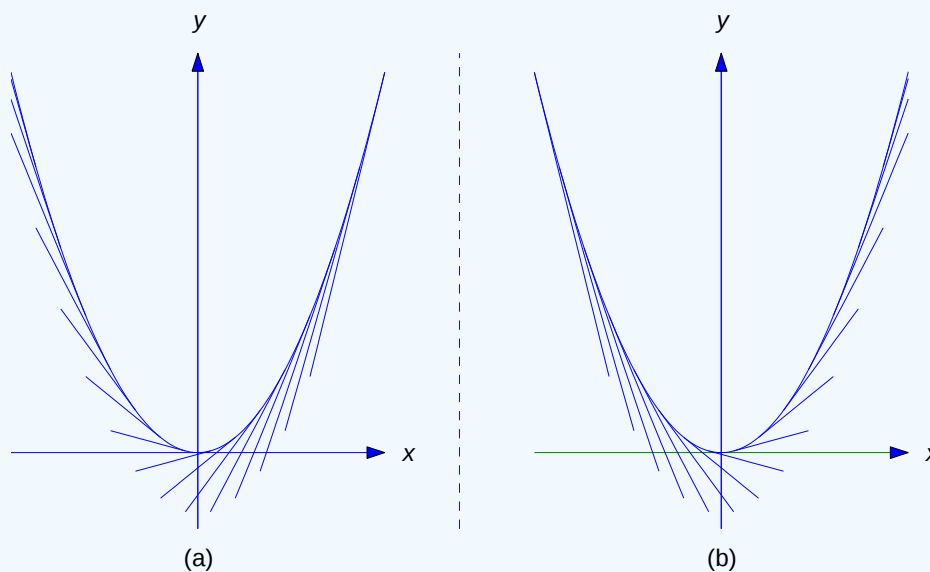


Figure 4.5.4

Solution b

From Equation 2.5.10 the point $(x, y) = (2, 3)$ is on the tangent line through (x_0, x_0^2) if and only if

$$3 = -x_0^2 + 4x_0,$$

which is equivalent to

$$x_0^2 - 4x_0 + 3 = (x_0 - 3)(x_0 - 1) = 0.$$

Letting $x_0 = 3$ in Equation 2.5.10 shows that $(2, 3)$ is on the line

$$y = -9 + 6x,$$

which is tangent to the parabola at $(x_0, x_0^2) = (3, 9)$, as shown in Figure 4.5.5

Letting $x_0 = 1$ in Equation 2.5.10 shows that $(2, 3)$ is on the line

$$y = -1 + 2x,$$

which is tangent to the parabola at $(x_0, x_0^2) = (1, 1)$, as shown in Figure 4.5.5 .

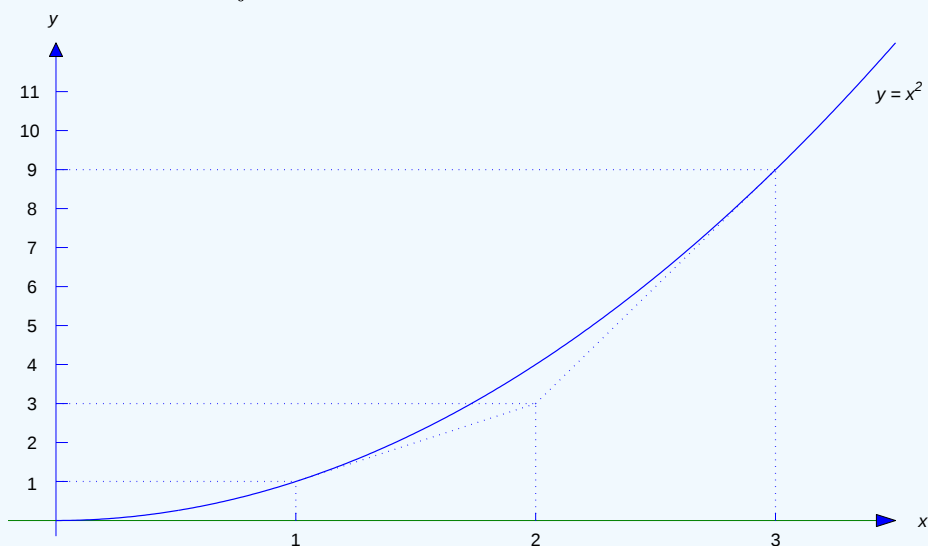


Figure 4.5.5

Geometric Problems

We now consider some geometric problems that can be solved by means of differential equations.

✓ Example 4.5.7

Find curves $y = y(x)$ such that every point $(x_0, y(x_0))$ on the curve is the midpoint of the line segment with endpoints on the coordinate axes and tangent to the curve at $(x_0, y(x_0))$ (Figure 4.5.6).

Solution

The equation of the line tangent to the curve at $P = (x_0, y(x_0))$ is

$$y = y(x_0) + y'(x_0)(x - x_0).$$

If we denote the x and y intercepts of the tangent line by x_I and y_I (Figure 4.5.6), then

$$0 = y(x_0) + y'(x_0)(x_I - x_0) \tag{2.5.14}$$

and

$$y_I = y(x_0) - y'(x_0)x_0. \tag{2.5.15}$$

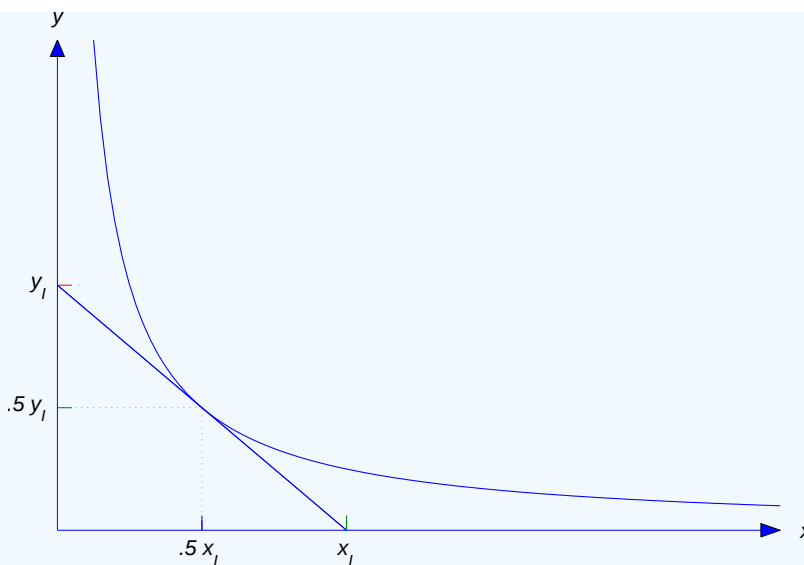


Figure 4.5.6

From Figure 4.5.6 , P is the midpoint of the line segment connecting $(x_I, 0)$ and $(0, y_I)$ if and only if $x_I = 2x_0$ and $y_I = 2y(x_0)$. Substituting the first of these conditions into Equation 2.5.14 or the second into Equation 2.5.15 yields

$$y(x_0) + y'(x_0)x_0 = 0.$$

Since x_0 is arbitrary we drop the subscript and conclude that $y = y(x)$ satisfies

$$y + xy' = 0,$$

which can be rewritten as

$$(xy)' = 0.$$

Integrating yields $xy = c$, or

$$y = \frac{c}{x}.$$

If $c = 0$ this curve is the line $y = 0$, which does not satisfy the geometric requirements imposed by the problem; thus, $c \neq 0$, and the solutions define a family of hyperbolas (Figure 4.5.7).

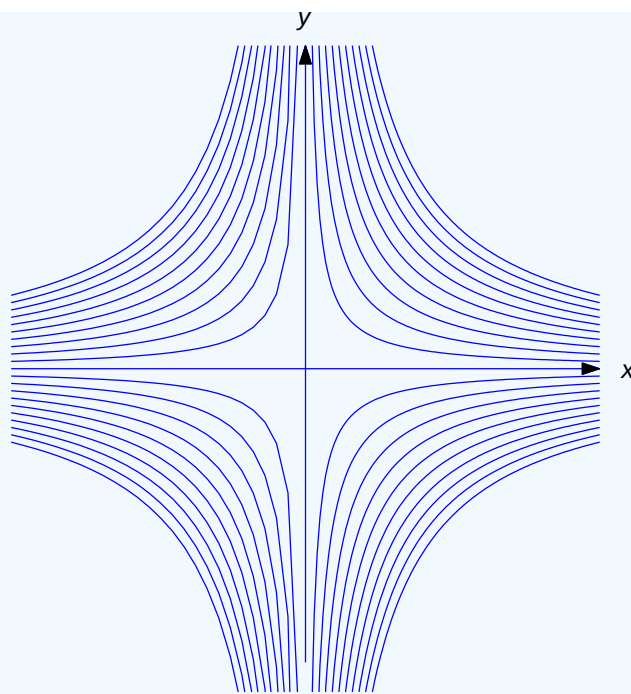


Figure 4.5.7

✓ Example 4.5.8

Find curves $y = y(x)$ such that the tangent line to the curve at any point $(x_0, y(x_0))$ intersects the x -axis at $(x_0^2, 0)$. Figure 4.5.8 illustrates the situation in the case where the curve is in the first quadrant and $0 < x < 1$.

Solution

The equation of the line tangent to the curve at $(x_0, y(x_0))$ is

$$y = y(x_0) + y'(x_0)(x - x_0).$$

Since $(x_0^2, 0)$ is on the tangent line,

$$0 = y(x_0) + y'(x_0)(x_0^2 - x_0).$$

Since x_0 is arbitrary we drop the subscript and conclude that $y = y(x)$ satisfies

$$y + y'(x^2 - x) = 0.$$

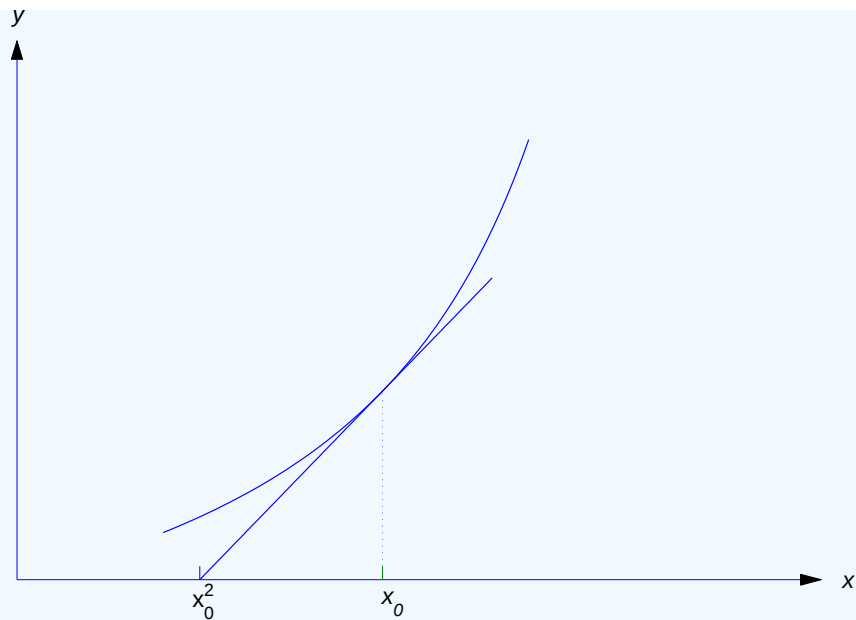


Figure 4.5.8

Therefore

$$\frac{y'}{y} = -\frac{1}{x^2 - x} = -\frac{1}{x(x-1)} = \frac{1}{x} - \frac{1}{x-1},$$

so

$$\ln|y| = \ln|x| - \ln|x-1| + k = \ln\left|\frac{x}{x-1}\right| + k,$$

and

$$y = \frac{cx}{x-1}.$$

If $c = 0$, the graph of this function is the x -axis. If $c \neq 0$, it is a hyperbola with vertical asymptote $x = 1$ and horizontal asymptote $y = c$. Figure 4.5.9 shows the graphs for $c \neq 0$.

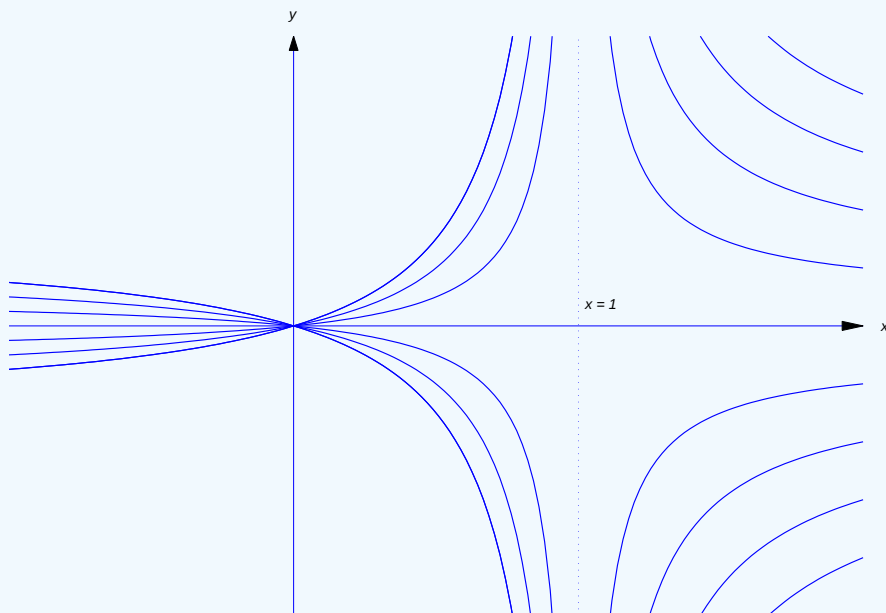


Figure 4.5.9

Orthogonal Trajectories

Two curves C_1 and C_2 are said to be orthogonal at a point of intersection (x_0, y_0) if they have perpendicular tangents at (x_0, y_0) . (Figure 4.5.10). A curve is said to be an **orthogonal trajectory** of a given family of curves if it is orthogonal to every curve in the family. For example, every line through the origin is an orthogonal trajectory of the family of circles centered at the origin. Conversely, any such circle is an orthogonal trajectory of the family of lines through the origin (Figure 4.5.11).

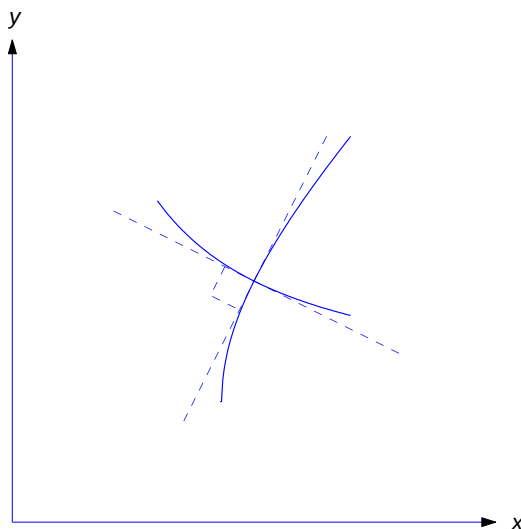


Figure 4.5.10

Orthogonal trajectories occur in many physical applications. For example, if $u = u(x, y)$ is the temperature at a point (x, y) , the curves defined by

$$u(x, y) = c \quad (2.5.16)$$

are called **isothermal** curves. The orthogonal trajectories of this family are called **heat-flow** lines, because at any given point the direction of maximum heat flow is perpendicular to the isothermal through the point. If u represents the potential energy of an object moving under a force that depends upon (x, y) , the curves Equation 2.5.16 are called **equipotentials**, and the orthogonal trajectories are called **lines of force**.

From analytic geometry we know that two nonvertical lines L_1 and L_2 with slopes m_1 and m_2 , respectively, are perpendicular if and only if $m_2 = -1/m_1$; therefore, the integral curves of the differential equation

$$y' = -\frac{1}{f(x, y)}$$

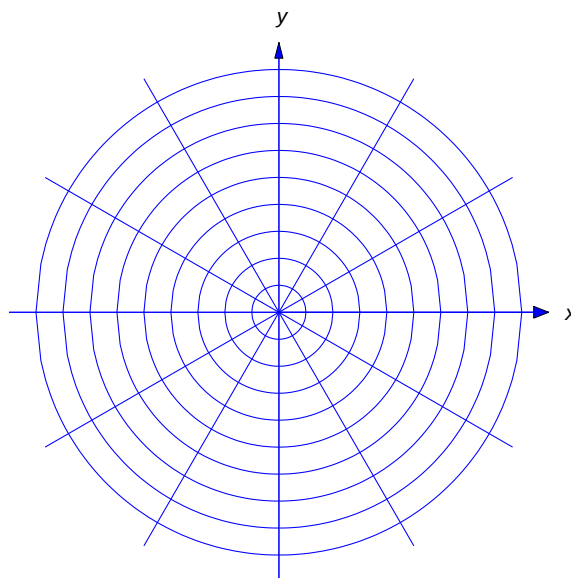


Figure 4.5.11

are orthogonal trajectories of the integral curves of the differential equation

$$y' = f(x, y),$$

because at any point (x_0, y_0) where curves from the two families intersect the slopes of the respective tangent lines are

$$m_1 = f(x_0, y_0) \quad \text{and} \quad m_2 = -\frac{1}{f(x_0, y_0)}.$$

This suggests a method for finding orthogonal trajectories of a family of integral curves of a first order equation.

✚ Finding Orthogonal Trajectories

- **Step 1.** Find a differential equation

$$y' = f(x, y)$$

for the given family.

- **Step 2.** Solve the differential equation

$$y' = -\frac{1}{f(x, y)}$$

to find the orthogonal trajectories.

✓ Example 4.5.9

Find the orthogonal trajectories of the family of circles

$$x^2 + y^2 = c^2 \quad (c > 0). \quad (2.5.17)$$

Solution

To find a differential equation for the family of circles we differentiate Equation 2.5.17 implicitly with respect to x to obtain

$$2x + 2yy' = 0,$$

or

$$y' = -\frac{x}{y}.$$

Therefore the integral curves of

$$y' = \frac{y}{x}$$

are orthogonal trajectories of the given family. We leave it to you to verify that the general solution of this equation is

$$y = kx,$$

where k is an arbitrary constant. This is the equation of a nonvertical line through $(0, 0)$. The y axis is also an orthogonal trajectory of the given family. Therefore every line through the origin is an orthogonal trajectory of the given family Equation 2.5.17 (Figure 4.5.11). This is consistent with the theorem of plane geometry which states that a diameter of a circle and a tangent line to the circle at the end of the diameter are perpendicular.

✓ Example 4.5.10

Find the orthogonal trajectories of the family of hyperbolas

$$xy = c \quad (c \neq 0) \quad (2.5.18)$$

(Figure 4.5.7).

Solution

Differentiating Equation 2.5.18 implicitly with respect to x yields

$$y + xy' = 0,$$

or

$$y' = -\frac{y}{x};$$

thus, the integral curves of

$$y' = \frac{x}{y}$$

are orthogonal trajectories of the given family. Separating variables yields

$$y'y = x$$

and integrating yields

$$y^2 - x^2 = k,$$

which is the equation of a hyperbola if $k \neq 0$, or of the lines $y = x$ and $y = -x$ if $k = 0$ (Figure 4.5.12).

✓ Example 4.5.11

Find the orthogonal trajectories of the family of circles defined by

$$(x - c)^2 + y^2 = c^2 \quad (c \neq 0). \quad (2.5.19)$$

These circles are centered on the x -axis and tangent to the y -axis (Figure 4.5.13 a).

Solution

Multiplying out the left side of Equation 2.5.19 yields

$$x^2 - 2cx + y^2 = 0, \quad (2.5.20)$$

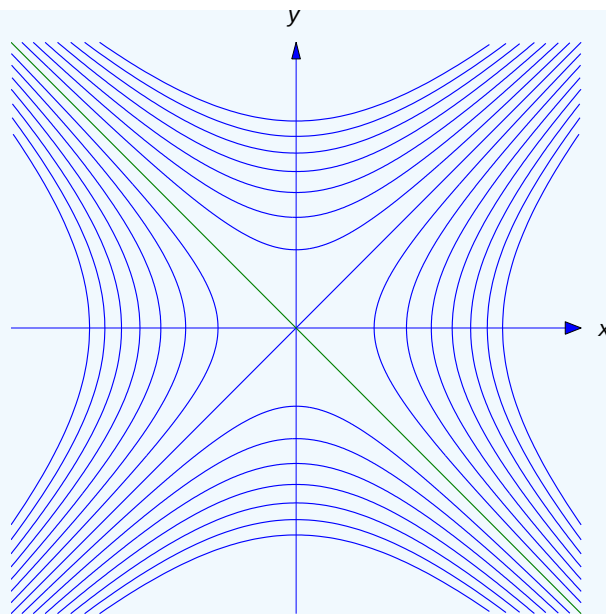


Figure 4.5.12

and differentiating this implicitly with respect to x yields

$$2(x - c) + 2yy' = 0. \quad (2.5.21)$$

From Equation 2.5.20,

$$c = \frac{x^2 + y^2}{2x},$$

so

$$x - c = x - \frac{x^2 + y^2}{2x} = \frac{x^2 - y^2}{2x}.$$

Substituting this into Equation 2.5.21 and solving for y' yields

$$y' = \frac{y^2 - x^2}{2xy}. \quad (2.5.22)$$

The curves defined by Equation 2.5.19 are integral curves of Equation 2.5.22, and the integral curves of

$$y' = \frac{2xy}{x^2 - y^2}$$

are orthogonal trajectories of the family Equation 2.5.19. This is a homogeneous nonlinear equation, which we studied in Section 2.4. Substituting $y = ux$ yields

$$u'x + u = \frac{2x(ux)}{x^2 - (ux)^2} = \frac{2u}{1 - u^2},$$

so

$$u'x = \frac{2u}{1 - u^2} - u = \frac{u(u^2 + 1)}{1 - u^2},$$

Separating variables yields

$$\frac{1 - u^2}{u(u^2 + 1)} u' = \frac{1}{x},$$

or, equivalently,

$$\left[\frac{1}{u} - \frac{2u}{u^2 + 1} \right] u' = \frac{1}{x}.$$

Therefore

$$\ln |u| - \ln(u^2 + 1) = \ln |x| + k.$$

By substituting $u = y/x$, we see that

$$\ln |y| - \ln |x| - \ln(x^2 + y^2) + \ln(x^2) = \ln |x| + k,$$

which, since $\ln(x^2) = 2 \ln |x|$, is equivalent to

$$\ln |y| - \ln(x^2 + y^2) = k,$$

or

$$|y| = e^k(x^2 + y^2).$$

To see what these curves are we rewrite this equation as

$$x^2 + |y|^2 - e^{-k}|y| = 0$$

and complete the square to obtain

$$x^2 + (|y| - e^{-k}/2)^2 = (e^{-k}/2)^2.$$

This can be rewritten as

$$x^2 + (y - h)^2 = h^2,$$

where

$$h = \begin{cases} \frac{e^{-k}}{2} & \text{if } y \geq 0, \\ -\frac{e^{-k}}{2} & \text{if } y \leq 0. \end{cases}$$

Thus, the orthogonal trajectories are circles centered on the y axis and tangent to the x axis (Figure 4.5.13 (b)). The circles for which $h > 0$ are above the x -axis, while those for which $h < 0$ are below.

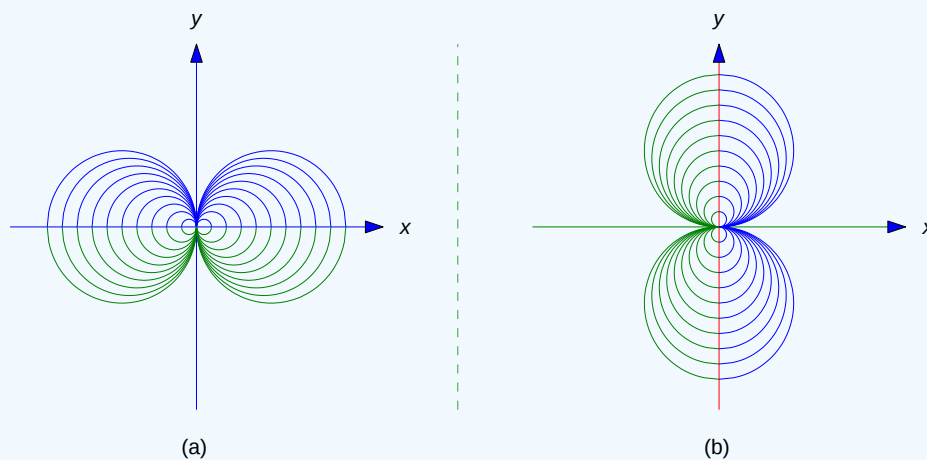


Figure 4.5.13 : (a) The circles $(x - c)^2 + y^2 = c^2$ (b) The circles $x^2 + (y - b)^2 = b^2$

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2.5E: Applications to Curves (Exercises)

Q4.5.1

In **Exercises 4.5.1-4.5.8** find a first order differential equation for the given family of curves.

1. $y(x^2 + y^2) = c$

2. $e^{xy} = cy$

3. $\ln |xy| = c(x^2 + y^2)$

4. $y = x^{1/2} + cx$

5. $y = e^{x^2} + ce^{-x^2}$

6. $y = x^3 + \frac{c}{x}$

7. $y = \sin x + ce^x$

8. $y = e^x + c(1 + x^2)$

Q4.5.2

9. Show that the family of circles

$$(x - x_0)^2 + y^2 = 1, \quad -\infty < x_0 < \infty,$$

can be obtained by joining integral curves of two first order differential equations. More specifically, find differential equations for the families of semicircles

$$(x - x_0)^2 + y^2 = 1, \quad x_0 < x < x_0 + 1, \quad -\infty < x_0 < \infty,$$

$$(x - x_0)^2 + y^2 = 1, \quad x_0 - 1 < x < x_0, \quad -\infty < x_0 < \infty.$$

10. Suppose f and g are differentiable for all x . Find a differential equation for the family of functions $y = f + cg$ (c =constant).

Q4.5.3

In **Exercises 4.5.11-4.5.13** find a first order differential equation for the given family of curves.

11. Lines through a given point (x_0, y_0) .

12. Circles through $(-1, 0)$ and $(1, 0)$.

13. Circles through $(0, 0)$ and $(0, 2)$.

Q4.5.4

14. Use the method Example 4.5.6 (a) to find the equations of lines through the given points tangent to the parabola $y = x^2$. Also, find the points of tangency.

a. $(5, 9)$

b. $(6, 11)$

c. $(-6, 20)$

d. $(-3, 5)$

15.

a. Show that the equation of the line tangent to the circle

$$x^2 + y^2 = 1 \tag{A}$$

at a point (x_0, y_0) on the circle is

$$y = \frac{1 - x_0 x}{y_0} \quad \text{if } x_0 \neq \pm 1. \tag{B}$$

b. Show that if y' is the slope of a nonvertical tangent line to the circle (A) and (x, y) is a point on the tangent line then

$$(y')^2(x^2 - 1) - 2xyy' + y^2 - 1 = 0. \quad (C)$$

c. Show that the segment of the tangent line (B) on which $(x - x_0)/y_0 > 0$ is an integral curve of the differential equation

$$y' = \frac{xy - \sqrt{x^2 + y^2 - 1}}{x^2 - 1}, \quad (D)$$

while the segment on which $(x - x_0)/y_0 < 0$ is an integral curve of the differential equation

$$y' = \frac{xy + \sqrt{x^2 + y^2 - 1}}{x^2 - 1}. \quad (E)$$

HINT: Use the quadratic formula to solve (C) for y' . Then substitute (B) for y and choose the $\pm$ sign in the quadratic formula so that the resulting expression for y' reduces to the known slope $y' = -x_0/y_0$

d. Show that the upper and lower semicircles of (A) are also integral curves of (D) and (E).

e. Find the equations of two lines through (5,5) tangent to the circle (A), and find the points of tangency.

16.

a. Show that the equation of the line tangent to the parabola

$$x = y^2 \quad (A)$$

at a point $(x_0, y_0) \neq (0, 0)$ on the parabola is

$$y = \frac{y_0}{2} + \frac{x}{2y_0}. \quad (B)$$

b. Show that if y' is the slope of a nonvertical tangent line to the parabola (A) and (x, y) is a point on the tangent line then

$$4x^2(y')^2 - 4xyy' + x = 0. \quad (C)$$

c. Show that the segment of the tangent line defined in (a) on which $x > x_0$ is an integral curve of the differential equation

$$y' = \frac{y + \sqrt{y^2 - x}}{2x}, \quad (D)$$

while the segment on which $x < x_0$ is an integral curve of the differential equation

$$y' = \frac{y - \sqrt{y^2 - x}}{2x}, \quad (E)$$

HINT: Use the quadratic formula to solve (c) for y' . Then substitute (B) for y and choose the $\pm$ sign in the quadratic formula so that the resulting expression for y' reduces to the known slope of $y' = \frac{1}{2y_0}$

d. Show that the upper and lower halves of the parabola (A), given by $y = \sqrt{x}$ and $y = -\sqrt{x}$ for $x > 0$, are also integral curves of (D) and (E).

17. Use the results of **Exercise 4.5.16** to find the equations of two lines tangent to the parabola $x = y^2$ and passing through the given point. Also find the points of tangency.

- $(-5, 2)$
- $(-4, 0)$
- $(7, 4)$
- $(5, -3)$

18. Find a curve $y = y(x)$ through (1,2) such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the x axis at $x_I = x_0/2$.

19. Find all curves $y = y(x)$ such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the x axis at $x_I = x_0^3$.

20. Find all curves $y = y(x)$ such that the tangent to the curve at any point passes through a given point (x_1, y_1) .

21. Find a curve $y = y(x)$ through $(1, -1)$ such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the y axis at $y_I = x_0^3$.
22. Find all curves $y = y(x)$ such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the y axis at $y_I = x_0$.
23. Find a curve $y = y(x)$ through $(0, 2)$ such that the normal to the curve at any point $(x_0, y(x_0))$ intersects the x axis at $x_I = x_0 + 1$.
24. Find a curve $y = y(x)$ through $(2, 1)$ such that the normal to the curve at any point $(x_0, y(x_0))$ intersects the y axis at $y_I = 2y(x_0)$.

Q4.5.5

In **Exercises 4.5.25-2.5.29** find the orthogonal trajectories of the given family of curves.

25. $x^2 + 2y^2 = c^2$
26. $x^2 + 4xy + y^2 = c$
27. $y = ce^{2x}$
28. $xye^{x^2} = c$
29. $y = \frac{ce^x}{x}$

Q4.5.6

30. Find a curve through $(-1, 3)$ orthogonal to every parabola of the form

$$y = 1 + cx^2$$

that it intersects. Which of these parabolas does the desired curve intersect?

31. Show that the orthogonal trajectories of

$$x^2 + 2axy + y^2 = c$$

satisfy

$$|y - x|^{a+1} |y + x|^{a-1} = k.$$

32. If lines L and L_1 intersect at (x_0, y_0) and α is the smallest angle through which L must be rotated counterclockwise about (x_0, y_0) to bring it into coincidence with L_1 , we say that α is the **angle from L to L_1** ; thus, $0 \leq \alpha < \pi$. If L and L_1 are tangents to curves C and C_1 , respectively, that intersect at (x_0, y_0) , we say that C_1 intersects C at the angle α . Use the identity

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

to show that if C and C_1 are intersecting integral curves of

$$y' = f(x, y) \quad \text{and} \quad y' = \frac{f(x, y) + \tan \alpha}{1 - f(x, y) \tan \alpha} \quad \left(\alpha \neq \frac{\pi}{2} \right),$$

respectively, then C_1 intersects C at the angle α .

33. Use the result of **Exercise 4.5.32** to find a family of curves that intersect every nonvertical line through the origin at the angle $\alpha = \pi/4$.
34. Use the result of **Exercise 4.5.32** to find a family of curves that intersect every circle centered at the origin at a given angle $\alpha \neq \pi/2$.

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CHAPTER OVERVIEW

3: Systems of Equations

- [3.1: Systems of Equations, Geometry](#)
- [3.2: Systems of Equations, Algebraic Procedures](#)
- [3.3: Gaussian Elimination](#)
- [3.4: Uniqueness of the Reduced Row-Echelon Form](#)
- [3.5: Rank and Homogeneous Systems](#)
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3.1: Systems of Equations, Geometry

Systems of Equations, Geometry

Outcomes

- A. Relate the types of solution sets of a system of two (three) variables to the intersections of lines in a plane (the intersection of planes in three space)

As you may remember, linear equations like $2x + 3y = 6$ can be graphed as straight lines in the coordinate plane. We say that this equation is in two variables, in this case x and y . Suppose you have two such equations, each of which can be graphed as a straight line, and consider the resulting graph of two lines. What would it mean if there exists a point of intersection between the two lines? This point, which lies on *both* graphs, gives x and y values for which both equations are true. In other words, this point gives the ordered pair (x, y) that satisfy both equations. If the point (x, y) is a point of intersection, we say that (x, y) is a **solution** to the two equations. In linear algebra, we often are concerned with finding the solution(s) to a system of equations, if such solutions exist. First, we consider graphical representations of solutions and later we will consider the algebraic methods for finding solutions.

When looking for the intersection of two lines in a graph, several situations may arise. The following picture demonstrates the possible situations when considering two equations (two lines in the graph) involving two variables.

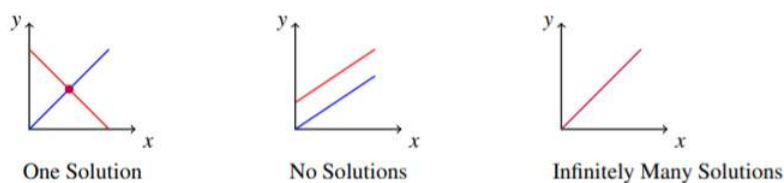


Figure 3.1.1

In the first diagram, there is a unique point of intersection, which means that there is only one (unique) solution to the two equations. In the second, there are no points of intersection and no solution. When no solution exists, this means that the two lines are parallel and they never intersect. The third situation which can occur, as demonstrated in diagram three, is that the two lines are really the same line. For example, $x + y = 1$ and $2x + 2y = 2$ are equations which when graphed yield the same line. In this case there are infinitely many points which are solutions of these two equations, as every ordered pair which is on the graph of the line satisfies both equations. When considering linear systems of equations, there are always three types of solutions possible; exactly one (unique) solution, infinitely many solutions, or no solution.

✓ Example 3.1.1: A Graphical Solution

Use a graph to find the solution to the following system of equations

$$\begin{aligned}x + y &= 3 \\ y - x &= 5\end{aligned}$$

Solution

Through graphing the above equations and identifying the point of intersection, we can find the solution(s). Remember that we must have either one solution, infinitely many, or no solutions at all. The following graph shows the two equations, as well as the intersection. Remember, the point of intersection represents the solution of the two equations, or the (x, y) which satisfy both equations. In this case, there is one point of intersection at $(-1, 4)$ which means we have one unique solution, $x = -1, y = 4$.

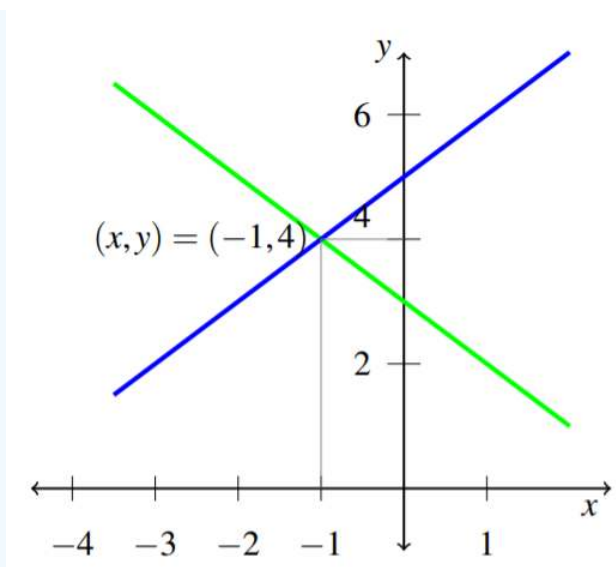


Figure 3.1.2

In the above example, we investigated the intersection point of two equations in two variables, x and y . Now we will consider the graphical solutions of three equations in two variables.

Consider a system of three equations in two variables. Again, these equations can be graphed as straight lines in the plane, so that the resulting graph contains three straight lines. Recall the three possible types of solutions; no solution, one solution, and infinitely many solutions. There are now more complex ways of achieving these situations, due to the presence of the third line. For example, you can imagine the case of three intersecting lines having no common point of intersection. Perhaps you can also imagine three intersecting lines which do intersect at a single point. These two situations are illustrated below.

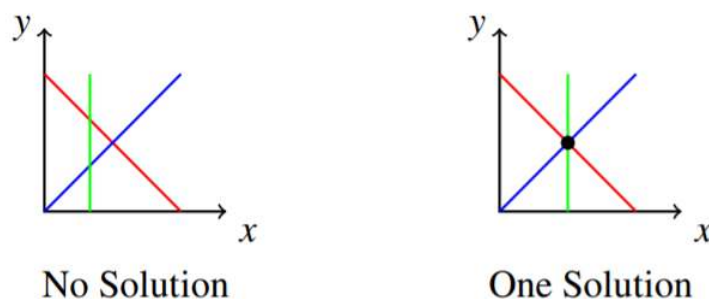


Figure 3.1.3

Consider the first picture above. While all three lines intersect with one another, there is no common point of intersection where all three lines meet at one point. Hence, there is no solution to the three equations. Remember, a solution is a point (x, y) which satisfies **all** three equations. In the case of the second picture, the lines intersect at a common point. This means that there is one solution to the three equations whose graphs are the given lines. You should take a moment now to draw the graph of a system which results in three parallel lines. Next, try the graph of three identical lines. Which type of solution is represented in each of these graphs?

We have now considered the graphical solutions of systems of two equations in two variables, as well as three equations in two variables. However, there is no reason to limit our investigation to equations in two variables. We will now consider equations in three variables.

Below is a video on solving systems of equations graphically



You may recall that equations in three variables, such as $2x + 4y - 5z = 8$, form a plane. Above, we were looking for intersections of lines in order to identify any possible solutions. When graphically solving systems of equations in three variables, we look for intersections of planes. These points of intersection give the (x, y, z) that satisfy all the equations in the system. What types of solutions are possible when working with three variables? Consider the following picture involving two planes, which are given by two equations in three variables.

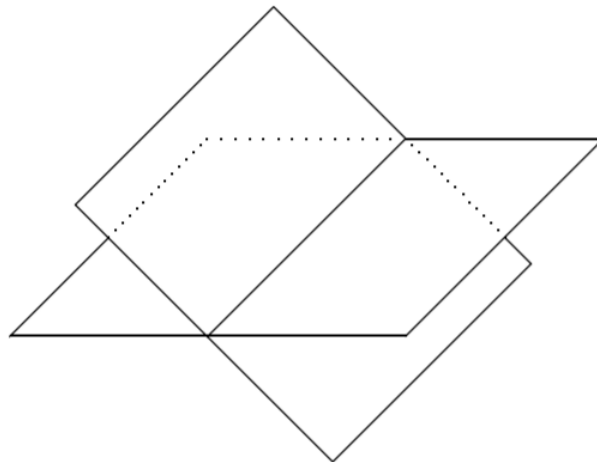


Figure 3.1.4

Notice how these two planes intersect in a line. This means that the points (x, y, z) on this line satisfy both equations in the system. Since the line contains infinitely many points, this system has infinitely many solutions.

It could also happen that the two planes fail to intersect. However, is it possible to have two planes intersect at a single point? Take a moment to attempt drawing this situation, and convince yourself that it is not possible! This means that when we have only two equations in three variables, there is no way to have a unique solution! Hence, the types of solutions possible for two equations in three variables are no solution or infinitely many solutions.

Now imagine adding a third plane. In other words, consider three equations in three variables. What types of solutions are now possible? Consider the following diagram.

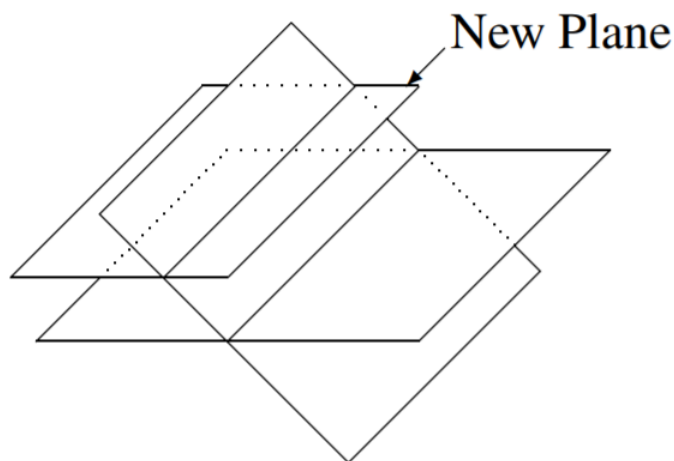


Figure 3.1.5

In this diagram, there is no point which lies in all three planes. There is no intersection between **all** planes so there is no solution. The picture illustrates the situation in which the line of intersection of the new plane with one of the original planes forms a line parallel to the line of intersection of the first two planes. However, in three dimensions, it is possible for two lines to fail to intersect even though they are not parallel. Such lines are called **skew lines**.

Recall that when working with two equations in three variables, it was not possible to have a unique solution. Is it possible when considering three equations in three variables? In fact, it is possible, and we demonstrate this situation in the following picture.

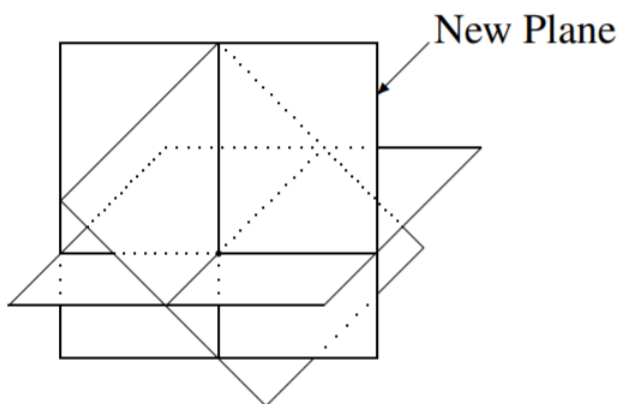


Figure 3.1.6

In this case, the three planes have a single point of intersection. Can you think of other types of solutions possible? Another is that the three planes could intersect in a line, resulting in infinitely many solutions, as in the following diagram.

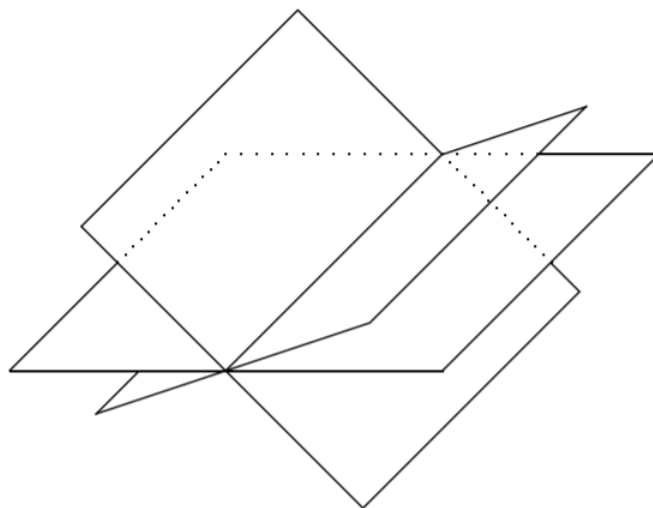


Figure 3.1.7

We have now seen how three equations in three variables can have no solution, a unique solution, or intersect in a line resulting in infinitely many solutions. It is also possible that the three equations graph the same plane, which also leads to infinitely many solutions.

You can see that when working with equations in three variables, there are many more ways to achieve the different types of solutions than when working with two variables. It may prove enlightening to spend time imagining (and drawing) many possible scenarios, and you should take some time to try a few.

You should also take some time to imagine (and draw) graphs of systems in more than three variables. Equations like $x + y - 2z + 4w = 8$ with more than three variables are often called **hyper-planes**. You may soon realize that it is tricky to draw the graphs of hyper-planes! Through the tools of linear algebra, we can algebraically examine these types of systems which are difficult to graph. In the following section, we will consider these algebraic tools.

More Videos:

[Solving Systems of Equations Graphically](#)

[Solving Systems of Equations by Graphing](#)

[Determining the Intersection of Two Graph on the TI83/84](#)

[Ex: Identify the Solution to a System of Equation Given a Graph, Then Verify](#)

[olve a System of Linear Equations by Graphing \(One Solution\)](#)

[Solve a System of Linear Equations by Graphing \(No Solution\)](#)

[x 1: Solve a System of Equations by Graphing](#)

[Ex 2: Solve a System of Equations by Graphing](#)

[Ex: Solve a System of Equations by Graphing \(Infinite Solutions\)](#)

[Ex: Solve a System of Equations by Graphing \(No Solution\)](#)

? Exercise 3.1.1

Graphically, find the point (x_1, y_1) which lies on both lines, $x + 3y = 1$ and $4x - y = 3$. That is, graph each line and see where they intersect.

? Exercise 3.1.2

Graphically, find the point of intersection of the two lines, $3x + y = 3$ and $x + 2y = 1$. That is, graph each line and see where they intersect

? Exercise 3.1.3

You have a system of k equations in two variables, $k \geq 2$. Explain the geometric significance of

- No solution.
- A unique solution.
- An infinite number of solutions.

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3.2: Systems of Equations, Algebraic Procedures

Algebraic Procedures

Outcomes

- A. Use elementary operations to find the solution to a linear system of equations.
- B. Find the row-echelon form and reduced row-echelon form of a matrix.
- C. Determine whether a system of linear equations has no solution, a unique solution or an infinite number of solutions from its .
- D. Solve a system of equations using Gaussian Elimination and Gauss-Jordan Elimination.
- E. Model a physical system with linear equations and then solve.

We have taken an in depth look at graphical representations of systems of equations, as well as how to find possible solutions graphically. Our attention now turns to working with systems algebraically.

Definition 3.2.1: System of Linear Equations

A **system of linear equations** is a list of equations,

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

where a_{ij} and b_j are real numbers. The above is a system of m equations in the n variables, $x_1, x_2, \dots, x_n$. Written more simply in terms of summation notation, the above can be written in the form

$$\sum_{j=1}^n a_{ij}x_j = b_i, \quad i = 1, 2, 3, \dots, m$$

The relative size of m and n is not important here. Notice that we have allowed a_{ij} and b_j to be any real number. We can also call these numbers **scalars**. We will use this term throughout the text, so keep in mind that the term **scalar** just means that we are working with real numbers.

Now, suppose we have a system where $b_i = 0$ for all i . In other words every equation equals 0. This is a special type of system.

Definition 3.2.2: Homogeneous System of Equations

A system of equations is called **homogeneous** if each equation in the system is equal to 0. A homogeneous system has the form

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0\end{aligned}$$

where a_{ij} are scalars and x_i are variables.

Recall from the previous section that our goal when working with systems of linear equations was to find the point of intersection of the equations when graphed. In other words, we looked for the solutions to the system. We now wish to find these solutions algebraically. We want to find values for $x_1, \dots, x_n$ which solve all of the equations. If such a set of values exists, we call $(x_1, \dots, x_n)$ the **solution set**.

Recall the above discussions about the types of solutions possible. We will see that systems of linear equations will have one unique solution, infinitely many solutions, or no solution. Consider the following definition.

 Definition 3.2.3: Consistent and Inconsistent Systems

A system of linear equations is called **consistent** if there exists at least one solution. It is called **inconsistent** if there is no solution.

If you think of each equation as a condition which must be satisfied by the variables, consistent would mean there is some choice of variables which can satisfy **all** the conditions. Inconsistent would mean there is no choice of the variables which can satisfy all of the conditions.

The following sections provide methods for determining if a system is consistent or inconsistent, and finding solutions if they exist.

Elementary Operations

We begin this section with an example. Recall from [Example 1.1.1](#) that the solution to the given system was $(x, y) = (-1, 4)$.

 Example 3.2.1: Verifying an Ordered Pair is a Solution

Algebraically verify that $(x, y) = (-1, 4)$ is a solution to the following system of equations.

$$x + y = 3$$

$$y - x = 5$$

Solution

By graphing these two equations and identifying the point of intersection, we previously found that $(x, y) = (-1, 4)$ is the unique solution.

We can verify algebraically by substituting these values into the original equations, and ensuring that the equations hold. First, we substitute the values into the first equation and check that it equals 3.

$$x + y = (-1) + (4) = 3$$

This equals 3 as needed, so we see that $(-1, 4)$ is a solution to the first equation. Substituting the values into the second equation yields

$$y - x = (4) - (-1) = 4 + 1 = 5$$

which is true. For $(x, y) = (-1, 4)$ each equation is true and therefore, this is a solution to the system.

Now, the interesting question is this: If you were not given these numbers to verify, how could you algebraically determine the solution? Linear algebra gives us the tools needed to answer this question. The following basic operations are important tools that we will utilize.

 Definition 3.2.4: Elementary Operations

Elementary operations are those operations consisting of the following.

1. Interchange the order in which the equations are listed.
2. Multiply any equation by a nonzero number.
3. Replace any equation with itself added to a multiple of another equation.

It is important to note that none of these operations will change the set of solutions of the system of equations. In fact, elementary operations are the *key tool* we use in linear algebra to find solutions to systems of equations.

Consider the following example.

✓ Example 3.2.2: Effects of an Elementary Operation

Show that the system

$$\begin{aligned}x + y &= 7 \\ 2x - y &= 8\end{aligned}$$

has the same solution as the system

$$\begin{aligned}x + y &= 7 \\ -3y &= -6\end{aligned}$$

Solution

Notice that the second system has been obtained by taking the second equation of the first system and adding -2 times the first equation, as follows:

$$2x - y + (-2)(x + y) = 8 + (-2)(7)$$

By simplifying, we obtain

$$-3y = -6$$

which is the second equation in the second system. Now, from here we can solve for y and see that $y = 2$. Next, we substitute this value into the first equation as follows

$$x + y = x + 2 = 7$$

Hence $x = 5$ and so $(x, y) = (5, 2)$ is a solution to the second system. We want to check if $(5, 2)$ is also a solution to the first system. We check this by substituting $(x, y) = (5, 2)$ into the system and ensuring the equations are true.

$$\begin{aligned}x + y &= (5) + (2) = 7 \\ 2x - y &= 2(5) - (2) = 8\end{aligned}$$

Hence, $(5, 2)$ is also a solution to the first system.

This example illustrates how an elementary operation applied to a system of two equations in two variables does not affect the solution set. However, a linear system may involve many equations and many variables and there is no reason to limit our study to small systems. For any size of system in any number of variables, the solution set is still the collection of solutions to the equations. In every case, the above operations of [Definition 3.2.4](#) do not change the set of solutions to the system of linear equations.

In the following theorem, we use the notation E_i to represent an equation, while b_i denotes a constant.

Theorem 3.2.1: Elementary Operations and Solutions

Suppose you have a system of two linear equations

$$\begin{aligned}E_1 &= b_1 \\ E_2 &= b_2\end{aligned}\tag{3.2.1}$$

Then the following systems have the same solution set as (3.2.1):

$$\begin{aligned}1. \quad & \begin{aligned}E_2 &= b_2 \\ E_1 &= b_1\end{aligned}\end{aligned}\tag{3.2.2}$$

$$\begin{aligned}2. \quad & \begin{aligned}E_1 &= b_1 \\ kE_2 &= kb_2\end{aligned}\end{aligned}\tag{3.2.3}$$

for any scalar k , provided $k \neq 0$.

$$\begin{aligned}3. \quad & \begin{aligned}E_1 &= b_1 \\ E_2 + kE_1 &= b_2 + kb_1\end{aligned}\end{aligned}\tag{3.2.4}$$

for any scalar k (including $k = 0$).

Before we proceed with the proof of [Theorem 3.2.1](#), let us consider this theorem in context of [Example 3.2.2](#). Then,

$$\begin{aligned} E_1 &= x + y, & b_1 &= 7 \\ E_2 &= 2x - y, & b_2 &= 8 \end{aligned}$$

Recall the elementary operations that we used to modify the system in the solution to the example. First, we added (-2) times the first equation to the second equation. In terms of [Theorem 3.2.1](#), this action is given by

$$E_2 + (-2) E_1 = b_2 + (-2) b_1$$

or

$$2x - y + (-2)(x + y) = 8 + (-2) 7$$

This gave us the second system in [Example 3.2.2](#), given by

$$\begin{aligned} E_1 &= b_1 \\ E_2 + (-2) E_1 &= b_2 + (-2) b_1 \end{aligned}$$

From this point, we were able to find the solution to the system. [Theorem 3.2.1](#) tells us that the solution we found is in fact a solution to the original system.

We will now prove [Theorem 3.2.1](#).

Proof

1. The proof that the systems [\(3.2.1\)](#) and [\(3.2.2\)](#) have the same solution set is as follows. Suppose that $(x_1, \dots, x_n)$ is a solution to $E_1 = b_1, E_2 = b_2$. We want to show that this is a solution to the system in [\(3.2.2\)](#) above. This is clear, because the system in [\(3.2.2\)](#) is the original system, but listed in a different order. Changing the order does not effect the solution set, so $(x_1, \dots, x_n)$ is a solution to [\(3.2.2\)](#).

2. Next we want to prove that the systems [\(3.2.1\)](#) and [\(3.2.3\)](#) have the same solution set. That is $E_1 = b_1, E_2 = b_2$ has the same solution set as the system $E_1 = b_1, kE_2 = kb_2$ provided $k \neq 0$. Let $(x_1, \dots, x_n)$ be a solution of $E_1 = b_1, E_2 = b_2$. We want to show that it is a solution to $E_1 = b_1, kE_2 = kb_2$. Notice that the only difference between these two systems is that the second involves multiplying the equation, $E_2 = b_2$ by the scalar k . Recall that when you multiply both sides of an equation by the same number, the sides are still equal to each other. Hence if $(x_1, \dots, x_n)$ is a solution to $E_2 = b_2$, then it will also be a solution to $kE_2 = kb_2$. Hence, $(x_1, \dots, x_n)$ is also a solution to [\(3.2.3\)](#).

Similarly, let $(x_1, \dots, x_n)$ be a solution of $E_1 = b_1, kE_2 = kb_2$. Then we can multiply the equation $kE_2 = kb_2$ by the scalar $1/k$, which is possible only because we have required that $k \neq 0$. Just as above, this action preserves equality and we obtain the equation $E_2 = b_2$. Hence $(x_1, \dots, x_n)$ is also a solution to $E_1 = b_1, E_2 = b_2$.

3. Finally, we will prove that the systems [\(3.2.1\)](#) and [\(3.2.4\)](#) have the same solution set. We will show that any solution of $E_1 = b_1, E_2 = b_2$ is also a solution of [\(3.2.4\)](#). Then, we will show that any solution of [\(3.2.4\)](#) is also a solution of $E_1 = b_1, E_2 = b_2$. Let $(x_1, \dots, x_n)$ be a solution to $E_1 = b_1, E_2 = b_2$. Then in particular it solves $E_1 = b_1$. Hence, it solves the first equation in [\(3.2.4\)](#). Similarly, it also solves $E_2 = b_2$. By our proof of [\(3.2.3\)](#), it also solves $kE_1 = kb_1$. Notice that if we add E_2 and kE_1 , this is equal to $b_2 + kb_1$. Therefore, if $(x_1, \dots, x_n)$ solves $E_1 = b_1, E_2 = b_2$ it must also solve $E_2 + kE_1 = b_2 + kb_1$.

Now suppose $(x_1, \dots, x_n)$ solves the system $E_1 = b_1, E_2 + kE_1 = b_2 + kb_1$. Then in particular it is a solution of $E_1 = b_1$. Again by our proof of [\(3.2.3\)](#), it is also a solution to $kE_1 = kb_1$. Now if we subtract these equal quantities from both sides of $E_2 + kE_1 = b_2 + kb_1$ we obtain $E_2 = b_2$, which shows that the solution also satisfies $E_1 = b_1, E_2 = b_2$.

Stated simply, the above theorem shows that the elementary operations do not change the solution set of a system of equations.

We will now look at an example of a system of three equations and three variables. Similarly to the previous examples, the goal is to find values for x, y, z such that each of the given equations are satisfied when these values are substituted in.

✓ Example 3.2.3: Solving a System of Equations with Elementary Operations

Find the solutions to the system,

$$\begin{aligned}x + 3y + 6z &= 25 \\ 2x + 7y + 14z &= 58 \\ 2y + 5z &= 19\end{aligned}\tag{3.2.5}$$

Solution

We can relate this system to [Theorem 3.2.1](#) above. In this case, we have

$$\begin{aligned}E_1 &= x + 3y + 6z, & b_1 &= 25 \\ E_2 &= 2x + 7y + 14z, & b_2 &= 58 \\ E_3 &= 2y + 5z, & b_3 &= 19\end{aligned}$$

[Theorem 3.2.1](#) claims that if we do elementary operations on this system, we will not change the solution set. Therefore, we can solve this system using the elementary operations given in [Definition 3.2.4](#). First, replace the second equation by (-2) times the first equation added to the second. This yields the system

$$\begin{aligned}x + 3y + 6z &= 25 \\ y + 2z &= 8 \\ 2y + 5z &= 19\end{aligned}\tag{3.2.6}$$

Now, replace the third equation with (-2) times the second added to the third. This yields the system

$$\begin{aligned}x + 3y + 6z &= 25 \\ y + 2z &= 8 \\ z &= 3\end{aligned}\tag{3.2.7}$$

At this point, we can easily find the solution. Simply take $z = 3$ and substitute this back into the previous equation to solve for y , and similarly to solve for x .

$$\begin{aligned}x + 3y + 6(3) &= x + 3y + 18 = 25 \\ y + 2(3) &= y + 6 = 8 \\ z &= 3\end{aligned}$$

The second equation is now

$$y + 6 = 8$$

You can see from this equation that $y = 2$. Therefore, we can substitute this value into the first equation as follows:

$$x + 3(2) + 18 = 25$$

By simplifying this equation, we find that $x = 1$. Hence, the solution to this system is $(x, y, z) = (1, 2, 3)$. This process is called **back substitution**.

Alternatively, in [\(3.2.7\)](#) you could have continued as follows. Add (-2) times the third equation to the second and then add (-6) times the second to the first. This yields

$$\begin{aligned}x + 3y &= 7 \\ y &= 2 \\ z &= 3\end{aligned}$$

Now add (-3) times the second to the first. This yields

$$\begin{aligned}x &= 1 \\ y &= 2 \\ z &= 3\end{aligned}$$

a system which has the same solution set as the original system. This avoided back substitution and led to the same solution set. It is your decision which you prefer to use, as both methods lead to the correct solution, $(x, y, z) = (1, 2, 3)$.

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3.3: Gaussian Elimination

The work we did in the previous section will always find the solution to the system. In this section, we will explore a less cumbersome way to find the solutions. First, we will represent a linear system with an **augmented matrix**. A **matrix** is simply a rectangular array of numbers. The size or dimension of a matrix is defined as $m \times n$ where m is the number of rows and n is the number of columns. In order to construct an augmented matrix from a linear system, we create a **coefficient matrix** from the coefficients of the variables in the system, as well as a **constant matrix** from the constants. The coefficients from one equation of the system create one row of the augmented matrix.

For example, consider the linear system in [Example 1.2.3](#)

$$\begin{aligned}x + 3y + 6z &= 25 \\2x + 7y + 14z &= 58 \\2y + 5z &= 19\end{aligned}$$

This system can be written as an augmented matrix, as follows

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array} \right]$$

Notice that it has exactly the same information as the original system. Here it is understood that the first column contains the coefficients from x in each equation, in order, $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Similarly, we create a column from the coefficients on y in each equation,

$\begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}$ and a column from the coefficients on z in each equation, $\begin{bmatrix} 6 \\ 14 \\ 5 \end{bmatrix}$. For a system of more than three variables, we would continue in this way constructing a column for each variable. Similarly, for a system of less than three variables, we simply construct a column for each variable.

Finally, we construct a column from the constants of the equations, $\begin{bmatrix} 25 \\ 58 \\ 19 \end{bmatrix}$.

The rows of the augmented matrix correspond to the equations in the system. For example, the top row in the augmented matrix, $[1 \ 3 \ 6 \ | \ 25]$ corresponds to the equation

$$x + 3y + 6z = 25.$$

Consider the following definition.

Definition 3.3.1: Augmented Matrix of a Linear System

For a linear system of the form

$$\begin{aligned}a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\&\vdots \\a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

where the x_i are variables and the a_{ij} and b_i are constants, the augmented matrix of this system is given by

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array} \right]$$

Now, consider elementary operations in the context of the augmented matrix. The elementary operations in [Definition 1.2.4](#) can be used on the rows just as we used them on equations previously. Changes to a system of equations in as a result of an elementary operation are equivalent to changes in the augmented matrix resulting from the corresponding row operation. Note that [Theorem 1.2.1](#) implies that any elementary row operations used on an augmented matrix will not change the solution to the corresponding system of equations. We now formally define elementary row operations. These are the *key tool* we will use to find solutions to systems of equations.

Definition 3.3.2: Elementary Row Operations

The **elementary row operations** (also known as **row operations**) consist of the following

1. Switch two rows.
2. Multiply a row by a nonzero number.
3. Replace a row by any multiple of another row added to it.

Recall how we solved [Example 1.2.3](#). We can do the exact same steps as above, except now in the context of an augmented matrix and using row operations. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array} \right]$$

Thus the first step in solving the system given by [\(1.2.5\)](#) would be to take (-2) times the first row of the augmented matrix and add it to the second row,

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 19 \end{array} \right]$$

Note how this corresponds to [\(1.2.6\)](#). Next take (-2) times the second row and add to the third,

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

This augmented matrix corresponds to the system

$$\begin{aligned} x + 3y + 6z &= 25 \\ y + 2z &= 8 \\ z &= 3 \end{aligned}$$

which is the same as [\(1.2.7\)](#). By back substitution you obtain the solution $x = 1$, $y = 2$, and $z = 3$.

Through a systematic procedure of row operations, we can simplify an augmented matrix and carry it to **row-echelon form** or **reduced row-echelon form**, which we define next. These forms are used to find the solutions of the system of equations corresponding to the augmented matrix.

In the following definitions, the term **leading entry** refers to the first nonzero entry of a row when scanning the row from left to right.

Definition 3.3.3: Row-Echelon Form

An augmented matrix is in **row-echelon form** if

1. All nonzero rows are above any rows of zeros.
2. Each leading entry of a row is in a column to the right of the leading entries of any row above it.
3. Each leading entry of a row is equal to 1.

We also consider another reduced form of the augmented matrix which has one further condition.

Definition 3.3.4: Reduced Row-Echelon Form

An augmented matrix is in **reduced row-echelon form** if

1. All nonzero rows are above any rows of zeros.
2. Each leading entry of a row is in a column to the right of the leading entries of any rows above it.
3. Each leading entry of a row is equal to 1.
4. All entries in a column above and below a leading entry are zero.

Notice that the first three conditions on a reduced row-echelon form matrix are the same as those for row-echelon form.

Hence, every reduced row-echelon form matrix is also in row-echelon form. The converse is not necessarily true; we cannot assume that every matrix in row-echelon form is also in reduced row-echelon form. However, it often happens that the row-echelon form is sufficient to provide information about the solution of a system.

The following examples describe matrices in these various forms. As an exercise, take the time to carefully verify that they are in the specified form.

✓ Example 3.3.1: Not in Row-Echelon Form

The following augmented matrices are not in row-echelon form (and therefore also not in reduced row-echelon form).

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 & 3 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{ccc|c} 1 & 2 & & 3 \\ 2 & 4 & & -6 \\ 4 & 0 & & 7 \end{array} \right], \left[\begin{array}{ccc|c} 0 & 2 & 3 & 3 \\ 1 & 5 & 0 & 2 \\ 7 & 5 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

✓ Example 3.3.2: Matrices in Row-Echelon Form

The following augmented matrices are in row-echelon form, but not in reduced row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 6 & 5 & 8 & 2 \\ 0 & 0 & 1 & 2 & 7 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{ccc|c} 1 & 3 & 5 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Notice that we could apply further row operations to these matrices to carry them to reduced row-echelon form. Take the time to try that on your own. Consider the following matrices, which are in reduced row-echelon form.

✓ Example 3.3.3: Matrices in Reduced Row-Echelon Form

The following augmented matrices are in reduced row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

One way in which the row-echelon form of a matrix is useful is in identifying the pivot positions and pivot columns of the matrix.

Definition 3.3.5: Pivot Position and Pivot Column

A **pivot position** in a matrix is the location of a leading entry in the row-echelon form of a matrix.

A **pivot column** is a column that contains a pivot position.

For example consider the following.

✓ Example 3.3.4: Pivot Position

Let

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 4 & 10 \end{array} \right]$$

Where are the pivot positions and pivot columns of the augmented matrix A ?

Solution

The row-echelon form of this matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This is all we need in this example, but note that this matrix is not in reduced row-echelon form.

In order to identify the pivot positions in the original matrix, we look for the leading entries in the row-echelon form of the matrix. Here, the entry in the first row and first column, as well as the entry in the second row and second column are the leading entries. Hence, these locations are the pivot positions. We identify the pivot positions in the original matrix, as in the following:

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 4 \\ 3 & \boxed{2} & 1 & 6 \\ 4 & 4 & 4 & 10 \end{array} \right]$$

Thus the pivot columns in the matrix are the first two columns.

The following is an algorithm for carrying a matrix to row-echelon form and reduced row-echelon form. You may wish to use this algorithm to carry the above matrix to row-echelon form or reduced row-echelon form yourself for practice.

Algorithm 3.3.1: Reduced Row-Echelon Form Algorithm

This algorithm provides a method for using row operations to take a matrix to its reduced row-echelon form. We begin with the matrix in its original form.

1. Starting from the left, find the first nonzero column. This is the first pivot column, and the position at the top of this column is the first pivot position. Switch rows if necessary to place a nonzero number in the first pivot position.
2. Use row operations to make the entries below the first pivot position (in the first pivot column) equal to zero.
3. Ignoring the row containing the first pivot position, repeat steps 1 and 2 with the remaining rows. Repeat the process until there are no more rows to modify.
4. Divide each nonzero row by the value of the leading entry, so that the leading entry becomes 1. The matrix will then be in row-echelon form.

The following step will carry the matrix from row-echelon form to reduced row-echelon form.

5. Moving from right to left, use row operations to create zeros in the entries of the pivot columns which are above the pivot positions. The result will be a matrix in reduced row-echelon form.

Most often we will apply this algorithm to an augmented matrix in order to find the solution to a system of linear equations. However, we can use this algorithm to compute the reduced row-echelon form of any matrix which could be useful in other applications.

Consider the following example of [Algorithm 3.3.1](#).

✓ **Example 3.3.5: Finding Row-Echelon Form and Reduced Row-Echelon Form of a Matrix**

Let

$$A = \begin{bmatrix} 0 & -5 & -4 \\ 1 & 4 & 3 \\ 5 & 10 & 7 \end{bmatrix}$$

Find the row-echelon form of A . Then complete the process until A is in reduced row-echelon form.

Solution

In working through this example, we will use the steps outlined in [Algorithm 3.3.1](#).

1. The first pivot column is the first column of the matrix, as this is the first nonzero column from the left. Hence the first pivot position is the one in the first row and first column. Switch the first two rows to obtain a nonzero entry in the first pivot position, outlined in a box below.

$$\begin{bmatrix} \boxed{1} & 4 & 3 \\ 0 & -5 & -4 \\ 5 & 10 & 7 \end{bmatrix}$$

2. Step two involves creating zeros in the entries below the first pivot position. The first entry of the second row is already a zero. All we need to do is subtract 5 times the first row from the third row. The resulting matrix is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & 10 & 8 \end{bmatrix}$$

3. Now ignore the top row. Apply steps 1 and 2 to the smaller matrix

$$\begin{bmatrix} -5 & -4 \\ 10 & 8 \end{bmatrix}$$

In this matrix, the first column is a pivot column, and -5 is in the first pivot position. Therefore, we need to create a zero below it. To do this, add 2 times the first row (of this matrix) to the second. The resulting matrix is

$$\begin{bmatrix} -5 & -4 \\ 0 & 0 \end{bmatrix}$$

Our original matrix now looks like

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

We can see that there are no more rows to modify.

4. Now, we need to create leading 1s in each row. The first row already has a leading 1 so no work is needed here. Divide the second row by -5 to create a leading 1. The resulting matrix is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is now in row-echelon form.

5. Now create zeros in the entries above pivot positions in each column, in order to carry this matrix all the way to reduced row-echelon form. Notice that there is no pivot position in the third column so we do not need to create any zeros in this column! The column in which we need to create zeros is the second. To do so, subtract 4 times the second row from the first row. The resulting matrix is

$$\begin{bmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is now in reduced row-echelon form.

The above algorithm gives you a simple way to obtain the row-echelon form and reduced row-echelon form of a matrix. The main idea is to do row operations in such a way as to end up with a matrix in row-echelon form or reduced row-echelon form. This process is important because the resulting matrix will allow you to describe the solutions to the corresponding linear system of equations in a meaningful way.

In the next example, we look at how to solve a system of equations using the corresponding augmented matrix.

✓ Example 3.3.6: Finding the Solution to a System

Give the complete solution to the following system of equations

$$2x + 4y - 3z = -1$$

$$5x + 10y - 7z = -2$$

$$3x + 6y + 5z = 9$$

Solution

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 5 & 10 & -7 & -2 \\ 3 & 6 & 5 & 9 \end{array} \right]$$

In order to find the solution to this system, we wish to carry the augmented matrix to reduced row-echelon form. We will do so using [Algorithm 3.3.1](#). Notice that the first column is nonzero, so this is our first pivot column. The first entry in the first row, 2, is the first leading entry and it is in the first pivot position. We will use row operations to create zeros in the entries below the 2. First, replace the second row with -5 times the first row plus 2 times the second row. This yields

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 6 & 5 & 9 \end{array} \right]$$

Now, replace the third row with -3 times the first row plus 2 times the third row. This yields

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 21 \end{array} \right]$$

Now the entries in the first column below the pivot position are zeros. We now look for the second pivot column, which in this case is column three. Here, the 1 in the second row and third column is in the pivot position. We need to do just one row operation to create a zero below the 1.

Taking -1 times the second row and adding it to the third row yields

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 20 \end{array} \right]$$

We could proceed with the algorithm to carry this matrix to row-echelon form or reduced row-echelon form. However, remember that we are looking for the solutions to the system of equations. Take another look at the third row of the matrix. Notice that it corresponds to the equation

$$0x + 0y + 0z = 20$$

There is no solution to this equation because for all x, y, z , the left side will equal 0 and $0 \neq 20$. This shows there is no solution to the given system of equations. In other words, this system is inconsistent.

The following is another example of how to find the solution to a system of equations by carrying the corresponding augmented matrix to reduced row-echelon form.

✓ Example 3.3.7: An Infinite Set of Solutions

Give the complete solution to the system of equations

$$\begin{aligned} 3x - y - 5z &= 9 \\ y - 10z &= 0 \\ -2x + y &= -6 \end{aligned} \tag{3.3.1}$$

Solution

The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ -2 & 1 & 0 & -6 \end{array} \right]$$

In order to find the solution to this system, we will carry the augmented matrix to reduced row-echelon form, using [Algorithm 3.3.1](#). The first column is the first pivot column. We want to use row operations to create zeros beneath the first entry in this column, which is in the first pivot position. Replace the third row with 2 times the first row added to 3 times the third row. This gives

$$\left[\begin{array}{ccc|c} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 1 & -10 & 0 \end{array} \right]$$

Now, we have created zeros beneath the 3 in the first column, so we move on to the second pivot column (which is the second column) and repeat the procedure. Take -1 times the second row and add to the third row.

$$\left[\begin{array}{ccc|c} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The entry below the pivot position in the second column is now a zero. Notice that we have no more pivot columns because we have only two leading entries.

At this stage, we also want the leading entries to be equal to one. To do so, divide the first row by 3.

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & -\frac{5}{3} & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This matrix is now in row-echelon form.

Let's continue with row operations until the matrix is in reduced row-echelon form. This involves creating zeros above the pivot positions in each pivot column. This requires only one step, which is to add $\frac{1}{3}$ times the second row to the first row.

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This is in reduced row-echelon form, which you should verify using [Definition 3.3.4](#). The equations corresponding to this reduced row-echelon form are

$$\begin{aligned} x - 5z &= 3 \\ y - 10z &= 0 \end{aligned}$$

or

$$\begin{aligned} x &= 3 + 5z \\ y &= 10z \end{aligned}$$

Observe that z is not restrained by any equation. In fact, z can equal any number. For example, we can let $z = t$, where we can choose t to be any number. In this context t is called a **parameter**. Therefore, the solution set of this system is

$$\begin{aligned} x &= 3 + 5t \\ y &= 10t \\ z &= t \end{aligned}$$

where t is arbitrary. The system has an infinite set of solutions which are given by these equations. For any value of t we select, x , y , and z will be given by the above equations. For example, if we choose $t = 4$ then the corresponding solution would be

$$\begin{aligned} x &= 3 + 5(4) = 23 \\ y &= 10(4) = 40 \\ z &= 4 \end{aligned}$$

In [Example 3.3.7](#) the solution involved one parameter. It may happen that the solution to a system involves more than one parameter, as shown in the following example.

✓ Example 3.3.8: A Two Parameter Set of Solutions

Find the solution to the system

$$\begin{aligned} x + 2y - z + w &= 3 \\ x + y - z + w &= 1 \\ x + 3y - z + w &= 5 \end{aligned}$$

Solution

The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{array} \right]$$

We wish to carry this matrix to row-echelon form. Here, we will outline the row operations used. However, make sure that you understand the steps in terms of [Algorithm 3.3.1](#).

Take -1 times the first row and add to the second. Then take -1 times the first row and add to the third. This yields

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 \end{array} \right]$$

Now add the second row to the third row and divide the second row by -1 .

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad (3.3.2)$$

This matrix is in row-echelon form and we can see that x and y correspond to pivot columns, while z and w do not. Therefore, we will assign parameters to the variables z and w . Assign the parameter s to z and the parameter t to w . Then the first row yields the equation $x + 2y - s + t = 3$, while the second row yields the equation $y = 2$. Since $y = 2$, the first equation becomes $x + 4 - s + t = 3$ showing that the solution is given by

$$\begin{aligned} x &= -1 + s - t \\ y &= 2 \\ z &= s \\ w &= t \end{aligned}$$

It is customary to write this solution in the form

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 + s - t \\ 2 \\ s \\ t \end{bmatrix} \quad (3.3.3)$$

This example shows a system of equations with an infinite solution set which depends on two parameters. It can be less confusing in the case of an infinite solution set to first place the augmented matrix in reduced row-echelon form rather than just row-echelon form before seeking to write down the description of the solution.

In the above steps, this means we don't stop with the row-echelon form in equation (3.3.2). Instead we first place it in reduced row-echelon form as follows.

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Then the solution is $y = 2$ from the second row and $x = -1 + z - w$ from the first. Thus letting $z = s$ and $w = t$, the solution is given by (3.3.3).

You can see here that there are two paths to the correct answer, which both yield the same answer. Hence, either approach may be used. The process which we first used in the above solution is called **Gaussian Elimination**. This process involves carrying the matrix to row-echelon form, converting back to equations, and using back substitution to find the solution. When you do row operations until you obtain reduced row-echelon form, the process is called **Gauss-Jordan Elimination**.

We have now found solutions for systems of equations with no solution and infinitely many solutions, with one parameter as well as two parameters. Recall the three types of solution sets which we discussed in the previous section; no solution, one solution, and infinitely many solutions. Each of these types of solutions could be identified from the graph of the system. It turns out that we can also identify the type of solution from the reduced row-echelon form of the augmented matrix.

- **No Solution:** In the case where the system of equations has no solution, the row-echelon form of the augmented matrix will have a row of the form

$$[0 \quad 0 \quad 0 \quad | \quad 1]$$

This row indicates that the system is inconsistent and has no solution.

- **One Solution:** In the case where the system of equations has one solution, every column of the coefficient matrix is a pivot column. The following is an example of an augmented matrix in reduced row-echelon form for a system of equations with one solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

- *Infinitely Many Solutions:* In the case where the system of equations has infinitely many solutions, the solution contains parameters. There will be columns of the coefficient matrix which are not pivot columns. The following are examples of augmented matrices in reduced row-echelon form for systems of equations with infinitely many solutions.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

or

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \end{array} \right]$$

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3.4: Uniqueness of the Reduced Row-Echelon Form

As we have seen in earlier sections, we know that every matrix can be brought into reduced row-echelon form by a sequence of elementary row operations. Here we will prove that the resulting matrix is unique; in other words, the resulting matrix in reduced row-echelon does not depend upon the particular sequence of elementary row operations or the order in which they were performed.

Let A be the augmented matrix of a homogeneous system of linear equations in the variables $x_1, x_2, \dots, x_n$ which is also in reduced row-echelon form. The matrix A divides the set of variables in two different types. We say that x_i is a *basic variable* whenever A has a leading 1 in column number i , in other words, when column i is a pivot column. Otherwise we say that x_i is a *free variable*.

Recall [Example 1.3.8](#).

✓ Example 3.4.1: Basic and Free Variables

Find the basic and free variables in the system

$$\begin{aligned}x + 2y - z + w &= 3 \\x + y - z + w &= 1 \\x + 3y - z + w &= 5\end{aligned}$$

Solution

Recall from the solution of [Example 1.3.8](#) that the row-echelon form of the augmented matrix of this system is given by

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

You can see that columns 1 and 2 are pivot columns. These columns correspond to variables x and y , making these the basic variables. Columns 3 and 4 are not pivot columns, which means that z and w are free variables.

We can write the solution to this system as

$$\begin{aligned}x &= -1 + s - t \\y &= 2 \\z &= s \\w &= t\end{aligned}$$

Here the free variables are written as parameters, and the basic variables are given by linear functions of these parameters.

In general, all solutions can be written in terms of the free variables. In such a description, the free variables can take any values (they become parameters), while the basic variables become simple linear functions of these parameters. Indeed, a basic variable x_i is a linear function of *only* those free variables x_j with $j > i$. This leads to the following observation.

Proposition 3.4.1: Basic and Free Variables

If x_i is a basic variable of a homogeneous system of linear equations, then any solution of the system with $x_j = 0$ for all those free variables x_j with $j > i$ must also have $x_i = 0$.

Using this proposition, we prove a lemma which will be used in the proof of the main result of this section below.

Lemma 3.4.1: Solutions and the Reduced Row-Echelon Form of a Matrix

Let A and B be two distinct augmented matrices for two homogeneous systems of m equations in n variables, such that A and B are each in reduced row-echelon. Then, the two systems do not have exactly the same solutions.

Proof

With respect to the linear systems associated with the matrices A and B , there are two cases to consider:

- Case 1: the two systems have the same basic variables
- Case 2: the two systems do not have the same basic variables

In case 1, the two matrices will have exactly the same pivot positions. However, since A and B are not identical, there is some row of A which is different from the corresponding row of B and yet the rows each have a pivot in the same column position. Let i be the index of this column position. Since the matrices are in reduced row-echelon form, the two rows must differ at some entry in a column $j > i$. Let these entries be a in A and b in B , where $a \neq b$. Since A is in reduced row-echelon form, if x_j were a basic variable for its linear system, we would have $a = 0$. Similarly, if x_j were a basic variable for the linear system of the matrix B , we would have $b = 0$. Since a and b are unequal, they cannot both be equal to 0, and hence x_j cannot be a basic variable for both linear systems. However, since the systems have the same basic variables, x_j must then be a free variable for each system. We now look at the solutions of the systems in which x_j is set equal to 1 and all other free variables are set equal to 0. For this choice of parameters, the solution of the system for matrix A has $x_j = -a$, while the solution of the system for matrix B has $x_j = -b$, so that the two systems have different solutions.

In case 2, there is a variable x_i which is a basic variable for one matrix, let's say A , and a free variable for the other matrix B . The system for matrix B has a solution in which $x_i = 1$ and $x_j = 0$ for all other free variables x_j . However, by [Proposition 3.4.1](#), this cannot be a solution of the system for the matrix A . This completes the proof of case 2.

Now, we say that the matrix B is **equivalent** to the matrix A provided that B can be obtained from A by performing a sequence of elementary row operations beginning with A . The importance of this concept lies in the following result.

Theorem 3.4.1: Equivalent Matrices

The two linear systems of equations corresponding to two equivalent augmented matrices have exactly the same solutions.

Proof

The proof of this theorem is left as an exercise.

Now, we can use [Lemma 3.4.1](#) and [Theorem 3.4.1](#) to prove the main result of this section.

Theorem 3.4.2: Uniqueness of the Reduced Row-Echelon Form

Every matrix A is equivalent to a unique matrix in reduced row-echelon form.

Proof

Let A be an $m \times n$ matrix and let B and C be matrices in reduced row-echelon form, each equivalent to A . It suffices to show that $B = C$.

Let A^+ be the matrix A augmented with a new rightmost column consisting entirely of zeros. Similarly, augment matrices B and C each with a rightmost column of zeros to obtain B^+ and C^+ . Note that B^+ and C^+ are matrices in reduced row-echelon form which are obtained from A^+ by respectively applying the same sequence of elementary row operations which were used to obtain B and C from A .

Now, A^+ , B^+ , and C^+ can all be considered as augmented matrices of homogeneous linear systems in the variables $x_1, x_2, \dots, x_n$. Because B^+ and C^+ are each equivalent to A^+ , [Theorem 3.4.1](#) ensures that all three homogeneous linear systems have exactly the same solutions. By [Lemma 3.4.1](#) we conclude that $B^+ = C^+$. By construction, we must also have $B = C$.

According to this theorem we can say that each matrix A has a unique reduced row-echelon form.

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3.5: Rank and Homogeneous Systems

Rank and Homogeneous Systems

There is a special type of system which requires additional study. This type of system is called a homogeneous system of equations, which we defined above in [Definition 1.2.3](#). Our focus in this section is to consider what types of solutions are possible for a homogeneous system of equations.

Consider the following definition.

Consider the homogeneous system of equations given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Then, $x_1 = 0, x_2 = 0, \dots, x_n = 0$ is always a solution to this system. We call this the **trivial solution**.

If the system has a solution in which not all of the $x_1, \dots, x_n$ are equal to zero, then we call this solution **nontrivial**. The trivial solution does not tell us much about the system, as it says that $0 = 0$! Therefore, when working with homogeneous systems of equations, we want to know when the system has a nontrivial solution.

Suppose we have a homogeneous system of m equations, using n variables, and suppose that $n > m$. In other words, there are more variables than equations. Then, it turns out that this system always has a nontrivial solution. Not only will the system have a nontrivial solution, but it also will have infinitely many solutions. It is also possible, but not required, to have a nontrivial solution if $n = m$ and $n < m$.

Consider the following example.

✓ Example 3.5.1: Solutions to a Homogeneous System of Equations

Find the nontrivial solutions to the following homogeneous system of equations

$$\begin{aligned} 2x + y - z &= 0 \\ x + 2y - 2z &= 0 \end{aligned}$$

Solution

Notice that this system has $m = 2$ equations and $n = 3$ variables, so $n > m$. Therefore by our previous discussion, we expect this system to have infinitely many solutions.

The process we use to find the solutions for a homogeneous system of equations is the same process we used in the previous section. First, we construct the augmented matrix, given by

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 2 & -2 & 0 \end{array} \right]$$

Then, we carry this matrix to its reduced row-echelon form, given below.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right]$$

The corresponding system of equations is

$$\begin{aligned} x &= 0 \\ y - z &= 0 \end{aligned}$$

Since z is not restrained by any equation, we know that this variable will become our parameter. Let $z = t$ where t is any number. Therefore, our solution has the form

$$\begin{aligned}x &= 0 \\y &= z = t \\z &= t\end{aligned}$$

Hence this system has infinitely many solutions, with one parameter t .

Suppose we were to write the solution to the previous example in another form. Specifically,

$$\begin{aligned}x &= 0 \\y &= 0 + t \\z &= 0 + t\end{aligned}$$

can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Notice that we have constructed a column from the constants in the solution (all equal to 0), as well as a column corresponding to the coefficients on t in each equation. While we will discuss this form of solution more in further chapters, for now consider the

column of coefficients of the parameter t . In this case, this is the column $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

There is a special name for this column, which is **basic solution**. The basic solutions of a system are columns constructed from the coefficients on parameters in the solution. We often denote basic solutions by X_1, X_2 etc., depending on how many solutions

occur. Therefore, [Example 3.5.1](#) has the basic solution $X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

We explore this further in the following example.

✓ Example 3.5.2: Basic Solutions of a Homogeneous System

Consider the following homogeneous system of equations.

$$\begin{aligned}x + 4y + 3z &= 0 \\3x + 12y + 9z &= 0\end{aligned}$$

Find the basic solutions to this system.

Solution

The augmented matrix of this system and the resulting reduced row-echelon form are

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 3 & 12 & 9 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

When written in equations, this system is given by

$$x + 4y + 3z = 0$$

Notice that only x corresponds to a pivot column. In this case, we will have two parameters, one for y and one for z . Let $y = s$ and $z = t$ for any numbers s and t . Then, our solution becomes

$$\begin{aligned}x &= -4s - 3t \\y &= s \\z &= t\end{aligned}$$

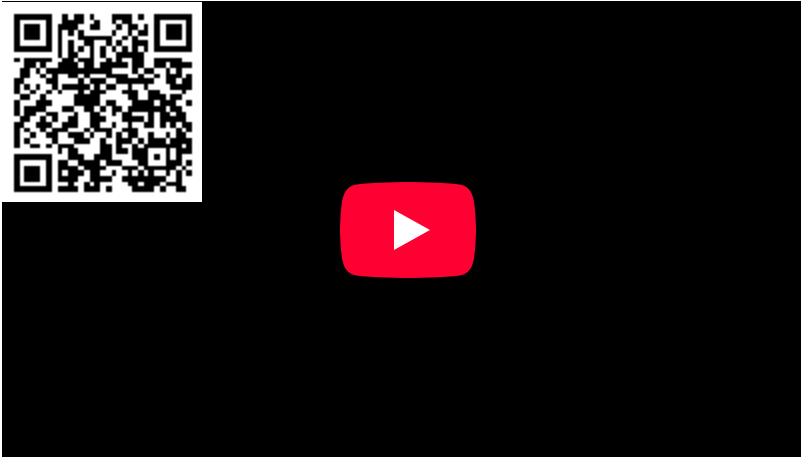
which can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

You can see here that we have two columns of coefficients corresponding to parameters, specifically one for s and one for t . Therefore, this system has two basic solutions! These are

$$X_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Below is a video on homogeneous systems of equations



Below is a video on solving a system of equations and determining the particular solution.



Below is a video on solving $Ax = 0$.



Below is another video on solving $Ax = 0$.



We now present a new definition.

Definition 3.5.2: Linear Combination

Let $X_1, \dots, X_n, V$ be column matrices. Then V is said to be a **linear combination** of the columns $X_1, \dots, X_n$ if there exist scalars, $a_1, \dots, a_n$ such that

$$V = a_1 X_1 + \dots + a_n X_n$$

A remarkable result of this section is that a linear combination of the basic solutions is again a solution to the system. Even more remarkable is that every solution can be written as a linear combination of these solutions. Therefore, if we take a linear combination of the two solutions to [Example 3.5.2](#), this would also be a solution. For example, we could take the following linear combination

$$3 \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}$$

You should take a moment to verify that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}$$

is in fact a solution to the system in [Example 3.5.2](#).

Another way in which we can find out more information about the solutions of a homogeneous system is to consider the **rank** of the associated coefficient matrix. We now define what is meant by the rank of a matrix.

Definition 3.5.3: Rank of a Matrix

Let A be a matrix and consider any row-echelon form of A . Then, the number r of leading entries of A does not depend on the row-echelon form you choose, and is called the **rank** of A . We denote it by $\text{Rank}(A)$.

Similarly, we could count the number of pivot positions (or pivot columns) to determine the rank of A .

Example 3.5.3: Finding the Rank of a Matrix

Consider the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 9 \\ 2 & 4 & 6 \end{bmatrix}$$

What is its rank?

Solution

First, we need to find the reduced row-echelon form of A . Through the usual algorithm, we find that this is

$$\begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Here we have two leading entries, or two pivot positions, shown above in boxes. The rank of A is $r = 2$.

Notice that we would have achieved the same answer if we had found the row-echelon form of A instead of the reduced row-echelon form.

Suppose we have a homogeneous system of m equations in n variables, and suppose that $n > m$. From our above discussion, we know that this system will have infinitely many solutions. If we consider the rank of the coefficient matrix of this system, we can find out even more about the solution. Note that we are looking at just the coefficient matrix, not the entire augmented matrix.

Theorem 3.5.1: Rank and Solutions to a Homogeneous System

Let A be the $m \times n$ coefficient matrix corresponding to a homogeneous system of equations, and suppose A has rank r . Then, the solution to the corresponding system has $n - r$ parameters.

Consider our above [Example 3.5.2](#) in the context of this theorem. The system in this example has $m = 2$ equations in $n = 3$ variables. First, because $n > m$, we know that the system has a nontrivial solution, and therefore infinitely many solutions. This tells us that the solution will contain at least one parameter. The rank of the coefficient matrix can tell us even more about the solution! The rank of the coefficient matrix of the system is 1, as it has one leading entry in row-echelon form. [Theorem 3.5.1](#) tells us that the solution will have $n - r = 3 - 1 = 2$ parameters. You can check that this is true in the solution to [Example 3.5.2](#).

Notice that if $n = m$ or $n < m$, it is possible to have either a unique solution (which will be the trivial solution) or infinitely many solutions.

We are not limited to homogeneous systems of equations here. The rank of a matrix can be used to learn about the solutions of any system of linear equations. In the previous section, we discussed that a system of equations can have no solution, a unique solution, or infinitely many solutions. Suppose the system is consistent, whether it is homogeneous or not. The following theorem tells us how we can use the rank to learn about the type of solution we have.

Below is a video on Rank, Basic Variables, and Free Variables of a coefficient matrix.



Theorem 3.5.2: Rank and Solutions to a Consistent System of Equations

Let A be the $m \times (n + 1)$ augmented matrix corresponding to a consistent system of equations in n variables, and suppose A has rank r . Then

1. the system has a unique solution if $r = n$
2. the system has infinitely many solutions if $r < n$

We will not present a formal proof of this, but consider the following discussions.

1. *No Solution* The above theorem assumes that the system is consistent, that is, that it has a solution. It turns out that it is possible for the augmented matrix of a system with no solution to have any rank r as long as $r > 1$. Therefore, we must know that the system is consistent in order to use this theorem!
2. *Unique Solution* Suppose $r = n$. Then, there is a pivot position in every column of the coefficient matrix of A . Hence, there is a unique solution.
3. *Infinitely Many Solutions* Suppose $r < n$. Then there are infinitely many solutions. There are less pivot positions (and hence less leading entries) than columns, meaning that not every column is a pivot column. The columns which are *not* pivot columns correspond to parameters. In fact, in this case we have $n - r$ parameters.

Below is a video on the geometry of $Ax = 0$ and $Ax = b$.



Below is another video on the geometry of $Ax = 0$ and $Ax = b$.



Below is a video on the geometry of $Ax = 0$ and $Ax = b$, this one with two free variables.



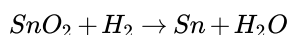
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3.6: Balancing Chemical Reactions

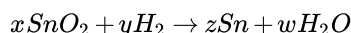
The tools of linear algebra can also be used in the subject area of Chemistry, specifically for balancing chemical reactions.

Consider the chemical reaction



Here the elements involved are tin (Sn), oxygen (O), and hydrogen (H). A chemical reaction occurs and the result is a combination of tin (Sn) and water (H_2O). When considering chemical reactions, we want to investigate how much of each element we began with and how much of each element is involved in the result.

An important theory we will use here is the mass balance theory. It tells us that we cannot create or delete elements within a chemical reaction. For example, in the above expression, we must have the same number of oxygen, tin, and hydrogen on both sides of the reaction. Notice that this is not currently the case. For example, there are two oxygen atoms on the left and only one on the right. In order to fix this, we want to find numbers x, y, z, w such that



where both sides of the reaction have the same number of atoms of the various elements.

This is a familiar problem. We can solve it by setting up a system of equations in the variables x, y, z, w . Thus you need

$$\begin{aligned} \text{Sn} : \quad x &= z \\ \text{O} : \quad 2x &= w \\ \text{H} : \quad 2y &= 2w \end{aligned}$$

We can rewrite these equations as

$$\begin{aligned} \text{Sn} : \quad x - z &= 0 \\ \text{O} : \quad 2x - w &= 0 \\ \text{H} : \quad 2y - 2w &= 0 \end{aligned}$$

The augmented matrix for this system of equations is given by

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 \end{array} \right]$$

The reduced row-echelon form of this matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{array} \right]$$

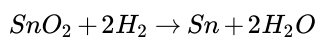
The solution is given by

$$\begin{aligned} x - \frac{1}{2}w &= 0 \\ y - w &= 0 \\ z - \frac{1}{2}w &= 0 \end{aligned}$$

which we can write as

$$\begin{aligned} x &= \frac{1}{2}t \\ y &= t \\ z &= \frac{1}{2}t \\ w &= t \end{aligned}$$

For example, let $w = 2$ and this would yield $x = 1, y = 2$, and $z = 1$. We can put these values back into the expression for the reaction which yields

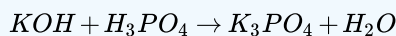


Observe that each side of the expression contains the same number of atoms of each element. This means that it preserves the total number of atoms, as required, and so the chemical reaction is balanced.

Consider another example.

✓ Example 3.6.1: Balancing a Chemical Reaction

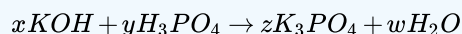
Potassium is denoted by K , oxygen by O , phosphorus by P and hydrogen by H . Consider the reaction given by



Balance this chemical reaction.

Solution

We will use the same procedure as above to solve this problem. We need to find values for x, y, z, w such that



preserves the total number of atoms of each element.

Finding these values can be done by finding the solution to the following system of equations.

$$\begin{aligned} K: \quad x &= 3z \\ O: \quad x + 4y &= 4z + w \\ H: \quad x + 3y &= 2w \\ P: \quad y &= z \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 1 & 4 & -4 & -1 & 0 \\ 1 & 3 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array} \right]$$

and the reduced row-echelon form is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

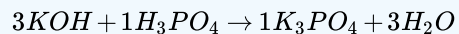
The solution is given by

$$\begin{aligned} x - w &= 0 \\ y - \frac{1}{3}w &= 0 \\ z - \frac{1}{3}w &= 0 \end{aligned}$$

which can be written as

$$\begin{aligned} x &= t \\ y &= \frac{1}{3}t \\ z &= \frac{1}{3}t \\ w &= t \end{aligned}$$

Choose a value for t , say 3. Then $w = 3$ and this yields $x = 3, y = 1, z = 1$. It follows that the balanced reaction is given by



Note that this results in the same number of atoms on both sides.

Of course these numbers you are finding would typically be the number of moles of the molecules on each side. Thus three moles of KOH added to one mole of H_3PO_4 yields one mole of K_3PO_4 and three moles of H_2O .

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3.7: Dimensionless Variables

This section shows how solving systems of equations can be used to determine appropriate dimensionless variables. It is only an introduction to this topic and considers a specific example of a simple airplane wing shown below. We assume for simplicity that it is a flat plane at an angle to the wind which is blowing against it with speed V as shown.

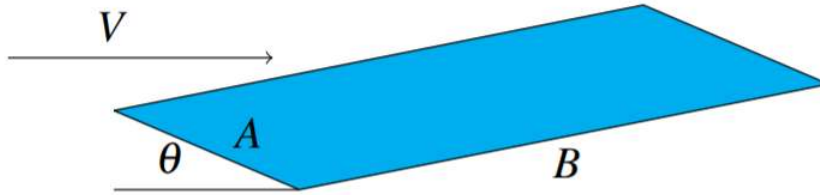


Figure 3.7.1

The angle θ is called the angle of incidence, B is the span of the wing and A is called the chord. Denote by l the lift. Then this should depend on various quantities like θ, V, B, A and so forth. Here is a table which indicates various quantities on which it is reasonable to expect l to depend.

Table 3.7.1

Variable	Symbol	Units
chord	A	m
span	B	m
angle incidence	θ	$m^0 kg^0 sec^0$
speed of wind	V	$msec^{-1}$
speed of sound	V_0	$msec^{-1}$
density of air	ρ	kgm^{-3}
viscosity	μ	$kgsec^{-1}m^{-1}$
lift	l	$kgsec^{-2}m$

Here m denotes meters, sec refers to seconds and kg refers to kilograms. All of these are likely familiar except for μ , which we will discuss in further detail now.

Viscosity is a measure of how much internal friction is experienced when the fluid moves. It is roughly a measure of how “sticky” the fluid is. Consider a piece of area parallel to the direction of motion of the fluid. To say that the viscosity is large is to say that the tangential force applied to this area must be large in order to achieve a given change in speed of the fluid in a direction normal to the tangential force. Thus

$$\mu (\text{area}) (\text{velocity gradient}) = \text{tangential force}$$

Hence

$$(\text{units on } \mu) m^2 \left(\frac{m}{sec\ m} \right) = kgsec^{-2} m$$

Thus the units on μ are

$$kgsec^{-1} m^{-1}$$

as claimed above.

Returning to our original discussion, you may think that we would want

$$l = f(A, B, \theta, V, V_0, \rho, \mu)$$

This is very cumbersome because it depends on seven variables. Also, it is likely that without much care, a change in the units such as going from meters to feet would result in an incorrect value for l . The way to get around this problem is to look for l as a function of dimensionless variables multiplied by something which has units of force. It is helpful because first of all, you will likely have fewer independent variables and secondly, you could expect the formula to hold independent of the way of specifying length, mass and so forth. One looks for

$$l = f(g_1, \dots, g_k) \rho V^2 AB$$

where the units on $\rho V^2 AB$ are

$$\frac{kg}{m^3} \left(\frac{m}{sec} \right)^2 m^2 = \frac{kg \times m}{sec^2}$$

which are the units of force. Each of these g_i is of the form

$$A^{x_1} B^{x_2} \theta^{x_3} V^{x_4} V_0^{x_5} \rho^{x_6} \mu^{x_7} \quad (3.7.1)$$

and each g_i is independent of the dimensions. That is, this expression must not depend on meters, kilograms, seconds, etc. Thus, placing in the units for each of these quantities, one needs

$$m^{x_1} m^{x_2} (m^{x_4} sec^{-x_4}) (m^{x_5} sec^{-x_5}) (kg m^{-3})^{x_6} (kg sec^{-1} m^{-1})^{x_7} = m^0 kg^0 sec^0$$

Notice that there are no units on θ because it is just the radian measure of an angle. Hence its dimensions consist of length divided by length, thus it is dimensionless. Then this leads to the following equations for the x_i .

$$\begin{aligned} m : \quad & x_1 + x_2 + x_4 + x_5 - 3x_6 - x_7 = 0 \\ sec : \quad & -x_4 - x_5 - x_7 = 0 \\ kg : \quad & x_6 + x_7 = 0 \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & 1 & 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array} \right]$$

The reduced row-echelon form is given by

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array} \right]$$

and so the solutions are of the form

$$\begin{aligned} x_1 &= -x_2 - x_7 \\ x_3 &= x_3 \\ x_4 &= -x_5 - x_7 \\ x_6 &= -x_7 \end{aligned}$$

Thus, in terms of vectors, the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} -x_2 - x_7 \\ x_2 \\ x_3 \\ -x_5 - x_7 \\ x_5 \\ -x_7 \\ x_7 \end{bmatrix}$$

Thus the free variables are x_2, x_3, x_5, x_7 . By assigning values to these, we can obtain dimensionless variables by placing the values obtained for the x_i in the formula (3.7.1). For example, let $x_2 = 1$ and all the rest of the free variables are 0. This yields

$$x_1 = -1, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0$$

The dimensionless variable is then $A^{-1}B^1$. This is the ratio between the span and the chord. It is called the aspect ratio, denoted as AR . Next let $x_3 = 1$ and all others equal zero. This gives for a dimensionless quantity the angle θ . Next let $x_5 = 1$ and all others equal zero. This gives

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = -1, x_5 = 1, x_6 = 0, x_7 = 0$$

Then the dimensionless variable is $V^{-1}V_0^1$. However, it is written as V/V_0 . This is called the Mach number $\mathcal{M}$. Finally, let $x_7 = 1$ and all the other free variables equal 0. Then

$$x_1 = -1, x_2 = 0, x_3 = 0, x_4 = -1, x_5 = 0, x_6 = -1, x_7 = 1$$

then the dimensionless variable which results from this is $A^{-1}V^{-1}\rho^{-1}\mu$. It is customary to write it as $Re = (AV\rho)/\mu$. This one is called the Reynold's number. It is the one which involves viscosity. Thus we would look for

$$l = f(Re, AR, \theta, \mathcal{M}) \text{ kg} \times \text{m} / \text{sec}^2$$

This is quite interesting because it is easy to vary Re by simply adjusting the velocity or A but it is hard to vary things like μ or ρ . Note that all the quantities are easy to adjust. Now this could be used, along with wind tunnel experiments to get a formula for the lift which would be reasonable. You could also consider more variables and more complicated situations in the same way.

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3.8: An Application to Resistor Networks

The tools of linear algebra can be used to study the application of resistor networks. An example of an electrical circuit is below.

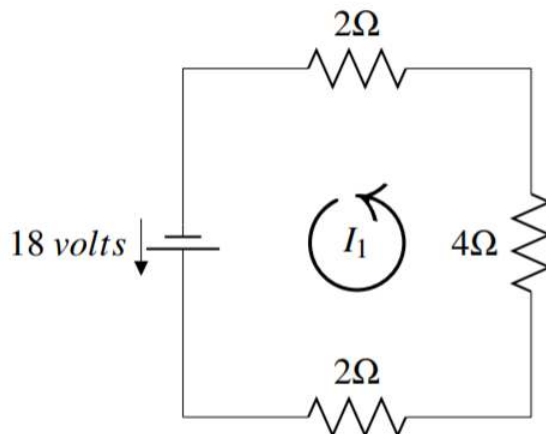


Figure 3.8.1

The jagged lines ($\text{---}\text{---}\text{---}$) denote resistors and the numbers next to them give their resistance in ohms, written as Ω . The voltage source ($\text{---}| \text{---}$) causes the current to flow in the direction from the shorter of the two lines toward the longer (as indicated by the arrow). The current for a circuit is labeled I_k .

In the above figure, the current I_1 has been labeled with an arrow in the counter clockwise direction. This is an entirely arbitrary decision and we could have chosen to label the current in the clockwise direction. With our choice of direction here, we define a positive current to flow in the counter clockwise direction and a negative current to flow in the clockwise direction.

The goal of this section is to use the values of resistors and voltage sources in a circuit to determine the current. An essential theorem for this application is Kirchhoff's law.

Theorem 3.8.1: Kirchhoff's Law

The sum of the resistance (R) times the amps (I) in the counter clockwise direction around a loop equals the sum of the voltage sources (V) in the same direction around the loop.

Kirchhoff's law allows us to set up a system of linear equations and solve for any unknown variables. When setting up this system, it is important to trace the circuit in the counter clockwise direction. If a resistor or voltage source is crossed against this direction, the related term must be given a negative sign.

We will explore this in the next example where we determine the value of the current in the initial diagram.

✓ Example 3.8.1: Solving for Current

Applying Kirchhoff's Law to the diagram below, determine the value for I_1 .

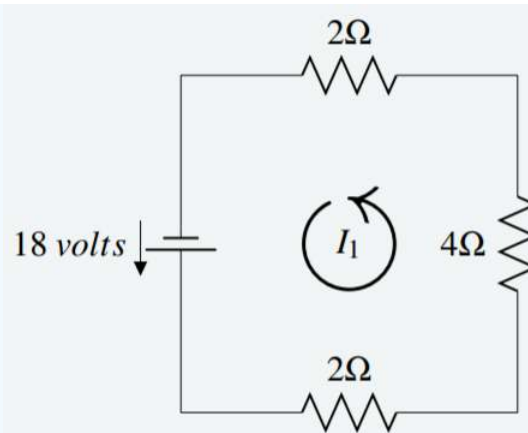


Figure 3.8.2

Solution

Begin in the bottom left corner, and trace the circuit in the counter clockwise direction. At the first resistor, multiplying resistance and current gives $2I_1$. Continuing in this way through all three resistors gives $2I_1 + 4I_1 + 2I_1$. This must equal the voltage source in the same direction. Notice that the direction of the voltage source matches the counter clockwise direction specified, so the voltage is positive.

Therefore the equation and solution are given by

$$\begin{aligned} 2I_1 + 4I_1 + 2I_1 &= 18 \\ 8I_1 &= 18 \\ I_1 &= \frac{9}{4} A \end{aligned}$$

Since the answer is positive, this confirms that the current flows counter clockwise.

✓ Example 3.8.2 Solving for Current

Applying Kirchhoff's Law to the diagram below, determine the value for I_1 .

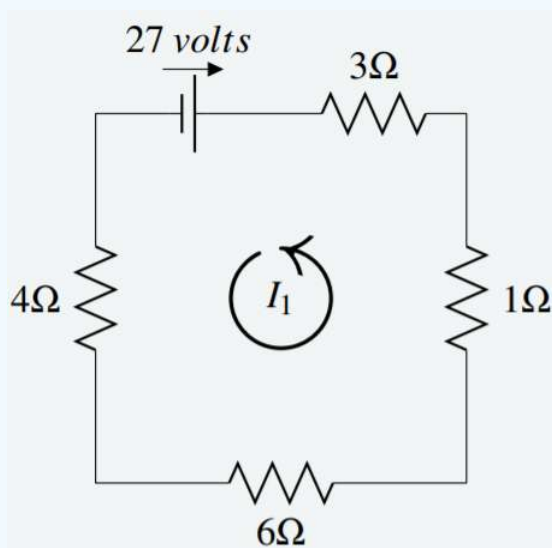


Figure 3.8.3

Solution

Begin in the top left corner this time, and trace the circuit in the counter clockwise direction. At the first resistor, multiplying resistance and current gives $4I_1$. Continuing in this way through the four resistors gives $4I_1 + 6I_1 + 1I_1 + 3I_1$. This must

equal the voltage source in the same direction. Notice that the direction of the voltage source is opposite to the counter clockwise direction, so the voltage is negative.

Therefore the equation and solution are given by

$$\begin{aligned} 4I_1 + 6I_1 + 1I_1 + 3I_1 &= -27 \\ 14I_1 &= -27 \\ I_1 &= -\frac{27}{14} A \end{aligned}$$

Since the answer is negative, this tells us that the current flows clockwise.

A more complicated example follows. Two of the circuits below may be familiar; they were examined in the examples above. However as they are now part of a larger system of circuits, the answers will differ.

✓ Example 3.8.3: Unknown Currents

The diagram below consists of four circuits. The current (I_k) in the four circuits is denoted by I_1, I_2, I_3, I_4 . Using Kirchhoff's Law, write an equation for each circuit and solve for each current.

Solution

The circuits are given in the following diagram.

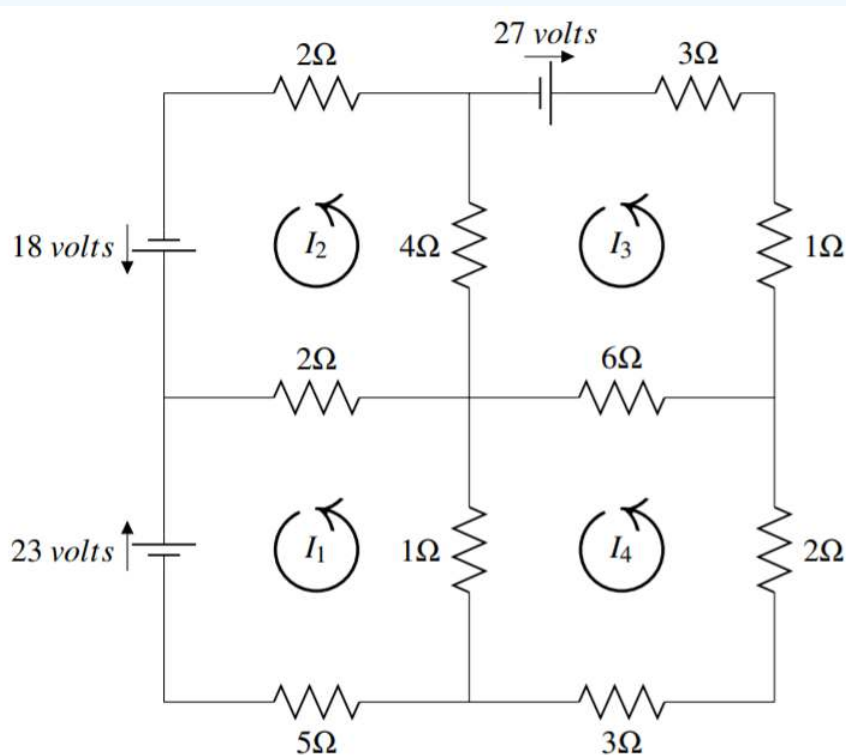


Figure 3.8.4

Starting with the top left circuit, multiply the resistance by the amps and sum the resulting products. Specifically, consider the resistor labeled 2Ω that is part of the circuits of I_1 and I_2 . Notice that current I_2 runs through this in a positive (counter clockwise) direction, and I_1 runs through in the opposite (negative) direction. The product of resistance and amps is then $2(I_2 - I_1) = 2I_2 - 2I_1$. Continue in this way for each resistor, and set the sum of the products equal to the voltage source to write the equation:

$$2I_2 - 2I_1 + 4I_2 - 4I_3 + 2I_2 = 18$$

The above process is used on each of the other three circuits, and the resulting equations are:

Upper right circuit:

$$4I_3 - 4I_2 + 6I_3 - 6I_4 + I_3 + 3I_3 = -27$$

Lower right circuit:

$$3I_4 + 2I_4 + 6I_4 - 6I_3 + I_4 - I_1 = 0$$

Lower left circuit:

$$5I_1 + I_1 - I_4 + 2I_1 - 2I_2 = -23$$

Notice that the voltage for the upper right and lower left circuits are negative due to the clockwise direction they indicate.

The resulting system of four equations in four unknowns is

$$\begin{aligned} 2I_2 - 2I_1 + 4I_2 - 4I_3 + 2I_2 &= 18 \\ 4I_3 - 4I_2 + 6I_3 - 6I_4 + I_3 + I_3 &= -27 \\ 2I_4 + 3I_4 + 6I_4 - 6I_3 + I_4 - I_1 &= 0 \\ 5I_1 + I_1 - I_4 + 2I_1 - 2I_2 &= -23 \end{aligned}$$

Simplifying and rearranging with variables in order, we have:

$$\begin{aligned} -2I_1 + 8I_2 - 4I_3 &= 18 \\ -4I_2 + 14I_3 - 6I_4 &= -27 \\ -I_1 - 6I_3 + 12I_4 &= 0 \\ 8I_1 - 2I_2 - I_4 &= -23 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{cccc|c} -2 & 8 & -4 & 0 & 18 \\ 0 & -4 & 14 & -6 & -27 \\ -1 & 0 & -6 & 12 & 0 \\ 8 & -2 & 0 & -1 & -23 \end{array} \right]$$

The solution to this matrix is

$$\begin{aligned} I_1 &= -3A \\ I_2 &= \frac{1}{4}A \\ I_3 &= -\frac{5}{2}A \\ I_4 &= -\frac{3}{2}A \end{aligned}$$

This tells us that currents I_1 , I_3 , and I_4 travel clockwise while I_2 travels counter clockwise.

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3.9: Exercises

Exercises

? Exercise 3.9.1

Graphically, find the point (x_1, y_1) which lies on both lines, $x + 3y = 1$ and $4x - y = 3$. That is, graph each line and see where they intersect.

Answer

$$\begin{array}{l} x + 3y = 1 \\ 4x - y = 3 \end{array}, \text{ Solution is: } \left[x = \frac{10}{13}, y = \frac{1}{13} \right].$$

? Exercise 3.9.2

Graphically, find the point of intersection of the two lines $3x + y = 3$ and $x + 2y = 1$. That is, graph each line and see where they intersect.

Answer

$$\begin{array}{l} 3x + y = 3 \\ x + 2y = 1 \end{array}, \text{ Solution is: } [x = 1, y = 0]$$

? Exercise 3.9.3

Do the three lines, $x + 2y = 1$, $2x - y = 1$, and $4x + 3y = 3$ have a common point of intersection? If so, find the point and if not, tell why they don't have such a common point of intersection.

Answer

$$\begin{array}{l} x + 2y = 1 \\ 2x - y = 1 \\ 4x + 3y = 3 \end{array}, \text{ Solution is: } \left[x = \frac{3}{5}, y = \frac{1}{5} \right]$$

? Exercise 3.9.4

Do the three planes, $x + y - 3z = 2$, $2x + y + z = 1$, and $3x + 2y - 2z = 0$ have a common point of intersection? If so, find one and if not, tell why there is no such point.

Answer

No solution exists. You can see this by writing the augmented matrix and doing row operations.
$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 2 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & -2 & 0 \end{array} \right], \text{ row}$$

echelon form:
$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right].$$
 Thus one of the equation says $0 = 1$ in an equivalent system of equations.

? Exercise 3.9.5

Four times the weight of Gaston is 150 pounds more than the weight of Ichabod. Four times the weight of Ichabod is 660 pounds less than seventeen times the weight of Gaston. Four times the weight of Gaston plus the weight of Siegfried equals 290 pounds. Brunhilde would balance all three of the others. Find the weights of the four people.

Answer

$$\begin{aligned} 4g - I &= 150 \\ 4I - 17g &= -660 \\ 4g + s &= 290 \\ g + I + s - b &= 0 \end{aligned} \quad , \text{ Solution is: } \{g = 60, I = 90, b = 200, s = 50\}$$

? Exercise 3.9.6

Consider the following augmented matrix in which * denotes an arbitrary number and ■ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array} \right]$$

Answer

The solution exists but is not unique.

? Exercise 3.9.7

Consider the following augmented matrix in which * denotes an arbitrary number and ■ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{array} \right]$$

Answer

A solution exists and is unique.

? Exercise 3.9.8

Consider the following augmented matrix in which * denotes an arbitrary number and ■ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & 0 & * & 0 & * \\ 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array} \right]$$

? Exercise 3.9.9

Consider the following augmented matrix in which * denotes an arbitrary number and ■ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & 0 & * & \blacksquare \end{array} \right]$$

Answer

There might be a solution. If so, there are infinitely many.

? Exercise 3.9.10

Suppose a system of equations has fewer equations than variables. Will such a system necessarily be consistent? If so, explain why and if not, give an example which is not consistent.

Answer

No. Consider $x + y + z = 2$ and $x + y + z = 1$.

? Exercise 3.9.11

If a system of equations has more equations than variables, can it have a solution? If so, give an example and if not, tell why not.

Answer

These can have a solution. For example, $x + y = 1$, $2x + 2y = 2$, $3x + 3y = 3$ even has an infinite set of solutions.

? Exercise 3.9.12

Find h such that

$$\left[\begin{array}{cc|c} 2 & h & 4 \\ 3 & 6 & 7 \end{array} \right]$$

is the augmented matrix of an *inconsistent* system.

Answer

$$h = 4$$

? Exercise 3.9.13

Find h such that

$$\left[\begin{array}{cc|c} 1 & h & 3 \\ 2 & 4 & 6 \end{array} \right]$$

is the augmented matrix of a *consistent* system.

Answer

Any h will work.

? Exercise 3.9.14

Find h such that

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & h & 12 \end{array} \right]$$

is the augmented matrix of a *consistent* system.

Answer

Any h will work.

? Exercise 3.9.15

Choose h and k such that the augmented matrix shown has each of the following:

- one solution
- no solution
- infinitely many solutions

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 2 & 4 & k \end{array} \right]$$

Answer

If $h \neq 2$ there will be a unique solution for any k . If $h = 2$ and $k \neq 4$, there are no solutions. If $h = 2$ and $k = 4$, then there are infinitely many solutions.

? Exercise 3.9.16

Choose h and k such that the augmented matrix shown has each of the following:

- one solution
- no solution
- infinitely many solutions

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & h & k \end{array} \right]$$

Answer

If $h \neq 4$, then there is exactly one solution. If $h = 4$ and $k \neq 4$, then there are no solutions. If $h = 4$ and $k = 4$, then there are infinitely many solutions.

? Exercise 3.9.17

Determine if the system is consistent. If so, is the solution unique?

$$x + 2y + z - w = 2$$

$$x - y + z + w = 1$$

$$2x + y - z = 1$$

$$4x + 2y + z = 5$$

Answer

There is no solution. The system is inconsistent. You can see this from the augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 0 & 1 \\ 4 & 2 & 1 & 0 & 5 \end{array} \right],$$

reduced row-echelon form: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right].$

? Exercise 3.9.18

Determine if the system is consistent. If so, is the solution unique?

$$x + 2y + z - w = 2$$

$$x - y + z + w = 0$$

$$2x + y - z = 1$$

$$4x + 2y + z = 3$$

Answer

Solution is: $[w = \frac{3}{2}y - 1, x = \frac{2}{3} - \frac{1}{2}y, z = \frac{1}{3}]$

? Exercise 3.9.19

Determine which matrices are in reduced row-echelon form.

a. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 7 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

Answer

- a. This one is not.
- b. This one is.
- c. This one is.

? Exercise 3.9.20

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 2 & -1 & 3 & -1 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 1 & -2 \end{bmatrix}$$

? Exercise 3.9.21

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

? Exercise 3.9.22

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 3 & -6 & -7 & -8 \\ 1 & -2 & -2 & -2 \\ 1 & -2 & -3 & -4 \end{bmatrix}$$

? Exercise 3.9.23

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 2 & 4 & 5 & 15 \\ 1 & 2 & 3 & 9 \\ 1 & 2 & 2 & 6 \end{bmatrix}$$

? Exercise 3.9.24

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 4 & -1 & 7 & 10 \\ 1 & 0 & 3 & 3 \\ 1 & -1 & -2 & 1 \end{bmatrix}$$

? Exercise 3.9.25

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 3 & 5 & -4 & 2 \\ 1 & 2 & -1 & 1 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

? Exercise 3.9.26

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} -2 & 3 & -8 & 7 \\ 1 & -2 & 5 & -5 \\ 1 & -3 & 7 & -8 \end{bmatrix}$$

? Exercise 3.9.27

Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 1 & 3 & 4 & 2 \\ 1 & 0 & 2 & 1 \end{array} \right]$$

? Exercise 3.9.28

Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 3 & 2 & 1 & 3 \end{array} \right]$$

Answer

The reduced row-echelon form is $\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 \end{array} \right]$. Therefore, the solution is of the form

$z = t, y = \frac{3}{4} + t\left(\frac{1}{4}\right), x = \frac{1}{2} - \frac{1}{2}t$ where $t \in \mathbb{R}$.

? Exercise 3.9.29

Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 4 & 2 \end{array} \right]$$

Answer

The reduced row-echelon form is $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 1 & -4 & -1 \end{array} \right]$ and so the solution is $z = t, y = 4t, x = 2 - 4t$.

? Exercise 3.9.30

Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 0 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 & 2 \end{array} \right]$$

Answer

The reduced row-echelon form is $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 9 & 3 \\ 0 & 1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & -7 & -1 \\ 0 & 0 & 0 & 1 & 6 & 1 \end{array} \right]$ and so $x_5 = t, x_4 = 1 - 6t, x_3 = -1 + 7t, x_2 = 4t, x_1 = 3 - 9t$.

? Exercise 3.9.31

Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 & 1 & 3 \\ 1 & -1 & 2 & 2 & 2 & 0 \end{array} \right]$$

Answer

The reduced row-echelon form is $\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -\frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$. Therefore, let $x_5 = t, x_3 = s$. Then the other variables are given by $x_4 = -\frac{1}{2} - \frac{3}{2}t, x_2 = \frac{3}{2} - t\frac{1}{2}, x_1 = \frac{5}{2} + \frac{1}{2}t - 2s$.

? Exercise 3.9.32

Find the solution to the system of equations, $7x + 14y + 15z = 22$, $2x + 4y + 3z = 5$, and $3x + 6y + 10z = 13$.

Answer

Solution is: $[x = 1 - 2t, z = 1, y = t]$

? Exercise 3.9.33

Find the solution to the system of equations, $3x - y + 4z = 6$, $y + 8z = 0$, and $-2x + y = -4$.

Answer

Solution is: $[x = 2 - 4t, y = -8t, z = t]$

? Exercise 3.9.34

Find the solution to the system of equations, $9x - 2y + 4z = -17$, $13x - 3y + 6z = -25$, and $-2x - z = 3$.

Answer

Solution is: $[x = -1, y = 2, z = -1]$

? Exercise 3.9.35

Find the solution to the system of equations, $65x + 84y + 16z = 546$, $81x + 105y + 20z = 682$, and $84x + 110y + 21z = 713$.

Answer

Solution is: $[x = 2, y = 4, z = 5]$

? Exercise 3.9.36

Find the solution to the system of equations, $8x + 2y + 3z = -3$, $8x + 3y + 3z = -1$, and $4x + y + 3z = -9$.

Answer

Solution is: $[x = 1, y = 2, z = -5]$

? Exercise 3.9.37

Find the solution to the system of equations, $-8x + 2y + 5z = 18$, $-8x + 3y + 5z = 13$, and $-4x + y + 5z = 19$.

Answer

Solution is: $[x = -1, y = -5, z = 4]$

? Exercise 3.9.38

Find the solution to the system of equations, $3x - y - 2z = 3$, $y - 4z = 0$, and $-2x + y = -2$.

Answer

Solution is: $[x = 2t + 1, y = 4t, z = t]$

? Exercise 3.9.39

Find the solution to the system of equations, $-9x + 15y = 66$, $-11x + 18y = 79$, $-x + y = 4$, and $z = 3$.

Answer

Solution is: $[x = 1, y = 5, z = 3]$

? Exercise 3.9.40

Find the solution to the system of equations, $-19x + 8y = -108$, $-71x + 30y = -404$, $-2x + y = -12$, $4x + z = 14$.

Answer

Solution is: $[x = 4, y = -4, z = -2]$

? Exercise 3.9.41

Suppose a system of equations has fewer equations than variables and you have found a solution to this system of equations. Is it possible that your solution is the only one? Explain.

Answer

No. Consider $x + y + z = 2$ and $x + y + z = 1$.

? Exercise 3.9.42

Suppose a system of linear equations has a 2×4 augmented matrix and the last column is a pivot column. Could the system of linear equations be consistent? Explain.

Answer

No. This would lead to $0 = 1$.

? Exercise 3.9.43

Suppose the coefficient matrix of a system of n equations with n variables has the property that every column is a pivot column. Does it follow that the system of equations must have a solution? If so, must the solution be unique? Explain.

Answer

Yes. It has a unique solution.

? Exercise 3.9.44

Suppose there is a unique solution to a system of linear equations. What must be true of the pivot columns in the augmented matrix?

Answer

The last column must not be a pivot column. The remaining columns must each be pivot columns.

? Exercise 3.9.45

The steady state temperature, u , of a plate solves Laplace's equation, $\Delta u = 0$. One way to approximate the solution is to divide the plate into a square mesh and require the temperature at each node to equal the average of the temperature at the four adjacent nodes. In the following picture, the numbers represent the observed temperature at the indicated nodes. Find the temperature at the interior nodes, indicated by x, y, z , and w . One of the equations is $z = 0.05in\frac{1}{4}(10 + 0 + w + x)$.

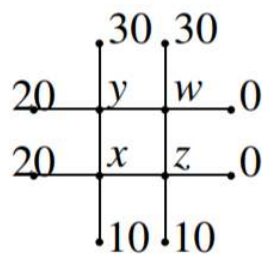


Figure 3.9.1

Answer

You need $\frac{1}{4}(20 + 30 + w + x) - y = 0$
 $\frac{1}{4}(y + 30 + 0 + z) - w = 0$, Solution is: $[w = 15, x = 15, y = 20, z = 10]$.
 $\frac{1}{4}(20 + y + z + 10) - x = 0$
 $\frac{1}{4}(x + w + 0 + 10) - z = 0$

? Exercise 3.9.46

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & -16 & -1 & -5 \\ 1 & -4 & 0 & -1 \\ 1 & -4 & -1 & -2 \end{bmatrix}$$

? Exercise 3.9.47

Find the rank of the following matrix.

$$\begin{bmatrix} 3 & 6 & 5 & 12 \\ 1 & 2 & 2 & 5 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

? Exercise 3.9.48

Find the rank of the following matrix.

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 3 \\ 1 & 4 & 1 & 0 & -8 \\ 1 & 4 & 0 & 1 & 2 \\ -1 & -4 & 0 & -1 & -2 \end{bmatrix}$$

? Exercise 3.9.49

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & -4 & 3 & -9 \\ 1 & -1 & 1 & -2 \\ 1 & -1 & 0 & -3 \end{bmatrix}$$

? Exercise 3.9.50

Find the rank of the following matrix.

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & 1 & 7 \end{bmatrix}$$

? Exercise 3.9.51

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & 15 & 29 \\ 1 & 4 & 8 \\ 1 & 3 & 5 \\ 3 & 9 & 15 \end{bmatrix}$$

? Exercise 3.9.52

Find the rank of the following matrix.

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 & -18 \\ 1 & 2 & 2 & -1 & -11 \\ -1 & -2 & -2 & 1 & 11 \end{bmatrix}$$

? Exercise 3.9.53

Find the rank of the following matrix.

$$\begin{bmatrix} 1 & -2 & 0 & 3 & 11 \\ 1 & -2 & 0 & 4 & 15 \\ 1 & -2 & 0 & 3 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

? Exercise 3.9.54

Find the rank of the following matrix.

$$\begin{bmatrix} -2 & -3 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ -3 & 0 & -3 \end{bmatrix}$$

? Exercise 3.9.55

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & 4 & 20 & -1 & 17 \\ 1 & 1 & 5 & 0 & 5 \\ 1 & 1 & 5 & -1 & 2 \\ 3 & 3 & 15 & -3 & 6 \end{bmatrix}$$

? Exercise 3.9.56

Find the rank of the following matrix.

$$\begin{bmatrix} -1 & 3 & 4 & -3 & 8 \\ 1 & -3 & -4 & 2 & -5 \\ 1 & -3 & -4 & 1 & -2 \\ -2 & 6 & 8 & -2 & 4 \end{bmatrix}$$

? Exercise 3.9.57

Suppose A is an $m \times n$ matrix. Explain why the rank of A is always no larger than $\min(m, n)$.

Answer

It is because you cannot have more than $\min(m, n)$ nonzero rows in the reduced row-echelon form. Recall that the number of pivot columns is the same as the number of nonzero rows from the description of this reduced row-echelon form.

? Exercise 3.9.58

State whether each of the following sets of data are possible for the matrix equation $AX = B$. If possible, describe the solution set. That is, tell whether there exists a unique solution, no solution or infinitely many solutions. Here, $[A|B]$ denotes the augmented matrix.

- A is a 5×6 matrix, $\text{rank}(A) = 4$ and $\text{rank}[A|B] = 4$.
- A is a 3×4 matrix, $\text{rank}(A) = 3$ and $\text{rank}[A|B] = 2$.
- A is a 4×2 matrix, $\text{rank}(A) = 4$ and $\text{rank}[A|B] = 4$.
- A is a 5×5 matrix, $\text{rank}(A) = 4$ and $\text{rank}[A|B] = 5$.
- A is a 4×2 matrix, $\text{rank}(A) = 2$ and $\text{rank}[A|B] = 2$.

Answer

- This says B is in the span of four of the columns. Thus the columns are not independent. Infinite solution set.
- This surely can't happen. If you add in another column, the rank does not get smaller.
- This says B is in the span of the columns and the columns must be independent. You can't have the rank equal 4 if you only have two columns.
- This says B is not in the span of the columns. In this case, there is no solution to the system of equations represented by the augmented matrix.
- In this case, there is a unique solution since the columns of A are independent.

? Exercise 3.9.59

Consider the system $-5x + 2y - z = 0$ and $-5x - 2y - z = 0$. Both equations equal zero and so $-5x + 2y - z = -5x - 2y - z$ which is equivalent to $y = 0$. Does it follow that x and z can equal anything? Notice that when $x = 1$, $z = -4$, and $y = 0$ are plugged in to the equations, the equations do not equal 0. Why?

Answer

These are not legitimate row operations. They do not preserve the solution set of the system.

? Exercise 3.9.60

Balance the following chemical reactions.

- $KNO_3 + H_2CO_3 \rightarrow K_2CO_3 + HNO_3$
- $AgI + Na_2S \rightarrow Ag_2S + NaI$

- c. $Ba_3N_2 + H_2O \rightarrow Ba(OH)_2 + NH_3$
 d. $CaCl_2 + Na_3PO_4 \rightarrow Ca_3(PO_4)_2 + NaCl$

? Exercise 3.9.61

In the section on dimensionless variables it was observed that $\rho V^2 AB$ has the units of force. Describe a systematic way to obtain such combinations of the variables which will yield something which has the units of force.

? Exercise 3.9.62

Consider the following diagram of four circuits.

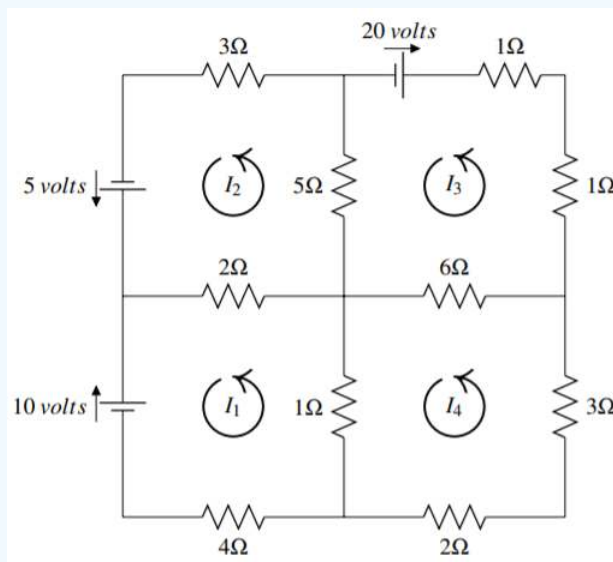


Figure 3.9.2

The current in amps in the four circuits is denoted by I_1, I_2, I_3, I_4 and it is understood that the motion is in the counter clockwise direction. If I_k ends up being negative, then it just means the current flows in the clockwise direction.

In the above diagram, the top left circuit should give the equation

$$2I_2 - 2I_1 + 5I_2 - 5I_3 + 3I_2 = 5$$

For the circuit on the lower left, you should have

$$4I_1 + I_1 - I_4 + 2I_1 - 2I_2 = -10$$

Write equations for each of the other two circuits and then give a solution to the resulting system of equations.

Answer

The other two equations are

$$\begin{aligned} 6I_3 - 6I_4 + I_3 + I_3 + 5I_3 - 5I_2 &= -20 \\ 2I_4 + 3I_4 + 6I_4 - 6I_3 + I_4 - I_1 &= 0 \end{aligned}$$

Then the system is

$$\begin{aligned} 2I_2 - 2I_1 + 5I_2 - 5I_3 + 3I_2 &= 5 \\ 4I_1 + I_1 - I_4 + 2I_1 - 2I_2 &= -10 \\ 6I_3 - 6I_4 + I_3 + I_3 + 5I_3 - 5I_2 &= -20 \\ 2I_4 + 3I_4 + 6I_4 - 6I_3 + I_4 - I_1 &= 0 \end{aligned}$$

The solution is:

$$I_1 = -\frac{750}{373}$$

$$I_2 = -\frac{1421}{1119}$$

$$I_3 = -\frac{3061}{1119}$$

$$I_4 = -\frac{1718}{1119}$$

? Exercise 3.9.63

Consider the following diagram of three circuits.

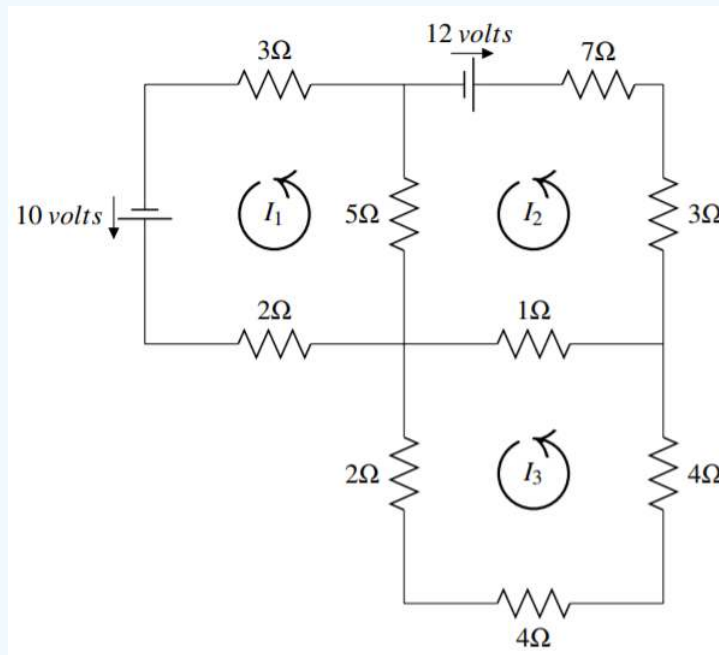


Figure 3.9.3

The current in amps in the four circuits is denoted by I_1 , I_2 , I_3 and it is understood that the motion is in the counter clockwise direction. If I_k ends up being negative, then it just means the current flows in the clockwise direction.

Find I_1 , I_2 , I_3 .

Answer

You have

$$\begin{aligned} 2I_1 + 5I_1 + 3I_1 - 5I_2 &= 10 \\ I_2 - I_3 + 3I_2 + 7I_2 + 5I_2 - 5I_1 &= -12 \\ 2I_3 + 4I_3 + 4I_3 + I_3 - I_2 &= 0 \end{aligned}$$

Simplifying this yields

$$\begin{aligned} 10I_1 - 5I_2 &= 10 \\ -5I_1 + 16I_2 - I_3 &= -12 \\ -I_2 + 11I_3 &= 0 \end{aligned}$$

The solution is given by

$$I_1 = \frac{218}{295}, I_2 = -\frac{154}{295}, I_3 = -\frac{14}{295}$$

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CHAPTER OVERVIEW

4: Matrices

- 4.1: Matrix Arithmetic
- 4.2: Multiplication of Matrices
- 4.3: The ij th Entry of a Product
- 4.4: Properties of Matrix Multiplication
- 4.5: The Transpose
- 4.6: The Identity and Inverses
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4.1: Matrix Arithmetic

Outcomes

- A. Perform the matrix operations of matrix addition, scalar multiplication, transposition and matrix multiplication. Identify when these operations are not defined. Represent these operations in terms of the entries of a matrix.
- B. Prove algebraic properties for matrix addition, scalar multiplication, transposition, and matrix multiplication. Apply these properties to manipulate an algebraic expression involving matrices.
- C. Compute the inverse of a matrix using row operations, and prove identities involving matrix inverses.
- D. Solve a linear system using matrix algebra.
- E. Use multiplication by an elementary matrix to apply row operations.
- F. Write a matrix as a product of elementary matrices.

You have now solved systems of equations by writing them in terms of an augmented matrix and then doing row operations on this augmented matrix. It turns out that matrices are important not only for systems of equations but also in many applications.

Recall that a **matrix** is a rectangular array of numbers. Several of them are referred to as **matrices**. For example, here is a matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix} \quad (4.1.1)$$

Recall that the size or dimension of a matrix is defined as $m \times n$ where m is the number of rows and n is the number of columns. The above matrix is a 3×4 matrix because there are three rows and four columns. You can remember the columns are like columns in a Greek temple. They stand upright while the rows lay flat like rows made by a tractor in a plowed field.

When specifying the size of a matrix, you always list the number of rows before the number of columns. You might remember that you always list the rows before the columns by using the phrase **Rowman Catholic**.

Consider the following definition.

A matrix A which has size $n \times n$ is called a **square matrix**. In other words, A is a square matrix if it has the same number of rows and columns.

There is some notation specific to matrices which we now introduce. We denote the columns of a matrix A by A_j as follows

$$A = [A_1 \quad A_2 \quad \cdots \quad A_n]$$

Therefore, A_j is the j^{th} column of A , when counted from left to right.

The individual elements of the matrix are called **entries** or **components** of A . Elements of the matrix are identified according to their position. The **(i,j)-entry** of a matrix is the entry in the i^{th} row and j^{th} column. For example, in the matrix (4.1.1) above, 8 is in position (2,3) (and is called the (2,3)-entry) because it is in the second row and the third column.

In order to remember which matrix we are speaking of, we will denote the entry in the i^{th} row and the j^{th} column of matrix A by a_{ij} . Then, we can write A in terms of its entries, as $A = [a_{ij}]$. Using this notation on the matrix in (4.1.1), $a_{23} = 8$, $a_{32} = -9$, $a_{12} = 2$, etc.

There are various operations which are done on matrices of appropriate sizes. Matrices can be added to and subtracted from other matrices, multiplied by a scalar, and multiplied by other matrices. We will never divide a matrix by another matrix, but we will see later how matrix inverses play a similar role.

In doing arithmetic with matrices, we often define the action by what happens in terms of the entries (or components) of the matrices. Before looking at these operations in depth, consider a few general definitions.

The $m \times n$ **zero matrix** is the $m \times n$ matrix having every entry equal to zero. It is denoted by 0 .

One possible zero matrix is shown in the following example.

✓ Example 4.1.1: The Zero Matrix

The 2×3 zero matrix is $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Note there is a 2×3 zero matrix, a 3×4 zero matrix, etc. In fact there is a zero matrix for every size!

Let A and B be two $m \times n$ matrices. Then $A = B$ means that for $A = [a_{ij}]$ and $B = [b_{ij}]$, $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

In other words, two matrices are equal exactly when they are the same size and the corresponding entries are identical. Thus

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

because they are different sizes. Also,

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

because, although they are the same size, their corresponding entries are not identical.

In the following section, we explore addition of matrices.

Addition of Matrices

When adding matrices, all matrices in the sum need have the same size. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} -1 & 4 & 8 \\ 2 & 8 & 5 \end{bmatrix}$$

cannot be added, as one has size 3×2 while the other has size 2×3 .

However, the addition

$$\begin{bmatrix} 4 & 6 & 3 \\ 5 & 0 & 4 \\ 11 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 0 \\ 4 & -4 & 14 \\ 1 & 2 & 6 \end{bmatrix}$$

is possible.

The formal definition is as follows.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then $A + B = C$ where C is the $m \times n$ matrix $C = [c_{ij}]$ defined by

$$c_{ij} = a_{ij} + b_{ij}$$

This definition tells us that when adding matrices, we simply add corresponding entries of the matrices. This is demonstrated in the next example.

Add the following matrices, if possible.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix}$$

Solution

Notice that both A and B are of size 2×3 . Since A and B are of the same size, the addition is possible. Using [Definition 4.1.4](#), the addition is done as follows.

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+2 & 3+3 \\ 1+(-6) & 0+2 & 4+1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 6 \\ -5 & 2 & 5 \end{bmatrix}$$

Addition of matrices obeys very much the same properties as normal addition with numbers. Note that when we write for example $A + B$ then we assume that both matrices are of equal size so that the operation is indeed possible.

Proposition 4.1.1: Properties of Matrix Addition

Let A , B and C be matrices. Then, the following properties hold.

- Commutative Law of Addition

$$A + B = B + A \quad (4.1.2)$$

- Associative Law of Addition

$$(A + B) + C = A + (B + C) \quad (4.1.3)$$

- Existence of an Additive Identity

There exists a zero matrix 0 such that

$$A + 0 = A \quad (4.1.4)$$

- Existence of an Additive Inverse

There exists a matrix $-A$ such that

$$A + (-A) = 0 \quad (4.1.5)$$

Proof

Consider the Commutative Law of Addition given in [\(4.1.2\)](#). Let A , B , C , and D be matrices such that $A + B = C$ and $B + A = D$. We want to show that $D = C$. To do so, we will use the definition of matrix addition given in [Definition 4.1.4](#). Now,

$$c_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = d_{ij}$$

Therefore, $C = D$ because the i_j^{th} entries are the same for all i and j . Note that the conclusion follows from the commutative law of addition of numbers, which says that if a and b are two numbers, then $a + b = b + a$. The proof of the other results are similar, and are left as an exercise.

We call the zero matrix in [\(4.1.4\)](#) the **additive identity**. Similarly, we call the matrix $-A$ in [\(4.1.5\)](#) the **additive inverse**. $-A$ is defined to equal $(-1)A = [-a_{ij}]$. In other words, every entry of A is multiplied by -1 .

In the next section we will study scalar multiplication in more depth to understand what is meant by $(-1)A$.

Scalar Multiplication of Matrices

Recall that we use the word *scalar* when referring to numbers. Therefore, *scalar multiplication of a matrix* is the multiplication of a matrix by a number. To illustrate this concept, consider the following example in which a matrix is multiplied by the scalar 3.

$$3 \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 & 12 \\ 15 & 6 & 24 & 21 \\ 18 & -27 & 3 & 6 \end{bmatrix}$$

The new matrix is obtained by multiplying every entry of the original matrix by the given scalar.

The formal definition of scalar multiplication is as follows.

 Definition 4.1.5: Scalar Multiplication of Matrices

If $A = [a_{ij}]$ and k is a scalar, then $kA = [ka_{ij}]$.

Below is a video on matrix addition, subtraction, and scalar multiplication.



Consider the following example.

 Example 4.1.3: Effect of Multiplication by a Scalar

Find the result of multiplying the following matrix A by 7.

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix}$$

Solution

By [Definition 4.1.5](#), we multiply each element of A by 7. Therefore,

$$7A = 7 \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 7(2) & 7(0) \\ 7(1) & 7(-4) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 7 & -28 \end{bmatrix}$$

Similarly to addition of matrices, there are several properties of scalar multiplication which hold.

 Proposition 4.1.2: Properties of Scalar Multiplication

Let A, B be matrices, and k, p be scalars. Then, the following properties hold.

- Distributive Law over Matrix Addition

$$k(A + B) = kA + kB$$

- Distributive Law over Scalar Addition

$$(k + p)A = kA + pA$$

- Associative Law for Scalar Multiplication

$$k(pA) = (kp)A$$

- Rule for Multiplication by 1

$$1A = A$$

Proof

The proof of this proposition is similar to the proof of [Proposition 4.1.1](#) and is left an exercise to the reader.

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4.2: Multiplication of Matrices

The next important matrix operation we will explore is multiplication of matrices. The operation of matrix multiplication is one of the most important and useful of the matrix operations. Throughout this section, we will also demonstrate how matrix multiplication relates to linear systems of equations.

First, we provide a formal definition of row and column vectors.

Definition 4.2.1: Row and Column Vectors

Matrices of size $n \times 1$ or $1 \times n$ are called **vectors**. If X is such a matrix, then we write x_i to denote the entry of X in the i^{th} row of a column matrix, or the i^{th} column of a row matrix.

The $n \times 1$ matrix

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is called a **column vector**. The $1 \times n$ matrix

$$X = [x_1 \quad \cdots \quad x_n]$$

is called a **row vector**.

We may simply use the term **vector** throughout this text to refer to either a column or row vector. If we do so, the context will make it clear which we are referring to.

In this chapter, we will again use the notion of linear combination of vectors as in [Definition 9.2.2](#). In this context, a linear combination is a sum consisting of vectors multiplied by scalars. For example,

$$\begin{bmatrix} 50 \\ 122 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

is a linear combination of three vectors.

It turns out that we can express any system of linear equations as a linear combination of vectors. In fact, the vectors that we will use are just the columns of the corresponding augmented matrix!

Definition 4.2.2: The Vector Form of a System of Linear Equations

Suppose we have a system of equations given by

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

$$\vdots$$

$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

We can express this system in **vector form** which is as follows:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Notice that each vector used here is one column from the corresponding augmented matrix. There is one vector for each variable in the system, along with the constant vector.

The first important form of matrix multiplication is multiplying a matrix by a vector. Consider the product given by

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

We will soon see that this equals

$$7 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 50 \\ 122 \end{bmatrix}$$

In general terms,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}$$

Thus you take x_1 times the first column, add to x_2 times the second column, and finally x_3 times the third column. The above sum is a linear combination of the columns of the matrix. When you multiply a matrix on the left by a vector on the right, the numbers making up the vector are just the scalars to be used in the linear combination of the columns as illustrated above.

Here is the formal definition of how to multiply an $m \times n$ matrix by an $n \times 1$ column vector.

Definition 4.2.3: Multiplication of Vector by Matrix

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let X be an $n \times 1$ matrix given by

$$A = [A_1 \cdots A_n], X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Then the product AX is the $m \times 1$ column vector which equals the following linear combination of the columns of A :

$$x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = \sum_{j=1}^n x_j A_j$$

If we write the columns of A in terms of their entries, they are of the form

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

Then, we can write the product AX as

$$AX = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Note that multiplication of an $m \times n$ matrix and an $n \times 1$ vector produces an $m \times 1$ vector.

Here is an example.

✓ Example 4.2.1: A Vector Multiplied by a Matrix

Compute the product AX for

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & -2 \\ 2 & 1 & 4 & 1 \end{bmatrix}, X = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Solution

We will use [Definition 4.2.3](#) to compute the product. Therefore, we compute the product AX as follows.

$$\begin{aligned} & 1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix} \end{aligned}$$

Using the above operation, we can also write a system of linear equations in **matrix form**. In this form, we express the system as a matrix multiplied by a vector. Consider the following definition.

Definition 4.2.4: The Matrix Form of a System of Linear Equations

Suppose we have a system of equations given by

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

Then we can express this system in **matrix form** as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The expression $AX = B$ is also known as the **Matrix Form** of the corresponding system of linear equations. The matrix A is simply the coefficient matrix of the system, the vector X is the column vector constructed from the variables of the system, and finally the vector B is the column vector constructed from the constants of the system. It is important to note that any system of linear equations can be written in this form.

Notice that if we write a homogeneous system of equations in matrix form, it would have the form $AX = 0$, for the zero vector 0 .

You can see from this definition that a vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

will satisfy the equation $AX = B$ only when the entries $x_1, x_2, \dots, x_n$ of the vector X are solutions to the original system.

Now that we have examined how to multiply a matrix by a vector, we wish to consider the case where we multiply two matrices of more general sizes, although these sizes still need to be appropriate as we will see. For example, in [Example 4.2.1](#), we multiplied a 3×4 matrix by a 4×1 vector. We want to investigate how to multiply other sizes of matrices.

We have not yet given any conditions on when matrix multiplication is possible! For matrices A and B , in order to form the product AB , the number of columns of A must equal the number of rows of B . Consider a product AB where A has size $m \times n$ and B has size $n \times p$. Then, the product in terms of size of matrices is given by

$$(m \times \overbrace{n}^{\text{these must match!}} (n \times p)) = m \times p$$

Note the two outside numbers give the size of the product. One of the most important rules regarding matrix multiplication is the following. If the two middle numbers don't match, you can't multiply the matrices!

When the number of columns of A equals the number of rows of B the two matrices are said to be **conformable** and the product AB is obtained as follows.

Definition 4.2.5: Multiplication of Two Matrices

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix of the form

$$B = [B_1 \cdots B_p]$$

where $B_1, \dots, B_p$ are the $n \times 1$ columns of B . Then the $m \times p$ matrix AB is defined as follows:

$$AB = A[B_1 \cdots B_p] = [(AB)_1 \cdots (AB)_p]$$

where $(AB)_k$ is an $m \times 1$ matrix or column vector which gives the k^{th} column of AB .

Below is a video on matrix multiplication.



Consider the following example.

Example 4.2.2: Multiplying Two Matrices

Find AB if possible.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

Solution

The first thing you need to verify when calculating a product is whether the multiplication is possible. The first matrix has size 2×3 and the second matrix has size 3×3 . The inside numbers are equal, so A and B are conformable matrices. According to

the above discussion AB will be a 2×3 matrix. [Definition 4.2.5](#) gives us a way to calculate each column of AB , as follows.

$$\left[\begin{array}{c|c|c} \text{First column} & \text{Second column} & \text{Third column} \\ \hline \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \end{array} \right]$$

You know how to multiply a matrix times a vector, using [Definition 4.2.3](#) for each of the three columns. Thus

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 9 & 3 \\ -2 & 7 & 3 \end{bmatrix}$$

Since vectors are simply $n \times 1$ or $1 \times m$ matrices, we can also multiply a vector by another vector.

✓ Example 4.2.3: Vector Times Vector Multiplication

Multiply if possible $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} [1 \ 2 \ 1 \ 0]$.

Solution

In this case we are multiplying a matrix of size 3×1 by a matrix of size 1×4 . The inside numbers match so the product is defined. Note that the product will be a matrix of size 3×4 . Using [Definition 4.2.5](#), we can compute this product as follows

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} [1 \ 2 \ 1 \ 0] = \left[\begin{array}{c|c|c|c} \text{First column} & \text{Second column} & \text{Third column} & \text{Fourth column} \\ \hline \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} [1], & \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} [2], & \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} [1], & \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} [0] \end{array} \right]$$

You can use [Definition 4.2.3](#) to verify that this product is

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

✓ Example 4.2.4: A Multiplication Which is Not Defined

Find BA if possible.

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

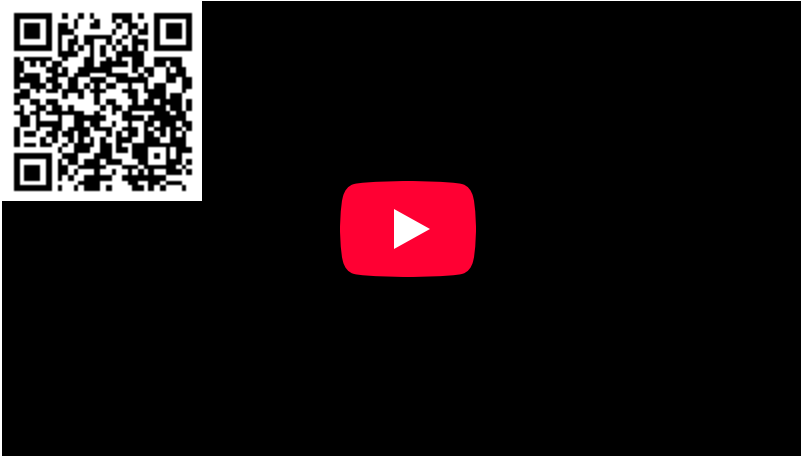
Solution

First check if it is possible. This product is of the form $(3 \times 3)(2 \times 3)$. The inside numbers do not match and so you can't do this multiplication.

In this case, we say that the multiplication is not defined. Notice that these are the same matrices which we used in [Example 4.2.2](#). In this example, we tried to calculate BA instead of AB . This demonstrates another property of matrix multiplication. While the product AB maybe be defined, we cannot assume that the product BA will be possible. Therefore, it is important to always check that the product is defined before carrying out any calculations.

Earlier, we defined the zero matrix 0 to be the matrix (of appropriate size) containing zeros in all entries. Consider the following example for multiplication by the zero matrix.

Below is a video determining if matrix multiplication is possible.



✓ Example 4.2.5: Multiplication by the Zero Matrix

Compute the product $A0$ for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and the 2×2 zero matrix given by

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution

In this product, we compute

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $A0 = 0$.

Notice that we could also multiply A by the 2×1 zero vector given by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The result would be the 2×1 zero vector. Therefore, it is always the case that $A0 = 0$, for an appropriately sized zero matrix or vector.

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4.3: The ij th Entry of a Product

In previous sections, we used the entries of a matrix to describe the action of matrix addition and scalar multiplication. We can also study matrix multiplication using the entries of matrices.

What is the ij th entry of AB ? It is the entry in the i th row and the j th column of the product AB .

Now if A is $m \times n$ and B is $n \times p$, then we know that the product AB has the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}$$

The j th column of AB is of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

which is an $m \times 1$ column vector. It is calculated by

$$b_{1j} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + b_{2j} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + b_{nj} \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Therefore, the ij th entry is the entry in row i of this vector. This is computed by

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^n a_{ik}b_{kj}$$

The following is the formal definition for the ij th entry of a product of matrices.

Definition 4.3.1: The ij th Entry of a Product

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let $B = [b_{ij}]$ be an $n \times p$ matrix. Then AB is an $m \times p$ matrix and the (i, j) -entry of AB is defined as

$$(AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$$

Another way to write this is

$$(AB)_{ij} = [a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

In other words, to find the (i, j) -entry of the product AB , or $(AB)_{ij}$, you multiply the i th row of A , on the left by the j th column of B . To express AB in terms of its entries, we write $AB = [(AB)_{ij}]$.

Consider the following example.

✓ Example 4.3.1: The Entries of a Product

Compute AB if possible. If it is, find the $(3, 2)$ -entry of AB using [Definition 4.3.1](#).

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 1 \\ 7 & 6 & 2 \end{bmatrix}$$

Solution

First check if the product is possible. It is of the form $(3 \times 2)(2 \times 3)$ and since the inside numbers match, it is possible to do the multiplication. The result should be a 3×3 matrix. We can first compute AB :

$$\left[\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]$$

where the commas separate the columns in the resulting product. Thus the above product equals

$$\begin{bmatrix} 16 & 15 & 5 \\ 13 & 15 & 5 \\ 46 & 42 & 14 \end{bmatrix}$$

which is a 3×3 matrix as desired. Thus, the $(3, 2)$ -entry equals 42.

Now using [Definition 4.3.1](#), we can find that the $(3, 2)$ -entry equals

$$\begin{aligned} \sum_{k=1}^2 a_{3k}b_{k2} &= a_{31}b_{12} + a_{32}b_{22} \\ &= 2 \times 3 + 6 \times 6 = 42 \end{aligned}$$

Consulting our result for AB above, this is correct!

You may wish to use this method to verify that the rest of the entries in AB are correct.

Here is another example.

✓ Example 4.3.2: Finding the Entries of a Product

Determine if the product AB is defined. If it is, find the $(2, 1)$ -entry of the product.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 7 & 6 & 2 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}$$

Solution

This product is of the form $(3 \times 3)(3 \times 2)$. The middle numbers match so the matrices are conformable and it is possible to compute the product.

We want to find the $(2, 1)$ -entry of AB , that is, the entry in the second row and first column of the product. We will use [Definition 4.3.1](#), which states

$$(AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$$

In this case, $n = 3$, $i = 2$ and $j = 1$. Hence the $(2, 1)$ -entry is found by computing

$$(AB)_{21} = \sum_{k=1}^3 a_{2k}b_{k1} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$$

Substituting in the appropriate values, this product becomes

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 7 & 6 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 1 \times 7 + 3 \times 6 + 2 \times 2 = 29$$

Hence, $(AB)_{21} = 29$.

You should take a moment to find a few other entries of AB . You can multiply the matrices to check that your answers are correct. The product AB is given by

$$AB = \begin{bmatrix} 13 & 13 \\ 29 & 32 \\ 0 & 0 \end{bmatrix}$$

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