CALCULUS



Robert Reed
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MIT & University of Wisconsin-Stevens Point

Calculus

Gilbert Strang & Edwin Herman

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5.0: Prelude to Integration

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5.1: APPROXIMATING AREAS

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5.1E: Exercises for Section 5.1

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5.2: THE DEFINITE INTEGRAL

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5.2E: Exercises for Section 5.2

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5.2E: Exercises for Section 5.2

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5.3: THE FUNDAMENTAL THEOREM OF CALCULUS

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5.3E: Exercises for Section 5.3

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5.3E: Exercises for Section 5.3

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5.4: INTEGRATION FORMULAS AND THE NET CHANGE THEOREM

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5.4E: Exercises for Section 5.4

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5.5: SUBSTITUTION

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5.5E: Exercises for Section 5.5

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5.5E: Exercises for Section 5.5

1) Why is u-substitution referred to as a *change of variable*?

2) If $f = g \circ h$, when reversing the chain rule, $\frac{d}{dx}(g \circ h)(x) = g'(h(x))h'(x)$, should you take $u = g(x)$ or $u = h(x)$?

Answer

$u = h(x)$

In exercises 3 - 7, verify each identity using differentiation. Then, using the indicated u-substitution, identify f such that the integral takes the form $\int f(u)\ du$.

3) $\int x\sqrt{x+1}\ dx = \frac{2}{15}(x+1)^{3/2}(3x-2)+C; \quad u = x+1$

4) $\int \frac{x^2}{\sqrt{x-1}}\ dx = \frac{2}{15}\sqrt{x-1}(3x^2+4x+8)+C, \quad (x>1); \quad u = x-1$

Answer

$f(u) = \frac{(u+1)^2}{\sqrt{u}}$

5) $\int x\sqrt{4x^2+9}\ dx = \frac{1}{12}(4x^2+9)^{3/2}+C; \quad u = 4x^2+9$

6) $\int \frac{x}{\sqrt{4x^2+9}}\ dx = \frac{1}{4}\sqrt{4x^2+9}+C; \quad u = 4x^2+9$

Answer

$du = 8x\ dx; \quad f(u) = \frac{1}{8\sqrt{u}}$

7) $\int \frac{x}{(4x^2+9)^2}\ dx = -\frac{1}{8(4x^2+9)}+C; \quad u = 4x^2+9$

In exercises 8 - 17, find the antiderivative using the indicated substitution.

8) $\int (x+1)^4\ dx; \quad u = x+1$

Answer

$\int (x+1)^4\ dx = \frac{1}{5}(x+1)^5+C$

9) $\int (x-1)^5\ dx; \quad u = x-1$

10) $\int (2x-3)^{-7}\ dx; \quad u = 2x-3$

Answer

$\int (2x-3)^{-7}\ dx = -\frac{1}{12(2x-3)^6}+C$

11) $\int (3x-2)^{-11}\ dx; \quad u = 3x-2$

12) $\int \frac{x}{\sqrt{x^2+1}}\ dx; \quad u = x^2+1$

Answer

$$\int \frac{x}{\sqrt{x^2+1}}\ dx = \sqrt{x^2+1} + C$$

13) $\int \frac{x}{\sqrt{1-x^2}}\ dx; \quad u = 1 - x^2$

14) $\int (x-1)(x^2-2x)^3\ dx; \quad u = x^2 - 2x$

Answer

$$\int (x-1)(x^2-2x)^3\ dx = \frac{1}{8}(x^2-2x)^4 + C$$

15) $\int (x^2-2x)(x^3-3x^2)^2\ dx; \quad u = x^3 = 3x^2$

16) $\int \cos^3\theta\ d\theta; \quad u = \sin\theta$ (Hint: $\cos^2\theta = 1 - \sin^2\theta$)

Answer

$$\int \cos^3\theta\ d\theta = \sin\theta - \frac{\sin^3\theta}{3} + C$$

17) $\int \sin^3\theta\ d\theta; \quad u = \cos\theta$ (Hint: $\sin^2\theta = 1 - \cos^2\theta$)

In exercises 18 - 34, use a suitable change of variables to determine the indefinite integral.

18) $\int t(1-t^2)^{10} dt$

Answer

$$\int t(1-t^2)^{10} dt = -\frac{1}{22}(1-t^2)^{11} + C$$

19) $\int x(1-x)^{99}\ dx$

Answer

$$\int x(1-x)^{99}\, dx = \frac{(1-x)^{101}}{101} - \frac{(1-x)^{100}}{100} + C$$
$$= -\frac{(1-x)^{100}}{10100}\left[100x+1\right] + C$$

20) $\int (11x-7)^{-3}\ dx$

Answer

$$\int (11x-7)^{-3}\ dx = -\frac{1}{22(11x-7)^2} + C$$

21) $\int (7x-11)^4\ dx$

22) $\int \cos^3\theta \sin\theta\ d\theta$

Answer

$$\int \cos^3\theta \sin\theta\ d\theta = -\frac{\cos^4\theta}{4} + C$$

23) $\int \sin^7\theta \cos\theta\ d\theta$

24) $\int \cos^2(\pi t)\sin(\pi t)\ dt$

Answer

$$\int \cos^2(\pi t)\sin(\pi t)\ dt = -\frac{cos^3(\pi t)}{3\pi} + C$$

25) $\int \sin^2 x \cos^3 x\ dx$ (Hint: $\sin^2 x + \cos^2 x = 1$)

26) $\int t\sin(t^2)\cos(t^2)\ dt$

Answer

$$\int t\sin(t^2)\cos(t^2)\ dt = -\frac{1}{4}\cos^2(t^2) + C$$

27) $\int t^2\cos^2(t^3)\sin(t^3)\ dt$

28) $\int \frac{x^2}{(x^3-3)^2}\ dx$

Answer

$$\int \frac{x^2}{(x^3-3)^2}\ dx = -\frac{1}{3(x^3-3)} + C$$

29) $\int \frac{x^3}{\sqrt{1-x^2}}\ dx$

Answer

$$\int \frac{x^3}{\sqrt{1-x^2}}\ dx = \frac{1}{3}(1-x^2) - (1-x^2)^{1/2} + C$$

30) $\int \frac{y^5}{(1-y^3)^{3/2}}\ dy$

Answer

$$\int \frac{y^5}{(1-y^3)^{3/2}}\ dy = \frac{2(2-y^3)}{3\sqrt{1-y^3}} + C$$

31) $\int \cos\theta(1-\cos\theta)^{99}\sin\theta\ d\theta$

32) $\int (1-\cos^3\theta)^{10}\cos^2\theta\sin\theta\ d\theta$

Answer

$$\int (1-\cos^3\theta)^{10}\cos^2\theta\sin\theta\ d\theta = \frac{1}{33}(1-\cos^3\theta)^{11} + C$$

33) $\int (\cos\theta - 1)(\cos^2\theta - 2\cos\theta)^3\sin\theta\ d\theta$

34) $\int (\sin^2\theta - 2\sin\theta)(\sin^3\theta - 3\sin^2\theta)^3\cos\theta\ d\theta$

Answer

$$\int (\sin^2\theta - 2\sin\theta)(\sin^3\theta - 3\sin^2\theta)^3\cos\theta\ d\theta = \frac{1}{12}(\sin^3\theta - 3\sin^2\theta)^4 + C$$

In exercises 35 - 38, use a calculator to estimate the area under the curve using left Riemann sums with 50 terms, then use substitution to solve for the exact answer.

35) [T] $y = 3(1-x)^2$ over $[0, 2]$

36) [T] $y = x(1-x^2)^3$ over $[-1, 2]$

Answer

$L_{50} = -8.5779$. The exact area is $\frac{-81}{8}$ units^2.

37) [T] $y = \sin x(1-\cos x)^2$ over $[0, \pi]$

38) [T] $y = \dfrac{x}{(x^2+1)^2}$ over $[-1, 1]$

Answer

$L_{50} = -0.006399$. The exact area is 0.

In exercises 39 - 44, use a change of variables to evaluate the definite integral.

39) $\displaystyle\int_0^1 x\sqrt{1-x^2}\ dx$

40) $\displaystyle\int_0^1 \frac{x}{\sqrt{1+x^2}}\ dx$

Answer

$$u = 1+x^2, \quad du = 2x\ dx, \quad \int_0^1 \frac{x}{\sqrt{1+x^2}}\ dx = \frac{1}{2}\int_1^2 u^{-1/2}du = \sqrt{2}-1$$

41) $\displaystyle\int_0^2 \frac{t}{1+t^2}\ dt$

Answer

$$\int_0^2 \frac{t}{1+t^2}\ dt = \ln\sqrt{5}$$

42) $\displaystyle\int_0^1 \frac{t^2}{\sqrt{1+t^3}}\ dt$

Answer

$$u = 1+t^3, \quad du = 3t^2, \quad \int_0^1 \frac{t^2}{\sqrt{1+t^3}}\ dt = \frac{1}{3}\int_1^2 u^{-1/2}du = \frac{2}{3}(\sqrt{2}-1)$$

43) $\displaystyle\int_0^{\pi/4} \sec^2\theta\tan\theta\ d\theta$

Answer

$$u = \tan\theta, \quad du = \sec^2\theta\ d\theta, \quad \int_0^{\pi/4} \sec^2\theta\tan\theta\ d\theta = \frac{1}{2}$$

44) $\displaystyle\int_0^{\pi/4} \frac{\sin\theta}{\cos^4\theta}\ d\theta$

Answer

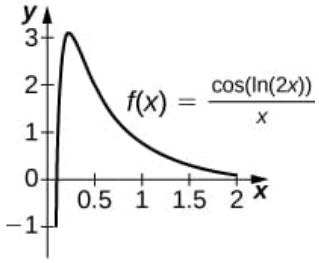
$$u = \cos\theta, \quad du = -\sin\theta\ d\theta, \quad \int_0^{\pi/4} \frac{\sin\theta}{\cos^4\theta}\ d\theta = -\int_1^{\sqrt{2}/2} u^{-4}\ du = \int_{\sqrt{2}/2}^1 u^{-4}\ du = \frac{1}{3}(2\sqrt{2}-1)$$

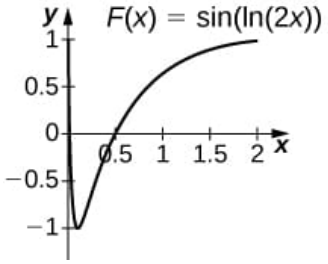
In exercises 45 - 50, evaluate the indefinite integral $\int f(x)\ dx$ with constant $C = 0$ using u-substitution. Then, graph the function and the antiderivative over the indicated interval. If possible, estimate a value of C that would need to be added to the antiderivative to make it equal to the definite integral $F(x) = \int_a^x f(t)\ dt$, with a the left endpoint of the given interval.

45) [T] $\int (2x+1)e^{x^2+x-6}\ dx$ over $[-3,2]$

46) [T] $\int \dfrac{\cos(\ln(2x))}{x}\ dx$ on $[0,2]$

Answer



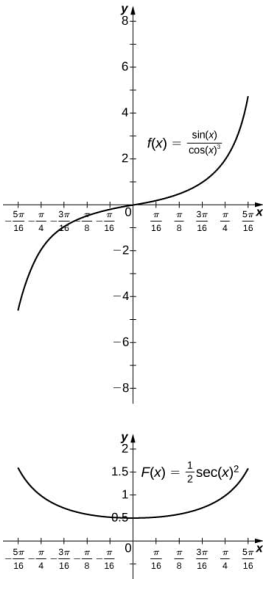


The antiderivative is $y=\sin(\ln(2x))$. Since the antiderivative is not continuous at $x=0$, one cannot find a value of C that would make $y=\sin(\ln(2x))-C$ work as a definite integral.

47) [T] $\int \dfrac{3x^2+2x+1}{\sqrt{x^3+x^2+x+4}}\ dx$ over $[-1,2]$

48) [T] $\int \dfrac{\sin x}{\cos^3 x}\ dx$ over $[-\frac{\pi}{3},\frac{\pi}{3}]$

Answer

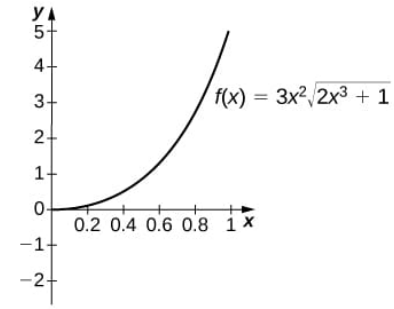


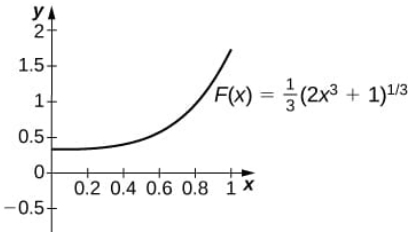
The antiderivative is $y=\frac{1}{2}\sec^2 x$. You should take $C=-2$ so that $F(-\frac{\pi}{3})=0$.

49) [T] $\int (x+2)e^{-x^2-4x+3}\ dx$ over $[-5,1]$

50) [T] $\int 3x^2\sqrt{2x^3+1}\ dx$ over $[0,1]$

Answer





The antiderivative is $y=\frac{1}{3}(2x^3+1)^{3/2}$. One should take $C=-\frac{1}{3}$.

51) If $h(a)=h(b)$ in $\int_a^b g'(h(x))h(x)\ dx$, what can you say about the value of the integral?

52) Is the substitution $u=1-x^2$ in the definite integral $\int_0^2 \frac{x}{1-x^2}\ dx$ okay? If not, why not?

Answer

No, because the integrand is discontinuous at $x=1$.

In exercises 53 - 59, use a change of variables to show that each definite integral is equal to zero.

53) $\int_0^{\pi} \cos^2(2\theta)\sin(2\theta)\ d\theta$

54) $\int_0^{\sqrt{\pi}} t\cos(t^2)\sin(t^2)\ dt$

Answer

$u=\sin(t^2)$; the integral becomes $\frac{1}{2}\int_0^0 u\ du.$

55) $\int_0^1 (1-2t)\ dt$

56) $\int_0^1 \frac{1-2t}{1+(t-\frac{1}{2})^2}\ dt$

Answer

$u=1+(t-\frac{1}{2})^2$; the integral becomes $-\int_{5/4}^{5/4}\frac{1}{u}\ du.$

57) $\int_0^{\pi} \sin\left(\left(t-\frac{\pi}{2}\right)^3\right)\cos\left(t-\frac{\pi}{2}\right)\ dt$

58) $\int_0^2 (1-t)\cos(\pi t)\ dt$

Answer

$u = 1-t$; Since the integrand is odd, the integral becomes

$$\int_1^{-1} u\cos\left(\pi(1-u)\right)\,du = \int_1^{-1} u[\cos\pi\cos u - \sin\pi\sin u]\,du = -\int_1^{-1} u\cos u\,du = \int_{-1}^{1} u\cos u\,du = 0$$

59) $\int_{\pi/4}^{3\pi/4} \sin^2 t\cos t\ dt$

60) Show that the average value of $f(x)$ over an interval $[a,b]$ is the same as the average value of $f(cx)$ over the interval $\left[\frac{a}{c},\frac{b}{c}\right]$ for $c>0$.

Answer

Setting $u=cx$ and $du = c\ dx$ gets you $\frac{1}{\frac{b}{c}-\frac{a}{c}}\int_{a/c}^{b/c} f(cx)\ dx = \frac{c}{b-a}\int_{u=a}^{u=b} f(u)\frac{du}{c} = \frac{1}{b-a}\int_a^b f(u)\ du.$

61) Find the area under the graph of $f(t) = \dfrac{t}{(1+t^2)^a}$ between $t=0$ and $t=x$ where $a>0$ and $a\neq 1$ is fixed, and evaluate the limit as $x\to\infty$.

62) Find the area under the graph of $g(t) = \dfrac{t}{(1-t^2)^a}$ between $t=0$ and $t=x$, where $0<x<1$ and $a>0$ is fixed. Evaluate the limit as $x\to 1$.

Answer

$\int_0^x g(t)\ dt = \frac{1}{2}\int_{u=1-x^2}^{1}\frac{du}{u^a} = \frac{1}{2(1-a)}u^{1-a}\mid 1u = \frac{1}{2(1-a)}(1-(1-x^2)^{1-a})$ As $x\to 1$ the limit is $\frac{1}{2(1-a)}$ if $a<1$, and the limit diverges to $+\infty$ if $a>1$.

63) The area of a semicircle of radius 1 can be expressed as $\int_{-1}^{1}\sqrt{1-x^2}\ dx$. Use the substitution $x=\cos t$ to express the area of a semicircle as the integral of a trigonometric function. Evaluate the integral.

Answer

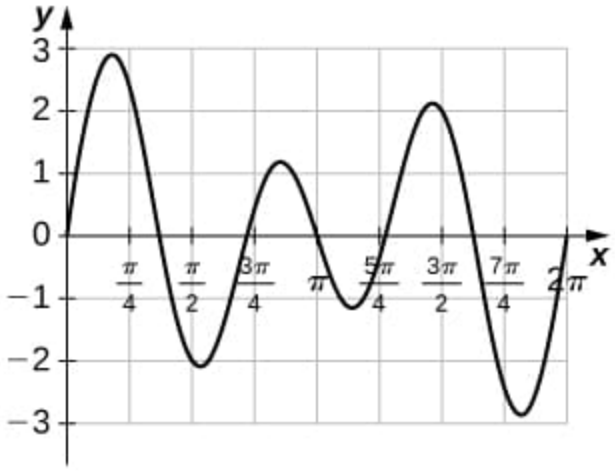
$\int_0^{\pi}\sin^2 t\ dt = \frac{1}{2}\pi$

64) The area of the top half of an ellipse with a major axis that is the x-axis from $x=-1$ to a and with a minor axis that is the y-axis from $y=-b$ to $y=b$ can be written as $\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}\ dx$. Use the substitution $x=a\cos t$ to express this area in terms of an integral of a trigonometric function. You do not need to compute the integral.

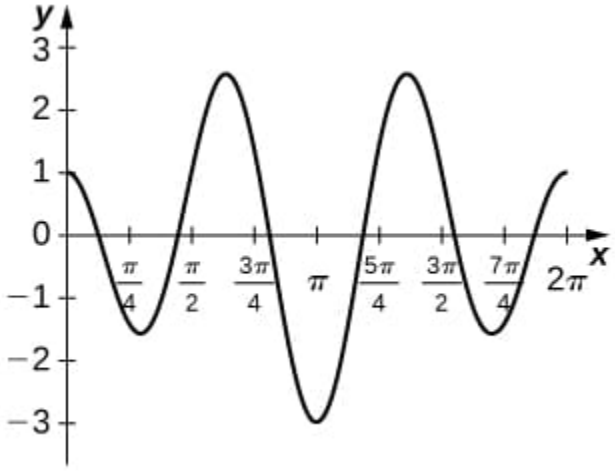
Answer

$\int_{t=\pi}^{t=0} b\sqrt{1-\cos^2 t}\times(-a\sin t)\ dt = \int_{t=0}^{t=\pi} ab\sin^2 t\ dt$

65) [T] The following graph is of a function of the form $f(t) = a\sin(nt)+b\sin(mt)$. Estimate the coefficients a and b and the frequency parameters n and m. Use these estimates to approximate $\int_0^{\pi} f(t)\ dt$.



66) [T] The following graph is of a function of the form $f(x) = a\cos(nt) + b\cos(mt)$. Estimate the coefficients a and b and the frequency parameters n and m. Use these estimates to approximate $\int_0^{\pi} f(t)\ dt$.



Answer

$$f(t) = 2\cos(3t) - \cos(2t); \quad \int_0^{\pi/2} (2\cos(3t) - \cos(2t))\ dt = -\frac{2}{3}$$

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5.6: INTEGRALS INVOLVING EXPONENTIAL AND LOGARITHMIC FUNCTIONS

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5.6E: Exercises for Section 5.6

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5.6E: Exercises for Section 5.6

For exercises 1 - 8, compute each indefinite integral.

1) $\int e^{2x}\ dx$

2) $\int e^{-3x}\ dx$

Answer

$$\int e^{-3x}\ dx \quad = \quad \frac{-1}{3}e^{-3x} + C$$

3) $\int 2^{x}\ dx$

4) $\int 3^{-x}\ dx$

Answer

$$\int 3^{-x}\ dx \quad = \quad -\frac{3^{-x}}{\ln 3} + C$$

5) $\int \frac{1}{2x}\ dx$

6) $\int \frac{2}{x}\ dx$

Answer

$$\int \frac{2}{x}\ dx \quad = \quad 2\ln x + C \quad = \quad \ln(x^2) + C$$

7) $\int \frac{1}{x^2}\ dx$

8) $\int \frac{1}{\sqrt{x}}\ dx$

Answer

$$\int \frac{1}{\sqrt{x}}\ dx \quad = \quad 2\sqrt{x} + C$$

In exercises 9 - 16, find each indefinite integral by using appropriate substitutions.

9) $\int \frac{\ln x}{x}\ dx$

10) $\int \frac{dx}{x(\ln x)^2}$

Answer

$$\int \frac{dx}{x(\ln x)^2} \quad = \quad -\frac{1}{\ln x} + C$$

11) $\int \frac{dx}{x\ln x} \quad (x > 1)$

12) $\int \frac{dx}{x\ln x\ln(\ln x)}$

Answer

$$\int \frac{dx}{x \ln x \ln(\ln x)} \quad = \quad \ln(\ln(\ln x)) + C$$

13) $\int \tan\theta \ d\theta$

14) $\int \frac{\cos x - x \sin x}{x \cos x} \ dx$

Answer

$$\int \frac{\cos x - x \sin x}{x \cos x} \ dx \quad = \quad \ln(x \cos x) + C$$

15) $\int \frac{\ln(\sin x)}{\tan x} \ dx$

16) $\int \ln(\cos x) \tan x \ dx$

Answer

$$\int \ln(\cos x) \tan x \ dx \quad = \quad -\frac{1}{2}(\ln(\cos(x)))^2 + C$$

17) $\int x e^{-x^2} \ dx$

18) $\int x^2 e^{-x^3} \ dx$

Answer

$$\int x^2 e^{-x^3} \ dx \quad = \quad \frac{-e^{-x^3}}{3} + C$$

19) $\int e^{\sin x} \cos x \ dx$

20) $\int e^{\tan x} \sec^2 x \ dx$

Answer

$$\int e^{\tan x} \sec^2 x \ dx \quad = \quad e^{\tan x} + C$$

21) $\int \frac{e^{\ln x}}{x} \ dx$

22) $\int \frac{e^{\ln(1-t)}}{1-t} \ dt$

Answer

$$\int \frac{e^{\ln(1-t)}}{1-t} \ dt = \int \frac{1-t}{1-t} \ dt = \int 1 \ dt \quad = \quad t + C$$

In exercises 23 - 28, verify by differentiation that $\int \ln x \ dx = x(\ln x - 1) + C$, then use appropriate changes of variables to compute the integral.

23) $\int \ln x \ dx$ (Hint: $\int \ln x \ dx = \frac{1}{2} \int x \ln(x^2) \ dx$)

24) $\int x^2 \ln^2 x \ dx$

Answer

$$\int x^2 \ln^2 x \; dx \quad = \quad \frac{1}{9}x^3(\ln(x^3) - 1) + C$$

25) $\int \frac{\ln x}{x^2} \; dx$ (Hint: Set $u = \frac{1}{x}$.)

26) $\int \frac{\ln x}{\sqrt{x}} \; dx$ (Hint: Set $u = \sqrt{x}$.)

Answer

$$\int \frac{\ln x}{\sqrt{x}} \; dx \quad = \quad 2\sqrt{x}(\ln x - 2) + C$$

27) Write an integral to express the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to e^x and evaluate the integral.

28) Write an integral to express the area under the graph of $y = e^t$ between $t = 0$ and $t = \ln x$, and evaluate the integral.

Answer

$$\int_0^{\ln x} e^t \; dt = e^t \Big|_0^{\ln x} = e^{\ln x} - e^0 = x - 1$$

In exercises 29 - 35, use appropriate substitutions to express the trigonometric integrals in terms of compositions with logarithms.

29) $\int \tan(2x) \; dx$

30) $\int \frac{\sin(3x) - \cos(3x)}{\sin(3x) + \cos(3x)} \; dx$

Answer

$$\int \frac{\sin(3x) - \cos(3x)}{\sin(3x) + \cos(3x)} \; dx \quad = \quad -\frac{1}{3}\ln|\sin(3x) + \cos(3x)| + C$$

31) $\int \frac{x \sin(x^2)}{\cos(x^2)} \; dx$

32) $\int x \csc(x^2) \; dx$

Answer

$$\int x \csc(x^2) \; dx \quad = \quad -\frac{1}{2}\ln|\csc(x^2) + \cot(x^2)| + C$$

33) $\int \ln(\cos x) \tan x \; dx$

34) $\int \ln(\csc x) \cot x \; dx$

Answer

$$\int \ln(\csc x) \cot x \; dx \quad = \quad -\frac{1}{2}(\ln(\csc x))^2 + C$$

35) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} \; dx$

In exercises 36 - 40, evaluate the definite integral.

36) $\int_1^2 \frac{1 + 2x + x^2}{3x + 3x^2 + x^3} \; dx$

Answer

$$\int_1^2 \frac{1+2x+x^2}{3x+3x^2+x^3}\,dx \quad = \quad \frac{1}{3}\ln\left(\tfrac{26}{7}\right)$$

37) $\int_0^{\pi/4} \tan x\ dx$

Answer

$$\int_0^{\pi/4} \tan x\ dx = \ln\sqrt{2}$$

38) $\int_0^{\pi/3} \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$

Answer

$$\int_0^{\pi/3} \frac{\sin x - \cos x}{\sin x + \cos x}\,dx \quad = \quad \ln(\sqrt{3}-1)$$

39) $\int_{\pi/4}^{\pi/3} \sec x\ dx$

Answer

$$\int_{\pi/4}^{\pi/3} \sec x\ dx \quad = \quad \ln\left(\frac{3}{\sqrt{6}+\sqrt{3}}\right)$$

40) $\int_{\pi/4}^{\pi/3} \cot x\ dx$

Answer

$$\int_{\pi/4}^{\pi/3} \cot x\ dx \quad = \quad \frac{1}{2}\ln\frac{3}{2}$$

In exercises 41 - 46, integrate using the indicated substitution.

41) $\int \frac{x}{x-100}\,dx; \quad u = x-100$

42) $\int \frac{y-1}{y+1}\,dy; \quad u = y+1$

Answer

$$\int \frac{y-1}{y+1}\,dy \quad = \quad y - 2\ln|y+1| + C$$

43) $\int \frac{1-x^2}{3x-x^3}\,dx; \quad u = 3x-x^3$

44) $\int \frac{\sin x + \cos x}{\sin x - \cos x}\,dx; \quad u = \sin x - \cos x$

Answer

$$\int \frac{\sin x + \cos x}{\sin x - \cos x}\,dx \quad = \quad \ln|\sin x - \cos x| + C$$

45) $\int e^{2x}\sqrt{1-e^{2x}}\,dx; \quad u = 1-e^{2x}$

46) $\int \ln(x)\frac{\sqrt{1-(\ln x)^2}}{x}\,dx; \quad u = 1-(\ln x)^2$

Answer

$$\int \ln(x)\frac{\sqrt{1-(\ln x)^2}}{x}\,dx \quad = \quad -\frac{1}{3}(1-(\ln x^2))^{3/2}+C$$

47) $\int \dfrac{\sqrt{x}}{\sqrt{x}+2}\,dx; \quad u=\sqrt{x}+2$

Answer

$$\int \frac{\sqrt{x}}{\sqrt{x}+2}\,dx \quad = \quad \left(\sqrt{x}+2\right)^2-8\left(\sqrt{x}+2\right)+8\ln\left(\sqrt{x}+2\right)+C$$

48) $\int e^x \sec(e^x+1)\tan(e^x+1)\;dx; \quad u=e^x+1$

Answer

$$\int e^x \sec(e^x+1)\tan(e^x+1)\;dx \quad = \quad \sec(e^x+1)+C$$

In exercises 49 - 54, state whether the right-endpoint approximation overestimates or underestimates the exact area. Then calculate the right endpoint estimate R_{50} and solve for the exact area.

49) [T] $y=e^x$ over $[0,1]$

50) [T] $y=e^{-x}$ over $[0,1]$

Answer

Since f is decreasing, the right endpoint estimate underestimates the area.
Exact solution: $\dfrac{e-1}{e}, \quad R_{50}=0.6258.$

51) [T] $y=\ln(x)$ over $[1,2]$

52) [T] $y=\dfrac{x+1}{x^2+2x+6}$ over $[0,1]$

Answer

Since f is increasing, the right endpoint estimate overestimates the area.
Exact solution: $\dfrac{2\ln(3)-\ln(6)}{2}, \quad R_{50}=0.2033.$

53) [T] $y=2^x$ over $[-1,0]$

54) [T] $y=-2^{-x}$ over $[0,1]$

Answer

Since f is increasing, the right endpoint estimate overestimates the area (the actual area is a larger negative number).
Exact solution: $-\dfrac{1}{\ln(4)}, \quad R_{50}=-0.7164.$

In exercises 55 - 58, $f(x)\geq 0$ for $a\leq x\leq b$. Find the area under the graph of $f(x)$ between the given values a and b by integrating.

55) $f(x)=\dfrac{\log_{10}(x)}{x}; \quad a=10, b=100$

56) $f(x)=\dfrac{\log_2(x)}{x}; \quad a=32, b=64$

Answer

$$\frac{11}{2}\ln 2$$

57) $f(x) = 2^{-x}; \quad a = 1, b = 2$

58) $f(x) = 2^{-x}; \quad a = 3, b = 4$

Answer

$$\frac{1}{\ln(65,536)}$$

59) Find the area under the graph of the function $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

60) Compute the integral of $f(x) = xe^{-x^2}$ and find the smallest value of N such that the area under the graph $f(x) = xe^{-x^2}$ between $x = N$ and $x = N + 10$ is, at most, 0.01.

Answer

$\int_N^{N+1} xe^{-x^2}\ dx = \frac{1}{2}(e^{-N^2} - e^{-(N+1)^2})$. The quantity is less than 0.01 when $N = 2$.

61) Find the limit, as N tends to infinity, of the area under the graph of $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

62) Show that $\int_a^b \frac{dt}{t} = \int_{1/b}^{1/a} \frac{dt}{t}$ when $0 < a \leq b$.

Answer

$$\int_a^b \frac{dx}{x} = \ln(b) - \ln(a) = \ln(\frac{1}{a}) - \ln(\frac{1}{b}) = \int_{1/b}^{1/a} \frac{dx}{x}$$

63) Suppose that $f(x) > 0$ for all x and that f and g are differentiable. Use the identity $f^g = e^{g \ln f}$ and the chain rule to find the derivative of f^g.

64) Use the previous exercise to find the antiderivative of $h(x) = x^x(1 + \ln x)$ and evaluate $\int_2^3 x^x(1 + \ln x)\ dx$.

Answer

23

65) Show that if $c > 0$, then the integral of $\frac{1}{x}$ from ac to bc $(\text{for } 0 < a < b)$ is the same as the integral of $\frac{1}{x}$ from a to b.

The following exercises are intended to derive the fundamental properties of the natural log starting from the definition $\ln(x) = \int_1^x \frac{dt}{t}$, using properties of the definite integral and making no further assumptions.

66) Use the identity $\ln(x) = \int_1^x \frac{dt}{t}$ to derive the identity $\ln\left(\frac{1}{x}\right) = -\ln x$.

Answer

We may assume that $x > 1$,so $\frac{1}{x} < 1$. Then, $\int_1^{1/x} \frac{dt}{t}$. Now make the substitution $u = \frac{1}{t}$, so $du = -\frac{dt}{t^2}$ and $\frac{du}{u} = -\frac{dt}{t}$, and change endpoints: $\int_1^{1/x} \frac{dt}{t} = -\int_1^x \frac{du}{u} = -\ln x$.

67) Use a change of variable in the integral $\int_1^{xy} \frac{1}{t}\ dt$ to show that $\ln xy = \ln x + \ln y$ for $x, y > 0$.

68) Use the identity $\ln x = \int_1^x \frac{dt}{x}$ to show that $\ln(x)$ is an increasing function of x on $[0, \infty)$, and use the previous exercises to show that the range of $\ln(x)$ is $(-\infty, \infty)$. Without any further assumptions, conclude that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$.

69) Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln(x)$, but keep in mind that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$. Call it E. Use the identity $\ln xy = \ln x + \ln y$ to deduce that $E(a + b) = E(a)E(b)$ for any real numbers a, b.

70) Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln x$, but keep in mind that $\ln x$ has an inverse function defined on $(-\infty, \infty)$. Call it E. Show that $E'(t) = E(t)$.

Answer

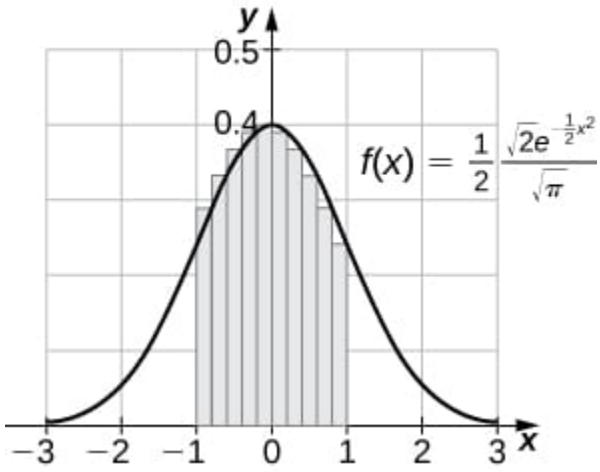
$x = E(\ln(x))$. Then, $1 = \dfrac{E'(\ln x)}{x}$ or $x = E'(\ln x)$. Since any number t can be written $t = \ln x$ for some x, and for such t we have $x = E(t)$, it follows that for any t, $E'(t) = E(t)$.

71) The sine integral, defined as $S(x) = \displaystyle\int_0^x \frac{\sin t}{t}\, dt$ is an important quantity in engineering. Although it does not have a simple closed formula, it is possible to estimate its behavior for large x. Show that for $k \geq 1$, $\quad |S(2\pi k) - S(2\pi(k+1))| \leq \dfrac{1}{k(2k+1)\pi}$. (Hint: $\sin(t+\pi) = -\sin t$)

72) [T] The normal distribution in probability is given by $p(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$, where σ is the standard deviation and μ is the average. The standard normal distribution in probability, p_s, corresponds to $\mu = 0$ and $\sigma = 1$. Compute the left endpoint estimates R_{10} and R_{100} of $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$.

Answer

$R_{10} = 0.6811, \quad R_{100} = 0.6827$



73) [T] Compute the right endpoint estimates R_{50} and R_{100} of $\displaystyle\int_{-3}^{5} \frac{1}{2\sqrt{2\pi}} e^{-(x-1)^2/8}$.

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- Paul Seeburger (Monroe Community College) added problems 47-48 to Section 5.6 exercises.

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5.7: INTEGRALS RESULTING IN INVERSE TRIGONOMETRIC FUNCTIONS

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5.7E: Exercises for Section 5.7

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5.7E: Exercises for Section 5.7

In exercises 1 - 6, evaluate each integral in terms of an inverse trigonometric function.

1) $\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

Answer

$$\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big|_0^{\sqrt{3}/2} = \frac{\pi}{3}$$

2) $\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}}$

3) $\int_{\sqrt{3}}^{1} \frac{dx}{1+x^2}$

Answer

$$\int_{\sqrt{3}}^{1} \frac{dx}{1+x^2} = \tan^{-1} x\Big|_{\sqrt{3}}^{1} = -\frac{\pi}{12}$$

4) $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$

5) $\int_1^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}}$

Answer

$$\int_1^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x\Big|_1^{\sqrt{2}} = \frac{\pi}{4}$$

6) $\int_1^{\frac{2}{\sqrt{3}}} \frac{dx}{|x|\sqrt{x^2-1}}$

In exercises 7 - 12, find each indefinite integral, using appropriate substitutions.

7) $\int \frac{dx}{\sqrt{9-x^2}}$

Answer

$$\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$$

8) $\int \frac{dx}{\sqrt{1-16x^2}}$

9) $\int \frac{dx}{9+x^2}$

Answer

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

10) $\int \frac{dx}{25+16x^2}$

Answer

$$\int \frac{dx}{25+16x^2} = \frac{1}{20}\arctan\left(\frac{4x}{5}\right)+C$$

11) $\int \frac{dx}{|x|\sqrt{x^2-9}}$

Answer

$$\int \frac{dx}{x\sqrt{x^2-9}} = \frac{1}{3}\sec^{-1}\left(\frac{|x|}{3}\right)+C$$

12) $\int \frac{dx}{|x|\sqrt{4x^2-16}}$

Answer

$$\int \frac{dx}{x\sqrt{4x^2-16}} = \frac{1}{4}\sec^{-1}\left(\frac{|x|}{2}\right)+C$$

13) Explain the relationship $-\cos^{-1}t+C=\int \frac{dt}{\sqrt{1-t^2}}=\sin^{-1}t+C$. Is it true, in general, that $\cos^{-1}t=-\sin^{-1}t$?

Answer

$\cos(\frac{\pi}{2}-\theta)=\sin\theta$. So, $\sin^{-1}t=\frac{\pi}{2}-\cos^{-1}t$. They differ by a constant.

14) Explain the relationship $\sec^{-1}t+C=\int \frac{dt}{|t|\sqrt{t^2-1}}=-\csc^{-1}t+C$. Is it true, in general, that $\sec^{-1}t=-\csc^{-1}t$?

15) Explain what is wrong with the following integral: $\int_1^2 \frac{dt}{\sqrt{1-t^2}}$.

Answer

$\sqrt{1-t^2}$ is not defined as a real number when $t>1$.

16) Explain what is wrong with the following integral: $\int_{-1}^1 \frac{dt}{|t|\sqrt{t^2-1}}$.

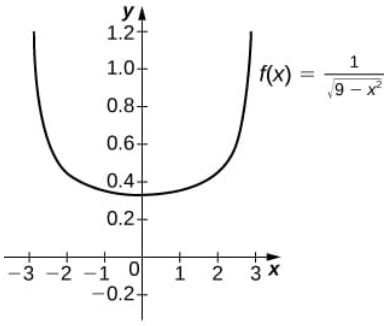
Answer

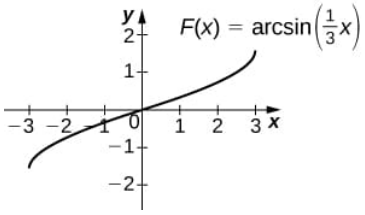
$\sqrt{t^2-1}$ is not defined as a real number when $-1<t<1$, and the integrand is undefined when $t=-1$ or $t=1$.

In exercises 17 - 20, solve for the antiderivative of f with $C=0$, then use a calculator to graph f and the antiderivative over the given interval $[a,b]$. Identify a value of C such that adding C to the antiderivative recovers the definite integral $F(x)=\int_a^x f(t)\ dt$.

17) [T] $\int \frac{1}{\sqrt{9-x^2}}\ dx$ over $[-3,3]$

Answer



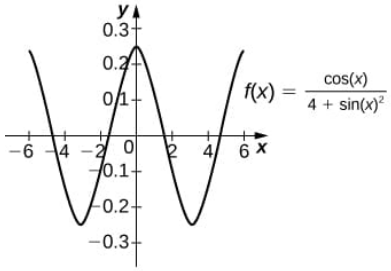


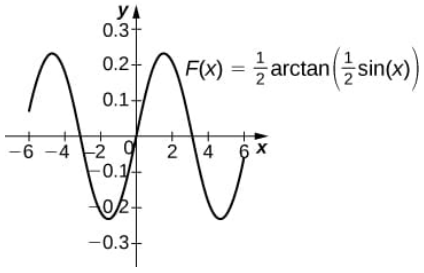
The antiderivative is $\sin^{-1}(\frac{x}{3}) + C$. Taking $C = \frac{\pi}{2}$ recovers the definite integral.

18) [T] $\int \frac{9}{9+x^2}\, dx$ over $[-6, 6]$

19) [T] $\int \frac{\cos x}{4+\sin^2 x}\, dx$ over $[-6, 6]$

Answer





The antiderivative is $\frac{1}{2}\tan^{-1}(\frac{\sin x}{2}) + C$. Taking $C = \frac{1}{2}\tan^{-1}(\frac{\sin(6)}{2})$ recovers the definite integral.

20) [T] $\int \frac{e^x}{1+e^{2x}}\, dx$ over $[-6, 6]$

In exercises 21 - 26, compute the antiderivative using appropriate substitutions.

21) $\int \frac{\sin^{-1} t}{\sqrt{1-t^2}}\, dt$

Answer

$$\int \frac{\sin^{-1} t\, dt}{\sqrt{1-t^2}} \quad = \quad \tfrac{1}{2}(\sin^{-1} t)^2 + C$$

22) $\int \frac{dt}{\sin^{-1} t\sqrt{1-t^2}}$

23) $\int \frac{\tan^{-1}(2t)}{1+4t^2}\, dt$

Answer

$$\int \frac{\tan^{-1}(2t)}{1+4t^2}\, dt \quad = \quad \frac{1}{4}(\tan^{-1}(2t))^2 + C$$

24) $\int \frac{t\tan^{-1}(t^2)}{1+t^4}\, dt$

25) $\int \frac{\sec^{-1}\left(\frac{t}{2}\right)}{|t|\sqrt{t^2-4}}\, dt$

Answer

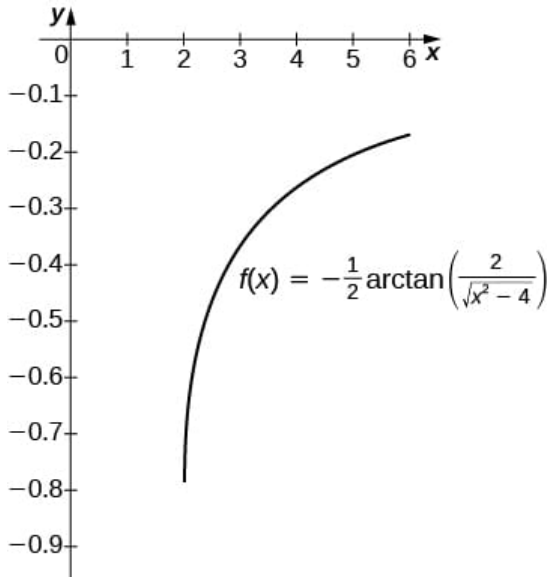
$$\int \frac{\sec^{-1}\left(\frac{t}{2}\right)}{|t|\sqrt{t^2-4}}\, dt \quad = \quad \tfrac{1}{4}(\sec^{-1}\left(\tfrac{t}{2}\right))^2 + C$$

26) $\int \frac{t\sec^{-1}(t^2)}{t^2\sqrt{t^4-1}}\, dt$

In exercises 27 - 32, use a calculator to graph the antiderivative of f with $C = 0$ over the given interval $[a, b]$. Approximate a value of C, if possible, such that adding C to the antiderivative gives the same value as the definite integral $F(x) = \int_a^x f(t)\, dt$.

27) [T] $\int \frac{1}{x\sqrt{x^2-4}}\, dx$ over $[2, 6]$

Answer

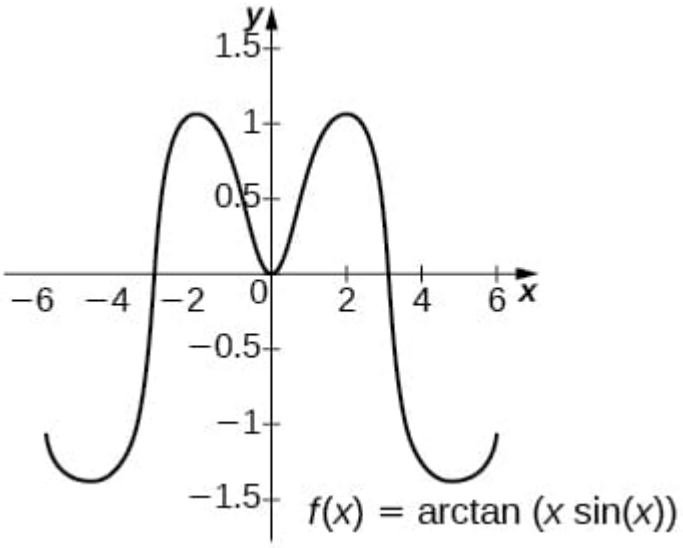


The antiderivative is $\frac{1}{2}\sec^{-1}(\frac{x}{2}) + C$. Taking $C = 0$ recovers the definite integral over $[2, 6]$.

28) [T] $\int \frac{1}{(2x+2)\sqrt{x}}\, dx$ over $[0, 6]$

29) [T] $\int \frac{(\sin x + x\cos x)}{1 + x^2 \sin^2 x\, dx}$ over $[-6, 6]$

Answer

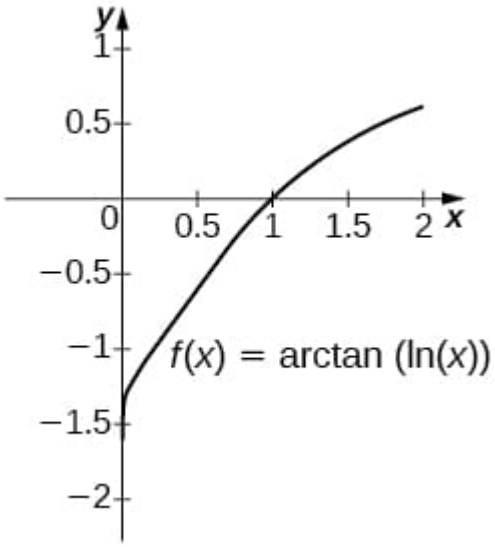


The general antiderivative is $\tan^{-1}(x \sin x) + C$. Taking $C = -\tan^{-1}(6\sin(6))$ recovers the definite integral.

30) [T] $\int \frac{2e^{-2x}}{\sqrt{1 - e^{-4x}}}\, dx$ over $[0, 2]$

31) [T] $\int \frac{1}{x + x\ln 2x}$ over $[0, 2]$

Answer



The general antiderivative is $\tan^{-1}(\ln x)+C$. Taking $C=\frac{\pi}{2}=\lim_{t\to\infty}\tan^{-1}t$ recovers the definite integral.

32) [T] $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}$ over $[-1,1]$

In exercises 33 - 38, compute each integral using appropriate substitutions.

33) $\int \frac{e^t}{\sqrt{1-e^{2t}}}\,dt$

Answer

$$\int \frac{e^t}{\sqrt{1-e^{2t}}}\,dt \quad = \quad \sin^{-1}(e^t)+C$$

34) $\int \frac{e^t}{1+e^{2t}}\,dt$

35) $\int \frac{dt}{t\sqrt{1-\ln^2 t}}$

Answer

$$\int \frac{dt}{t\sqrt{1-\ln^2 t}} \quad = \quad \sin^{-1}(\ln t)+C$$

36) $\int \frac{dt}{t(1+\ln^2 t)}$

37) $\int \frac{\cos^{-1}(2t)}{\sqrt{1-4t^2}}\,dt$

Answer

$$\int \frac{\cos^{-1}(2t)}{\sqrt{1-4t^2}}\,dt \quad = \quad -\frac{1}{2}(\cos^{-1}(2t))^2+C$$

38) $\int \frac{e^t\cos^{-1}(e^t)}{\sqrt{1-e^{2t}}}\,dt$

In exercises 39 - 42, compute each definite integral.

39) $\int_0^{1/2} \frac{\tan(\sin^{-1}t)}{\sqrt{1-t^2}}\,dt$

Answer

$$\int_0^{1/2} \frac{\tan(\sin^{-1}t)}{\sqrt{1-t^2}}\,dt \quad = \quad \frac{1}{2}\ln\left(\frac{4}{3}\right)$$

40) $\int_{1/4}^{1/2} \frac{\tan(\cos^{-1}t)}{\sqrt{1-t^2}}\,dt$

41) $\int_0^{1/2} \frac{\sin(\tan^{-1}t)}{1+t^2}\,dt$

Answer

$$\int_0^{1/2} \frac{\sin(\tan^{-1}t)}{1+t^2}\,dt \quad = \quad 1-\frac{2}{\sqrt{5}}$$

42) $\int_0^{1/2} \frac{\cos(\tan^{-1}t)}{1+t^2}\,dt$

43) For $A > 0$, compute $I(A) = \int_{-A}^{A} \frac{dt}{1+t^2}$ and evaluate $\lim_{a\to\infty} I(A)$, the area under the graph of $\frac{1}{1+t^2}$ on $[-\infty, \infty]$.

Answer

$2\tan^{-1}(A) \to \pi$ as $A \to \infty$

44) For $1 < B < \infty$, compute $I(B) = \int_{1}^{B} \frac{dt}{t\sqrt{t^2-1}}$ and evaluate $\lim_{B\to\infty} I(B)$, the area under the graph of $\frac{1}{t\sqrt{t^2-1}}$ over $[1, \infty)$.

45) Use the substitution $u = \sqrt{2}\cot x$ and the identity $1 + \cot^2 x = \csc^2 x$ to evaluate $\int \frac{dx}{1+\cos^2 x}$. (Hint: Multiply the top and bottom of the integrand by $\csc^2 x$.)

Answer

Using the hint, one has $\int \frac{\csc^2 x}{\csc^2 x + \cot^2 x}\, dx = \int \frac{\csc^2 x}{1 + 2\cot^2 x}\, dx$. Set $u = \sqrt{2}\cot x$. Then, $du = -\sqrt{2}\csc^2 x$ and the integral is $-\frac{1}{\sqrt{2}} \int \frac{du}{1+u^2} = -\frac{\sqrt{2}}{2}\tan^{-1} u + C = \frac{\sqrt{2}}{2}\tan^{-1}(\sqrt{2}\cot x) + C$. If one uses the identity $\tan^{-1} s + \tan^{-1}(\frac{1}{s}) = \frac{\pi}{2}$, then this can also be written $\frac{\sqrt{2}}{2}\tan^{-1}(\frac{\tan x}{\sqrt{2}}) + C$.

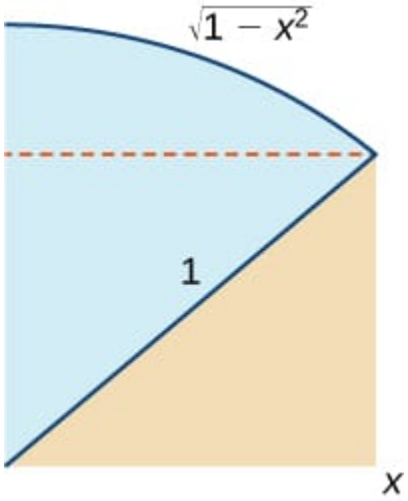
46) [T] Approximate the points at which the graphs of $f(x) = 2x^2 - 1$ and $g(x) = (1 + 4x^2)^{-3/2}$ intersect, and approximate the area between their graphs accurate to three decimal places.

47) [T] Approximate the points at which the graphs of $f(x) = x^2 - 1$ and $f(x) = x^2 - 1$ intersect, and approximate the area between their graphs accurate to three decimal places.

Answer

$x \approx \pm 1.13525$. The left endpoint estimate with $N = 100$ is 2.796 and these decimals persist for $N = 500$.

48) Use the following graph to prove that $\int_0^x \sqrt{1-t^2}\ dt = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x$.



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5.8: CALCULUS OF THE HYPERBOLIC FUNCTIONS

6.09: Calculus of the Hyperbolic Functions

5.8E: Exercises for Section 5.8

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5.8E: Exercises for Section 5.8

1) [T] Find expressions for $\cosh x + \sinh x$ and $\cosh x - \sinh x$. Use a calculator to graph these functions and ensure your expression is correct.

Answer

e^x and e^{-x}

2) From the definitions of $\cosh(x)$ and $\sinh(x)$, find their antiderivatives.

3) Show that $\cosh(x)$ and $\sinh(x)$ satisfy $y'' = y$.

Answer

Answers may vary

4) Use the quotient rule to verify that $\frac{d}{dx}\big(\tanh(x)\big) = \operatorname{sech}^2(x)$.

5) Derive $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$ from the definitions of $\cosh(x)$ and $\sinh(x)$.

6) Take the derivative of the previous expression to find an expression for $\sinh(2x)$.

7) Prove $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$ by changing the expression to exponentials.

Answer

Answers may vary

8) Take the derivative of the previous expression to find an expression for $\cosh(x+y)$.

In exercises 9 - 18, find the derivatives of the given functions and graph along with the function to ensure your answer is correct.

9) [T] $\cosh(3x+1)$

Answer

$3\sinh(3x+1)$

10) [T] $\sinh(x^2)$

11) [T] $\dfrac{1}{\cosh(x)}$

Answer

$-\tanh(x)\operatorname{sech}(x)$

12) [T] $\sinh(\ln(x))$

13) [T] $\cosh^2(x) + \sinh^2(x)$

Answer

$4\cosh(x)\sinh(x)$

14) [T] $\cosh^2(x) - \sinh^2(x)$

15) [T] $\tanh(\sqrt{x^2+1})$

Answer

$$\frac{x\text{sech}^2(\sqrt{x^2+1})}{\sqrt{x^2+1}}$$

16) [T] $\dfrac{1+\tanh(x)}{1-\tanh(x)}$

17) [T] $\sinh^6(x)$

Answer

$$6\sinh^5(x)\cosh(x)$$

18) [T] $\ln(\text{sech}(x)+\tanh(x))$

In exercises 19 - 28, find the antiderivatives for the given functions.

19) $\cosh(2x+1)$

Answer

$$\tfrac{1}{2}\sinh(2x+1)+C$$

20) $\tanh(3x+2)$

21) $x\cosh(x^2)$

Answer

$$\tfrac{1}{2}\sinh(x^2)+C$$

22) $3x^3\tanh(x^4)$

23) $\cosh^2(x)\sinh(x)$

Answer

$$\tfrac{1}{3}\cosh^3(x)+C$$

24) $\tanh^2(x)\text{sech}^2(x)$

25) $\dfrac{\sinh(x)}{1+\cosh(x)}$

Answer

$$\ln(1+\cosh(x))+C$$

26) $\coth(x)$

27) $\cosh(x)+\sinh(x)$

Answer

$$\cosh(x)+\sinh(x)+C$$

28) $(\cosh(x)+\sinh(x))^n$

In exercises 29 - 35, find the derivatives for the functions.

29) $\tanh^{-1}(4x)$

Answer

$$\frac{4}{1-16x^2}$$

30) $\sinh^{-1}(x^2)$

31) $\sinh^{-1}(\cosh(x))$

Answer

$$\frac{\sinh(x)}{\sqrt{\cosh^2(x)+1}}$$

32) $\cosh^{-1}(x^3)$

33) $\tanh^{-1}(\cos(x))$

Answer

$$-\csc(x)$$

34) $e^{\sinh^{-1}(x)}$

35) $\ln(\tanh^{-1}(x))$

Answer

$$-\frac{1}{(x^2-1)\tanh^{-1}(x)}$$

In exercises 36 - 42, find the antiderivatives for the functions.

36) $\int \frac{dx}{4-x^2}$

37) $\int \frac{dx}{a^2-x^2}$

Answer

$$\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+C$$

38) $\int \frac{dx}{\sqrt{x^2+1}}$

39) $\int \frac{xdx}{\sqrt{x^2+1}}$

Answer

$$\sqrt{x^2+1}+C$$

40) $\int -\frac{dx}{x\sqrt{1-x^2}}$

41) $\int \frac{e^x}{\sqrt{e^{2x}-1}}$

Answer

$$\cosh^{-1}(e^x)+C$$

42) $\int -\frac{2x}{x^4-1}$

In exercises 43 - 45, use the fact that a falling body with friction equal to velocity squared obeys the equation $\frac{dv}{dt}=g-v^2$.

43) Show that $v(t)=\sqrt{g}\tanh(\sqrt{g}t)$ satisfies this equation.

Answer

Answers may vary

44) Derive the previous expression for $v(t)$ by integrating $\frac{dv}{g-v^2} = dt$.

45) [T] Estimate how far a body has fallen in 12 seconds by finding the area underneath the curve of $v(t)$.

Answer

37.30

In exercises 46 - 48, use this scenario: A cable hanging under its own weight has a slope $S = \frac{dy}{dx}$ that satisfies $\frac{dS}{dx} = c\sqrt{1+S^2}$. The constant c is the ratio of cable density to tension.

46) Show that $S = \sinh(cx)$ satisfies this equation.

47) Integrate $\frac{dy}{dx} = \sinh(cx)$ to find the cable height $y(x)$ if $y(0) = 1/c$.

Answer

$y = \frac{1}{c}\cosh(cx)$

48) Sketch the cable and determine how far down it sags at $x = 0$.

In exercises 49 - 52, solve each problem.

49) [T] A chain hangs from two posts 2m apart to form a catenary described by the equation $y = 2\cosh(x/2) - 1$. Find the slope of the catenary at the left fence post.

Answer

-0.521095

50) [T] A chain hangs from two posts four meters apart to form a catenary described by the equation $y = 4\cosh(x/4) + 3$. Find the total length of the catenary (arc length).

Answer

$8\sinh(0.5) = 4.1688$ meters

51) [T] A high-voltage power line is a catenary described by $y = 10\cosh(x/10)$. Find the ratio of the area under the catenary to its arc length. What do you notice?

Answer

10

52) A telephone line is a catenary described by $y = a\cosh(x/a)$. Find the ratio of the area under the catenary to its arc length. Does this confirm your answer for the previous question?

53) Prove the formula for the derivative of $y = \sinh^{-1}(x)$ by differentiating $x = \sinh(y)$.

(Hint: Use hyperbolic trigonometric identities.)

54) Prove the formula for the derivative of $y = \cosh^{-1}(x)$ by differentiating $x = \cosh(y)$.

(Hint: Use hyperbolic trigonometric identities.)

55) Prove the formula for the derivative of $y = \operatorname{sech}^{-1}(x)$ by differentiating $x = \operatorname{sech}(y)$.

(Hint: Use hyperbolic trigonometric identities.)

56) Prove that $\cosh(x) + \sinh(x))^n = \cosh(nx) + \sinh(nx)$.

57) Prove the expression for $\sinh^{-1}(x)$. Multiply $x = \sinh(y) = \frac{e^y - e^{-y}}{2}$ by $2e^y$ and solve for y. Does your expression match the textbook?

58) Prove the expression for $\cosh^{-1}(x)$. Multiply $x = \cosh(y) = \frac{e^y + e^{-y}}{2}$ by $2e^y$ and solve for y. Does your expression match the textbook?

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5.9: Chapter 5 Review Exercises

5.08: Chapter 5 Review Exercises

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6: INTRODUCTION TO DIFFERENTIAL EQUATIONS

Many real-world phenomena can be modeled mathematically by using differential equations. Population growth, radioactive decay, predator-prey models, and spring-mass systems are four examples of such phenomena. In this chapter we study some of these applications. A goal of this chapter is to develop solution techniques for different types of differential equations. As the equations become more complicated, the solution techniques also become more complicated, and in fact an entire course could be dedicated to the study of these equations. In this chapter we study several types of differential equations and their corresponding methods of solution.

6.0: Prelude to Differential Equations

6.1: Basics of Differential Equations

6.1E: Exercises for Section 6.1

6.2: Direction Fields and Numerical Methods

6.2E: Exercises for Section 6.2

6.3: Separable Equations

6.3E: Exercises for Section 6.3

6.4: The Logistic Equation

6.4E: Exercises for Section 6.4

6.5: First-order Linear Equations

6.5E: Exercises for Section 6.5

6.6: Chapter 6 Review Exercises

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6.0: Prelude to Differential Equations

8.00: Prelude to Differential Equations

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6.1: BASICS OF DIFFERENTIAL EQUATIONS

8.01: Basics of Differential Equations

6.1E: Exercises for Section 6.1

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6.1E: Exercises for Section 6.1

In exercises 1 - 7, determine the order of each differential equation.

1) $y'+y=3y^2$

Answer

1st-order

2) $(y')^2=y'+2y$

3) $y'''+y''y'=3x^2$

Answer

3rd-order

4) $y'=y''+3t^2$

5) $\dfrac{dy}{dt}=t$

Answer

1st-order

6) $\dfrac{dy}{dx}+\dfrac{d^2y}{dx^2}=3x^4$

7) $\left(\dfrac{dy}{dt}\right)^2+8\dfrac{dy}{dt}+3y=4t$

Answer

1st-order

In exercises 8 - 17, verify that the given function is a solution to the given differential equation.

8) $y=\dfrac{x^3}{3}$ solves $y'=x^2$

9) $y=2e^{-x}+x-1$ solves $y'=x-y$

10) $y=e^{3x}-\dfrac{e^x}{2}$ solves $y'=3y+e^x$

11) $y=\dfrac{1}{1-x}$ solves $y'=y^2$

12) $y=\dfrac{e^{x^2}}{2}$ solves $y'=xy$

13) $y=4+\ln x$ solves $xy'=1$

14) $y=3-x+x\ln x$ solves $y'=\ln x$

15) $y=2e^x-x-1$ solves $y'=y+x$

16) $y=e^x+\dfrac{\sin x}{2}-\dfrac{\cos x}{2}$ solves $y'=\cos x+y$

17) $y=\pi e^{-\cos x}$ solves $y'=y\sin x$

In exercises 18 - 27, verify the given general solution and find the particular solution.

18) Find the particular solution to the differential equation $y' = 4x^2$ that passes through $(-3, -30)$, given that $y = C + \dfrac{4x^3}{3}$ is a general solution.

19) Find the particular solution to the differential equation $y' = 3x^3$ that passes through $(1, 4.75)$, given that $y = C + \dfrac{3x^4}{4}$ is a general solution.

Answer

$$y = 4 + \frac{3x^4}{4}$$

20) Find the particular solution to the differential equation $y' = 3x^2 y$ that passes through $(0, 12)$, given that $y = Ce^{x^3}$ is a general solution.

21) Find the particular solution to the differential equation $y' = 2xy$ that passes through $\left(0, \frac{1}{2}\right)$, given that $y = Ce^{x^2}$ is a general solution.

Answer

$$y = \tfrac{1}{2}e^{x^2}$$

22) Find the particular solution to the differential equation $y' = \left(2xy\right)^2$ that passes through $\left(1, -\frac{1}{2}\right)$, given that $y = -\dfrac{3}{C + 4x^3}$ is a general solution.

23) Find the particular solution to the differential equation $y'x^2 = y$ that passes through $\left(1, \frac{2}{e}\right)$, given that $y = Ce^{-1/x}$ is a general solution.

Answer

$$y = 2e^{-1/x}$$

24) Find the particular solution to the differential equation $8\dfrac{dx}{dt} = -2\cos(2t) - \cos(4t)$ that passes through (π, π), given that $x = C - \frac{1}{8}\sin(2t) - \frac{1}{32}\sin(4t)$ is a general solution.

25) Find the particular solution to the differential equation $\dfrac{du}{dt} = \tan u$ that passes through $\left(1, \frac{\pi}{2}\right)$, given that $u = \sin^{-1}\left(e^{C+t}\right)$ is a general solution.

Answer

$$u = \sin^{-1}\left(e^{-1+t}\right)$$

26) Find the particular solution to the differential equation $\dfrac{dy}{dt} = e^{t+y}$ that passes through $(1, 0)$, given that $y = -\ln(C - e^t)$ is a general solution.

27) Find the particular solution to the differential equation $y'(1 - x^2) = 1 + y$ that passes through $(0, -2)$, given that $y = C\dfrac{\sqrt{x+1}}{\sqrt{1-x}} - 1$ is a general solution.

Answer

$$y = -\frac{\sqrt{x+1}}{\sqrt{1-x}} - 1$$

In exercises 28 - 37, find the general solution to the differential equation.

28) $y' = 3x + e^x$

29) $y' = \ln x + \tan x$

Answer

$$y = C - x + x \ln x - \ln(\cos x)$$

30) $y' = \sin x e^{\cos x}$

31) $y' = 4^x$

Answer

$$y = C + \frac{4^x}{\ln 4}$$

32) $y' = \sin^{-1}(2x)$

33) $y' = 2t\sqrt{t^2 + 16}$

Answer

$$y = \tfrac{2}{3}\sqrt{t^2 + 16}\left(t^2 + 16\right) + C$$

34) $x' = \coth t + \ln t + 3t^2$

35) $x' = t\sqrt{4 + t}$

Answer

$$x = \tfrac{2}{15}\sqrt{4 + t}\left(3t^2 + 4t - 32\right) + C$$

36) $y' = y$

37) $y' = \dfrac{y}{x}$

Answer

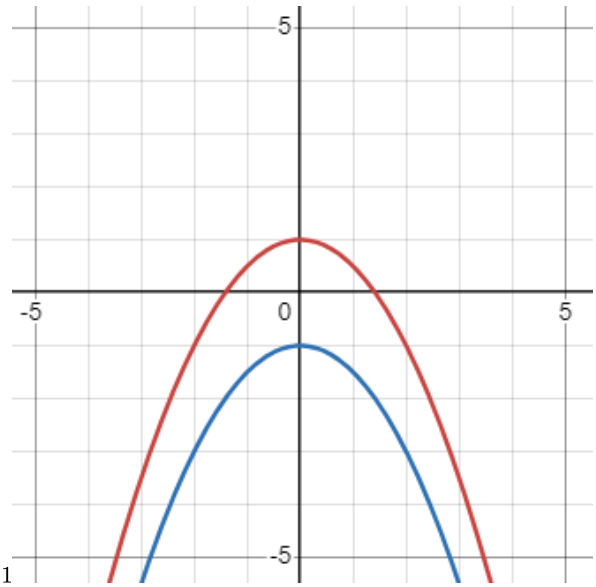
$$y = Cx$$

In exercises 38 - 42, solve the initial-value problems starting from $y(0) = 1$ and $y(0) = -1$. Draw both solutions on the same graph.

38) $\dfrac{dy}{dt} = 2t$

39) $\dfrac{dy}{dt} = -t$

Answer

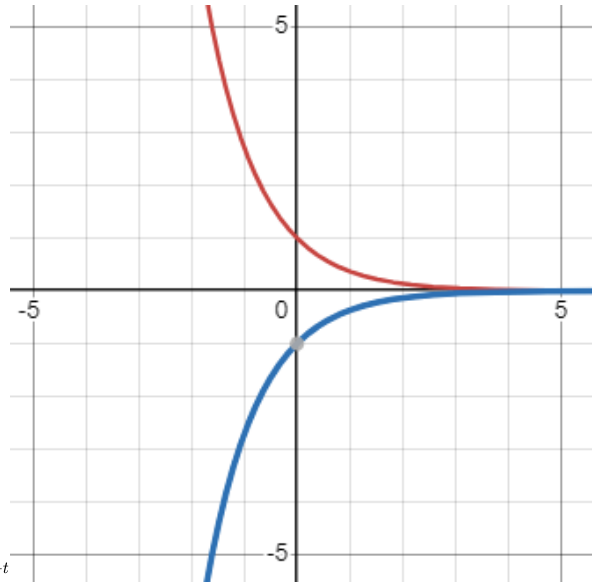


$y = 1 - \frac{t^2}{2}$, and $y = -\frac{t^2}{2} - 1$

40) $\frac{dy}{dt} = 2y$

41) $\frac{dy}{dt} = -y$

Answer



$y = e^{-t}$ and $y = -e^{-t}$

42) $\frac{dy}{dt} = 2$

In exercises 43 - 47, solve the initial-value problems starting from $y_0 = 10$.

43) $\frac{dy}{dt} = 4t$

Answer

$y = 2(t^2 + 5)$

44) $\frac{dy}{dt} = 4y$

45) $\frac{dy}{dt} = -2y$

Answer

$y = 10e^{-2t}$

46) $\frac{dy}{dt} = e^{4t}$

47) $\frac{dy}{dt} = e^{-4t}$

Answer

$y = \frac{1}{4}(41 - e^{-4t})$, Neither condition will ever happen.

Recall that a family of solutions includes solutions to a differential equation that differ by a constant. For exercises 48 - 52, use your calculator to graph a family of solutions to the given differential equation. Use initial conditions from $y(t = 0) = -10$ to $y(t = 0) = 10$ increasing by 2. Is there some critical point where the behavior of the solution begins to change?

48) [T] $y' = y(x)$

49) [T] $xy' = y$

Answer

Solution changes from increasing to decreasing at $y(0) = 0$.

50) [T] $y' = t^3$

51) [T] $y' = x + y$ (Hint: $y = Ce^x - x - 1$ is the general solution)

Answer

Solution changes from increasing to decreasing at $y(0) = 0$.

52) [T] $y' = x \ln x + \sin x$

53) Find the general solution to find the velocity of a ball, $v(t)$, of mass 1 kg that is thrown upward at a rate of v_0 m/sec. Assume the acceleration of gravity on Earth is $a = -9.8$ m/s^2. Assume the ball is thrown is from an initial height of 0 meters. Hint: Integrate acceleration to find the velocity.

Answer

$v(t) = -9.8t + v_0$

54) In the preceding problem, if the initial velocity of the ball thrown into the air is $v_0 = 25$ m/s, find the particular solution of the velocity of the ball. Solve to find the time when the ball hits the ground.

Answer

$v(t) = -9.8t + 25, \quad t = 5.1$ sec

55) In the preceding problem, if the initial height of the ball is $h_0 = 1$ m, find the particular solution of the height of the ball.

Answer

$h(t) = -4.9t + 25t + 1$

56) You throw two objects with differing masses m_1 and m_2 upward into the air with the same initial velocity of v_0 ft/s. What is the difference in their velocity after 1 second?

Answer

0 ft/s

57) [T] You throw a ball of mass 1 kilogram upward with a velocity of $v_0 = 25$ m/s on Mars, where the acceleration of gravity is $a = -3.711$ m/s^2. Use your calculator to approximate how much longer the ball is in the air on Mars.

58) [T] For the previous problem, use your calculator to approximate how much higher the ball went on Mars.

Answer

4.86 meters

59) [T] A car on the freeway accelerates according to $a = 15\cos(\pi t)$, where t is measured in hours. Set up and solve the differential equation to determine the velocity of the car if it has an initial speed of 50 mph. After 40 minutes of driving, what is the driver's velocity?

Answer

$v(t) = \frac{15}{\pi}\sin(\pi t) + 50$

60) [T] For the car in the preceding problem, find the expression for the distance the car has traveled in time t, assuming an initial distance of 0 miles. How long does it take the car to travel 100 miles? Use Desmos and round your answer to hours and minutes.

Answer

$x(t) = -\frac{15}{\pi^2}\cos(\pi t) + 50t,\ 2$ hours 1 minute

61) [T] For the previous problem, find the total distance traveled in the first hour.

62) Substitute $y = Be^{3t}$ into $y' - y = 8e^{3t}$ to find a particular solution.

Answer

$y = 4e^{3t}$

63) Substitute $y = a\cos(2t) + b\sin(2t)$ into $y' + y = 4\sin(2t)$ to find a particular solution.

64) Substitute $y = a + bt + ct^2$ into $y' + y = 1 + t^2$ to find a particular solution.

Answer

$y = 1 - 2t + t^2$

65) Substitute $y = ae^t\cos t + be^t\sin t$ into $y' = 2e^t\cos t$ to find a particular solution.

66) Solve $y' = e^{kt}$ with the initial condition $y(0) = 0$ and solve $y' = 1$ with the same initial condition. As k approaches 0, what do you notice?

Answer

$y = \frac{1}{k}(e^{kt} - 1)$ and $y = t$

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6.2: DIRECTION FIELDS AND NUMERICAL METHODS

8.02: Direction Fields and Numerical Methods

6.2E: Exercises for Section 6.2

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6.2E: Exercises for Section 6.2

8.2E: Exercises for Section 8.2

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6.3: SEPARABLE EQUATIONS

8.03: Separable Equations

6.3E: Exercises for Section 6.3

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6.3E: Exercises for Section 6.3

In exercises 1 - 4, solve the following initial-value problems with the initial condition $y_0 = 0$ and graph the solution.

1) $\frac{dy}{dt} = y + 1$

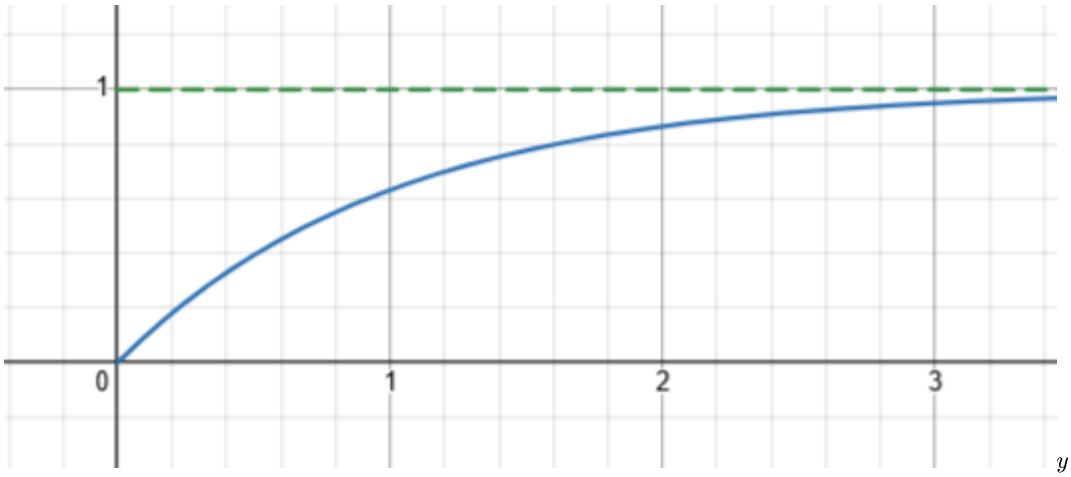
Answer

$y = e^t - 1$

2) $\frac{dy}{dt} = y - 1$

3) $\frac{dy}{dt} = -y + 1$

Answer



$y = 1 - e^{-t}$

4) $\frac{dy}{dt} = -y - 1$

In exercises 5 - 14, find the general solution to the differential equation.

5) $x^2 y' = (x + 1)y$

Answer

$y = Cxe^{-1/x}$

6) $y' = \tan(y)x$

7) $y' = 2xy^2$

Answer

$y = \frac{1}{C - x^2}$

8) $\frac{dy}{dt} = y\cos(3t + 2)$

9) $2x\frac{dy}{dx} = y^2$

Answer

$y = -\frac{2}{C + \ln|x|}$

10) $y' = e^y x^2$

11) $(1+x)y' = (x+2)(y-1)$

Answer

$$y = Ce^x(x+1)+1$$

12) $\dfrac{dx}{dt} = 3t^2(x^2+4)$

13) $t\dfrac{dy}{dt} = \sqrt{1-y^2}$

Answer

$$y = \sin(\ln|t| + C)$$

14) $y' = e^x e^y$

In exercises 15 - 24, find the solution to the initial-value problem.

15) $y' = e^{y-x}, \quad y(0) = 0$

Answer

$y = -\ln(e^{-x})$ which simplifies to $y = x$

16) $y' = y^2(x+1), \quad y(0) = 2$

17) $\dfrac{dy}{dx} = y^3 x e^{x^2}, \quad y(0) = 1$

Answer

$$y = \frac{1}{\sqrt{2-e^{x^2}}}$$

18) $\dfrac{dy}{dt} = y^2 e^x \sin(3x), \quad y(0) = 1$

19) $y' = \dfrac{x}{\operatorname{sech}^2 y}, \quad y(0) = 0$

Answer

$$y = \tanh^{-1}\left(\frac{x^2}{2}\right)$$

20) $y' = 2xy(1+2y), \quad y(0) = -1$

21) $\dfrac{dx}{dt} = \ln(t)\sqrt{1-x^2}, \quad x(1) = 0$

Answer

$$x = \sin(1 - t + t\ln t)$$

22) $y' = 3x^2(y^2+4), \quad y(0) = 0$

23) $y' = e^y 5^x, \quad y(0) = \ln(\ln(5))$

Answer

$$y = \ln(\ln(5)) - \ln(2-5^x)$$

24) $y' = -2x\tan(y), \quad y(0) = \dfrac{\pi}{2}$

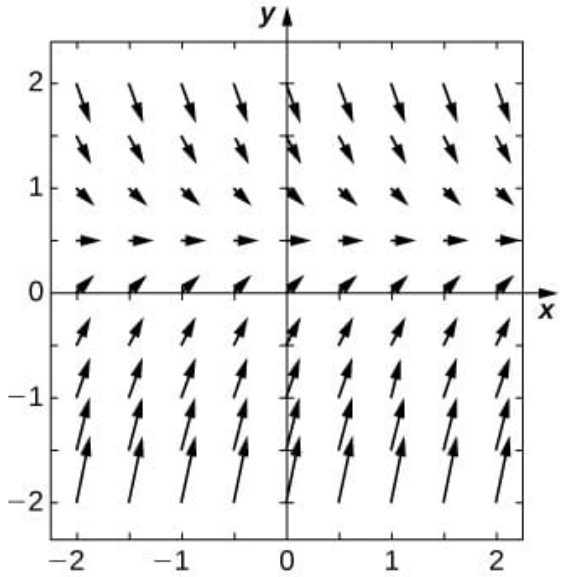
For problems 25 - 29, use a software program or your calculator to generate the directional fields. Solve explicitly and

draw solution curves for several initial conditions. Are there some critical initial conditions that change the behavior of the solution?

25) [T] $y' = 1 - 2y$

Answer

$$y = Ce^{-2}x + \frac{1}{2}$$

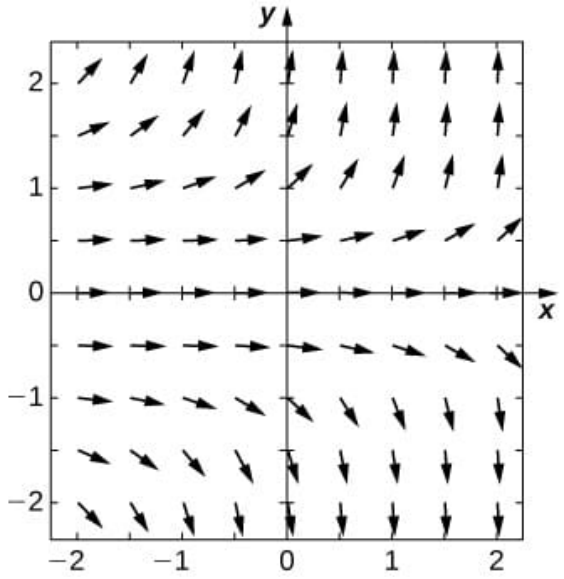


26) [T] $y' = y^2x^3$

27) [T] $y' = y^3e^x$

Answer

$$y = \frac{1}{\sqrt{2}\sqrt{C - e^x}}$$

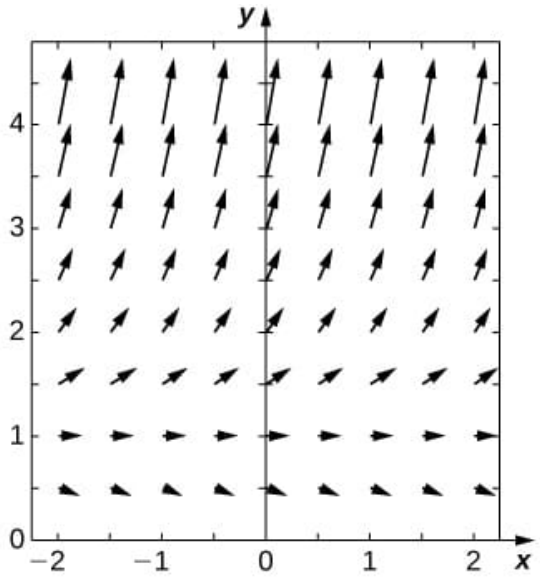


28) [T] $y' = e^y$

29) [T] $y' = y\ln(x)$

Answer

$y = Ce^{-x}x^x$



30) Most **drugs in the bloodstream** decay according to the equation $y' = ky$, where y is the concentration of the drug in the bloodstream. If the half-life of a drug is 2 hours, what fraction of the initial dose remains after 6 hours?

Answer

$\frac{y(6)}{y(0)} = \frac{1}{8}$

31) A drug is administered intravenously to a patient at a rate r mg/h and is cleared from the body at a rate proportional to the amount of drug still present in the body, y. Set up and solve the differential equation for y, assuming there is no drug initially present in the body.

Answer

$y' = r - ky, \quad y = \frac{r}{k}(1 - e^{-kt})$

32) [T] How often should a drug be taken if its dose is 3 mg, it is cleared at a rate $c = 0.1$ mg/h, and 1 mg is required to be in the bloodstream at all times?

33) A tank contains 1 kilogram of salt dissolved in 100 liters of water. A salt solution of 0.1 kg salt/L is pumped into the tank at a rate of 2 L/min and is drained at the same rate. Solve for the amount of salt in the tank, y, at time t. Assume the tank is well mixed. Example: https://youtu.be/oW-yuGNaxq0

Answer

$y' = 0.2 - 0.02y, \quad y(t) = 10 - 9e^{-t/50}$

34) A tank containing 10 kilograms of salt dissolved in 1000 liters of water has two salt solutions pumped in. The first solution of 0.2 kg salt/L is pumped in at a rate of 20 L/min and the second solution of 0.05 kg salt/L is pumped in at a rate of 5 L/min. The tank drains at 25 L/min. Assume the tank is well mixed. Solve for the salt concentration at time t.

35) [T] For the preceding problem, find how much salt is in the tank 1 hour after the process begins.

Answer

134.3 kilograms

36) Torricelli's law states that for a water tank with a hole in the bottom that has a cross-section of A and with a height of water h

above the bottom of the tank, the rate of change of volume of water flowing from the tank is proportional to the square root of the height of water, according to $\frac{dV}{dt} = -A\sqrt{2gh}$, where g is the acceleration due to gravity. Note that $\frac{dV}{dt} = A\frac{dh}{dt}$. Solve the resulting initial-value problem for the height of the water, assuming a tank with a hole of radius 2 ft. The initial height of water is 100 ft.

37) For the preceding problem, determine how long it takes the tank to drain.

Answer

720 seconds

For problems 38 - 44, use Newton's law of cooling.

38) The liquid base of an ice cream has an initial temperature of $200\,°F$ before it is placed in a freezer with a constant temperature of $0\,°F$. After 1 hour, the temperature of the ice-cream base has decreased to $140\,°F$. Formulate and solve the initial-value problem to determine the temperature of the ice cream.

39) [T] The liquid base of an ice cream has an initial temperature of $210\,°F$ before it is placed in a freezer with a constant temperature of $20\,°F$. After 2 hours, the temperature of the ice-cream base has decreased to $170\,°F$. At what time will the ice cream be ready to eat? (Assume $30\,°F$ is the optimal eating temperature.)

Answer

$T(t) = 20 + 190e^{-0.118t}$, $\quad 24$ hours 55 minutes

40) [T] You are organizing an ice cream social. The outside temperature is $80\,°F$ and the ice cream is at $10\,°F$. After 10 minutes, the ice cream temperature has risen by $10\,°F$. How much longer can you wait before the ice cream melts at $40\,°F$?

41) You have a cup of coffee at temperature $70\,°C$ and the ambient temperature in the room is $20\,°C$. Assuming a cooling rate k of 0.125, write and solve the differential equation to describe the temperature of the coffee with respect to time.

Answer

$T(t) = 20 + 50e^{-0.125t}$

42) [T] You have a cup of coffee at temperature $70\,°C$ that you put outside, where the ambient temperature is $0\,°C$. After 5 minutes, how much colder is the coffee?

43) You have a cup of coffee at temperature $70\,°C$ and you immediately pour in 1 part milk to 5 parts coffee. The milk is initially at temperature $1\,°C$. Write and solve the differential equation that governs the temperature of this coffee.

Answer

$T(t) = 20 + 38.5e^{-0.125t}$

44) You have a cup of coffee at temperature $70\,°C$, which you let cool 10 minutes before you pour in the same amount of milk at $1\,°C$ as in the preceding problem. How does the temperature compare to the previous cup after 10 minutes?

45) Solve the generic problem $y' = ay + b$ with initial condition $y(0) = c$.

Answer

$y = (c + ba)e^{ax} - \frac{b}{a}$

46) Prove the basic continual compounded interest equation. Assuming an initial deposit of P_0 and an interest rate of r, set up and solve an equation for continually compounded interest.

47) Assume an initial nutrient amount of I kilograms in a tank with L liters. Assume a concentration of c kg/L being pumped in at a rate of r L/min. The tank is well mixed and is drained at a rate of r L/min. Find the equation describing the amount of nutrient in the tank.

Answer

$y(t) = cL + (I - cL)e^{-rt/L}$

48) Leaves accumulate on the forest floor at a rate of 2 g/cm^2/yr and also decompose at a rate of 90% per year. Write a differential

equation governing the number of grams of leaf litter per square centimeter of forest floor, assuming at time 0 there is no leaf litter on the ground. Does this amount approach a steady value? What is that value?

49) Leaves accumulate on the forest floor at a rate of 4 g/cm^2/yr. These leaves decompose at a rate of 10% per year. Write a differential equation governing the number of grams of leaf litter per square centimeter of forest floor. Does this amount approach a steady value? What is that value?

Answer

Differential Equation: $\frac{dy}{dt} = 4 - 0.1y$

Solution, the model for this situation: $y = 40(1 - e^{-0.1t})$,

Amount approaches a steady value of 40 g/cm^2

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6.4: THE LOGISTIC EQUATION

8.04: The Logistic Equation

6.4E: Exercises for Section 6.4

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6.4E: Exercises for Section 6.4

8.4E: Exercises for Section 8.4

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6.5: FIRST-ORDER LINEAR EQUATIONS

8.05: First-order Linear Equations

6.5E: Exercises for Section 6.5

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6.5E: Exercises for Section 6.5

8.5E: Exercises for Section 8.5

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6.6: Chapter 6 Review Exercises

8.06: Chapter 8 Review Exercises

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7: APPLICATIONS OF INTEGRATION

In this chapter, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.

7.0: Prelude to Applications of Integration

7.1: Areas between Curves

7.1E: Exercises for Section 7.1

7.2: Determining Volumes by Slicing

7.2E: Exercises for Section 7.2

7.3: Volumes of Revolution - Cylindrical Shells

7.3E: Exercises for Section 7.3

7.4: Arc Length of a Curve and Surface Area

7.4E: Exercises for Section 7.4

7.5: Physical Applications of Integration

7.5E: Exercises for Section 7.5

7.6: Moments and Centers of Mass

7.6E: Exercises for Section 7.6

7.7: Integrals, Exponential Functions, and Logarithms

7.7E: Exercises for Section 7.7

7.8: Exponential Growth and Decay

7.8E: Exercises for Section 7.8

7.9: Chapter 6 Review Exercises

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7.0: Prelude to Applications of Integration

6.00: Prelude to Applications of Integration

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7.1: AREAS BETWEEN CURVES

6.01: Areas between Curves

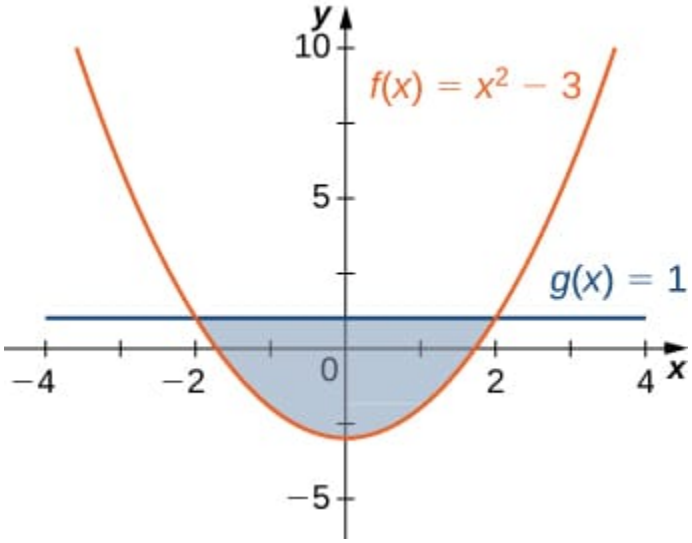
7.1E: Exercises for Section 7.1

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7.1E: Exercises for Section 7.1

For exercises 1 - 2, determine the area of the region between the two curves in the given figure by integrating over the x-axis.

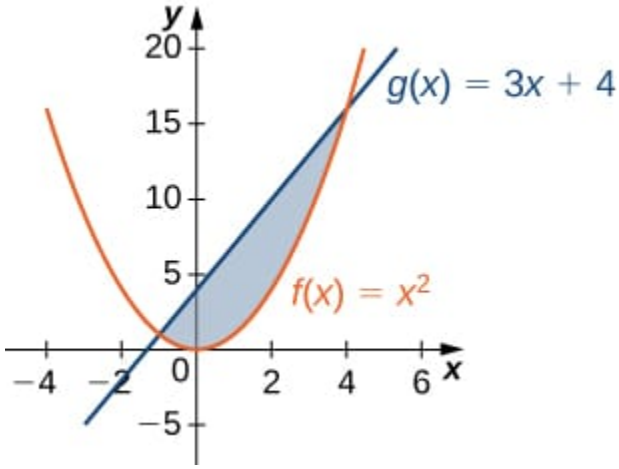
1) $y = x^2 - 3$ and $y = 1$



Answer

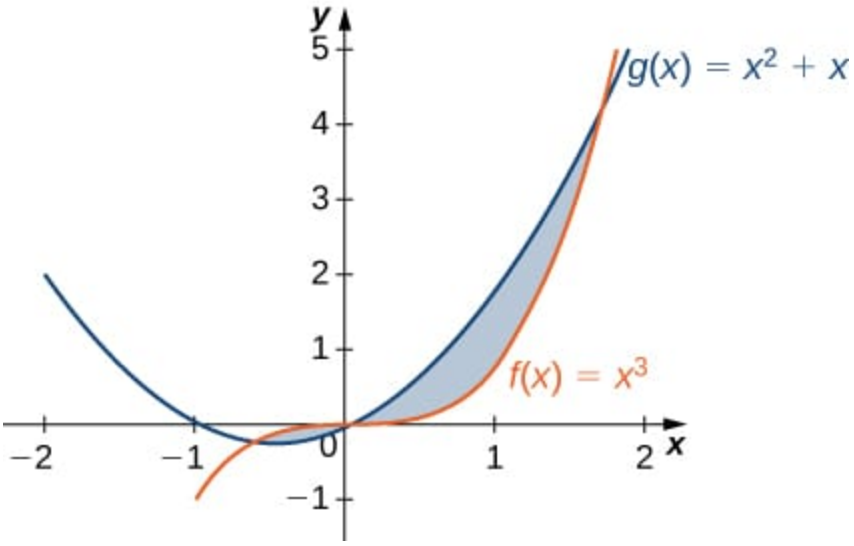
$\frac{32}{3}$ units^2

2) $y = x^2$ and $y = 3x + 4$



For exercises 3 - 4, split the region between the two curves into two smaller regions, then determine the area by integrating over the x-axis. Note that you will have two integrals to solve.

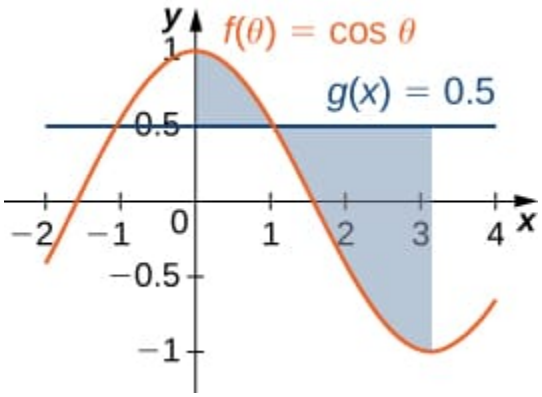
3) $y = x^3$ and $y = x^2 + x$



Answer

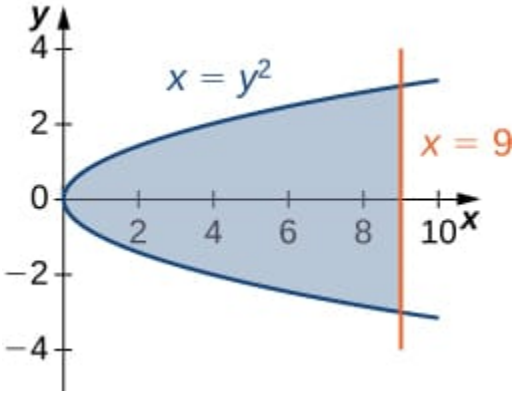
$\frac{13}{12}$ units^2

4) $y = \cos\theta$ and $y = 0.5$, for $0 \le \theta \le \pi$



For exercises 5-6, determine the area of the region between the two curves by integrating over the y-axis.

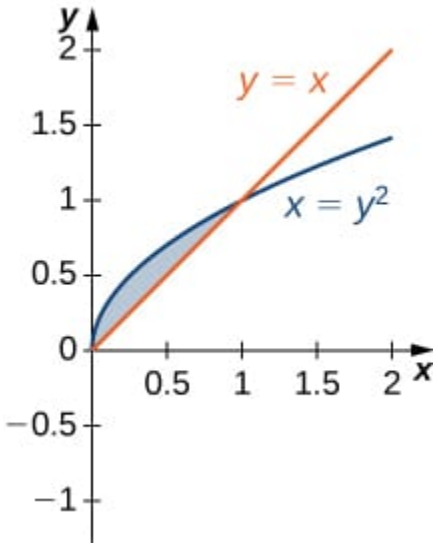
5) $x = y^2$ and $x = 9$



Answer

36 units^2

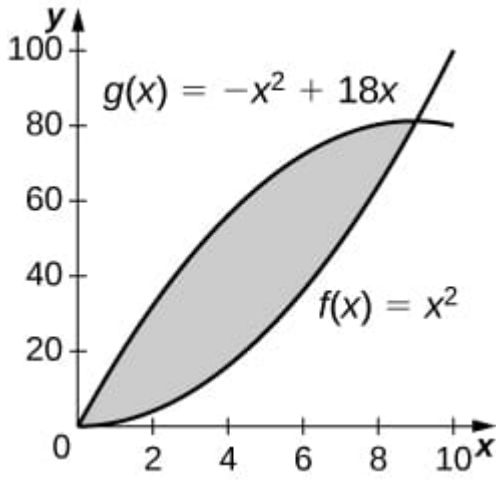
6) $y = x$ and $x = y^2$



For exercises 7 - 13, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis.

7) $y = x^2$ and $y = -x^2 + 18x$

Answer



243 square units

8) $y = \dfrac{1}{x}$, $y = \dfrac{1}{x^2}$, and $x = 3$

9) $y = \cos x$ and $y = \cos^2 x$ on $x \in [-\pi, \pi]$

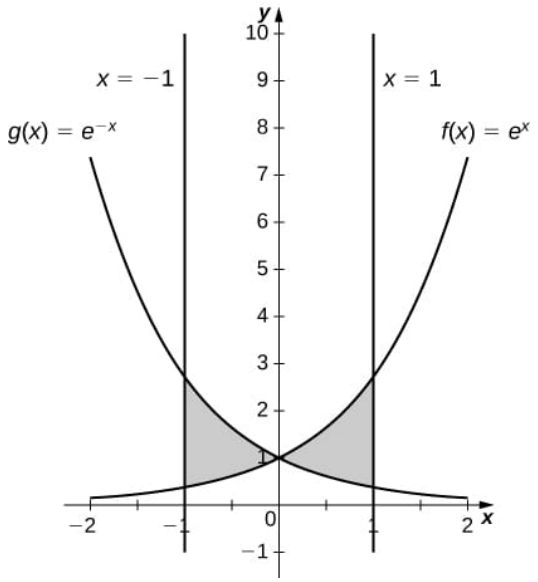
Answer

4 square units

10) $y = e^x$, $y = e^{2x-1}$, and $x = 0$

11) $y = e^x, \quad y = e^{-x}, \quad x = -1$ and $x = 1$

Answer

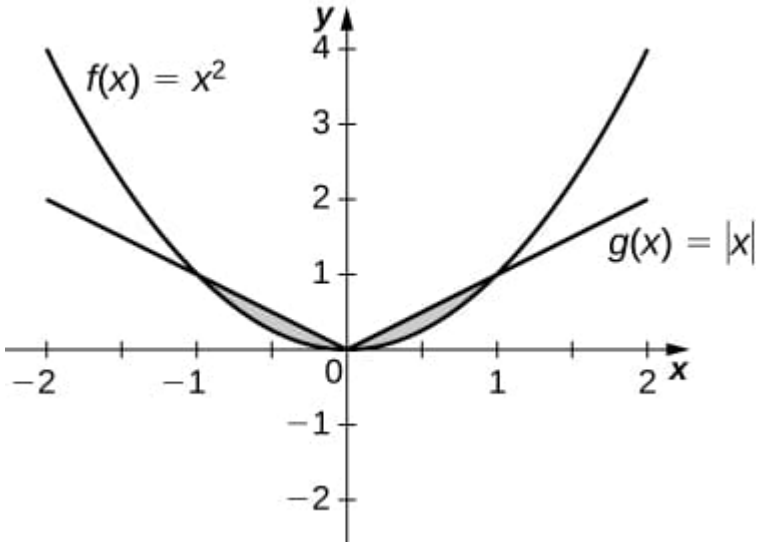


$\frac{2(e-1)^2}{e}$ units^2

12) $y = e, \quad y = e^x$, and $y = e^{-x}$

13) $y = |x|$ and $y = x^2$

Answer



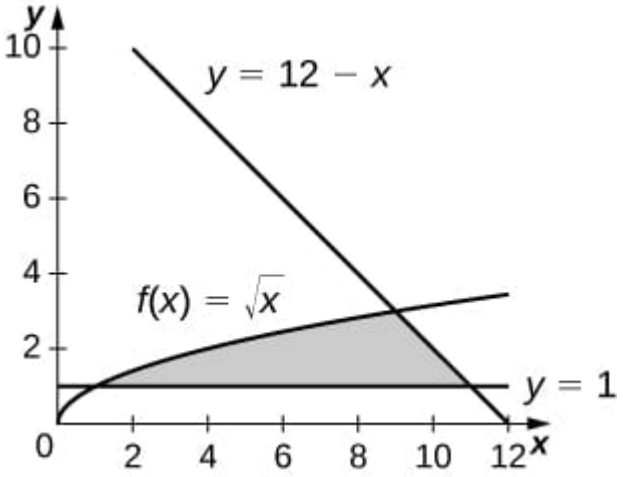
$\frac{1}{3}$ units^2

For exercises 14 - 19, graph the equations and shade the area of the region between the curves. If necessary, break the region into sub-regions to determine its entire area.

14) $y = \sin(\pi x), \quad y = 2x$, and $x > 0$

15) $y = 12 - x, \quad y = \sqrt{x}, \text{ and } y = 1$

Answer

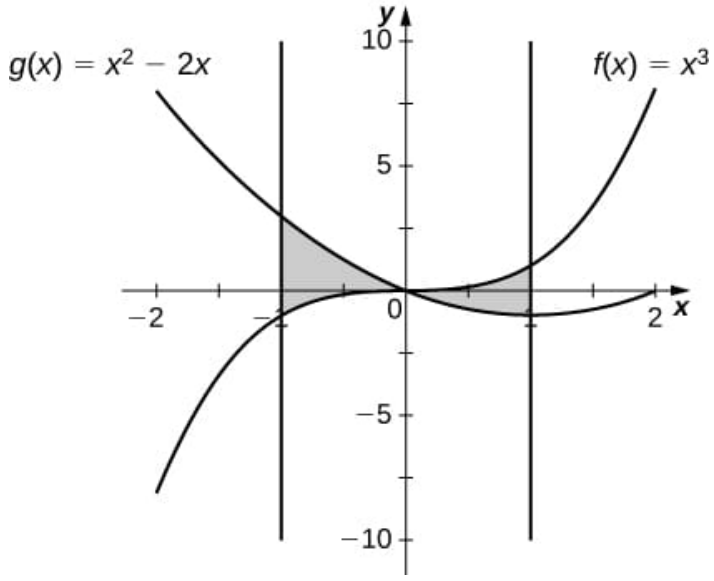


$\frac{34}{3}$ units^2

16) $y = \sin x$ and $y = \cos x$ over $x \in [-\pi, \pi]$

17) $y = x^3$ and $y = x^2 - 2x$ over $x \in [-1, 1]$

Answer

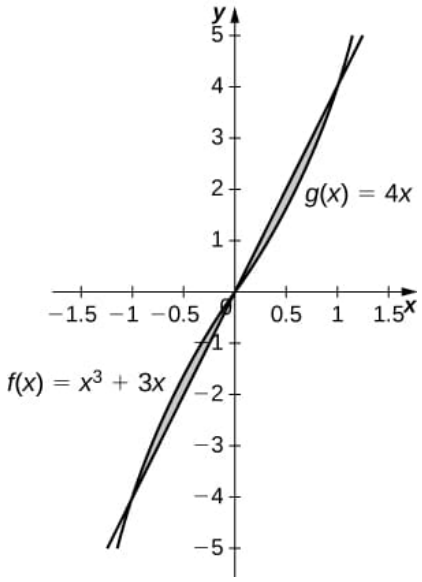


$\frac{5}{2}$ units^2

18) $y = x^2 + 9$ and $y = 10 + 2x$ over $x \in [-1, 3]$

19) $y = x^3 + 3x$ and $y = 4x$

Answer



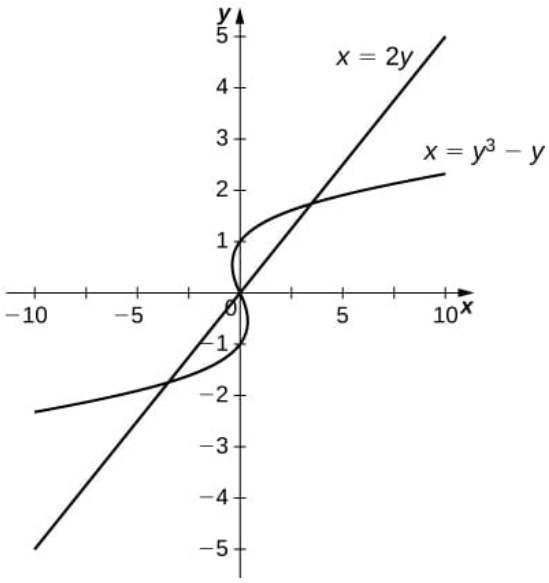
$\frac{1}{2}$ units^2

For exercises 20 -25, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the y-axis.

20) $x = y^3$ and $x = 3y - 2$

21) $x = 2y$ and $x = y^3 - y$

Answer

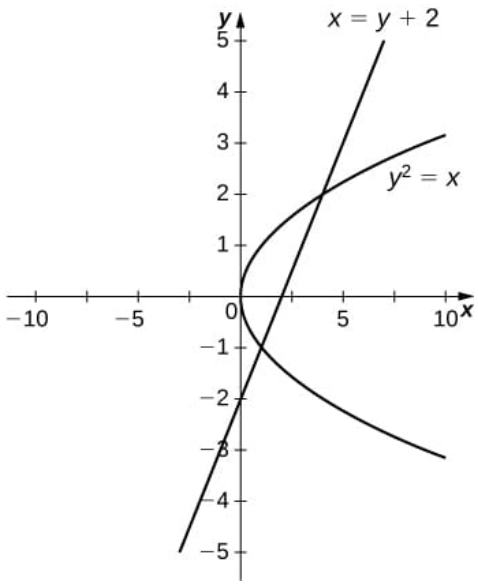


$\frac{9}{2}$ units^2

22) $x = -3 + y^2$ and $x = y - y^2$

23) $y^2 = x$ and $x = y + 2$

Answer

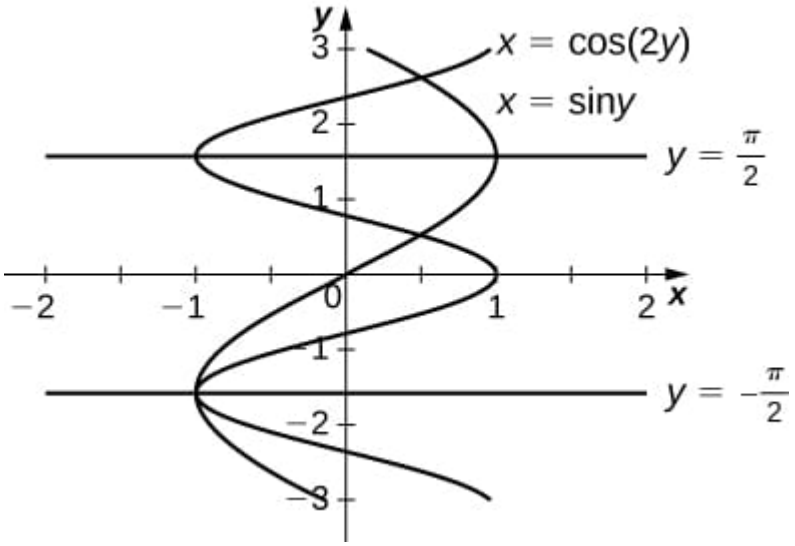


$\frac{9}{2}$ units^2

24) $x = |y|$ and $2x = -y^2 + 2$

25) $x = \sin y, \quad x = \cos(2y), \quad y = \pi/2$, and $y = -\pi/2$

Answer



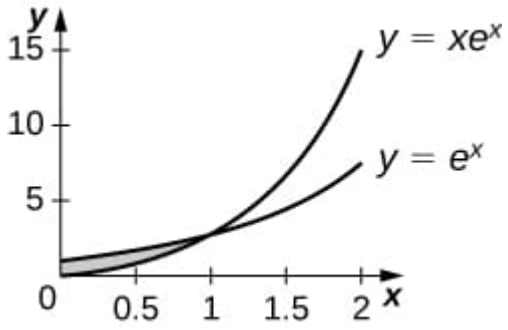
$\frac{3\sqrt{3}}{2}$ units^2

For exercises 26 - 37, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis or y-axis, whichever seems more convenient.

26) $x = y^4$ and $x = y^5$

27) $y = xe^x, \quad y = e^x,$ and $x = 0$

Answer

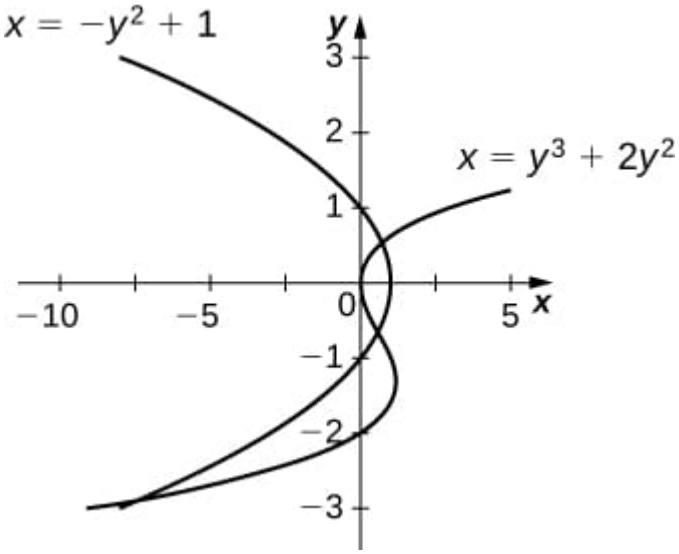


$e - 2$ units^2

28) $y = x^6$ and $y = x^4$

29) $x = y^3 + 2y^2 + 1$ and $x = -y^2 + 1$

Answer

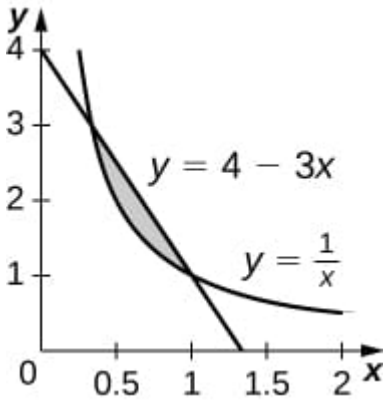


$\frac{27}{4}$ units^2

30) $y = |x|$ and $y = x^2 - 1$

31) $y = 4 - 3x$ and $y = \frac{1}{x}$

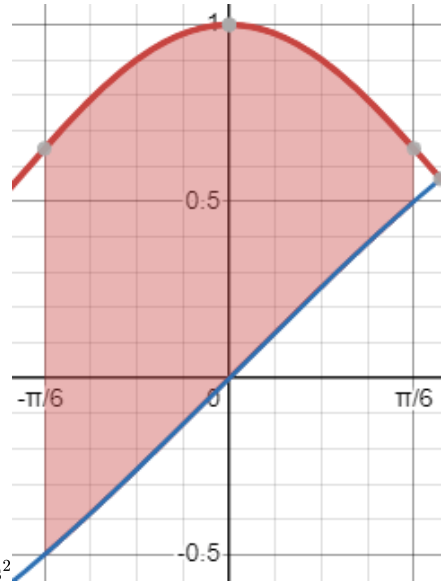
Answer



$$\left(\frac{4}{3} - \ln(3)\right) \text{ units}^2$$

32) $y = \sin x, \quad x = -\pi/6, \quad x = \pi/6$, and $y = \cos^3 x$

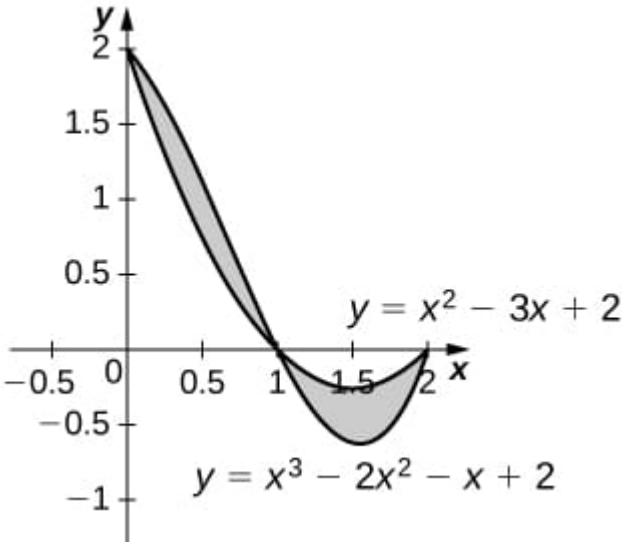
Answer



$\frac{11}{12}\text{units}^2$

33) $y = x^2 - 3x + 2$ and $y = x^3 - 2x^2 - x + 2$

Answer

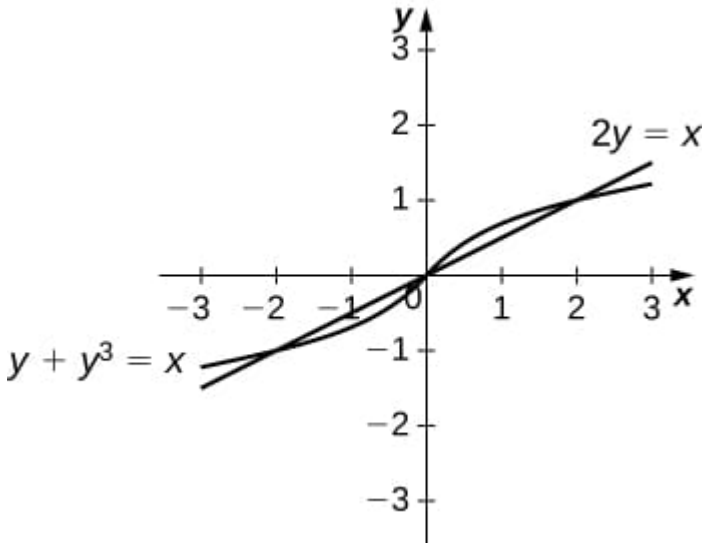


$\frac{1}{2}$ square units

34) $y = 2\cos^3(3x), \quad y = -1, \quad x = \frac{\pi}{4}$, and $x = -\frac{\pi}{4}$

35) $y + y^3 = x$ and $2y = x$

Answer

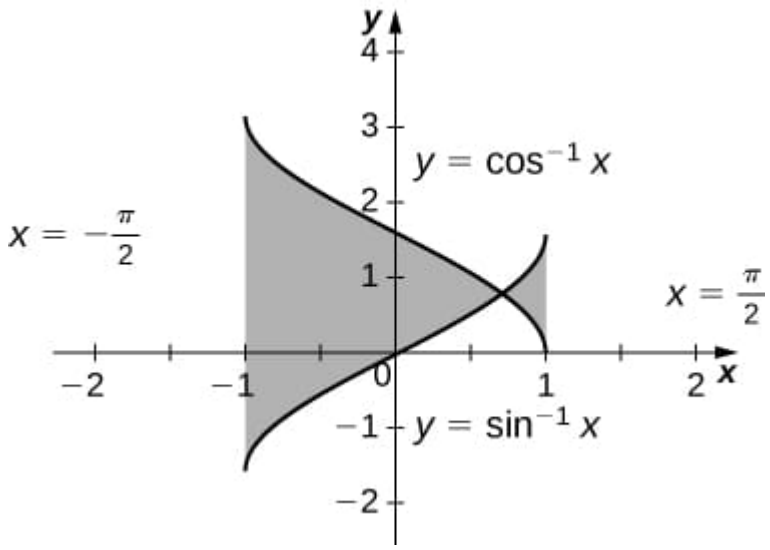


$\frac{1}{2}$ square units

36) $y = \sqrt{1 - x^2}$ and $y = x^2 - 1$

37) $y = \cos^{-1} x, \quad y = \sin^{-1} x, \quad x = -1$, and $x = 1$

Answer



$-2(\sqrt{2} - \pi)$ square units

For exercises 38 - 47, find the exact area of the region bounded by the given equations if possible. If you are unable to determine the intersection points analytically, use a calculator to approximate the intersection points with three decimal places and determine the approximate area of the region.

38) [T] $x = e^y$ and $y = x - 2$

39) [T] $y = x^2$ and $y = \sqrt{1 - x^2}$

Answer

1.067 square units

40) [T] $y = 3x^2 + 8x + 9$ and $3y = x + 24$

41) [T] $x = \sqrt{4 - y^2}$ and $y^2 = 1 + x^2$

Answer

0.852 square units

42) [T] $x^2 = y^3$ and $x = 3y$

43) [T] $y = \sin^3 x + 2, \quad y = \tan x, \quad x = -1.5$, and $x = 1.5$

Answer

7.523 square units

44) [T] $y = \sqrt{1 - x^2}$ and $y^2 = x^2$

45) [T] $y = \sqrt{1 - x^2}$ and $y = x^2 + 2x + 1$

Answer

$\frac{3\pi - 4}{12}$ square units

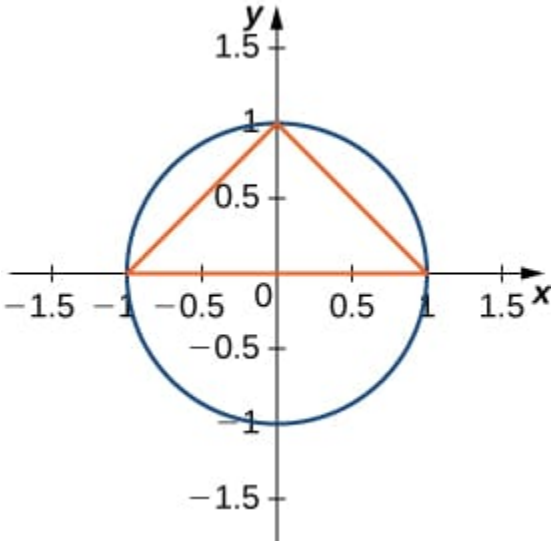
46) [T] $x = 4 - y^2$ and $x = 1 + 3y + y^2$

47) [T] $y = \cos x, \quad y = e^x, \quad x = -\pi, \quad$ and $\quad x = 0$

Answer

1.429 square units

48) The largest triangle with a base on the x-axis that fits inside the upper half of the unit circle $y^2 + x^2 = 1$ is given by $y = 1 + x$ and $y = 1 - x$. See the following figure. What is the area inside the semicircle but outside the triangle?



49) A factory selling cell phones has a marginal cost function $C'(x) = 0.01x^2 - 3x + 229$, where x represents the number of cell phones, and a marginal revenue function given by $R'(x) = 429 - 2x$. Find the area between the graphs of these curves and $x = 0$. What does this area represent?

Answer

$33,333.33 total profit for **200** cell phones sold

50) An amusement park has a marginal cost function $C(x) = 1000e - x + 5$, where x represents the number of tickets sold, and a marginal revenue function given by $R(x) = 60 - 0.1x$. Find the total profit generated when selling 550 tickets. Use a calculator to determine intersection points, if necessary, to two decimal places.

51) The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = 1 - \cos((\pi t)/2)$ whereas the speed of the tortoise is $T(t) = (1/2)\tan^{-1}(t/4)$, where t is time measured in hours and the speed is measured in miles per hour. Find the area between the curves from time $t = 0$ to the first time after one hour when the tortoise and hare are traveling at the same speed. What does it represent? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

Answer

3.263 mi represents how far ahead the hare is from the tortoise

52) The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = (1/2) - (1/2)\cos(2\pi t)$ whereas the speed of the tortoise is $T(t) = \sqrt{t}$, where t is time measured in hours and speed is measured in kilometers per hour. If the race is over in 1 hour, who won the race and by how much?

Answer

The tortoise traveled $\frac{2}{3}$ mile and the hare traveled $\frac{1}{2}$ mile, so the tortoise won!

For exercises 53 - 55, find the area between the curves by integrating with respect to x and then with respect to y. Is one method easier than the other? Do you obtain the same answer?

53) $y = x^2 + 2x + 1$ and $y = -x^2 - 3x + 4$

Answer

$\frac{343}{24}$ square units

54) $y = x^4$ and $x = y^5$

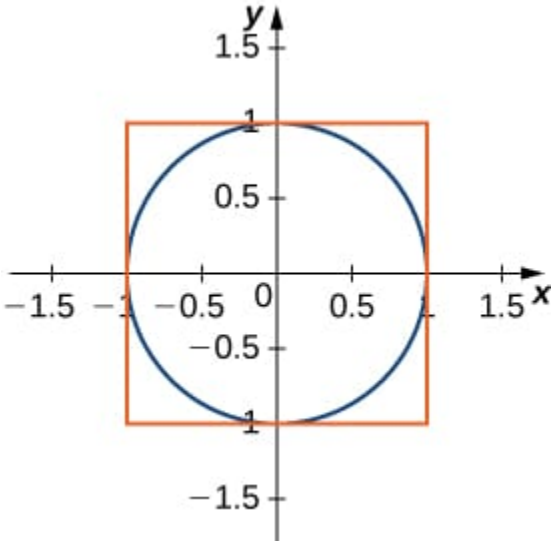
55) $x = y^2 - 2$ and $x = 2y$

Answer

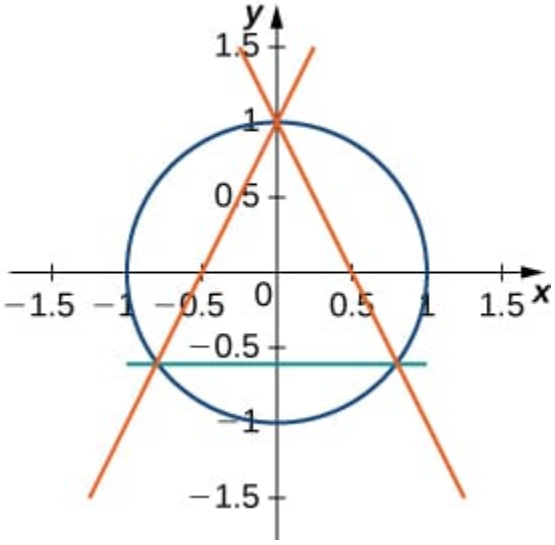
$4\sqrt{3}$ square units

For exercises 56 - 57, solve using calculus, then check your answer with geometry.

56) Determine the equations for the sides of the square that touches the unit circle on all four sides, as seen in the following figure. Find the area between the perimeter of this square and the unit circle. Is there another way to solve this without using calculus?



57) Find the area between the perimeter of the unit circle and the triangle created from $y = 2x + 1,\ y = 1 - 2x$ and $y = -\frac{3}{5}$, as seen in the following figure. Is there a way to solve this without using calculus?



Answer

$\left(\pi - \frac{32}{25}\right)$ square units

Contributors

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7.2: DETERMINING VOLUMES BY SLICING

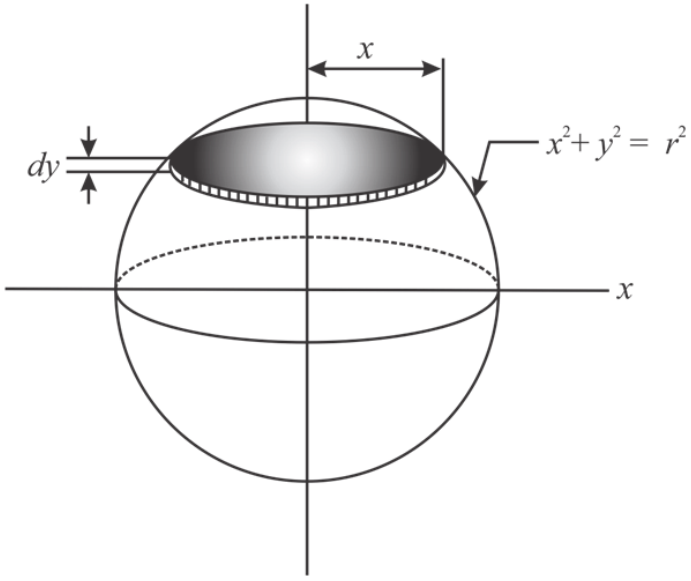
6.02: Determining Volumes by Slicing

7.2E: Exercises for Section 7.2

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7.2E: Exercises for Section 7.2

1) Derive the formula for the volume of a sphere of radius r using the slicing method.



Answer

$V = \frac{4\pi}{3}r^3$ units^3

2) Use the slicing method to derive the formula for the volume of a cone with height h and radius r.

Answer

$V = \frac{\pi}{3}r^2h$ units^3

3) Use the slicing method to derive the formula for the volume of a tetrahedron with side length a.

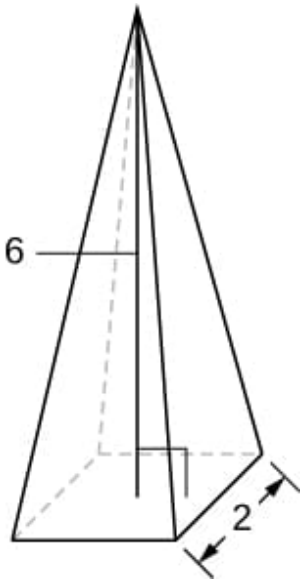
4) Use the disk method to derive the formula for the volume of a trapezoidal cylinder.

5) Explain when you would use the disk method versus the washer method. When are they interchangeable?

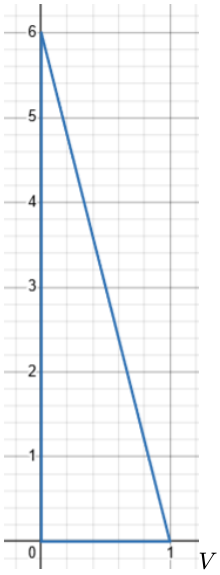
Volumes by Slicing

For exercises 6 - 10, draw a typical slice and find the volume using the slicing method for the given volume.

6) A pyramid with height 6 units and square base of side 2 units, as pictured here.



Answer:



$V = 8$ units^3

Solution:

Here the cross-sections are squares taken perpendicular to the y-axis.
We use the vertical cross-section of the pyramid through its center to obtain an equation relating x and y.
Here this would be the equation, $y = 6 - 6x$. Since we need the dimensions of the square at each y-level, we solve this equation for x to get, $x = 1 - \frac{y}{6}$.
This is half the distance across the square cross-section at the y-level, so the side length of the square cross-section is, $s = 2\left(1 - \frac{y}{6}\right)$.
Thus, we have the area of a cross-section is,

$$A(y) = \left[2\left(1 - \frac{y}{6}\right)\right]^2 = 4\left(1 - \frac{y}{6}\right)^2.$$

$$\text{Then,} \quad V = \int_0^6 4\left(1 - \tfrac{y}{6}\right)^2 \, dy$$

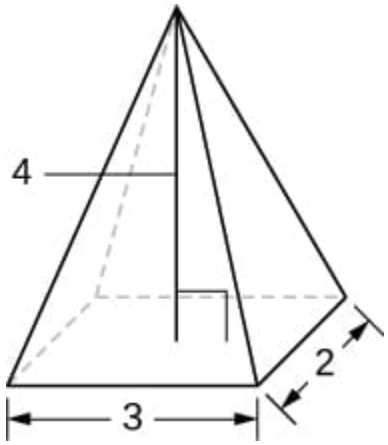
$$= -24 \int_1^0 u^2 \, du, \quad \text{where } u = 1 - \tfrac{y}{6}, \text{ so } du = -\tfrac{1}{6}\, dy, \quad \Longrightarrow \quad -6\, du = dy$$

$$= 24 \int_0^1 u^2 \, du = 24 \frac{u^3}{3}\Big|_0^1$$

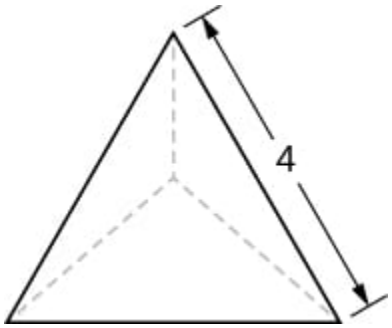
$$= 8u^3\Big|_0^1$$

$$= 8\left(1^3 - 0^3\right) \quad = \quad 8 \text{ units}^3$$

7) A pyramid with height 4 units and a rectangular base with length 2 units and width 3 units, as pictured here.



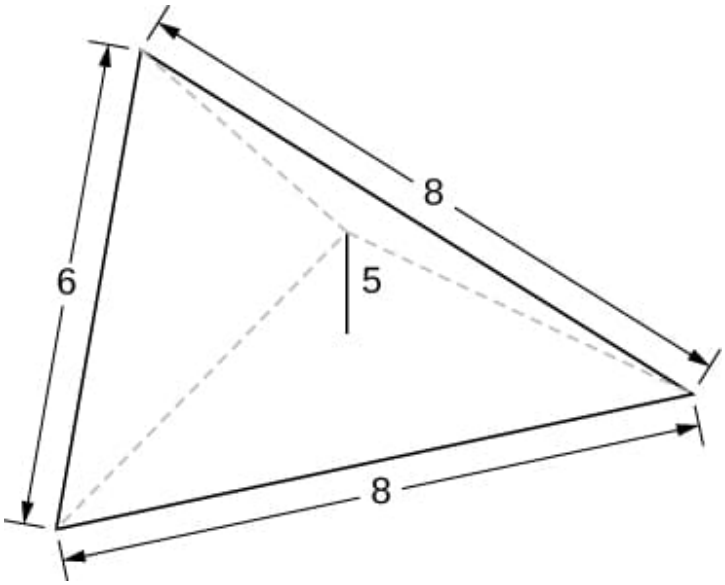
8) A tetrahedron with a base side of 4 units, as seen here.



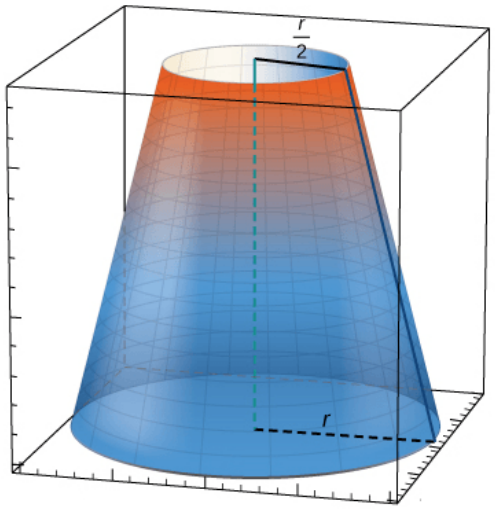
Answer

$V = \frac{32}{3\sqrt{2}} = \frac{16\sqrt{2}}{3}$ units^3

9) A pyramid with height 5 units, and an isosceles triangular base with lengths of 6 units and 8 units, as seen here.



10) A cone of radius r and height h has a smaller cone of radius $r/2$ and height $h/2$ removed from the top, as seen here. The resulting solid is called a frustum.



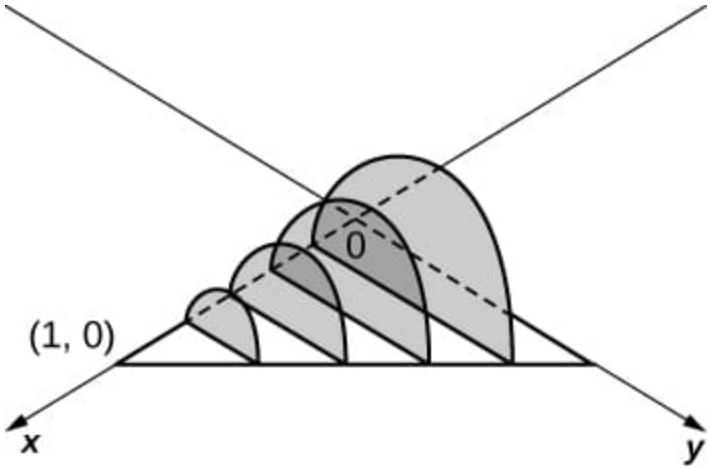
Answer

$V = \frac{7\pi}{24} r^2 h$ units^3

For exercises 11 - 16, draw an outline of the solid and find the volume using the slicing method.

11) The base is a circle of radius a. The slices perpendicular to the base are squares.

12) The base is a triangle with vertices $(0, 0), (1, 0)$, and $(0, 1)$. Slices perpendicular to the x-axis are semicircles.

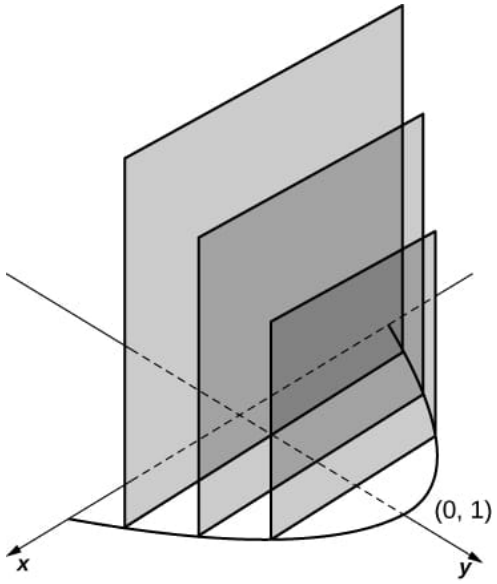


Answer

$$V = \int_0^1 \frac{\pi(1-x)^2}{8}\, dx \quad = \quad \frac{\pi}{24} \text{ units}^3$$

13) The base is the region under the parabola $y = 1 - x^2$ in the first quadrant. Slices perpendicular to the xy-plane are squares.

14) The base is the region under the parabola $y = 1 - x^2$ and above the x**-axis**. Slices perpendicular to the y**-axis** are squares.



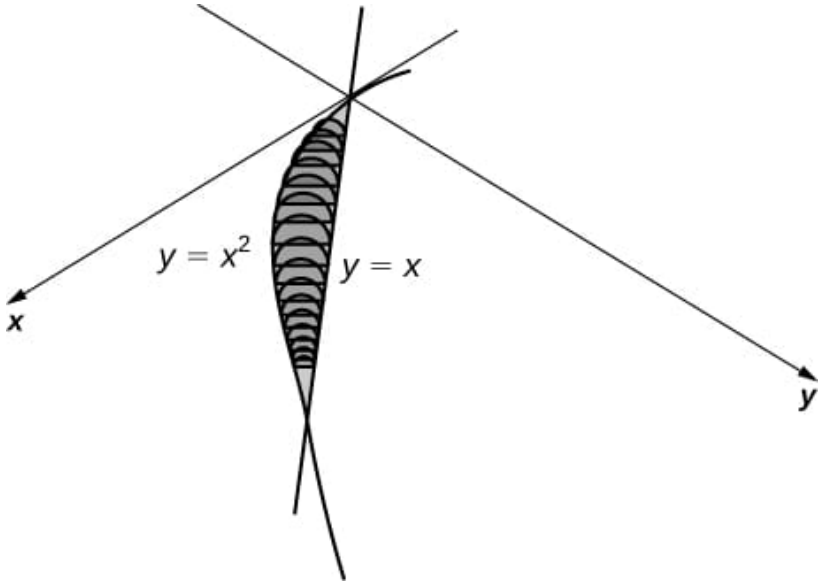
Answer

$$V = \int_0^1 4(1-y)\, dy \quad = \quad 2 \text{ units}^3$$

15) The base is the region enclosed by $y = x^2$ and $y = 9$. Slices perpendicular to the x-axis are right isosceles triangles.

16) The base is the area between $y = x$ and $y = x^2$. Slices perpendicular to the x-axis are semicircles.

Answer



$$V = \int_0^1 \frac{\pi}{8}\left(x - x^2\right)^2 \, dx \quad = \quad \frac{\pi}{240} \text{ units}^3$$

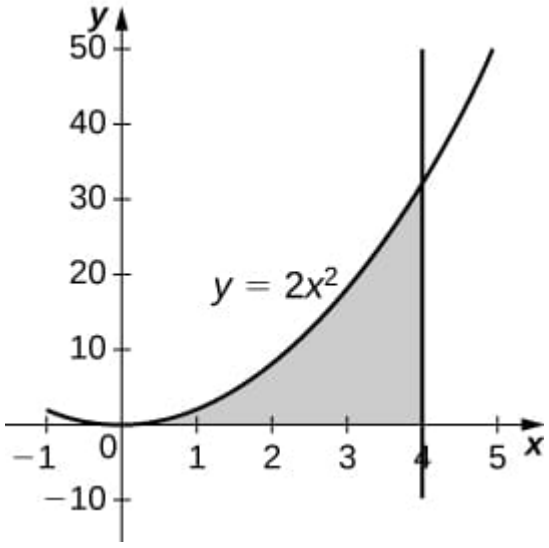
Disk and Washer Method

For exercises 17 - 24, draw the region bounded by the curves. Then, use the disk or washer method to find the volume when the region is rotated around the x-axis.

17) $x + y = 8, \quad x = 0$, and $y = 0$

18) $y = 2x^2, \quad x = 0, \quad x = 4, \quad$ and $\quad y = 0$

Answer

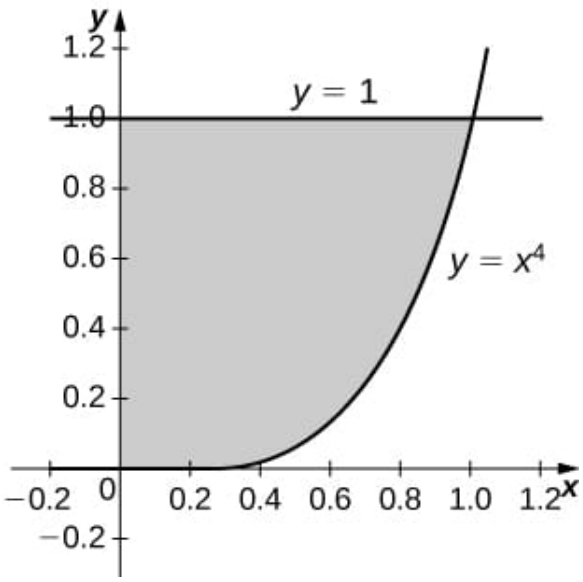


$$V = \int_0^4 4\pi x^4 \, dx \quad = \quad \frac{4096\pi}{5} \text{ units}^3$$

19) $y = e^x + 1, \quad x = 0, \quad x = 1$, and $y = 0$

20) $y = x^4, \quad x = 0$, and $y = 1$

Answer

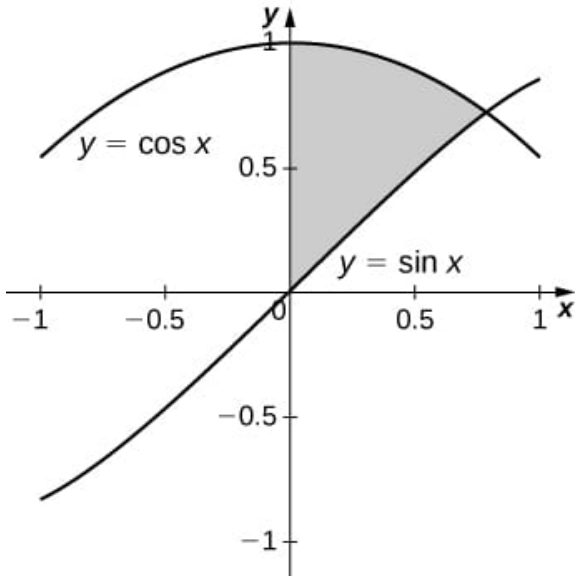


$$V = \int_0^1 \pi\left(1^2 - \left(x^4\right)^2\right)\ dx = \int_0^1 \pi\left(1 - x^8\right)\ dx \quad = \quad \frac{8\pi}{9} \text{ units}^3$$

21) $y = \sqrt{x}, \quad x = 0, \quad x = 4$, and $y = 0$

22) $y = \sin x, \quad y = \cos x,$ and $x = 0$

Answer

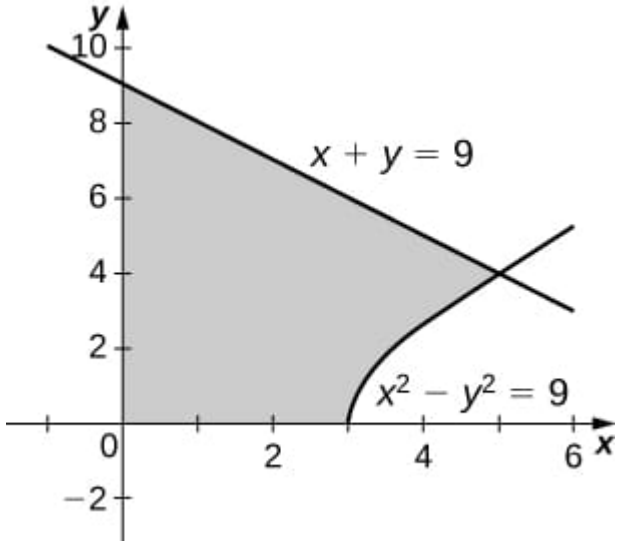


$$V = \int_0^{\pi/4} \pi\left(\cos^2 x - \sin^2 x\right)\ dx = \int_0^{\pi/4} \pi \cos 2x\ dx \quad = \quad \frac{\pi}{2} \text{ units}^3$$

23) $y = \dfrac{1}{x}, \quad x = 2$, and $y = 3$

24) $x^2 - y^2 = 9$ and $x + y = 9, \quad y = 0$ and $x = 0$

Answer



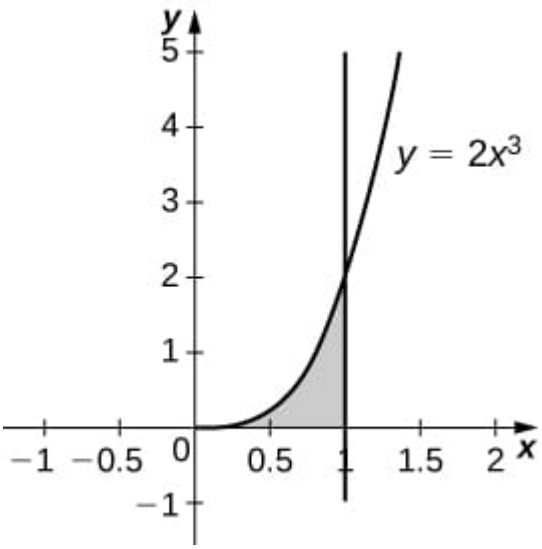
$V = 207\pi$ units^3

For exercises 25 - 32, draw the region bounded by the curves. Then, find the volume when the region is rotated around the y-axis.

25) $y = 4 - \frac{1}{2}x, \quad x = 0, \quad$ and $\quad y = 0$

26) $y = 2x^3, \quad y = 0, \quad$ and $\quad x = 1$

Answer

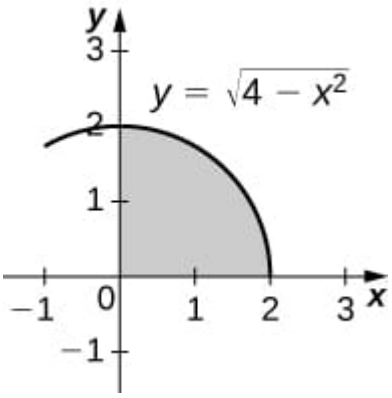


$V = \frac{4\pi}{5}$ units^3

27) $y = 3x^2, \quad x = 0$, and $y = 3$

28) $y = \sqrt{4 - x^2}, \quad y = 0$, and $x = 0$

Answer

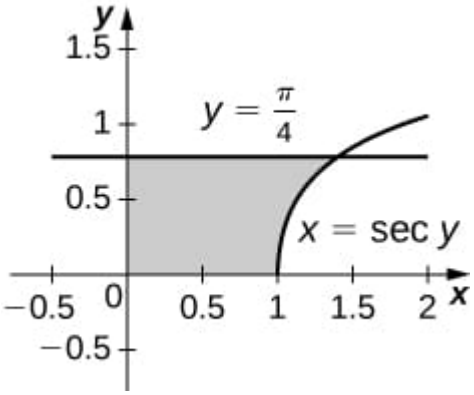


$V = \frac{16\pi}{3}$ units^3

29) $y = \dfrac{1}{\sqrt{x+1}}, \quad x = 0$, and $x = 3$

30) $x = \sec(y)$ and $y = \dfrac{\pi}{4}, \quad y = 0$ and $x = 0$

Answer

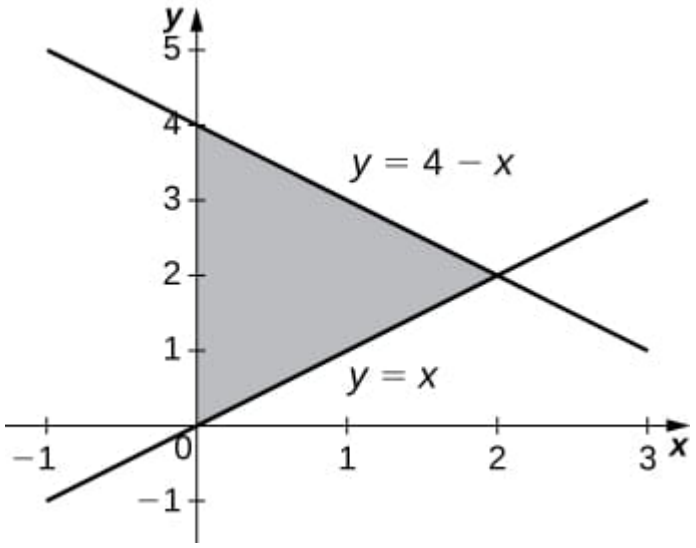


$V = \pi$ units^3

31) $y = \dfrac{1}{x+1}, \quad x = 0$, and $x = 2$

32) $y = 4 - x, \quad y = x$, and $x = 0$

Answer



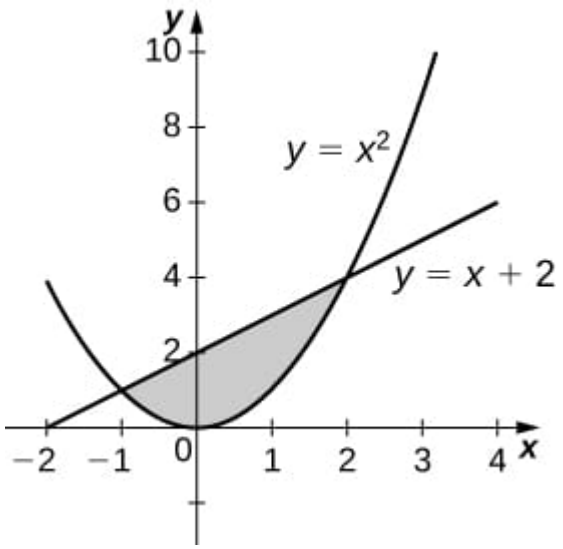
$V = 16\pi$ units^3

For exercises 33 - 40, draw the region bounded by the curves. Then, find the volume when the region is rotated around the x-axis.

33) $y = x + 2, \quad y = x + 6, \quad x = 0$, and $x = 5$

34) $y = x^2$ and $y = x + 2$

Answer

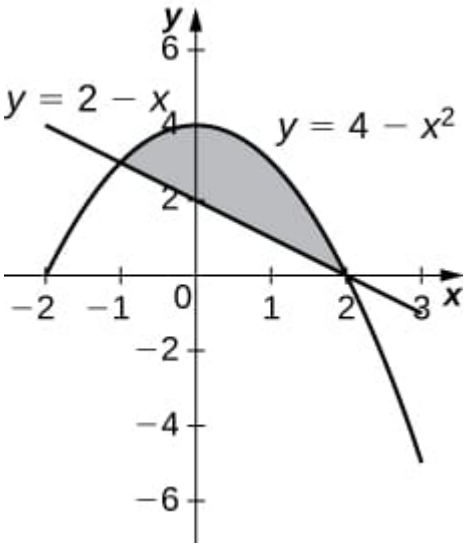


$V = \frac{72\pi}{5}$ units^3

35) $x^2 = y^3$ and $x^3 = y^2$

36) $y = 4 - x^2$ and $y = 2 - x$

Answer

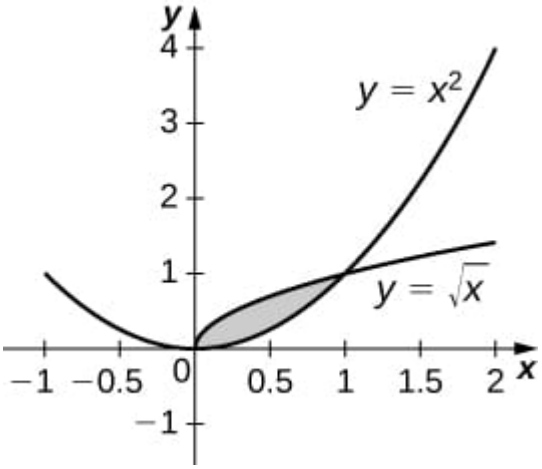


$V = \frac{108\pi}{5}$ units^3

37) [T] $y = \cos x, \quad y = e^{-x}, \quad x = 0$, and $x = 1.2927$

38) $y = \sqrt{x}$ and $y = x^2$

Answer

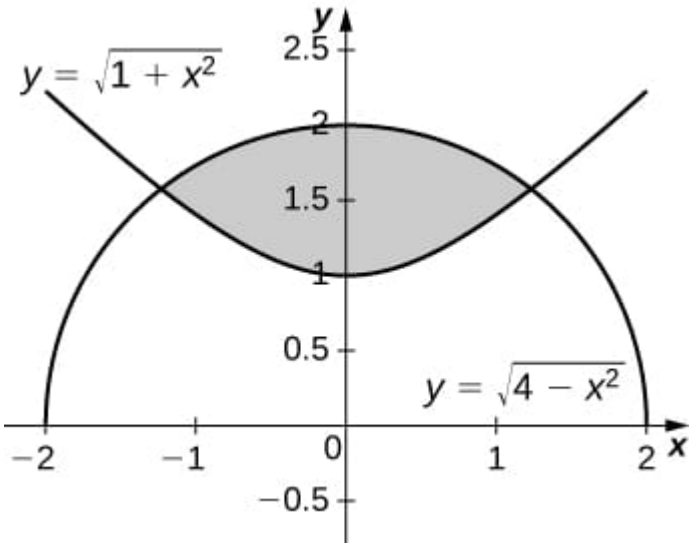


$V = \frac{3\pi}{10}$ units^3

39) $y = \sin x, \quad y = 5\sin x, \quad x = 0$ and $x = \pi$

40) $y = \sqrt{1 + x^2}$ and $y = \sqrt{4 - x^2}$

Answer



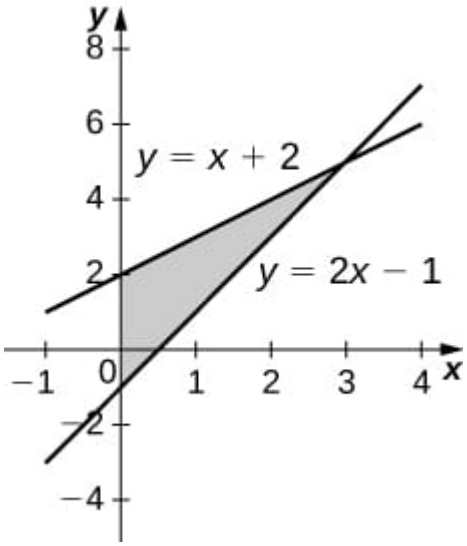
$V = 2\sqrt{6}\pi$ units^3

For exercises 41 - 45, draw the region bounded by the curves. Then, use the washer method to find the volume when the region is revolved around the y-axis.

41) $y = \sqrt{x}, \quad x = 4$, and $y = 0$

42) $y = x + 2, \quad y = 2x - 1$, and $x = 0$

Answer

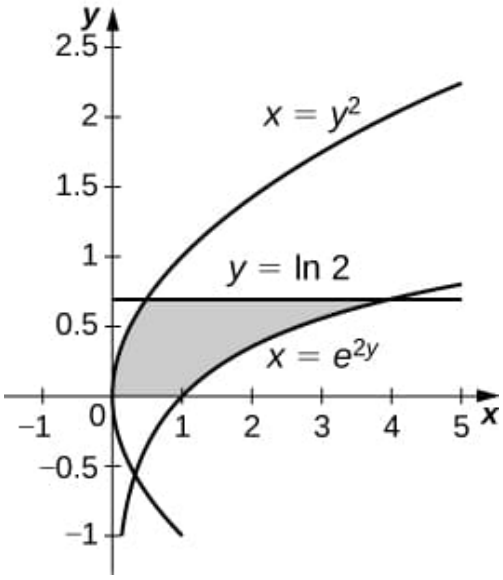


$V = 9\pi$ units^3

43) $y = \dfrac{3}{x}$ and $y = x^3$

44) $x = e^{2y}, \quad x = y^2, \quad y = 0$, and $y = \ln(2)$

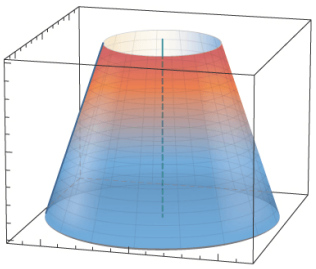
Answer



$$V = \frac{\pi}{20}(75 - 4\ln^5(2)) \text{ units}^3$$

45) $x = \sqrt{9 - y^2}$, $x = e^{-y}$, $y = 0$, and $y = 3$

46) Yogurt containers can be shaped like frustums. Rotate the line $y = mx$ around the y-axis to find the volume between $y = a$ and $y = b$.

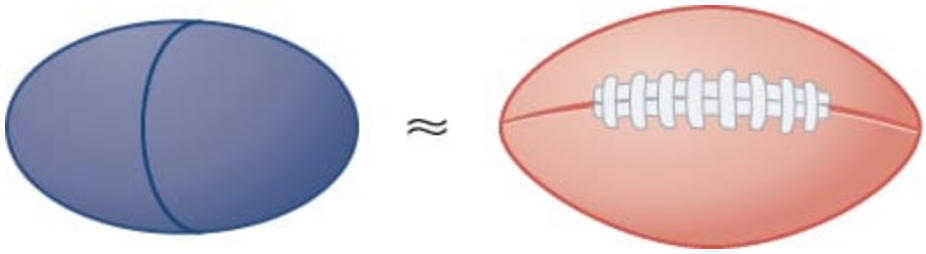




Answer

$$V = \frac{\pi}{3m^2}(b^3 - a^3) \text{ units}^3$$

47) Rotate the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ around the x-axis to approximate the volume of a football, as seen here.



Answer

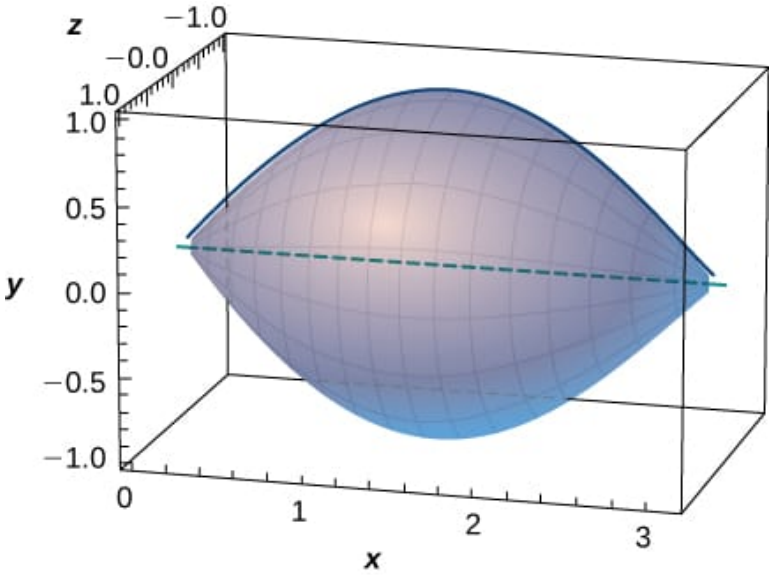
$V = \frac{4\pi ab^2}{3}$ units^3

48) Rotate the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ around the y-axis to approximate the volume of a football.

Answer

$V = \frac{4\pi a^2 b}{3}$ units^3

49) A better approximation of the volume of a football is given by the solid that comes from rotating $y = \sin x$ around the x-axis from $x = 0$ to $x = \pi$. What is the volume of this football approximation, as seen here?



For exercises 51 - 56, find the volume of the solid described.

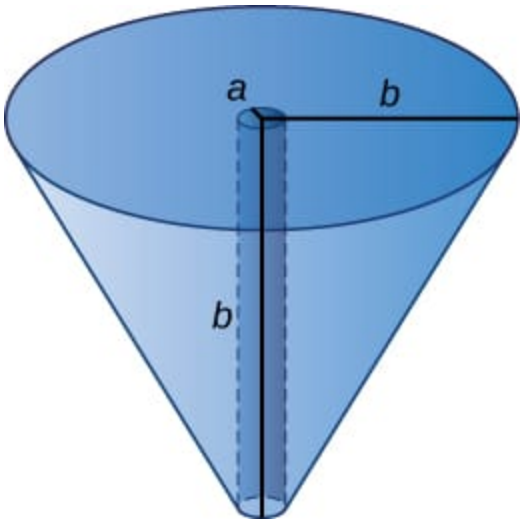
51) The base is the region between $y = x$ and $y = x^2$. Slices perpendicular to the x-axis are semicircles.

52) The base is the region enclosed by the generic ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Slices perpendicular to the x-axis are semicircles.

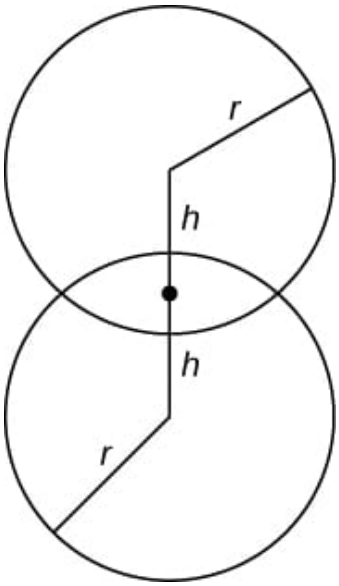
Answer

$V = \frac{2ab^2\pi}{3}$ units^3

53) Bore a hole of radius a down the axis of a right cone and through the base of radius b, as seen here.



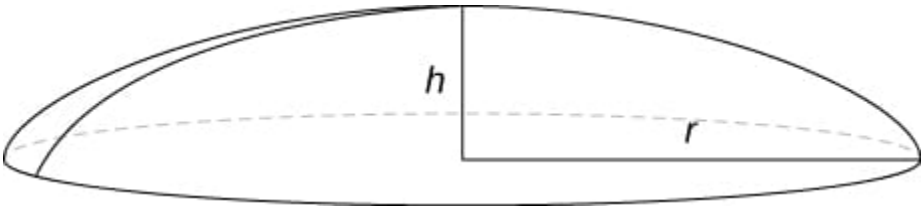
54) Find the volume common to two spheres of radius r with centers that are $2h$ apart, as shown here.



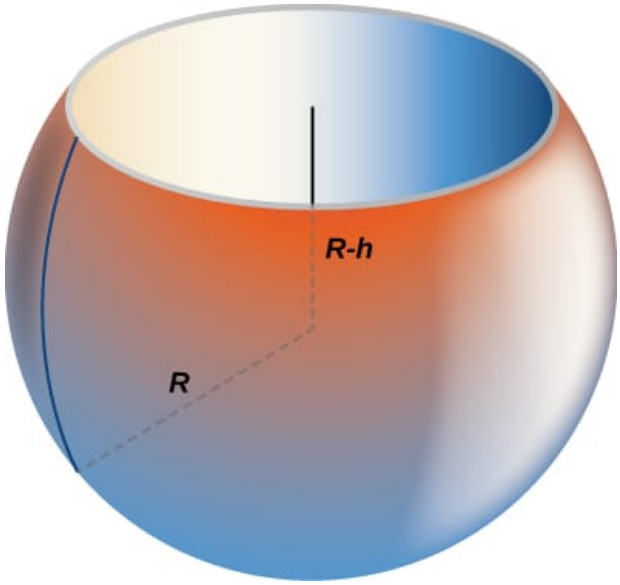
Answer

$V = \frac{\pi}{12}(r+h)^2(6r-h)$ units^3

55) Find the volume of a spherical cap of height h and radius r where $h < r$, as seen here.



56) Find the volume of a sphere of radius R with a cap of height h removed from the top, as seen here. Hint: Use the method used in #1.



Answer

$$V = \frac{\pi}{3}(h + R)(h - 2R)^2 \text{ units}^3$$

Contributors

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7.3: VOLUMES OF REVOLUTION - CYLINDRICAL SHELLS

6.03: Volumes of Revolution - Cylindrical Shells

7.3E: Exercises for Section 7.3

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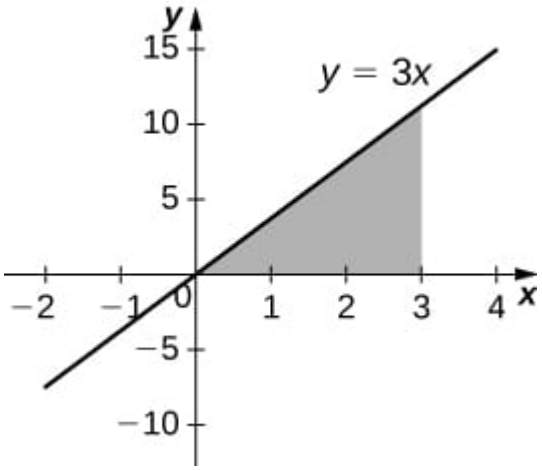
7.3E: Exercises for Section 7.3

For exercises 1 - 6, find the volume generated when the region between the two curves is rotated around the given axis. Use both the shell method and the washer method. Graph the functions and draw and label a rectangular slice of the washer.

1) [T] Over the curve of $y = 3x$, $x = 0$, and $y = 3$ rotated around the y-axis.

2) [T] Under the curve of $y = 3x$, $x = 0$, and $x = 3$ rotated around the y-axis.

Answer

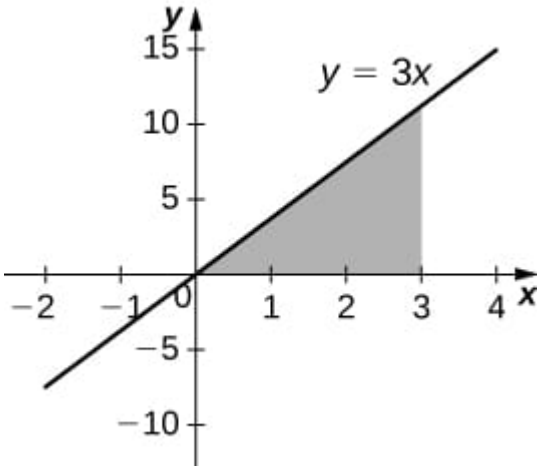


$V = 54\pi$ units^3

3) [T] Over the curve of $y = 3x$, $x = 0$, and $y = 3$ rotated around the x-axis.

4) [T] Under the curve of $y = 3x$, $x = 0$, and $x = 3$ rotated around the x-axis.

Answer

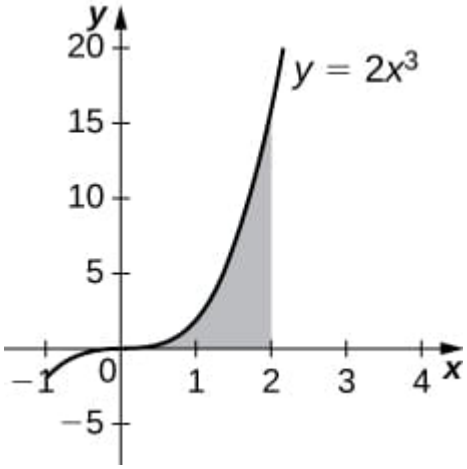


$V = 81\pi$ units^3

5) [T] Under the curve of $y = 2x^3$, $x = 0$, and $x = 2$ rotated around the y-axis.

6) [T] Under the curve of $y = 2x^3,\ \ x = 0$, and $x = 2$ rotated around the x-axis.

Answer



$V = \frac{512\pi}{7}$ units^3

For exercises 7 - 16, use shells to find the volumes of the given solids rotated around the y-axis. Note that the rotated regions lie between the curve and the x-axis.

7) $y = 1 - x^2$, $x = 0$, and $x = 1$

8) $y = 5x^3$, $x = 0$, and $x = 1$

Answer

$V = 2\pi$ units^3

9) $y = \dfrac{1}{x}$, $x = 1$, and $x = 100$

10) $y = \sqrt{1 - x^2}$, $x = 0$, and $x = 1$

Answer

$V = \frac{2\pi}{3}$ units^3

11) $y = \dfrac{1}{1 + x^2}$, $x = 0$,and $x = 3$

12) $y = \sin(x^2), x = 0$, and $x = \sqrt{\pi}$

Answer

$V = 2\pi$ units^3

13) $y = \dfrac{1}{\sqrt{1 - x^2}}$, $x = 0$, and $x = \frac{1}{2}$

14) $y = \sqrt{x}$, $x = 0$, and $x = 1$

Answer

$V = \frac{4\pi}{5}$ units^3

15) $y = (1 + x^2)^3$, $x = 0$, and $x = 1$

16) $y = 5x^3 - 2x^4$, $x = 0$, and $x = 2$

Answer

$V = \frac{64\pi}{3}$ units^3

For exercises 17 - 26, use shells to find the volume generated by rotating the regions between the given curve and $y = 0$ around the x-axis.

17) $y = \sqrt{1 - x^2}$, $x = 0$, and $x = 1$

18) $y = x^2$, $y = 0$, and $x = 2$

Answer

$V = \frac{32\pi}{5}$ units^3

19) $y = e^x$, $x = 0$, and $x = 1$

20) $y = \ln(x)$, $x = 1$, and $x = e$

Answer

$V = \pi(e - 2)$ units^3

21) $x = \dfrac{1}{1 + y^2}$, $y = 1$, and $y = 4$

22) $x = \dfrac{1 + y^2}{y}$, $y = 0$, and $y = 2$

Answer

$V = \frac{28\pi}{3}$ units^3

23) $x = \cos y$, $y = 0$, and $y = \pi$

24) $x = y^3 - 4y^2$, $x = -1$, and $x = 2$

Answer

$V = \frac{84\pi}{5}$ units^3

25) $x = ye^y$, $x = -1$, and $x = 2$

26) $x = e^y \cos y$, $x = 0$, and $x = \pi$

Answer

$V = e^\pi \pi^2$ units^3

For exercises 27 - 36, find the volume generated when the region between the curves is rotated around the given axis.

27) $y = 3 - x$, $y = 0$, $x = 0$, and $x = 2$ rotated around the y-axis.

28) $y = x^3$, $x = 0$, and $y = 8$ rotated around the y-axis.

Answer

$V = \frac{64\pi}{5}$ units^3

29) $y = x^2$, $y = x$, rotated around the y-axis.

30) $y = \sqrt{x}$, $x = 0$, and $x = 1$ rotated around the line $x = 2$.

Answer

$V = \frac{28\pi}{15}$ units^3

31) $y = \dfrac{1}{4-x}$, $x = 1$, and $x = 2$ rotated around the line $x = 4$.

Answer

$V = 2\pi$ units^3

32) $y = \sqrt{x}$ and $y = x^2$ rotated around the y-axis.

Answer

$V = \frac{3\pi}{10}$ units^3

33) $y = \sqrt{x}$ and $y = x^2$ rotated around the line $x = 2$.

Answer

$V = \frac{31\pi}{30}$ units^3

34) $x = y^3$, $y = \dfrac{1}{x}$, $x = 1$, and $y = 2$ rotated around the x-axis.

Answer

$\frac{52\pi}{5}$ units^3

35) $x = y^2$ and $y = x$ rotated around the line $y = 2$.

Answer

$\frac{\pi}{2}$ units^3

36) [T] Left of $x = \sin(\pi y)$, right of $y = x$, around the y-axis.

Answer

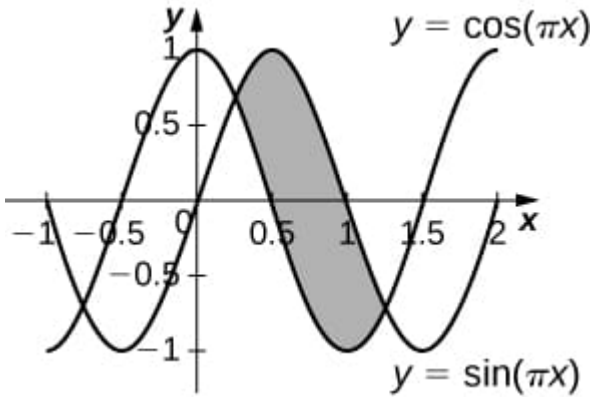
$V \approx 0.9876$ units^3

For exercises 37 - 44, use technology to graph the region. Determine which method you think would be easiest to use to calculate the volume generated when the function is rotated around the specified axis. Then, use your chosen method to find the volume.

37) [T] $y = x^2$ and $y = 4x$ rotated around the y-axis.

38) [T] $y = \cos(\pi x), y = \sin(\pi x), x = \frac{1}{4}$, and $x = \frac{5}{4}$ rotated around the y-axis.

Answer

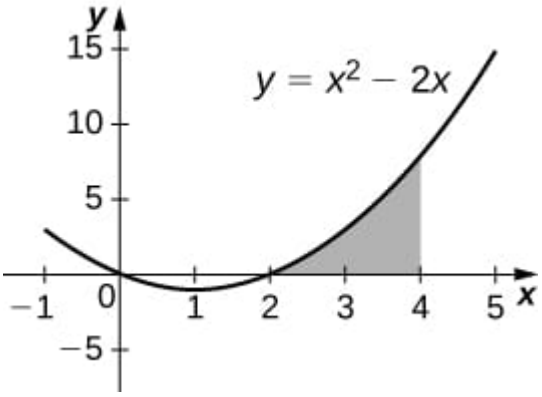


$V = 3\sqrt{2}$ units^3

39) [T] $y = x^2 - 2x,\ \ x = 2$, and $x = 4$ rotated around the y-axis.

40) [T] $y = x^2 - 2x,\ \ x = 2$, and $x = 4$ rotated around the x-axis.

Answer

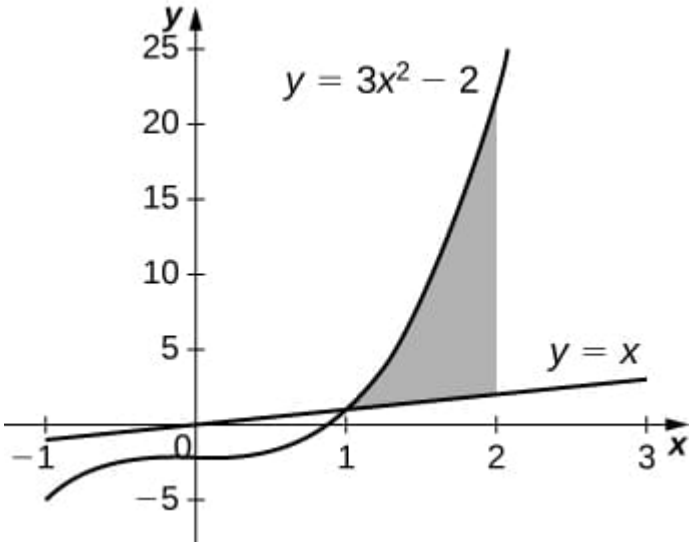


$V = \frac{496\pi}{15}$ units^3

41) [T] $y = 3x^3 - 2,\ \ y = x$, and $x = 2$ rotated around the x-axis.

42) [T] $y = 3x^3 - 2,\ \ y = x$, and $x = 2$ rotated around the y-axis.

Answer

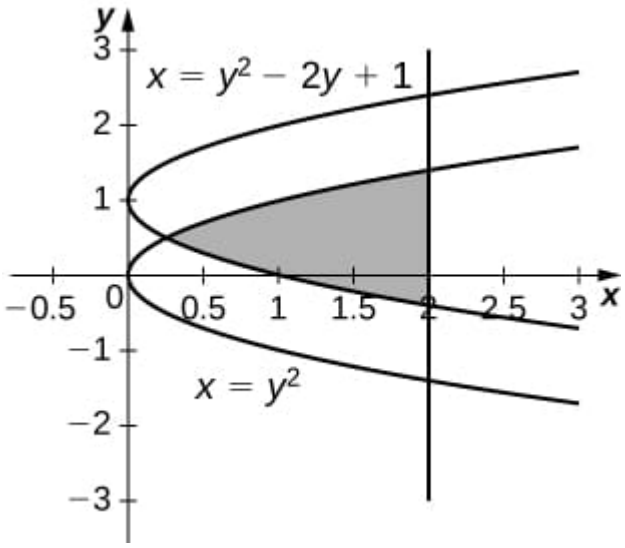


$V = \frac{398\pi}{15}$ units^3

43) [T] $x = \sin(\pi y^2)$ and $x = \sqrt{2}y$ rotated around the x-axis.

44) [T] $x = y^2,\ \ x = y^2 - 2y + 1$, and $x = 2$ rotated around the y-axis.

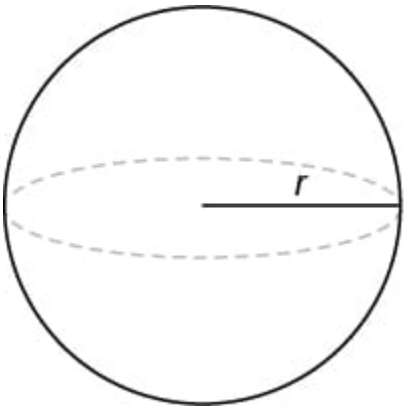
Answer



$V = 15.9074$ units^3

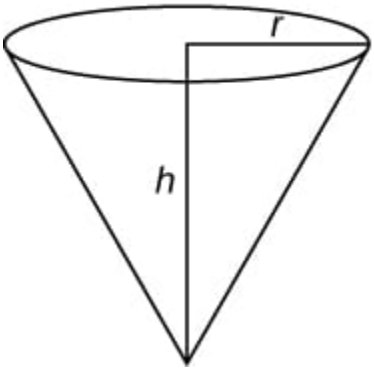
For exercises 45 - 51, use the method of shells to approximate the volumes of some common objects, which are pictured in accompanying figures.

45) Use the method of shells to find the volume of a sphere of radius r.





46) Use the method of shells to find the volume of a cone with radius r and height h.

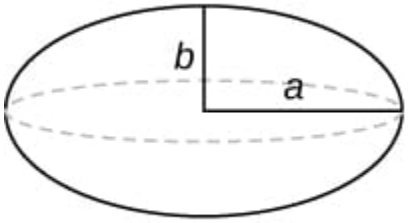


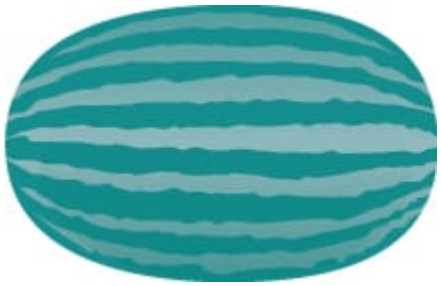


Answer

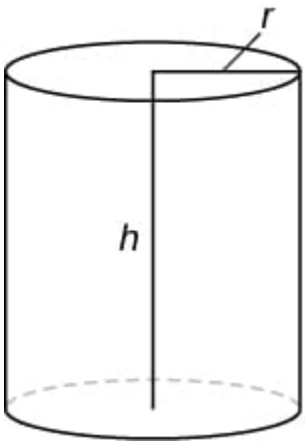
$V = \frac{1}{3}\pi r^2 h$ units^3

47) Use the method of shells to find the volume of an ellipse $(x^2/a^2) + (y^2/b^2) = 1$ rotated around the x-axis.





48) Use the method of shells to find the volume of a cylinder with radius r and height h.

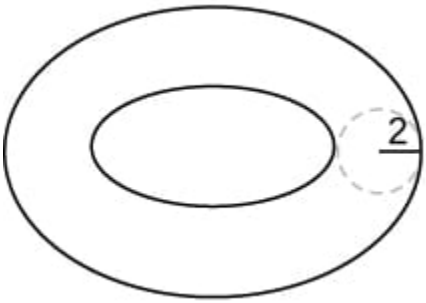




Answer

$V = \pi r^2 h$ units^3

49) Use the method of shells to find the volume of the donut created when the circle $x^2 + y^2 = 4$ is rotated around the line $x = 4$.





Answer

$V = 32\pi^2$ units^3

50) Consider the region enclosed by the graphs of $y = f(x)$, $y = 1 + f(x)$, $x = 0$, $y = 0$, and $x = a > 0$. What is the volume of the solid generated when this region is rotated around the y-axis? Assume that the function is defined over the interval $[0, a]$.

Answer

$V = \pi a^2$ units^3

51) Consider the function $y = f(x)$, which decreases from $f(0) = b$ to $f(1) = 0$. Set up the integrals for determining the volume, using both the shell method and the disk method, of the solid generated when this region, with $x = 0$ and $y = 0$, is rotated around the y-axis. Prove that both methods approximate the same volume. Which method is easier to apply? (Hint: Since $f(x)$ is one-to-one, there exists an inverse $f^{-1}(y)$.)

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7.4: ARC LENGTH OF A CURVE AND SURFACE AREA

6.04: Arc Length of a Curve and Surface Area

7.4E: Exercises for Section 7.4

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7.4E: Exercises for Section 7.4

For exercises 1 - 3, find the length of the functions over the given interval.

1) $y = 5x$ from $x = 0$ to $x = 2$

Answer

$s = 2\sqrt{26}$ units

2) $y = -\frac{1}{2}x + 25$ from $x = 1$ to $x = 4$

3) $x = 4y$ from $y = -1$ to $y = 1$

Answer

$s = 2\sqrt{17}$ units

4) Pick an arbitrary linear function $x = g(y)$ over any interval of your choice (y_1, y_2). Determine the length of the function and then prove the length is correct by using geometry.

5) Find the surface area of the volume generated when the curve $y = \sqrt{x}$ revolves around the x-axis from $(1, 1)$ to $(4, 2)$.

Answer

$A = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$ units^2

6) Find the surface area of the volume generated when the curve $y = x^2$ revolves around the y-axis from $(1, 1)$ to $(3, 9)$.

For exercises 7 - 16, find the lengths of the functions of x over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

7) $y = x^{3/2}$ from $(0, 0)$ to $(1, 1)$

Answer

$s = \frac{13\sqrt{13}-8}{27}$ units

8) $y = x^{2/3}$ from $(1, 1)$ to $(8, 4)$

9) $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 1$

Answer

$s = \frac{4}{3}$ units

10) $y = \frac{1}{3}(x^2 - 2)^{3/2}$ from $x = 2$ to $x = 4$

11) [T] $y = e^x$ on $x = 0$ to $x = 1$

Answer

$s \approx 2.0035$ units

12) $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$ from $x = 1$ to $x = 3$

13) $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$ from $x = 1$ to $x = 2$

Answer

$s = \frac{123}{32}$ units

14) $y = \dfrac{2x^{3/2}}{3} - \dfrac{x^{1/2}}{2}$ from $x = 1$ to $x = 4$

15) $y = \frac{1}{27}(9x^2 + 6)^{3/2}$ from $x = 0$ to $x = 2$

Answer

$s = 10$ units

16) [T] $y = \sin x$ on $x = 0$ to $x = \pi$

For exercises 17 - 26, find the lengths of the functions of y over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

17) $y = \dfrac{5 - 3x}{4}$ from $y = 0$ to $y = 4$

Answer

$s = \frac{20}{3}$ units

18) $x = \frac{1}{2}(e^y + e^{-y})$ from $y = -1$ to $y = 1$

19) $x = 5y^{3/2}$ from $y = 0$ to $y = 1$

Answer

$s = \frac{1}{675}(229\sqrt{229} - 8)$ units

20) [T] $x = y^2$ from $y = 0$ to $y = 1$

21) $x = \sqrt{y}$ from $y = 0$ to $y = 1$

Answer

$s = \frac{\sqrt{5}}{2} + \frac{1}{4}(\ln(2 + \sqrt{5}))$ units

22) $x = \frac{2}{3}(y^2 + 1)^{3/2}$ from $y = 1$ to $y = 3$

23) [T] $x = \tan y$ from $y = 0$ to $y = \frac{3}{4}$

Answer

$s \approx 1.201$ units

24) [T] $x = \cos^2 y$ from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$

25) [T] $x = 4^y$ from $y = 0$ to $y = 2$

Answer

$s \approx 15.2341$ units

26) [T] $x = \ln(y)$ on $y = \dfrac{1}{e}$ to $y = e$

For exercises 27 - 34, find the surface area of the surface generated when the following curves revolve around the x -axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.

27) $y = \sqrt{x}$ from $x = 2$ to $x = 6$

Answer

$A = \frac{49\pi}{3}$ units^2

28) $y = x^3$ from $x = 0$ to $x = 1$

29) $y = 7x$ from $x = 0$ to $x = 2$

Answer

$A = 140\pi\sqrt{2}$ units^2

30) [T] $y = \frac{1}{x^2}$ from $x = 1$ to $x = 3$

31) $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$

Answer

$A = 8\pi$ units^2

32) $y = \sqrt{4 - x^2}$ from $x = -1$ to $x = 1$

33) $y = 5x$ from $x = 1$ to $x = 5$

Answer

$A = 120\pi\sqrt{26}$ units^2

34) [T] $y = \tan x$ from $x = -\frac{\pi}{4}$ to $x = \frac{\pi}{4}$

For exercises 35 - 42, find the surface area of the surface generated when the following curves revolve around the y -axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.

35) $y = x^2$ from $x = 0$ to $x = 2$

Answer

$A = \frac{\pi}{6}(17\sqrt{17} - 1)$ units^2

36) $y = \frac{1}{2}x^2 + \frac{1}{2}$ from $x = 0$ to $x = 1$

37) $y = x + 1$ from $x = 0$ to $x = 3$

Answer

$A = 15\sqrt{2}\pi$ units^2

38) [T] $y = \dfrac{1}{x}$ from $x = \dfrac{1}{2}$ to $x = 1$

39) $y = \sqrt[3]{x}$ from $x = 1$ to $x = 27$

Answer

$A = \frac{10\sqrt{10}\pi}{27}(73\sqrt{73} - 1)$ units^2

40) [T] $y = 3x^4$ from $x = 0$ to $x = 1$

41) [T] $y = \dfrac{1}{\sqrt{x}}$ from $x = 1$ to $x = 3$

Answer

$A \approx 25.645$ units^2

42) [T] $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$

43) The base of a lamp is constructed by revolving a quarter circle $y = \sqrt{2x - x^2}$ around the y-axis from $x = 1$ to $x = 2$, as seen here. Create an integral for the surface area of this curve and compute it.

Answer

$A = 2\pi$ units^2

44) A light bulb is a sphere with radius $1/2$ in. with the bottom sliced off to fit exactly onto a cylinder of radius $1/4$ in. and length

$1/3$ in., as seen here. The sphere is cut off at the bottom to fit exactly onto the cylinder, so the radius of the cut is $1/4$ in. Find the surface area (not including the top or bottom of the cylinder).

45) [T] A lampshade is constructed by rotating $y = 1/x$ around the x-axis from $y = 1$ to $y = 2$. Determine how much material you would need to construct this lampshade—that is, the surface area—accurate to four decimal places. Use Desmos to numerically integrate.

Answer

10.5017 units^2

46) [T] An anchor drags behind a boat according to the function $y = 24e^{-x/2} - 24$, where y represents the depth beneath the boat and x is the horizontal distance of the anchor from the back of the boat. If the anchor is 23 ft below the boat, how much rope do you have to pull to reach the anchor? Round your answer to three decimal places.

47) [T] You are building a bridge that will span 10 ft. You intend to add decorative rope in the shape of $y = 5|\sin((x\pi)/5)|$, where x is the distance in feet from one end of the bridge. Find out how much rope you need to buy, rounded to the nearest foot.

Answer

23 ft

48) [T] A chain hangs from two posts four meters apart to form a catenary described by the equation $y = 4\cosh(x/4) + 3$. Find the exact value of the total length of the chain.

Answer

$8\sinh(0.5) \approx 4.1688$ meters

For exercise 48, find the exact arc length for the following problems over the given interval.

49) $y = \ln(\sin x)$ from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$. (Hint: Recall trigonometric identities.)

50) Draw graphs of $y = x^2, y = x^6$, and $y = x^{10}$. For $y = x^n$, as n increases, formulate a prediction on the arc length from $(0, 0)$ to $(1, 1)$. Now, compute the lengths of these three functions and determine whether your prediction is correct.

Answer

2

51) Compare the lengths of the parabola $x = y^2$ and the line $x = by$ from $(0, 0)$ to (b^2, b) as b increases. What do you notice?

52) Solve for the length of $x = y^2$ from $(0, 0)$ to $(1, 1)$. Show that $x = \dfrac{y^2}{2}$ from $(0, 0)$ to $(2, 2)$ is twice as long. Graph both functions and explain why this is so.

Answer

Answers may vary

53) [T] Which is longer between $(1, 1)$ and $\left(2, \frac{1}{2}\right)$: the hyperbola $y = \dfrac{1}{x}$ or the graph of $x + 2y = 3$?

54) Explain why the surface area is infinite when $y = 1/x$ is rotated around the x-axis for $1 \leq x < \infty$, but the volume is finite.

Answer

For more information, look up Gabriel's Horn.

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7.5: PHYSICAL APPLICATIONS OF INTEGRATION

6.05: Physical Applications of Integration

7.5E: Exercises for Section 7.5

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7.5E: Exercises for Section 7.5

Basic Work Problems

For exercises 1 - 6, find the work done.

1) Find the work done when a constant force $F = 12$ lb moves a chair from $x = 0.9$ to $x = 1.1$ ft.

2) How much work is done when a person lifts a 50 lb box of comics onto a truck that is 3 ft off the ground?

Answer

$W = 150$ ft-lb

3) What is the work done lifting a 20 kg child from the floor to a height of 2 m? (Note that 1 kg equates to 9.8 N)

Answer

$W = 392$ J (1 joule = 1 newton-meter)

4) Find the work done when you push a box along the floor 2 m, when you apply a constant force of $F = 100$ N.

Answer

$W = 200$ J

5) Compute the work done for a force $F = \dfrac{12}{x^2}$ N from $x = 1$ to $x = 2$ m.

6) What is the work done moving a particle from $x = 0$ to $x = 1$ m if the force acting on it is $F = 3x^2$ N?

Answer

$W = 1$ J

Density Problems

In exercises 7 - 11, find the mass of the one-dimensional object.

7) A wire that is 2 ft long (starting at $x = 0$) and has a density function of $\rho(x) = x^2 + 2x$ lb/ft

8) A car antenna that is 3 ft long (starting at $x = 0$) and has a density function of $\rho(x) = 3x + 2$ lb/ft

Answer

$\frac{39}{2}$

9) A metal rod that is 8 in. long (starting at $x = 0$) and has a density function of $\rho(x) = e^{1/2x}$ lb/in.

10) A pencil that is 4 in. long (starting at $x = 2$) and has a density function of $\rho(x) = \dfrac{5}{x}$ oz/in.

Answer

$\ln(243)$

11) A ruler that is 12 in. long (starting at $x = 5$) and has a density function of $\rho(x) = \ln(x) + (1/2)x^2$ oz/in.

In exercises 12 - 16, find the mass of the two-dimensional object that is centered at the origin.

12) An oversized hockey puck of radius 2 in. with density function $\rho(x) = x^3 - 2x + 5$

Answer

$\frac{332\pi}{15}$

13) A frisbee of radius 6 in. with density function $\rho(x) = e^{-x}$

14) A plate of radius 10 in. with density function $\rho(x) = 1 + \cos(\pi x)$

Answer

100π

15) A jar lid of radius 3 in. with density function $\rho(x) = \ln(x + 1)$

16) A disk of radius 5 cm with density function $\rho(x) = \sqrt{3x}$

Answer

$20\pi\sqrt{15}$

Spring Work Problems

17) A force of 6 pounds is required to hold a spring stretched 0.3 feet beyond its natural length. How much work (in foot-pounds) is done in stretching the spring from its natural length to 0.9 feet beyond its natural length?

Answer

$W = 8.1$ ft-lb

18) A spring has a natural length of 10 cm. It takes 2 J of work to stretch the spring to 15 cm. How much work would it take to stretch the spring from 15 cm to 20 cm? Example: http://youtu.be/Fd65ZxqpktE

Answer

$W = 6$ J

19) A 1-m spring requires 10 J to stretch the spring to 1.1 m. How much work would it take to stretch the spring from 1 m to 1.2 m?

20) A spring requires 5 J to stretch the spring from 8 cm to 12 cm, and an additional 4 J to stretch the spring from 12 cm to 14 cm. What is the natural length of the spring?

Answer

The natural length is5 cm.

21) A shock absorber is compressed 1 in. by a weight of 1 ton. What is the spring constant?

22) A force of $F = (20x - x^3)$ N stretches a nonlinear spring by x meters. What work is required to stretch the spring from $x = 0$ to $x = 2$ m?

Answer

$W = 36$ J

Cable and Chain Work Problems

23) Find the work done by winding up a hanging cable of length 100 ft and weight-density 5 lb/ft.

Answer

$W = 25,000$ ft-lb

24) For the cable in the preceding exercise, how much work is done to lift the cable 50 ft?

Answer

$W = 18,750$ ft-lb

25) For the cable in the preceding exercise, how much additional work is done by hanging a 200 lb weight at the end of the cable?

Answer

$W = 28,750$ ft-lb

26) A bucket that has a mass of 30 kg when filled with sand needs to be lifted to the top of a 20 meter tall building. You have a rope that has a mass of 0.4 kg/m that must be secured to the bucket. It takes 1 meter of rope to secure the bucket. Once the bucket reaches the top of the building it only has mass 22 kg because there was a hole in the bottom and sand was leaking out at a constant rate while it was being lifted to the top of the building. Find the work done lifting the bucket (sand and rope) to the top of the building. Example: http://youtu.be/XEwdZqz93z4

Answer

$W \approx 5958.4$ Joules

Pyramid & Satellite/Rocket Work Problems

27) [T] A pyramid of height 500 ft has a square base 800 ft by 800 ft. Find the area A at height h. If the rock used to build the pyramid weighs approximately $w = 100 \text{ lb/ft}^3$, how much work did it take to lift all the rock?

Answer

$W = \frac{4}{3} \times 10^{10}$ ft-lb

28) [T] For the pyramid in the preceding exercise, assume there were 1000 workers each working 10 hours a day, 5 days a week, 50 weeks a year. If each of the workers, on average, lifted ten 100-lb rocks 2 ft/hr, how long did it take to build the pyramid?

Answer

$2\frac{2}{3}$ years

29) [T] The force of gravity on a mass m is $F = -((GMm)/x^2)$ newtons. For a rocket of mass $m = 1000$ kg, compute the work to lift the rocket from $x = 6400$ to $x = 6500$ km. (Note: $G = 6 \times 10^{-17} \text{ N m}^2/\text{kg}^2$ and $M = 6 \times 10^{24}$ kg.)

Answer

$W = 8.65 \times 10^5$ J

30) [T] For the rocket in the preceding exercise, find the work to lift the rocket from $x = 6400$ to $x = \infty$.

Pumping Work Problems

31) [T] Find the work required to pump all the water out of a cylinder that has a circular base of radius 5ft and height 200 ft. Use the fact that the density of water is 62 lb/ft^3.

32) [T] Find the work required to pump all the water out of the cylinder in the preceding exercise if the cylinder is only half full.

Answer

$W = 23.25\pi$ million ft-lb

33) [T] How much work is required to pump out a swimming pool if the area of the base is 800 ft^2, the water is 4 ft deep, and the top is 1 ft above the water level? Assume that the density of water is 62 lb/ft^3.

34) A cylinder of depth H and cross-sectional area A stands full of water at density ρ. Compute the work to pump all the water to the top.

Answer

$W = \frac{A\rho H^2}{2}$

35) For the cylinder in the preceding exercise, compute the work to pump all the water to the top if the cylinder is only half full.

36) A cone-shaped tank has a cross-sectional area that increases with its depth: $A = \frac{\pi r^2 h^2}{H^3}$. Show that the work to empty it is half the work for a cylinder with the same height and base.

Answer

Answers may vary.

37) A trough is 9 meters long, 1.5 meters wide, and 3 meters deep. The vertical cross-section of the trough parallel to an end is shaped like an isosceles triangle (with height 3 meters, and base, on top, of length 1.5 meters). The trough is full of water (density $9800\ N/m^3$). Find the amount of work in joules required to empty the trough by pumping the water over the top. Example: http://youtu.be/1x-xMf4TFNM

Answer

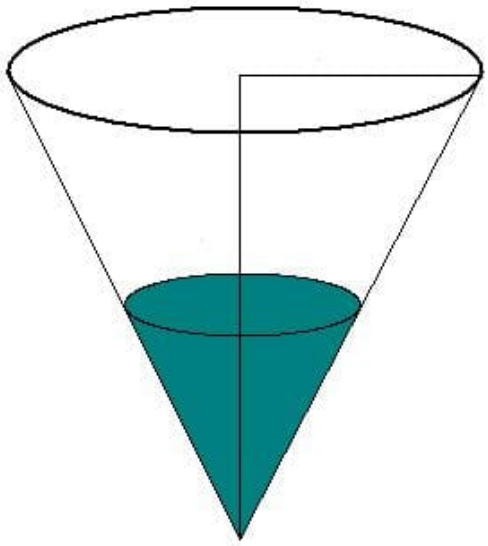
$W = 198,450$ J

38) A trough is 3 feet long and 1 foot high. The vertical cross-section of the trough parallel to an end is shaped like the graph of $y = x^8$ from $x = -1$ to $x = 1$. The trough is full of water. Find the amount of work in foot-pounds required to empty the trough by pumping the water over the top. Note: In this problem, use 62 pounds per cubic foot as the weight of water.

Answer

$W \approx 155.6078$ ft-lb

39) A tank, shaped like a cone has height 6 meters and base radius 5 meters long. It is placed so that the circular part is upward. Assume that a cubic meter of water weighs 9800 N, i.e. the density of water is $9800\ N/m^3$. How much work does it require to pump all of the water out of the top of the tank, assuming it is full? Enter the exact value of your answer.



Answer

$W = 735,000\pi$ J

Hydrostatic Force and Pressure

40) [T] A rectangular dam is 40 ft high and 60 ft wide. Assume water weighs 62 pounds per cubic foot. Compute the total force F on the dam in pounds when

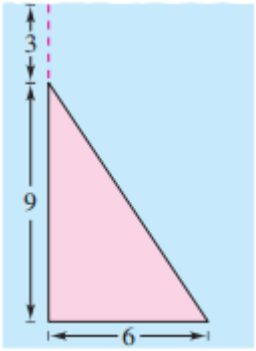
a. the surface of the water is at the top of the dam and

b. the surface of the water is halfway down the dam.

Answer

a. $3,000,000$ lb,
b. $749,000$ lb

41) [T] Find the fluid force on the vertical plate submerged in water, where the dimensions are given in meters and the weight-density of water is 9800 newtons per cubic meter.



Answer

$2,381,400$ newtons

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7.6: MOMENTS AND CENTERS OF MASS

6.06: Moments and Centers of Mass

7.6E: Exercises for Section 7.6

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7.6E: Exercises for Section 7.6

In exercises 1 - 6, calculate the center of mass for the collection of masses given.

1) $m_1 = 2$ at $x_1 = 1$ and $m_2 = 4$ at $x_2 = 2$

2) $m_1 = 1$ at $x_1 = -1$ and $m_2 = 3$ at $x_2 = 2$

Answer

$x = \frac{5}{4}$

3) $m = 3$ at $x = 0, 1, 2, 6$

4) Unit masses at $(x, y) = (1, 0), (0, 1), (1, 1)$

Answer

$\left(\frac{2}{3}, \frac{2}{3}\right)$

5) $m_1 = 1$ at $(1, 0)$ and $m_2 = 4$ at $(0, 1)$

6) $m_1 = 1$ at $(1, 0)$ and $m_2 = 3$ at $(2, 2)$

Answer

$\left(\frac{7}{4}, \frac{3}{2}\right)$

In exercises 7 - 16, compute the center of mass $\bar{x}$.

7) $\rho = 1$ for $x \in (-1, 3)$

8) $\rho = x^2$ for $x \in (0, L)$

Answer

$\frac{3L}{4}$

9) $\rho = 1$ for $x \in (0, 1)$ and $\rho = 2$ for $x \in (1, 2)$

10) $\rho = \sin x$ for $x \in (0, \pi)$

Answer

$\frac{\pi}{2}$

11) $\rho = \cos x$ for $x \in (0, \frac{\pi}{2})$

12) $\rho = e^x$ for $x \in (0, 2)$

Answer

$\frac{e^2 + 1}{e^2 - 1}$

13) $\rho = x^3 + xe^{-x}$ for $x \in (0, 1)$

14) $\rho = x \sin x$ for $x \in (0, \pi)$

Answer

$\frac{\pi^2 - 4}{\pi}$

15) $\rho = \sqrt{x}$ for $x \in (1, 4)$

16) $\rho = \ln x$ for $x \in (1, e)$

Answer

$\frac{1}{4}(1 + e^2)$

In exercises 17 - 19, compute the center of mass $(\bar{x}, \bar{y})$. Use symmetry to help locate the center of mass whenever possible.

17) $\rho = 7$ in the square $0 \le x \le 1, \ \ 0 \le y \le 1$

18) $\rho = 3$ in the triangle with vertices $(0, 0), \ \ (a, 0)$, and $(0, b)$

Answer

$\left(\frac{a}{3}, \ \frac{b}{3}\right)$

19) $\rho = 2$ for the region bounded by $y = \cos(x), \ \ y = -\cos(x), \ \ x = -\frac{\pi}{2}$, and $x = \frac{\pi}{2}$

In exercises 20 - 26, draw the region, then compute the center of mass $(\bar{x}, \bar{y})$. Use symmetry to help locate the center of mass whenever possible.

20) [T] The region bounded by $y = \cos(2x), \ \ x = -\frac{\pi}{4}$, and $x = \frac{\pi}{4}$

Answer

$\left(0, \frac{\pi}{8}\right)$

21) [T] The region between $y = 2x^2, \ \ y = 0, \ \ x = 0$, and $x = 1$

Answer

$\left(\frac{3}{4}, \ \frac{3}{5}\right)$

22) [T] The region between $y = \frac{5}{4}x^2$ and $y = 5$

Answer

$(0, 3)$

23) [T] Region between $y = \sqrt{x}, \ \ y = \ln x, \ \ x = 1$, and $x = 4$

24) [T] The region bounded by $y = 0$ and $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

Answer

$\left(0, \frac{4}{\pi}\right)$

25) [T] The region bounded by $y = 0, \ \ x = 0$, and $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

26) [T] The region bounded by $y = x^2$ and $y = x^4$ in the first quadrant

Answer

$\left(\frac{5}{8}, \ \frac{1}{3}\right)$

In exercises 27 - 31, use the theorem of Pappus to determine the volume of the shape.

27) Rotating $y = mx$ around the x-axis between $x = 0$ and $x = 1$

28) Rotating $y = mx$ around the y-axis between $x = 0$ and $x = 1$

Answer

$V = \frac{m\pi}{3}$ units^3

29) A general cone created by rotating a triangle with vertices $(0,0),\ (a,0)$, and $(0,b)$ around the y-axis. Does your answer agree with the volume of a cone?

30) A general cylinder created by rotating a rectangle with vertices $(0,0),\ (a,0),\ (0,b)$, and (a,b) around the y-axis. Does your answer agree with the volume of a cylinder?

Answer

$V = \pi a^2 b$ units^3

31) A sphere created by rotating a semicircle with radius a around the y-axis. Does your answer agree with the volume of a sphere?

In exercises 32 - 36, draw the region enclosed by the curve. Find the mass m and the centroid $(\bar{x},\bar{y})$ for the given shapes. Use symmetry to help locate the centroid whenever possible. Assume the densities of the laminas are all one.

32) [T] Quarter-circle: $y = \sqrt{1-x^2},\ \ y = 0$, and $x = 0$

Answer

$\left(\frac{4}{3\pi},\ \frac{4}{3\pi}\right)$

33) [T] Triangle: $y = x,\ \ y = 2 - x$, and $x = 0$

Answer

$\left(\frac{1}{3},\ 1\right)$

34) [T] Lens: $y = x^2$ and $y = x$

Answer

$\left(\frac{1}{2},\ \frac{2}{5}\right)$

35) [T] Ring: $y^2 + x^2 = 1$ and $y^2 + x^2 = 4$

36) [T] Half-ring: $y^2 + x^2 = 1,\ \ y^2 + x^2 = 4$, and $y = 0$

Answer

$\left(0,\ \frac{28}{9\pi}\right)$

37) Find the generalized center of mass in the sliver between $y = x^a$ and $y = x^b$ with $a > b$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the y-axis.

38) Find the generalized center of mass between $y = a^2 - x^2,\ \ x = 0$, and $y = 0$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the y-axis.

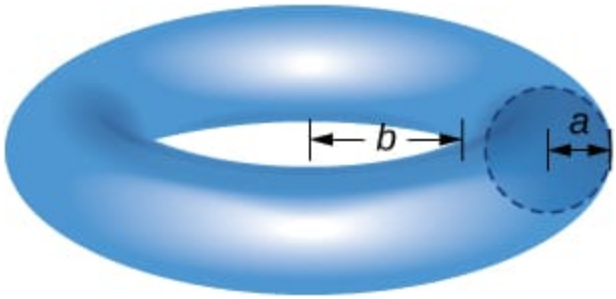
Answer

Center of mass: $\left(\frac{a}{6},\ \frac{4a^2}{5}\right)$,

Volume: $\dfrac{2\pi a^4}{9}$ units^3

39) Find the generalized center of mass between $y = b\sin(ax),\ \ x = 0$, and $x = \dfrac{\pi}{a}$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the y-axis.

40) Use the theorem of Pappus to find the volume of a torus (pictured here). Assume that a disk of radius a is positioned with the left end of the circle at $x = b,\ \ b > 0$, and is rotated around the y-axis.



Answer

Volume: $V = 2\pi^2 a^2(b + a)$

41) Find the center of mass $(\bar{x}, \bar{y})$ for a thin wire along the semicircle $y = \sqrt{1 - x^2}$ with unit mass. (Hint: Use the theorem of Pappus.)

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7.7: INTEGRALS, EXPONENTIAL FUNCTIONS, AND LOGARITHMS

6.07: Integrals Exponential Functions and Logarithms

7.7E: Exercises for Section 7.7

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7.7E: Exercises for Section 7.7

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7.8: EXPONENTIAL GROWTH AND DECAY

6.08: Exponential Growth and Decay

7.8E: Exercises for Section 7.8

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7.8E: Exercises for Section 7.8

6.8E: Exercises for Section 6.8

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7.9: Chapter 6 Review Exercises

6.10: Chapter 6 Review Exercises

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8: TECHNIQUES OF INTEGRATION

We saw in the previous chapter how important integration can be for all kinds of different topics—from calculations of volumes to flow rates, and from using a velocity function to determine a position to locating centers of mass. It is no surprise, then, that techniques for finding antiderivatives (or indefinite integrals) are important to know for everyone who uses them. We have already discussed some basic integration formulas and the method of integration by substitution. In this chapter, we study some additional techniques, including some ways of approximating definite integrals when normal techniques do not work.

8.0: Prelude to Techniques of Integration

8.1: Integration by Parts

8.1E: Exercises for Section 8.1

8.2: Trigonometric Integrals

8.2E: Exercises for Section 8.2

8.3: Trigonometric Substitution

8.3E: Exercises for Section 8.3

8.4: Partial Fractions

8.4E: Exercises for Section 8.4

8.5: Other Strategies for Integration

8.5E: Exercises for Section 8.5

8.6: Numerical Integration

8.6E: Exercises for Section 8.6

8.7: Improper Integrals

8.7E: Exercises for Section 8.7

8.8: Chapter 8 Review Exercises

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8.0: Prelude to Techniques of Integration

7.00: Prelude to Techniques of Integration

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8.1: INTEGRATION BY PARTS

7.01: Integration by Parts

8.1E: Exercises for Section 8.1

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8.1E: Exercises for Section 8.1

In using the technique of integration by parts, you must carefully choose which expression is u. For each of the following problems, use the guidelines in this section to choose u. Do not evaluate the integrals.

1) $\int x^3 e^{2x}\ dx$

Answer

$u = x^3$

2) $\int x^3 \ln(x)\ dx$

3) $\int y^3 \cos y\ dy$

Answer

$u = y^3$

4) $\int x^2 \arctan x\ dx$

5) $\int e^{3x} \sin(2x)\ dx$

Answer

$u = \sin(2x)$

In exercises 6 - 37, find the integral by using the simplest method. Not all problems require integration by parts.

6) $\int v \sin v\ dv$

7) $\int \ln x\ dx$ (Hint: $\int \ln x\ dx$ is equivalent to $\int 1 \cdot \ln(x)\ dx$.)

Answer

$$\int \ln x\ dx \quad = \quad -x + x \ln x + C$$

8) $\int x \cos x\ dx$

9) $\int \tan^{-1} x\ dx$

Answer

$$\int \tan^{-1} x\ dx \quad = \quad x \tan^{-1} x - \tfrac{1}{2}\ln(1 + x^2) + C$$

10) $\int x^2 e^x\ dx$

11) $\int x \sin(2x)\ dx$

Answer

$$\int x \sin(2x)\ dx \quad = \quad -\tfrac{1}{2} x \cos(2x) + \tfrac{1}{4}\sin(2x) + C$$

12) $\int xe^{4x}\ dx$

13) $\int xe^{-x}\ dx$

Answer

$$\int xe^{-x}\ dx \quad = \quad e^{-x}(-1-x)+C$$

14) $\int x\cos 3x\ dx$

15) $\int x^2\cos x\ dx$

Answer

$$\int x^2\cos x\ dx \quad = \quad 2x\cos x+(-2+x^2)\sin x+C$$

16) $\int x\ln x\ dx$

17) $\int \ln(2x+1)\ dx$

Answer

$$\int \ln(2x+1)\ dx \quad = \quad \tfrac{1}{2}(1+2x)(-1+\ln(1+2x))+C$$

18) $\int x^2e^{4x}\ dx$

19) $\int e^x\sin x\ dx$

Answer

$$\int e^x\sin x\ dx \quad = \quad \tfrac{1}{2}e^x(-\cos x+\sin x)+C$$

20) $\int e^x\cos x\ dx$

21) $\int xe^{-x^2}\ dx$

Answer

$$\int xe^{-x^2}\ dx \quad = \quad -\frac{e^{-x^2}}{2}+C$$

22) $\int x^2e^{-x}\ dx$

23) $\int \sin(\ln(2x))\ dx$

Answer

$$\int \sin(\ln(2x))\ dx \quad = \quad -\tfrac{1}{2}x\cos[\ln(2x)]+\tfrac{1}{2}x\sin[\ln(2x)]+C$$

24) $\int \cos(\ln x)\ dx$

25) $\int (\ln x)^2\ dx$

Answer

$$\int (\ln x)^2 \; dx \quad = \quad 2x - 2x \ln x + x(\ln x)^2 + C$$

26) $\int \ln(x^2) \; dx$

27) $\int x^2 \ln x \; dx$

Answer

$$\int x^2 \ln x \; dx \quad = \quad -\frac{x^3}{9} + \tfrac{1}{3} x^3 \ln x + C$$

28) $\int \sin^{-1} x \; dx$

29) $\int \cos^{-1}(2x) \; dx$

Answer

$$\int \cos^{-1}(2x) \; dx \quad = \quad -\tfrac{1}{2}\sqrt{1 - 4x^2} + x \cos^{-1}(2x) + C$$

30) $\int x \arctan x \; dx$

31) $\int x^2 \sin x \; dx$

Answer

$$\int x^2 \sin x \; dx \quad = \quad -(-2 + x^2) \cos x + 2x \sin x + C$$

32) $\int x^3 \cos x \; dx$

33) $\int x^3 \sin x \; dx$

Answer

$$\int x^3 \sin x \; dx \quad = \quad -x(-6 + x^2) \cos x + 3(-2 + x^2) \sin x + C$$

34) $\int x^3 e^x \; dx$

35) $\int x \sec^{-1} x \; dx$

Answer

$$\int x \sec^{-1} x \; dx \quad = \quad \tfrac{1}{2} x \left(-\sqrt{1 - \frac{1}{x^2}} + x \cdot \sec^{-1} x \right) + C$$

36) $\int x \sec^2 x \; dx$

37) $\int x \cosh x \; dx$

Answer

$$\int x \cosh x \; dx \quad = \quad -\cosh x + x \sinh x + C$$

In exercises 38 - 46, compute the definite integrals. Use a graphing utility to confirm your answers.

38) $\int_{1/e}^{1} \ln x \ dx$

39) $\int_{0}^{1} xe^{-2x} \ dx$ (Express the answer in exact form.)

Answer

$$\int_{0}^{1} xe^{-2x} \ dx \quad = \quad \frac{1}{4} - \frac{3}{4e^2}$$

40) $\int_{0}^{1} e^{\sqrt{x}} \ dx \quad (\text{let } u = \sqrt{x})$

41) $\int_{1}^{e} \ln(x^2) \ dx$

Answer

$$\int_{1}^{e} \ln(x^2) \ dx \quad = \quad 2$$

42) $\int_{0}^{\pi} x \cos x \ dx$

43) $\int_{-\pi}^{\pi} x \sin x \ dx$ (Express the answer in exact form.)

Answer

$$\int_{-\pi}^{\pi} x \sin x \ dx \quad = \quad 2\pi$$

44) $\int_{0}^{3} \ln(x^2 + 1) \ dx$ (Express the answer in exact form.)

45) $\int_{0}^{\pi/2} x^2 \sin x \ dx$ (Express the answer in exact form.)

Answer

$$\int_{0}^{\pi/2} x^2 \sin x \ dx \quad = \quad -2 + \pi$$

46) $\int_{0}^{1} x5^x \ dx$ (Express the answer using five significant digits.)

47) Evaluate $\int \cos x \ln(\sin x) \ dx$

Answer

$$\int \cos x \ln(\sin x) \ dx \quad = \quad -\sin(x) + \ln[\sin(x)] \sin x + C$$

In exercises 48 - 50, derive the following formulas using the technique of integration by parts. Assume that n is a positive integer. These formulas are called *reduction formulas* because the exponent in the x term has been reduced by one in each case. The second integral is simpler than the original integral.

48) $\int x^n e^x \ dx = x^n e^x - n \int x^{n-1} e^x \ dx$

49) $\int x^n \cos x \ dx = x^n \sin x - n \int x^{n-1} \sin x \ dx$

Answer

Answers vary

50) $\int x^n \sin x \ dx =$_____

51) Integrate $\int 2x\sqrt{2x-3} \ dx$ using two methods:

a. Using parts, letting $dv = \sqrt{2x-3} \ dx$

b. Substitution, letting $u = 2x - 3$

Challenge: Show the two answers are equivalent.

Answer

a. $\int 2x\sqrt{2x-3} \ dx = \frac{2}{3}x(2x-3)^{3/2} - \frac{2}{15}(2x-3)^{5/2} + C$

b. $\int 2x\sqrt{2x-3} \ dx = (2x-3)^{3/2} + \frac{1}{5}(2x-3)^{5/2} + C$

In exercises 52 - 57, state whether you would use integration by parts to evaluate the integral. If so, identify u and dv. If not, describe the technique used to perform the integration without actually doing the problem.

52) $\int x \ln x \ dx$

53) $\int \frac{\ln^2 x}{x} \ dx$

Answer

Do not use integration by parts. Choose u to be $\ln x$, and the integral is of the form $\int u^2 \ du$.

54) $\int xe^x \ dx$

55) $\int xe^{x^2-3} \ dx$

Answer

Do not use integration by parts. Let $u = x^2 - 3$, and the integral can be put into the form $\int e^u \ du$.

56) $\int x^2 \sin x \ dx$

57) $\int x^2 \sin(3x^3 + 2) \ dx$

Answer

Do not use integration by parts. Choose u to be $u = 3x^3 + 2$ and the integral can be put into the form $\int \sin(u) \ du$.

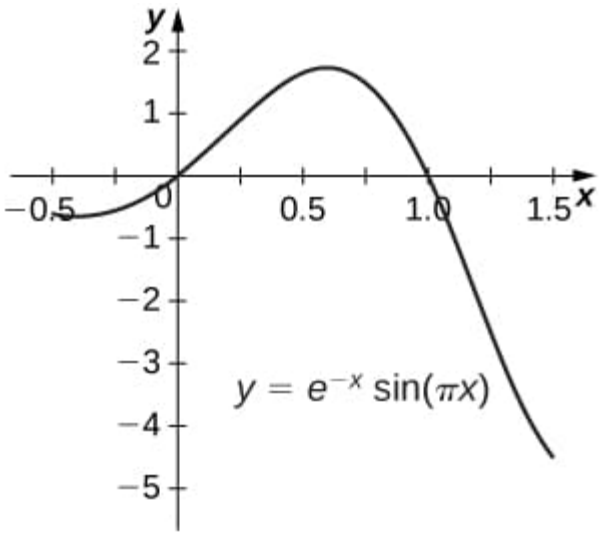
In exercises 58-59, sketch the region bounded above by the curve, the x-axis, and $x = 1$, and find the area of the region. Provide the exact form or round answers to the number of places indicated.

58) $y = 2xe^{-x}$ (Approximate answer to four decimal places.)

59) $y = e^{-x}\sin(\pi x)$

Answer

The area under graph is $\frac{\pi}{\pi^2+1}(e^{-1}+1)$ units^2.



In exercises 60 - 61, find the volume generated by rotating the region bounded by the given curves about the specified line. Express the answers in exact form or approximate to the number of decimal places indicated.

60) $y = \sin x,\ y = 0,\ x = 2\pi,\ x = 3\pi$; about the y-axis (Express the answer in exact form.)

61) $y = e^{-x},\ y = 0,\ x = -1,\ x = 0$; about $x = 1$ (Express the answer in exact form.)

Answer

$V = 2\pi e \text{ units}^3$

62) A particle moving along a straight line has a velocity of $v(t) = t^2e^{-t}$ after t sec. How far does it travel in the first 2 sec? (Assume the units are in feet and express the answer in exact form.)

63) Find the area under the graph of $y = \sec^3 x$ from $x = 0$ to $x = 1$. (Round the answer to two significant digits.)

Answer

$A = \frac{1}{2}(sec(1)tan(1) + \ln(sec(1) + tan(1))) \approx 2.05 \text{ units}^2$

64) Find the area between $y = (x - 2)e^x$ and the x-axis from $x = 2$ to $x = 5$. (Express the answer in exact form.)

65) Find the area of the region enclosed by the curve $y = x\cos x$ and the x-axis for $\frac{11\pi}{2} \le x \le \frac{13\pi}{2}$. (Express the answer in exact form.)

Answer

$A = 12\pi \text{ units}^2$

66) Find the volume of the solid generated by revolving the region bounded by the curve $y = \ln x$, the x-axis, and the vertical line $x = e^2$ about the x-axis. (Express the answer in exact form.)

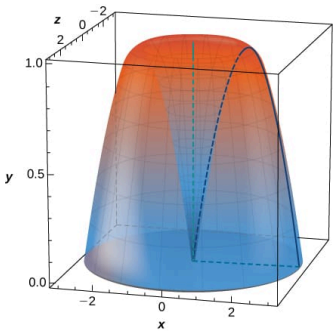
67) Find the volume of the solid generated by revolving the region bounded by the curve $y = 4\cos x$ and the x-axis, $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$, about the x-axis. (Express the answer in exact form.)

Answer

$V = 8\pi^2 \text{ units}^3$

68) Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = e^x$ and the x-axis, from $x = 0$ to $x = \ln(7)$, about the y-axis. (Express the answer in exact form.)

69) What is the volume of the Bundt cake that comes from rotating $y = \sin x$ around the y-axis from $x = 0$ to $x = \pi$?





Answer

$V = 2\pi^2$ units^3

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8.2: TRIGONOMETRIC INTEGRALS

7.02: Trigonometric Integrals

8.2E: Exercises for Section 8.2

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8.2E: Exercises for Section 8.2

Fill in the blank to make a true statement.

1) $\sin^2 x +$_____ $= 1$

Answer

$\cos^2 x$

2) $\sec^2 x - 1 =$_____

Answer

$\tan^2 x$

Use an identity to reduce the power of the trigonometric function to a trigonometric function raised to the first power.

3) $\sin^2 x =$_____

Answer

$\dfrac{1-\cos(2x)}{2}$

4) $\cos^2 x =$_____

Answer

$\dfrac{1+\cos(2x)}{2}$

Evaluate each of the following integrals by u-substitution.

5) $\int \sin^3 x \cos x \ dx$

Answer

$\int \sin^3 x \cos x \ dx \quad = \quad \dfrac{\sin^4 x}{4} + C$

6) $\int \sqrt{\cos x} \sin x \ dx$

7) $\int \tan^5(x) \sec^2(x) \ dx$

Answer

$\int \tan^5(x) \sec^2(x) \ dx \quad = \quad \frac{1}{6}\tan^6(2x) + C$

8) $\int \sin^7(2x) \cos(2x) \ dx$

9) $\int \tan(\frac{x}{2}) \sec^2(\frac{x}{2}) \ dx$

Answer

$\int \tan(\frac{x}{2}) \sec^2(\frac{x}{2}) \ dx \quad = \quad \tan^2(\frac{x}{2}) + C$

10) $\int \tan^2 x \sec^2 x \ dx$

Compute the following integrals using the guidelines for integrating powers of trigonometric functions. Use a CAS to check the solutions. (*Note*: Some of the problems may be done using techniques of integration learned previously.)

11) $\int \sin^3 x \ dx$

Answer

$$\int \sin^3 x \ dx = -\frac{3\cos x}{4} + \tfrac{1}{12}\cos(3x) + C = -\cos x + \frac{\cos^3 x}{3} + C$$

12) $\int \cos^3 x \ dx$

13) $\int \sin x \cos x \ dx$

Answer

$$\int \sin x \cos x \ dx = -\tfrac{1}{2}\cos^2 x + C$$

14) $\int \cos^5 x \ dx$

15) $\int \sin^5 x \cos^2 x \ dx$

Answer

$$\int \sin^5 x \cos^2 x \ dx = -\frac{5\cos x}{64} - \tfrac{1}{192}\cos(3x) + \tfrac{3}{320}\cos(5x) - \tfrac{1}{448}\cos(7x) + C$$

16) $\int \sin^3 x \cos^3 x \ dx$

17) $\int \sqrt{\sin x}\cos x \ dx$

Answer

$$\int \sqrt{\sin x}\cos x \ dx = \tfrac{2}{3}(\sin x)^{3/2} + C$$

18) $\int \sqrt{\sin x}\cos^3 x \ dx$

19) $\int \sec x \tan x \ dx$

Answer

$$\int \sec x \tan x \ dx = \sec x + C$$

20) $\int \tan(5x) \ dx$

21) $\int \tan^2 x \sec x \ dx$

Answer

$$\int \tan^2 x \sec x \ dx = \tfrac{1}{2}\sec x \tan x - \tfrac{1}{2}\ln(\sec x + \tan x) + C$$

22) $\int \tan x \sec^3 x \ dx$

Answer

$$\int \tan x \sec^3 x \; dx \quad = \quad \tfrac{1}{3}(\sec x)^3 + C$$

23) $\int \sec^4 x \; dx$

Answer

$$\int \sec^4 x \; dx \quad = \quad \frac{2\tan x}{3} + \tfrac{1}{3}\sec^2 x \tan x = \tan x + \frac{\tan^3 x}{3} + C$$

24) $\int \cot x \; dx$

25) $\int \csc x \; dx$

Answer

$$\int \csc x \; dx \quad = \quad -\ln|\cot x + \csc x| + C$$

26) $\int \frac{\tan^3 x}{\sqrt{\sec x}} \; dx$

For exercises 27 - 28, find a general formula for the integrals.

27) $\int \sin^2 ax \cos ax \; dx$

Answer

$$\int \sin^2 ax \cos ax \; dx \quad = \quad \frac{\sin^3(ax)}{3a} + C$$

28) $\int \sin ax \cos ax \; dx.$

Use power-reducing formulas or double-angle formulas to evaluate the integrals in exercises 29 - 34.

29) $\int_0^{\pi} \sin^2 x \; dx$

Answer

$$\int_0^{\pi} \sin^2 x \; dx \quad = \quad \frac{\pi}{2}$$

30) $\int_0^{\pi} \sin^4 x \; dx$

Answer

$$\int_0^{\pi} \sin^4 x \; dx \quad = \quad \frac{3\pi}{8}$$

31) $\int \cos^2 3x \; dx$

Answer

$$\int \cos^2 3x \; dx \quad = \quad \frac{x}{2} + \tfrac{1}{12}\sin(6x) + C$$

32) $\int \sin^2 x \cos^2 x \; dx$

33) $\int \sin^2 x \; dx + \int \cos^2 x \; dx$

Answer

$$\int \sin^2 x \; dx + \int \cos^2 x \; dx \quad = \quad x + C$$

34) $\int \sin^2 x \cos^2(2x) \; dx$

For exercises 35 - 43, evaluate the definite integrals. Express answers in exact form whenever possible.

35) $\int_0^{2\pi} \cos x \sin 2x \; dx$

Answer

$$\int_0^{2\pi} \cos x \sin 2x \; dx \quad = \quad 0$$

36) $\int_0^{\pi} \sin 3x \sin 5x \; dx$

37) $\int_0^{\pi} \cos(99x) \sin(101x) \; dx$

Answer

$$\int_0^{\pi} \cos(99x) \sin(101x) \; dx \quad = \quad 0$$

38) $\int_{-\pi}^{\pi} \cos^2(3x) \; dx$

Answer

$$\int_{-\pi}^{\pi} \cos^2(3x) \; dx \quad = \quad \pi$$

39) $\int_0^{2\pi} \sin x \sin(2x) \sin(3x) \; dx$

Answer

$$\int_0^{2\pi} \sin x \sin(2x) \sin(3x) \; dx \quad = \quad 0$$

40) $\int_0^{4\pi} \cos(x/2) \sin(x/2) \; dx$

41) $\int_{\pi/6}^{\pi/3} \dfrac{\cos^3 x}{\sqrt{\sin x}} \; dx$ (Round this answer to three decimal places.)

Answer

$$\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} \; dx \quad \approx \quad 0.239$$

42) $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 x - 1} \; dx$

43) $\int_0^{\pi/2} \sqrt{1 - \cos(2x)} \; dx$

Answer

$$\int_0^{\pi/2} \sqrt{1 - \cos(2x)} \; dx \quad = \quad \sqrt{2}$$

44) Find the area of the region bounded by the graphs of the equations $y = \sin x, \;\; y = \sin^3 x, \;\; x = 0$, and $x = \frac{\pi}{2}$.

45) Find the area of the region bounded by the graphs of the equations $y = \cos^2 x,\ y = \sin^2 x,\ x = -\frac{\pi}{4}$, and $x = \frac{\pi}{4}$.

Answer

$A = 1 \text{ unit}^2$

46) A particle moves in a straight line with the velocity function $v(t) = \sin(\omega t)\cos^2(\omega t)$. Find its position function $x = f(t)$ if $f(0) = 0$.

47) Find the average value of the function $f(x) = \sin^2 x \cos^3 x$ over the interval $[-\pi, \pi]$.

Answer

0

For exercises 48 - 49, solve the differential equations.

48) $\frac{dy}{dx} = \sin^2 x$. The curve passes through point $(0, 0)$.

49) $\frac{dy}{d\theta} = \sin^4(\pi\theta)$

Answer

$f(x) = \frac{3\theta}{8} - \frac{1}{4\pi}\sin(2\pi\theta) + \frac{1}{32\pi}\sin(4\pi\theta) + C$

50) Find the length of the curve $y = \ln(\csc x),\ \text{for } \frac{\pi}{4} \le x \le \frac{\pi}{2}$.

51) Find the length of the curve $y = \ln(\sin x),\ \text{for } \frac{\pi}{3} \le x \le \frac{\pi}{2}$.

Answer

$s = \ln(\sqrt{3})$

52) Find the volume generated by revolving the curve $y = \cos(3x)$ about the x-axis, for $0 \le x \le \frac{\pi}{36}$.

For exercises 53 - 54, use this information: The inner product of two functions f and g over $[a, b]$ is defined by $f(x) \cdot g(x) = \langle f, g\rangle = \int_a^b f \cdot g\ dx$. Two distinct functions f and g are said to be orthogonal if $\langle f, g\rangle = 0$.

53) Show that $\sin(2x),\ \cos(3x)$ are orthogonal over the interval $[-\pi,\ \pi]$.

Answer

$\int_{-\pi}^{\pi} \sin(2x)\cos(3x)\ dx = 0$

54) Evaluate $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\ dx$.

55) Integrate $y' = \sqrt{\tan x}\sec^4 x$.

Answer

$y = \int \sqrt{\tan x}\sec^4 x\ dx \quad = \quad \frac{2}{3}(\tan x)^{3/2} + \frac{2}{7}(\tan x)^{7/2} + C = \frac{2}{21}(\tan x)^{3/2}\left[7 + 3\tan^2 x\right] + C$

For each pair of integrals in exercises 56 - 57, determine which one is more difficult to evaluate. Explain your reasoning.

56) $\int \sin^{456} x \cos x\ dx$ or $\int \sin^2 x \cos^2 x\ dx$

57) $\int \tan^{350} x \sec^2 x\ dx$ or $\int \tan^{350} x \sec x\ dx$

Answer

The second integral is more difficult because the first integral is simply a u-substitution type.

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8.3: TRIGONOMETRIC SUBSTITUTION

7.03: Trigonometric Substitution

8.3E: Exercises for Section 8.3

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8.3E: Exercises for Section 8.3

Simplify the expressions in exercises 1 - 5 by writing each one using a single trigonometric function.

1) $4 - 4\sin^2\theta$

2) $9\sec^2\theta - 9$

Answer

$$9\sec^2\theta - 9 \quad = \quad 9\tan^2\theta$$

3) $a^2 + a^2\tan^2\theta$

4) $a^2 + a^2\sinh^2\theta$

Answer

$$a^2 + a^2\sinh^2\theta \quad = \quad a^2\cosh^2\theta$$

5) $16\cosh^2\theta - 16$

Use the technique of completing the square to express each trinomial in exercises 6 - 8 as the square of a binomial.

6) $4x^2 - 4x + 1$

Answer

$4(x - \frac{1}{2})^2$

7) $2x^2 - 8x + 3$

8) $-x^2 - 2x + 4$

Answer

$-(x+1)^2 + 5$

In exercises 9 - 28, integrate using the method of trigonometric substitution. Express the final answer in terms of the original variable.

9) $\int \frac{dx}{\sqrt{4 - x^2}}$

10) $\int \frac{dx}{\sqrt{x^2 - a^2}}$

Answer

$$\int \frac{dx}{\sqrt{x^2 - a^2}} \quad = \quad \ln|\, x + \sqrt{-a^2 + x^2}\,| + C$$

11) $\int \sqrt{9 - x^2}\ dx$

Answer

$$\int \sqrt{9 - x^2}\ dx \quad = \quad \tfrac{9}{2}\arcsin(\frac{x}{3}) + \tfrac{1}{2}x\sqrt{9 - x^2} + C$$

12) $\int \frac{dx}{\sqrt{1 + 9x^2}}$

Answer

$$\int \frac{dx}{\sqrt{1+9x^2}} \quad = \quad \tfrac{1}{3}\ln\left|\sqrt{1+9x^2}+3x\right|+C$$

13) $\int \frac{x^2\,dx}{\sqrt{1-x^2}}$

Answer

$$\int \frac{x^2\,dx}{\sqrt{1-x^2}} \quad = \quad \tfrac{1}{2}\arcsin(x)+x\sqrt{1-x^2}+C$$

14) $\int \frac{dx}{x^2\sqrt{1-x^2}}$

Answer

$$\int \frac{dx}{x^2\sqrt{1-x^2}} \quad = \quad -\frac{\sqrt{1-x^2}}{x}+C$$

15) $\int \frac{dx}{(1+x^2)^2}$

16) $\int \sqrt{x^2+9}\;dx$

Answer

$$\int \sqrt{x^2+9}\;dx \quad = \quad 9\left[\frac{x\sqrt{x^2+9}}{18}+\tfrac{1}{2}\ln\left|\frac{\sqrt{x^2+9}}{3}+\frac{x}{3}\right|\right]+C$$

17) $\int \frac{\sqrt{x^2-25}}{x}\;dx$

18) $\int \frac{\theta^3}{\sqrt{9-\theta^2}}\;d\theta$

Answer

$$\int \frac{\theta^3 d\theta}{\sqrt{9-\theta^2}}\;d\theta \quad = \quad -\tfrac{1}{3}\sqrt{9-\theta^2}(18+\theta^2)+C$$

19) $\int \frac{dx}{\sqrt{x^6-x^2}}$

20) $\int \sqrt{x^6-x^8}\;dx$

Answer

$$\int \sqrt{x^6-x^8}\;dx \quad = \quad \frac{(-1+x^2)(2+3x^2)\sqrt{x^6-x^8}}{15x^3}+C$$

21) $\int \frac{dx}{(1+x^2)^{3/2}}$

22) $\int \frac{dx}{(x^2-9)^{3/2}}$

Answer

$$\int \frac{dx}{(x^2-9)^{3/2}} \quad = \quad -\frac{x}{9\sqrt{x^2-9}}+C$$

23) $\int \frac{\sqrt{1+x^2}}{x}\;dx$

24) $\int \frac{x^2}{\sqrt{x^2-1}}\,dx$

Answer

$$\int \frac{x^2}{\sqrt{x^2-1}}\,dx \quad = \quad \tfrac{1}{2}(\ln\,|\,x+\sqrt{x^2-1}\,|\,+x\sqrt{x^2-1})+C$$

25) $\int \frac{x^2}{x^2+4}\,dx$

26) $\int \frac{dx}{x^2\sqrt{x^2+1}}$

Answer

$$\int \frac{dx}{x^2\sqrt{x^2+1}} \quad = \quad -\frac{\sqrt{1+x^2}}{x}+C$$

27) $\int \frac{x^2}{\sqrt{1+x^2}}\,dx$

28) $\int_{-1}^{1}(1-x^2)^{3/2}\,dx$

Answer

$$\int_{-1}^{1}(1-x^2)^{3/2}\,dx \quad = \quad \tfrac{1}{8}\left(x(5-2x^2)\sqrt{1-x^2}+3\arcsin x\right) = \quad \tfrac{3\pi}{8}$$

In exercises 29 - 34, use the substitutions $x=\sinh\theta$, $\cosh\theta$, or $\tanh\theta$. Express the final answers in terms of the variable x.

29) $\int \frac{dx}{\sqrt{x^2-1}}$

30) $\int \frac{dx}{x\sqrt{1-x^2}}$

Answer

$$\int \frac{dx}{x\sqrt{1-x^2}} \quad = \quad \ln x-\ln\,|\,1+\sqrt{1-x^2}\,|\,+C$$

31) $\int \sqrt{x^2-1}\,dx$

32) $\int \frac{\sqrt{x^2-1}}{x^2}\,dx$

Answer

$$\int \frac{\sqrt{x^2-1}}{x^2}\,dx \quad = \quad -\frac{\sqrt{-1+x^2}}{x}+\ln\left|x+\sqrt{-1+x^2}\right|+C$$

33) $\int \frac{dx}{1-x^2}$

34) $\int \frac{\sqrt{1+x^2}}{x^2}\,dx$

Answer

$$\int \frac{\sqrt{1+x^2}}{x^2}\,dx \quad = \quad -\frac{\sqrt{1+x^2}}{x}+\operatorname{arcsinh}\,x+C$$

Use the technique of completing the square to evaluate the integrals in exercises 35 - 39.

35) $\int \frac{1}{x^2-6x}\,dx$

36) $\int \frac{1}{x^2+2x+1}\,dx$

Answer

$$\int \frac{1}{x^2+2x+1}\,dx \quad = \quad -\frac{1}{1+x}+C$$

37) $\int \frac{1}{\sqrt{-x^2+2x+8}}\,dx$

38) $\int \frac{1}{\sqrt{-x^2+10x}}\,dx$

Answer

$$\int \frac{1}{\sqrt{-x^2+10x}}\,dx \quad = \quad \int \frac{1}{\sqrt{25-(x-5)^2}}\,dx \quad = \quad \arcsin\left(\frac{x-5}{5}\right)+C$$

39) $\int \frac{1}{\sqrt{x^2+4x-12}}\,dx$

40) Evaluate the integral without using calculus: $\int_{-3}^{3}\sqrt{9-x^2}\,dx.$

Answer

$\int_{-3}^{3}\sqrt{9-x^2}\,dx \quad = \quad \frac{9\pi}{2}$; area of a semicircle with radius 3

41) Find the area enclosed by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1.$

42) Evaluate the integral $\int \frac{dx}{\sqrt{1-x^2}}$ using two different substitutions. First, let $x=\cos\theta$ and evaluate using trigonometric substitution. Second, let $x=\sin\theta$ and use trigonometric substitution. Are the answers the same?

Answer

$\int \frac{dx}{\sqrt{1-x^2}} \quad = \quad \arcsin(x)+C$ is the common answer.

43) Evaluate the integral $\int \frac{dx}{x\sqrt{x^2-1}}$ using the substitution $x=\sec\theta$. Next, evaluate the same integral using the substitution $x=\csc\theta$. Show that the results are equivalent.

44) Evaluate the integral $\int \frac{x}{x^2+1}\,dx$ using the form $\int \frac{1}{u}\,du$. Next, evaluate the same integral using $x=\tan\theta$. Are the results the same?

Answer

$\int \frac{x}{x^2+1}\,dx \quad = \quad \frac{1}{2}\ln(1+x^2)+C$ is the result using either method.

45) State the method of integration you would use to evaluate the integral $\int x\sqrt{x^2+1}\,dx$. Why did you choose this method?

46) State the method of integration you would use to evaluate the integral $\int x^2\sqrt{x^2-1}\,dx$. Why did you choose this method?

Answer

Use trigonometric substitution. Let $x=\sec(\theta)$.

47) Evaluate $\int_{-1}^{1} \frac{x}{x^2+1}\, dx$

48) Find the length of the arc of the curve over the specified interval: $y = \ln x, \quad [1, 5]$. Round the answer to three decimal places.

Answer

$s = 4.367$ units

49) Find the surface area of the solid generated by revolving the region bounded by the graphs of $y = x^2,\ y = 0,\ x = 0$, and $x = \sqrt{2}$ about the x-axis. (Round the answer to three decimal places).

50) The region bounded by the graph of $f(x) = \frac{1}{1+x^2}$ and the x-axis between $x = 0$ and $x = 1$ is revolved about the x-axis. Find the volume of the solid that is generated.

Answer

$V = \left(\frac{\pi^2}{8} + \frac{\pi}{4}\right)$ units^3

In exercises 51 - 52, solve the initial-value problem for y as a function of x.

51) $(x^2 + 36)\frac{dy}{dx} = 1, \quad y(6) = 0$

52) $(64 - x^2)\frac{dy}{dx} = 1, \quad y(0) = 3$

Answer

$y = \frac{1}{16}\ln\left|\frac{x+8}{x-8}\right| + 3$

53) Find the area bounded by $y = \frac{2}{\sqrt{64-4x^2}},\ x = 0,\ y = 0$, and $x = 2$.

54) An oil storage tank can be described as the volume generated by revolving the area bounded by $y = \frac{16}{\sqrt{64+x^2}},\ x = 0,\ y = 0,\ x = 2$ about the x-axis. Find the volume of the tank (in cubic meters).

Answer

$V = 24.6$ m^3

55) During each cycle, the velocity v (in feet per second) of a robotic welding device is given by $v = 2t - \frac{14}{4+t^2}$, where t is time in seconds. Find the expression for the displacement s (in feet) as a function of t if $s = 0$ when $t = 0$.

56) Find the length of the curve $y = \sqrt{16 - x^2}$ between $x = 0$ and $x = 2$.

Answer

$s = \frac{2\pi}{3}$ units

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8.4: PARTIAL FRACTIONS

7.04: Partial Fractions

8.4E: Exercises for Section 8.4

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8.4E: Exercises for Section 8.4

Use partial fraction decomposition (or a simpler technique) to express the rational function as a sum or difference of two or more simpler rational expressions.

1) $\dfrac{1}{(x-3)(x-2)}$

2) $\dfrac{x^2+1}{x(x+1)(x+2)}$

Answer

$$\frac{x^2+1}{x(x+1)(x+2)} = -\frac{2}{x+1}+\frac{5}{2(x+2)}+\frac{1}{2x}$$

3) $\dfrac{1}{x^3-x}$

4) $\dfrac{3x+1}{x^2}$

Answer

$$\frac{3x+1}{x^2} = \frac{1}{x^2}+\frac{3}{x}$$

5) $\dfrac{3x^2}{x^2+1}$ (Hint: Use long division first.)

6) $\dfrac{2x^4}{x^2-2x}$

Answer

$$\frac{2x^4}{x^2-2x} = 2x^2+4x+8+\frac{16}{x-2}$$

7) $\dfrac{1}{(x-1)(x^2+1)}$

8) $\dfrac{1}{x^2(x-1)}$

Answer

$$\frac{1}{x^2(x-1)} = -\frac{1}{x^2}-\frac{1}{x}+\frac{1}{x-1}$$

9) $\dfrac{x}{x^2-4}$

10) $\dfrac{1}{x(x-1)(x-2)(x-3)}$

Answer

$$\frac{1}{x(x-1)(x-2)(x-3)} = -\frac{1}{2(x-2)}+\frac{1}{2(x-1)}-\frac{1}{6x}+\frac{1}{6(x-3)}$$

11) $\dfrac{1}{x^4-1} = \dfrac{1}{(x+1)(x-1)(x^2+1)}$

12) $\dfrac{3x^2}{x^3-1}$ Hint: Use the difference of cubes formula first

Answer

$$\frac{3x^2}{x^3-1} = \frac{1}{x-1}+\frac{2x+1}{x^2+x+1}$$

13) $\dfrac{2x}{(x+2)^2}$

14) $\dfrac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}$

Answer

$$\frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2} = \frac{2}{x+1}+\frac{x}{x^2+4}-\frac{1}{(x^2+4)^2}$$

In exercises 15 - 25, use the method of partial fractions to evaluate each of the following integrals.

15) $\displaystyle\int \frac{dx}{(x-3)(x-2)}$

16) $\displaystyle\int \frac{3x}{x^2+2x-8}\,dx$

Answer

$$\int \frac{3x}{x^2+2x-8}\,dx = 2\ln|x+4|+\ln|x-2|+C = \ln\left|(x+4)^2(x-2)\right|+C$$

17) $\displaystyle\int \frac{dx}{x^3-x}$

18) $\displaystyle\int \frac{x}{x^2-4}\,dx$

Answer

Note that you don't need Partial Fractions here. You can use u-substitution.

$$\int \frac{x}{x^2-4}\,dx = \tfrac{1}{2}\ln|4-x^2|+C$$

19) $\displaystyle\int \frac{dx}{x(x-1)(x-2)(x-3)}$

20) $\displaystyle\int \frac{2x^2+4x+22}{x^2+2x+10}\,dx$

Answer

Note that since the degree of the numerator is equal to the degree of the denominator, we need to start with long division.

Then note that we will need to use completing the square to continue since we cannot factor the trinomial in the denominator.

$$\int \frac{2x^2+4x+22}{x^2+2x+10}\,dx = 2\left(x+\tfrac{1}{3}\arctan\left(\frac{1+x}{3}\right)\right)+C$$

21) $\displaystyle\int \frac{dx}{x^2-5x+6}$

22) $\displaystyle\int \frac{2-x}{x^2+x}\,dx$

Answer

$$\int \frac{2-x}{x^2+x}\,dx = 2\ln|x|-3\ln|1+x|+C = \ln\left|\frac{x^2}{(1+x)^3}\right|+C$$

23) $\displaystyle\int \frac{2}{x^2-x-6}\,dx$

24) $\int \frac{dx}{x^3-2x^2-4x+8}$ Hint: Factor the denominator first using grouping

Answer

$$\int \frac{dx}{x^3-2x^2-4x+8} = \frac{1}{16}\left(-\frac{4}{-2+x}-\ln|-2+x|+\ln|2+x|\right)+C = \frac{1}{16}\left(-\frac{4}{-2+x}+\ln\left|\frac{x+2}{x-2}\right|\right)+C$$

25) $\int \frac{dx}{x^4-10x^2+9}$

In exercises 26 - 29, evaluate the integrals with irreducible quadratic factors in the denominators.

26) $\int \frac{2}{(x-4)(x^2+2x+6)}\,dx$

Answer

$$\int \frac{2}{(x-4)(x^2+2x+6)}\,dx = \frac{1}{30}(-2\sqrt{5}\arctan\left[\frac{1+x}{\sqrt{5}}\right]+2\ln|-4+x|-\ln|6+2x+x^2|)+C$$

27) $\int \frac{x^2}{x^3-x^2+4x-4}\,dx$

28) $\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\,dx$

Answer

Note that we need to use long division first, since the degree of the numerator is greater than the degree of the denominator.

$$\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\,dx = -\frac{3}{x}+4\ln|x+2|+x+C$$

29) $\int \frac{x}{(x-1)(x^2+2x+2)^2}\,dx$

In exercises 30 - 32, use the method of partial fractions to evaluate the integrals.

30) $\int \frac{3x+4}{(x^2+4)(3-x)}\,dx$

Answer

$$\int \frac{3x+4}{(x^2+4)(3-x)}\,dx = -\ln|3-x|+\frac{1}{2}\ln|x^2+4|+C$$

31) $\int \frac{2}{(x+2)^2(2-x)}\,dx$

32) $\int \frac{3x+4}{x^3-2x-4}\,dx$ (Hint: Use the rational root theorem.)

Answer

$$\int \frac{3x+4}{x^3-2x-4}\,dx = \ln|x-2|-\frac{1}{2}\ln|x^2+2x+2|+C$$

In exercises 33 - 46, use substitution to convert the integrals to integrals of rational functions. Then use partial fractions to evaluate the integrals.

33) $\int_0^1 \frac{e^x}{36-e^{2x}}\,dx$ (Give the exact answer and the decimal equivalent. Round to five decimal places.)

34) $\int \frac{e^x\,dx}{e^{2x}-e^x}\,dx$

Answer

$$\int \frac{e^x\,dx}{e^{2x}-e^x}\,dx \quad = \quad -x+\ln|1-e^x|+C$$

35) $\int \frac{\sin x\,dx}{1-\cos^2 x}$

36) $\int \frac{\sin x}{\cos^2 x+\cos x-6}\,dx$

Answer

$$\int \frac{\sin x}{\cos^2 x+\cos x-6}\,dx \quad = \quad \tfrac{1}{5}\ln\left|\frac{\cos x+3}{\cos x-2}\right|+C$$

37) $\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

38) $\int \frac{dt}{(e^t-e^{-t})^2}$

Answer

$$\int \frac{dt}{(e^t-e^{-t})^2} \quad = \quad \frac{1}{2-2e^{2t}}+C$$

39) $\int \frac{1+e^x}{1-e^x}\,dx$

40) $\int \frac{dx}{1+\sqrt{x+1}}$

Answer

$$\int \frac{dx}{1+\sqrt{x+1}} \quad = \quad 2\sqrt{1+x}-2\ln|1+\sqrt{1+x}|+C$$

41) $\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}}$

42) $\int \frac{\cos x}{\sin x(1-\sin x)}\,dx$

Answer

$$\int \frac{\cos x}{\sin x(1-\sin x)}\,dx \quad = \quad \ln\left|\frac{\sin x}{1-\sin x}\right|+C$$

43) $\int \frac{e^x}{(e^{2x}-4)^2}\,dx$

44) $\int_1^2 \frac{1}{x^2\sqrt{4-x^2}}\,dx$

Answer

$$\int_1^2 \frac{1}{x^2\sqrt{4-x^2}}\,dx \quad = \quad \frac{\sqrt{3}}{4}$$

45) $\int \frac{1}{2+e^{-x}}\,dx$

46) $\int \frac{1}{1+e^x}\,dx$

Answer

$$\int \frac{1}{1+e^x}\,dx \quad = \quad x-\ln(1+e^x)+C$$

In exercises 47 - 48, use the given substitution to convert the integral to an integral of a rational function, then evaluate.

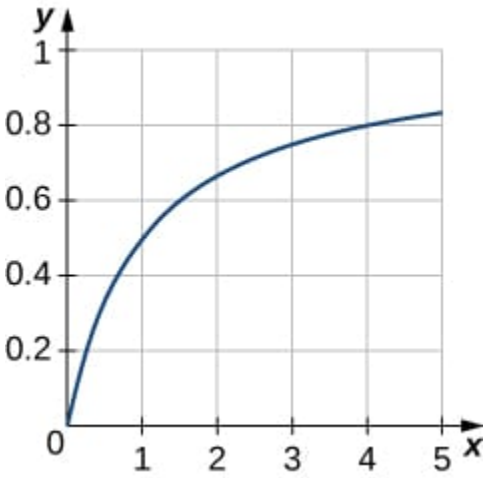
47) $\int \frac{1}{t-\sqrt[3]{t}}\,dt; \quad t=x^3$

48) $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx; \quad x=u^6$

Answer

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx \quad = \quad 6x^{1/6}-3x^{1/3}+2\sqrt{x}-6\ln(1+x^{1/6})+C$$

49) Graph the curve $y=\frac{x}{1+x}$ over the interval $[0,5]$. Then, find the area of the region bounded by the curve, the x-axis, and the line $x=4$.



50) Find the volume of the solid generated when the region bounded by $y=\frac{1}{\sqrt{x(3-x)}}$, $y=0$, $x=1$, and $x=2$ is revolved about the x-axis.

Answer

$$V=\tfrac{4}{3}\pi\operatorname{arctanh}\left[\tfrac{1}{3}\right]=\tfrac{1}{3}\pi\ln 4 \text{ units}^3$$

51) The velocity of a particle moving along a line is a function of time given by $v(t)=\frac{t^2}{t^2+1}$ meters per second. Find the distance that the particle has traveled after $t=5$ sec.

Answer

$$V=5-\arctan(5)\approx 3.63\text{meters}$$

In exercises 52 - 54, solve the initial-value problem for x as a function of t.

52) $(t^2-7t+12)\frac{dx}{dt}=1, \quad t>4, \; x(5)=0$

Answer

$$x=-\ln|t-3|+\ln|t-4|+\ln 2=\ln\left|\frac{2(t-4)}{t-3}\right|$$

53) $(t+5)\frac{dx}{dt}=x^2+1, \quad t>-5, \; x(1)=\tan 1$

54) $(2t^3-2t^2+t-1)\frac{dx}{dt}=3, \quad x(2)=0$

Answer

$$x = \ln|t-1| - \sqrt{2}\arctan(\sqrt{2}t) - \tfrac{1}{2}\ln(t^2+\tfrac{1}{2}) + \sqrt{2}\arctan(2\sqrt{2}) + \tfrac{1}{2}\ln 4.5$$

55) Find the x-coordinate of the centroid of the area bounded by $y(x^2-9)=1,\ y=0,\ x=4$, and $x=5$. (Round the answer to two decimal places.)

56) Find the volume generated by revolving the area bounded by $y=\dfrac{1}{x^3+7x^2+6x},\ x=1,\ x=7$, and $y=0$ about the y-axis.

Answer

$$V = \tfrac{2}{5}\pi \ln \tfrac{28}{13}\ \text{units}^3$$

57) Find the area bounded by $y=\dfrac{x-12}{x^2-8x-20},\ y=0,\ x=2$, and $x=4$. (Round the answer to the nearest hundredth.)

58) Evaluate the integral $\displaystyle\int \frac{dx}{x^3+1}$.

Answer

$$\int \frac{dx}{x^3+1} = \frac{\arctan[\frac{-1+2x}{\sqrt{3}}]}{\sqrt{3}} + \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln|1-x+x^2| + C$$

For problems 59 - 62, use the substitutions $\tan(\frac{x}{2})=t$, $dx=\dfrac{2}{1+t^2}\,dt$, $\sin x=\dfrac{2t}{1+t^2}$, and $\cos x=\dfrac{1-t^2}{1+t^2}$.

59) $\displaystyle\int \frac{dx}{3-5\sin x}$

60) Find the area under the curve $y=\dfrac{1}{1+\sin x}$ between $x=0$ and $x=\pi$. (Assume the dimensions are in inches.)

Answer

2.0 in.^{2}

61) Given $\tan\left(\frac{x}{2}\right)=t$, derive the formulas $dx=\dfrac{2}{1+t^2}dt,\ \sin x=\dfrac{2t}{1+t^2}$, and $\cos x=\dfrac{1-t^2}{1+t^2}$.

62) Evaluate $\displaystyle\int \frac{\sqrt[3]{x-8}}{x}\,dx$.

Answer

$$\int \frac{\sqrt[3]{x-8}}{x}\,dx = 3(-8+x)^{1/3} - 2\sqrt{3}\arctan\left[\frac{-1+(-8+x)^{1/3}}{\sqrt{3}}\right] - 2\ln\left[2+(-8+x)^{1/3}\right] + \ln\left[4-2(-8+x)^{1/3}+(-8+x)^{2/3}\right] + C$$

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8.5: OTHER STRATEGIES FOR INTEGRATION

7.05: Other Strategies for Integration

8.5E: Exercises for Section 8.5

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8.5E: Exercises for Section 8.5

7.5E: Exercises for Section 7.5

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8.6: NUMERICAL INTEGRATION

7.06: Numerical Integration

8.6E: Exercises for Section 8.6

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8.6E: Exercises for Section 8.6

7.6E: Exercises for Section 7.6

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8.7: IMPROPER INTEGRALS

7.07: Improper Integrals

8.7E: Exercises for Section 8.7

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8.7E: Exercises for Section 8.7

In exercises 1 - 8, evaluate the following integrals. If the integral is not convergent, answer "It diverges."

1) $\int_2^4 \frac{dx}{(x-3)^2}$

Answer

It diverges.

2) $\int_0^\infty \frac{1}{4+x^2}\ dx$

3) $\int_0^2 \frac{1}{\sqrt{4-x^2}}\ dx$

Answer

Converges to $\frac{\pi}{2}$

4) $\int_1^\infty \frac{1}{x \ln x}\ dx$

5) $\int_1^\infty xe^{-x}\ dx$

Answer

Converges to $\frac{2}{e}$

6) $\int_{-\infty}^\infty \frac{x}{x^2+1}\ dx$

Answer

It diverges.

7) Without integrating, determine whether the integral $\int_1^\infty \frac{1}{\sqrt{x^3+1}}\ dx$ converges or diverges by comparing the function $f(x) = \frac{1}{\sqrt{x^3+1}}$ with $g(x) = \frac{1}{\sqrt{x^3}}$.

Answer

It converges since $\frac{1}{\sqrt{x^3+1}} < \frac{1}{\sqrt{x^3}}$, and $\int_1^\infty \frac{1}{\sqrt{x^3}}\ dx$ converges.

8) Without integrating, determine whether the integral $\int_1^\infty \frac{1}{\sqrt{x+1}}\ dx$ converges or diverges.

In exercises 9 - 25, determine whether the improper integrals converge or diverge. If possible, determine the value of the integrals that converge.

9) $\int_0^\infty e^{-x}\cos x\ dx$

Answer

Converges to $\frac{1}{2}$.

10) $\int_1^\infty \frac{\ln x}{x}\ dx$

11) $\int_0^1 \frac{\ln x}{\sqrt{x}}\ dx$

Answer

Converges to -4.

12) $\int_0^1 \ln x\ dx$

13) $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\ dx$

Answer

Converges to π.

14) $\int_1^5 \frac{dx}{\sqrt{x-1}}$

15) $\int_{-2}^2 \frac{dx}{(1+x)^2}$

Answer

It diverges.

16) $\int_0^{\infty} e^{-x}\ dx$

Answer

Converges to 1.

17) $\int_0^{\infty} \sin x\ dx$

Answer

It diverges.

18) $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\ dx$

19) $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

Answer

Converges to 1.5.

20) $\int_0^2 \frac{dx}{x^3}$

21) $\int_{-1}^2 \frac{dx}{x^3}$

Answer

It diverges.

22) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

23) $\int_0^3 \frac{1}{x-1}\ dx$

Answer

It diverges.

24) $\int_1^{\infty} \frac{5}{x^3}\, dx$

Answer

Converges to 2.5.

25) $\int_3^5 \frac{5}{(x-4)^2}\, dx$

Answer

It diverges.

In exercises 26 and 27, determine the convergence of each of the following integrals by comparison with the given integral. If the integral converges, find the number to which it converges.

26) $\int_1^{\infty} \frac{dx}{x^2+4x}$; compare with $\int_1^{\infty} \frac{dx}{x^2}$.

Answer

Both integrals converge since $\int_1^{\infty} \frac{dx}{x^2+4x} < \int_1^{\infty} \frac{dx}{x^2}$ which converges. If an integral is smaller than another convergent integral, it must converge.

27) $\int_1^{\infty} \frac{dx}{\sqrt{x}-1}$; compare with $\int_1^{\infty} \frac{dx}{\sqrt{x}}$.

Answer

Both integrals diverge since $\int_1^{\infty} \frac{dx}{\sqrt{x}-1} > \int_1^{\infty} \frac{dx}{\sqrt{x}}$ which diverges. If an integral is bigger than another divergent integral, it must diverge.

In exercises 28 - 38, evaluate the integrals. If the integral diverges, answer "It diverges."

28) $\int_1^{\infty} \frac{dx}{x^e}$

Answer

Converges to $\frac{1}{e-1}$.

29) $\int_0^1 \frac{dx}{x^{\pi}}$

Answer

It diverges.

30) $\int_0^1 \frac{dx}{\sqrt{1-x}}$

31) $\int_0^1 \frac{dx}{1-x}$

Answer

It diverges.

32) $\int_{-\infty}^0 \frac{dx}{x^2+1}$

33) $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$

Answer

Converges to π.

34) $\int_{0}^{1} \frac{\ln x}{x}\ dx$

35) $\int_{0}^{e} \ln(x)\ dx$

Answer

Converges to 0.

36) $\int_{0}^{\infty} xe^{-x}\ dx$

37) $\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2}\ dx$

Answer

Converges to 0.

38) $\int_{0}^{\infty} e^{-x}\ dx$

In exercises 39 - 44, evaluate the improper integrals. Each of these integrals has an infinite discontinuity either at an endpoint or at an interior point of the interval.

39) $\int_{0}^{9} \frac{dx}{\sqrt{9-x}}$

Answer

Converges to 6.

40) $\int_{-27}^{1} \frac{dx}{x^{2/3}}$

41) $\int_{0}^{3} \frac{dx}{\sqrt{9-x^2}}$

Answer

Converges to $\frac{\pi}{2}$.

42) $\int_{6}^{24} \frac{dt}{t\sqrt{t^2-36}}$

43) $\int_{0}^{4} x\ln(4x)\ dx$

Answer

Converges to $8\ln(16)-4$.

44) $\int_{0}^{3} \frac{x}{\sqrt{9-x^2}}\ dx$

45) Evaluate $\int_{.5}^{t} \frac{dx}{\sqrt{1-x^2}}$. (Be careful!) (Express your answer using three decimal places.)

Answer

Converges to about 1.047.

46) Evaluate $\int_1^4 \frac{dx}{\sqrt{x^2-1}}$. (Express the answer in exact form.)

47) Evaluate $\int_2^\infty \frac{dx}{(x^2-1)^{3/2}}$.

Answer

Converges to $-1+\frac{2}{\sqrt{3}}$.

48) Find the area of the region in the first quadrant between the curve $y=e^{-6x}$ and the x-axis.

49) Find the area of the region bounded by the curve $y=\frac{7}{x^2}$, the x-axis, and on the left by $x=1$.

Answer

$A=7.0$ units.^{2}

50) Find the area under the curve $y=\frac{1}{(x+1)^{3/2}}$, bounded on the left by $x=3$.

51) Find the area under $y=\frac{5}{1+x^2}$ in the first quadrant.

Answer

$A=\frac{5\pi}{2}$ units.^{2}

52) Find the volume of the solid generated by revolving about the x-axis the region under the curve $y=\frac{3}{x}$ from $x=1$ to $x=\infty$.

53) Find the volume of the solid generated by revolving about the y-axis the region under the curve $y=6e^{-2x}$ in the first quadrant.

Answer

$V=3\pi$ units^3

54) Find the volume of the solid generated by revolving about the x-axis the area under the curve $y=3e^{-x}$ in the first quadrant.

The **Laplace transform** of a continuous function over the interval $[0,\infty)$ is defined by $F(s)=\int_0^\infty e^{-sx}f(x)\ dx$ (see the Student Project). This definition is used to solve some important initial-value problems in differential equations, as discussed later. The domain of F is the set of all real numbers s such that the improper integral converges. Find the Laplace transform F of each of the following functions and give the domain of F.

55) $f(x)=1$

Answer

$\frac{1}{s}, \quad s>0$

56) $f(x)=x$

57) $f(x)=\cos(2x)$

Answer

$\frac{s}{s^2+4}, \quad s>0$

58) $f(x)=e^{ax}$

59) Use the formula for arc length to show that the circumference of the circle $x^2+y^2=1$ is 2π.

Answer

Answers will vary.

A function is a **probability density function** if it satisfies the following definition: $\int_{-\infty}^{\infty} f(t)\ dt = 1$. The probability that a random variable x lies between a and b is given by $P(a \leq x \leq b) = \int_{a}^{b} f(t)\ dt$.

60) Show that $f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 7e^{-7x}, & \text{if } x \geq 0 \end{cases}$ is a probability density function.

61) Find the probability that x is between 0 and 0.3. (Use the function defined in the preceding problem.) Use four-place decimal accuracy.

Answer

0.8775

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8.8: Chapter 8 Review Exercises

7.08: Chapter 7 Review Exercises

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9: SEQUENCES AND SERIES

The topic of infinite series may seem unrelated to differential and integral calculus. In fact, an infinite series whose terms involve powers of a variable is a powerful tool that we can use to express functions as “infinite polynomials.” We can use infinite series to evaluate complicated functions, approximate definite integrals, and create new functions. In addition, infinite series are used to solve differential equations that model physical behavior, from tiny electronic circuits to Earth-orbiting satellites.

9.0: Prelude to Sequence and Series

9.1: Sequences

9.1E: Exercises for Section 9.1

9.2: Infinite Series

9.2E: Exercises for Section 9.2

9.3: The Divergence and Integral Tests

9.3E: Exercises for Section 9.3

9.4: Comparison Tests

9.4E: Exercises for Section 9.4

9.5: Alternating Series

9.5E: Exercises for Section 9.5

9.6: Ratio and Root Tests

9.6E: Exercises for Section 9.6

9.7: Chapter 9 Review Exercises

9.8: Power Series

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9.0: Prelude to Sequence and Series

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9.1: SEQUENCES

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9.1E: Exercises for Section 9.1

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9.1E: Exercises for Section 9.1

In exercises 1 - 4, find the first six terms of each sequence, starting with $n = 1$.

1) $a_n = 1 + (-1)^n$ for $n \geq 1$

Answer

$a_n = 0$ if n is odd and $a_n = 2$ if n is even

2) $a_n = n^2 - 1$ for $n \geq 1$

3) $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$

Answer

$a_n = 1, 3, 6, 10, 15, 21, \ldots$

4) $a_1 = 1, a_2 = 1$ and $a_n + 2 = a_n + a_{n+1}$ for $n \geq 1$

5) Find an explicit formula for a_n where $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$.

Answer

$$a_n = \frac{n(n+1)}{2}$$

6) Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = 1$ such that $a_{n-1} - a_n = 17$ for $n \geq 1$.

7) Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = -3$ such that $a_{n-1} - a_n = 4$ for $n \geq 1$.

Answer

$a_n = 4n - 7$

8) Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 1$ such that $\frac{a_{n+1}}{a_n} = 10$ for $n \geq 1$.

9) Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 3$ such that $\frac{a_{n+1}}{a_n} = 1/10$ for $n \geq 1$.

Answer

$a_n = 3.10^{1-n} = 30.10^{-n}$

10) Find an explicit formula for the nth term of the sequence whose first several terms are $0, 3, 8, 15, 24, 35, 48, 63, 80, 99, \ldots$. (Hint: First add one to each term.)

11) Find an explicit formula for the nth term of the sequence satisfying $a_1 = 0$ and $a_n = 2a_{n-1} + 1$ for $n \geq 2$.

Answer

$a_n = 2^n - 1$

In exercises 12 and 13, find a formula for the general term a_n of each of the following sequences.

12) $1, 0, -1, 0, 1, 0, -1, 0, \ldots$ (Hint: Find where $\sin x$ takes these values)

13) $1, -1/3, 1/5, -1/7, \ldots$

Answer

$$a_n = \frac{(-1)^{n-1}}{2n-1}$$

In exercises 14-18, find a function $f(n)$ that identifies the nth term a_n of the following recursively defined sequences, as

$a_n = f(n)$.

14) $a_1 = 1$ and $a_{n+1} = -a_n$ for $n \geq 1$

15) $a_1 = 2$ and $a_{n+1} = 2a_n$ for $n \geq 1$

Answer

$$f(n) = 2^n$$

16) $a_1 = 1$ and $a_{n+1} = (n+1)a_n$ for $n \geq 1$

17) $a_1 = 2$ and $a_{n+1} = (n+1)a_n/2$ for $n \geq 1$

Answer

$$f(n) = \frac{n!}{2^{n-2}}$$

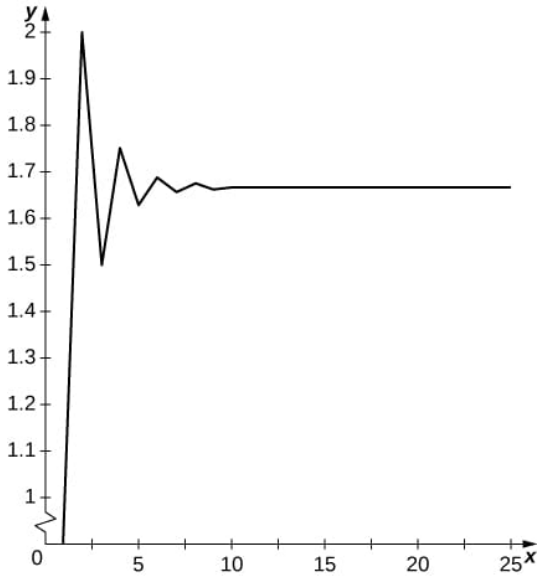
18) $a_1 = 1$ and $a_{n+1} = a_n/2^n$ for $n \geq 1$

In exercises 19 - 22, Use DESMOS to plot the first N terms of the given sequence. State whether the graphical evidence suggests that the sequence converges or diverges.

19) [T] $a_1 = 1, a_2 = 2$, and for $n \geq 2, a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$; $N = 30$

Answer

Terms oscillate above and below $5/3$ and appear to converge to $5/3$.

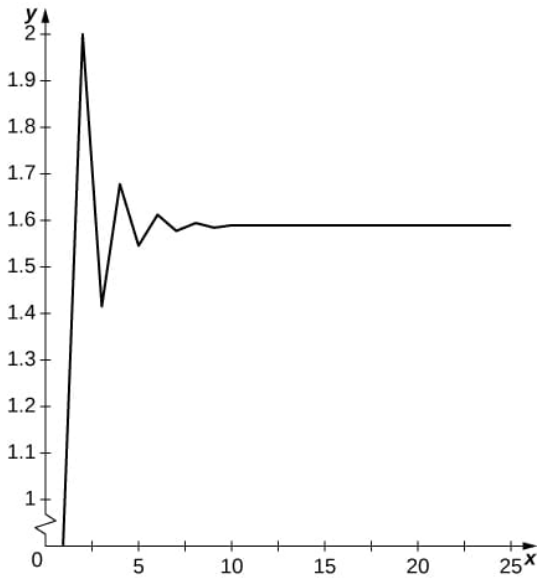


20) [T] $a_1 = 1, a_2 = 2, a_3 = 3$ and for $n \geq 4, a_n = \frac{1}{3}(a_{n-1} + a_{n-2} + a_{n-3}), N = 30$

21) [T] $a_1 = 1, a_2 = 2$, and for $n \geq 3, a_n = \sqrt{a_{n-1}a_{n-2}}; N = 30$

Answer

Terms oscillate above and below $y \approx 1.57..$ and appear to converge to a limit.



22) [T] $a_1 = 1, a_2 = 2, a_3 = 3$, and for $n \geq 4, a_n = \sqrt{a_{n-1}a_{n-2}a_{n-3}}; N = 30$

In exercises 23 - 16, suppose that $\lim_{n\to\infty} a_n = 1$, $\lim_{n\to\infty} b_n = -1$, and $0 < -b_n < a_n$ for all n.

Using this information, evaluate each of the following limits, state that the limit does not exist, or state that there is not enough information to determine whether the limit exists.

23) $\lim_{n\to\infty} 3a_n - 4b_n$

Answer

$$\lim_{n\to\infty} 3a_n - 4b_n \quad = \quad 7$$

24) $\lim_{n\to\infty} \frac{1}{2}b_n - \frac{1}{2}a_n$

25) $\lim_{n\to\infty} \frac{a_n + b_n}{a_n - b_n}$

Answer

$$\lim_{n\to\infty} \frac{a_n + b_n}{a_n - b_n} \quad = \quad 0$$

26) $\lim_{n\to\infty} \frac{a_n - b_n}{a_n + b_n}$

In exercises 27 - 30, find the limit of each of the following sequences, using L'Hôpital's rule when appropriate.

27) $\frac{n^2}{2^n}$

Answer

$$\lim_{n\to\infty} \frac{n^2}{2^n} \quad = \quad 0$$

28) $n\sin(1/n)$ Hint: Rewrite by dividing by 1/n

Answer

$$\lim_{n\to\infty} n\sin(1/n) \quad = \quad 1$$

29) $\dfrac{\sqrt{n}}{\sqrt{n+1}}$

Answer

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} \quad = \quad 1$$

30) $n^{1/n}$ (Hint: $n^{1/n} = e^{\frac{1}{n}\ln n}$)

In exercises 31 - 37, state whether each sequence is bounded and whether it is eventually monotone increasing or monotone decreasing.

31) $n/2^n, n \geq 2$

Answer

bounded, monotone decreasing for $n \geq 2$

32) $\ln\left(1+\dfrac{1}{n}\right)$

33) $\sin n$

Answer

bounded, not monotone

34) $\cos(n^2)$

35) $n^{1/n}, \quad n \geq 3$

Answer

bounded, decreasing

36) $n^{-1/n}, \quad n \geq 3$

37) $\tan n$

Answer

not monotone, not bounded

In exercises 38 - 39, determine whether the given sequence has a limit. If it does, find the limit.

38) $a_1 = \sqrt{2}, a_2 = \sqrt{2\sqrt{2}}. a_3 = \sqrt{2\sqrt{2\sqrt{2}}}$ etc.

39) $a_1 = 3, a_n = \sqrt{2a_{n-1}}, n = 2, 3, \ldots.$

Answer

a_n is decreasing and bounded below by 2. The limit a must satisfy $a = \sqrt{2a}$ so $a = 2$, independent of the initial value.

Use the Squeeze Theorem to find the limit of each sequence in exercises 40 - 43.

40) $n\sin(1/n)$

41) $\dfrac{\cos(1/n)-1}{1/n}$

Answer

0

42) $a_n = \dfrac{n!}{n^n}$

Answer

0

43) $a_n = \sin(n)\sin(1/n)$

Answer

0

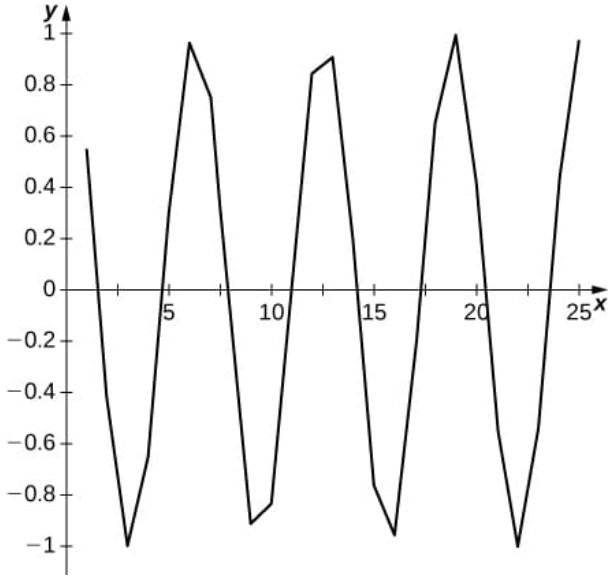
For the sequences in exercises 44 and 45, plot the first 25 terms of the sequence and state whether the graphical evidence suggests that the sequence converges or diverges.

44) [T] $a_n = \sin n$

45) [T] $a_n = \cos n$

Answer

Graph oscillates and suggests no limit.



In exercises 46 - 52, determine the limit of the sequence or show that the sequence diverges. If it converges, find its limit.

46) $a_n = \tan^{-1}(n^2)$

47) $a_n = (2n)^{1/n} - n^{1/n}$

Answer

$n^{1/n} \to 1$ and $2^{1/n} \to 1$, so $a_n \to 0$

48) $a_n = \dfrac{\ln(n^2)}{\ln(2n)}$

Answer

2

49) $a_n = \left(1 - \frac{2}{n}\right)^n$

Answer

Since $(1 + 1/n)^n \to e$, one has $(1 - 2/n)^n \approx (1 + k)^{-2k} \to e^{-2}$ as $k \to \infty$.

50) $a_n = \ln\left(\dfrac{n+2}{n^2-3}\right)$

51) $a_n = \dfrac{2^n+3^n}{4^n}$

Answer

$2^n + 3^n \le 2 \cdot 3^n$ and $3^n/4^n \to 0$ as $n \to \infty$, so $a_n \to 0$ as $n \to \infty$.

52) $a_n = \dfrac{2^n}{n!}$

Answer

0

53) $a_n = \dfrac{(n!)^2}{(2n)!}$

Answer

$\dfrac{a_{n+1}}{a_n} = n!/(n+1)(n+2)\cdots(2n) = \dfrac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)\cdots(2n)} < 1/2^n$. In particular, $a_{n+1}/a_n \le 1/2$, so $a_n \to 0$ as $n \to \infty$.

Newton's method seeks to approximate a solution $f(x) = 0$ that starts with an initial approximation x_0 and successively defines a sequence $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. For the given choice of f and x_0, write out the formula for x_{n+1}. If the sequence appears to converge, give an exact formula for the solution x, then identify the limit x accurate to four decimal places and the smallest n such that x_n agrees with x up to four decimal places.

54) [T] $f(x) = x^2 - 2, \quad x_0 = 1$

55) [T] $f(x) = (x-1)^2 - 2, \quad x_0 = 2$

Answer

$x_{n+1} = x_n - ((x_n-1)^2 - 2)/2(x_n - 1); \;\; x = 1 + \sqrt{2}, \;\; x \approx 2.4142, \;\; n = 5$

56) [T] $f(x) = e^x - 2, \quad x_0 = 1$

57) [T] $f(x) = \ln x - 1, \quad x_0 = 2$

Answer

$x_{n+1} = x_n - x_n(\ln(x_n) - 1); \;\; x = e, \;\; x \approx 2.7183, \;\; n = 5$

58) [T] Suppose you start with one liter of vinegar and repeatedly remove 0.1 L, replace with water, mix, and repeat.

a. Find a formula for the concentration after n steps.

b. After how many steps does the mixture contain less than 10% vinegar?

Answer

$a. C_n = 0.9^n$ b. 22 steps

59) [T] A lake initially contains 2000 fish. Suppose that in the absence of predators or other causes of removal, the fish population increases by 6% each month. However, factoring in all causes, 150 fish are lost each month.

a. Explain why the fish population after n months is modeled by $P_n = 1.06P_{n-1} - 150$ with $P_0 = 2000$.

b. How many fish will be in the pond after one year?

Answer

a. Without losses, the population would obey $P_n = 1.06P_{n-1}$. The subtraction of 150 accounts for fish losses.
b. After 12 months, we have $P_{12} \approx 1494$.

60) [T] A bank account earns 5% interest compounded monthly. Suppose that $\$1000$ is initially deposited into the account, but that $\$10$ is withdrawn each month.

a. Find a recursive formula that gives the amount in the account after n months, A_n, in terms of the amount in the account after $n-1$ months, A_{n-1}.

b. How much money will be in the account after $\frac{1}{2}$ of a year?

c. Is the amount increasing or decreasing?

d. Suppose that instead of $\$10$, a fixed amount d dollars is withdrawn each month. Find the value of d such that the amount in the account after each month remains $\$1000$.

e. What happens if d is greater than this amount?

Answer

a. $A_n = (1 + .05/12)A_{n-1} - 10;\ A_0 = 1000$ b. The account has $\$974.74$ in it after 6 months.
c. The amount is decreasing since the withdrawal amount is greater than the interest earned on the principal.
d. $d = 1000(0.05/12) \approx \$4.17$; e. If the amount withdrawn is greater than this amount, the account balance will decrease, if it is less it will increase

61) [T] A student takes out a college loan of $\$10,000$ at an annual percentage rate of 6%, compounded monthly. Suppose the student makes payments of $\$100$ per month on the loan.

a. Find a recursive formula that gives the amount left on the loan after n months, A_n, in terms of the amount left on the loan after $n-1$ months, A_{n-1}.

b. If the student makes payments of $\$100$ per month, how much does the student owe after 1 year?

c. If the amount left on the loan after n months is given by $A_n = 20,000 - 10,000(1.005)^n$, After how many months will the loan be paid off?

Answer

a. $A_n = (1.005)A_{n-1} - 100;\ A_0 = 10,000$
b. The student owes $\$9383$ after 12 months.
c. The loan will be paid in full after 139 months or about eleven and a half years.

62) [T] Consider a series combining geometric growth and arithmetic decrease. Let $a_1 = 1$. Fix $a > 1$ and $0 < b < a$. Set $a_{n+1} = a.a_n - b$. Find a formula for a_{n+1} in terms of a_n, a, and b and a relationship between a and b such that a_n converges.

63) [T] The binary representation $x = 0.b_1b_2b_3...$ of a number x between 0 and 1 can be defined as follows. Let $b_1 = 0$ if $x < 1/2$ and $b_1 = 1$ if $1/2 \leq x < 1$. Let $x_1 = 2x - b_1$. Let $b_2 = 0$ if $x_1 < 1/2$ and $b_2 = 1$ if $1/2 \leq x < 1$. Let $x_2 = 2x_1 - b_2$ and in general, $x_n = 2x_{n-1} - b_n$ and $b_n{-}1 = 0$ if $x_n < 1/2$ and $b_{n-1} = 1$ if $1/2 \leq x_n < 1$. Find the binary expansion of $1/3$.

Answer

$b_1 = 0, x_1 = 2/3, b_2 = 1, x_2 = 4/3 - 1 = 1/3$, so the pattern repeats, and $1/3 = 0.010101\ldots$.

64) [T] To find an approximation for π, set $a_0 = \sqrt{2+1}, a_1 = \sqrt{2+a_0}$, and, in general, $a_{n+1} = \sqrt{2+a_n}$. Finally, set $p_n = 3.2^n\sqrt{2-a_n}$. Find the first ten terms of p_n and compare the values to π.

For the following two exercises, assume that you have access to a computer program or Internet source that can generate a list of zeros and ones of any desired length. Pseudo-random number generators (PRNGs) play an important role in simulating random noise in physical systems by creating sequences of zeros and ones that appear like the result of flipping a coin repeatedly. One of the simplest types of PRNGs recursively defines a random-looking sequence of N integers $a_1, a_2, \ldots, a_N$ by fixing two special integers (K and M and letting a_{n+1} be the remainder after dividing $K.a_n$ into M, then creates a bit sequence of zeros and ones whose n^{th} term b_n is equal to one if a_n is odd and equal to zero if a_n is even. If the bits b_n are pseudo-random, then the behavior

of their average $(b_1 + b_2 + \cdots + b_N)/N$ should be similar to behavior of averages of truly randomly generated bits.

65) [T] Starting with $K = 16,807$ and $M = 2,147,483,647$, using ten different starting values of a_1, compute sequences of bits b_n up to $n = 1000$, and compare their averages to ten such sequences generated by a random bit generator.

Answer

For the starting values $a_1 = 1, a_2 = 2, \ldots, a_1 = 10$, the corresponding bit averages calculated by the method indicated are $0.5220, 0.5000, 0.4960, 0.4870, 0.4860, 0.4680, 0.5130, 0.5210, 0.5040,$ and 0.4840. Here is an example of ten corresponding averages of strings of 1000 bits generated by a random number generator: $0.4880, 0.4870, 0.5150, 0.5490, 0.5130, 0.5180, 0.4860, 0.5030, 0.5050, 0.4980$. There is no real pattern in either type of average. The random-number-generated averages range between 0.4860 and 0.5490, a range of 0.0630, whereas the calculated PRNG bit averages range between 0.4680 and 0.5220, a range of 0.0540.

66) [T] Find the first 1000 digits of π using either a computer program or Internet resource. Create a bit sequence b_n by letting $b_n = 1$ if the nth digit of π is odd and $b_n = 0$ if the nth digit of π is even. Compute the average value of b_n and the average value of $d_n = |b_{n+1} - b_n|, n = 1, \ldots, 999$. Does the sequence b_n appear random? Do the differences between successive elements of b_n appear random?

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9.2: INFINITE SERIES

Learning Objectives

- Explain the meaning of the sum of an infinite series.
- Calculate the sum of a geometric series.
- Evaluate a telescoping series.

We have seen that a sequence is an ordered set of terms. If you add these terms together, you get a series. In this section we define an infinite series and show how series are related to sequences. We also define what it means for a series to converge or diverge. We introduce one of the most important types of series: the geometric series. We will use geometric series in the next chapter to write certain functions as polynomials with an infinite number of terms. This process is important because it allows us to evaluate, differentiate, and integrate complicated functions by using polynomials that are easier to handle. We also discuss the harmonic series, arguably the most interesting divergent series because it just fails to converge.

Sums and Series

An infinite series is a sum of infinitely many terms and is written in the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

But what does this mean? We cannot add an infinite number of terms in the same way we can add a finite number of terms. Instead, the value of an infinite series is defined in terms of the limit of partial sums. A partial sum of an infinite series is a finite sum of the form

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$

To see how we use partial sums to evaluate infinite series, consider the following example. Suppose oil is seeping into a lake such that 1000 gallons enters the lake the first week. During the second week, an additional 500 gallons of oil enters the lake. The third week, 250 more gallons enters the lake. Assume this pattern continues such that each week half as much oil enters the lake as did the previous week. If this continues forever, what can we say about the amount of oil in the lake? Will the amount of oil continue to get arbitrarily large, or is it possible that it approaches some finite amount? To answer this question, we look at the amount of oil in the lake after k weeks. Letting S_k denote the amount of oil in the lake (measured in thousands of gallons) after k weeks, we see that

$$S_1 = 1$$

$$S_2 = 1 + 0.5 = 1 + \tfrac{1}{2}$$

$$S_3 = 1 + 0.5 + 0.25 = 1 + \tfrac{1}{2} + \tfrac{1}{4}$$

$$S_4 = 1 + 0.5 + 0.25 + 0.125 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}$$

$$S_5 = 1 + 0.5 + 0.25 + 0.125 + 0.0625 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16}.$$

Looking at this pattern, we see that the amount of oil in the lake (in thousands of gallons) after k weeks is

$$S_k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{k-1}} = \sum_{n=1}^{k} \left(\frac{1}{2}\right)^{n-1}.$$

We are interested in what happens as $k \to \infty$. Symbolically, the amount of oil in the lake as $k \to \infty$ is given by the infinite series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots.$$

At the same time, as $k\to\infty$, the amount of oil in the lake can be calculated by evaluating $\lim_{k\to\infty} S_k$. Therefore, the behavior of the infinite series can be determined by looking at the behavior of the sequence of partial sums S_k. If the sequence of partial sums S_k converges, we say that the infinite series converges, and its sum is given by $\lim_{k\to\infty} S_k$. If the sequence S_k diverges, we say the infinite series diverges. We now turn our attention to determining the limit of this sequence S_k.

First, simplifying some of these partial sums, we see that

$$S_1=1$$

$$S_2=1+\tfrac{1}{2}=\tfrac{3}{2}$$

$$S_3=1+\tfrac{1}{2}+\tfrac{1}{4}=\tfrac{7}{4}$$

$$S_4=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}=\tfrac{15}{8}$$

$$S_5=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}=\tfrac{31}{16}.$$

Plotting some of these values in Figure, it appears that the sequence S_k could be approaching 2.

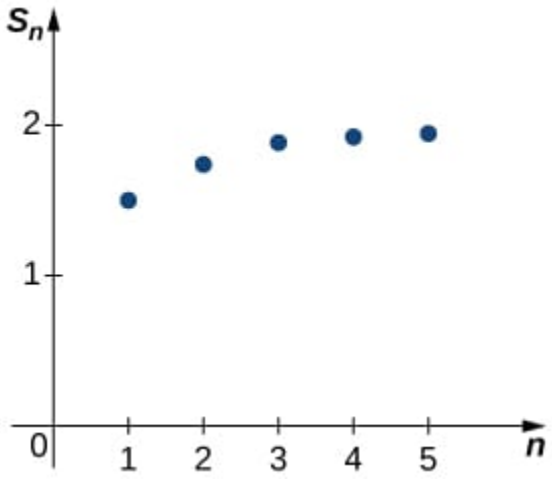


Figure $9.2.1$: The graph shows the sequence of partial sums S_k. It appears that the sequence is approaching the value 2.

Let's look for more convincing evidence. In the following table, we list the values of S_k for several values of k.

k	5	10	15	20
S_k	1.9375	1.998	1.999939	1.999998

These data supply more evidence suggesting that the sequence S_k converges to 2. Later we will provide an analytic argument that can be used to prove that $\lim_{k\to\infty} S_k=2$. For now, we rely on the numerical and graphical data to convince ourselves that the sequence of partial sums does actually converge to 2. Since this sequence of partial sums converges to 2, we say the infinite series converges to 2 and write

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=2.$$

Returning to the question about the oil in the lake, since this infinite series converges to 2, we conclude that the amount of oil in the lake will get arbitrarily close to 2000 gallons as the amount of time gets sufficiently large.

This series is an example of a geometric series. We discuss geometric series in more detail later in this section. First, we

summarize what it means for an infinite series to converge.

Definition

An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

For each positive integer k, the sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

is called the kth **partial sum** of the infinite series. The partial sums form a sequence S_k. If the sequence of partial sums converges to a real number S, the infinite series converges. If we can describe the **convergence of a series** to S, we call S the sum of the series, and we write

$$\sum_{n=1}^{\infty} a_n = S.$$

If the sequence of partial sums diverges, we have the **divergence of a series.**

Note that the index for a series need not begin with $n = 1$ but can begin with any value. For example, the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$$

can also be written as

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \text{ or } \sum_{n=5}^{\infty} \left(\frac{1}{2}\right)^{n-5}.$$

Often it is convenient for the index to begin at 1, so if for some reason it begins at a different value, we can re-index by making a change of variables. For example, consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

By introducing the variable $m = n - 1$, so that $n = m + 1$, we can rewrite the series as

$$\sum_{m=1}^{\infty} \frac{1}{(m+1)^2}.$$

Example 9.2.1: Evaluating Limits of Sequences of Partial Sums

For each of the following series, use the sequence of partial sums to determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

b. $\displaystyle\sum_{n=1}^{\infty} (-1)^n$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solution

a. The sequence of partial sums S_k satisfies

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{2}{3}$$

$$S_3 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4}$$

$$S_4 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}.$$

Notice that each term added is greater than $1/2$. As a result, we see that

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{2}{3} > \frac{1}{2} + \frac{1}{2} = 2\left(\frac{1}{2}\right)$$

$$S_3 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3\left(\frac{1}{2}\right)$$

$$S_4 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4\left(\frac{1}{2}\right).$$

From this pattern we can see that $S_k > k\left(\frac{1}{2}\right)$ for every integer k. Therefore, S_k is unbounded and consequently, diverges. Therefore, the infinite series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

b. The sequence of partial sums S_k satisfies

$$S_1 = -1$$

$$S_2 = -1 + 1 = 0$$

$$S_3 = -1 + 1 - 1 = -1$$

$$S_4 = -1 + 1 - 1 + 1 = 0.$$

From this pattern we can see the sequence of partial sums is

$$S_k = -1, 0, -1, 0, \ldots.$$

Since this sequence diverges, the infinite series $\sum_{n=1}^{\infty} (-1)^n$ diverges.

c. The sequence of partial sums S_k satisfies

$$S_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$$

$$S_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$S_3 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$

$$S_4 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} = \frac{4}{5}$$

$$S_5 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} = \frac{5}{6}.$$

From this pattern, we can see that the kth partial sum is given by the explicit formula

$$S_k = \frac{k}{k+1}$$

.

Since $k/(k+1) \to 1$, we conclude that the sequence of partial sums converges, and therefore the infinite series converges to 1. We have

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Exercise 9.2.1

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n}$ converges or diverges.

Hint available at the end of this section.

Answer available at the end of this section.

The Harmonic Series

A useful series to know about is the **harmonic series.** The harmonic series is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

This series is interesting because it diverges, but it diverges very slowly. By this we mean that the terms in the sequence of partial sums S_k approach infinity, but do so very slowly. We will show that the series diverges, but first we illustrate the slow growth of the terms in the sequence S_k in the following table.

k	10	100	1000	10,00	100,000	1,000,000
S_k	2.92897	5.18738	7.48547	9.78761	12.09015	14.39273

Even after $1,000,000$ terms, the partial sum is still relatively small. From this table, it is not clear that this series actually diverges. However, we can show analytically that the sequence of partial sums diverges, and therefore the series diverges.

To show that the sequence of partial sums diverges, we show that the sequence of partial sums is unbounded. We begin by writing the first several partial sums:

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{3}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$$

Notice that for the last two terms in S_4,

$$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}$$

Therefore, we conclude that

$$S_4>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)=1+\frac{1}{2}+\frac{1}{2}=1+2\left(\frac{1}{2}\right).$$

Using the same idea for S_8, we see that

$$S_8=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=1+3\left(\frac{1}{2}\right).$$

From this pattern, we see that $S_1=1, S_2=1+1/2, S_4>1+2(1/2)$, and $S_8>1+3(1/2)$. More generally, it can be shown that $S_{2^j}>1+j(1/2)$ for all $j>1$. Since $1+j(1/2)\to\infty$, we conclude that the sequence S_k is unbounded and therefore diverges. In the previous section, we stated that convergent sequences are bounded. Consequently, since S_k is unbounded, it diverges. Thus, the harmonic series diverges.

Algebraic Properties of Convergent Series

Since the sum of a convergent infinite series is defined as a limit of a sequence, the algebraic properties for series listed below follow directly from the algebraic properties for sequences.

Note 9.2.1: Algebraic Properties of Convergent Series

Let $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ be convergent series. Then the following algebraic properties hold.

i. The series $\sum_{n=1}^{\infty}(a_n+b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n$. (Sum Rule)

ii. The series $\sum_{n=1}^{\infty}(a_n-b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n-b_n)=\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}b_n$. (Difference Rule)

iii. For any real number c, the series $\sum_{n=1}^{\infty}ca_n$ converges, and $\sum_{n=1}^{\infty}ca_n=c\sum_{n=1}^{\infty}a_n$. (Constant Multiple Rule)

Example 9.2.2: Using Algebraic Properties of Convergent Series

Evaluate $\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)}+\left(\frac{1}{2}\right)^{n-2}\right]$.

Solution

We showed earlier that

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$$

and

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=2.$$

Since both of those series converge, we can apply the properties of Note 9.2.1 to evaluate

$$\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)}+\left(\frac{1}{2}\right)^{n-2}\right].$$

Using the sum rule, write

$$\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)}+\left(\frac{1}{2}\right)^{n-2}\right]=\sum_{n=1}^{\infty}\frac{3}{n(n+1)}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-2}.$$

Then, using the constant multiple rule and the sums above, we can conclude that

$$\sum_{n=1}^{\infty}\frac{3}{n(n+1)}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-2}=3\sum_{n=1}^{\infty}\frac{1}{n(n+1)}+\left(\frac{1}{2}\right)^{-1}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=3(1)+\left(\frac{1}{2}\right)^{-1}(2)=3+2(2)=7.$$

Exercise 9.2.2

Evaluate $\sum_{n=1}^{\infty}\frac{5}{2^{n-1}}$.

Hint available at the end of this section.

Answer available at the end of this section.

Geometric Series

A **geometric series** is any series that we can write in the form

$$a+ar+ar^2+ar^3+\cdots=\sum_{n=1}^{\infty}ar^{n-1}.$$

Because the ratio of each term in this series to the previous term is *r*, the number *r* is called the ratio. We refer to a as the initial term because it is the first term in the series. For example, the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

is a geometric series with initial term $a=1$ and ratio $r=1/2$.

In general, when does a geometric series converge? Consider the geometric series

$$\sum_{n=1}^{\infty}ar^{n-1}$$

when $a>0$. Its sequence of partial sums S_k is given by

$$S_k=\sum_{n=1}^{k}ar^{n-1}=a+ar+ar^2+\cdots+ar^{k-1}.$$

Consider the case when $r=1$. In that case,

$$S_k=a+a(1)+a(1)^2+\cdots+a(1)^{k-1}=ak.$$

Since $a>0$, we know $ak\to\infty$ as $k\to\infty$. Therefore, the sequence of partial sums is unbounded and thus diverges. Consequently, the infinite series diverges for $r=1$. For $r\neq 1$, to find the limit of S_k, multiply Equation by $1-r$. Doing so, we see that

$$(1-r)S_k=a(1-r)(1+r+r^2+r^3+\cdots+r^{k-1})=a[(1+r+r^2+r^3+\cdots+r^{k-1})-(r+r^2+r^3+\cdots+r^k)]=a(1-r^k).$$

All the other terms cancel out.

Therefore,

$$S_k = \frac{a(1-r^k)}{1-r} \text{ for } r \neq 1.$$

From our discussion in the previous section, we know that the geometric sequence $r^k \to 0$ if $|r| < 1$ and that r^k diverges if $|r| > 1$ or $r = \pm 1$. Therefore, for $|r| < 1, S_k \to \frac{a}{1-r}$ and we have

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ if } |r| < 1.$$

If $|r| \geq 1, S_k$ diverges, and therefore

$$\sum_{n=1}^{\infty} ar^{n-1} \text{ diverges if } |r| \geq 1.$$

Theorem: The Convergence and Divergence of Geometric Series

A geometric series is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots.$$

If $|r| < 1$, the series **converges**, and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1.$$

If $|r| \geq 1$, the series **diverges**.

Geometric series sometimes appear in slightly different forms. For example, sometimes the index begins at a value other than $n = 1$ or the exponent involves a linear expression for n other than $n - 1$. As long as we can rewrite the series in the form given by Equation, it is a geometric series. For example, consider the series

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2}.$$

To see that this is a geometric series, we write out the first several terms:

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2} = \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \cdots = \frac{4}{9} + \frac{4}{9} \cdot \left(\frac{2}{3}\right) + \frac{4}{9} \cdot \left(\frac{2}{3}\right)^2 + \cdots.$$

We see that the initial term is $a = 4/9$ and the ratio is $r = 2/3$. Therefore, the series can be written as

$$\sum_{n=1}^{\infty} \frac{4}{9} \cdot \left(\frac{2}{3}\right)^{n-1}.$$

Since $r = 2/3 < 1$, this series converges, and its sum is given by

$$\sum_{n=1}^{\infty} \frac{4}{9} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{4/9}{1-2/3} = \frac{4}{3}.$$

Example 9.2.3: Determining Convergence or Divergence of a Geometric Series

Determine whether each of the following geometric series converges or diverges, and if it converges, find its sum.

a. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}}$

b. $\sum_{n=1}^{\infty} e^{2n}$

Solution

a. Writing out the first several terms in the series, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}} &= \frac{(-3)^2}{4^0} + \frac{(-3)^3}{4} + \frac{(-3)^4}{4^2} + \cdots \\ &= (-3)^2 + (-3)^2 \cdot \left(\frac{-3}{4}\right) + (-3)^2 \cdot \left(\frac{-3}{4}\right)^2 + \cdots \\ &= 9 + 9 \cdot \left(\frac{-3}{4}\right) + 9 \cdot \left(\frac{-3}{4}\right)^2 + \cdots .\end{aligned}$$

The initial term $a = -3$ and the ratio $r = -3/4$. Since $|r| = 3/4 < 1$, the series converges to

$$\frac{9}{1-(-3/4)} = \frac{9}{7/4} = \frac{36}{7}.$$

b. Writing this series as

$$e^2 \sum_{n=1}^{\infty} (e^2)^{n-1}$$

we can see that this is a geometric series where $r = e^2 > 1$. Therefore, the series diverges.

Exercise 9.2.3

Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{-2}{5}\right)^{n-1}$ converges or diverges. If it converges, find its sum.

Hint available at the end of this section.

Answer available at the end of this section.

We now turn our attention to a nice application of geometric series. We show how they can be used to write repeating decimals as fractions of integers.

Example 9.2.4: Writing Repeating Decimals as Fractions of Integers

Use a geometric series to write $3.2\overline{6}$ as a fraction of integers.

Solution

Since $3.2\overline{6} = 3.262626\ldots$, first we write

$$3.262626\ldots = 3 + \frac{26}{100} + \frac{26}{10,000} + \frac{26}{1,000,000} + \cdots$$
$$= 3 + \frac{26}{10^2} + \frac{26}{10^4} + \frac{26}{10^6} + \cdots.$$

Ignoring the term 3, the rest of this expression is a geometric series with initial term $a = 26/10^2$ and ratio $r = 1/10^2$. Therefore, the sum of this series is

$$\frac{26/10^2}{1-(1/10^2)} = \frac{26/10^2}{99/10^2} = \frac{26}{99}.$$

Thus,

$$3.262626\ldots = 3 + \tfrac{26}{99} = \tfrac{323}{99}.$$

Exercise 9.2.4

Write $5.2\overline{7}$ as a fraction of integers.

Hint available at the end of this section.

Answer available at the end of this section.

Example 9.2.5: Finding the Area of the Koch Snowflake

Define a sequence of figures $\{F_n\}$ recursively as follows (Figure 9.2.2). Let F_0 be an equilateral triangle with sides of length 1. For $n \geq 1$, let F_n be the curve created by removing the middle third of each side of F_{n-1} and replacing it with an equilateral triangle pointing outward. The limiting figure as $n \to \infty$ is known as **Koch's snowflake.**

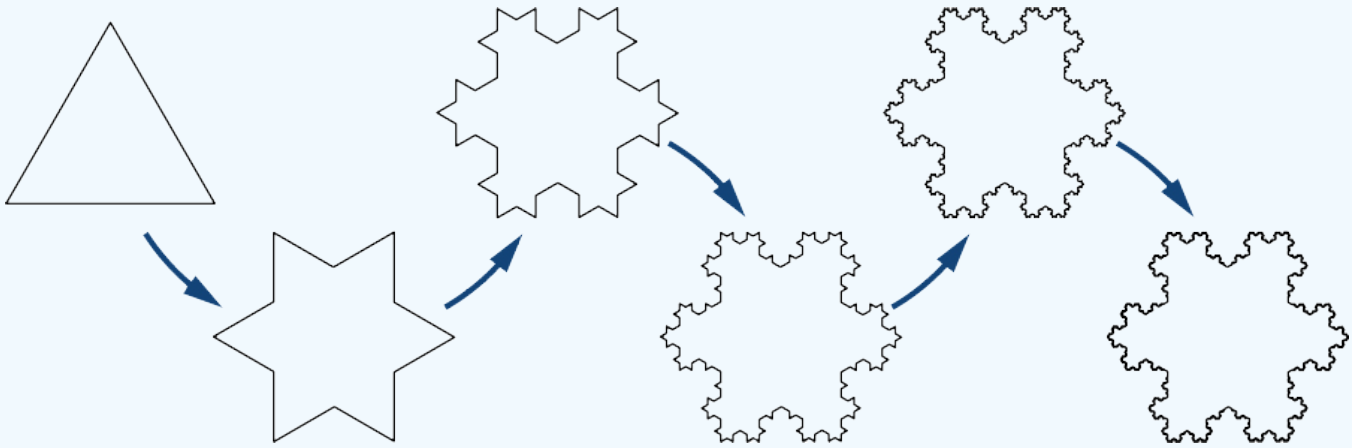


Figure 9.2.2: The first four figures, F_0, F_1, F_2, and F_3, in the construction of the Koch snowflake.

a. Find the length L_n of the perimeter of F_n. Evaluate $\lim_{n\to\infty} L_n$ to find the length of the perimeter of Koch's snowflake.

b. Find the area A_n of figure F_n. Evaluate $\lim_{n\to\infty} A_n$ to find the area of Koch's snowflake.

Solution

a. Let N_n denote the number of sides of figure F_n. Since F_0 is a triangle, $N_0 = 3$. Let ln denote the length of each side of F_n. Since F_0 is an equilateral triangle with sides of length $l_0 = 1$, we now need to determine N_1 and l_1. Since F_1 is created by removing the middle third of each side and replacing that line segment with two line segments, for each side of F_0, we get four sides in F_1. Therefore, the number of sides for F_1 is

$$N_1 = 4 \cdot 3.$$

Since the length of each of these new line segments is $1/3$ the length of the line segments in F_0, the length of the line segments for F_1 is given by

$$l_1 = \tfrac{1}{3} \cdot 1 = \tfrac{1}{3}.$$

Similarly, for F_2, since the middle third of each side of F_1 is removed and replaced with two line segments, the number of sides in F_2 is given by

$$N_2 = 4N_1 = 4(4 \cdot 3) = 4^2 \cdot 3.$$

Since the length of each of these sides is $1/3$ the length of the sides of F_1, the length of each side of figure F_2 is given by

$$l_2 = \tfrac{1}{3} \cdot l_1 = \tfrac{1}{3} \cdot \tfrac{1}{3} = \left(\tfrac{1}{3}\right)^2.$$

More generally, since F_n is created by removing the middle third of each side of F_{n-1} and replacing that line segment with two line segments of length $\frac{1}{3}l_{n-1}$ in the shape of an equilateral triangle, we know that $N_n = 4N_{n-1}$ and $l_n = \dfrac{l_{n-1}}{3}$. Therefore, the number of sides of figure F_n is

$$N_n = 4^n \cdot 3$$

and the length of each side is

$$l_n = \left(\frac{1}{3}\right)^n.$$

Therefore, to calculate the perimeter of F_n, we multiply the number of sides N_n and the length of each side l_n. We conclude that the perimeter of F_n is given by

$$L_n = N_n \cdot l_n = 3 \cdot \left(\frac{4}{3}\right)^n$$

Therefore, the length of the perimeter of Koch's snowflake is

$$L = \lim_{n \to \infty} L_n = \infty.$$

b. Let T_n denote the area of each new triangle created when forming F_n. For $n = 0, T_0$ is the area of the original equilateral triangle. Therefore, $T_0 = A_0 = \sqrt{3}/4$. For $n \geq 1$, since the lengths of the sides of the new triangle are $1/3$ the length of the sides of F_{n-1}, we have

$$T_n = \left(\frac{1}{3}\right)^2 \cdot T_{n-1} = \frac{1}{9} \cdot T_{n-1}.$$

Therefore, $T_n = \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4}$. Since a new triangle is formed on each side of F_{n-1},

$$A_n = A_{n-1} + N_{n-1} \cdot T_n = A_{n-1} + (3 \cdot 4_{n-1}) \cdot \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4} = A_{n-1} + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^n \cdot \frac{\sqrt{3}}{4}.$$

Writing out the first few terms A_0, A_1, A_2, we see that

$A_0 = \frac{\sqrt{3}}{4}$

$A_1 = A_0 + \frac{3}{4} \cdot \left(\frac{4}{9}\right) \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4} + \frac{3}{4} \cdot \left(\frac{4}{9}\right) \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)\right]$

$A_2 = A_1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)\right] + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right) + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2]$.

More generally,

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4}\left(\frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^n\right)\right].$$

Factoring $4/9$ out of each term inside the inner parentheses, we rewrite our expression as

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}\right)\right].$$

The expression $1 + \left(\frac{4}{9}\right) + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}$ is a geometric sum. As shown earlier, this sum satisfies

$$1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1} = \frac{1-(4/9)^n}{1-(4/9)}.$$

Substituting this expression into the expression above and simplifying, we conclude that

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3}\left(\frac{1-(4/9)^n}{1-(4/9)}\right)\right] = \frac{\sqrt{3}}{4}\left[\frac{8}{5} - \frac{3}{5}\left(\frac{4}{9}\right)^n\right].$$

Therefore, the area of Koch's snowflake is

$$A = \lim_{n\to\infty} A_n = \frac{2\sqrt{3}}{5}.$$

Analysis

The Koch snowflake is interesting because it has finite area, yet infinite perimeter. Although at first this may seem impossible, recall that you have seen similar examples earlier in the text. For example, consider the region bounded by the curve $y = 1/x^2$ and the x-axis on the interval $[1, \infty)$. Since the improper integral

$$\int_1^\infty \frac{1}{x^2}\,dx$$

converges, the area of this region is finite, even though the perimeter is infinite.

Telescoping Series

Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. We discussed this series in Example, showing that the series converges by writing out the first several partial sums $S_1, S_2, \ldots, S_6$ and noticing that they are all of the form $S_k = \dfrac{k}{k+1}$. Here we use a different technique to show that this series converges. By using partial fractions, we can write

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore, the series can be written as

$$\sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+1}\right] = \left(1 + \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots.$$

Writing out the first several terms in the sequence of partial sums S_k, we see that

$$S_1 = 1 - \tfrac{1}{2}$$

$$S_2 = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) = 1 - \tfrac{1}{3}$$

$$S_3 = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) = 1 - \tfrac{1}{4}.$$

In general,

$$S_k = (1 - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + (\tfrac{1}{3} - \tfrac{1}{4}) + \cdots + (\tfrac{1}{k} - \tfrac{1}{k+1}) = 1 - \frac{1}{k+1}.$$

We notice that the middle terms cancel each other out, leaving only the first and last terms. In a sense, the series collapses like a spyglass with tubes that disappear into each other to shorten the telescope. For this reason, we call a series that has this property a telescoping series. For this series, since $S_k = 1 - 1/(k+1)$ and $1/(k+1) \to 0$ as $k \to \infty$, the sequence of partial sums converges to 1, and therefore the series converges to 1.

Definition

A **telescoping series** is a series in which most of the terms cancel in each of the partial sums, leaving only some of the first terms and some of the last terms.

For example, any series of the form

$$\sum_{n=1}^{\infty}[b_n - b_{n+1}] = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots$$

is a telescoping series. We can see this by writing out some of the partial sums. In particular, we see that

$$S_1 = b_1 - b_2$$

$$S_2 = (b_1 - b_2) + (b_2 - b_3) = b_1 - b_3$$

$$S_3 = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) = b_1 - b_4.$$

In general, the *kth* partial sum of this series is

$$S_k = b_1 - b_{k+1}.$$

Since the *kth* partial sum can be simplified to the difference of these two terms, the sequence of partial sums S_k will converge if and only if the sequence b_{k+1} converges. Moreover, if the sequence b_{k+1} converges to some finite number B, then the sequence of partial sums converges to $b_1 - B$, and therefore

$$\sum_{n=1}^{\infty}[b_n - b_{n+1}] = b_1 - B.$$

In the next example, we show how to use these ideas to analyze a telescoping series of this form.

Example 9.2.6: Evaluating a Telescoping Series

Determine whether the telescoping series

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right]$$

converges or diverges. If it converges, find its sum.

Solution

By writing out terms in the sequence of partial sums, we can see that

$$S_1 = \cos(1) - \cos(\tfrac{1}{2})$$

$$S_2 = (\cos(1) - \cos(\tfrac{1}{2})) + (\cos(\tfrac{1}{2}) - \cos(\tfrac{1}{3})) = \cos(1) - \cos(\tfrac{1}{3})$$

$$S_3 = (\cos(1) - \cos(\tfrac{1}{2})) + (\cos(\tfrac{1}{2}) - \cos(\tfrac{1}{3})) + (\cos(\tfrac{1}{3}) - \cos(\tfrac{1}{4}))$$

$= \cos(1) - \cos(\frac{1}{4})$.

In general,

$$S_k = \cos(1) - \cos\left(\tfrac{1}{k+1}\right).$$

Since $1/(k+1) \to 0$ as $k \to \infty$ and $\cos x$ is a continuous function, $\cos(1/(k+1)) \to \cos(0) = 1$. Therefore, we conclude that $S_k \to \cos(1) - 1$. The telescoping series converges and the sum is given by

$$\sum_{n=1}^{\infty} \left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right] = \cos(1) - 1.$$

Exercise 9.2.5

Determine whether $\sum_{n=1}^{\infty}[e^{1/n} - e^{1/(n+1)}]$ converges or diverges. If it converges, find its sum.

Hint available at the end of this section.

Answer available at the end of this section.

Euler's Constant

We have shown that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Here we investigate the behavior of the partial sums S_k as $k \to \infty$. In particular, we show that they behave like the natural logarithm function by showing that there exists a constant γ such that

$$\sum_{n=1}^{k} \left(\frac{1}{n} - \ln k\right) \to \gamma \text{ as } k \to \infty.$$

This constant γ is known as **Euler's constant.**

1. Let $T_k = \sum_{n=1}^{k} \left(\frac{1}{n} - \ln k\right)$. Evaluate T_k for various values of k.

2. For T_k as defined in part 1. show that the sequence T_k converges by using the following steps.

   a. Show that the sequence T_k is monotone decreasing. (Hint: Show that $\ln(1 + 1/k > 1/(k+1))$

   b. Show that the sequence T_k is bounded below by zero. (Hint: Express $\ln k$ as a definite integral.)

   c. Use the Monotone Convergence Theorem to conclude that the sequence T_k converges. The limit γ is Euler's constant.

3. Now estimate how far T_k is from γ for a given integer k. Prove that for $k \geq 1, 0 < T_k - \gamma \leq 1/k$ by using the following steps.

   a. Show that $\ln(k+1) - \ln k < 1/k$.

   b. Use the result from part a. to show that for any integer k,

   $$T_k - T_{k+1} < \frac{1}{k} - \frac{1}{k+1}.$$

   c. For any integers k and j such that $j > k$, express $T_k - T_j$ as a telescoping sum by writing

$$T_k - T_j = (T_k - T_{k+1}) + (T_{k+1} - T_{k+2}) + (T_{k+2} - T_{k+3}) + \cdots + (T_{j-1} - T_j).$$

Use the result from part b. combined with this telescoping sum to conclude that

$$T_k - T_j < \frac{1}{k} - \frac{1}{j}.$$

a. Apply the limit to both sides of the inequality in part c. to conclude that

$$T_k - \gamma \leq \frac{1}{k}.$$

e. Estimate γ to an accuracy of within **0.001**.

Key Concepts

- Given the infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

and the corresponding sequence of partial sums S_k where

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k,$$

the series converges if and only if the sequence S_k converges.

- The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. For $|r| < 1$,

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

- The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges.

- A series of the form $\sum_{n=1}^{\infty} [b_n - b_{n+1}] = [b_1 - b_2] + [b_2 - b_3] + [b_3 - b_4] + \cdots + [b_n - b_{n+1}] + \cdots$ is a telescoping series. The kth partial sum of this series is given by $S_k = b_1 - b_{k+1}$. The series will converge if and only if $\lim_{k\to\infty} b_{k+1}$ exists. In that case,

$$\sum_{n=1}^{\infty} [b_n - b_{n+1}] = b_1 - \lim_{k\to\infty} (b_{k+1}).$$

Key Equations

- **Harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- **Sum of a geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1$$

Glossary

convergence of a series
: a series converges if the sequence of partial sums for that series converges

divergence of a series
: a series diverges if the sequence of partial sums for that series diverges

geometric series
: a geometric series is a series that can be written in the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

harmonic series
: the harmonic series takes the form

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

infinite series
: an infinite series is an expression of the form

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

partial sum
: the kth partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$ is the finite sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

telescoping series
: a telescoping series is one in which most of the terms cancel in each of the partial sums

Answers

Exercise 9.2.1
Look at the sequence of partial sums.

Exercise 9.2.1
The series diverges because the kth partial sum $S_k > k$.

Exercise 9.2.2
Rewrite as $\sum_{n=1}^{\infty} 5\left(\frac{1}{2}\right)^{n-1}$.

Exercise 9.2.2
10

Exercise 9.2.3
$r = -2/5$

Exercise 9.2.3
$5/7$

Exercise 9.2.4
By expressing this number as a series, find a geometric series with initial term $a = 7/100$ and ratio $r = 1/10$.

Exercise 9.2.4

$475/90$

Exercise 9.2.5

Write out the sequence of partial sums to see which terms cancel.

Exercise 9.2.5

$e - 1$

9.2E: Exercises for Section 9.2

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9.2E: Exercises for Section 9.2

In exercises 1 - 4, use sigma notation to write each expressions as an infinite series.

1) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

Answer

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

2) $1-1+1-1+\cdots$

3) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$

Answer

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

4) $\sin 1+\sin\frac{1}{2}+\sin\frac{1}{3}+\sin\frac{1}{4}+\cdots$

In exercises 5 - 8, compute the first four partial sums $S_1,\ldots,S_4$ for the series having n^{th} term a_n starting with $n=1$ as follows.

5) $a_n=n$

Answer

$1,3,6,10$

6) $a_n=1/n$

Answer

$1,\frac{3}{2},\frac{11}{6},\frac{25}{12}$

7) $a_n=\sin(\frac{n\pi}{2})$

Answer

$1,1,0,0$

8) $a_n=(-1)^n$

In exercises 9 - 12, compute the general term a_n of the series with the given partial sum S_n. If the sequence of partial sums converges, find its limit S.

9) $S_n=1-\frac{1}{n},\quad n\geq 2$

Answer

$a_n=S_n-S_{n-1}=\dfrac{1}{n(n-1)}$. Since $S=\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)=1$, the series converges to $S=1$.

10) $S_n=\dfrac{n(n+1)}{2},\quad n\geq 1$

11) $S_n=\sqrt{n},\quad n\geq 2$

Answer

$a_n=S_n-S_{n-1}=\sqrt{n}-\sqrt{n-1}=\dfrac{1}{\sqrt{n-1}+\sqrt{n}}$.

The series diverges because the partial sums are unbounded.

That is, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n} = \infty$.

12) $S_n = 2 - \dfrac{n+2}{2^n}, \quad n \geq 1$

For each series in exercises 13 - 16, use the sequence of partial sums to determine whether the series converges or diverges.

13) $\sum_{n=1}^{\infty} \dfrac{n}{n+2}$

Answer

$S_1 = 1/3,$
$S_2 = 1/3 + 2/4 > 1/3 + 1/3 = 2/3,$
$S_3 = 1/3 + 2/4 + 3/5 > 3 \cdot (1/3) = 1.$
In general $S_k > k/3$, so the series diverges.
Note that the nth Term Test for Divergence could also be used to prove that this series diverges.

14) $\sum_{n=1}^{\infty}(1 - (-1)^n))$

15) $\sum_{n=1}^{\infty} \dfrac{1}{(n+1)(n+2)}$ Hint: Use a partial fraction decomposition to see the series is a telescoping series.)

Answer

The series converges to $1/2$.

16) $\sum_{n=1}^{\infty} \dfrac{1}{2n+1}$ Hint: Follow the reasoning for $\sum_{n=1}^{\infty} \dfrac{1}{n}$.)

Suppose that $\sum_{n=1}^{\infty} a_n = 1$, $\sum_{n=1}^{\infty} b_n = -1$, $a_1 = 2$, and $b_1 = -3$. Use this information to find the sum of the indicated series in exercises 17 - 20.

17) $\sum_{n=1}^{\infty}(a_n + b_n)$

Answer

$$\sum_{n=1}^{\infty}(a_n + b_n) \quad = \quad \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \quad = \quad 1 + (-1) \quad = \quad 0$$

18) $\sum_{n=1}^{\infty}(a_n - 2b_n)$

19) $\sum_{n=2}^{\infty}(a_n - b_n)$

Answer

$$\sum_{n=2}^{\infty}(a_n - b_n) \quad = \quad \sum_{n=2}^{\infty} a_n - \sum_{n=2}^{\infty} b_n \quad = \quad \left(\sum_{n=1}^{\infty} a_n - a_1\right) - \left(\sum_{n=1}^{\infty} b_n - b_1\right) \quad = \quad (1-2) - (-1-(-3)) = -1 - 2 \quad = \quad -3$$

20) $\sum_{n=1}^{\infty}(3a_{n+1} - 4b_{n+1})$

In exercises 21 - 26, state whether the given series converges or diverges and explain why.

21) $\sum_{n=1}^{\infty} \frac{1}{n+1000}$ (Hint: Rewrite using a change of index.)

Answer

The series diverges, $\sum_{n=1001}^{\infty} \frac{1}{n}$

22) $\sum_{n=1}^{\infty} \frac{1}{n+10^{80}}$ (Hint: Rewrite using a change of index.)

23) $1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots$

Answer

This is a convergent geometric series, since $r=\frac{1}{10}<1$

24) $1+\frac{e}{\pi}+\frac{e^2}{\pi^2}+\frac{e^3}{\pi^3}+\cdots$

25) $1+\frac{\pi}{e^2}+\frac{\pi^2}{e^4}+\frac{\pi^3}{e^6}+\frac{\pi^4}{e^8}+\cdots$

Answer

This is a convergent geometric series, since $r=\pi/e^2<1$

26) $1-\sqrt{\frac{\pi}{3}}+\sqrt{\frac{\pi^2}{9}}-\sqrt{\frac{\pi^3}{27}}+\cdots$

For each a_n in exercises 27 - 30, write its sum as a geometric series of the form $\sum_{n=1}^{\infty} ar^n$. State whether the series converges and if it does, find the exact value of its sum.

27) $a_1=-1$ and $\frac{a_n}{a_{n+1}}=-5$ for $n\geq 1$.

Answer

$\sum_{n=1}^{\infty} 5\cdot(-1/5)^n$, converges to $-5/6$

28) $a_1=2$ and $\frac{a_n}{a_{n+1}}=1/2$ for $n\geq 1$.

29) $a_1=10$ and $\frac{a_n}{a_{n+1}}=10$ for $n\geq 1$.

Answer

$\sum_{n=1}^{\infty} 100\cdot(1/10)^n$, converges to $\frac{100}{9}$

30) $a_1=\frac{1}{10}$ and $a_n/a_{n+1}=-10$ for $n\geq 1$.

In exercises 31 - 34, use the identity $\frac{1}{1-y}=\sum_{n=0}^{\infty} y^n$ (which is true for $|y|<1$) to express each function as a geometric series in the indicated term.

31) $\frac{x}{1+x}$ in x

Answer

$x\sum_{n=0}^{\infty}(-x)^n=\sum_{n=1}^{\infty}(-1)^{n-1}x^n$

32) $\frac{\sqrt{x}}{1-x^{3/2}}$ in $\sqrt{x}$

33) $\dfrac{1}{1+\sin^2 x}$ in $\sin x$

Answer

$$\sum_{n=0}^{\infty}(-1)^n \sin^{2n}(x)$$

34) $\sec^2 x$ in $\sin x$

In exercises 35 - 38, evaluate the telescoping series or state whether the series diverges.

35) $\sum_{n=1}^{\infty} 2^{1/n} - 2^{1/(n+1)}$

Answer

$S_k = 2 - 2^{1/(k+1)} \to 1$ as $k \to \infty$.

36) $\sum_{n=1}^{\infty} \dfrac{1}{n^{13}} - \dfrac{1}{(n+1)^{13}}$

37) $\sum_{n=1}^{\infty}(\sqrt{n} - \sqrt{n+1})$

Answer

$S_k = 1 - \sqrt{k+1}$ diverges

38) $\sum_{n=1}^{\infty}(\sin n - \sin(n+1))$

Express each series in exercises 39 - 42 as a telescoping sum and evaluate its nth partial sum.

39) $\sum_{n=1}^{\infty} \ln\left(\dfrac{n}{n+1}\right)$

Answer

$$\sum_{n=1}^{\infty}[\ln n - \ln(n+1)], \quad S_k = -\ln(k+1)$$

40) $\sum_{n=1}^{\infty} \dfrac{2n+1}{(n^2+n)^2}$ (Hint: Factor denominator and use partial fractions.)

41) $\sum_{n=2}^{\infty} \dfrac{\ln\left(1+\frac{1}{n}\right)}{(\ln n)\ln(n+1)}$

Answer

$a_n = \frac{1}{\ln n} - \frac{1}{\ln(n+1)}$ and $S_k = \frac{1}{\ln(2)} - \frac{1}{\ln(k+1)} \to \frac{1}{\ln(2)}$

42) $\sum_{n=1}^{\infty} \dfrac{(n+2)}{n(n+1)2^{n+1}}$ (Hint: Look at $1/(n2^n)$.

A general telescoping series is one in which all but the first few terms cancel out after summing a given number of successive terms.

43) Let $a_n = f(n) - 2f(n+1) + f(n+2)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\sum_{n=1}^{\infty} a_n$.

Answer

$$\sum_{n=1}^{\infty} a_n = f(1) - f(2)$$

44) $a_n = f(n) - f(n+1) - f(n+2) + f(n+3)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\sum_{n=1}^{\infty} a_n$.

45) Suppose that $a_n = c_0 f(n) + c_1 f(n+1) + c_2 f(n+2) + c_3 f(n+3) + c_4 f(n+4)$, where $f(n) \to 0$ as $n \to \infty$. Find a condition on the coefficients $c_0, \ldots, c_4$ that make this a general telescoping series.

Answer

$$c_0 + c_1 + c_2 + c_3 + c_4 = 0$$

46) Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ (Hint: $\frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$)

47) Evaluate $\sum_{n=2}^{\infty} \frac{2}{n^3 - n}$.

Answer

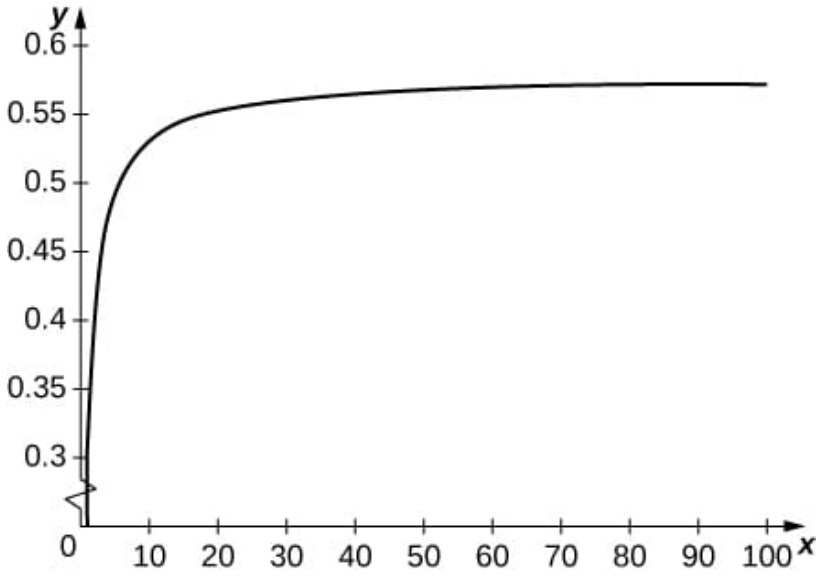
$$\frac{2}{n^3-1} = \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1},$$
$$S_n = (1 - 1 + 1/3) + (1/2 - 2/3 + 1/4) + (1/3 - 2/4 + 1/5) + (1/4 - 2/5 + 1/6) + \cdots = 1/2$$

48) Find a formula for $\sum_{n=1}^{\infty} \frac{1}{n(n+N)}$ where N is a positive integer.

49) [T] Define a sequence $t_k = \sum_{n=1}^{k-1} (1/k) - \ln k$. Use the graph of $1/x$ to verify that t_k is increasing. Plot t_k for $k = 1 \ldots 100$ and state whether it appears that the sequence converges.

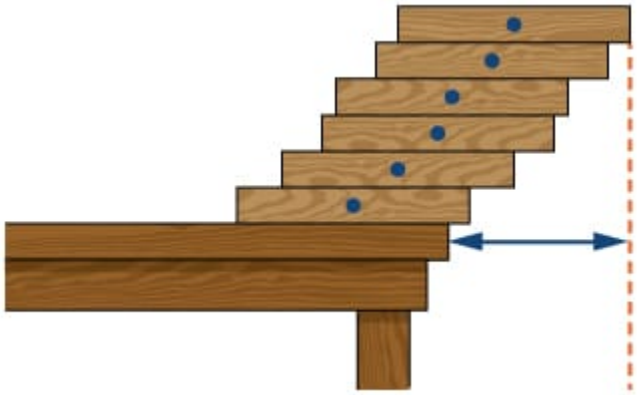
Answer

t_k converges to $0.57721 \ldots t_k$ is a sum of rectangles of height $1/k$ over the interval $[k, k+1]$ which lie above the graph of $1/x$.



50) [T] Suppose that N equal uniform rectangular blocks are stacked one on top of the other, allowing for some overhang. Archimedes' law of the lever implies that the stack of N blocks is stable as long as the center of mass of the top $(N-1)$ blocks lies at the edge of the bottom block. Let x denote the position of the edge of the bottom block, and think of its position as relative

to the center of the next-to-bottom block. This implies that $(N-1)x = \left(\frac{1}{2} - x\right)$ or $x = 1/(2N)$. Use this expression to compute the maximum overhang (the position of the edge of the top block over the edge of the bottom block.) See the following figure.



Each of the following infinite series converges to the given multiple of π or $1/\pi$.

In each case, find the minimum value of N such that the Nth partial sum of the series accurately approximates the left-hand side to the given number of decimal places, and give the desired approximate value. Up to 15 decimals place, $\pi = 3.141592653589793....$

51) [T] $\pi = -3 + \sum_{n=1}^{\infty} \frac{n2^n n!^2}{(2n)!}$, error < 0.0001

Answer

$N = 22,$
$S_N = 6.1415$

52) [T] $\frac{\pi}{2} = \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!} = \sum_{k=0}^{\infty} \frac{2^k k!^2}{(2k+1)!}$, error $< 10^{-4}$

53) [T] $\frac{9801}{2\pi} = \frac{4}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$, error $< 10^{-12}$

Answer

$N = 3,$
$S_N = 1.559877597243667...$

54) [T] $\frac{1}{12\pi} = \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}$, error $< 10^{-15}$

55) [T] A fair coin is one that has probability $1/2$ of coming up heads when flipped.

a. What is the probability that a fair coin will come up tails n times in a row?

b. Find the probability that a coin comes up heads for the first time after an even number of coin flips.

Answer

a. The probability of any given ordered sequence of outcomes for n coin flips is $1/2^n$.
b. The probability of coming up heads for the first time on the n th flip is the probability of the sequence $TT...TH$ which is $1/2^n$. The probability of coming up heads for the first time on an even flip is $\sum_{n=1}^{\infty} 1/2^{2n}$ or $1/3$.

56) [T] Find the probability that a fair coin is flipped a multiple of three times before coming up heads.

57) [T] Find the probability that a fair coin will come up heads for the second time after an even number of flips.

Answer

$5/9$

58) Use a geometric series to write $7.485\overline{5}$ as a fraction of integers.

Answer

$7\frac{437}{900} = \frac{6737}{900}$

59) A ball dropped from a height of 15 feet bounces $\frac{4}{5}$ of the height of each previous bounce.

a. Use a geometric partial sum formula to find the total distance the ball has traveled vertically after 20 bounces.

b. Use a geometric infinite sum formula to find the total distance the ball travels vertically.

Answer

a. $S_{20} = 15 + 120(1 - (0.8^{20})) \approx 133.6165$ feet b. $S = 135$ feet

60) Annuity: A person deposits $\$100$ at the end of each quarter into a bank account that earns 6% annual interest compounded quarterly (four times a year).

a. Find the amount in the account (accrual) after n quarters.

b. Find the accrual after 2 years using a geometric partial sum formula.

Answer

a. $\frac{100(1.015^n - 1)}{0.015}$ b. $\approx \$843.28$

61) [T] Suppose that the amount of a drug in a patient's system diminishes by a multiplicative factor $r < 1$ each hour. Suppose that a new dose is administered every N hours. Find an expression that gives the amount $A(n)$ in the patient's system after n hours for each n in terms of the dosage d and the ratio r. (Hint: Write $n = mN + k$, where $0 \le k < N$, and sum over values from the different doses administered.)

Answer

The part of the first dose after n hours is dr^n, the part of the second dose is dr^{n-N}, and, in general, the part remaining of the mth dose is dr^{n-mN}, so $A(n) = \sum_{l=0}^{m} dr^{n-lN} = \sum_{l=0}^{m} dr^{k+(m-l)N} = \sum_{q=0}^{m} dr^{k+qN} = dr^k \sum_{q=0}^{m} r^{Nq}$

$= dr^k \frac{1 - r^{(m+1)N}}{1 - r^N}$, where $n = k + mN$.

62) [T] A certain drug is effective for an average patient only if there is at least 1 mg per kg in the patient's system, while it is safe only if there is at most 2 mg per kg in an average patient's system. Suppose that the amount in a patient's system diminishes by a multiplicative factor of 0.9 each hour after a dose is administered. Find the maximum interval N of hours between doses, and corresponding dose range d (in mg/kg) for this N that will enable use of the drug to be both safe and effective in the long term.

63) Suppose that $a_n \ge 0$ is a sequence of numbers. Explain why the sequence of partial sums of a_n is increasing.

Answer

$S_{N+1} = a_{N+1} + S_N \ge S_N$

64) [T] Suppose that a_n is a sequence of positive numbers and the sequence S_n of partial sums of a_n is bounded above. Explain why $\sum_{n=1}^{\infty} a_n$ converges. Does the conclusion remain true if we remove the hypothesis $a_n \ge 0$?

65) [T] Suppose that $a_1 = S_1 = 1$ and that, for given numbers $S > 1$ and $0 < k < 1$, one defines $a_{n+1} = k(S - S_n)$ and $S_{n+1} = a_{n+1} + S_n$. Does S_n converge? If so, to what? (Hint: First argue that $S_n < S$ for all n and S_n is increasing.)

Answer

Since $S > 1, a_2 > 0,$ and since $k < 1, S_2 = 1 + a_2 < 1 + (S - 1) = S$. If $S_n > S$ for some n, then there is a smallest n. For this $n, S > S_{n-1}$, so $S_n = S_{n-1} + k(S - S_{n-1}) = kS + (1 - k)S_{n-1} < S$, a contradiction. Thus S_n

$< S$ and $a_{n+1} > 0$ for all n, so S_n is increasing and bounded by S. Let $S_* = \lim S_n$. If $S_* < S$, then $\delta = k(S - S_*) > 0$, but we can find n such that $S_* - S_n < \delta/2$, which implies that $S_{n+1} = S_n + k(S - S_n) > S_* + \delta/2$, contradicting that Sn is increasing to S_*. Thus $S_n \to S$.

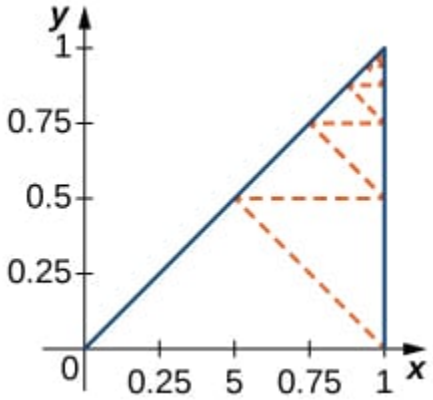
66) [T] A version of **von Bertalanffy growth** can be used to estimate the age of an individual in a homogeneous species from its length if the annual increase in year $n+1$ satisfies $a_{n+1} = k(S - S_n)$, with S_n as the length at year n, S as a limiting length, and k as a relative growth constant. If $S_1 = 3, S = 9$, and $k = 1/2$, numerically estimate the smallest value of n such that $S_n \geq 8$. Note that $S_{n+1} = S_n + a_{n+1}$. Find the corresponding n when $k = 1/4$.

67) [T] Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms. Explain why $\lim_{N\to\infty} \sum_{n=N+1}^{\infty} a_n = 0$.

Answer

Let $S_k = \sum_{n=1}^{k} a_n$ and $S_k \to L$. Then S_k eventually becomes arbitrarily close to L, which means that $L - S_N = \sum_{n=N+1}^{\infty} a_n$ becomes arbitrarily small as $N \to \infty$.

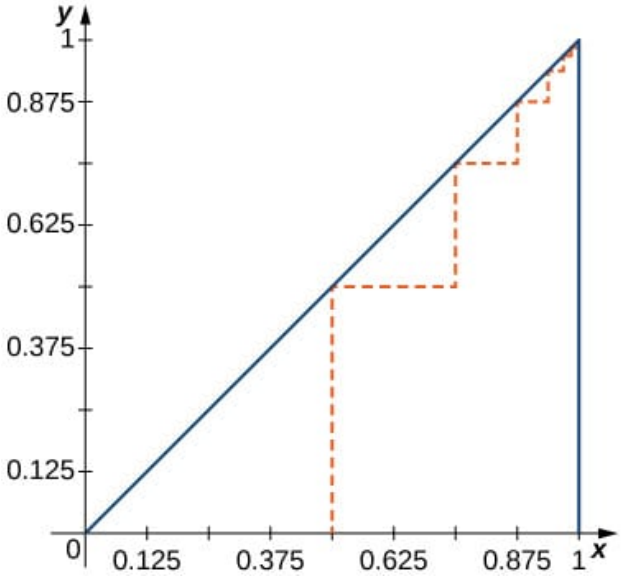
68) [T] Find the length of the dashed zig-zag path in the following figure.



Answer

$1 + \sqrt{2}$

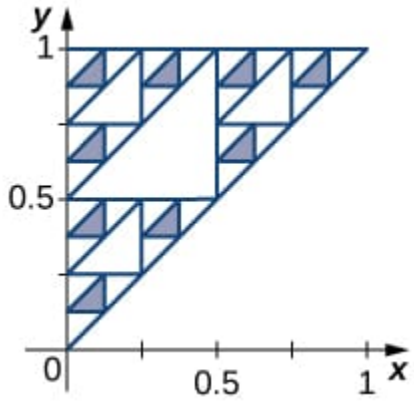
69) [T] Find the total length of the dashed path in the following figure.



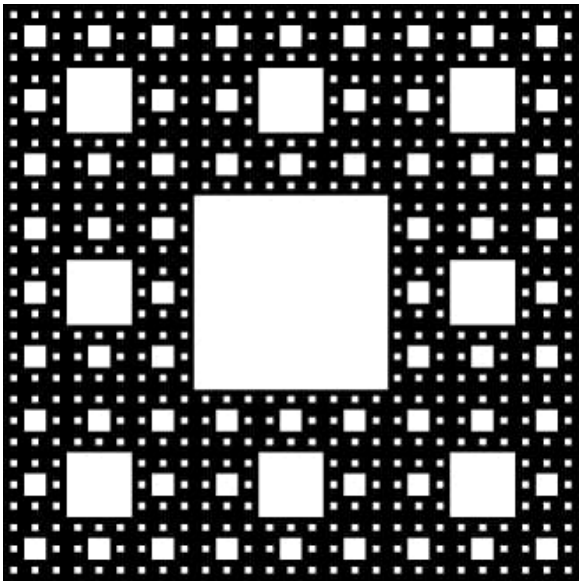
Answer

$$L = \left(1 + \frac{1}{2}\right) \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{3}{2}.$$

70) [T] The **Sierpinski triangle** is obtained from a triangle by deleting the middle fourth as indicated in the first step, by deleting the middle fourths of the remaining three congruent triangles in the second step, and in general deleting the middle fourths of the remaining triangles in each successive step. Assuming that the original triangle is shown in the figure, find the areas of the remaining parts of the original triangle after N steps and find the total length of all of the boundary triangles after N steps.



71) [T] The Sierpinski gasket is obtained by dividing the unit square into nine equal sub-squares, removing the middle square, then doing the same at each stage to the remaining sub-squares. The figure shows the remaining set after four iterations. Compute the total area removed after N stages, and compute the length the total perimeter of the remaining set after N stages.



Answer

At stage one a square of area $1/9$ is removed, at stage 2 one removes 8 squares of area $1/9^2$, at stage three one removes 8^2 squares of area $1/9^3$, and so on. The total removed area after N stages is $\sum_{n=0}^{N-1} \frac{8^N}{9^{N+1}} = \frac{1}{8} \cdot \frac{1-(8/9)^N}{1-8/9} \to 1$ as $N \to \infty$. The total perimeter is $4 + 4\sum_{n=0}^{\infty} \frac{8^N}{3^{N+1}} \to \infty$.

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9.3: THE DIVERGENCE AND INTEGRAL TESTS

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9.3E: Exercises for Section 9.3

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9.3E: Exercises for Section 9.3

Divergence Test Problems

Consider the sequence for each series in exercises 1 - 14, if the divergence test applies, either state that $\lim_{n\to\infty} a_n$ does not exist or find $\lim_{n\to\infty} a_n$. If the divergence test does not apply, state why.

1) $\sum_{n=1}^{\infty} \frac{n}{n+2}$

2) $\sum_{n=1}^{\infty} \frac{n}{5n^2-3}$

Answer

$\lim_{n\to\infty} a_n = 0$. The Divergence Test does not apply.

3) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{3n^2+2n+1}}$

4) $\sum_{n=1}^{\infty} \frac{(2n+1)(n-1)}{(n+1)^2}$

Answer

$\lim_{n\to\infty} a_n = 2$. So the series diverges by the nth-Term Test for Divergence.

5) $\sum_{n=1}^{\infty} \frac{(2n+1)^{2n}}{(3n^2+1)^n}$

6) $\sum_{n=1}^{\infty} \frac{2^n}{3^{n/2}}$

Answer

$\lim_{n\to\infty} a_n = \infty$ (does not exist). So the series diverges by the nth-Term Test for Divergence.

7) $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$

Answer

The is a geometric series with $r = \frac{2}{3}$, so the series converges. Therefore $\lim_{n\to\infty} a_n = 0$, and the Divergence Test does not apply.

8) $\sum_{n=1}^{\infty} e^{-2/n}$

Answer

$\lim_{n\to\infty} a_n = 1$. So the series diverges by the nth-Term Test for Divergence.

9) $\sum_{n=1}^{\infty} \cos n$

10) $\sum_{n=1}^{\infty} \tan n$

Answer

$\lim_{n\to\infty} a_n$ does not exist. So the series diverges by the nth-Term Test for Divergence.

11) $\sum_{n=1}^{\infty} \frac{1-\cos^2(1/n)}{\sin^2(2/n)}$

12) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ (Hint: Thank Euler)

Answer

$\lim_{n\to\infty} a_n = e > 1$. So the series diverges by the nth-Term Test for Divergence.

13) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

14) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{\sqrt{n}}$

Answer

$\lim_{n\to\infty} a_n = 0$. The Divergence Test does not apply.

p-Series Problems & Integral Test Problems

In exercises 15 - 20, state whether the given p-series converges.

15) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

16) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

Answer

The series converges, since $p = 3/2 > 1$.

17) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$

Answer

The series diverges, since $p = 2/3 < 1$.

18) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$

Answer

The series converges, since $p = 4/3 > 1$.

19) $\sum_{n=1}^{\infty} \frac{n^e}{n^\pi}$

Answer

The series converges, since $p = \pi - e > 1$.

20) $\sum_{n=1}^{\infty} \frac{n^\pi}{n^{2e}}$

Answer

The series converges, since $p = 2e - \pi > 1$.

In exercises 21 - 27, use the integral test to determine whether the following sums converge.

21) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$

22) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+5}}$

Answer

The series diverges by the Integral Test since $\int_1^{\infty} \frac{dx}{(x+5)^{1/3}}$ can be shown to diverge.

23) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

24) $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$

Answer

The series diverges by the Integral Test since $\int_1^{\infty} \frac{x}{1+x^2}\ dx$ can be shown to diverge.

25) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

26) $\sum_{n=1}^{\infty} \frac{2n}{1+n^4}$

Answer

The series converges by the Integral Test since $\int_1^{\infty} \frac{2x}{1+x^4}\ dx$ converges.

27) $\sum_{n=2}^{\infty} \frac{1}{n \ln^2(n)}$

Answer

The series converges by the Integral Test since $\int_2^{\infty} \frac{1}{n \ln^2(n)}\ dx$ converges.

Express the sums in exercises 28 - 31 as p-series and determine whether each converges.

28) $\sum_{n=1}^{\infty} 2^{-\ln n}$ Hint: $2^{-\ln n} = \frac{1}{n^{\ln 2}}.$)

Answer

$2^{-\ln n} = 1/n^{\ln 2}$. Since $p = \ln 2 < 1$, this series diverges by the p-series test.

29) $\sum_{n=1}^{\infty} 3^{-\ln n}$ Hint: $3^{-\ln n} = \frac{1}{n^{\ln 3}}.$)

30) $\sum_{n=1}^{n} 2^{-2 \ln n}$

Answer

$2^{-2 \ln n} = 1/n^{2 \ln 2}$. Since $p = 2 \ln 2 - 1 < 1$, this series diverges by the p-series test.

31) $\sum_{n=1}^{\infty} n 3^{-2 \ln n}$

In exercises 32 - 35, use the estimate $R_N \le \int_N^\infty f(t)\ dt$ to find a bound for the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ where $a_n = f(n)$.

32) $\sum_{n=1}^{1000} \frac{1}{n^2}$

Answer

$$R_{1000} \le \int_{1000}^{\infty} \frac{dt}{t^2} = \lim_{b\to\infty} -\frac{1}{t}\Big|_{1000}^{b} = \lim_{b\to\infty}\left(-\frac{1}{b} + \frac{1}{1000}\right) = 0.001$$

33) $\sum_{n=1}^{1000} \frac{1}{n^3}$

34) $\sum_{n=1}^{1000} \frac{1}{1+n^2}$

Answer

$$R_{1000} \le \int_{1000}^{\infty} \frac{dt}{1+t^2} = \lim_{b\to\infty}\left(\tan^{-1} b - \tan^{-1}(1000)\right) = \pi/2 - \tan^{-1}(1000) \approx 0.000999$$

35) $\sum_{n=1}^{100} \frac{n}{2^n}$

[T] In exercises 36 - 40, find the minimum value of N such that the remainder estimate $\int_{N+1}^{\infty} f(x)\ dx < R_N < \int_N^{\infty} f(x)\ dx$ guarantees that $\sum_{n=1}^{N} a_n$ estimates $\sum_{n=1}^{\infty} a_n$, accurate to within the given error.

36) $a_n = \frac{1}{n^2}$, error $< 10^{-4}$

Answer

$$R_N < \int_N^{\infty} \frac{dx}{x^2} = 1/N, \text{ for } N > 10^4$$

37) $a_n = \frac{1}{n^{1.1}}$, error $< 10^{-4}$

38) $a_n = \frac{1}{n^{1.01}}$, error $< 10^{-4}$

Answer

$$R_N < \int_N^{\infty} \frac{dx}{x^{1.01}} = 100N^{-0.01}, \text{ for } N > 10^{600}$$

39) $a_n = \frac{1}{n\ln^2 n}$, error $< 10^{-3}$

40) $a_n = \frac{1}{1+n^2}$, error $< 10^{-3}$

Answer

$$R_N < \int_N^{\infty} \frac{dx}{1+x^2} = \pi/2 - \tan^{-1}(N), \text{ for } N > \tan(\pi/2 - 10^{-3}) \approx 1000$$

In exercises 41 - 45, find a value of N such that R_N is smaller than the desired error. Compute the corresponding sum $\sum_{n=1}^{N} a_n$ and compare it to the given estimate of the infinite series.

41) $a_n = \dfrac{1}{n^{11}}$, error $< 10^{-4}, \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{11}} = 1.000494\ldots$

42) $a_n = \dfrac{1}{e^n}$, error $< 10^{-5}, \displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{1}{e-1} = 0.581976\ldots$

Answer

$R_N < \displaystyle\int_N^{\infty} \frac{dx}{e^x} = e^{-N}$, for $N > 5\ln(10)$, okay if $N = 12$; $\displaystyle\sum_{n=1}^{12} e^{-n} = 0.581973\ldots$. Estimate agrees with $1/(e-1)$ to five decimal places.

43) $a_n = \dfrac{1}{e^{n^2}}$, error $< 10^{-5}, \displaystyle\sum_{n=1}^{\infty} \frac{1}{e^{n^2}} = 0.40488139857\ldots$

44) $a_n = \dfrac{1}{n^4}$, error $< 10^{-4}, \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} = 1.08232\ldots$

Answer

$R_N < \displaystyle\int_N^{\infty} \frac{dx}{x^4} = 4/N^3$, for $N > (4.10^4)^{1/3}$, okay if $N = 35$; $\displaystyle\sum_{n=1}^{35} \frac{1}{n^4} = 1.08231\ldots$. Estimate agrees with the sum to four decimal places.

45) $a_n = \dfrac{1}{n^6}$, error $< 10^{-6}, \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} = 1.01734306\ldots,$

46) Find the limit as $n \to \infty$ of $\dfrac{1}{n} + \dfrac{1}{n+1} + \cdots + \dfrac{1}{2n}$. Hint: Compare to $\displaystyle\int_n^{2n} \frac{1}{t}\, dt.\Big)$

Answer

$\ln(2)$

47) Find the limit as $n \to \infty$ of $\dfrac{1}{n} + \dfrac{1}{n+1} + \cdots + \dfrac{1}{3n}$

The next few exercises are intended to give a sense of applications in which partial sums of the harmonic series arise.

48) In certain applications of probability, such as the so-called Watterson estimator for predicting mutation rates in population genetics, it is important to have an accurate estimate of the number $H_k = (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k})$. Recall that $T_k = H_k - \ln k$ is decreasing. Compute $T = \lim\limits_{k\to\infty} T_k$ to four decimal places.

Hint: $\dfrac{1}{k+1} < \displaystyle\int_k^{k+1} \frac{1}{x}\, dx.\Big)$

Answer

$T = 0.5772\ldots$

49) [T] Complete sampling with replacement, sometimes called the **coupon collector's problem**, is phrased as follows: Suppose you have N unique items in a bin. At each step, an item is chosen at random, identified, and put back in the bin. The problem asks what is the expected number of steps $E(N)$ that it takes to draw each unique item at least once. It turns out that $E(N) = N.H_N = N\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}\right)$. Find $E(N)$ for $N = 10, 20,$ and 50.

50) [T] The simplest way to shuffle cards is to take the top card and insert it at a random place in the deck, called top random insertion, and then repeat. We will consider a deck to be randomly shuffled once enough top random insertions have been made that the card originally at the bottom has reached the top and then been randomly inserted. If the deck has n cards, then the probability that the insertion will be below the card initially at the bottom (call this card B) is $1/n$. Thus the expected number of top random insertions before B is no longer at the bottom is n. Once one card is below B, there are two places below B and the probability that a randomly inserted card will fall below B is $2/n$. The expected number of top random insertions before this happens is $n/2$. The two cards below B are now in random order. Continuing this way, find a formula for the expected number of top random insertions needed to consider the deck to be randomly shuffled.

Answer

The expected number of random insertions to get B to the top is $n + n/2 + n/3 + \cdots + n/(n-1)$. Then one more insertion puts B back in at random. Thus, the expected number of shuffles to randomize the deck is $n(1 + 1/2 + \cdots + 1/n)$.

51) Suppose a scooter can travel 100 km on a full tank of fuel. Assuming that fuel can be transferred from one scooter to another but can only be carried in the tank, present a procedure that will enable one of the scooters to travel $100H_N$ km, where $H_N = 1 + 1/2 + \cdots + 1/N$.

52) Show that for the remainder estimate to apply on $[N, \infty)$ it is sufficient that $f(x)$ be decreasing on $[N, \infty)$, but f need not be decreasing on $[1, \infty)$.

Answer

Set $b_n = a_{n+N}$ and $g(t) = f(t + N)$ such that f is decreasing on $[t, \infty)$.

53) [T] Use the remainder estimate and integration by parts to approximate $\sum_{n=1}^{\infty} \frac{n}{e^n}$ within an error smaller than 0.0001.

54) Does $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converge if p is large enough? If so, for which p?

Answer

The series converges for $p > 1$ by integral test using change of variable.

55) [T] Suppose a computer can sum one million terms per second of the divergent series $\sum_{n=1}^{N} \frac{1}{n}$. Use the integral test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed 100.

56) [T] A fast computer can sum one million terms per second of the divergent series $\sum_{n=2}^{N} \frac{1}{n \ln n}$. Use the integral test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed 100.

Answer

$N = e^{e^{100}} \approx e^{10^{43}}$ terms are needed.

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9.4: COMPARISON TESTS

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9.4E: Exercises for Section 9.4

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9.4E: Exercises for Section 9.4

Use the Comparison Test to determine whether each series in exercises 1 - 13 converges or diverges.

1) $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{2}{n(n+1)}$

2) $\sum_{n=1}^{\infty} \dfrac{1}{n(n+3)}$

Answer

Converges by comparison with $\dfrac{1}{n^2}$ since $\dfrac{1}{n(n+3)} < \dfrac{1}{n^2}$.

3) $\sum_{n=1}^{\infty} \dfrac{1}{2(n+1)}$

4) $\sum_{n=1}^{\infty} \dfrac{1}{2n-1}$

Answer

Diverges since $\dfrac{1}{2n-1} > \dfrac{1}{2n}$, and $\sum_{n=1}^{\infty} \dfrac{1}{2n} = \dfrac{1}{2}\sum_{n=1}^{\infty} \dfrac{1}{n}$ is a divergent harmonic series.

5) $\sum_{n=2}^{\infty} \dfrac{1}{(n \ln n)^2}$

Answer

Converges by comparison with $\dfrac{1}{n^2}$ since $\dfrac{1}{(n \ln n)^2} < \dfrac{1}{n^2}$ for $n \geq 3$.

6) $\sum_{n=1}^{\infty} \dfrac{n!}{(n+2)!}$

Answer

$a_n = 1/(n+1)(n+2) < 1/n^2$. Converges by comparison with p-series, $p = 2 > 1$.

7) $\sum_{n=1}^{\infty} \dfrac{\ln(n)}{n}$

Answer

Diverges since $\dfrac{\ln(n)}{n} > \dfrac{1}{n}$, and $\sum_{n=1}^{\infty} \dfrac{1}{n}$ is a divergent harmonic series.

8) $\sum_{n=1}^{\infty} \dfrac{\sin(1/n)}{n^3}$

Answer

Converges by comparison with p-series, $p = 3 > 1$.

9) $\sum_{n=1}^{\infty} \dfrac{\sin^2 n}{n^2}$

10) $\sum_{n=1}^{\infty} \dfrac{\sin(1/n)}{\sqrt{n}}$

Answer

$\sin(1/n) \le 1/n$, so converges by comparison with p-series, $p = 3/2 > 1$.

11) $\sum_{n=1}^{\infty} \frac{n^{1.2} - 1}{n^{2.3} + 1}$

Answer

Converges by comparison with p-series, $p = 1.1 > 1$ (must show).

12) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}$

Answer

Since $\sqrt{n+1} - \sqrt{n} = 1/(\sqrt{n+1} + \sqrt{n}) \le 2/\sqrt{n}$, series converges by comparison with p-series for $p = 1.5 > 1$.

13) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n^4 + n^2}}$

Answer

Since $\frac{\sqrt{n}}{\sqrt{n^4+n^2}} \le \frac{1}{\sqrt{n^3}}$, series converges by comparison with p-series for $p = 1.5 > 1$.

Use the Limit Comparison Test to determine whether each series in exercises 14 - 28 converges or diverges.

14) $\sum_{n=1}^{\infty} \frac{1}{n^2 - n}$ Can you use a direct comparison to test this series? Why or why not?

Answer

$a_n = \frac{1}{n^2-n} > \frac{1}{n^2}$, but the terms of the series must be less than the terms of the comparison series in order to converge. By limit comparison to the p-series for $p = 2$, the series converges.

15) $\sum_{n=1}^{\infty} \frac{1}{n!}$ Hint: Write out factorial and compare to $\frac{1}{n(n-1)}$)

Answer

Since $a_n = \frac{1}{n!} < \frac{1}{n(n-1)} = \frac{1}{n^2-n}$, the series converges since $\sum_{n=1}^{\infty} \frac{1}{n^2 - n}$ converges (must show).

16) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Answer

Diverges by direct comparison with p-series for $p = 1$.

17) $\sum_{n=1}^{\infty} \frac{\ln\left(1 + \frac{1}{n}\right)}{n}$

Answer

Converges by limit comparison with p-series, $p = 2 > 1$.

18) $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$

19) $\sum_{n=1}^{\infty} \frac{1}{4^n - 3^n}$

Answer

Converges by limit comparison with 4^{-n}.

20) $\sum_{n=1}^{\infty} \frac{1}{n^2 - n \sin n}$

Answer

Converges by limit comparison with p-series for $p = 2 > 1$.

21) $\sum_{n=1}^{\infty} \frac{1}{e^{(1.1)n} - 3^n}$

Answer

Converges by limit comparison with $1/e^{1.1n}$.

22) $\sum_{n=1}^{\infty} \frac{1}{e^{(1.01)n} - 3^n}$

23) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

Answer

Diverges by limit comparison with harmonic series.

24) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right)$

Answer

Converges by limit comparison with p-series, $p = 3 > 1$.

25) $\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$

26) $\sum_{n=1}^{\infty} \frac{1}{n}\left(\tan^{-1} n - \frac{\pi}{2}\right)$

Answer

Converges by limit comparison with p-series, $p = 3 > 1$.

27) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$ Hint: $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{n} = \frac{1}{e}$

28) $\sum_{n=1}^{\infty} \left(1 - e^{-1/n}\right)$ (Hint: $1/e \approx (1 - 1/n)^n$, so $1 - e^{-1/n} \approx 1/n$.)

Answer

Diverges by limit comparison with $1/n$.

29) Does $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$ converge if p is large enough? If so, for which p?

30) Does $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^p$ converge if p is large enough? If so, for which p?

Answer

Converges for $p > 1$ by comparison with a p series for slightly smaller p.

31) For which p does the series $\sum_{n=1}^{\infty} 2^{pn}/3^n$ converge?

32) For which $p > 0$ does the series $\sum_{n=1}^{\infty} \frac{n^p}{2^n}$ converge?

Answer

Converges for all $p > 0$.

33) For which $r > 0$ does the series $\sum_{n=1}^{\infty} \frac{r^{n^2}}{2^n}$ converge?

34) For which $r > 0$ does the series $\sum_{n=1}^{\infty} \frac{2^n}{r^{n^2}}$ converge?

Answer

Converges for all $r > 1$. If $r > 1$ then $r^n > 4$, say, once $n > \ln(2)/\ln(r)$ and then the series converges by limit comparison with a geometric series with ratio $1/2$.

35) Find all values of p and q such that $\sum_{n=1}^{\infty} \frac{n^p}{(n!)^q}$ converges.

36) Does $\sum_{n=1}^{\infty} \frac{\sin^2(nr/2)}{n}$ converge or diverge? Explain.

Answer

The numerator is equal to 1 when n is odd and 0 when n is even, so the series can be rewritten $\sum_{n=1}^{\infty} \frac{1}{2n+1}$, which diverges by limit comparison with the harmonic series.

37) Explain why, for each n, at least one of $|\sin n|, |\sin(n+1)|, ..., |\sin(n+6)|$ is larger than $1/2$. Use this relation to test convergence of $\sum_{n=1}^{\infty} \frac{|\sin n|}{\sqrt{n}}$.

38) Suppose that $a_n \geq 0$ and $b_n \geq 0$ and that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges and $\sum_{n=1}^{\infty} a_n b_n \leq \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right)$.

Answer

$(a-b)^2 = a^2 - 2ab + b^2$ or $a^2 + b^2 \geq 2ab$, so convergence follows from comparison of $2a_n b_n$ with $a_n^2 + b_n^2$. Since the partial sums on the left are bounded by those on the right, the inequality holds for the infinite series.

39) Does $\sum_{n=1}^{\infty} 2^{-\ln\ln n}$ converge? (Hint: Write $2^{\ln\ln n}$ as a power of $\ln n$.)

40) Does $\sum_{n=1}^{\infty} (\ln n)^{-\ln n}$ converge? (Hint: Use $t = e^{\ln(t)}$ to compare to a p–series.)

Answer

$(\ln n)^{-\ln n} = e^{-\ln(n)\ln\ln(n)}$. If n is sufficiently large, then $\ln\ln n > 2$, so $(\ln n)^{-\ln n} < 1/n^2$, and the series converges by comparison to a p–series.

41) Does $\sum_{n=2}^{\infty} (\ln n)^{-\ln\ln n}$ converge? (Hint: Compare a_n to $1/n$.)

42) Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges. If $\sum_{n=1}^{\infty} a_n^2$ converges, does $\sum_{n=1}^{\infty} a_n$ necessarily converge?

Answer

$a_n \to 0$, so $a_n^2 \le |a_n|$ for large n. Convergence follows from limit comparison. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n}$ does not, so the fact that $\sum_{n=1}^{\infty} a_n^2$ converges does not imply that $\sum_{n=1}^{\infty} a_n$ converges.

43) Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ converges. Suppose that b_n is an arbitrary sequence of zeros and ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

44) Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ diverges. Suppose that b_n is an arbitrary sequence of zeros and ones with infinitely many terms equal to one. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily diverge?

Answer

No. $\sum_{n=1}^{\infty} 1/n$ diverges. Let $b_k = 0$ unless $k = n^2$ for some n. Then $\sum_k b_k/k = \sum 1/k^2$ converges.

45) Complete the details of the following argument: If $\sum_{n=1}^{\infty} \frac{1}{n}$ converges to a finite sum s, then $\frac{1}{2}s = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$ and $s - \frac{1}{2}s = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$. Why does this lead to a contradiction?

46) Show that if $a_n \ge 0$ and $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} \sin^2(a_n)$ converges.

Answer

$|\sin t| \le |t|$, so the result follows from the comparison test.

47) Suppose that $a_n/b_n \to 0$ in the comparison test, where $a_n \ge 0$ and $b_n \ge 0$. Prove that if $\sum b_n$ converges, then $\sum a_n$ converges.

48) Let b_n be an infinite sequence of zeros and ones. What is the largest possible value of $x = \sum_{n=1}^{\infty} b_n/2^n$?

Answer

By the comparison test, $x = \sum_{n=1}^{\infty} b_n/2^n \le \sum_{n=1}^{\infty} 1/2^n = 1$.

49) Let d_n be an infinite sequence of digits, meaning d_n takes values in $\{0, 1, \ldots, 9\}$. What is the largest possible value of $x = \sum_{n=1}^{\infty} d_n/10^n$ that converges?

50) Explain why, if $x > 1/2$, then x cannot be written $x = \sum_{n=2}^{\infty} \frac{b_n}{2^n} \quad (b_n = 0 \text{ or } 1, \ b_1 = 0)$.

Answer

If $b_1 = 0$, then, by comparison, $x \le \sum_{n=2}^{\infty} 1/2^n = 1/2$.

51) [T] Evelyn has a perfect balancing scale, an unlimited number of 1-kg weights, and one each of $1/2$-kg, $1/4$-kg, $1/8$-kg, and so on weights. She wishes to weigh a meteorite of unspecified origin to arbitrary precision. Assuming the scale is big enough, can she do it? What does this have to do with infinite series?

52) [T] Robert wants to know his body mass to arbitrary precision. He has a big balancing scale that works perfectly, an unlimited collection of 1-kg weights, and nine each of 0.1-kg, 0.01-kg, 0.001-kg, and so on weights. Assuming the scale is big enough, can

he do this? What does this have to do with infinite series?

Answer

Yes. Keep adding 1-kg weights until the balance tips to the side with the weights. If it balances perfectly, with Robert standing on the other side, stop. Otherwise, remove one of the 1-kg weights, and add 0.1-kg weights one at a time. If it balances after adding some of these, stop. Otherwise if it tips to the weights, remove the last 0.1-kg weight. Start adding 0.01-kg weights. If it balances, stop. If it tips to the side with the weights, remove the last 0.01-kg weight that was added. Continue in this way for the 0.001-kg weights, and so on. After a finite number of steps, one has a finite series of the form $A + \sum_{n=1}^{N} s_n/10^n$ where A is the number of full kg weights and d_n is the number of $1/10^n$-kg weights that were added. If at some state this series is Robert's exact weight, the process will stop. Otherwise it represents the Nth partial sum of an infinite series that gives Robert's exact weight, and the error of this sum is at most $1/10^N$.

53) The series $\sum_{n=1}^{\infty} \frac{1}{2n}$ is half the harmonic series and hence diverges. It is obtained from the harmonic series by deleting all terms in which n is odd. Let $m > 1$ be fixed. Show, more generally, that deleting all terms $1/n$ where $n = mk$ for some integer k also results in a divergent series.

54) In view of the previous exercise, it may be surprising that a subseries of the harmonic series in which about one in every five terms is deleted might converge. A depleted harmonic series is a series obtained from $\sum_{n=1}^{\infty} \frac{1}{n}$ by removing any term $1/n$ if a given digit, say 9, appears in the decimal expansion of n. Argue that this depleted harmonic series converges by answering the following questions.

a. How many whole numbers n have d digits?

b. How many d-digit whole numbers $h(d)$. do not contain 9 as one or more of their digits?

c. What is the smallest d-digit number $m(d)$?

d. Explain why the deleted harmonic series is bounded by $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$.

e. Show that $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$ converges.

Answer

a. $10^d - 10^{d-1} < 10^d$
b. $h(d) < 9^d$
c. $m(d) = 10^{d-1} + 1$
d. Group the terms in the deleted harmonic series together by number of digits. $h(d)$ bounds the number of terms, and each term is at most $\frac{1}{m(d)}$.
Then $\sum_{d=1}^{\infty} h(d)/m(d) \leq \sum_{d=1}^{\infty} 9^d/(10)^{d-1} \leq 90$. One can actually use comparison to estimate the value to smaller than 80. The actual value is smaller than 23.

55) Suppose that a sequence of numbers $a_n > 0$ has the property that $a_1 = 1$ and $a_{n+1} = \frac{1}{n+1} S_n$, where $S_n = a_1 + \cdots + a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (Hint: S_n is monotone.)

56) Suppose that a sequence of numbers $a_n > 0$ has the property that $a_1 = 1$ and $a_{n+1} = \frac{1}{(n+1)^2} S_n$, where $S_n = a_1 + \cdots + a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (Hint: $S_2 = a_2 + a_1 = a_2 + S_1 = a_2 + 1 = 1 + 1/4 = (1 + 1/4)S_1, S_3 = \frac{1}{3^2} S_2 + S_2 = (1 + 1/9)S_2 = (1 + 1/9)(1 + 1/4)S_1$, etc. Look at $\ln(S_n)$, and use $\ln(1 + t) \leq t, t > 0$.)

Answer

Continuing the hint gives $S_N = (1 + 1/N^2)(1 + 1/(N-1)^2 \ldots (1 + 1/4))$. Then $\ln(S_N) = \ln(1 + 1/N^2) + \ln(1 + 1/(N-1)^2) + \cdots + \ln(1 + 1/4)$. Since $\ln(1 + t)$ is bounded by a constant times t, when $0 < t < 1$ one has $\ln(S_N) \leq C \sum_{n=1}^{N} \frac{1}{n^2}$, which converges by comparison to the p-series for $p = 2$.

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9.5: ALTERNATING SERIES

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9.5E: Exercises for Section 9.5

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9.5E: Exercises for Section 9.5

In exercises 1 - 30, determine whether each of the following series converges or diverges.

1) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{n}{n+3}$

2) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sqrt{n}+1}{\sqrt{n}+3}$

Answer

This series diverges by the divergence test. Terms do not tend to zero.

3) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{\sqrt{n+3}}$

4) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sqrt{n+3}}{n}$

Answer

Converges conditionally by alternating series test, since $\sqrt{n+3}/n$ is decreasing and its limit is 0. Does not converge absolutely by comparison with p-series, $p=1/2$.

5) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{n!}$

6) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{3^n}{n!}$

Answer

Converges absolutely by limit comparison to $3^n/4^n$, for example.

7) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\dfrac{n-1}{n}\right)^n$

8) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\dfrac{n+1}{n}\right)^n$

Answer

Diverges by divergence test since $\lim_{n\to\infty}|a_n| = e$ and not 0.

9) $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2 n$

10) $\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2 n$

Answer

Diverges by the divergence test, since its terms do not tend to zero. The limit of the sequence of its terms does not exist.

11) $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2(1/n)$

12) $\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2(1/n)$

Answer

$\lim_{n\to\infty}\cos^2(1/n)=1$. Diverges by divergence test.

13) $\sum_{n=1}^{\infty}(-1)^{n+1}\ln(1/n)$

14) $\sum_{n=1}^{\infty}(-1)^{n+1}\ln(1+\frac{1}{n})$

Answer

Converges conditionally by AST, but not absolutely, since $a_n=\ln(n+1)-\ln(n)$ are terms of a telescoping series which add to infinity.

15) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{1+n^4}$

Answer

Converges absolutely by comparison with p-series, $p=2$.

16) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^e}{1+n^\pi}$

Answer

Converges conditionally by alternating series test. Does not converge absolutely by limit comparison with p -series, $p=\pi-e$

Solution:

17) $\sum_{n=1}^{\infty}(-1)^{n+1}2^{1/n}$

18) $\sum_{n=1}^{\infty}(-1)^{n+1}n^{1/n}$

Answer

Diverges; terms do not tend to zero.

19) $\sum_{n=1}^{\infty}(-1)^{n}(1-n^{1/n})$ (Hint: $n^{1/n}\approx 1+\ln(n)/n$ for large n.)

20) $\sum_{n=1}^{\infty}(-1)^{n+1}n\left(1-\cos\left(\frac{1}{n}\right)\right)$ (Hint: $\cos(1/n)\approx 1-1/n^2$ for large n.)

Answer

Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

21) $\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt{n+1}-\sqrt{n})$ (Hint: Rationalize the numerator.)

22) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ (Hint: Cross-multiply then rationalize numerator.)

Answer

Converges absolutely by limit comparison with p-series, $p=3/2$, after applying the hint.

23) $\sum_{n=1}^{\infty}(-1)^{n+1}(\ln(n+1)-\ln n)$

24) $\sum_{n=1}^{\infty}(-1)^{n+1}n(\tan^{-1}(n+1)-\tan^{-1}n)$ (Hint: Use Mean Value Theorem.)

Answer

Converges by alternating series test since $n(\tan^{-1}(n+1)-\tan^{-1}n)$ is decreasing to zero for large n.Does not converge absolutely by limit comparison with harmonic series after applying hint.

25) $\sum_{n=1}^{\infty}(-1)^{n+1}((n+1)^2-n^2)$

26) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

Answer

Converges absolutely, since $a_n=\frac{1}{n}-\frac{1}{n+1}$ are terms of a convergent telescoping series.

27) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$

28) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^{1/n}}$

Answer

Terms do not tend to zero. Series diverges by divergence test.

29) $\sum_{n=1}^{\infty}\frac{1}{n}\sin(\frac{n\pi}{2})$

30) $\sum_{n=1}^{\infty}\sin(n\pi/2)\sin(1/n)$

Answer

Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

In exercises 31 - 36, use the estimate $|R_N|\le b_{N+1}$ to find a value of N that guarantees that the sum of the first N terms of the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ differs from the infinite sum by at most the given error. Calculate the partial sum S_N for this N.

31) [T] $b_n=1/n$, error $<10^{-5}$

32) [T] $b_n=1/\ln(n), n\ge 2$, error $<10^{-1}$

Answer

$\ln(N+1)>10, N+1>e^{10}, N\ge 22026; S_{22026}=0.0257\ldots$

33) [T] $b_n=1/\sqrt{n}$, error $<10^{-3}$

34) [T] $b_n=1/2^n$, error $<10^{-6}$

Answer

$2^{N+1}>10^6$ or $N+1>6\ln(10)/\ln(2)=19.93.$ or $N\ge 19; S_{19}=0.333333969\ldots$

35) [T] $b_n=ln(1+\frac{1}{n})$, error $<10^{-3}$

36) [T] $b_n = 1/n^2$, error $< 10^{-6}$

Answer

$(N+1)^2 > 10^6$ or $N > 999; S_{1000} \approx 0.822466.$

For exercises 37 - 45, indicate whether each of the following statements is true or false. If the statement is false, provide an example in which it is false.

37) If $b_n \geq 0$ is decreasing and $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty}(b_{2n-1} - b_{2n})$ converges absolutely.

38) If $b_n \geq 0$ is decreasing, then $\sum_{n=1}^{\infty}(b_{2n-1} - b_{2n})$ converges absolutely.

Answer

True. b_n need not tend to zero since if $c_n = b_n - \lim b_n$, then $c_{2n-1} - c_{2n} = b_{2n-1} - b_{2n}$.

39) If $b_n \geq 0$ and $\lim_{n\to\infty} b_n = 0$ then $\sum_{n=1}^{\infty}(\frac{1}{2}(b_{3n-2} + b_{3n-1}) - b_{3n})$ converges.

40) If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty}(b_{3n-2} + b_{3n-1} - b_{3n})$ converges then $\sum_{n=1}^{\infty} b_{3n-2}$ converges.

Answer

True. $b_{3n-1} - b_{3n} \geq 0$, so convergence of $\sum b_{3n-2}$ follows from the comparison test.

41) If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges conditionally but not absolutely, then b_n does not tend to zero.

42) Let $a_n^+ = a_n$ if $a_n \geq 0$ and $a_n^- = -a_n$ if $a_n < 0$. (Also, $a_n^+ = 0$ if $a_n < 0$ and $a_n^- = 0$ if $a_n \geq 0$.) If $\sum_{n=1}^{\infty} a_n$ converges conditionally but not absolutely, then neither $\sum_{n=1}^{\infty} a_n^+$ nor $\sum_{n=1}^{\infty} a_n^-$ converge.

Answer

True. If one converges, then so must the other, implying absolute convergence.

43) Suppose that a_n is a sequence of positive real numbers and that $\sum_{n=1}^{\infty} a_n$ converges.

44) Suppose that b_n is an arbitrary sequence of ones and minus ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

45) Suppose that a_n is a sequence such that $\sum_{n=1}^{\infty} a_n b_n$ converges for every possible sequence b_n of zeros and ones. Does $\sum_{n=1}^{\infty} a_n$ converge absolutely?

Answer

Yes. Take $b_n = 1$ if $a_n \geq 0$ and $b_n = 0$ if $a_n < 0$. Then $\sum_{n=1}^{\infty} a_n b_n = \sum_{n:a_n\geq 0} a_n$ converges. Similarly, one can show $\sum_{n:a_n<0} a_n$ converges. Since both series converge, the series must converge absolutely.

In exercises 46 - 49, the series do not satisfy the hypotheses of the alternating series test as stated. In each case, state which hypothesis is not satisfied. State whether the series converges absolutely.

46) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin^2(n)}{n^2}$

Answer

Not strictly decreasing. Converges absolutely by limit comparison test since $\sin^2(n) \leq 1$

47) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos^2 n}{n}$

Answer

Not decreasing. Does not converge absolutely.

48) $1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots$

Answer

Not alternating. Does not converge absolutely.

49) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$

Answer

Not alternating. Can be expressed as $\sum_{n=1}^{\infty} \left(\frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n} \right)$, which diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{3n-2}$.

50) Show that the alternating series $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{8} + \cdots$ does not converge. What hypothesis of the alternating series test is not met?

51) Suppose that $\sum a_n$ converges absolutely. Show that the series consisting of the positive terms a_n also converges.

Answer

Let $a_n^+ = a_n$ if $a_n \geq 0$ and $a_n^+ = 0$ if $a_n < 0$. Then $a_n^+ \leq |a_n|$ for all n so the sequence of partial sums of a_n^+ is increasing and bounded above by the sequence of partial sums of $|a_n|$, which converges; hence, $\sum_{n=1}^{\infty} a_n^+$ converges.

52) Show that the alternating series $\frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \frac{5}{9} + \cdots$ does not converge. What hypothesis of the alternating series test is not met?

53) The formula $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$ will be derived in the next chapter. Use the remainder $|R_N| \leq b_{N+1}$ to find a bound for the error in estimating $\cos\theta$ by the fifth partial sum $1 - \theta^2/2! + \theta^4/4! - \theta^6/6! + \theta^8/8!$ for $\theta = 1, \theta = \pi/6$, and $\theta = \pi$.

Answer

For $N = 5$ one has $\mid R_N \mid b_6 = \theta^{10}/10!$. When $\theta = 1, R_5 \leq 1/10! \approx 2.75 \times 10^{-7}$. When $\theta = \pi/6$, $R_5 \leq (\pi/6)^{10}/10! \approx 4.26 \times 10^{-10}$. When $\theta = \pi, R_5 \leq \pi^{10}/10! = 0.0258$.

54) The formula $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ will be derived in the next chapter. Use the remainder $|R_N| \leq b_{N+1}$ to find a bound for the error in estimating $\sin\theta$ by the fifth partial sum $\theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \theta^9/9!$ for $\theta = 1, \theta = \pi/6$, and $\theta = \pi$.

55) How many terms in $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$ are needed to approximate $\cos 1$ accurate to an error of at most 0.00001?

Answer

Let $b_n = 1/(2n-2)!$. Then $R_N \leq 1/(2N)! < 0.00001$ when $(2N)! > 10^5$ or $N = 5$ and $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} = 0.540325\ldots$, whereas $\cos 1 = 0.5403023\ldots$

56) How many terms in $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ are needed to approximate $\sin 1$ accurate to an error of at most 0.00001?

Answer

$N = 4$ and $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} = 0.841468\ldots$, whereas $\sin 1 = 0.841471\ldots$

57) Sometimes the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges to a certain fraction of an absolutely convergent series $\sum_{n=1}^{\infty} b_n$ at a faster rate. Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find $S = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$. Which of the series $6\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $S\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ gives a better estimation of π^2 using 1000 terms?

Answer

Let $T = \sum \frac{1}{n^2}$. Then $T - S = \frac{1}{2}T$, so $S = T/2$. $\sqrt{6 \times \sum_{n=1}^{1000} 1/n^2} = 3.140638\ldots$; $\sqrt{\frac{1}{2} \times \sum_{n=1}^{1000} (-1)^{n-1}/n^2} = 3.141591\ldots$; $\pi = 3.141592\ldots$. The alternating series is more accurate for 1000 terms.

The alternating series in exercises 58 & 59 converge to given multiples of π. Find the value of N predicted by the remainder estimate such that the Nth partial sum of the series accurately approximates the left-hand side to within the given error. Find the minimum N for which the error bound holds, and give the desired approximate value in each case. Up to 15 decimals places, $\pi = 3.141592653589793\ldots$.

58) [T] $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, error < 0.001 (Use Desmos to find the first term less than 0.001)

Answer

$N = 500, S_{499} = 0.784898$

59) [T] $\frac{\pi}{\sqrt{12}} = \sum_{k=0}^{\infty} \frac{(-3)^{-k}}{2k+1}$, error < 0.0001

Answer

$N = 6, S_N = 0.9068$

60) [T] The series $\sum_{n=0}^{\infty} \frac{\sin(x+\pi n)}{x+\pi n}$ plays an important role in signal processing. Show that $\sum_{n=0}^{\infty} \frac{\sin(x+\pi n)}{x+\pi n}$ converges whenever $0 < x < \pi$. (Hint: Use the formula for the sine of a sum of angles.)

61) [T] If $\sum_{n=1}^{N} (-1)^{n-1} \frac{1}{n} \to ln2$, what is $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} + \cdots$?

Answer

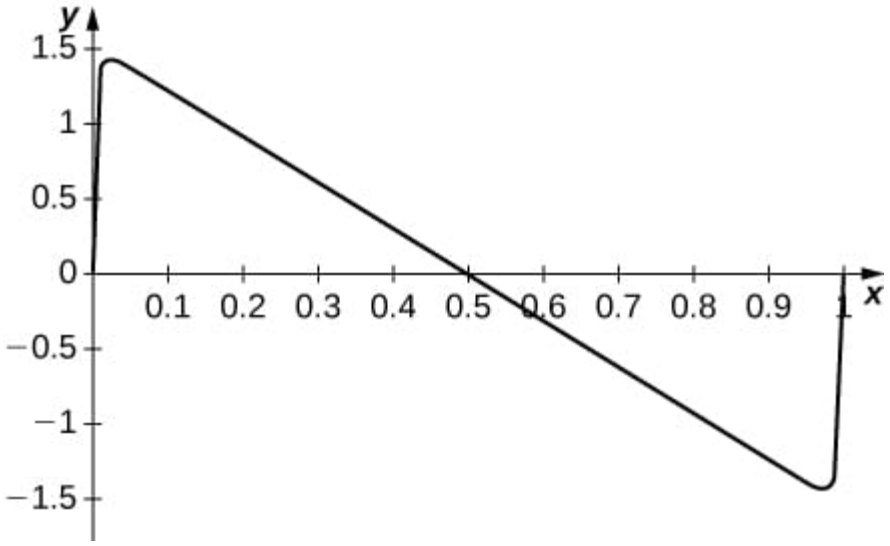
$\ln(2)$. The nth partial sum is the same as that for the alternating harmonic series.

62) [T] Plot the series $\sum_{n=1}^{100} \frac{\cos(2\pi n x)}{n}$ for $0 \le x < 1$. Explain why $\sum_{n=1}^{100} \frac{\cos(2\pi n x)}{n}$ diverges when $x = 0, 1$. How does the series behave for other x?

63) [T] Plot the series $\sum_{n=1}^{100} \frac{\sin(2\pi n x)}{n}$ for $0 \le x < 1$ and comment on its behavior

Answer

The series jumps rapidly near the endpoints. For x away from the endpoints, the graph looks like $\pi(1/2 - x)$.

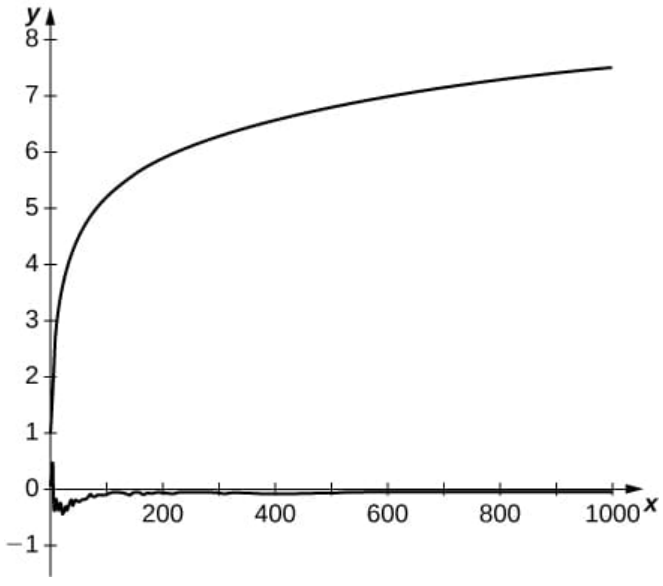


64) [T] Plot the series $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n^2}$ for $0 \leq x < 1$ and describe its graph.

65) [T] The alternating harmonic series converges because of cancelation among its terms. Its sum is known because the cancelation can be described explicitly. A random harmonic series is one of the form $\sum_{n=1}^{\infty} \frac{S_n}{n}$, where s_n is a randomly generated sequence of $\pm 1's$ in which the values ± 1 are equally likely to occur. Use a random number generator to produce 1000 random $\pm 1's$ and plot the partial sums $S_N = \sum_{n=1}^{N} \frac{s_n}{n}$ of your random harmonic sequence for $N = 1$ to 1000. Compare to a plot of the first 1000 partial sums of the harmonic series.

Answer

Here is a typical result. The top curve consists of partial sums of the harmonic series. The bottom curve plots partial sums of a random harmonic series.



66) [T] Estimates of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ can be accelerated by writing its partial sums as $\sum_{n=1}^{N} \frac{1}{n^2} = \sum_{n=1}^{N} \frac{1}{n(n+1)} + \sum_{n=1}^{N} \frac{1}{n^2(n+1)}$ and recalling

that $\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}$ converges to one as $N \to \infty$. Compare the estimate of $\pi^2/6$ using the sums $\sum_{n=1}^{1000} \frac{1}{n^2}$ with the estimate using $1 + \sum_{n=1}^{1000} \frac{1}{n^2(n+1)}$.

67) [T] The **Euler transform** rewrites $S = \sum_{n=0}^{\infty} (-1)^n b_n$ as $S = \sum_{n=0}^{\infty} (-1)^n 2^{-n-1} \sum_{m=0}^{n} \binom{n}{m} b_{n-m}$. For the alternating harmonic series, it takes the form $\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{1}{n2^n}$. Compute partial sums of $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ until they approximate $\ln(2)$ accurate to within 0.0001. How many terms are needed? Compare this answer to the number of terms of the alternating harmonic series are needed to estimate $\ln(2)$.

Answer

By the alternating series test, $|S_n - S| \le b_{n+1}$, so one needs 10^4 terms of the alternating harmonic series to estimate $\ln(2)$ to within 0.0001. The first 10 partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ are (up to four decimals) $0.5000, 0.6250, 0.6667, 0.6823, 0.6885, 0.6911, 0.6923, 0.6928, 0.6930, 0.6931$ and the tenth partial sum is within 0.0001 of $\ln(2) = 0.6931\ldots$.

68) [T] In the text it was stated that a conditionally convergent series can be rearranged to converge to any number. Here is a slightly simpler, but similar, fact. If $a_n \ge 0$ is such that $a_n \to 0$ as $n \to \infty$ but $\sum_{n=1}^{\infty} a_n$ diverges, then, given any number A there is a sequence s_n of $\pm 1's$ such that $\sum_{n=1}^{\infty} a_n s_n \to A$. Show this for $A > 0$ as follows.

a. Recursively define s_n by $s_n = 1$ if $S_{n-1} = \sum_{k=1}^{n-1} a_k s_k < A$ and $s_n = -1$ otherwise.

b. Explain why eventually $S_n \ge A$, and for any m larger than this n, $A - a_m \le S_m \le A + a_m$.

c. Explain why this implies that $S_n \to A$ as $n \to \infty$.

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9.6: RATIO AND ROOT TESTS

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9.6E: Exercises for Section 9.6

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9.6E: Exercises for Section 9.6

In exercises 1 - 11, use the ratio test to determine whether each series $\sum_{n=1}^{\infty} a_n$ converges or diverges. State if the ratio test is inconclusive.

1) $\sum_{n=1}^{\infty} \frac{1}{n!}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$. Converges by the Ratio Test.

2) $\sum_{n=1}^{\infty} \frac{10^n}{n^2}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 10 > 1$. Diverges by the Ratio Test.

3) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2} < 1$. Converges by the Ratio Test.

4) $\sum_{n=1}^{\infty} \frac{n^{10}}{2^n}$

5) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{27} < 1$. Converges by the Ratio Test.

6) $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n)!}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{8}{27} < 1$. Converges by the Ratio Test.

7) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{4}{e^2} < 1$. Converges by the Ratio Test.

8) $\sum_{n=1}^{\infty} \frac{(2n)!}{(2n)^n}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty > 1$. Diverges by the Ratio Test.

9) $\sum_{n=1}^{\infty} \frac{n!}{(n/e)^n}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Ratio test is inconclusive.

10) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n/e)^{2n}}$

11) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{e} < 1$. Converges by the Ratio Test.

In exercises 12 - 21, use the root test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, where a_n is as follows.

12) $a_k = \left(\frac{k-1}{2k+3}\right)^k$

Answer

$\lim_{k\to\infty} (a_k)^{1/k} = \frac{1}{2} < 1$. Converges by the Root Test.

13) $a_k = \left(\frac{2k^2-1}{k^2+3}\right)^k$

Answer

$\lim_{k\to\infty} (a_k)^{1/k} = 2 > 1$. Diverges by the Root Test.

14) $a_n = \frac{(\ln n)^{2n}}{n^n}$

Answer

$\lim_{n\to\infty} (a_n)^{1/n} = 0 < 1$. Converges by the Root Test.

15) $a_n = n/2^n$

Answer

$\lim_{n\to\infty} (a_n)^{1/n} = 1/2 < 1$. Converges by the Root Test.

16) $a_n = n/e^n$

17) $a_k = \frac{k^e}{e^k}$

Answer

$\lim_{k\to\infty} (a_k)^{1/k} = 1/e < 1$. Converges by the Root Test.

18) $a_k = \frac{\pi^k}{k^\pi}$

19) $a_n = \left(\frac{1}{e} + \frac{1}{n}\right)^n$

Answer

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{1}{e} + \frac{1}{n} = \frac{1}{e} < 1$. Converges by the Root Test.

20) $a_k = \dfrac{1}{(1+\ln k)^k}$

21) $a_n = \dfrac{(\ln(1+\ln n))^n}{(\ln n)^n}$

Answer

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \dfrac{(\ln(1+\ln n))}{(\ln n)} = 0$ by L'Hôpital's rule. Converges by the Root Test.

In exercises 22 - 28, use either the ratio test or the root test as appropriate to determine whether the series $\sum_{k=1}^{\infty} a_k$ with given terms a_k converges, or state if the test is inconclusive.

22) $a_k = \dfrac{k!}{1\cdot 3\cdot 5\cdots(2k-1)}$

23) $a_k = \dfrac{2\cdot 4\cdot 6\cdots 2k}{(2k)!}$

Answer

$\lim_{k\to\infty} \dfrac{a_{k+1}}{a_k} = \lim_{k\to\infty} \dfrac{1}{2k+1} = 0$. Converges by the Ratio Test.

24) $a_k = \dfrac{1\cdot 4\cdot 7\cdots(3k-2)}{3^k k!}$

25) $a_n = \left(1-\dfrac{1}{n}\right)^{n^2}$

Answer

$\lim_{n\to\infty} (a_n)^{1/n} = 1/e$. Converges by the Root Test.

26) $a_k = \left(\dfrac{1}{k+1} + \dfrac{1}{k+2} + \cdots + \dfrac{1}{2k}\right)^k$ (Hint: Compare $a_k^{1/k}$ to $\int_k^{2k} \dfrac{dt}{t}$.)

27) $a_k = \left(\dfrac{1}{k+1} + \dfrac{1}{k+2} + \cdots + \dfrac{1}{3k}\right)^k$

Answer

$\lim_{k\to\infty} a_k^{1/k} = \ln(3) > 1$. Diverges by the Root Test.

28) $a_n = \left(n^{1/n} - 1\right)^n$

In exercises 29 - 30, use the ratio test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, or state if the ratio test is inconclusive.

29) $\sum_{n=1}^{\infty} \dfrac{3^{n^2}}{2^{n^3}}$

Answer

$\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \dfrac{3^{2n+1}}{2^{3n^2+3n+1}} = 0$. Converges by the Ratio Test.

30) $\sum_{n=1}^{\infty} \dfrac{2^{n^2}}{n^n n!}$

In exercises 31, use the root and limit comparison tests to determine whether $\sum_{n=1}^{\infty} a_n$ converges.

31) $\sum_{n=1}^{\infty} \frac{1}{x_n^n}$ where $x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}, x_1 = 1$ (Hint: Find limit of x_n.)

Answer

Converges by the Root Test and Limit Comparison Test since $\lim_{n\to\infty} x_n = \sqrt{2}$.

In exercises 32 - 43, use an appropriate test to determine whether the series converges.

32) $\sum_{n=1}^{\infty} \frac{n+1}{n^3+n^2+n+1}$

33) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n^3+3n^2+3n+1}$

Answer

Converges absolutely by limit comparison with p**–series,** $p = 2$.

34) $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^3+(1.1)^n}$

35) $\sum_{n=1}^{\infty} \frac{(n-1)^n}{(n+1)^n}$

Answer

$\lim_{n\to\infty} a_n = 1/e^2 \neq 0$. Series diverges by the Divergence Test.

36) $a_n = \left(1+\frac{1}{n^2}\right)^n$ $\left(\text{Hint: } \left(1+\frac{1}{n^2}\right)^{n^2} \approx e.\right)$

37) $a_k = 1/2^{\sin^2 k}$

Answer

Terms do not tend to zero: $a_k \geq 1/2$, since $\sin^2 x \leq 1$.

38) $a_k = 2^{-\sin(1/k)}$

39) $a_n = 1/\binom{n+2}{n}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Answer

$a_n = \frac{2}{(n+1)(n+2)}$, which converges by comparison with p**–series** for $p = 2$.

40) $a_k = 1/\binom{2k}{k}$

41) $a_k = 2^k/\binom{3k}{k}$

Answer

$a_k = \frac{2^k 1\cdot 2\cdots k}{(2k+1)(2k+2)\cdots 3k} \leq (2/3)^k$ converges by comparison with geometric series.

42) $a_k = \left(\frac{k}{k+\ln k}\right)^k$ $\left(\text{Hint: } a_k = \left(1+\frac{\ln k}{k}\right)^{-(k/\ln k)\ln k} \approx e^{-\ln k}.\right)$

43) $a_k = \left(\frac{k}{k+\ln k}\right)^{2k}$ $\left(\text{Hint: } a_k = \left(1+\frac{\ln k}{k}\right)^{-(k/\ln k)\ln k^2}.\right)$

Answer

$a_k \approx e^{-\ln k^2} = 1/k^2$. Series converges by limit comparison with p**–series**, $p = 2$.

The series in exercises 44 - 47 converge by the ratio test. Use summation by parts, $\sum_{k=1}^{n} a_k(b_{k+1}-b_k) = [a_{n+1}b_{n+1} - a_1b_1] - \sum_{k=1}^{n} b_{k+1}(a_{k+1}-a_k)$, to find the sum of the given series.

44) $\sum_{k=1}^{\infty} \frac{k}{2^k}$ (Hint: Take $a_k = k$ and $b_k = 2^{1-k}$.)

45) $\sum_{k=1}^{\infty} \frac{k}{c^k}$, where $c > 1$ (Hint: Take $a_k = k$ and $b_k = c^{1-k}/(c-1)$.)

Answer

If $b_k = c^{1-k}/(c-1)$ and $a_k = k$, then $b_{k+1} - b_k = -c^{-k}$ and $\sum_{n=1}^{\infty} \frac{k}{c^k} = a_1b_1 + \frac{1}{c-1}\sum_{k=1}^{\infty} c^{-k} = \frac{c}{(c-1)^2}$.

46) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

47) $\sum_{n=1}^{\infty} \frac{(n+1)^2}{2^n}$

Answer

$6 + 4 + 1 = 11$

The kth term of each of the following series has a factor x^k. Find the range of x for which the ratio test implies that the series converges.

48) $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$

Answer

$|x| \le 1$

49) $\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2}$

Answer

$|x| \le 1$

50) $\sum_{k=1}^{\infty} \frac{x^{2k}}{3^k}$

Answer

$|x| \le \sqrt{3}$

51) $\sum_{k=1}^{\infty} \frac{x^k}{k!}$

Answer

$|x| < \infty$

52) Does there exist a number p such that $\sum_{n=1}^{\infty} \frac{2^n}{n^p}$ converges?

53) Let $0 < r < 1$. For which real numbers p does $\sum_{n=1}^{\infty} n^p r^n$ converge?

Answer

All real numbers p by the Ratio Test.

54) Suppose that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = p$. For which values of p must $\sum_{n=1}^{\infty} 2^n a_n$ converge?

55) Suppose that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = p$. For which values of $r > 0$ is $\sum_{n=1}^{\infty} r^n a_n$ guaranteed to converge?

Answer

$r < 1/p$

56) Suppose that $\left|\frac{a_{n+1}}{a_n}\right| \le (n+1)^p$ for all $n = 1, 2, \ldots$ where p is a fixed real number. For which values of p is $\sum_{n=1}^{\infty} n! a_n$ guaranteed to converge?

57) For which values of $r > 0$, if any, does $\sum_{n=1}^{\infty} r^{\sqrt{n}}$ converge? $\left(\text{Hint: } sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty}\sum_{n=k^2}^{(k+1)^2-1} a_n.\right)$

Answer

$0 < r < 1$. Note that the ratio and root tests are inconclusive. Using the hint, there are $2k$ terms $r^{\sqrt{n}}$ for $k^2 \le n < (k+1)^2$, and for $r < 1$ each term is at least r^k. Thus, $\sum_{n=1}^{\infty} r^{\sqrt{n}} = \sum_{k=1}^{\infty}\sum_{n=k^2}^{(k+1)^2-1} r^{\sqrt{n}} \ge \sum_{k=1}^{\infty} 2kr^k$, which converges by the ratio test for $r < 1$. For $r \ge 1$ the series diverges by the divergence test.

58) Suppose that $\left|\frac{a_{n+2}}{a_n}\right| \le r < 1$ for all n. Can you conclude that $\sum_{n=1}^{\infty} a_n$ converges?

59) Let $a_n = 2^{-[n/2]}$ where $[x]$ is the greatest integer less than or equal to x. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and justify your answer.

Answer

One has $a_1 = 1, a_2 = a_3 = 1/2, \ldots a_{2n} = a_{2n+1} = 1/2^n$. The ratio test does not apply because $a_{n+1}/a_n = 1$ if n is even. However, $a_{n+2}/a_n = 1/2$, so the series converges according to the previous exercise. Of course, the series is just a duplicated geometric series.

The following *advanced exercises* use a generalized ratio test to determine convergence of some series that arise in particular applications when tests in this chapter, including the ratio and root test, are not powerful enough to determine their convergence. The test states that if $\lim_{n\to\infty}\frac{a_{2n}}{a_n} < 1/2$, then $\sum a_n$ converges, while if $\lim_{n\to\infty}\frac{a_{2n+1}}{a_n} > 1/2$, then $\sum a_n$ diverges.

60) Let $a_n = \frac{1}{4}\frac{3}{6}\frac{5}{8}\cdots\frac{2n-1}{2n+2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(n+1)!}$. Explain why the ratio test cannot determine convergence of $\sum_{n=1}^{\infty} a_n$. Use the fact that $1 - 1/(4k)$ is increasing k to estimate $\lim_{n\to\infty}\frac{a_{2n}}{a_n}$.

61) Let $a_n = \frac{1}{1+x}\frac{2}{2+x}\cdots\frac{n}{n+x}\frac{1}{n} = \frac{(n-1)!}{(1+x)(2+x)\cdots(n+x).}$ Show that $a_{2n}/a_n \le e^{-x/2}/2$. For which $x > 0$ does the generalized ratio test imply convergence of $\sum_{n=1}^{\infty} a_n$? (Hint: Write $2a_{2n}/a_n$ as a product of n factors each smaller than $1/(1 + x/(2n)).$)

Answer

$a_{2n}/a_n = \frac{1}{2}\cdot\frac{n+1}{n+1+x}\frac{n+2}{n+2+x}\cdots\frac{2n}{2n+x}$. The inverse of the kth factor is $(n+k+x)/(n+k) > 1 + x/(2n)$ so the product is less than $(1 + x/(2n))^{-n} \approx e^{-x/2}$. Thus for $x > 0, \frac{a_{2n}}{a_n} \le \frac{1}{2}e^{-x/2}$. The series converges for $x > 0$.

62) Let $a_n = \dfrac{n^{\ln n}}{(\ln n)^n}$. Show that $\dfrac{a_{2n}}{a_n} \to 0$ as $n \to \infty$.

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9.7: Chapter 9 Review Exercises

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9.8: Power Series

A power series (in one variable) is an infinite series. Any polynomial can be easily expressed as a power series around any center c, although most of the coefficients will be zero since a power series has infinitely many terms by definition. One can view power series as being like "polynomials of infinite degree," although power series are not polynomials. The content in this Textmap's chapter is complemented by Guichard's Calculus Textmap.

Thumbnail: The graph shows the function $y = sinx$ and the Maclaurin polynomials p_1, p_3 and p_5. (CC BY-SA 3.0; OpenStax).

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10: POWER SERIES

A power series (in one variable) is an infinite series. Any polynomial can be easily expressed as a power series around any center c, although most of the coefficients will be zero since a power series has infinitely many terms by definition. One can view power series as being like "polynomials of infinite degree," although power series are not polynomials. The content in this Textmap's chapter is complemented by Guichard's Calculus Textmap.

10.0: Prelude to Power Series

10.1: Power Series and Functions

10.1E: Exercises for Section 10.1

10.2: Properties of Power Series

10.2E: Exercises for Section 10.2

10.3: Taylor and Maclaurin Series

10.3E: Exercises for Section 10.3

10.4: Working with Taylor Series

10.4E: Exercises for Section 10.4

10.5: Chapter 10 Review Exercises

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10.0: Prelude to Power Series

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10.1: POWER SERIES AND FUNCTIONS

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10.1E: Exercises for Section 10.1

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10.1E: Exercises for Section 10.1

In exercises 1 - 4, state whether each statement is true, or give an example to show that it is false.

1) If $\sum_{n=1}^{\infty} a_n x^n$ converges, then $a_n x^n \to 0$ as $n \to \infty$.

Answer

True. If a series converges then its terms tend to zero.

2) $\sum_{n=1}^{\infty} a_n x^n$ converges at $x = 0$ for any real numbers a_n.

3) Given any sequence a_n, there is always some $R > 0$, possibly very small, such that $\sum_{n=1}^{\infty} a_n x^n$ converges on $(-R, R)$.

Answer

False. It would imply that $a_n x^n \to 0$ for $|x| < R$. If $a_n = n^n$, then $a_n x^n = (nx)^n$ does not tend to zero for any $x \neq 0$.

4) If $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence $R > 0$ and if $|b_n| \leq |a_n|$ for all n, then the radius of convergence of $\sum_{n=1}^{\infty} b_n x^n$ is greater than or equal to R.

5) Suppose that $\sum_{n=0}^{\infty} a_n (x-3)^n$ converges at $x = 6$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.

a. $x = 1$

b. $x = 2$

c. $x = 3$

d. $x = 0$

e. $x = 5.99$

f. $x = 0.000001$

Answer

It must converge on $(0, 6]$ and hence at: a. $x = 1$; b. $x = 2$; c. $x = 3$; e. $x = 5.99$; and f. $x = 0.000001$.

6) Suppose that $\sum_{n=0}^{\infty} a_n (x+1)^n$ converges at $x = -2$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.

a. $x = 2$

b. $x = -1$

c. $x = -3$

d. $x = 0$

e. $x = 0.99$

f. $x = 0.000001$

In the following exercises, suppose that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$. Find the radius of convergence for each series.

7) $\sum_{n=0}^{\infty} a_n 2^n x^n$

Answer

$\left|\frac{a_{n+1}2^{n+1}x^{n+1}}{a_n 2^n x^n}\right| = 2|x|\left|\frac{a_{n+1}}{a_n}\right| \to 2|x|$ so $R = \frac{1}{2}$

8) $\sum_{n=0}^{\infty} \frac{a_n x^n}{2^n}$

9) $\sum_{n=0}^{\infty} \frac{a_n \pi^n x^n}{e^n}$

Answer

$\left|\frac{a_{n+1}(\frac{\pi}{e})^{n+1}x^{n+1}}{a_n(\frac{\pi}{e})^n x^n}\right| = \frac{\pi|x|}{e}\left|\frac{a_{n+1}}{a_n}\right| \to \frac{\pi|x|}{e}$ so $R = \frac{e}{\pi}$

10) $\sum_{n=0}^{\infty} \frac{a_n(-1)^n x^n}{10^n}$

11) $\sum_{n=0}^{\infty} a_n(-1)^n x^{2n}$

Answer

$\left|\frac{a_{n+1}(-1)^{n+1}x^{2n+2}}{a_n(-1)^n x^{2n}}\right| = |x^2|\left|\frac{a_{n+1}}{a_n}\right| \to |x^2|$ so $R = 1$

12) $\sum_{n=0}^{\infty} a_n(-4)^n x^{2n}$

In exercises 13 - 22, find the radius of convergence R and interval of convergence for $\sum a_n x^n$ with the given coefficients a_n.

13) $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$

Answer

$a_n = \frac{2^n}{n}$ so $\frac{a_{n+1}x}{a_n} \to 2x$. so $R = \frac{1}{2}$. When $x = \frac{1}{2}$ the series is harmonic and diverges. When $x = -\frac{1}{2}$ the series is alternating harmonic and converges. The interval of convergence is $I = \left[-\frac{1}{2}, \frac{1}{2}\right)$.

14) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt{n}}$

15) $\sum_{n=1}^{\infty} \frac{nx^n}{2^n}$

Answer

$a_n = \frac{n}{2^n}$ so $\frac{a_{n+1}x}{a_n} \to \frac{x}{2}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

16) $\sum_{n=1}^{\infty} \frac{n^2 x^n}{e^n}$

Answer

$a_n = \frac{n^2}{e^n}$ so $R = e$. When $x = \pm e$ the series diverges by the divergence test. The interval of convergence is $I = (-e, e)$.

17) $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}$

Answer

$a_n = \frac{n^2}{2^n}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

18) $\sum_{k=1}^{\infty} \frac{k^e x^k}{e^k}$

19) $\sum_{k=1}^{\infty} \frac{\pi^k x^k}{k^\pi}$

Answer

$a_k = \frac{\pi^k}{k^\pi}$ so $R = \frac{1}{\pi}$. When $x = \pm\frac{1}{\pi}$ the series is an absolutely convergent p-series. The interval of convergence is $I = \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$.

20) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

21) $\sum_{n=1}^{\infty} \frac{10^n x^n}{n!}$

Answer

$a_n = \frac{10^n}{n!}, \frac{a_{n+1}x}{a_n} = \frac{10x}{n+1} \to 0 < 1$ so the series converges for all x by the ratio test and $I = (-\infty, \infty)$.

22) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\ln(2n)}$

In exercises 23 - 28, find the radius of convergence of each series.

23) $\sum_{k=1}^{\infty} \frac{(k!)^2 x^k}{(2k)!}$

Answer

$a_k = \frac{(k!)^2}{(2k)!}$ so $\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{(2k+2)(2k+1)} \to \frac{1}{4}$ so $R = 4$

24) $\sum_{n=1}^{\infty} \frac{(2n)! x^n}{n^{2n}}$

25) $\sum_{k=1}^{\infty} \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)} x^k$

Answer

$a_k = \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$ so $\frac{a_{k+1}}{a_k} = \frac{k+1}{2k+1} \to \frac{1}{2}$ so $R = 2$

26) $\sum_{k=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots 2k}{(2k)!} x^k$

27) $\sum_{n=1}^{\infty} \frac{x^n}{\binom{2n}{n}}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Answer

$$a_n = \frac{1}{\binom{2n}{n}} \text{ so } \frac{a_{n+1}}{a_n} = \frac{\big((n+1)!\big)^2}{(2n+2)!} \frac{2n!}{(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} \to \frac{1}{4} \text{ so } R = 4$$

28) $\sum_{n=1}^{\infty} \sin^2 n x^n$

In exercises 29 - 32, use the ratio test to determine the radius of convergence of each series.

29) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n$

Answer

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \to \frac{1}{27} \text{ so } R = 27$$

30) $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n)!} x^n$

31) $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

Answer

$$a_n = \frac{n!}{n^n} \text{ so } \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} = (\frac{n}{n+1})^n \to \frac{1}{e} \text{ so } R = e$$

32) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}} x^n$

In the following exercises, given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with convergence in $(-1, 1)$, find the power series for each function with the given center a, and identify its interval of convergence.

33) $f(x) = \frac{1}{x}; a = 1$ (Hint: $\frac{1}{x} = \frac{1}{1-(1-x)}$)

Answer

$$f(x) = \sum_{n=0}^{\infty} (1-x)^n \text{ on } I = (0, 2)$$

34) $f(x) = \frac{1}{1-x^2}; a = 0$

35) $f(x) = \frac{x}{1-x^2}; a = 0$

Answer

$$\sum_{n=0}^{\infty} x^{2n+1} \text{ on } I = (-1, 1)$$

36) $f(x) = \frac{1}{1+x^2}; a = 0$

37) $f(x) = \dfrac{x^2}{1+x^2}; a = 0$

Answer

$\sum_{n=0}^{\infty} (-1)^n x^{2n+2}$ on $I = (-1, 1)$

38) $f(x) = \dfrac{1}{2-x}$; a) centered around $a = 0$ b) centered around $a = 1$

Answer

$f(x) = \sum_{n=0}^{\infty} \dfrac{x^n}{2^n}$ on $I = (-2, 2)$ around $a = 0$; $f(x) = \sum_{n=0}^{\infty} (x-1)^n$ on $I = (0, 2)$ around $a = 1$

39) $f(x) = \dfrac{1}{1-2x}; a = 0.$

Answer

$\sum_{n=0}^{\infty} 2^n x^n$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$

40) $f(x) = \dfrac{1}{1-4x^2}; a = 0$

41) $f(x) = \dfrac{x^2}{1-4x^2}; a = 0$

Answer

$\sum_{n=0}^{\infty} 4^n x^{2n+2}$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$

42) $f(x) = \dfrac{x^2}{5-4x+x^2}; a = 2$

Use the result of exercise 43 to find the radius of convergence of the given series in the subsequent exercises (44 - 47).

43) Explain why, if $|a_n|^{1/n} \to r > 0$, then $|a_n x^n|^{1/n} \to |x| r < 1$ whenever $|x| < \frac{1}{r}$ and, therefore, the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$ is $R = \frac{1}{r}$.

Answer

$|a_n x^n|^{1/n} = |a_n|^{1/n}|x| \to |x| r$ as $n \to \infty$ and $|x| r < 1$ when $|x| < \frac{1}{r}$. Therefore, $\sum_{n=1}^{\infty} a_n x^n$ converges when $|x| < \frac{1}{r}$ by the nth root test.

44) $\sum_{n=1}^{\infty} \dfrac{x^n}{n^n}$

Answer

$\lim_{n\to\infty} (a_n)^{1/n} = \dfrac{|x|}{n} \to 0 < 1$, so the series converges for all x by the root test and $I = (-\infty, \infty)$.

45) $\sum_{k=1}^{\infty} \left(\dfrac{k-1}{2k+3}\right)^k x^k$

Answer

$a_k = \left(\dfrac{k-1}{2k+3}\right)^k$ so $(a_k)^{1/k} \to \frac{1}{2} < 1$ so $R = 2$

46) $\sum_{k=1}^{\infty}(\frac{2k^2-1}{k^2+3})^k x^k$

47) $\sum_{n=1}^{\infty} a_n = (n^{1/n}-1)^n x^n$

Answer

$a_n = (n^{1/n}-1)^n$ so $(a_n)^{1/n} \to 0$ so $R = \infty$

48) Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $a_n = 0$ if n is even. Explain why $p(x) = p(-x)$.

49) Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $a_n = 0$ if n is odd. Explain why $p(x) = -p(-x)$.

Answer

We can rewrite $p(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$ and $p(x) = p(-x)$ since $x^{2n+1} = -(-x)^{2n+1}$.

50) Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-1, 1]$. Find the interval of convergence of $p(Ax)$.

51) Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-1, 1]$. Find the interval of convergence of $p(2x-1)$.

Answer

If $x \in [0, 1]$, then $y = 2x - 1 \in [-1, 1]$ so $p(2x-1) = p(y) = \sum_{n=0}^{\infty} a_n y^n$ converges.

In the following exercises, suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfies $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ where $a_n \geq 0$ for each n. State whether each series converges on the full interval $(-1, 1)$, or if there is not enough information to draw a conclusion. Use the comparison test when appropriate.

52) $\sum_{n=0}^{\infty} a_n x^{2n}$

53) $\sum_{n=0}^{\infty} a_{2n} x^{2n}$

Answer

Converges on $(-1, 1)$ by the ratio test

54) $\sum_{n=0}^{\infty} a_{2n} x^n$ (Hint:$x = \pm\sqrt{x^2}$)

55) $\sum_{n=0}^{\infty} a_{n^2} x^{n^2}$ (Hint: Let $b_k = a_k$ if $k = n^2$ for some n, otherwise $b_k = 0$.)

Answer

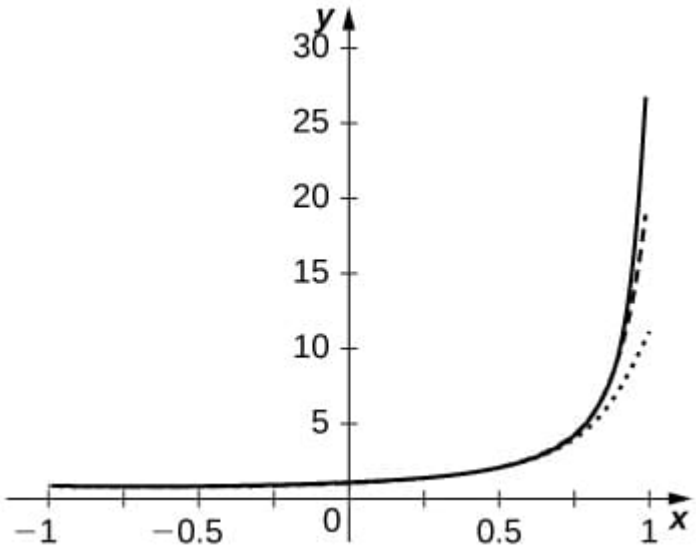
Consider the series $\sum b_k x^k$ where $b_k = a_k$ if $k = n^2$ and $b_k = 0$ otherwise. Then $b_k \leq a_k$ and so the series converges on $(-1, 1)$ by the comparison test.

56) Suppose that $p(x)$ is a polynomial of degree N. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} p(n) x^n$.

57) [T] Plot the graphs of $\frac{1}{1-x}$ and of the partial sums $S_N = \sum_{n=0}^{N} x^n$ for $n = 10, 20, 30$ on the interval $[-0.99, 0.99]$. Comment on the approximation of $\frac{1}{1-x}$ by S_N near $x = -1$ and near $x = 1$ as N increases.

Answer

The approximation is more accurate near $x = -1$. The partial sums follow $\frac{1}{1-x}$ more closely as N increases but are never accurate near $x = 1$ since the series diverges there.

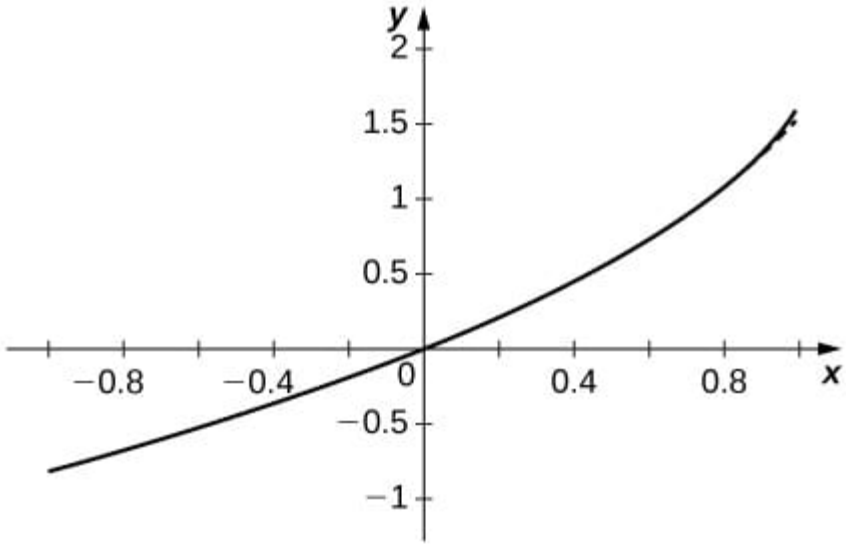


58) [T] Plot the graphs of $-\ln(1-x)$ and of the partial sums $S_N = \sum_{n=1}^{N} \frac{x^n}{n}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

59) [T] Plot the graphs of the partial sums $S_n = \sum_{n=1}^{N} \frac{x^n}{n^2}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

Answer

The approximation appears to stabilize quickly near both $x = \pm 1$.

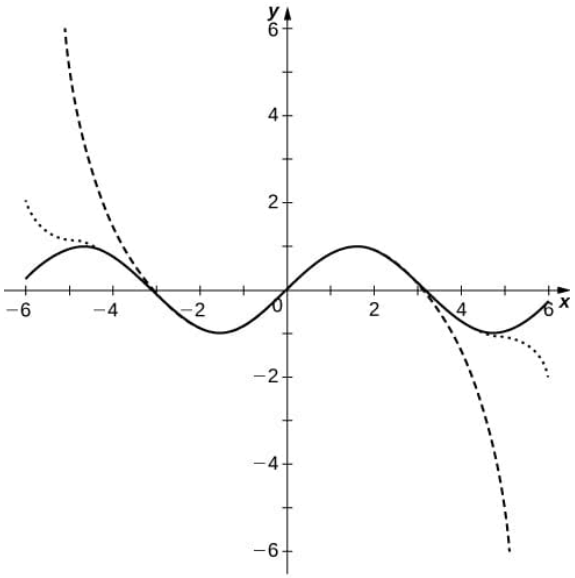


60) [T] Plot the graphs of the partial sums $S_N = \sum_{n=1}^{N} (\sin n) x^n$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

61) [T] Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $\sin x$ as N increases.

Answer

The polynomial curves have roots close to those of $\sin x$ up to their degree and then the polynomials diverge from $\sin x$.



62) [T] Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \dfrac{x^{2n}}{(2n)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $\cos x$ as N increases.

Contributors

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10.2: PROPERTIES OF POWER SERIES

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10.2E: Exercises for Section 10.2

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10.2E: Exercises for Section 10.2

1) If $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and $g(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$, find the power series of $\frac{1}{2}\big(f(x)+g(x)\big)$ and of $\frac{1}{2}\big(f(x)-g(x)\big)$.

Answer

$$\frac{1}{2}\big(f(x)+g(x)\big)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\text{ and }\frac{1}{2}\big(f(x)-g(x)\big)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}.$$

2) If $C(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ and $S(x)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$, find the power series of $C(x)+S(x)$ and of $C(x)-S(x)$.

In exercises 3 - 6, use partial fractions to find the power series of each function.

3) $\frac{4}{(x-3)(x+1)}$

Answer

$$\frac{4}{(x-3)(x+1)}=\frac{1}{x-3}-\frac{1}{x+1}=-\frac{1}{3(1-\frac{x}{3})}-\frac{1}{1-(-x)}=-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n-\sum_{n=0}^{\infty}(-1)^nx^n$$
$$=\sum_{n=0}^{\infty}\left((-1)^{n+1}-\frac{1}{3n+1}\right)x^n$$

4) $\frac{3}{(x+2)(x-1)}$

5) $\frac{5}{(x^2+4)(x^2-1)}$

Answer

$$\frac{5}{(x^2+4)(x^2-1)}=\frac{1}{x^2-1}-\frac{1}{4}\frac{1}{1+(\frac{x}{2})^2}=-\sum_{n=0}^{\infty}x^{2n}-\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}\left((-1)+(-1)^{n+1}\frac{1}{2^{n+2}}\right)x^{2n}$$

6) $\frac{30}{(x^2+1)(x^2-9)}$

In exercises 7 - 10, express each series as a rational function.

7) $\sum_{n=1}^{\infty}\frac{1}{x^n}$

Answer

$$\frac{1}{x}\sum_{n=0}^{\infty}\frac{1}{x^n}=\frac{1}{x}\cdot\frac{1}{1-\frac{1}{x}}=\frac{1}{x-1}$$

8) $\sum_{n=1}^{\infty}\frac{1}{x^{2n}}$

9) $\sum_{n=1}^{\infty}\frac{1}{(x-3)^{2n-1}}$

Answer

$$\frac{1}{x-3}\cdot\frac{1}{1-\frac{1}{(x-3)^2}}=\frac{x-3}{(x-3)^2-1}$$

10) $\sum_{n=1}^{\infty}\left(\frac{1}{(x-3)^{2n-1}}-\frac{1}{(x-2)^{2n-1}}\right)$

Exercises 11 - 16 explore applications of *annuities*.

11) Calculate the present values P of an annuity in which \$10,000 is to be paid out annually for a period of 20 years, assuming interest rates of $r=0.03,\ r=0.05$, and $r=0.07$.

Answer

$P=P_1+\cdots+P_{20}$ where $P_k=10,000\frac{1}{(1+r)^k}$. Then $P=10,000\sum_{k=1}^{20}\frac{1}{(1+r)^k}=10,000\frac{1-(1+r)^{-20}}{r}$. When $r=0.03,\ P\approx 10,000\times 14.8775=148,775$. When $r=0.05,\ P\approx 10,000\times 12.4622=124,622$. When $r=0.07,\ P\approx 105,940$.

12) Calculate the present values P of annuities in which \$9,000 is to be paid out annually perpetually, assuming interest rates of $r=0.03,\ r=0.05$ and $r=0.07$.

13) Calculate the annual payouts C to be given for 20 years on annuities having present value \$100,000 assuming respective interest rates of $r=0.03,\ r=0.05$, and $r=0.07$.

Answer

In general, $P=\frac{C(1-(1+r)^{-N})}{r}$ for N years of payouts, or $C=\frac{Pr}{1-(1+r)^{-N}}$. For $N=20$ and $P=100,000$, one has $C=6721.57$ when $r=0.03;\ C=8024.26$ when $r=0.05$; and $C\approx 9439.29$ when $r=0.07$.

14) Calculate the annual payouts C to be given perpetually on annuities having present value \$100,000 assuming respective interest rates of $r=0.03,\ r=0.05$, and $r=0.07$.

15) Suppose that an annuity has a present value $P=1$ **million dollars.** What interest rate r would allow for perpetual annual payouts of \$50,000?

Answer

In general, $P=\frac{C}{r}$. Thus, $r=\frac{C}{P}=5\times\frac{10^4}{10^6}=0.05$.

16) Suppose that an annuity has a present value $P=10$ **million dollars.** What interest rate r would allow for perpetual annual payouts of \$100,000?

In exercises 17 - 20, express the sum of each power series in terms of geometric series, and then express the sum as a rational function.

17) $x+x^2-x^3+x^4+x^5-x^6+\cdots$ (Hint: Group powers $x^{3k},\ x^{3k-1}$, and x^{3k-2}.)

Answer

$$(x+x^2-x^3)(1+x^3+x^6+\cdots)=\frac{x+x^2-x^3}{1-x^3}$$

18) $x+x^2-x^3-x^4+x^5+x^6-x^7-x^8+\cdots$ (Hint: Group powers $x^{4k},\ x^{4k-1}$, etc.)

19) $x-x^2-x^3+x^4-x^5-x^6+x^7-\cdots$ (Hint: Group powers $x^{3k},\ x^{3k-1}$, and x^{3k-2}.)

Answer

$$(x-x^2-x^3)(1+x^3+x^6+\cdots)=\frac{x-x^2-x^3}{1-x^3}$$

20) $\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{16}+\frac{x^5}{32}-\frac{x^6}{64}+\cdots$ (Hint: Group powers $\left(\frac{x}{2}\right)^{3k},\ \left(\frac{x}{2}\right)^{3k-1}$, and $\left(\frac{x}{2}\right)^{3k-2}$.)

In exercises 21 - 24, find the power series of $f(x)g(x)$ given f and g as defined.

21) $f(x) = 2\sum_{n=0}^{\infty} x^n, g(x) = \sum_{n=0}^{\infty} nx^n$

Answer

$a_n = 2,\ b_n = n$ so $c_n = \sum_{k=0}^{n} b_k a_{n-k} = 2\sum_{k=0}^{n} k = (n)(n+1)$ and $f(x)g(x) = \sum_{n=1}^{\infty} n(n+1)x^n$

22) $f(x) = \sum_{n=1}^{\infty} x^n,\ \ g(x) = \sum_{n=1}^{\infty} \frac{1}{n}x^n$. Express the coefficients of $f(x)g(x)$ in terms of $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

23) $f(x) = g(x) = \sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n$

Answer

$a_n = b_n = 2^{-n}$ so $c_n = \sum_{k=1}^{n} b_k a_{n-k} = 2^{-n}\sum_{k=1}^{n} 1 = \frac{n}{2^n}$ and $f(x)g(x) = \sum_{n=1}^{\infty} n\left(\frac{x}{2}\right)^n$

24) $f(x) = g(x) = \sum_{n=1}^{\infty} nx^n$

In exercises 25 - 26, differentiate the given series expansion of f term-by-term to obtain the corresponding series expansion for the derivative of f.

25) $f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

Answer

The derivative of f is $-\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^n n x^{n-1}$.

26) $f(x) = \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$

In exercises 27 - 28, integrate the given series expansion of f term-by-term from zero to x to obtain the corresponding series expansion for the indefinite integral of f.

27) $f(x) = \frac{2x}{(1+x^2)^2} = \sum_{n=1}^{\infty} (-1)^n (2n) x^{2n-1}$

Answer

The indefinite integral of f is $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

28) $f(x) = \frac{2x}{1+x^2} = 2\sum_{n=0}^{\infty} (-1)^n x^{2n+1}$

In exercises 29 - 32, evaluate each infinite series by identifying it as the value of a derivative or integral of geometric series.

29) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$ as $f\,'(\frac{1}{2})$ where $f(x) = \sum_{n=0}^{\infty} x^n$.

Answer

$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x};\ \ f\,'(\frac{1}{2}) = \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{d}{dx}(1-x)^{-1}\Big|_{x=1/2} = \frac{1}{(1-x)^2}\Big|_{x=1/2} = 4$ so $\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$.

30) Evaluate $\sum_{n=1}^{\infty} \frac{n}{3^n}$ as $f\,'(\frac{1}{3})$ where $f(x) = \sum_{n=0}^{\infty} x^{6n}$.

31) Evaluate $\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}$ as $f''\left(\frac{1}{2}\right)$ where $f(x) = \sum_{n=0}^{\infty} x^n$.

Answer

$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$; $f''\left(\frac{1}{2}\right) = \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n-2}} = \frac{d^2}{dx^2}(1-x)^{-1}\Big|_{x=1/2} = \frac{2}{(1-x)^3}\Big|_{x=1/2} = 16$ so $\sum_{n=2}^{\infty} n\frac{(n-1)}{2^n} = 4$.

32) Evaluate $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ as $\int_0^1 f(t)\ dt$ where $f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$.

In exercises 33 - 39, given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, use term-by-term differentiation or integration to find power series for each function centered at the given point.

33) $f(x) = \ln x$ centered at $x = 1$ (Hint: $x = 1 - (1-x)$)

Answer

$f(x) \quad = \ln x \quad = \int \frac{1}{x} \quad dx \quad = \int \frac{1}{1-(1-x)} \quad dx \quad = \int \sum_{n=0}^{\infty} (1-x)^n \quad dx \quad = \int \sum_{n=0}^{\infty} (-1)^n (x-1)^n \quad dx$
$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$

34) $\ln(1-x)$ at $x = 0$

35) $\ln(1-x^2)$ at $x = 0$

Answer

$-\int_{t=0}^{x^2} \frac{1}{1-t} dt = -\sum_{n=0}^{\infty} \int_0^{x^2} t^n dx = -\sum_{n=0}^{\infty} \frac{x^{2(n+1)}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^{2n}}{n}$

36) $f(x) = \frac{2x}{(1-x^2)^2}$ at $x = 0$

Answer

$f(x) = \frac{2x}{(1-x^2)^2} = \frac{d}{dx}\frac{1}{1-x^2} = \frac{d}{dx}\sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} 2nx^{2n-1}$

37) $f(x) = \tan^{-1}(x^2)$ at $x = 0$

Answer

$\int_0^{x^2} \frac{dt}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n \int_0^{x^2} t^{2n} dt = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1}\Big|_{t=0}^{x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$

38) $f(x) = \ln(1+x^2)$ at $x = 0$

39) $f(x) = \int_0^x \ln t\ dt$ where $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$

Answer

Term-by-term integration gives $\int_0^x \ln t\ \ dt\ = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^{n+1}}{n(n+1)}\ = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n} - \frac{1}{n+1}\right)(x-1)^{n+1}$
$= (x-1)\ln x + \sum_{n=2}^{\infty} (-1)^n \frac{(x-1)^n}{n} = x\ln x - x.$

40) [T] Evaluate the power series expansion $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ at $x = 1$ to show that $\ln(2)$ is the sum of the alternating harmonic series. Use the alternating series test to determine how many terms of the sum are needed to estimate $\ln(2)$ accurate

to within 0.001, and find such an approximation.

41) [T] Subtract the infinite series of $\ln(1-x)$ from $\ln(1+x)$ to get a power series for $\ln\left(\frac{1+x}{1-x}\right)$. Evaluate at $x=\frac{1}{3}$. What is the smallest N such that the Nth partial sum of this series approximates $\ln(2)$ with an error less than 0.001?

Answer

We have $\ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}$ so $\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$. Thus, $\ln\left(\frac{1+x}{1-x}\right)=\sum_{n=1}^{\infty}\left(1+(-1)^{n-1}\right)\frac{x^n}{n}$ $=2\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1}$. When $x=\frac{1}{3}$ we obtain $\ln(2)=2\sum_{n=1}^{\infty}\frac{1}{3^{2n-1}(2n-1)}$. We have $2\sum_{n=1}^{3}\frac{1}{3^{2n-1}(2n-1)}=0.69300\ldots$, while $2\sum_{n=1}^{4}\frac{1}{3^{2n-1}(2n-1)}=0.69313\ldots$ and $\ln(2)=0.69314\ldots$; therefore, $N=4$.

In exercises 42 - 45, using a substitution if indicated, express each series in terms of elementary functions and find the radius of convergence of the sum.

42) $\sum_{k=0}^{\infty}(x^k-x^{2k+1})$

43) $\sum_{k=1}^{\infty}\frac{x^{3k}}{6k}$

Answer

$\sum_{k=1}^{\infty}\frac{x^k}{k}=-\ln(1-x)$ so $\sum_{k=1}^{\infty}\frac{x^{3k}}{6k}=-\frac{1}{6}\ln(1-x^3)$. The radius of convergence is equal to 1 by the ratio test.

44) $\sum_{k=1}^{\infty}(1+x^2)^{-k}$ using $y=\frac{1}{1+x^2}$

45) $\sum_{k=1}^{\infty}2^{-kx}$ using $y=2^{-x}$

Answer

If $y=2^{-x}$, then $\sum_{k=1}^{\infty}y^k=\frac{y}{1-y}=\frac{2^{-x}}{1-2^{-x}}=\frac{1}{2^x-1}$. If $a_k=2^{-kx}$, then $\frac{a_{k+1}}{a_k}=2^{-x}<1$ when $x>0$. So the series converges for all $x>0$.

46) Show that, up to powers x^3 and y^3, $E(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ satisfies $E(x+y)=E(x)E(y)$.

47) Differentiate the series $E(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ term-by-term to show that $E(x)$ is equal to its derivative.

Answer

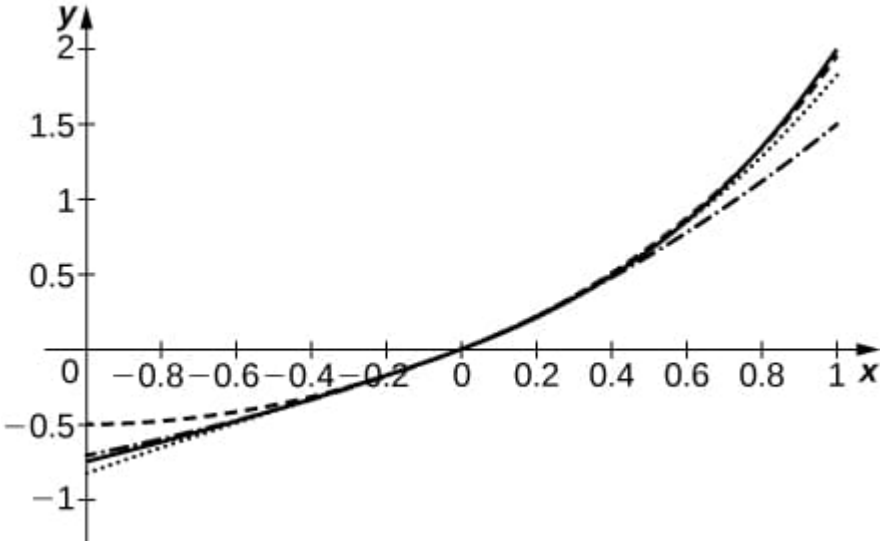
Answers will vary.

48) Show that if $f(x)=\sum_{n=0}^{\infty}a_nx^n$ is a sum of even powers, that is, $a_n=0$ if n is odd, then $F=\int_0^x f(t)\,dt$ is a sum of odd powers, while if I is a sum of odd powers, then F is a sum of even powers.

49) [T] Suppose that the coefficients an of the series $\sum_{n=0}^{\infty}a_nx^n$ are defined by the recurrence relation $a_n=\frac{a_{n-1}}{n}+\frac{a_{n-2}}{n(n-1)}$. For $a_0=0$ and $a_1=1$, compute and plot the sums $S_N=\sum_{n=0}^{N}a_nx^n$ for $N=2,3,4,5$ on $[-1,1]$.

Answer

The solid curve is S_5. The dashed curve is S_2, dotted is S_3, and dash-dotted is S_4



50) [T] Suppose that the coefficients an of the series $\sum_{n=0}^{\infty} a_n x^n$ are defined by the recurrence relation $a_n = \frac{a_{n-1}}{\sqrt{n}} - \frac{a_{n-2}}{\sqrt{n(n-1)}}$. For $a_0 = 1$ and $a_1 = 0$, compute and plot the sums $S_N = \sum_{n=0}^{N} a_n x^n$ for $N = 2, 3, 4, 5$ on $[-1, 1]$.

51) [T] Given the power series expansion $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$, determine how many terms N of the sum evaluated at $x = -1/2$ are needed to approximate $\ln(2)$ accurate to within $1/1000$. Evaluate the corresponding partial sum $\sum_{n=1}^{N} (-1)^{n-1} \frac{x^n}{n}$.

Answer

When $x = -\frac{1}{2}$, $-\ln(2) = \ln\left(\frac{1}{2}\right) = -\sum_{n=1}^{\infty} \frac{1}{n2^n}$. Since $\sum_{n=11}^{\infty} \frac{1}{n2^n} < \sum_{n=11}^{\infty} \frac{1}{2^n} = \frac{1}{2^{10}}$, one has $\sum_{n=1}^{10} \frac{1}{n2^n} = 0.69306\ldots$ whereas $\ln(2) = 0.69314\ldots$; therefore, $N = 10$.

52) [T] Given the power series expansion $\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$, use the alternating series test to determine how many terms N of the sum evaluated at $x = 1$ are needed to approximate $\tan^{-1}(1) = \frac{\pi}{4}$ accurate to within $1/1000$. Evaluate the corresponding partial sum $\sum_{k=0}^{N} (-1)^k \frac{x^{2k+1}}{2k+1}$.

53) [T] Recall that $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. Assuming an exact value of $\frac{1}{\sqrt{3}}$), estimate $\frac{\pi}{6}$ by evaluating partial sums $S_N\left(\frac{1}{\sqrt{3}}\right)$ of the power series expansion $\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$ at $x = \frac{1}{\sqrt{3}}$. What is the smallest number N such that $6S_N\left(\frac{1}{\sqrt{3}}\right)$ approximates π accurately to within 0.001? How many terms are needed for accuracy to within 0.00001?

Answer

$6S_N\left(\frac{1}{\sqrt{3}}\right) = 2\sqrt{3}\sum_{n=0}^{N} (-1)^n \frac{1}{3^n(2n+1).}$ One has $\pi - 6S_4\left(\frac{1}{\sqrt{3}}\right) = 0.00101\ldots$ and $\pi - 6S_5\left(\frac{1}{\sqrt{3}}\right) = 0.00028\ldots$ so $N = 5$ is the smallest partial sum with accuracy to within 0.001. Also, $\pi - 6S_7\left(\frac{1}{\sqrt{3}}\right) = 0.00002\ldots$ while $\pi - 6S_8\left(\frac{1}{\sqrt{3}}\right) = -0.000007\ldots$ so $N = 8$ is the smallest N to give accuracy to within 0.00001.

Contributors

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10.3: TAYLOR AND MACLAURIN SERIES

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10.3E: Exercises for Section 10.3

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10.3E: Exercises for Section 10.3

Taylor Polynomials

In exercises 1 - 8, find the Taylor polynomials of degree two approximating the given function centered at the given point.

1) $f(x) = 1 + x + x^2$ at $a = 1$

2) $f(x) = 1 + x + x^2$ at $a = -1$

Answer

$f(-1) = 1; \;\; f\,'(-1) = -1; \;\; f''(-1) = 2; \quad p_2(x) = 1 - (x+1) + (x+1)^2$

3) $f(x) = \cos(2x)$ at $a = \pi$

4) $f(x) = \sin(2x)$ at $a = \frac{\pi}{2}$

Answer

$f\,'(x) = 2\cos(2x); \;\; f''(x) = -4\sin(2x); \quad p_2(x) = -2(x - \frac{\pi}{2})$

5) $f(x) = \sqrt{x}$ at $a = 4$

6) $f(x) = \ln x$ at $a = 1$

Answer

$f\,'(x) = \dfrac{1}{x}; \;\; f''(x) = -\dfrac{1}{x^2}; \quad p_2(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2$

7) $f(x) = \dfrac{1}{x}$ at $a = 1$

8) $f(x) = e^x$ at $a = 1$

Answer

$p_2(x) = e + e(x-1) + \dfrac{e}{2}(x-1)^2$

Taylor Remainder Theorem

In exercises 9 - 14, verify that the given choice of n in the remainder estimate $|R_n| \le \dfrac{M}{(n+1)!}(x-a)^{n+1}$, where M is the maximum value of $|\, f^{(n+1)}(z)\,|$ on the interval between a and the indicated point, yields $|R_n| \le \frac{1}{1000}$. Find the value of the Taylor polynomial p_n of f at the indicated point.

9) [T] $\sqrt{10}; \;\; a = 9, \;\; n = 3$

10) [T] $(28)^{1/3}; \;\; a = 27, \;\; n = 1$

Answer

$\dfrac{d^2}{dx^2}x^{1/3} = -\dfrac{2}{9x^{5/3}} \ge -0.00092\ldots$ when $x \ge 28$ so the remainder estimate applies to the linear approximation $x^{1/3} \approx p_1(27) = 3 + \dfrac{x-27}{27}$, which gives $(28)^{1/3} \approx 3 + \frac{1}{27} = 3.0\bar{3}7$, while $(28)^{1/3} \approx 3.03658.$

11) [T] $\sin(6); \;\; a = 2\pi, \;\; n = 5$

12) [T] $e^2; \;\; a = 0, \;\; n = 9$

Answer

Using the estimate $\frac{2^{10}}{10!} < 0.000283$ we can use the Taylor expansion of order 9 to estimate e^x at $x = 2$. as $e^2 \approx p_9(2) = 1 + 2 + \frac{2^2}{2} + \frac{2^3}{6} + \cdots + \frac{2^9}{9!} = 7.3887\ldots$ whereas $e^2 \approx 7.3891$.

13) [T] $\cos(\frac{\pi}{5})$; $a = 0$, $n = 4$

14) [T] $\ln(2)$; $a = 1$, $n = 1000$

Answer

Since $\frac{d^n}{dx^n}(\ln x) = (-1)^{n-1}\frac{(n-1)!}{x^n}, R_{1000} \approx \frac{1}{1001}$. One has $p_{1000}(1) = \sum_{n=1}^{1000}\frac{(-1)^{n-1}}{n} \approx 0.6936$ whereas $\ln(2) \approx 0.6931\cdots$.

Approximating Definite Integrals Using Taylor Series

15) Integrate the approximation $\sin t \approx t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040}$ evaluated at πt to approximate $\int_0^1 \frac{\sin \pi t}{\pi t}\, dt$.

16) Integrate the approximation $e^x \approx 1 + x + \frac{x^2}{2} + \cdots + \frac{x^6}{720}$ evaluated at $-x^2$ to approximate $\int_0^1 e^{-x^2}\, dx$.

Answer

$\int_0^1 \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \frac{x^{12}}{720}\right)\, dx = 1 - \frac{1^3}{3} + \frac{1^5}{10} - \frac{1^7}{42} + \frac{1^9}{9\cdot 24} - \frac{1^{11}}{120\cdot 11} + \frac{1^{13}}{720\cdot 13} \approx 0.74683$ whereas $\int_0^1 e^{-x^2}dx \approx 0.74682$.

More Taylor Remainder Theorem Problems

In exercises 17 - 20, find the smallest value of n such that the remainder estimate $|R_n| \le \frac{M}{(n+1)!}(x-a)^{n+1}$, where M is the maximum value of $|\, f^{(n+1)}(z)\,|$ on the interval between a and the indicated point, yields $|R_n| \le \frac{1}{1000}$ on the indicated interval.

17) $f(x) = \sin x$ on $[-\pi, \pi]$, $a = 0$

18) $f(x) = \cos x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $a = 0$

Answer

Since $f^{(n+1)}(z)$ is $\sin z$ or $\cos z$, we have $M = 1$. Since $|x - 0| \le \frac{\pi}{2}$, we seek the smallest n such that $\frac{\pi^{n+1}}{2^{n+1}(n+1)!} \le 0.001$. The smallest such value is $n = 7$. The remainder estimate is $R_7 \le 0.00092$.

19) $f(x) = e^{-2x}$ on $[-1, 1], a = 0$

20) $f(x) = e^{-x}$ on $[-3, 3], a = 0$

Answer

Since $f^{(n+1)}(z) = \pm e^{-z}$ one has $M = e^3$. Since $|x - 0| \le 3$, one seeks the smallest n such that $\frac{3^{n+1}e^3}{(n+1)!} \le 0.001$. The smallest such value is $n = 14$. The remainder estimate is $R_{14} \le 0.000220$.

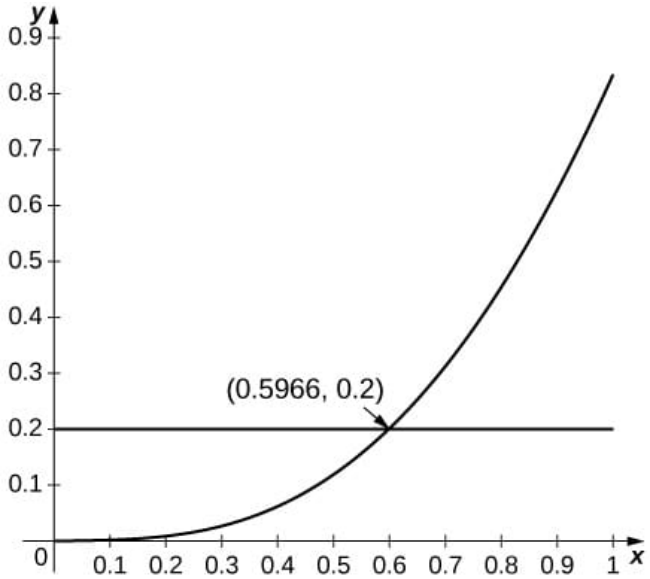
In exercises 21 - 24, the maximum of the right-hand side of the remainder estimate $|R_1| \le \frac{max|f''(z)|}{2}R^2$ on $[a - R, a + R]$ occurs at a or $a \pm R$. Estimate the maximum value of R such that $\frac{max|f''(z)|}{2}R^2 \le 0.1$ on $[a - R, a + R]$ by plotting this maximum as a function of R.

21) [T] e^x approximated by $1+x,\ \ a=0$

22) [T] $\sin x$ approximated by $x,\ \ a=0$

Answer

Since $\sin x$ is increasing for small x and since $\frac{d^2}{dx^2}(\sin x) = -\sin x$, the estimate applies whenever $R^2 \sin(R) \leq 0.2$, which applies up to $R = 0.596$.

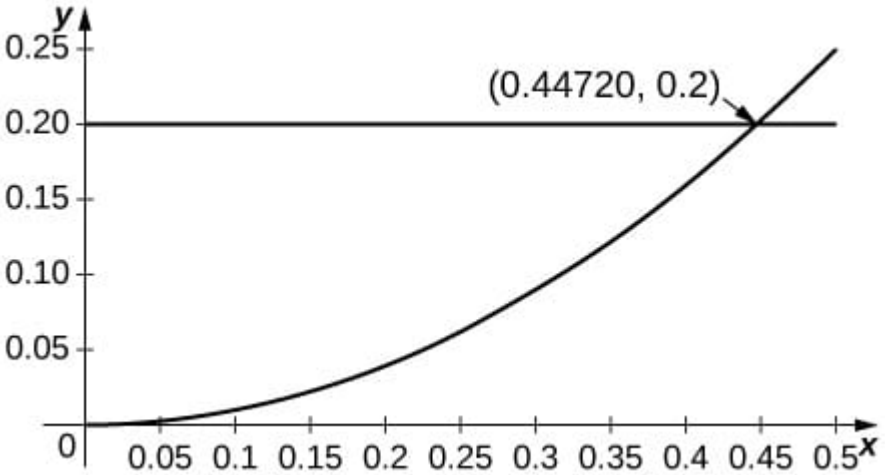


23) [T] $\ln x$ approximated by $x-1,\ \ a=1$

24) [T] $\cos x$ approximated by $1,\ \ a=0$

Answer

Since the second derivative of $\cos x$ is $-\cos x$ and since $\cos x$ is decreasing away from $x=0$, the estimate applies when $R^2 \cos R \leq 0.2$ or $R \leq 0.447$.



Taylor Series

In exercises 25 - 35, find the Taylor series of the given function centered at the indicated point.

25) $f(x) = x^4$ at $a = -1$

26) $f(x) = 1 + x + x^2 + x^3$ at $a = -1$

Answer

$(x+1)^3 - 2(x+1)^2 + 2(x+1)$

27) $f(x) = \sin x$ at $a = \pi$

28) $f(x) = \cos x$ at $a = 2\pi$

Answer

Values of derivatives are the same as for $x = 0$ so $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-2\pi)^{2n}}{(2n)!}$

29) $f(x) = \sin x$ at $x = \frac{\pi}{2}$

30) $f(x) = \cos x$ at $x = \frac{\pi}{2}$

Answer

$\cos(\frac{\pi}{2}) = 0, \ \ -\sin(\frac{\pi}{2}) = -1$ so $\cos x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$, which is also $-\cos(x - \frac{\pi}{2})$.

31) $f(x) = e^x$ at $a = -1$

32) $f(x) = e^x$ at $a = 1$

Answer

The derivatives are $f^{(n)}(1) = e$, so $e^x = e \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$.

33) $f(x) = \frac{1}{(x-1)^2}$ at $a = 0$ (Hint: Differentiate the Taylor Series for $\frac{1}{1-x}$.)

34) $f(x) = \frac{1}{(x-1)^3}$ at $a = 0$

Answer

$$\frac{1}{(x-1)^3} = -\frac{1}{2}\frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = -\sum_{n=0}^{\infty}\left(\frac{(n+2)(n+1)x^n}{2}\right)$$

35) $F(x) = \int_0^x \cos(\sqrt{t}) \ dt; \quad$ where $f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{(2n)!}$ at a=0 (Note: f is the Taylor series of $\cos(\sqrt{t})$.)

In exercises 36 - 44, compute the Taylor series of each function around $x = 1$.

36) $f(x) = 2 - x$

Answer

$2 - x = 1 - (x-1)$

37) $f(x) = x^3$

38) $f(x) = (x-2)^2$

Answer

$((x-1)-1)^2 = (x-1)^2 - 2(x-1) + 1$

39) $f(x) = \ln x$

40) $f(x) = \dfrac{1}{x}$

Answer

$$\frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

41) $f(x) = \dfrac{1}{2x - x^2}$

42) $f(x) = \dfrac{x}{4x - 2x^2 - 1}$

Answer

$$x \sum_{n=0}^{\infty} 2^n (1-x)^{2n} = \sum_{n=0}^{\infty} 2^n (x-1)^{2n+1} + \sum_{n=0}^{\infty} 2^n (x-1)^{2n}$$

43) $f(x) = e^{-x}$

44) $f(x) = e^{2x}$

Answer

$$e^{2x} = e^{2(x-1)+2} = e^2 \sum_{n=0}^{\infty} \frac{2^n (x-1)^n}{n!}$$

Maclaurin Series

[T] In exercises 45 - 48, identify the given series $\sum_{n=0}^{\infty} a_n$ as the Maclaurin series of some function $f(x)$ at some x. Approximate the value of $f(x)$ using the given series $S_{10} = \sum_{n=0}^{10} a_n$.

45) $\sum_{n=0}^{\infty} \dfrac{1}{n!}$

46) $\sum_{n=0}^{\infty} \dfrac{2^n}{n!}$

Answer

$$f(x) = e^x; \quad x = 2; \quad S_{10} = \frac{34,913}{4725} \approx 7.3889947 \approx e^2$$

47) $\sum_{n=0}^{\infty} \dfrac{(-1)^n (2\pi)^{2n}}{(2n)!}$

48) $\sum_{n=0}^{\infty} \dfrac{(-1)^n (2\pi)^{2n+1}}{(2n+1)!}$

Answer

$$f(x) = sin(x); \quad x = 2\pi; \quad S_{10} = 8.27 \times 10^{-5} \approx sin(2\pi)$$

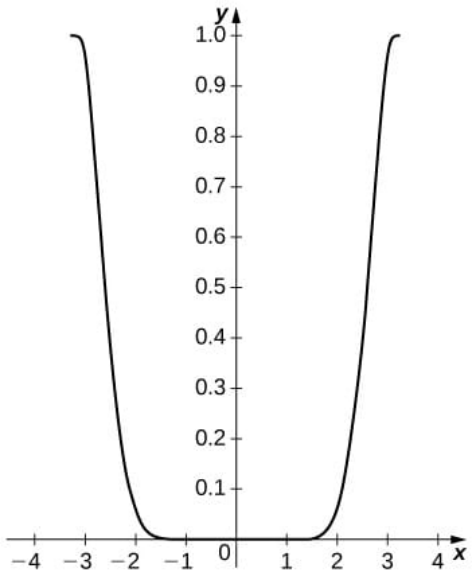
In exercises 49 - 52 use the functions $S_5(x) = x - \dfrac{x^3}{6} + \dfrac{x^5}{120}$ and $C_4(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$ on $[-\pi, \pi]$.

49) [T] Plot $\sin^2 x - (S_5(x))^2$ on $[-\pi, \pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $\sin x$.

50) [T] Plot $\cos^2 x - (C_4(x))^2$ on $[-\pi, \pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $\cos x$.

Answer

The difference is small on the interior of the interval but approaches 1 near the endpoints. The remainder estimate is $|R_4| = \frac{\pi^5}{120} \approx 2.552$.

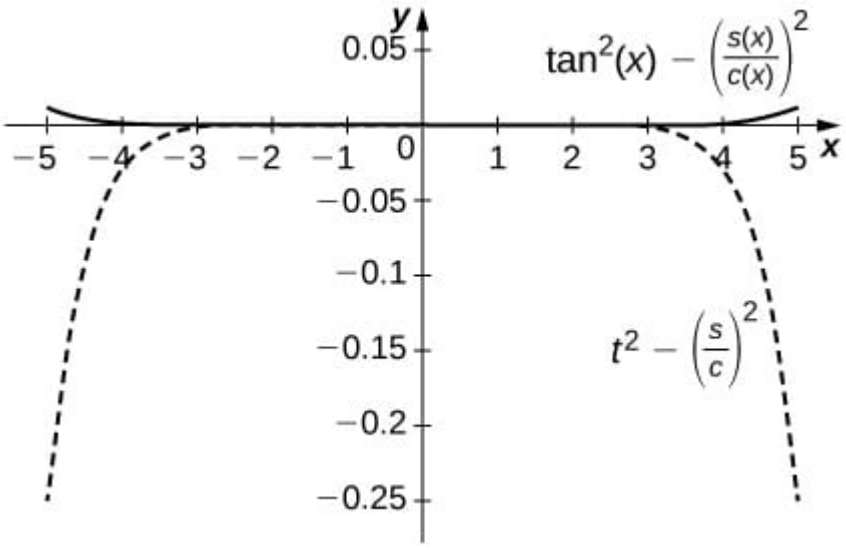


51) [T] Plot $|2S_5(x)C_4(x) - \sin(2x)|$ on $[-\pi, \pi]$.

52) [T] Compare $\dfrac{S_5(x)}{C_4(x)}$ on $[-1, 1]$ to $\tan x$. Compare this with the Taylor remainder estimate for the approximation of $\tan x$ by $x + \dfrac{x^3}{3} + \dfrac{2x^5}{15}$.

Answer

The difference is on the order of 10^{-4} on $[-1, 1]$ while the Taylor approximation error is around 0.1 near ± 1. The top curve is a plot of $\tan^2 x - \left(\dfrac{S_5(x)}{C_4(x)}\right)^2$ and the lower dashed plot shows $t^2 - \left(\dfrac{S_5}{C_4}\right)^2$.



53) [T] Plot $e^x - e_4(x)$ where $e_4(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24}$ on $[0, 2]$. Compare the maximum error with the Taylor remainder

estimate.

54) (Taylor approximations and root finding.) Recall that Newton's method $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ approximates solutions of $f(x) = 0$ near the input x_0.

a. If f and g are inverse functions, explain why a solution of $g(x) = a$ is the value $f(a)$ of f.

b. Let $p_N(x)$ be the Nth degree Maclaurin polynomial of e^x. Use Newton's method to approximate solutions of $p_N(x) - 2 = 0$ for $N = 4, 5, 6$.

c. Explain why the approximate roots of $p_N(x) - 2 = 0$ are approximate values of $\ln(2)$.

Answer

a. Answers will vary.
b. The following are the x_n values after 10 iterations of Newton's method to approximation a root of $p_N(x) - 2 = 0$: for $N = 4, x = 0.6939...$; for $N = 5, x = 0.6932...$; for $N = 6, x = 0.69315...$; . (Note: $\ln(2) = 0.69314...$)
c. Answers will vary.

Evaluating Limits using Taylor Series

In exercises 55 - 58, use the fact that if $q(x) = \sum_{n=1}^{\infty} a_n(x-c)^n$ converges in an interval containing c, then $\lim_{x\to c} q(x) = a_0$ to evaluate each limit using Taylor series.

55) $\lim_{x\to 0} \dfrac{\cos x - 1}{x^2}$

56) $\lim_{x\to 0} \dfrac{\ln(1-x^2)}{x^2}$

Answer

$$\frac{\ln(1-x^2)}{x^2} \to -1$$

57) $\lim_{x\to 0} \dfrac{e^{x^2} - x^2 - 1}{x^4}$

58) $\lim_{x\to 0^+} \dfrac{\cos(\sqrt{x}) - 1}{2x}$

Answer

$$\frac{\cos(\sqrt{x}) - 1}{2x} \approx \frac{(1 - \frac{x}{2} + \frac{x^2}{4!} - \cdots) - 1}{2x} \to -\frac{1}{4}$$

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10.4: WORKING WITH TAYLOR SERIES

Learning Objectives

- Write the terms of the binomial series.
- Recognize the Taylor series expansions of common functions.
- Recognize and apply techniques to find the Taylor series for a function.
- Use Taylor series to solve differential equations.
- Use Taylor series to evaluate non-elementary integrals.

In the preceding section, we defined Taylor series and showed how to find the Taylor series for several common functions by explicitly calculating the coefficients of the Taylor polynomials. In this section we show how to use those Taylor series to derive Taylor series for other functions. We then present two common applications of power series. First, we show how power series can be used to solve differential equations. Second, we show how power series can be used to evaluate integrals when the antiderivative of the integrand cannot be expressed in terms of elementary functions. In one example, we consider $\int e^{-x^2}dx$, an integral that arises frequently in probability theory.

The Binomial Series

Our first goal in this section is to determine the Maclaurin series for the function $f(x) = (1+x)^r$ for all real numbers r. The Maclaurin series for this function is known as the **binomial series.** We begin by considering the simplest case: r is a nonnegative integer. We recall that, for $r = 0,\ 1,\ 2,\ 3,\ 4,\ \ f(x) = (1+x)^r$ can be written as

$$\begin{aligned} f(x) &= (1+x)^0 = 1, \\ f(x) &= (1+x)^1 = 1+x, \\ f(x) &= (1+x)^2 = 1+2x+x^2, \\ f(x) &= (1+x)^3 = 1+3x+3x^2+x^3 \\ f(x) &= (1+x)^4 = 1+4x+6x^2+4x^3+x^4. \end{aligned}$$

The expressions on the right-hand side are known as binomial expansions and the coefficients are known as binomial coefficients. More generally, for any nonnegative integer r, the binomial coefficient of x^n in the binomial expansion of $(1+x)^r$ is given by

$$\binom{r}{n} = \frac{r!}{n!(r-n)!} \tag{10.4.1}$$

and

$$\begin{aligned} f(x) &= (1+x)^r \\ &= \binom{r}{0} + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \cdots + \binom{r}{r-1}x^{r-1} + \binom{r}{r}x^r \\ &= \sum_{n=0}^{r} \binom{r}{n}x^n. \end{aligned} \tag{10.4.2}$$

For example, using this formula for $r = 5$, we see that

$$\begin{aligned} f(x) &= (1+x)^5 \\ &= \binom{5}{0}1 + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5 \\ &= \frac{5!}{0!5!}1 + \frac{5!}{1!4!}x + \frac{5!}{2!3!}x^2 + \frac{5!}{3!2!}x^3 + \frac{5!}{4!1!}x^4 + \frac{5!}{5!0!}x^5 \\ &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5. \end{aligned}$$

We now consider the case when the exponent r.

is any real number, not necessarily a nonnegative integer. If r is not a nonnegative integer, then $f(x) = (1+x)^r$ cannot be written as a finite polynomial. However, we can find a power series for f. Specifically, we look for the Maclaurin series for f. To do this, we find the derivatives of f and evaluate them at $x = 0$.

$$\begin{aligned} f(x) &= (1+x)^r & f(0) &= 1 \\ f'(x) &= r(1+x)^{r-1} & f'(0) &= r \\ f''(x) &= r(r-1)(1+x)^{r-2} & f''(0) &= r(r-1) \\ f'''(x) &= r(r-1)(r-2)(1+x)^{r-3} & f'''(0) &= r(r-1)(r-2) \\ f^{(n)}(x) &= r(r-1)(r-2)\cdots(r-n+1)(1+x)^{r-n} & f^{(n)}(0) &= r(r-1)(r-2)\cdots(r-n+1) \end{aligned}$$

We conclude that the coefficients in the binomial series are given by

$$\frac{f^{(n)}(0)}{n!} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}. \tag{10.4.3}$$

We note that if r is a nonnegative integer, then the $(r+1)^{\text{st}}$ derivative $f^{(r+1)}$ is the zero function, and the series terminates. In addition, if r is a nonnegative integer, then Equation 10.4.3 for the coefficients agrees with Equation 10.4.1 for the coefficients, and the formula for the binomial series agrees with Equation 10.4.2 for the finite binomial expansion. More generally, to denote the binomial coefficients for any real number r, we define

$$\binom{r}{n} = \frac{(r-1)(r-2)\cdots(r-n+1)}{n!}.$$

With this notation, we can write the binomial series for $(1+x)^r$ as

$$\sum_{n=0}^{\infty}\binom{r}{n}x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots. \tag{10.4.4}$$

We now need to determine the interval of convergence for the binomial series Equation 10.4.4. We apply the ratio test. Consequently, we consider

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|r(r-1)(r-2)\cdots(r-n)||x||^{n+1}}{(n+1)!} \cdot \frac{n}{|r(r-1)(r-2)\cdots(r-n+1)||x|^n} \\ &= \frac{|r-n||x|}{|n+1|} \end{aligned}$$

.

Since

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x| < 1$$

if and only if $|x| < 1$, we conclude that the interval of convergence for the binomial series is $(-1, 1)$. The behavior at the endpoints depends on r. It can be shown that for $r \geq 0$ the series converges at both endpoints; for $-1 < r < 0$, the series converges at $x = 1$ and diverges at $x = -1$; and for $r < -1$, the series diverges at both endpoints. The binomial series does converge to $(1+x)^r$ in $(-1, 1)$ for all real numbers r, but proving this fact by showing that the remainder $R_n(x) \to 0$ is difficult.

Definition: binomial series

For any real number r, the Maclaurin series for $f(x) = (1+x)^r$ is the binomial series. It converges to f for $|x| < 1$, and we write

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + r\frac{(r-1)\cdots(r-n+1)}{n!}x^n + \cdots$$

for $|x| < 1$.

We can use this definition to find the binomial series for $f(x) = \sqrt{1+x}$ and use the series to approximate $\sqrt{1.5}$.

Example 10.4.1: Finding Binomial Series

a. Find the binomial series for $f(x) = \sqrt{1+x}$.
b. Use the third-order Maclaurin polynomial $p_3(x)$ to estimate $\sqrt{1.5}$. Use Taylor's theorem to bound the error. Use a graphing utility to compare the graphs of f and p_3.

Solution

a. Here $r = \frac{1}{2}$. Using the definition for the binomial series, we obtain

Erroneous nesting of equation structures

b. From the result in part a. the third-order Maclaurin polynomial is

$$p_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3.$$

Therefore,

$$\sqrt{1.5} = \sqrt{1+0.5} \approx 1 + \frac{1}{2}(0.5) - \frac{1}{8}(0.5)^2 + \frac{1}{16}(0.5)^3 \approx 1.2266.$$

From Taylor's theorem, the error satisfies

$$R_3(0.5) = \frac{f^{(4)}(c)}{4!}(0.5)^4$$

for some c between 0 and 0.5. Since $f^{(4)}(x) = -\dfrac{15}{2^4(1+x)^{7/2}}$, and the maximum value of $|\ f^{(4)}(x)\ |$ on the interval $(0, 0.5)$ occurs at $x = 0$, we have

$$|R_3(0.5)| \leq \frac{15}{4!2^4}(0.5)^4 \approx 0.00244.$$

The function and the Maclaurin polynomial p_3 are graphed in Figure 10.4.1.

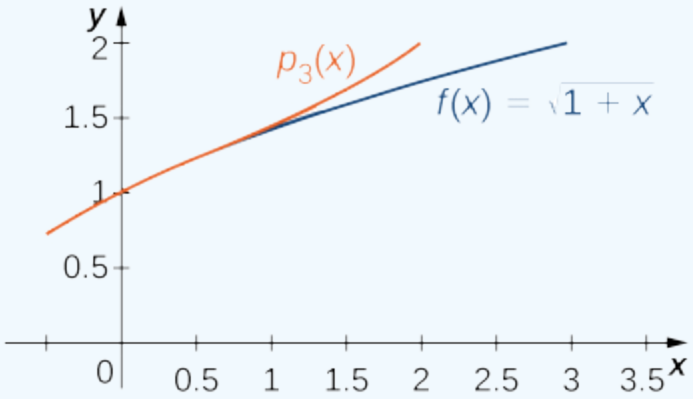


Figure 10.4.1: The third-order Maclaurin polynomial $p_3(x)$ provides a good approximation for $f(x) = \sqrt{1+x}$ for x near zero.

Exercise 10.4.1

Find the binomial series for $f(x) = \dfrac{1}{(1+x)^2}$.

Hint available at the end of this section.

Answer available at the end of this section.

Common Functions Expressed as Taylor Series

At this point, we have derived Maclaurin series for exponential, trigonometric, and logarithmic functions, as well as functions of the form $f(x) = (1+x)^r$. In Table 10.4.1, we summarize the results of these series. We remark that the convergence of the Maclaurin series for $f(x) = \ln(1+x)$ at the endpoint $x = 1$ and the Maclaurin series for $f(x) = \tan^{-1} x$ at the endpoints $x = 1$ and $x = -1$ relies on a more advanced theorem than we present here. (Refer to Abel's theorem for a discussion of this more technical point.)

Table 10.4.1: Maclaurin Series for Common Functions

Function	Maclaurin Series	Interval of Convergence
$f(x) = \dfrac{1}{1-x}$	$\displaystyle\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$f(x) = e^x$	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$f(x) = \sin x$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$f(x) = \cos x$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$f(x) = \ln(1+x)$	$\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$-1 < x < 1$
$f(x) = \tan^{-1} x$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 < x < 1$
$f(x) = (1+x)^r$	$\displaystyle\sum_{n=0}^{\infty} \binom{r}{n} x^n$	$-1 < x < 1$

Earlier in the chapter, we showed how you could combine power series to create new power series. Here we use these properties,

combined with the Maclaurin series in Table 10.4.1, to create Maclaurin series for other functions.

Example 10.4.2: Deriving Maclaurin Series from Known Series

Find the Maclaurin series of each of the following functions by using one of the series listed in Table 10.4.1.

a. $f(x) = \cos\sqrt{x}$
b. $f(x) = \sinh x$

Solution

a. Using the Maclaurin series for $\cos x$ we find that the Maclaurin series for $\cos\sqrt{x}$ is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots.$$

This series converges to $\cos\sqrt{x}$ for all x in the domain of $\cos\sqrt{x}$; that is, for all $x \geq 0$.

b. To find the Maclaurin series for $\sinh x$, we use the fact that

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Using the Maclaurin series for e^x, we see that the nth term in the Maclaurin series for $\sinh x$ is given by

$$\frac{x^n}{n!} - \frac{(-x)^n}{n!}.$$

For n even, this term is zero. For n odd, this term is $\frac{2x^n}{n!}$. Therefore, the Maclaurin series for $\sinh x$ has only odd-order terms and is given by

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

Exercise 10.4.2

Find the Maclaurin series for $\sin(x^2)$.

Hint available at the end of this section.

Answer available at the end of this section.

We also showed previously in this chapter how power series can be differentiated term by term to create a new power series. In Example 10.4.3, we differentiate the binomial series for $\sqrt{1+x}$ term by term to find the binomial series for $\frac{1}{\sqrt{1+x}}$. Note that we could construct the binomial series for $\frac{1}{\sqrt{1+x}}$ directly from the definition, but differentiating the binomial series for $\sqrt{1+x}$ is an easier calculation.

Example 10.4.3: Differentiating a Series to Find a New Series

Use the binomial series for $\sqrt{1+x}$ to find the binomial series for $\frac{1}{\sqrt{1+x}}$.

Solution

The two functions are related by

$$\frac{d}{dx}\sqrt{1+x}=\frac{1}{2\sqrt{1+x}},$$

so the binomial series for $\dfrac{1}{\sqrt{1+x}}$ is given by

$$\frac{1}{\sqrt{1+x}}=2\frac{d}{dx}\sqrt{1+x}=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}x^n.$$

Exercise 10.4.3

Find the binomial series for $f(x)=\dfrac{1}{(1+x)^{3/2}}$

Hint available at the end of this section.

Answer available at the end of this section.

In this example, we differentiated a known Taylor series to construct a Taylor series for another function. The ability to differentiate power series term by term makes them a powerful tool for solving differential equations. We now show how this is accomplished.

Solving Differential Equations with Power Series

Consider the differential equation

$$y'(x)=y.$$

Recall that this is a first-order separable equation and its solution is $y=Ce^x$. This equation is easily solved using techniques discussed earlier in the text. For most differential equations, however, we do not yet have analytical tools to solve them. Power series are an extremely useful tool for solving many types of differential equations. In this technique, we look for a solution of the form $y=\sum_{n=0}^{\infty}c_nx^n$ and determine what the coefficients would need to be. In the next example, we consider an initial-value problem involving $y'=y$ to illustrate the technique.

Example 10.4.4: Power Series Solution of a Differential Equation

Use power series to solve the initial-value problem $y'=y, \quad y(0)=3.$

Solution

Suppose that there exists a power series solution

$$y(x)=\sum_{n=0}^{\infty}c_nx^n=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+\cdots.$$

Differentiating this series term by term, we obtain

$$y'=c_1+2c_2x+3c_3x^2+4c_4x^3+\cdots.$$

If y satisfies the differential equation, then

$$c_0+c_1x+c_2x^2+c_3x^3+\cdots=c_1+2c_2x+3c_3x^2+4c_3x^3+\cdots.$$

Using the uniqueness of power series representations, we know that these series can only be equal if their coefficients are equal. Therefore,

$$c_0 = c_1,$$

$$c_1 = 2c_2,$$

$$c_2 = 3c_3,$$

$$c_3 = 4c_4,$$

$$\vdots$$

Using the initial condition $y(0) = 3$ combined with the power series representation

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$

we find that $c_0 = 3$. We are now ready to solve for the rest of the coefficients. Using the fact that $c_0 = 3$, we have

$$\begin{aligned} c_1 &= c_0 = 3 = \frac{3}{1!}, \\ c_2 &= \frac{c_1}{2} = \frac{3}{2} = \frac{3}{2!}, \\ c_3 &= \frac{c_2}{3} = \frac{3}{3 \cdot 2} = \frac{3}{3!}, \\ c_4 &= \frac{c_3}{4} = \frac{3}{4 \cdot 3 \cdot 2} = \frac{3}{4!}. \end{aligned}$$

Therefore,

$$y = 3\left[1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \frac{1}{4!}x^4 + \cdots\right] = 3\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

You might recognize

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

as the Taylor series for e^x. Therefore, the solution is $y = 3e^x$.

Exercise 10.4.4

Use power series to solve $y' = 2y, \quad y(0) = 5.$

Hint available at the end of this section.

Answer available at the end of this section.

We now consider an example involving a differential equation that we cannot solve using previously discussed methods. This differential equation

$$y' - xy = 0$$

is known as **Airy's equation**. It has many applications in mathematical physics, such as modeling the diffraction of light. Here we show how to solve it using power series.

Example 10.4.5: Power Series Solution of Airy's Equation

Use power series to solve $y'' - xy = 0$ with the initial conditions $y(0) = a$ and $y'(0) = b$.

Solution

We look for a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

Differentiating this function term by term, we obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots,$$
$$y'' = 2 \cdot 1 c_2 + 3 \cdot 2 c_3 x + 4 \cdot 3 c_4 x^2 + \cdots.$$

If y satisfies the equation $y'' = xy$, then

$$2 \cdot 1 c_2 + 3 \cdot 2 c_3 x + 4 \cdot 3 c_4 x^2 + \cdots = x(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots).$$

Using [link] on the uniqueness of power series representations, we know that coefficients of the same degree must be equal. Therefore,

$$2 \cdot 1 c_2 = 0,$$

$$3 \cdot 2 c_3 = c_0,$$

$$4 \cdot 3 c_4 = c_1,$$

$$5 \cdot 4 c_5 = c_2,$$

$$\vdots$$

More generally, for $n \geq 3$, we have $n \cdot (n-1) c_n = c_{n-3}$. In fact, all coefficients can be written in terms of c_0 and c_1. To see this, first note that $c_2 = 0$. Then

$$c_3 = \frac{c_0}{3 \cdot 2},$$

$$c_4 = \frac{c_1}{4 \cdot 3}.$$

For $c_5,\ c_6,\ c_7$, we see that

$$c_5 = \frac{c_2}{5 \cdot 4} = 0,$$
$$c_6 = \frac{c_3}{6 \cdot 5} = \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2},$$
$$c_7 = \frac{c_4}{7 \cdot 6} = \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}.$$

Therefore, the series solution of the differential equation is given by

$$y = c_0 + c_1 x + 0 \cdot x^2 + \frac{c_0}{3 \cdot 2} x^3 + \frac{c_1}{4 \cdot 3} x^4 + 0 \cdot x^5 + \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2} x^6 + \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3} x^7 + \cdots.$$

The initial condition $y(0) = a$ implies $c_0 = a$. Differentiating this series term by term and using the fact that $y'(0) = b$, we conclude that $c_1 = b$.

Therefore, the solution of this initial-value problem is

$$y = a\left(1 + \frac{x^3}{3\cdot 2} + \frac{x}{6\cdot 5\cdot 3\cdot 2} + \cdots\right) + b\left(x + \frac{x^4}{4\cdot 3} + \frac{x^7}{7\cdot 6\cdot 4\cdot 3} + \cdots\right).$$

Exercise 10.4.5

Use power series to solve $y'' + x^2 y = 0$ with the initial condition $y(0) = a$ and $y'(0) = b$.

Hint available at the end of this section.

Answer available at the end of this section.

Evaluating Non-elementary Integrals

Solving differential equations is one common application of power series. We now turn to a second application. We show how power series can be used to evaluate integrals involving functions whose antiderivatives cannot be expressed using elementary functions.

One integral that arises often in applications in probability theory is $\int e^{-x^2}\, dx$. Unfortunately, the antiderivative of the integrand e^{-x^2} is not an elementary function. By elementary function, we mean a function that can be written using a finite number of algebraic combinations or compositions of exponential, logarithmic, trigonometric, or power functions. We remark that the term "elementary function" is not synonymous with noncomplicated function. For example, the function $f(x) = \sqrt{x^2 - 3x} + e^{x^3} - \sin(5x + 4)$ is an elementary function, although not a particularly simple-looking function. Any integral of the form $\int f(x)\, dx$ where the antiderivative of f cannot be written as an elementary function is considered a **non-elementary integral.**

non-elementary integrals cannot be evaluated using the basic integration techniques discussed earlier. One way to evaluate such integrals is by expressing the integrand as a power series and integrating term by term. We demonstrate this technique by considering $\int e^{-x^2}\, dx$.

Example 10.4.6: Using Taylor Series to Evaluate a Definite Integral

a. Express $\int e^{-x^2} dx$ as an infinite series.

b. Evaluate $\int_0^1 e^{-x^2} dx$ to within an error of 0.01.

Solution

a. The Maclaurin series for e^{-x^2} is given by

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}. \end{aligned}$$

Therefore,

$$\int e^{-x^2}\,dx = \int \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots\right) dx$$
$$= C + x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots.$$

b. Using the result from part a. we have

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots.$$

The sum of the first four terms is approximately 0.74. By the alternating series test, this estimate is accurate to within an error of less than $\frac{1}{216} \approx 0.0046296 < 0.01$.

Exercise 10.4.6

Express $\int \cos\sqrt{x}\ dx$ as an infinite series. Evaluate $\int_0^1 \cos\sqrt{x}\ dx$ to within an error of 0.01.

Hint available at the end of this section.

Answer available at the end of this section.

As mentioned above, the integral $\int e^{-x^2}\ dx$ arises often in probability theory. Specifically, it is used when studying data sets that are normally distributed, meaning the data values lie under a bell-shaped curve. For example, if a set of data values is normally distributed with mean μ and standard deviation σ, then the probability that a randomly chosen value lies between $x = a$ and $x = b$ is given by

$$\frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-(x-\mu)^2/(2\sigma^2)}\,dx. \tag{10.4.5}$$

(See Figure 10.4.2.)

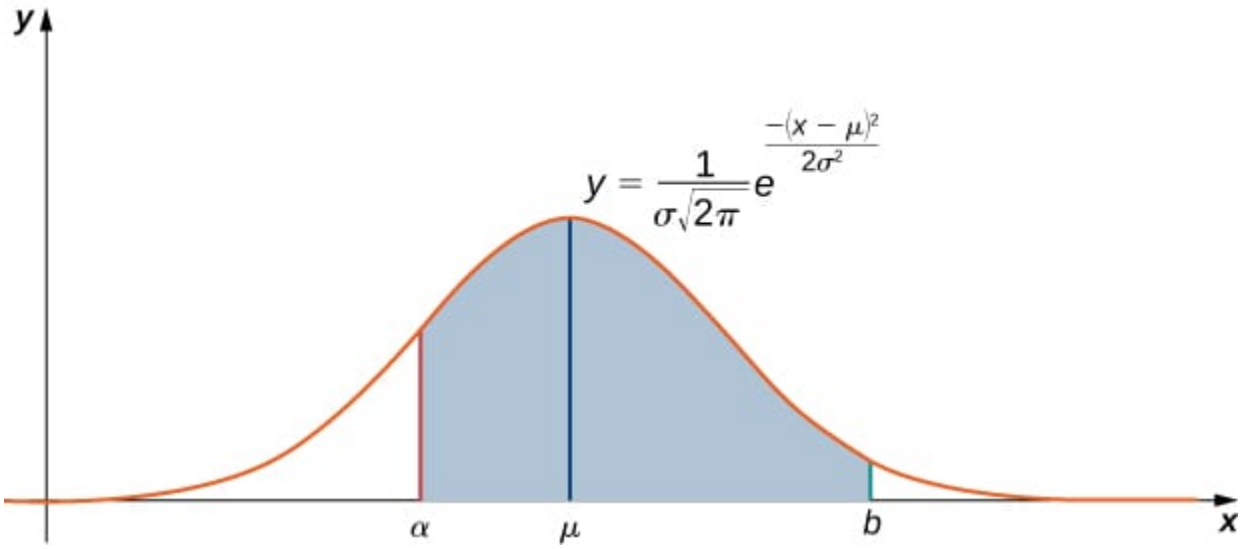


Figure 10.4.2: If data values are normally distributed with mean μ and standard deviation σ, the probability that a randomly selected data value is between a and b is the area under the curve $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$ between $x = a$ and $x = b$.

To simplify this integral, we typically let $z = \frac{x-\mu}{\sigma}$. This quantity z is known as the z score of a data value. With this simplification,

integral Equation 10.4.5 becomes

$$\frac{1}{\sqrt{2\pi}}\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-z^2/2}\,dz.$$

In Example 10.4.7, we show how we can use this integral in calculating probabilities.

Example 10.4.7: Using Maclaurin Series to Approximate a Probability

Suppose a set of standardized test scores are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 50$. Use Equation 10.4.5 and the first six terms in the Maclaurin series for $e^{-x^2/2}$ to approximate the probability that a randomly selected test score is between $x = 100$ and $x = 200$. Use the alternating series test to determine how accurate your approximation is.

Solution

Since $\mu = 100, \sigma = 50,$ and we are trying to determine the area under the curve from $a = 100$ to $b = 200$, integral Equation 10.4.5 becomes

$$\frac{1}{\sqrt{2\pi}}\int_0^2 e^{-z^2/2}\,dz.$$

The Maclaurin series for $e^{-x^2/2}$ is given by

$$\begin{aligned} e^{-x^2/2} &= \sum_{n=0}^{\infty}\frac{\left(-\frac{x^2}{2}\right)^n}{n!} \\ &= 1 - \frac{x^2}{2^1\cdot 1!} + \frac{x^4}{2^2\cdot 2!} - \frac{x^6}{2^3\cdot 3!} + \cdots + (-1)^n\frac{x^{2n}}{2^n\cdot n}! + \cdots \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n\cdot n!}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}\int e^{-z^2/2}\,dz &= \frac{1}{\sqrt{2\pi}}\int\left(1 - \frac{z^2}{2^1\cdot 1!} + \frac{z^4}{2^2\cdot 2!} - \frac{z^6}{2^3\cdot 3!} + \cdots + (-1)^n\frac{z^{2n}}{2^n\cdot n!} + \cdots\right)dz \\ &= \frac{1}{\sqrt{2\pi}}\left(C + z - \frac{z^3}{3\cdot 2^1\cdot 1!} + \frac{z^5}{5\cdot 2^2\cdot 2!} - \frac{z^7}{7\cdot 2^3\cdot 3!} + \cdots + (-1)^n\frac{z^{2n+1}}{(2n+1)2^n\cdot n!} + \cdots\right) \end{aligned}$$

$$\frac{1}{\sqrt{2\pi}}\int_0^2 e^{-z^2/2}\,dz = \frac{1}{\sqrt{2\pi}}\left(2 - \frac{8}{6} + \frac{32}{40} - \frac{128}{336} + \frac{512}{3456} - \frac{2^{11}}{11\cdot 2^5\cdot 5!} + \cdots\right)$$

Using the first five terms, we estimate that the probability is approximately **0.4922.** By the alternating series test, we see that this estimate is accurate to within

$$\frac{1}{\sqrt{2\pi}}\frac{2^{13}}{13\cdot 2^6\cdot 6!} \approx 0.00546.$$

Analysis

If you are familiar with probability theory, you may know that the probability that a data value is within two standard deviations of the mean is approximately 95%. Here we calculated the probability that a data value is between the mean and two standard deviations above the mean, so the estimate should be around 47.5%. The estimate, combined with the bound on the accuracy, falls within this range.

Exercise 10.4.7

Use the first five terms of the Maclaurin series for $e^{-x^2/2}$ to estimate the probability that a randomly selected test score is between 100 and 150. Use the alternating series test to determine the accuracy of this estimate.

Hint available at the end of this section.

Answer available at the end of this section.

Another application in which a non-elementary integral arises involves the period of a pendulum. The integral is

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

.

An integral of this form is known as an **elliptic integral** of the first kind. Elliptic integrals originally arose when trying to calculate the arc length of an ellipse. We now show how to use power series to approximate this integral.

Example 10.4.8: Period of a Pendulum

The period of a pendulum is the time it takes for a pendulum to make one complete back-and-forth swing. For a pendulum with length L that makes a maximum angle θ_{max} with the vertical, its period T is given by

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

where g is the acceleration due to gravity and $k = \sin\left(\frac{\theta_{max}}{2}\right)$ (see Figure $10.4.3$). (We note that this formula for the period arises from a non-linearized model of a pendulum. In some cases, for simplification, a linearized model is used and $\sin\theta$ is approximated by θ.)

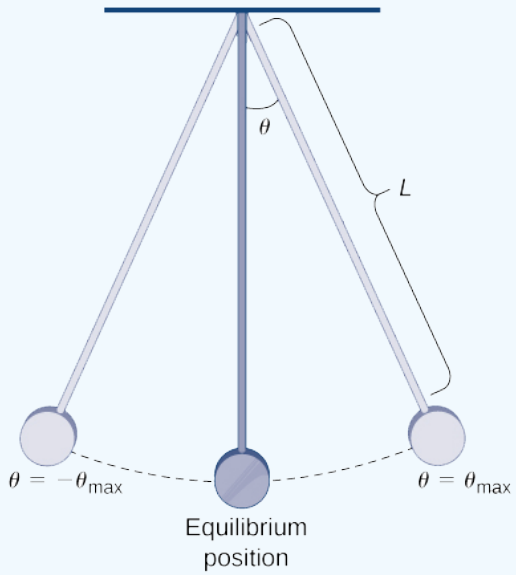


Figure $10.4.3$: This pendulum has length L and makes a maximum angle θ_{max} with the vertical.

Use the binomial series

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n$$

to estimate the period of this pendulum. Specifically, approximate the period of the pendulum if

a. you use only the first term in the binomial series, and
b. you use the first two terms in the binomial series.

Solution

We use the binomial series, replacing x with $-k^2 \sin^2 \theta$. Then we can write the period as

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \left(1 + \frac{1}{2}k^2 \sin^2 \theta + \frac{1 \cdot 3}{2!2^2} k^4 \sin^4 \theta + \cdots\right) d\theta.$$

a. Using just the first term in the integrand, the first-order estimate is

$$T \approx 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} d\theta = 2\pi\sqrt{\frac{L}{g}}.$$

If θ_{max} is small, then $k = \sin\left(\frac{\theta_{max}}{2}\right)$ is small. We claim that when k is small, this is a good estimate. To justify this claim, consider

$$\int_0^{\pi/2} \left(1 + \frac{1}{2}k^2 \sin^2 \theta + \frac{1 \cdot 3}{2!2^2} k^4 \sin^4 \theta + \cdots\right) d\theta.$$

Since $|\sin x| \leq 1$, this integral is bounded by

$$\int_0^{\pi/2} \left(\frac{1}{2}k^2 + \frac{1.3}{2!2^2}k^4 + \cdots\right) d\theta \; < \; \frac{\pi}{2}\left(\frac{1}{2}k^2 + \frac{1 \cdot 3}{2!2^2}k^4 + \cdots\right).$$

Furthermore, it can be shown that each coefficient on the right-hand side is less than 1 and, therefore, that this expression is bounded by

$$\frac{\pi k^2}{2}(1 + k^2 + k^4 + \cdots) = \frac{\pi k^2}{2} \cdot \frac{1}{1-k^2},$$

which is small for k small.

b. For larger values of θ_{max}, we can approximate T by using more terms in the integrand. By using the first two terms in the integral, we arrive at the estimate

$$T \approx 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \left(1 + \frac{1}{2}k^2 \sin^2 \theta\right) d\theta = 2\pi\sqrt{\frac{L}{g}} \left(1 + \frac{k^2}{4}\right).$$

The applications of Taylor series in this section are intended to highlight their importance. In general, Taylor series are useful because they allow us to represent known functions using polynomials, thus providing us a tool for approximating function values and estimating complicated integrals. In addition, they allow us to define new functions as power series, thus providing us with a powerful tool for solving differential equations.

Key Concepts

- The binomial series is the Maclaurin series for $f(x) = (1+x)^r$. It converges for $|x| < 1$.
- Taylor series for functions can often be derived by algebraic operations with a known Taylor series or by differentiating or integrating a known Taylor series.
- Power series can be used to solve differential equations.

- Taylor series can be used to help approximate integrals that cannot be evaluated by other means.

Glossary

binomial series
the Maclaurin series for $f(x) = (1+x)^r$; it is given by $(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots$ for $|x| < 1$

non-elementary integral
an integral for which the antiderivative of the integrand cannot be expressed as an elementary function

Answers

Exercise 10.4.1
Use the definition of binomial series for $r = -2$.

Exercise 10.4.1
$\sum_{n=0}^{\infty} (-1)^n (n+1) x^n$

Exercise 10.4.2
Use the Maclaurin series for $\sin x$.

Exercise 10.4.2
$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$

Exercise 10.4.3
Differentiate the series for $\frac{1}{\sqrt{1+x}}$

Exercise 10.4.3
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n$

Exercise 10.4.4
The equations for the first several coefficients c_n will satisfy $c_0 = 2c_1,\ c_1 = 2 \cdot 2c_2,\ c_2 = 2 \cdot 3c_3,\ \ldots$. In general, for all $n \geq 0,\ c_n = 2(n+1)C_{n+1}$.

Exercise 10.4.4
$y = 5e^{2x}$

Exercise 10.4.5
The coefficients satisfy $c_0 = a,\ c_1 = b,\ c_2 = 0,\ c_3 = 0$, and for $n \geq 4,\ n(n-1)c_n = -c_{n-4}$.

Exercise 10.4.5
$y = a\left(1 - \frac{x^4}{3 \cdot 4} + \frac{x^8}{3 \cdot 4 \cdot 7 \cdot 8} - \cdots\right) + b\left(x - \frac{x^5}{4 \cdot 5} + \frac{x^9}{4 \cdot 5 \cdot 8 \cdot 9} - \cdots\right)$

Exercise 10.4.6
Use the series found in Example 10.4.6.

Exercise 10.4.6
$C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n(2n-2)!}$ The definite integral is approximately 0.514 to within an error of 0.01.

Exercise 10.4.7
Evaluate $\int_0^1 e^{-z^2/2}\ dz$ using the first five terms of the Maclaurin series for $e^{-z^2/2}$.

Exercise 10.4.7
The estimate is approximately 0.3414. This estimate is accurate to within 0.0000094.

10.4E: Exercises for Section 10.4

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10.4E: Exercises for Section 10.4

In exercises 1 - 4, use appropriate substitutions to write down the Maclaurin series for the given binomial.

1) $(1-x)^{1/3}$

2) $(1+x^2)^{-1/3}$

Answer

$$(1+x^2)^{-1/3}=\sum_{n=0}^{\infty}\binom{-1/3}{n}x^{2n}$$

3) $(1-x)^{1.01}$

4) $(1-2x)^{2/3}$

Answer

$$(1-2x)^{2/3}=\sum_{n=0}^{\infty}(-1)^n 2^n\binom{2/3}{n}x^n$$

In exercises 5 - 12, use the substitution $(b+x)^r=(b+a)^r\left(1+\dfrac{x-a}{b+a}\right)^r$ in the binomial expansion to find the Taylor series of each function with the given center.

5) $\sqrt{x+2}$ at $a=0$

6) $\sqrt{x^2+2}$ at $a=0$

Answer

$$\sqrt{2+x^2}=\sum_{n=0}^{\infty}2^{(1/2)-n}\binom{1/2}{n}x^{2n};(|x^2|<2)$$

7) $\sqrt{x+2}$ at $a=1$

8) $\sqrt{2x-x^2}$ at $a=1$ (Hint: $2x-x^2=1-(x-1)^2$)

Answer

$\sqrt{2x-x^2}=\sqrt{1-(x-1)^2}$ so $\sqrt{2x-x^2}=\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(n^{\frac{1}{2}}\right)(x-1)^{2n}$

9) $(x-8)^{1/3}$ at $a=9$

10) $\sqrt{x}$ at $a=4$

Answer

$\sqrt{x}=2\sqrt{1+\frac{x-4}{4}}$ so $\sqrt{x}=\displaystyle\sum_{n=0}^{\infty}2^{1-2n}\left(n^{\frac{1}{2}}\right)(x-4)^n$

11) $x^{1/3}$ at $a=27$

12) $\sqrt{x}$ at $x=9$

Answer

$$\sqrt{x}=\sum_{n=0}^{\infty}3^{1-3n}\left(n^{\frac{1}{2}}\right)(x-9)^n$$

In exercises 13 - 14, use the binomial theorem to estimate each number, computing enough terms to obtain an estimate

accurate to an error of at most 1/1000.

13) [T] $(15)^{1/4}$ using $(16-x)^{1/4}$

14) [T] $(1001)^{1/3}$ using $(1000+x)^{1/3}$

Answer

$10(1+\frac{x}{1000})^{1/3} = \sum_{n=0}^{\infty} 10^{1-3n}(\frac{1}{n})x^n$. Using, for example, a fourth-degree estimate at $x=1$ gives $(1001)^{1/3}$ $\approx 10\left(1+\left(1^{\frac{1}{3}}\right)10^{-3}+\left(2^{\frac{1}{3}}\right)10^{-6}+\left(3^{\frac{1}{3}}\right)10^{-9}+\left(3^{\frac{1}{3}}\right)10^{-12}\right)$ $= 10\left(1+\frac{1}{3.10^3}-\frac{1}{9.10^6}+\frac{5}{81.10^9}-\frac{10}{243.10^{12}}\right)$ $= 10.00333222...$ whereas $(1001)^{1/3} = 10.00332222839093....$ Two terms would suffice for three-digit accuracy.

In exercises 15 - 18, use the binomial approximation $\sqrt{1-x} \approx 1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-\frac{7x^5}{256}$ for $|x|<1$ to approximate each number. Compare this value to the value given by a scientific calculator.

15) [T] $\frac{1}{\sqrt{2}}$ using $x=\frac{1}{2}$ in $(1-x)^{1/2}$

16) [T] $\sqrt{5} = 5\times\frac{1}{\sqrt{5}}$ using $x=\frac{4}{5}$ in $(1-x)^{1/2}$

Answer

The approximation is 2.3152; the CAS value is $2.23\ldots$.

17) [T] $\sqrt{3} = \frac{3}{\sqrt{3}}$ using $x=\frac{2}{3}$ in $(1-x)^{1/2}$

18) [T] $\sqrt{6}$ using $x=\frac{5}{6}$ in $(1-x)^{1/2}$

Answer

The approximation is $2.583\ldots$; the CAS value is $2.449\ldots$.

19) Integrate the binomial approximation of $\sqrt{1-x}$ to find an approximation of $\int_0^x \sqrt{1-t}\ dt$.

20) [T] Recall that the graph of $\sqrt{1-x^2}$ is an upper semicircle of radius 1. Integrate the binomial approximation of $\sqrt{1-x^2}$ up to order 8 from $x=-1$ to $x=1$ to estimate $\frac{\pi}{2}$.

Answer

$\sqrt{1-x^2} = 1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}+\cdots$. Thus $\int_{-1}^{1}\sqrt{1-x^2}\ dx = \left[x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{7\cdot 16}-\frac{5x^9}{9\cdot 128}+\cdots\right]\Big|_{-1}^{1} \approx 2$ $-\frac{1}{3}-\frac{1}{20}-\frac{1}{56}-\frac{10}{9\cdot 128}+error = 1.590...$ whereas $\frac{\pi}{2} = 1.570...$

In exercises 21 - 24, use the expansion $(1+x)^{1/3} = 1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots$ to write the first five terms (not necessarily a quartic polynomial) of each expression.

21) $(1+4x)^{1/3};\ \ a=0$

22) $(1+4x)^{4/3};\ \ a=0$

Answer

$(1+x)^{4/3} = (1+x)(1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots) = 1+\frac{4x}{3}+\frac{2x^2}{9}-\frac{4x^3}{81}+\frac{5x^4}{243}+\cdots$

23) $(3+2x)^{1/3};\ \ a=-1$

24) $(x^2+6x+10)^{1/3};\ \ a=-3$

Answer

$(1+(x+3)^2)^{1/3} = 1+\frac{1}{3}(x+3)^2-\frac{1}{9}(x+3)^4+\frac{5}{81}(x+3)^6-\frac{10}{243}(x+3)^8+\cdots$

25) Use $(1+x)^{1/3} = 1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots$ with $x=1$ to approximate $2^{1/3}$.

26) Use the approximation $(1-x)^{2/3} = 1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} + \cdots$ for $|x| < 1$ to approximate $2^{1/3} = 2.2^{-2/3}$.

Answer

Twice the approximation is $1.260\ldots$ whereas $2^{1/3} = 1.2599\ldots$.

27) Find the 25th derivative of $f(x) = (1+x^2)^{13}$ at $x = 0$.

28) Find the 99th derivative of $f(x) = (1+x^4)^{25}$.

Answer

$f^{(99)}(0) = 0$

In exercises 29 - 36, find the Maclaurin series of each function.

29) $f(x) = xe^{2x}$

Answer

$$xe^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}$$

30) $f(x) = 2^x$

Answer

$$\sum_{n=0}^{\infty} \frac{(\ln(2)x)^n}{n!}$$

31) $f(x) = \dfrac{\sin x}{x}$

32) $f(x) = \dfrac{\sin(\sqrt{x})}{\sqrt{x}}, (x > 0),$

Answer

For $x > 0,\ \sin(\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{(2n+1)/2}}{\sqrt{x}(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^n}{(2n+1)!}.$

33) $f(x) = \sin(x^2)$

34) $f(x) = e^{x^3}$

Answer

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

35) $f(x) = \cos^2 x$ using the identity $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

36) $f(x) = \sin^2 x$ using the identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$

Answer

$$\sin^2 x = -\sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1} x^{2k}}{(2k)!}$$

In exercises 37 - 44, find the Maclaurin series of $F(x) = \displaystyle\int_0^x f(t)\ dt$ by integrating the Maclaurin series of f term by term. If f is not strictly defined at zero, you may substitute the value of the Maclaurin series at zero.

37) $F(x) = \displaystyle\int_0^x e^{-t^2}\ dt;\ \ f(t) = e^{-t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}$

Answer

$$\int_0^x e^{-t^2}\ dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}$$

38) $F(x) = \tan^{-1} x;\ \ f(t) = \dfrac{1}{1+t^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n t^{2n}$

Answer

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

39) $F(x) = \tanh^{-1} x;\ \ f(t) = \dfrac{1}{1-t^2} = \displaystyle\sum_{n=0}^{\infty} t^{2n}$

40) $F(x) = \sin^{-1} x;\ \ f(t) = \dfrac{1}{\sqrt{1-t^2}} = \displaystyle\sum_{n=0}^{\infty} \binom{1/2}{n} \frac{t^{2n}}{n!}$

Answer

$$\sin^{-1} x = \sum_{n=0}^{\infty} \binom{1/2}{n} \frac{x^{2n+1}}{(2n+1)n!}$$

41) $F(x) = \displaystyle\int_0^x \frac{\sin t}{t}\ dt;\ \ f(t) = \frac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}$

42) $F(x) = \displaystyle\int_0^x \cos\left(\sqrt{t}\right)\ dt;\ \ f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$

Answer

$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)(2n)!}$$

43) $F(x) = \displaystyle\int_0^x \frac{1-\cos t}{t^2}\ dt;\ \ f(t) = \frac{1-\cos t}{t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+2)!}$

44) $F(x) = \displaystyle\int_0^x \frac{\ln(1+t)}{t}\ dt;\ \ f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n+1}$

Answer

$$F(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n^2}$$

In exercises 45 - 52, compute at least the first three nonzero terms (not necessarily a quadratic polynomial) of the Maclaurin series of f.

45) $f(x) = \sin\left(x + \frac{\pi}{4}\right) = \sin x \cos\left(\frac{\pi}{4}\right) + \cos x \sin\left(\frac{\pi}{4}\right)$

46) $f(x) = \tan x$

Answer

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

47) $f(x) = \ln(\cos x)$

48) $f(x) = e^x \cos x$

Answer

$$1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

49) $f(x) = e^{\sin x}$

50) $f(x) = \sec^2 x$

Answer

$$1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \cdots$$

51) $f(x) = \tanh x$

52) $f(x) = \dfrac{\tan\sqrt{x}}{\sqrt{x}}$ (see expansion for $\tan x$)

Answer

Using the expansion for $\tan x$ gives $1 + \dfrac{x}{3} + \dfrac{2x^2}{15}$.

In exercises 53 - 56, find the radius of convergence of the Maclaurin series of each function.

53) $\ln(1 + x)$

54) $\dfrac{1}{1 + x^2}$

Answer

$\dfrac{1}{1 + x^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n}$ so $R = 1$ by the ratio test.

55) $\tan^{-1} x$

56) $\ln(1 + x^2)$

Answer

$\ln(1 + x^2) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{2n}$ so $R = 1$ by the ratio test.

57) Find the Maclaurin series of $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

58) Find the Maclaurin series of $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Answer

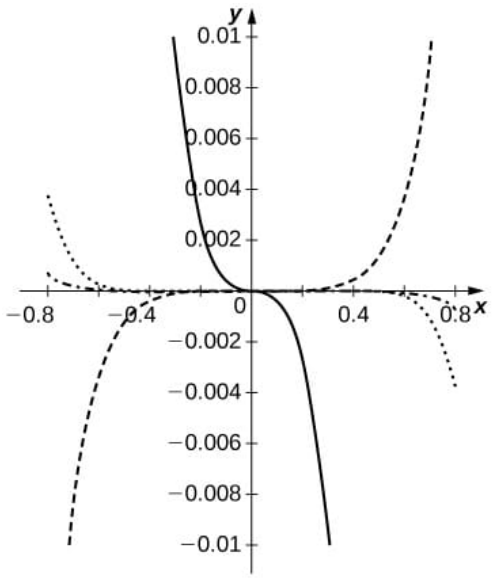
Add series of e^x and e^{-x} term by term. Odd terms cancel and $\cosh x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$.

59) Differentiate term by term the Maclaurin series of $\sinh x$ and compare the result with the Maclaurin series of $\cosh x$.

60) [T] Let $S_n(x) = \displaystyle\sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ and $C_n(x) = \displaystyle\sum_{n=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$ denote the respective Maclaurin polynomials of degree $2n + 1$ of $\sin x$ and degree $2n$ of $\cos x$. Plot the errors $\dfrac{S_n(x)}{C_n(x)} - \tan x$ for $n = 1, .., 5$ and compare them to $x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315} - \tan x$ on $(-\frac{\pi}{4}, \frac{\pi}{4})$.

Answer

The ratio $\dfrac{S_n(x)}{C_n(x)}$ approximates $\tan x$ better than does $p_7(x) = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315}$ for $N \geq 3$. The dashed curves are $\dfrac{S_n}{C_n} - \tan x$ for $n = 1,\ 2$. The dotted curve corresponds to $n = 3$, and the dash-dotted curve corresponds to $n = 4$. The solid curve is $p_7 - \tan x$.



61) Use the identity $2\sin x\cos x = \sin(2x)$ to find the power series expansion of $\sin^2 x$ at $x = 0$. (Hint: Integrate the Maclaurin series of $\sin(2x)$ term by term.)

62) If $y = \sum_{n=0}^{\infty} a_n x^n$, find the power series expansions of $xy\,'$ and $x^2 y''$.

Answer

By the term-by-term differentiation theorem, $y\,' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ so $y\,' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ $xy\,' = \sum_{n=1}^{\infty} n a_n x^n$, whereas $y\,' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ so $xy'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n$.

63) [T] Suppose that $y = \sum_{k=0}^{\infty} a^k x^k$ satisfies $y\,' = -2xy$ and $y(0) = 0$. Show that $a_{2k+1} = 0$ for all k and that $a_{2k+2} = \dfrac{-a_{2k}}{k+1}$. Plot the partial sum S_{20} of y on the interval $[-4, 4]$.

64) [T] Suppose that a set of standardized test scores is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 10$. Set up an integral that represents the probability that a test score will be between 90 and 110 and use the integral of the degree 10 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ to estimate this probability.

Answer

The probability is $p = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-x^2/2}\ dx$ where $a = 90$ and $b = 100$, that is, $p = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-1}^{1} e^{-x^2/2}\ dx$ $= \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-1}^{1} \sum_{n=0}^{5} (-1)^n \frac{x^{2n}}{2^n n!}\ dx = \frac{2}{\sqrt{2\pi}} \sum_{n=0}^{5} (-1)^n \frac{1}{(2n+1)2^n n!} \approx 0.6827.$

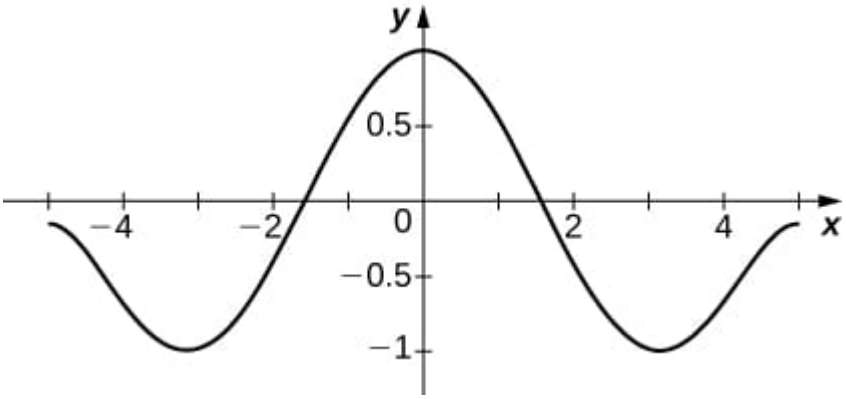
65) [T] Suppose that a set of standardized test scores is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 10$. Set up an integral that represents the probability that a test score will be between 70 and 130 and use the integral of the degree 50 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ to estimate this probability.

66) [T] Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0) = 1,\ \ f\,'(0) = 0$, and $f''(x) = -f(x)$. Find a formula for a_n and plot the partial sum S_N for $N = 20$ on $[-5, 5]$.

Answer

As in the previous problem one obtains $a_n = 0$ if n is odd and $a_n = -(n+2)(n+1)a_{n+2}$ if n is even, so $a_0 = 1$

leads to $a_{2n} = \dfrac{(-1)^n}{(2n)!}$.



67) [T] Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0) = 0, \quad f\,'(0) = 1$, and $f''(x) = -f(x)$. Find a formula for an and plot the partial sum S_N for $N = 10$ on $[-5, 5]$.

68) Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y'' - y\,' + y = 0$ where $y(0) = 1$ and $y'(0) = 0$. Find a formula that relates $a_{n+2}, \quad a_{n+1}$, and an and compute $a_0, \ldots, a_5$.

Answer

$y'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$ and $y\,' = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$ so $y'' - y\,' + y = 0$ implies that $(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + a_n = 0$ or $a_n = \dfrac{a_{n-1}}{n} - \dfrac{a_{n-2}}{n(n-1)}$ for all $n \cdot y(0) = a_0 = 1$ and $y\,'(0) = a_1 = 0$, so $a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{6}$, $a_4 = 0$, and $a_5 = -\frac{1}{120}$.

69) Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y'' - y\,' + y = 0$ where $y(0) = 0$ and $y\,'(0) = 1$. Find a formula that relates $a_{n+2}, \quad a_{n+1}$, and an and compute $a_1, \ldots, a_5$.

The error in approximating the integral $\int_a^b f(t)\ dt$ by that of a Taylor approximation $\int_a^b Pn(t)\ dt$ is at most $\int_a^b R_n(t)\ dt$. In exercises 70 - 71, the Taylor remainder estimate $R_n \le \frac{M}{(n+1)!}|x-a|^{n+1}$ guarantees that the integral of the Taylor polynomial of the given order approximates the integral of f with an error less than $\frac{1}{10}$.

a. Evaluate the integral of the appropriate Taylor polynomial and verify that it approximates the CAS value with an error less than $\frac{1}{100}$.

b. Compare the accuracy of the polynomial integral estimate with the remainder estimate.

70) [T] $\int_0^{\pi} \dfrac{\sin t}{t}\ dt; \quad P_s = 1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \dfrac{x^6}{7!} + \dfrac{x^8}{9!}$ (You may assume that the absolute value of the ninth derivative of $\frac{\sin t}{t}$ is bounded by 0.1.)

Answer

a. (Proof)

b. We have $R_s \le \frac{0.1}{(9)!}\pi^9 \approx 0.0082 < 0.01$. We have $\int_0^{\pi}\left(1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \dfrac{x^6}{7!} + \dfrac{x^8}{9!}\right)\ dx = \pi - \dfrac{\pi^3}{3 \cdot 3!} + \dfrac{\pi^5}{5 \cdot 5!} - \dfrac{\pi^7}{7 \cdot 7!} + \dfrac{\pi^9}{9 \cdot 9!} = 1.852\ldots$, whereas $\int_0^{\pi} \dfrac{\sin t}{t}\ dt = 1.85194\ldots$, so the actual error is approximately 0.00006.

71) [T] $\int_0^2 e^{-x^2}\ dx; \quad p_{11} = 1 - x^2 + \dfrac{x^4}{2} - \dfrac{x^6}{3!} + \cdots - \dfrac{x^{22}}{11!}$ (You may assume that the absolute value of the 23rd derivative of e^{-x^2} is

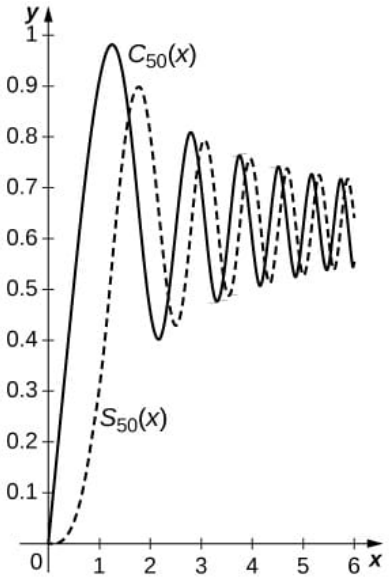
less than 2×10^{14}.)

The following exercises (72-73) deal with *Fresnel integrals*.

72) The Fresnel integrals are defined by $C(x) = \int_0^x \cos(t^2)\ dt$ and $S(x) = \int_0^x \sin(t^2)\ dt$. Compute the power series of $C(x)$ and $S(x)$ and plot the sums $C_N(x)$ and $S_N(x)$ of the first $N = 50$ nonzero terms on $[0, 2\pi]$.

Answer

Since $\cos(t^2) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n}}{(2n)!}$ and $\sin(t^2) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+2}}{(2n+1)!}$, one has $S(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!}$ and $C(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(4n+1)(2n)!}$. The sums of the first 50 nonzero terms are plotted below with $C_{50}(x)$ the solid curve and $S_{50}(x)$ the dashed curve.



73) [T] The Fresnel integrals are used in design applications for roadways and railways and other applications because of the curvature properties of the curve with coordinates $(C(t), S(t))$. Plot the curve (C_{50}, S_{50}) for $0 \le t \le 2\pi$, the coordinates of which were computed in the previous exercise.

74) Estimate $\int_0^{1/4} \sqrt{x - x^2}\ dx$ by approximating $\sqrt{1-x}$ using the binomial approximation $1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{2128} - \frac{7x^5}{256}$.

Answer

$\int_0^{1/4} \sqrt{x}\left(1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256}\right)\ dx = \frac{2}{3}2^{-3} - \frac{1}{2}\frac{2}{5}2^{-5} - \frac{1}{8}\frac{2}{7}2^{-7} - \frac{1}{16}\frac{2}{9}2^{-9} - \frac{5}{128}\frac{2}{11}2^{-11} - \frac{7}{256}\frac{2}{13}2^{-13} = 0.0767732\ldots$ whereas $\int_0^{1/4} \sqrt{x - x^2}\ dx = 0.076773.$

75) [T] Use Newton's approximation of the binomial $\sqrt{1-x^2}$ to approximate π as follows. The circle centered at $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$ has upper semicircle $y = \sqrt{x}\sqrt{1-x}$. The sector of this circle bounded by the x-axis between $x = 0$ and $x = \frac{1}{2}$ and by the line joining $(\frac{1}{4}, \frac{\sqrt{3}}{4})$ corresponds to $\frac{1}{6}$ of the circle and has area $\frac{\pi}{24}$. This sector is the union of a right triangle with height $\frac{\sqrt{3}}{4}$ and base $\frac{1}{4}$ and the region below the graph between $x = 0$ and $x = \frac{1}{4}$. To find the area of this region you can write $y = \sqrt{x}\sqrt{1-x}$ $= \sqrt{x} \times (\text{binomial expansion of} \sqrt{1-x})$ and integrate term by term. Use this approach with the binomial approximation from the previous exercise to estimate π.

76) Use the approximation $T \approx 2\pi\sqrt{\frac{L}{g}}(1 + \frac{k^2}{4})$ to approximate the period of a pendulum having length 10 meters and maximum angle $\theta_{max} = \frac{\pi}{6}$ where $k = \sin\left(\frac{\theta_{max}}{2}\right)$. Compare this with the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$.

Answer

$T \approx 2\pi\sqrt{\frac{10}{9.8}}\left(1+\frac{\sin^2(\theta/12)}{4}\right) \approx 6.453$ seconds. The small angle estimate is $T \approx 2\pi\sqrt{\frac{10}{9.8}} \approx 6.347$. The relative error is around 2 percent.

77) Suppose that a pendulum is to have a period of 2 seconds and a maximum angle of $\theta_{max} = \frac{\pi}{6}$. Use $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}\right)$ to approximate the desired length of the pendulum. What length is predicted by the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$?

78) Evaluate $\int_0^{\pi/2} \sin^4\theta \; d\theta$ in the approximation $T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1+\frac{1}{2}k^2\sin^2\theta+\frac{3}{8}k^4\sin^4\theta+\cdots\right) \; d\theta$ to obtain an improved estimate for T.

Answer

$\int_0^{\pi/2} \sin^4\theta \; d\theta = \frac{3\pi}{16}$. Hence $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}+\frac{9}{256}k^4\right)$.

79) [T] An equivalent formula for the period of a pendulum with amplitude θ_{max} is $T(\theta_{max}) = 2\sqrt{2}\sqrt{\frac{L}{g}}\int_0^{\theta_{max}} \frac{d\theta}{\sqrt{\cos\theta-\cos(\theta_{max})}}$ where L is the pendulum length and g is the gravitational acceleration constant. When $\theta_{max} = \frac{\pi}{3}$ we get $\frac{1}{\sqrt{\cos t - 1/2}} \approx \sqrt{2}\left(1+\frac{t^2}{2}+\frac{t^4}{3}+\frac{181t^6}{720}\right)$. Integrate this approximation to estimate $T(\frac{\pi}{3})$ in terms of L and g. Assuming $g = 9.806$ meters per second squared, find an approximate length L such that $T(\frac{\pi}{3}) = 2$ seconds.

Contributors

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10.5: Chapter 10 Review Exercises

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11: PARAMETRIC EQUATIONS AND POLAR COORDINATES

Parametric equations define a group of quantities as functions of one or more independent variables called parameters. Parametric equations are commonly used to express the coordinates of the points that make up a geometric object such as a curve or surface, in which case the equations are collectively called a parametric representation or parameterization. The polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction. The reference point (analogous to the origin of a Cartesian system) is called the pole, and the ray from the pole in the reference direction is the polar axis. The distance from the pole is called the radial coordinate or radius, and the angle is called the angular coordinate, polar angle, or azimuth

11.0: Prelude to Parametric Equations and Polar Coordinates

11.1: Conic Sections

11.1E: Exercises for Section 11.1

11.2: Parametric Equations

11.2E: Exercises for Section 11.2

11.3: Calculus of Parametric Curves

11.3E: Exercises for Section 11.3

11.4: Polar Coordinates

11.4E: Exercises for Section 11.4

11.5: Area and Arc Length in Polar Coordinates

11.5E: Exercises for Section 11.5

11.6: Chapter 11 Review Exercises

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11.1: CONIC SECTIONS

11.05: Conic Sections

11.1E: Exercises for Section 11.1

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11.1E: Exercises for Section 11.1

In exercises 1 - 8, determine the equation of the parabola using the information given.

1) Focus $(4,0)$ and directrix $x=-4$

Answer

$y^2=16x$

2) Focus $(0,-3)$ and directrix $y=3$

3) Focus $(0,0.5)$ and directrix $y=-0.5$

Answer

$x^2=2y$

4) Focus $(2,3)$ and directrix $x=-2$

5) Focus $(0,2)$ and directrix $y=4$

Answer

$x^2=-4(y-3)$

6) Focus $(-1,4)$ and directrix $x=5$

7) Focus $(-3,5)$ and directrix $y=1$

Answer

$(x+3)^2=8(y-3)$

8) Focus $\left(\frac{5}{2},-4\right)$ and directrix $x=\frac{7}{2}$

In exercises 9 - 16, determine the equation of the ellipse using the information given.

9) Endpoints of major axis at $(4,0),\ (-4,0)$ and foci located at $(2,0),\ (-2,0)$

Answer

$\frac{x^2}{16}+\frac{y^2}{12}=1$

10) Endpoints of major axis at $(0,5),\ (0,-5)$ and foci located at $(0,3),\ (0,-3)$

11) Endpoints of major axis at $(0,2),\ (0,-2)$ and foci located at $(3,0),\ (-3,0)$

Answer

$\frac{x^2}{13}+\frac{y^2}{4}=1$

12) Endpoints of major axis at $(-3,3),\ (7,3)$ and foci located at $(-2,3),\ (6,3)$

13) Endpoints of major axis at $(-3,5),\ (-3,-3)$ and foci located at $(-3,3),\ (-3,-1)$

Answer

$\frac{(y-1)^2}{16}+\frac{(x+3)^2}{12}=1$

14) Endpoints of major axis at $(0,0),\ (0,4)$ and foci located at $(5,2),\ (-5,2)$

15) Foci located at $(2,0),\ (-2,0)$ and eccentricity of $\frac{1}{2}$

Answer

$$\frac{x^2}{16}+\frac{y^2}{12}=1$$

16) Foci located at $(0,-3),\ (0,3)$ and eccentricity of $\frac{3}{4}$

In exercises 17 - 24, determine the equation of the hyperbola using the information given.

17) Vertices located at $(5,0),\ (-5,0)$ and foci located at $(6,0),\ (-6,0)$

Answer

$\frac{x^2}{25}-\frac{y^2}{11}=1$

18) Vertices located at $(0,2),\ (0,-2)$ and foci located at $(0,3),\ (0,-3)$

19) Endpoints of the conjugate axis located at $(0,3),\ (0,-3)$ and foci located $(4,0),\ (-4,0)$

Answer

$$\frac{x^2}{7}-\frac{y^2}{9}=1$$

20) Vertices located at $(0,1),\ (6,1)$ and focus located at $(8,1)$

21) Vertices located at $(-2,0),\ (-2,-4)$ and focus located at $(-2,-8)$

Answer

$$\frac{(y+2)^2}{4}-\frac{(x+2)^2}{32}=1$$

22) Endpoints of the conjugate axis located at $(3,2),\ (3,4)$ and focus located at $(3,7)$

23) Foci located at $(6,-0),\ (6,0)$ and eccentricity of 3

Answer

$$\frac{x^2}{4}-\frac{y^2}{32}=1$$

24) $(0,10),\ (0,-10)$ and eccentricity of 2.5

In exercises 25 - 30, consider the following polar equations of conics. Determine the eccentricity and identify the conic.

25) $r=\dfrac{-1}{1+\cos\theta}$

Answer

$e=1$, parabola

26) $r=\dfrac{8}{2-\sin\theta}$

27) $r=\dfrac{5}{2+\sin\theta}$

Answer

$e=\frac{1}{2}$, ellipse

28) $r=\dfrac{5}{-1+2\sin\theta}$

29) $r=\dfrac{3}{2-6\sin\theta}$

Answer

$e = 3$, hyperbola

30) $r = \dfrac{3}{-4 + 3\sin\theta}$

In exercises 31 - 34, find a polar equation of the conic with focus at the origin and eccentricity and directrix as given.

31) Directrix: $x = 4;\ \ e = \frac{1}{5}$

Answer

$$r = \frac{4}{5 + \cos\theta}$$

32) Directrix: $x = -4;\ \ e = 5$

33) Directrix: $y = 2;\ \ e = 2$

Answer

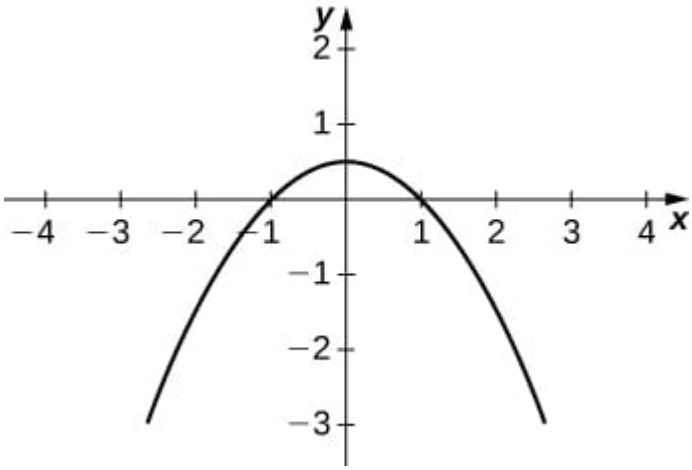
$$r = \frac{4}{1 + 2\sin\theta}$$

34) Directrix: $y = -2;\ \ e = \frac{1}{2}$

In exercises 35 - 51, sketch the graph of each conic.

35) $r = \dfrac{1}{1 + \sin\theta}$

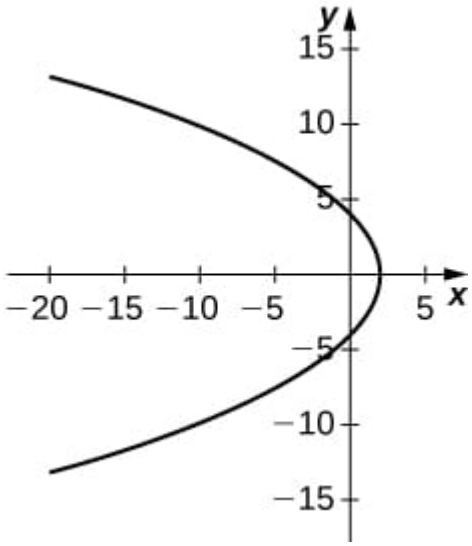
Answer



36) $r = \dfrac{1}{1 - \cos\theta}$

37) $r = \dfrac{4}{1 + \cos\theta}$

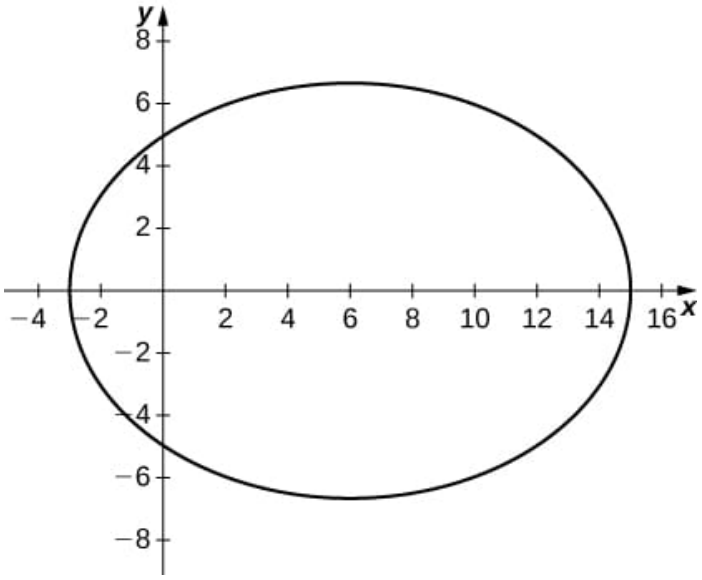
Answer



38) $r = \dfrac{10}{5 + 4\sin\theta}$

39) $r = \dfrac{15}{3 - 2\cos\theta}$

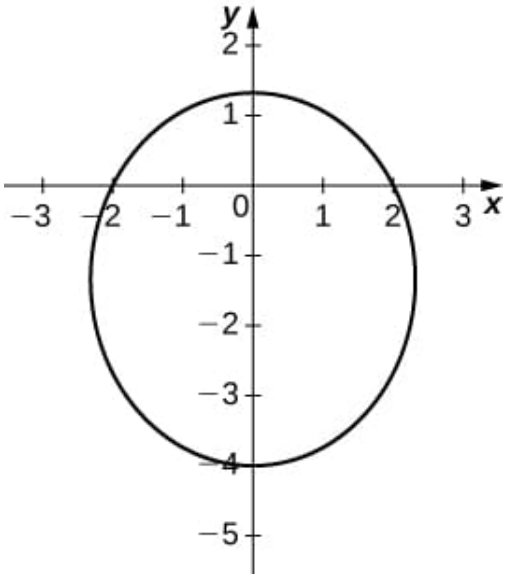
Answer



40) $r = \dfrac{32}{3 + 5\sin\theta}$

41) $r(2 + \sin\theta) = 4$

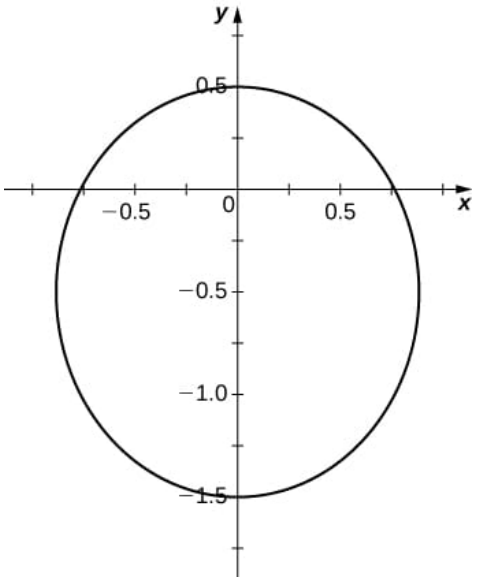
Answer



42) $r = \dfrac{3}{2 + 6\sin\theta}$

43) $r = \dfrac{3}{-4 + 2\sin\theta}$

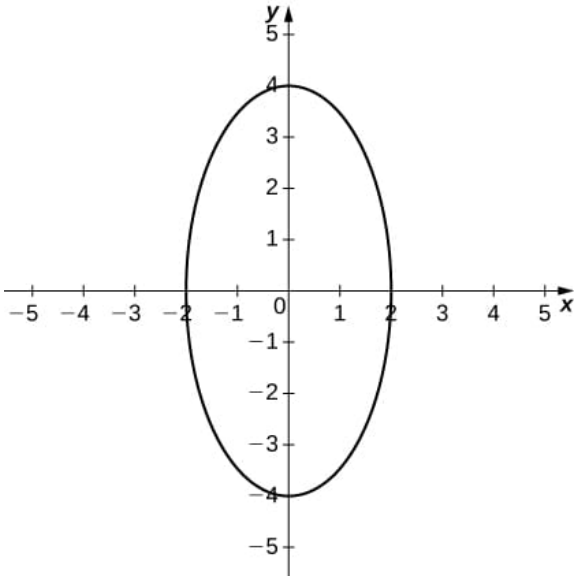
Answer



44) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

45) $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$

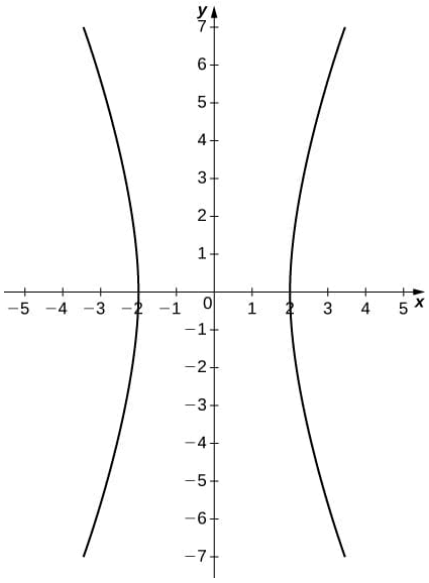
Answer



46) $4x^2 + 9y^2 = 36$

47) $25x^2 - 4y^2 = 100$

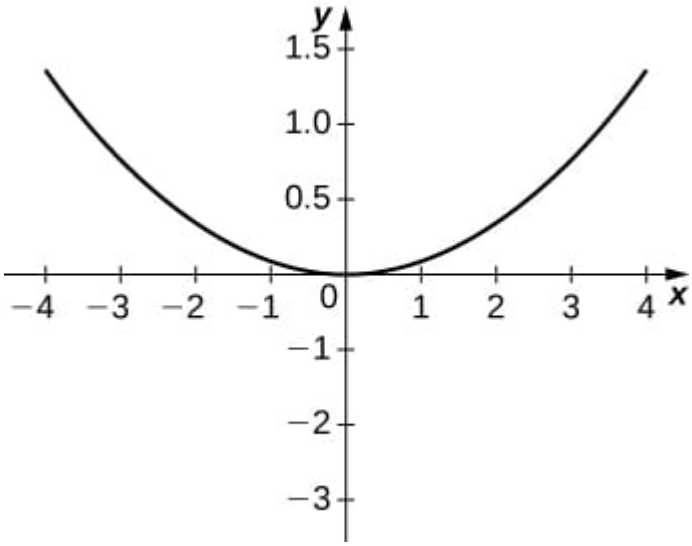
Answer



48) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

49) $x^2 = 12y$

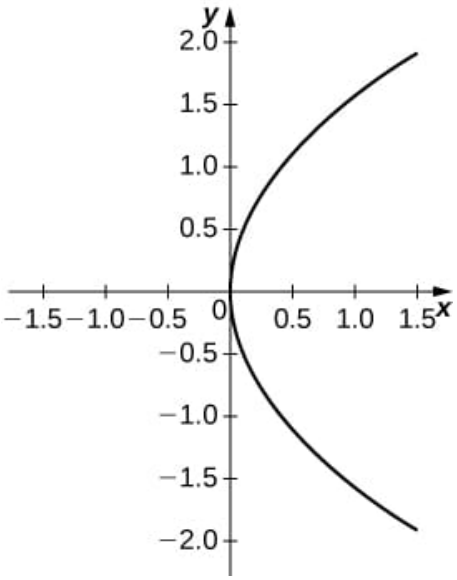
Answer



50) $y^2 = 20x$

51) $12x = 5y^2$

Answer



For the equations in exercises 52 - 57, determine which of the conic sections is described.

52) $xy = 4$

Answer

Rotated Hyperbola

53) $x^2 + 4xy - 2y^2 - 6 = 0$

Answer

Hyperbola

54) $x^2 + 2\sqrt{3}xy + 3y^2 - 6 = 0$

55) $x^2 - xy + y^2 - 2 = 0$

Answer

Ellipse

56) $34x^2 - 24xy + 41y^2 - 25 = 0$

57) $52x^2 - 72xy + 73y^2 + 40x + 30y - 75 = 0$

Answer

Ellipse

58) The mirror in an automobile headlight has a parabolic cross section, with the lightbulb at the focus. On a schematic, the equation of the parabola is given as $x^2 = 4y$. At what coordinates should you place the lightbulb?

59) A satellite dish is shaped like a paraboloid of revolution. The receiver is to be located at the focus. If the dish is 12 feet across at its opening and 4 feet deep at its center, where should the receiver be placed?

Answer

At the point 2.25 feet above the vertex.

60) Consider the satellite dish of the preceding problem. If the dish is 8 feet across at the opening and 2 feet deep, where should we place the receiver?

61) A searchlight is shaped like a paraboloid of revolution. A light source is located 1 foot from the base along the axis of symmetry. If the opening of the searchlight is 3 feet across, find the depth.

Answer

0.5625 feet

62) Whispering galleries are rooms designed with elliptical ceilings. A person standing at one focus can whisper and be heard by a person standing at the other focus because all the sound waves that reach the ceiling are reflected to the other person. If a whispering gallery has a length of 120 feet and the foci are located 30 feet from the center, find the height of the ceiling at the center.

63) A person is standing 8 feet from the nearest wall in a whispering gallery. If that person is at one focus and the other focus is 80 feet away, what is the length and the height at the center of the gallery?

Answer

Length is 96 feet and height is approximately 26.53 feet.

In exercises 64 - 67, determine the polar equation form of the orbit given the length of the major axis and eccentricity for the orbits of the comets or planets. Distance is given in astronomical units (AU).

64) Halley's Comet: length of major axis = 35.88, eccentricity = 0.967

65) Hale-Bopp Comet: length of major axis = 525.91, eccentricity = 0.995

Answer

$$r = \frac{2.616}{1 + 0.995\cos\theta}$$

66) Mars: length of major axis = 3.049, eccentricity = 0.0934

67) Jupiter: length of major axis = 10.408, eccentricity = 0.0484

Answer

$$r = \frac{5.192}{1 + 0.0484\cos\theta}$$

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11.2: PARAMETRIC EQUATIONS

11.01: Parametric Equations

11.2E: Exercises for Section 11.2

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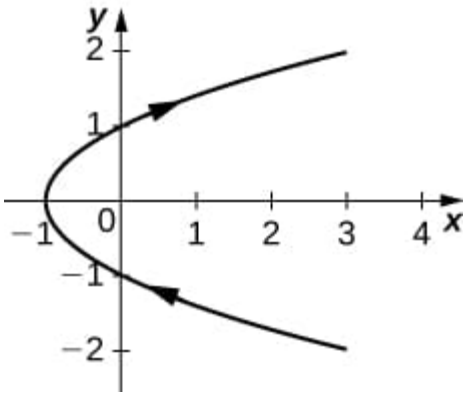
11.2E: Exercises for Section 11.2

In exercises 1 - 4, sketch the curves below by eliminating the parameter t. Give the orientation of the curve.

1) $x = t^2 + 2t, \quad y = t + 1$

Answer

Orientation: bottom to top

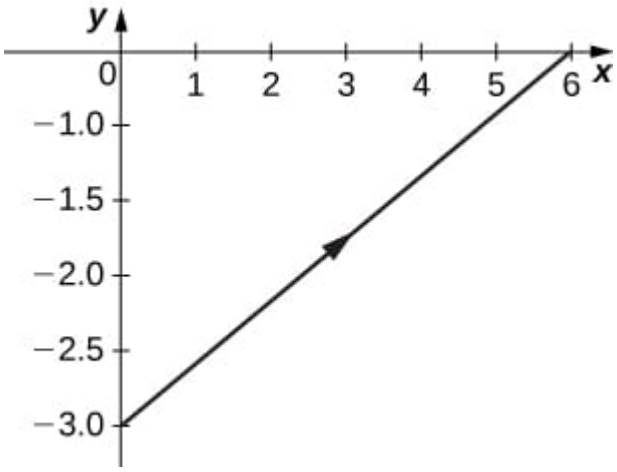


2) $x = \cos(t), \quad y = \sin(t), \quad \text{for } (0, 2\pi]$

3) $x = 2t + 4, \quad y = t - 1$

Answer

Orientation: left to right



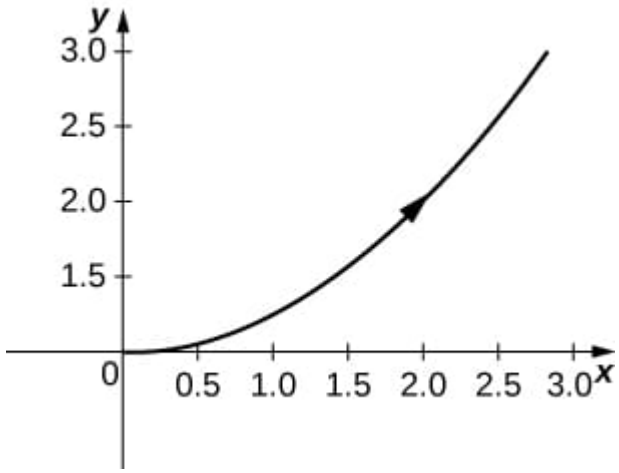
4) $x = 3 - t, \quad y = 2t - 3, \quad \text{for } 1.5 \le t \le 3$

In exercise 5, eliminate the parameter and sketch the graph.

5) $x = 2t^2, \quad y = t^4 + 1$

Answer

$$y = \frac{x^2}{4} + 1$$

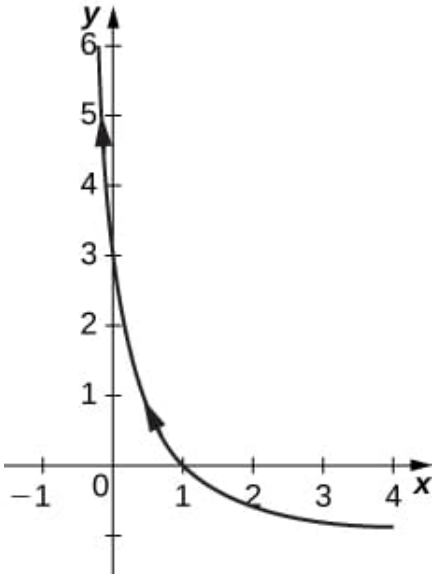


In exercises 6 - 9, use technology (CAS or calculator) to sketch the parametric equations.

6) [T] $x = t^2 + t, \quad y = t^2 - 1$

7) [T] $x = e^{-t}, \quad y = e^{2t} - 1$

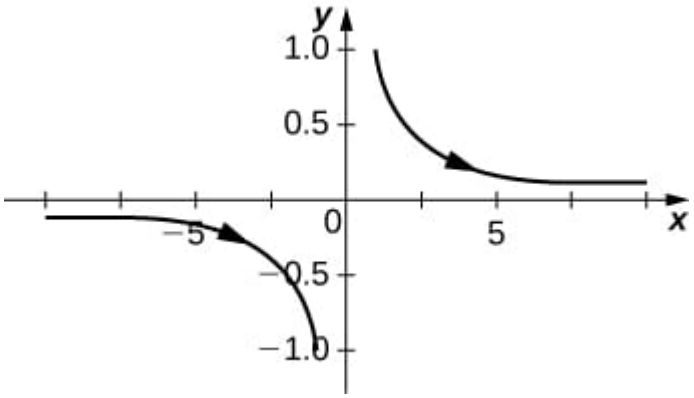
Answer



8) [T] $x = 3\cos t, \quad y = 4\sin t$

9) [T] $x = \sec t, \quad y = \cos t$

Answer

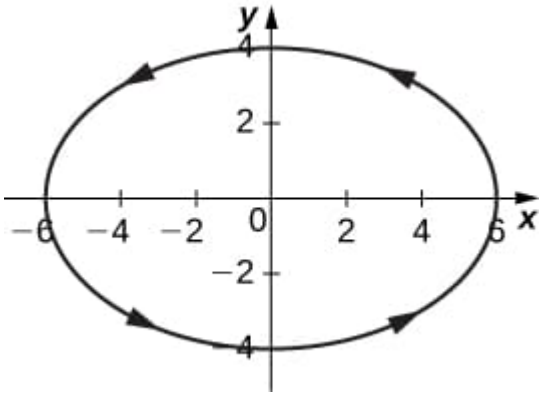


In exercises 10 - 20, sketch the parametric equations by eliminating the parameter. Indicate any asymptotes of the graph.

10) $x = e^t, \quad y = e^{2t} + 1$

11) $x = 6\sin(2\theta), \quad y = -4\cos(2\theta)$

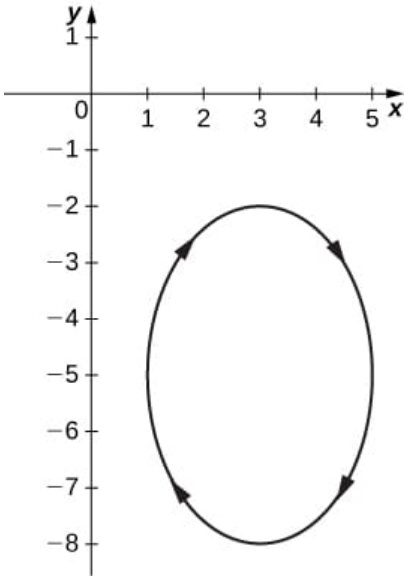
Answer



12) $x = \cos\theta, \quad y = 2\sin(2\theta)$

13) $x = 3 - 2\cos\theta, \quad y = -5 + 3\sin\theta$

Answer

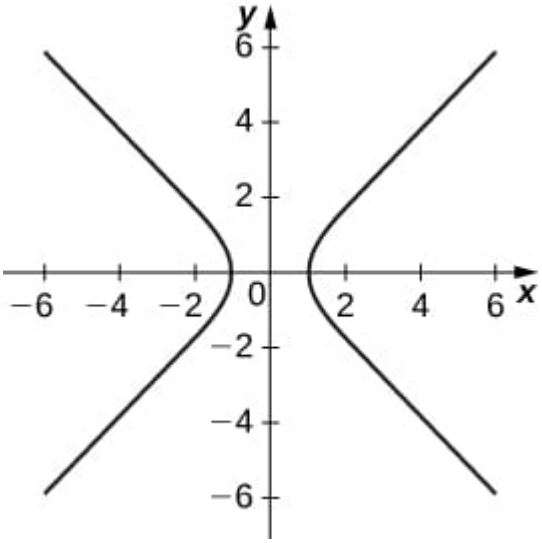


14) $x = 4 + 2\cos\theta, \quad y = -1 + \sin\theta$

15) $x = \sec t, \quad y = \tan t$

Answer

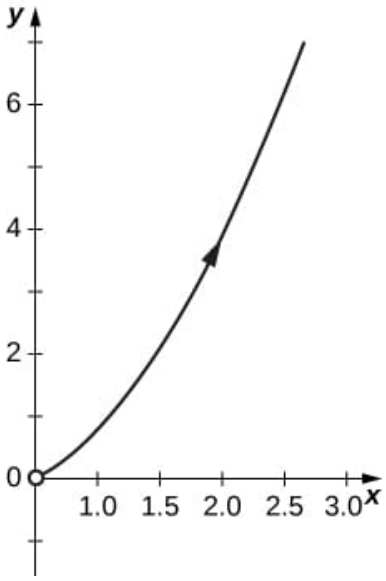
Asymptotes are $y = x$ and $y = -x$



16) $x = \ln(2t), \quad y = t^2$

17) $x = e^t, \quad y = e^{2t}$

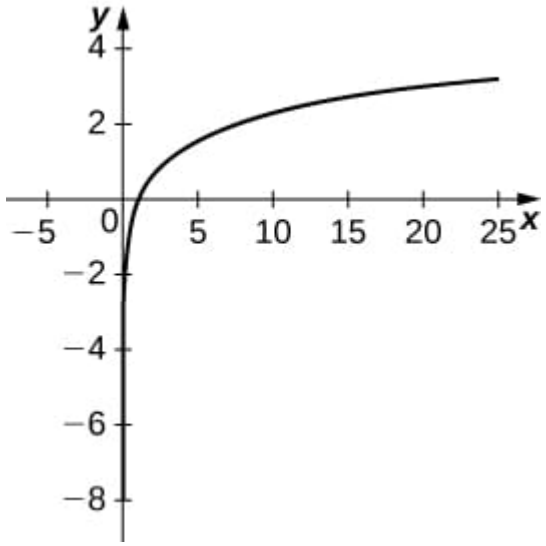
Answer



18) $x = e^{-2t}, \quad y = e^{3t}$

19) $x = t^3, \quad y = 3 \ln t$

Answer



20) $x = 4 \sec \theta, \quad y = 3 \tan \theta$

In exercises 21 - 38, convert the parametric equations of a curve into rectangular form. No sketch is necessary. State the domain of the rectangular form.

21) $x = t^2 - 1, \quad y = \dfrac{t}{2}$

Answer

$x = 4y^2 - 1$; domain: $x \in [1, \infty)$.

22) $x = \dfrac{1}{\sqrt{t+1}}, \quad y = \dfrac{t}{1+t}, \quad \text{for } t > -1$

23) $x = 4 \cos \theta, \quad y = 3 \sin \theta, \quad \text{for } t \in (0, 2\pi]$

Answer

$\frac{x^2}{16} + \frac{y^2}{9} = 1$; domain $x \in [-4, 4]$.

24) $x = \cosh t, \quad y = \sinh t$

25) $x = 2t - 3, \quad y = 6t - 7$

Answer

$y = 3x + 2$; domain: all real numbers.

26) $x = t^2, \quad y = t^3$

27) $x = 1 + \cos t, \quad y = 3 - \sin t$

Answer

$(x-1)^2 + (y-3)^2 = 1$; domain: $x \in [0, 2]$.

28) $x = \sqrt{t}, \quad y = 2t + 4$

29) $x = \sec t, \quad y = \tan t, \quad$ for $\pi \leq t < \frac{3\pi}{2}$

Answer

$y = \sqrt{x^2 - 1}$; domain: $x \in (-\infty, -1]$.

30) $x = 2\cosh t, \quad y = 4\sinh t$

31) $x = \cos(2t), \quad y = \sin t$

Answer

$y^2 = \frac{1 - x}{2}$; domain: $x \in [-1, 1]$.

32) $x = 4t + 3, \quad y = 16t^2 - 9$

33) $x = t^2, \quad y = 2\ln t, \quad$ for $t \geq 1$

Answer

$y = \ln x$; domain: $x \in [1, \infty)$.

34) $x = t^3, \quad y = 3\ln t, \quad$ for $t \geq 1$

35) $x = t^n, \quad y = n\ln t, \quad$ for $t \geq 1$, where n is a natural number

Answer

$y = \ln x$; domain: $x \in (0, \infty)$.

36) $x = \ln(5t), \quad y = \ln(t^2)$ where $1 \leq t \leq e$

37) $x = 2\sin(8t), \quad y = 2\cos(8t)$

Answer

$x^2 + y^2 = 4$; domain: $x \in [-2, 2]$.

38) $x = \tan t, \quad y = \sec^2 t - 1$

In exercises 39 - 48, the pairs of parametric equations represent lines, parabolas, circles, ellipses, or hyperbolas. Name the type of basic curve that each pair of equations represents.

39) $x = 3t + 4, \quad y = 5t - 2$

Answer

line

40) $x - 4 = 5t, \quad y + 2 = t$

41) $x = 2t + 1, \quad y = t^2 - 3$

Answer

parabola

42) $x = 3\cos t, \quad y = 3\sin t$

43) $x = 2\cos(3t), \quad y = 2\sin(3t)$

Answer

circle

44) $x = \cosh t, \quad y = \sinh t$

45) $x = 3\cos t, \quad y = 4\sin t$

Answer

ellipse

46) $x = 2\cos(3t), \quad y = 5\sin(3t)$

47) $x = 3\cosh(4t) \quad y = 4\sinh(4t)$

Answer

the right branch of a horizontally opening hyperbola

48) $x = 2\cosh t, \quad y = 2\sinh t$

49) Show that $x = h + r\cos\theta, \quad y = k + r\sin\theta$ represents the equation of a circle.

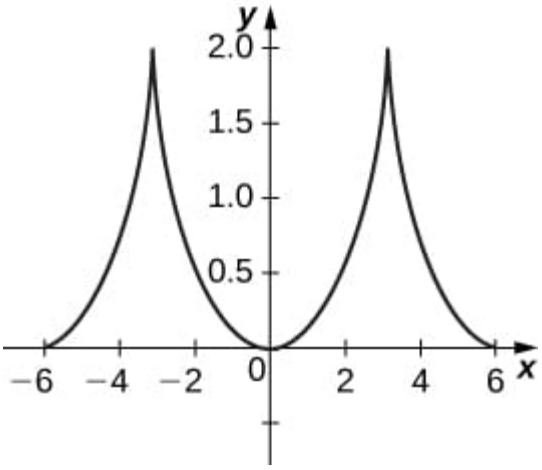
50) Use the equations in the preceding problem to find a set of parametric equations for a circle whose radius is 5 and whose center is $(-2, 3)$.

In exercises 51 - 53, use a graphing utility to graph the curve represented by the parametric equations and identify the curve from its equation.

51) [T] $x = \theta + \sin\theta, \quad y = 1 - \cos\theta$

Answer

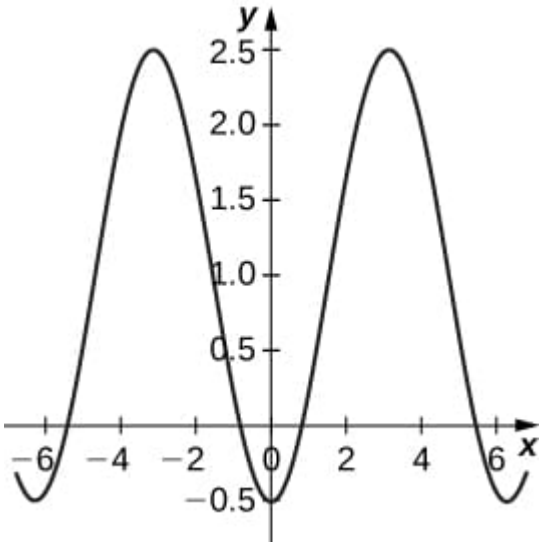
The equations represent a cycloid.



52) [T] $x = 2t - 2\sin t, \quad y = 2 - 2\cos t$

53) [T] $x = t - 0.5\sin t, \quad y = 1 - 1.5\cos t$

Answer



54) An airplane traveling horizontally at 100 m/s over flat ground at an elevation of 4000 meters must drop an emergency package on a target on the ground. The trajectory of the package is given by $x = 100t, \quad y = -4.9t^2 + 4000, \quad$ where $t \geq 0$ where the origin is the point on the ground directly beneath the plane at the moment of release. How many horizontal meters before the target should the package be released in order to hit the target?

Answer

approximately 2857 meters

55) The trajectory of a bullet is given by $x = v_0(\cos\alpha)t, \quad y = v_0(\sin\alpha)t - \frac{1}{2}gt^2$ where $v_0 = 500$ m/s, $g = 9.8 = 9.8\ \text{m/s}^2$, and $\alpha = 30$ degrees. When will the bullet hit the ground? How far from the gun will the bullet hit the ground?

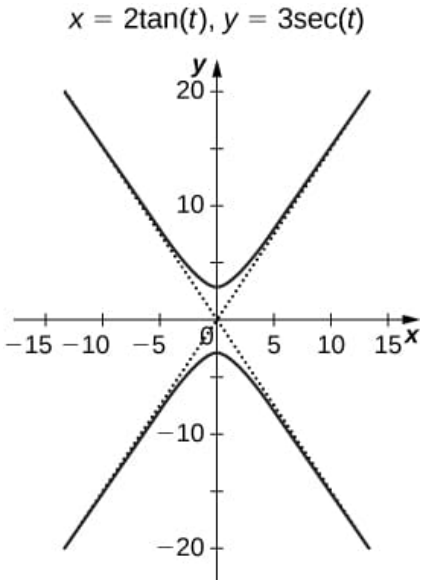
Answer

22,092 meters at approximately 51 seconds.

56) [T] Use technology to sketch the curve represented by $x = \sin(4t), \quad y = \sin(3t), \quad$ for $0 \leq t \leq 2\pi$.

57) [T] Use technology to sketch $x = 2\tan(t), \quad y = 3\sec(t), \quad$ for $-\pi < t < \pi$.

Answer



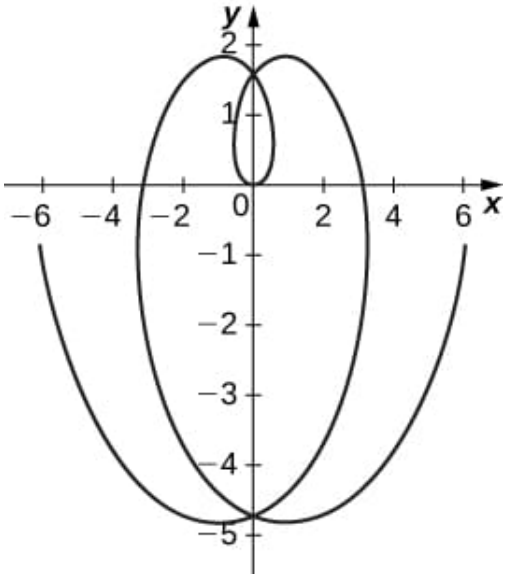
58) Sketch the curve known as an **epitrochoid**, which gives the path of a point on a circle of radius b as it rolls on the outside of a circle of radius a. The equations are

$x = (a+b)\cos t - c \cdot \cos\left[\frac{(a+b)t}{b}\right], \quad y = (a+b)\sin t - c \cdot \sin\left[\frac{(a+b)t}{b}\right].$

Let $a = 1, \ \ b = 2, \ \ c = 1.$

59) [T] Use technology to sketch the spiral curve given by $x = t\cos(t), \quad y = t\sin(t)$ for $-2\pi \le t \le 2\pi.$

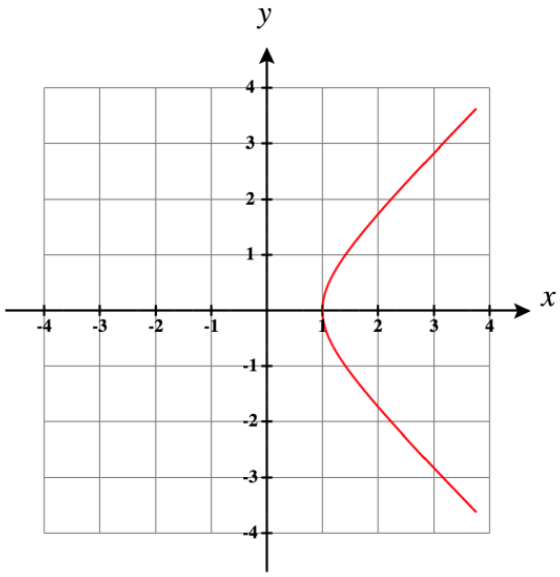
Answer



60) [T] Use technology to graph the curve given by the parametric equations $x = 2\cot(t), \quad y = 1 - \cos(2t), \quad \text{for } -\pi/2 \le t \le \pi/2.$ This curve is known as the witch of Agnesi.

61) [T] Sketch the curve given by parametric equations $x = \cosh(t), \quad y = \sinh(t)$, for $-2 \le t \le 2.$

Answer



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11.3: CALCULUS OF PARAMETRIC CURVES

11.02: Calculus of Parametric Curves

11.3E: Exercises for Section 11.3

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11.3E: Exercises for Section 11.3

In exercises 1 - 4, each set of parametric equations represents a line. Without eliminating the parameter, find the slope of each line.

1) $x = 3 + t, \quad y = 1 - t$

2) $x = 8 + 2t, \quad y = 1$

Answer

$m = 0$

3) $x = 4 - 3t, \quad y = -2 + 6t$

4) $x = -5t + 7, \quad y = 3t - 1$

Answer

$m = -\frac{3}{5}$

In exercises 5 - 9, determine the slope of the tangent line, then find the equation of the tangent line at the given value of the parameter.

5) $x = 3\sin t, \quad y = 3\cos t, \quad \text{for } t = \frac{\pi}{4}$

6) $x = \cos t, \quad y = 8\sin t, \quad \text{for } t = \frac{\pi}{2}$

Answer

Slope $= 0; y = 8.$

7) $x = 2t, \quad y = t^3, \quad \text{for } t = -1$

8) $x = t + \dfrac{1}{t}, \quad y = t - \dfrac{1}{t}, \quad \text{for } t = 1$

Answer

Slope is undefined; $x = 2.$

9) $x = \sqrt{t}, \quad y = 2t, \quad \text{for } t = 4$

In exercises 10 - 13, find all points on the curve that have the given slope.

10) $x = 4\cos t, \quad y = 4\sin t$, slope = 0.5

Solution

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4\cos t}{-4\sin t} = -\cot t.$

Setting this derivative equal to 0.5, we obtain the equation, $\tan t = -2.$

$\tan t = -2 \implies \dfrac{y}{x} = -2 \implies y = -2x.$

Note also that this pair of parametric equations represents the circle $x^2 + y^2 = 16.$

By substitution, we find that this curve has a slope of 0.5 at the points:

$\left(\frac{4\sqrt{5}}{5}, \frac{-8\sqrt{5}}{5}\right)$ and $\left(\frac{-4\sqrt{5}}{5}, \frac{8\sqrt{5}}{5}\right)$.

11) $x = 2\cos t, \quad y = 8\sin t$, slope= -1

12) $x = t + \dfrac{1}{t}, \quad y = t - \dfrac{1}{t}$, slope= 1

Answer

No points possible; undefined expression.

13) $x = 2 + \sqrt{t}, \quad y = 2 - 4t$, slope= 0

In exercises 14 - 16, write the equation of the tangent line in Cartesian coordinates for the given parameter t.

14) $x = e^{\sqrt{t}}, \quad y = 1 - \ln t^2, \quad$ for $t = 1$

Answer

$y = -\left(\frac{4}{e}\right)x + 5$

15) $x = t \ln t, \quad y = \sin^2 t, \quad$ for $t = \frac{\pi}{4}$

16) $x = e^t, \quad y = (t-1)^2$, at $(1, 1)$

Answer

$y = -2x + 3$

17) For $x = \sin(2t), \quad y = 2 \sin t$ where $0 \le t < 2\pi$. Find all values of t at which a horizontal tangent line exists.

18) For $x = \sin(2t), \quad y = 2 \sin t$ where $0 \le t < 2\pi$. Find all values of t at which a vertical tangent line exists.

Answer

A vertical tangent line exists at $t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

19) Find all points on the curve $x = 4\cos(t), \quad y = 4\sin(t)$ that have the slope of $\frac{1}{2}$.

20) Find $\dfrac{dy}{dx}$ for $x = \sin(t), \quad y = \cos(t)$.

Answer

$$\frac{dy}{dx} = -\tan(t)$$

21) Find the equation of the tangent line to $x = \sin(t), \quad y = \cos(t)$ at $t = \frac{\pi}{4}$.

22) For the curve $x = 4t, \quad y = 3t - 2$, find the slope and concavity of the curve at $t = 3$.

Answer

$\dfrac{dy}{dx} = \dfrac{3}{4}$ and $\dfrac{d^2y}{dx^2} = 0$, so the curve is neither concave up nor concave down at $t = 3$. Therefore the graph is linear and has a constant slope but no concavity.

23) For the parametric curve whose equation is $x = 4\cos\theta, \quad y = 4\sin\theta$, find the slope and concavity of the curve at $\theta = \frac{\pi}{4}$.

24) Find the slope and concavity for the curve whose equation is $x = 2 + \sec\theta, \quad y = 1 + 2\tan\theta$ at $\theta = \frac{\pi}{6}$.

Answer

$\dfrac{dy}{dx} = 4, \quad \dfrac{d^2y}{dx^2} = -4\sqrt{3}$; the curve is concave down at $\theta = \frac{\pi}{6}$.

25) Find all points on the curve $x = t + 4, \quad y = t^3 - 3t$ at which there are vertical and horizontal tangents.

26) Find all points on the curve $x = \sec\theta, \quad y = \tan\theta$ at which horizontal and vertical tangents exist.

Answer

No horizontal tangents. Vertical tangents at $(1, 0)$ and $(-1, 0)$.

In exercises 27 - 29, find d^2y/dx^2.

27) $x = t^4 - 1, \quad y = t - t^2$

28) $x = \sin(\pi t), \quad y = \cos(\pi t)$

Answer

$d^2y/dx^2 = -\sec^3(\pi t)$

29) $x = e^{-t}, \quad y = te^{2t}$

In exercises 30 - 31, find points on the curve at which tangent line is horizontal or vertical.

30) $x = t(t^2 - 3), \quad y = 3(t^2 - 3)$

Answer

Horizontal $(0, -9)$;
Vertical $(\pm 2, -6)$.

31) $x = \dfrac{3t}{1 + t^3}, \quad y = \dfrac{3t^2}{1 + t^3}$

In exercises 32 - 34, find dy/dx at the value of the parameter.

32) $x = \cos t, y = \sin t, \quad \text{for } t = \frac{3\pi}{4}$

Answer

$dy/dx = 1$

33) $x = \sqrt{t}, \quad y = 2t + 4, t = 9$

34) $x = 4\cos(2\pi s), \quad y = 3\sin(2\pi s), \quad \text{for } s = -\frac{1}{4}$

Answer

$dy/dx = 0$

In exercises 35 - 36, find d^2y/dx^2 at the given point without eliminating the parameter.

35) $x = \frac{1}{2}t^2, \quad y = \frac{1}{3}t^3, \quad \text{for } t = 2$

36) $x = \sqrt{t}, \quad y = 2t + 4, \quad \text{for } t = 1$

Answer

$d^2y/dx^2 = 4$

37) Find intervals for t on which the curve $x = 3t^2, \quad y = t^3 - t$ is concave up as well as concave down.

38) Determine the concavity of the curve $x = 2t + \ln t, \quad y = 2t - \ln t$.

Answer

Concave up on $t > 0$.

39) Sketch and find the area under one arch of the cycloid $x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta)$.

40) Find the area bounded by the curve $x = \cos t, \quad y = e^t, \quad \text{for } 0 \le t \le \frac{\pi}{2}$ and the lines $y = 1$ and $x = 0$.

Answer

1 unit^2

41) Find the area enclosed by the ellipse $x = a\cos\theta, \quad y = b\sin\theta, \quad \text{for } 0 \le \theta < 2\pi$.

42) Find the area of the region bounded by $x = 2\sin^2\theta, \quad y = 2\sin^2\theta\tan\theta$, for $0 \le \theta \le \frac{\pi}{2}$.

Answer

$\frac{3\pi}{2} \text{ units}^2$

In exercises 43 - 46, find the area of the regions bounded by the parametric curves and the indicated values of the parameter.

43) $x = 2\cot\theta, \quad y = 2\sin^2\theta, \quad \text{for } 0 \le \theta \le \pi$

Answer

$-4\pi \text{ units}^2$

44) [T] $x = 2a\cos t - a\cos(2t), \quad y = 2a\sin t - a\sin(2t), \quad \text{for } 0 \le t < 2\pi$

Answer

$6\pi a^2 \text{ units}^2$

45) [T] $x = a\sin(2t), \quad y = b\sin(t), \quad \text{for } 0 \le t < 2\pi$ (the "hourglass")

Answer

$\frac{28}{3} \text{ units}^2$

46) [T] $x = 2a\cos t - a\sin(2t), \quad y = b\sin t, \quad \text{for } 0 \le t < 2\pi$ (the "teardrop")

Answer

$2\pi ab \text{ units}^2$

In exercises 47 - 52, find the arc length of the curve on the indicated interval of the parameter.

47) $x = 4t + 3, \quad y = 3t - 2, \quad \text{for } 0 \le t \le 2$

48) $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2, \quad \text{for } 0 \le t \le 1$

Answer

$s = \frac{1}{3}(2\sqrt{2} - 1)$ units

49) $x = \cos(2t), \quad y = \sin(2t), \quad \text{for } 0 \le t \le \frac{\pi}{2}$

50) $x = 1 + t^2, \quad y = (1 + t)^3, \quad \text{for } 0 \le t \le 1$

Answer

$s = 7.075$ units

51) $x = e^t\cos t, \quad y = e^t\sin t, \quad \text{for } 0 \le t \le \frac{\pi}{2}$ (express answer as a decimal rounded to three places)

52) $x = a\cos^3\theta, \quad y = a\sin^3\theta$ on the interval $[0, 2\pi)$ (the hypocycloid) Hint: Use symmetry by finding 4 times integral with limits from 0 to π/2

Answer

$s = 6a$ units

53) Find the length of one arch of the cycloid $x = 4(t - \sin t), \quad y = 4(1 - \cos t).$

54) Find the distance traveled by a particle with position (x, y) as t varies in the given time interval: $x = \sin^2 t, \quad y = \cos^2 t, \quad \text{for } 0 \le t \le 3\pi.$

Answer

$6\sqrt{2}$ units

55) Find the length of one arch of the cycloid $x = \theta - \sin\theta, \quad y = 1 - \cos\theta.$

56) Show that the total length of the ellipse $x = 4\sin\theta, \quad y = 3\cos\theta$ is $L = 16\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\, d\theta$, where $e = \frac{c}{a}$ and $c = \sqrt{a^2 - b^2}$

.

57) Find the length of the curve $x = e^t - t, \quad y = 4e^{t/2}, \quad \text{for } -8 \le t \le 3.$

In exercises 58 - 59, find the area of the surface obtained by rotating the given curve about the x-axis.

58) $x = t^3, \quad y = t^2, \quad \text{for } 0 \le t \le 1$

Answer

$$\frac{2\pi(247\sqrt{13}+64)}{1215} \text{ units}^2$$

59) $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad \text{for } 0 \le \theta \le \frac{\pi}{2}$

60) [T] Use a CAS to find the area of the surface generated by rotating $x = t + t^3, \quad y = t - \frac{1}{t^2}, \quad \text{for } 1 \le t \le 2$ about the x-axis. (Answer to three decimal places.)

Answer

$$59.101 \text{ units}^2$$

61) Find the surface area obtained by rotating $x = 3t^2, \quad y = 2t^3, \quad \text{for } 0 \le t \le 5$ about the y-axis.

62) Find the area of the surface generated by revolving $x = t^2, \quad y = 2t, \quad \text{for } 0 \le t \le 4$ about the x-axis.

Answer

$$\frac{8\pi}{3}(17\sqrt{17}-1) \text{ units}^2$$

63) Find the surface area generated by revolving $x = t^2, \quad y = 2t^2, \quad \text{for } 0 \le t \le 1$ about the y-axis.

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11.4: POLAR COORDINATES

11.03: Polar Coordinates

11.4E: Exercises for Section 11.4

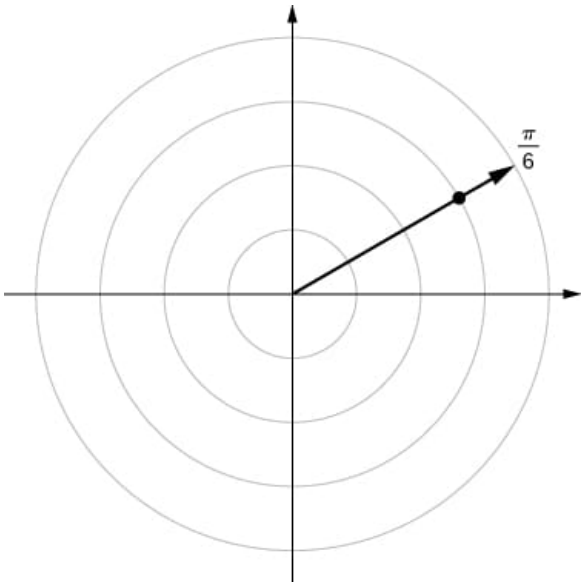
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11.4E: Exercises for Section 11.4

In exercises 1 - 7, plot the point whose polar coordinates are given by first constructing the angle θ and then marking off the distance r along the ray.

1) $\left(3, \frac{\pi}{6}\right)$

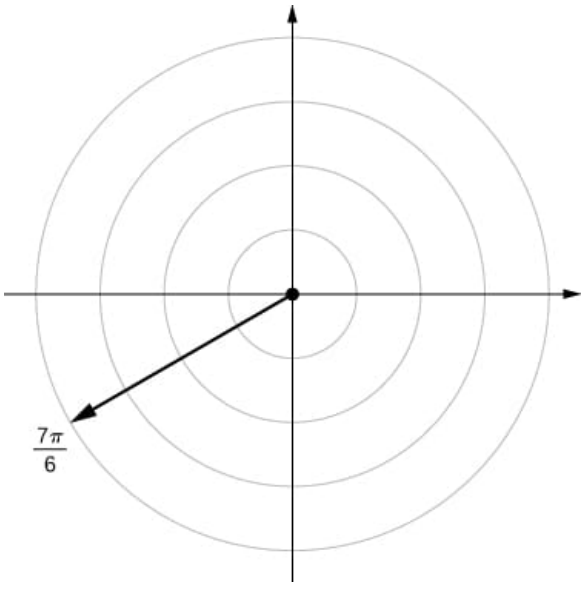
Answer



2) $\left(-2, \frac{5\pi}{3}\right)$

3) $\left(0, \frac{7\pi}{6}\right)$

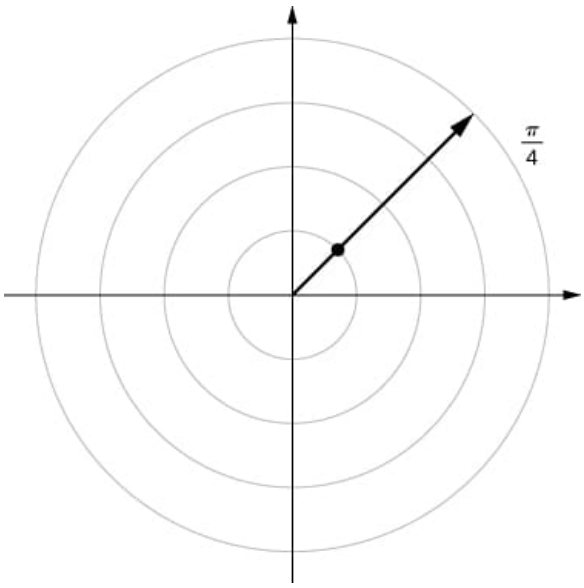
Answer



4) $\left(-4, \frac{3\pi}{4}\right)$

5) $\left(1, \frac{\pi}{4}\right)$

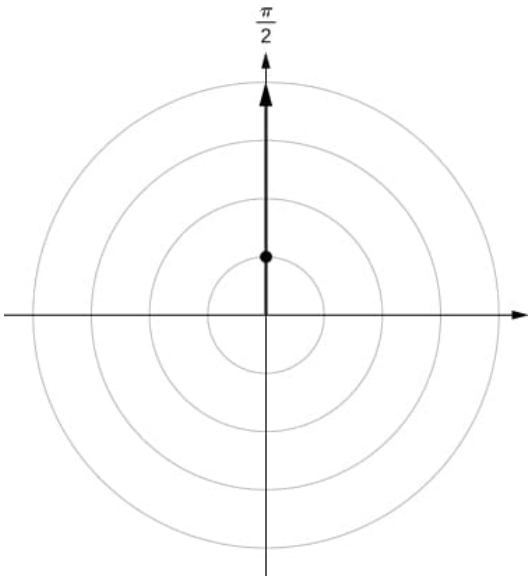
Answer



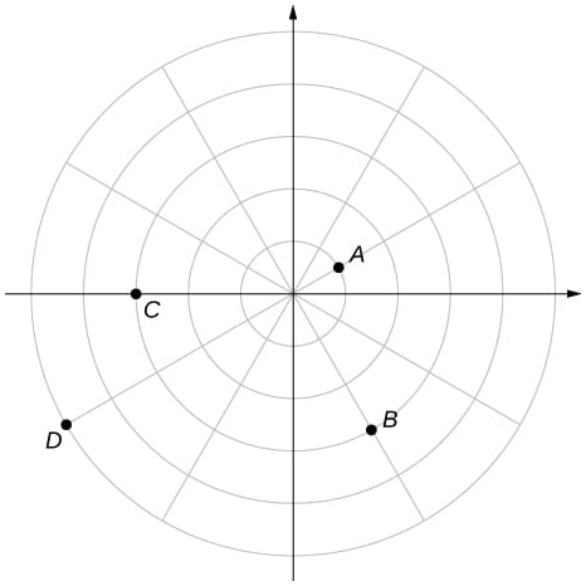
6) $(2, \frac{5\pi}{6})$

7) $(1, \frac{\pi}{2})$

Answer



In exercises 8 - 11, consider the polar graph below. Give two sets of polar coordinates for each point.



8) Coordinates of point A.

9) Coordinates of point B.

Answer

$B\left(3, \frac{-\pi}{3}\right)\, B\left(-3, \frac{2\pi}{3}\right)$

10) Coordinates of point C.

11) Coordinates of point D.

Answer

$D\left(5, \frac{7\pi}{6}\right)\, D\left(-5, \frac{\pi}{6}\right)$

In exercises 12 - 17, the rectangular coordinates of a point are given. Find two sets of polar coordinates for the point in $(0, 2\pi]$. Round to three decimal places.

12) $(2, 2)$

13) $(3, -4)$

Answer

$(5, -0.927),\ \ (-5, -0.927 + \pi)$

14) $(8, 15)$

15) $(-6, 8)$

Answer

$(10, -0.927),\ \ (-10, -0.927 + \pi)$

16) $(4, 3)$

17) $(3, -\sqrt{3})$

Answer

$(2\sqrt{3}, -0.524),\ \ (-2\sqrt{3}, -0.524 + \pi)$

In exercises 18 - 24, find rectangular coordinates for the given point in polar coordinates.

18) $\left(2, \frac{5\pi}{4}\right)$

19) $\left(-2, \frac{\pi}{6}\right)$

Answer

$(-\sqrt{3}, -1)$

20) $\left(5, \frac{\pi}{3}\right)$

21) $\left(1, \frac{7\pi}{6}\right)$

Answer

$\left(-\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

22) $\left(-3, \frac{3\pi}{4}\right)$

23) $\left(0, \frac{\pi}{2}\right)$

Answer

$(0, 0)$

24) $(-4.5, 6.5)$

In exercises 25 - 29, determine whether the graphs of the polar equation are symmetric with respect to the x-axis, the y-axis, or the origin.

25) $r = 3\sin(2\theta)$

Answer

Symmetry with respect to the *x*-axis, *y*-axis, and origin.

26) $r^2 = 9\cos\theta$

27) $r = \cos\left(\frac{\theta}{5}\right)$

Answer

Symmetric with respect to *x*-axis only.

28) $r = 2\sec\theta$

29) $r = 1 + \cos\theta$

Answer

Symmetry with respect to *x*-axis only.

In exercises 30 - 33, describe the graph of each polar equation. Confirm each description by converting into a rectangular equation.

30) $r = 3$

31) $\theta = \frac{\pi}{4}$

Answer

Line $y = x$

32) $r = \sec\theta$

33) $r = \csc\theta$

Answer

$y = 1$

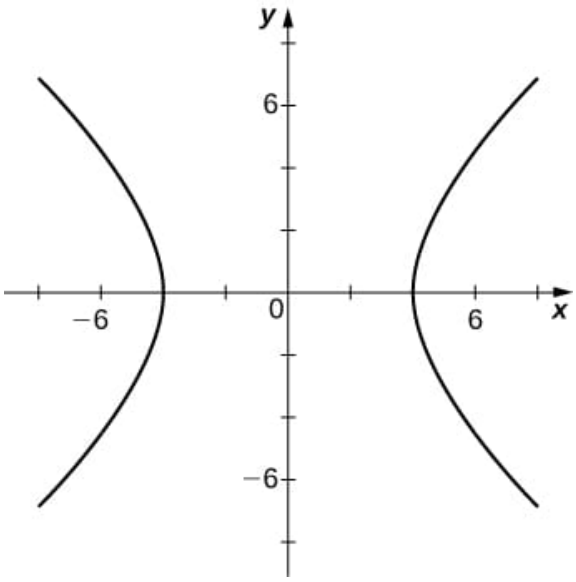
In exercises 34 - 36, convert the rectangular equation to polar form and sketch its graph.

34) $x^2 + y^2 = 16$

35) $x^2 - y^2 = 16$

Answer

Hyperbola; polar form $r^2 \cos(2\theta) = 16$ or $r^2 = 16 \sec \theta$.



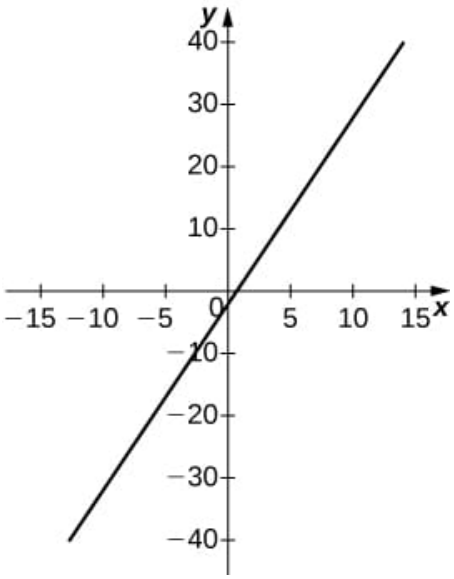
36) $x = 8$

In exercises 37 - 38, convert the rectangular equation to polar form and sketch its graph.

37) $3x - y = 2$

Answer

$r = \frac{2}{3\cos\theta - \sin\theta}$

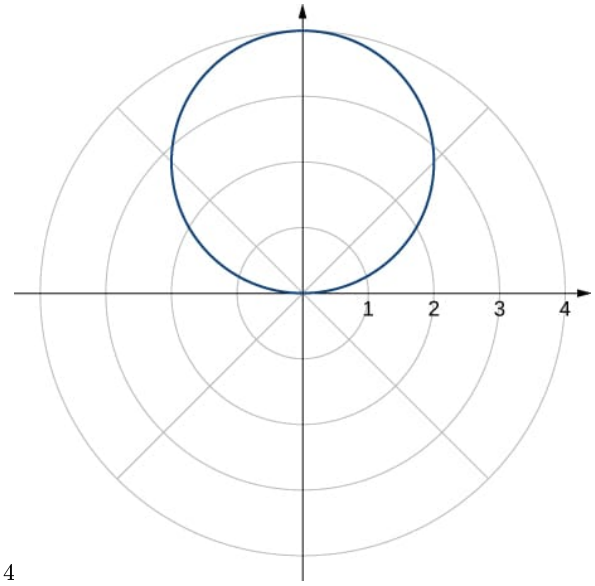


38) $y^2 = 4x$

In exercises 39 - 43, convert the polar equation to rectangular form and sketch its graph.

39) $r = 4\sin\theta$

Answer



$x^2 + (y-2)^2 = 4$

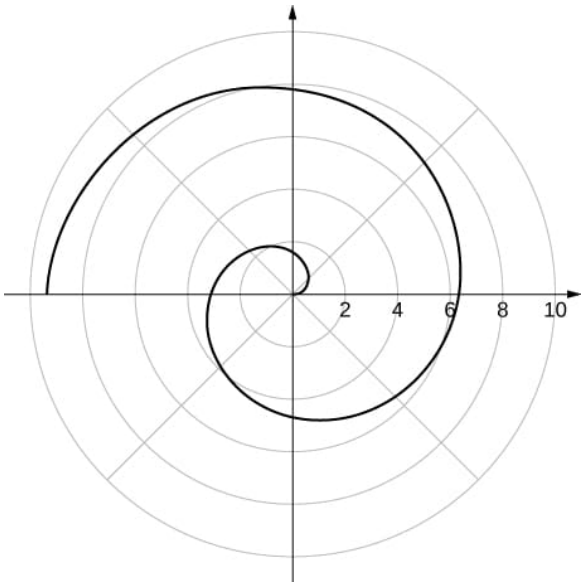
40)

41) $r = 6\cos\theta$

42) $r = \theta$

Answer

$x \tan \sqrt{x^2 + y^2} = y$



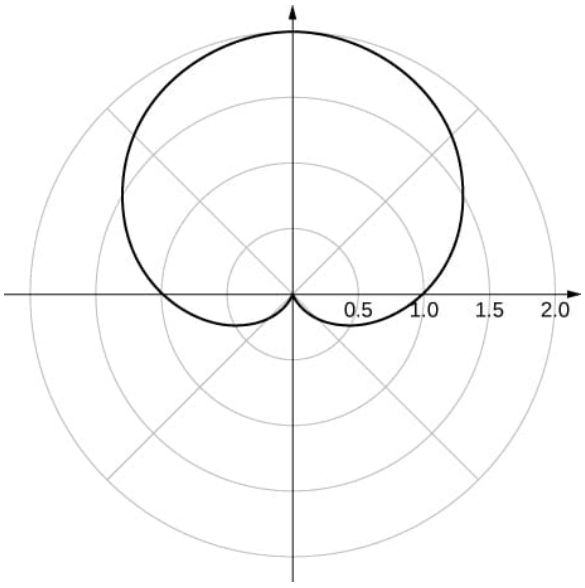
43) $r = \cot\theta \csc\theta$

In exercises 44 - 54, sketch a graph of the polar equation and identify any symmetry.

44) $r = 1 + \sin\theta$

Answer

y-axis symmetry

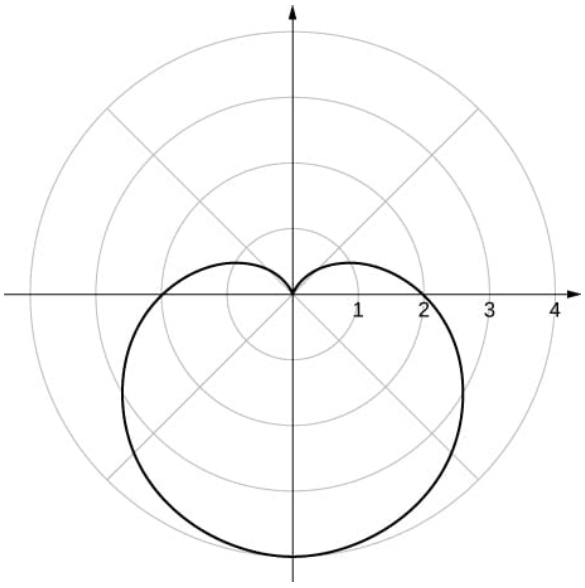


45) $r = 3 - 2\cos\theta$

46) $r = 2 - 2\sin\theta$

Answer

y-axis symmetry

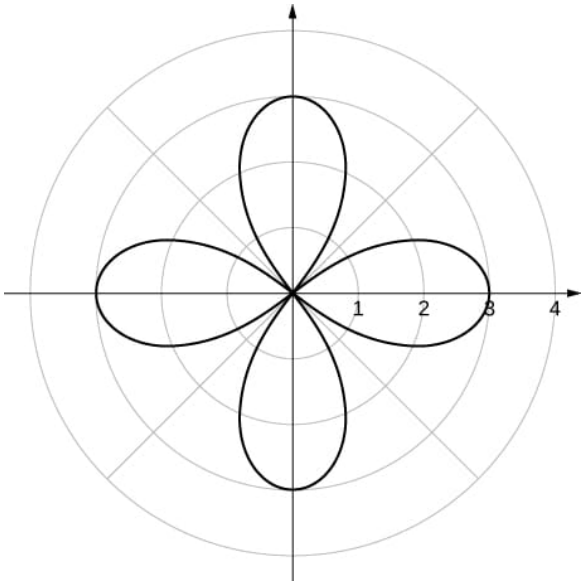


47) $r = 5 - 4\sin\theta$

48) $r = 3\cos(2\theta)$

Answer

x-and y-axis symmetry and symmetry about the pole

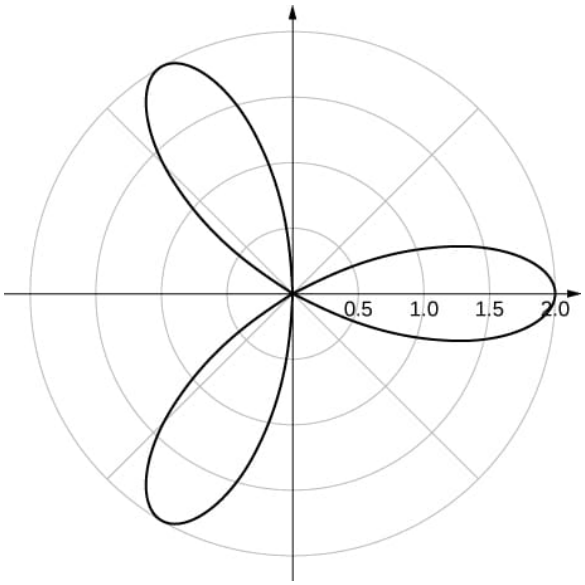


49) $r = 3\sin(2\theta)$

50) $r = 2\cos(3\theta)$

Answer

x-axis symmetry

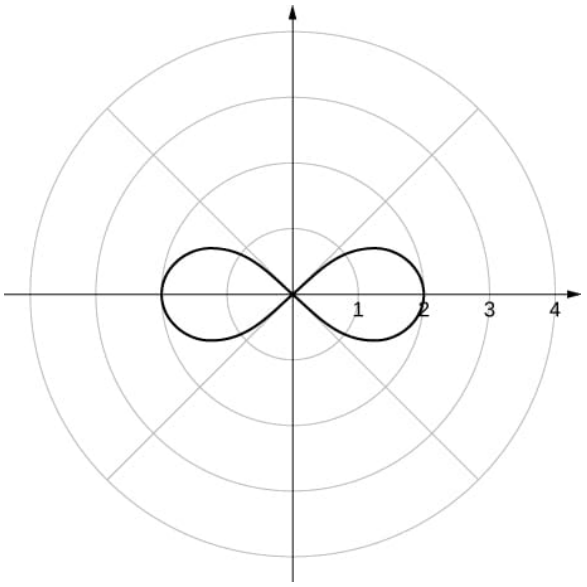


51) $r = 3\cos\left(\frac{\theta}{2}\right)$

52) $r^2 = 4\cos\left(\frac{2}{\theta}\right)$

Answer

x-and y-axis symmetry and symmetry about the pole

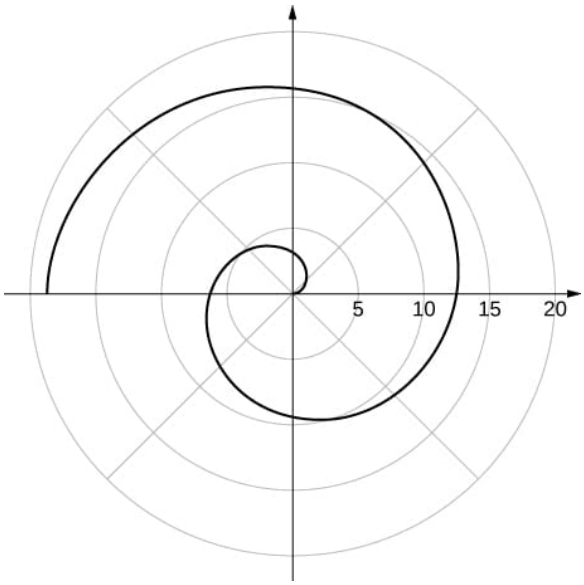


53) $r^2 = 4\sin\theta$

54) $r = 2\theta$

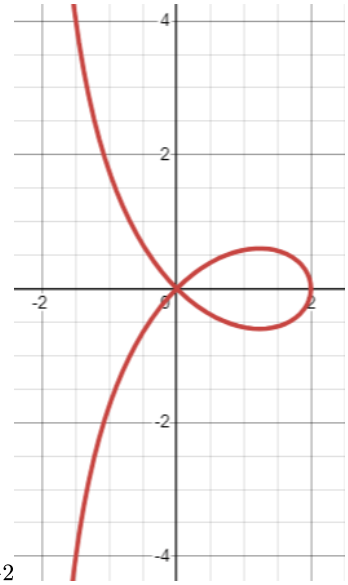
Answer

no symmetry



55) [T] The graph of $r = 2\cos(2\theta)\sec(\theta).$ is called a strophoid. Use a graphing utility to sketch the graph, and, from the graph, determine the asymptote.

Answer

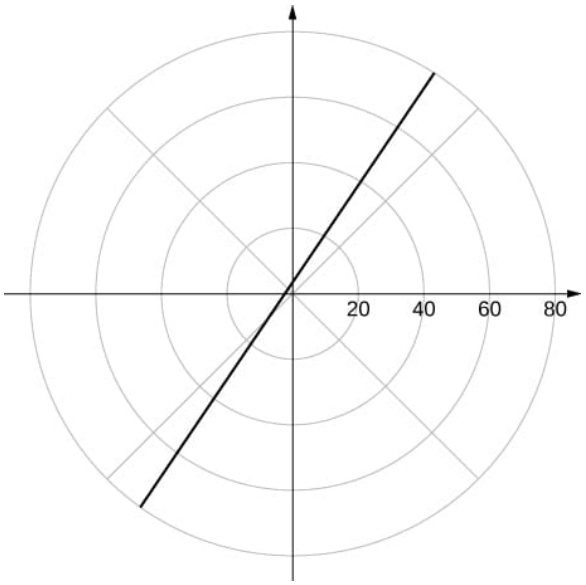


VA at $x = -2$

56) [T] Use a graphing utility and sketch the graph of $r = \dfrac{6}{2\sin\theta - 3\cos\theta}$.

Answer

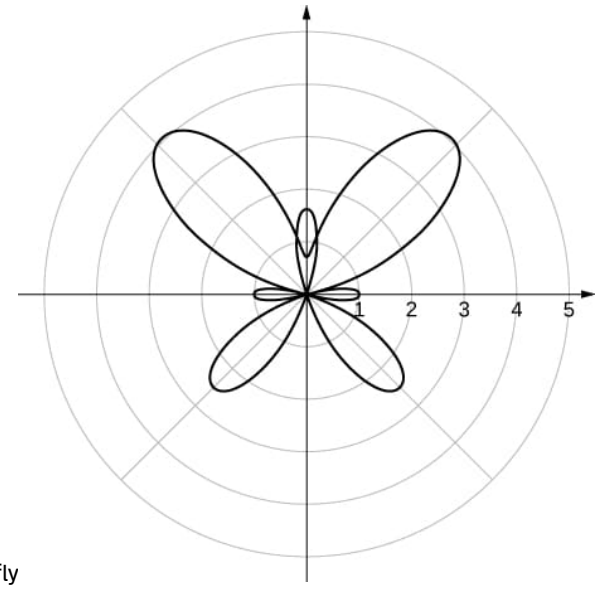
a line



57) [T] Use a graphing utility to graph $r = \frac{1}{1-\cos\theta}$.

58) [T] Use technology to graph $r = e^{\sin(\theta)} - 2\cos(4\theta)$.

Answer

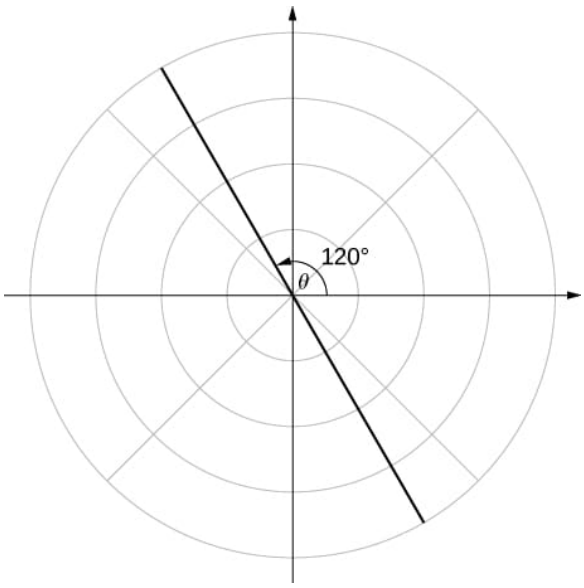


the rare polar butterfly

59) [T] Use technology to plot $r = \sin(\frac{3\theta}{7})$ (use the interval $0 \leq \theta \leq 14\pi$).

60) Without using technology, sketch the polar curve $\theta = \frac{2\pi}{3}$.

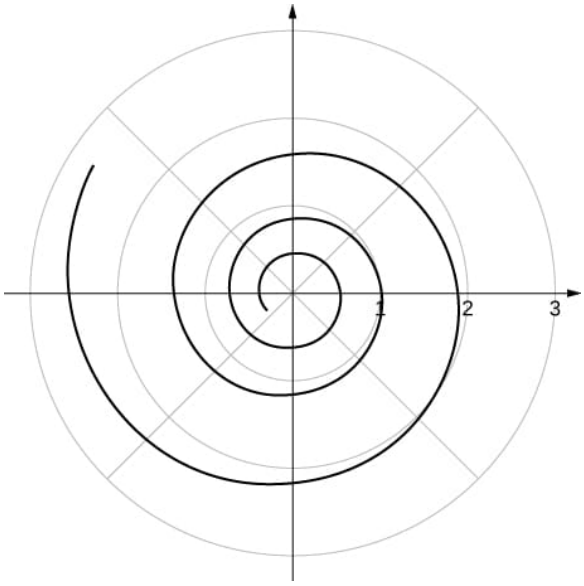
Answer



61) [T] Use a graphing utility to plot $r = \theta \sin \theta$ for $-\pi \leq \theta \leq \pi$.

62) [T] Use technology to plot $r = e^{-0.1\theta}$ for $-10 \leq \theta \leq 10$.

Answer



63) [T] There is a curve known as the "Black Hole." Use technology to plot $r = e^{-0.01\theta}$ for $-100 \leq \theta \leq 100$.

64) [T] Use the results of the preceding two problems to explore the graphs of $r = e^{-0.001\theta}$ and $r = e^{-0.0001\theta}$ for $|\theta| > 100$.

Answer

Answers vary. One possibility is the spiral lines become closer together and the total number of spirals increases.

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11.5: AREA AND ARC LENGTH IN POLAR COORDINATES

11.04: Area and Arc Length in Polar Coordinates

11.5E: Exercises for Section 11.5

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11.5E: Exercises for Section 11.5

In exercises 1 -13, determine a definite integral that represents the area.

1) Region enclosed by $r = 4$

2) Region enclosed by $r = 3\sin\theta$

Answer

$$\frac{9}{2}\int_0^{\pi} \sin^2\theta \; d\theta$$

3) Region in the first quadrant within the cardioid $r = 1 + \sin\theta$

4) Region enclosed by one petal of $r = 8\sin(2\theta)$

Answer

$$32\int_0^{\pi/2} \sin^2(2\theta) \; d\theta$$

5) Region enclosed by one petal of $r = cos(3\theta)$

6) Region below the polar axis and enclosed by $r = 1 - \sin\theta$

Answer

$$\frac{1}{2}\int_{\pi}^{2\pi} (1 - \sin\theta)^2 \; d\theta$$

7) Region in the first quadrant enclosed by $r = 2 - \cos\theta$

8) Region enclosed by the inner loop of $r = 2 - 3\sin\theta$

Answer

$$\int_{\sin^{-1}(2/3)}^{\pi/2} (2 - 3\sin\theta)^2 \; d\theta$$

9) Region enclosed by the inner loop of $r = 3 - 4\cos\theta$

10) Region enclosed by $r = 1 - 2\cos\theta$ and outside the inner loop

Answer

$$\int_0^{\pi} (1 - 2\cos\theta)^2 \; d\theta - \int_0^{\pi/3} (1 - 2\cos\theta)^2 \; d\theta$$

11) Region common to $r = 3\sin\theta$ and $r = 2 - \sin\theta$

12) Region common to $r = 2$ and $r = 4\cos\theta$

Answer

$$4\int_0^{\pi/3} \; d\theta + 16\int_{\pi/3}^{\pi/2} (\cos^2\theta) \; d\theta$$

13) Region common to $r = 3\cos\theta$ and $r = 3\sin\theta$

In exercises 14 -26, find the area of the described region.

14) Enclosed by $r = 6\sin\theta$

Answer

$9\pi \text{ units}^2$

15) Above the polar axis enclosed by $r = 2 + \sin\theta$

16) Below the polar axis and enclosed by $r = 2 - \cos\theta$

Answer

$\frac{9\pi}{4} \text{ units}^2$

17) Enclosed by one petal of $r = 4\cos(3\theta)$

18) Enclosed by one petal of $r = 3\cos(2\theta)$

Answer

$\frac{9\pi}{8} \text{ units}^2$

19) Enclosed by $r = 1 + \sin\theta$

20) Enclosed by the inner loop of $r = 3 + 6\cos\theta$

Answer

$\frac{18\pi - 27\sqrt{3}}{2} \text{ units}^2$

21) Enclosed by $r = 2 + 4\cos\theta$ and outside the inner loop

22) Common interior of $r = 4\sin(2\theta)$ and $r = 2$

Answer

$\frac{4}{3}(4\pi - 3\sqrt{3}) \text{ units}^2$

23) Common interior of $r = 3 - 2\sin\theta$ and $r = -3 + 2\sin\theta$

24) Common interior of $r = 6\sin\theta$ and $r = 3$

Answer

$\frac{3}{2}(4\pi - 3\sqrt{3}) \text{ units}^2$

25) Inside $r = 1 + \cos\theta$ and outside $r = \cos\theta$

26) Common interior of $r = 2 + 2\cos\theta$ and $r = 2\sin\theta$

Answer

$(2\pi - 4) \text{ units}^2$

In exercises 27 - 30, find a definite integral that represents the arc length.

27) $r = 4\cos\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

28) $r = 1 + \sin\theta$ on the interval $0 \le \theta \le 2\pi$

Answer

$$\int_0^{2\pi} \sqrt{(1 + \sin\theta)^2 + \cos^2\theta}\, d\theta$$

29) $r = 2\sec\theta$ on the interval $0 \le \theta \le \frac{\pi}{3}$

30) $r = e^{\theta}$ on the interval $0 \le \theta \le 1$

Answer

$$\sqrt{2}\int_0^1 e^\theta \, d\theta$$

In exercises 31 - 35, find the length of the curve over the given interval.

31) $r = 6$ on the interval $0 \le \theta \le \frac{\pi}{2}$

32) $r = e^{3\theta}$ on the interval $0 \le \theta \le 2$

Answer

$\frac{\sqrt{10}}{3}(e^6 - 1)$ units

33) $r = 6\cos\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

34) $r = 8 + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

Answer

32 units

35) $r = 1 - \sin\theta$ on the interval $0 \le \theta \le 2\pi$

In exercises 36 - 40, use the integration capabilities of a calculator to approximate the length of the curve.

36) [T] $r = 3\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

Answer

6.238 units

37) [T] $r = \dfrac{2}{\theta}$ on the interval $\pi \le \theta \le 2\pi$

38) [T] $r = \sin^2\left(\frac{\theta}{2}\right)$ on the interval $0 \le \theta \le \pi$

Answer

2 units

39) [T] $r = 2\theta^2$ on the interval $0 \le \theta \le \pi$

40) [T] $r = \sin(3\cos\theta)$ on the interval $0 \le \theta \le \pi$

Answer

4.39 units

In exercises 41 - 43, use the familiar formula from geometry to find the area of the region described and then confirm by using the definite integral.

41) $r = 3\sin\theta$ on the interval $0 \le \theta \le \pi$

42) $r = \sin\theta + \cos\theta$ on the interval $0 \le \theta \le \pi$

Answer

$$A = \pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{2} \text{ units}^2 \text{ and } \frac{1}{2}\int_0^\pi (1 + 2\sin\theta\cos\theta) \, d\theta = \frac{\pi}{2} \text{ units}^2$$

43) $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

In exercises 44 - 46, use the familiar formula from geometry to find the length of the curve and then confirm using the definite integral.

44) $r = 3\sin\theta$ on the interval $0 \le \theta \le \pi$

Answer

$C = 2\pi\left(\frac{3}{2}\right) = 3\pi$ units and $\int_0^{\pi} 3 \; d\theta = 3\pi$ units

45) $r = \sin\theta + \cos\theta$ on the interval $0 \leq \theta \leq \pi$

46) $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \leq \theta \leq \pi$

Answer

$C = 2\pi(5) = 10\pi$ units and $\int_0^{\pi} 10 \; d\theta = 10\pi$ units

47) Verify that if $y = r\sin\theta = f(\theta)\sin\theta$ then $\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$.

In exercises 48 - 56, find the slope of a tangent line to a polar curve $r = f(\theta)$. Let $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$, so the polar equation $r = f(\theta)$ is now written in parametric form.

48) Use the definition of the derivative $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ and the product rule to derive the derivative of a polar equation.

Answer

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

49) $r = 1 - \sin\theta; \;\; \left(\frac{1}{2}, \frac{\pi}{6}\right)$

50) $r = 4\cos\theta; \;\; \left(2, \frac{\pi}{3}\right)$

Answer

The slope is $\frac{1}{\sqrt{3}}$.

51) $r = 8\sin\theta; \;\; \left(4, \frac{5\pi}{6}\right)$

52) $r = 4 + \sin\theta; \;\; \left(3, \frac{3\pi}{2}\right)$

Answer

The slope is 0.

53) $r = 6 + 3\cos\theta; \;\; (3, \pi)$

54) $r = 4\cos(2\theta)$; tips of the leaves

Answer

At $(4, 0)$, the slope is undefined. At $\left(-4, \frac{\pi}{2}\right)$, the slope is 0.

55) $r = 2\sin(3\theta)$; tips of the leaves

56) $r = 2\theta; \;\; \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$

Answer

The slope is undefined at $\theta = \frac{\pi}{4}$.

57) Find the points on the interval $-\pi \leq \theta \leq \pi$ at which the cardioid $r = 1 - \cos\theta$ has a vertical or horizontal tangent line.

58) For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \frac{\pi}{3}$.

Answer

Slope = –1.

In exercises 59 - 62, find the slope of the tangent line to the given polar curve at the point given by the value of θ.

59) $r = 3\cos\theta, \quad \theta = \frac{\pi}{3}$

60) $r = \theta, \quad \theta = \frac{\pi}{2}$

Answer

Slope is $\frac{-2}{\pi}$.

61) $r = \ln\theta, \quad \theta = e$

62) [T] Use technology: $r = 2 + 4\cos\theta$ at $\theta = \frac{\pi}{6}$

Answer

Calculator answer: –0.836.

In exercises 63 - 66, find the points at which the following polar curves have a horizontal or vertical tangent line.

63) $r = 4\cos\theta$

64) $r^2 = 4\cos(2\theta)$

Answer

Horizontal tangent at $\left(\pm\sqrt{2}, \frac{\pi}{6}\right), \quad \left(\pm\sqrt{2}, -\frac{\pi}{6}\right)$.

65) $r = 2\sin(2\theta)$

66) The cardioid $r = 1 + \sin\theta$

Answer

Horizontal tangents at $\frac{\pi}{2}, \quad \frac{7\pi}{6}, \quad \frac{11\pi}{6}$.
Vertical tangents at $\frac{\pi}{6}, \quad \frac{5\pi}{6}$ and also at the pole $(0, 0)$.

67) Show that the curve $r = \sin\theta\tan\theta$ (called a **cissoid of Diocles**) has the line $x = 1$ as a vertical asymptote.

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11.6: Chapter 11 Review Exercises

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APPENDICES

A: Table of Derivatives

B: Table of Integrals

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A: Table of Derivatives

General Formulas

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(f(x) + g(x)) = f\,'(x) + g\,'(x)$
3. $\frac{d}{dx}(f(x)g(x)) = f\,'(x)g(x) + f(x)g\,'(x)$
4. $\frac{d}{dx}(x^n) = nx^{n-1}, \quad$ for real numbers n
5. $\frac{d}{dx}(cf(x)) = cf\,'(x)$
6. $\frac{d}{dx}(f(x) - g(x)) = f\,'(x) - g\,'(x)$
7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
8. $\frac{d}{dx}[f(g(x))] = f\,'(g(x)) \cdot g\,'(x)$

Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$
10. $\frac{d}{dx}(\tan x) = \sec^2 x$
11. $\frac{d}{dx}(\sec x) = \sec x \tan x$
12. $\frac{d}{dx}(\cos x) = -\sin x$
13. $\frac{d}{dx}(\cot x) = -\csc^2 x$
14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Inverse Trigonometric Functions

15. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
17. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

18. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$

19. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$

20. $\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$

Exponential and Logarithmic Functions

21. $\frac{d}{dx}\left(e^x\right) = e^x$

22. $\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}$

23. $\frac{d}{dx}\left(b^x\right) = b^x \ln b$

24. $\frac{d}{dx}\left(\log_b x\right) = \frac{1}{x \ln b}$

Hyperbolic Functions

25. $\frac{d}{dx}\left(\sinh x\right) = \cosh x$

26. $\frac{d}{dx}\left(\tanh x\right) = \operatorname{sech}^2 x$

27. $\frac{d}{dx}\left(\operatorname{sech} x\right) = -\operatorname{sech} x \tanh x$

28. $\frac{d}{dx}\left(\cosh x\right) = \sinh x$

29. $\frac{d}{dx}\left(\coth x\right) = -\operatorname{csch}^2 x$

30. $\frac{d}{dx}\left(\operatorname{csch} x\right) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic Functions

31. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$

32. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2} \quad (|x|<1)$

33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}} \quad (0<x<1)$

34. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}} \quad (x>1)$

35. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2} \quad (|x|>1)$

36. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}} \quad (x \neq 0)$

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B: Table of Integrals

Basic Integrals

1. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
2. $\int \frac{du}{u} = \ln |u| + C$
3. $\int e^u \, du = e^u + C$
4. $\int a^u \, du = \frac{a^u}{\ln a} + C$
5. $\int \sin u \, du = -\cos u + C$
6. $\int \cos u \, du = \sin u + C$
7. $\int \sec^2 u \, du = \tan u + C$
8. $\int \csc^2 u \, du = -\cot u + C$
9. $\int \sec u \tan u \, du = \sec u + C$
10. $\int \csc u \cot u \, du = -\csc u + C$
11. $\int \tan u \, du = \ln |\sec u| + C$
12. $\int \cot u \, du = \ln |\sin u| + C$
13. $\int \sec u \, du = \ln |\sec u + \tan u| + C$
14. $\int \csc u \, du = \ln |\csc u - \cot u| + C$
15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
17. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$

Trigonometric Integrals

18. $\int \sin^2 u \, du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

19. $\int \cos^2 u \ du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

20. $\int \tan^2 u \ du = \tan u - u + C$

21. $\int \cot^2 u \ du = -\cot u - u + C$

22. $\int \sin^3 u \ du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$

23. $\int \cos^3 u \ du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$

24. $\int \tan^3 u \ du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

25. $\int \cot^3 u \ du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

26. $\int \sec^3 u \ du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

27. $\int \csc^3 u \ du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

28. $\int \sin^n u \ du = \frac{-1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u \ du$

29. $\int \cos^n u \ du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u \ du$

30. $\int \tan^n u \ du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u \ du$

31. $\int \cot^n u \ du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u \ du$

32. $\int \sec^n u \ du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u \ du$

33. $\int \csc^n u \ du = \frac{-1}{n-1}\cot u \csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u \ du$

34. $\int \sin au \sin bu \ du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

35. $\int \cos au \cos bu \ du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

36. $\int \sin au \cos bu \ du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

37. $\int u \sin u \ du = \sin u - u \cos u + C$

38. $\int u \cos u \ du = \cos u + u \sin u + C$

39. $\int u^n \sin u \ du = -u^n \cos u + n\int u^{n-1} \cos u \ du$

40. $\int u^n \cos u \ du = u^n \sin u - n\int u^{n-1} \sin u \ du$

41. Erroneous nesting of equation structures

Exponential and Logarithmic Integrals

42. $\int ue^{au}\ du = \frac{1}{a^2}(au-1)e^{au} + C$

43. $\int u^n e^{au}\ du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\ du$

44. $\int e^{au}\sin bu\ du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

45. $\int e^{au}\cos bu\ du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

46. $\int \ln u\ du = u\ln u - u + C$

47. $\int u^n \ln u\ du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

48. $\int \frac{1}{u\ln u}\ du = \ln|\ln u| + C$

Hyperbolic Integrals

49. $\int \sinh u\ du = \cosh u + C$

50. $\int \cosh u\ du = \sinh u + C$

51. $\int \tanh u\ du = \ln\cosh u + C$

52. $\int \coth u\ du = \ln|\sinh u| + C$

53. $\int \operatorname{sech}\ u\ du = \tan^{-1}|\sinh u| + C$

54. $\int \operatorname{csch}\ u\ du = \ln|\tanh\frac{1}{2}u| + C$

55. $\int \operatorname{sech}^2 u\ du = \tanh\ u + C$

56. $\int \operatorname{csch}^2 u\ du = -\coth\ u + C$

57. $\int \operatorname{sech}\ u\tanh u\ du = -\operatorname{sech}\ u + C$

58. $\int \operatorname{csch}\ u\coth u\ du = -\operatorname{csch}\ u + C$

Inverse Trigonometric Integrals

59. $\int \sin^{-1} u\ du = u\sin^{-1} u + \sqrt{1-u^2} + C$

60. $\int \cos^{-1} u \; du = u\cos^{-1} u - \sqrt{1-u^2} + C$

61. $\int \tan^{-1} u \; du = u\tan^{-1} u - \frac{1}{2}\ln(1+u^2) + C$

62. $\int u\sin^{-1} u \; du = \frac{2u^2-1}{4}\sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$

63. $\int u\cos^{-1} u \; du = \frac{2u^2-1}{4}\cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$

64. $\int u\tan^{-1} u \; du = \frac{u^2+1}{2}\tan^{-1} u - \frac{u}{2} + C$

65. $\int u^n \sin^{-1} u \; du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1} u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

66. $\int u^n \cos^{-1} u \; du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

67. $\int u^n \tan^{-1} u \; du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1} u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$

Integrals Involving $a^2 + u^2$, $a > 0$

68. $\int \sqrt{a^2+u^2} \; du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

69. $\int u^2\sqrt{a^2+u^2} \; du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

70. $\int \frac{\sqrt{a^2+u^2}}{u} \; du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

71. $\int \frac{\sqrt{a^2+u^2}}{u^2} \; du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

72. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$

73. $\int \frac{u^2}{\sqrt{a^2+u^2}} \; du = \frac{u}{2}\left(\sqrt{a^2+u^2}\right) - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

74. $\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{-1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

75. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

76. $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

Integrals Involving $u^2 - a^2$, $a > 0$

77. $\int \sqrt{u^2-a^2} \; du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

78. $$\int u^2\sqrt{u^2-a^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$$

79. $$\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C$$

80. $$\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$$

81. $$\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$$

82. $$\int \frac{u^2}{\sqrt{u^2-a^2}}\,du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$$

83. $$\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$$

84. $$\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$$

Integrals Involving $a^2 - u^2$, $a > 0$

85. $$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

86. $$\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$$

87. $$\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

88. $$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = \frac{-1}{u}\sqrt{a^2-u^2} - \sin^{-1}\frac{u}{a} + C$$

89. $$\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = \frac{1}{2}\left(-u\sqrt{a^2-u^2} + a^2\sin^{-1}\frac{u}{a}\right) + C$$

90. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

91. $$\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2u}\sqrt{a^2-u^2} + C$$

92. $$\int \left(a^2-u^2\right)^{3/2}\,du = -\frac{u}{8}\left(2u^2-5a^2\right)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$$

93. $$\int \frac{du}{(a^2-u^2)^{3/2}} = -\frac{u}{a^2\sqrt{a^2-u^2}} + C$$

Integrals Involving $2au - u^2$, $a > 0$

94. $$\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$$

95. $$\int \frac{du}{\sqrt{2au-u^2}} = \cos^{-1}\left(\frac{a-u}{a}\right) + C$$

96. $$\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$$

97. $$\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$$

Integrals Involving $a + bu$, $a \neq 0$

98. $$\int \frac{u}{a+bu}\,du = \frac{1}{b^2}(a+bu-a\ln|a+bu|) + C$$

99. $$\int \frac{u^2}{a+bu}\,du = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$$

100. $$\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$$

101. $$\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

102. $$\int \frac{u}{(a+bu)^2}\,du = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$$

103. $$\int \frac{u}{u(a+bu)^2}\,du = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

104. $$\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(a+bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$$

105. $$\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$$

106. $$\int \frac{u}{\sqrt{a+bu}}\,du = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$$

107. $$\int \frac{u^2}{\sqrt{a+bu}}\,du = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$$

108. $$\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, & \text{if } a > 0 \\ \frac{\sqrt{2}}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C, & \text{if } a < 0 \end{cases}$$

109. $$\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$$

110. $$\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$$

111. $$\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$$

112. $$\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du$$

113. $$\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$$

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