# MATH 230- MATHEMATICS FOR LIBERAL ARTS STUDENTS



Los Angeles City College

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# Math 230-Mathematics for Liberal Arts Students

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# Licensing

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# CHAPTER OVERVIEW

## 1: Linear Equations

**Learning Objectives**

*In this chapter, you will learn to:*

1. Graph a linear equation.
2. Find the slope of a line.
3. Determine an equation of a line.
4. Solve linear systems.
5. Do application problems using linear equations.


1.1: Graphing a Linear Equation

1.1.1: Graphing a Linear Equation (Exercises)

1.2: Slope of a Line

1.2.1: Slope of a Line (Exercises)

1.3: Determining the Equation of a Line

1.3.1: Determining the Equation of a Line (Exercises)

1.4: Applications

1.4.1: Applications (Exercises)

1.5: More Applications

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1.6: Chapter Review



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# 1.1: Graphing a Linear Equation

**Learning Objectives**

In this section, you will learn to:

1. Graph a line when you know its equation
2. Graph a line when you are given its equation in parametric form
3. Graph and find equations of vertical and horizontal lines

## Graphing a Line from it Equation

Equations whose graphs are straight lines are called **linear equations.** The following are some examples of linear equations:

$2x-3y=6, \quad 3x=4y-7, \quad y=2x-5, \quad 2y=3, \quad \text{and} \quad x-2=0$

A line is completely determined by two points. Therefore, to graph a linear equation we need to find the coordinates of two points. This can be accomplished by choosing an arbitrary value for x or y and then solving for the other variable.

**Example 1.1.1**

Graph the line: $y=3x+2$

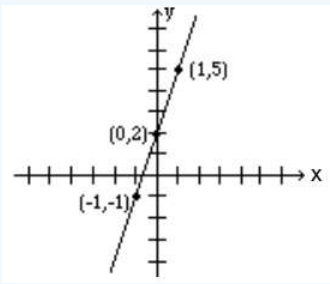
### Solution

We need to find the coordinates of at least two points. We arbitrarily choose x = - 1, x = 0, and x = 1.

- If x = -1, then y = 3(-1) + 2 or -1. Therefore, (-1, -1) is a point on this line.
- If x = 0, then y = 3(0) + 2 or y = 2. Hence the point (0, 2).
- If x = 1, then y = 5, and we get the point (1, 5).

Below, the results are summarized, and the line is graphed.

| x | -1 | 0 | 1 |
|---|---|---|---|
| y | -1 | 2 | 5 |




**Example 1.1.2**

Graph the line: $2x+y=4$

### Solution

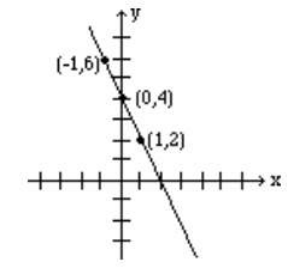
Again, we need to find coordinates of at least two points.

We arbitrarily choose x = -1, x = 0, and y = 2.

- If x = -1, then 2(-1) + y = 4 which results in y = 6. Therefore, (-1, 6) is a point on this line.
- If x = 0, then 2(0) + y = 4, which results in y = 4. Hence the point (0, 4).
- If y = 2, then 2x + 2 = 4, which yields x = 1, and gives the point (1, 2).

The table below shows the points, and the line is graphed.

| | | | |
|---|---|---|---|
| x | -1 | 0 | 1 |
| y | 6 | 4 | 2 |




## Intercepts

The points at which a line crosses the coordinate axes are called the **intercepts**.
When graphing a line by plotting two points, using the intercepts is often preferred because they are easy to find.

- To find the value of the x-intercept, we let y = 0
- To find the value of the y-intercept, we let x = 0.

### Example 1.1.3

Find the intercepts of the line: $2x - 3y = 6$, and graph.

**Solution**

To find the x-intercept, let y = 0 in the equation, and solve for x.

$$\begin{aligned} 2x - 3(0) &= 6 \\ 2x - 0 &= 6 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

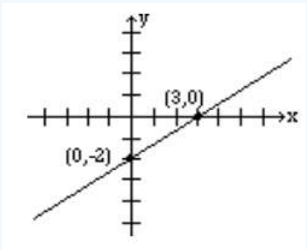
Therefore, the x-intercept is the point (3,0).

To find the y-intercept, let x = 0 in the equation, and solve for y.

$$\begin{aligned} 2(0) - 3y &= 6 \\ 0 - 3y &= 6 \\ -3y &= 6 \\ y &= -2 \end{aligned}$$

Therefore, the y-intercept is the point (0, -2).

To graph the line, plot the points for the x-intercept (3,0) and the y-intercept (0, -2), and use them to draw the line.



## Graphing a Line from its Equation in Parametric Form

In higher math, equations of lines are sometimes written in parametric form. For example, $x = 3 + 2t$ , $y = 1 + t$ . The letter $t$ is called the **parameter**, or the dummy variable.

Parametric lines can be graphed by finding values for x and y by substituting numerical values for t. Plot the points using their (x,y) coordinates and use the points to draw the line.

### Example 1.1.4

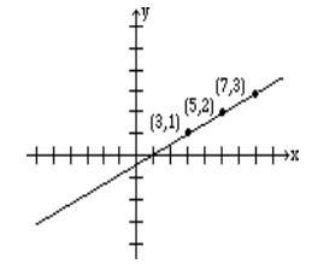
Graph the line given by the parametric equations: $x = 3 + 2t$ , $y = 1 + t$

**Solution**

Let t = 0, 1 and 2; for each value of t find the corresponding values for x and y.

The results are given in the table below.

| t | x | y | Point on Line |
|---|---|---|---|
| 0 | 3 | 1 | (3, 1) |
| 1 | 5 | 2 | (5, 2) |
| 2 | 7 | 3 | (7, 3) |




## Horizontal and Vertical Lines

When an equation of a line has only one variable, the resulting graph is a horizontal or a vertical line.

- The graph of the line $x = a$, where $a$ is a constant, is a vertical line that passes through the point $(a, 0)$. Every point on this line has the x-coordinate equal to a, regardless of the y-coordinate.
- The graph of the line $y = b$, where $b$ is a constant, is a horizontal line that passes through the point $(0, b)$. Every point on this line has the y-coordinate equal to b, regardless of the x-coordinate.

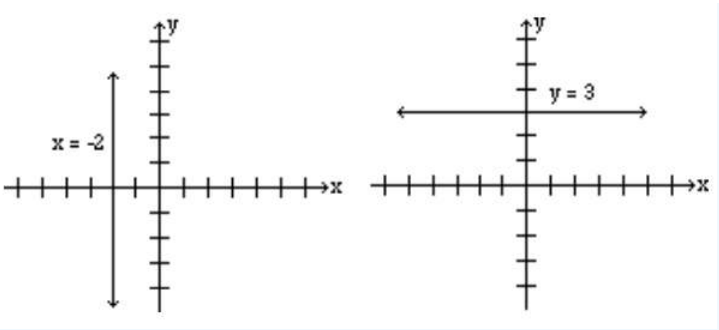
### Example 1.1.5

Graph the lines: x = -2 , and y = 3.

**Solution**

The graph of the line x = -2 is a vertical line that has the x-coordinate -2 no matter what the y-coordinate is. The graph is a vertical line passing through point (-2, 0).

The graph of the line y = 3, is a horizontal line that has the y-coordinate 3 regardless of what the x-coordinate is. Therefore, the graph is a horizontal line that passes through point (0, 3).




Note: Most students feel that the coordinates of points must always be integers. This is not true, and in real life situations, not always possible. Do not be intimidated if your points include numbers that are fractions or decimals.

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# 1.1.1: Graphing a Linear Equation (Exercises)

## SECTION 1.1 PROBLEM SET: GRAPHING A LINEAR EQUATION

Work the following problems.

| | |
|---|---|
| 1) Is the point (2, 3) on the line 5x - 2y = 4? | 2) Is the point (1, - 2) on the line 6x - y = 4? |
| 3) For the line 3x - y = 12, complete the following ordered pairs.<br>(2, ) ( , 6)<br>(0, ) ( , 0) | 4) For the line 4x + 3y = 24, complete the following ordered pairs.<br>(3, ) ( , 4)<br>(0, ) ( , 0) |
| 5) Graph y = 2x + 3 | 6) Graph y = - 3x + 5 |
| 7) Graph y = 4x - 3 | 8) Graph x - 2y = 8 |
| 9) Graph 2x + y = 4 | 10) Graph 2x - 3y = 6 |

***SECTION 1.1 PROBLEM SET: GRAPHING A LINEAR EQUATION***

| | |
|---|---|
| 11) Graph 2x + 4 = 0 | 12) Graph 2y - 6 = 0 |
| 13) Graph the following three equations on the same set of coordinate axes.<br>$y = x + 1 \quad y = 2x + 1 \quad y = -x + 1$ | 14) Graph the following three equations on the same set of coordinate axes.<br>$y = 2x + 1 \quad y = 2x \quad y = 2x - 1$ |
| 15) Graph the line using the parametric equations<br>x = 1 + 2t, y = 3 + t | 16) Graph the line using the parametric equations<br>x = 2 - 3t, y = 1 + 2t |

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# 1.2: Slope of a Line

**Learning Objectives**

In this section, you will learn to:

1. Find the slope of a line.
2. Graph the line if a point and the slope are given.

In the last section, we learned to graph a line by choosing two points on the line. A graph of a line can also be determined if one point and the "steepness" of the line is known. The number that refers to the steepness or inclination of a line is called the **slope** of the line.

From previous math courses, many of you remember slope as the "rise over run," or "the vertical change over the horizontal change" and have often seen it expressed as:

$$\frac{\text{rise}}{\text{run}}, \frac{\text{vertical change}}{\text{horizontal change}}, \frac{\Delta y}{\Delta x} \text{ etc.}$$

We give a precise definition.

**Definition: Slope**

If $(x_1, y_1)$ and $(x_2, y_2)$ are two different points on a line, the **slope** of the line is

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.2.1}$$

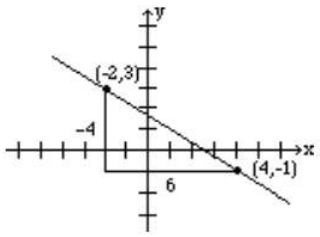
**Example 1.2.1**

Find the slope of the line passing through points (−2, 3) and (4, −1), and graph the line.

**Solution**

Let $(x_1, y_1)$ = (−2, 3) and $(x_2, y_2)$ = (4, −1), then the slope (via Equation 1.2.1) is

$$m = \frac{-1-3}{4-(-2)} = \frac{-4}{6} = -\frac{2}{3}$$




To give the reader a better understanding, both the vertical change, -4, and the horizontal change, 6, are shown in the above figure.

When two points are given, it does not matter which point is denoted as $(x_1, y_1)$ and which $(x_2, y_2)$. The value for the slope will be the same.

In Example 1.2.1, if we instead choose $(x_1, y_1)$ = (4, −1) and $(x_2, y_2)$ = (−2, 3), then we will get the same value for the slope as we obtained earlier.

The steps involved are as follows.

$$m = \frac{3-(-1)}{-2-4} = \frac{4}{-6} = -\frac{2}{3}$$

The student should further observe that

- if a line rises when going from left to right, then it has a positive slope. In this situation, as the value of $x$ increases, the value of $y$ also increases
- if a line falls going from left to right, it has a negative slope; as the value of $x$ increases, the value of $y$ decreases.

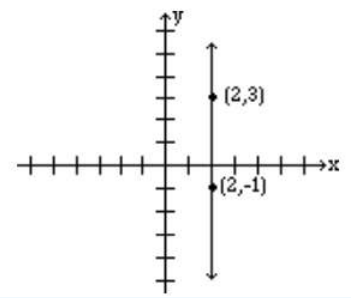
### Example 1.2.2

Find the slope of the line that passes through the points (2, 3) and (2, −1), and graph.

**Solution**

Let $(x_1, y_1)$ = (2, 3) and $(x_2, y_2)$ = (2, −1) then the slope is

$$m = \frac{-1-3}{2-2} = \frac{4}{0} = \text{undefined.}$$




*Note: The slope of a vertical line is undefined.*

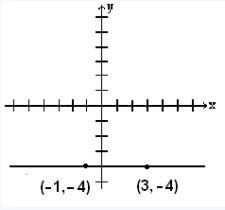
### Example 1.2.3

Find the slope of the line that passes through the points (−1, −4) and (3, −4)

**Solution**

Let $(x_1, y_1)$ = (−1, −4) and $(x_2, y_2)$ = (3, −4), then the slope is

$$m = \frac{-4-(-4)}{3-(-1)} = \frac{0}{4} = 0$$



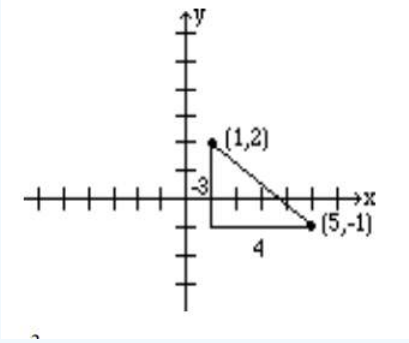

*Note: The slope of a horizontal line is 0*

### Example 1.2.4

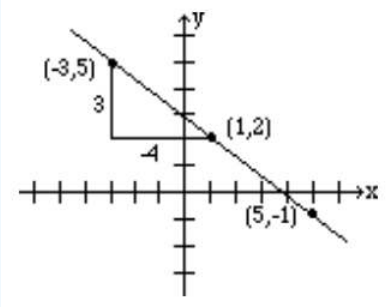
Graph the line that passes through the point (1, 2) and has slope $-\frac{3}{4}$.

**Solution**

Slope equals $\frac{\text{rise}}{\text{run}}$. The fact that the slope is $\frac{-3}{4}$, means that for every rise of -3 units (fall of 3 units) there is a run of 4. So if from the given point (1, 2) we go down 3 units and go right 4 units, we reach the point (5, -1). The graph is obtained by connecting these two points.




Alternatively, since $\frac{3}{-4}$ represents the same number, the line can be drawn by starting at the point (1,2) and choosing a rise of 3 units followed by a run of -4 units. So from the point (1, 2), we go up 3 units, and to the left 4, thus reaching the point (-3, 5) which is also on the same line. See figure below.




**Example 1.2.5**

Find the slope of the line $2x + 3y = 6$.

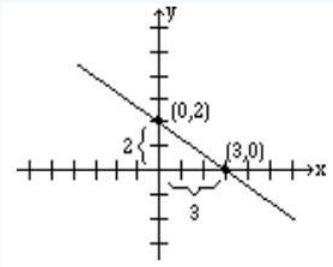
**Solution**

In order to find the slope of this line, we will choose any two points on this line.

Again, the selection of $x$ and $y$ intercepts seems to be a good choice. The $x$-intercept is (3, 0), and the $y$-intercept is (0, 2). Therefore, the slope is

$$m = \frac{2-0}{0-3} = -\frac{2}{3}.$$

The graph below shows the line and the $x$-intercepts and $y$-intercepts:




**Example 1.2.6**

Find the slope of the line $y = 3x + 2$.

**Solution**

We again find two points on the line, e.g., (0, 2) and (1, 5). Therefore, the slope is

$$m = \frac{5-2}{1-0} = \frac{3}{1} = 3.$$

Look at the slopes and the $y$-intercepts of the following lines.

| The line | slope | $y$-intercept |
|---|---|---|
| $y = 3x + 2$ | 3 | 2 |
| $y = -2x + 5$ | -2 | 5 |
| $y = \frac{3}{2}x - 4$ | $\frac{3}{2}$ | -4 |

It is no coincidence that when an equation of the line is solved for $y$, the coefficient of the $x$ term represents the slope, and the constant term represents the $y$-intercept.

In other words, for the line $y = mx + b$, $m$ is the slope, and $b$ is the $y$-intercept.

**Example 1.2.7**

Determine the slope and $y$-intercept of the line $2x + 3y = 6$.

**Solution**

We solve for $y$:

$$\begin{aligned} 2x + 3y &= 6 \\ 3y &= -2x + 6 \\ y &= (-2/3)x + 2 \end{aligned}$$

The slope = the coefficient of the $x$ term = − 2/3.

The $y$-intercept = the constant term = 2.

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# 1.2.1: Slope of a Line (Exercises)

## SECTION 1.2 PROBLEM SET: SLOPE OF A LINE

Find the slope of the line passing through the following pair of points.

| | |
|---|---|
| 1) (2, 3) and (5, 9) | 2) (4, 1) and (2, 5) |
| 3) (- 1, 1) and (1, 3) | 4) (4, 3) and (- 1, 3) |
| 5) (6, - 5) and (4, - 1) | 6) (5, 3) and (- 1, - 4) |
| 7) (3, 4) and (3, 7) | 8) (- 2, 4) and (- 3, - 2) |
| 9) (- 3, - 5) and (- 1, - 7) | 10) (0, 4) and (3, 0) |

***SECTION 1.2 PROBLEM SET: SLOPE OF A LINE***

Determine the slope of the line from the given equation of the line.

| | |
|---|---|
| 11) y = - 2x + 1 | 12) y = 3x - 2 |
| 13) 2x - y = 6 | 14) x + 3y = 6 |
| 15) 3x - 4y = 12 | 16) What is the slope of the x-axis?<br>What is the slope of the y-axis? |

Graph the line that passes through the given point and has the given slope.

| | |
|---|---|
| 17) (1, 2) and m = - 3/4 | 18) (2, - 1) and m = 2/3 |
| 19) (0, 2) and m = - 2 | 20) (2, 3) and m = 0 |

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# 1.3: Determining the Equation of a Line

**Learning Objectives**

In this section, you will learn to:

1. Find an equation of a line if a point and the slope are given.
2. Find an equation of a line if two points are given.

So far, we were given an equation of a line and were asked to give information about it. For example, we were asked to find points on the line, find its slope and even find intercepts. Now we are going to reverse the process. That is, we will be given either two points, or a point and the slope of a line, and we will be asked to find its equation.

An equation of a line can be written in three forms, the **slope-intercept form**, the **point-slope form,** or the **standard form**. We will discuss each of them in this section.

A line is completely determined by two points, or by a point and slope. The information we are given about a particular line will influence which form of the equation is most convenient to use. Once we know any form of the equation of a line, it is easy to re-express the equation in the other forms if needed.

## THE SLOPE-INTERCEPT FORM OF A LINE: $y = mx + b$

In the last section we learned that the equation of a line whose slope = $m$ and $y$-intercept = $b$ is

$$\mathbf{y = mx + b}.$$

This is called the **slope-intercept form** of the line and is the most commonly used form.

**Example 1.3.1**

Find an equation of a line whose slope is 5, and $y$-intercept is 3.

**Solution**

Since the slope is $m = 5$, and the $y$- intercept is $b = 3$, the equation is $y = 5x + 3$.

**Example 1.3.2**

Find the equation of the line that passes through the point $(2, 7)$ and has slope $3$.

**Solution**

Since $m = 3$, the partial equation is $y = 3x + b$.

Now $b$ can be determined by substituting the point $(2, 7)$ in the equation $y = 3x + b$.

$$\begin{aligned} 7 &= 3(2) + b \\ b &= 1 \end{aligned}$$

Therefore, the equation is $y = 3x + 1$.

**Example 1.3.3**

Find an equation of the line that passes through the points (-1, 2), and (1, 8).

**Solution**

$m = \frac{8-2}{1-(-1)} = \frac{6}{2} = 3$ . So the partial equation is $y = 3x + b$.

We can use either of the two points (-1, 2) or (1, 8), to find $b$. Substituting (-1, 2) gives

$$2 = 3(-1) + b$$
$$5 = b$$

So the equation is $y = 3x + 5$.

**Example 1.3.4**

Find an equation of the line that has $x$-intercept 3, and $y$-intercept 4.

**Solution**

$x$-intercept = 3, and $y$-intercept = 4 correspond to the points (3, 0), and (0, 4), respectively.

$$m = \frac{4-0}{0-3} = -\frac{4}{3}$$

We are told the $y$-intercept is 4; thus $b$ = 4

Therefore, the equation is $y = -\frac{4}{3}x + 4$.

## THE POINT-SLOPE FORM OF A LINE: $y - y_1 = m(x - x_1)$

The **point-slope** form is useful when we know two points on the line and want to find the equation of the line.

Let $L$ be a line with slope $m$, and known to contain a specific point $(x_1, y_1)$. If $(x, y)$ is any other point on the line $L$, then the definition of a slope leads us to the **point-slope form** or point-slope formula.

The slope is $\frac{y-y_1}{x-x_1} = m$

Multiplying both sides by $(x - x_1)$ gives the point-slope form:

$$\mathbf{y - y_1 = m(x - x_1)}$$

**Example 1.3.5**

Find the point-slope form of the equation of a line that has slope 1.5 and passes through the point (12,4).

**Solution**

Substituting the point $(x_1, y_1) = (12, 4)$ and $m = 1.5$ in the point-slope formula, we get

$$\mathbf{y - y_1 = m(x - x_1)}$$
$$y - 4 = 1.5(x - 12)$$

The student may be tempted to simplify this into the slope intercept form $y = mx + b$. But since the problem specifically requests **point-slope** form we will not simplify it.

## THE STANDARD FORM OF A LINE: $Ax + By = C$

Another useful form of the equation of a line is the standard form.

If we know the equation of a line in point-slope form, $y - y_1 = m(x - x_1)$ , or if we know the equation of the line in slope-intercept form $y = mx + b$, we can simplify the formula to have all terms for the $x$ and $y$ variables on one side of the equation, and the constant on the other side of the equation.

The result is referred to as the **standard form** of the line:

$$\mathbf{Ax + By = C}.$$

**Example 1.3.6**

Using the point-slope formula, find the standard form of an equation of the line that passes through the point (2, 3) and has slope $-3/5$.

***Solution:*** Substituting the point (2, 3) and $m = -3/5$ in the point-slope formula, we get

$$y - 3 = -3/5(x - 2)$$

Multiplying both sides by 5 gives us

$$\begin{aligned} 5(y-3) &= -3(x-2) \\ 5y - 15 &= -3x + 6 \\ 3x + 5y &= 21 \quad \text{Standard Form} \end{aligned}$$

**Example 1.3.7**

Find the standard form of the line that passes through the points (1, -2), and (4, 0).

**Solution**

First we find the slope: $m = \frac{0-(-2)}{4-1} = \frac{2}{3}$

Then, the point-slope form is: $y - (-2) = \frac{2}{3}(x - 1)$

Multiplying both sides by 3 gives us

$$\begin{aligned} 3(y+2) &= 2(x-1) \\ 3y + 6 &= 2x - 2 \\ -2x + 3y &= -8 \\ 2x - 3y &= 8 \quad \text{Standard Form} \end{aligned}$$

We should always be able to convert from one form of an equation to another. For example, if we are given a line in the slope-intercept form, we should be able to express it in the standard form, and vice versa.

**Example 1.3.8**

Write the equation $y = -\frac{2}{3}x + 3$ in the standard form.

**Solution**

Multiplying both sides of the equation by 3, we get

$$\begin{aligned} 3y &= -2x + 9 \\ 2x + 3y &= 9 \quad \text{Standard Form} \end{aligned}$$

**Example 1.3.9**

Write the equation $3x - 4y = 10$ in the slope-intercept form.

**Solution**

Solving for $y$, we get

$$\begin{aligned} -4y &= -3x + 10 \\ y &= \frac{3}{4}x - \frac{5}{2} \quad \text{Standard Form} \end{aligned}$$

Finally, we learn a very quick and easy way to write an equation of a line in the standard form. But first we must learn to find the slope of a line in the standard form by inspection.

By solving for $y$, it can easily be shown that the slope of the line $Ax + By = C$ is $-A/B$.
The reader should verify this.

**Example 1.3.10**

Find the slope of the following lines, by inspection.

a. $3x - 5y = 10$
b. $2x + 7y = 20$
c. $4x - 3y = 8$

**Solution**

a. $A = 3$, $B = -5$, therefore, $m = -\frac{3}{-5} = \frac{3}{5}$
b. $A = 2$, $B = 7$, therefore, $m = -\frac{2}{7}$
c. $m = -\frac{4}{-3} = \frac{4}{3}$

Now that we know how to find the slope of a line in the standard form by inspection, our job in finding the equation of a line is going to be easy.

**Example 1.3.11**

Find an equation of the line that passes through (2, 3) and has slope - 4/5.

**Solution**

Since the slope of the line is - 4/5, we know that the left side of the equation is $4x + 5y$, and the partial equation is going to be

$$4x + 5y = c$$

Of course, $c$ can easily be found by substituting for $x$ and $y$.

$$4(2) + 5(3) = c$$
$$23 = c$$

The desired equation is

$$4x + 5y = 23.$$

If you use this method often enough, you can do these problems very quickly.

We summarize the forms for equations of a line below:

**Slope Intercept form:** $\mathbf{y = mx + b}$,
where $m$ = slope, $b$ = $y$-intercept

**Point Slope form:** $\mathbf{y - y_1 = m(x - x_1)}$,
where $m$ = slope, $(x_1, y_1)$ is a point on the line

**Standard form:** $\mathbf{Ax + By = C}$

**Horizontal Line:** $\mathbf{y = b}$
where $b$ = $y$-intercept

**Vertical Line:** $\mathbf{x = a}$
where $a$ = $x$-intercept

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# 1.3.1: Determining the Equation of a Line (Exercises)

## SECTION 1.3 PROBLEM SET: DETERMINING THE EQUATION OF A LINE

Write an equation of the line satisfying the following conditions.
Write the equation in the form y = mx + b.

| | |
|---|---|
| 1) It passes through the point (3, 10) and has slope = 2. | 2) It passes through point (4,5) and has m = 0. |
| 3) It passes through (3, 5) and (2, - 1). | 4) It has slope 3, and its y-intercept equals 2. |
| 5) It passes through (5, - 2) and m = 2/5. | 6) It passes through (- 5, - 3) and (10, 0). |
| 7) It passes through (4, - 4) and (5, 3). | 8) It passes through (7, - 2) ; its y-intercept is 5. |
| 9) It passes through (2, - 5) and its x-intercept is 4. | 10) Its a horizontal line through the point (2, - 1). |

***SECTION 1.3 PROBLEM SET: DETERMINING THE EQUATION OF A LINE***

Write an equation of the line satisfying the following conditions.
Write the equation in the form y = mx + b.

| | |
|---|---|
| 11) It passes through (5, - 4) and (1, - 4). | 12) It is a vertical line through the point (3, - 2). |
| 13) It passes through (3, - 4) and (3, 4). | 14) It has x-intercept = 3 and y-intercept = 4. |

Write an equation of the line satisfying the following conditions.
Write the equation in the form Ax + By = C.

| | |
|---|---|
| 15) It passes through (3, - 1) and m = 2. | 16) It passes through (- 2, 1) and m = - 3/2. |
| 17) It passes through (- 4, - 2) and m = 3/4. | 18) Its x-intercept equals 3, and m = - 5/3. |

***SECTION 1.3 PROBLEM SET: DETERMINING THE EQUATION OF A LINE***

Write an equation of the line satisfying the following conditions.
Write the equation in the form Ax + By = C.

| | |
|---|---|
| 19) It passes through (2, - 3) and (5, 1). | 20) It passes through (1, - 3) and (- 5, 5). |

Write an equation of the line satisfying the following conditions.
Write the equation in point slope form $y-y_1 = m (x-x_1)$

| | |
|---|---|
| 21) It passes through (2, - 3) and (5, 1). | 22) It passes through (1, - 3) and (- 5, 2). |
| 23) It passes through (6, -2) and (0, 2). | 24) It passes through (8, 2) and (-7, -4). |
| 25) It passes through (-12, 7) and has slope = -1/3. | 26) It passes through (8, - 7) and has slope 3/4. |

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# 1.4: Applications

**Learning Objectives**

In this section, you will learn to use linear functions to model real-world applications

Now that we have learned to determine equations of lines, we get to apply these ideas in a variety of real-life situations.

Read the problem carefully. Highlight important information. Keep track of which values correspond to the independent variable (x) and which correspond to the dependent variable (y).

**Example 1.4.1**

A taxi service charges \$0.50 per mile plus a \$5 flat fee. What will be the cost of traveling 20 miles? What will be cost of traveling $x$ miles?

**Solution**

$x$ = distance traveled, in miles and $y$ = cost in dollars

The cost of traveling 20 miles is

$$y = (0.50)(20) + 5 = 10 + 5 = 15$$

The cost of traveling $x$ miles is

$$y = (0.50)(x) + 5 = 0.50x + 5$$

In this problem, \$0.50 per mile is referred to as the **variable cost**, and the flat charge \$5 as the **fixed cost**. Now if we look at our cost equation $y = .50x + 5$, we can see that the variable cost corresponds to the slope and the fixed cost to the $y$-intercept.

**Example 1.4.2**

The variable cost to manufacture a product is \$10 per item and the fixed cost \$2500. If $x$ represents the number of items manufactured and $y$ represents the total cost, write the cost function.

**Solution**

- The variable cost of \$10 per item tells us that $m = 10$.
- The fixed cost represents the $y$-intercept. So $b = 2500$.

Therefore, the cost equation is $y = 10x + 2500$.

**Example 1.4.3**

It costs \$750 to manufacture 25 items, and \$1000 to manufacture 50 items. Assuming a linear relationship holds, find the cost equation, and use this function to predict the cost of 100 items.

**Solution**

We let $x$ = the number of items manufactured, and let $y$ = the cost.

Solving this problem is equivalent to finding an equation of a line that passes through the points (25, 750) and (50, 1000).

$$m = \frac{1000 - 750}{50 - 25} = 10$$

Therefore, the partial equation is $y = 10x + b$

By substituting one of the points in the equation, we get $b = 500$

Therefore, the cost equation is $y = 10x + 500$

To find the cost of 100 items, substitute $x = 100$ in the equation $y = 10x + 500$

So the cost is

$$y = 10(100) + 500 = 1500$$

It costs $1500 to manufacture 100 items.

### Example 1.4.4

The freezing temperature of water in Celsius is 0 degrees and in Fahrenheit 32 degrees. And the boiling temperatures of water in Celsius, and Fahrenheit are 100 degrees, and 212 degrees, respectively. Write a conversion equation from Celsius to Fahrenheit and use this equation to convert 30 degrees Celsius into Fahrenheit.

**Solution**

Let us look at what is given.

| Celsius | Fahrenheit |
|---|---|
| 0 | 32 |
| 100 | 212 |

Again, solving this problem is equivalent to finding an equation of a line that passes through the points (0, 32) and (100, 212).

Since we are finding a linear relationship, we are looking for an equation $y = mx + b$, or in this case $F = mC + b$, where $x$ or $C$ represent the temperature in Celsius, and y or F the temperature in Fahrenheit.

$$\text{slope m } = \frac{312 - 32}{100 - 0} = \frac{9}{5}$$

The equation is $F = \frac{9}{5}C + b$

Substituting the point (0, 32), we get

$$F = \frac{9}{5}C + 32.$$

To convert 30 degrees Celsius into Fahrenheit, substitute $C = 30$ in the equation

$$\mathrm{F} = \frac{9}{5}\mathrm{C} + 32$$
$$\mathrm{F} = \frac{9}{5}(30) + 32 = 86$$

### Example 1.4.5

The population of Canada in the year 1980 was 24.5 million, and in the year 2010 it was 34 million. The population of Canada over that time period can be approximately modelled by a linear function. Let x represent time as the number of years after 1980 and let y represent the size of the population.

a. Write the linear function that gives a relationship between the time and the population.
b. Assuming the population continues to grow linearly in the future, use this equation to predict the population of Canada in the year 2025.

**Solution**

The problem can be made easier by using 1980 as the base year, that is, we choose the year 1980 as the year zero. This will mean that the year 2010 will correspond to year 30. Now we look at the information we have:

| Year | Population |
|---|---|
| 0 (1980) | 24.5 million |

| | |
|---|---|
| 30 (2010) | 34 million |

a. Solving this problem is equivalent to finding an equation of a line that passes through the points (0, 24.5) and (30, 34). We use these two points to find the slope:

$$m = \frac{34 - 24.5}{30 - 0} = \frac{9.5}{30} = 0.32$$

The $y$-intercept occurs when $x = 0$, so $b = 24.5$

$$y = 0.32x + 24.5$$

b. Now to predict the population in the year 2025, we let $x = 2025 - 1980 = 45$

$$y = 0.32x + 24.5$$
$$y = 0.32(45) + 24.5 = 38.9$$

In the year 2025, we predict that the population of Canada will be 38.9 million people.

Note that we assumed the population trend will continue to be linear. Therefore if population trends change and this assumption does not continue to be true in the future, this prediction may not be accurate.

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# 1.4.1: Applications (Exercises)

## SECTION 1.4 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

| | |
|---|---|
| 1) The variable cost to manufacture a product is $25 per item, and the fixed costs are $1200. If x is the number of items manufactured and y is the cost, write the cost function. | 2) It costs $90 to rent a car driven 100 miles and $140 for one driven 200 miles. If x is the number of miles driven and y the total cost of the rental, write the cost function. |
| 3) The variable cost to manufacture an item is $20 per item, and it costs a total of $750 to produce 20 items. If x represents the number of items manufactured and y is the cost, write the cost function. | 4) To manufacture 30 items, it costs $2700, and to manufacture 50 items, it costs $3200. If x represents the number of items manufactured and y the cost, write the cost function. |
| 5) To manufacture 100 items, it costs $32,000, and to manufacture 200 items, it costs $40,000. If x is the number of items manufactured and y is the cost, write the cost function. | 6) It costs $1900 to manufacture 60 items, and the fixed costs are $700. If x represents the number of items manufactured and y the cost, write the cost function. |

***SECTION 1.4 PROBLEM SET: APPLICATIONS***

In the following application problems, assume a linear relationship holds.

| | |
|---|---|
| 7) A person who weighs 150 pounds has 60 pounds of muscles; a person that weighs 180 pounds has 72 pounds of muscles. If x represents body weight and y is muscle weight, write an equation describing their relationship. Use this relationship to determine the muscle weight of a person that weighs 170 pounds. | 8) A spring on a door stretches 6 inches if a force of 30 pounds is applied. It stretches 10 inches if a 50 pound force is applied. If x represents the number of inches stretched, and y is the force, write a linear equation describing the relationship. Use it to determine the amount of force required to stretch the spring 12 inches. |
| 9). A male college student who is 64 inches tall weighs 110 pounds. Another student who is 74 inches tall weighs 180 pounds. Assuming the relationship between male students' heights (x), and weights (y) is linear, write a function to express weights in terms of heights, and use this function to predict the weight of a student who is 68 inches tall. | 10) EZ Clean company has determined that if it spends $30,000 on advertising, it can hope to sell 12,000 of its Minivacs a year, but if it spends $50,000, it can sell 16,000. Write an equation that gives a relationship between the number of dollars spent on advertising (x) and the number of minivacs sold(y). |
| 11) The freezing temperatures for water for Celsius and Fahrenheit scales are 0ºC and 32ºF. The boiling temperatures for water are 100 ºC and 212 ºF. Let C denote the temperature in Celsius and F in Fahrenheit. Write the conversion function from Celsius to Fahrenheit. Use the function to convert 25 ºC into ºF. | 12) By reversing the coordinates in the previous problem, find a conversion function that converts Fahrenheit into Celsius, and use this conversion function to convert 72 ºF into an equivalent Celsius measure. |

***SECTION 1.4 PROBLEM SET: APPLICATIONS***

In the following application problems, assume a linear relationship holds.

| | |
|---|---|
| 13) California's population was 29.8 million in the year 1990, and 37.3 million in 2010. Assume that the population trend was and continues to be linear, write the population function. Use this function to predict the population in 2025. *Hint: Use 1990 as the base year (year 0); then 2010 and 2025 are years 20, and 35, respectively.)* | 14) Use the population function for California in the previous problem to find the year in which the population will be 40 million people. |

| | |
|---|---|
| 15) A college's enrollment was 13,200 students in the year 2000, and 15,000 students in 2015. Enrollment has followed a linear pattern.<br>Write the function that models enrollment as a function of time. Use the function to find the college's enrollment in the year 2010.<br>*Hint: Use year 2000 as the base year.* | 16) If the college's enrollment continues to follow this pattern, in what year will the college have 16,000 students enrolled. |
| 17) The cost of electricity in residential homes is a linear function of the amount of energy used. In Grove City, a home using 250 kilowatt hours (kwh) of electricity per month pays $55.<br>A home using 600 kwh per month pays $118. Write the cost of electricity as a function of the amount used. Use the function to find the cost for a home using 400 kwh of electricity per month. | 18) Find the level of electricity use that would correspond to a monthly cost of $100. |

***SECTION 1.4 PROBLEM SET: APPLICATIONS***

In the following application problems, assume a linear relationship holds.

| | |
|---|---|
| 19) At ABC Co., sales revenue is $170,000 when it spends $5000 on advertising.<br>Sales revenue is $254,000 when $12,000 is spent on advertising.<br>a) Find a linear function for<br>y = amount of sales revenue as a function of<br>x = amount spent on advertising.<br>b) Find revenue if $10,000 is spent on advertising.<br>c) Find the amount that should be spent on advertising to achieve $200,000 in revenue. | 20) For problem 19, explain the following:<br>a. Explain what the slope of the line tells us about the effect on sales revenue of money spent on advertising. Be specific, explaining both the number and the sign of the slope in the context of this problem.<br>b. Explain what the y intercept of the line tells us about the sales revenue in the context of this problem |
| 21) Mugs Café sells 1000 cups of coffee per week if it does not advertise. For every $50 spent in advertising per week, it sells an additional 150 cups of coffee.<br>a) Find a linear function that gives<br>y = number of cups of coffee sold per week<br>x = amount spent on advertising per week.<br>b) How many cups of coffee does Mugs Café expect to sell if $100 per week is spent on advertising? | 22) Party Sweets makes baked goods that can be ordered for special occasions. The price is $24 to order one dozen (12 cupcakes) and $9 for each additional 6 cupcakes.<br>a. Find a linear function that gives the total price of a cupcake order as a function of the number of cupcakes ordered<br>b. Find the price for an order of 5 dozen (60) cupcakes |

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# 1.5: More Applications

**Learning Objectives**

In this section, you will learn to:

1. Solve a linear system in two variables.
2. Find the equilibrium point when a demand and a supply equation are given.
3. Find the break-even point when the revenue and the cost functions are given.

## Finding the Point of Intersection of Two Lines

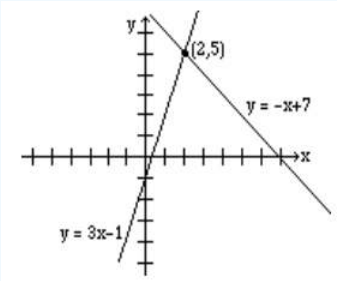
In this section, we will do application problems that involve the intersection of lines. Therefore, before we proceed any further, we will first learn how to find the intersection of two lines.

**Example 1.5.1**

Find the intersection of the line $y = 3x - 1$ and the line $y = -x + 7$.

**Solution**

We graph both lines on the same axes, as shown below, and read the solution (2, 5).




Finding an intersection of two lines graphically is not always easy or practical; therefore, we will now learn to solve these problems algebraically.

At the point where two lines intersect, the $x$ and $y$ values for both lines are the same. So in order to find the intersection, we either let the $x$-values or the $y$-values equal.

If we were to solve the above example algebraically, it will be easier to let the y-values equal. Since $y = 3x - 1$ for the first line, and $y = -x + 7$ for the second line, by letting the y-values equal, we get

$$\begin{aligned} 3x - 1 &= -x + 7 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

By substituting $x = 2$ in any of the two equations, we obtain $y = 5$.

Hence, the solution (2, 5).

A common algebraic method used to solve systems of equations is called the **elimination method**. The object is to eliminate one of the two variables by adding the left and right sides of the equations together. Once one variable is eliminated, we have an equation with only one variable for can be solved. Finally, by substituting the value of the variable that has been found in one of the original equations, we get the value of the other variable.

**Example 1.5.2**

Find the intersection of the lines $2x + y = 7$ and $3x - y = 3$ by the elimination method.

**Solution**

We add the left and right sides of the two equations.

$$\begin{aligned} 2x + y &= 7 \\ 3x - y &= 3 \\ \hline 5x &= 10 \\ x &= 2 \end{aligned}$$

Now we substitute $x = 2$ in any of the two equations and solve for $y$.

$$\begin{aligned} 2(2) + y &= 7 \\ y &= 3 \end{aligned}$$

Therefore, the solution is (2, 3).

**Example 1.5.3**

Solve the system of equations $x + 2y = 3$ and $2x + 3y = 4$ by the elimination method.

**Solution**

If we add the two equations, none of the variables are eliminated. But the variable $x$ can be eliminated by multiplying the first equation by -2, and leaving the second equation unchanged.

$$\begin{aligned} -2x - 4y &= -6 \\ 2x + 3y &= 4 \\ \hline -y &= -2 \\ y &= 2 \end{aligned}$$

Substituting $y = 2$ in $x + 2y = 3$, we get

$$\begin{aligned} x + 2(2) &= 3 \\ x &= -1 \end{aligned} \tag{1.5.1}$$

Therefore, the solution is (-1, 2).

**Example 1.5.4**

Solve the system of equations $3x - 4y = 5$ and $4x - 5y = 6$.

**Solution**

This time, we multiply the first equation by - 4 and the second by 3 before adding. (The choice of numbers is not unique.)

$$\begin{aligned} -12x + 16y &= -20 \\ 12x - 15y &= 18 \\ \hline y &= -2 \end{aligned}$$

By substituting y = - 2 in any one of the equations, we get x = -1.
Hence the solution is (-1, -2).

## SUPPLY, DEMAND AND THE EQUILIBRIUM MARKET PRICE

In a free market economy the supply curve for a commodity is the number of items of a product that can be made available at different prices, and the demand curve is the number of items the consumer will buy at different prices.

As the price of a product increases, its demand decreases and supply increases. On the other hand, as the price decreases the demand increases and supply decreases. The **equilibrium price** is reached when the demand equals the supply.

### Example 1.5.5

The supply curve for a product is $y = 3.5x - 14$ and the demand curve for the same product is $y = -2.5x + 34$, where x is the price and y the number of items produced. Find the following.

a. How many items will be supplied at a price of $10?
b. How many items will be demanded at a price of $10?
c. Determine the equilibrium price.
d. How many items will be produced at the equilibrium price?

**Solution**

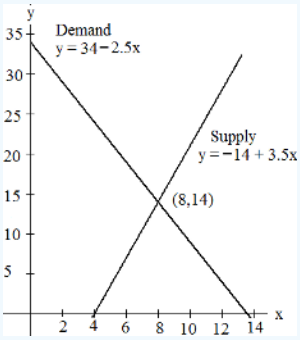
a) We substitute $x = 10$ in the supply equation, $y = 3.5x - 14$; the answer is $y = 3.5(10) - 14 = 21$.

b) We substitute $x = 10$ in the demand equation, $y = -2.5x + 34$; the answer is $y = -2.5(10) + 34 = 9$.

c) By letting the supply equal the demand, we get

$$\begin{aligned} 3.5x - 14 &= -25x + 34 \\ 6x &= 48 \\ x &= \$8 \end{aligned}$$

d) We substitute $x = 8$ in either the supply or the demand equation; we get $y = 14$.




The graph shows the intersection of the supply and the demand functions and their point of intersection, (8,14).

Interpretation: At equilibrium, the price is $8 per item, and 14 items are produced by suppliers and purchased by consumers.

## The Break-Even Point

In a business, the profit is generated by selling products.

- If a company sells x number of items at a price P, then the **revenue R** is the price multiplied by number of items sold: $R = P \cdot x$.
- The **production costs C** are the sum of the variable costs and the fixed costs, and are often written as C = mx + b, where x is the number of items manufactured.
  - The slope m is the called marginal cost and represents the cost to produce one additional item or unit.
  - The variable cost, mx, depends on how much is being produced
  - The fixed cost b is constant; it does not change no matter how much is produced.
- **Profit** is equal to Revenue minus Cost: Profit = R − C

A company makes a profit if the revenue is greater than the cost. There is a loss if the cost is greater than the revenue. The point on the graph where the revenue equals the cost is called the **break-even point**. At the break-even point, profit is 0.

## Example 1.5.6

If the revenue function of a product is $R = 5x$ and the cost function is $y = 3x + 12$, find the following.

a. If 4 items are produced, what will the revenue be?
b. What is the cost of producing 4 items?
c. How many items should be produced to break even?
d. What will be the revenue and the cost at the break-even point?

### Solution

a) We substitute $x = 4$ in the revenue equation $R = 5x$, and the answer is $R = 20$.

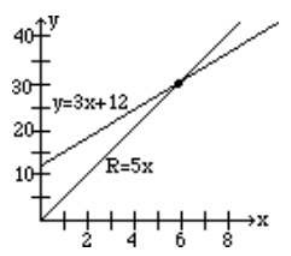
b) We substitute $x = 4$ in the cost equation $C = 3x + 12$, and the answer is $C = 24$.

c) By letting the revenue equal the cost, we get

$$\begin{aligned} 5x &= 3x + 12 \\ x &= 6 \end{aligned}$$

d) Substitute $x = 6$ in either the revenue or the cost equation: we get $R = C = 30$.

The graph below shows the intersection of the revenue and cost functions and their point of intersection, (6, 30).




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# 1.5.1: More Applications (Exercises)

## SECTION 1.5 PROBLEM SET: MORE APPLICATIONS

Solve the following problems.

| | |
|---|---|
| 1) Solve for x and y.<br>y = 3x + 4<br>y = 5x - 2 | 2) Solve for x and y.<br>2x - 3y = 4<br>3x - 4y = 5 |
| 3) The supply and demand curves for a product are: Supply y = 2000x - 6500<br>Demand y = - 1000x + 28000,<br>where x is price and y is the number of items. At what price will supply equal demand and how many items will be produced at that price? | 4) The supply and demand curves for a product are<br>Supply y = 300x - 18000 and<br>Demand y = - 100x + 14000,<br>where x is price and y is the number of items. At what price will supply equal demand, and how many items will be produced at that price? |
| 5) A car rental company offers two plans for one way rentals.<br>Plan I charges $36 per day and 17 cents per mile. Plan II charges $24 per day and 25 cents per mile.<br>a. If you were to drive 300 miles in a day, which plan is better?<br>b. For what mileage are both rates equal? | |

***SECTION 1.5 PROBLEM SET: MORE APPLICATIONS***

Solve the following problems.

| | |
|---|---|
| 6) A demand curve for a product is the number of items the consumer will buy at different prices. At a price of $2 a store can sell 2400 of a particular type of toy truck. At a price of $8 the store can sell 600 such trucks. If x represents the price of trucks and y the number of items sold, write an equation for the demand curve. | 7) A supply curve for a product is the number of items that can be made available at different prices. A manufacturer of toy trucks can supply 2000 trucks if they are sold for $8 each; it can supply only 400 trucks if they are sold for $4 each. If x is the price and y the number of items, write an equation for the supply curve. |
| 8) The equilibrium price is the price where the supply equals the demand. From the demand and supply curves obtained in the previous two problems, find the equilibrium price, and determine the number of items that can be sold at that price. | 9) A break-even point is the intersection of the cost function and the revenue function, that is, where total cost equals revenue, and profit is zero. Mrs. Jones Cookies Store's cost and revenue, in dollars, for x number of cookies is given by C = .05x + 3000 and R = .80x. Find the number of cookies that must be sold to break even. |

***SECTION 1.5 PROBLEM SET: MORE APPLICATIONS***

Solve the following problems.

| | |
|---|---|
| 10) A company's revenue and cost in dollars are given by R = 225x and C = 75x + 6000, where x is the number of items. Find the number of items that must be produced to break-even. | 11) A firm producing socks has a fixed cost of $20,000 and variable cost of $2 per pair of socks. Let x = the number of pairs of socks. Find the break-even point if the socks sell for $4.50 per pair. |

| | |
|---|---|
| 12) Whackemhard Sports is planning to introduce a new line of tennis rackets. The fixed costs for the new line are $25,000 and the variable cost of producing each racket is $60.<br>x is the number of rackets; y is in dollars.<br>If the racket sells for $80, how many rackets must be sold in order to break even? | 13) It costs $1,200 to produce 50 pounds of a chemical and it costs $2,200 to produce 150 pounds. The chemical sells for $15 per pound<br>x is the amount of chemical; y is in dollars.<br>a. Find the cost function.<br>b. What is the fixed cost?<br>c. How many pounds must be sold to break even?<br>d. Find the cost and revenue at the break-even point. |

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# 1.6: Chapter Review

## SECTION 1.6 PROBLEM SET: CHAPTER REVIEW

1) Find an equation of the x-axis.

2) Find the slope of the line whose equation is $2x + 3y = 6$.

3) Find the slope of the line whose equation is $y = - 3x + 5$.

4) Find both the x and y intercepts of the line $3x - 2y = 12$.

5) Find an equation of the line whose slope is 3 and y-intercept 5.

6) Find an equation of the line whose x-intercept is 2 and y-intercept 3.

7) Find an equation of the line that has slope 3 and passes through the point (2, 15).

8) Find an equation of the line that has slope -3/2 and passes through the point (4, 3).

9) Find an equation of the line that passes through the points (0, 32) and (100, 212).

10) Find an equation of the line that passes through the point (2, 5) and is parallel to the line $y = 3x + 4$.

11) Find the point of intersection of the lines $2x - 3y = 9$ and $3x + 4y = 5$.

12) Is the point (3, - 2) on the line $5x - 2y = 11$?

13) Find two points on the line given by the parametric equations, $x = 2 + 3t$, $y = 1 - 2t$.

14) Find two points on the line $2x - 6 = 0$.

15) Graph the line $2x - 3y + 6 = 0$.

16) Graph the line $y = - 2x + 3$.

17) A female college student who is 60 inches tall weighs 100 pounds. Another female student who is 66 inches tall weighs 124 pounds. Assume the relationship between the female students' weights and heights is linear. Find an equation for weight as a function of height. Use this relationship to predict the weight of a female student who is 70 inches tall.

18) In deep-sea diving, the pressure exerted by water plays a great role in designing underwater equipment. If at a depth of 10 feet there is a pressure of 21 $lb/in^2$, and at a depth of 50 ft there is a pressure of 75 $lb/in^2$, write a linear equation giving a relationship between depth and pressure. Use this relationship to predict pressure at a depth of 100 ft.

19) The variable cost to manufacture an item is \$30 per item; the fixed costs are \$2750. Find the cost function.

20) The variable cost to manufacture an item is \$10 per item, and it costs \$2,500 to produce 100 items. Write the cost function, and use this function to estimate the cost of manufacturing 300 items.

21) It costs \$2,700 to manufacture 100 items of a product, and \$4,200 to manufacture 200 items.
x= the number of items; y= cost. Find the cost function; use it to predict the cost to produce 1000 items.

22) In 1990, the average house in Emerald City cost \$280,000 and in 2007 the same house cost \$365,000. Assuming a linear relationship, write an equation that will give the price of the house in any year, and use this equation to predict the price of a similar house in the year 2020.

23) The population of Mexico in 1995 was 95.4 million and in 2010 it was 117.9 million. Assuming a linear relationship, write an equation that will give the population of Mexico in any year, and use this equation to predict the population of Mexico in the year 2025.

***SECTION 1.6 PROBLEM SET: CHAPTER REVIEW***

24) At Nuts for Soup Lunch Bar, they sell 150 bowls of soup if the high temperature for the day is 40 ºF. For every 5 ºF increase in high temperature for the day, they sell 10 fewer bowls of soup.

a. Assuming a linear relationship, write an equation that will give y = the number of bowls of soup sold as a function of x = the daily high temperature.
b. How many bowls of soup are sold when the temperature is 75 ºF?

c. What is the temperature when 100 bowls of soup are sold?

25) Two-hundred items are demanded at a price of $5, and 300 items are demanded at a price of $3. If x represents the price, and y the number of items, write the demand function.

26) A supply curve for a product is the number of items of the product that can be made available at different prices. A doll manufacturer can supply 2000 dolls if the dolls are sold for $30 each, but he can supply only 400 dolls if the dolls are sold for $10 each. If x represents the price of dolls and y the number of items, write an equation for the supply curve.

27) Suppose you are trying to decide on a price for your latest creation - a coffee mug that never tips. Through a survey, you have determined that at a price of $2, you can sell 2100 mugs, but at a price of $12 you can only sell 100 mugs. Furthermore, your supplier can supply you 3100 mugs if you charge your customers $12, but only 100 mugs if you charge $2. What price should you charge so that the supply equals demand, and at that price how many coffee mugs will you be able to sell?

28) A car rental company offers two plans. Plan I charges $16 a day and 25 cents a mile, while Plan II charges $45 a day but no charge for miles. If you were to drive 200 miles in a day, which plan is better? For what mileage are both rates the same?

29) The supply curve for a product is y = 250x - 1000. The demand curve for the same product is
y = - 350x + 8,000, where x is the price and y the number of items produced. Find the following.

a. At a price of $10, how many items will be in demand?
b. At what price will 4,000 items be supplied?
c. What is the equilibrium price for this product?
d. How many items will be manufactured at the equilibrium price?

30) The supply curve for a product is y = 625x - 600 and the demand curve for the same product is
y = - 125x + 8,400, where x is the price and y the number of items produced.
Find the equilibrium price and determine the number of items that will be produced at that price.

31) Both Jenny and Masur work in the sales department for Sports Supply. Jenny gets paid $120 per day plus 4% commission on the sales. Masur gets paid $132 per day plus 8% commission on the sales in excess of $1,000. For what sales amount would they both earn the same daily amounts?

32) A company's revenue and cost in dollars are given by R = 25x and C = 10x + 9,000, where x represents the number of items. Find the number of items that must be produced to break-even.

33) A firm producing a certain type of CFL lightbulb has fixed costs of $6,800, and a variable cost of $2.30 per bulb. The bulbs sell for $4 each. How many bulbs must be produced to break-even?

34) A company producing tire pressure gauges has fixed costs of $7,500, and variable cost of $1.50 cents per item. If the gauges sell for $4.50, how many must be produced to break-even?

35) A company is introducing a new cordless travel shaver before the Christmas holidays. It hopes to sell 15,000 of these shavers in December. The variable cost is $11 per item and the fixed costs $100,000. If the shavers sell for $19 each, how many must be produced and sold to break-even?

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# CHAPTER OVERVIEW

## 2: Matrices

**Learning Objectives**

In this chapter, you will learn to:

1. Do matrix operations.
2. Solve linear systems using the Gauss-Jordan method.
3. Solve linear systems using the matrix inverse method.
4. Do application problems.


2.1: Introduction to Matrices

2.1.1: Introduction to Matrices (Exercises)

2.2: Systems of Linear Equations and the Gauss-Jordan Method

2.2.1: Systems of Linear Equations and the Gauss-Jordan Method (Exercises)

2.3: Systems of Linear Equations – Special Cases

2.3.1: Systems of Linear Equations – Special Cases (Exercises)

2.4: Inverse Matrices

2.4.1: Inverse Matrices (Exercises)

2.5: Application of Matrices in Cryptography

2.5.1: Application of Matrices in Cryptography (Exercises)

2.6: Applications – Leontief Models

2.6.1: Applications – Leontief Models (Exercises)

2.7: Chapter Review


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# 2.1: Introduction to Matrices

**Learning Objectives**

In this section, you will learn to:

1. Add and subtract matrices.
2. Multiply a matrix by a scalar.
3. Multiply two matrices.

A matrix is a 2 dimensional array of numbers arranged in rows and columns. Matrices provide a method of organizing, storing, and working with mathematical information. Matrices have an abundance of applications and use in the real world. Matrices provide a useful tool for working with models based on systems of linear equations. We'll use matrices in sections 2.2, 2.3, and 2.4 to solve systems of linear equations with several variables in this chapter.

Matrices are used in encryption, which we will explore in section 2.5 and in economic modelling, explored in section 2.6. We use matrices again in chapter 4, in optimization problems such as maximizing profit or revenue, or minimizing cost. Matrices are used in business for scheduling, routing transportation and shipments, and managing inventory.

Just about any application that collects and manages data can apply matrices. Use of matrices has grown as the availability of data in many areas of life and business has increased. They are important tools for organizing data and solving problems in all fields of science, from physics and chemistry, to biology and genetics, to meteorology, and economics. In computer science, matrix mathematics lies behind animation of images in movies and video games.

Computer science analyzes diagrams of networks to understand how things are connected to each other, such as relationships between people on a social website, and relationships between results in line search and how people link from one website to another. The mathematics to work with network diagrams comprise the field of "graph theory"; it relies on matrices to organize the information in the graphs that diagram connections and associations in a network. For example, if you use Facebook or Linked-In, or other social media sites, these sites use network graphs and matrices to organize your relationships with other users.

## Introduction to Matrices

A matrix is a rectangular array of numbers. Matrices are useful in organizing and manipulating large amounts of data. In order to get some idea of what matrices are all about, we will look at the following example.

**Example 2.1.1**

Fine Furniture Company makes chairs and tables at its San Jose, Hayward, and Oakland factories. The total production, in hundreds, from the three factories for the years 2014 and 2015 is listed in the table below.

| | 2014 | | 2015 | |
|---|---|---|---|---|
| | CHAIRS | TABLES | CHAIRS | TABLES |
| SAN JOSE | 30 | 18 | 36 | 20 |
| HAYWARD | 20 | 12 | 24 | 18 |
| OAKLAND | 16 | 10 | 20 | 12 |

a. Represent the production for the years 2014 and 2015 as the matrices A and B.
b. Find the difference in sales between the years 2014 and 2015.
c. The company predicts that in the year 2020 the production at these factories will be double that of the year 2014. What will the production be for the year 2020?

**Solution**

a) The matrices are as follows:

$$A = \begin{bmatrix} 30 & 18 \\ 20 & 12 \\ 16 & 10 \end{bmatrix}$$

$$\mathrm{B} = \begin{bmatrix} 36 & 20 \\ 24 & 18 \\ 20 & 12 \end{bmatrix}$$

b) We are looking for the matrix $B - A$. When two matrices have the same number of rows and columns, the matrices can be added or subtracted entry by entry. Therefore, we get

$$\mathrm{B} - \mathrm{A} = \begin{bmatrix} 36-30 & 20-18 \\ 24-20 & 18-12 \\ 20-16 & 12-10 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 4 & 6 \\ 4 & 2 \end{bmatrix}$$

c) We would like a matrix that is twice the matrix of 2014, i.e., $2A$.

Whenever a matrix is multiplied by a number, each entry is multiplied by the number.

$$2\mathrm{A} = 2\begin{bmatrix} 30 & 18 \\ 20 & 12 \\ 16 & 10 \end{bmatrix} = \begin{bmatrix} 60 & 36 \\ 40 & 24 \\ 32 & 20 \end{bmatrix}$$

Before we go any further, we need to familiarize ourselves with some terms that are associated with matrices. The numbers in a matrix are called the **entries** or the **elements** of a matrix.

Whenever we talk about a matrix, we need to know the **size** or the **dimension** of the matrix. The dimension of a matrix is the number of rows and columns it has. When we say a matrix is a "3 by 4 matrix", we are saying that it has 3 rows and 4 columns. The rows are always mentioned first and the columns second. This means that a $3 \times 4$ matrix does not have the same dimension as a $4 \times 3$ matrix.

$$A = \begin{bmatrix} 1 & 4 & -2 & 0 \\ 3 & -1 & 7 & 9 \\ 6 & 2 & 0 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 9 & 8 \\ -3 & 0 & 1 \\ 6 & 5 & -2 \\ -4 & 7 & 8 \end{bmatrix}$$

Matrix $A$ has dimensions $3 \times 4$ and matrix $B$ has dimensions $4 \times 3$.

A matrix that has the same number of rows as columns is called a **square matrix**. A matrix with all entries zero is called a **zero matrix.** A square matrix with 1's along the main diagonal and zeros everywhere else, is called an **identity matrix**. When a square matrix is multiplied by an identity matrix of same size, the matrix remains the same.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix $I$ is a $3 \times 3$ identity matrix

A matrix with only one row is called a row matrix or a **row vector,** and a matrix with only one column is called a column matrix or a **column vector**. Two matrices are **equal** if they have the same size and the corresponding entries are equal.

We can perform arithmetic operations with matrices. Next we will define and give examples illustrating the operations of matrix addition and subtraction, scalar multiplication, and matrix multiplication. Note that matrix multiplication is quite different from what you would intuitively expect, so pay careful attention to the explanation. Note also that the ability to perform matrix

operations depends on the matrices involved being compatible in size, or dimensions, for that operation. The definition of compatible dimensions is different for different operations, so note the requirements carefully for each.

## Matrix Addition and Subtraction

If two matrices have the same size, they can be added or subtracted. The operations are performed on corresponding entries.

**Example 2.1.2**

Given the matrices $A$, $B$, $C$ and $D$, below

$$A=\begin{bmatrix}1&2&4\\2&3&1\\5&0&3\end{bmatrix}\quad B=\begin{bmatrix}2&-1&3\\2&4&2\\3&6&1\end{bmatrix}\quad C=\begin{bmatrix}4\\2\\3\end{bmatrix}\quad D=\begin{bmatrix}-2\\-3\\4\end{bmatrix}$$

Find, if possible.

a. $A+B$
b. $C-D$
c. $A+D$.

**Solution**

As we mentioned earlier, matrix addition and subtraction involves performing these operations entry by entry.

a) We add each element of $A$ to the corresponding entry of $B$.

$$A+B=\begin{bmatrix}3&1&7\\4&7&3\\8&6&4\end{bmatrix}$$

b) Just like the problem above, we perform the subtraction entry by entry.

$$\mathrm{C}-\mathrm{D}=\begin{bmatrix}6\\5\\-1\end{bmatrix}$$

c) The sum $A+D$ cannot be found because the two matrices have different sizes.

Note: Two matrices can *only* be added or subtracted if they have the same dimension.

## Multiplying a Matrix by a Scalar

If a matrix is multiplied by a scalar, each entry is multiplied by that scalar. We can consider scalar multiplication as multiplying a number and a matrix to obtain a new matrix as the product.

**Example 2.1.3**

Given the matrix $A$ and $C$ in the example above, find $2A$ and $-3C$.

**Solution**

To find $2A$, we multiply each entry of matrix $A$ by 2, and to find $-3C$, we multiply each entry of C by -3. The results are given below.

a) We multiply each entry of A by 2.

$$2\mathrm{A}=\begin{bmatrix}2&4&8\\4&6&2\\10&0&6\end{bmatrix}$$

b) We multiply each entry of C by -3.

$$-3C = \begin{bmatrix} -12 \\ -6 \\ -9 \end{bmatrix}$$

## Multiplication of Two Matrices

To multiply a matrix by another is not as easy as the addition, subtraction, or scalar multiplication of matrices. Because of its wide use in application problems, it is important that we learn it well. Therefore, we will try to learn the process in a step by step manner. We first begin by finding a product of a row matrix and a column matrix.

### Example 2.1.4

Find the product $AB$, given

$$A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

**Solution**

The product is a $1 \times 1$ matrix whose entry is obtained by multiplying the corresponding entries and then forming the sum.

$$\begin{aligned} \mathrm{AB} &= \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{bmatrix} \\ &= [2(\mathrm{a}+3\mathrm{b}+4\mathrm{c})] \end{aligned}$$

Note that $AB$ is a $1 \times 1$ matrix, and its only entry is $2a+3b+4c$.

### Example 2.1.5

Find the product $AB$, given

$$A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

**Solution**

Again, we multiply the corresponding entries and add.

$$\begin{aligned} \mathrm{AB} &= \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \\ &= [2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7] \\ &= [10 + 18 + 28] \\ &= [56] \end{aligned}$$

Note: In order for a product of a row matrix and a column matrix to exist, the number of entries in the row matrix must be the same as the number of entries in the column matrix.

**Example 2.1.6**

Find the product AB, given

$$A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}.$$

**Solution**

We know how to multiply a row matrix by a column matrix. To find the product $AB$, in this example, we will multiply the row matrix $A$ to both the first and second columns of matrix $B$, resulting in a $1 \times 2$ matrix.

$$\mathrm{AB} = \begin{bmatrix} 2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 & 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 \end{bmatrix} = \begin{bmatrix} 56 & 38 \end{bmatrix}$$

We multiplied a $1 \times 3$ matrix by a matrix whose size is $3 \times 2$. So unlike addition and subtraction, it is possible to multiply two matrices with different dimensions, if the number of entries in the rows of the first matrix is the same as the number of entries in the columns of the second matrix.

**Example 2.1.7**

Find the product $AB$, given:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

and

$$\mathrm{B} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}$$

**Solution**

This time we are multiplying two rows of the matrix $A$ with two columns of the matrix $B$. Since the number of entries in each row of $A$ is the same as the number of entries in each column of $B$, the product is possible. We do exactly what we did in the last example. The only difference is that the matrix $A$ has one more row.

We multiply the first row of the matrix $A$ with the two columns of $B$, one at a time, and then repeat the process with the second row of A. We get

$$\mathrm{AB} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 & 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 \\ 1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 & 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 \end{bmatrix}$$

$$\mathrm{AB} = \begin{bmatrix} 56 & 38 \\ 38 & 26 \end{bmatrix}$$

**Example 2.1.8**

Find, if possible:

a. $EF$
b. $FE$
c. $FH$
d. $GH$
e. $HG$

$$\mathrm{E}=\begin{bmatrix}1 & 2\\ 4 & 2\\ 3 & 1\end{bmatrix} \quad \mathrm{F}=\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix} \quad \mathrm{G}=\begin{bmatrix}4 & 1\end{bmatrix} \quad \mathrm{H}=\begin{bmatrix}-3\\ -1\end{bmatrix}$$

**Solution**

a) To find $EF$, we multiply the first row $\begin{bmatrix}1 & 2\end{bmatrix}$ of E with the columns $\begin{bmatrix}2\\ 3\end{bmatrix}$ and $\begin{bmatrix}1\\ -2\end{bmatrix}$ of the matrix F, and then repeat the process by multiplying the other two rows of E with these columns of F. The result is as follows:

$$\mathrm{EF}=\begin{bmatrix}1 & 2\\ 4 & 2\\ 3 & 1\end{bmatrix}\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}=\begin{bmatrix}1\cdot 2+2\cdot 3 & 1\cdot -1+2\cdot 2\\ 4\cdot 2+2\cdot 3 & 4\cdot -1+2\cdot 2\\ 3\cdot 2+1\cdot 3 & 3\cdot -1+1\cdot 2\end{bmatrix}=\begin{bmatrix}8 & 3\\ 14 & 0\\ 9 & -1\end{bmatrix}$$

b) Product $FE$ is not possible because the matrix F has two entries in each row, while the matrix E has three entries in each column. In other words, the matrix F has two columns, while the matrix E has three rows.

c)

$$\mathrm{FH}=\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}\begin{bmatrix}-3\\ -1\end{bmatrix}=\begin{bmatrix}2\cdot -3+-1\cdot -1\\ 3\cdot -3+2\cdot -1\end{bmatrix}=\begin{bmatrix}-5\\ -11\end{bmatrix}$$

d)

$$\mathrm{GH}=\begin{bmatrix}4 & 1\end{bmatrix}\begin{bmatrix}-3\\ -1\end{bmatrix}=\begin{bmatrix}4\cdot -3+1\cdot -1\\ -1\end{bmatrix}=[-13]$$

e)

$$\mathrm{HG}=\begin{bmatrix}-3\\ -1\end{bmatrix}\begin{bmatrix}4 & 1\end{bmatrix}=\begin{bmatrix}-3\cdot 4 & -3\cdot 1\\ -1\cdot 4 & -1\cdot 1\end{bmatrix}=\begin{bmatrix}-12 & -3\\ -4 & -1\end{bmatrix}$$

We summarize some important properties of matrix multiplication that we observed in the previous examples.

**In order for product $\mathbf{AB}$ to exist:**

- **the number of columns of $\mathbf{A}$ must equal the number of rows of $\mathbf{B}$**
- **if matrix $\mathbf{A}$ has dimension $\mathbf{m\times n}$ and matrix $\mathbf{B}$ has dimension $\mathbf{n\times p}$, then the product $\mathbf{AB}$ will be a matrix with dimension $\mathbf{m\times p}$.**

**Matrix multiplication is not commutative: if both matrix products $\mathbf{AB}$ and $\mathbf{BA}$ exist, most of the time $\mathbf{AB}$ will not equal $\mathbf{BA}$.**

**Example 2.1.9**

Given matrices $R$, $S$, and $T$ below, find $2RS-3ST$.

$$R=\begin{bmatrix}1 & 0 & 2\\ 2 & 1 & 5\\ 2 & 3 & 1\end{bmatrix} \quad S=\begin{bmatrix}0 & -1 & 2\\ 3 & 1 & 0\\ 4 & 2 & 1\end{bmatrix} \quad T=\begin{bmatrix}-2 & 3 & 0\\ -3 & 2 & 2\\ -1 & 1 & 0\end{bmatrix}$$

**Solution**

We multiply the matrices R and S.

$$\mathrm{RS} = \begin{bmatrix} 8 & 3 & 4 \\ 23 & 9 & 9 \\ 13 & 3 & 5 \end{bmatrix}$$

$$2\mathrm{RS} = 2\begin{bmatrix} 8 & 3 & 4 \\ 23 & 9 & 9 \\ 13 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 6 & 8 \\ 46 & 18 & 18 \\ 26 & 6 & 10 \end{bmatrix}$$

$$\mathrm{ST} = \begin{bmatrix} 1 & 0 & -2 \\ -9 & 11 & 2 \\ -15 & 17 & 4 \end{bmatrix}$$

$$3\mathrm{ST} = 3\begin{bmatrix} 1 & 0 & -2 \\ -9 & 11 & 2 \\ -15 & 17 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -6 \\ -27 & 33 & 6 \\ -45 & 51 & 12 \end{bmatrix}$$

Thus

$$2\mathrm{RS} - 3\mathrm{ST} = \begin{bmatrix} 16 & 6 & 8 \\ 46 & 18 & 18 \\ 26 & 6 & 10 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -6 \\ -27 & 33 & 6 \\ -45 & 51 & 12 \end{bmatrix} = \begin{bmatrix} 13 & 6 & 14 \\ 73 & -15 & 12 \\ 71 & -45 & -2 \end{bmatrix}$$

**Example 2.1.10**

Find $F^2$ given matrix

$$\mathrm{F} = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$$

**Solution**

$F^2$ is found by multiplying matrix $F$ by itself, using matrix multiplication.

$$\mathrm{F}^2 = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 2+(-1)\cdot 3 & 2\cdot(-1)+(-1)\cdot 2 \\ 3\cdot 2+2\cdot 3 & 3\cdot(-1)+2\cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 12 & 1 \end{bmatrix}$$

Note that $F^2$ is not found by squaring each entry of matrix $F$. The process of raising a matrix to a power, such as finding $F^2$, is only possible if the matrix is a square matrix.

## USING MATRICES TO REPRESENT A SYSTEM OF LINEAR EQUATIONS

In this chapter, we will be using matrices to solve linear systems. In section 2.4, we will be asked to express linear systems as the **matrix equation $\mathbf{AX} = \mathbf{B}$**, where $A$, $X$, and $B$ are matrices.

- Matrix $A$ is called the **coefficient matrix.**
- Matrix $X$ is a matrix with 1 column that contains the variables.
- Matrix $B$ is a matrix with 1 column that contains the constants.

**Example 2.1.11**

Verify that the system of two linear equations with two unknowns:

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned} \tag{2.1.1}$$

can be written as $AX = B$, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} h \\ k \end{bmatrix}$$

**Solution**

If we multiply the matrices $A$ and $X$, we get

$$AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

If $AX = B$ then

$$\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}$$

If two matrices are equal, then their corresponding entries are equal. It follows that

$$\begin{aligned} ax+by &= h \\ cx+dy &= k \end{aligned} \tag{2.1.2}$$

**Example 2.1.12**

Express the following system as a matrix equation in the form $AX = B$.

$$\begin{aligned} 2x+3y-4z &= 5 \\ 3x+4y-5z &= 6 \\ 5x \quad -6z &= 7 \end{aligned} \tag{2.1.3}$$

**Solution**

This system of equations can be expressed in the form $AX = B$ as shown below.

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & 4 & -5 \\ 5 & 0 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

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# 2.1.1: Introduction to Matrices (Exercises)

A vendor sells hot dogs and corn dogs at three different locations. His total sales(in hundreds) for January and February from the three locations are given in the table below.

| | JANUARY | | | FEBRUARY | |
|---|---|---|---|---|---|
| | HOT DOGS | CORN DOGS | | HOT DOGS | CORN DOGS |
| PLACE I | 10 | 8 | | 8 | 7 |
| PLACE II | 8 | 6 | | 6 | 7 |
| PLACE III | 6 | 4 | | 6 | 5 |

Represent these tables as $3 \times 2$ matrices $J$ and $F$, and answer problems 1 - 5.

| | |
|---|---|
| 1) Determine total sales for the two months, that is, find $J+F$. | 2) Find the difference in sales, $J-F$. |
| 3) If hot dogs sell for \$3 and corn dogs for \$2, find the revenue from the sale of hot dogs and corn dogs. *Hint: Let $P$ be a $2 \times 1$ matrix. Find $(J+F)P$.* | 4) If March sales will be up from February by 10%, 15%, and 20% at Place I, Place II, and Place III, respectively, find the expected number of hot dogs and corn dogs to be sold in March. *Hint: Let $R$ be a $1 \times 3$ matrix with entries 1.10, 1.15, and 1.20. Find $M = RF$.* |
| 5) Hots dogs sell for \$3 and corn dogs sell for \$2. Using matrix M that predicts the number of hot dogs and corn dogs expected to be sold in March from problem (4), find the $1 \times 1$ matrix that predicts total revenue in March. *Hint: Use $2 \times 1$ price matrix $P$ from problem (3) and find $MP$.* | |

Determine the sums and products in problems 6-13. Given the matrices $A$, $B$, $C$, and $D$ as follows:

$$\mathrm{A} = \begin{bmatrix} 3 & 6 & 1 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix} \quad \mathrm{B} = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 4 & 2 \\ 3 & 1 & 1 \end{bmatrix} \quad \mathrm{C} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathrm{D} = \begin{bmatrix} 2 & 3 & 2 \end{bmatrix}$$

| | |
|---|---|
| 6) $3A-2B$ | 7) $AB$ |
| 8) $BA$ | 9) $AB+BA$ |
| 10) $A^2$ | 11) $2BC$ |
| 12) $2CD+3AB$ | 13) $A^2B$ |

| | |
|---|---|
| 14) Let $E = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$ and $\mathrm{F} = \begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$, find EF. | 15) Let $E = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$ and $\mathrm{F} = \begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$, find FE. |
| 16) Let $G = \begin{bmatrix} 3 & 6 & 1 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix}$ and $H = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, find GH. | 17) Let $G = \begin{bmatrix} 3 & 6 & 1 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix}$ and $H = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Explain why the product $HG$ does not exist. |

Express the following systems as $AX = B$, where $A$, $X$, and $B$ are matrices.

18)

$$\begin{aligned} 4x - 5y &= 6 \\ 5x - 6y &= 7 \end{aligned} \qquad (2.1.1.1)$$

19)

$$\begin{aligned} x - 2y + 2z &= 3 \\ x - 3y + 4z &= 7 \\ x - 2y - 3z &= -12 \end{aligned} \qquad (2.1.1.2)$$

20)

$$\begin{aligned} 2x+3z &= 17 \\ 3x-2y &= 10 \\ 5y+2z &= 11 \end{aligned} \tag{2.1.1.3}$$

21)

$$\begin{aligned} x &+ 2y &+ 3z &+ 2w &&= 14 \\ x &- 2y &- z & &&= -5 \\ & y &- 2z &+ 4w &&= 9 \\ x & & + 3z &+ 3w &&= 15 \end{aligned} \tag{2.1.1.4}$$

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# 2.2: Systems of Linear Equations and the Gauss-Jordan Method

**Learning Objectives**

In this section you will learn to

1. Represent a system of linear equations as an augmented matrix
2. Solve the system using elementary row operations.

In this section, we learn to solve systems of linear equations using a process called the Gauss-Jordan method. The process begins by first expressing the system as a matrix, and then reducing it to an equivalent system by simple row operations. The process is continued until the solution is obvious from the matrix. The matrix that represents the system is called the **augmented matrix**, and the arithmetic manipulation that is used to move from a system to a reduced equivalent system is called a **row operation**.

**Example 2.2.1**

Write the following system as an augmented matrix.

$$\begin{aligned} 2x+3y-4z&=5\\ 3x+4y-5z&=-6\\ 4x+5y-6z&=7 \end{aligned}$$

**Solution**

We express the above information in matrix form. Since a system is entirely determined by its coefficient matrix and by its matrix of constant terms, the augmented matrix will include only the coefficient matrix and the constant matrix. So the augmented matrix we get is as follows:

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 5 \\ 3 & 4 & -5 & -6 \\ 4 & 5 & -6 & 7 \end{array}\right]$$

In the last section, we expressed the system of equations as $AX=B$, where $A$ represented the coefficient matrix, and $B$ the matrix of constant terms. As an augmented matrix, we write the matrix as $\left[\begin{array}{c|c} A & B \end{array}\right]$. It is clear that all of the information is maintained in this matrix form, and only the letters $x$, $y$ and $z$ are missing. A student may choose to write $x$, $y$ and $z$ on top of the first three columns to help ease the transition.

**Example 2.2.2**

For the following augmented matrix, write the system of equations it represents.

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 2 & 0 & -3 & -5 \\ 3 & 2 & -3 & -1 \end{array}\right]$$

**Solution**

The system is readily obtained as below.

$$\begin{aligned} x+3y-5z&=2\\ 2x-3z&=-5\\ 3x+2y-3z&=-1 \end{aligned}$$

Once a system is expressed as an augmented matrix, the Gauss-Jordan method reduces the system into a series of equivalent systems by using the row operations. This row reduction continues until the system is expressed in what is called the **reduced row**

**echelon form**. The reduced row echelon form of the coefficient matrix has 1's along the main diagonal and zeros elsewhere. The solution is readily obtained from this form.

The method is not much different form the algebraic operations we employed in the elimination method in the first chapter. The basic difference is that it is algorithmic in nature, and, therefore, can easily be programmed on a computer.

We will next solve a system of two equations with two unknowns, using the elimination method, and then show that the method is analogous to the Gauss-Jordan method.

### Example 2.2.3

Solve the following system by the elimination method.

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

**Solution**

We multiply the first equation by – 3, and add it to the second equation.

$$\begin{array}{r} -3x-9y = -21 \\ \underline{3x+4y = 11} \\ -5y = -10 \end{array}$$

By doing this we transformed our original system into an equivalent system:

$$\begin{aligned} x+3y &= 7 \\ -5y &= -10 \end{aligned}$$

We divide the second equation by – 5, and we get the next equivalent system.

$$\begin{aligned} x+3y &= 7 \\ y &= 2 \end{aligned}$$

Now we multiply the second equation by – 3 and add to the first, we get

$$\begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

### Example 2.2.4

Solve the following system from Example 3 by the Gauss-Jordan method, and show the similarities in both methods by writing the equations next to the matrices.

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned} \tag{2.2.1}$$

**Solution**

The augmented matrix for the system is as follows.

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 3 & 4 & 11 \end{array}\right] \quad \left[\begin{array}{r} x+3y=7 \\ 3x+4y=11 \end{array}\right]$$

We multiply the first row by – 3, and add to the second row.

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -5 & -10 \end{array}\right] \quad \left[\begin{array}{rr} x+3y & =7 \\ -5y & =-10 \end{array}\right]$$

We divide the second row by – 5, we get,

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{rr} x+3y & =7 \\ y & =2 \end{array}\right]$$

Finally, we multiply the second row by – 3 and add to the first row, and we get,

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{c} x=1 \\ y=2 \end{array}\right]$$

Now we list the three row operations the Gauss-Jordan method employs.

**Row Operations**

1. Any two rows in the augmented matrix may be interchanged.
2. Any row may be multiplied by a non-zero constant.
3. A constant multiple of a row may be added to another row.

One can easily see that these three row operation may make the system look different, but they do not change the solution of the system.

The first row operation states that if any two rows of a system are interchanged, the new system obtained has the same solution as the old one. Let us look at an example in two equations with two unknowns. Consider the system

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

We interchange the rows, and we get,

$$\begin{aligned} 3x+4y &= 11 \\ x+3y &= 7 \end{aligned}$$

Clearly, this system has the same solution as the one above.

The second operation states that if a row is multiplied by any non-zero constant, the new system obtained has the same solution as the old one. Consider the above system again,

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

We multiply the first row by –3, we get,

$$\begin{aligned} -3x-9y &= -21 \\ 3x+4y &= 11 \end{aligned}$$

Again, it is obvious that this new system has the same solution as the original.

The third row operation states that any constant multiple of one row added to another preserves the solution. Consider our system,

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

If we multiply the first row by –3, and add it to the second row, we get,

$$\begin{aligned} x+3y &= 7 \\ -5y &= -10 \end{aligned}$$

And once again, the same solution is maintained.

Now that we understand how the three row operations work, it is time to introduce the Gauss-Jordan method to solve systems of linear equations. As mentioned earlier, the Gauss-Jordan method starts out with an augmented matrix, and by a series of row operations ends up with a matrix that is in the **reduced row echelon form**.

A matrix is in the **reduced row echelon form** if the first nonzero entry in each row is a 1, and the columns containing these 1's have all other entries as zeros. The reduced row echelon form also requires that the leading entry in each row be to the right of the leading entry in the row above it, and the rows containing all zeros be moved down to the bottom. We state the Gauss-Jordan method as follows.

## Gauss-Jordan Method

1. Write the augmented matrix.
2. Interchange rows if necessary to obtain a non-zero number in the first row, first column.
3. Use a row operation to get a 1 as the entry in the first row and first column.
4. Use row operations to make all other entries as zeros in column one.
5. Interchange rows if necessary to obtain a nonzero number in the second row, second column. Use a row operation to make this entry 1. Use row operations to make all other entries as zeros in column two.
6. Repeat step 5 for row 3, column 3. Continue moving along the main diagonal until you reach the last row, or until the number is zero.

The final matrix is called the reduced row-echelon form.

## Example 2.2.5

Solve the following system by the Gauss-Jordan method.

$$\begin{aligned} 2x+y+2z &= 10 \\ x+2y+z &= 8 \\ 3x+y-z &= 2 \end{aligned}$$

**Solution**

We write the augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 10 \\ 1 & 2 & 1 & 8 \\ 3 & 1 & -1 & 2 \end{array}\right]$$

We want a 1 in row one, column one. This can be obtained by dividing the first row by 2, or interchanging the second row with the first. Interchanging the rows is a better choice because that way we avoid fractions.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 2 & 10 \\ 3 & 1 & -1 & 2 \end{array}\right] \quad \text{we interchanged row 1 (R1) and row 2 (R2)}$$

We need to make all other entries zeros in column 1. To make the entry (2) a zero in row 2, column 1, we multiply row 1 by - 2 and add it to the second row. We get,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 3 & 1 & -1 & 2 \end{array}\right] \quad -2R1+R2$$

To make the entry (3) a zero in row 3, column 1, we multiply row 1 by - 3 and add it to the third row. We get,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & -5 & -4 & -22 \end{array}\right] \quad -3R1+R3$$

So far we have made a 1 in the left corner and all other entries zeros in that column. Now we move to the next diagonal entry, row 2, column 2. We need to make this entry(–3) a 1 and make all other entries in this column zeros. To make row 2, column 2 entry a 1, we divide the entire second row by –3.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & -22 \end{array}\right] \quad \text{R2} \div (-3)$$

Next, we make all other entries zeros in the second column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right] \quad -2R2+R1 \text{ and } 5R2+R3$$

We make the last diagonal entry a 1, by dividing row 3 by – 4.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R3 \div (-4)$$

Finally, we make all other entries zeros in column 3.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad -\mathrm{R3}+\mathrm{R1}$$

Clearly, the solution reads $x = 1$, $y = 2$, and $z = 3$.

Before we leave this section, we mention some terms we may need in the fourth chapter.

The process of obtaining a 1 in a location, and then making all other entries zeros in that column, is called **pivoting**.

The number that is made a 1 is called the **pivot element,** and the row that contains the pivot element is called the **pivot row**.

We often multiply the pivot row by a number and add it to another row to obtain a zero in the latter. The row to which a multiple of pivot row is added is called the **target row**.

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# 2.2.1: Systems of Linear Equations and the Gauss-Jordan Method (Exercises)

## SECTION 2.2 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS

Solve the following by the Gauss-Jordan Method. Show all work.

| | |
|---|---|
| 1) $$\begin{aligned} x+3y &= 1 \\ 2x-5y &= 13 \end{aligned}$$ | 2) $$\begin{aligned} x-y-z &= -1 \\ x-3y+2z &= 7 \\ 2x-y+z &= 3 \end{aligned}$$ |
| 3) $$\begin{aligned} x+2y+3z &= 9 \\ 3x+4y+z &= 5 \\ 2x-y+2z &= 11 \end{aligned}$$ | 4) $$\begin{aligned} x+2y &= 0 \\ y+z &= 3 \\ x+3z &= 14 \end{aligned}$$ |

***SECTION 2.2 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS***

Solve the following by the Gauss-Jordan Method. Show all work.

| | |
|---|---|
| 5) Two apples and four bananas cost $2.00 and three apples and five bananas cost $2.70. Find the price of each. | 6) A bowl of corn flakes, a cup of milk, and an egg provide 16 grams of protein. A cup of milk and two eggs provide 21 grams of protein.<br>Two bowls of corn flakes with two cups of milk provide 16 grams of protein. How much protein is provided by one unit of food? |
| 7) $$\begin{aligned} x+2y &= 10 \\ y+z &= 5 \\ z+w &= 3 \\ x+w &= 5 \end{aligned}$$ | 8) $$\begin{aligned} x+w &= 6 \\ 2x+y+w &= 16 \\ x-2z &= 0 \\ z+w &= 5 \end{aligned}$$ |

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# 2.3: Systems of Linear Equations – Special Cases

**Learning Objectives**

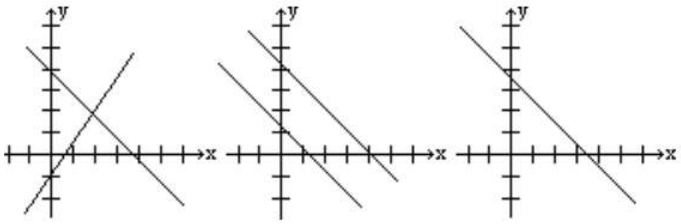
In this section, you will learn to:

1. Determine the linear systems that have no solution.
2. Solve the linear systems that have infinitely many solutions.

If we consider the intersection of two lines in a plane, three things can happen.

1. The lines intersect in exactly one point. This is called an **independent system**.
2. The lines are parallel, so they do not intersect. This is called an **inconsistent system**.
3. The lines coincide; they intersect at infinitely many points. This is a **dependent system**.

The figures below show all three cases.




Every system of equations has either one solution, no solution, or infinitely many solutions.

In the last section, we used the Gauss-Jordan method to solve systems that had exactly one solution. In this section, we will determine the systems that have no solution, and solve the systems that have infinitely many solutions.

**Example 2.3.1**

Solve the following system of equations:

$$\begin{aligned} x+y &= 7 \\ x+y &= 9 \end{aligned}$$

**Solution**

Let us use the Gauss-Jordan method to solve this system. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 1 & 1 & 9 \end{array}\right] \quad \left[\begin{array}{l} x+y=7 \\ x+y=9 \end{array}\right]$$

If we multiply the first row by - 1 and add to the second row, we get

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 0 & 2 \end{array}\right] \quad \left[\begin{array}{l} x+y=7 \\ 0x+0y=2 \end{array}\right]$$

Since 0 cannot equal 2, the last equation cannot be true for any choices of x and y.

Alternatively, it is clear that the two lines are parallel; therefore, they do not intersect.

In the examples that follow, we are going to start using a calculator to row reduce the augmented matrix, in order to focus on understanding the answer rather than focusing on the process of carrying out the row operations.

**Example 2.3.2**

Solve the following system of equations.

$$2x+3y-4z=7$$
$$3x+4y-2z=9$$
$$5x+7y-6z=20$$

**Solution**

We enter the following augmented matrix in the calculator.

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 7 \\ 3 & 4 & -2 & 9 \\ 5 & 7 & -6 & 20 \end{array}\right]$$

Now by pressing the key to obtain the reduced row-echelon form, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 10 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last row states that $0x+0y+0z=1$ . But the left side of the equation is equal to 0. So this last row states 0 = 1, which is a contradiction, a false statement.

This bottom row indicates that the system is inconsistent; therefore, there is no solution.

## Example 2.3.3

Solve the following system of equations.

$$x+y=7$$
$$x+y=7$$

**Solution**

The problem clearly asks for the intersection of two lines that are the same; that is, the lines coincide. This means the lines intersect at an infinite number of points.

A few intersection points are listed as follows: (3, 4), (5, 2), (-1, 8), (-6, 13) etc. However, when a system has an infinite number of solutions, the solution is often expressed in the parametric form. This can be accomplished by assigning an arbitrary constant, t, to one of the variables, and then solving for the remaining variables. Therefore, if we let $y=t$, then $x=7-t$ . Or we can say all ordered pairs of the form $(7-t,t)$ satisfy the given system of equations.

Alternatively, while solving the Gauss-Jordan method, we will get the reduced row-echelon form given below.

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 0 & 0 \end{array}\right]$$

The row of all zeros, can simply be ignored. This row says $0x+0y=0$; it provides no further information about the values of x and y that solve this system.

This leaves us with only one equation but two variables. And whenever there are more variables than the equations, the solution must be expressed as a parametric solution in terms of an arbitrary constant, as above.

**Parametric Solution: $\mathbf{x=7-t, y=t}$ .**

## Example 2.3.4

Solve the following system of equations.

$$x+y+z=2$$
$$2x+y-z=3$$
$$3x+2y=5$$

**Solution**

The augmented matrix and the reduced row-echelon form are given below.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 3 \\ 3 & 2 & 0 & 5 \end{array}\right] \text{ Augmented Matrix for this system}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ Reduced Row Echelon Form}$$

Since the last equation dropped out, we are left with two equations and three variables. This means the system has infinite number of solutions. We express those solutions in the parametric form by letting the last variable $z$ equal the parameter $t$.

The first equation reads $x-2z=1$ , therefore, $x=1+2z$ .

The second equation reads $y+3z=1$ , therefore, $y=1-3z$ .

And now if we let $z=t$ , the parametric solution is expressed as follows:

**Parametric Solution: $\mathbf{x=1+2t,\quad y=1-3t,\quad z=t}$ .**

The reader should note that particular solutions, or specific solutions, to the system can be obtained by assigning values to the parameter t. For example:

- if we let $t=2$ , we have the solution $x=5, y=-5, z=2 : (5,-5,2)$
- if we let $t=0$ , we have the solution $x=1, y=1, z=0 : (1,1,0)$.

## Example 2.3.5

Solve the following system of equations.

$$\begin{aligned} x+2y-3z &= 5 \\ 2x+4y-6z &= 10 \\ 3x+6y-9z &= 15 \end{aligned}$$

**Solution**

The reduced row-echelon form is given below.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This time the last two equations drop out. We are left with one equation and three variables. Again, there are an infinite number of solutions. But this time the answer must be expressed in terms of two arbitrary constants.
If we let $z=t$ and let $y=s$, the first equation $x+2y-3z=5$ results in $x=5-2s+3t$ .

We rewrite the **parametric solution: $\mathbf{x=5-2s+3t,\quad y=s,\quad z=t}$ .**

We summarize our discussion in the following table.

1. If any row of the reduced row-echelon form of the matrix gives a false statement such as 0 = 1, the system is inconsistent and has no solution.
2. If the reduced row echelon form has fewer equations than the variables and the system is consistent, then the system has an infinite number of solutions. Remember the rows that contain all zeros are dropped.
   a. If a system has an infinite number of solutions, the solution must be expressed in the parametric form.
   b. The number of arbitrary parameters equals the number of variables minus the number of equations.

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# 2.3.1: Systems of Linear Equations – Special Cases (Exercises)

## SECTION 2.3 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS - SPECIAL CASES

Solve the following inconsistent or dependent systems by using the Gauss-Jordan method.

| | |
|---|---|
| 1. $\begin{aligned} 2x+6y &= 8 \\ x+3y &= 4 \end{aligned}$ | 2. The sum of digits of a two digit number is 9. The sum of the number and the number obtained by interchanging the digits is 99. Find the number. |
| 3. $\begin{aligned} 2x-y &= 10 \\ -4x+2y &= 15 \end{aligned}$ | 4. $\begin{aligned} x+y+z &= 6 \\ 3x+2y+z &= 14 \\ 4x+3y+2z &= 20 \end{aligned}$ |
| 5. $\begin{aligned} x+2y-4z &= 1 \\ 2x-3y+8z &= 9 \end{aligned}$ | 6. Jessica has a collection of 15 coins consisting of nickels, dimes and quarters. If the total worth of the coins is $1.80, how many are there of each? Find all three solutions. |

***SECTION 2.3 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS - SPECIAL CASES***

Solve the following inconsistent or dependent systems by using the Gauss-Jordan method.

| | |
|---|---|
| 7. A company is analyzing sales reports for three products: products X, Y, Z. One report shows that a combined total of 20,000 of items X, Y, and Z were sold. Another report shows that the sum of the number of item Z sold and twice the number of item X sold equals 10,000. Also item X has 5,000 more items sold than item Y. Are these reports consistent? | 8. $\begin{aligned} x+y+2z &= 0 \\ x+2y+z &= 0 \\ 2x+3y+3z &= 0 \end{aligned}$ |
| 9. Find three solutions to the following system of equations. $\begin{aligned} x+2y+z &= 12 \\ y &= 3 \end{aligned}$ | 10. $\begin{aligned} x+2y &= 5 \\ 2x+4y &= k \end{aligned}$ For what values of k does this system of equations have<br>a. No solution?<br>b. Infinitely many solutions? |
| 11. $x+3y-z=5$ | 12. Why is it not possible for a linear system to have exactly two solutions? Explain geometrically. |

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# 2.4: Inverse Matrices

**Learning Objectives**

In this section, you will learn to:

1. Find the inverse of a matrix, if it exists.
2. Use inverses to solve linear systems.

In this section, we will learn to find the inverse of a matrix, if it exists. Later, we will use matrix inverses to solve linear systems.

Definition of an Inverse: An $n \times n$ matrix has an inverse if there exists a matrix $B$ such that $AB = BA = I_n$, where $I_n$ is an $n \times n$ identity matrix. The inverse of a matrix $A$, if it exists, is denoted by the symbol $A^{-1}$.

**Example 2.4.1**

Given matrices $A$ and $B$ below, verify that they are inverses.

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

**Solution**

The matrices are inverses if the product $AB$ and $BA$ both equal the identity matrix of dimension $2 \times 2$: $I_2$,

$$\mathrm{AB} = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}_2$$

and

$$\mathrm{BA} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}_2$$

Clearly that is the case; therefore, the matrices A and B are inverses of each other.

**Example 2.4.2**

Find the inverse of the matrix $\mathrm{A} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

**Solution**

Suppose $A$ has an inverse, and it is

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then $AB = I_2$: $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

After multiplying the two matrices on the left side, we get

$$\begin{bmatrix} 3a+c & 3b+d \\ 5a+2c & 5b+2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding entries, we get four equations with four unknowns:

$$\begin{aligned} 3a+c &= 1 \quad & 3b+d &= 0 \\ 5a+2c &= 0 \quad & 5b+2d &= 1 \end{aligned}$$

Solving this system, we get: $a=2 \quad b=-1 \quad c=-5 \quad d=3$

Therefore, the inverse of the matrix $A$ is $B=\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

In this problem, finding the inverse of matrix $A$ amounted to solving the system of equations:

$$\begin{aligned} 3a+c&=1 \qquad 3b+d=0 \\ 5a+2c&=0 \qquad 5b+2d=1 \end{aligned}$$

Actually, it can be written as two systems, one with variables $a$ and $c$, and the other with $b$ and $d$. The augmented matrices for both are given below.

$$\left[\begin{array}{cc|c} 3 & 1 & 1 \\ 5 & 2 & 0 \end{array}\right] \text{ and } \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 5 & 2 & 1 \end{array}\right]$$

As we look at the two augmented matrices, we notice that the coefficient matrix for both the matrices is the same. This implies the row operations of the Gauss-Jordan method will also be the same. A great deal of work can be saved if the two right hand columns are grouped together to form one augmented matrix as below.

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right]$$

And solving this system, we get

The matrix on the right side of the vertical line is the $A^{-1}$ matrix.

What you just witnessed is no coincidence. This is the method that is often employed in finding the inverse of a matrix. We list the steps, as follows:

**The Method for Finding the Inverse of a Matrix**

1. Write the augmented matrix $[A|I_n]$.
2. Write the augmented matrix in step 1 in reduced row echelon form.
3. If the reduced row echelon form in 2 is $[I_n|B]$, then $B$ is the inverse of $A$.
4. If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.

**Example 2.4.3**

Given the matrix A below, find its inverse.

$$A=\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

**Solution**

We write the augmented matrix as follows.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

We will reduce this matrix using the Gauss-Jordan method.

Multiplying the first row by -2 and adding it to the second row, we get

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

If we swap the second and third rows, we get

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \\ 0 & 5 & -2 & -2 & 1 & 0 \end{array}\right]$$

Divide the second row by -2. The result is

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 5 & -2 & -2 & 1 & 0 \end{array}\right]$$

Let us do two operations here. 1) Add the second row to first, 2) Add -5 times the second row to the third. And we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1/2 & -2 & 1 & 5/2 \end{array}\right]$$

Multiplying the third row by 2 results in

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

Multiply the third row by 1/2 and add it to the second.
Also, multiply the third row by -1/2 and add it to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -3 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

Therefore, the inverse of matrix $A$ is $\mathrm{A}^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$

One should verify the result by multiplying the two matrices to see if the product does, indeed, equal the identity matrix.

Now that we know how to find the inverse of a matrix, we will use inverses to solve systems of equations. The method is analogous to solving a simple equation like the one below.

$$\frac{2}{3}x = 4$$

**Example 2.4.4**

Solve the following equation: $\frac{2}{3}x = 4$

**Solution**

To solve the above equation, we multiply both sides of the equation by the multiplicative inverse of $\frac{2}{3}$ which happens to be $\frac{3}{2}$. We get

$$\frac{3}{2} \cdot \frac{2}{3}x = 4 \cdot \frac{3}{2}$$
$$x = 6$$

We use the Example 2.4.4 as an analogy to show how linear systems of the form $AX = B$ are solved.

To solve a linear system, we first write the system in the matrix equation $AX = B$, where $A$ is the coefficient matrix, $X$ the matrix of variables, and $B$ the matrix of constant terms.
We then multiply both sides of this equation by the multiplicative inverse of the matrix $A$.

Consider the following example.

**Example 2.4.5**

Solve the following system

$$\begin{aligned} 3x+y &= 3 \\ 5x+2y &= 4 \end{aligned}$$

**Solution**

To solve the above equation, first we express the system as

$$AX = B$$

where A is the coefficient matrix, and B is the matrix of constant terms. We get

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

To solve this system, we multiply both sides of the matrix equation $AX = B$ by $A^{-1}$. Matrix multiplication is not commutative, so we need to multiply by $A^{-1}$ on the left on both sides of the equation.

Matrix $A$ is the same matrix $A$ whose inverse we found in Example 2.4.2, so $\mathrm{A}^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

Multiplying both sides by $A^{-1}$, we get

$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Therefore, $x = 2$, and $y = -3$.

**Example 2.4.6**

Solve the following system:

$$\begin{aligned} x-y+z &= 6 \\ 2x+3y &= 1 \\ -2y+z &= 5 \end{aligned}$$

**Solution**

To solve the above equation, we write the system in matrix form $AX = B$ as follows:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

To solve this system, we need inverse of $A$. From Example 2.4.3, $\mathrm{A}^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$

Multiplying both sides of the matrix equation $AX = B$ on the left by $A^{-1}$, we get

$$\begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

After multiplying the matrices, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

We remind the reader that not every system of equations can be solved by the matrix inverse method. Although the Gauss-Jordan method works for every situation, the matrix inverse method works only in cases where the inverse of the square matrix exists. In such cases the system has a unique solution.

**The Method for Finding the Inverse of a Matrix**

1. Write the augmented matrix $[\mathrm{A}|\mathrm{I_n}]$.
2. Write the augmented matrix in step 1 in reduced row echelon form.
3. If the reduced row echelon form in 2 is $[\mathrm{I_n}|\mathrm{B}]$, then $B$ is the inverse of $A$.
4. If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.

**The Method for Solving a System of Equations When a Unique Solution Exists**

1. Express the system in the matrix equation $AX = B$.

2. To solve the equation $AX = B$, we multiply on both sides by $A^{-1}$.

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B \text{ where } I \text{ is the identity matrix}$$

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# 2.4.1: Inverse Matrices (Exercises)

## SECTION 2.4 PROBLEM SET: INVERSE MATRICES

In problems 1- 2, verify that the given matrices are inverses of each other.

1. $\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 3 & -4 \end{bmatrix} \begin{bmatrix} 3 & -4 & 1 \\ 2 & -4 & 1 \\ 3 & -5 & 1 \end{bmatrix}$

In problems 3- 6, find the inverse of each matrix by the row-reduction method.

3. $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

***SECTION 2.4 PROBLEM SET: INVERSE MATRICES***

In problems 5 - 6, find the inverse of each matrix by the row-reduction method.

5. $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Problems 7 -10: Express the system as $AX = B$; then solve using matrix inverses found in problems 3 - 6.

7. $\begin{aligned} 3x - 5y &= 2 \\ -x + 2y &= 0 \end{aligned}$

8. $\begin{aligned} x + \quad 2z &= 8 \\ y + 4z &= 8 \\ z &= 3 \end{aligned}$

***SECTION 2.4 PROBLEM SET: INVERSE MATRICES***

Problems 9 -10: Express the system as $AX = B$; then solve using matrix inverses found in problems 3 - 6.

9. $\begin{aligned} x + y - z &= 2 \\ x + z &= 7 \\ 2x + y + z &= 13 \end{aligned}$

10. $\begin{aligned} x + y + z &= 2 \\ 3x + y &= 7 \\ x + y + 2z &= 3 \end{aligned}$

11. Why is it necessary that a matrix be a square matrix for its inverse to exist? Explain by relating the matrix to a system of equations.

12. Suppose we are solving a system $AX = B$ by the matrix inverse method, but discover $A$ has no inverse. How else can we solve this system? What can be said about the solutions of this system?

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# 2.5: Application of Matrices in Cryptography

**Learning Objectives**

In this section, we will learn to find the inverse of a matrix, if it exists. Later, we will use matrix inverses to solve linear systems. In this section you will learn to

1. encode a message using matrix multiplication.
2. decode a coded message using the matrix inverse and matrix multiplication

Encryption dates back approximately 4000 years. Historical accounts indicate that the Chinese, Egyptians, Indian, and Greek encrypted messages in some way for various purposes. One famous encryption scheme is called the Caesar cipher, also called a substitution cipher, used by Julius Caesar, involved shifting letters in the alphabet, such as replacing A by C, B by D, C by E, etc, to encode a message. Substitution ciphers are too simple in design to be considered secure today.

In the middle ages, European nations began to use encryption. A variety of encryption methods were used in the US from the Revolutionary War, through the Civil War, and on into to modern times.

Applications of mathematical theory and methods to encryption became widespread in military usage in the $20^{th}$ century. The military would encode messages before sending and the recipient would decode the message, in order to send information about military operations in a manner that kept the information safe if the message was intercepted. In World War II, encryption played an important role, as both Allied and Axis powers sent encrypted messages and devoted significant resources to strengthening their own encryption while also trying to break the opposition's encryption.

In this section we will examine a method of encryption that uses matrix multiplication and matrix inverses. This method, known as the Hill Algorithm, was created by Lester Hill, a mathematics professor who taught at several US colleges and also was involved with military encryption. The Hill algorithm marks the introduction of modern mathematical theory and methods to the field of cryptography.

These days, the Hill Algorithm is not considered a secure encryption method; it is relatively easy to break with modern technology. However, in 1929 when it was developed, modern computing technology did not exist. This method, which we can handle easily with today's technology, was too cumbersome to use with hand calculations. Hill devised a mechanical encryption machine to help with the mathematics; his machine relied on gears and levers, but never gained widespread use. Hill's method was considered sophisticated and powerful in its time and is one of many methods influencing techniques in use today. Other encryption methods at that time also utilized special coding machines. Alan Turing, a computer scientist pioneer in the field of artificial intelligence, invented a machine that was able to decrypt messages encrypted by the German Enigma machine, helping to turn the tide of World War II.

With the advent of the computer age and internet communication, the use of encryption has become widespread in communication and in keeping private data secure; it is no longer limited to military uses. Modern encryption methods are more complicated, often combining several steps or methods to encrypt data to keep it more secure and harder to break. Some modern methods make use of matrices as part of the encryption and decryption process; other fields of mathematics such as number theory play a large role in modern cryptography.

To use matrices in encoding and decoding secret messages, our procedure is as follows.

We first convert the secret message into a string of numbers by arbitrarily assigning a number to each letter of the message. Next we convert this string of numbers into a new set of numbers by multiplying the string by a square matrix of our choice that has an inverse. This new set of numbers represents the coded message.

To decode the message, we take the string of coded numbers and multiply it by the inverse of the matrix to get the original string of numbers. Finally, by associating the numbers with their corresponding letters, we obtain the original message.

In this section, we will use the correspondence shown below where letters A to Z correspond to the numbers 1 to 26, a space is represented by the number 27, and punctuation is ignored.

| A | B | C | D | E | F | G | H | I | J | K | L | M |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| N | O | P | Q | R | S | T | U | V | W | X | Y | Z |
| 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |

**Example 2.5.1**

Use matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ to encode the message: ATTACK NOW!

**Solution**

We divide the letters of the message into groups of two.

$$\text{AT} \quad \text{TA} \quad \text{CK} \quad \text{-N} \quad \text{OW}$$

We assign the numbers to these letters from the above table, and convert each pair of numbers into $2 \times 1$ matrices. In the case where a single letter is left over on the end, a space is added to make it into a pair.

$$\begin{bmatrix} \text{A} \\ \text{T} \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix} \quad \begin{bmatrix} \text{T} \\ \text{A} \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \end{bmatrix} \quad \begin{bmatrix} \text{C} \\ \text{K} \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} - \\ \text{N} \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \end{bmatrix} \quad \begin{bmatrix} \text{O} \\ \text{W} \end{bmatrix} = \begin{bmatrix} 15 \\ 23 \end{bmatrix}$$

So at this stage, our message expressed as $2 \times 1$ matrices is as follows.

$$\begin{bmatrix} 1 \\ 20 \end{bmatrix} \begin{bmatrix} 20 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 11 \end{bmatrix} \begin{bmatrix} 27 \\ 14 \end{bmatrix} \begin{bmatrix} 15 \\ 23 \end{bmatrix} \quad \textbf{(I)}$$

Now to encode, we multiply, on the left, each matrix of our message by the matrix $A$. For example, the product of $A$ with our first matrix is: $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} 41 \\ 61 \end{bmatrix}$

And the product of $A$ with our second matrix is: $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 23 \end{bmatrix}$

Multiplying each matrix in $\textbf{(I)}$ by matrix $A$, in turn, gives the desired coded message:

$$\begin{bmatrix} 41 \\ 61 \end{bmatrix} \begin{bmatrix} 22 \\ 23 \end{bmatrix} \begin{bmatrix} 25 \\ 36 \end{bmatrix} \begin{bmatrix} 55 \\ 69 \end{bmatrix} \begin{bmatrix} 61 \\ 84 \end{bmatrix}$$

**Example 2.5.2**

Decode the following message that was encoded using matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 21 \\ 26 \end{bmatrix} \begin{bmatrix} 37 \\ 53 \end{bmatrix} \begin{bmatrix} 45 \\ 54 \end{bmatrix} \begin{bmatrix} 74 \\ 101 \end{bmatrix} \begin{bmatrix} 53 \\ 69 \end{bmatrix} \quad \textbf{(II)}$$

**Solution**

Since this message was encoded by multiplying by the matrix $A$ in Example 2.5.1, we decode this message by first multiplying each matrix, on the left, by the inverse of matrix $A$ given below.

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

For example: $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 21 \\ 26 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$

By multiplying each of the matrices in $\textbf{(II)}$ by the matrix $A^{-1}$, we get the following.

$$\begin{bmatrix}11\\5\end{bmatrix}\begin{bmatrix}5\\16\end{bmatrix}\begin{bmatrix}27\\9\end{bmatrix}\begin{bmatrix}20\\27\end{bmatrix}\begin{bmatrix}21\\16\end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain:

$$\begin{bmatrix}\mathrm{K}\\\mathrm{E}\end{bmatrix}\begin{bmatrix}\mathrm{E}\\\mathrm{P}\end{bmatrix}\begin{bmatrix}-\\\mathrm{I}\end{bmatrix}\begin{bmatrix}\mathrm{T}\\-\end{bmatrix}\begin{bmatrix}\mathrm{U}\\\mathrm{P}\end{bmatrix}$$

And the message reads: KEEP IT UP.

Now suppose we wanted to use a $3\times 3$ matrix to encode a message, then instead of dividing the letters into groups of two, we would divide them into groups of three.

**Example 2.5.3**

Using the matrix $B=\begin{bmatrix}1&1&-1\\1&0&1\\2&1&1\end{bmatrix}$, encode the message: ATTACK NOW!

**Solution**

We divide the letters of the message into groups of three.

ATT ACK -NO W- -

Note that since the single letter "W" was left over on the end, we added two spaces to make it into a triplet.

Now we assign the numbers their corresponding letters from the table, and convert each triplet of numbers into $3\times 1$ matrices. We get

$$\begin{bmatrix}\mathrm{A}\\\mathrm{T}\\\mathrm{T}\end{bmatrix}=\begin{bmatrix}1\\20\\20\end{bmatrix}\begin{bmatrix}\mathrm{A}\\\mathrm{C}\\\mathrm{K}\end{bmatrix}=\begin{bmatrix}1\\3\\11\end{bmatrix}\begin{bmatrix}-\\\mathrm{N}\\\mathrm{O}\end{bmatrix}=\begin{bmatrix}27\\14\\15\end{bmatrix}\begin{bmatrix}\mathrm{W}\\-\\-\end{bmatrix}=\begin{bmatrix}23\\27\\27\end{bmatrix}$$

So far we have,

$$\begin{bmatrix}1\\20\\20\end{bmatrix}\begin{bmatrix}1\\3\\11\end{bmatrix}\begin{bmatrix}27\\14\\15\end{bmatrix}\begin{bmatrix}23\\27\\27\end{bmatrix}\quad\textbf{(III)}$$

We multiply, on the left, each matrix of our message by the matrix $B$. For example,

$$\begin{bmatrix}1&1&-1\\1&0&1\\2&1&1\end{bmatrix}\begin{bmatrix}1\\20\\20\end{bmatrix}=\begin{bmatrix}1\\21\\42\end{bmatrix}$$

By multiplying each of the matrices in $\textbf{(III)}$ by the matrix $B$, we get the desired coded message as follows:

$$\begin{bmatrix}1\\21\\42\end{bmatrix}\begin{bmatrix}-7\\12\\16\end{bmatrix}\begin{bmatrix}26\\42\\83\end{bmatrix}\begin{bmatrix}23\\50\\100\end{bmatrix}$$

If we need to decode this message, we simply multiply the coded message by $B^{-1}$, and associate the numbers with the corresponding letters of the alphabet.

In Example 2.5.4 we will demonstrate how to use matrix $B^{-1}$ to decode an encrypted message.

**Example 2.5.4**

Decode the following message that was encoded using matrix $B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix} \begin{bmatrix} 25 \\ 10 \\ 41 \end{bmatrix} \begin{bmatrix} 22 \\ 14 \\ 41 \end{bmatrix} \quad \textbf{(IV)}$$

**Solution**

Since this message was encoded by multiplying by the matrix $B$. We first determine inverse of $B$.

$$\mathrm{B}^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

To decode the message, we multiply each matrix, on the left, by $B^{-1}$. For example,

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix}$$

Multiplying each of the matrices in $\textbf{(IV)}$ by the matrix $B^{-1}$ gives the following.

$$\begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix} \begin{bmatrix} 4 \\ 27 \\ 6 \end{bmatrix} \begin{bmatrix} 9 \\ 18 \\ 5 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain

$$\begin{bmatrix} \mathrm{H} \\ \mathrm{O} \\ \mathrm{L} \end{bmatrix} \begin{bmatrix} \mathrm{D} \\ - \\ \mathrm{F} \end{bmatrix} \begin{bmatrix} \mathrm{I} \\ \mathrm{R} \\ \mathrm{E} \end{bmatrix} \quad \text{The message reads: HOLD FIRE}$$

The message reads: HOLD FIRE.

We summarize:

**TO ENCODE A MESSAGE**

1. Divide the letters of the message into groups of two or three.

2. Convert each group into a string of numbers by assigning a number to each letter of the message. Remember to assign letters to blank spaces.

3. Convert each group of numbers into column matrices.

3. Convert these column matrices into a new set of column matrices by multiplying them with a compatible square matrix of your choice that has an inverse. This new set of numbers or matrices represents the coded message.

**TO DECODE A MESSAGE**

1. Take the string of coded numbers and multiply it by the inverse of the matrix that was used to encode the message.

2. Associate the numbers with their corresponding letters.

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# 2.5.1: Application of Matrices in Cryptography (Exercises)

## SECTION 2.5 PROBLEM SET: APPLICATION OF MATRICES IN CRYPTOGRAPHY

In problems 1 - 8, the letters A to Z correspond to the numbers 1 to 26, as shown below, and a space is represented by the number 27.

| A | B | C | D | E | F | G | H | I | J | K | L | M |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| N | O | P | Q | R | S | T | U | V | W | X | Y | Z |
| 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |

In problems 1 - 2, use the matrix $A$, given below, to encode the given messages.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

In problems 3 - 4, decode the messages that were encoded using matrix $A$.
Make sure to consider the spaces between words, but ignore all punctuation. Add a final space if necessary.

| | |
|---|---|
| 1. Encode the message: WATCH OUT! | 2. Encode the message: HELP IS ON THE WAY. |
| 3. Decode the following message:<br>64 23 102 41 82 32 97 35 71 28 69 32 | 4. Decode the following message:<br>105 40 117 48 39 19 69 32 72 27 37 15 114 47 |

***SECTION 2.5 PROBLEM SET: APPLICATION OF MATRICES IN CRYPTOGRAPHY***

In problems 5 - 6, use the matrix $B$, given below, to encode the given messages.

$$\mathrm{B} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

In problems 7 - 8, decode the messages that were encoded using matrix $B$.

Make sure to consider the spaces between words, but ignore all punctuation. Add a final space if necessary.

| | |
|---|---|
| 5. Encode the message using matrix $B$:<br>LUCK IS ON YOUR SIDE. | 6. Encode the message using matrix $B$:<br>MAY THE FORCE BE WITH YOU. |
| 7. Decode the following message that was encoded using matrix $B$:<br>8 23 7 4 47 - 2 15 102 -12 20 58 15 27 80 18 12 74 -7 | 8. Decode the following message that was encoded using matrix $B$:<br>12 69 - 3 11 53 9 5 46 -10 18 95 - 9 25 107 4 27 76 22 1 72 -26 |

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# 2.6: Applications – Leontief Models

**Learning Objectives**

In this section we will examine an application of matrices to model economic systems.

In the 1930's, Wassily Leontief used matrices to model economic systems. His models, often referred to as the input-output models, divide the economy into sectors where each sector produces goods and services not only for itself but also for other sectors. These sectors are dependent on each other and the total input always equals the total output. In 1973, he won the Nobel Prize in Economics for his work in this field. In this section we look at both the closed and the open models that he developed.

## The Closed Model

As an example of the closed model, we look at a very simple economy, where there are only three sectors: food, shelter, and clothing.

**Example 2.6.1**

We assume that in a village there is a farmer, carpenter, and a tailor, who provide the three essential goods: food, shelter, and clothing. Suppose the farmer himself consumes 40% of the food he produces, and gives 40% to the carpenter, and 20% to the tailor. Thirty percent of the carpenter's production is consumed by himself, 40% by the farmer, and 30% by the carpenter. Fifty percent of the tailor's production is used by himself, 30% by the farmer, and 20% by the tailor. Write the matrix that describes this closed model.

**Solution**

The table below describes the above information.

| | Proportion produced by the farmer | Proportion produced by the carpenter | Proportion produced by the tailor |
|---|---|---|---|
| The proportion used by the farmer | .40 | .40 | .30 |
| The proportion used by the carpenter | .40 | .30 | .20 |
| The proportion used by the tailor | .20 | .30 | .50 |

In a matrix form it can be written as follows.

$$A = \begin{bmatrix} .40 & .40 & .30 \\ .40 & .30 & .20 \\ .20 & .30 & .50 \end{bmatrix}$$

This matrix is called the **input-output matrix.** It is important that we read the matrix correctly. For example the entry $A_{23}$, the entry in row 2 and column 3, represents the following.

$A_{23}$ = 20% of the tailor's production is used by the carpenter.

$A_{33}$ = 50% of the tailor's production is used by the tailor.

**Example 2.6.2**

In Example 2.6.1 above, how much should each person get for his efforts?

**Solution**

We choose the following variables.

$x$ = Farmer's pay $y$ = Carpenter's pay $z$ = Tailor's pay

As we said earlier, in this model input must equal output. That is, the amount paid by each equals the amount received by each.

Let us say the farmer gets paid $x$ dollars. Let us now look at the farmer's expenses. The farmer uses up 40% of his own production, that is, of the x dollars he gets paid, he pays himself .40x dollars, he pays .40y dollars to the carpenter, and .30z to the tailor. Since the expenses equal the wages, we get the following equation.

$$x = .40x + .40y + .30z$$

In the same manner, we get

$$y = .40x + .30y + .20z$$
$$z = .20x + .30y + .50z$$

The above system can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .40 & .40 & 30 \\ .40 & .30 & .20 \\ .20 & .30 & .50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This system is often referred to as $X = AX$

Simplification results in the system of equations $(I - A)X = 0$

$$\begin{aligned} .60x - .40y - .30z &= 0 \\ -.40x + .70y - .20z &= 0 \\ -.20x - .30y + .50z &= 0 \end{aligned}$$

Solving for $x$, $y$, and $z$ using the Gauss-Jordan method, we get

$$x = \frac{29}{26}t \quad y = \frac{12}{13}t \quad \text{and } z = t$$

Since we are only trying to determine the proportions of the pay, we can choose t to be any value. Suppose we let \(t\(= $2600, then we get

$$x = \$2900 \quad y = \$2400 \quad \text{and } z = \$2600$$

Note: The use of a graphing calculator or computer application in solving the systems of linear matrix equations in these problems is strongly recommended.

## The Open Model

The open model is more realistic, as it deals with the economy where sectors of the economy not only satisfy each other's' needs, but they also satisfy some outside demands. In this case, the outside demands are put on by the consumer. But the basic assumption is still the same; that is, whatever is produced is consumed.

Let us again look at a very simple scenario. Suppose the economy consists of three people, the farmer F, the carpenter C, and the tailor T. A part of the farmer's production is used by all three, and the rest is used by the consumer. In the same manner, a part of the carpenter's and the tailor's production is used by all three, and rest is used by the consumer.

Let us assume that whatever the farmer produces, 20% is used by him, 15% by the carpenter, 10% by the tailor, and the consumer uses the other 40 billion dollars worth of the food. Ten percent of the carpenter's production is used by him, 25% by the farmer, 5% by the tailor, and 50 billion dollars worth by the consumer. Fifteen percent of the clothing is used by the tailor, 10% by the farmer, 5% by the carpenter, and the remaining 60 billion dollars worth by the consumer. We write the internal consumption in the following table, and express the demand as the matrix D.

| | F produces | C produces | T produces |
|---|---|---|---|
| F uses | .20 | .25 | .10 |
| C uses | .15 | .10 | .05 |

| | | | |
|---|---|---|---|
| T uses | .10 | .05 | .15 |

The consumer demand for each industry in billions of dollars is given below.

$$\mathrm{D} = \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$$

**Example 2.6.3**

In the example above, what should be, in billions of dollars, the required output by each industry to meet the demand given by the matrix $D$?

**Solution**

We choose the following variables.

x = Farmer's output

y = Carpenter's output

z = Tailor's output

In the closed model, our equation was $X = AX$, that is, the total input equals the total output. This time our equation is similar with the exception of the demand by the consumer.

So our equation for the open model should be $X = AX + D$ , where $D$ represents the demand matrix.

We express it as follows:

$$X = AX + D$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .20 & .25 & .10 \\ .15 & .10 & .05 \\ .10 & .05 & .15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$$

To solve this system, we write it as

$$\begin{aligned} &X = AX + D \\ &(I - A)X = D \quad \text{where I is a 3 by 3 identity matrix} \\ &X = (I - A)^{-1} D \end{aligned}$$

$$\mathrm{I - A} = \begin{bmatrix} .80 & -.25 & -.10 \\ -.15 & .90 & -.05 \\ -.10 & -.05 & .85 \end{bmatrix}$$

$$(\mathrm{I - A})^{-1} = \begin{bmatrix} 1.3445 & .3835 & .1807 \\ .2336 & 1.1814 & .097 \\ .1719 & .1146 & 1.2034 \end{bmatrix}$$

$$X = \begin{bmatrix} 1.3445 & .3835 & .1807 \\ .2336 & 1.1814 & .097 \\ .1719 & .1146 & 1.2034 \end{bmatrix} \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$$

$$X = \begin{bmatrix} 83.7999 \\ 74.2341 \\ 84.8138 \end{bmatrix}$$

The three industries must produce the following amount of goods in billions of dollars.

Farmer = \$83.7999 Carpenter = \$74.2341 Tailor = \$84.813

We will do one more problem like the one above, except this time we give the amount of internal and external consumption in dollars and ask for the proportion of the amounts consumed by each of the industries. In other words, we ask for the matrix $A$.

**Example 2.6.4**

Suppose an economy consists of three industries F, C, and T. Each of the industries produces for internal consumption among themselves, as well as for external demand by the consumer. The table shows the use of each industry's production ,in dollars.

| | F | C | T | Demand | Total |
|---|---|---|---|---|---|
| F | 40 | 50 | 60 | 100 | 250 |
| C | 30 | 40 | 40 | 110 | 220 |
| T | 20 | 30 | 30 | 120 | 200 |

The first row says that of the $250 dollars worth of production by the industry F, $40 is used by F, $50 is used by C, $60 is used by T, and the remainder of $100 is used by the consumer. The other rows are described in a similar manner.

Once again, the total input equals the total output. Find the proportion of the amounts consumed by each of the industries. In other words, find the matrix $A$.

**Solution**

We are being asked to determine the following:

How much of the production of each of the three industries, F, C, and T is required to produce one unit of F? In the same way, how much of the production of each of the three industries, F, C, and T is required to produce one unit of C? And finally, how much of the production of each of the three industries, F, C, and T is required to produce one unit of T?

Since we are looking for proportions, we need to divide the production of each industry by the total production for each industry.

We analyze as follows:

To produce 250 units of F, we need to use 40 units of F, 30 units of C, and 20 units of T.

Therefore, to produce 1 unit of F, we need to use 40/250 units of F, 30/250 units of C, and 20/250 units of T.

To produce 220 units of C, we need to use 50 units of F, 40 units of C, and 30 units of T.

Therefore, to produce 1 unit of C, we need to use 50/220 units of F, 40/220 units of C, and 30/220 units of T.

To produce 200 units of T, we need to use 60 units of F, 40 units of C, and 30 units of T.

Therefore, to produce 1 unit of T, we need to use 60/200 units of F, 40/200 units of C, and 30/200 units of T.

We obtain the following matrix.

$$\mathrm{A} = \begin{bmatrix} 40/250 & 50/220 & 60/200 \\ 30/250 & 40/220 & 40/200 \\ 20/250 & 30/220 & 30/200 \end{bmatrix} = \begin{bmatrix} .1600 & .2273 & .3000 \\ .1200 & .1818 & .2000 \\ .0800 & .1364 & .1500 \end{bmatrix}$$

Clearly $AX + D = X$

$$\begin{bmatrix} 40/250 & 50/220 & 60/200 \\ 30/250 & 40/220 & 40/200 \\ 20/250 & 30/220 & 30/200 \end{bmatrix} \begin{bmatrix} 250 \\ 220 \\ 200 \end{bmatrix} + \begin{bmatrix} 100 \\ 110 \\ 120 \end{bmatrix} = \begin{bmatrix} 250 \\ 220 \\ 200 \end{bmatrix}$$

We summarize as follows:

**LEONTIEF'S CLOSED MODEL**

1. All consumption is within the industries. There is no external demand.
2. Input = Output

3. $X = AX$ or $(I - A)X = 0$

**LEONTIEF'S OPEN MODEL**

1. In addition to internal consumption, there is an outside demand by the consumer.
2. Input = Output
3. $X = AX + D$ or $X = (I - A)^{-1}D$

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# 2.6.1: Applications – Leontief Models (Exercises)

## SECTION 2.6 PROBLEM SET: APPLICATIONS - LEONTIEF MODELS

1) Solve the following homogeneous system.

$$\begin{aligned} x+y+z &= 0 \\ 3x+2y+z &= 0 \\ 4x+3y+2z &= 0 \end{aligned}$$

2) Solve the following homogeneous system.

$$\begin{aligned} x-y-z &= 0 \\ x-3y+2z &= 0 \\ 2x-4y+z &= 0 \end{aligned}$$

3) Chris and Ed decide to help each other by doing repairs on each others houses. Chris is a carpenter, and Ed is an electrician. Chris does carpentry work on his house as well as on Ed's house. Similarly, Ed does electrical repairs on his house and on Chris' house. When they are all finished they realize that Chris spent 60% of his time on his own house, and 40% of his time on Ed's house. On the other hand Ed spent half of his time on his house and half on Chris's house. If they originally agreed that each should get about a $1000 for their work, how much money should each get for their work?

4) Chris, Ed, and Paul decide to help each other by doing repairs on each others houses. Chris is a carpenter, Ed is an electrician, and Paul is a plumber. Each does work on his own house as well as on the others houses. When they are all finished they realize that Chris spent 30% of his time on his own house, 40% of his time on Ed's house, and 30% on Paul's house. Ed spent half of his time on his own house, 30% on Chris' house, and remaining on Paul's house. Paul spent 40% of the time on his own house, 40% on Chris' house, and 20% on Ed's house. If they originally agreed that each should get about a $1000 for their work, how much money should each get for their work?

***SECTION 2.6 PROBLEM SET: APPLICATIONS - LEONTIEF MODELS***

5) Given the internal consumption matrix $A$, and the external demand matrix $D$ as follows.

$$A = \begin{bmatrix} .30 & .20 & .10 \\ .20 & .10 & .30 \\ .10 & .20 & .30 \end{bmatrix} \quad D = \begin{bmatrix} 100 \\ 150 \\ 200 \end{bmatrix}$$

Solve the system using the open model: $X = AX + D$ or $X = (I - A)^{-1} D$

6) Given the internal consumption matrix $A$, and the external demand matrix $D$ as follows.

$$A = \begin{bmatrix} .05 & .10 & .10 \\ .10 & .15 & .05 \\ .05 & .20 & .20 \end{bmatrix} \quad D = \begin{bmatrix} 50 \\ 100 \\ 80 \end{bmatrix}$$

Solve the system using the open model: $X = AX + D$ or $X = (I - A)^{-1} D$

7) An economy has two industries, farming and building. For every $1 of food produced, the farmer uses $.20 and the builder uses $.15. For every $1 worth of building, the builder uses $.25 and the farmer uses $.20. If the external demand for food is $100,000, and for building $200,000, what should be the total production for each industry in dollars?

***SECTION 2.6 PROBLEM SET: APPLICATIONS - LEONTIEF MODELS***

8) An economy has three industries, farming, building, and clothing. For every $1 of food produced, the farmer uses $.20, the builder uses $.15, and the tailor $.05. For every $1 worth of building, the builder uses $.25, the farmer uses $.20, and the tailor $.10. For every $1 worth of clothing, the tailor uses $.10, the builder uses $.20, the farmer uses $.15. If the external demand for food is $100 million, for building $200 million, and for clothing $300 million, what should be the total production for each in dollars?

9) Suppose an economy consists of three industries F, C, and T. The following table gives information about the internal use of each industry's production and external demand in dollars.

| | F | C | T | Demand | Total |
|---|---|---|---|---|---|
| F | 30 | 10 | 20 | 40 | 100 |
| C | 20 | 30 | 20 | 50 | 120 |
| T | 10 | 10 | 30 | 60 | 110 |

Find the proportion of the amounts consumed by each of the industries; that is, find the matrix $A$.

10) If in problem 9, the consumer demand for F, C, and T becomes 60, 80, and 100, respectively, find the total output and the internal use by each industry to meet that demand.

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# 2.7: Chapter Review

## SECTION 2.7 PROBLEM SET: CHAPTER REVIEW

1. To reinforce her diet, Mrs. Tam bought a bottle containing 30 tablets of Supplement A and a bottle containing 50 tablets of Supplement B. Each tablet of supplement A contains 1000 mg of calcium, 400 mg of magnesium, and 15 mg of zinc, and each tablet of supplement B contains 800 mg of calcium, 500 mg of magnesium, and 20 mg of zinc.}
   a. Represent the amount of calcium, magnesium and zinc in each tablet as a $2 \times 3$ matrix.
   b. Represent the number of tablets in each bottle as a row matrix.
   c. Use matrix multiplication to find the total amount of calcium, magnesium, and zinc in both bottles.
2. Let matrix $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 3 & -1 \\ 1 & 4 & -3 \end{bmatrix}$. Find the following.
   a. $\frac{1}{2}(A+B)$
   b. $3A = 2B$
3. Let matrix $\mathrm{C} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & -3 & -1 \\ 3 & -1 & -2 \\ 3 & -3 & -2 \end{bmatrix}$. Find the following.
   a. $2(C-D)$
   b. $C - 3D$
4. Let matrix $E = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $\mathrm{F} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -3 \end{bmatrix}$. Find the following.
   a. $2EF$
   b. $3FE$
5. Let matrix $G = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ and $H = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Find the following.
   a. $2GH$
   b. $HG$
6. Solve the following systems using the Gauss-Jordan Method.
   a.
   $$\begin{aligned} x + 3y - 2z &= 7 \\ 2x + 7y - 5z &= 16 \\ x + 5y - 3z &= 10 \end{aligned}$$
   b.
   $$\begin{aligned} 2x - 4y + 4z &= 2 \\ 2x + y + 9z &= 17 \\ 3x - 2y + 2z &= 7 \end{aligned}$$
7. An apple, a banana and three oranges or two apples, two bananas, and an orange, or four bananas and two oranges cost $2. Find the price of each.
8. Solve the following systems. If a system has an infinite number of solutions, first express the solution in parametric form, and then determine one particular solution.
   a.
   $$\begin{aligned} x + y + z &= 6 \\ 2x - 3y + 2z &= 12 \\ 3x - 2y + 3z &= 18 \end{aligned}$$
   b.
   $$\begin{aligned} x + y + 3z &= 4 \\ x + z &= 1 \\ 2x - y &= 2 \end{aligned}$$

9. Elise has a collection of 12 coins consisting of nickels, dimes and quarters. If the total worth of the coins is $1.80, how many are there of each? Find all possible solutions.

***SECTION 2.7 PROBLEM SET: CHAPTER REVIEW***

10. Solve the following systems. If a system has an infinite number of solutions, first express the solution in parametric form, and then find a particular solution.

    a. $$\begin{aligned} 2x+y-2z&=0\\ 2x+2y-3z&=0\\ 6x+4y-7z&=0 \end{aligned}$$

    b. $$\begin{aligned} 3x+4y-3z&=5\\ 2x+3y-z&=4\\ x+2y+z&=1 \end{aligned}$$

11. Solve the following systems. If a system has an infinite number of solutions, first express the solution in parametric form, and then provide one particular solution.

    a. $$\begin{aligned} 2x+y-2z&=0\\ 2x+2y-3z&=0\\ 6x+4y-7z&=0 \end{aligned}$$

    b. $$\begin{aligned} 3x+4y-3z&=5\\ 2x+3y-z&=4\\ x+2y+z&=1 \end{aligned}$$

12. Find the inverse of the following matrices:

    a. $$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

    b. $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

13. Solve the following systems using the matrix inverse method.

    a. $$\begin{aligned} 2x+3y+z&=12\\ x+2y+z&=9\\ x+y+z&=5 \end{aligned}$$

    b. $$\begin{aligned} x+2y-3z+w&=\\ x-z&=4\\ x-2y+z&=0\\ y-2z+w=&-11=0 \end{aligned}$$

14. Use matrix $A$ to encode the following messages. The space between the letters is represented by the number 27, and all punctuation is ignored.

$$A=\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

    a. TAKE IT AND RUN
    b. GET OUT QUICK

15. Decode the following messages that were encoded using matrix $A$ in the above problem.
    a. 44, 71, 15, 18, 27, 1, 68, 82, 27, 69, 76, 27, 19, 33, 9
    b. 37, 64, 15, 36, 54, 15, 67, 75, 20, 59, 66, 27, 39, 43, 12

16. Chris, Bob, and Matt decide to help each other study during the final exams. Chris's favorite subject is chemistry, Bob loves biology, and Matt knows his math. Each studies his own subject as well as helps the others learn their subjects. After the finals, they realize that Chris spent 40% of his time studying his own subject chemistry, 30% of his time helping Bob learn chemistry, and 30% of the time helping Matt learn chemistry. Bob spent 30% of his time studying his own subject biology, 30% of his time helping Chris learn biology, and 40% of the time helping Matt learn biology. Matt spent 20% of his time studying his own subject math, 40% of his time helping Chris learn math, and 40% of the time helping Bob learn math. If they originally agreed that each should work about 33 hours, how long did each work?
17. As in the previous problem, Chris, Bob, and Matt decide to not only help each other study during the final exams, but also tutor others to make a little money. Chris spends 30% of his time studying chemistry, 15% of his time helping Bob with chemistry, and 25% helping Matt with chemistry. Bob spends 25% of his time studying biology, 15% helping Chris with biology, and 30% helping Matt. Similarly, Matt spends 20% of his time on his own math, 20% helping Chris, and 20% helping Bob. If they spend respectively, 12, 12, and 10 hours tutoring others, how many total hours are they going to end up working?

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# CHAPTER OVERVIEW

## 3: Linear Programming - A Geometric Approach

**Learning Objectives**

In this chapter, you will learn to:

1. Solve linear programming problems that maximize the objective function.
2. Solve linear programming problems that minimize the objective function.


3.1: Maximization Applications

3.1.1: Maximization Applications (Exercises)

3.2: Minimization Applications

3.2.1: Minimization Applications (Exercises)

3.3: Chapter Review


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# 3.1: Maximization Applications

**Learning Objectives**

In this section, you will learn to:

1. Recognize the typical form of a linear programing problem
2. Formulate maximization linear programming problems
3. Graph feasibility regions for maximization linear programming problems
4. Determine optimal solutions for maximization linear programming problems.

Application problems in business, economics, and social and life sciences often ask us to make decisions on the basis of certain conditions. The conditions or constraints often take the form of inequalities. In this section, we will begin to formulate, analyze, and solve such problems, at a simple level, to understand the many components of such a problem.

A typical **linear programming** problem consists of finding an extreme value of a linear function subject to certain constraints. We are either trying to maximize or minimize the value of this linear function, such as to maximize profit or revenue, or to minimize cost. That is why these linear programming problems are classified as **maximization** or **minimization problems**, or just **optimization problems**. The function we are trying to optimize is called an **objective function**, and the conditions that must be satisfied are called **constraints**.

A typical example is to maximize profit from producing several products, subject to limitations on materials or resources needed for producing these items; the problem requires us to determine the amount of each item produced. Another type of problem involves scheduling; we need to determine how much time to devote to each of several activities in order to maximize income from (or minimize cost of) these activities, subject to limitations on time and other resources available for each activity.

In this chapter, we will work with problems that involve only two variables, and therefore, can be solved by graphing. In the next chapter, we'll learn an algorithm to find a solution numerically. That will provide us with a tool to solve problems with more than two variables. At that time, with a little more knowledge about linear programming, we'll also explore the many ways these techniques are used in business and wide variety of other fields.

We begin by solving a maximization problem.

**Example 3.1.1**

Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation.

If Nikki makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

**Solution**

We start by choosing our variables.

- Let $x$ = The number of hours per week Niki will work at Job I.
- Let $y$ = The number of hours per week Niki will work at Job II.

Now we write the objective function. Since Niki gets paid $40 an hour at Job I, and $30 an hour at Job II, her total income I is given by the following equation.

$$I = 40x + 30y$$

Our next task is to find the constraints. The second sentence in the problem states, "She never wants to work more than a total of 12 hours a week." This translates into the following constraint:

$$x + y \leq 12$$

The third sentence states, "For every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation." The translation follows.

$$2x + y \leq 16$$

The fact that $x$ and $y$ can never be negative is represented by the following two constraints:

$$x \geq 0, \text{ and } y \geq 0.$$

Well, good news! We have formulated the problem. We restate it as

$$\begin{aligned} &\textbf{Maximize} && \mathrm{I} = 40\mathrm{x} + 30\mathrm{y} \\ &\textbf{Subject to:} && \mathrm{x} + \mathrm{y} \leq 12 \\ &&& 2\mathrm{x} + \mathrm{y} \leq 16 \\ &&& \mathrm{x} \geq 0; \mathrm{y} \geq 0 \end{aligned} \tag{3.1.1}$$

In order to solve the problem, we graph the constraints and shade the region that satisfies **all** the inequality constraints.

Any appropriate method can be used to graph the lines for the constraints. However often the easiest method is to graph the line by plotting the x-intercept and y-intercept.

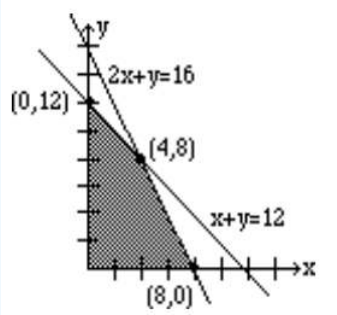
The line for a constraint will divide the plane into two region, one of which satisfies the inequality part of the constraint. A test point is used to determine which portion of the plane to shade to satisfy the inequality. Any point on the plane that is not on the line can be used as a test point.

- If the test point satisfies the inequality, then the region of the plane that satisfies the inequality is the region that contains the test point.
- If the test point does not satisfy the inequality, then the region that satisfies the inequality lies on the opposite side of the line from the test point.

In the graph below, after the lines representing the constraints were graphed using an appropriate method from Chapter 1, the point (0,0) was used as a test point to determine that

- (0,0) satisfies the constraint $x + y \leq 12$ because $0 + 0 < 12$
- (0,0) satisfies the constraint $2x + y \leq 16$ because $2(0) + 0 < 16$

Therefore, in this example, we shade the region that is below and to the left of **both** constraint lines, but also above the x axis and to the right of the y axis, in order to further satisfy the constraints $x \geq 0$ and $y \geq 0$.




The shaded region where all conditions are satisfied is called the **feasibility region** or the feasibility polygon.

The **Fundamental Theorem of Linear Programming** states that the maximum (or minimum) value of the objective function always takes place at the vertices of the feasibility region.

Therefore, we will identify all the vertices (corner points) of the feasibility region. We call these points **critical points.** They are listed as (0, 0), (0, 12), (4, 8), (8, 0). To maximize Niki's income, we will substitute these points in the objective function to see which point gives us the highest income per week. We list the results below.

| Critical Points | Income |
|---|---|
| | |

| Critical Points | Income |
|---|---|
| (0, 0) | 40(0) + 30(0) = $0 |
| (0, 12) | 40(0) + 30(12) = $360 |
| (4, 8) | 40(4) + 30(8) = $400 |
| (8, 0) | 40(8) + 30(0) = $320 |

Clearly, the point (4, 8) gives the most profit: $400.

Therefore, we conclude that Niki should work 4 hours at Job I, and 8 hours at Job II.

### Example 3.1.2

A factory manufactures two types of gadgets, regular and premium. Each gadget requires the use of two operations, assembly and finishing, and there are at most 12 hours available for each operation. A regular gadget requires 1 hour of assembly and 2 hours of finishing, while a premium gadget needs 2 hours of assembly and 1 hour of finishing. Due to other restrictions, the company can make at most 7 gadgets a day. If a profit of $20 is realized for each regular gadget and $30 for a premium gadget, how many of each should be manufactured to maximize profit?

**Solution**

We choose our variables.

- Let $x$ = The number of regular gadgets manufactured each day.
- and $y$ = The number of premium gadgets manufactured each day.

The objective function is

$$P = 20x + 30y$$

We now write the constraints. The fourth sentence states that the company can make at most 7 gadgets a day. This translates as

$$x + y \leq 7$$

Since the regular gadget requires one hour of assembly and the premium gadget requires two hours of assembly, and there are at most 12 hours available for this operation, we get

$$x + 2y \leq 12$$

Similarly, the regular gadget requires two hours of finishing and the premium gadget one hour. Again, there are at most 12 hours available for finishing. This gives us the following constraint.

$$2x + y \leq 12$$

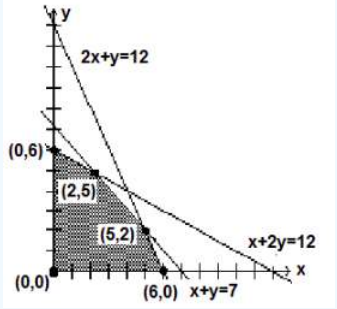
The fact that $x$ and $y$ can never be negative is represented by the following two constraints:

$$x \geq 0, \text{ and } y \geq 0.$$

We have formulated the problem as follows:

$$\begin{aligned} &\textbf{Maximize} && \mathrm{P = 20x + 30y} \\ &\textbf{Subject to:} && \mathrm{x + y \leq 7} \\ & && \mathrm{x + 2y \leq 12} \\ & && \mathrm{2x + y \leq 12} \\ & && \mathrm{x \geq 0; y \geq 0} \end{aligned}$$

In order to solve the problem, we next graph the constraints and feasibility region.




Again, we have shaded the feasibility region, where all constraints are satisfied.

Since the extreme value of the objective function always takes place at the vertices of the feasibility region, we identify all the critical points. They are listed as (0, 0), (0, 6), (2, 5), (5, 2), and (6, 0). To maximize profit, we will substitute these points in the objective function to see which point gives us the maximum profit each day. The results are listed below.

| Critical Point | Income |
|---|---|
| (0, 0) | 20(0) + 30(0) = $0 |
| (0, 6) | 20(0) + 30(6) = $180 |
| (2, 5) | 20(2) + 30(5) = $190 |
| (5, 2) | 20(5) + 30(2) = $160 |
| (6,0) | 20(6) + 30(0) = $120 |

The point (2, 5) gives the most profit, and that profit is $190.
Therefore, we conclude that we should manufacture 2 regular gadgets and 5 premium gadgets daily to obtain the maximum profit of $190.

So far we have focused on "**standard maximization problems**" in which

1. The objective function is to be maximized
2. All constraints are of the form $ax + by \leq c$
3. All variables are constrained to by non-negative ($x \geq 0$, $y \geq 0$)

We will next consider an example where that is not the case. Our next problem is said to have "**mixed constraints**", since some of the inequality constraints are of the form $ax + by \leq c$ and some are of the form $ax + by \geq c$. The non-negativity constraints are still an important requirement in any linear program.

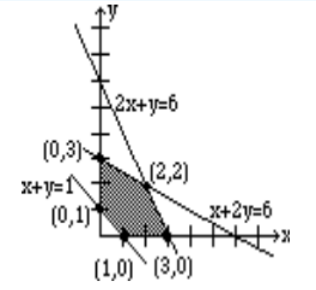
**Example 3.1.3**

Solve the following maximization problem graphically.

$$\begin{aligned} \textbf{Maximize} \quad & P = 10x + 15y \\ \textbf{Subject to:} \quad & x + y \geq 1 \\ & x + 2y \leq 6 \\ & 2x + y \leq 6 \\ & x \geq 0; y \geq 0 \end{aligned}$$

**Solution**

The graph is shown below.




The five critical points are listed in the above figure. The reader should observe that the first constraint $x+y \geq 1$ requires that the feasibility region must be bounded below by the line $x+y=1$; the test point (0,0) does not satisfy $x+y \geq 1$, so we shade the region on the opposite side of the line from test point (0,0).

| Critical point | Income |
|---|---|
| (1, 0) | 10(1) + 15(0) = $10 |
| (3, 0) | 10(3) + 15(0) = $30 |
| (2, 2) | 10(2) + 15(2) = $50 |
| (0, 3) | 10(0) + 15(3) = $45 |
| (0,1) | 10(0) + 15(1) = $15 |

Clearly, the point (2, 2) maximizes the objective function to a maximum value of 50.

It is important to observe that that if the point (0,0) lies on the line for a constraint, then (0,0) could not be used as a test point. We would need to select any other point we want that does not lie on the line to use as a test point in that situation.

Finally, we address an important question. Is it possible to determine the point that gives the maximum value without calculating the value at each critical point?

The answer is yes.

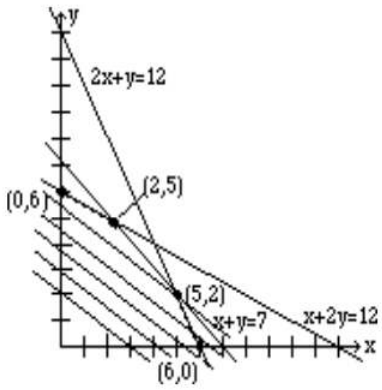
For example 3.1.2, we substituted the points (0, 0), (0, 6), (2, 5), (5, 2), and (6, 0), in the objective function $P=20x+30y$, and we got the values $0, $180, $190, $160, $120, respectively.

Sometimes that is not the most efficient way of finding the optimum solution. Instead we could find the optimal value by also graphing the objective function.

To determine the largest $P$, we graph $P=20x+30y$ for any value $P$ of our choice. Let us say, we choose $P=60$. We graph $20x+30y=60$.

Now we move the line parallel to itself, that is, keeping the same slope at all times. Since we are moving the line parallel to itself, the slope is kept the same, and the only thing that is changing is the P. As we move away from the origin, the value of P increases. The largest possible value of P is realized when the line touches the last corner point of the feasibility region.

The figure below shows the movements of the line, and the optimum solution is achieved at the point (2, 5). In maximization problems, as the line is being moved away from the origin, this optimum point is the farthest critical point.




We summarize:

**The Maximization Linear Programming Problems**

1. Write the objective function.
2. Write the constraints.
    1. For the standard maximization linear programming problems, constraints are of the form: $ax + by \leq c$
    2. Since the variables are non-negative, we include the constraints: $x \geq 0$; $y \geq 0$.
3. Graph the constraints.
4. Shade the feasibility region.
5. Find the corner points.
6. Determine the corner point that gives the maximum value.
    a. This is done by finding the value of the objective function at each corner point.
    b. This can also be done by moving the line associated with the objective function.

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# 3.1.1: Maximization Applications (Exercises)

For the following maximization problems, choose your variables, write the objective function and the constraints, graph the constraints, shade the feasibility region, label all critical points, and determine the solution that optimizes the objective function.

1) A farmer has 100 acres of land on which she plans to grow wheat and corn. Each acre of wheat requires 4 hours of labor and $20 of capital, and each acre of corn requires 16 hours of labor and $40 of capital. The farmer has at most 800 hours of labor and $2400 of capital available. If the profit from an acre of wheat is $80 and from an acre of corn is $100, how many acres of each crop should she plant to maximize her profit?

2) Mr. Tran has $24,000 to invest, some in bonds and the rest in stocks. He has decided that the money invested in bonds must be at least twice as much as that in stocks. But the money invested in bonds must not be greater than $18,000. If the bonds earn 6%, and the stocks earn 8%, how much money should he invest in each to maximize profit?

3) A factory manufactures chairs and tables, each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 40 hours; the second at most 42 hours; and the third at most 25 hours. A chair requires 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; a table needs 2 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is $20 per unit for a chair and $30 for a table, how many units of each should be manufactured to maximize revenue?

*4)* The Silly Nut Company makes two mixtures of nuts: Mixture A and Mixture B. A pound of Mixture A contains 12 oz of peanuts, 3 oz of almonds and 1 oz of cashews and sells for $4. A pound of Mixture B contains 12 oz of peanuts, 2 oz of almonds and 2 oz of cashews and sells for $5. The company has 1080 lb. of peanuts, 240 lb. of almonds, 160 lb. of cashews. How many pounds of each of mixtures A and B should the company make to maximize profit?
*(Hint: Use consistent units. Work the entire problem in pounds by converting all values given in ounces into fractions of pounds).*

5)

$$\begin{aligned} \text{Maximize:} \quad & Z = 4x + 10y \\ \text{Subject to:} \quad & x + y \leq 5 \\ & 2x + y \leq 8 \\ & x + 2y \leq 8 \\ & x \geq 0, y \geq 0 \end{aligned}$$

6) This maximization linear programming problem is not in “standard” form. It has mixed constraints, some involving $\leq$ inequalities and some involving $\geq$ inequalities. However with careful graphing, we can solve this using the techniques we have learned in this section.

$$\begin{aligned} \text{Maximize:} \quad & Z = 5x + 7y \\ \text{Subject to:} \quad & x + y \leq 30 \\ & 2x + y \leq 50 \\ & 4x + 3y \geq 60 \\ & 2x \geq y \\ & x \geq 0, y \geq 0 \end{aligned}$$

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# 3.2: Minimization Applications

**Learning Objectives**

In this section, you will learn to:

1. Formulate minimization linear programming problems
2. Graph feasibility regions for maximization linear programming problems
3. Determine optimal solutions for maximization linear programming problems.

Minimization linear programming problems are solved in much the same way as the maximization problems.

For the **standard minimization linear program**, the constraints are of the form $ax + by \geq c$, as opposed to the form $ax + by \leq c$ for the standard maximization problem. As a result, the feasible solution extends indefinitely to the upper right of the first quadrant, and is unbounded. But that is not a concern, since in order to minimize the objective function, the line associated with the objective function is moved towards the origin, and the critical point that minimizes the function is closest to the origin.

However, one should be aware that in the case of an unbounded feasibility region, the possibility of no optimal solution exists.

**Example 3.2.1**

At a university, Professor Symons wishes to employ two people, John and Mary, to grade papers for his classes. John is a graduate student and can grade 20 papers per hour; John earns $15 per hour for grading papers. Mary is an post-doctoral associate and can grade 30 papers per hour; Mary earns $25 per hour for grading papers. Each must be employed at least one hour a week to justify their employment.
If Prof. Symons has at least 110 papers to be graded each week, how many hours per week should he employ each person to minimize the cost?

**Solution**

We choose the variables as follows:

Let $x$ = The number of hours per week John is employed.

and $y$ = The number of hours per week Mary is employed.

The objective function is

$$C = 15x + 25y$$

The fact that each must work at least one hour each week results in the following two constraints:

$$x \geq 1$$
$$y \geq 1$$

Since John can grade 20 papers per hour and Mary 30 papers per hour, and there are at least 110 papers to be graded per week, we get

$$20x + 30y \geq 110$$

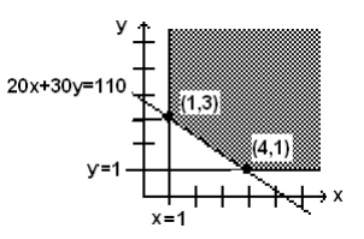
The fact that $x$ and $y$ are non-negative, we get

$$x \geq 0, \text{and } y \geq 0.$$

The problem has been formulated as follows.

$$\begin{aligned} &\textbf{Minimize} && \mathrm{C} = 15\mathrm{x} + 25\mathrm{y} \\ &\textbf{Subject to:} && \mathrm{x} \geq 1 \\ &&& \mathrm{y} \geq 1 \\ &&& 20\mathrm{x} + 30\mathrm{y} \geq 110 \\ &&& \mathrm{x} \geq 0; \mathrm{y} \geq 0 \end{aligned}$$

To solve the problem, we graph the constraints as follows:




Again, we have shaded the feasibility region, where all constraints are satisfied.

If we used test point (0,0) that does not lie on any of the constraints, we observe that (0, 0) **does not** satisfy any of the constraints $x \geq 1$, $y \geq 1$, $20x + 30y \geq 110$. Thus all the shading for the feasibility region lies on the opposite side of the constraint lines from the point (0,0).

Alternatively we could use test point (4,6), which also does not lie on any of the constraint lines. We'd find that (4,6) **does** satisfy all of the inequality constraints. Consequently all the shading for the feasibility region lies on the same side of the constraint lines as the point (4,6).

Since the extreme value of the objective function always takes place at the vertices of the feasibility region, we identify the two critical points, (1, 3) and (4, 1). To minimize cost, we will substitute these points in the objective function to see which point gives us the minimum cost each week. The results are listed below.

| Critical points | Income |
|---|---|
| (1, 3) | 15(1) + 25(3) = $90 |
| (4, 1) | 15(4) + 25(1) = $85 |

The point (4, 1) gives the least cost, and that cost is $85. Therefore, we conclude that in order to minimize grading costs, Professor Symons should employ John 4 hours a week, and Mary 1 hour a week at a cost of $85 per week.

## Example 3.2.2

Professor Hamer is on a low cholesterol diet. During lunch at the college cafeteria, he always chooses between two meals, Pasta or Tofu. The table below lists the amount of protein, carbohydrates, and vitamins each meal provides along with the amount of cholesterol he is trying to minimize. Mr. Hamer needs at least 200 grams of protein, 960 grams of carbohydrates, and 40 grams of vitamins for lunch each month. Over this time period, how many days should he have the Pasta meal, and how many days the Tofu meal so that he gets the adequate amount of protein, carbohydrates, and vitamins and at the same time minimizes his cholesterol intake?

| | **PASTA** | **TOFU** |
|---|---|---|
| PROTEIN | 8g | 16g |
| CARBOHYDRATES | 60g | 40g |
| VITAMIN C | 2g | 2g |
| CHOLESTEROL | 60mg | 50mg |

**Solution**

We choose the variables as follows.

Let $x$ = The number of days Mr. Hamer eats Pasta.

and $y$ = The number of days Mr. Hamer eats Tofu.

Since he is trying to minimize his cholesterol intake, our objective function represents the total amount of cholesterol C provided by both meals.

$$C = 60x + 50y$$

The constraint associated with the total amount of protein provided by both meals is

$$8x + 16y \geq 200$$

Similarly, the two constraints associated with the total amount of carbohydrates and vitamins are obtained, and they are

$$60x + 40y \geq 960$$
$$2x + 2y \geq 40$$

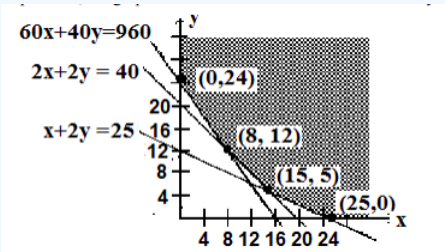
The constraints that state that x and y are non-negative are

$$x \geq 0, \text{ and } y \geq 0.$$

We summarize all information as follows:

$$\begin{array}{ll} \textbf{Minimize} & \mathrm{C} = 60\mathrm{x} + 50\mathrm{y} \\ \textbf{Subject to:} & 8\mathrm{x} + 16\mathrm{y} \geq 200 \\ & 60\mathrm{x} + 40\mathrm{y} \geq 960 \\ & 2\mathrm{x} + 2\mathrm{y} \geq 40 \\ & \mathrm{x} \geq 0; \mathrm{y} \geq 0 \end{array}$$

To solve the problem, we graph the constraints and shade the feasibility region.




We have shaded the unbounded feasibility region, where all constraints are satisfied.

To minimize the objective function, we find the vertices of the feasibility region. These vertices are (0, 24), (8, 12), (15, 5) and (25, 0). To minimize cholesterol, we will substitute these points in the objective function to see which point gives us the smallest value. The results are listed below.

| Critical points | Income |
|---|---|
| (0, 24) | 60(0) + 50(24) = 1200 |
| (8, 12) | 60(8) + 50(12) = 1080 |
| (15, 5) | 60(15) + 50(5) = 1150 |
| (25, 0) | 60(25) + 50(0) = 1500 |

The point (8, 12) gives the least cholesterol, which is 1080 mg. This states that for every 20 meals, Professor Hamer should eat Pasta 8 days, and Tofu 12 days.

We must be aware that in some cases, a linear program may not have an optimal solution.

- A linear program can fail to have an optimal solution is if there is not a feasibility region. If the inequality constraints are not compatible, there may not be a region in the graph that satisfies **all** the constraints. If the linear program does not have a feasible solution satisfying all constraints, then it can not have an optimal solution.
- A linear program can fail to have an optimal solution if the feasibility region is unbounded.

- The two minimization linear programs we examined had unbounded feasibility regions. The feasibility region was bounded by constraints on some sides but was not entirely enclosed by the constraints. Both of the minimization problems had optimal solutions.
- However, if we were to consider a maximization problem with a similar unbounded feasibility region, the linear program would have no optimal solution. No matter what values of x and y were selected, we could always find other values of $x$ and $y$ that would produce a higher value for the objective function. In other words, if the value of the objective function can be increased without bound in a linear program with an unbounded feasible region, there is no optimal maximum solution.

Although the method of solving minimization problems is similar to that of the maximization problems, we still feel that we should summarize the steps involved.

**Minimization Linear Programming Problems**

1. Write the objective function.
2. Write the constraints.
   a. For standard minimization linear programming problems, constraints are of the form: $ax + by \geq c$
   b. Since the variables are non-negative, include the constraints: $x \geq 0$; $y \geq 0$.
3. Graph the constraints.
4. Shade the feasibility region.
5. Find the corner points.
6. Determine the corner point that gives the minimum value.
   a. This can be done by finding the value of the objective function at each corner point.
   b. This can also be done by moving the line associated with the objective function.
   c. There is the possibility that the problem has no solution

---

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# 3.2.1: Minimization Applications (Exercises)

For each of the following minimization problems, choose your variables, write the objective function and the constraints, graph the constraints, shade the feasibility region, label all critical points, and determine the solution that optimizes the objective function.

1) A diet is to contain at least 2400 units of vitamins, 1800 units of minerals, and 1200 calories. Two foods, Food A and Food B are to be purchased. Each unit of Food A provides 50 units of vitamins, 30 units of minerals, and 10 calories. Each unit of Food B provides 20 units of vitamins, 20 units of minerals, and 40 calories. Food A costs $2 per unit and Food B cost $1 per unit. How many units of each food should be purchased to keep costs at a minimum?

2) A computer store sells two types of computers, laptops and desktops. The supplier demands that at least 150 computers be sold a month. Experience shows that most consumers prefer laptops, but some business customers require desktops. The result is that the number of laptops sold is at least twice of the number of desktops. The store pays its sales staff a $60 commission for each laptop, and a $40 commission for each desktop. Let $x$ = the number of laptops and $y$ = the number of desktop computers. How many of each type must be sold to minimize commission to its sales people?
What is the minimum commission?

3) An oil company has two refineries. Each day, Refinery A produces 200 barrels of high-grade oil, 300 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs $12,000 to operate. Each day, Refinery B produces 100 barrels of high-grade oil, 100 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs $10,000 to operate. The company must produce at least 800 barrels of high-grade oil, 900 barrels of medium-grade oil, and 1,000 barrels of low-grade oil.
How many days should each refinery be operated to meet the goals at a minimum cost?

4) A print shop at a community college in Cupertino, California, employs two different contractors to maintain its copying machines. The print shop needs to have 12 IBM, 18 Xerox, and 20 Canon copying machines serviced. Contractor A can repair 2 IBM, 1 Xerox, and 2 Canon machines at a cost of $800 per month, while Contractor B can repair 1 IBM, 3 Xerox, and 2 Canon machines at a cost of $1000 per month. How many months should each of the two contractors be employed to minimize the cost?

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# 3.3: Chapter Review

Solve the following linear programming problems by the graphical method.

1) Mr. Shoemacher has $20,000 to invest in two types of mutual funds: a High-Yield Fund and an Equity Fund. The High-Yield fund has an annual yield of 12%, while the Equity fund earns 8%. He would like to invest at least $3000 in the High-Yield fund and at least $4000 in the Equity fund. How much should he invest in each to maximize his annual yield, and what is the maximum yield?

2) Dr. Lum teaches part-time at two community colleges, Hilltop College and Serra College. Dr. Lum can teach up to 5 classes per semester. For every class he teaches at Hilltop College, he needs to spend 3 hours per week preparing lessons and grading papers. For each class at Serra College, he must do 4 hours of work per week. He has determined that he cannot spend more than 18 hours per week preparing lessons and grading papers. If he earns $6,000 per class at Hilltop College and $7,500 per class at Serra College, how many classes should he teach at each college to maximize his income, and what will be his income?

3) Mr. Shamir employs two part-time typists, Inna and Jim, for his typing needs. Inna charges $15 an hour and can type 6 pages an hour, while Jim charges $18 an hour and can type 8 pages per hour. Each typist must be employed at least 8 hours per week to keep them on the payroll. If Mr. Shamir has at least 208 pages to be typed, how many hours per week should he employ each typist to minimize his typing costs, and what will be the total cost?

4) Mr. Boutros wants to invest up to $20,000 in two stocks, Cal Computers and Texas Tools. The Cal Computers stock is expected to yield a 16% annual return, while the Texas Tools stock promises a 12% yield. Mr. Boutros would like to earn at least $2,880 this year. According to Value Line Magazine's safety index (1 highest to 5 lowest), Cal Computers has a safety number of 3 and Texas Tools has a safety number of 2. How much money should he invest in each to minimize the safety number? Note: A lower safety number means less risk.

5) A store sells two types of copy machines: compact (low capacity) and standard (which takes more space). The store can sell up to 90 copiers a month. A maximum of 1080 cubic feet of storage space is available. A compact copier requires 6 cu. ft. of storage space, and a standard copier requires 18 cu. ft.. The compact and standard copy machines take, respectively, 1 and 1.5 sales hours of labor.
A maximum of 99 hours of labor is available. The profit from each of these copiers is $60 and $80, respectively, how many of each type should be sold to maximize profit, and what is the maximum profit?

6) A company manufactures two types of cell phones, a Basic model and a Pro model. The Basic model generates a profit of $100 per phone and the Pro model has a profit of $150 per phone. On the assembly line the Basic phone requires 7 hours, while the Pro model takes 11 hours. The Basic phone requires one hour and the Pro phone needs 3 hours for finishing, which includes loading software. Both phones require one hour for testing. On a particular production run the company has available 1,540 work hours on the assembly line, 360 work hours for finishing, and 200 work hours in the testing department. How many cell phones of each type should be produced to maximize profit, and what is that maximum profit?

7) John wishes to choose a combination of two types of cereals for breakfast - Cereal A and Cereal B. A small box (one serving) of Cereal A costs $0.50 and contains 10 units of vitamins, 5 units of minerals, and 15 calories. A small box(one serving) of Cereal B costs $0.40 and contains 5 units of vitamins, 10 units of minerals, and 15 calories. John wants to buy enough boxes to have at least 500 units of vitamins, 600 units of minerals, and 1200 calories. How many boxes of each food should he buy to minimize his cost, and what is the minimum cost?

8) Jessica needs at least 60 units of vitamin A, 40 units of vitamin B, and 140 units of vitamin C each week. She can choose between Costless brand or Savemore brand tablets. A Costless tablet costs 5 cents and contains 3 units of vitamin A, 1 unit of vitamin B, and 2 units of vitamin C. A Savemore tablet costs 7 cents and contains 1 unit of A, 1 of B, and 5 of C. How many tablets of each kind should she buy to minimize cost, and what is the minimum cost?

9) A small company manufactures two products: A and B. Each product requires three operations: Assembly, Finishing and Testing. Product A requires 1 hour of Assembly, 3 hours of Finishing, and 1 hour of Testing. Product B requires 3 hours of Assembly, 1 hour of Finishing, and 1 hour of Testing. The total work-hours available per week in the Assembly division is 60, in Finishing is 60, and in Testing is 24. Each item of product A has a profit of $50, and each item of Product B has a profit of $75. How many of each should be made to maximize profit? What is the maximum profit?

10) A factory manufactures two products, A and B. Each product requires the use of three machines, Machine I, Machine II, and Machine III. The time requirements and total hours available on each machine are listed below.

| | Machine I | Machine II | Machine III |
|---|---|---|---|
| Product A | 1 | 2 | 4 |
| Product B | 2 | 2 | 2 |
| Total hours | 70 | 90 | 160 |

If product A generates a profit of $60 per unit and product B a profit of $50 per unit, how many units of each product should be manufactured to maximize profit, and what is the maximum profit?

11) A company produces three types of shoes, formal, casual, and athletic, at its two factories, Factory I and Factory II. The company must produce at least 6000 pairs of formal shoes, 8000 pairs of casual shoes, and 9000 pairs of athletic shoes. Daily production of each factory for each type of shoe is:

| | Factory I | Factory II |
|---|---|---|
| Formal | 100 | 100 |
| Casual | 100 | 200 |
| Athletic | 300 | 100 |

Operating Factory I costs $1500 per day and it costs $2000 per day to operate Factory II. How many days should each factory operate to complete the order at a minimum cost, and what is the minimum cost?

12) A professor gives two types of quizzes, objective and recall. He plans to give at least 15 quizzes this quarter. The student preparation time for an objective quiz is 15 minutes and for a recall quiz 30 minutes. The professor would like a student to spend at least 5 hours (300 minutes) preparing for these quizzes above and beyond the normal study time. The average score on an objective quiz is 7, and on a recall type 5, and the professor would like the students to score at least 85 points on all quizzes. It takes the professor one minute to grade an objective quiz, and 1.5 minutes to grade a recall type quiz. How many of each type should he give in order to minimize his grading time?

13) A company makes two mixtures of nuts: Mixture A and Mixture B. Mixture A contains 30% peanuts, 30% almonds and 40% cashews and sells for $5 per pound. Mixture B contains 30% peanuts, 60% almonds and 10% cashews and sells for $3 a pound. The company has 540 pounds of peanuts, 900 pounds of almonds, 480 pounds of cashews. How many pounds of each of mixtures A and B should the company make to maximize profit, and what is the maximum profit?

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# CHAPTER OVERVIEW

## 4: Mathematics of Finance

**Learning Objectives**

In this chapter, you will learn to:

1. Solve financial problems that involve simple interest.
2. Solve problems involving compound interest.
3. Find the future value of an annuity, and the amount of payments to a sinking fund.
4. Find the future value of an annuity, and an installment payment on a loan.


4.1: Simple Interest and Discount

4.1.1: Simple Interest and Discount (Exercises)

4.2: Compound Interest

4.2.1: Compound Interest (Exercises)

4.3: Annuities and Sinking Funds

4.3.1: Annuities and Sinking Funds (Exercises)

4.4: Present Value of an Annuity and Installment Payment

4.4.1: Present Value of an Annuity and Installment Payment (Exercises)

4.5: Miscellaneous Application Problems

4.5.1: Miscellaneous Application Problems (Exercises)

4.6: Classification of Finance Problems

4.6.1: Classification of Finance Problems (Exercises)

4.7: Chapter Review



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# 4.1: Simple Interest and Discount

**Learning Objectives**

In this section, you will learn to:

1. Find simple interest.
2. Find present value.
3. Find discounts and proceeds.

## Simple Interest

It costs to borrow money. The rent one pays for the use of money is called the **interest**. The amount of money that is being borrowed or loaned is called the **principal** or **present value**. Simple interest is paid only on the original amount borrowed. When the money is loaned out, the person who borrows the money generally pays a fixed rate of interest on the principal for the time period he keeps the money. Although the interest rate is often specified for a year, it may be specified for a week, a month, or a quarter, etc. The credit card companies often list their charges as monthly rates, sometimes it is as high as 1.5% a month.

**Definition: Simple Interest**

If an amount $P$ is borrowed for a time $t$ at an interest rate of $r$ per time period, then the simple interest is given by

$$I = P \cdot r \cdot t$$

**Definition: Accumulated Value**

The total amount $A$, also called the **accumulated value** or the future value, is given by

$$\begin{aligned} A &= P + I \\ &= P + Prt \end{aligned}$$

or

$$A = P(1 + rt) \tag{4.1.1}$$

where interest rate $r$ is expressed in decimals.

**Example 4.1.1**

Ursula borrows \$600 for 5 months at a simple interest rate of 15% per year. Find the interest, and the total amount she is obligated to pay?

**Solution**

The interest is computed by multiplying the principal with the interest rate and the time.

$$\begin{aligned} \mathrm{I} &= \mathrm{Prt} \\ &= \$600(0.15)\frac{5}{12} \\ &= \$37.50 \end{aligned}$$

The total amount is

$$\begin{aligned} \mathrm{A} &= \mathrm{P} + \mathrm{I} \\ &= \$600 + \$37.50 \\ &= \$637.50 \end{aligned}$$

Incidentally, the total amount can be computed directly via Equation 4.1.1 as

$$\begin{aligned} A &= P(1+rt) \\ &= \$600[1+(0.15)(5/12)] \\ &= \$600(1+0.0625) \\ &= \$637.50 \end{aligned}$$

**Example 4.1.2**

Jose deposited $2500 in an account that pays 6% simple interest. How much money will he have at the end of 3 years?

**Solution**

The total amount or the future value is given by Equation 4.1.1.

$$\begin{aligned} A &= P(1+rt) \\ &= \$2500[1+(.06)(3)] \\ \mathrm{A} &= \$2950 \end{aligned}$$

**Example 4.1.3**

Darnel owes a total of $3060 which includes 12% interest for the three years he borrowed the money. How much did he originally borrow?

**Solution**

This time we are asked to compute the principal $P$ via Equation 4.1.1.

$$\begin{aligned} \$3060 &= \mathrm{P}[1+(0.12)(3)] \\ \$3060 &= \mathrm{P}(1.36) \\ \frac{\$3060}{1.36} &= \mathrm{P} \\ \$2250 &= \mathrm{P} \quad \text{Darnel originally borrowed } \$2250. \end{aligned}$$

**Example 4.1.4**

A Visa credit card company charges a 1.5% finance charge each month on the unpaid balance. If Martha owed $2350 and has not paid her bill for three months, how much does she owe now?

**Solution**

Before we attempt the problem, the reader should note that in this problem the rate of finance charge is given per month and not per year.

The total amount Martha owes is the previous unpaid balance plus the finance charge.

$$A = \$2350 + \$2350(.015)(3) = \$2350 + \$105.75 = \$2455.75$$

Alternatively, again, we can compute the amount directly by using formula $A = P(1+rt)$

$$A = \$2350[1+(.015)(3)] = \$2350(1.045) = \$2455.75$$

## Discount and Proceeds

Banks often deduct the simple interest from the loan amount at the time that the loan is made. When this happens, we say the loan has been **discounted.** The interest that is deducted is called the **discount**, and the actual amount that is given to the borrower is called the **proceeds**. The amount the borrower is obligated to repay is called the **maturity value.**

### Discount and Proceeds

If an amount $M$ is borrowed for a time $t$ at a discount rate of $r$ per year, then the discount $D$ is

$$D = M \cdot r \cdot t$$

The proceeds $P$, the actual amount the borrower gets, is given by

$$\begin{aligned} P &= M - D \\ &= M - Mrt \end{aligned}$$

or

$$P = M(1 - rt)$$

where interest rate $r$ is expressed in decimals.

### Example 4.1.5

Francisco borrows \$1200 for 10 months at a simple interest rate of 15% per year. Determine the discount and the proceeds.

**Solution**

The discount $D$ is the interest on the loan that the bank deducts from the loan amount.

$$\begin{aligned} \mathrm{D} &= \mathrm{Mrt} \\ \mathrm{D} &= \$1200(0.15)\left(\tfrac{10}{12}\right) = \$150 \end{aligned}$$

Therefore, the bank deducts \$150 from the maturity value of \$1200, and gives Francisco \$1050. Francisco is obligated to repay the bank \$1200.

In this case, the discount $D$ = \$150, and the proceeds

$$P = \$1200 - \$150 = \$1050.$$

### Example 4.1.6

If Francisco wants to receive \$1200 for 10 months at a simple interest rate of 15% per year, what amount of loan should he apply for?

**Solution**

In this problem, we are given the proceeds $P$ and are being asked to find the maturity value $M$.

We have $P$ = \$1200, $r = 0.15$, $t$ = 10/12 . We need to find $M$.

We know $P = M - D$

but also $D = Mrt$

therefore

$$\begin{aligned} \mathrm{P} &= \mathrm{M} - \mathrm{Mrt} \\ &= \mathrm{M}(1 - \mathrm{rt}) \\ \$1200 &= \mathrm{M}\left[1 - (0.15)\left(\frac{10}{12}\right)\right] \end{aligned}$$

We need to solve for $M$.

$$\$1200 = \mathrm{M}(1-0.125)$$
$$\$1200 = \mathrm{M}(0.875)$$
$$\frac{\$1200}{0.875} = \mathrm{M}$$
$$\$1371.43 = \mathrm{M}$$

Therefore, Francisco should ask for a loan for $1371.43.

The bank will discount $171.43 and Francisco will receive $1200.

## Summary

Below is a summary of the formulas we developed for calculations involving simple interest:

### Simple interest

If an amount $P$ is borrowed for a time $t$ at an interest rate of $r$ per time period, then the simple interest is given by

$$I = P \cdot r \cdot t$$

The total amount $A$, also called the accumulated value or the future value, is given by

$$A = P + I = P + Prt$$

or

$$A = P(1 + rt)$$

where interest rate $r$ is expressed in decimals.

### Discount and Proceeds

If an amount $M$ is borrowed for a time $t$ at a discount rate of $r$ per year, then the discount $D$ is

$$D = M \cdot r \cdot t$$

The proceeds $P$, the actual amount the borrower gets, is given by

$$P = M - D$$
$$P = M - Mrt$$

or

$$P = M(1 - rt)$$

where interest rate $r$ is expressed in decimals.

At the end of the loan's term, the borrower repays the entire maturity amount $M$.

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# 4.1.1: Simple Interest and Discount (Exercises)

## SECTION 6.1 PROBLEM SET: SIMPLE INTEREST AND DISCOUNT

Do the following simple interest problems.

| | |
|---|---|
| 1) If an amount of $2,000 is borrowed at a simple interest rate of 10% for 3 years, how much is the interest? | 2) You borrow $4,500 for six months at a simple interest rate of 8%. How much is the interest? |
| 3) John borrows $2400 for 3 years at 9% simple interest. How much will he owe at the end of 3 years? | 4) Jessica takes a loan of $800 for 4 months at 12% simple interest. How much does she owe at the end of the 4-month period? |
| 5) If an amount of $2,160, which includes a 10% simple interest for 2 years, is paid back, how much was borrowed 2 years earlier? | 6) Jamie just paid off a loan of $2,544, the principal and simple interest. If he took out the loan six months ago at 12% simple interest, what was the amount borrowed? |
| 7) Shanti charged $800 on her charge card and did not make a payment for six months. If there is a monthly charge of 1.5%, how much does she owe? | 8) A credit card company charges 18% interest on the unpaid balance. If you owed $2000 three months ago and have been delinquent since, how much do you owe? |

***SECTION 6.1 PROBLEM SET: SIMPLE INTEREST AND DISCOUNT***

Do the following simple interest problems.

| | |
|---|---|
| 9) An amount of $2000 is borrowed for 3 years. At the end of the three years, $2660 is paid back. What was the simple interest rate? | 10) Nancy borrowed $1,800 and paid back $1,920, four months later. What was the simple interest rate? |
| 11) Jose agrees to pay $2,000 in one year at an interest rate of 12%. The bank subtracts the discount of 12% of $2,000, and gives the rest to Jose. Find the amount of the discount and the proceeds to Jose. | 12) Tasha signs a note for a discounted loan agreeing to pay $1200 in 8 months at an 18% discount rate. Determine the amount of the discount and the proceeds to her. |
| 13) An amount of $8,000 is borrowed at a discount rate of 12%, find the proceeds if the length of the loan is 7 months. | 14) An amount of $4,000 is borrowed at a discount rate of 10%, find the proceeds if the length of the loan is 180 days. |
| 15) Derek needs $2400 new equipment for his shop. He can borrow this money at a discount rate of 14% for a year. Find the amount of the loan he should ask for so that his proceeds are $2400. | 16) Mary owes Jim $750, and wants to repay him. Mary decides to borrow the amount from her bank at a discount rate of 16%. If she borrows the money for 10 months, find the amount of the loan she should ask for so that her proceeds are $750? |

## SECTION 6.2 PROBLEM SET: COMPOUND INTEREST

Do the following compound interest problems involving a lump-sum amount.

| | |
|---|---|
| 1) What will the final amount be in 4 years if $8,000 is invested at 9.2% compounded monthly.? | 2) How much should be invested at 10.3% for it to amount to $10,000 in 6 years? |
| 3) Lydia's aunt Rose left her $5,000. Lydia spent $1,000 on her wardrobe and deposited the rest in an account that pays 6.9% compounded daily. How much money will she have in 5 years? | 4) Thuy needs $1,850 in eight months for her college tuition. How much money should she deposit lump sum in an account paying 8.2% compounded monthly to achieve that goal? |
| 5) Bank A pays 5% compounded daily, while Bank B pays 5.12% compounded monthly. Which bank pays more? Explain. | 6) EZ Photo Company needs five copying machines in 2 1/2 years for a total cost of $15,000. How much money should be deposited now to pay for these machines, if the interest rate is 8% compounded semiannually? |

| | |
|---|---|
| 7) Jon's grandfather was planning to give him $12,000 in 10 years. Jon has convinced his grandfather to pay him $6,000 now, instead. If Jon invests this $6,000 at 7.5% compounded continuously, how much money will he have in 10 years? | 8) What will be the price of a $20,000 car in 5 years if the inflation rate is 6%? |

***SECTION 6.2 PROBLEM SET: COMPOUND INTEREST***

Do the following compound interest problems.

| | |
|---|---|
| 9) At an interest rate of 8% compounded continuously, how many years will it take to double your money? | 10) If an investment earns 10% compounded continuously, in how many years will it triple? . |
| 11) The City Library ordered a new computer system costing $158,000; it will be delivered in 6 months, and the full amount will be due 30 days after delivery. How much must be deposited today into an account paying 7.5% compounded monthly to have $158,000 in 7 months? | 12) Mr. and Mrs. Tran are expecting a baby girl in a few days. They want to put away money for her college education now. How much money should they deposit in an account paying 10.2% so they will have $100,000 in 18 years to pay for their daughter's educational expenses? |
| 13) Find the effective interest rate for an account paying 7.2% compounded quarterly. | 14) If a bank pays 5.75% compounded monthly, what is the effective interest rate? |
| 15) The population of the African nation of Cameroon was 12 million people in the year 2015; it has been growing at the rate of 2.5% per year. If the population continues to grow that rate,what will the population be in 2030? (http://databank.worldbank.org/data on 4/26/2016) | 16) According to the Law of 70, if an amount grows at an annual rate of 1%, then it doubles every seventy years. Suppose a bank pays 5% interest, how long will it take for you to double your money? How about at 15%? |

## SECTION 6.3 PROBLEM SET:ANNUITIES AND SINKING FUNDS

Each of the following problems involve an annuity - a sequence of payments.

| | |
|---|---|
| 1) Find the future value of an annuity of $200 per month for 5 years at 6% compounded monthly. | 2) How much money should be deposited at the end of each month in an account paying 7.5% for it to amount to $10,000 in 5 years? |
| 3) At the end of each month Rita deposits $300 in an account that pays 5%. What will the final amount be in 4 years? | 4) Mr. Chang wants to retire in 10 years and can save $650 every three months. If the interest rate is 7.8%, how much will he have (a) at the end of 5 years? (b) at the end of 10 years? |
| 5) A firm needs to replace most of its machinery in five years at a cost of $500,000. The company wishes to create a sinking fund to have this money available in five years. How much should the quarterly deposits be if the fund earns 8%? | 6) Mrs. Brown needs $5,000 in three years. If the interest rate is 9%, how much should she save at the end of each month to have that amount in three years? |

***SECTION 6.3 PROBLEM SET: ANNUITIES AND SINKING FUNDS***

Each of the following problems involve an annuity - a sequence of payments.

| | |
|---|---|
| 7) A company has a $120,000 note due in 4 years. How much should be deposited at the end of each quarter in a sinking fund to payoff the note in four years if the interest rate is 8%? | 8) You are now 20 years of age and decide to save $100 at the end of each month until you are 65. If the interest rate is 9.2%, how much money will you have when you are 65? |
| 9) Is it better to receive $400 at the beginning of each month for six years, or a lump sum of $25,000 today if the interest rate is 7%? Explain. | 10) To save money for a vacation, Jill decided to save $125 at the beginning of each month for the next 8 months. If the interest rate is 7%, how much money will she have at the end of 8 months? |

| | |
|---|---|
| 11) Mrs. Gill puts $2200 at the end of each year in her IRA account that earns 9% per year. How much total money will she have in this account after 20 years? | 12) If the inflation rate stays at 6% per year for the next five years, how much will the price be of a $15,000 car in five years? How much must you save at the end of each month at an interest rate of 7.3% to buy that car in 5 years? |

## SECTION 6.4 PROBLEM SET: PRESENT VALUE OF AN ANNUITY AND INSTALLMENT PAYMENT

For the following problems, show all work.

| | |
|---|---|
| 1) Shawn has won a lottery paying him $10,000 per month for the next 20 years. He'd rather have the whole amount in one lump sum today. If the current interest rate is 8.2%, how much money can he hope to get? | 2) Sonya bought a car for $15,000. Find the monthly payment if the loan is to be amortized over 5 years at a rate of 10.1%. |
| 3) You determine that you can afford $250 per month for a car. What is the maximum amount you can afford to pay for a car if the interest rate is 9% and you want to repay the loan in 5 years? | 4) Compute the monthly payment for a house loan of $200,000 to be financed over 30 years at an interest rate of 10%. |
| 5) If the $200,000 loan in the previous problem is financed over 15 years rather than 30 years at 10%, what will the monthly payment be? | 6) Friendly Auto offers Jennifer a car for $2000 down and $300 per month for 5 years. Jason wants to buy the same car but wants to pay cash. How much must Jason pay if the interest rate is 9.4%? |

***SECTION 6.4 PROBLEM SET: PRESENT VALUE OF AN ANNUITY AND INSTALLMENT PAYMENT***

For the following problems, show all work.

| | |
|---|---|
| 7) The Gomez family bought a house for $450,000. They paid 20% down and amortized the rest at 5.2% over a 30-year period. Find their monthly payment. | 8) Mr. and Mrs. Wong purchased their new house for $350,000. They made a down payment of 15%, and amortized the rest over 30 years. If the interest rate is 5.8%, find their monthly payment. |
| 9) A firm needs a piece of machinery that has a useful life of 5 years. It has an option of leasing it for $10,000 a year, or buying it for $40,000 cash. If the interest rate is 10%, which choice is better? | 10) Jackie wants to buy a $19,000 car, but she can afford to pay only $300 per month for 5 years. If the interest rate is 6%, how much does she need to put down? |
| 11) Vijay's tuition at college for the next year is $32,000. His parents have decided to pay the tuition by making nine monthly payments. If the interest rate is 6%, what is the monthly payment? | 12) Glen borrowed $10,000 for his college education at 8% compounded quarterly. Three years later, after graduating and finding a job, he decided to start paying off his loan. If the loan is amortized over five years at 9%, find his monthly payment for the next five years. |

## SECTION 6.5 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS

For problems 1 - 4, assume a $200,000 house loan is amortized over 30 years at an interest rate of 5.4%.

| | |
|---|---|
| 1) Find the monthly payment. | 2) Find the balance owed after 20 years. |
| 3) Find the balance of the loan after 100 payments.<br>\ | 4) Find the monthly payment if the original loan were amortized over 15 years. |

| | |
|---|---|
| 5) Mr. Patel wants to pay off his car loan. The monthly payment for his car is $365, and he has 16 payments left. If the loan was financed at 6.5%, how much does he owe? | 6) An amount of $2000 is borrowed for a year at a rate of 7%. Make an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the portion of the payment going toward reducing the debt, and the balance. |

***SECTION 6.5 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS***

| | |
|---|---|
| 7) Fourteen months after Dan bought his new car he lost his job. His car was repossessed by his lender after he made only 14 monthly payments of $376 each. If the loan was financed over a 4-year period at an interest rate of 6.3%, how much did the car cost the lender? In other words, how much did Dan still owe on the car? | 8) You have a choice of either receiving $5,000 at the end of each year for the next 5 years or receiving $3000 per year for the next 10 years. If the current interest rate is 9%, which is better? |
| 9) Mr. Smith is planning to retire in 25 years and would like to have $250,000 then. What monthly payment made at the end of each month to an account that pays 6.5% will achieve his objective? | 10) Assume Mr. Smith has reached retirement and has $250,000 in an account which is earning 6.5%. He would now like to make equal monthly withdrawals for the next 15 years to completely deplete this account. Find the withdrawal payment. |
| 11) Mrs. Garcia is planning to retire in 20 years. She starts to save for retirement by depositing $2000 each quarter into a retirement investment account that earns 6% interest compounded quarterly. Find the accumulated value of her retirement savings at the end of 20 years. | 12) Assume Mrs. Garcia has reached retirement and has accumulated the amount found in question 13 in a retirement savings account. She would now like to make equal monthly withdrawals for the next 15 years to completely deplete this account. Find the withdrawal payment. Assume the account now pays 5.4% compounded monthly. |

***SECTION 6.5 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS***

| | |
|---|---|
| 13) A ten-year $1,000 bond pays $35 every six months. If the current interest rate is 8.2%, find the fair market value of the bond.<br>Hint: You must do the following.<br>a) Find the present value of $1000.<br>b) Find the present value of the $35 payments.<br>c) The fair market value of the bond = a + b | 14) Find the fair market value of the ten-year $1,000 bond that pays $35 every six months, if the current interest rate has dropped to 6%.<br>Hint: You must do the following.<br>a) Find the present value of $1000.<br>b) Find the present value of the $35 payments.<br>c) The fair market value of the bond = a + b |
| 15) A twenty-year $1,000 bond pays $30 every six months. If the current interest rate is 4.2%, find the fair market value of the bond.<br>Hint: You must do the following.<br>a) Find the present value of $1000.<br>b) Find the present value of the $30 payments.<br>c) The fair market value of the bond = a + b | 16) Find the fair market value of the twenty-year $1,000 bond that pays $30 every six months, if the current interest rate has increased to 7.5%. |

***SECTION 6.5 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS***

| | |
|---|---|
| 17) Mr. and Mrs. Nguyen deposit $10,000 into a college investment account when their new baby grandchild is born. The account earns 6.25% interest compounded quarterly.<br>a, When their grandchild reaches the age of 18, what is the accumulated value of the college investment account?<br>b, The Nguyen's grandchild has just reached the age of 18 and started college. If she is to withdraw the money in the college savings account n equal monthly payments over the next 4 years, how much money will be withdrawn each month? | 18) Mr. Singh is 38 and plans to retire at age 65. He opens a retirement savings account.<br>a. Mr. Singh wants to save enough money to accumulate $500,000 by the time he retires. The retirement investment account pays 7% interest compounded monthly. How much does he need to deposit each month to achieve this goal?<br>b. Mr. Singh has now reached at 65 and retires. How much money can he withdraw each month for 25 years if the retirement investment account now pays 5.2% interest, compounded monthly? |

## SECTION 6.6 PROBLEM SET: CLASSIFICATION OF FINANCE PROBLEMS

Let the letters A, B, C, D, E and F be represented as follows:

A = FV of a lump-sum C = FV of an annuity E = Installment payment

B = PV of a lump-sum D = Sinking fund payment F = PV of an annuity

Classify each by writing the appropriate letter in the box, and write an equation for solution.

1) What monthly deposits made to an account paying 9% will grow to $10,000 in 4 years?

2) An amount of $4000 is invested at 6% compounded daily. What will the final amount be in 5 years?

3) David has won a lottery paying him $10,000 per month for the next 20 years. He'd rather have the whole amount in one lump sum now. If the current interest rate is 7%, how much money can he hope to get?

4) Each month Linda deposits $250 in an account that pays 9%. How much money will she have in 4 years?

5) Find the monthly payment for a $15,000 car if the loan is amortized over 4 years at a rate of 10%.

6) What lump-sum deposited in an account paying 7% compounded daily will grow to $10,000 in 5 years?

7) What amount of quarterly payments will amount to $250,000 in 5 years at a rate of 8%?

8) The Chang family bought their house 25 years ago. They had their loan financed for 30 years at an interest rate of 11% resulting in a payment of $1350 a month. Find the balance of the loan.

9-10) A 10-year $1000 bond pays $35 every six months. If the current interest rate is 8%, in order to find the fair market value of the bond, we need to find the following.

9) The present value of $1000. 10) The present value of the $35 per six month payments.

***SECTION 6.6 PROBLEM SET: CLASSIFICATION OF FINANCE PROBLEMS***

A = FV of a lump-sum C = FV of an annuity E = Installment payment

B = PV of a lump-sum D = Sinking fund payment F = PV of an annuity

11) What lump-sum deposit made today is equal to 33 monthly deposits of $500 if the interest rate is 8%?

12) What monthly deposits made to an account paying 10% will accumulated to $10,000 in six years?

13) A department store charges a finance charge of 1.5% per month on the outstanding balance.
If Ned charged $400 three months ago and has not paid his bill, how much does he owe?

14) What will the value of $300 monthly deposits be in 10 years if the account pays 12% compounded monthly?

15) What lump-sum deposited at 6% compounded daily will grow to $2000 in three years?

16) A company buys an apartment complex for $5,000,000 and amortizes the loan over 10 years.
What is the yearly payment if the interest rate is 14%?

17) In 2002, a house in Rock City cost $300,000. Real estate in Rock City has been increasing in value at the annual rate of 5.3%.. Find the price of that house in 2016.

18) You determine that you can afford to pay $400 per month for a car. What is the maximum price you can pay for a car if the interest rate is 11% and you want to repay the loan in 4 years?

19) A business needs $350,000 in 5 years. How much lump-sum should be put aside in an account that pays 9% so that five years from now the company will have $350,000?

20) A person wishes to have $500,000 in a pension fund 20 years from now. How much should he deposit each month in an account paying 9% compounded monthly?

## SECTION 6.7 PROBLEM SET: CHAPTER REVIEW

1) Manuel borrows $800 for 6 months at 18% simple interest. How much does he owe at the end of 6 months?

2) The population of a city is 65,000 and expects to grow at a rate of 2.3% per year for the next 10 years. What will the population of this city be in 10 years?

3) The Gill family is buying a $250,000 house with a 10% down payment. If the loan is financed over a 30 year period at an interest rate of 4.8%, what is the monthly payment?

4) Find the monthly payment for the house in the above problem if the loan was amortized over 15 years.

5) You look at your budget and decide that you can afford $250 per month for a car. What is the maximum amount you can afford to pay for the car if the interest rate is 8.6% and you want to finance the loan over 5 years?

6) Mr. Nakahama bought his house in the year 1998. He had his loan financed for 30 years at an interest rate of 6.2% resulting in a monthly payment of $1500. In 2015, 17 years later, he paid off the balance of the loan. How much did he pay?

7) Lisa buys a car for $16,500, and receives $2400 for her old car as a trade-in value. Find the monthly payment for the balance if the loan is amortized over 5 years at 8.5%.

8) A car is sold for $3000 cash down and $400 per month for the next 4 years. Find the cash value of the car today if the money is worth 8.3% compounded monthly.

9) An amount of $2300 is borrowed for 7 months at a simple interest rate of 16%. Find the discount and the proceeds.

10) Marcus has won a lottery paying him $5000 per month for the next 25 years. He'd rather have the whole amount in one lump sum today. If the current interest rate is 7.3%, how much money can he hope to get?

11) In the year 2000, an average house in Star City cost $250,000. If the average annual inflation rate for the past years has been about 4.7%, what was the price of that house in 2015?

12) Find the 'fair market' value of a ten-year $1000 bond which pays $30 every six months if the current interest rate is 7%. What if the current interest rate is 5%?

13) A Visa credit card company has a finance charge of 1.5% per month (18% per year) on the outstanding balance. John owed $3200 and has been delinquent for 5 months. How much total does he owe, now?

14) You want to purchase a home for $200,000 with a 30-year mortgage at 9.24% interest. Find a) the monthly payment and b) the balance owed after 20 years.

15) When Jose bought his car, he amortized his loan over 6 years at a rate of 9.2%, and his monthly payment came out to be $350 per month. He has been making these payments for the past 40 months and now wants to pay off the remaining balance. How much does he owe?

16) A lottery pays $10,000 per month for the next 20 years. If the interest rate is 7.8%, find both its present and future values.

***SECTION 6.7 PROBLEM SET: CHAPTER REVIEW***

17) A corporation estimates it will need $300,000 in 8 years to replace its existing machinery. How much should it deposit each quarter in a sinking fund earning 8.4% compounded quarterly to meet this obligation?

18) Our national debt in 1992 was about $4 trillion. If the annual interest rate was 7% then, what was the daily interest on the national debt?

19) A business must raise $400,000 in 10 years. What should be the size of the owners' monthly payments to a sinking fund paying 6.5% compounded monthly?

20) The population of a city of 80,000 is growing at a rate of 3.2% per year. What will the population be at the end of 10 years?

21) A sum of $5000 is deposited in a bank today. What will the final amount be in 20 months if the bank pays 9% and the interest is compounded monthly?

22) A manufacturing company buys a machine for $500 cash and $50 per month for the next 3 years. Find the cash value of the machine today if the money is worth 6.2% compounded monthly.

23) The United States paid about 4 cents an acre for the Louisiana Purchase in 1803. Suppose the value of this property grew at a rate of 5.5% annually. What would an acre be worth in the year 2000?

24) What amount should be invested per month at 9.1% compounded monthly so that it will become $5000 in 17 months?

25) A machine costs $8000 and has a life of 5 years. It can be leased for $160 per month for 5 years with a cash down payment of $750. The current interest rate is 8.3%. Is it cheaper to lease or to buy?

26) If inflation holds at 5.2% per year for 5 years, what will be the cost in 5 years of a car that costs $16,000 today? How much will you need to deposit each quarter in a sinking fund earning 8.7% per year to purchase the new car in 5 years?

27) City Bank pays an interest rate of 6%, while Western Bank pays 5.8% compounded continuously. Which one is a better deal?

28) Ali has inherited $20,000 and is planning to invest this amount at 7.9% interest. At the same time he wishes to make equal monthly withdrawals to use up the entire sum in 5 years. How much can he withdraw each month?

29) Jason has a choice of receiving $300 per month for the next 5 years or $500 per month for the next 3 years. Which one is worth more if the current interest rate is 7.7%?

30) If a bank pays 6.8% compounded continuously, how long will it take to double your money?

31) A mutual fund claims a growth rate of 8.3% per year. If $500 per month is invested, what will the final amount be in 15 years?

32) Mr. Vasquez has been given two choices for his compensation. He can have $20,000 cash plus $500 per month for 10 years, or he can receive $12,000 cash plus $1000 per month for 5 years. If the interest rate is 8%, which is the better offer?

***SECTION 6.7 PROBLEM SET: CHAPTER REVIEW***

33) How much should Mr. Shackley deposit in a trust account so that his daughter can withdraw $400 per month for 4 years if the interest rate is 8%?

34) Mr. Albers borrowed $425,000 from the bank for his new house at an interest rate of 4.7%. He will make equal monthly payments for the next 30 years. How much money will he end up paying the bank over the life of the loan, and how much is the interest?

35) Mr. Tong puts away $500 per month for 10 years in an account that earns 9.3%. After 10 years, he decides to withdraw $1,000 per month. If the interest rate stays the same, how long will it take Mr. Tong to deplete the account?

36) An amount of $5000 is borrowed for 15 months at an interest rate of 9%. Find the monthly payment and construct an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the amount of payment contributing towards debt, and the outstanding debt.

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# 4.2: Compound Interest

**Learning Objectives**

In this section, you will learn to:

1. Find the future value of a lump-sum.
2. Find the present value of a lump-sum.
3. Find the effective interest rate.

## Compound Interest

In the last section, we examined problems involving simple interest. Simple interest is generally charged when the lending period is short and often less than a year. When the money is loaned or borrowed for a longer time period, if the interest is paid (or charged) not only on the principal, but also on the past interest, then we say the interest is **compounded.**

Suppose we deposit \$200 in an account that pays 8% interest. At the end of one year, we will have \$200 + \$200(.08) = \$200(1 + .08) = \$216.

Now suppose we put this amount, \$216, in the same account. After another year, we will have \$216 + \$216(.08) = \$216(1 + .08) = \$233.28.

So an initial deposit of \$200 has accumulated to \$233.28 in two years. Further note that had it been simple interest, this amount would have accumulated to only \$232. The reason the amount is slightly higher is because the interest (\$16) we earned the first year, was put back into the account. And this \$16 amount itself earned for one year an interest of \$16(.08) = \$1.28, thus resulting in the increase. So we have earned interest on the principal as well as on the past interest, and that is why we call it compound interest.

Now suppose we leave this amount, \$233.28, in the bank for another year, the final amount will be \$233.28 + \$233.28(.08) = \$233.28(1 + .08) = \$251.94.

Now let us look at the mathematical part of this problem so that we can devise an easier way to solve these problems.

After one year, we had \$200(1 + .08) = \$216

After two years, we had \$216(1 + .08)

But \$216 = \$200(1 + .08), therefore, the above expression becomes

$$\$200(1+.08)(1+.08) = \$200(1+.08)^2 = \$233.28$$

After three years, we get

$$\$233.28(1+.08) = \$200(1+.08)(1+.08)(1+.08)$$

which can be written as

$$\$200(1+.08)^3 = \$251.94$$

Suppose we are asked to find the total amount at the end of 5 years, we will get

$$200(1+.08)^5 = \$293.87$$

We summarize as follows:

| | | |
|---|---|---|
| The original amount | \$200 | = \$200 |
| The amount after one year | \$200(1 + .08) | = \$216 |
| The amount after two years | \$200(1 + .08)$^2$ | = \$233.28 |
| The amount after three years | \$200(1 + .08)$^3$ | = \$251.94 |
| The amount after five years | \$200(1 + .08)$^5$ | = \$293.87 |

| | | |
|---|---|---|
| The amount after t years | $200(1 + .08)^t | |

## COMPOUNDING PERIODS

Banks often compound interest more than one time a year. Consider a bank that pays 8% interest but compounds it four times a year, or quarterly. This means that every quarter the bank will pay an interest equal to one-fourth of 8%, or 2%.

Now if we deposit $200 in the bank, after one quarter we will have $\$200\left(1+\frac{.08}{4}\right)$ or $204.

After two quarters, we will have $\$200\left(1+\frac{.08}{4}\right)^2$ or $208.08.

After one year, we will have $\$200\left(1+\frac{.08}{4}\right)^4$ or $216.49.

After three years, we will have $\$200\left(1+\frac{.08}{4}\right)^{12}$ or $253.65, etc.

| | | |
|---|---|---|
| The original amount | $200 | = $200 |
| The amount after one quarter | $\$200\left(1+\frac{.08}{4}\right)$ | = $204 |
| The amount after two quarters | $\$200\left(1+\frac{.08}{4}\right)^2$ | = $208.08 |
| The amount after one year | $\$200\left(1+\frac{.08}{4}\right)^4$ | = $216.49 |
| The amount after two years | $\$200\left(1+\frac{.08}{4}\right)^8$ | = $234.31 |
| The amount after three years | $\$200\left(1+\frac{.08}{4}\right)^{12}$ | = $253.65 |
| The amount after five years | $\$200\left(1+\frac{.08}{4}\right)^{20}$ | = $297.19 |
| The amount after t years | $\$200\left(1+\frac{.08}{4}\right)^{4t}$ | |

Therefore, if we invest a lump-sum amount of $P$ dollars at an interest rate $r$, compounded $n$ times a year, then after $t$ years the final amount is given by

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

The following examples use the compound interest formula $A = P\left(1+\frac{r}{n}\right)^{nt}$

### Example 4.2.1

If $3500 is invested at 9% compounded monthly, what will the future value be in four years?

**Solution**

Clearly an interest of .09/12 is paid every month for four years. The interest is compounded $4 \times 12 = 48$ times over the four-year period. We get

$$\mathrm{A} = \$3500\left(1+\frac{.09}{12}\right)^{48} = \$3500(1.0075)^{48} = \$5009.92$$

$3500 invested at 9% compounded monthly will accumulate to $5009.92 in four years.

### Example 4.2.2

How much should be invested in an account paying 9% compounded daily for it to accumulate to $5,000 in five years?

**Solution**

We know the future value, but need to find the principal.

$$\$5000 = P\left(1+\tfrac{.09}{365}\right)^{365\times 5}$$
$$\$5000 = P(1.568225)$$
$$\$3188.32 = P$$

$3188,32 invested into an account paying 9% compounded daily will accumulate to $5,000 in five years.

### Example 4.2.3

If $4,000 is invested at 4% compounded annually, how long will it take to accumulate to $6,000?

**Solution**

$n = 1$ because annual compounding means compounding only once per year. The formula simplifies to $A = (1+r)^t$ when $n = 1$.

$$\begin{aligned} \$6000 &= 4000(1+.04)^t \\ \frac{6000}{4000} &= 1.04^t \\ 1.5 &= 1.04^t \end{aligned}$$

We use logarithms to solve for the value of $t$ because the variable $t$ is in the exponent.

$$t = \log_{1.04}(1.5)$$

Using the change of base formula we can solve for $t$:

$$t = \frac{\ln(1.5)}{\ln(1.04)} = 10.33 \text{ years}$$

It takes 10.33 years for $4000 to accumulate to $6000 if invested at 4% interest, compounded annually

### Example 4.2.4

If $5,000 is invested now for 6 years what interest rate compounded quarterly is needed to obtain an accumulated value of $8000.

**Solution**

We have $n = 4$ for quarterly compounding.

$$\begin{aligned} \$8000 &= \$5000\left(1+\frac{r}{4}\right)^{4\times 6} \\ \frac{\$8000}{\$5000} &= \left(1+\frac{r}{4}\right)^{24} \\ 1.6 &= \left(1+\frac{r}{4}\right)^{24} \end{aligned}$$

We use roots to solve for $t$ because the variable $r$ is in the base, whereas the exponent is a known number.

$$\sqrt[24]{1.6} = 1+\frac{\mathrm{r}}{4}$$

Many calculators have a built in “nth root” key or function. In the TI-84 calculator, this is found in the Math menu. Roots can also be calculated as fractional exponents; if necessary, the previous step can be rewritten as

$$1.6^{1/24} = 1+\frac{\mathrm{r}}{4}$$

Evaluating the left side of the equation gives

$$1.0197765 = 1 + \frac{r}{4}$$
$$0.0197765 = \frac{r}{4}$$
$$r = 4(0.0197765) = 0.0791$$

An interest rate of 7.91% is needed in order for $5000 invested now to accumulate to $8000 at the end of 6 years, with interest compounded quarterly.

## Effective Interest Rate

Banks are required to state their interest rate in terms of an **"effective yield"** " or **"effective interest rate"**, for comparison purposes. The effective rate is also called the Annual Percentage Yield (APY) or Annual Percentage Rate (APR).

The effective rate is the interest rate compounded annually would be equivalent to the stated rate and compounding periods. The next example shows how to calculate the effective rate.

To examine several investments to see which has the best rate, we find and compare the effective rate for each investment.

Example $4.2.5$ illustrates how to calculate the effective rate.

**Example 4.2.5**

If Bank A pays 7.2% interest compounded monthly, what is the effective interest rate?
If Bank B pays 7.25% interest compounded semiannually, what is the effective interest rate? Which bank pays more interest?

**Solution**

Bank A: Suppose we deposit $1 in this bank and leave it for a year, we will get

$$1\left(1+\frac{0.072}{12}\right)^{12} = 1.0744$$
$$r_{EFF} = 1.0744 - 1 = 0.0744$$

We earned interest of $1.0744 - $1.00 = $.0744 on an investment of $1.

**The effective interest rate is 7.44%,** often referred to as the APY or APR.

Bank B: The effective rate is calculated as

$$r_{EFF} = 1\left(1+\frac{0.072}{2}\right)^{2} - 1 = .0738$$

**The effective interest rate is 7.38%**.

Bank A pays slightly higher interest, with an effective rate of 7.44%, compared to Bank B with effective rate 7.38%.

## Continuous Compounding

Interest can be compounded yearly, semiannually, quarterly, monthly, and daily. Using the same calculation methods, we could compound every hour, every minute, and even every second. As the compounding period gets shorter and shorter, we move toward the concept of continuous compounding.

But what do we mean when we say the interest is compounded continuously, and how do we compute such amounts? When interest is compounded "infinitely many times", we say that the interest is **compounded continuously**. Our next objective is to derive a formula to model continuous compounding.

Suppose we put $1 in an account that pays 100% interest. If the interest is compounded once a year, the total amount after one year will be $\$1(1+1) = \$2$.

- If the interest is compounded semiannually, in one year we will have $\$1(1+1/2)^2 = \$2.25$
- If the interest is compounded quarterly, in one year we will have $\$1(1+1/4)^4 = \$2.44$
- If the interest is compounded monthly, in one year we will have $\$1(1+1/12)^{12} = \$2.61$
- If the interest is compounded daily, in one year we will have $\$1(1+1/365)^{365} = \$2.71$

We show the results as follows:

| Frequency of compounding | Formula | Total amount |
|---|---|---|
| Annually | $\$1(1+1)$ | \$2 |
| Semiannually | $\$1(1+1/2)^2$ | \$2.25 |
| Quarterly | $\$1(1+1/4)^4 = \$2.44$ | \$2.44140625 |
| Monthly | $\$1(1+1/12)^{12}$ | \$2.61303529 |
| Daily | $\$1(1+1/365)^{365}$ | \$2.71456748 |
| Hourly | $\$1(1+1/8760)^{8760}$ | \$2.71812699 |
| Every minute | $\$1(1+1/525600)^{525600}$ | \$2.71827922 |
| Every Second | $\$1(1+1/31536000)^{31536000}$ | \$2.71828247 |
| Continuously | $\$1(2.718281828\ldots)$ | \$2.718281828... |

We have noticed that the \$1 we invested does not grow without bound. It starts to stabilize to an irrational number 2.718281828... given the name "$e$" after the great mathematician Euler.

In mathematics, we say that as $n$ becomes infinitely large the expression equals $\left(1+\frac{1}{n}\right)^n$ $e$.

Therefore, it is natural that the number $e$ play a part in continuous compounding.
It can be shown that as $n$ becomes infinitely large the expression $\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$

Therefore, it follows that if we invest $\$P$ at an interest rate $r$ per year, compounded continuously, after $t$ years the final amount will be given by

$$A = P \cdot e^{rt}$$

**Example 4.2.6**

\$3500 is invested at 9% compounded continuously. Find the future value in 4 years.

**Solution**

Using the formula for the continuous compounding, we get $A = Pe^{rt}$ .

$$A = \$3500e^{0.09\times 4}$$
$$A = \$3500e^{0.36}$$
$$A = \$5016.65$$

**Example 4.2.7**

If an amount is invested at 7% compounded continuously, what is the effective interest rate?

**Solution**

If we deposit \$1 in the bank at 7% compounded continuously for one year, and subtract that \$1 from the final amount, we get the effective interest rate in decimals.

$$\mathrm{r_{EFF}} = 1\mathrm{e}^{0.07} - 1$$
$$\mathrm{r_{EFF}} = 1.0725 - 1$$
$$\mathrm{r_{EFF}} = .0725 \text{ or } 7.25\%$$

**Example 4.2.8**

If an amount is invested at 7% compounded continuously, how long will it take to double?

We offer two solutions.

Solution 1 uses logarithms to calculate the exact answer, so it is preferred. We already used this method in Example $4.2.3$ to solve for time needed for an investment to accumulate to a specified future value.

Solution 2 provides an estimated solution that is applicable only to doubling time, but not to other multiples. Students should find out from their instructor if there is a preference as to which solution method is to be used for doubling time problems.

***Solution: Solution 1:*** **Calculating the answer exactly:** $Pe^{0.07t} = A$.

We don't know the initial value of the prinicipal but we do know that the accumulated value is double (twice) the principal.

$$\mathrm{P}_{e}^{0.07t} = 2\mathrm{P}$$

We divide both sides by $\mathrm{P}$

$$e^{.07t} = 2$$

Using natural logarithm:

$$.07\mathrm{t} = \ln(2)$$
$$\mathrm{t} = \ln(2)/.07 = 9.9 \text{ years}$$

It takes 9.9 years for money to double if invested at 7% continuous interest.

***Solution 2:*** **Estimating the answer using the Law of 70:**

The Law of 70 is a useful tool for estimating the time needed for an investment to double in value. It is an approximation and is not exact and comes from our previous solution. We calculated that

$$\mathrm{t} = \ln(2)/\mathrm{r} \text{ where r was } 0.07 \text{ in that solution.}$$

Evaluating $\ln(2) = 0.693$, gives $t = 0.693/\mathrm{r}$. Multiplying numerator and denominator by 100 gives $t = 69.3/(100\mathrm{r})$

If we estimate 69.3 by 70 and state the interest rate as a percent instead of a decimal, we obtain the Law of 70:

**Law of 70:** The number of years required to double money ≈ 70 ÷ interest rate

- Note that this is an approximate estimate only.
- The interest rate is stated as a percent (not decimal) in the Law of 70.

Using the Law of 70 gives us $t$ ≈ 70/7=10 which is close to but not exactly the value of 9.9 years calculated in Solution 1.

Approximate Doubling Time in Years as a Function of Interest Rate

| Annual interest rate | 1% | 2% | 3% | 4% | 5% | 6% | 7% | 8% | 9% | 10% |
|---|---|---|---|---|---|---|---|---|---|---|
| Number of years to double money | 70 | 35 | 23 | 18 | 14 | 12 | 10 | 9 | 8 | 7 |

The pattern in the table approximates the Law of 70.

With technology available to do calculations using logarithms, we would use the Law of 70 only for quick estimates of doubling times. Using the Law of 70 as an estimate works only for doubling times, but not other multiples, so it's not a replacement for knowing how to find exact solutions.

However, the Law of 70 can be useful to help quickly estimate many "doubling time" problems mentally, which can be useful in compound interest applications as well as other applications involving exponential growth.

**Example 4.2.9**

a. At the peak growth rate in the 1960's the worlds population had a doubling time of 35 years. At that time, approximately what was the growth rate?
b. As of 2015, the world population's annual growth rate was approximately 1.14%. Based on that rate, find the approximate doubling time.

**Solution**

a. According to the law of 70,

$$\text{doubling time} = 35 \approx 70 \div r$$

$$r \approx 2 \text{ expressed as a percent}$$

Therefore, the world population was growing at an approximate rate of 2% in the 1960's.

b.. According to the law of 70,

$$\text{doubling time } t \approx 70 \div r = 70 \div 1.14 \approx 61 \text{ years}$$

If the world population were to continue to grow at the annual growth rate of 1.14% , it would take approximately 61 years for the population to double.

## SECTION 6.2 SUMMARY

Below is a summary of the formulas we developed for calculations involving compound interest:

**COMPOUND INTEREST n times per year**

1. If an amount P is invested for $t$ years at an interest rate $r$ per year, compounded $n$ times a year, then the future value is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

**P is called the principal and is also called the present value.**

2. If a bank pays an interest rate $r$ per year, compounded $n$ times a year, then the effective interest rate is given by

$$\mathrm{r_{EFF}} = \left(1 + \frac{r}{n}\right)^{n} - 1$$

**CONTINUOUSLY COMPOUNDED INTEREST**

3. If an amount P is invested for $t$ years at an interest rate $r$ per year, compounded continuously, then the future value is given by

$$\mathrm{A} = \mathrm{P}e^{rt}$$

4. If a bank pays an interest rate $r$ per year, compounded $n$ times a year, then the effective interest rate is given by

$$\mathrm{r_{EFF}} = e^{\mathrm{r}} - 1$$

5. The Law of 70 states that

**The number of years to double money is approximately 70 ÷ interest rate**

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# 4.2.1: Compound Interest (Exercises)

## SECTION 6.2 PROBLEM SET: COMPOUND INTEREST

Do the following compound interest problems involving a lump-sum amount.

| | |
|---|---|
| 1) What will the final amount be in 4 years if $8,000 is invested at 9.2% compounded monthly.? | 2) How much should be invested at 10.3% for it to amount to $10,000 in 6 years? |
| 3) Lydia's aunt Rose left her $5,000. Lydia spent $1,000 on her wardrobe and deposited the rest in an account that pays 6.9% compounded daily. How much money will she have in 5 years? | 4) Thuy needs $1,850 in eight months for her college tuition. How much money should she deposit lump sum in an account paying 8.2% compounded monthly to achieve that goal? |
| 5) Bank A pays 5% compounded daily, while Bank B pays 5.12% compounded monthly. Which bank pays more? Explain. | 6) EZ Photo Company needs five copying machines in 2 1/2 years for a total cost of $15,000. How much money should be deposited now to pay for these machines, if the interest rate is 8% compounded semiannually? |
| 7) Jon's grandfather was planning to give him $12,000 in 10 years. Jon has convinced his grandfather to pay him $6,000 now, instead. If Jon invests this $6,000 at 7.5% compounded continuously, how much money will he have in 10 years? | 8) What will be the price of a $20,000 car in 5 years if the inflation rate is 6%? |

***SECTION 6.2 PROBLEM SET: COMPOUND INTEREST***

Do the following compound interest problems.

| | |
|---|---|
| 9) At an interest rate of 8% compounded continuously, how many years will it take to double your money? | 10) If an investment earns 10% compounded continuously, in how many years will it triple? . |
| 11) The City Library ordered a new computer system costing $158,000; it will be delivered in 6 months, and the full amount will be due 30 days after delivery. How much must be deposited today into an account paying 7.5% compounded monthly to have $158,000 in 7 months? | 12) Mr. and Mrs. Tran are expecting a baby girl in a few days. They want to put away money for her college education now. How much money should they deposit in an account paying 10.2% so they will have $100,000 in 18 years to pay for their daughter's educational expenses? |
| 13) Find the effective interest rate for an account paying 7.2% compounded quarterly. | 14) If a bank pays 5.75% compounded monthly, what is the effective interest rate? |
| 15) The population of the African nation of Cameroon was 12 million people in the year 2015; it has been growing at the rate of 2.5% per year. If the population continues to grow that rate,what will the population be in 2030? (http://databank.worldbank.org/data on 4/26/2016) | 16) According to the Law of 70, if an amount grows at an annual rate of 1%, then it doubles every seventy years. Suppose a bank pays 5% interest, how long will it take for you to double your money? How about at 15%? |

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# 4.3: Annuities and Sinking Funds

**Learning Objectives**

In this section, you will learn to:

1. Find the future value of an annuity.
2. Find the amount of payments to a sinking fund.

## Ordinary Annuity

In the first two sections of this chapter, we examined problems where an amount of money was deposited lump sum in an account and was left there for the entire time period. Now we will do problems where timely payments are made in an account. When a sequence of payments of some fixed amount are made in an account at equal intervals of time, we call that an **annuity**. And this is the subject of this section.

To develop a formula to find the value of an annuity, we will need to recall the formula for the sum of a geometric series. A geometric series is of the form:

$$a + ax + ax^2 + ax^3 + \ldots + ax^n. \tag{4.3.1}$$

In a geometric series, each subsequent term is obtained by multiplying the preceding term by a number, called the **common ratio**. A geometric series is completely determined by knowing its first term, the common ratio, and the number of terms.

The first term of the series in Equation 4.3.1 is $a$, the common ratio is $x$, and the number of terms is $n+1$. The following are some examples of geometric series.

$$3 + 6 + 12 + 24 + 48$$

This above series has first term $a = 3$ and common ratio $x = 2$

$$2 + 6 + 18 + 54 + 162$$

This above series has first term $a = 2$ and common ratio $x = 3$

$$37 + 3.7 + 0.37 + 0.037 + 0.0037$$

This above series has first term $a = 35$ and common ratio $x = 0.1$

In your algebra class, you developed a formula for finding the sum of a geometric series. You probably used $r$ as the symbol for the ratio, but we are using $x$ because $r$ is the symbol we have been using for the interest rate. The formula for the sum of the terms of a geometric series with first term $a$, common ratio $x$ and last term $ax^n$ is:

$$\frac{a\left(x^{n+1} - 1\right)}{x - 1}$$

We will use this formula to find the value of an annuity. Consider the following example.

**Example 4.3.1**

If at the end of each month a deposit of $500 is made in an account that pays 8% compounded monthly, what will the final amount be after five years?

**Solution**

There are 60 deposits made in this account. The first payment stays in the account for 59 months, the second payment for 58 months, the third for 57 months, and so on.

- The first payment of $500 will accumulate to an amount of $500(1 + 0.08/12)$^{59}$.
- The second payment of $500 will accumulate to an amount of $500(1 + 0.08/12)$^{58}$.
- The third payment will accumulate to $500(1 + 0.08/12)$^{57}$.
- The fourth payment will accumulate to $500(1 + 0.08/12)$^{56}$.

And so on . . .

Finally the next to last ($59^{th}$) payment will accumulate to $\$500(1+0.08/12)^1$.

The last payment is taken out the same time it is made, and will not earn any interest.

To find the total amount in five years, we need to add the accumulated value of these sixty payments.

In other words, we need to find the sum of the following series.

$$\$500(1+0.08/12)^{59}+\$500(1+0.08/12)^{58}+\$500(1+0.08/12)^{57}+\ldots+\$500$$

Written backwards, we have

$$\$500+\$500(1+0.08/12)+\$500(1+0.08/12)^2+\ldots+\$500(1+0.08/12)^{59}$$

This is a geometric series with $a=\$500$, $r=(1+0.08/12$, and $n=59$. The sum is

$$\begin{aligned}\text{sum} &= \frac{\$500\left[(1+0.08/12)^{60}-1\right]}{0.08/12}\\ &= \$500(73.47686)\\ &= \$36,738.43\end{aligned}$$

When the payments are made at the end of each period rather than at the beginning, we call it an **ordinary annuity.**

## Future Value of an Ordinary Annuity

If a payment of m dollars is made in an account n times a year at an interest $r$, then the final amount $A$ after $t$ years is

$$\mathbf{A=\frac{m\left[(1+r/n)^{nt}-1\right]}{r/n}}$$

The future value is also called the accumulated value

### Example 4.3.2

Tanya deposits $300 at the end of each quarter in her savings account. If the account earns 5.75% compounded quarterly, how much money will she have in 4 years?

**Solution**

The future value of this annuity can be found using the above formula.

$$\begin{aligned}A &= \frac{\$300\left[(1+.0575/4)^{16}-1\right]}{0.0575/4}\\ &= \$300(17.8463)\\ &= \$5353.89\end{aligned}$$

If Tanya deposits $300 into a savings account earning 5.75% compounded quarterly for 4 years, then at the end of 4 years she will have $5,353.89

### Example 4.3.3

Robert needs $5,000 in three years. How much should he deposit each month in an account that pays 8% compounded monthly in order to achieve his goal?

**Solution**

If Robert saves m dollars per month, after three years he will have

$$\frac{m\left[(1+.08/12)^{36}-1\right]}{0.08/12}$$

But we'd like this amount to be \$5,000. Therefore,

$$\begin{aligned}\frac{\mathrm{m}\left[(1+.08/12)^{36}-1\right]}{.08/12} &= \$5000\\ \mathrm{m}(40.5356) &= \$5000\\ \mathrm{m} &= \frac{5000}{40.5356}\\ &= \$123.35\end{aligned}$$

Robert needs to deposit \$123.35 at the end of each month for 3 years into an account paying 8% compounded monthly in order to have \$5,000 at the end of 5 years.

## Sinking Fund

When a business deposits money at regular intervals into an account in order to save for a future purchase of equipment, the savings fund is referred to as a "**sinking fund**". Calculating the sinking fund deposit uses the same method as the previous problem.

**Example 4.3.4**

A business needs \$450,000 in five years. How much should be deposited each quarter in a sinking fund that earns 9% compounded quarterly to have this amount in five years?

**Solution**

Again, suppose that $m$ dollars are deposited each quarter in the sinking fund. After five years, the future value of the fund should be \$450,000. This suggests the following relationship:

$$\begin{aligned}\frac{\mathrm{m}\left[(1+0.09/4)^{20}-1\right]}{0.09/4} &= \$450,000\\ \mathrm{m}(24.9115) &= 450,000\\ \mathrm{m} &= \frac{450000}{24.9115}\\ &= \$18,063.93\end{aligned}$$

The business needs to deposit \$18,063.93 at the end of each quarter for 5 years into an sinking fund earning interest of 9% compounded quarterly in order to have \$450,000 at the end of 5 years.

## Annuity Due

If the payment is made at the beginning of each period, rather than at the end, we call it an **annuity due**. The formula for the annuity due can be derived in a similar manner. Reconsider Example 1, with the change that the deposits are made at the beginning of each month.

**Example 4.3.5**

If at the beginning of each month a deposit of \$500 is made in an account that pays 8% compounded monthly, what will the final amount be after five years?

**Solution**

There are 60 deposits made in this account. The first payment stays in the account for 60 months, the second payment for 59 months, the third for 58 months, and so on.

- The first payment of \$500 will accumulate to an amount of $\$500(1+0.08/12)^{60}$.

- The second payment of \$500 will accumulate to an amount of $\$500(1+.08/12)^{59}$.
- The third payment will accumulate to $\$500(1+0.08/12)^{58}$.

And so on . . .

The last payment is in the account for a month and accumulates to $\$500(1+0.08/12)$

To find the total amount in five years, we need to find the sum of the series:

$$\$500(1+0.08/12)^{60}+\$500(1+0.08/12)^{59}+\$500(1+0.08/12)^{58}+\ldots+\$500(1+0.08/12)$$

Written backwards, we have

$$\$500(1+0.08/12)+\$500(1+0.08/12)^{2}+\ldots+\$500(1+0.08/12)^{60}$$

If we add \$500 to this series, and later subtract that \$500, the value will not change. We get

$$\mathbf{\$500}+\$500(1+0.08/12)+\$500(1+0.08/12)^{2}+\ldots+\$500(1+0.08/12)^{60}-\mathbf{\$500}$$

Except for the last term, we have a geometric series with $a$ = \$500, $r$ = (1 + .08/12), and $n$ = 60. Therefore the sum is

$$\begin{aligned} \mathrm{A} &= \frac{\$500\left[(1+0.08/12)^{61}-1\right]}{0.08/12}-\$500 \\ &= \$500(74.9667)-\$500 \\ &= \$37483.35-\$500 \\ &= \$36983.35 \end{aligned}$$

So, in the case of an annuity due, to find the future value, we increase the number of periods $n$ by 1, and subtract one payment.

**Future Value of an "Annuity Due"**

$$\mathrm{A}=\frac{\mathrm{m}\left[(1+\mathrm{r}/\mathrm{n})^{\mathrm{nt}+1}-1\right]}{\mathrm{r}/\mathrm{n}}-\mathrm{m}$$

Most of the problems we are going to do in this chapter involve ordinary annuities, therefore, we will down play the significance of the last formula for the annuity due. We mentioned the formula for the annuity due only for completeness.

## Summary

Finally, it is the author's wish that the student learn the concepts in a way that he or she will not have to memorize every formula. It is for this reason formulas are kept at a minimum. But before we conclude this section we will once again mention one single equation that will help us find the future value, as well as the sinking fund payment.

If a payment of $m$ dollars is made in an account $n$ times a year at an interest $r$, then the future value $\mathrm{A}$ after $t$ years is

$$\mathbf{A}=\frac{\mathbf{m}\left[(\mathbf{1}+\mathbf{r}/\mathbf{n})^{\mathbf{nt}}-\mathbf{1}\right]}{\mathbf{r}/\mathbf{n}}$$

Note that the formula assumes that the payment period is the same as the compounding period. If these are not the same, then this formula does not apply.

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# 4.3.1: Annuities and Sinking Funds (Exercises)

## SECTION 6.3 PROBLEM SET:ANNUITIES AND SINKING FUNDS

Each of the following problems involve an annuity - a sequence of payments.

| | |
|---|---|
| 1) Find the future value of an annuity of $200 per month for 5 years at 6% compounded monthly. | 2) How much money should be deposited at the end of each month in an account paying 7.5% for it to amount to $10,000 in 5 years? |
| 3) At the end of each month Rita deposits $300 in an account that pays 5%. What will the final amount be in 4 years? | 4) Mr. Chang wants to retire in 10 years and can save $650 every three months. If the interest rate is 7.8%, how much will he have (a) at the end of 5 years? (b) at the end of 10 years? |
| 5) A firm needs to replace most of its machinery in five years at a cost of $500,000. The company wishes to create a sinking fund to have this money available in five years. How much should the quarterly deposits be if the fund earns 8%? | 6) Mrs. Brown needs $5,000 in three years. If the interest rate is 9%, how much should she save at the end of each month to have that amount in three years? |
| 7) A company has a $120,000 note due in 4 years. How much should be deposited at the end of each quarter in a sinking fund to payoff the note in four years if the interest rate is 8%? | 8) You are now 20 years of age and decide to save $100 at the end of each month until you are 65. If the interest rate is 9.2%, how much money will you have when you are 65? |
| 9) Is it better to receive $400 at the beginning of each month for six years, or a lump sum of $25,000 today if the interest rate is 7%? Explain. | 10) To save money for a vacation, Jill decided to save $125 at the beginning of each month for the next 8 months. If the interest rate is 7%, how much money will she have at the end of 8 months? |
| 11) Mrs. Gill puts $2200 at the end of each year in her IRA account that earns 9% per year. How much total money will she have in this account after 20 years? | 12) If the inflation rate stays at 6% per year for the next five years, how much will the price be of a $15,000 car in five years? How much must you save at the end of each month at an interest rate of 7.3% to buy that car in 5 years? |

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# 4.4: Present Value of an Annuity and Installment Payment

**Learning Objectives**

In this section, you will learn to:

1. Find the present value of an annuity.
2. Find the amount of installment payment on a loan.

## PRESENT VALUE OF AN ANNUITY

In Section 6.2, we learned to find the future value of a lump sum, and in Section 6.3, we learned to find the future value of an annuity. With these two concepts in hand, we will now learn to amortize a loan, and to find the present value of an annuity.

The **present value** of an annuity is the amount of money we would need now in order to be able to make the payments in the annuity in the future. In other word, the present value is the value now of a future stream of payments.

We start by breaking this down step by step to understand the concept of the present value of an annuity. After that, the examples provide a more efficient way to do the calculations by working with concepts and calculations we have already explored in Sections 6.2 and 6.3.

Suppose Carlos owns a small business and employs an assistant manager to help him run the business. Assume it is January 1 now. Carlos plans to pay his assistant manager a $1000 bonus at the end of this year and another $1000 bonus at the end of the following year. Carlos' business had good profits this year so he wants to put the money for his assistant's future bonuses into a savings account now. The money he puts in now will earn interest at the rate of 4% per year compounded annually while in the savings account.

How much money should Carlos put into the savings account now so that he will be able to withdraw $1000 one year from now and another $1000 two years from now?

At first, this sounds like a sinking fund. But it is different. In a sinking fund, we put money into the fund with periodic payments to save to accumulate to a specified lump sum that is the future value at the end of a specified time period.

In this case we want to put a lump sum into the savings account now, so that lump sum is our principal, P. Then we want to withdraw that amount as a series of period payments; in this case the withdrawals are an annuity with $1000 payments at the end of each of two years.

We need to determine the amount we need in the account now, the present value, to be able to make withdraw the periodic payments later.




We use the compound interest formula from Section 6.2 with $r$ = 0.04 and $n$ = 1 for annual compounding to determine the present value of each payment of $1000.

Consider the first payment of $1000 at the end of year 1. Let $P_1$ be its present value

$$\$1000 = P_1(1.04)^1 \text{ so } P_1 = \$961.54$$

Now consider the second payment of $1000 at the end of year 2. Let $P_2$ is its present value

$$\$1000 = P_2(1.04)^2 \text{ so } P_2 = \$924.56$$

To make the $1000 payments at the specified times in the future, the amount that Carlos needs to deposit now is the present value $P = P_1 + P_2 = \$961.54 + \$924.56 = \$1886.10$

The calculation above was useful to illustrate the meaning of the present value of an annuity.
But it is not an efficient way to calculate the present value. If we were to have a large number of annuity payments, the step by step calculation would be long and tedious.

Example 4.4.1 investigates and develops an efficient way to calculate the present value of an annuity, by relating the future (accumulated) value of an annuity and its present value.

### Example 4.4.1

Suppose you have won a lottery that pays $1,000 per month for the next 20 years. But, you prefer to have the entire amount now. If the interest rate is 8%, how much will you accept?

**Solution**

This classic present value problem needs our complete attention because the rationalization we use to solve this problem will be used again in the problems to follow.

Consider, for argument purposes, that two people Mr. Cash, and Mr. Credit have won the same lottery of $1,000 per month for the next 20 years. Mr. Credit is happy with his $1,000 monthly payment, but Mr. Cash wants to have the entire amount now.

Our job is to determine how much Mr. Cash should get. We reason as follows:

If Mr. Cash accepts P dollars, then the P dollars deposited at 8% for 20 years should yield the same amount as the $1,000 monthly payments for 20 years. In other words, we are comparing the future values for both Mr. Cash and Mr. Credit, and we would like the future values to equal.

Since Mr. Cash is receiving a lump sum of $x$ dollars, its future value is given by the lump sum formula we studied in Section 6.2, and it is

$$\mathrm{A} = \mathrm{P}(1 + .08/12)^{240}$$

Since Mr. Credit is receiving a sequence of payments, or an annuity, of $1,000 per month, its future value is given by the annuity formula we learned in Section 6.3. This value is

$$\mathrm{A} = \frac{\$1000\left[(1 + .08/12)^{240} - 1\right]}{.08/12}$$

The only way Mr. Cash will agree to the amount he receives is if these two future values are equal. So we set them equal and solve for the unknown.

$$\begin{aligned}
\mathrm{P}(1 + .08/12)^{240} &= \frac{\$1000\left[(1+.08/12)^{240} - 1\right]}{.08/12} \\
\mathrm{P}(4.9268) &= \$1000(589.02041) \\
\mathrm{P}(4.9268) &= \$589020.41 \\
\mathrm{P} &= \$119,554.36
\end{aligned}$$

The present value of an ordinary annuity of $1,000 each month for 20 years at 8% is $119,554.36

The reader should also note that if Mr. Cash takes his lump sum of P = $119,554.36 and invests it at 8% compounded monthly, he will have an accumulated value of A=$589,020.41 in 20 years.

## INSTALLMENT PAYMENT ON A LOAN

If a person or business needs to buy or pay for something now (a car, a home, college tuition, equipment for a business) but does not have the money, they can borrow the money as a loan.

They receive the loan amount called the principal (or present value) now and are obligated to pay back the principal in the future over a stated amount of time (term of the loan), as regular periodic payments with interest.

Example 4.4.2 examines how to calculate the loan payment, using reasoning similar to Example 4.4.1.

**Example 4.4.2**

Find the monthly payment for a car costing $15,000 if the loan is amortized over five years at an interest rate of 9%.

**Solution**

Again, consider the following scenario:

Two people, Mr. Cash and Mr. Credit, go to buy the same car that costs $15,000. Mr. Cash pays cash and drives away, but Mr. Credit wants to make monthly payments for five years.

Our job is to determine the amount of the monthly payment. We reason as follows:

If Mr. Credit pays m dollars per month, then the m dollar payment deposited each month at 9% for 5 years should yield the same amount as the $15,000 lump sum deposited for 5 years.

Again, we are comparing the future values for both Mr. Cash and Mr. Credit, and we would like them to be the same.

Since Mr. Cash is paying a lump sum of $15,000, its future value is given by the lump sum formula, and it is

$$\$15,000(1+.09/12)^{60}$$

Mr. Credit wishes to make a sequence of payments, or an annuity, of $x$ dollars per month, and its future value is given by the annuity formula, and this value is

$$\frac{\mathrm{x}\left[(1+.09/12)^{60}-1\right]}{.09/12}$$

We set the two future amounts equal and solve for the unknown.

$$\begin{aligned}\$15,000(1+.09/12)^{60} &= \frac{m\left[(1+.09/12)^{60}-1\right]}{.09/12}\\ \$15,000(1.5657) &= m(75.4241)\\ \$311.38 &= m\end{aligned}$$

Therefore, the monthly payment needed to repay the loan is $311.38 for five years.

## SECTION 6.4 SUMMARY

We summarize the method used in examples 4.4.1 and 4.4.2 below.

**The Equation to Find the Present Value of an Annuity,**

**Or the Installment Payment for a Loan**

If a payment of $m$ dollars is made in an account $n$ times a year at an interest $r$, then the present value P of the annuity after $t$ years is

$$\mathbf{P(1+r/n)^{nt}=\frac{m\left[(1+r/n)^{nt}-1\right]}{r/n}}$$

When used for a loan, the amount P is the loan amount, and $m$ is the periodic payment needed to repay the loan over a term of $t$ years with $n$ payments per year.

If the present value or loan amount is needed, solve for $P$

If the periodic payment is needed, solve for $m$.

Note that the formula assumes that the payment period is the same as the compounding period. If these are not the same, then this formula does not apply.

Finally, we note that many finite mathematics and finance books develop the formula for the present value of an annuity differently.

Instead of using the formula:

$$\mathrm{P}(1+\mathrm{r}/\mathrm{n})^{\mathrm{nt}}=\frac{\mathrm{m}\left[(1+\mathrm{r}/\mathrm{n})^{\mathrm{nt}}-1\right]}{\mathrm{r}/\mathrm{n}} \tag{4.4.1}$$

and solving for the present value $\mathrm{P}$ after substituting the numerical values for the other items in the formula, many textbooks first solve the formula for $\mathrm{P}$ in order to develop a new formula for the present value. Then the numerical information can be substituted into the present value formula and evaluated, without needing to solve algebraically for $\mathrm{P}$.

### Alternate Method to find Present Value of an Annuity

Starting with formula 4.4.1: $\mathrm{P}(1+\mathrm{r}/\mathrm{n})^{\mathrm{nt}}=\frac{\mathrm{m}\left[(1+\mathrm{r}/\mathrm{n})^{\mathrm{nt}}-1\right]}{\mathrm{r}/\mathrm{n}}$

Divide both sides by $(1+r/n)^{nt}$ to isolate $\mathrm{P}$, and simplify

$$P=\frac{m\left[(1+r/n)^{nt}-1\right]}{r/n}\cdot\frac{1}{(1+r/n)^{nt}}$$

$$P=\frac{m\left[1-(1+r/n)^{-nt}\right]}{r/n} \tag{4.4.2}$$

The authors of this book believe that it is easier to use formula 4.4.1 at the top of this page and solve for $\mathrm{P}$ or $m$ as needed. In this approach there are fewer formulas to understand, and many students find it easier to learn. In the problems the rest of this chapter, when a problem requires the calculation of the present value of an annuity, formula 4.4.1 will be used.

However, some people prefer formula 4.4.2, and it is mathematically correct to use that method. Note that if you choose to use formula 4.4.2, you need to be careful with the negative exponents in the formula. And if you needed to find the periodic payment, you would still need to do the algebra to solve for the value of m.

It would be a good idea to check with your instructor to see if he or she has a preference. In fact, you can usually tell your instructor's preference by noting how he or she explains and demonstrates these types of problems in class.

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# 4.4.1: Present Value of an Annuity and Installment Payment (Exercises)

## SECTION 6.4 PROBLEM SET: PRESENT VALUE OF AN ANNUITY AND INSTALLMENT PAYMENT

For the following problems, show all work.

| | |
|---|---|
| 1) Shawn has won a lottery paying him $10,000 per month for the next 20 years. He'd rather have the whole amount in one lump sum today. If the current interest rate is 8.2%, how much money can he hope to get? | 2) Sonya bought a car for $15,000. Find the monthly payment if the loan is to be amortized over 5 years at a rate of 10.1%. |
| 3) You determine that you can afford $250 per month for a car. What is the maximum amount you can afford to pay for a car if the interest rate is 9% and you want to repay the loan in 5 years? | 4) Compute the monthly payment for a house loan of $200,000 to be financed over 30 years at an interest rate of 10%. |
| 5) If the $200,000 loan in the previous problem is financed over 15 years rather than 30 years at 10%, what will the monthly payment be? | 6) Friendly Auto offers Jennifer a car for $2000 down and $300 per month for 5 years. Jason wants to buy the same car but wants to pay cash. How much must Jason pay if the interest rate is 9.4%? |
| 7) The Gomez family bought a house for $450,000. They paid 20% down and amortized the rest at 5.2% over a 30-year period. Find their monthly payment. | 8) Mr. and Mrs. Wong purchased their new house for $350,000. They made a down payment of 15%, and amortized the rest over 30 years. If the interest rate is 5.8%, find their monthly payment. |
| 9) A firm needs a piece of machinery that has a useful life of 5 years. It has an option of leasing it for $10,000 a year, or buying it for $40,000 cash. If the interest rate is 10%, which choice is better? | 10) Jackie wants to buy a $19,000 car, but she can afford to pay only $300 per month for 5 years. If the interest rate is 6%, how much does she need to put down? |
| 11) Vijay's tuition at college for the next year is $32,000. His parents have decided to pay the tuition by making nine monthly payments. If the interest rate is 6%, what is the monthly payment? | 12) Glen borrowed $10,000 for his college education at 8% compounded quarterly. Three years later, after graduating and finding a job, he decided to start paying off his loan. If the loan is amortized over five years at 9%, find his monthly payment for the next five years. |

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# 4.5: Miscellaneous Application Problems

**Learning Objectives**

In this section, you will learn to apply to concepts for compound interest for savings and annuities to:

1. Find the outstanding balance, partway through the term of a loan, of the future payments still remaining on the loan.
2. Perform financial calculations in situations involving several stages of savings and/or annuities.
3. Find the fair market value of a bond.
4. Construct an amortization schedule for a loan.

We have already developed the tools to solve most finance problems. Now we use these tools to solve some application problems.

## OUTSTANDING BALANCE ON A LOAN

One of the most common problems deals with finding the balance owed at a given time during the life of a loan. Suppose a person buys a house and amortizes the loan over 30 years, but decides to sell the house a few years later. At the time of the sale, he is obligated to pay off his lender, therefore, he needs to know the balance he owes. Since most long term loans are paid off prematurely, we are often confronted with this problem.

To find the outstanding balance of a loan at a specified time, we need to find the present value $\mathrm{P}$ of all future payments that have not yet been paid. In this case t does not represent the entire term of the loan. Instead:

- $t$ represents the time that still remains on the loan
- $nt$ represents the total number of future payments.

**Example 4.5.1**

Mr. Jackson bought his house in 1995, and financed the loan for 30 years at an interest rate of 7.8%. His monthly payment was $1260. In 2015, Mr. Jackson decides to pay off the loan. Find the balance of the loan he still owes.

**Solution**

The reader should note that the original amount of the loan is not mentioned in the problem. That is because we don't need to know that to find the balance.

The original loan was for 30 years. 20 years have past so there are years still remaining. 12(10) = 120 payments still remain to be paid on this loan.

As for the bank or lender is concerned, Mr. Jackson is obligated to pay $1260 each month for 10 more years; he still owes a total of 120 payments. But since Mr. Jackson wants to pay it all off now, we need to find the present value $\mathrm{P}$ at the time of repayment of the remaining 10 years of payments of $1260 each month. Using the formula we get for the present value of an annuity, we get

$$\begin{aligned} \mathrm{P}(1+.078/12)^{120} &= \frac{\$1260\left[(1+.078/12)^{120}-1\right)]}{(.078/12)} \\ \mathrm{P}(2.17597) &= \$227957.85 \\ \mathrm{P} &= \$104761.48 \end{aligned}$$

**to find the outstanding balance of a loan**

If a loan has a payment of $m$ dollars made $n$ times a year at an interest $r$, then the outstanding value of the loan when there are $t$ years still remaining on the loan is given by $\mathrm{P}$:

$$\mathbf{P(1+r/n)^{nt}=\frac{m\left[(1+r/n)^{nt}-1\right|}{r/n}}$$

IMPORTANT: Note that $t$ is not the original term of the loan but instead $t$ is the amount of time still remaining in the future $nt$ is the number of payments still remaining in the future

If the problem does not directly state the amount of time still remaining in the term of the loan, then it must be calculated BEFORE using the above formula as $t$ = original term of loan - time already passed since the start date of the loan.

Note that there are other methods to find the outstanding balance on a loan, but the method illustrated above is the easiest.

One alternate method would be to use an amortization schedule, as illustrated toward the end of this section. An amortization schedule shows the payments, interest, and outstanding balance step by step after each loan payment. An amortization schedule is tedious to calculate by hand but can be easily constructed using spreadsheet software.

Another way to find the outstanding balance, that we will not illustrate here, is to find the difference A - B, where

A = the original loan amount (principal) accumulated to the date on which we want to find the outstanding balance (using compound interest formula)

B = the accumulated value of all payments that have been made as of the date on which we want to find the outstanding balance (using formula for accumulated value of an annuity)

In this case we would need do a compound interest calculation and an annuity calculation; we then need to find the difference between them. Three calculations are needed instead of one.

It is a mathematically acceptable way to calculate the outstanding balance. However, **it is very strongly recommended that students use the method explained in box above** and illustrated in Example 4.5.1, as it is much simpler.

## PROBLEMS INVOLVING MULTIPLE STAGES OF SAVINGS AND/OR ANNUITIES

Consider the following situations:

a. Suppose a baby, Aisha, is born and her grandparents invest $5000 in a college fund. The money remains invested for 18 years until Aisha enters college, and then is withdrawn in equal semiannual payments over the 4 years that Aisha expects to need to finish college. The college investment fund earns 5% interest compounded semiannually. How much money can Aisha withdraw from the account every six months while she is in college?
b. Aisha graduates college and starts a job. She saves $1000 each quarter, depositing it into a retirement savings account. Suppose that Aisha saves for 30 years and then retires. At retirement she wants to withdraw money as an annuity that pays a constant amount every month for 25 years. During the savings phase, the retirement account earns 6% interest compounded quarterly. During the annuity payout phase, the retirement account earns 4.8% interest compounded monthly. Calculate Aisha's monthly retirement annuity payout.

These problems appear complicated. But each can be broken down into two smaller problems involving compound interest on savings or involving annuities. Often the problem involves a savings period followed by an annuity period. ; the accumulated value from first part of the problem may become a present value in the second part. Read each problem carefully to determine what is needed.

### Example 4.5.2

Suppose a baby, Aisha, is born and her grandparents invest $8000 in a college fund. The money remains invested for 18 years until Aisha enters college, and then is withdrawn in equal semiannual payments over the 4 years that Aisha expects to attend college. The college investment fund earns 5% interest compounded semiannually. How much money can Aisha withdraw from the account every six months while she is in college?

**Solution**

**Part 1: Accumulation of College Savings:** Find the accumulated value at the end of 18 years of a sum of $8000 invested at 5% compounded semiannually.

$$A = \$8000(1 + .05/2)^{(2\times 18)} = \$8000(1.025)^{36} = \$8000(2.432535)$$
$$A = \$19460.28$$

**Part 2: Seminannual annuity payout from savings to put toward college expenses.** Find the amount of the semiannual payout for four years using the accumulated savings from part 1 of the problem with an interest rate of 5% compounded semiannually.

$A$= \$19460.28 in Part 1 is the accumulated value at the end of the savings period. This becomes the present value $P$=\$19460.28 when calculating the semiannual payments in Part 2.

$$\$19460.28\left(1+\frac{.05}{2}\right)^{2\times 4}=\frac{m\left[\left(1+\frac{.05}{2}\right)^{2\times 4}-1\right]}{(.05/2)}$$
$$\$23710.46=m(8.73612)$$
$$m=\$2714.07$$

Aisha will be able to withdraw \$2714.07 semiannually for her college expenses.

**Example 4.5.3**

Aisha graduates college and starts a job. She saves \$1000 each quarter, depositing it into a retirement savings account. Suppose that Aisha saves for 30 years and then retires. At retirement she wants to withdraw money as an annuity that pays a constant amount every month for 25 years. During the savings phase, the retirement account earns 6% interest compounded quarterly. During the annuity payout phase, the retirement account earns 4.8% interest compounded monthly. Calculate Aisha's monthly retirement annuity payout.

**Solution**

**Part 1: Accumulation of Retirement Savings:** Find the accumulated value at the end of 30 years of \$1000 deposited at the end of each quarter into a retirement savings account earning 6% interest compounded quarterly.

$$A=\frac{\$1000\left[(1+.06/4)^{4\times 30}-1\right]}{(.06/4)}$$
$$A=\$331288.19$$

**Part 2: Monthly retirement annuity payout:** Find the amount of the monthly annuity payments for 25 years using the accumulated savings from part 1 of the problem with an interest rate of 4.8% compounded monthly.

$A$= \$331288.19 in Part 1 is the accumulated value at the end of the savings period. This amount will become the present value P =\$331288.19 when calculating the monthly retirement annuity payments in Part 2.

$$\$331288.19(1+.048/12)^{12\times 25}=\frac{m\left[(1+.048/12)^{12\times 25}-1\right]}{(.048/12)}$$
$$\$1097285.90=m(578.04483)$$
$$m=\$1898.27$$

Aisha will have a monthly retirement annuity income of \$1898.27 when she retires.

## FAIR MARKET VALUE OF A BOND

Whenever a business, and for that matter the U. S. government, needs to raise money it does it by selling bonds. A **bond** is a certificate of promise that states the terms of the agreement. Usually the business sells bonds for the **face amount** of \$1,000 each for a stated **term**, a period of time ending at a specified **maturity** date.

The person who buys the bond, the **bondholder**, pays \$1,000 to buy the bond.

The bondholder is promised two things: First that he will get his \$1,000 back at the maturity date, and second that he will receive a fixed amount of interest every six months.

As the market interest rates change, the price of the bond starts to fluctuate. The bonds are bought and sold in the market at their **fair market value**.

The interest rate a bond pays is fixed, but if the market interest rate goes up, the value of the bond drops since the money invested in the bond could earn more if invested elsewhere. When the value of the bond drops, we say it is trading at a **discount**.

On the other hand, if the market interest rate drops, the value of the bond goes up since the bond now yields a higher return than the market interest rate, and we say it is trading at a **premium**.

## Example 4.5.4

The Orange Computer Company needs to raise money to expand. It issues a 10-year $1,000 bond that pays $30 every six months. If the current market interest rate is 7%, what is the fair market value of the bond?

**Solution**

The bond certificate promises us two things - An amount of $1,000 to be paid in 10 years, and a semi-annual payment of $30 for ten years. Therefore, to find the fair market value of the bond, we need to find the present value of the lump sum of $1,000 we are to receive in 10 years, as well as, the present value of the $30 semi-annual payments for the 10 years.

We will let $P_1$ = the present value of the face amount of $1,000

$$P_1(1+.07/2)^{20} = \$1,000$$

Since the interest is paid twice a year, the interest is compounded twice a year and $nt$ = 2(10)=20

$$P_1(1.9898) = \$1,000$$
$$P_1 = \$502.56$$

We will let $P_2$ = the present value of the $30 semi-annual payments is

$$P_2(1+.07/2)^{20} = \frac{\$30\left[(1+.07/2)^{20}-1\right]}{(.07/2)}$$
$$P_2(1.9898) = 848.39$$
$$P_2 = \$426.37$$

The present value of the lump-sum $1,000 = $502.56

The present value of the $30 semi-annual payments = $426.37

The fair market value of the bond is P = $P_{1+}$ $P_2$ = $502.56 + $426.37 = $928.93

Note that because the market interest rate of 7% is higher than the bond's implied interest rate of 6% implied by the semiannual payments, the bond is selling at a discount; its fair market value of $928.93 is less than its face value of $1000.

## Example 4.5.5

A state issues a 15 year $1000 bond that pays $25 every six months. If the current market interest rate is 4%, what is the fair market value of the bond?

**Solution**

The bond certificate promises two things - an amount of $1,000 to be paid in 15 years, and semi-annual payments of $25 for 15 years. To find the fair market value of the bond, we find the present value of the $1,000 face value we are to receive in 15 years and add it to the present value of the $25 semi-annual payments for the 15 years. In this example, $nt = 2(15) = 30$.

We will let $P_1$ = the present value of the lump-sum $1,000

$$P_1(1+.04/2)^{30} = \$1,000$$
$$P_1 = \$552.07$$

We will let $P_2$ = the present value of the $25 semi-annual payments is

$$P_2(1+.04/2)^{30} = \frac{\$25\left[(1+.04/2)^{30}-1\right]}{(.04/2)}$$
$$P_2(1.18114) = \$1014.20$$
$$P_2 = \$559.90$$

The present value of the lump-sum $1,000 = $552.07

The present value of the $30 semi-annual payments = $559.90

Therefore, the fair market value of the bond is

$$P = P_1 + P_2 = \$552.07 + \$559.90 = \$1111.97$$

Because the market interest rate of 4% is lower than the interest rate of 5% implied by the semiannual payments, the bond is selling at a premium: the fair market value of $1,111.97 is more than the face value of $1,000.

To summarize:

**to find the Fair Market Value of a Bond**

Find the present value of the face amount $\mathrm{A}$ that is payable at the maturity date:

$$\mathbf{A} = \mathbf{P}_1(\mathbf{1} + \mathbf{r}/\mathbf{n})^{\mathbf{nt}};\ \text{solve to find}\ \mathrm{P}_1$$

Find the present value of the semiannually payments of $\$m$ over the term of the bond:

$$\mathbf{P}_2(\mathbf{1} + \mathbf{r}/\mathbf{n})^{\mathbf{nt}} = \frac{\mathbf{m}\left[(\mathbf{1} + \mathbf{r}/\mathbf{n})^{\mathbf{nt}} - \mathbf{1}\right]}{\mathbf{r}/\mathbf{n}} \quad ;\ \text{solve to find}\ \mathbf{P}_2$$

The fair market value (or present value or price or current value) of the bond is the sum of the present values calculated above:

$$\mathrm{P} = \mathbf{P}_1 + \mathbf{P}_2$$

## AMORTIZATION SCHEDULE FOR A LOAN

An amortization schedule is a table that lists all payments on a loan, splits them into the portion devoted to interest and the portion that is applied to repay principal, and calculates the outstanding balance on the loan after each payment is made.

**Example 4.5.6**

An amount of $500 is borrowed for 6 months at a rate of 12%. Make an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the portion of the payment contributing toward reducing the debt, and the outstanding balance.

**Solution**

The reader can verify that the monthly payment is $86.27.

The first month, the outstanding balance is $500, and therefore, the monthly interest on the outstanding balance is

(outstanding balance)(the monthly interest rate) = ($500)(.12/12) = $5

This means, the first month, out of the $86.27 payment, $5 goes toward the interest and the remaining $81.27 toward the balance leaving a new balance of $500 - $81.27 = $418.73.

Similarly, the second month, the outstanding balance is $418.73, and the monthly interest on the outstanding balance is ($418.73)(.12/12) = $4.19. Again, out of the $86.27 payment, $4.19 goes toward the interest and the remaining $82.08 toward the balance leaving a new balance of $418.73 - $82.08 = $336.65. The process continues in the table below.

| Payment # | Payment | Interest | Debt Payment | Balance |
|---|---|---|---|---|
| 1 | $86.27 | $5 | $81.27 | $418.73 |
| 2 | $86.27 | $4.19 | $82.08 | $336.65 |
| 3 | $86.27 | $3.37 | $82.90 | $253.75 |
| 4 | $86.27 | $2.54 | $83.73 | $170.02 |
| 5 | $86.27 | $1.70 | $84.57 | $85.45 |
| 6 | $86.27 | $0.85 | $85.42 | $0.03 |

> Note that the last balance of 3 cents is due to error in rounding off.

An amortization schedule is usually lengthy and tedious to calculate by hand. For example, an amortization schedule for a 30 year mortgage loan with monthly payments would have (12)(30)=360 rows of calculations in the amortization schedule table. A car loan with 5 years of monthly payments would have 12(5)=60 rows of calculations in the amortization schedule table. However it would be straightforward to use a spreadsheet application on a computer to do these repetitive calculations by inputting and copying formulas for the calculations into the cells.

Most of the other applications in this section's problem set are reasonably straightforward, and can be solved by taking a little extra care in interpreting them. And remember, there is often more than one way to solve a problem.

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# 4.5.1: Miscellaneous Application Problems (Exercises)

## SECTION 6.5 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS

For problems 1 - 4, assume a $200,000 house loan is amortized over 30 years at an interest rate of 5.4%.

| | |
|---|---|
| 1) Find the monthly payment. | 2) Find the balance owed after 20 years. |
| 3) Find the balance of the loan after 100 payments. | 4) Find the monthly payment if the original loan were amortized over 15 years. |

***6.5 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS***

| | |
|---|---|
| 5) Mr. Patel wants to pay off his car loan. The monthly payment for his car is $365, and he has 16 payments left. If the loan was financed at 6.5%, how much does he owe? | 6) An amount of $2000 is borrowed for a year at a rate of 7%. Make an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the portion of the payment going toward reducing the debt, and the balance. |
| 7) Fourteen months after Dan bought his new car he lost his job. His car was repossessed by his lender after he made only 14 monthly payments of $376 each. If the loan was financed over a 4-year period at an interest rate of 6.3%, how much did the car cost the lender? In other words, how much did Dan still owe on the car? | 8) You have a choice of either receiving $5,000 at the end of each year for the next 5 years or receiving $3000 per year for the next 10 years. If the current interest rate is 9%, which is better? |
| 9) Mr. Smith is planning to retire in 25 years and would like to have $250,000 then. What monthly payment made at the end of each month to an account that pays 6.5% will achieve his objective? | 10) Assume Mr. Smith has reached retirement and has $250,000 in an account which is earning 6.5%. He would now like to make equal monthly withdrawals for the next 15 years to completely deplete this account. Find the withdrawal payment. |
| 11) Mrs. Garcia is planning to retire in 20 years. She starts to save for retirement by depositing $2000 each quarter into a retirement investment account that earns 6% interest compounded quarterly. Find the accumulated value of her retirement savings at the end of 20 years. | 12) Assume Mrs. Garcia has reached retirement and has accumulated the amount found in question 13 in a retirement savings account. She would now like to make equal monthly withdrawals for the next 15 years to completely deplete this account. Find the withdrawal payment. Assume the account now pays 5.4% compounded monthly. |

***SECTION 6.5 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS***

| | |
|---|---|
| 13) A ten-year $1,000 bond pays $35 every six months. If the current interest rate is 8.2%, find the fair market value of the bond.<br>Hint: You must do the following.<br>a) Find the present value of $1000.<br>b) Find the present value of the $35 payments.<br>c) The fair market value of the bond = a + b | 14) Find the fair market value of the ten-year $1,000 bond that pays $35 every six months, if the current interest rate has dropped to 6%.<br>Hint: You must do the following.<br>a) Find the present value of $1000.<br>b) Find the present value of the $35 payments.<br>c) The fair market value of the bond = a + b |
| 15) A twenty-year $1,000 bond pays $30 every six months. If the current interest rate is 4.2%, find the fair market value of the bond.<br>Hint: You must do the following.<br>a) Find the present value of $1000.<br>b) Find the present value of the $30 payments.<br>c) The fair market value of the bond = a + b | 16) Find the fair market value of the twenty-year $1,000 bond that pays $30 every six months, if the current interest rate has increased to 7.5%. |

***SECTION 6.5 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS***

| | |
|---|---|
| 17) Mr. and Mrs. Nguyen deposit $10,000 into a college investment account when their new baby grandchild is born. The account earns 6.25% interest compounded quarterly.<br>a) When their grandchild reaches the age of 18, what is the accumulated value of the college investment account?<br>b) The Nguyen's grandchild has just reached the age of 18 and started college. If she is to withdraw the money in the college savings account n equal monthly payments over the next 4 years, how much money will be withdrawn each month? | 18) Mr. Singh is 38 and plans to retire at age 65. He opens a retirement savings account.<br>a) Mr. Singh wants to save enough money to accumulate $500,000 by the time he retires. The retirement investment account pays 7% interest compounded monthly. How much does he need to deposit each month to achieve this goal?<br>b) Mr. Singh has now reached at 65 and retires. How much money can he withdraw each month for 25 years if the retirement investment account now pays 5.2% interest, compounded monthly? |

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# 4.6: Classification of Finance Problems

**Learning Objectives**

In this section, you will review the concepts of chapter 6 to:

1. Re-examine the types of financial problems and classify them.
2. Re-examine the vocabulary words used in describing financial calculations

We'd like to remind the reader that the hardest part of solving a finance problem is determining the category it falls into. So in this section, we will emphasize the classification of problems rather than finding the actual solution.

We suggest that the student read each problem carefully and look for the word or words that may give clues to the kind of problem that is presented. For instance, students often fail to distinguish a lump-sum problem from an annuity. Since the payments are made each period, an annuity problem contains words such as each, every, per etc.. One should also be aware that in the case of a lump-sum, only a single deposit is made, while in an annuity numerous deposits are made at equal spaced time intervals. To help interpret the vocabulary used in the problems, we include a glossary at the end of this section.

Students often confuse the present value with the future value. For example, if a car costs $15,000, then this is its present value. Surely, you cannot convince the dealer to accept $15,000 in some future time, say, in five years. Recall how we found the installment payment for that car. We assumed that two people, Mr. Cash and Mr. Credit, were buying two identical cars both costing $15, 000 each. To settle the argument that both people should pay exactly the same amount, we put Mr. Cash's cash of $15,000 in the bank as a lump-sum and Mr. Credit's monthly payments of x dollars each as an annuity. Then we make sure that the future values of these two accounts are equal. As you remember, at an interest rate of 9%

$$\text{the future value of Mr. Cash's lump-sum was } \$15,000(1+.09/12)^{60}\text{, and}$$

$$\text{the future value of Mr. Credit's annuity was } \frac{x\left[(1+.09/12)^{60}-1\right]}{.09/12}.$$

To solve the problem, we set the two expressions equal and solve for $m$.

The present value of an annuity is found in exactly the same way. For example, suppose Mr. Credit is told that he can buy a particular car for $311.38 a month for five years, and Mr. Cash wants to know how much he needs to pay. We are finding the present value of the annuity of $311.38 per month, which is the same as finding the price of the car. This time our unknown quantity is the price of the car. Now suppose the price of the car is P, then

$$\text{the future value of Mr. Cash's lump-sum is } \mathrm{P}(1+.09/12)^{60}\text{, and}$$

$$\text{the future value of Mr. Credit's annuity is } \frac{\$311.38\left[(1+.09/12)^{60}-1\right]}{.09/12}.$$

Setting them equal we get,

$$\begin{aligned} P(1+.09/12)^{60} &= \frac{\$311.38\left[(1+.09/12)^{60}-1\right]}{.09/12} \\ P(1.5657) &= (\$311.38)(75.4241) \\ P(1.5657) &= \$23,485.57 \\ P &= \$15,000.04 \end{aligned}$$

## CLASSIFICATION OF PROBLEMS AND EQUATIONS FOR SolutionS

We now list six problems that form a basis for all finance problems.
Further, we classify these problems and give an equation for the solution.

**Example 4.6.1**

If $2,000 is invested at 7% compounded quarterly, what will the final amount be in 5 years?

***Classification:*** **Future (accumulated) Value of a Lump-sum** or FV of a lump-sum.

***Equation:***

$$\text{FV} = \text{A} = \$2000(1 + .07/4)^{20}$$

**Example 4.6.2**

How much should be invested at 8% compounded yearly, for the final amount to be $5,000 in five years?

***Classification:*** **Present Value of a Lump-sum** or PV of a lump-sum.

***Equation:***

$$\text{PV}(1 + .08)^5 = \$5,000$$

**Example 4.6.3**

If $200 is invested *each* month at 8.5% compounded monthly, what will the final amount be in 4 years?

***Classification:*** **Future (accumulated) Value of an Annuity** or FV of an annuity.

***Equation:***

$$\text{FV} = \text{A} = \frac{\$200\left[(1 + .085/12)^{48} - 1\right]}{.085/12}$$

**Example 4.6.4**

How much should be invested *each* month at 9% for it to accumulate to $8,000 in three years?

***Classification:*** **Sinking Fund Payment**

***Equation:***

$$\frac{m\left[(1 + .09/12)^{36} - 1\right]}{.09/12} = \$8,000$$

**Example 4.6.5**

Keith has won a lottery paying him $2,000 *per* month for the next 10 years. He'd rather have the entire sum now. If the interest rate is 7.6%, how much should he receive?

***Classification:*** **Present Value of an Annuity** or PV of an annuity.

***Equation:***

$$\text{PV}(1 + .076/12)^{120} = \frac{\$2000\left[(1 + .076/12)^{120} - 1\right]}{.076/12}$$

**Example 4.6.6**

Mr. A has just donated $25,000 to his alma mater. Mr. B would like to donate an equivalent amount, but would like to pay by monthly payments over a five year period. If the interest rate is 8.2%, determine the size of the monthly payment?

***Classification:*** **Installment Payment**.

***Equation:***

$$\frac{m\left[(1 + .082/12)^{60} - 1\right]}{.082/12} = \$25,000(1 + .082/12)^{60}$$

## GLOSSARY: VOCABULARY AND SYMBOLS USED IN FINANCIAL CALCULATIONS

As we've seen in these examples, it's important to read the problems carefully to correctly identify the situation. It is essential to understand to vocabulary for financial problems. Many of the vocabulary words used are listed in the glossary below for easy reference.

| | | |
|---|---|---|
| $t$ | Term | Time period for a loan or investment. In this book $t$ is represented in years and should be converted into years when it is stated in months or other units. |
| P | Principal | Principal is the amount of money borrowed in a loan.<br>If a sum of money is invested for a period of time, the sum invested at the start is the Principal. |
| P | Present Value | Value of money at the beginning of the time period. |
| A | Accumulated Value<br>Future Value | Value of money at the end of the time period |
| $D$ | Discount | In loans involving simple interest, a discount occurs if the interest is deducted from the loan amount at the beginning of the loan period, rather than being repaid at the end of the loan period. |
| $m$ | Periodic Payment | The amount of a constant periodic payment that occurs at regular intervals during the time period under consideration (examples: periodic payments made to repay a loan, regular periodic payments into a bank account as savings, regular periodic payment to a retired person as an annuity,) |
| $n$ | Number of payment periods and compounding periods per year | In this book, when we consider periodic payments, we will always have the compounding period be the same as the payment period.<br>In general the compounding and payment periods do not have to be the same, but the calculations are more complicated if they are different. If the periods differ, formulas for the calculations can be found in finance textbooks or various online resources. Calculations can easily be done using technology such as an online financial calculator, or financial functions in a spreadsheet, or a financial pocket calculator. |

| | | |
|---|---|---|
| $nt$ | Number of periods | $nt$ = (number of periods per year) $\times$(number of years)<br>$nt$ gives the total number of payment and compounding periods<br>In some situations we will calculate $nt$ as the multiplication shown above. In other situations the problem may state $nt$, such as a problem describing an investment of 18 months duration compounded monthly. In this example: $nt$ = 18 months and $n$ = 12; then $t$ = 1.5 years but $t$ is not stated explicitly in the problem. The TI-84+ calculators built in TVM solver uses $N = nt$. |
| $r$ | Annual interest rate<br>Nominal rate | The stated annual interest rate. This is stated as a percent but converted to decimal form when using financial calculation formulas.<br>If a bank account pays 3% interest compounded quarterly, then 3% is the nominal rate, and it is included in the financial formulas as $r$ = 0.03 |
| $r/n$ | Interest rate per compounding period | If a bank account pays 3% interest compounded quarterly, then $r/n$ = 0.03/4 = 0. 075, corresponding to a rate of 0.75% per quarter. Some Finite Math books use the symbol $i$ to represent $r/n$ |

| | | |
|---|---|---|
| $r_{EFF}$ | Effective Rate<br>Effective Annual Interest Rate<br>APY Annual Percentage Yield<br>APR Annual Percentage Rate | The effective rate is the interest rate compounded annually that would give the same interest rate as the compounded rate stated for the investment.<br>The effective rate provides a uniform way for investors or borrowers to compare different interest rates with different compounding periods. |
| $I$ | Interest | Money paid by a borrower for the use of money borrowed as a loan.<br>Money earned over time when depositing money into a savings account, certificate of deposit, or money market account. When a person deposits money in a bank account, the person depositing the funds is essentially temporarily lending the money to the bank and the bank pays interest to the depositor. |

| | | |
|---|---|---|
| | Sinking Fund | A fund set up by making payments over a period of time into a savings or investment account in order to save to fund a future purchase. Businesses use sinking funds to save for a future purchase of equipment at the end of the savings period by making periodic installment payments into a sinking fund. |
| | Annuity | An annuity is a stream of periodic payments. In this book it refers to a stream of constant periodic payments made at the end of each compounding period for a specific amount of time.<br>In common use the term annuity generally refers to a constant stream of periodic payments received by a person as retirement income, such as from a pension.<br>Annuity payments in general may be made at the end of each payment period (ordinary annuity) or at the start of each period (annuity due).<br>The compounding periods and payment periods do not need to be equal, but in this textbook we only consider situations when these periods are equal. |
| | Lump Sum | A single sum of money paid or deposited at one time, rather than being spread out over time.<br>An example is lottery winnings if the recipient chooses to receive a single “lump sum” one-time payment, instead of periodic payments over a period of time or as.<br>Use of the word lump sum indicates that this is a one time transaction and is not a stream of periodic payments. |
| | Loan | An amount of money that is borrowed with the understanding that the borrower needs to repay the loan to the lender in the future by the end of a period of time that is called the term of the loan.<br>The repayment is most often accomplished through periodic payments until the loan has been completely repaid over the term of the loan.<br>However there are also loans that can be repaid as a single sum at the end of the term of the loan, with interest paid either periodically over the term or in a lump sum at the end of the loan or as a discount at the start of the loan. |

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# 4.6.1: Classification of Finance Problems (Exercises)

## SECTION 6.6 PROBLEM SET: CLASSIFICATION OF FINANCE PROBLEMS

Let the letters A, B, C, D, E and F be represented as follows:

$$A = FV \text{ of a lump-sum} \quad C = FV \text{ of an annuity} \quad E = \text{ Installment payment}$$
$$B = PV \text{ of a lump-sum} \quad D = \text{ sinking fund payment} \quad F = PV \text{ of an annuity}$$

Classify each by writing the appropriate letter in the box, and write an equation for solution.

1) What monthly deposits made to an account paying 9% will grow to $10,000 in 4 years?

2) An amount of $4000 is invested at 6% compounded daily. What will the final amount be in 5 years?

3) David has won a lottery paying him $10,000 per month for the next 20 years. He'd rather have the whole amount in one lump sum now. If the current interest rate is 7%, how much money can he hope to get?

4) Each month Linda deposits $250 in an account that pays 9%. How much money will she have in 4 years?

5) Find the monthly payment for a $15,000 car if the loan is amortized over 4 years at a rate of 10%.

6) What lump-sum deposited in an account paying 7% compounded daily will grow to $10,000 in 5 years?

7) What amount of quarterly payments will amount to $250,000 in 5 years at a rate of 8%?

8) The Chang family bought their house 25 years ago. They had their loan financed for 30 years at an interest rate of 11% resulting in a payment of $1350 a month. Find the balance of the loan.

A 10-year $1000 bond pays $35 every six months. If the current interest rate is 8%, in order to find the fair market value of the bond, we need to find the following.

9) The present value of $1000.

10) The present value of the $35 per six month payments.

***SECTION 6.6 PROBLEM SET: CLASSIFICATION OF FINANCE PROBLEMS***

$$A = FV \text{ of a lump-sum} \quad C = FV \text{ of an annuity} \quad E = \text{ Installment payment}$$
$$B = PV \text{ of a lump-sum} \quad D = \text{ sinking fund payment} \quad F = PV \text{ of an annuity}$$

11) What lump-sum deposit made today is equal to 33 monthly deposits of $500 if the interest rate is 8%?

12) What monthly deposits made to an account paying 10% will accumulated to $10,000 in six years?

13) A department store charges a finance charge of 1.5% per month on the outstanding balance.
If Ned charged $400 three months ago and has not paid his bill, how much does he owe?

14) What will the value of $300 monthly deposits be in 10 years if the account pays 12% compounded monthly?

15) What lump-sum deposited at 6% compounded daily will grow to $2000 in three years?

16) A company buys an apartment complex for $5,000,000 and amortizes the loan over 10 years.
What is the yearly payment if the interest rate is 14%?

17) In 2002, a house in Rock City cost $300,000. Real estate in Rock City has been increasing in value at the annual rate of 5.3%.. Find the price of that house in 2016.

18) You determine that you can afford to pay $400 per month for a car. What is the maximum price you can pay for a car if the interest rate is 11% and you want to repay the loan in 4 years?

19) A business needs $350,000 in 5 years. How much lump-sum should be put aside in an account that pays 9% so that five years from now the company will have $350,000?

20) A person wishes to have $500,000 in a pension fund 20 years from now. How much should he deposit each month in an account paying 9% compounded monthly?

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# 4.7: Chapter Review

## SECTION 6.7 PROBLEM SET: CHAPTER REVIEW

1. Manuel borrows $800 for 6 months at 18% simple interest. How much does he owe at the end of 6 months?
2. The population of a city is 65,000 and expects to grow at a rate of 2.3% per year for the next 10 years. What will the population of this city be in 10 years?
3. The Gill family is buying a $250,000 house with a 10% down payment. If the loan is financed over a 30 year period at an interest rate of 4.8%, what is the monthly payment?
4. Find the monthly payment for the house in the above problem if the loan was amortized over 15 years.
5. You look at your budget and decide that you can afford $250 per month for a car. What is the maximum amount you can afford to pay for the car if the interest rate is 8.6% and you want to finance the loan over 5 years?
6. Mr. Nakahama bought his house in the year 1998. He had his loan financed for 30 years at an interest rate of 6.2% resulting in a monthly payment of $1500. In 2015, 17 years later, he paid off the balance of the loan. How much did he pay?
7. Lisa buys a car for $16,500, and receives $2400 for her old car as a trade-in value. Find the monthly payment for the balance if the loan is amortized over 5 years at 8.5%.
8. A car is sold for $3000 cash down and $400 per month for the next 4 years. Find the cash value of the car today if the money is worth 8.3% compounded monthly.
9. An amount of $2300 is borrowed for 7 months at a simple interest rate of 16%. Find the discount and the proceeds.
10. Marcus has won a lottery paying him $5000 per month for the next 25 years. He'd rather have the whole amount in one lump sum today. If the current interest rate is 7.3%, how much money can he hope to get?
11. In the year 2000, an average house in Star City cost $250,000. If the average annual inflation rate for the past years has been about 4.7%, what was the price of that house in 2015?
12. Find the 'fair market' value of a ten-year $1000 bond which pays $30 every six months if the current interest rate is 7%. What if the current interest rate is 5%?
13. A Visa credit card company has a finance charge of 1.5% per month (18% per year) on the outstanding balance. John owed $3200 and has been delinquent for 5 months. How much total does he owe, now?
14. You want to purchase a home for $200,000 with a 30-year mortgage at 9.24% interest. Find
    a. the monthly payment
    b. the balance owed after 20 years.
15. When Jose bought his car, he amortized his loan over 6 years at a rate of 9.2%, and his monthly payment came out to be $350 per month. He has been making these payments for the past 40 months and now wants to pay off the remaining balance. How much does he owe?
16. A lottery pays $10,000 per month for the next 20 years. If the interest rate is 7.8%, find both its present and future values.
17. A corporation estimates it will need $300,000 in 8 years to replace its existing machinery. How much should it deposit each quarter in a sinking fund earning 8.4% compounded quarterly to meet this obligation?
18. Our national debt in 1992 was about $4 trillion. If the annual interest rate was 7% then, what was the daily interest on the national debt?
19. A business must raise $400,000 in 10 years. What should be the size of the owners' monthly payments to a sinking fund paying 6.5% compounded monthly?
20. The population of a city of 80,000 is growing at a rate of 3.2% per year. What will the population be at the end of 10 years?
21. A sum of $5000 is deposited in a bank today. What will the final amount be in 20 months if the bank pays 9% and the interest is compounded monthly?
22. A manufacturing company buys a machine for $500 cash and $50 per month for the next 3 years. Find the cash value of the machine today if the money is worth 6.2% compounded monthly.
23. The United States paid about 4 cents an acre for the Louisiana Purchase in 1803. Suppose the value of this property grew at a rate of 5.5% annually. What would an acre be worth in the year 2000?
24. What amount should be invested per month at 9.1% compounded monthly so that it will become $5000 in 17 months?
25. A machine costs $8000 and has a life of 5 years. It can be leased for $160 per month for 5 years with a cash down payment of $750. The current interest rate is 8.3%. Is it cheaper to lease or to buy?
26. If inflation holds at 5.2% per year for 5 years, what will be the cost in 5 years of a car that costs $16,000 today? How much will you need to deposit each quarter in a sinking fund earning 8.7% per year to purchase the new car in 5 years?

27. City Bank pays an interest rate of 6%, while Western Bank pays 5.8% compounded continuously. Which one is a better deal?
28. Ali has inherited $20,000 and is planning to invest this amount at 7.9% interest. At the same time he wishes to make equal monthly withdrawals to use up the entire sum in 5 years. How much can he withdraw each month?
29. Jason has a choice of receiving $300 per month for the next 5 years or $500 per month for the next 3 years. Which one is worth more if the current interest rate is 7.7%?
30. If a bank pays 6.8% compounded continuously, how long will it take to double your money?
31. A mutual fund claims a growth rate of 8.3% per year. If $500 per month is invested, what will the final amount be in 15 years?
32. Mr. Vasquez has been given two choices for his compensation. He can have $20,000 cash plus $500 per month for 10 years, or he can receive $12,000 cash plus $1000 per month for 5 years. If the interest rate is 8%, which is the better offer?
33. How much should Mr. Shackley deposit in a trust account so that his daughter can withdraw $400 per month for 4 years if the interest rate is 8%?
34. Mr. Albers borrowed $425,000 from the bank for his new house at an interest rate of 4.7%. He will make equal monthly payments for the next 30 years. How much money will he end up paying the bank over the life of the loan, and how much is the interest?
35. Mr. Tong puts away $500 per month for 10 years in an account that earns 9.3%. After 10 years, he decides to withdraw $1,000 per month. If the interest rate stays the same, how long will it take Mr. Tong to deplete the account?
36. An amount of $5000 is borrowed for 15 months at an interest rate of 9%. Find the monthly payment and construct an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the amount of payment contributing towards debt, and the outstanding debt.

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# CHAPTER OVERVIEW

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# 5.1: Introduction

What is logic? Logic is the study of reasoning, and it has applications in many fields, including philosophy, law, psychology, digital electronics, and computer science.

In law, constructing a well-reasoned, logical argument is extremely important. The main goal of arguments made by lawyers is to convince a judge and jury that their arguments are valid and well-supported by the facts of the case, so the case should be ruled in their favor. Think about Thurgood Marshall arguing for desegregation in front of the U.S. Supreme Court during the *Brown v. Board of Education of Topeka* lawsuit in 1954, or Ruth Bader Ginsburg arguing for equality in social security benefits for both men and women under the law during the mid-1970s. Both these great minds were known for the preparation and thoroughness of their logical legal arguments, which resulted in victories that advanced the causes they fought for. Thurgood Marshall and Ruth Bader Ginsburg would later become well respected justices on the U.S. Supreme Court themselves.



Figure 5.1.1: Logic is key to a well-reasoned argument, in both math and law. (credit: modification of work "Lady Justicia holding sword and scale bronze figurine with judge hammer on wooden table" by Jernej Furman/Flickr, CC BY 2.0)

In this chapter, we will explore how to construct well-reasoned logical arguments using varying structures. Your ability to form and comprehend logical arguments is a valuable tool in many areas of life, whether you're planning a dinner date, negotiating the purchase of a new car, or persuading your boss that you deserve a raise.

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# 5.2: Statements and Quantifiers



Figure 5.2.1Construction of a logical argument, like that of a house, requires you to begin with the right parts. (credit: modification of work “Barn Raising” by Robert Stinnett/Flickr, CC BY 2.0)

## Learning Objectives

After completing this section, you should be able to:

1. Identify logical statements.
2. Represent statements in symbolic form.
3. Negate statements in words.
4. Negate statements symbolically.
5. Translate negations between words and symbols.
6. Express statements with quantifiers of all, some, and none.
7. Negate statements containing quantifiers of all, some, and none.

Have you ever built a club house, tree house, or fort with your friends? If so, you and your friends likely started by gathering some tools and supplies to work with, such as hammers, saws, screwdrivers, wood, nails, and screws. Hopefully, at least one member of your group had some knowledge of how to use the tools correctly and helped to direct the construction project. After all, if your house isn't built on a strong foundation, it will be weak and could possibly fall apart during the next big storm. This same foundation is important in logic.

In this section, we will begin with the parts that make up all logical arguments. The building block of any logical argument is a logical statement, or simply a statement. A logical statement has the form of a complete sentence, and it must make a claim that can be identified as being true or false.

When making arguments, sometimes people make false claims. When evaluating the strength or validity of a logical argument, you must also consider the truth values, or the identification of true or false, of all the statements used to support the argument. While a false statement is still considered a logical statement, a strong logical argument starts with true statements.

## Identifying Logical Statements



Figure 5.2.2Not all roses are red. (credit: “assorted pink yellow white red roses macro” by ProFlowers/Flickr, CC BY 2.0)

An example of logical statement with a false truth value is, “All roses are red.” It is a logical statement because it has the form of a complete sentence and makes a claim that can be determined to be either true or false. It is a false statement because not all roses are red: some roses are red, but there are also roses that are pink, yellow, and white. Requests, questions, or directives may be complete sentences, but they are not logical statements because they cannot be determined to be true or false. For example, suppose someone said to you, “Please, sit down over there.” This request does not make a claim and it cannot be identified as true or false; therefore, it is not a logical statement.

### Example 5.2.1

#### Identifying Logical Statements

Determine whether each of the following sentences represents a logical statement. If it is a logical statement, determine whether it is true or false.

1. Tiger Woods won the Master’s golf championship at least five times.
2. Please, sit down over there.
3. All cats dislike dogs.

**Answer**

1. This is a logical statement because it is a complete sentence that makes a claim that can be identified as being true or false. As of 2021, this statement is true: Tiger Woods won the Master’s in 1997, 2001, 2002, 2005 and 2019.
2. This is not a logical statement because, although it is a complete sentence, this request does not make a claim that can be identified as being either true or false.

3. This is a logical statement because it is a complete sentence that makes a claim that can be identified as being true or false. This statement is false because some cats do like some dogs.

**Your Turn 5.2.1**

Determine whether each of the following sentences represents a logical statement. If it is a logical statement, determine whether it is true or false.

1. The Buffalo Bills defeated the New York Giants in Super Bowl XXV.
2. Michael Jackson's album Thriller was released in 1982.
3. Would you like some coffee or tea?

## Representing Statements in Symbolic Form

When analyzing logical arguments that are made of multiple logical statements, symbolic form is used to reduce the amount of writing involved. Symbolic form also helps visualize the relationship between the statements in a more concise way in order to determine the strength or validity of an argument. Each logical statement is represented symbolically as a single lowercase letter, usually starting with the letter $p$.

To begin, you will practice how to write a single logical statement in symbolic form. This skill will become more useful as you work with compound statements in later sections.

**Example 5.2.1**

### Representing Statements Using Symbolic Form

Write each of the following logical statements in symbolic form.

1. Barry Bonds holds the Major League Baseball record for total career home runs.
2. Some mammals live in the ocean.
3. Ruth Bader Ginsburg served on the U.S. Supreme Court from 1993 to 2020.

**Answer**

1. $p$: Barry Bonds holds the Major League Baseball record for total career home runs. The statement is labeled with a $p$. Once the statement is labeled, use $p$ as a replacement for the full written statement to write and analyze the argument symbolically.
2. $q$: Some mammals live in the ocean. The letter $q$ is used to distinguish this statement from the statement in question 1, but any lower-case letter may be used.
3. $r$: Ruth Bader Ginsburg served on the U.S. Supreme Court from 1993 to 2020. When multiple statements are present in later sections, you will want to be sure to use a different letter for each separate logical statement.

**Your Turn 5.2.2**

Write each of the following logical statements in symbolic form.

1. The movie *Gandhi* won the Academy Award for Best Picture in 1982.
2. Soccer is the most popular sport in the world.
3. All oranges are citrus fruits.

**Who Knew?**

Mathematics is not the only language to use symbols to represent thoughts or ideas. The Chinese and Japanese languages use symbols known as Hanzi and Kanji, respectively, to represent words and phrases. At one point, the American musician Prince famously changed his name to a symbol representing love.

BBC Prince Symbol Article

### Negating Statements

Consider the false statement introduced earlier, “All roses are red.” If someone said to you, “All roses are red,” you might respond with, “Some roses are not red.” You could then strengthen your argument by providing additional statements, such as, “There are also white roses, yellow roses, and pink roses, to name a few.”

The negation of a logical statement has the opposite truth value of the original statement. If the original statement is false, its negation is true, and if the original statement is true, its negation is false. Most logical statements can be negated by simply adding or removing the word *not*. For example, consider the statement, “Emma Stone has green eyes.” The negation of this statement would be, “Emma Stone does not have green eyes.” The table below gives some other examples.

Table 2.1

| Logical Statement | Negation |
|---|---|
| Gordon Ramsey is a chef. | Gordon Ramsey is not a chef. |
| Tony the Tiger does not have spots. | Tony the Tiger has spots. |

The way you phrase your argument can impact its success. If someone presents you with a false statement, the ability to rebut that statement with its negation will provide you with the tools necessary to emphasize the correctness of your position.

**Example 5.2.3**

### Negating Logical Statements

Write the negation of each logical statement in words.

1. Michael Phelps was an Olympic swimmer.
2. Tom is a cat.
3. Jerry is not a mouse.

**Answer**

1. Add the word *not* to negate the affirmative statement: “Michael Phelps was not an Olympic swimmer.”
2. Add the word *not* to negate the affirmative statement: “Tom is not a cat.”
3. Remove the word *not* to negate the negative statement: “Jerry is a mouse.”

**Your Turn 5.2.3**

Write the negation of each logical statement in words.

1. Ted Cruz was not born in Texas.
2. Adele has a beautiful singing voice.
3. Leaves convert carbon dioxide to oxygen during the process of photosynthesis.

### Negating Logical Statements Symbolically

The symbol for negation, or not, in logic is the tilde, ~. So, not $p$ is represented as $\sim p$. To negate a statement symbolically, remove or add a tilde. The negation of not (not $p$) is $p$. Symbolically, this equation is $\sim(\sim p) = p$.

**Example 5.2.4**

### Negating Logical Statements Symbolically

Write the negation of each logical statement symbolically.

1. $p$: Michael Phelps was an Olympic swimmer.
2. $r$: Tom is not a cat.
3. $\sim q$: Jerry is not a mouse.

**Answer**

1. To negate an affirmative logical statement symbolically, add a tilde: $\sim p$.
2. Because the symbol for this statement is $r$, its negation is $\sim r$.
3. The symbol for this statement is $\sim q$, so to negate it we simply remove the $\sim$, because $\sim(\sim q) = q$. The answer is $q$.

**Your Turn 5.2.4**

Write the negation of each logical statement symbolically.

1. Ted Cruz was not born in Texas.
2. Adele has a beautiful singing voice.
3. Leaves convert carbon dioxide to oxygen during the process of photosynthesis.

### Translating Negations Between Words and Symbols

In order to analyze logical arguments, it is important to be able to translate between the symbolic and written forms of logical statements.

**Example 5.2.5**

### Translating Negations Between Words and Symbols

Given the statements:

$r$: Elmo is a red Muppet.

$p$: Ketchup is not a vegetable.

1. Write the symbolic form of the statement, “Elmo is not a red Muppet.”
2. Translate the statement $\sim p$ into words.

**Answer**

1. “Elmo is not a red muppet” is the negation of “Elmo is a red muppet,” which is represented as $r$. Thus, we would write “Elmo is not a red muppet” symbolically as $\sim r$.
2. Because $p$ is the symbol representing the statement, “Ketchup is not a vegetable,” $\sim p$ is equivalent to the statement, “Ketchup is a vegetable.”

### Your Turn 5.2.5

Given the statements:
r: Woody and Buzz Lightyear are best friends.
~p: Wonder Woman is not stronger than Captain Marvel.

Write the symbolic form of the statement, "Wonder Woman is stronger than Captain Marvel."

Translate the statement ~r into words.

## Expressing Statements with Quantifiers of All, Some, or None

A quantifier is a term that expresses a numerical relationship between two sets or categories. For example, all squares are also rectangles, but only some rectangles are squares, and no squares are circles. In this example, *all*, *some*, and *none* are quantifiers. In a logical argument, the logical statements made to support the argument are called premises, and the judgment made based on the premises is called the conclusion. Logical arguments that begin with specific premises and attempt to draw more general conclusions are called inductive arguments.

Consider, for example, a parent walking with their three-year-old child. The child sees a cardinal fly by and points it out. As they continue on their walk, the child notices a robin land on top of a tree and a duck flying across to land on a pond. The child recognizes that cardinals, robins, and ducks are all birds, then excitedly declares, "All birds fly!" The child has just made an inductive argument. They noticed that three different specific types of birds all fly, then synthesized this information to draw the more general conclusion that all birds can fly. In this case, the child's conclusion is false.

The specific premises of the child's argument can be paraphrased by the following statements:

- Premise: Cardinals are birds that fly.
- Premise: Robins are birds that fly.
- Premise: Ducks are birds that fly.

The general conclusion is: “All birds fly!”

All inductive arguments should include at least three specific premises to establish a pattern that supports the general conclusion. To counter the conclusion of an inductive argument, it is necessary to provide a counter example. The parent can tell the child about penguins or emus to explain why that conclusion is false.

On the other hand, it is usually impossible to prove that an inductive argument is true. So, inductive arguments are considered either strong or weak. Deciding whether an inductive argument is strong or weak is highly subjective and often determined by the background knowledge of the person making the judgment. Most hypotheses put forth by scientists using what is called the “scientific method” to conduct experiments are based on inductive reasoning.

In the following example, we will use quantifiers to express the conclusion of a few inductive arguments.

### Example 5.2.6

#### Drawing General Conclusions to Inductive Arguments Using Quantifiers

For each series of premises, draw a logical conclusion to the argument that includes one of the following quantifiers: all, some, or none.

1. Squares and rectangles have four sides. A square is a parallelogram, and a rectangle is a parallelogram. What conclusion can be drawn from these premises?
2. $1+2=3,\ 6+7=13,$ and $23+24=47$. Of these, 1 and 2, 6 and 7, and 23 and 24 are consecutive integers; 3, 13, and 47 are odd numbers. What conclusion can be drawn from these premises?
3. Sea urchins live in the ocean, and they do not breathe air. Sharks live in the ocean, and they do not breathe air. Eels live in the ocean, and they do not breath air. What conclusion can be drawn from these premises?

**Answer**

1. The conclusion you would likely come to here is “Some four-sided figures are parallelograms.” However, it would be incorrect to say that all four-sided figures are parallelograms because there are some four-sided figures, such as trapezoids, that are not parallelograms. This is a false conclusion.
2. From these premises, you may draw the conclusion “All sums of two consecutive counting numbers result in an odd number.” Most inductive arguments cannot be proven true, but several mathematical properties can be. If we let $n$ represent our first counting number, then $n+1$ would be the next counting number and $n+(n+1)=2n+1$ . Because 2n2n is divisible by two, it is an even number, and if you add one to any even number the result is always an odd number. Thus, the conclusion is true!
3. Based on the premises provided, with no additional knowledge about whales or dolphins, you might conclude “No creatures that live in the ocean breathe air.” Even though this conclusion is false, it still follows from the known premises and thus is a logical conclusion based on the evidence presented. Alternatively, you could conclude “Some creatures that live in the ocean do not breathe air.” The quantifier you choose to write your conclusion with may vary from another person’s based on how persuasive the argument is. There may be multiple acceptable answers.

### Your Turn 5.2.6

For each series of premises, draw a logical conclusion to the argument that includes one of the following quantifiers: all, some, or none.

1 + 2 = 3, 5 + 6 = 11, and 14 + 15 = 29. Of these, 1 and 2 are consecutive integers, 5 and 6 are consecutive integers, and 14 and 15 are consecutive integers. Also, their sums, 3, 11, and 29 are all prime numbers. Prime numbers are positive integers greater than one that are only divisible by one and the number itself. What conclusion can you draw from these premises?

A robin is a bird that lays blue eggs. A chicken is a bird that typically lays white and brown eggs. An ostrich is a bird that lays exceptionally large eggs. If a bird lays eggs, then they do not give live birth to their young. What conclusion can you draw from these premises?

All parallelograms have four sides. All rectangles are parallelograms. All squares are rectangles. What additional conclusion can you make about squares from these premises?

### Checkpoint

*It is not possible to prove definitively that an inductive argument is true or false in most cases.*

### Video

Logic Part 1A: Logic Statements and Quantifiers

## Check Your Understanding

1. A _____ _____ is a complete sentence that makes a claim that may be either true or false.

2. The _____ of a logical statement has the opposite truth value of the original statement.

3. If /**/p/**/ represents the logical statement, “Marigolds are yellow flowers,” then _____ represents the statement, “Marigolds are not yellow flowers.”

4. The statement /**/\text{~}(\text{~}p)/**/ has the same truth value as the statement _____.

5. The logical statements used to support the conclusion of an argument are called _____.

6. _____ arguments attempt to draw a general conclusion from specific premises.

7. All, some, and none are examples of _____, words that assign a numerical relationship between two or more groups.

8. The negation of the statement, “All giraffes are tall,” is _____.

## Section 2.1 Exercises

For the following exercises, determine whether the sentence represents a logical statement. If it is a logical statement, determine whether it is true or false.

1. A loan used to finance a house is called a mortgage.
2. All odd numbers are divisible by 2.
3. Please, bring me that notebook.
4. Robot, what’s your function?
5. In English, a conjunction is a word that connects two phrases or parts of a sentence together.
6. 8 - 3 = 5.
7. 7 + 3 = 11.
8. What is 7 plus 3?

For the following exercises, write each statement in symbolic form.

9. Grammy award winning singer, Lady Gaga, was not born in Houston, Texas.
10. Bruno Mars performed during the Super Bowl halftime show twice.
11. Coco Chanel said, “The most courageous act is still to think for yourself. Aloud.”
12. Bruce Wayne is not Superman.

For the following exercises, write the negation of each statement in words.

13. Bozo is not a clown.
14. Ash is Pikachu’s trainer and friend.
15. Vanilla is the most popular flavor of ice cream.
16. Smaug is a fire breathing dragon.
17. Elephant and Piggy are not best friends.
18. Some dogs like cats.
19. Some donuts are not round.
20. All cars have wheels.
21. No circles are squares.
22. Nature’s first green is not gold.
23. The ancient Greek philosopher Plato said, “The greatest wealth is to live content with little.”
24. All trees produce nuts.

For the following exercises, write the negation of each statement symbolically and in words.

25. p: Their hair is red.
26. ~q: My favorite superhero does not wear a cape.
27. Wolves howl at the moon.
28. ~u: I do not love New York.
29. ~v: Some cats are not tigers.
30. t: Nobody messes with Texas.
31. ~q: No squares are not parallelograms.
32. ~p: The President does not like broccoli.

For the following exercises, write each of the following symbolic statements in words.

33. Given: p: Kermit is a green frog; translate ~p into words.
34. Given: ~r: Mick Jagger is not the lead singer for The Rolling Stones; translate r into words.
35. Given: q: All dogs go to heaven; translate ~q into words.
36. Given: ~s: Some pizza does not come with pepperoni on it; translate s into words.
37. Given: ~p: No pizza comes with pineapple on it; translate ~~p into words.
38. Given: r: Not all roses are red; translate ~~r into words.

39. Given: ~t: Thelonious Monk is not a famous jazz pianist; translate ~~t into words.
40. Given: ~v: Not all violets are blue; translate ~~v into words.

For the following exercises, draw a logical conclusion from the premises that includes one of the following quantifiers: all, some, or none.

41. The Ford Motor Company builds cars in Michigan. General Motors builds cars in Michigan. Chrysler builds cars in Michigan. What conclusion can be drawn from these premises?
42. Michelangelo Buonarroti was a great Renaissance artist from Italy. Raphael Sanzio was a great Renaissance artist from Italy. Sandro Botticelli was a great Renaissance artist from Italy. What conclusion can you draw from these premises?
43. Four is an even number and it is divisible by 2. Six is an even number and it is divisible by 2. Eight is an even number and it is divisible by 2. What conclusion can you draw from these premises?
44. Three is an odd number and it is not divisible by 2. Seven is an odd number and it is not divisible by 2. Twenty-one is an odd number and it is not divisible by 2. What conclusion can you draw from these premises?
45. The odd number 5 is not divisible by 3. The odd number 7 is not divisible by 3. The odd number 29 is not divisible by 3. What conclusion can you draw from these premises?
46. Penguins are flightless birds. Emus are flightless birds. Ostriches are flightless birds. What conclusion can you draw from these premises?
47. Plants need water to survive. Animals need water to survive. Bacteria need water to survive. What conclusion can you draw from these premises?
48. A chocolate chip cookie is not sour. An oatmeal cookie is not sour. An Oreo cookie is not sour. What conclusion can you draw from these premises?

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# 5.3: Compound Statements



Figure 5.3.1A person seeking their driver's license must pass two tests. A compound statement can be used to explain performance on both tests at once. (credit: modification of work “Drivers License -Teen driver” by State Farm/Flickr, CC BY 2.0)

## Learning Objectives

After completing this section, you should be able to:

1. Translate compound statements into symbolic form.
2. Translate compound statements in symbolic form with parentheses into words.
3. Apply the dominance of connectives.

Suppose your friend is trying to get a license to drive. In most places, they will need to pass some form of written test proving their knowledge of the laws and rules for driving safely. After passing the written test, your friend must also pass a road test to demonstrate that they can perform the physical task of driving safely within the rules of the law.

Consider the statement: "My friend passed the written test, but they did not pass the road test." This is an example of a compound statement, a statement formed by using a connective to join two independent clauses or logical statements. The statement, “My friend passed the written test,” is an independent clause because it is a complete thought or idea that can stand on its own. The second independent clause in this compound statement is, “My friend did not pass the road test.” The word "but" is the connective used to join these two statements together, forming a compound statement. So, did your friend acquire their driving license?.

This section introduces common logical connectives and their symbols, and allows you to practice translating compound statements between words and symbols. It also explores the order of operations, or dominance of connectives, when a single compound statement uses multiple connectives.

## Common Logical Connectives

Understanding the following logical connectives, along with their properties, symbols, and names, will be key to applying the topics presented in this chapter. The chapter will discuss each connective introduced here in more detail.

The joining of two logical statements with the word "and" or "but" forms a compound statement called a conjunction. In logic, for a conjunction to be true, all the independent logical statements that make it up must be true. The symbol for a conjunction is $\wedge$. Consider the compound statement, “Derek Jeter played professional baseball for the New York Yankees, and he was a shortstop.” If $p$ represents the statement, “Derrick Jeter played professional baseball for the New York Yankees,” and if $q$ represents the statement, “Derrick Jeter was a short stop,” then the conjunction will be written symbolically as $p \wedge q$.

The joining of two logical statements with the word “or” forms a compound statement called a disjunction. Unless otherwise specified, a disjunction is an inclusive *or* statement, which means the compound statement formed by joining two independent clauses with the word *or* will be true if a least one of the clauses is true. Consider the compound statement, "The office manager ordered cake for for an employee’s birthday or they ordered ice cream.” This is a disjunction because it combines the independent clause, “The office manager ordered cake for an employee’s birthday,” with the independent clause, “The office manager ordered ice cream,” using the connective, *or*. This disjunction is true if the office manager ordered only cake, only ice cream, or they ordered both cake and ice cream. Inclusive *or* means you can have one, or the other, or both!

Joining two logical statements with the word *implies*, or using the phrase “if *first statement*, then *second statement*,” is called a conditional or *implication*. The clause associated with the "if" statement is also called the hypothesis or *antecedent*, while the clause following the "then" statement or the word *implies* is called the **conclusion** or *consequent*. The conditional statement is like a one-way contract or promise. The only time the conditional statement is false, is if the hypothesis is true and the conclusion is false. Consider the following conditional statement, “If Pedro does his homework, then he can play video games.” The hypothesis/antecedent is the statement following the word *if*, which is “Pedro does/did his homework.” The conclusion/consequent is the statement following the word *then*, which is “Pedro can play his video games.”

Joining two logical statements with the connective phrase “if and only if” is called a biconditional. The connective phrase "if and only if" is represented by the symbol, $\leftrightarrow$. In the biconditional statement, $p \leftrightarrow q$, $p$ is called the hypothesis or antecedent and $q$ is called the conclusion or consequent. For a biconditional statement to be true, the truth values of $p$ and $q$ must match. They must both be true, or both be false.

The table below lists the most common connectives used in logic, along with their symbolic forms, and the common names used to describe each connective.

Table 5.3.1

| Connective | Symbol | Name |
|---|---|---|
| and<br>but | $\wedge$ | conjunction |
| or | $\vee$ | disjunction, inclusive or |
| not | ~ | negation |
| if . . ., then implies | $\rightarrow$ | conditional, implication |
| if and only if | $\leftrightarrow$ | biconditional |

**Checkpoint**

*These connectives, along with their names, symbols, and properties, will be used throughout this chapter and should be memorized.*

**Example 5.3.1**

### Associate Connectives with Symbols and Names

For each of the following connectives, write its name and associated symbol.

1. or
2. implies
3. but

**Solution**

1. A compound statement formed with the connective word *or* is called a disjunction, and it is represented by the $\vee$ symbol.
2. A compound statement formed with the connective word *implies* or phrase “*if …, then*” is called a conditional statement or implication and is represented by the $\rightarrow$ symbol.

3. A compound statement formed with the connective words *but* or *and* is called a conjunction, and it is represented by the $\wedge$ symbol.

**Your Turn 5.3.1**

For each connective write its name and associated symbol.

1. not
2. and
3. if and only if

### Translating Compound Statements to Symbolic Form

To translate a compound statement into symbolic form, we take the following steps.

1. Identify and label all independent affirmative logical statements with a lower case letter, such as $p$, $q$, or $r$.
2. Identify and label any negative logical statements with a lowercase letter preceded by the negation symbol, such as $\sim p$, $\sim q$, or $\sim r$.
3. Replace the connective words with the symbols that represent them, such as $\wedge$, $\vee$, $\rightarrow$, or $\leftrightarrow$.

Consider the previous example of your friend trying to get their driver's license. Your friend passed the written test, but they did not pass the road test. Let $p$ represent the statement, "My friend passed the written test." And, let $\sim q$ represent the statement, "My friend did not pass the road test." Because the connective *but* is logically equivalent to the word *and*, the symbol for *but* is the same as the symbol for *and*; replace *but* with the symbol $\wedge$. The compound statement is symbolically written as: $p \wedge \sim q$. My friend passed the written test, but they did not pass the road test.

**Checkpoint**

*When translating English statements into symbolic form, do not assume that every connective "and" means you are dealing with a compound statement. The statement, "The stripes on the U.S. flag are red and white," is a simple statement. The word "white" doesn't express a complete thought, so it is not an independent clause and does not get its own symbol. This statement should be represented by a single letter, such as $s$: The stripes on the U.S. flag are red and white.*

**Example 5.3.2**

### Translating Compound Statements into Symbolic Form

Let $p$ represent the statement, "It is a warm sunny day," and let $q$ represent the statement, "the family will go to the beach." Write the symbolic form of each of the following compound statements.

1. If it is a warm sunny day, then the family will go to the beach.
2. The family will go to the beach, and it is a warm sunny day.
3. The family will not go to the beach if and only if it is not a warm sunny day.
4. The family not go to the beach, or it is a warm sunny day.

**Answer**

1. Replace "it is a warm sunny day" with $p$. Replace "the family will go to the beach." with $q$. Next. Next, because the connective is *if …, then* place the conditional symbol, $\rightarrow$, between $p$ and $q$. The compound statement is written symbolically as: $p \rightarrow q$.
2. Replace "The family will go to the beach" with $q$. Replace "it is a warm sunny day." with $p$. Next, because the connective is *and*, place the $\wedge$ symbol between $q$ and $p$. The compound statement is written symbolically as: $q \wedge p$.

3. Replace "The family will not go to the beach. with $\sim q$. Replace "it is not a warm sunny day" with $\sim p$. Next, because the connective is *or*, *if and only if*, place the biconditional symbol, $\leftrightarrow$ between $\sim q$ and $\sim p$. The compound statement is written symbolically as: $\sim q \leftrightarrow \sim p$.
4. Replace "The family will not go to the beach" with $\sim q$. Replace "it is a warm sunny day" with $p$. Next, because the connective is *or*, place the $\vee$ symbol between $\sim q$ and $p$. The compound statement is written symbolically as: $\sim q \vee p$.

**Your Turn 5.3.2**

Let p represent the statement, "Last night it snowed," and let q represent the statement, "Today we will go skiing." Write the symbolic form of each of the following compound statements:

1. Today we will go skiing, but last night it did not snow.
2. Today we will go skiing if and only if it snowed last night.
3. Last night is snowed or today we will not go skiing.
4. If it snowed last night, then today we will go skiing.

### Translating Compound Statements in Symbolic Form with Parentheses into Words

When parentheses are written in a logical argument, they group a compound statement together just like when calculating numerical or algebraic expressions. Any statement in parentheses should be treated as a single component of the expression. If multiple parentheses are present, work with the inner most parentheses first.

Consider your friend's struggles to get their license to drive. Let $p$ represent the statement, "My friend passed the written test," let $q$ represent the statement, "My friend passed the road test," and let $r$ represent the statement, "My friend received a driver's license." The statement $(p \wedge q) \rightarrow r$ can be translated into words as follows: the statement $p \wedge q$ is grouped together to form the hypothesis of the conditional statement and $r$ is the conclusion. The conditional statement has the form "*if* $p \wedge q$, *then* $r$." Therefore, the written form of this statement is: "If my friend passed the written test and they passed the road test, then my friend received a driver's license."

Sometimes a compound statement within parentheses may need to be negated as a group. To accomplish this, add the phrase, "it is not the case that" before the translation of the phrase in parentheses. For example, using $p$, $q$, and $r$ of your friend obtaining a license, let's translate the statement $\sim(p \wedge q) \rightarrow \sim r$ into words.

In this case, the hypothesis of the conditional statement is $\sim (p \wedge q)$ and the conclusion is $\sim r$. To negate the hypothesis, add the phrase "it is not the case" before translating what is in parentheses. The translation of the hypothesis is the sentence, "It is not the case that my friend passed the written test and they passed the road test," and the translation of the conclusion is, "My friend did not receive a driver's license." So, a translation of the complete conditional statement, $\sim (p \wedge q) \rightarrow \sim r$ is: "If it is not the case that my friend passed the written test and the road test, then my friend did not receive a driver's license."

**Checkpoint**

*It is acceptable to interchange proper names with pronouns and remove repeated phrases to make the written statement more readable, as long the meaning of the logical statement is not changed.*

**Video**

Logic Part 1B: Compound Statements, Connectives and Symbols

**Example 5.3.3**

### Translating Compound Statements in Symbolic Form with Parentheses into Words

Let $p$ represent the statement, "My child finished their homework," let $q$ represent the statement, "My child cleaned her room," let $r$ represent the statement, "My child played video games," and let $s$ represent the statement, "My child streamed a movie."

Translate each of the following symbolic statements into words.

1. ~ $(p \wedge q)$
2. $(p \wedge q) \rightarrow (r \vee s)$
3. ~ $(r \vee s) \leftrightarrow$ ~ $(p \wedge q)$

**Answer**

1. Replace ~ with "It is not the case," and $\wedge$ with "and." One possible translation is: "It is not the case that my child finished their homework and cleaned their room."
2. The hypothesis of the conditional statement is, "My child finished their homework and cleaned their room." The conclusion of the conditional statement is, "My child played video games or streamed a movie." One possible translation of the entire statement is: "If my child finished their homework and cleaned their room, then they played video games or streamed a movie."
3. The hypothesis of the biconditional statement is ~$(r \vee s)$ and is written in words as: "It is not the case that my child played video games or streamed a movie." The conclusion of the biconditional statement is ~$(p \wedge q)$, which translates to: "It is not the case that my child finished their homework and cleaned their room." Because the biconditional, $\leftrightarrow$ translates to *if and only if*, one possible translation of the statement is: "It is not the case that my child played video games or streamed a movie if and only if it is not the case that my child finished their homework and cleaned their room."

**Your Turn 5.3.3**

Let p represent the statement, "My roommates ordered pizza," let q represent the statement, "I ordered wings," and let r be the statement, "Our friends came over to watch the game." Translate the following statements into words.

1. ~r→(p∨q)
2. /**/\text{~}(p \vee r)/**/

## The Dominance of Connectives

The order of operations for working with algebraic and arithmetic expressions provides a set of rules that allow consistent results. For example, if you were presented with the problem $1+3 \times 2$, and you were not familiar with the order of operation, you might assume that you calculate the problem from left to right. If you did so, you would add 1 and 3 to get 4, and then multiply this answer by 2 to get 8, resulting in an incorrect answer. Try inputting this expression into a scientific calculator. If you do, the calculator should return a value of 7, not 8.

### Scientific Calculator

The order of operations for algebraic and arithmetic operations states that all multiplication must be applied prior to any addition. Parentheses are used to indicate which operation—addition or multiplication—should be done first. Adding parentheses can change and/or clarify the order. The parentheses in the expression $1+(3 \times 2)$ indicate that 3 should be multiplied by 2 to get 6, and then 1 should be added to 6 to get 7: $1+(3 \times 2)=7$.

As with algebraic expressions, there is a set of rules that must be applied to compound logical statements in order to evaluate them with consistent results. This set of rules is called the dominance of connectives. When evaluating compound logical statements, connectives are evaluated from least dominant to most dominant as follows:

1. Parentheses are the least dominant connective. So, any expression inside parentheses must be evaluated first. Add as many parentheses as needed to any statement to specify the order to evaluate each connective.
2. Next, we evaluate negations.
3. Then, we evaluate conjunctions and disjunctions from left to right, because they have equal dominance.
4. After evaluating all conjunctions and disjunctions, we evaluate conditionals.
5. Lastly, we evaluate the most dominant connective, the biconditional. If a statement includes multiple connectives of equal dominance, then we will evaluate them from left to right.

See Figure 2.7 for a visual breakdown of the dominance of connectives.

| Dominance | Connective | Symbol | Evaluate |
|---|---|---|---|
| Least Dominant | Parentheses | ( ) | First |
| ↓ | Negation | ~ | ↓ |
| ↓ | Disjunction/Conjunction | ∨, ∧ | Left to right or add parentheses to specify order because or/and have equal dominance. |
| ↓ | Conditional | → | ↓ |
| Most Dominant | Biconditional | ↔ | Last |

Figure 5.3.2

Let's revisit your friend's struggles to get their driver's license. Let $p$ represent the statement, "My friend passed the written test," let $q$ represent the statement, "My friend passed the road test," and let $r$ represent the statement, "My friend received a driver's license." Let's use the dominance of connectives to determine how the compound statement $p \wedge \sim q \rightarrow r$ should be evaluated.

**Step 1:** There are no parentheses, which is least dominant of all connectives, so we can skip over that.

**Step 2:** Because negation is the next least dominant, we should evaluate $\sim q$ first. We could add parentheses to the statement to make it clearer which connecting needs to be evaluated first: $p \wedge \sim q \rightarrow r$ is equivalent to $p \wedge (\sim q) \rightarrow r$.

**Step 3:** Next, we evaluate the conjunction, $\wedge$. $p \wedge (\sim q) \rightarrow r$ is equivalent to $(p \wedge (\sim q)) \rightarrow r$.

**Step 4:** Finally, we evaluate the conditional, $\rightarrow$, as this is the most dominant connective in this compound statement.

### Who Knew?

When using spreadsheet applications, like Microsoft Excel or Google Sheets, have you noticed that core functions, such as sum, average, or rate, have the same name and syntax for use? This is not a coincidence. Various standards organizations have developed requirements that software developers must implement to meet the conditions set by governments and large customers worldwide. The OpenDocument Format OASIS Standard enables transferring data between different office productivity applications and was approved by the International Standards Organization (ISO) and IEC on May 6, 2006.

Prior to adopting these standards, a government entity, business, or individual could lose access to their own data simply because it was saved in a format no longer supported by a proprietary software product—even data they were required by law to preserve, or data they simply wished to maintain access to.

Just as rules for applying the dominance of connectives help maintain understanding and consistency when writing and interpreting compound logical statements and arguments, the standards adopted for database software maintain global integrity and accessibility across platforms and user bases.

## Example 5.3.4

### Applying the Dominance of Connectives

For each of the following compound logical statements, add parentheses to indicate the order to evaluate the statement. Recall that parentheses are evaluated innermost first.

1. $p \wedge \sim q \vee r$
2. $q \rightarrow \sim p \wedge r$
3. $\sim (p \vee q) \leftrightarrow \sim p \wedge \sim q$

**Answer**

1. Because negation is the least dominant connective, we evaluate it first: $p \wedge (\sim q) \vee r$. Because conjunction and disjunction have the same dominance, we evaluate them left to right. So, we evaluate the conjunction next, as indicated by the additional set of parentheses: $(p \wedge (\sim q)) \vee r$. The only remaining connective is the disjunction, so it is evaluated last, as indicated by the third set of parentheses. The complete solution is: $((p \wedge (\sim q)) \vee r)$.
2. Negation has the lowest dominance, so it is evaluated first: $q \rightarrow (\sim p) \wedge r$. The remaining connectives are the conditional and the conjunction. Because conjunction has a lower precedence than the conditional, it is evaluated next, as indicated by the second set of parentheses: $q \rightarrow ((\sim p) \wedge r)$. The last step is to evaluate the conditional, as indicated by the third set of parentheses: $(q \rightarrow ((\sim p) \wedge r))$.
3. This statement is known as De Morgan's Law for the negation of a disjunction. It is always true. Section 2.6 of this chapter will explore De Morgan's Laws in more detail.
   - First, we evaluate the negations on the right side of the biconditional prior to the conjunction.
   - Then, we evaluate the disjunction on the left side of the biconditional, followed by the negation of the disjunction on the left side.
   - Lastly, after completely evaluating each side of the biconditional, we evaluate the biconditional. It does not matter which side you begin with.

   The final solution is: $(\sim (p \vee q)) \leftrightarrow ((\sim p) \wedge (\sim q))$.

## Your Turn 5.3.4

For each of the following compound logical statements, add parentheses to indicate the order in which to evaluate the statement. Recall that parentheses are evaluated innermost first.

1. $p \vee q \wedge \sim r$
2. $\sim p \rightarrow q \vee r$
3. $\sim p \vee \sim q \leftrightarrow \sim (p \wedge q)$

## WORK IT OUT

### Logic Terms and Properties – Matching Game

Materials: For every group of four students, include 30 cards (game, trading, or playing cards), 30 individual clear plastic gaming card sleeves, and 30 card-size pieces of paper. Prepare a list of 60 questions and answers ahead of time related to definitions and problems in Statements and Quantifiers and Compound Statements. Provide each group of four students with 20 questions and their associated answers. Instruct each group to select 15 of the 20 questions, and then, for each problem selected, create one card with the question and one card with the answer. Instruct the groups to then place each problem and answer in a separate card sleeve. Once they create 15 problem cards and 15 answer cards, have students shuffle the set of cards.

To play the game, groups should exchange their card decks to ensure no team begins playing with the deck that they created. Split each four-person group into teams of two students. After shuffling the cards, one team lays cards face down on their desk in a five-by-six grid. The other team will go first.

> Each player will have a turn to try matching the question with the correct answer by flipping two cards to the face up position. If a team makes a match, they get to flip another two cards; if they do not get a match, they flip the cards face down and it is the other team's turn to flip over two cards. The game continues in this manner until teams match all question cards with their corresponding answer cards. The team with the most set of matching cards wins.

In the first module of this chapter, we evaluated the truth value of negations. In the following modules, we will discuss how to evaluate conjunctions, disjunctions, conditionals, and biconditionals, and then evaluate compound logical statements using truth tables and our knowledge of the dominance of connectives.

### Check Your Understanding

1. A _____ _____ is a logical statement formed by combining two or more statements with connecting words, such as and, or, but, not, and if …, then.

2. A _____ is a word or symbol used to join two or more logical statements together to form a compound statement.

3. The most dominant connective is the _____.

4._____ are used to specify which logical connective should be evaluated first when evaluating a compound statement.

5. Both _____ and _____ have equal dominance and are evaluated from left to right when no parentheses are present in a compound logical statement.

### Section 2.2 Exercises

For the following exercises, translate each compound statement into symbolic form.

For the following exercises, translate each compound statement into symbolic form.
Given $p$ : "Layla has two weeks for vacation," $q$ : "Marcus is Layla's friend," $r$ : "Layla will travel to Paris, France," and $s$ : "Layla and Marcus will travel together to Niagara Falls, Ontario."
1. If Layla has two weeks for vacation, then she will travel to Paris, France.
2. Layla and Marcus will travel together to Niagara Falls, Ontario or Layla will travel to Paris, France.
3. If Marcus is not Layla's friend, then they will not travel to Niagara Falls, Ontario together."
4. Layla and Marcus will travel to Niagara Falls, Ontario together if and only if Layla and Marcus are friends.
5. If Layla does not have two weeks for vacation and Marcus is Layla's friend, then Marcus and Layla will travel together to Niagara Falls, Ontario.
6. If Layla has two weeks for vacation and Marcus is not her friend, then she will travel to Paris, France.

For the following exercises, translate each compound statement into symbolic form.
Given $p$ : "Tom is a cat," $q$ : "Jerry is a mouse," $r$ : "Spike is a dog," $s$ : "Tom chases Jerry," and $t$ : "Spike catches Tom."
7. Jerry is a mouse and Tom is a cat.
8. If Tom chases Jerry, then Spike will catch Tom.
9. If Spike does not catch Tom, then Tom did not chase Jerry.
10. Tom is a cat and Spike is a dog, or Jerry is not a Mouse.
11. It is not the case that Tom is not a cat and Jerry is not a mouse.
12. Spike is not a dog and Jerry is a mouse if and only if Tom chases Jerry, but Spike does not catch Tom.

For the following exercises, translate the symbolic form of each compound statement into words.
Given $p$ : "Tracy Chapman plays guitar," $q$ : "Joan Jett plays guitar," $r$ : "All rock bands include guitarists," and $s$ : "Elton John plays the piano."
13. $p \vee r$
14. $\sim s \rightarrow \sim q$
15. $(p \wedge q) \leftrightarrow r$
16. $\sim r \wedge (q \vee s)$
17. $\sim (p \wedge \sim q)$
18. $(q \rightarrow \sim r) \leftrightarrow (\sim p \vee \sim r)$

For the following exercises, translate the symbolic form of each compound statement into words.
Given $p$ : "The median is the middle number," $q$ : "The mode is the most frequent number," $r$ : "The mean is the average number," $s$ : "The median, mean, and mode are equal," and $t$ : "The data set is symmetric."

19. $t \rightarrow s$
20. $p \wedge (q \wedge r)$
21. $\sim t \rightarrow \sim s$
22. $(r \wedge p) \leftrightarrow q$
23. $(t \rightarrow \sim q) \vee (r \rightarrow s)$
24. $\sim (q \vee r) \rightarrow t$

For the following exercises, apply the proper dominance of connectives by adding parentheses to indicate the order in which the statement must be evaluated.

25. $p \rightarrow q \vee r$
26. $p \wedge q \leftrightarrow \sim r$
27. $p \vee r \vee \sim q$
28. $p \wedge \sim q \wedge r$
29. $p \wedge r \vee s \wedge t$
30. $q \rightarrow \sim r \leftrightarrow \sim p \vee \sim r$
31. $p \rightarrow r \vee s \leftrightarrow \sim t$
32. $\sim (t \wedge s) \vee (p \rightarrow q) \wedge \sim r$

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# 5.4: Constructing Truth Tables



Figure 5.4.1: Just like solving a puzzle, a computer programmer must consider all potential solutions in order to account for each possible outcome. (credit: modification of work “Deadline Xmas 2010” by Allan Henderson/Flickr, CC BY 2.0)

**Learning Objectives**

After completing this section, you should be able to:

1. Interpret and apply negations, conjunctions, and disjunctions.
2. Construct a truth table using negations, conjunctions, and disjunctions.
3. Construct a truth table for a compound statement and interpret its validity.

Are you familiar with the Choose Your Own Adventure book series written by Edward Packard? These gamebooks allow the reader to become one of the characters and make decisions that affect what happens next, resulting in different sequences of events in the story and endings based on the choices made. Writing a computer program is a little like what it must be like to write one of these books. The programmer must consider all the possible inputs that a user can put into the program and decide what will happen in each case, then write their program to account for each of these possible outcomes.

A truth table is a graphical tool used to analyze all the possible truth values of the component logical statements to determine the validity of a statement or argument along with all its possible outcomes. The rows of the table correspond to each possible outcome for the given logical statement identified at the top of each column. A single logical statement $p$ has two possible truth values, true or false. In truth tables, a capital T will represent true values, and a capital F will represent false values.

In this section, you will use the knowledge built in Statements and Quantifiers and Compound Statements to analyze arguments and determine their truth value and validity. A logical argument is valid if its conclusion follows from its premises, regardless of whether those premises are true or false. You will then explore the truth tables for negation, conjunction, and disjunction, and use these truth tables to analyze compound logical statements containing these connectives.

## Interpret and Apply Negations, Conjunctions, and Disjunctions

The negation of a statement will have the opposite truth value of the original statement. When $p$ is true, $\sim p$ is false, and when $p$ is false, $\sim p$ is true.

**? Exercise 5.4.1**

For each logical statement, determine the truth value of its negation.

1. $p$: $3+5=8$.
2. $q$: All horses are mustangs.
3. $\sim r$: New Delhi is not the capital of India.

**Answer**

$p$ is true because 3 + 5 does equal 8; therefore, the negation of p, ~ p : 3 + 5 ≠ 8 ~ p : 3 + 5 ≠ 8 , is false.

$q$ is false because there are other types of horses besides mustangs, such as Clydesdales or Arabians; therefore, the negation of $q$, $\sim q$ , is true.

$\sim r$ is false because New Delhi is the capital of India; therefore, the negation of $\sim r$, $r$ , is true.

**Your Turn 5.4.1**

For each logical statement, determine the truth value of its negation.

1. $\sim p$: $3 \times 5 = 14$ .
2. $\sim q$: Some houses are built with bricks
3. $r$: Abuja is the capital of Nigeria.

A conjunction is a logical *and* statement. For a conjunction to be true, both statements that make up the conjunction must be true. If at least one of the statements is false, the *and* statement is false.

1. $p \wedge q$
2. $\sim q \wedge r$
3. $\sim p \wedge q$

**Answer**

1. $p$ is true, and $q$ is false. Because one statement is true, and the other statement is false, this makes the complete conjunction false.
2. $q$ is false, so $\sim q$ is true, and $r$ is true. Therefore, both statements are true, making the complete conjunction true.
3. $\sim p$ is false, and $q$ is false. Because both statements are false, the complete conjunction is false.

**Your Turn 5.4.2**

Given $p$ : Yellow is a primary color, $q$ : Blue is a primary color, and $r$ : Green is a primary color, determine the truth value of each conjunction.

1. $p \wedge q$
2. $q \wedge r$
3. $\sim r \wedge p$ .

**Checkpoint**

*The only time a conjunction is true is if both statements that make up the conjunction are true.*

A disjunction is a logical inclusive *or* statement, which means that a disjunction is true if one or both statements that form it are true. The only way a logical inclusive *or* statement is false is if both statements that form the disjunction are false.

1. $p \vee q$
2. $\sim q \vee r$
3. $\sim p \vee q$

**Answer**

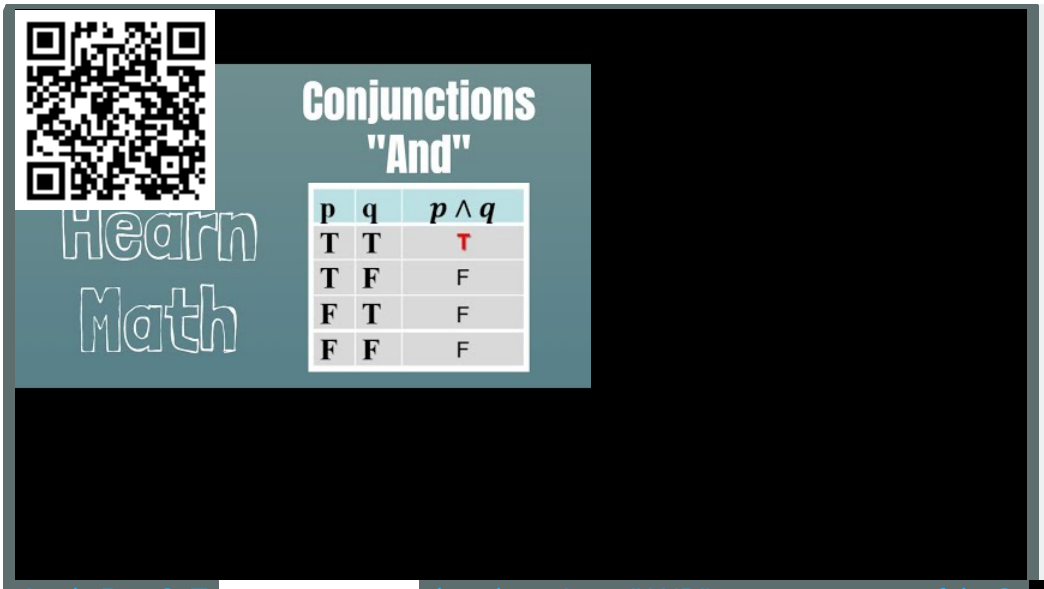
1. **$p$ is true, and $q$ is false. One statement is true, and one statement is false, which makes the complete disjunction true.**
2. **$q$ is false, so $\sim q$ is true, and $r$ is true. Therefore, one statement is true, and the other statement is true, which makes the complete disjunction true.**
3. **$\sim p$ is false, and $q$ is false. When all of the component statements are false, the disjunction is false.**

**Your Turn 5.4.3**

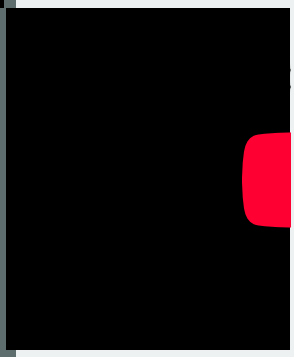
Given $p$ : Yellow is a primary color, $q$ : Blue is a primary color, and $r$ : Green is a primary color, determine the truth value of each disjunction.

1. $p \vee q$
2. $\sim p \vee r$
3. $q \vee r$

In the next example, you will apply the dominance of connectives to find the truth values of compound statements containing negations, conjunctions, and disjunctions and use a table to record the results. When constructing a truth table to analyze an argument where you can determine the truth value of each component statement, the strategy is to create a table with two rows. The first row contains the symbols representing the components that make up the compound statement. The second row contains the truth values of each component below its symbol. Include as many columns as necessary to represent the value of each compound statement. The last column includes the complete compound statement with its truth value below it.




Logic Part 2: Tr njunctions: Is an "AND" statement true or false?



Logic Part 3: Truth Values of Disjunctions: Is an "OR" statement true or false?

1. $\sim p \wedge q \vee r$
2. $\sim p \vee q \wedge r$
3. $\sim (p \wedge r) \vee q$

**Answer**

1. **Step 1:** The statement " $\sim p \wedge q \vee r$ " contains three basic logical statements, $p, q$, and $r$, and three connectives, $\sim, \wedge, \vee$. When we place parentheses in the statement to indicate the dominance of connectives, the statement becomes $(((\sim p) \wedge q) \vee r)$.
**Step 2:** After we have applied the dominance of connectives, we create a two row table that includes a column for each basic statement that makes up the compound statement, and an additional column for the contents of each parentheses. Because we have three sets of parentheses, we include a column for $\sim p$, the innermost parentheses, a column for $(\sim p) \wedge q$, the next set of parentheses, and $((\sim p) \wedge q) \vee r$ in the last column for the third parentheses.
**Step 3:** Once the table is created, we determine the truth value of each statement starting from left to right. The truth values of $p, q$, and $r$ are true, false, and true, respectively, so we place T, F, and $\mathrm{T}$ in the second row of the table. Because $p$ is true, $\sim p$ is false.
**Step 4:** Next, evaluate $\sim p \wedge q$ from the table: $\sim p$ is false, and $q$ is also false, so $\sim p \wedge q$ is false, because a conjunction is only true if both of the statements that make it are true. Place an $\mathrm{F}$ in the table below its heading.
**Step 5:** Finally, using the table, we understand that $(\sim p \wedge q)$ is false and $r$ is true, so the complete statement $(\sim p \wedge q) \vee r$ is false or true, which is true (because a disjunction is true whenever at least one of the statements that make it is true). Place a $\mathrm{T}$ in the last column of the table. The complete statement $\sim p \wedge q \vee r$ is true.

| | | | | | |
|---|---|---|---|---|---|
| | | | | | |

2. **Step 1**: Applying the dominance of connectives to the original compound statement $\sim p \vee q \wedge r$, we get $(((\sim p) \vee q) \wedge r)$.
**Step 2**: The table needs columns for $p, q, r, \sim p, (\sim p) \vee q$, and $((\sim p) \vee q) \wedge r$.
**Step 3**:The truth values of $p, q, r$, and $\sim p$ are the same as in Question 1.
**Step 4**: Next, $\sim p \vee q$ is false or false, which is false, so we place an $\mathrm{F}$ below this statement in the table. This is the only time that a disjunction is false.
**Step 5**: Finally, $\sim p \vee q$ and $r$ are the conjunction of the statements, $\sim p \vee q$ and $r$, and so the expressions evaluate to false and true, which is false. Recall that the only time an "and" statement is true is when both statements that form it are also true. The complete statement $\sim p \vee q \wedge r$ is false.

| | | | | | |
|---|---|---|---|---|---|
| | | | | | |

3. **Step 1**: Applying the dominance of connectives to the original statement, we have: $((\sim (p \wedge r)) \vee q)$.
**Step 2**: So, the table needs the following columns: $p, q, r, p \wedge r, \sim (p \wedge r)$, and $\sim (p \wedge r) \vee q$.
**Step 3**: The truth values of $p, q$, and $r$ are the same as in Questions 1 and 2.
**Step 4**: From the table it can be seen that $p \wedge r$ is true and true, which is true. So the negation of $p$ and $r$ is false, because the negation of a statement always has the opposite truth value of the original statement.
**Step 5**: Finally, $\sim (p \wedge r) \vee q$ is the disjunction of $\sim (p \wedge r)$ with $q$, and so we have false or false, which makes the complete statement false.

| | | | | | |
|---|---|---|---|---|---|
| T | F | T | T | F | F |

**Your Turn 5.4.4**

Given $p$ : Yellow is a primary color, $q$ : Blue is a primary color, and $r$ : Green is a primary color, determine the truth value of each compound statement, by constructing a truth table.

1. $\sim q \wedge p \vee r$
2. $p \vee q \wedge \sim r$
3. $\sim (p \wedge r) \wedge q$

## Construct Truth Tables to Analyze All Possible Outcomes

Recall from Statements and Questions that the negation of a statement will always have the opposite truth value of the original statement; if a statement $p$ is false, then its negation $\sim p$ is true, and if a statement $p$ is true, then its negation $\sim p$ is false. To create a truth table for the negation of statement $p$, add a column with a heading of $\sim p$, and for each row, input the truth value with the opposite value of the value listed in the column for $p$, as depicted in the table below.

| Negation | |
|---|---|
| p | **~ p** |
| T | F |
| F | T |

Conjunctions and disjunctions are compound statements formed when two logical statements combine with the connectives “and” and “or” respectively. How does that change the number of possible outcomes and thus determine the number of rows in our truth table? The multiplication principle, also known as the *fundamental counting principle*, states that the number of ways you can select an item from a group of $n$ items and another item from a group with $m$ items is equal to the product of $m$ and $n$. Because each proposition $p$ and $q$ has two possible outcomes, true or false, the multiplication principle states that there will be $2 \times 2 = 4$ possible outcomes: {TT, TF, FT, FF}.

The tree diagram and table in Figure $5.4.2$ demonstrate the four possible outcomes for two propositions $p$ and $q$.

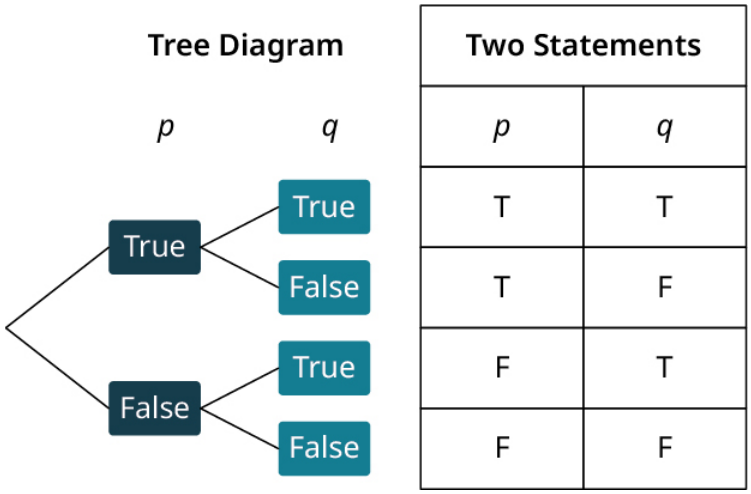


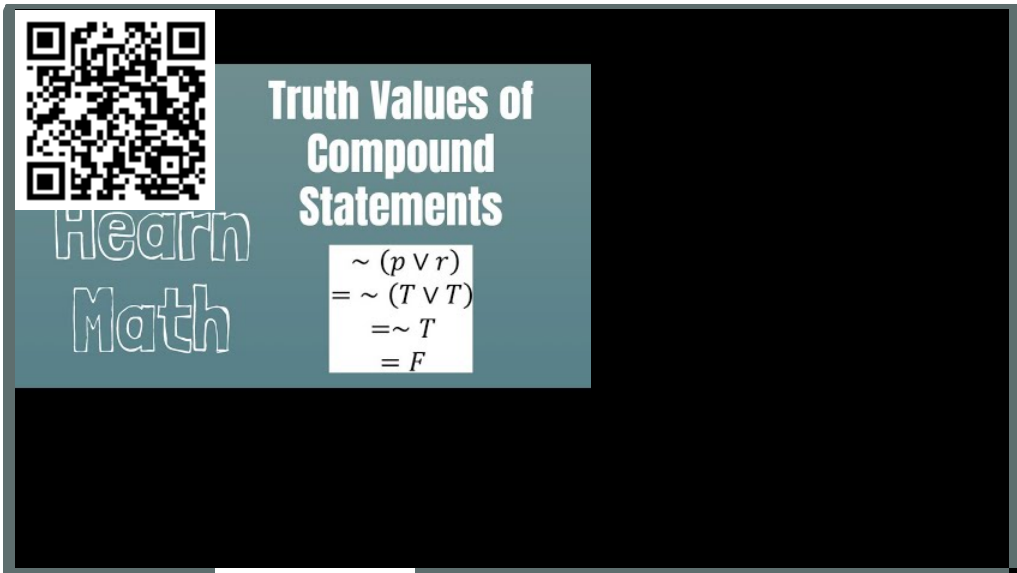

Figure $5.4.2$:

A conjunction is a logical *and* statement. For a conjunction to be true, both the statements that make up the conjunction must be true. If at least one of the statements is false, the *and* statement is false.

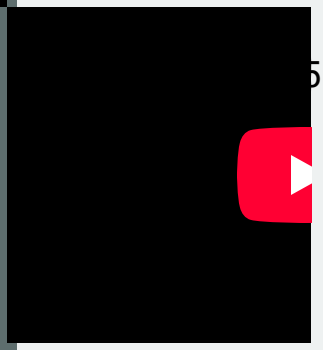
A disjunction is a logical inclusive *or* statement. Which means that a disjunction is true if one or both statements that make it are true. The only way a logical inclusive *or* statement is false is if both statements that make up the disjunction are false.

| Conjunction (AND) | Disjunction (OR) |
|---|---|

| Conjunction (AND) | | | | Disjunction (OR) | | |
|---|---|---|---|---|---|---|
| $p$ | $q$ | $p \wedge q$ | | $p$ | $q$ | $p \vee q$ |
| $p$ | $q$ | $p \wedge q$ | | $p$ | $q$ | $p \vee q$ |
| T | T | T | | T | T | T |
| T | F | F | | T | F | T |
| F | T | F | | F | T | T |
| F | F | F | | F | F | F |




Logic Part 4: Tr ompound Statements with "and", "or", and "not"



Logic Part 5: What are truth tables? How do you set them up?

**? Exercise 5.4.5: Constructing Truth Tables to Analyze Compound Statements**

Construct a truth table to analyze all possible outcomes for each of the following arguments.

1. $p \vee \sim q$
2. $\sim (p \wedge q)$
3. $(p \vee \sim q) \wedge r$

**Answer**

**1. Step 1**: Because there are two basic statements, $p$ and $q$, and each of these has two possible outcomes, we will have $2(2) = 4$ rows in our table to represent all possible outcomes: TT, TF, FT, and FF. The columns will include $p, q, \sim q$, and $p \vee \sim q$.
**Step 2:** Every value in column $\sim q$ will have the opposite truth value of the corresponding value in column $q$ : F, T, F, and T.

**Step 3:** To complete the last column, evaluate each element in column $p$ with the corresponding element in column $\sim q$ using the connective or.

| $p$ | $q$ | $\sim q$ | $p \vee \sim q$ . |
|---|---|---|---|
| T | T | F | T |
| T | F | T | T |
| F | T | F | F |
| F | F | T | T |

**2. Step 1:** The columns will include $p, q, p \wedge q$, and $\sim (p \wedge q)$ . Because there are two basic statements, $p$ and $q$, the table will have four rows to account for all possible outcomes.
**Step 2**: The $p \wedge q$ column will be completed by evaluating the corresponding elements in columns $p$ and $q$ respectively with the and connective.
**Step 3**: The final column, $\sim (p \wedge q)$ , will be the negation of the $p \wedge q$ column.

| $p$ | $q$ | $p \wedge q$ | $\sim (p \wedge q)$ |
|---|---|---|---|
| T | T | T | F |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

**3. Step 1:** This statement has three basic statements, $p, q$, and $r$. Because each basic statement has two possible truth values, true or false, the multiplication principal indicates there are $2(2)(2) = 8$ possible outcomes. So eight rows of outcomes are needed in the truth table to account for each possibility. Half of the eight possibilities must be true for the first statement, and half must be false.
**Step 2:** So, the first column for statement $p$, will have four T's followed by four F's. In the second column for statement $q$, when $p$ is true, half the outcomes for $q$ must be true and the other half must be false, and the same pattern will repeat for when $p$ is false. So, column $q$ will have TT, FF, FF, FF.

**Step 3:** The column for the third statement, $r$, must alternate between $T$ and $F$. Once, the three basic propositions are listed, you will need a column for $\sim, p \vee \sim q$ , and $(p \vee \sim q) \wedge r$ .
**Step 4:** The column for the negation of $q, \sim q$, will have the opposite truth value of each value in column $q$.
**Step 5:** Next, fill in the truth values for the column containing the statement $p \vee \sim q$ . The or statement is true if at least one of $p$ or $\sim q$ is true, otherwise it is false.
**Step 6:** Finally, fill in the column containing the conjunction $(p \vee \sim q) \wedge r$ . To evaluate this statement, combine column $p \vee \sim q$ and column $r$ with the and connective. Recall, that only time "and" is true is when both values are true, otherwise the statement is false. The complete truth table is:

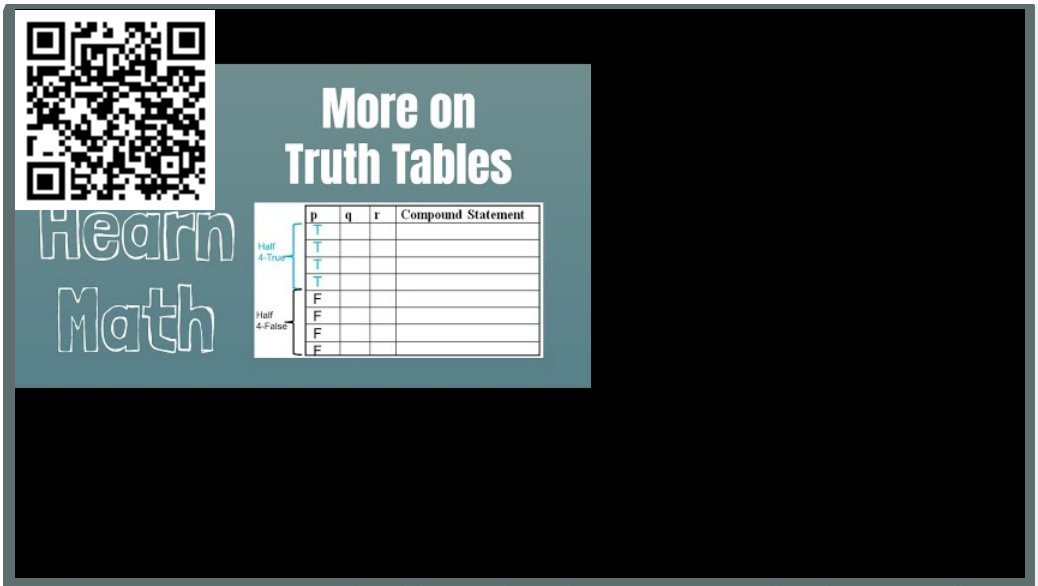
| $p$ | $q$ | $r$ | $\sim q$ | $p \vee \sim q$ | $(p \vee \sim q) \wedge r$ |
|---|---|---|---|---|---|
| T | T | T | F | T | T |
| T | T | F | F | T | F |
| T | F | T | T | T | T |
| T | F | F | T | T | F |
| F | T | T | F | F | F |
| F | T | F | F | F | F |
| F | F | T | T | T | T |

| $p$ | $q$ | $r$ | $\sim q$ | $p \vee \sim q$ | $(p \vee \sim q) \wedge r$ |
|---|---|---|---|---|---|
| F | F | F | T | T | F |

## Your Turn 5.4.5

Construct a truth table to analyze all possible outcomes for each of the following arguments.

1. $p \wedge \sim q$
2. $\sim (p \vee q)$
3. $(p \wedge \sim q) \vee r$




Logic Part 6: More on Truth Tables and Setting Up Rows and Column Headings

### Determine the Validity of a Truth Table for a Compound Statement

A logical statement is valid if it is always true regardless of the truth values of its component parts. To test the validity of a compound statement, construct a truth table to analyze all possible outcomes. If the last column, representing the complete statement, contains only true values, the statement is valid.

## Exercise 5.4.6: Determining the Validity of Compound Statements

Construct a truth table to determine the validity of each of the following statements.

1. $\sim p \wedge q$
2. $\sim (p \wedge \sim p)$

**Answer**

**1. Step 1**: Because there are two statements, $p$ and $q$, and each of these has two possible outcomes, there will be $2(2) = 4$ rows in our table to represent all possible outcomes: TT, TF, FT, and FF.
**Step 2**: The columns, will include $p, q, \sim p$ and $\sim p \wedge q$ . Every value in column $\sim p$ will have the opposite truth value of the corresponding value in column $p$.
**Step 3:** To complete the last column, evaluate each element in column $\sim p$ with the corresponding element in column $q$ using the connective and. The last column contains at least one false, therefore the statement $\sim p \wedge q$ is not valid.

| $p$ | $q$ | $\sim p$ | $\boldsymbol{\sim p \wedge q}$ |
|---|---|---|---|
| T | T | F | F |

| $p$ | $q$ | $\sim p$ | $\sim p \wedge q$ |
|---|---|---|---|
| T | F | F | F |
| F | T | T | T |
| F | F | T | F |

**2. Step 1**: Because the statement $\sim (p \wedge \sim p)$ only contains one basic proposition, the truth table will only contain two rows. Statement $p$ may be either true or false.
**Step 2:** The columns will include $p, \sim p, p \wedge \sim p$, and $\sim (p \wedge \sim p)$. Evaluate column $p \wedge \sim p$ with the and connective, because the symbol $\wedge$ represents a conjunction or logical and statement. True and false is false, and false and true is also false.
**Step 3:** The final column is the negation of each entry in the third column, both of which are false, so the negation of false is true. Because all the truth values in the final column are true, the statement $\sim (p \wedge \sim p)$ is valid.

| $p$ | $\sim p$ | $p \wedge \sim p$ | $\sim (p \wedge \sim p)$ |
|---|---|---|---|
| T | F | F | T |
| F | T | F | T |

**Your Turn 5.4.6**

Construct a truth table to determine the validity of each of the following statements.
1. $p \vee \sim p$
2. $\sim p \vee \sim q$

## Check Your Understanding

1. A logical argument is _____ if its conclusion follows from its premises.
2. A logical statement is valid if it is always _____.
3. A _____ _____ is a tool used to analyze all the possible outcomes for a logical statement.
4. The truth table for the conjunction, $p \wedge q$, has _____ rows of truth values.
5. The truth table for the negation of a logical statement, $\sim p$, has _____ rows of truth values.

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# 5.5: Truth Tables for the Conditional and Biconditional

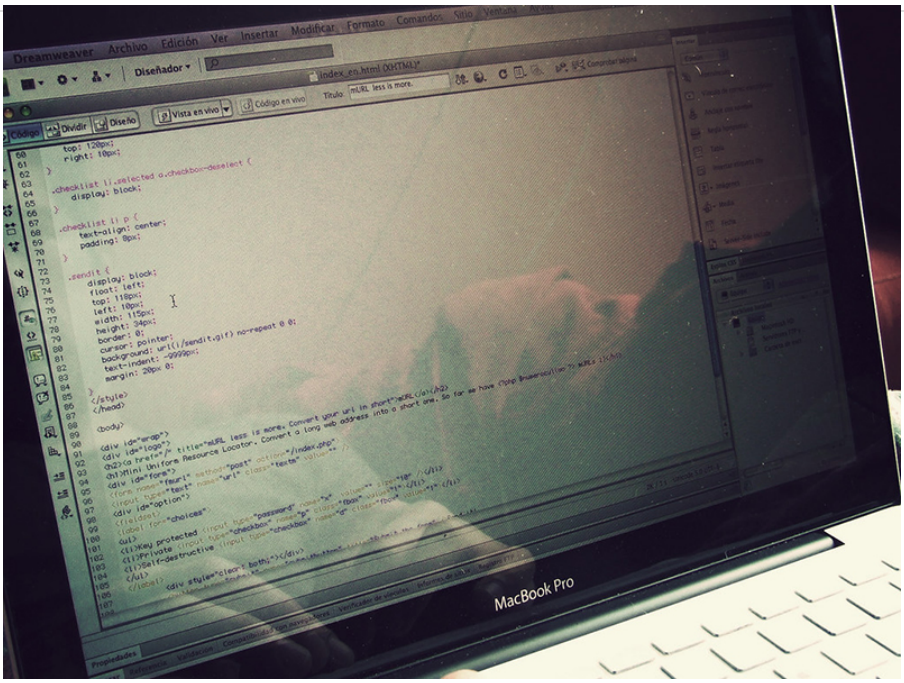



Figure 5.5.1: If-then statements use logic to execute directions. (credit: “Coding” by Carlos Varela/Flickr, CC BY 2.0)

**Learning Objectives**

After completing this section, you should be able to:

1. Use and apply the conditional to construct a truth table.
2. Use and apply the biconditional to construct a truth table.
3. Use truth tables to determine the validity of conditional and biconditional statements.

Computer languages use if-then or if-then-else statements as decision statements:

- *If* the hypothesis is true, *then* do something.
- Or, *if* the hypothesis is true, *then* do something; else do something else.

For example, the following representation of computer code creates an if-then-else decision statement:

Check value of variable $i$.

If $i < 1$, then print "Hello, World!" else print "Goodbye".

In this imaginary program, the if-then statement evaluates and acts on the value of the variable $i$. For instance, if $i = 0$, the program would consider the statement $i < 1$ as true and “Hello, World!” would appear on the computer screen. If instead, $i = 3$, the program would consider the statement $i < 1$ as false (because 3 is greater than 1), and print “Goodbye” on the screen.

In this section, we will apply similar reasoning without the use of computer programs.

**People in Mathematics**

The Countess of Lovelace, Ada Lovelace, is credited with writing the first computer program. She wrote an algorithm to work with Charles Babbage’s Analytical Engine that could compute the Bernoulli numbers in 1843. In doing so, she became the first person to write a program for a machine that would produce more than just a simple calculation. The computer programming language ADA is named after her.

Reference: Posamentier, Alfred and Spreitzer Christian, “Chapter 34 Ada Lovelace: English (1815-1852)” pp. 272-278, *Math Makers: The Lives and Works of 50 Famous Mathematicians*, Prometheus Books, 2019.

## Use and Apply the Conditional to Construct a Truth Table

A conditional is a logical statement of the form if $p$, then $q$. The conditional statement in logic is a promise or contract. The only time the conditional, $p \rightarrow q$, is false is when the contract or promise is broken.

For example, consider the following scenario. A child's parent says, "If you do your homework, then you can play your video games." The child really wants to play their video games, so they get started right away, finish within an hour, and then show their parent the completed homework. The parent thanks the child for doing a great job on their homework and allows them to play video games. Both the parent and child are happy. The contract was satisfied; true implies true is true.

Now, suppose the child does not start their homework right away, and then struggles to complete it. They eventually finish and show it to their parent. The parent again thanks the child for completing their homework, but then informs the child that it is too late in the evening to play video games, and that they must begin to get ready for bed. Now, the child is really upset. They held up their part of the contract, but they did not receive the promised reward. The contract was broken; true implies false is false.

So, what happens if the child does not do their homework? In this case, the hypothesis is false. No contract has been entered, therefore, no contract can be broken. If the conclusion is false, the child does not get to play video games and might not be happy, but this outcome is expected because the child did not complete their end of the bargain. They did not complete their homework. False implies false is true. The last option is not as intuitive. If the parent lets the child play video games, even if they did not do their homework, neither parent nor child are going to be upset. False implies true is true.

The truth table for the conditional statement below summarizes these results.

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | **T** |
| T | F | **F** |
| F | T | **T** |
| F | F | **T** |

**Checkpoint**

*Notice that the only time the conditional statement, $p \rightarrow q$, is false is when the hypothesis, $p$, is true and the conclusion, $q$, is false.*

**Video**



Logic Part 8: The Conditional and Tautologies

**? Exercise 5.5.1: Constructing Truth Tables for Conditional Statements**

Assume both of the following statements are true: $p$: My sibling washed the dishes, and $q$: My parents paid them $5.00. Create a truth table to determine the truth value of each of the following conditional statements.

1. $p \rightarrow q$
2. $p \rightarrow \sim q$

3. $\sim p \rightarrow q$

**Answer**

Because p is true and q is true, the statement p → q is, “If my sibling washed the dishes, then my parents paid them $5.00.” My sibling did wash the dishes, since p p is true, and the parents did pay the sibling $5.00, so the contract was entered and completed. The conditional statement is true, as indicated by the truth table representing this case:
T → T = T.

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | **T** |

p → ~ q p

2. p → ~ q p → ~ q translates to the statement, “If my sibling washed the dishes, then my parents did not pay them $5.00.” p p is true, but ~ q ~ q is false. The sibling completed their end of the contract, but they did not get paid. The contract was broken by the parents. The conditional statement is false, as indicated by the truth table representing this case:

T → F = F.

| $p$ | $q$ | $\sim q$ | $p \rightarrow \sim q$ |
|---|---|---|---|
| T | T | F | **F** |

3. ~ p → q translates to the statement, “If my sibling did not wash the dishes, then my parents paid them $5.00.” ~ p is false, but q is true. The sibling did not do the dishes. No contract was entered, so it could not be broken. The parents decided to pay them $5.00 anyway. The conditional statement is true, as indicated by the truth table representing this case: F → T = T.

| $p$ | $q$ | $\sim p$ | $\sim p \rightarrow q$ |
|---|---|---|---|
| T | T | F | **T** |

**Your Turn 5.5.1**

Assume p is true and q is false. p: Kevin vacuumed the living room, and q: Kevin's parents did not let him borrow the car. Create a truth table to determine the truth value of each of the following conditional statements.

1. $p \rightarrow q$
2. $p \rightarrow \sim q$
3. $\sim p \rightarrow q$

**Exercise 5.5.2: Determining Validity of Conditional Statements**

Construct a truth table to analyze all possible outcomes for each of the following statements then determine whether they are valid.

1. $p \wedge q \rightarrow \sim q$
2. $p \rightarrow \sim p \vee q$

**Answer**

1. Applying the dominance of connectives, the statement $p \wedge q \rightarrow \sim q$ is equivalent to $(p \wedge q) \rightarrow (\sim q)$. So, the columns of the truth table will include $p$, $q$, $p \wedge q$, $\sim q$, and $p \wedge q \rightarrow \sim q$. Because there are only two basic propositions, $p$ and $q$, the table will have 2 ( 2 ) = 4 2 ( 2 ) = 4 rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

| $p$ | $q$ | $p \wedge q$ | $\sim q$ | $p \wedge q \rightarrow \sim q$ |
|---|---|---|---|---|
| T | T | T | F | **F** |

| $p$ | $q$ | $p \wedge q$ | $\sim q$ | $p \wedge q \rightarrow \sim q$ |
|---|---|---|---|---|
| T | F | F | T | **T** |
| F | T | F | F | **T** |
| F | F | F | T | **T** |

Applying the dominance of connectives, the statement $p \rightarrow \sim p \vee q$ is equivalent to $(p) \rightarrow ((\sim p) \vee q)$. So, the columns of the truth table will include p , q , ~ p, ~ p ∨ q, and $p \rightarrow (\sim p \vee q)$. Because there are only two basic propositions, p and q , the table will have 2 ( 2 ) = 4 2 ( 2 ) = 4 rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

| $p$ | $q$ | $\sim p$ | $\sim p \vee q$ | $p \rightarrow (\sim p \vee q)$ |
|---|---|---|---|---|
| T | T | F | T | **T** |
| T | F | F | F | **F** |
| F | T | T | T | **T** |
| F | F | T | T | **T** |

**Your Turn 5.5.2**

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $q \rightarrow \sim p \vee q$
2. $\sim p \rightarrow q \wedge p$

## Use and Apply the Biconditional to Construct a Truth Table

The biconditional, $p \leftrightarrow q$, is a two way contract; it is equivalent to the statement $(p \rightarrow q) \wedge (q \rightarrow p)$. A biconditional statement, $p \leftrightarrow q$, is true whenever the truth value of the hypothesis matches the truth value of the conclusion, otherwise it is false.

The truth table for the biconditional is summarized below.

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | **T** |
| T | F | **F** |
| F | T | **F** |
| F | F | **T** |

Video



## Exercise 5.5.3: Constructing Truth Tables for Biconditional Statements

Assume both of the following statements are true: $p$ : The plumber fixed the leak, and $q$ : The homeowner paid the plumber $150.00. Create a truth table to determine the truth value of each of the following biconditional statements.

1. $p \leftrightarrow q$
2. $p \leftrightarrow \sim q$
3. $\sim p \leftrightarrow \sim q$

**Answer**

1. Because p p is true and q q is true, the statement p ↔ q p ↔ q is "The plumber fixed the leak if and only if the homeowner paid them $150.00." Because both p p and q q are true, the leak was fixed and the plumber was paid, meaning both parties satisfied their end of the bargain. The biconditional statement is true, as indicated by the truth table representing this case: T ↔ T = T.

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | **T** |

2. p → ~ q p → ~ q translates to the statement, "The plumber fixed the leak if and only if the homeowner did not pay them $150." If the plumber fixed the leak and the homeowner did not pay them, the homeowner will have broken their end of the contract. The biconditional statement is false, as indicated by the truth table representing this case:
T ↔ F = F.

| $p$ | $q$ | $\sim q$ | $p \leftrightarrow \sim q$ |
|---|---|---|---|
| T | T | F | **F** |

3. ~ p ↔ ~ q ~ p ↔ ~ q translates to the statement, "The plumber did not fix the leak if and only if the homeowner did not pay them $150." In this case, neither party—the plumber nor the homeowner—entered into the contract. The leak was not repaired, and the plumber was not paid. No agreement was broken. The biconditional statement is true, as indicated by the truth table representing this case: F ↔ F = T.

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \leftrightarrow \sim q$ |
|---|---|---|---|---|
| T | T | F | F | **T** |

## Your Turn 5.5.3

Assume $p$ is true and $q$ is false: $p$ : The contractor fixed the broken window, and $q$ : The homeowner paid the contractor $200. Create a truth table to determine the truth value of each of the following biconditional statements.

1. $p \leftrightarrow q$
2. $p \leftrightarrow \sim q$
3. $\sim p \leftrightarrow q$

## Checkpoint

*The biconditional, $p \leftrightarrow q$, is true whenever the truth values of $p$ and $q$ match, otherwise it is false.*

## Video



Logic Part 13: Truth Tables to Determine if Argument is Valid or Invalid

### Exercise 5.5.1: Determining Validity of Biconditional Statements

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $p \wedge q \leftrightarrow p \wedge \sim q$
2. $p \vee q \leftrightarrow \sim p \vee q$
3. $p \rightarrow q \leftrightarrow \sim q \rightarrow \sim p$
4. $p \wedge q \rightarrow \sim r \leftrightarrow p \wedge q \wedge r$

**Answer**

1. Applying the dominance of connectives, the statement $p \wedge q \leftrightarrow p \wedge \sim q$ is equivalent to $(p \wedge q) \leftrightarrow (p \wedge (\sim q))$. So, the columns of the truth table will include $p, q, p \wedge q, \sim q, p \wedge \sim q$ and $(p \wedge q) \leftrightarrow (p \wedge \sim q)$. Because there are only two basic propositions, $p$ and $q$, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

| $p$ | $q$ | $p \wedge q$ | $\sim q$ | $p \wedge \sim q$ | $(p \wedge q) \leftrightarrow (p \wedge \sim q)$ |
|---|---|---|---|---|---|
| T | T | T | F | F | **F** |
| T | F | F | T | T | **F** |
| F | T | F | F | F | **T** |
| F | F | F | T | F | **T** |

2. Applying the dominance of connectives, the statement $p \vee q \leftrightarrow \sim p \vee q$ is equivalent to $(p \vee q) \leftrightarrow ((\sim p) \vee q)$. So, the columns of the truth table will include $p, q, p \vee q, \sim p, \sim p \vee q$, and $(p \vee q) \leftrightarrow (\sim p \vee q)$. Because there are only two basic propositions, $p$ and $q$, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

| $p$ | $q$ | $p \vee q$ | $\sim p$ | $\sim p \vee q$ | $(p \vee q) \leftrightarrow (\sim p \vee q)$ |
|---|---|---|---|---|---|
| T | T | T | F | T | **T** |
| T | F | T | F | F | **F** |
| F | T | T | T | T | **T** |
| F | F | F | T | T | **F** |

3. Applying the dominance of connectives, the statement $p \rightarrow q \leftrightarrow \sim q \rightarrow \sim p$ is equivalent to $(p \rightarrow q) \leftrightarrow ((\sim q) \rightarrow (\sim p))$. So, the columns of the truth table will include $p, q, p \rightarrow q, \sim q, \sim p, \sim q \rightarrow \sim p$, and $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$. Because there are only two basic propositions, $p$ and $q$ the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is valid because the last column is all true.

| $p$ | $q$ | $p \to q$ | $\sim q$ | $\sim p$ | $\sim q \to \sim p$ | $(p \to \mathbf{q}) \leftrightarrow (\sim q \to \sim p)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | T | **T** |
| T | F | F | T | F | F | **T** |
| F | T | T | F | T | T | **T** |
| F | F | T | T | T | T | **T** |

4. Applying the dominance of connectives, the statement $p \wedge q \to \sim r \leftrightarrow p \wedge q \wedge r$ is equivalent to $((p \wedge q) \to (\sim r)) \leftrightarrow ((p \wedge q) \wedge r)$. So, the columns of the truth table will include $p, q, r, \sim r, p \wedge q, (p \wedge q) \wedge \sim r, (p \wedge q) \wedge r$, and $((p \wedge q) \to (\sim r)) \leftrightarrow ((p \wedge q) \wedge r)$. Because there are three basic propositions, $p, q$, and $r$, the table will have $2(2)(2) = 8$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

| $p$ | $q$ | $r$ | $\sim r$ | $p \wedge q$ | $(p \wedge q) \to \sim r$ | $(p \wedge q) \wedge r$ | $(p \wedge q \to \sim r) \leftrightarrow (p \wedge$ |
|---|---|---|---|---|---|---|---|
| T | T | T | F | T | F | T | **T** |
| T | T | F | T | T | T | F | **T** |
| T | F | T | F | F | T | F | **F** |
| T | F | F | T | F | T | F | **F** |
| F | T | T | F | F | T | F | **F** |
| F | T | F | T | F | T | F | **F** |
| F | F | T | F | F | T | F | **F** |
| F | F | F | T | F | T | F | **F** |

### Your Turn 5.5.1

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
2. $\sim p \leftrightarrow q \wedge p$
3. $p \to q \leftrightarrow \sim p \vee q$
4. $p \wedge q \to r \leftrightarrow \sim p \vee \sim q \vee r$

## Check Your Understanding

1. 1. In logic, a conditional statement can be thought of as a _____.
2. If the hypothesis, $p$, of a conditional statement is true, the _____, $q$, must also be true for the conditional statement \(p \to q to be true.
3. If the _____ of a conditional statement is false, the conditional statement is true.
4. The symbolic form of the _____ statement is $p \leftrightarrow q$
5. The _____ statement is equivalent to the statement $(p \to q) \wedge (q \to p)$.
6. $p$ if and only if $q$ is _____ whenever the truth value of $p$ matches the truth value of $q$, otherwise it is false.

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# 5.6: Equivalent Statements



Figure 5.6.1: How your logical argument is stated affects the response, just like how you speak when holding a conversation can affect how your words are received. (credit: modification of work by Goelshivi/Flickr, Public Domain Mark 1.0)

**Learning Objectives**

After completing this section, you should be able to:

1. Determine whether two statements are logically equivalent using a truth table.
2. Compose the converse, inverse, and contrapositive of a conditional statement

Have you ever had a conversation with or sent a note to someone, only to have them misunderstand what you intended to convey? The way you choose to express your ideas can be as, or even more, important than what you are saying. If your goal is to convince someone that what you are saying is correct, you will not want to alienate them by choosing your words poorly.

Logical arguments can be stated in many different ways that still ultimately result in the same valid conclusion. Part of the art of constructing a persuasive argument is knowing how to arrange the facts and conclusion to elicit the desired response from the intended audience.

In this section, you will learn how to determine whether two statements are **logically equivalent** using truth tables, and then you will apply this knowledge to compose logically equivalent forms of the conditional statement. Developing this skill will provide the additional skills and knowledge needed to construct well-reasoned, persuasive arguments that can be customized to address specific audiences.

**Checkpoint**

*An alternate way to think about logical equivalence is that the truth values have to match. That is, whenever $p$ is true, $q$ is also true, and whenever $p$ is false, $q$ is also false.*

## Determine Logical Equivalence

Two statements, $p$ and $q$, are logically equivalent when $p \leftrightarrow q$ is a valid argument, or when the last column of the truth table consists of only true values. When a logical statement is always true, it is known as a tautology. To determine whether two statements $p$ and $q$ are logically equivalent, construct a truth table for $p \leftrightarrow q$ and determine whether it valid. If the last column is all true, the argument is a tautology, it is valid, and $p$ is logically equivalent to $q$; otherwise, $p$ is not logically equivalent to $q$.

## Exercise 5.6.1: Determining Logical Equivalence with a Truth Table

Create a truth table to determine whether the following compound statements are logically equivalent.

1. $p \rightarrow q$; $\sim p \rightarrow \sim q$
2. $p \rightarrow q$; $\sim p \vee q$

**Answer**

1. Construct a truth table for the biconditional formed by using the first statement as the hypothesis and the second statement as the conclusion, $( p \rightarrow q ) \leftrightarrow ( \sim p \rightarrow \sim q )$.

| $p$ | $q$ | $p \rightarrow q$ | $\sim p$ | $\sim q$ | $\sim p \rightarrow \sim q$ | $(p \rightarrow q) \leftrightarrow (\sim p \rightarrow \sim q$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | T | **T** |
| T | F | F | F | T | T | **F** |
| F | T | T | T | F | F | **F** |
| F | F | T | T | T | T | **T** |

Because the last column it not all true, the biconditional is not valid and the statement $p \rightarrow q$ is not logically equivalent to the statement $\sim p \rightarrow \sim q$.

2. Construct a truth table for the biconditional formed by using the first statement as the hypothesis and the second statement as the conclusion,
$( p \rightarrow q) \leftrightarrow ( \sim p \vee q )$.

| $p$ | $q$ | $p \rightarrow q$ | $\sim p$ | $\sim p \vee q$ | $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$ |
|---|---|---|---|---|---|
| T | T | T | F | T | **T** |
| T | F | F | F | F | **T** |
| F | T | T | T | T | **T** |
| F | F | T | T | T | **T** |

Because the last column is true for every entry, the biconditional is valid and the statement $p \rightarrow q$ is logically equivalent to the statement $\sim p \vee q$. Symbolically, $p \rightarrow q \equiv \sim p \vee q$.

## Your Turn 5.6.1

Create a truth table to determine whether the following compound statements are logically equivalent.

1. $p \rightarrow q$; $q \rightarrow \sim p$
2. $p \rightarrow q$; $p \vee \sim q$

## Compose the Converse, Inverse, and Contrapositive of a Conditional Statement

The converse, inverse, and contrapositive are variations of the conditional statement, $p \rightarrow q$.

- The converse is if $q$ then $p$, and it is formed by interchanging the hypothesis and the conclusion. The converse is logically equivalent to the inverse.
- The inverse is if $\sim p$ then $\sim q$, and it is formed by negating both the hypothesis and the conclusion. The inverse is logically equivalent to the converse.
- The contrapositive is if $\sim q$ then $\sim p$, and it is formed by interchanging and negating both the hypothesis and the conclusion. The contrapositive is logically equivalent to the conditional.

The table below shows how these variations are presented symbolically.

| | | | | Conditional | Contrapositive | Converse | Inverse |
|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \rightarrow q$ | $\sim q \rightarrow \sim p$ | $q \rightarrow p$ | $\sim p \rightarrow \sim q$ |
| T | T | F | F | T | T | T | T |
| T | F | F | T | F | F | T | T |
| F | T | T | F | T | T | F | F |
| F | F | T | T | T | T | T | T |

### Exercise 5.6.2: Writing the Converse, Inverse, and Contrapositive of a Conditional Statement

Use the statements, $p$: Harry is a wizard and $q$: Hermione is a witch, to write the following statements:

1. Write the conditional statement, $p \rightarrow q$, in words.
2. Write the converse statement, $q \rightarrow p$, in words.
3. Write the inverse statement, $\sim p \rightarrow \sim q$, in words.
4. Write the contrapositive statement, $\sim q \rightarrow \sim p$, in words.

**Answer**

1. The conditional statement takes the form, “if p p , then q q ,” so the conditional statement is: “If Harry is a wizard, then Hermione is a witch.” Remember the *if … then …* words are the connectives that form the conditional statement.
2. The converse swaps or interchanges the hypothesis, p p , with the conclusion, q q . It has the form, “if q q , then p p .” So, the converse is: "If Hermione is a witch, then Harry is a wizard."
3. To construct the inverse of a statement, negate both the hypothesis and the conclusion. The inverse has the form, “if ~ p ~ p , then ~ q ~ q ,” so the inverse is: "If Harry is not a wizard, then Hermione is not a witch."
4. The contrapositive is formed by negating and interchanging both the hypothesis and conclusion. It has the form, “if ~ q ~ q , then ~ p ~ p ," so the contrapositive statement is: "If Hermione is not a witch, then Harry is not a wizard."

### Your Turn 5.6.2

Use the statements, $p$ : Elvis Presley wore capes and $q$ : Some superheroes wear capes, to write the following statements:

1. Write the conditional statement, $p \rightarrow q$, in words.
2. Write the converse statement, $q \rightarrow p$, in words.
3. Write the inverse statement, $\sim p \rightarrow \sim q$, in words.
4. Write the contrapositive statement, $\sim q \rightarrow \sim p$, in words.

### Exercise 5.6.3

Use the conditional statement, “If all dogs bark, then Lassie likes to bark,” to identify the following.

1. Write the hypothesis of the conditional statement and label it with a $p$.
2. Write the conclusion of the conditional statement and label it with a $q$.
3. Identify the following statement as the converse, inverse, or contrapositive: “If Lassie likes to bark, then all dogs bark.”
4. Identify the following statement as the converse, inverse, or contrapositive: “If Lassie does not like to bark, then some dogs do not bark.”
5. Which statement is logically equivalent to the conditional statement?

**Answer**

1. The hypothesis is the phrase following the *if*. The answer is $p$: All dogs bark. Notice, the word *if* is not included as part of the hypothesis.
2. The conclusion of a conditional statement is the phrase following the *then*. The word *then* is not included when stating the conclusion. The answer is: q: Lassie likes to bark.

3. "Lassie likes to bark" is $q$ and "All dogs bark" is $p$. So, "If Lassie likes to bark, then all dogs bark," has the form "if $q$, then $p$," which is the form of the converse. "Lassie does not like to bark" is $\sim q$ and "Some dogs do not bark" is $\sim p$. The statement, "If Lassie does not like to bark, then some dogs do not bark," has the form "if $\sim q$, then $\sim p$," which is the form of the contrapositive.
4. The contrapositive $\sim q \rightarrow \sim p$ is logically equivalent to the conditional statement $p \rightarrow q$ .

### Your Turn 5.6.3

Use the conditional statement, "If Dora is an explorer, then Boots is a monkey," to identify the following:

1. Write the hypothesis of the conditional statement and label it with a p$\hat{}$.

2. Write the conclusion of the conditional statement and label it with a q$\hat{}$.

3. Identify the following statement as the converse, inverse, or contrapositive: "If Dora is not an explorer, then Boots is not a monkey."

4. Identify the following statement as the converse, inverse, or contrapositive: "If Boots is a monkey, then Dora is an explorer."

5. Which statement is logically equivalent to the inverse?

### Exercise 5.6.4

Assume the conditional statement, p→q: "If Chadwick Boseman was an actor, then Chadwick Boseman did not star in the movie *Black Panther*" is false, and use it to answer the following questions.

1. Write the converse of the statement in words and determine its truth value.
2. Write the inverse of the statement in words and determine its truth value.
3. Write the contrapositive of the statement in words and determine its truth value.

**Answer**

The only way the conditional statement can be false is if the hypothesis, $p$: Chadwick Boseman was an actor, is true and the conclusion, q: Chadwick Boseman did not star in the movie *Black Panther*, is false. The converse is q → p, and it is written in words as: "If Chadwick Boseman did not star in the movie *Black Panther*, then Chadwick Boseman was an actor." This statement is true, because false → true is true.

The inverse has the form " $\sim p \rightarrow \sim q$." The written form is: "If Chadwick Boseman was not an actor, then Chadwick Boseman starred in the movie *Black Panther*." Because $p$ is true, and $q$ is false, $\sim p$ is false, and $\sim q$ is true. This means the inverse is false → true, which is true. Alternatively, from Question 1, the converse is true, and because the inverse is logically equivalent to the converse it must also be true.

The contrapositive is logically equivalent to the conditional. Because the conditional is false, the contrapositive is also false. This can be confirmed by looking at the truth values of the contrapositive statement. The contrapositive has the form " $\sim q \rightarrow \sim p$." Because q is false and p is true, ~ q is true and $\sim p$ is false. Therefore, $\sim q \rightarrow \sim p$ is true → false, which is false. The written form of the contrapositive is "If Chadwick Boseman starred in the movie *Black Panther*, then Chadwick Boseman was not an actor."

### Your Turn 5.6.4

Assume the conditional statement $p \rightarrow q$: "If my friend lives in San Francisco, then my friend does not live in California" is false, and use it to answer the following questions.

1. Write the converse of the statement in words and determine its truth value.

2. Write the inverse of the statement in words and determine its truth value.

3. Write the contrapositive of the statement in words and determine its truth value.

## Check Your Understanding

1. Two statements $p$ and $q$ are logically equivalent to each other if the biconditional statement, $p \leftrightarrow q$ is _____.
2. The _____ statement has the form, “$p$ then $q$.”
3. The contrapositive is _____ _____ to the conditional statement, and has the form, "if $\sim q$, then $\sim p$."
4. The _____ of the conditional statement has the form, "if $\sim p$, then $\sim q$."
5. The _____ of the conditional statement is logically equivalent to the _____ and has the form, "if $q$ then $p$."

---

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# 5.7: De Morgan's Laws

Figure 5.7.1: De Morgan's Laws were key to the rise of logical mathematical expression and helped serve as a bridge for the invention of the computer. (credit: modification of work "Golden Gate Bridge (San Francisco Bay, California, USA)" by James St. John/Flickr, CC BY 2.0)

**Learning Objectives**

After completing this section, you should be able to:

1. Use De Morgan's Laws to negate conjunctions and disjunctions.
2. Construct the negation of a conditional statement.
3. Use truth tables to evaluate De Morgan's Laws.

The contributions to logic made by Augustus De Morgan and George Boole during the 19th century acted as a bridge to the development of computers, which may be the greatest invention of the 20th century. **Boolean** logic is the basis for computer science and digital electronics, and without it the technological revolution of the late 20th and early 21st centuries—including the creation of computer chips, microprocessors, and the Internet—would not have been possible. Every modern computer language uses Boolean logic statements, which are translated into commands understood by the underlying electronic circuits enabling computers to operate. But how did this logic get its name?

**People in Mathematics: George Boole**

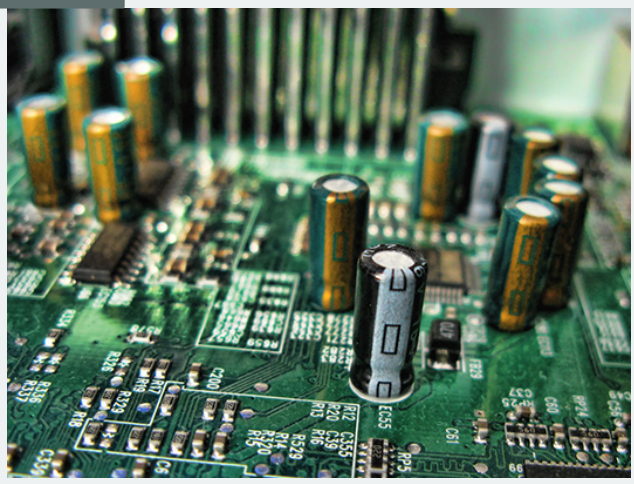


Figure 5.7.2: Boole's algebra of logic was foundational in the design of digital computer circuits. (credit: "Circuit Board" by Squeezyboy/Flickr, CC BY 2.0)

George Boole was born in Lincolnshire, England in 1815. He was the son of a cobbler who provided him some initial education, but Boole was mostly self-taught. He began teaching at 16 years of age, and opened his own school at the age of 20. In 1849, at the age of 34, he was appointed Professor of Mathematics at Queens College in Cork, Ireland. In 1853, he published the paper, *An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities*, which is the treatise that the field of Boolean algebra and digital circuitry was built on.

Reference: Posamentier, Alfred and Spreitzer Christian, "Chapter 35 George Boole: English (1815-1864)" pp. 279-283, *Math Makers: The Lives and Works of 50 Famous Mathematicians*, Prometheus Books, 2019.

## Negation of Conjunctions and Disjunctions

In Chapter 1, Example 1.37 used a Venn diagram to prove De Morgan's Law for set complement over union. Because the complement of a set is analogous to negation and union is analogous to an *or* statement, there are equivalent versions of De Morgan's Laws for logic.

**FORMULA**

De Morgan's Law for negation of a conjunction: $\sim(p \wedge q) \equiv \sim p \vee \sim q$

De Morgan's Law for the negation of a disjunction: $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Negation of a conditional: $\sim(p \rightarrow q) \equiv p \wedge \sim q$

Writing conditional as a disjunction: $p \rightarrow q \equiv \sim p \vee q$

**Checkpoint**

*Recall that the symbol for logical equivalence is:* $\equiv$.

De Morgan's Laws allow us to write the negation of conjunctions and disjunctions without using the phrase, "It is not the case that …" to indicate the parentheses. Avoiding this phrase often results in a written or verbal statement that is clearer or easier to understand.

**Exercise 5.7.1: Applying De Morgan's Law for Negation of Conjunctions and Disjunctions**

Write the negation of each statement in words without using the phrase, "It is not the case that."

1. Kristin is a biomedical engineer and Thomas is a chemical engineer.
2. A person had cake or they had ice cream.

**Answer**

1. Kristin is a biomedical engineer and Thomas is a chemical engineer has the form " $p \wedge q$ ," where $p$ is the statement, "Kristin is a biomedical engineer," and $q$ is the statement, "Thomas is a chemical engineer." By De Morgan's Law, the negation of a conjunction, $\sim(p \wedge q)$ , is logically equivalent to $\sim p \vee \sim q$. $\sim p$ is "Kristen is not a biomedical engineer," and $\sim q$ is "Thomas is not a chemical engineer." By De Morgan's Law, the solution has the form " $\sim p \vee \sim q$ ," so the answer is: "Kristin is not a biomedical engineer or Tom is not a chemical engineer."
2. A person had cake or they had ice cream has the form " $p \vee q$ ," where $p$ is the statement, "A person had cake," and $q$ is the statement, "A person had ice cream." By De Morgan's Law for the negations of a disjunction, $\sim(p \vee q) \equiv \sim p \wedge \sim q$ . The solution is the statement: "A person did not have cake and they did not have ice cream."

**Your Turn 5.7.1**

Write the negation of each statement in words without using the phrase, *it is not the case that*.

1. Jackie played softball or she ran track.
2. Brandon studied for his certification exam, and he passed his exam.

is not a chemical engineer."

## Negation of a Conditional Statement

The negation of any statement has the opposite truth values of the original statement. The **negation of a conditional**, $\sim(p \rightarrow q)$, is the conjunction of $p$ and not $q$, $p \wedge \sim q$. Consider the truth table below for the negation of the conditional.

| $p$ | $q$ | $p \rightarrow q$ | $\sim(p \rightarrow q)$ |
|---|---|---|---|
| T | T | T | F |
| T | F | F | T |
| F | T | T | F |
| F | F | T | F |

The only time the negation of the conditional statement is true is when $p$ is true, and $q$ is false. This means that $\sim(p \rightarrow q)$ is logically equivalent to $p \wedge \sim q$, as the following truth table demonstrates.

| $p$ | $q$ | $p \rightarrow q$ | $\sim(p \rightarrow q)$ | $\sim q$ | $p \wedge \sim q$ | $\sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | **T** |
| T | F | F | T | T | T | **T** |
| F | T | T | F | F | F | **T** |
| F | F | T | F | T | F | **T** |

### Exercise 5.7.2: Constructing the Negation of a Conditional Statement

Write the negation of each conditional statement.

1. If Adele won a Grammy, then she is a singer.
2. If Henrik Lundqvist played professional hockey, then he did not win the Stanley Cup.

**Answer**

1. The negation of the conditional statement, $p \rightarrow q$, is the statement, $p \wedge \sim q$. The hypothesis of the conditional statement is $p$ : "Adele won a Grammy," and conclusion of the conditional statement is $q$ : "Adele is a singer." The negation of the conclusion, $\sim q$, is the statement: "She is not a singer." Therefore, the answer is $p \wedge \sim q$ : "Adele won a Grammy, and she is not a singer."

2. The hypothesis is $p$ : "Henrik Lundqvist played professional hockey," and the conclusion of the conditional statement is $q$ : "He did not win the Stanley Cup." The negation of $q$ is the statement: "He won the Stanley Cup." The negation of the conditional statement is equal to $p \wedge \sim q$ : "Henrick Lundqvist played professional hockey, and he won the Stanley Cup."

### Your Turn 5.7.2

Write the negation of each conditional statement.

1. If Edna Mode makes a new superhero costume, then it will not include a cape.
2. If I had pancakes for breakfast, then I used maple syrup.

### Exercise 5.7.3: Constructing the Negation of a Conditional Statement with Quantifiers

Write the negation of each conditional statement.

1. If all cats purr, then my partner's cat purrs.
2. If a penguin is a bird, then some birds do not fly.

**Answer**

1. The negation of the conditional statement $p \rightarrow q$ is the statement $p \wedge \sim q$. The hypothesis of the conditional statement is $p$ : "All cats purr," and the conclusion of the conditional statement is $q$ : "My partner's cat purrs." The negation of the conclusion, $\sim q$, is the statement: "My partner's cat does not purr." Therefore, the answer is $p \wedge \sim q$ : "All cats purr, but my partner's cat does not purr."

2. The hypothesis is $p$ : "A penguin is a bird," and the conclusion of the conditional statement is $q$ : "Some birds do not fly." The negation of $q$ is the statement: "All birds fly." Therefore, the negation of the conditional statement is equal to $p \wedge \sim q$ : "A penguin is a bird, and all birds fly."

### Your Turn 5.7.3

Write the negation of each conditional statement.

1. If some people like ice cream, then ice cream makers will make a profit.
2. If Raquel cannot play video games, then nobody will play video games.

Many of the properties that hold true for number systems and sets also hold true for logical statements. The following table summarizes some of the most useful properties for analyzing and constructing logical arguments. These properties can be verified using a truth table.

| Property | Conjunction (AND) | Disjunction (OR) |
|---|---|---|
| **Commutative** | $p \wedge q \equiv q \wedge p$ | $p \vee q \equiv q \vee p$ |
| **Associative** | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ |
| **Distributive** | $p \wedge (q \wedge r) \equiv (p \wedge q) \vee (p \wedge r)$ | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ |
| **De Morgan's** | $\sim (p \wedge q) \equiv \sim p \vee \sim q$ | $\sim (p \vee q) \equiv \sim p \wedge \sim q$ |
| **Conditional** | $\sim (p \rightarrow q) \equiv p \wedge \sim q$ | $p \rightarrow q \equiv \sim p \vee q$ |

**? Exercise 5.7.4**

### Negating a Conditional Statement with a Conjunction or Disjunction

Write the negation of each conditional statement applying De Morgan's Law.

1. If mom needs to buy chips, then Mike had friends over and Bob was hungry.
2. If Juan had pizza or Chris had wings, then dad watched the game.

**Answer**

1. The conditional has the form "If $p$ then $q$ or $r$," where $p$ is "Mom needs to buy chips," $q$ is "Mike had friends over," and $r$ is "Bob was hungry." The negation of $p \rightarrow (q \wedge r)$ is $p \wedge \sim (q \wedge r)$. Applying De Morgan's Law to the statement $\sim (q \wedge r)$ the result is $\sim q \vee \sim r$, so our conditional statement becomes $p \wedge (\sim q \vee \sim r)$. By the distributive property for conjunction over disjunction, this statement is equivalent to $(p \wedge \sim q) \vee (p \wedge \sim r)$. Translating the statement $(p \wedge \sim q) \vee (p \wedge \sim r)$ into words, the solution is: "Mom needs to buy chips and Mike did not have friends over, or Mom needs to buy chips and Bob was not hungry."

2. The conditional has the form "If $p$ or $q$, then $r$," where $p$ is "Juan had pizza," $q$ is "Chris had wings," and $r$ is "Dad watched the game." The negation of $(p \vee q) \rightarrow r$ is $(p \vee q) \wedge \sim r$. By the distributive property for disjunction over conjuction, the statement is equivalent to $(p \vee \sim r) \wedge (q \vee \sim r)$. Translating the statement $(p \vee \sim r) \wedge (q \vee \sim r)$ into words, the solution is: "Juan had pizza or dad did not watch the game, and Chris had wings or dad did not watch the game."

**✓ Your Turn 5.7.4**

Write the negation of each conditional statement applying De Morgan's Law.

1. If Eric needs to replace the light bulb, then Marcus left the light on all night or Dan broke the bulb.
2. If Trenton went to school and Regina went to work, then Merika cleaned the house.

## Evaluating De Morgan's Laws with Truth Tables

In Chapter 1, you learned that you could prove the validity of De Morgan's Laws using Venn diagrams. Truth tables can also be used to prove that two statements are logically equivalent. If two statements are logically equivalent, you can use the form of the statement that is clearer or more persuasive when constructing a logical argument.

The next example will prove the validity of one of De Morgan's Laws using a truth table. The same procedure can be applied to any two logical statement that you believe are equivalent. If the last column of the truth table is a tautology, then the two statements are logically equivalent.

**? Exercise 5.7.5: Verifying De Morgan's Law for Negation of a Conjunction**

Construct a truth table to verify De Morgan's Law for the negation of a conjunction, $\sim (p \wedge q) \equiv \sim p \vee \sim q$, is valid.

**Answer**

**Step 1**: To verify any logical equivalence, you must first replace the logical equivalence symbol, $\equiv$, with the biconditional symbol, $\leftrightarrow$. The statement $\sim (p \wedge q) \equiv \sim p \vee \sim q$ becomes $\sim (p \wedge q) \leftrightarrow \sim p \vee \sim q$

**Step 2:** Next, you create a truth table for the statement. Because we have two basic statements, $p$, and $q$, the truth table will have four rows to account for all the possible outcomes. The columns will be $p, q, \sim p, \sim q, p \wedge q, \sim (p \wedge q), \sim p \vee \sim q$, and the biconditional statement is $\sim (p \wedge q) \leftrightarrow \sim p \vee \sim q$.

| $p$ | $q$ | $p \wedge q$ | $\sim (p \wedge q)$ | $\sim p$ | $\sim q$ | $\sim p \vee \sim q$ | $\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim$ |
|---|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F | **T** |
| T | F | F | T | F | T | T | **T** |
| F | T | F | T | T | F | T | **T** |
| F | F | F | T | T | T | T | **T** |

**Step 3:** Finally, verify that the statement is valid by confirming it is a tautology. In this instance, the last column is all true. Therefore, the statement is valid and De Morgan's Law for the negation of a conjunction is verified.

### Your Turn 5.7.5

1. Construct a truth table to verify De Morgan's Law for the negation of a disjunction, $\sim (p \vee q) \equiv \sim p \wedge \sim q$, is valid.

## Check Your Understanding

1. De Morgan's Law for the negation of a conjunction states that $\sim (p \wedge q)$ is logically equivalent to _____.
2. De Morgan's Law for the negation of a disjunction states that $\sim (p \vee q)$ is logically equivalent to _____.
3. The negation of the conditional statement, $\sim (p \rightarrow q)$, is logically equivalent to _____.

$\sim (\sim (p \rightarrow q)) \equiv p \rightarrow q$, which means the conditional statement is logically equivalent to $\sim (p \wedge \sim q)$. Apply _____ to the statement $\sim (p \wedge \sim q)$ to show that the conditional statement $p \rightarrow q \equiv \sim p \vee q$.

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# 5.8: Logical Arguments

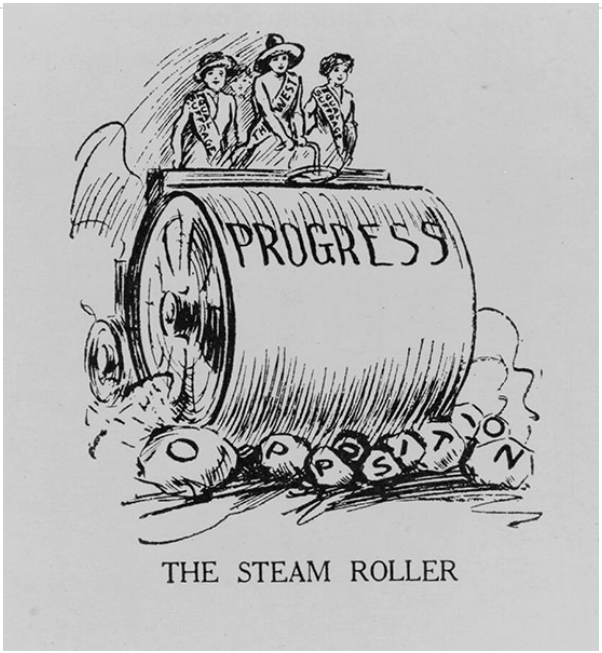



Figure 5.8.1: Not all logical arguments are valid, and the ongoing fight for equal rights proves that much progress has yet to be made. (credit: "The Steam Roller" by Library of Congress Prints and Photographs Division, public domain) ;

**Learning Objectives**

After completing this section, you should be able to:

1. Apply the law of detachment to determine the conclusion of a pair of statements.
2. Apply the law of denying the consequent to determine the conclusion for pairs of statements.
3. Apply the chain rule to determine valid conclusions for pairs of true statements.

The previous sections of this chapter provide the foundational skills for constructing and analyzing logical arguments. All logical arguments include a set of premises that support a claim or conclusion; but not all logical arguments are valid and sound. A logical argument is valid if its conclusion follows from the premises, and it is sound if it is valid and all of its premises are true. A false or deceptive argument is called a fallacy. Many types of fallacies are so common that they have been named.

**Who Knew?**

In 1936, Dale Carnegie published his first book, titled *How to Win Friends and Influence People*. It was marketed as training materials for the improvement of public speaking and negotiation skills, and the methods it presented are still used today. Carnegie famously said, “When dealing with people, remember you are not dealing with creatures of logic, but creatures of emotion.”

People who put forth fallacious logical arguments often take advantage of our susceptibility to emotional appeals, to try to convince us that what they are saying is true. The study of logic helps us combat this weakness through recognition and learning to focus on the facts and structure of the argument.

This section focuses on the two main forms that logical arguments can take. While inductive arguments attempt to draw a more general conclusion from a pattern of specific premises, deductive arguments attempt to draw specific conclusions from at least one or more general premises. Deductive arguments can be proven to be valid using Venn diagrams or truth tables.

Inductive arguments generally cannot be proven to be true. They are judged as being strong or weak, but, like any opinion, whether you believe an argument is strong or weak often depends on your knowledge of the topic being discussed along with the evidence being provided in the premises. *Hasty generalization* is the name given to any fallacy that presents a weak inductive argument.

**Checkpoint**

*Be careful! Premises may be true or false. If a premise is false, the claims made by the argument should be questioned.*

## Law of Detachment

The law of detachment is a valid form of a conditional argument that asserts that if both the conditional, $p \to q$, and the hypothesis, $p$, are true, then the conclusion $q$ must also be true. The law of detachment is also called affirming the hypothesis (or antecedent) and modus ponens. Symbolically, it has the form $((p \to q) \wedge p) \to q$ .

| Law of Detachment | |
|---|---|
| Premise: | $p \to q$ |
| Premise: | $p$ |
| Conclusion: | $\therefore q$ |

**Checkpoint**

*The $\therefore$ is read as the word, "therefore."*

Looking at the truth table for the conditional statement, the only time the conditional is true is when the hypothesis $p$ is also true. The only place this happens is in the first row, where $q$ is also true, confirming that the law of detachment is a valid argument.

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | **T** |
| T | F | **F** |
| F | T | **T** |
| F | F | **T** |

Another way to verify that the law of detachment is a valid argument is to construct a truth table for the argument $((p \to q) \wedge p) \to q$ and verify that it is a tautology.

| $p$ | $q$ | $p \to q$ | $(p \to q) \wedge p$ | $((p \to q) \wedge p) \to q$ |
|---|---|---|---|---|
| T | T | T | T | **T** |
| T | F | F | F | **T** |
| F | T | T | F | **T** |
| F | F | T | F | **T** |

Venn diagrams may also be used to verify deductive arguments, which include conditional premises. Consider the statement $p \to q$ :Then verify if the conclusion is also represented by the Venn diagram of the premises. If it is, the argument is valid. If it is not, the argument is not valid. The set of guitarists is drawn as a subset of the set of musicians to represent the premise.

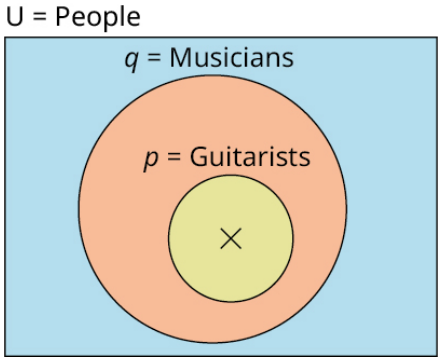



Figure 5.8.2

Now, examine the Venn diagram to verify if the conclusion is included in the picture. The conclusion is $q$. Because the $\times$ is in the set $p$, and $p$ is a subset of $q$, $\times$ is also in $q$; therefore, the law of detachment is a valid argument.

**Checkpoint**

*Remember that an argument can be valid without being true. For the argument to be proven true, it must be both valid and sound. An argument is sound if all its premises are true.*

**Video**



Logic Part 14: Common Argument Forms like Modus Ponens and Tollens

**Exercise 5.8.1: Applying the Law of Detachment to Determine a Valid Conclusion**

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of detachment to determine a valid conclusion.

1. If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*. Leonardo da Vinci was an artist.
2. If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player. Michael Jordan played for the Chicago Bulls.
3. If all fish have gills, then clown fish have gills. All fish have gills.

**Answer**

1. The premises are $p \rightarrow q$ : If Leonardo da Vinci was an artist, then he painted the Mona Lisa, and $p$ : Leonardo da Vinci was an artist. This argument has the form of the law of detachment, so, the conclusion is $q$ : Leonardo da Vinci painted the Mona Lisa.

2. The premises follow the form of the law of detachment, so a valid conclusion would be $q$. The premises are $p \rightarrow q$ : If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player, and $p$ : Michael Jordan played for the Chicago Bulls. The conclusion that follows from the premises is $q$ : Michael Jordan was not a soccer player.

3. The premises are $p \rightarrow q$ : If all fish have gills, then clown fish have gills, and $p$ : All fish have gills. This argument has the form of the law of detachment, so the conclusion is $q$ : clown fish have gills.

**Your Turn 5.8.1**

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of detachment to determine a valid conclusion.

1. If my classmate likes history, then some people like history. My classmate likes history.
2. If you do not like to read, then some people do not like reading. You do not like to read.
3. If the polygon has five sides, then it is not an octagon. The polygon has five sides.

Another form of a valid conditional argument is called the law of denying the consequent, or modus tollens. Recall, that the conditional statement, $p \rightarrow q$, is logically equivalent to the contrapositive, $\sim q \rightarrow \sim p$. So, if the conditional statement is true, then the contrapositive statement is also true. By the law of detachment, if $\sim q$ is also true, then it follows that $\sim p$ must also be true. Symbolically, it has the form $((p \rightarrow q) \wedge \sim q) \rightarrow \sim p$ .

| Law of Denying the Consequent | |
|---|---|
| Premise: | $p \rightarrow q$ |
| Premise: | $\sim q$ |

| Law of Denying the Consequent | |
|---|---|
| Conclusion: | $\therefore \sim p$ |

**Checkpoint**

*The conditional statement can also be described as, "If antecedent, then consequent."This is where the law of denying the consequent gets its name.*

To verify if the law of denying the consequent is a valid argument, construct a truth table for the argument, $((p \rightarrow q) \wedge \sim q) \rightarrow \sim p$ , and verify that it is a tautology.

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \rightarrow q$ | $(p \rightarrow q) \wedge \sim q$ | $((p \rightarrow q) \wedge \sim q) \rightarrow \sim p$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | **T** |
| T | F | F | T | F | F | **T** |
| F | T | T | F | T | F | **T** |
| F | F | T | T | T | T | **T** |

To verify an argument of this form using a Venn diagram, again consider the premise: $p \rightarrow q$

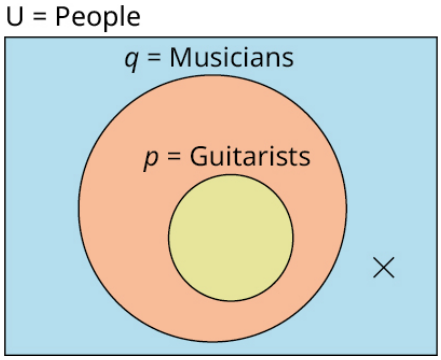



Figure 5.8.3

Because the $\times$ is also outside the set of guitarists, the statement $\sim p$ follows from the premises and the argument is valid.

**Exercise 5.8.2: Applying the Law of Denying the Consequent to Determine a Valid Conclusion**

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of denying the consequent to determine a valid conclusion.

1. If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*. Leonardo da Vinci did not paint the *Mona Lisa*.
2. If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player. Michael Jordan was a soccer player.
3. If all fish have gills, then clown fish have gills. Clown fish do not have gills.

**Answer**

1. The premises are $p \rightarrow q$ : If Leonardo da Vinci was an artist, then he painted the Mona Lisa, and $\sim q$ : Leonardo da Vinci did not paint the Mona Lisa. This argument has the form of the law of denying the consequent, so the conclusion is $\sim p$ : Leonardo da Vinci was not an artist.
2. The premises follow the form of the law of denying the consequent, so a valid conclusion would be $\sim p$. The premises are: $p \rightarrow q$ : If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player, and $\sim q$ : Michael Jordan was a soccer player. The conclusion that follows from the premises is $\sim p$ : Michael Jordan did not play for the Chicago Bulls.
3. The premises are $p \rightarrow q$ : If all fish have gills, then clown fish have gills, and $\sim q$ : Clown fish do not have gills. This argument has the form of the law denying the consequent, so the conclusion is $\sim p$ : Some fish do not have gills.

**Your Turn 5.8.2**

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of denying the consequent to determine a valid conclusion.

1. If my classmate likes history, then some people like history. Nobody likes history.
2. If Homer does not like to read, then some people do not like reading. All people like reading.
3. If the polygon has five sides, then it is not an octagon. The polygon is an octagon.

## Chain Rule for Conditional Arguments

The chain rule for conditional arguments is another form of a valid conditional argument. It is also called hypothetical syllogism or the transitivity of implication. Recall that the conditional statement $p \to q$ can also be read as $p$ implies $q$. This is where the name *transitivity of implication* comes from. The transitive property for numbers states that, if $3 < 4$ and $4 < 5$, then it follows that $3 < 5$. The chain rule extends this property to conditional statements. If the premises of the argument consist of two conditional statements, with the form "$p \to q$" and "$q \to r$," then it follows that $p \to r$. Symbolically, it has the form $((p \to q) \wedge (q \to r)) \to (p \to r)$ .

| Chain Rule for Conditional Arguments | |
|---|---|
| Premise: | $p \to q$ |
| Premise: | $q \to r$ |
| Conclusion: | $\therefore p \to r$ |

To verify the chain rule for conditional arguments, construct a truth table for the argument, $((p \to q) \wedge (q \to r)) \to (p \to r)$ , and verify that it is a tautology.

| $p$ | $q$ | $r$ | $p \to q$ | $q \to r$ | $(p \to q) \wedge (p \to r)$ | $p \to r$ | $((p \to q) \wedge (q \to r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | **T** |
| T | T | F | T | F | F | F | **T** |
| T | F | T | F | T | F | T | **T** |
| T | F | F | F | T | F | F | **T** |
| F | T | T | T | T | T | T | **T** |
| F | T | F | T | F | F | T | **T** |
| F | F | T | T | T | T | T | **T** |
| F | F | F | T | T | T | T | **T** |

To verify an argument of this form using a Venn diagram, again consider the premise $p \to q$:

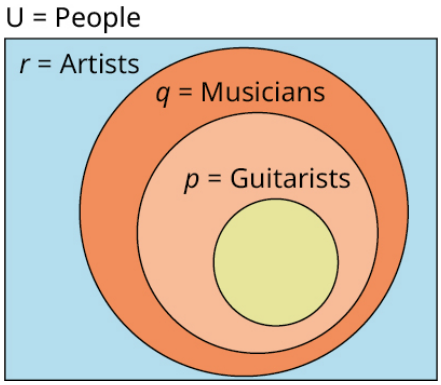



Figure 5.8.4

## Exercise 5.8.3: Applying the Chain Rule for Conditional Arguments to Determine a Valid and Sound Conclusion

Each pair of statements represents true premises in a logical argument. Based on these premises, apply the chain rule for conditional arguments to determine a valid and sound conclusion.

1. If my roommate goes to work, then my roommate will get paid. If my roommate gets paid, then my roommate will pay their bills.
2. If robins can fly, then some birds can fly. If some birds can fly, then we will watch birds fly.
3. If Irma is a teacher, then Irma has a college degree. If Irma has a college degree, then Irma graduated from college.

**Answer**

1. The premises are $p \to q$ : "If my roommate goes to work, then they will get paid," and $q \to r$ : "If my roommate gets paid, then my roommate will pay their bills." This argument has the form of the chain rule for conditional arguments, so the valid conclusion will have the form " $p \to r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If my roommate goes to work, then my roommate will pay their bills."

2. The premises are $p \to q$ : "If robins can fly, then some birds can fly," and $q \to r$ : "If some birds can fly, then we will watch them fly." This argument has the form of the chain rule for conditional arguments, so, the valid conclusion will have the form " $p \to r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If robins can fly, then we will watch birds fly."

3. The premises are $p \to q$ : (see line 1 of solution 1 and 2 above) "If Irma is a teacher, then Irma has a college degree," and $q \to r$ : "If Irma has a college degree, then Irma graduated from college." This argument has the form of the chain rule for conditional arguments, so the valid conclusion will have the form " $p \to r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If Irma is a teacher, then Irma graduated from college."

## Your Turn 5.8.3

Each pair of statements represent true premises in a logical argument. Based on these premises, apply the chain rule for conditional arguments to determine a valid and sound conclusion.

1. If my roommate does not go to work, then my roommate will not get paid. If my roommate does not get paid, then they will not be able to pay their bills.

2. If penguins cannot fly, then some birds cannot fly. If some birds cannot fly, then we will watch the news.

3. If Marcy goes to the movies, then Marcy will buy popcorn. If Marcy buys popcorn, then she will buy water.

## Check Your Understanding

1. A _____ is a logical statement used as a fact to support the conclusion of an argument.
2. A logical argument is _____ if its conclusion follows from the premises.
3. A logical argument that attempts to draw a more general conclusion from a pattern of specific premises is called an _____ argument.
4. A _____ argument draws specific conclusions from more general premises.
5. Not all arguments are true. A false or deceptive argument is called a _____.
6. If an argument is valid and all of its premises are true, then it is considered _____.

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# SECTION OVERVIEW

## 5.9: Chapter Summary


### 5.9.1: Key Terms

### 5.9.2: Key Concepts

### 5.9.3: Videos

### 5.9.4: Projects

### 5.9.5: Chapter Review

### 5.9.6: Chapter Test


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# 5.9.1: Key Terms

## Key Terms

### 2.1 Statements and Quantifiers

- logic
- logical statement
- truth values
- symbolic form
- negation of a logical statement
- quantifier
- premises
- conclusion
- inductive logical arguments

### 2.2 Compound Statements

- compound statement
- connective
- conjunction
- disjunction
- conditional
- hypothesis
- conclusion
- biconditional
- dominance of connectives

### 2.3 Constructing Truth Tables

- Truth table
- Multiplication principle
- Valid

### 2.5 Equivalent Statements

- logically equivalent
- tautology
- inverse
- converse
- contrapositive

### 2.6 De Morgan's Laws

- Boolean logic
- negation of a conditional

### 2.7 Logical Arguments

- sound
- fallacy
- deductive arguments
- law of detachment
- law of denying the consequent
- chain rule for conditional arguments

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# 5.9.2: Key Concepts

## Key Concepts

### 2.1 Statements and Quantifiers

- Logical statements have the form of a complete sentence and make claims that can be identified as true or false.
- Logical statements are represented symbolically using a lowercase letter.
- The negation of a logical statement has the opposite truth value of the original statement.
- Be able to
  - Determine whether a sentence represents a logical statement.
  - Write and translate logical statements between words and symbols.
  - Negate logical statements, including logical statements containing quantifiers of *all*, *some*, *and none*.

### 2.2 Compound Statements

- Logical connectives are used to form compound logical statements by using words such as *and, or, and if …, then*.
- A conjunction is a compound logical statement formed by combining two statements with the words “and” or “but.” If the two independent clauses are represented by $p$ and $q$, respectively, then the conjunction is written symbolically as $p \wedge q$. For the conjunction to be true, both $p$ and $q$ must be true.
- A disjunction joins two logical statements with the *or* connective. In, logic *or* is inclusive. For an *or* statement to be true at least one statement must be true, but both may also be true.
- A conditional statement has the form if $p$, then $q$, where $p$ and $q$ are logical statements. The only time the conditional statement is false is when $p$ is true, and $q$ is false.
- The biconditional statement is formed using the connective $\text{if and only}$ for the biconditional statement to be true, the true values of $p$ and $q$, must match. If $p$ is true then $q$ must be true, if $p$ is false, then $q$ must be false.
- Translate compound statements between words and symbolic form.

| Connective | Symbol | Name |
|---|---|---|
| and<br>but | $\wedge$ | conjunction |
| or | $\vee$ | disjunction, inclusive or |
| not | ~ | negation |
| if . . ., then implies | $\rightarrow$ | conditional, implication |
| if and only if | $\leftrightarrow$ | biconditional |

- The dominance of connectives explains the order in which compound logical statements containing multiple connectives should be interpreted.
- The dominance of connectives should be applied in the following order
- Parentheses
- Negations
- Disjunctions/Conjunctions, left to right
- Conditionals
- Biconditionals

| Dominance | Connective | Symbol | Evaluate |
|---|---|---|---|
| Least Dominant | Parentheses | ( ) | First |
| ↓ | Negation | ~ | ↓ |
| | Disjunction/Conjunction | ∨, ∧ | Left to right or add parentheses to specify order because or/and have equal dominance. |
| | Conditional | → | ↓ |
| Most Dominant | Biconditional | ↔ | Last |

Figure 2.18

## 2.3 Constructing Truth Tables

- Determine the true values of logical statements involving negations, conjunctions, and disjunctions.
  - The negation of a logical statement has the opposite true value of the original statement.
  - A conjunction is true when both $p$ and $q$ are true, otherwise it is false.
  - A disjunction is false when both $p$ and $q$ are false, otherwise it is true.
- Know how to construct a truth table involving negations, conjunctions, and disjunctions and apply the dominance of connectives to determine the truth value of a compound logical statement containing, negations, conjunctions, and disjunctions.

| Negation | | | Conjunction (AND) | | | | Disjunction (OR) | | |
|---|---|---|---|---|---|---|---|---|---|
| $p$ | $\sim p$ | | $p$ | $q$ | $p \wedge q$ | | $p$ | $q$ | $p \vee q$ |
| T | F | | T | T | T | | T | T | T |
| F | T | | T | F | F | | T | F | T |
| | | | F | T | F | | F | T | T |
| | | | F | F | F | | F | F | F |

- A logical statement is valid if it is always true. Know how to construct a truth table for a compound statement and use it to determine the validity of compound statements involving negations, conjunctions, and disjunctions.

## 2.4 Truth Tables for the Conditional and Biconditional

- The conditional statement, if $p$ then $q$, is like a contract. The only time it is false is when the contract has been broken. That is, when $p$ is true, and $q$ is false.

| Conditional | | |
|---|---|---|
| $p$ | $q$ | $p \rightarrow q$ |
| T | T | **T** |
| T | F | **F** |
| F | T | **T** |
| F | F | **T** |

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# 5.9.3: Videos

## Videos

### 2.1 Statements and Quantifiers



- Logic Part 1A: Logic Statements and Quantifiers

### 2.2 Compound Statements



- Logic Part 1B: Compound Statements, Connectives and Symbols

### 2.3 Constructing Truth Tables



- Logic Part 2: Truth Values of Conjunctions: Is an "AND" statement true or false?



- Logic Part 3: Truth Values of Disjunctions: Is an "OR" statement true or false?



- Logic Part 4: Truth Values of Compound Statements with "and", "or", and "not"



- Logic Part 5: What are truth tables? How do you set them up?



- Logic Part 6: More on Truth Tables and Setting Up Rows and Column Headings

## 2.4 Truth Tables for the Conditional and Biconditional



- Logic Part 8: The Conditional and Tautologies



- Logic Part 11B Biconditional and Summary of Truth Value Rules in Logic



- Logic Part 13: Truth Tables to Determine if Argument is Valid or Invalid

## 2.7 Logical Arguments



- Logic Part 14: Common Argument Forms like Modus Ponens and Tollens

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# 5.9.4: Projects

## Projects

### Logic Gates

Logic gates are the basis for all digital circuits.

1. Research and document the following terms: logic gate, OR gate, AND gate, and NOT gate.
2. Construct a diagram of a NAND gate, NOR gate, and a XOR gate by using at least two of the following gates: AND, OR, and NOT.
3. Digital electronics use a 1 for true or on, and a 0 for false or off. Create a truth table documenting all possible cases using 0s and 1s for the NAND gate, NOR gate and XOR gate.
4. Use a truth table to explain how XOR is related to the biconditional statement.

### Logical Fallacies

Fallacies are false or deceptive logical arguments.

1. Research and document the structure of five of the following named fallacies: hasty generalization, limited choice, false cause, appeal to popularity, appeal to emotion, appeal to authority, personal attack, gamblers' ruin, slippery slope, and circular reasoning.
2. Create a presentation highlighting one of the five fallacies researched in the previous question. The presentation must include an introductory slide with the title of the fallacy and the form or structure of the argument. The second slide must include an example of this fallacy as used in a commercial, a political cartoon or a current event or new article. The third slide must include an explanation of why the example on slide to is a representative example of the fallacy. The last slide must include citations for any materials used. No textbooks should be used as reference.

### Careers in Logic

Lawyers, mathematicians, and computer programmers are a few of the careers that require knowledge of logic.

1. What career are you interested in? Research how knowledge of logic applies to your chosen field of study. Then, write a cover letter for a position in your field you'd like to apply to. In the cover letter, include how your knowledge of logic qualifies you for the position you are applying for. If you do not think logic is important for your given career choice, find a position where logic is an essential element of the position and complete the project by pretending you are writing a cover letter for that job.

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# 5.9.5: Chapter Review

## Statements and Quantifiers

Fill in the blanks to complete the following sentences.

**Exercise 5.9.5.1**

The _____ of a logical statement has the opposite truth value of the original statement.

**Exercise 5.9.5.2**

_____ are logical statements presented as the facts used to support the conclusion of a logical argument.

Determine whether each of the following sentences represents a logical statement, also called a proposition. If it is a logical statement, determine whether it is true or false.

**Exercise 5.9.5.3**

Where is the restroom?

**Exercise 5.9.5.4**

No even numbers are odd numbers.

**Exercise 5.9.5.5**

4 + 3 = 8.

Write the negation of each following statement symbolically and in words.

**Exercise 5.9.5.6**

$\sim p$: Pink Floyd's album *The Wall* is not a rock opera.

**Exercise 5.9.5.7**

$q$: Some dogs are Labrador retrievers.

**Exercise 5.9.5.8**

$\sim r$: Some universities are not expensive.

Draw a logical conclusion to the following arguments, and include in both one of the following quantifiers: all, some, or none.

**Exercise 5.9.5.9**

Spaghetti noodles are made with wheat, ramen noodles are made with wheat, and lo mein noodles are made with wheat.

**Exercise 5.9.5.10**

A Porsche Boxster does not have four doors, a Volkswagen Beetle does not have four doors, and a Mazda Miata does not have four doors.

## Compound Statements

Fill in the blanks to complete the following sentences.

**Exercise 5.9.5.11**

_____ are words or symbols used to join two or more logical statement together to from a compound statement.

**Exercise 5.9.5.12**

_____ and _____ have equal dominance and are evaluated from left to right when no parentheses are present in a compound logical statement.

Translate each compound statement below into symbolic form. Given: $p$: "Tweety Bird is a bird," $q$: "Bugs is a bunny," $r$: "Bugs says, 'What's up, Doc?'," $s$: "Sylvester is a cat," and $t$: "Sylvester chases Tweety Bird."

**Exercise 5.9.5.13**

If Tweety Bird is a bird, then Sylvester will not chase him.

**Exercise 5.9.5.14**

Tweety Bird is a bird and Sylvester chases him if and only if Bugs says, "What's up Doc?"

Translate the symbolic form of each compound logical statement below into words.

Given: $p$: "Tweety Bird is a bird," $q$: "Bugs is a bunny," $r$: "Bugs says, 'What's up, Doc?'," $s$: "Sylvester is a cat," and $t$: "Sylvester chases Tweety Bird."

**Exercise 5.9.5.15**

$\sim q \vee p \rightarrow \sim s$

**Exercise 5.9.5.16**

$\sim (p \wedge \sim s) \leftrightarrow q \rightarrow r$

For each of the following compound logical statements, apply the proper dominance of connectives by adding parentheses to indicate the order to evaluate the statement.

**Exercise 5.9.5.17**

$p \vee q \wedge r \rightarrow \sim s \wedge t$

**Exercise 5.9.5.18**

$\sim p \rightarrow q \vee r \leftrightarrow p \wedge s \rightarrow \sim t$

## Constructing Truth Tables

Fill in the blanks to complete the sentences.

**Exercise 5.9.5.19**

A _____ is true if at least one of its component statements is true.

**Exercise 5.9.5.20**

For a _____ to be true, all of its component statements must be true.

Given the statements, $p$: "No fish are mammals," $q$: "All lions are cats," and $r$: "Some birds do not lay eggs," construct a truth table to determine the truth value of each compound statement below.

**Exercise 5.9.5.21**

$p \wedge \sim r$

**Exercise 5.9.5.22**

$\sim (p \vee \sim r)$

**Exercise 5.9.5.23**

$\sim p \vee q \wedge \sim (\sim r)$

Construct a truth table to analyze all the possible outcomes of the following statements, and determine whether the statements are valid.

**Exercise 5.9.5.24**

$\sim p \vee q \wedge p$

**Exercise 5.9.5.25**

$\sim p \vee q \vee \sim q$

## Truth Tables for the Conditional and Biconditional

Fill in the blanks to complete the following sentences.

**Exercise 5.9.5.26**

If the , $p$, of a conditional statement is true, then the conclusion, $q$, must also be true for the conditional statement $p \rightarrow q$ to be true.

**Exercise 5.9.5.27**

The biconditional statement $p \leftrightarrow q$ is whenever the truth value of $p$ matches the truth value of $q$, otherwise it is

Complete the truth tables below to determine the truth value of the proposition in the last column.

**Exercise 5.9.5.28**

| $p$ | $q$ | $r$ | $p \vee q$ | $\sim (p \vee q)$ | $\sim r$ | $\sim (p \vee q) \rightarrow \sim r$ |
|---|---|---|---|---|---|---|
| F | F | T | | | | |

**Exercise 5.9.5.29**

| $p$ | $q$ | $\sim q$ | $p \rightarrow q$ | $\sim (p \rightarrow q)$ | $p \wedge \sim q$ | $\sim (p \rightarrow q) \leftrightarrow (p \wedge \sim q$ |
|---|---|---|---|---|---|---|
| T | F | | | | | |

Assume the following statements are true. $p$: “Poof is a baby fairy,” $q$: “Timmy Turner has fairly odd parents,” $r$: “Cosmo and Wanda will grant Timmy’s wishes,” and $t$: “Timmy Turner is 10 years old.” Translate each of the following statements into symbolic form, then determine its truth value.

**Exercise 5.9.5.30**

If Timmy Turner is 10 years old and Poof is not a baby fairy, then Timmy Turner has fairly odd parents.

**Exercise 5.9.5.31**

Cosmos and Wanda will not grant Timmy’s wishes if and only if Timmy Turner is 10 years old or he does not have fairly odd parents.

**Exercise 5.9.5.32**

Construct a truth table to analyze all the possible outcomes and determine the validity of the following argument.

$\sim p \vee q \leftrightarrow \sim q \rightarrow \sim p$

## Equivalent Statements

Fill in the blanks to complete the sentences below. 33. The _____ is logically equivalent to the inverse $\sim p \rightarrow \sim q$ $\sim p \rightarrow \sim q$.

**Exercise 5.9.5.34**

The _____ is logically equivalent to the conditional $p \rightarrow qp \rightarrow q$.

Use the conditional statement, $p \rightarrow qp \rightarrow q$: “If Novak makes the basket, then Novak’s team will win the game," to answer the following questions. 35. Write the conclusion of the conditional statement in words and label it appropriately.

**Exercise 5.9.5.36**

Write the hypothesis of the conditional statement in words and label it appropriately.

**Exercise 5.9.5.37**

Identify the following statement as the converse, inverse, or contrapositive: “If Novak does not make the basket, then his team will not win the game.”

**Exercise 5.9.5.38**

Identify the following statement as the converse, inverse, or contrapositive: “If Novak’s team wins the game, then he made the basket.”

## De Morgan’s Laws

Fill in the blanks to complete the sentences.

**Exercise 5.9.5.39**

De Morgan’s Law for the negation of a disjunction states that \(\sim(p \vee q) \equiv\)_____.

**Exercise 5.9.5.40**

De Morgan’s Law for the negation of a conjunctions states that _____\( \equiv \sim p \vee \sim q\).

**Exercise 5.9.5.41**

Apply De Morgan's Law to write the statement without parentheses: \(~ (~p \wedge q).

**Exercise 5.9.5.42**

Apply the property for the negation of a conditional to write the statement as a conjunction or disjunction: \(~(~p \wedge q \rightarrow ~r).

**Exercise 5.9.5.43**

Write the negation of the conditional statement in words: If Thomas Edison invented the phonograph, then albums are made of vinyl, or the transistor radio was the first portable music device.

**Exercise 5.9.5.44**

Construct a truth table to verify that the logical property is valid: $\sim (\sim p \rightarrow \sim q) \equiv \sim p \wedge q$.

## Logical Arguments

Fill in the blanks to complete the sentences below.

**Exercise 5.9.5.45**

The _____ ___ _____ is a valid logical argument with premises, $p \rightarrow q$ and $p$, used to support the conclusion, $q$.

**Exercise 5.9.5.46**

The chain rule for conditional arguments states that the _____ property applies to conditional arguments, so that: \( (p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r).

Assume each pair of statements represents true premises in a logical argument. Based on these premises, state a valid conclusion that is consistent with the form of the argument.

**Exercise 5.9.5.47**

If the Tampa Bay Buccaneers did not win Super Bowl LV, then Tom Brady was not their quarterback. Tom Brady was the Tampa Bay Buccaneers quarterback.

**Exercise 5.9.5.48**

If $\sim q$, then $p$ and if $r$, then $\sim q$.

**Exercise 5.9.5.49**

If Kamala Harris is the vice president of the United States, then Kamala Harris is the president of the U.S. Senate. Kamala Harris is the vice president of the United States.

**Exercise 5.9.5.50**

Construct a truth table or Venn diagram to prove whether the following argument is valid. If the argument is valid, determine whether it is sound.

If all frogs are brown, then Kermit is not a frog. Kermit is a frog. Therefore, some frogs are not brown.

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# 5.9.6: Chapter Test

## Chapter Test

Determine whether each of the following sentences represent a logical statement. If it is a logical statement, determine whether it is true or false.

1. $1+2-3=0$

2. Please, sit down over there.

3. All mammals lay eggs.

Write the negation of each statement below.

4. Some monkeys do not have tails.

5. $p \wedge \sim q$

6. If the plumber does not remove the clog, then the homeowner will not pay the plumber.

Given: $p$ : Frodo is a hobbit, $q$ : Gandalf is a wizard, $r$ : Frodo and Samwise will take the ring to Mordor, and $s$ : Gollum will help Frodo get into Mordor.
Translate the symbolic form of each compound logical statement into words.

7. $p \vee q \leftrightarrow r$

8. $\sim(\sim r \vee \sim s)$

Translate the written form of each compound logical statement into symbolic form.

9. Frodo and Samwise will take the ring to Mordor or Gandalf is not a wizard and Frodo is a hobbit.

10. If Gollum will not help Frodo get into Mordor, then Gandalf is not a wizard and Frodo is not a hobbit.

For each of the following compound logical statements, apply the proper dominance of connectives by adding parentheses to indicate the order in which the statement must be evaluated

11. $\sim p \rightarrow q \leftrightarrow r$

12. $\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q$

13. Complete the truth table to determine the truth value of the proposition in the last column.

| $p$ | $q$ | $p \wedge q$ | $\sim q$ | $p \vee \sim q$ | $(p \wedge q) \leftrightarrow (p \vee \sim q)$ |
|---|---|---|---|---|---|
| F | T | | | | |

Given the true statements $p$ : "A right triangle has one 90-degree angle," $q$ : "The triangle is a right triangle," $r$ : " $a^2+b^2=c^2$, " and $s$ : "The longest side of a triangle is $c$ implies $a+b$ must be $>c$." Write each of the following compound statements in symbolic form, then construct a truth table to determine the truth value of the compound statement.

14. If a triangle is a right triangle, then it does not have one 90-degree angle or $a^2+b^2=c^2$ .

15. The triangle is a right triangle, or a right triangle does not have a 90-degree angle, if and only if it is not the case that the longest side of a triangle is $c$ implies $a+b$ must be $>c$.

Use the conditional statement, $p \rightarrow q$ : "If Phil Mickelson is 50 years old, then Phil Mickelson won the Player's Championship," to answer the following questions.

16. Write the converse statement in words.

17. If the conditional statement is true, and the hypothesis is true, what is a valid conclusion to the argument?

18. If the conditional statement is true, and the conclusion is false, what is a valid conclusion to the argument?

19. Construct a truth table to analyze all the possible outcomes and determine the validity of the following argument.

$$\sim p \vee q \leftrightarrow q \rightarrow p$$

20. Construct a truth table or Venn diagram to prove whether the following argument is valid. If the argument is valid, determine whether it is sound.
If John Mayer played MTV unplugged, then some guitars are acoustic. John Mayer played MTV unplugged. Therefore, some guitars are acoustic.

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# CHAPTER OVERVIEW

## 6: Sets and Counting

**Learning Objectives**

In this chapter, you will learn to:

- Use set theory and Venn diagrams to solve counting problems.
- Use the Multiplication Axiom to solve counting problems.
- Use Permutations to solve counting problems.
- Use Combinations to solve counting problems.
- Use the Binomial Theorem to expand $(x+y)^n$


6.1: Sets and Counting

6.1.1: Sets and Counting (Exercises)

6.2: Tree Diagrams and the Multiplication Axiom

6.2.1: Tree Diagrams and the Multiplication Axiom (Exercises)

6.3: Permutations

6.3.1: Permutations (Exercises)

6.4: Circular Permutations and Permutations with Similar Elements

6.4.1: Circular Permutations and Permutations with Similar Elements (Exercises)

6.5: Combinations

6.5.1: Combinations (Exercises)

6.6: Combinations- Involving Several Sets

6.6.1: Combinations- Involving Several Sets (Exercises)

6.7: Binomial Theorem

6.7.1: Binomial Theorem (Exercises)

6.8: Chapter Review



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# 6.1: Sets and Counting

**Learning Objectives**

In this section, you will learn to:

1. Use set notation to represent unions, intersections, and complements of sets
2. Use Venn diagrams to solve counting problems.

## Introduction to Sets

In this section, we will familiarize ourselves with set operations and notations, so that we can apply these concepts to both counting and probability problems. We begin by defining some terms.

**Definition: Set and Elements**

A **set** is a collection of objects, and its members are called the **elements** of the set.

We name the set by using capital letters, and enclose its members in braces. Suppose we need to list the members of the chess club. We use the following set notation.

C = { Ken, Bob, Tran, Shanti, Eric }

**Definition: Empty Set**

A set that has no members is called an **empty set**. The empty set is denoted by the symbol $\varnothing$.

**Definition: Set Equality**

Two sets are **equal** if they have the same elements.

**Definition: Subset**

Set $A$ is a **subset** of a set $B$ if every member of $A$ is also a member of $B$.

Suppose C = { Al, Bob, Chris, David, Ed } and A = { Bob, David }. Then A is a subset of C, written as A $\subseteq$ C.

Every set is a subset of itself, and the empty set is a subset of every set.

**Definition: Union of Two Sets**

Let $A$ and $B$ be two sets, then the **union** of $A$ and $B$, written as $A \cup B$, is the set of all elements that are either in $A$ or in $B$, or in both $A$ and $B$.

**Definition: Intersection of Two Sets**

Let $A$ and \(B\0 be two sets, then the **intersection** of $A$ and $B$, written as $A \cap B$, is the set of all elements that are common to both sets $A$ and $B$.

**Definition: Universal Set**

A **universal set** $U$ is the set consisting of all elements under consideration.

### Definition: Complement of a Set and Disjoint Sets

Let $A$ be any set, then the **complement** of set $A$, written as $\bar{A}$, is the set consisting of elements in the universal set $U$ that are not in $A$.

Two sets A and B are called **disjoint sets** if their intersection is an empty set. Clearly, a set and its complement are disjoint; however two sets can be disjoint and not be complements.

### Example 6.1.1

List all the subsets of the set of primary colors { red, yellow, blue }.

**Solution**

The subsets are ∅, {red}, {yellow}, {blue}, {red, yellow}, {red, blue}, {yellow, blue}, {red, yellow, blue}

Note that the empty set is a subset of every set, and a set is a subset of itself.

### Example 6.1.2

Let F = { Aikman, Jackson, Rice, Sanders, Young }, and let B = { Griffey, Jackson, Sanders, Thomas }.
Find the intersection of the sets F and B.

**Solution**

The intersection of the two sets is the set whose elements belong to both sets. Therefore, F ∩ B = { Jackson, Sanders }

### Example 6.1.3

Find the union of the sets F and B given as follows.

- F = { Aikman, Jackson, Rice, Sanders, Young }
- B = { Griffey, Jackson, Sanders, Thomas }

**Solution**

The union of two sets is the set whose elements are either in A or in B or in both A and B. Observe that when writing the union of two sets, the repetitions are avoided.

$$F \cup B = \{ \text{Aikman, Griffey, Jackson, Rice, Sanders, Thomas, Young} \}$$

### Example 6.1.4

Let the universal set U = { red, orange, yellow, green, blue, indigo, violet }, and P = { red, yellow, blue }. Find the complement of P.

**Solution**

The complement of a set P is the set consisting of elements in the universal set U that are not in P. Therefore,

$$\bar{P} = \{ \text{orange, green, indigo, violet} \}$$

To achieve a better understanding, let us suppose that the universal set U represents the colors of the spectrum, and P the primary colors, then $\bar{P}$ represents those colors of the spectrum that are not primary colors.

### Example 6.1.5

Let the universal set U = { red, orange, yellow, green, blue, indigo, violet }, and P = { red, yellow, blue } . Find a set R so that R is not the complement of P but R and P are disjoint.

**Solution**

R = { orange, green } and P = { red, yellow, blue } are disjoint because the intersection of the two sets is the empty set. The sets have no elements in common. However they are not complements because their union P ∪ R = { red, yellow, blue, orange, green } is not equal to the universal set U.

**Example 6.1.6**

Let U = { red, orange, yellow, green, blue, indigo, violet }, P = { red, yellow, blue }, Q = { red, green }, and R = { orange, green, indigo }. Find $\overline{P \cup Q} \cap \overline{R}$.

**Solution**

We do the problems in steps:

P ∪ Q = { red, yellow, blue, green }
$\overline{P \cup Q}$ = { orange, indigo, violet }
$\overline{R}$ = { red, yellow, blue, violet }
$\overline{P \cup Q} \cap \overline{R}$ = {violet}

## Venn Diagrams

We now use **Venn diagrams** to illustrate the relations between sets. In the late 1800s, an English logician named John Venn developed a method to represent relationship between sets. He represented these relationships using diagrams, which are now known as Venn diagrams.

A Venn diagram represents a set as the interior of a circle. Often two or more circles are enclosed in a rectangle where the rectangle represents the universal set. To visualize an intersection or union of a set is easy. In this section, we will mainly use Venn diagrams to sort various populations and count objects.

**Example 6.1.7**

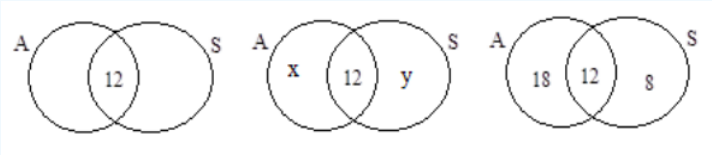
Suppose a survey of car enthusiasts showed that over a certain time period, 30 drove cars with automatic transmissions, 20 drove cars with standard transmissions, and 12 drove cars of both types. If everyone in the survey drove cars with one of these transmissions, how many people participated in the survey?

**Solution**

We will use Venn diagrams to solve this problem.

Let the set A represent those car enthusiasts who drove cars with automatic transmissions, and set S represent the car enthusiasts who drove the cars with standard transmissions. Now we use Venn diagrams to sort out the information given in this problem.

Since 12 people drove both cars, we place the number 12 in the region common to both sets.




Because 30 people drove cars with automatic transmissions, the circle A must contain 30 elements. This means that

x + 12 = 30, or x = 18.

Similarly, since 20 people drove cars with standard transmissions, the circle B must contain 20 elements.

Thus, y + 12 = 20 which in turn makes y = 8.

Now that all the information is sorted out, it is easy to read from the diagram that 18 people drove cars with automatic transmissions only, 12 people drove both types of cars, and 8 drove cars with standard transmissions only.

Therefore, 18 + 12 + 8 = 38 people took part in the survey.

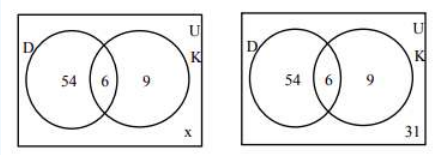
## Example 6.1.8

A survey of 100 people in California indicates that 60 people have visited Disneyland, 15 have visited Knott's Berry Farm, and 6 have visited both. How many people have visited neither place?

**Solution**

The problem is similar to the one in the previous example.

Let the set D represent the people who have visited Disneyland, and K the set of people who have visited Knott's Berry Farm.




We fill the three regions associated with the sets D and K in the same manner as before. Since 100 people participated in the survey, the rectangle representing the universal set U must contain 100 objects. Let x represent those people in the universal set that are neither in the set D nor in K. This means 54 + 6 + 9 + x = 100, or x = 31.

Therefore, there are 31 people in the survey who have visited neither place.

## Example 6.1.9

A survey of 100 exercise conscious people resulted in the following information:

- 50 jog, 30 swim, and 35 cycle
- 14 jog and swim
- 7 swim and cycle
- 9 jog and cycle
- 3 people take part in all three activities

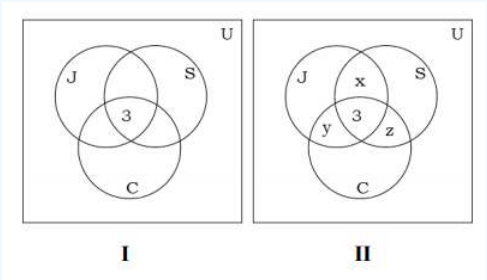
a. How many jog but do not swim or cycle?
b. How many take part in only one of the activities?
c. How many do not take part in any of these activities?

**Solution**

Let J represent the set of people who jog, S the set of people who swim, and C who cycle.

In using Venn diagrams, our ultimate aim is to assign a number to each region. We always begin by first assigning the number to the innermost region and then working our way out.

We'll show the solution step by step. As you practice working out such problems, you will find that with practice you will not need to draw multiple copies of the diagram.



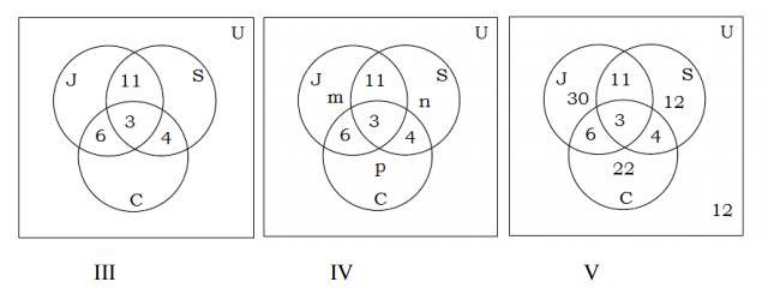

We place a 3 in the innermost region of figure I because it represents the number of people who participate in all three activities. Next we use figure II to compute x, y and z.

Since 14 people jog and swim, x + 3 = 14, or x = 11.

The fact that 9 people jog and cycle results in y + 3 = 9, or y = 6.

Since 7 people swim and cycle, z + 3 = 7, or z = 4.

This information is depicted in figure III.




Now we proceed to find the unknowns m, n and p, as shown in figure IV

Since 50 people jog, m + 11 + 6 + 3 = 50, or m = 30.

30 people swim, therefore, n + 11 + 4 + 3 = 30, or n = 12.

35 people cycle, therefore, p + 6 + 4 + 3 = 35, or p = 22.

By adding all the entries in all three sets, we get a sum of 88.
Since 100 people were surveyed, the number inside the universal set but outside of all three sets is 100 - 88, or 12.

In figure V, all the information is sorted out, and the questions can readily be answered.

a. The number of people who jog but do not swim or cycle is 30.
b. The number who take part in only one of these activities is 30 + 12 + 22 = 64.
c. The number of people who do not take part in any of these activities is 12.

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# 6.1.1: Sets and Counting (Exercises)

## SECTION 7.1 PROBLEM SET: SETS AND COUNTING

Find the indicated sets.

| | |
|---|---|
| 1. List all subsets of the following set.<br>{ Al, Bob } | 2. List all subsets of the following set.<br>{ Al, Bob, Chris } |
| 3. List the elements of the following set.<br>{ Al, Bob, Chris, Dave } ∩ { Bob, Chris, Dave, Ed } | 4. List the elements of the following set.<br>{ Al, Bob, Chris, Dave } ∪ {Bob, Chris, Dave, Ed } |

Problems 5 – 8: Let Universal set U = { a, b, c, d, e, f, g, h, i, j }, sets V = { a, e, i, f, h }, W = { a, c, e, g, i }. List the members of the following sets.

| | |
|---|---|
| 5. V ∪ W | 6. V ∩ W |
| 7. $\overline{V \cup W}$ | 8. $\overline{V} \cap \overline{W}$ |

Problems 9 – 12: Let Universal set U = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 } and sets A = { 1, 2, 3, 4, 5 }, B = { 1, 3, 4, 6 }, and C = { 2, 4, 6 }.

List the members of the following sets.

| | |
|---|---|
| 9. A ∪ B | 10. A ∩ C |
| 11. $\overline{A \cup B} \cap C$ | 12. $\bar{A} \cup \overline{B \cap C}$ |

Use Venn Diagrams to find the number of elements in the following sets.

| | |
|---|---|
| 13. In Mrs. Yamamoto's class of 35 students, 12 students are taking history, 18 are taking English, and 4 are taking both. Draw a Venn diagram and use it determine how many students are taking neither history nor English. | 14. In a survey of 1200 college students, 700 used Spotify to listen to music and 400 used iTunes to listen to music; of these, 100 used both.<br>a. Draw a Venn Diagram and find the number of people in each region of the diagram.<br>b. How many used either Spotify or iTunes? |
| 15. A survey of athletes revealed that for their minor aches and pains, 30 used aspirin, 50 used ibuprofen, and 15 used both. How many athletes were surveyed? | 16. In 2016, 80 college students were surveyed about what video services they subscribed to. Suppose the survey showed that 50 use Amazon Prime, 30 use Netflix, 20 use Hulu. Of those, 13 use Amazon Prime and Netflix, 9 use Amazon Prime and Hulu, 7 use Netflix and Hulu. 3 students use all three services.<br>a. Draw a Venn Diagram and use it to determine the number of people in each region of the diagram.<br>b. How many use at least one of these?<br>c. How many use none of these? |

17. A survey of 100 students at a college finds that 50 take math, 40 take English, and 30 take history. Of these 15 take English and math, 10 take English and history, 10 take math and history, and 5 take all three subjects. Draw a Venn diagram and find the numbers in each region. Use the diagram to answer the questions below.
    a. Find the number of students taking math but not the other two subjects.
    b. The number of students taking English or math but not history.
    c. The number of students taking none of these subjects.

18. In a survey of investors it was found that 100 invested in stocks, 60 in mutual funds, and 50 in bonds. Of these, 35 invested in stocks and mutual funds, 30 in mutual funds and bonds, 28 in stocks and bonds, and 20 in all three. Draw a Venn diagram and find the numbers in each region. Use the diagram to answer the questions below.
    a. Find the number of investors that participated in the survey.
    b. How many invested in stocks or mutual funds but not in bonds?
    c. How many invested in exactly one type of investment?

19. A survey of 100 students at a college finds that 50 take math, 40 take English, and 30 take history. Of these 15 take English and math, 10 take English and history, 10 take math and history, and 5 take all three subjects. (This question relates back to question #17.) For each of the following draw a Venn Diagram and shade the indicated sets and determine the number of students in the set.

| | |
|---|---|
| a. Students who take at least one of these classes | b. Students who take exactly one of these classes |
| c. Students who take at least two of these classes | d. Students who take exactly two of these classes |
| e. Students who take at most two of these classes | f. Students who take English or Math but not both |
| g. Students who take Math or History but not English | h. Students who take all of these classes |

20. In a survey of investors it was found that 100 invested in stocks, 60 in mutual funds, and 50 in bonds. Of these, 35 invested in stocks and mutual funds, 30 in mutual funds and bonds, 28 in stocks and bonds, and 20 in all three. (This question relates back to question #18.) For each of the following draw a Venn Diagram and shade the indicated sets and determine the number of students in the set.

| | |
|---|---|
| a. Investors who invested in mutual funds only | b. Investors who invested in stocks and bonds but not mutual funds |
| c. Investors who invested in exactly one of these investments | d. Investors who invested in exactly two of these investments |
| e. Investors who invested in at least two of these investments | f. Investors who invested in at most two of these investments |
| g. Investors who did not invest in bonds | h. Investors who invested in all three investments |

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# 6.2: Tree Diagrams and the Multiplication Axiom

**Learning Objectives**

In this section you will learn to

1. Use trees to count possible outcomes in a multi-step process
2. Use the multiplication axiom to count possible outcomes in a multi-step process.

In this chapter, we are trying to develop counting techniques that will be used in the next chapter to study probability. One of the most fundamental of such techniques is called the Multiplication Axiom. Before we introduce the multiplication axiom, we first look at some examples.

**Example 6.2.1**

If a woman has two blouses and three skirts, how many different outfits consisting of a blouse and a skirt can she wear?

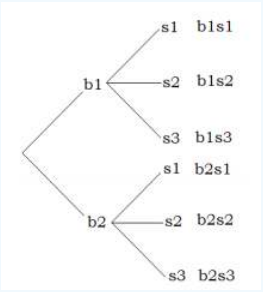
**Solution**

Suppose we call the blouses $b_1$ and $b_2$, and skirts $s_1$, $s_2$, and $s_3$.

We can have the following six outfits.

$$b_1s_1, b_1s_2, b_1s_3, b_2s_1, b_2s_2, b_2s_3$$

Alternatively, we can draw a tree diagram:




The tree diagram gives us all six possibilities. The method involves two steps. First the woman chooses a blouse. She has two choices: blouse one or blouse two. If she chooses blouse one, she has three skirts to match it with; skirt one, skirt two, or skirt three. Similarly if she chooses blouse two, she can match it with each of the three skirts, again. The tree diagram helps us visualize these possibilities.

The reader should note that the process involves two steps. For the first step of choosing a blouse, there are two choices, and for each choice of a blouse, there are three choices of choosing a skirt. So altogether there are $2 \cdot 3 = 6$ possibilities.

If, in the previous example, we add the shoes to the outfit, we have the following problem.

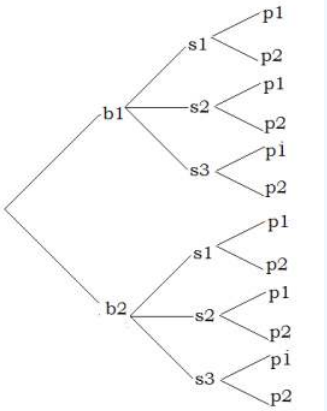
**Example 6.2.2**

If a woman has two blouses, three skirts, and two pumps, how many different outfits consisting of a blouse, a skirt, and a pair of pumps can she wear?

**Solution**

Suppose we call the blouses $b_1$ and $b_2$, the skirts $s_1$, $s_2$, and $s_3$, and the pumps $p_1$, and $p_2$.

The following tree diagram results.




We count the number of branches in the tree, and see that there are 12 different possibilities.

This time the method involves three steps. First, the woman chooses a blouse. She has two choices: blouse one or blouse two. Now suppose she chooses blouse one. This takes us to step two of the process which consists of choosing a skirt. She has three choices for a skirt, and let us suppose she chooses skirt two. Now that she has chosen a blouse and a skirt, we have moved to the third step of choosing a pair of pumps. Since she has two pairs of pumps, she has two choices for the last step. Let us suppose she chooses pumps two. She has chosen the outfit consisting of blouse one, skirt two, and pumps two, or $b_1 s_2 p_2$. By looking at the different branches on the tree, one can easily see the other possibilities.

The important thing to observe here, again, is that this is a three step process. There are two choices for the first step of choosing a blouse. For each choice of a blouse, there are three choices of choosing a skirt, and for each combination of a blouse and a skirt, there are two choices of selecting a pair of pumps.

All in all, we have $2 \cdot 3 \cdot 2 = 12$ different possibilities.

Tree diagrams help us visualize the different possibilities, but they are not practical when the possibilities are numerous. Besides, we are mostly interested in finding the number of elements in the set and not the actual list of all possibilities; once the problem is envisioned, we can solve it without a tree diagram. The two examples we just solved may have given us a clue to do just that.

Let us now try to solve Example $6.2.2$ without a tree diagram. The problem involves three steps: choosing a blouse, choosing a skirt, and choosing a pair of pumps. The number of ways of choosing each are listed below. By multiplying these three numbers we get 12, which is what we got when we did the problem using a tree diagram.

| The number of ways of choosing a blouse | The number of ways of choosing a skirt | The number of ways of choosing pumps |
|---|---|---|
| 2 | 3 | 2 |

The procedure we just employed is called the multiplication axiom.

### The Multiplication Axiom

If a task can be done in $m$ ways, and a second task can be done in $n$ ways, then the operation involving the first task followed by the second can be performed in $m \cdot n$ ways.

The general multiplication axiom is not limited to just two tasks and can be used for any number of tasks.

### Example $6.2.3$

A truck license plate consists of a letter followed by four digits. How many such license plates are possible?

**Solution**

Since there are 26 letters and 10 digits, we have the following choices for each.

| Letter | Digit | Digit | Digit | Digit |
|---|---|---|---|---|
| 26 | 10 | 10 | 10 | 10 |

Therefore, the number of possible license plates is $26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 260,000$.

### Example 6.2.4

In how many different ways can a 3-question true-false test be answered?

**Solution**

Since there are two choices for each question, we have

| Question 1 | Question 2 | Question 3 |
|---|---|---|
| 2 | 2 | 2 |

Applying the multiplication axiom, we get $2 \cdot 2 \cdot 2 = 8$ different ways.

We list all eight possibilities: TTT, TTF, TFT, TFF, FTT, FTF, FFT, FFF

The reader should note that the first letter in each possibility is the answer corresponding to the first question, the second letter corresponds to the answer to the second question, and so on. For example, TFF, says that the answer to the first question is given as true, and the answers to the second and third questions false.

### Example 6.2.5

In how many different ways can four people be seated in a row?

**Solution**

Suppose we put four chairs in a row, and proceed to put four people in these seats.

There are four choices for the first chair we choose. Once a person sits down in that chair, there are only three choices for the second chair, and so on. We list as shown below.

| | | | |
|---|---|---|---|
| 4 | 3 | 2 | 1 |

So there are altogether $4 \cdot 3 \cdot 2 \cdot 1 = 24$ different ways.

### Example 6.2.6

How many three-letter word sequences can be formed using the letters { A, B, C } if no letter is to be repeated?

**Solution**

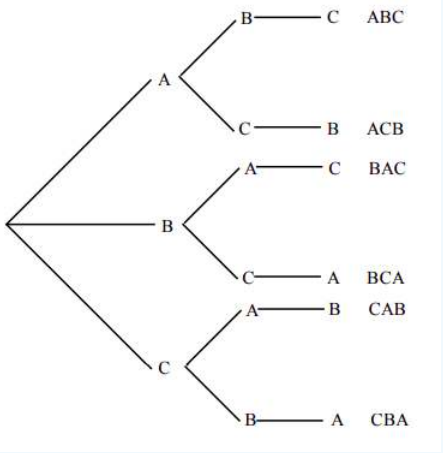
The problem is very similar to the previous example.

Imagine a child having three building blocks labeled A, B, and C. Suppose he puts these blocks on top of each other to make word sequences. For the first letter he has three choices, namely A, B, or C. Let us suppose he chooses the first letter to be a B, then for the second block which must go on top of the first, he has only two choices: A or C. And for the last letter he has only one choice. We list the choices below.

| | | |
|---|---|---|
| 3 | 2 | 1 |

Therefore, 6 different word sequences can be formed.

Finally, we'd like to illustrate this with a tree diagram showing all six possibilities.




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# 6.2.1: Tree Diagrams and the Multiplication Axiom (Exercises)

Do the following problems using a tree diagram or the multiplication axiom.

| | |
|---|---|
| 1. A man has 3 shirts, and 2 pairs of pants. Use a tree diagram to determine the number of possible outfits. | 2. In a city election, there are 2 candidates for mayor, and 3 for supervisor. Use a tree diagram to find the number of ways to fill the two offices. |
| 3. There are 4 roads from Town A to Town B, 2 roads from Town B to Town C. Use a tree diagram to find the number of ways one can travel from Town A to Town C. | 4. Brown Home Construction offers a selection of 3 floor plans, 2 roof types, and 2 exterior wall types. Use a tree diagram to determine the number of possible homes available. |
| 5. For lunch, a small restaurant offers 2 types of soups, three kinds of sandwiches, and two types of soft drinks. Use a tree diagram to determine the number of possible meals consisting of a soup, sandwich, and a soft drink. | 6. A California license plate consists of a number from 1 to 5, then three letters followed by three digits. How many such plates are possible? |

Do the following problems using the Multiplication Axiom

| | |
|---|---|
| 7. A license plate consists of three letters followed by three digits. How many license plates are possible if no letter may be repeated? | 8. How many different 4-letter radio station call letters can be made if the first letter must be K or W and no letters can be repeated? |
| 9. How many seven-digit telephone numbers are possible if the first two digits cannot be ones or zeros? | 10. How many 3-letter word sequences can be formed using the letters {a, b, c, d} if no letter is to be repeated? |

Use a tree diagram for questions 11 and 12:

| | |
|---|---|
| 11. A family has two children, use a tree diagram to determine all four possibilities of outcomes by gender. | 12. A coin is tossed three times and the sequence of heads and tails is recorded. Use a tree diagram to list all the possible outcomes. |

Do the following problems using the Multiplication Axiom

| | |
|---|---|
| 13. In how many ways can a 4-question true-false test be answered? | 14. In how many ways can three people be arranged to stand in a straight line? |
| 15. A combination lock is opened by first turning to the left, then to the right, and then to the left again. If there are 30 digits on the dial, how many possible combinations are there? | 16. How many different answers are possible for a multiple-choice test with 10 questions and five possible answers for each question? |
| 17. In the past, a college required students to use a 4 digit PIN (Personal Identification Number) as their password for its registration system. How many different PINs are possible if each must have 4 digits with no restrictions on selection or arrangement of the digits used? | 18. The college decided that a more secure password system is needed. New passwords must have 3 numerical digits followed by 6 letters. There are no restrictions on the selection of the numerical digits. However, the letters I and O are not permitted. How many different passwords are possible? |

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# 6.3: Permutations

**Learning Objectives**

In this section you will learn to

1. Count the number of possible permutations (ordered arrangement) of n items taken r at a time
2. Count the number of possible permutations when there are conditions imposed on the arrangements
3. Perform calculations using factorials

In Example 7.2.6 of section 7.2, we were asked to find the word sequences formed by using the letters { A, B, C } if no letter is to be repeated. The tree diagram gave us the following six arrangements.

ABC, ACB, BAC, BCA, CAB, and CBA.

Arrangements like these, where order is important and no element is repeated, are called permutations.

**Definition: Permutations**

A permutation of a set of elements is an ordered arrangement where each element is used once.

**Example 6.3.1**

How many three-letter word sequences can be formed using the letters { A, B, C, D }?

***Solution***

There are four choices for the first letter of our word, three choices for the second letter, and two choices for the third.

| | | |
|---|---|---|
| 4 | 3 | 2 |

Applying the multiplication axiom, we get $4 \cdot 3 \cdot 2 = 24$ different arrangements.

**Example 6.3.2**

How many permutations of the letters of the word ARTICLE have consonants in the first and last positions?

***Solution***

In the word ARTICLE, there are 4 consonants.

Since the first letter must be a consonant, we have four choices for the first position, and once we use up a consonant, there are only three consonants left for the last spot. We show as follows:

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | | | | | | 3 |

Since there are no more restrictions, we can go ahead and make the choices for the rest of the positions.

So far we have used up 2 letters, therefore, five remain. So for the next position there are five choices, for the position after that there are four choices, and so on. We get

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | 5 | 4 | 3 | 2 | 1 | 3 |

So the total permutations are $4 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 = 1440$ .

**Example 6.3.3**

Given five letters { A, B, C, D, E }. Find the following:

a. The number of four-letter word sequences.
b. The number of three-letter word sequences.
c. The number of two-letter word sequences.

***Solution***

The problem is easily solved by the multiplication axiom, and answers are as follows:

a. The number of four-letter word sequences is $5 \cdot 4 \cdot 3 \cdot 2 = 120$ .
b. The number of three-letter word sequences is $5 \cdot 4 \cdot 3 = 60$ .
c. The number of two-letter word sequences is $5 \cdot 4 = 20$.

We often encounter situations where we have a set of n objects and we are selecting r objects to form permutations. We refer to this as **permutations of n objects taken r at a time**, and we write it as **nPr**.

Therefore, the above example can also be answered as listed below.

a. The number of four-letter word sequences is 5P4 = 120.
b. The number of three-letter word sequences is 5P3 = 60.
c. The number of two-letter word sequences is 5P2 = 20.

Before we give a formula for nPr, we'd like to introduce a symbol that we will use a great deal in this as well as in the next chapter.

**Definition: Factorial**

$$n! = n(n-1)(n-2)(n-3)\cdots 3 \cdot 2 \cdot 1$$

where $n$ is a natural number.

$$0! = 1$$

Now we define nPr.

**Definition: nPr**

**The Number of Permutations of n Objects Taken r at a Time**

$$nPr = n(n-1)(n-2)(n-3)\cdots(n-r+1)$$

or

$$nPr = \frac{n!}{(n-r)!}$$

where $n$ and $r$ are natural numbers.

The reader should become familiar with both formulas and should feel comfortable in applying either.

**Example 6.3.4**

Compute the following using both formulas.

a. 6P3
b. 7P2

**Solution**

We will identify $n$ and $r$ in each case and solve using the formulas provided.

a. $6P3 = 6 \cdot 5 \cdot 4 = 120$, alternately

$$6P3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 120$$

b. $7P2 = 7 \cdot 6 = 42$, or

$$7P2 = \frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 42$$

Next we consider some more permutation problems to get further insight into these concepts.

### Example 6.3.5

In how many different ways can 4 people be seated in a straight line if two of them insist on sitting next to each other?

**Solution**

Let us suppose we have four people A, B, C, and D. Further suppose that A and B want to sit together. For the sake of argument, we tie A and B together and treat them as one person.

The four people are $\boxed{AB}$ CD. Since $\boxed{AB}$ is treated as one person, we have the following possible arrangements.

$$\boxed{AB}CD, \boxed{AB}DC, C\boxed{AB}D, D\boxed{AB}C, CD\boxed{AB}, DC\boxed{AB}$$

Note that there are six more such permutations because A and B could also be tied in the order BA. And they are

$$\boxed{BA}CD, \boxed{BA}DC, C\boxed{BA}D, D\boxed{BA}C, CD\boxed{BA}, DC\boxed{BA}$$

So altogether there are 12 different permutations.

Let us now do the problem using the multiplication axiom.

After we tie two of the people together and treat them as one person, we can say we have only three people. The multiplication axiom tells us that three people can be seated in 3! ways. Since two people can be tied together 2! ways, there are 3! 2! = 12 different arrangements

### Example 6.3.6

You have 4 math books and 5 history books to put on a shelf that has 5 slots. In how many ways can the books be shelved if the first three slots are filled with math books and the next two slots are filled with history books?

***Solution***

We first do the problem using the multiplication axiom.

Since the math books go in the first three slots, there are 4 choices for the first slot,
3 choices for the second and 2 choices for the third.

The fourth slot requires a history book, and has five choices. Once that choice is made, there are 4 history books left, and therefore, 4 choices for the last slot. The choices are shown below.

| | | | | |
|---|---|---|---|---|
| 4 | 3 | 2 | 5 | 4 |

Therefore, the number of permutations are $4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 = 480$ .

Alternately, we can see that $4 \cdot 3 \cdot 2$ is really same as 4P3, and $5 \cdot 4$ is 5P2.

So the answer can be written as (4P3) (5P2) = 480.

Clearly, this makes sense. For every permutation of three math books placed in the first three slots, there are 5P2 permutations of history books that can be placed in the last two slots. Hence the multiplication axiom applies, and we have the answer (4P3) (5P2).

We summarize the concepts of this section:

Note

**1. Permutations**

A permutation of a set of elements is an ordered arrangement where each element is used once.

**2. Factorial**

$$n! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$$

Where $n$ is a natural number.

$$0! = 1$$

**3. Permutations of n Objects Taken r at a Time**

$$nPr = n(n-1)(n-2)(n-3)\cdots(n-r+1)$$

or

$$nPr = \frac{n!}{(n-r)!}$$

where $n$ and $r$ are natural numbers.

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# 6.3.1: Permutations (Exercises)

Do the following problems using permutations.

| | |
|---|---|
| 1. How many three-letter words can be made using the letters { a, b, c, d, e } if no repetitions are allowed? | 2. A grocery store has five checkout counters, and seven clerks. How many different ways can the 7 clerks be assigned to the 5 counters? |
| 3. A group of fifteen people who are members of an investment club wish to choose a president, and a secretary. How many different ways can this be done? | 4. Compute the following.<br>a. 9P2<br>b. 6P4<br>c. 8P3<br>d. 7P4 |
| 5. In how many ways can the letters of the word CUPERTINO be arranged if each letter is used only once in each arrangement? | 6. How many permutations of the letters of the word PROBLEM end in a vowel? |
| 7. How many permutations of the letters of the word SECURITY end in a consonant? | 8. How many permutations of the letters PRODUCT have consonants in the second and third positions? |
| 9. How many three-digit numbers are there? | 10. How many three-digit odd numbers are there? |
| 11. In how many different ways can five people be seated in a row if two of them insist on sitting next to each other? | 12. In how many different ways can five people be seated in a row if two of them insist on not sitting next to each other? |
| 13. In how many ways can 3 English, 3 history, and 2 math books be set on a shelf, if the English books are set on the left, history books in the middle, and math books on the right? | 14. In how many ways can 3 English, 3 history, and 2 math books be set on a shelf, if they are grouped by subject? |
| 15. You have 5 math books and 6 history books to put on a shelf with five slots. In how many ways can you put the books on the shelf if the first two slots are to be filled with math books and the next three with history books? | 16. You have 5 math books and 6 history books to put on a shelf with five slots. In how many ways can you put the books on the shelf if the first two slots are to be filled with the books of one subject and the next three slots are to be filled with the books of the other subject? |
| 17. A bakery has 9 different fancy cakes. In how many ways can 5 of the 9 fancy cakes be lined up in a row in the bakery display case? | 18. A landscaper has 6 different flowering plants. She needs to plant 4 of them in a row in a garden. How many different ways can 4 of the 6 plants be arranged in a row? |
| 19. At an auction of used construction vehicles, there are 7 different vehicles for sale. In how many orders could these 7 vehicles be listed in the auction program? | 20. A landscaper has 6 different flowering plants and 4 different non-flowering bushes. She needs to plant a row of 6 plants in a garden. There must be a bush at each end, and four flowering plants in a row in between the bushes. How many different arrangements in a row are possible? |
| 21. In how many ways can all 7 letters of the word QUIETLY be arranged if the letters Q and U must be next to each other in the order QU? | 22. a. In how many ways can the letters ABCDEXY be arranged if the X and Y must be next to each other in either order XY or YX?<br>b. In how many ways can the letters ABCDEXY be arranged if the X and Y can not be next to each other? |

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# 6.4: Circular Permutations and Permutations with Similar Elements

**Learning Objectives**

In this section you will learn to

1. Count the number of possible permutations of items arranged in a circle
2. Count the number of possible permutations when there are repeated items

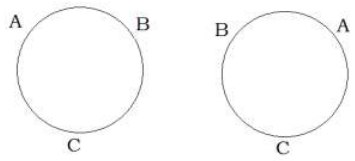
In this section we will address the following two problems.

1. In how many different ways can five people be seated in a circle?
2. In how many different ways can the letters of the word MISSISSIPPI be arranged?

The first problem comes under the category of Circular Permutations, and the second under Permutations with Similar Elements.

## Circular Permutations

Suppose we have three people named A, B, and C. We have already determined that they can be seated in a straight line in 3! or 6 ways. Our next problem is to see how many ways these people can be seated in a circle. We draw a diagram.




It happens that there are only two ways we can seat three people in a circle, relative to each other's positions. This kind of permutation is called a circular permutation. In such cases, no matter where the first person sits, the permutation is not affected. Each person can shift as many places as they like, and the permutation will not be changed. We are interested in the position of each person in relation to the others. Imagine the people on a merry-go-round; the rotation of the permutation does not generate a new permutation. So in circular permutations, the first person is considered a place holder, and where he sits does not matter.

**Definition: Circular Permutations**

The number of permutations of $n$ elements in a circle is $(n-1)!$

**Example 6.4.1**

In how many different ways can five people be seated at a circular table?

**Solution**

We have already determined that the first person is just a place holder. Therefore, there is only one choice for the first spot. We have

| | | | | |
|---|---|---|---|---|
| 1 | 4 | 3 | 2 | 1 |

So the answer is 24.

**Example 6.4.2**

In how many ways can four couples be seated at a round table if the men and women want to sit alternately?

**Solution**

We again emphasize that the first person can sit anywhere without affecting the permutation.

So there is only one choice for the first spot. Suppose a man sat down first. The chair next to it must belong to a woman, and there are 4 choices. The next chair belongs to a man, so there are three choices and so on. We list the choices below.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 4 | 3 | 3 | 2 | 2 | 1 | 1 |

So the answer is 144.

## PERMUTATIONS WITH SIMILAR ELEMENTS

Let us determine the number of distinguishable permutations of the letters ELEMENT.

Suppose we make all the letters different by labeling the letters as follows.

$$E_1LE_2ME_3NT$$

Since all the letters are now different, there are 7! different permutations.

Let us now look at one such permutation, say

$$LE_1ME_2NE_3T$$

Suppose we form new permutations from this arrangement by only moving the E's. Clearly, there are 3! or 6 such arrangements. We list them below.

$$\begin{array}{l}\mathrm{LE_1ME_2NE_3}\\ \mathrm{LE_1ME_3NE_2}\\ \mathrm{LE_2ME_1NE_3T}\\ \mathrm{LE_2ME_3NE_1T}\\ \mathrm{LE_3ME_2NE_1T}\\ \mathrm{LE_3ME_INE_2T}\end{array}$$

Because the E's are not different, there is only one arrangement LEMENET and not six. This is true for every permutation.

Let us suppose there are n different permutations of the letters ELEMENT.

Then there are $n \cdot 3!$ permutations of the letters $E_1LE_2ME_3NT$.

But we know there are 7! permutations of the letters $E_1LE_2ME_3NT$.

Therefore, $n \cdot 3! = 7!$

Or $n = \frac{7!}{3!}$.

This gives us the method we are looking for.

**Definition: Permutations with Similar Elements**

The number of permutations of n elements taken $n$ at a time, with $r_1$ elements of one kind, $r_2$ elements of another kind, and so on, is

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$

**Example 6.4.3**

Find the number of different permutations of the letters of the word MISSISSIPPI.

**Solution**

The word MISSISSIPPI has 11 letters. If the letters were all different there would have been 11! different permutations. But MISSISSIPPI has 4 S's, 4 I's, and 2 P's that are alike.

So the answer is $\frac{11!}{4!4!2!} = 34,650$.

**Example 6.4.4**

If a coin is tossed six times, how many different outcomes consisting of 4 heads and 2 tails are there?

**Solution**

Again, we have permutations with similar elements.

We are looking for permutations for the letters HHHHTT.

The answer is $\frac{6!}{4!2!} = 15$.

**Example 6.4.5**

In how many different ways can 4 nickels, 3 dimes, and 2 quarters be arranged in a row?

**Solution**

Assuming that all nickels are similar, all dimes are similar, and all quarters are similar, we have permutations with similar elements. Therefore, the answer is

$$\frac{9!}{4!3!2!} = 1260$$

**Example 6.4.6**

A stock broker wants to assign 20 new clients equally to 4 of its salespeople. In how many different ways can this be done?

**Solution**

This means that each sales person gets 5 clients. The problem can be thought of as an ordered partitions problem. In that case, using the formula we get

$$\frac{20!}{5!5!5!5!} = 11,732,745,024$$

**Example 6.4.7**

A shopping mall has a straight row of 5 flagpoles at its main entrance plaza. It has 3 identical green flags and 2 identical yellow flags. How many distinct arrangements of flags on the flagpoles are possible?

**Solution**

The problem can be thought of as distinct permutations of the letters GGGYY; that is arrangements of 5 letters, where 3 letters are similar, and the remaining 2 letters are similar:

$$\frac{5!}{3!2!} = 10$$

Just to provide a little more insight into the solution, we list all 10 distinct permutations:

GGGYY, GGYGY, GGYYG, GYGGY, GYGYG, GYYGG, YGGGY, YGGYG, YGYGY, YYGGG

We summarize.

**Summary**

**1. Circular Permutations**

The number of permutations of n elements in a circle is

$$(n-1)!$$

**2. Permutations with Similar Elements**

The number of permutations of n elements taken n at a time, with $r_1$ elements of one kind, $r_2$ elements of another kind, and so on, such that $\mathrm{n} = \mathrm{r}_1 + \mathrm{r}_2 + \ldots + \mathrm{r}_\mathrm{k}$ is

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$

This is also referred to as **ordered partitions**.

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# 6.4.1: Circular Permutations and Permutations with Similar Elements (Exercises)

Do the following problems using the techniques learned in this section.

| | |
|---|---|
| 1. In how many different ways can five children hold hands to play "Ring Around the Rosy"? | 2. In how many ways can three people be made to sit at a round table? |
| 3. In how many different ways can six children ride a "Merry Go Around" with six horses? | 4. In how many ways can three couples be seated at a round table, so that men and women sit alternately? |
| 5. In how many ways can six trinkets be arranged on a chain? | 6. In how many ways can five keys be put on a key ring? |
| 7. Find the number of different permutations of the letters of the word MASSACHUSETTS. | 8. Find the number of different permutations of the letters of the word MATHEMATICS. |
| 9. Seven flags are to be flown on seven poles: 3 flags are red, 2 are white, and 2 are blue,. How many different arrangements are possible? | 10. How many different ways can 3 pennies, 2 nickels and 5 dimes be arranged in a row? |
| 11. How many four-digit numbers can be made using two 2's and two 3's? | 12. How many five-digit numbers can be made using two 6's and three 7's? |
| 13. If a coin is tossed 5 times, how many different outcomes of 3 heads and 2 tails are possible? | 14. If a coin is tossed 10 times, how many different outcomes of 7 heads and 3 tails are possible? |
| 15. If a team plays ten games, how many different outcomes of 6 wins and 4 losses are possible? | 16. If a team plays ten games, how many different ways can the team have a winning season? |

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# 6.5: Combinations

**Learning Objectives**

In this section you will learn to

1. Count the number of combinations of r out of n items (selections without regard to arrangement )
2. Use factorials to perform calculations involving combinations

Suppose we have a set of three letters { A, B, C }, and we are asked to make two-letter word sequences. We have the following six permutations.

AB BA BC CB AC CA

Now suppose we have a group of three people { A, B, C } as Al, Bob, and Chris, respectively, and we are asked to form committees of two people each. This time we have only three committees, namely,

AB BC AC

When forming committees, the order is not important, because the committee that has Al and Bob is no different than the committee that has Bob and Al. As a result, we have only three committees and not six.

Forming word sequences is an example of permutations, while forming committees is an example of **combinations** - the topic of this section.

Permutations are those arrangements where order is important, while combinations are those arrangements where order is not significant. From now on, this is how we will tell permutations and combinations apart.

In the above example, there were six permutations, but only three combinations.

Just as the symbol nPr represents the number of permutations of n objects taken r at a time, nCr represents the number of combinations of n objects taken r at a time.

So in the above example, 3P2 = 6, and 3C2 = 3.

Our next goal is to determine the relationship between the number of combinations and the number of permutations in a given situation.

In the above example, if we knew that there were three combinations, we could have found the number of permutations by multiplying this number by 2!. That is because each combination consists of two letters, and that makes 2! permutations.

**Example 6.5.1**

Given the set of letters { A, B, C, D }. Write the number of combinations of three letters, and then from these combinations determine the number of permutations.

**Solution**

We have the following four combinations.

ABC BCD CDA BDA

Since every combination has three letters, there are 3! permutations for every combination. We list them below.

| | | | |
|---|---|---|---|
| ABC | BCD | CDA | BDA |
| ACB | BDC | CAD | BAD |
| BAC | CDB | DAC | DAB |
| BCA | CBD | DCA | DBA |
| CAB | DCB | ACD | ADB |
| CBA | DBC | ADC | ABD |

The number of permutations are 3! times the number of combinations; that is

4P3 = 3! · 4C3

or

$$4C3 = \frac{4P3}{3!}$$

In general,

$$nCr = \frac{nPr}{r!}$$

Since

$$nPr = \frac{n!}{(n-r)!}$$

We have,

$$nCr = \frac{n!}{(n-r)!r!}$$

Summarizing,

**Note**

**1. Combinations**

A combination of a set of elements is an arrangement where each element is used once, and order is not important.

**2. The Number of Combinations of n Objects Taken r at a Time**

$$nCr = \frac{n!}{(n-r)!r!}$$

where n and r are natural numbers.

## Example \(\PageIndex{3}\)Example 6.5.2

Compute:

a. 5C3
b. 7C3

**Solution**

We use the above formula.

$$5C3 = \frac{5!}{(5-3)!3!} = \frac{5!}{2!3!} = 10$$

$$7C3 = \frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = 35$$

## Example 6.5.3

In how many different ways can a student select to answer five questions from a test that has seven questions, if the order of the selection is not important?

**Solution**

Since the order is not important, it is a combination problem, and the answer is

7C5 = 21

### Example 6.5.4

How many line segments can be drawn by connecting any two of the six points that lie on the circumference of a circle?

***Solution***

Since the line that goes from point A to point B is same as the one that goes from B to A, this is a combination problem.

It is a combination of 6 objects taken 2 at a time. Therefore, the answer is

$$6C2 = \frac{6!}{4!2!} = 15$$

### Example 6.5.5

There are ten people at a party. If they all shake hands, how many hand-shakes are possible?

***Solution***

Note that between any two people there is only one hand shake. Therefore, we have

10C2 = 45 hand-shakes.

### Example 6.5.6

The shopping area of a town is in the shape of square that is 5 blocks by 5 blocks. How many different routes can a taxi driver take to go from one corner of the shopping area to the opposite cater-corner?

**Solution**

Let us suppose the taxi driver drives from the point A, the lower left hand corner, to the point B, the upper right hand corner as shown in the figure below.




To reach his destination, he has to travel ten blocks; five horizontal, and five vertical. So if out of the ten blocks he chooses any five horizontal, the other five will have to be the vertical blocks, and vice versa.

Therefore, all he has to do is to choose 5 out of ten to be the horizontal blocks

The answer is 10C5, or 252.

Alternately, the problem can be solved by permutations with similar elements.

The taxi driver's route consists of five horizontal and five vertical blocks. If we call a horizontal block H, and a vertical block a V, then one possible route may be as follows.

HHHHHVVVVV

Clearly there are $\frac{10!}{5!5!} = 252$ permutations.

Further note that by definition 10C5 = $\frac{10!}{5!5!}$.

### Example 6.5.7

If a coin is tossed six times, in how many ways can it fall four heads and two tails?

**Solution**

First we solve this problem using section 6.5 technique-permutations with similar elements.

We need 4 heads and 2 tails, that is

$$\text{HHHHTT}$$

There are $\frac{6!}{4!2!} = 15$ permutations.

Now we solve this problem using combinations.

Suppose we have six spots to put the coins on. If we choose any four spots for heads, the other two will automatically be tails. So the problem is simply

$$6C4 = 15.$$

Incidentally, we could have easily chosen the two tails, instead. In that case, we would have gotten

$$6C2 = 15.$$

Further observe that by definition

$$6C4 = \frac{6!}{2!4!}$$

and

$$6C2 = \frac{6!}{4!2!}$$

Which implies 6C4 = 6C2.

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# 6.5.1: Combinations (Exercises)

Do the following problems using combinations.

| | |
|---|---|
| 1. How many different 3-people committees can be chosen from ten people? | 2. How many different 5-player teams can be chosen from eight players? |
| 3. In how many ways can a person chose to vote for three out of five candidates on a ballot for a school board election? | 4. Compute the following:<br>a. 9C2<br>b. 6C4<br>c. 8C3<br>d. 7C4 |
| 5. How many 5-card hands can be chosen from a deck of cards? | 6. How many 13-card bridge hands can be chosen from a deck of cards? |
| 7. There are twelve people at a party. If they all shake hands, how many different hand-shakes are there? | 8. In how many ways can a student choose to do four questions out of five on a test? |
| 9. Five points lie on a circle. How many chords can be drawn through them? | 10. How many diagonals does a hexagon have? |
| 11. There are five team in a league. How many games are played if every team plays each other twice? | 12. A team plays 15 games a season. In how many ways can it have 8 wins and 7 losses? |
| 13. In how many different ways can a 4-child family have 2 boys and 2 girls? | 14. A coin is tossed five times. In how many ways can it fall three heads and two tails? |
| 15. The shopping area of a town is a square that is six blocks by six blocks. How many different routes can a taxi driver take to go from one corner of the shopping area to the opposite cater-corner? | 16. If the shopping area in the previous problem has a rectangular form of 5 blocks by 3 blocks, then how many different routes can a taxi driver take to drive from one end of the shopping area to the opposite kitty corner end? |
| 17. A team of 7 workers is assigned to a project. In how many ways can 3 of the 7 workers be selected to make a presentation to the management about their progress on the project? | 18. A real estate company has 12 houses listed for sale by their clients. In how many ways can 5 of the 12 houses be selected to be featured in an advertising brochures? |
| 19. A frozen yogurt store has 9 toppings to choose from. In how many ways can 3 of the 9 toppings be selected ? | 20. A kindergarten teacher has 14 books about a holiday. In how many ways can she select 4 of the books to read to her class in the week before the holiday? |

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# 6.6: Combinations- Involving Several Sets

**Learning Objectives**

In this section you will learn to

1. count the number of items selected from more than one set
2. count the number of items selected when there are restrictions on the selections

So far we have solved the basic combination problem of r objects chosen from n different objects. Now we will consider certain variations of this problem.

**Example 6.6.1**

How many five-people committees consisting of 2 men and 3 women can be chosen from a total of 4 men and 4 women?

**Solution**

We list 4 men and 4 women as follows:

$$M_1 M_2 M_3 M_4 W_1 W_2 W_3 W_4$$

Since we want 5-people committees consisting of 2 men and 3 women, we'll first form all possible two-man committees and all possible three-woman committees. Clearly there are 4C2 = 6 two-man committees, and 4C3 = 4 three-woman committees, we list them as follows:

| 2-Man Committees | 3-Woman Committees |
|---|---|
| $M_1M_2$<br>$M_1M_3$<br>$M_1M_4$<br>$M_2M_3$<br>$M_2M_4$<br>$M_3M_4$ | $W_1W_2W_3$<br>$W_1W_2W_4$<br>$W_1W_3W_4$<br>$W_2W_3W_4$ |

For every 2-man committee there are four 3-woman committees that can be chosen to make a 5-person committee. If we choose $M_1M_2$ as our 2-man committee, then we can choose any of $W_1W_2W_3$, $W_1W_2W_4$, $W_1W_3W_4$, or $W_2W_3W_4$ as our 3-woman committees. As a result, we get

$$\boxed{M_1M_2}W_1W_2W_3, \boxed{M_1M_2}W_1W_2W_4, \boxed{M_1M_2}W_1W_3W_4, \boxed{M_1M_2}W_2W_3W_4$$

Similarly, if we choose $M_1M_3$ as our 2-man committee, then, again, we can choose any of $W_1W_2W_3$, $W_1W_2W_4$, $W_1W_3W_4$, or $W_2W_3W_4$ as our 3-woman committees.

$$\boxed{M_1M_3}W_1W_2W_3, \boxed{M_1M_3}W_1W_2W_4, \boxed{M_1M_3}W_1W_3W_4, \boxed{M_1M_3}W_2W_3W_4$$

And so on.

Since there are six 2-man committees, and for every 2-man committee there are four 3-woman committees, there are altogether $6 \cdot 4 = 24$ five-people committees.

In essence, we are applying the multiplication axiom to the different combinations.

**Example 6.6.2**

A high school club consists of 4 freshmen, 5 sophomores, 5 juniors, and 6 seniors. How many ways can a committee of 4 people be chosen that includes

a. One student from each class?
b. All juniors?
c. Two freshmen and 2 seniors?
d. No freshmen?
e. At least three seniors?

**Solution**

a. Applying the multiplication axiom to the combinations involved, we get

$$( 4C1 ) ( 5C1 ) ( 5C1 ) ( 6C1 ) = 600$$

b. We are choosing all 4 members from the 5 juniors, and none from the others.

$$5C4 = 5$$

c. $4C2 \cdot 6C2 = 90$

d. Since we don't want any freshmen on the committee, we need to choose all members from the remaining 16. That is

$$16C4 = 1820$$

e. Of the 4 people on the committee, we want at least three seniors. This can be done in two ways. We could have three seniors, and one non-senior, or all four seniors.

$$( 6C3 ) ( 14C1 ) + 6C4 = 295$$

## Example 6.6.3

How many five-letter word sequences consisting of 2 vowels and 3 consonants can be formed from the letters of the word INTRODUCE?

**Solution**

First we select a group of five letters consisting of 2 vowels and 3 consonants.
Since there are 4 vowels and 5 consonants, we have

$$( 4C2 ) ( 5C3 )$$

Since our next task is to make word sequences out of these letters, we multiply these by 5!.

$$( 4C2 ) ( 5C3 ) ( 5! ) = 7200.$$

## Example 6.6.4

A standard deck of playing cards has 52 cards consisting of 4 suits each with 13 cards. In how many different ways can a 5-card hand consisting of four cards of one suit and one of another be drawn?

**Solution**

We will do the problem using the following steps.
Step 1. Select a suit.
Step 2. Select four cards from this suit.
Step 3. Select another suit.
Step 4. Select a card from that suit.

Applying the multiplication axiom, we have

| Ways of selecting the first suit | Ways of selecting 4 cards from this suit | Ways of selecting the next suit | Ways of selecting a card from that suit |
|---|---|---|---|
| 4C1 | 13C4 | 3C1 | 13C1 |

$( 4C1 ) ( 13C4 ) ( 3C1 ) ( 13C1 ) = 111,540.$

## A STANDARD DECK OF 52 PLAYING CARDS

As in the previous example, many examples and homework problems in this book refer to a standard deck of 52 playing cards. Before we end this section, we take a minute to describe a standard deck of playing cards, as some readers may not be familiar with this.

A standard deck of 52 playing cards has 4 suits with 13 cards in each suit.

♦ diamonds ♥ hearts ♠ spades ♣ clubs

Each suit is associated with a color, either black (spades, clubs) or red (diamonds, hearts)

Each suit contains 13 denominations (or values) for cards:

nine numbers 2, 3, 4, …., 10 and Jack(J), Queen (Q), King (K), Ace (A).

The Jack, Queen and King are called “face cards” because they have pictures on them. Therefore a standard deck has 12 face cards: (3 values JQK ) x (4 suits ♦♥♠ ♣ )

We can visualize the 52 cards by the following display

| Suit | Color | Values (Denominations) |
|---|---|---|
| ♦ Diamonds | Red | 2 3 4 5 6 7 8 9 10 J Q K A |
| ♥ Hearts | Red | 2 3 4 5 6 7 8 9 10 J Q K A |
| ♠ Spades | Black | 2 3 4 5 6 7 8 9 10 J Q K A |
| ♣ Clubs | Black | 2 3 4 5 6 7 8 9 10 J Q K A |

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# 6.6.1: Combinations- Involving Several Sets (Exercises)

Following problems involve combinations from several different sets.

| | |
|---|---|
| 1. How many 5-people committees consisting of three boys and two girls can be chosen from a group of four boys and four girls? | 2. A club has 4 men, 5 women, 8 boys and 10 girls as members. In how many ways can a group of 2 men, 3 women, 4 boys and 4 girls be chosen? |
| 3. How many 4-people committees chosen from 4 men and 6 women will have at least 3 men? | 4. A batch contains 10 transistors of which three are defective. If three are chosen, in how many ways can they be selected with two defective? |
| 5. In how many ways can five counters labeled A, B, C, D and E at a store be staffed by two men and three women chosen from a group of four men and six women? | 6. How many 4-letter word sequences consisting of two vowels and two consonants can be made from the letters of the word PHOENIX if no letter is repeated? |

Three marbles are chosen from an urn that contains 5 red, 4 white, and 3 blue marbles. How many samples of the following type are possible?

| | |
|---|---|
| 7. All three white. | 8. Two blue and one white |
| 9. One of each color. | 10. All three of the same color. |
| 11. At least two red. | 12. None red. |

The following problems involve combinations from several different sets.

Five coins are chosen from a bag that contains 4 dimes, 5 nickels, and 6 pennies. How many samples of five coins of the following types are possible?

| | |
|---|---|
| 13. At least four nickels. | 14. No pennies. |
| 15. Five of a kind. | 16. Four of a kind. |
| 17. Two of one kind and two of another kind. | 18. Three of one kind and two of another kind. |

Find the number of different ways draw a 5-card hand from a deck to have the following combinations.

| | |
|---|---|
| 19. Three face cards. | 20. A heart flush (all hearts). |
| 21. Two hearts and three diamonds | 22. Two cards of one suit, and three of another suit. |
| 23. Two kings and three queesns. | 24. 2 cards of one value and 3 of another value |

The party affiliation of the 100 United States Senators in the $114^{th}$ Congress, January 2015, was:

44 Democrats, 54 Republicans, and 2 Independents.

| | |
|---|---|
| 25. In how many ways could a 10 person committee be selected if it is to contain 4 Democrats, 5 Republicans, and 1 Independent? | 26. In how many different ways could a 10 person committee be selected with 6 or 7 Republicans and the Democrats (with no Independents)? |

The 100 United States Senators in the $114^{th}$ Congress, January 2015, included 80 men and 20 women. Suppose a committee senators is working on legislation about wage discrimination by gender.

| | |
|---|---|
| 27. In how many ways could a 12 person committee be selected to contain equal numbers of men and women. | 28. In how many ways could a 6 person committee be selected to contain fewer women than men? |

Jorge has 6 rock songs, 7 rap songs and 4 country songs that he likes to listen to while he exercises. He randomly selects six (6) of these songs to create a playlist to listen to today while he exercises.

How many different playlists of 6 songs can be selected that satisfy each of the following: (We care which songs are selected to be on the playlist, but not what order they are selected or listed in.)

| | |
|---|---|
| 29. Playlist has 2 songs of each type | 30. Playlist has no country songs |
| 31. Playlist has 3 rocks, 2 raps, and 1 country song | 32. Playlist has 3 or 4 rock songs and all the rest are rap songs |

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# 6.7: Binomial Theorem

We end this chapter with one more application of combinations. Combinations are used in determining the coefficients of a binomial expansion such as $(x+y)^n$. Expanding a binomial expression by multiplying it out is a very tedious task, and is not practiced. Instead, a formula known as the Binomial Theorem is utilized to determine such an expansion. Before we introduce the Binomial Theorem, however, consider the following expansions.

$$\begin{aligned}(x+y)^2 &= x^2+2xy+y^2\\ (x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\ (x+y)^4 &= x^4+4x^2y+6x^2y^2+4xy^3+y^4\\ (x+y)^5 &= x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5\\ (x+y)^6 &= x6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6\end{aligned}$$

We make the following observations.

1. There are $n+1$ terms in the expansion $(x+y)^n$
2. The sum of the powers of $x$ and $y$ is $n$.
3. The powers of $x$ begin with $n$ and decrease by one with each successive term.
   The powers of $y$ begin with 0 and increase by one with each successive term.

Suppose we want to expand $(x+y)^3$. We first write the expansion without the coefficients. We temporarily substitute a blank in place of the coefficients.

$$(x+y)^3 = \square x^3+\square x^2y+\square xy^2+\square y^3 \tag{6.7.1}$$

Our next job is to replace each of the blanks in equation (6.7.1) with the corresponding coefficients that belong to this expansion. Clearly,

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

If we multiply the right side and do not collect terms, we get the following.

$$xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy$$

Each product in the above expansion is the result of multiplying three variables by picking one from each of the factors $(x+y)(x+y)(x+y)$. For example, the product $xxy$ is gotten by choosing $x$ from the first factor, $x$ from the second factor, and $y$ from the third factor. There are three such products that simplify to $x^2y$, namely $xxy$, $xyx$, and $yxx$. These products take place when we choose an $x$ from two of the factors and choose a $y$ from the other factor. Clearly this can be done in 3C2, or 3 ways. Therefore, the coefficient of the term $x^2y$ is 3. The coefficients of the other terms are obtained in a similar manner.

We now replace the blanks with the coefficients in equation (6.7.1), and we get

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

**Example 6.7.1**

Find the coefficient of the term $x^2y^5$ in the expansion $(x+y)^7$.

**Solution**

The expansion $(x+y)^7 = (x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$

In multiplying the right side, each product is gotten by picking an $x$ or $y$ from each of the seven factors $(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$ .

The term $x^2y^5$ is obtained by choosing an $x$ from two of the factors and a $y$ from the other five factors. This can be done in 7C2, or 21 ways.

Therefore, the coefficient of the term $x^2y^5$ is 21.

**Example 6.7.2**

Expand $(x+y)^7$

**Solution**

We first write the expansion without the coefficients.

$$(x+y)^7 = \square x^7 + \square x^6 y + \square x^5 y^2 + \square x^4 y^3 + \square x^3 y^4 + \square x^2 y^5 + \square x y^6 + \square y^7$$

Now we determine the coefficient of each term as we did in Example 6.7.1.

The coefficient of the term $x^7$ is 7C7 or 7CO which equals 1.

The coefficient of the term $x^6 y$ is 7C6 or 7C1 which equals 7.

The coefficient of the term $x^5 y^2$ is 7C5 or 7C2 which equals 21.

The coefficient of the term $x^4 y^3$ is 7C4 or 7C3 which equals 35,

and so on.

Substituting, we get: $(x+y)^7 = x^7 + 7x^6 y + 21x^5 y^2 + 35x^4 y^3 + 35x^3 y^4 + 21x^2 y^5 + 7xy^6 + y^7$

We generalize the result.

**Binomial Theorem**

$$(x+y)^n = {}_n C_0 x^n + {}_n C_1 x^{n-1} y + {}_n C_2 x^{n-2} y^2 + \cdots + {}_n C_{n-1} x y_n^{n-1+} C_n y^n$$

**Example 6.7.3**

Expand $(3a-2b)^4$

**Solution**

If we let $x = 3a$ and $y = -2b$, and apply the Binomial Theorem, we get

$$\begin{aligned}(3a-2b)^4 &= 4\,\mathrm{Co}(3a)^4 + 4Cl(3a)^3(-2b) + 4C2(3a)^2(-2b)^2 + 4C3(3a)(-2b)^3 + 4C4(-2b)^4 \\ &= 1\left(81a^4\right) + 4\left(27a^3\right)(-2b) + 6\left(9a^2\right)\left(4b^2\right) + 4(3a)\left(-8b^3\right) + 1\left(16b^3\right) \\ &= 81a^4 - 216a^3 b + 216a^2 b^2 - 96ab^3 + 16b^4\end{aligned}$$

**Example 6.7.4**

Find the fifth term of the expansion $(3a-2b)^7$.

**Solution**

The Binomial theorem tells us that in the r-th term of an expansion, the exponent of the $y$ term is always one less than $r$, and, the coefficient of the term is ${}_{\mathrm{n}}\mathrm{C}_{\mathrm{r}-1}$.

$n = 7$ and $r - 1 = 5 - 1 = 4$ , so the coefficient is $7\mathrm{C}4 = 35$

Thus, the fifth term is $(7\mathrm{C}4)(3\mathrm{a})^3(-2\mathrm{b})^4 = 35\left(27\mathrm{a}^3\right)\left(16\mathrm{b}^4\right) = 15120\mathrm{a}^3\mathrm{b}^4$

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# 6.7.1: Binomial Theorem (Exercises)

Use the Binomial Theorem to do the following problems.

| | |
|---|---|
| 1. Expand $(a+b)^5$. | 2. Expand $(a-b)^6$. |
| 3. Expand $(x-2y)^5$. | 4. Expand $(2x-3y)^4$. |
| 5. Find the third term of $(2x-3y)^6$. | 6. Find the sixth term of $(5x+y)^8$. |
| 7. Find the coefficient of the $x^3y^4$ term in the expansion of $(2x+y)^7$. | 8. Find the coefficient of the $a^4b^6$ term in the expansion of $(3a-b)^{10}$. |
| 9. A coin is tossed 5 times, in how many ways is it possible to get three heads and two tails? | 10. A coin is tossed 10 times, in how many ways is it possible to get seven heads and three tails? |
| 11. How many subsets are there of a set that has 6 elements? | 12. How many subsets are there of a set that has $n$ elements? |

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## 6.8: Chapter Review

1. Suppose of the 4,000 freshmen at a college everyone is enrolled in a mathematics or an English class during a given quarter. If 2,000 are enrolled in a mathematics class, and 3,000 in an English class, how many are enrolled in both a mathematics class and an English class?
2. In a survey of 250 people, it was found that 125 had read Time magazine, 175 had read Newsweek, 100 had read U. S. News, 75 had read Time and Newsweek, 60 had read Newsweek and U. S. News, 55 had read Time and U. S. News, and 25 had read all three.
   a. How many had read Time but not the other two?
   b. How many had read Time or Newsweek but not the U. S. News And World Report?
   c. How many had read none of these three magazines?
3. At a manufacturing plant, a product goes through assembly, testing, and packing. If a plant has three assembly stations, two testing stations, and two packing stations, in how many different ways can a product achieve its completion?
4. Six people are to line up for a photograph. How many different lineups are possible if three of them insist on standing next to each other?
5. How many four-letter word sequences can be made from the letters of the word CUPERTINO?
6. In how many different ways can a 20-question multiple choice test be designed so that its answers contain 2 A's, 4 B's, 9 C's, 3 D's, and 2 E's?
7. The U. S. Supreme Court has nine judges. In how many different ways can the judges cast a six-tothree decision in favor of a ruling?
8. In how many different ways can a coach choose a linebacker, a guard, and a tackle from five players on the bench, if all five can play any of the three positions?
9. How many three digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if no repetitions are allowed?
10. Compute:
    a. $9C4$
    b. $8P3$
    c. $\frac{10!}{4!(10-4)!}$
11. In how many ways can 3 English, 3 Math, and 4 Spanish books be set on a shelf if the books are grouped by subject?
12. In how many ways can a 10-question multiple choice test with four possible answers for each question be answered?
13. On a soccer team three fullbacks can play any of the three fullback positions, left, center, and right. The three halfbacks can play any of the three halfback positions, the four forwards can play any of the four positions, and the goalkeeper plays only his position. How many different arrangements of the 11 players are possible?
14. From a group of 6 people, 3 are assigned to cleaning, 2 to hauling and one to garbage collecting. How many different ways can this be done?
15. How many three-letter word sequences can be made from the letters of the word OXYGEN?
16. In how many ways can 3 books be selected from 4 English and 2 History books if at least one English book must be chosen?
17. Five points lie on the rim of a circle. Choosing the points as vertices, how many different triangles can be drawn?
18. A club consists of six men and nine women. In how many ways can a president, a vice president and a treasurer be chosen if the two of the officers must be women?
19. Of its 12 sales people, a company wants to assign 4 to its Western territory, 5 to its Northern territory, and 3 to its Southern territory. How many ways can this be done?
20. How many permutations of the letters of the word OUTSIDE have consonants in the first and last place?
21. How many distinguishable permutations are there in the word COMMUNICATION?
22. How many five-card poker hands consisting of the following distribution are there?
    a. A flush(all five cards of a single suit)
    b. Three of a kind(e.g. three aces and two other cards)
    c. Two pairs(e.g. two aces, two kings and one other card)
    d. A straight(all five cards in a sequence)
23. Company stocks on an exchange are given symbols consisting of three letters. How many different three-letter symbols are possible?
24. How many four-digit odd numbers are there?

25. In how many ways can 7 people be made to stand in a straight line? In a circle?
26. A United Nations delegation consists of 6 Americans, 5 Russians, and 4 Chinese. Answer the following questions.
    a. How many committees of five people are there?
    b. How many committees of three people consisting of at least one American are there?
    c. How many committees of four people having no Russians are there?
    d. How many committees of three people have more Americans than Russians?
    e. How many committees of three people do not have all three Americans?
27. If a coin is flipped five times, in how many different ways can it show up three heads?
28. To reach his destination, a man is to walk three blocks north and four blocks west. How many different routes are possible?
29. All three players of the women's beach volleyball team, and all three players of the men's beach volleyball team are to line up for a picture with all members of the women's team lined together and all members of men's team lined up together. How many ways can this be done?
30. From a group of 6 Americans, 5 Japanese and 4 German delegates, two Americans, two Japanese and a German are chosen to line up for a photograph. In how many different ways can this be done?
31. Find the fourth term of the expansion $(2x-3y)^8$.
32. Find the coefficient of the $a^5b^4$ term in the expansion of $(a-2b)^9$.

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# CHAPTER OVERVIEW

## 7: Probability

**Learning Objectives**

In this chapter, you will learn to:

1. Write sample spaces.
2. Determine whether two events are mutually exclusive.
3. Use the Addition Rule.
4. Calculate probabilities using both tree diagrams and combinations.
5. Do problems involving conditional probability.
6. Determine whether two events are independent.


7.1: Sample Spaces and Probability

7.1.1: Sample Spaces and Probability (Exercises)

7.2: Mutually Exclusive Events and the Addition Rule

7.2.1: Mutually Exclusive Events and the Addition Rule (Exercises)

7.3: Probability Using Tree Diagrams and Combinations

7.3.1: Probability Using Tree Diagrams and Combinations (Exercises)

7.4: Conditional Probability

7.4.1: Conditional Probability (Exercises)

7.5: Independent Events

7.5.1: Independent Events (Exercises)

7.6: Chapter Review



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# 7.1: Sample Spaces and Probability

**Learning Objectives**

In this section, you will learn to:

1. Write sample spaces.
2. Calculate probabilities by examining simple events in sample spaces.

If two coins are tossed, what is the probability that both coins will fall heads? The problem seems simple enough, but it is not uncommon to hear the incorrect answer 1/3. A student may incorrectly reason that if two coins are tossed there are three possibilities, one head, two heads, or no heads. Therefore, the probability of two heads is one out of three. The answer is wrong because if we toss two coins there are four possibilities and not three. For clarity, assume that one coin is a penny and the other a nickel. Then we have the following four possibilities.

HH HT TH TT

The possibility HT, for example, indicates a head on the penny and a tail on the nickel, while TH represents a tail on the penny and a head on the nickel. It is for this reason, we emphasize the need for understanding sample spaces.

## Sample Spaces

An act of flipping coins, rolling dice, drawing cards, or surveying people are referred to as a probability **experiment**. A **sample space** of an experiment is the set of all possible outcomes.

**Example 7.1.1**

If a die is rolled, write a sample space.

**Solution**

A die has six faces each having an equally likely chance of appearing. Therefore, the set of all possible outcomes $S$ is

$$\{ 1, 2, 3, 4, 5, 6 \}.$$

**Example 7.1.2**

A family has three children. Write a sample space.

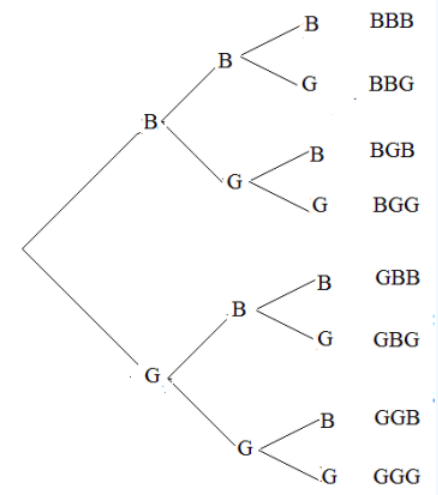
**Solution**

The sample space consists of eight possibilities.

$$\{ \text{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG} \}$$

The possibility BGB, for example, indicates that the first born is a boy, the second born a girl, and the third a boy.

We illustrate these possibilities with a tree diagram.




**Example 7.1.3**

Two dice are rolled. Write the sample space.

**Solution**

We assume one of the dice is red, and the other green. We have the following 36 possibilities.

| **Green** | | | | | | |
|---|---|---|---|---|---|---|
| **Red** | **1** | **2** | **3** | **4** | **5** | **6** |
| **1** | (1, 1) | (1, 2) | (1, 3) | (1, 4) | (1, 5) | (1, 6) |
| **2** | (2, 1) | (2, 2) | (2, 3) | (2, 4) | (2, 5) | (2, 6) |
| **3** | (3, 1) | (3, 2) | (3, 3) | (3, 4) | (3, 5) | (3, 6) |
| **4** | (4, 1) | (4, 2) | (4, 3) | (4, 4) | (4, 5) | (4, 6) |
| **5** | (5, 1) | (5, 2) | (5, 3) | (5, 4) | (5, 5) | (5, 6) |
| **6** | (6, 1) | (6, 2) | (6, 3) | (6, 4) | (6, 5) | (6, 6) |

The entry (2, 5), for example, indicates that the red die shows a 2, and the green a 5.

## Probability

Now that we understand the concept of a sample space, we will define probability.

**Definition: Probability**

For a sample space $S$, and an outcome $A$ of $S$, the following two properties are satisfied.

1. If $A$ is an outcome of a sample space, then the probability of $A$, denoted by $P(A)$, is between 0 and 1, inclusive.

$$0 \leq P(A) \leq 1$$

2. The sum of the probabilities of all the outcomes in $S$ equals 1.

The probability $P(A)$ of an event $A$ describes the chance or likelihood of that event occurring.

- If $P(A) = 0$, event A is certain not to occur. If $P(A) = 1$, event $A$ is certain to occur.
- If $P(A) = 0.5$, then event A is equally likely to occur or not occur.
- If we toss a fair coin that is equally likely to land on heads or tails, then P(Head) = 0.50.

- If the weather forecast says there is a 70% chance of rain today, then P(Rain) = 0.70, indicating is it more likely to rain than to not rain.

**Example 7.1.4**

If two dice, one red and one green, are rolled, find the probability that the red die shows a 3 and the green shows a six.

**Solution**

Since two dice are rolled, there are 36 possibilities. The probability of each outcome, listed in Example 7.1.3, is equally likely.

Since (3, 6) is one such outcome, the probability of obtaining (3, 6) is 1/36.

The example we just considered consisted of only one outcome of the sample space. We are often interested in finding probabilities of several outcomes represented by an event.

An **event** is a subset of a sample space. If an event consists of only one outcome, it is called a **simple event.**

**Example 7.1.5**

If two dice are rolled, find the probability that the sum of the faces of the dice is 7.

**Solution**

Let E represent the event that the sum of the faces of two dice is 7.

The possible cases for the sum to be equal to 7 are: (1, 6), (2,5), (3, 4), (4, 3), (5, 2), and (6, 1), so event E is

$$E = \{(1, 6), (2,5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

The probability of the event E is

$$P(E) = 6/36 \text{ or } 1/6.$$

**Example 7.1.6**

A jar contains 3 red, 4 white, and 3 blue marbles. If a marble is chosen at random, what is the probability that the marble is a red marble or a blue marble?

**Solution**

We assume the marbles are $r_1, r_2, r_3, w_1, w_2, w_3, w_4, b_1, b_2, b_3$. Let the event C represent that the marble is red or blue.

The sample space $S = \{r_1, r_2, r_3, w_1, w_2, w_3, w_4, b_1, b_2, b_3\}$.

And the event $C = \{r_1, r_2, r_3, b_1, b_2, b_3\}$

Therefore, the probability of C,

$$P(C) = 6/10 \text{ or } 3/5$$

**Example 7.1.7**

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **without replacement**, what is the probability that the sum of the numbers is 5?

Note: The two marbles in this example are drawn consecutively **without replacement**. That means that after a marble is drawn it is not replaced in the jar, and therefore is no longer available to select on the second draw.

**Solution**

Since two marbles are drawn without replacement, the sample space consists of the following six possibilities.

$$S = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$$

Note that (1,1), (2,2) and (3,3) are not listed in the sample space. These outcomes are not possible when drawing without replacement, because once the first marble is drawn but not replaced into the jar, that marble is not available in the jar to be selected again on the second draw.

Let the event $\mathrm{E}$ represent that the sum of the numbers is five. Then

$$\mathrm{E} = \{(2,3),(3,2)\}$$

Therefore, the probability of $\mathrm{E}$ is

$$\mathrm{P}(\mathrm{E}) = 2/6 \text{ or } 1/3.$$

### Example 7.1.8

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **without replacement**, what is the probability that the sum of the numbers is *at least* 4?

**Solution**

The sample space, as in Example $7.1.7$, consists of the following six possibilities.

$$\mathrm{S} = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$$

Let the event $\mathrm{F}$ represent that the sum of the numbers is at least four. Then

$$\mathrm{F} = \{(1,3),(3,1),(2,3),(3,2)\}$$

Therefore, the probability of $\mathrm{F}$ is

$$\mathrm{P}(\mathrm{F}) = 4/6 \text{ or } 2/3$$

### Example 7.1.9

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **with replacement**, what is the probability that the sum of the numbers is 5?

Note: The two marbles in this example are drawn consecutively **with replacement**. That means that after a marble is drawn it IS replaced in the jar, and therefore is available to select again on the second draw.

**Solution**

When two marbles are drawn with replacement, the sample space consists of the following nine possibilities.

$$\mathrm{S} = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$

Note that (1,1), (2,2) and (3,3) are listed in the sample space. These outcomes are possible when drawing with replacement, because once the first marble is drawn and replaced, that marble is not available in the jar to be drawn again.

Let the event $\mathrm{E}$ represent that the sum of the numbers is four. Then

$$\mathrm{E} = (2,3),(3,2)$$

Therefore, the probability of $\mathrm{F}$ is $\mathrm{P}(\mathrm{E}) = 2/9$

Note that in Example $7.1.9$ when we selected marbles with replacement, the probability has changed from Example $7.1.7$ where we selected marbles without replacement.

### Example 7.1.10

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **with replacement**, what is the probability that the sum of the numbers is *at least* 4?

**Solution**

The sample space when drawing with replacement consists of the following nine possibilities.

$$S = (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)$$

Let the event $F$ represent that the sum of the numbers is at least four. Then

$$F = (1,3), (3,1), (2,3), (3,2), (2,2), (3,3)$$

Therefore, the probability of $F$ is

$$P(F) = 6/9 \text{ or } 2/3.$$

Note that in Example $7.1.10$ when we selected marbles with replacement, the probability is the same as in Example $7.1.8$ where we selected marbles without replacement.

Thus sampling with or without replacement MAY change the probabilities, but may not, depending on the situation in the particular problem under consideration. We'll re-examine the concepts of sampling with and without replacement in Section 8.3.

### Example 7.1.11

One 6 sided die is rolled once. Find the probability that the result is greater than 4.

**Solution**

The sample space consists of the following six possibilities in set $S$: $S = 1, 2, 3, 4, 5, 6$

Let $E$ be the event that the number rolled is greater than four: $E = 5, 6$

Therefore, the probability of $E$ is: $P(E) = 2/6$ or $1/3$.

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# 7.1.1: Sample Spaces and Probability (Exercises)

## SECTION 8.1 PROBLEM SET: SAMPLE SPACES AND PROBABILITY

In problems 1 - 6, write a sample space for the given experiment.

| | |
|---|---|
| 1) A die is rolled. | 2) A penny and a nickel are tossed. |
| 3) A die is rolled, and a coin is tossed. | 4) Three coins are tossed. |
| 5) Two dice are rolled. | 6) A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn. |

In problems 7 - 12, one card is randomly selected from a deck. Find the following probabilities.

| | |
|---|---|
| 7) P( an ace) | 8) P( a red card) |
| 9) P( a club) | 10) P( a face card) |
| 11) P(a jack or a spade) | 12) P(a jack and a spade) |

For problems 13 - 16: A jar contains 6 red, 7 white, and 7 blue marbles. If one marble is chosen at random, find the following probabilities.

| | |
|---|---|
| 13) P(red) | 14) P(white) |
| 15) P(red or blue) | 16) P(red and blue) |

For problems 17 - 22: Consider a family of three children. Find the following probabilities.

| | |
|---|---|
| 17) P(two boys and a girl) | 18) P(at least one boy) |
| 19) P(children of both sexes) | 20) P(at most one girl) |
| 21) P(first and third children are male) | 22) P(all children are of the same gender) |

For problems 23 - 27: Two dice are rolled. Find the following probabilities.

| | |
|---|---|
| 23) P(the sum of the dice is 5) | 24) P(the sum of the dice is 8) |
| 25) P(the sum is 3 or 6) | 26) P(the sum is more than 10) |
| 27) P(the result is a double) (Hint: a double means that both dice show the same value) | |

For problems 28-31: A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn randomly WITHOUT REPLACEMENT. That means that after a marble is drawn it is NOT replaced in the jar before the second marble is selected. Find the following probabilities.

| | |
|---|---|
| 28) P(the sum of the numbers is 5) | 29) P(the sum of the numbers is odd) |
| 30) P(the sum of the numbers is 9) | 31) P(one of the numbers is 3) |

For problems 32-33: A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn randomly WITH REPLACEMENT. That means that after a marble is drawn it is replaced in the jar before the second marble is selected. Find the following probabilities.

| | |
|---|---|
| 32) P(the sum of the numbers is 5) | 33) P(the sum of the numbers is 2) |

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# 7.2: Mutually Exclusive Events and the Addition Rule

**Learning Objectives**

In this section, you will learn to:

1. Define compound events using union, intersection, and complement.
2. Identify mutually exclusive events
3. Use the Addition Rule to calculate probability for unions of events.

In the last chapter, we learned to find the union, intersection, and complement of a set. We will now use these set operations to describe events.

- The **union** of two events E and F, E ∪ F, is the set of outcomes that are in E or in F or in both.
- The **intersection** of two events E and F, E ∩ F, is the set of outcomes that are in both E and F.
- The **complement** of an event E, denoted by $E^c$, is the set of outcomes in the sample space S that are not in E.

It is worth noting that $P(E^c) = 1 - P(E)$. This follows from the fact that if the sample space has $n$ elements and E has $k$ elements, then $E^c$ has $n-k$ elements. Therefore,

$$P(E^c) = \frac{n-k}{n} = 1 - \frac{k}{n} = 1 - P(E)$$

Of particular interest to us are the events whose outcomes do not overlap. We call these events mutually exclusive.

Two events E and F are said to be **mutually exclusive** if they do not intersect: $E \cap F = \varnothing$.

Next we'll determine whether a given pair of events are mutually exclusive.

**Example 7.2.1**

A card is drawn from a standard deck. Determine whether the pair of events given below is mutually exclusive.

E = {The card drawn is an Ace}

F = {The card drawn is a heart}

**Solution**

Clearly the ace of hearts belongs to both sets. That is

$$E \cap F = \{\text{ Ace of hearts }\} \neq \varnothing$$

Therefore, the events E and F are not mutually exclusive.

**Example 7.2.2**

Two dice are rolled. Determine whether the pair of events given below is mutually exclusive.

G = {The sum of the faces is six}

H = {One die shows a four}

**Solution**

For clarity, we list the elements of both sets.

G = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)} and H = {(2, 4), (4, 2)}

Clearly, $G \cap H = \{(2, 4), (4, 2)\} \neq \varnothing$.

Therefore, the two sets are not mutually exclusive.

**Example 7.2.3**

A family has three children. Determine whether the following pair of events are mutually exclusive.

$$M = \{\text{The family has at least one boy}\}$$

$$N = \{\text{The family has all girls}\}$$

**Solution**

Although the answer may be clear, we list both the sets.

$$M = \{ BBB, BBG, BGB, BGG, GBB, GBG, GGB \} \text{ and } N = \{ GGG \}$$

Clearly, $M \cap N = \varnothing$

Therefore, events M and N are mutually exclusive.

We will now consider problems that involve the union of two events.

Given two events, E, F, then finding the probability of $E \cup F$, is the same as finding the probability that E will happen, or F will happen, or both will happen.

**Example 7.2.4**

If a die is rolled, what is the probability of obtaining an even number or a number greater than four?

**Solution**

Let E be the event that the number shown on the die is an even number, and let F be the event that the number shown is greater than four.

The sample space S = { 1, 2, 3, 4, 5, 6 }. The event E = { 2, 4, 6 }, and event F = { 5, 6 }

We need to find $P(E \cup F)$.

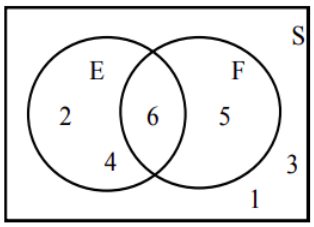
Since P(E) = 3/6, and P(F) = 2/6, a student may say $P(E \cup F) = 3/6 + 2/6$. This will be incorrect because the element 6, which is in both E and F has been counted twice, once as an element of E and once as an element of F. In other words, the set $E \cup F$ has only four elements and not five: set $E \cup F$ = { 2,4,5,6 }

Therefore, $P(E \cup F) = 4/6$ and not 5/6.

This can be illustrated by a Venn diagram. We'll use the Venn Diagram to re-examine Example 7.2.4 and derive a probability rule that we can use to calculate probabilities for unions of events.

The sample space S, the events E and F, and $E \cap F$ are listed below.

$$S = \{1, 2, 3, 4, 5, 6\}, E = \{2, 4, 6\}, F = \{5, 6\}, \text{ and } E \cap F = \{6\}.$$




The above figure shows S, E, F, and $E \cap F$.

Finding the probability of $E \cup F$, is the same as finding the probability that E will happen, or F will happen, or both will happen.

If we count the number of elements n(E) in E, and add to it the number of elements n(F) in F, the points in both E and F are counted twice, once as elements of E and once as elements of F. Now if we subtract from the sum, n(E) + n(F), the number $n(E \cap F)$, we remove the duplicity and get the correct answer. So as a rule,

$$n(E \cup F) = n(E) + n(F) - n(E \cap F)$$

By dividing the entire equation by n(S), we get

$$\frac{n(E \cup F)}{n(S)} = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(E \cap F)}{n(S)}$$

Since the probability of an event is the number of elements in that event divided by the number of all possible outcomes, we have

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Applying the above for Example 7.2.4, we get

$$P(E \cup F) = 3/6 + 2/6 - 1/6 = 4/6$$

This is because, when we add P(E) and P(F), we have added P(E ∩ F) twice. Therefore, we must subtract P(E ∩ F), once.

This gives us the general formula, called **the Addition Rule**, for finding the probability of the union of two events. Because event E ∪ F is the event that E will happen, OR F will happen, OR both will happen, we sometimes call this the **Addition Rule for OR Events.** It states

**Addition Rule**

$$\mathbf{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$$

If, and only if, two events $E$ and $F$ are mutually exclusive, then $E \cap F = \varnothing$ and $P(E \cap F) = 0$, and we get $P(E \cup F) = P(E) + P(F)$

**Example 7.2.5**

If a card is drawn from a deck, use the addition rule to find the probability of obtaining an ace or a heart.

**Solution**

Let A be the event that the card is an ace, and H the event that it is a heart.

Since there are four aces, and thirteen hearts in the deck,

P(A) = 4/52 and P(H) = 13/52.

Furthermore, since the intersection of two events consists of only one card, the ace of hearts, we now have:

P(A ∩ H) = 1/52

We need to find P(A ∪ H):

$$\begin{aligned} P(A \cup H) &= P(A) + P(H) - P(A \cap H) \\ &= 4/52 + 13/52 - 1/52 = 16/52 \end{aligned}$$

**Example 7.2.6**

Two dice are rolled, and the events F and T are as follows:

F = {The sum of the dice is four} and T = {At least one die shows a three}

Find P(F ∪ T).

**Solution**

We list F and T, and F ∩ T as follows:

F = {(1, 3), (2, 2), (3, 1)}

T = {(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (1, 3), (2, 3), (4, 3), (5, 3), (6, 3)}

F ∩ T = {(1, 3), (3, 1)}

Since P(F ∪ T) = P(F) + P(T) - P(F ∩ T)

We have P(F ∪ T) = 3/36 + 11/36 - 2/36 = 12/36.

### Example 7.2.7

Mr. Washington is seeking a mathematics instructor's position at his favorite community college in Cupertino. His employment depends on two conditions: whether the board approves the position, and whether the hiring committee selects him. There is a 80% chance that the board will approve the position, and there is a 70% chance that the hiring committee will select him. If there is a 90% chance that at least one of the two conditions, the board approval or his selection, will be met, what is the probability that Mr. Washington will be hired?

**Solution**

Let A be the event that the board approves the position, and S be the event that Mr. Washington gets selected. We have,

P(A) = .80, P(S) = .70, and P(A ∪ S) = .90.

We need to find, P(A ∩ S).

The addition formula states that,

$$P(A \cup S) = P(A) + P(S) - P(A \cap S)$$

Substituting the known values, we get

$$.90 = .80 + .70 - P(\mathrm{A} \cap \mathrm{S})$$

Therefore, P(A ∩ S) = .60.

### Example 7.2.8

The probability that this weekend will be cold is .6, the probability that it will be rainy is .7, and probability that it will be both cold and rainy is .5. What is the probability that it will be neither cold nor rainy?

**Solution**

Let C be the event that the weekend will be cold, and R be event that it will be rainy. We are given that

$$\mathrm{P(C)} = .6, \quad \mathrm{P(R)} = .7, \quad \mathrm{P(C \cap R)} = .5$$

First we find P(C ∪ R) using the Addition Rule.

$$P(C \cup R) = P(C) + P(R) - P(C \cap R) = .6 + .7 - .5 = .8$$

Then we find P((C ∪ R)$^c$) using the Complement Rule.

$$P\left((C \cup R)^c\right) = 1 - P(C \cup R) = 1 - .8 = .2$$

We summarize this section by listing the important rules.

### Summary

**The Addition Rule**

For Two Events E and F, $\mathrm{P(E \cup F)} = \mathrm{P(E)} + \mathrm{P(F)} - \mathrm{P(E \cap F)}$

**The Addition Rule for Mutually Exclusive Events**

If Two Events E and F are Mutually Exclusive, then $\mathrm{P(E \cup F)} = \mathrm{P(E)} + \mathrm{P(F)}$

**The Complement Rule**

If $\mathrm{E}^c$ is the Complement of Event E, then $\mathrm{P}(\mathrm{E}^c) = 1 - \mathrm{P(E)}$

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# 7.2.1: Mutually Exclusive Events and the Addition Rule (Exercises)

## SECTION 8.2 PROBLEM SET: MUTUALLY EXCLUSIVE EVENTS AND THE ADDITION RULE

Determine whether the following pair of events are mutually exclusive.

| | |
|---|---|
| 1) A = {A person earns more than $25,000}<br>B = {A person earns less than $20,000} | 2) A card is drawn from a deck.<br>C = {It is a King} D = {It is a heart}. |
| 3) A die is rolled.<br>E = {An even number shows}<br>F = {A number greater than 3 shows} | 4) Two dice are rolled.<br>G = {The sum of dice is 8}<br>H = {One die shows a 6} |
| 5) Three coins are tossed.<br>I = {Two heads come up}<br>J = {At least one tail comes up} | 6) A family has three children.<br>K = {First born is a boy}<br>L = {The family has children of both sexes} |

Use the Addition Rule to find the following probabilities.

| | |
|---|---|
| 7) A card is drawn from a deck. Events C and D are:<br>C = {It is a king}<br>D = {It is a heart}<br>Find P(C or D). | 8) A die is rolled. The events E and F are:<br>E = {An even number shows}<br>F = {A number greater than 3 shows}<br>Find P(E or F). |
| 9) Two dice are rolled. Events G and H are:<br>G = {The sum of dice is 8}<br>H ={Exactly one die shows a 6}<br>Find P(G or H). | 10) Three coins are tossed. Events I and J are:<br>I = {Two heads come up}<br>J = {At least one tail comes up}<br>Find P(I or J). |
| 11) At a college, 20% of the students take Finite Mathematics, 30% take Statistics and 10% take both. What percent of students take Finite Mathematics or Statistics? | 12) This quarter, there is a 50% chance that Jason will pass Accounting, a 60% chance that he will pass English, and 80% chance that he will pass at least one of these two courses. What is the probability that he will pass both Accounting and English? |

Questions 13 - 20 refer to the following: The table shows the distribution of Democratic and Republican U.S by gender in the 114$^{th}$ Congress as of January 2015.

| | MALE(M) | FEMALE(F) | TOTAL |
|---|---|---|---|
| DEMOCRATS (D) | 30 | 14 | 44 |
| REPUBLICANS(R) | 48 | 6 | 54 |
| OTHER (T) | 2 | 0 | 2 |
| TOTALS | 80 | 20 | 100 |

Use this table to determine the following probabilities.

| | |
|---|---|
| 13) P(M and D) | 14) P(F and R) |
| 15) P(M or D) | 16) P(F or R) |
| 17) P(Mc or R) | 18) P(M or F) |
| 19) Are the events F, R mutually exclusive?<br>Use probabilities to support your conclusions. | 20) Are the events F, T mutually exclusive?<br>Use probabilities to support your conclusion. |

***SECTION 8.2 PROBLEM SET: MUTUALLY EXCLUSIVE EVENTS AND THE ADDITION RULE***

Use the Addition Rule to find the following probabilities.

| | |
|---|---|
| 21) If P(E) = .5 , P(F) = .4 , E and F are mutually exclusive, find P(E and F). | 22) If P(E) = .4 , P(F) = .2 , E and F are mutually exclusive, find P(E or F). |
| 23) If P(E) = .3, P(E or F) = .6 , P(E and F) = .2, find P(F). | 24) If P(E) = .4, P(F) = .5 , P(E or F) = .7, find P(E and F). |

| | |
|---|---|
| 25) In a box of assorted cookies, 36% of cookies contain chocolate and 12% of cookies contain nuts. 8% of cookies have both chocolats and nuts. Sean is allergic to chocolate and nuts. Find the probability that a cookie has chocolate chips **or** nuts (he can't eat it). | 26) At a college, 72% of courses have final exams and 46% of courses require research papers.<br>32% of courses have both a research paper and a final exam. Let F be the event that a course has a final exam and R be the event that a course requires a research paper.<br>Find the probability that a course requires a final exam **or** a research paper. |

*Questions 25 and 26 are adapted from Introductory Statistics from OpenStax under a creative Commons Attribution 3.0 Unported License, available for download free atcnx.org/content/col11562/latest u*

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# 7.3: Probability Using Tree Diagrams and Combinations

**Learning Objectives**

In this section, you will learn to:

1. Use probability tree diagrams to calculate probabilities
2. Use combinations to calculate probabilities

In this section, we will apply previously learnt counting techniques in calculating probabilities, and use tree diagrams to help us gain a better understanding of what is involved.

## USING TREE DIAGRAMS TO CALCULATE PROBABILITIES

We already used tree diagrams to list events in a sample space. Tree diagrams can be helpful in organizing information in probability problems; they help provide a structure for understanding probability. In this section we expand our previous use of tree diagrams to situations in which the events in the sample space are not all equally likely.

We assign the appropriate probabilities to the events shown on the branches of the tree.
By multiplying probabilities along a path through the tree, we can find probabilities for “and” events, which are intersections of events.

We begin with an example.

**Example 7.3.1**

Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn with replacement, what is the probability that both marbles are red?

**Solution**

Let $\mathrm{E}$ be the event that the first marble drawn is red, and let $\mathrm{F}$ be the event that the second marble drawn is red.

We need to find $\mathrm{P}(\mathrm{E} \cap \mathrm{F})$.

By the statement, "two marbles are drawn with replacement," we mean that the first marble is replaced before the second marble is drawn.

There are 7 choices for the first draw. And since the first marble is replaced before the second is drawn, there are, again, seven choices for the second draw. Using the multiplication axiom, we conclude that the sample space $\mathrm{S}$ consists of 49 ordered pairs. Of the 49 ordered pairs, there are $3 \times 3 = 9$ ordered pairs that show red on the first draw and, also, red on the second draw. Therefore,

$$P(E \cap F) = \frac{9}{49}$$

Further note that in this particular case

$$P(E \cap F) = \frac{9}{49} = \frac{3}{7} \cdot \frac{3}{7}$$

giving us the result that in this example: $\mathbf{P}(\mathbf{E} \cap \mathbf{F}) = \mathbf{P}(\mathbf{E}) \cdot \mathbf{P}(\mathbf{F})$

**Example 7.3.2**

If in Example $7.3.1$, the two marbles are drawn without replacement, then what is the probability that both marbles are red?

**Solution**

By the statement, "two marbles are drawn without replacement," we mean that the first marble is not replaced before the second marble is drawn.

Again, we need to find $\mathrm{P}(\mathrm{E} \cap \mathrm{F})$.

There are, again, 7 choices for the first draw. And since the first marble is not replaced before the second is drawn, there are only six choices for the second draw. Using the multiplication axiom, we conclude that the sample space S consists of 42 ordered pairs. Of the 42 ordered pairs, there are $3 \times 2 = 6$ ordered pairs that show red on the first draw and red on the second draw. Therefore,

$$P(E \cap F) = \frac{6}{42}$$

Note that we can break this calculation down as

$$P(E \cap F) = \frac{6}{42} = \frac{3}{7} \cdot \frac{2}{6}$$

.

Here 3/7 represents $\mathrm{P(E)}$, and 2/6 represents the probability of drawing a red on the second draw, given that the first draw resulted in a red.

We write the latter as $\mathrm{P}$(red on the second | red on first) or $\mathrm{P(F|E)}$. The "|" represents the word "given" or “if”. This leads to the result that:

$$\mathbf{P(E \cap F) = P(E) \cdot P(F|E)}$$

The is an important result, called the **Multiplication Rule,** which will appear again in later sections.

We now demonstrate the above results with a tree diagram.

## Example 7.3.3

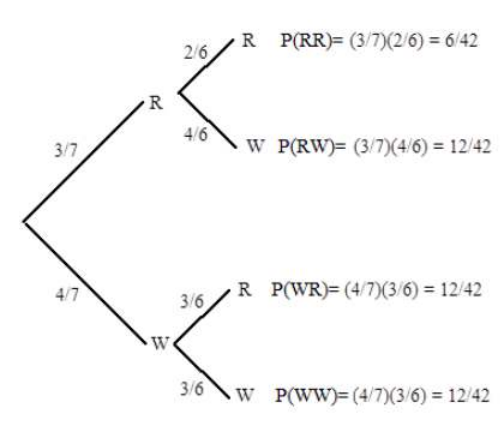
Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn without replacement, find the following probabilities using a tree diagram.

a. The probability that both marbles are red.
b. The probability that the first marble is red and the second white.
c. The probability that one marble is red and the other white.

**Solution**

Let $\mathrm{R}$ be the event that the marble drawn is red, and let W be the event that the marble drawn is white.

We draw the following tree diagram.




a. The probability that both marbles are red is $\mathrm{P(RR)} = 6/42$
b. The probability that the first marble is red and the second is white is $\mathrm{P(RW)} = 12/42$
c. For the probability that one marble is red and the other is white, we observe that this can be satisfied if the first is red and the second is white, **or** if the first is white and the second is red. The “or” tells us we’ll be using the Addition Rule from Section 7.2.

Furthermore events RW and WR are mutually exclusive events, so we use the form of the Addition Rule that applies to mutually exclusive events.

Therefore

P(one marble is red and the other marble is white)

$$\begin{aligned} &= \mathrm{P}(\mathrm{RW} \text{ or } \mathrm{WR}) \\ &= \mathrm{P}(\mathrm{RW}) + \mathrm{P}(\mathrm{WR}) \\ &= 12/42 + 12/42 = 24/42 \end{aligned}$$

## USING COMBINATIONS TO FIND PROBABILITIES

Although the tree diagrams give us better insight into a problem, they are not practical for problems where more than two or three things are chosen. In such cases, we use the concept of combinations that we learned in the last chapter. This method is best suited for problems where the order in which the objects are chosen is not important, and the objects are chosen without replacement.

### Example 7.3.4

Suppose a jar contains 3 red, 2 white, and 3 blue marbles. If three marbles are drawn without replacement, find the following probabilities.

a. P(Two red and one white)
b. P(One of each color)
c. P(None blue)
d. P(At least one blue)

**Solution**

Let us suppose the marbles are labeled as $R_1, R_2, R_3, W_1, W_2, B_1, B_2, B_3$.

a. P(Two red and one white)

Since we are choosing 3 marbles from a total of 8, there are 8C3 = 56 possible combinations. Of these 56 combinations, there are $3\mathrm{C}2 \times 2\mathrm{C}1 = 6$ combinations consisting of 2 red and one white. Therefore,

$$P(\text{ Two red and one white }) = \frac{3\mathrm{C}2 \times 2\mathrm{C}1}{8\mathrm{C}3} = \frac{6}{56}.$$

b. P(One of each color)

Again, there are 8C3 = 56 possible combinations. Of these 56 combinations, there are $3\mathrm{Cl} \times 2\mathrm{Cl} \times 3\mathrm{Cl} = 18$ combinations consisting of one red, one white, and one blue. Therefore,

$$P(\text{ One of each color }) = \frac{3\mathrm{C}1 \times 2\mathrm{C}1 \times 3\mathrm{C}1}{8\mathrm{C}3} = \frac{18}{56}$$

c. P(None blue)

There are 5 non-blue marbles, therefore

$$\mathrm{P}(\text{ None blue }) = \frac{5\mathrm{C}3}{8\mathrm{C}3} = \frac{10}{56} = \frac{5}{28}$$

d. P(At least one blue)

By "at least one blue marble," we mean the following: one blue marble and two non-blue marbles, *OR* two blue marbles and one non-blue marble, *OR* all three blue marbles. So we have to find the sum of the probabilities of all three cases.

$$\mathrm{P}(\text{At least one blue }) = \mathrm{P}(1 \text{ blue, } 2 \text{ non-blue}) + \mathrm{P}(2 \text{ blue, l non-blue}) + \mathrm{P}(3 \text{ blue})$$

$$P(\text{ At least one blue }) = \frac{3\mathrm{C}1 \times 5\mathrm{C}2}{8\mathrm{C}3} + \frac{3\mathrm{C}2 \times 5\mathrm{C}1}{8\mathrm{C}3} + \frac{3\mathrm{C}3}{8\mathrm{C}3}$$

$$P(\text{ At least one blue }) = 30/56 + 15/56 + 1/56 = 46/56 = 23/28$$

Alternately, we can use the fact that $\mathrm{P(E)} = 1 - \mathrm{P(E}^c)$ . If the event E = At least one blue, then $E^c$ = None blue.

But from part c of this example, we have $(\mathrm{E}^c) = 5/28$, so $\mathrm{P(E)} = 1 - 5/28 = 23/28$.

### Example 7.3.5

Five cards are drawn from a deck. Find the probability of obtaining two pairs, that is, two cards of one value, two of another value, and one other card.

**Solution**

Let us first do an easier problem-the probability of obtaining a pair of kings and queens.

Since there are four kings, and four queens in the deck, the probability of obtaining two kings, two queens and one other card is

$$\mathrm{P(A\ pair\ of\ kings\ and\ queens\ )} = \frac{4\mathrm{C}2 \times 4\mathrm{C}2 \times 44\mathrm{C}1}{52\mathrm{C}5}$$

To find the probability of obtaining two pairs, we have to consider all possible pairs.

Since there are altogether 13 values, that is, aces, deuces, and so on, there are $13\mathrm{C}2$ different combinations of pairs.

$$P(\ \mathrm{Two\ pairs}\ ) = 13\mathrm{C}2 \cdot \frac{4\mathrm{C}2 \times 4\mathrm{C}2 \times 44\mathrm{C}1}{52\mathrm{C}5} = .04754$$

### Example 7.3.6

A cell phone store receives a shipment of 15 cell phones that contains 8 iPhones and 7 Android phones. Suppose that 6 cell phones are randomly selected from this shipment. Find the probability that a randomly selected set of 6 cell phones consists of 2 iPhones and 4 Android phones.

**Solution**

There are $8\mathrm{C}2$ ways of selecting 2 out of the 8 iPhones.

and $7\mathrm{C}4$ ways of selecting 4 out of the 7 Android phones

But altogether there are $15\mathrm{C}6$ ways of selecting 6 out of 15 cell phones.

Therefore we have

$$P(2\ \mathrm{iPhones\ and\ 4\ Android\ phones}\ ) = \frac{8\mathrm{C}2 \times 7\mathrm{C}4}{15\mathrm{C}6} = \frac{(28)(35)}{5005} = \frac{980}{5005} = 0.1958$$

### Example 7.3.7

One afternoon, a bagel store still has 53 bagels remaining: 20 plain, 15 poppyseed, and 18 sesame seed bagels. Suppose that the store owner packages up a bag of 9 bagels to bring home for tomorrow's breakfast, and selects the bagels randomly. Find the probability that the bag contains 4 plain, 3 poppyseed, and 2 sesame seed.

**Solution**

There are $20\mathrm{C}4$ ways of selecting 4 out of the 20 plain bagels,

and $15\mathrm{C}3$ ways of selecting 3 out of the 15 poppyseed bagels,

and $18\mathrm{C}2$ ways of selecting 2 out of the 18 sesame seed bagels.

But altogether there are $53\mathrm{C}9$ ways of selecting 9 out of the 53 bagels.

$$\begin{aligned} \mathrm{P(4\ plain,\ 3\ poppyseed,\ and\ 2\ sesame\ seed)} &= \tfrac{20\mathrm{C}4 \times 15\mathrm{C}3 \times 18\mathrm{C}2}{53\mathrm{C}9} \\ &= \tfrac{(4845)(455)(153)}{4431613550} \\ &= 0.761 \end{aligned} \tag{7.3.1}$$

We end the section by solving a famous problem called the **Birthday Problem**.

### Example 7.3.8: Birthday Problem

If there are 25 people in a room, what is the probability that at least two people have the same birthday?

**Solution**

Let event $\mathrm{E}$ represent that at least two people have the same birthday.

We first find the probability that no two people have the same birthday.

We analyze as follows.

Suppose there are 365 days to every year. According to the multiplication axiom, there are $365^{25}$ possible birthdays for 25 people. Therefore, the sample space has $365^{25}$ elements. We are interested in the probability that no two people have the same birthday. There are 365 possible choices for the first person and since the second person must have a different birthday, there are 364 choices for the second, 363 for the third, and so on. Therefore,

$$\mathrm{P}(\text{No two have the same birthday }) = \frac{365 \cdot 364 \cdot 363 \cdots 341}{365^{25}} = \frac{365\mathrm{P}25}{365^{25}}$$

Since $\mathrm{P}$(at least two people have the same birthday) = 1 - $\mathrm{P}$(No two have the same birthday),

$$\mathrm{P} \text{ at least two people have the same birthday }) = 1 - \frac{365\mathrm{P}25}{365^{25}} = .5687$$

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# 7.3.1: Probability Using Tree Diagrams and Combinations (Exercises)

## SECTION 8.3 PROBLEM SET: PROBABILITIES USING TREE DIAGRAMS AND COMBINATIONS

Two apples are chosen from a basket containing five red and three yellow apples.
Draw a tree diagram below, and find the following probabilities.

| | |
|---|---|
| 1) P( both red) | 2) P(one red, one yellow) |
| 3) P(both yellow) | 4) P(First red and second yellow) |

A basket contains six red and four blue marbles. Three marbles are drawn at random.
Find the following probabilities using the method shown in Example 8.3.2. Do not use combinations.

| | |
|---|---|
| 5) P( All three red) | 6) P(two red, one blue) |
| 7) P(one red, two blue) | 8) P(first red, second blue, third red) |

Three marbles are drawn from a jar containing five red, four white, and three blue marbles.
Find the following probabilities using combinations.

| | |
|---|---|
| 9) P(all three red) | 10) P(two white and 1 blue) |
| 11) P(none white) | 12) P(at least one red) |

A committee of four is selected from a total of 4 freshmen, 5 sophomores, and 6 juniors. Find the probabilities for the following events.

| | |
|---|---|
| 13) At least three freshmen. | 14) No sophomores. |
| 15) All four of the same class. | 16) Not all four from the same class. |
| 17) Exactly three of the same class. | 18) More juniors than freshmen and sophomores combined. |

Five cards are drawn from a deck. Find the probabilities for the following events.

| | |
|---|---|
| 19) Two hearts, two spades, and one club. | 20) A flush of any suit (*all cards of a single suit*). |
| 21) A full house of nines and tens (*3 nines and 2 tens*). | 22) Any full house. |
| 23) A pair of nines and a pair of tens (*and the fifth card is not a nine or ten*). | 24) Any two pairs (*two cards of one value, two more cards of another value, and the fifth card does not have the same value as either pair*). |

Jorge has 6 rock songs, 7 rap songs and 4 country songs that he likes to listen to while he exercises.
He randomly selects six (6) of these songs to create a playlist to listen to today while he exercises.

Find the following probabilities:

| | |
|---|---|
| 25) P(playlist has 2 songs of each type) | 26) P(playlist has no country songs) |
| 27) P(playlist has 3 rock, 2 rap, and 1 country song) | 28) P(playlist has 3 or 4 rock songs and the rest are rap songs) |

A project is staffed 12 people: 5 engineers, 4 salespeople, and 3 customer service representatives.
A committee of 5 people is selected to make a presentation to senior management.

Find the probabilities of the following events.

| | |
|---|---|
| 29) The committee has 2 engineers, 2 salespeople, and 1 customer service representative. | 30) The committee contains 3 engineer and 2 salespeople. |
| 31) The committee has no engineers. | 32) The committee has all salespeople. |

Do the following birthday problems.

| | |
|---|---|
| 33) If there are 5 people in a room, what is the probability that no two have the same birthday? | 34) If there are 5 people in a room, find the probability that at least 2 have the same birthday. |

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# 7.4: Conditional Probability

**Learning Objectives**

In this section, you will learn to:

1. recognize situations involving conditional probability
2. calculate conditional probabilities

Suppose a friend asks you the probability that it will snow today.

If you are in Boston, Massachusetts in the winter, the probability of snow today might be quite substantial. If you are in Cupertino, California in summer, the probability of snow today is very tiny, this probability is pretty much 0.

Let:

- $\mathrm{A}$ = the event that it will snow today
- $\mathrm{B}$ = the event that today you are in Boston in wintertime
- $\mathrm{C}$ = the event that today you are in Cupertino in summertime

Because the probability of snow is affected by the location and time of year, we can't just write $\mathrm{P}(\mathrm{A})$ for the probability of snow. We need to indicate the other information we know -location and time of year. We need to use **conditional probability**.

The event we are interested in is event $\mathrm{A}$ for snow. The other event is called the condition, representing location and time of year in this case.

We represent conditional probability using a vertical line | that means "if", or "given that", or "if we know that". The event of interest appears on the left of the |. The condition appears on the right side of the |.

The probability it will snow given that (if) you are in Boston in the winter is represented by $\mathbf{P}(\mathbf{A}|\mathbf{B})$. In this case, the condition is $\mathrm{B}$.

The probability that it will snow given that (if) you are in Cupertino in the summer is represented by $\mathbf{P}(\mathbf{A}|\mathbf{C})$. In this case, the condition is $\mathrm{C}$.

Now, let's examine a situation where we can calculate some probabilities.

Suppose you and a friend play a game that involves choosing a single card from a well-shuffled deck. Your friend deals you one card, face down, from the deck and offers you the following deal: If the card is a king, he will pay you $5, otherwise, you pay him $1. Should you play the game?

You reason in the following manner. Since there are four kings in the deck, the probability of obtaining a king is 4/52 or 1/13. So, probability of not obtaining a king is 12/13. This implies that the ratio of your winning to losing is 1 to 12, while the payoff ratio is only $1 to $5. Therefore, you determine that you should not play.

But consider the following scenario. While your friend was dealing the card, you happened to get a glance of it and noticed that the card was a face card. Should you, now, play the game?

Since there are 12 face cards in the deck, the total elements in the sample space are no longer 52, but just 12. This means the chance of obtaining a king is 4/12 or 1/3. So your chance of winning is 1/3 and of losing 2/3. This makes your winning to losing ratio 1 to 2 which fares much better with the payoff ratio of $1 to $5. This time, you determine that you should play.

In the second part of the above example, we were finding the probability of obtaining a king knowing that a face card had shown. This is an example of **conditional probability**. Whenever we are finding the probability of an event $\mathrm{E}$ under the condition that another event $\mathrm{F}$ has happened, we are finding conditional probability.

The symbol $\mathrm{P}(\mathrm{E}|\mathrm{F})$ denotes the problem of finding the probability of $\mathrm{E}$ given that $\mathrm{F}$ has occurred. We read $\mathrm{P}(\mathrm{E}|\mathrm{F})$ as "the probability of $\mathrm{E}$, given $\mathrm{F}$."

### Example 7.4.1

A family has three children. Find the conditional probability of having two boys and a girl given that the first born is a boy.

**Solution**

Let event E be that the family has two boys and a girl, and F that the first born is a boy.

First, we the sample space for a family of three children as follows.

$$S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$$

Since we know the first born is a boy, our possibilities narrow down to four outcomes: BBB, BBG, BGB, and BGG.

Among the four, BBG and BGB represent two boys and a girl.

Therefore, $\mathrm{P}(\mathrm{E}|\mathrm{F})$ = 2/4 or 1/2.

### Example 7.4.2

One six sided die is rolled once.

a. Find the probability that the result is even.
b. Find the probability that the result is even given that the result is greater than three.

**Solution**

The sample space is $\mathrm{S} = 1, 2, 3, 4, 5, 6$

Let event E be that the result is even and T be that the result is greater than 3.

a. $\mathrm{P}(\mathrm{E})$ = 3/6 because $\mathrm{E} = 2, 4, 6$

b. Because $T = 4, 5, 6$, we know that 1, 2, 3 cannot occur; only outcomes 4, 5, 6 are possible. Therefore of the values in E, only 4, 6 are possible.

Therefore, $\mathrm{P}(\mathrm{E}|\mathrm{T})$ = 2/3

### Example 7.4.3

A fair coin is tossed twice.

a. Find the probability that the result is is two heads.
b. Find the probability that the result is two heads given that at least one head is obtained.

**Solution**

The sample space is $S = HH, HT, TH, TT$

Let event E be that the two heads are obtained and F be at least one head is obtained

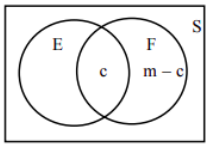
a. $\mathrm{P}(\mathrm{E})$ = 1/4 because $\mathrm{E} = HH$ and the sample space S has 4 outcomes.

b. $\mathrm{F} = HH, HT, TH$. Since at least one head was obtained, TT did not occur.
We are interested in the probability event $\mathrm{E} = HH$ out of the 3 outcomes in the reduced sample space F.

Therefore, $\mathrm{P}(\mathrm{E}|\mathrm{F})$ = 1/3

Let us now develop a formula for the conditional probability $\mathrm{P}(\mathrm{E}|\mathrm{F})$.

Suppose an experiment consists of $n$ equally likely events. Further suppose that there are $m$ elements in F, and $c$ elements in $\mathrm{E} \cap \mathrm{F}$, as shown in the following Venn diagram.




If the event $\mathrm{F}$ has occurred, the set of all possible outcomes is no longer the entire sample space, but instead, the subset $\mathrm{F}$. Therefore, we only look at the set $\mathrm{F}$ and at nothing outside of $\mathrm{F}$. Since $\mathrm{F}$ has $m$ elements, the denominator in the calculation of $\mathrm{P(E|F)}$ is $m$. We may think that the numerator for our conditional probability is the number of elements in $\mathrm{E}$. But clearly we cannot consider the elements of $\mathrm{E}$ that are not in $\mathrm{F}$. We can only count the elements of $\mathrm{E}$ that are in $\mathrm{F}$, that is, the elements in $\mathrm{E} \cap \mathrm{F}$. Therefore,

$$\mathrm{P(E|F)} = \frac{\mathrm{c}}{\mathrm{m}}$$

Dividing both the numerator and the denominator by $n$, we get

$$\mathrm{P(E|F)} = \frac{c/n}{m/n}$$

But $c/n = \mathrm{P(E \cap F)}$, and $m/n = \mathrm{P(F)}$.

Substituting, we derive the following formula for $\mathrm{P(E|F)}$.

**Conditional Probability Rule**

For two events $\mathrm{E}$ and $\mathrm{F}$, the probability of "$\mathrm{E}$ Given $\mathrm{F}$" is

$$\mathbf{P(E|F)} = \frac{\mathbf{P(E \cap F)}}{\mathbf{P(F)}}$$

**Example 7.4.4**

A single die is rolled. Use the above formula to find the conditional probability of obtaining an even number given that a number greater than three has shown.

**Solution**

Let $\mathrm{E}$ be the event that an even number shows, and $\mathrm{F}$ be the event that a number greater than three shows. We want $\mathrm{P(E|F)}$.

$\mathrm{E} = 2, 4, 6$ and $\mathrm{F} = 4, 5, 6$. Which implies, $\mathrm{E} \cap \mathrm{F} = 4, 6$

Therefore, $\mathrm{P(F)}$ = 3/6, and $\mathrm{P(E \cap F)}$ = 2/6

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{2/6}{3/6} = \frac{2}{3}.$$

**Example 7.4.5**

The following table shows the distribution by gender of students at a community college who take public transportation and the ones who drive to school.

| | Male(M) | Female(F) | Total |
|---|---|---|---|
| Public Transportation(T) | 8 | 13 | 21 |
| Drive(D) | 39 | 40 | 79 |
| Total | 47 | 53 | 100 |

The events $\mathrm{M}, \mathrm{F}, \mathrm{T}$, and $\mathrm{D}$ are self explanatory. Find the following probabilities.

a. $\mathrm{P(D|M)}$
b. $\mathrm{P(F|D)}$
c. $\mathrm{P(M|T)}$

**Solution 1**

Conditional probabilities can often be found directly from a contingency table. If the condition corresponds to only one row or only one column in the table, then you can ignore the rest of the table and read the conditional probability directly from the row or column indicated by the condition.

a. The condition is event $\mathrm{M}$; we can look at only the "Male" column of the table and ignore the rest of the table: $\mathrm{P(D|M)} = \frac{39}{47}$.
b. The condition is event $\mathrm{D}$; we can look at only the "Drive" row of the table and ignore the rest of the table: $\mathrm{P(F|D)} = \frac{40}{79}$.
c. The condition is event $\mathrm{T}$; we can look at only the "Public Transportation" row of the table and ignore the rest of the table: $\mathrm{P(M|T)} = \frac{8}{21}$.

**Solution 2**

We use the conditional probability formula $\mathrm{P(E|F)} = \frac{\mathrm{P(E \cap F)}}{\mathrm{P(F)}}$.

a. $$P(D|M) = \frac{P(D \cap M)}{P(M)} = \frac{39/100}{47/100} = \frac{39}{47}.$$

b. $$P(F|D) = \frac{P(F \cap D)}{P(D)} = \frac{40/100}{79/100} = \frac{40}{79}.$$

c. $$\mathrm{P(M|T)} = \frac{P(M \cap T)}{P(T)} = \frac{8/100}{21/100} = \frac{8}{21}$$

## Example 7.4.6

Given $\mathrm{P(E)}$ = .5, $\mathrm{P(F)}$ = .7, and $\mathrm{P(E \cap F)}$ = .3. Find the following:

a. $\mathrm{P(E|F)}$
b. $\mathrm{P(F|E)}$

**Solution**

We use the conditional probability formula.

a. $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{3}{7} = \frac{3}{7}$
b. $P(F|E) = \frac{P(E \cap F)}{P(E)} = .3/.5 = 3/5$

## Example 7.4.7

$\mathrm{E}$ and $\mathrm{F}$ are mutually exclusive events such that $\mathrm{P(E)}$ = .4, $\mathrm{P(F)}$ = .9. Find $\mathrm{P(E|F)}$.

**Solution**

$\mathrm{E}$ and $\mathrm{F}$ are mutually exclusive, so $\mathrm{P(E \cap F)}$ = 0.
Therefore $\mathrm{P(E|F)} = \frac{\mathrm{P(E \cap F)}}{\mathrm{P(F)}} = \frac{0}{9} = 0$.

## Example 7.4.8

Given $\mathrm{P(F|E)}$ = .5, and $\mathrm{P(E \cap F)}$ = .3. Find $\mathrm{P(E)}$.

**Solution**

Using the conditional probability formula $\mathrm{P(E|F)} = \frac{\mathrm{P(E \cap F)}}{\mathrm{P(F)}}$, we get

$$P(F|E) = \frac{P(E \cap F)}{P(E)}$$

Substituting and solving:

$$.5 = \frac{.3}{P(E)} \quad \text{or} \quad P(E) = 3/5$$

## Example 7.4.9

In a family of three children, find the conditional probability of having two boys and a girl, given that the family has at least two boys.

**Solution**

Let event E be that the family has two boys and a girl, and let F be the probability that the family has at least two boys. We want $P(E|F)$.

We list the sample space along with the events E and F.

$$S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$$
$$E = \{BBG, BGB, GBB\} \text{ and } F = \{BBB, BBG, BGB, GBB\}$$
$$E \cap F = \{BBG, BGB, GBB\}$$

Therefore, $P(F)$ = 4/8, and $P(E \cap F)$ = 3/8, and

$$P(E|F) = \frac{P(E \cap F)}{P(E)} = \frac{3/8}{4/8} = \frac{3}{4}.$$

## Example 7.4.10

At a community college 65% of the students subscribe to Amazon Prime, 50% subscribe to Netflix, and 20% subscribe to both. If a student is chosen at random, find the following probabilities:

a. the student subscribes to Amazon Prime given that he subscribes to Netflix
b. the student subscribes to Netflix given that he subscribes to Amazon Prime

**Solution**

Let A be the event that the student subscribes to Amazon Prime, and N be the event that the student subscribes to Netflix.

First identify the probabilities and events given in the problem.

P(student subscribes to Amazon Prime) = $P(A)$ = 0.65

P(student subscribes to Netflix) = $P(N)$ = 0.50

P(student subscribes to both Amazon Prime and Netflix) = $P(A \cap N)$ = 0.20

Then use the conditional probability rule:

a. $P(A|N) = \frac{P(A \cap N)}{P(N)} = \frac{.20}{.50} = \frac{2}{5}$
b. $P(N|A) = \frac{P(A \cap N)}{P(A)} = \frac{.20}{.65} = \frac{4}{13}$

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# 7.4.1: Conditional Probability (Exercises)

## SECTION 8.4 PROBLEM SET: CONDITIONAL PROBABILITY

Questions 1 - 4: Do these problems using the conditional probability formula: $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

| | |
|---|---|
| 1. A card is drawn from a deck. Find the conditional probability of $P$(a queen \| a face card). | 2. A card is drawn from a deck. Find the conditional probability of $P$(a queen \| a club). |
| 3. A die is rolled. Find the conditional probability that it shows a three if it is known that an odd number has shown. | 4. If $P(A)$ = .3 , $P(B)$ = .4, $P(A$ and $B)$ = .12, find:<br>a. $P(A\|B)$<br>b. $P(B\|A)$ |

Questions 5 - 8 refer to the following: The table shows the distribution of Democratic and Republican U.S. Senators by gender in the 114th Congress as of January 2015.

| | MALE(M) | FEMALE(F) | TOTAL |
|---|---|---|---|
| DEMOCRATS (D) | 30 | 14 | 44 |
| REPUBLICANS(R) | 48 | 6 | 54 |
| OTHER (T) | 2 | 0 | 2 |
| TOTALS | 80 | 20 | 100 |

Use this table to determine the following probabilities:

| | |
|---|---|
| 5. $P(M\|D)$ | 6. $P(D\|M)$ |
| 7. $P(F\|R)$ | 8. $P(R\|F)$ |

Do the following conditional probability problems.

| | |
|---|---|
| 9. At a college, 20% of the students take Finite Math, 30% take History, and 5% take both Finite Math and History. If a student is chosen at random, find the following conditional probabilities.<br>a. He is taking Finite Math given that he is taking History.<br>b. He is taking History assuming that he is taking Finite Math. | 10. At a college, 60% of the students pass Accounting, 70% pass English, and 30% pass both of these courses. If a student is selected at random, find the following conditional probabilities.<br>a. He passes Accounting given that he passed English.<br>b. He passes English assuming that he passed Accounting. |
| 11. If $P(F) = .4$, $P(E\|F) = .3$, find $P(E$ and $F)$. | 12. $P(E) = .3$, $P(F) = .3$; $E$ and $F$ are mutually exclusive. Find $P(E\|F)$. |
| 13. If $P(E) = .6$, $P(E$ and $F)$ = .24, find $P(F\|E)$. | 14. If $P(E$ and $F)$ = .04, $P(E\|F) = .1$, find $P(F)$. |

At a college, 72% of courses have final exams and 46% of courses require research papers. 32% of courses have both a research paper and a final exam. Let $F$ be the event that a course has a final exam and $R$ be the event that a course requires a research paper.

| | |
|---|---|
| 15. Find the probability that a course has a final exam given that it has a research paper. | 16. Find the probability that a course has a research paper if it has a final exam. |

***SECTION 8.4 PROBLEM SET: CONDITIONAL PROBABILITY***

Consider a family of three children. Find the following probabilities.

| | |
|---|---|
| 17. $P$(two boys \| first born is a boy) | 18. $P$(all girls \| at least one girl is born) |

| | |
|---|---|
| 19. $P$(children of both sexes \| first born is a boy) | 20. $P$(all boys \| there are children of both sexes) |

Questions 21 - 26 refer to the following:
The table shows highest attained educational status for a sample of US residents age 25 or over:

| | (D) Did not Complete High School | (H) High School Graduate | (C) Some College | (A) Associate Degree | (B) Bachelor Degree | (G) Graduate Degree | TOTAL |
|---|---|---|---|---|---|---|---|
| 25-44 (R) | 95 | 228 | 143 | 81 | 188 | 61 | 796 |
| 45-64 (S) | 83 | 256 | 136 | 80 | 150 | 67 | 772 |
| 65+ (T) | 96 | 191 | 84 | 36 | 80 | 41 | 528 |
| Total | 274 | 675 | 363 | 197 | 418 | 169 | 2096 |

Use this table to determine the following probabilities:

| | | |
|---|---|---|
| 21. $P(C\|T)$ | 22. $P(S\|A)$ | 23. $P(CandT)$ |
| 24. $P(R\|B)$ | 25. $P(B\|R)$ | 26. $P(G\|S)$ |

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# 7.5: Independent Events

**Learning Objectives**

In this section, you will:

1. Define independent events
2. Identify whether two events are independent or dependent

In the last section, we considered conditional probabilities. In some examples, the probability of an event changed when additional information was provided. This is not always the case. The additional information may or may not alter the probability of the event.

In Example 7.5.1 we revisit the discussion at the beginning of the previous section and then contrast that with Example 7.5.2.

**Example 7.5.1**

A card is drawn from a deck. Find the following probabilities.

a. The card is a king.
b. The card is a king given that the card is a face card.

**Solution**

a. Clearly, $P$(The card is a king) = 4/52 = 1/13.

b. To find $P$(The card is a king | The card is a face card), we reason as follows:

There are 12 face cards in a deck of cards. There are 4 kings in a deck of cards.

$$P(\text{The card is a king} \mid \text{The card is a face card}) = 4/12 = 1/3.$$

The reader should observe that in the above example,

$$P(\text{The card is a king} \mid \text{The card is a face card}) \neq P(\text{The card is a king})$$

In other words, the additional information, knowing that the card selected is a face card changed the probability of obtaining a king.

**Example 7.5.2**

A card is drawn from a deck. Find the following probabilities.

a. The card is a king.
b. The card is a king given that a red card has shown.

**Solution**

a. Clearly, $P$(The card is a king) = 4/52 = 1/13.

b. To find $P$(The card is a king | A red card has shown), we reason as follows:

Since a red card has shown, there are only twenty six possibilities. Of the 26 red cards, there are two kings. Therefore,

$$P(\text{The card is a king} \mid \text{A red card has shown}) = 2/26 = 1/13.$$

The reader should observe that in the above example,

$$P(\text{The card is a king} \mid \text{A red card has shown}) = P(\text{The card is a king})$$

In other words, the additional information, a red card has shown, did not affect the probability of obtaining a king.

Whenever the probability of an event $E$ is not affected by the occurrence of another event $F$, and vice versa, we say that the two events $E$ and $F$ are **independent.** This leads to the following definition.

**Definition: Independent**

Two Events $E$ and $F$ are **independent** if and only if at least one of the following two conditions is true.

1. $\mathbf{P(E|F) = P(E)}$ or
2. $\mathbf{P(F|E) = P(F)}$

If the events are not independent, then they are dependent.

If one of these conditions is true, then both are true.

We can use the definition of independence to determine if two events are independent.

We can use that definition to develop another way to test whether two events are independent.

Recall the conditional probability formula:

$$\mathrm{P(E|F)} = \frac{\mathrm{P(E \cap F)}}{\mathrm{P(F)}}$$

Multiplying both sides by $\mathrm{P(F)}$, we get

$$\mathrm{P(E \cap F) = P(E|F)P(F)}$$

Now if the two events are independent, then by definition

$$\mathrm{P(E|F) = P(E)}$$

Substituting, $P(E \cap F) = P(E)P(F)$

We state it formally as follows.

**Test For Independence**

Two events $E$ and $F$ are independent if and only if

$$\mathbf{P(E \cap F) = P(E)P(F)}$$

In the Examples $7.5.3$ and $7.5.4$, we'll examine how to check for independence using both methods:

- Examine the probability of intersection of events to check whether $P(E \cap F) = P(E)P(F)$
- Examine conditional probabilities to check whether $P(E|F) = P(E)$ or $P(F|E) = P(F)$

We need to use only **one** of these methods. Both methods, if used properly, will always give results that are consistent with each other.

Use the method that seems easier based on the information given in the problem.

**Example 7.5.3**

The table below shows the distribution of color-blind people by gender.

| | Male(M) | Female(F) | Total |
|---|---|---|---|
| Color-Blind(C) | 6 | 1 | 7 |
| Not Color-Blind(N) | 46 | 47 | 93 |
| Total | 52 | 48 | 100 |

where $M$ represents male, $F$ represents female, $C$ represents color-blind, and $N$ represents not color-blind. Are the events color-blind and male independent?

***Solution 1:*** According to the test for independence, $C$ and $M$ are independent if and only if $\mathrm{P(C \cap M) = P(C)P(M)}$ .

From the table: $P(C)$ = 7/100, $P(M)$ = 52/100 and $P(C \cap M)$ = 6/100

So $P(C)P(M)$ = (7/100)(52/100) = .0364

which is **not** equal to $P(C \cap M)$ = 6/100 = .06

Therefore, the two events are not independent. We may say they are dependent.

***Solution 2:*** $C$ and $M$ are independent if and only if $P(C|M) = P(C)$.

From the total column $P(C)$ = 7/100 = 0.07

From the male column $P(C|M)$ = 6/52= 0.1154

Therefore $P(C|M) \neq P(C)$, indicating that the two events are not independent.

### Example 7.5.4

In a city with two airports, 100 flights were surveyed. 20 of those flights departed late.

- 45 flights in the survey departed from airport A; 9 of those flights departed late.
- 55 flights in the survey departed from airport B; 11 flights departed late.

Are the events "depart from airport A" and "departed late" independent?

***Solution 1***

Let A be the event that a flight departs from airport A, and L the event that a flight departs late. We have

$$P(A \cap L) = 9/100,\ P(A) = 45/100 \text{ and } P(L) = 20/100$$

In order for two events to be independent, we must have $P(A \cap L) = P(A)P(L)$

Since $P(A \cap L)$ = 9/100 = 0.09

and $P(A)P(L)$ = (45/100)(20/100) = 900/10000 = 0.09

the two events "departing from airport A" and "departing late" are independent.

***Solution 2***

The definition of independent events states that two events are independent if $P(E|F) = P(E)$.

In this problem we are given that

$$P(L|A) = 9/45 = 0.2 \text{ and } P(L) = 20/100 = 0.2$$

$P(L|A) = P(L)$, so events "departing from airport A" and "departing late" are independent.

### Example 7.5.5

A coin is tossed three times, and the events $E$, $F$ and $G$ are defined as follows:

$E$: The coin shows a head on the first toss.

$F$: At least two heads appear.

$G$: Heads appear in two successive tosses.

Determine whether the following events are independent.

a. $E$ and $F$
b. $F$ and $G$
c. $E$ and $G$

**Solution**

We list the sample space, the events, their intersections and the probabilities.

$$
\begin{aligned}
&S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\\
&E = \{HHH, HHT, HTH, HTT\}, \quad P(E) = 4/8 \text{ or } 1/2\\
&F = \{HHH, HHT, HTH, THH\}, \quad P(F) = 4/8 \text{ or } 1/2\\
&G = \{HHT, THH\}, \quad P(G) = 2/8 \text{ or } 1/4\\
&E \cap F = \{HHH, HHT, HTH\}, \quad P(E \cap F) = 3/8\\
&F \cap G = \{HHT, THH\}, \quad P(F \cap G) = 2/8 \text{ or } 1/4\\
&E \cap G = \{HHT\} \quad P(E \cap G) = 1/8
\end{aligned}
$$

a. $E$ and $F$ will be independent if and only if $P(E \cap F) = P(E)P(F)$

$$P(E \cap F) = 3/8 \text{ and } P(E)P(F) = 1/2 \cdot 1/2 = 1/4.$$

Since 3/8 ≠ 1/4, we have $P(E \cap F) \neq P(E)P(F)$ .

Events $E$ and $F$ are not independent.

b. $F$ and $G$ will be independent if and only if $P(F \cap G) = P(F)P(G)$ .

$$P(F \cap G) = 1/4 \text{ and } P(F)P(G) = 1/2 \cdot 1/4 = 1/8.$$

Since 3/8 ≠ 1/4, we have $P(F \cap G) \neq P(F)P(G)$ .

Events $F$ and $G$ are not independent.

c. $E$ and $G$ will be independent if $P(E \cap G) = P(E)P(G)$

$$P(E \cap G) = 1/8 \text{ and } P(E)P(G) = 1/2 \cdot 1/4 = 1/8$$

Events $E$ and $G$ are independent events because $P(E \cap G) = P(E)P(G)$

### Example 7.5.6

The probability that Jaime will visit his aunt in Baltimore this year is .30, and the probability that he will go river rafting on the Colorado river is .50. If the two events are independent, what is the probability that Jaime will do both?

***Solution***

Let $A$ be the event that Jaime will visit his aunt this year, and $R$ be the event that he will go river rafting.

We are given $P(A)$ = .30 and $P(R)$ = .50, and we want to find $P(A \cap R)$.

Since we are told that the events $A$ and $R$ are independent,

$$P(A \cap R) = P(A)P(R) = (.30)(.50) = .15$$

### Example 7.5.7

Given $P(B|A) = .4$. If A and B are independent, find $P(B)$.

**Solution**

If $A$ and $B$ are independent, then by definition $P(B|A) = P(B)$

Therefore, $P(B) = .4$

### Example 7.5.8

Given $P(A) = .7$, $P(B|A) = .5$. Find $P(A \cap B)$.

***Solution 1***

By definition $P(B|A) = \frac{P(A \cap B)}{P(A)}$

Substituting, we have

$$.5 = \frac{\mathrm{P(A \cap B)}}{.7}$$

Therefore, $P(A \cap B) = .35$

***Solution 2***

Again, start with $P(B|A) = \frac{P(A \cap B)}{P(A)}$

Multiplying both sides by $P(A)$ gives

$$P(A \cap B) = P(B|A)P(A) = (.5)(.7) = .35$$

Both solutions to Example $7.5.8$ are actually the same, except that in Solution 2 we delayed substituting the values into the equation until after we solved the equation for $P(A \cap B)$. That gives the following result:

**Multiplication Rule for events that are NOT independent**

If events $E$ and $F$ are not independent

$$\mathbf{P(E \cap F) = P(E|F)P(F)} \quad \text{and} \quad \mathbf{P(E \cap F) = P(F|E)P(E)}$$

**Example 7.5.9**

Given $P(A) = .5$, $P(A \cup B) = .7$ , if $A$ and $B$ are independent, find $P(B)$.

**Solution**

The addition rule states that

$$\mathrm{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Since $A$ and $B$ are independent, $P(A \cap B) = P(A)P(B)$

We substitute for $P(A \cap B)$ in the addition formula and get

$$\mathrm{P(A \cup B) = P(A) + P(B) - P(A)P(B)}$$

By letting $P(B) = x$, and substituting values, we get

$$\begin{aligned} .7 &= .5 + x - .5x \\ .7 &= .5 + .5x \\ .2 &= .5x \\ .4 &= x \end{aligned}$$

Therefore, $P(B) = .4$

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# 7.5.1: Independent Events (Exercises)

## SECTION 8.5 PROBLEM SET: INDEPENDENT EVENTS

The distribution of the number of fiction and non-fiction books checked out at a city's main library and at a smaller branch on a given day is as follows.

| | MAIN (M) | BRANCH (B) | TOTAL |
|---|---|---|---|
| FICTION (F) | 300 | 100 | 400 |
| NON-FICTION (N) | 150 | 50 | 200 |
| TOTALS | 450 | 150 | 600 |

Use this table to determine the following probabilities:

| | |
|---|---|
| 1. $P(F)$ | 2. $P(M\|F)$ |
| 3. $P(N\|B)$ | 4. Is the fact that a person checks out a fiction book independent of the main library? Use probabilities to justify your conclusion. |

For a two-child family, let the events $E$, $F$, and $G$ be as follows.

$E$: The family has at least one boy
$F$: The family has children of both sexes
$G$: The family's first born is a boy

| | |
|---|---|
| 5. Find the following.<br>a. $P(E)$<br>b. $P(F)$<br>c. $P(E \cap F)$<br>d. Are $E$ and $F$ independent? Use probabilities to justify your conclusion. | 6. Find the following.<br>a. $P(F)$<br>b. $P(G)$<br>c. $P(F \cap G)$<br>d. Are $F$ and $G$ independent? Use probabilities to justify your conclusion. |

Do the following problems involving independence.

| | |
|---|---|
| 7. If $P(E) = .6$, $P(F) = .2$, and $E$ and $F$ are independent, find $P(E$ and $F)$. | 8. If $P(E) = .6$, $P(F) = .2$, and $E$ and $F$ are independent, find $P(E$ or $F)$. |
| 9. If $P(E) = .9$, $P(F\|E) = .36$, and $E$ and $F$ are independent, find $P(F)$. | 10. If $P(E) = .6$, $P(E$ or $F)$ = .8, and $E$ and $F$ are independent, find $P(F)$. |
| 11. In a survey of 100 people, 40 were casual drinkers, and 60 did not drink. Of the ones who drank, 6 had minor headaches. Of the non-drinkers, 9 had minor headaches. Are the events "drinkers" and "had headaches" independent? | 12. It is known that 80% of the people wear seat belts, and 5% of the people quit smoking last year. If 4% of the people who wear seat belts quit smoking, are the events, wearing a seat belt and quitting smoking, independent? |
| 13. John's probability of passing statistics is 40%, and Linda's probability of passing the same course is 70%. If the two events are independent, find the following probabilities.<br>a. $P($ both of them will pass statistics)<br>b. $P($at least one of them will pass statistics) | 14. Jane is flying home for the Christmas holidays. She has to change planes twice. There is an 80% chance that she will make the first connection, and a 90% chance that she will make the second connection. If the two events are independent, find the probabilities:<br>a. $P($ Jane will make both connections)<br>b. $P($Jane will make at least one connection) |

For a three-child family, let the events $E$, $F$, and $G$ be as follows.

$E$: The family has at least one boy
$F$: The family has children of both sexes
$G$: The family's first born is a boy

| | |
|---|---|
| 15. Find the following.<br>a. $P(E)$<br>b. $P(F)$<br>c. $P(E \cap F)$<br>d. Are $E$ and $F$ independent? | 16. Find the following.<br>a. $P(F)$<br>b. $P(G)$<br>c. $P(F \cap G)$<br>d. Are $F$ and $G$ independent? |

***SECTION 8.5 PROBLEM SET: INDEPENDENT EVENTS***

| | |
|---|---|
| 17. $P(K\|D) = 0.7$, $P(D) = 0.25$ and $P(K) = 0.7$<br>a. Are events $K$ and $D$ independent? Use probabilities to justify your conclusion.<br>b. Find $P(K \cap D)$ | 18. $P(R\|S) = 0.4$, $P(S) = 0.2$ and $P(R) = 0.3$<br>a. Are events $R$ and $S$ independent? Use probabilities to justify your conclusion.<br>b. Find $P(R \cap S)$ |
| 19. At a college:<br>54% of students are female<br>25% of students are majoring in engineering.<br>15% of female students are majoring in engineering.<br>Event $E$ = student is majoring in engineering<br>Event $F$ = student is female<br>a. Are events $E$ and $F$ independent? Use probabilities to justify your conclusion.<br>b. Find $P(E \cap F)$ | 20. At a college:<br>54% of all students are female<br>60% of all students receive financial aid.<br>60% of female students receive financial aid.<br>Event $A$ = student receives financial aid<br>Event $F$ = student is female<br>a. Are events $A$ and $F$ independent? Use probabilities to justify your conclusion.<br>b. Find $P(A \cap F)$ |

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# 7.6: Chapter Review

## SECTION 8.6 PROBLEM SET: CHAPTER REVIEW

1. Two dice are rolled. Find the probability that the sum of the dice is
   a. four
   b. five
2. A jar contains 3 red, 4 white, and 5 blue marbles. If a marble is chosen at random, find the following probabilities:
   a. P(red or blue)
   b. P(not blue)
3. A card is drawn from a standard deck. Find the following probabilities:
   a. P(a jack or a king)
   b. P(a jack or a spade)
4. A basket contains 3 red and 2 yellow apples. Two apples are chosen at random. Find the following probabilities:
   a. P(one red, one yellow)
   b. P(at least one red)
5. A basket contains 4 red, 3 white, and 3 blue marbles. Three marbles are chosen at random. Find the following probabilities:
   a. P(two red, one white)
   b. P(first red, second white, third blue)
   c. P(at least one red)
   d. P(none red)
6. Given a family of four children. Find the following probabilities:
   a. P(All boys)
   b. P(1 boy and 3 girls)
7. Consider a family of three children. Find the following:
   a. P(children of both sexes | first born is a boy)
   b. P(all girls | children of both sexes)
8. Mrs. Rossetti is flying from San Francisco to New York. On her way to the San Francisco Airport she encounters heavy traffic and determines that there is a 20% chance that she will be late to the airport and will miss her flight. Even if she makes her flight, there is a 10% chance that she will miss her connecting flight at Chicago. What is the probability that she will make it to New York as scheduled?
9. At a college, twenty percent of the students take history, thirty percent take math, and ten percent take both. What percent of the students take at least one of these two courses?
10. In a T-maze, a mouse may run to the right (R) or may run to the left (L). A mouse goes up the maze three times, and events E and F are described as follows:

    $$\text{E: Runs to the right on the first trial} \qquad \text{F: Runs to the left two consecutive times}$$

    Determine whether the events E and F are independent.
11. A college has found that 20% of its students take advanced math courses, 40% take advanced English courses and 15% take both advanced math and advanced English courses. If a student is selected at random, what is the probability that
    a. he is taking English given that he is taking math?
    b. he is taking math or English?
12. If there are 35 students in a class, what is the probability that at least two have the same birthday?
13. A student feels that her probability of passing accounting is .62, of passing mathematics is .45, and her passing accounting or mathematics is .85. Find the probability that the student passes both accounting and math.
14. There are nine judges on the U. S. Supreme Court. Suppose that five are conservative and four are liberal. This year the court will act on six major cases. What is the probability that out of six cases the court will favor the conservatives in at least four?
15. Five cards are drawn from a deck. Find the probability of obtaining
    a. four cards of a single suit

b. two cards of one suit, two of another suit, and one from the remaining
c. a pair(e.g. two aces and three other cards)
d. a straight flush(five in a row of a single suit but not a royal flush)

16. The following table shows a distribution of drink preferences by gender.

| | Coke(C) | Pepsi(P) | Seven Up(S) | TOTALS |
|---|---|---|---|---|
| Male(M) | 60 | 50 | 22 | 132 |
| Female(F) | 50 | 40 | 18 | 108 |
| TOTALS | 110 | 90 | 40 | 240 |

The events M, F, C, P and S are defined as Male, Female, Coca Cola, Pepsi, and Seven Up, respectively. Find the following:

a. P(F | S)
b. P( P | F)
c. P(C | M)
d. P(M | P ∪ C)
e. Are the events F and S mutually exclusive?
f. Are the events F and S independent?

17. At a clothing outlet 20% of the clothes are irregular, 10% have at least a button missing and 4% are both irregular and have a button missing. If Martha found a dress that has a button missing, what is the probability that it is irregular?
18. A trade delegation consists of four Americans, three Japanese and two Germans. Three people are chosen at random. Find the following probabilities:

a. P(two Americans and one Japanese)
b. P(at least one American)
c. P(One of each nationality)
d. P(no German)

19. A coin is tossed three times, and the events E and F are as follows.

E: It shows a head on the first toss  F: Never turns up a tail

Are the events E and F independent?

20. If $P(E) = .6$ and $P(F) = .4$ and E and F are mutually exclusive, find $P$(E and F).
21. If $P(E) = .5$ and $P(F) = .3$ and E and F are independent, find $P(E \cup F)$.
22. If $P(F) = .9$ and $P(E|F) = .36$ and E and F are independent, find $P(E)$.
23. If $P(E) = .4$ and $P$(E or F) =.9 and E and F are independent, find $P(F)$.
24. If $P(E) = .4$ and $P(F|E) = .5$, find $P$(E and F).
25. If $P(E) = .6$ and $P$(E and F) = .3, find $P(F|E)$.
26. If $P(E) = .3$ and $P(F) = .4$ and E and F are independent, find $P(E|F)$.

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# CHAPTER OVERVIEW

## 8: More Probability

**Learning Objectives**

In this chapter, you will learn to:

1. Find the probability of a binomial experiment.
2. Find probabilities using Bayes' Formula.
3. Find the expected value or payoff in a game of chance.
4. Find probabilities using tree diagrams.


8.1: Binomial Probability

8.1.1: Binomial Probability (Exercises)

8.2: Bayes' Formula

8.2.1: Bayes' Formula (Exercises)

8.3: Expected Value

8.3.1: Expected Value (Exercises)

8.4: Probability Using Tree Diagrams

8.4.1: Probability Using Tree Diagrams (Exercises)

8.5: The Normal Distribution

8.6: Applications of the Normal Distribution

8.7: Chapter Review



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# 8.1: Binomial Probability

**Learning Objectives**

In this section, you will learn to:

1. Recognize when to use the binomial probability distribution
2. Derive the formula for the binomial probability distribution
3. Calculate probabilities for a binomial probability experiment

In this section, we consider problems that involve a sequence of trials, where each trial has only two outcomes, a *success* or a *failure*. These trials are independent, that is, the outcome of one does not affect the outcome of any other trial. The probability of success, $p$, and the probability of failure, $(1-p)$, remains the same throughout the experiment. These problems are called **binomial probability** problems. Since these problems were researched by Swiss mathematician Jacques Bernoulli around 1700, they are also called **Bernoulli trials.**

We give the following definition:

**Binomial Experiment**

A binomial experiment satisfies the following four conditions:

1. There are only two outcomes, a success or a failure, for each trial.
2. The same experiment is repeated several times.
3. The trials are independent; that is, the outcome of a particular trial does not affect the outcome of any other trial.
4. The probability of success remains the same for every trial.

This probability model that will give us the tools to solve many real-life problems , such as:

1. If a coin is flipped 10 times, what is the probability that it will fall heads 3 times?
2. If a basketball player makes 3 out of every 4 free throws, what is the probability that he will make 7 out of 10 free throws in a game?
3. If a medicine cures 80% of the people who take it, what is the probability that among the ten people who take the medicine, 6 will be cured?
4. If a microchip manufacturer claims that only 4% of his chips are defective, what is the probability that among the 60 chips chosen, exactly three are defective?
5. If a telemarketing executive has determined that 15% of the people contacted will purchase the product, what is the probability that among the 12 people who are contacted, 2 will buy the product?

We now consider the following example to develop a formula for finding the probability of $k$ successes in $n$ Bernoulli trials.

**Example 8.1.1**

A baseball player has a batting average of .300. If he bats four times in a game, find the probability that he will have

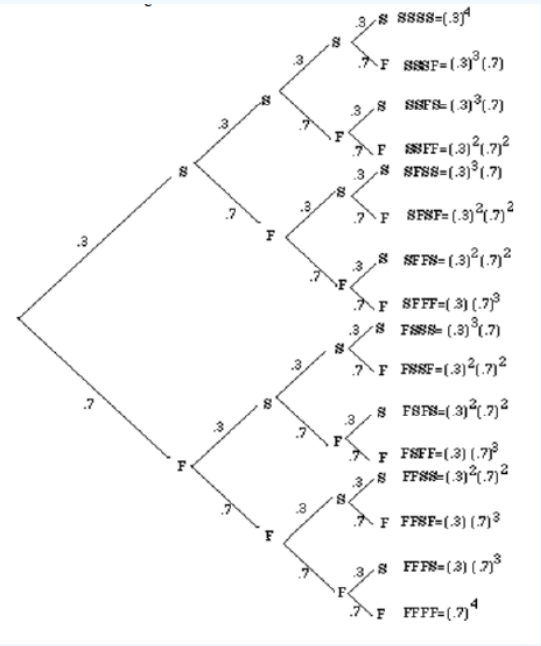
a. 4 hits
b. 3 hits
c. 2 hits
d. 1 hit
e. no hits.

**Solution**

Let S denote that the player gets a hit, and F denote that he does not get a hit.

This is a binomial experiment because it meets all four conditions. First, there are only two outcomes, S or F. Clearly the experiment is repeated four times. Lastly, if we assume that the player's skillfulness to get a hit does not change each time he comes to bat, the trials are independent with a probability of .3 of getting a hit during each trial.

We draw a tree diagram to show all situations.




Let us first find the probability of getting, for example, two hits. We will have to consider the six possibilities, SSFF, SFSF, SFFS, FSSF, FSFS, FFSS, as shown in the above tree diagram. We list the probabilities of each below.

$$\begin{aligned}
\mathrm{P(SSFF)} &= (.3)(.3)(.7)(.7) = (.3)^2(.7)^2 \\
\mathrm{P(SFSF)} &= (.3)(.7)(.3)(.7) = (.3)^2(.7)^2 \\
\mathrm{P(SFFS)} &= (.3)(.7)(.7)(.3) = (.3)^2(.7)^2 \\
\mathrm{P(FSSF)} &= (.7)(.3)(.3)(.7) = (.3)^2(.7)^2 \\
\mathrm{P(FSFS)} &= (.7)(.3)(.7)(.3) = (.3)^2(.7)^2 \\
\mathrm{P(FFSS)} &= (.7)(.7)(.3)(.3) = (.3)^2(.7)^2
\end{aligned}$$

Since the probability of each of these six outcomes is $(.3)^2(.7)^2$, the probability of obtaining two successes is $6(.3)^2(.7)^2$.

The probability of getting one hit can be obtained in the same way. Since each permutation has one S and three F's, there are four such outcomes: SFFF, FSFF, FFSF, and FFFS.

And since the probability of each of the four outcomes is $(.3)(.7)^3$, the probability of getting one hit is $4(.3)(.7)^3$.

The table below lists the probabilities for all cases, and shows a comparison with the binomial expansion of fourth degree. Again, $p$ denotes the probability of success, and $q = (1-p)$ the probability of failure.

| Outcome | Four Hits | Three Hits | Two Hits | One Hit | No Hits |
|---|---|---|---|---|---|
| Probability | $(.3)^4$ | $4(.3)^3(.7)$ | $6(.3)^2(.7)^2$ | $4(.3)(.7)^3$ | $(.7)^4$ |

$$\begin{aligned}
(.3+.7)^4 &= (.3)^4 + 4(.3)^3(.7) + 6(.3)^2(.7)^2 + 4(.3)(.7)^3 + (.7)^4 \\
(p+q)^4 &= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4
\end{aligned}$$

This gives us the following theorem:

**Binomial Probability Theorem**

The probability of obtaining $k$ successes in $n$ independent Bernoulli trials is given by

$$\mathbf{P(n,k;p) = nCk\ p^k q^{n-k}}$$

where $p$ denotes the probability of success and $q = (1 - p)$ the probability of failure.

We use the binomial probability formula to solve the following examples.

### Example 8.1.2

If a coin is flipped 10 times, what is the probability that it will fall heads 3 times?

**Solution**

Let S denote the probability of obtaining a head, and F the probability of getting a tail.

Clearly, $n = 10$, $k = 3$, $p = 1/2$, and $q = 1/2$.

Therefore, $\mathrm{b}(10, 3; 1/2) = 10\mathrm{C}3\ (1/2)^3(1/2)^7 = .1172$

### Example 8.1.3

If a basketball player makes 3 out of every 4 free throws, what is the probability that he will make 6 out of 10 free throws in a game?

**Solution**

The probability of making a free throw is 3/4.
Therefore, $p = 3/4$, $q = 1/4$, $n = 10$, and $k = 6$.

Therefore, $b(10, 6; 3/4) = 10\mathrm{C}6\ (3/4)^6(1/4)^4 = .1460$

### Example 8.1.4

If a medicine cures 80% of the people who take it, what is the probability that of the eight people who take the medicine, 5 will be cured?

**Solution**

Here $p = .80$, $q = .20$, $n = 8$, and $k = 5$.

$$b(8, 5; .80) = 8\mathrm{C}5\ (.80)^5(.20)^3 = .1468$$

### Example 8.1.5

If a microchip manufacturer claims that only 4% of his chips are defective, what is the probability that among the 60 chips chosen, exactly three are defective?

**Solution**

If S denotes the probability that the chip is defective, and F the probability that the chip is not defective, then $p = .04$, $q = .96$, $n = 60$, and $k = 3$.

$$b(60, 3; .04) = 60\mathrm{C}3\ (.04)^3(.96)^{57} = .2138$$

### Example 8.1.6

A telemarketing executive has determined that 15% of people contacted will purchase the product. 12 people are contacted about this product.

a. Find the probability that among 12 people contacted, 2 will buy the product.
b. Find the probability that among 12 people contacted, at most 2 will buy the product?

**Solution**

a. If S denoted the probability that a person will buy the product, and F the probability that the person will not buy the product, then $p = .15$, $q = .85$, $n = 12$, and $k = 2$.

$$b(12, 2; .15) = 12C2\ (.15)^2(.85)^{10} = .2924.$$

The probability that 2 people buy the product is 0.2924.

b. Again $p = .15$, $q = .85$, $n = 12$. But to find the probability that **at most 2** buy the product, we need to find the probabilities for $k = 0$, $k = 1$, $k = 2$ and add them together.

$$b(12, 0; .15) = 12C0\ (.15)^0(.85)^{12} = .1422$$
$$b(12, 1; .15) = 12C1\ (.15)^1(.85)^{11} = .3012$$

Adding all three probabilities gives: .1422 +0.3012 +.2924 = .7358

The probability that at most 2 people buy the product is 0.7358.

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# 8.1.1: Binomial Probability (Exercises)

## SECTION 9.1 PROBLEM SET: BINOMIAL PROBABILITY

Do the following problems using the binomial probability formula.

| | |
|---|---|
| 1. A coin is tossed ten times. Find the probability of getting six heads and four tails. | 2. A family has three children. Find the probability of having one boy and two girls. |
| 3. What is the probability of getting three aces(ones) if a die is rolled five times? | 4. A baseball player has a .250 batting average. What is the probability that he will have three hits in five times at bat? |
| 5. A basketball player has an 80% chance of sinking a basket on a free throw. What is the probability that he will sink at least three baskets in five free throws? | 6. With a new flu vaccination, 85% of the people in the high risk group can go through the entire winter without contracting the flu. In a group of six people who were vaccinated with this drug, what is the probabi |
| 7. A transistor manufacturer has known that 5% of the transistors produced are defective. What is the probability that a batch of twenty five transistors will have two defective? | 8. It has been determined that only 80% of the people wear seat belts. If a police officer stops a car with four people, what is the probability that at least one person will not be wearing a seat belt? |
| 9. What is the probability that a family of five children will have at least three boys? | 10. What is the probability that a toss of four coins will yield at most two heads? |
| 11. A telemarketing executive has determined that for a particular product, 20% of the people contacted will purchase the product. If 10 people are contacted, what is the probability that at most 2 will buy the product? | 12. To the problem: "Five cards are dealt from a deck of cards, find the probability that three of them are kings," the following incorrect answer was offered by a student. $$5C3\,(1/13)^3(12/13)^2$$ What change would you make in the wording of the problem for the given answer to be correct? |
| 13. 63% of all registered voters in a large city voted in the last election. 20 registered voters from this city are randomly selected. Find the probability that<br>a. exactly half of them voted in the last election<br>b. all of them voted | 14. 30% of customers at BigMart pay cash for their purchases. Suppose that 15 customers are randomly selected. Find the probability that<br>a. 5 or 6 of them pay cash<br>b. at most 1 pays cash |
| 15. 12% of all cars on Brighton Expressway exceed the speed limit. If 10 vehicles on this road are randomly selected and their speed is recorded by radar, find the probability that<br>a. none of them are exceeding the speed limit<br>b. 1 or 2 are exceeding the speed limit | 16. Suppose that 73% of all people taking a professional certification exam pass the exam. If 12 people who take this exam are randomly selected, find the probability that<br>a. exactly half of them pass the exam<br>b. all of them pass the exam<br>c. 8 or 9 of them pass the exam |

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# 8.2: Bayes' Formula

**Learning Objectives**

In this section, you will learn to:

1. Find probabilities using Bayes' formula
2. Use a probability tree to find and represent values needed when using Bayes' formula.

In this section, we will develop and use Bayes' Formula to solve an important type of probability problem. Bayes' formula is a method of calculating the conditional probability $P(F|E)$ from $P(E|F)$. The ideas involved here are not new, and most of these problems can be solved using a tree diagram. However, Bayes' formula does provide us with a tool with which we can solve these problems without a tree diagram.

We begin with an example.

**Example 8.2.1**

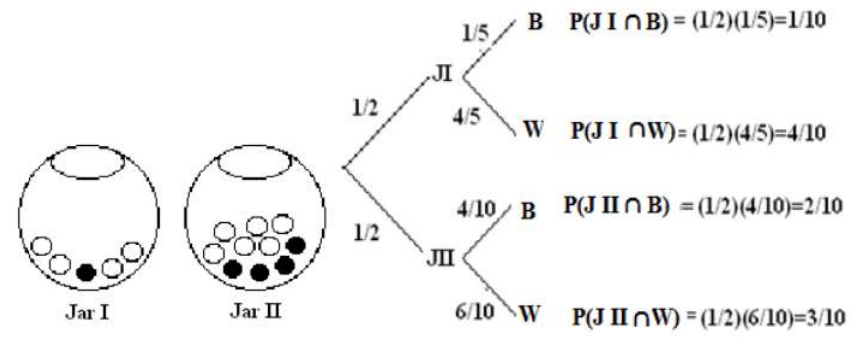
Suppose you are given two jars. Jar I contains one black and 4 white marbles, and Jar II contains 4 black and 6 white marbles. If a jar is selected at random and a marble is chosen,

a. What is the probability that the marble chosen is a black marble?
b. If the chosen marble is black, what is the probability that it came from Jar I?
c. If the chosen marble is black, what is the probability that it came from Jar II?

**Solution**

Let $JI$ be the event that Jar I is chosen, $JII$ be the event that Jar II is chosen, $B$ be the event that a black marble is chosen and $W$ the event that a white marble is chosen.

We illustrate using a tree diagram.




a. The probability that a black marble is chosen is $P(B)$ = 1/10 + 2/10 = 3/10.
b. To find $P(JI|B)$, we use the definition of conditional probability, and we get

$$P(JI|B) = \frac{P(JI \cap B)}{P(B)} = \frac{1/10}{3/10} = \frac{1}{3}$$

c. Similarly, $\mathrm{P(JII|B)} = \frac{\mathrm{P(JII \cap B)}}{\mathrm{P(B)}} = \frac{2/10}{3/10} = \frac{2}{3}$

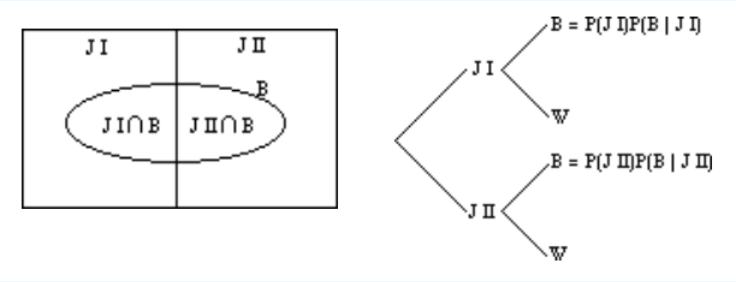
In parts b and c, the reader should note that the denominator is the sum of all probabilities of all branches of the tree that produce a black marble, while the numerator is the branch that is associated with the particular jar in question.

We will soon discover that this is a statement of Bayes' formula .

Let us first visualize the problem.

We are given a sample space S and two mutually exclusive events $JI$ and $JII$. That is, the two events, $JI$ and $JII$, divide the sample space into two parts such that $\mathrm{JI} \cup \mathrm{JII} = \mathrm{S}$. Furthermore, we are given an event $B$ that has elements in both $JI$ and

$JII$, as shown in the Venn diagram below.




From the Venn diagram, we can see that $\mathrm{B} = (\mathrm{B} \cap \mathrm{JI}) \cup (\mathrm{B} \cap \mathrm{JII})$ Therefore:

$$P(B) = P(B \cap JI) + P(B \cap JII) \tag{8.2.1}$$

But the product rule in chapter 7 gives us

$$P(B \cap JI) = P(JI) \cdot P(B|JI) \quad \text{and} \quad P(B \cap JII) = P(JII) \cdot P(B|JII)$$

Substituting in 8.2.1, we get

$$P(B) = P(JI) \cdot P(B|JI) + P(JII) \cdot P(B|JII)$$

The conditional probability formula gives us

$$P(JI|B) = \frac{P(JI \cap B)}{P(B)}$$

Therefore, $P(JI|B) = \frac{P(JI) \cdot P(B|JI)}{P(B)}$

or

$$P(JI|B) = \frac{P(JI) \cdot P(B|JI)}{P(JI) \cdot P(B|JI) + P(JII) \cdot P(B|JII)}$$

The last statement is Bayes' Formula for the case where the sample space is divided into two partitions.

The following is the generalization of Bayes' formula for n partitions.

### Bayes' Formula for $n$ partitions

Let S be a sample space that is divided into $n$ partitions, $A_1, A_2, \ldots A_n$. If $E$ is any event in S, then

$$\mathbf{P}(\mathbf{A_i}|\mathbf{E}) = \frac{\mathbf{P}(\mathbf{A_i})\,\mathbf{P}(\mathbf{E}|\mathbf{A_i})}{\mathbf{P}(\mathbf{A_1})\,\mathbf{P}(\mathbf{E}|\mathbf{A_1}) + \mathbf{P}(\mathbf{A_2})\,\mathbf{P}(\mathbf{E}|\mathbf{A_2}) + \cdots + \mathbf{P}(\mathbf{A_n})\,\mathbf{P}(\mathbf{E}|\mathbf{A_n})}$$

We begin with the following example.

### Example 8.2.2

A department store buys 50% of its appliances from Manufacturer A, 30% from Manufacturer B, and 20% from Manufacturer C. It is estimated that 6% of Manufacturer A's appliances, 5% of Manufacturer B's appliances, and 4% of Manufacturer C's appliances need repair before the warranty expires. An appliance is chosen at random. If the appliance chosen needed repair before the warranty expired, what is the probability that the appliance was manufactured by Manufacturer A? Manufacturer B? Manufacturer C?

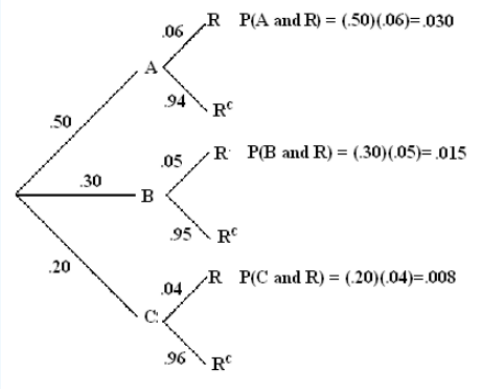
#### Solution

Let A, B and C be the events that the appliance is manufactured by Manufacturer A, Manufacturer B, and Manufacturer C, respectively. Further, suppose that the event R denotes that the appliance needs repair before the warranty expires.

We need to find P(A | R), P(B | R) and P(C | R).

We will do this problem both by using a tree diagram and by using Bayes' formula.

We draw a tree diagram.




The probability P(A | R), for example, is a fraction whose denominator is the sum of all probabilities of all branches of the tree that result in an appliance that needs repair before the warranty expires, and the numerator is the branch that is associated with Manufacturer A. P(B | R) and P(C | R) are found in the same way.

$$P(A|R) = \frac{.030}{(.030)+(.015)+(.008)} = \frac{.030}{.053} = .566$$
$$P(B|R) = \frac{.015}{.053} = .283 \text{ and } P(C|R) = \frac{.008}{.053} = .151$$

Alternatively, using Bayes' formula,

$$\begin{aligned} \mathrm{P(A|R)} &= \frac{\mathrm{P(A)P(R|A)}}{\mathrm{P(A)P(R|A)+P(B)P(R|B)+P(C)P(R|C)}} \\ &= \frac{.030}{(.030)+(.015)+(.008)} = \frac{.030}{.053} = .566 \end{aligned}$$

P(B | R) and P(C | R) can be determined in the same manner.

## Example 8.2.3

There are five Jacy's department stores in San Jose. The distribution of number of employees by gender is given in the table below.

| Store Number | Number of Employees | Percent of Women Employees |
|---|---|---|
| 1 | 300 | .40 |
| 2 | 150 | .65 |
| 3 | 200 | .60 |
| 4 | 250 | .50 |
| 5 | 100 | .70 |
| | Total = 1000 | |

If an employee chosen at random is a woman, what is the probability that the employee works at store III?

### Solution

Let $k$ = 1, 2, . . . , 5 be the event that the employee worked at store $k$, and W be the event that the employee is a woman. Since there are a total of 1000 employees at the five stores,

$$P(1) = .30 \quad P(2) = .15 \quad P(3) = .20 \quad P(4) = .25 \quad P(5) = .10$$

Using Bayes' formula,

$$\begin{aligned} \mathrm{P}(3|\mathrm{W}) &= \frac{\mathrm{P}(3)\mathrm{P}(\mathrm{W}|3)}{\mathrm{P}(1)\mathrm{P}(\mathrm{W}|1)+\mathrm{P}(2)\mathrm{P}(\mathrm{W}|2)+\mathrm{P}(3)\mathrm{P}(\mathrm{W}|3)+\mathrm{P}(4)\mathrm{P}(\mathrm{W}|4)+\mathrm{P}(5)\mathrm{P}(\mathrm{W}|5)} \\ &= \frac{(.20)(.60)}{(.30)(.40)+(.15)(.65)+(.20)(.60)+(.25)(.50)+(.10)(.70)} \\ &= .2254 \end{aligned}$$

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# 8.2.1: Bayes' Formula (Exercises)

## SECTION 9.2 PROBLEM SET: BAYES' FORMULA

1. Jar I contains five red and three white marbles, and Jar II contains four red and two white marbles. A jar is picked at random and a marble is drawn. Draw a tree diagram below, and find the following probabilities.
   a. P(marble is red)
   b. P(It came from Jar II | marble is white)
   c. P(Red | Jar I)

2. In Mr. Symons' class, if a student does homework most days, the chance of passing the course is 90%. On the other hand, if a student does not do homework most days, the chance of passing the course is only 20%.
   H = event that the student did homework
   C = event that the student passed the course
   Mr. Symons claims that 80% of his students do homework on a regular basis. If a student is chosen at random from Mr. Symons' class, find the following probabilities.
   a. P(C)
   b. P(H|C)
   c. P(C|H)

3. A city has 60% Democrats, and 40% Republicans. In the last mayoral election, 60% of the Democrats voted for their Democratic candidate while 95% of the Republicans voted for their candidate. Which party's mayor runs city hall?

4. In a certain population of 48% men and 52% women, 56% of the men and 8% of the women are color-blind.
   a. What percent of the people are color-blind?
   b. If a person is found to be color-blind, what is the probability that the person is a male?

5. A test for a certain disease gives a positive result 95% of the time if the person actually carries the disease. However, the test also gives a positive result 3% of the time when the individual is not carrying the disease. It is known that 10% of the population carries the disease. If a person tests positive, what is the probability that he or she has the disease?

6. A person has two coins: a fair coin and a two-headed coin. A coin is selected at random, and tossed. If the coin shows a head, what is the probability that the coin is fair?

7. A computer company buys its chips from three different manufacturers. Manufacturer I provides 60% of the chips and is known to produce 5% defective; Manufacturer II supplies 30% of the chips and makes 4% defective; while the rest are supplied by Manufacturer III with 3% defective chips. If a chip is chosen at random, find the following probabilities:
   a. P(the chip is defective)
   b. P(chip is from Manufacturer II | defective)
   c. P(defective |chip is from manufacturer III)

8. Lincoln Union High School District is made up of three high schools: Monterey, Fremont, and Kennedy, with an enrollment of 500, 300, and 200, respectively. On a given day, the percentage of students absent at Monterey High School is 6%, at Fremont 4%, and at Kennedy 5%. If a student is chosen at random, find the probabilities below: *Hint: Convert the enrollments into percentages.*
   a. P(the student is absent)
   b. P(student is from Kennedy | student is absent)
   c. P(student is absent | student is from Fremont)

9. At a retail store, 20% of customers use the store's online app to assist them when shopping in the store ; 80% of store shoppers don't use the app.
Of those customers that use the online app while in the store, 50% are very satisfied with their purchases, 40% are moderately satisfied, and 10% are dissatisfied.
Of those customers that do not use the online app while in the store, 30% are very satisfied with their purchases, 50% are moderately satisfied and 20% are dissatisfied.
Indicate the events by the following:
A = shopper uses the app in the store
N = shopper does not use the app in the store
V = very satisfied with purchase
M = moderately satisfied
D = dissatisfied
a. Find P(A and D), the probability that a store customer uses the app and is dissatisfied
b. Find P(A|D), the probability that a store customer uses the app if the customer is dissatisfied.

10. A medical clinic uses a pregnancy test to confirm pregnancy in patients who suspect they are pregnant. Historically data has shown that overall, 70% of the women at this clinic who are given the pregnancy test are pregnant, but 30% are not.
The test's manufacturer indicates that if a woman is pregnant, the test will be positive 92% of the time.
But if a woman is not pregnant, the test will be positive only 2% of the time and will be negative 98% of the time.
a. Find the probability that a woman at this clinic is pregnant and tests positive.
b. Find the probability that a woman at this clinic is actually pregnant given that she tests positive.

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# 8.3: Expected Value

**Learning Objectives**

In this section, you will learn to:

1. Find the expected value of a discrete probability distribution
2. Interpret expected value as a long-run average

An expected gain or loss in a game of chance is called **Expected Value**. The concept of expected value is closely related to a *weighted average*. Consider the following situations.

1. Suppose you and your friend play a game that consists of rolling a die. Your friend offers you the following deal: If the die shows any number from 1 to 5, he will pay you the face value of the die in dollars, that is, if the die shows a 4, he will pay you $4. But if the die shows a 6, you will have to pay him $18.

Before you play the game you decide to find the expected value. You analyze as follows.

Since a die will show a number from 1 to 6, with an equal probability of 1/6, your chance of winning $1 is 1/6, winning $2 is 1/6, and so on up to the face value of 5. But if the die shows a 6, you will lose $18. You write the expected value.

$$\mathrm{E} = \$1(1/6) + \$2(1/6) + \$3(1/6) + \$4(1/6) + \$5(1/6) - \$18(1/6) = -\$.50$$

This means that every time you play this game, you can expect to lose 50 cents. In other words, if you play this game 100 times, theoretically you will lose $50. Obviously, it is not to your interest to play.

2. Suppose of the ten quizzes you took in a course, on eight quizzes you scored 80, and on two you scored 90. You wish to find the average of the ten quizzes.
The average is

$$\mathrm{A} = \frac{(80)(8)+(90)(2)}{10} = (80)\frac{8}{10} + (90)\frac{2}{10} = 82$$

It should be observed that it would be incorrect to take the average of 80 and 90 because you scored 80 on eight quizzes, and 90 on only two of them. Therefore, you take a "weighted average" of 80 and 90. That is, the average of 8 parts of 80 and 2 parts of 90, which is 82.

In the first situation, to find the expected value, we multiplied each payoff by the probability of its occurrence, and then added up the amounts calculated for all possible cases. In the second example, if we consider our test score a payoff, we did the same. This leads us to the following definition.

**Expected Value**

If an experiment has the following probability distribution,

| Payoff | $x_1$ | $x_2$ | $x_3$ | $\ldots$ | $x_n$ |
|---|---|---|---|---|---|
| Probability | $p(x_1)$ | $p(x_2)$ | $p(x_3)$ | $\ldots$ | $p(x_n)$ |

then the expected value of the experiment is

$$\text{Expected Value } = x_1 p\left(x_1\right) + x_2 p\left(x_2\right) + x_3 p\left(x_3\right) + \cdots + x_n p\left(x_n\right)$$

**Example 8.3.1**

In a town, 10% of the families have three children, 60% of the families have two children, 20% of the families have one child, and 10% of the families have no children. What is the expected number of children to a family?

**Solution**

We list the information in the following table.

| Number of children | 3 | 2 | 1 | 0 |
|---|---|---|---|---|
| Probability | 0.10 | 0.60 | 0.20 | 0.10 |

Expected Value = $x_1p(x_1) + x_2p(x_2) + x_3p(x_3) + x_4p(x_4)$

$E = 3(.10) + 2(.60) + 1(.20) + 0(.10) = 1.7$

So on average, there are 1.7 children to a family.

### Example 8.3.2

To sell an average house, a real estate broker spends $1200 for advertisement expenses. If the house sells in three months, the broker makes $8,000. Otherwise, the broker loses the listing. If there is a 40% chance that the house will sell in three months, what is the expected payoff for the real estate broker?

**Solution**

The broker makes $8,000 with a probability of .40, but he loses $1200 whether the house sells or not.

$$E = (\$8000)(.40) - (\$1200) = \$2,000.$$

Alternatively, the broker makes $(8000 - 1200) with a probability of .40, but loses $1200 with a probability of .60. Therefore,

$$E = (\$6800)(.40) - (\$1200)(.60) = \$2,000.$$

### Example 8.3.3

In a town, the attendance at a football game depends on the weather. On a sunny day the attendance is 60,000, on a cold day the attendance is 40,000, and on a stormy day the attendance is 30,000. If for the next football season, the weatherman has predicted that 30% of the days will be sunny, 50% of the days will be cold, and 20% days will be stormy, what is the expected attendance for a single game?

**Solution**

Using the expected value formula, we get

$$E = (60,000)(.30) + (40,000)(.50) + (30,000)(.20) = 44,000.$$

### Example 8.3.4

A lottery consists of choosing 6 numbers from a total of 51 numbers. The person who matches all six numbers wins $2 million. If the lottery ticket costs $1, what is the expected payoff?

**Solution**

Since there are $51C6 = 18,009,460$combinations of six numbers from a total of 51 numbers, the chance of choosing the winning number is 1 out of 18,009,460.

So the expected payoff is: $\mathrm{E} = (\$2 \text{ million }) \left(\frac{1}{18009460}\right) - \$1 = -\$0.89$

This means that every time a person spends $1 to buy a ticket, he or she can expect to lose 89 cents.

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# 8.3.1: Expected Value (Exercises)

## SECTION 9.3 PROBLEM SET: EXPECTED VALUE

Do the following problems using the expected value concepts learned in this section,

1. You are about to make an investment which gives you a 30% chance of making $60,000 and 70% chance of losing $ 30,000. Should you invest? Explain.
2. In a town, 40% of the men and 30% of the women are overweight. If the town has 46% men and 54% women, what percent of the people are overweight?
3. A game involves rolling a Korean die (4 faces). If a one, two, or three shows, the player receives the face value of the die in dollars, but if a four shows, the player is obligated to pay $4. What is the expected value of the game?
4. A game involves rolling a single die. One receives the face value of the die in dollars. How much should one be willing to pay to roll the die to make the game fair?
5. In a European country, 20% of the families have three children, 40% have two children, 30% have one child, and 10% have no children. On average, how many children are there to a family?
6. A game involves drawing a single card from a standard deck. One receives 60 cents for an ace, 30 cents for a king, and 5 cents for a red card that is neither an ace nor a king. If the cost of each draw is 10 cents, should one play? Explain.
7. Hillview Church plans to raise money by raffling a television worth $500. A total of 3000 tickets are sold at $1 each. Find the expected value of the winnings for a person who buys a ticket in the raffle.
8. During her four years at college, Niki received A's in 30% of her courses, B's in 60% of her courses, and C's in the remaining 10%. If A = 4, B = 3, and C = 2, find her grade point average.
9. Attendance at a Stanford football game depends upon which team Stanford is playing against. If the game is against U. C. Berkeley,attendance will be 70,000; if it is against another California team, it will be 40,000; and if it is against an out of state team, it will be 30,000. If the probability of playing against U. C. Berkeley is 10%, against a California team 50% and against an out of state team 40%, how many fans are expected to attend a game?
10. A Texas oil drilling company has determined that it costs $25,000 to sink a test well. If oil is hit, the revenue for the company will be $500,000. If natural gas is found, the revenue will be $150,000. If the probability of hitting oil is 3% and of hitting gas is 6%, find the expected value of sinking a test well.
11. A $1 lottery ticket offers a grand prize of $10,000; 10 runner-up prizes each pay $1000; 100 third-place prizes each pay $100; and 1,000 fourth-place prizes each pay $10. Find the expected value of entering this contest if 1 million tickets are sold.
12. Assume that for the next heavyweight fight the odds of current champion winning are 15 to 2. A gambler bets $10 that the current champion will lose. If current champion loses, how much can the gambler hope to receive?
13. In a housing development, 35% of households have no school age children, 20% of households have 1 school age child, 25% of households have 2 school age children, 15% have 3, and 5% have 4 school age children.
    a. Find the average number of children per household
    b. If there are 300 homes in this housing development, what is the total number of children expected to attend school?
14. At a large community college, 30% of students take one course, 15% take two courses, 25% take three courses and 20% take four courses. The rest of the students take five courses.
    a. What percent of students take 5 courses?
    b. Find the average number of courses that students take.

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# 8.4: Probability Using Tree Diagrams

**Learning Objectives**

In this section, you will learn to:

1. Use probability trees to organize information in probability problems
2. Use probability trees to calculate probabilities

As we have already seen, tree diagrams play an important role in solving probability problems. A tree diagram helps us not only visualize, but also list all possible outcomes in a systematic fashion. Furthermore, when we list various outcomes of an experiment and their corresponding probabilities on a tree diagram, we gain a better understanding of when probabilities are multiplied and when they are added.

The meanings of the words *and* and *or* become clear when we learn to multiply probabilities horizontally across branches, and add probabilities vertically down the tree.

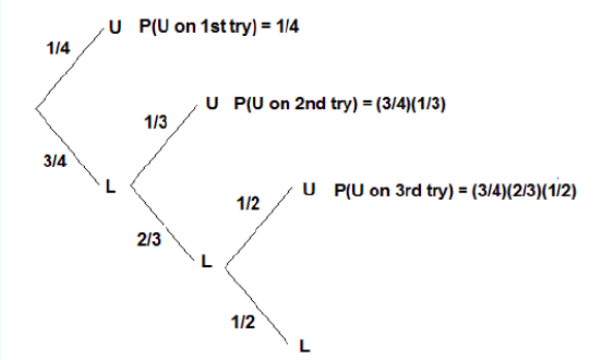
Although tree diagrams are not practical in situations where the possible outcomes become large, they are a significant tool in breaking the problem down in a schematic way. We consider some examples that may seem difficult at first, but with the help of a tree diagram, they can easily be solved.

**Example 8.4.1**

A person has four keys and only one key fits to the lock of a door. What is the probability that the locked door can be unlocked in at most three tries?

**Solution**

Let U be the event that the door has been unlocked and L be the event that the door has not been unlocked. We illustrate with a tree diagram.




The probability of unlocking the door in the first try = 1/4

The probability of unlocking the door in the second try = (3/4)(1/3) = 1/4

The probability of unlocking the door in the third try = (3/4)(2/3)(1/2) = 1/4

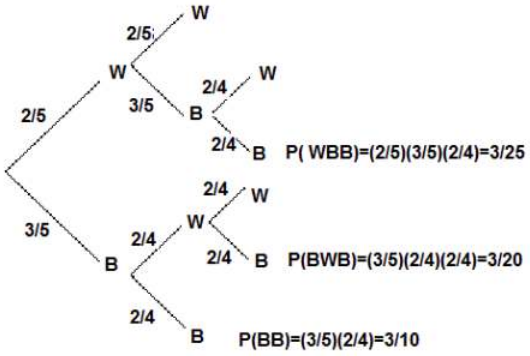
Therefore, the probability of unlocking the door in at most three tries = 1/4 + 1/4 + 1/4 = 3/4.

**Example 8.4.2**

A jar contains 3 black and 2 white marbles. We continue to draw marbles one at a time until two black marbles are drawn. If a white marble is drawn, the outcome is recorded and the marble is put back in the jar before drawing the next marble. What is the probability that we will get exactly two black marbles in at most three tries?

**Solution**

We illustrate using a tree diagram.




The probability that we will get two black marbles in the first two tries is listed adjacent to the lowest branch, and it = 3/10.

The probability of getting first black, second white, and third black = 3/20.

Similarly, the probability of getting first white, second black, and third black = 3/25.

Therefore, the probability of getting exactly two black marbles in at most three tries = 3/10 + 3/20 + 3/25 = 57/100.

### Example 8.4.3

A circuit consists of three resistors: resistor $R_1$, resistor $R_2$, and resistor $R_3$, joined in a series. If one of the resistors fails, the circuit stops working. The probabilities that resistors $R_1$, $R_2$, or $R_3$ will fail are .07, .10, and .08, respectively. Find the probability that at least one of the resistors will fail?

#### Solution

The probability that at least one of the resistors fails = 1 - none of the resistors fails.

It is quite easy to find the probability of the event that none of the resistors fails.
We don't even need to draw a tree because we can visualize the only branch of the tree that assures this outcome.

The probabilities that $R_1$, $R_2$, $R_3$ will not fail are .93, .90, and .92 respectively. Therefore, the probability that none of the resistors fails = (.93)(.90)(.92) = .77.

Thus, the probability that at least one of them will fail = 1 - .77 = .23.

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# 8.4.1: Probability Using Tree Diagrams (Exercises)

## SECTION 9.4 PROBLEM SET: PROBABILITY USING TREE DIAGRAM

Use a tree diagram to solve the following problems.

| | |
|---|---|
| 1. Suppose you have five keys and only one key fits to the lock of a door. What is the probability that you can open the door in at most three tries? | 2. A coin is tossed until a head appears. What is the probability that a head will appear in at most three tries? |
| 3. A basketball player has an 80% chance of making a basket on a free throw. If he makes the basket on the first throw, he has a 90% chance of making it on the second. However, if he misses on the first try, there is only a 70% chance he will make it on the second. If he gets two free throws, what is the probability that he will make at least one of them? | 4. You are to play three games. In the first game, you draw a card, and you win if the card is a heart. In the second game, you toss two coins, and you win if one head and one tail are shown. In the third game, two dice are rolled and you win if the sum of the dice is 7 or 11. What is the probability that you win all three games? What is the probability that you win exactly two games? |
| 5. John's car is in the garage, and he has to take a bus to get to school. He needs to make all three connections on time to get to his class. If the chance of making the first connection on time is 80%, the second 80%, and the third 70%, what is the chance that John will make it to his class on time? | 6. For a real estate exam the probability of a person passing the test on the first try is .70. The probability that a person who fails on the first try will pass on each of the successive attempts is .80. What is the probability that a person passes the test in at most three attempts? |
| 7. On a Christmas tree with lights, if one bulb goes out, the entire string goes out. If there are twelve bulbs on a string, and the probability of any one going out is .04, what is the probability that the string will not go out? | 8. The Long Life Light Bulbs claims that the probability that a light bulb will go out when first used is 15%, but if it does not go out on the first use the probability that it will last the first year is 95%, and if it lasts the first year, there is a 90% probability that it will last two years. Find the probability that a new bulb will last 2 years. |
| 9. A die is rolled until an ace (1) shows. What is the probability that an ace will show on the fourth try? | 10. If there are four people in a room, what is the probability that no two have the same birthday? |
| 11. Dan forgets to set his alarm 60% of the time. If he hears the alarm, he turns it off and goes back to sleep 20% of the time, and even if he does wake up on time, he is late getting ready 30% of the time. What is the probability that Dan will be late to school? | 12. It has been estimated that 20% of the athletes take some type of drugs. A drug test is 90% accurate, that is, the probability of a falsenegative is 10%. Furthermore, for this test the probability of a false-positive is 20%. If an athlete tests positive, what is the probability that he is a drug user? |

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# 8.5: The Normal Distribution

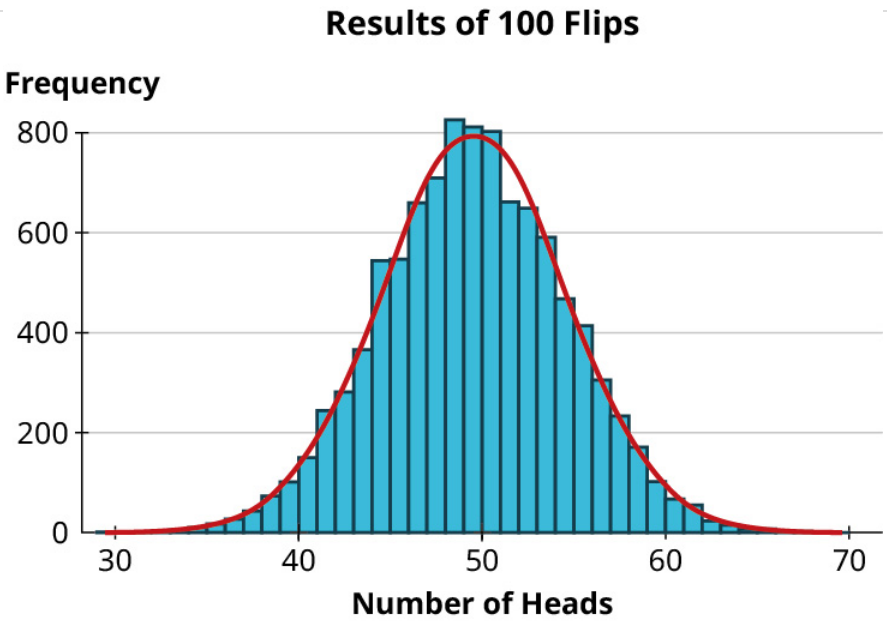



Figure 8.5.1: Symmetric, bell-shaped distributions arise naturally in many different situations, including coin flips.

**Learning Objectives**

1. Describe the characteristics of the normal distribution.
2. Apply the 68-95-99.7 percent groups to normal distribution datasets.
3. Use the normal distribution to calculate a $z$-score.
4. Find and interpret percentiles and quartiles.

Many datasets that result from natural phenomena tend to have histograms that are symmetric and bell-shaped. Imagine finding yourself with a whole lot of time on your hands, and nothing to keep you entertained but a coin, a pencil, and paper. You decide to flip that coin 100 times and record the number of heads. With nothing else to do, you repeat the experiment ten times total. Using a computer to simulate this series of experiments, here's a sample for the number of heads in each trial:

54, 51, 40, 42, 53, 50, 52, 52, 47, 54

It makes sense that we'd get somewhere around 50 heads when we flip the coin 100 times, and it makes sense that the result won't *always* be exactly 50 heads. In our results, we can see numbers that were generally near 50 and not always 50, like we thought.

## Moving Toward Normality

Let's take a look at a histogram for the dataset in our section opener:

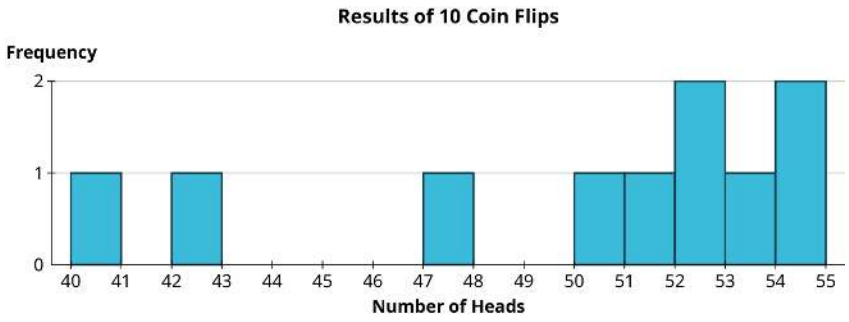



Figure 8.5.1

This is interesting, but the data seem pretty sparse. There were no trials where you saw between 43 and 47 heads, for example. Those results don't seem impossible; we just didn't flip enough times to give them a chance to pop up. So, let's do it again, but this

time we'll perform 100 coin flips 100 times. Rather than review all 100 results, which could be overwhelming, let's instead visualize the resulting histogram.

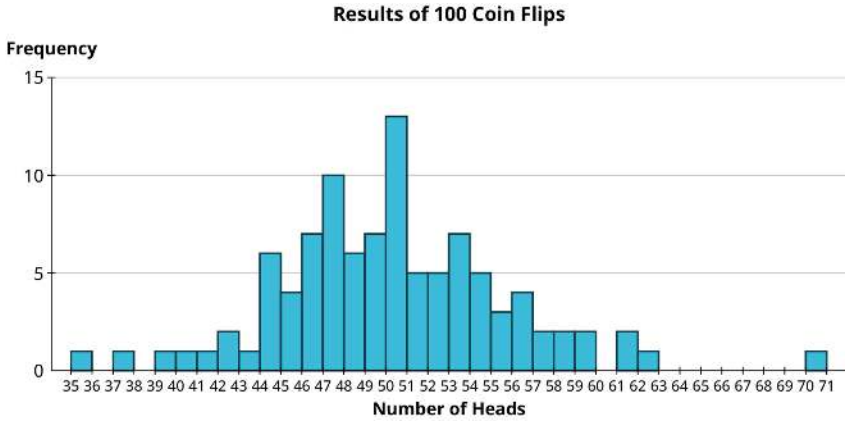



Figure 8.5.2

From the histogram, we see that most of the trials resulted in between, say, 44 and 56 heads. There were some more unusual results: one trial resulted in 70 heads, which seems really unlikely (though still possible!). But we're starting to maybe get a sense of the distribution. More data would help, though. Let's simulate another 900 trials and add them to the histogram!

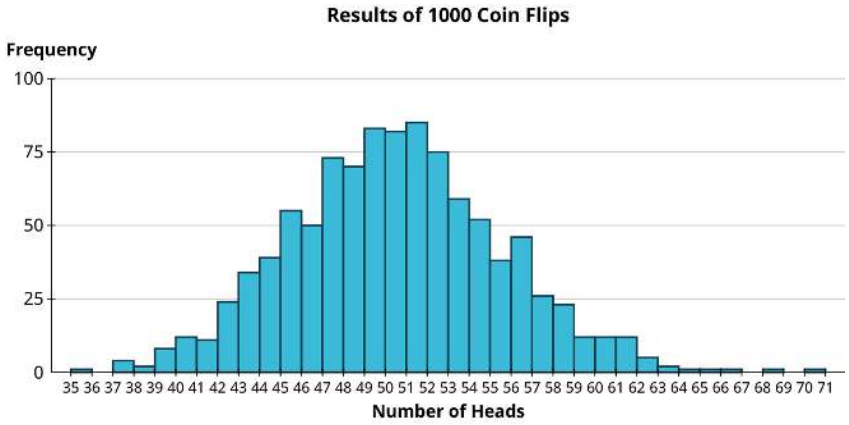



Figure 8.5.3

We can still see that 70 is a really unusual observation, though we came close in another trial (one that had 68 heads). Now, the distribution is coming more into focus: It looks quite symmetric and bell-shaped. Let's just go ahead and take this thought experiment to an extreme conclusion: 10,000 trials.

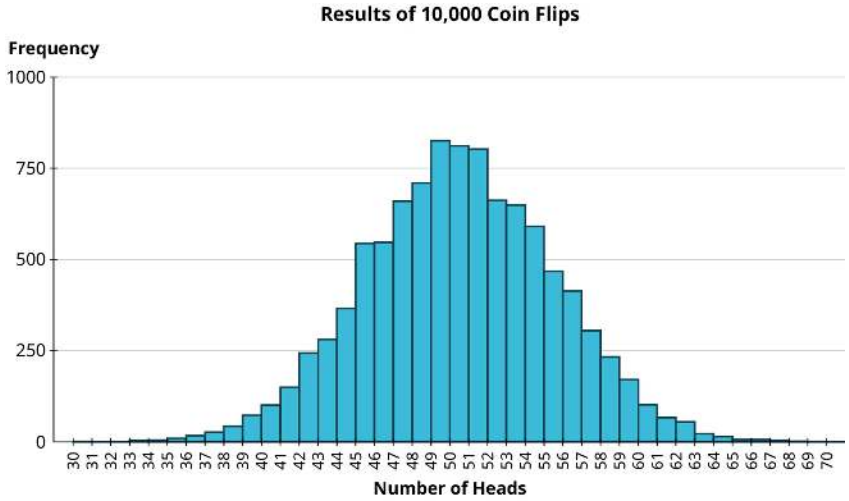



Figure 8.5.4

The distribution is pretty clear now. Distributions that are symmetric and bell-shaped like this pop up in all sorts of natural phenomena, such as the heights of people in a population, the circumferences of eggs of a particular bird species, and the numbers of leaves on mature trees of a particular species. All of these have bell-shaped distributions. Additionally, the results of many types of repeated experiments generally follow this same pattern, as we saw with the coin-flipping example; this fact is the basis for much of the work done by statisticians. It's a fact that's important enough to have its own name: the Central Limit Theorem.

**People in Mathematics: John Kerrich**

Having enough time on your hands to actually perform this coin-flipping experiment may sound far-fetched, but the English mathematician John Kerrich found himself in just such a situation. While he was studying abroad in Denmark in 1940, that country was invaded by the Germans. Kerrich was captured and placed in an internment camp, where he remained for the duration of the war. Kerrich knew that he had all kinds of time on his hands. He also studied statistics, so he knew what should happen *theoretically* if he flipped a coin many, many times. He also knew of nobody who had ever *tested* that theory with an actual, large-scale experiment. So, he did it: While he was incarcerated, Kerrich flipped a regular coin 10,000 times and recorded the results. Sure enough, the theory held up!

## The Normal Distribution

In the coin flipping example above, the distribution of the number of heads for 10,000 trials was close to perfectly symmetric and bell-shaped:

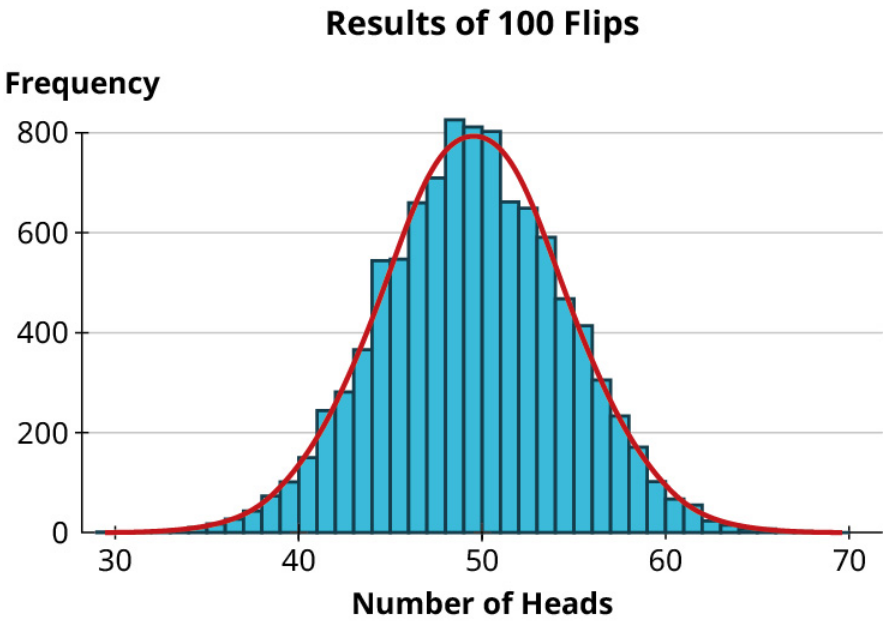



Figure 8.5.5

Because distributions with this shape appear so often, we have a special name for them: normal distributions. Normal distributions can be completely described using two numbers we've seen before: the *mean* of the data and the *standard deviation* of the data. You may remember that we described the mean as a measure of centrality; for a normal distribution, the mean tells us exactly where the center of the distribution falls. The peak of the distribution happens at the mean (and, because the distribution is symmetric, it's also the median). The standard deviation is a measure of dispersion; for a normal distribution, it tells us how spread out the histogram looks. To illustrate these points, let's look at some examples.

**Example 8.5.1: Identifying the Mean of a Normal Distribution**

This graph shows three normal distributions. What are their means?

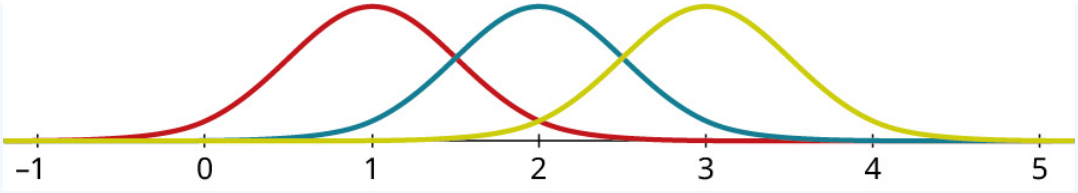



Figure ✔ 8.5.6

**Answer**

**Step 1:** Take a look at the three curves on the graph. Since the mean of a normal distribution occurs at the peak, we should look for the highest point on each distribution. Let's draw a line from each curve's peak down to the axis, so we can see where these peaks occur:

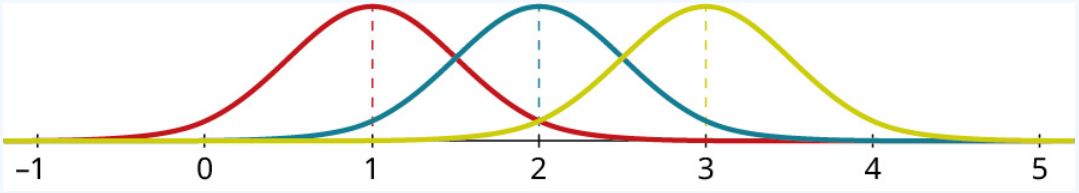



Figure ✔ 8.5.7

Step 2: The peak of the red (leftmost) distribution occurs over the number 1 on the horizontal axis. Thus, the mean of the red distribution is 1. Similarly, the mean of the blue (middle) distribution is 2, and the mean of the yellow (rightmost) distribution is 3.

## Your Turn 8.5.1

Identify the means of these three distributions:

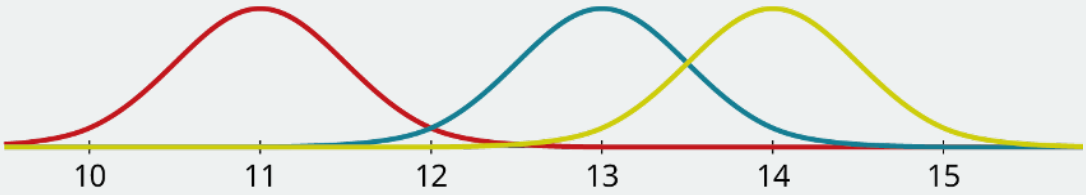



Figure 📌 8.5.8

## ✔ Example 8.5.2: Identifying the Standard Deviation of a Normal Distribution

This graph shows three distributions, all with mean 2. What are their standard deviations?

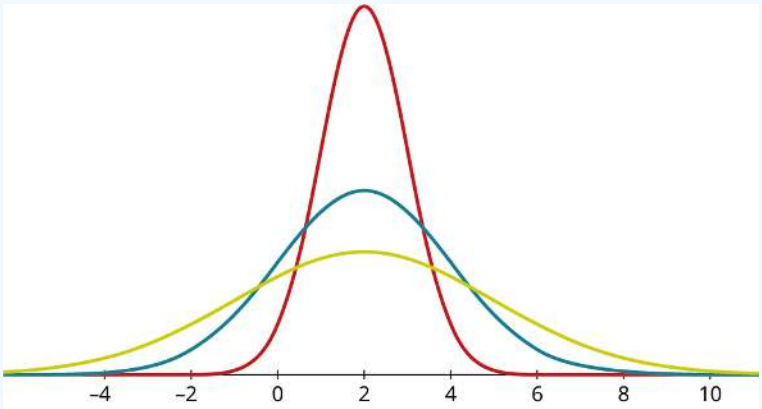



Figure ✔ 8.5.9

**Answer**

**Step 1:** Identifying the standard deviation from a graph can be a little bit tricky. Let's focus in on the yellow (lowest peaked) curve:

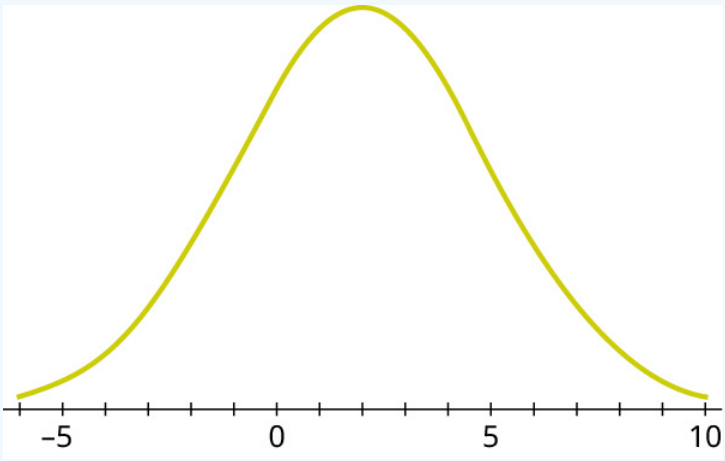



Figure ✔ 8.5.10

**Step 2:** Notice that the graph curves downward in the middle, and curves upwards on the ends. Highlighted in red is the part that curves downward and in green, the part that curves upward:

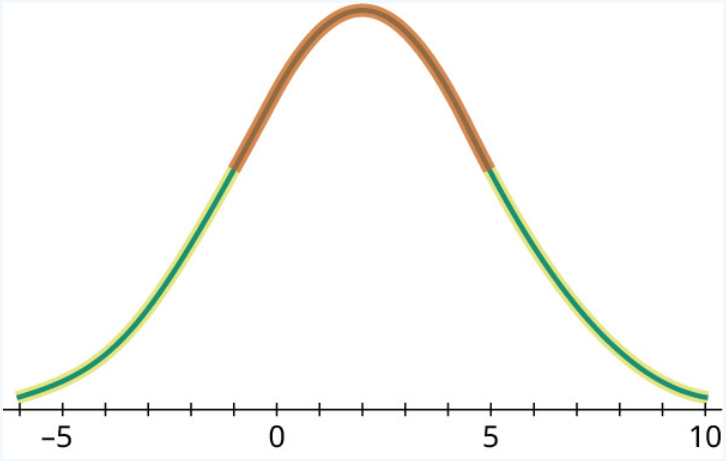



Figure ✔ 8.5.11

**Step 3:** The places where the graph changes from curving up to curving down (or vice versa) are called *inflection points*. Let's identify where those occur by dropping a line straight down from each:

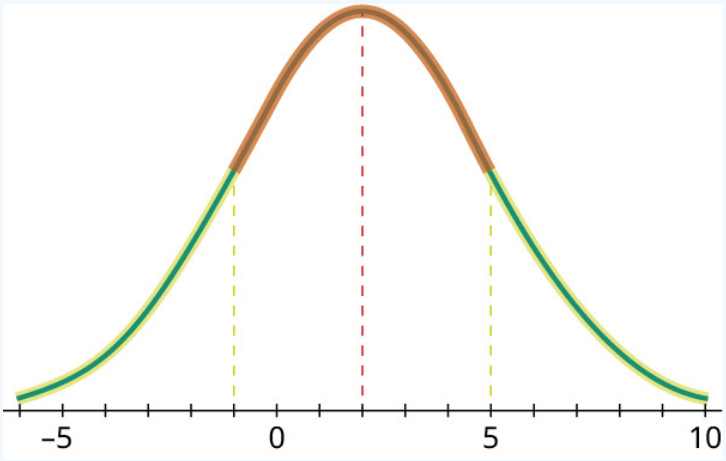



Figure ✔ 8.5.12

**Step 4:** We can estimate that the inflection points occur at $x = -1$ and $x = 5$; the mean is at $x = 2$ (as shown by the middle dotted line). The difference between the mean and the location of either inflection point is the standard deviation; since $5 - 2 = 2 - (-1) = 3$ , we conclude that the standard deviation of the green distribution is 3.

**Step 5:** Now, looking at the other two graphs, let's first identify the inflection points:

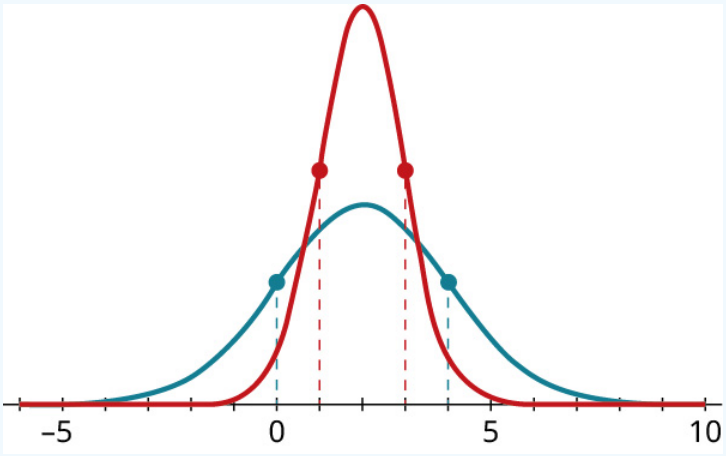



Figure ✔ 8.5.13

**Step 6:** The red (tallest peaked) distribution has inflection points at 1 and 3, and the mean is 2. Thus, the standard deviation of the red distribution is 3 - 2 = 2 - 1 = 1 3 - 2 = 2 - 1 = 1 . The blue (lower peaked) distribution has inflection points at 0 and 4, and its mean is also 2. So, the standard deviation of the blue distribution is 2.

## Your Turn 8.5.2

Estimate the standard deviations of this normal distribution, centered at 5:

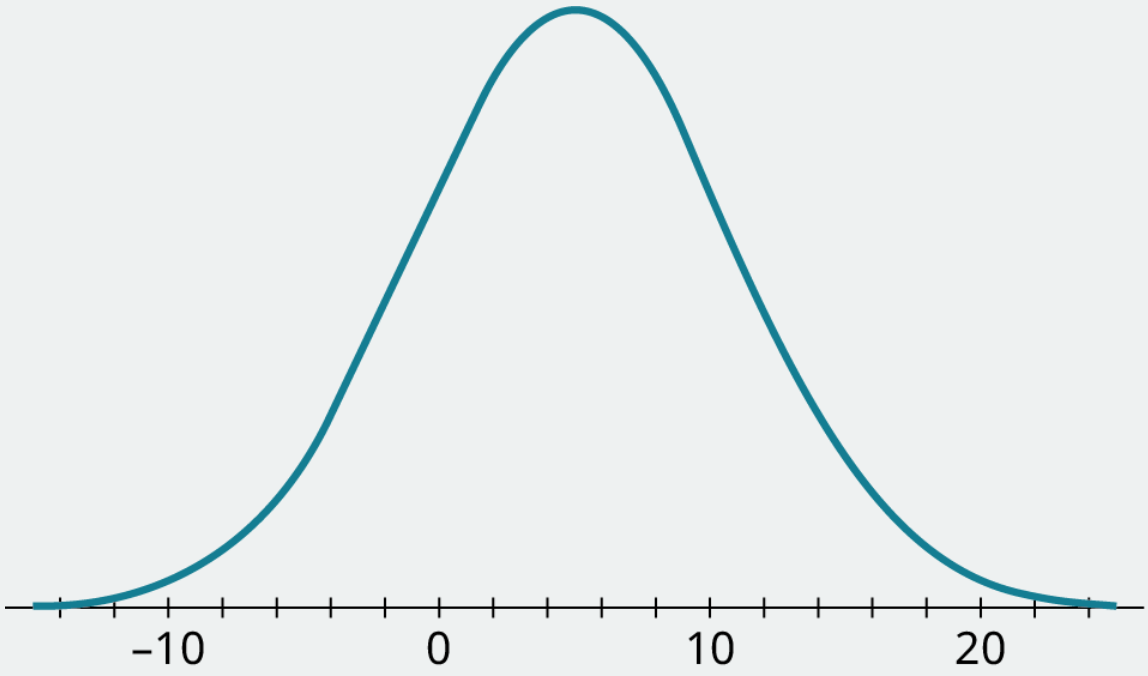



Figure 📌 8.5.14

Let’s put it all together to identify a completely unknown normal distribution.

## Example 8.5.3: Identifying the Mean and Standard Deviation of a Normal Distribution

Using the graph, identify the mean and standard deviation of the normal distribution.

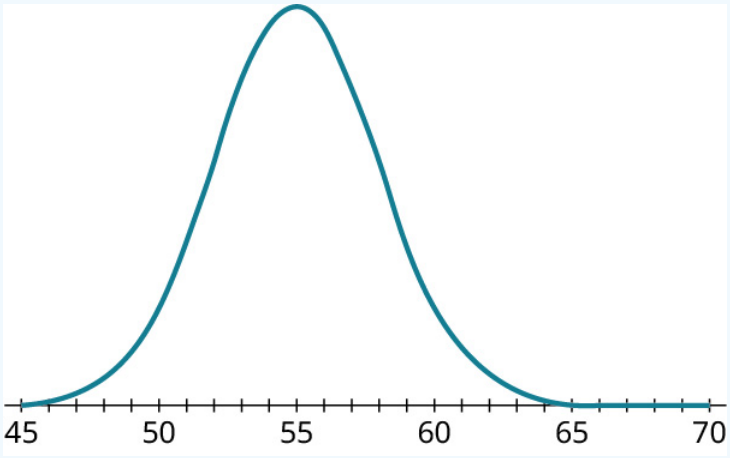



Figure 8.5.15

**Answer**

**Step 1:** Let's start by putting dots on the graph at the peak and at the inflection points, then drop lines from those points straight down to the axis:

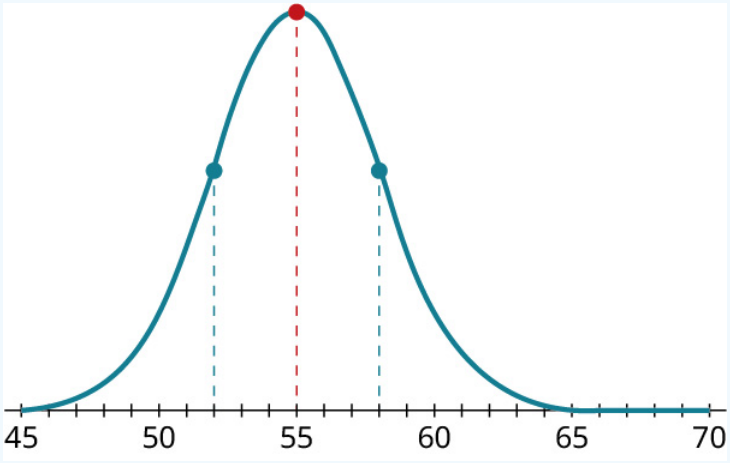



Figure 8.5.16

Step 2: From the red (middle) line, we can see that the mean of this distribution is 55. The blue (outermost) lines are each 3 units away from the mean (at 52 and 58), so the standard deviation is 3.

## Your Turn 8.5.3

Identify the mean and standard deviation of this distribution. Anything within 5 on the standard deviation is acceptable.

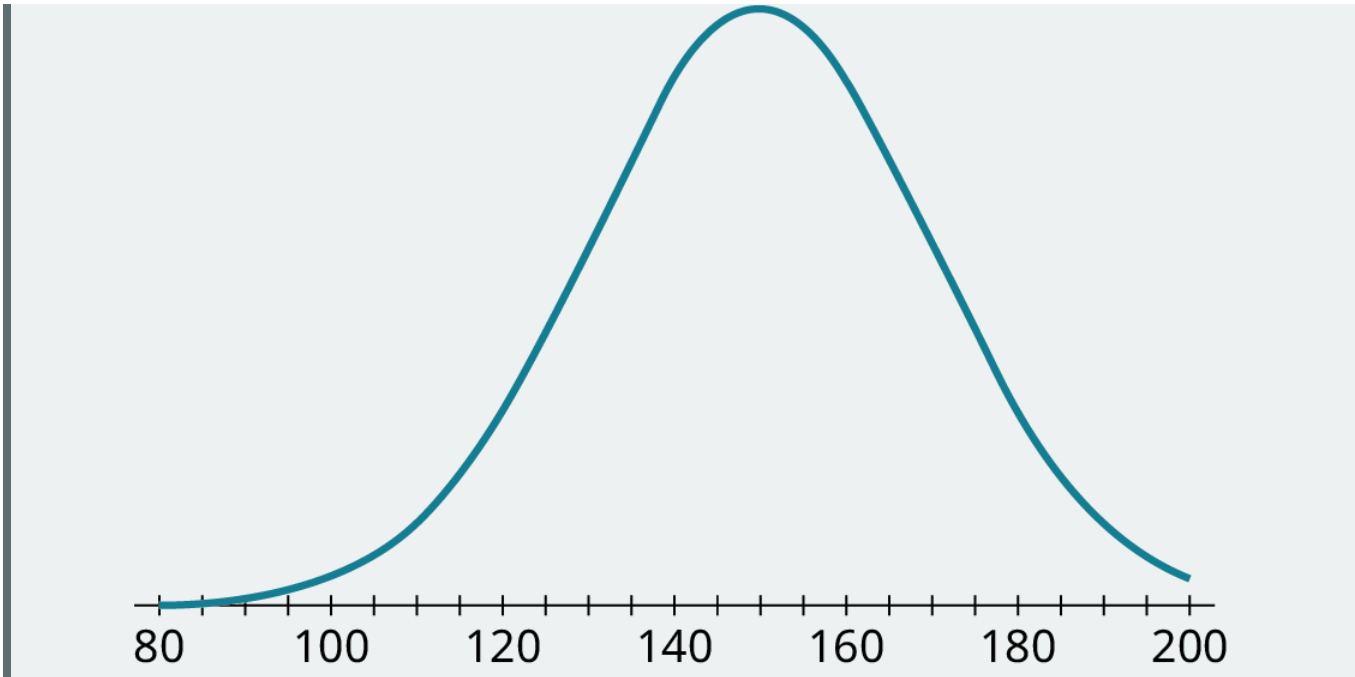



Figure 8.5.17

## Properties of Normal Distributions: The 68-95-99.7 Rule

The most important property of normal distributions is tied to its standard deviation. If a dataset is perfectly normally distributed, then 68% of the data values will fall within one standard deviation of the mean. For example, suppose we have a set of data that follows the normal distribution with mean 400 and standard deviation 100. This means 68% of the data would fall between the values of 300 (one standard deviation *below* the mean: $400-100=300$) and 500 (one standard deviation *above* the mean: $400+100=500$). Looking at the histogram below, the shaded area represents 68% of the total area under the graph and above the axis:

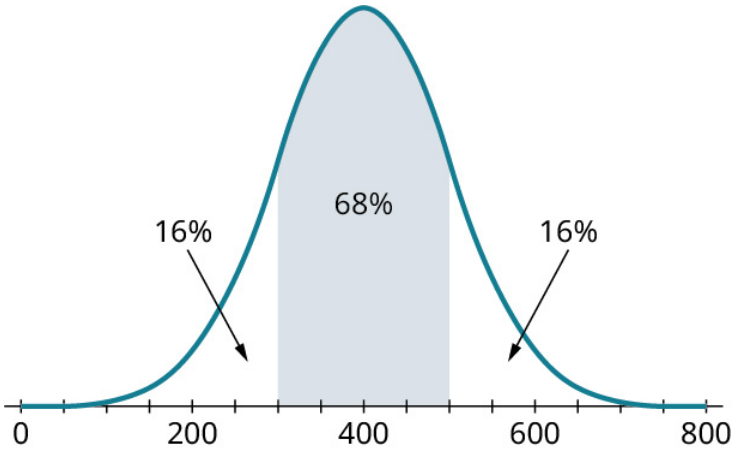



Figure 8.5.18

Since 68% of the area is in the shaded region, this means that $100\%-68\%=32\%$ of the area is found in the unshaded regions. We know that the distribution is symmetric, so that 32% must be divided evenly into the two unshaded tails: 16% in each.

Of course, datasets in the real world are never perfect; when dealing with actual data that seem to follow a symmetric, bell-shaped distribution, we'll give ourselves a little bit of wiggle room and say that *approximately* 68% of the data fall within one standard deviation of the mean.

The rule for one standard deviation can be extended to two standard deviations. Approximately 95% of a normally distributed dataset will fall within 2 standard deviations of the mean. If the mean is 400 and the standard deviation is 100, that means 95%

calculation describes the way we compute standardized scores. (two standard deviations *below* the mean: $400 - 2 \times 100 = 200$) and 600 (two standard deviations *above* the mean: $400 + 2 \times 100 = 600$). We can visualize this in the following histogram:

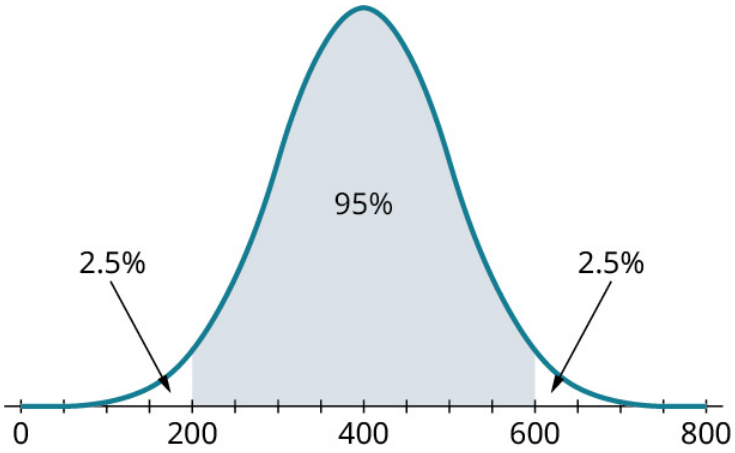



Figure 8.5.19

As before, since 95% of the data are in the shaded area, that leaves 5% of the data to go into the unshaded tails. Since the histogram is symmetric, half of the 5% (that's 2.5%) is in each.

We can even take this one step further: 99.7% of normally distributed data fall within 3 standard deviations of the mean. In this example, we'd see 99.7% of the data between 100 (calculated as $400 - 3 \times 100 = 100$) and 700 (calculated as $400 + 3 \times 100 = 700$). We can see this in the histogram below, although you may need to squint to find the unshaded bits in the tails!

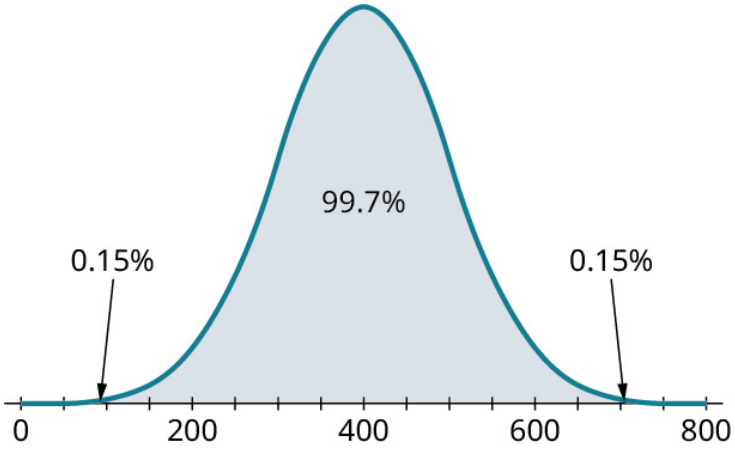



Figure 8.5.20

This observation is formally known as the 68-95-99.7 Rule.

1. If data are normally distributed with mean 8 and standard deviation 2, what percent of the data falls between 4 and 12?
2. If data are normally distributed with mean 25 and standard deviation 5, what percent of the data falls between 20 and 30?
3. If data are normally distributed with mean 200 and standard deviation 15, what percent of the data falls between 155 and 245?

**Answer**

1. **Let's look at a table that sets out the data values that are even multiples of the standard deviation (SD) above and below the mean:**

| mean−3×SDmean-3×SD | mean−2×SDmean-2×SD | mean−1×SDmean-1×SD | Mean | mean+1×SDmean+1×SD | mean+2×SDmean+2×SD | mean+3×SDmean+3×SD |
|---|---|---|---|---|---|---|
| **2** | **4** | **6** | **8** | **10** | **12** | **14** |

**Since 4 and 12 represent two standard deviations above and below the mean, we conclude that 95% of the data will fall between them.**

- Let's build another table:

| mean−3×SDmean-3×SD | mean−2×SDmean-2×SD | mean−1×SDmean-1×SD | Mean | mean+1×SDmean+1×SD | mean+2×SDmean+2×SD | mean+3×SDmean+3×SD |
|---|---|---|---|---|---|---|
| 10 | 15 | 20 | 25 | 30 | 35 | 40 |

We can see that 20 and 30 represent one standard deviation above and below the mean, so 68% of the data fall in that range.

- Let's make one more table:

| mean−3×SDmean-3×SD | mean−2×SDmean-2×SD | mean−1×SDmean−1×SD | Mean | mean+1×SDmean+1×SD | mean+2×SDmean+2×SD | mean+3×SDmean+3×SD |
|---|---|---|---|---|---|---|
| 155 | 170 | 185 | 200 | 215 | 230 | 245 |

Since 155 and 245 are three standard deviations above and below the mean, we know that 99.7% of the data will fall between them.

**Your Turn 8.5.4**

If data are distributed normally with mean 0 and standard deviation 3, what percent of the data fall between –9 and 9?

If data are distributed normally with mean 50 and standard deviation 10, what percent of the data fall between 30 and 70?

If data are distributed normally with mean 60 and standard deviation 5, what percent of the data fall between 55 and 65?

1. If data are distributed normally with mean 100 and standard deviation 20, between what two values will 68% of the data fall?
2. If data are distributed normally with mean 0 and standard deviation 15, between what two values will 95% of the data fall?
3. If data are distributed normally with mean 14 and standard deviation 2, between what two values will 99.7% of the data fall?

**Answer**

1. **The 68-95-99.7 Rule tells us that 68% of the data will fall within one standard deviation of the mean. So, to find the values we seek, we'll add and subtract one standard deviation from the mean:** $100-1\times 20=80$ **and** $100+1\times 20=120$**. Thus, we know that 68% of the data fall between 80 and 120.**
2. **Using the 68-95-99.7 Rule again, we know that 95% of the data will fall within 2 standard deviations of the mean. Let's add and subtract two standard deviations from that mean:** $0-2\times 15=-30$ **and** $0+2\times 15=30$**. So, 95% of the data will fall between -30 and 30.**
3. **Once again, the 68-95-99.7 Rule tells us that 99.7% of the data will fall within three standard deviations of the mean. So, let's add and subtract three standard deviations from the mean: and** $14+3\times 2=20$**. Thus, we conclude that 99.7% of the data will fall between 8 and 20.**

**Your Turn 8.5.5**

1. If data are distributed normally with mean 70 and standard deviation 5, between what two values will 68% of the data fall?
2. If data are distributed normally with mean 40 and standard deviation 7, between what two values will 95% of the data fall?
3. If data are distributed normally with mean 200 and standard deviation 30, between what two values will 99.7% of the data fall?

There are more problems we can solve using the 68-95-99.7 Rule. but first we must understand what the rule implies. Remember, the rule says that 68% of the data falls within one standard deviation of the mean. Thus, with normally distributed data with mean 100 and standard deviation 10, we have this distribution:

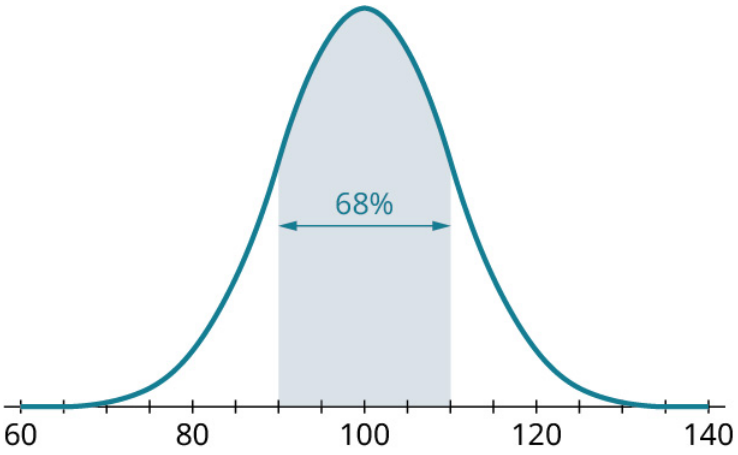



Figure 8.5.21

Since we know that 68% of the data lie within one standard deviation of the mean, the implication is that 32% of the data must fall *beyond* one standard deviation away from the mean. Since the histogram is symmetric, we can conclude that half of the 32% (or 16%) is more than one standard deviation *above* the mean and the other half is more than one standard deviation *below* the mean:

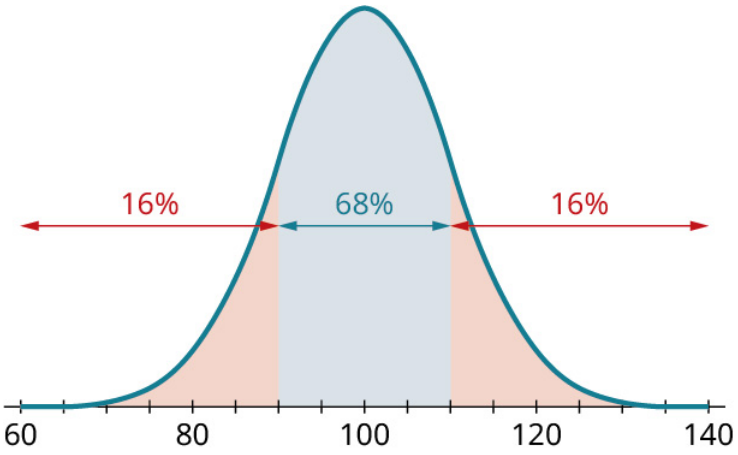



Figure 8.5.22

Further, we know that the middle 68% can be split in half at the peak of the histogram, leaving 34% on either side:

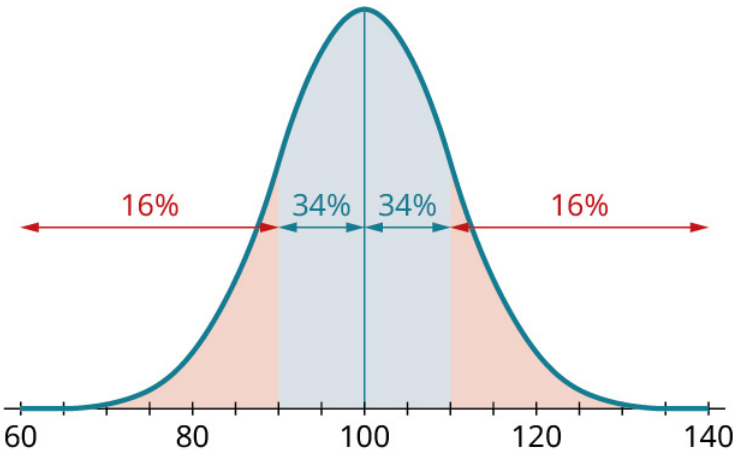



Figure 8.5.23

So, just the “68” part of the 68-95-99.7 Rule gives us four other proportions in addition to the 68% in the rule. Similarly, the “95” and “99.7” parts each give us four more proportions:

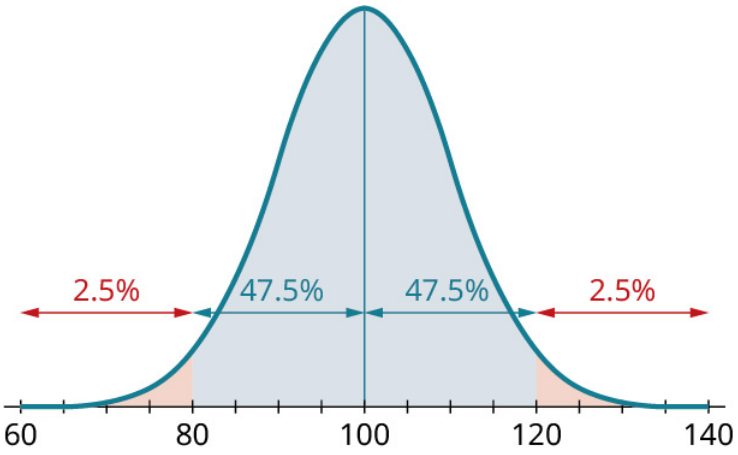



Figure 8.5.24

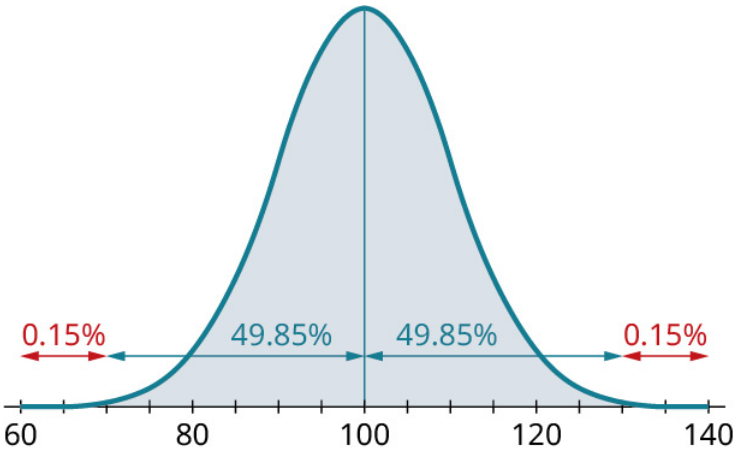



Figure 8.5.25: Copy and Paste Caption here. (Copyright; author via source)

We can put all these together to find even more complicated proportions. For example, since the proportion between 100 and 120 is 47.5% and the proportion between 100 and 110 is 34%, we can subtract to find that the proportion between 110 and 120 is $47.5 - 34 = 13.5\,\%$:

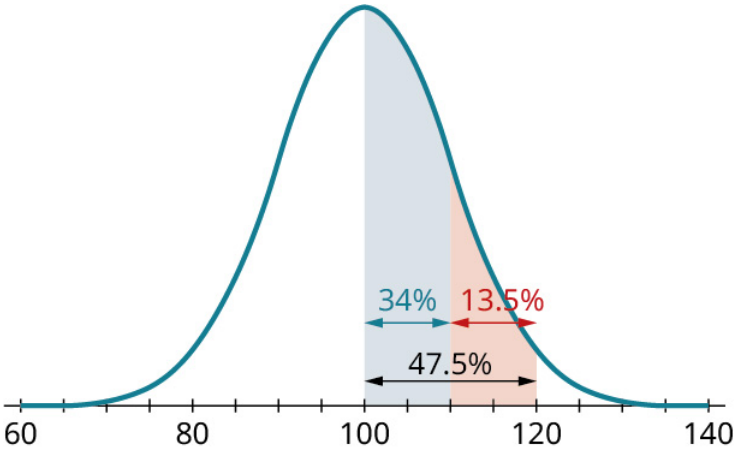



Figure 8.5.26

Assume that we have data that are normally distributed with mean 80 and standard deviation 3.

1. What proportion of the data will be greater than 86?
2. What proportion of the data will be between 74 and 77?
3. What proportion of the data will be between 74 and 83?

**Answer**

**Before we can answer these questions, we must mark off sections that are multiples of the standard deviation away from the mean:**

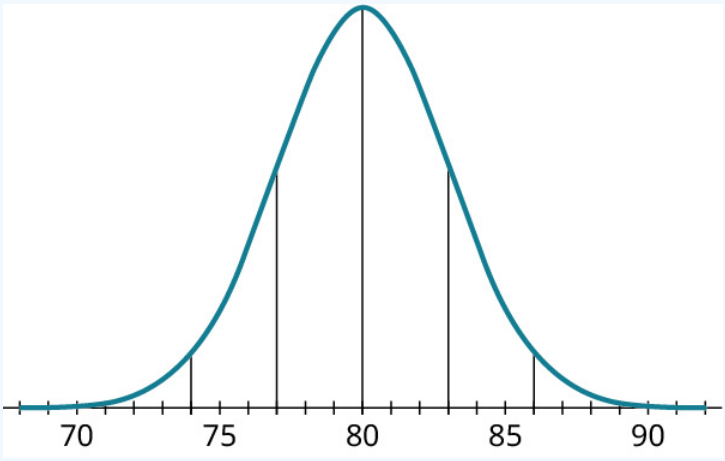



**Figure ✔** 8.5.27

1. To figure out what proportion of the data will be greater than 86, let's start by shading in the area of data that are above 86 in our figure, or the data more than two standard deviations above the mean.

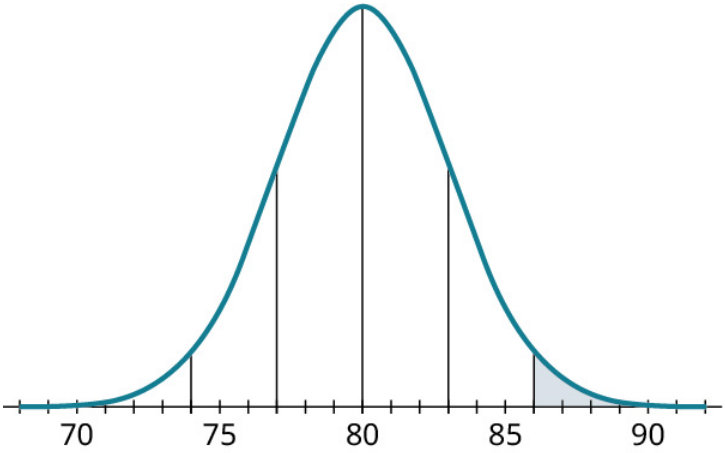



Figure 8.5.28

We saw in Figure 8.51 that this proportion is 2.5%.

2. To figure out what proportion of the data will be between 74 and 77, let's start by shading in that area of data. These are data that are more than one but less than two standard deviations below the mean.

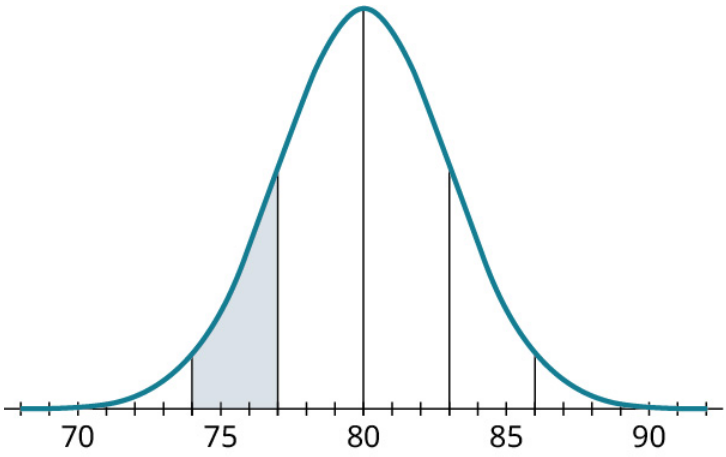



Figure 8.5.29

From Figure 8.51, we know that the proportion of data less than two standard deviations below the mean is 47.5%. And, from YOUR TURN 8.33, we know that 34% of the data is less than one standard deviation below the mean:

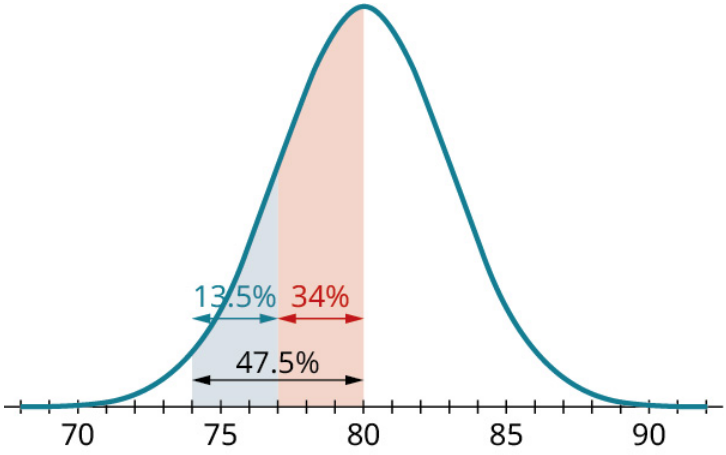



Figure 8.5.30

Subtracting, we see that the proportion of data between 74 and 77 is 13.5%.

3. To figure out what proportion of the data will be between 74 and 83, let's start by shading in that area of data in our figure.

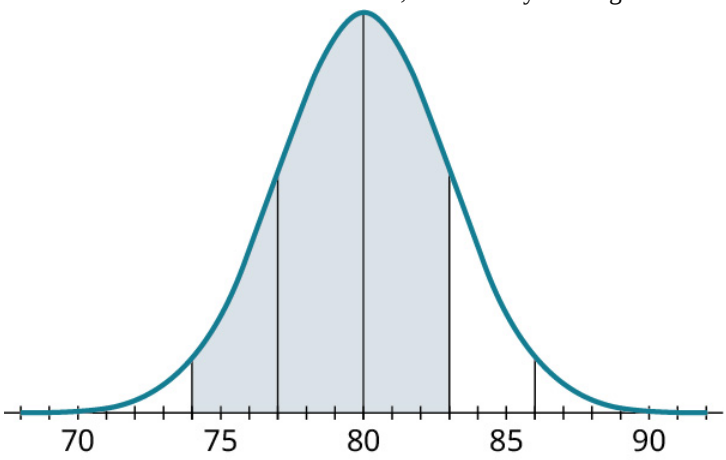



Figure 8.5.31

Next, we'll break this region into two pieces at the mean:

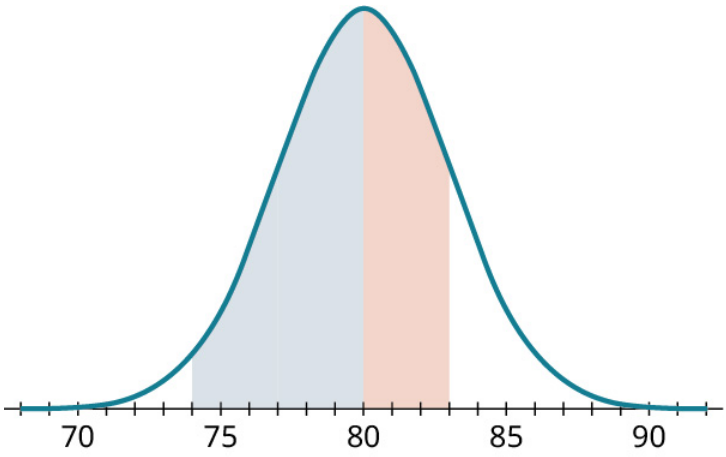



Figure 8.5.32

From Figure 8.51, we know the blue (leftmost) region represents 47.5% of the data. And, using YOUR TURN 8.33, we get that the red (rightmost) region covers 34% of the data. Adding those together, the proportion we want is 81.5%.

**Your Turn 8.5.6**

Suppose we have data that are normally distributed with mean 500 and standard deviation 100. What proportions of the data fall in these ranges?

300 to 500

600 to 800

400 to 700

## Standardized Scores

When we want to apply the 68-95-99.7 Rule, we must first figure out how many standard deviations above or below the mean our data fall. This calculation is common enough that it has its own name: the standardized score. Values above the mean have positive standardized scores, while those below the mean have negative standardized scores. Since it's common to use the letter $z$ to represent a standard score, this value is also often referred to as a ***z*-score**.

So far, we've only really considered $z$-scores that are whole numbers, but in general they can be any number at all. For example, if we have data that are normally distributed with mean 80 and standard deviation 6, the value 85 is five units above the mean, which is less than one standard deviation. Dividing by the standard deviation, we get $\frac{5}{6}$. Since 85 is $\frac{5}{6}$ of one standard deviation above the mean, we'd say that the standardized score for 85 is $z = 56$z=56 (which is positive, since $85 > 80$). This calculation describes the way we compute standardized scores.

**FORMULA**

If $x$ is a member of a normally distributed dataset with mean $\mu$ and standard deviation $\sigma$, then the standardized score for $x$ is

$$z = \frac{x - \mu}{\sigma}.$$

If you know a $z$-score but not the original data value $x$, you can find it by solving the previous equation for $x$ :

$$x = \mu + z \times \sigma.$$

z×σ.

**Checkpoint**

The symbols μμ and σσ are the Greek letters *mu* and *sigma*. They are the analogues of the English letters $m$ and $s$, which stand for *mean* and *standard deviation*.

If you convert every data value in a dataset into its $z$-score, the resulting set of data will have mean 0 and standard deviation 1. This is why we call these *standardized scores*: the normal distribution with mean 0 and standard deviation 1 is often called the *standard normal distribution*.

**Example 8.5.7: Standardizing Data**

Suppose we have data that are normally distributed with mean 50 and standard deviation 6. Compute the standardized scores (rounded to three decimal places) for these data values:

1. 52
2. 40
3. 68

**Answer**

For each of these, we'll plug the given values into the formula. Remember, the mean is $\mu = 50$ and the standard deviation is $\sigma = 6$ :

1. $z = \frac{x-\mu}{\sigma} = \frac{52-50}{6} = 0.333$
2. $z = \frac{x-\mu}{\sigma} = \frac{40-50}{6} = -1.667$
3. $z = \frac{x-\mu}{\sigma} = \frac{68-50}{6} = 3$

**Your Turn 8.5.7**

Suppose we have data that are normally distributed with mean 75 and standard deviation 5. Compute standardized scores for each of these data values: 66, 83, and 72.

**Example 8.5.8: Converting Standardized Scores to Original Values**

Suppose we have data that are normally distributed with mean 10 and standard deviation 2. Convert the following standardized scores into data values.

1. 1.4
2. −0.9
3. 3.5

**Answer**

We'll use the formula previously introduced to convert $z$-scores into $x$-values. In this case, the mean is $\mu = 10$ and the standard deviation is $\sigma = 2$ :

1. $x = \mu + z \times \sigma = 10 + 1.4 \times 2 = 12.8$
2. $x = \mu + z \times \sigma = 10 + (-0.9) \times 2 = 8.2$
3. $x = \mu + z \times \sigma = 10 + 3.5 \times 2 = 17$ 17x=μ+z×σ=10+3.5×2=17

**Your Turn 8.5.8**

Suppose you have a normally distributed dataset with mean 2 and standard deviation 20. Convert these standardized scores to data values: –2.3, 1.4, and 0.2.

## Using Google Sheets to Find Normal Percentiles

The 68-95-99.7 Rule is great when we're dealing with whole-number $z$-scores. However, if the $z$-score is not a whole number, the Rule isn't going to help us. Luckily, we can use technology to help us out. We'll talk here about the built-in functions in Google Sheets, but other tools work similarly.

Let's say we're working with normally distributed data with mean 40 and standard deviation 7, and we want to know at what percentile a data value of 50 would fall. That corresponds to finding the proportion of the data that are less than 50. If we create our histogram and mark off whole-number multiples of the standard deviation like we did before, we'll see why the 68-95-99.7 Rule isn't going to help:

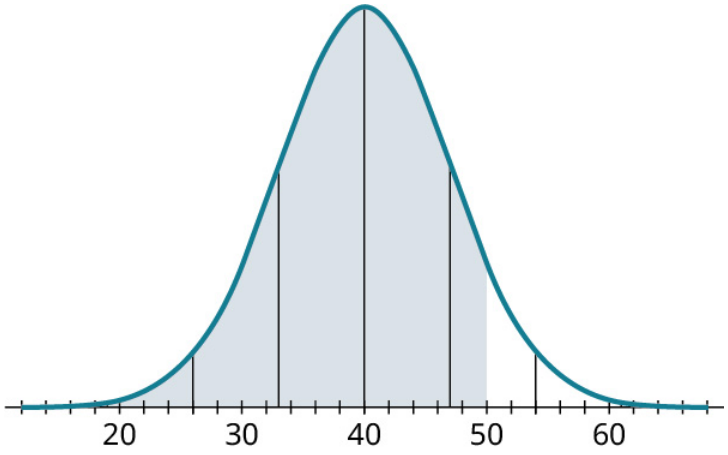



Figure 8.5.33

Since 50 doesn't line up with one of our lines, the 68-95-99.7 Rule fails us. Looking back at Figure 8.51 and Figure 8.52, the best we can say is that 50 is between the 84th and 99.5th percentiles, but that's a pretty wide range. Google Sheets has a function that can help; it's called NORM.DIST. Here's how to use it:

1. Click in an empty cell in your worksheet.
2. Type "=NORM.DIST("
3. Inside the parentheses, we must enter a list of four things, separated by commas: the data value, the mean, the standard deviation, and the word "TRUE". These have to be entered in this order!
4. Close the parentheses, and hit Enter. The result is then displayed in the cell; convert it to a percent to get the percentile.

So, for our example, we should type "=NORM.DIST(50, 40, 7, TRUE)" into an empty cell, and hit Enter. The result is 0.9234362745; converting to a percent and rounding, we can conclude that 50 is at the 92nd percentile. Let's walk through a few more examples.

**Example 8.5.9: Using Google Sheets to Find Percentiles**

Suppose we have data that are normally distributed with mean 28 and standard deviation 4. At what percentile do each of the following data values fall?

1. 30
2. 23
3. 35

**Answer**

1. By entering "=NORM.DIST(30, 28, 4, TRUE)" we find that 30 is at the 69th percentile.
2. By entering "=NORM.DIST(23, 28, 4, TRUE)" we find that 23 is at the 11th percentile.

3. By entering “=NORM.DIST(35, 28, 4, TRUE)” we find that 35 is at the 96th percentile.

### Your Turn 8.5.9

Suppose you have data that are normally distributed with mean 20 and standard deviation 6. Determine at what percentiles these data values fall: 25, 12, and 31.

Google Sheets can also help us go the other direction: If we want to find the data value that corresponds to a given percentile, we can use the NORM.INV function. For example, if we have normally distributed data with mean 150 and standard deviation 25, we can find the data value at the 30th percentile as follows:

1. Click on an empty cell in your worksheet.
2. Type “=NORM.INV(“
3. Inside the parentheses, we’ll enter a list of three numbers, separated by commas: the percentile in question expressed as a decimal, the mean, and the standard deviation. These must be entered in this order!
4. Close the parentheses and hit Enter. The desired data value will be in the cell!

In our example, we want the 30th percentile; converting 30% to a decimal gives us 0.3. So, we’ll type “=NORM.INV(0.3, 150, 25)” to get 136.8899872; let’s round that off to 137.

### Example 8.5.10: Using Google Sheets to Find the Data Value Corresponding to a Percentile

Suppose we have data that are normally distributed with mean 47 and standard deviation 9. Find the data values (rounded to the nearest tenth) corresponding to these percentiles:

1. 75th (that’s the third quartile)
2. 12th
3. 90th

**Answer**

1. By entering “=NORM.INV(0.75, 47, 9)” we find that 53.1 is at the 75th percentile.
2. By entering “=NORM.INV(0.12, 47, 9)” we find that 36.4 is at the 12th percentile.
3. By entering “=NORM.INV(0.9, 47, 9)” we find that 58.5 is at the 90th percentile.

### Your Turn 8.5.10

Suppose we have data that are normally distributed with mean 5 and standard deviation 1.6. Identify which data values (rounded to the nearest tenth) correspond to these percentiles: 25th (the first quartile), 80th, and 10th.

## Check Your Understanding

For each of these problems, assume we’re working with normally distributed data with mean 100 and standard deviation 12.

1. What percentage of the data falls between 76 and 124? Use the 68-95-99.7 Rule.
2. What percentage of the data falls between 100 and 112? Use the 68-95-99.7 Rule.
3. At what percentile does 112 fall? Use the 68-95-99.7 Rule.
4. What’s the $z$-score of the data value 107? Round to three decimal places.
5. What data value’s $z$-score is –2.4?
6. At what percentile does 107 fall? Use Google Sheets (or another technology).
7. What data value is at the 90th percentile? Use Google Sheets (or another technology), and round to the nearest hundredth.

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# 8.6: Applications of the Normal Distribution

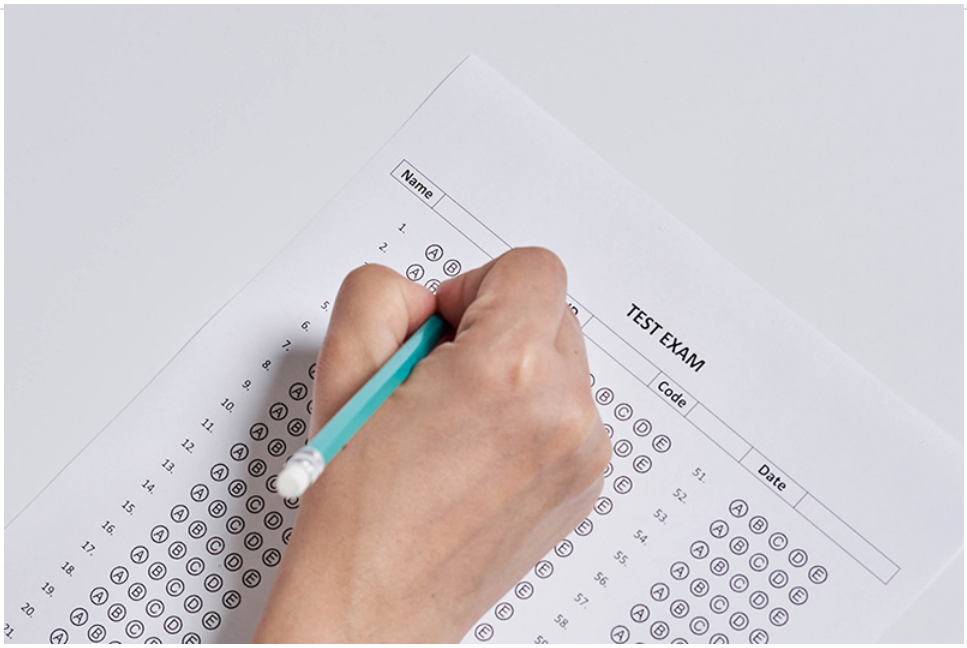



Figure 8.6.1: Standardized test results generally adhere to the normal distribution. (credit: “Taking a Test” by Marco Verch Professional Photographer/Flickr, CC BY 2.0)

**Learning Objectives**

Apply the normal distribution to real-world scenarios.

As we saw in The Normal Distribution, the word “standardized” is closely associated with the normal distribution. This is why tests like college entrance exams, state achievement tests for K–12 students, and Advanced Placement tests are often called “standardized tests”: scores are assigned in a way that forces them to follow a normal distribution, with a mean and standard deviation that are consistent from year to year. Standardization also allows people like college admissions officers to directly compare an applicant who took the ACT (a college entrance exam) to an applicant who instead chose to take the SAT (a different college entrance exam). Standardization allows us to compare individuals from different groups; this is among the most important applications of the normal distribution. We’ll explore this and other real-world uses of the normal distribution in this section.

## College Entrance Exams

There are two good ways to compare two data values from different groups: using $z$-scores and using percentiles. The two methods will always give consistent results (meaning that we won’t find, for example, that the first value is better using $z$-scores but the second value is better using percentiles), so use whichever method is more comfortable for you.

**Example 8.6.1: Evaluating College Entrance Exam Scores**

According to the Digest of Education Statistics, composite scores on the SAT have mean 1060 and standard deviation 195, while composite scores on the ACT have mean 21 and standard deviation 5.

1. At what percentile would an SAT score of 990 fall?
2. What is the $z$-score of an ACT score of 27?
3. Which is better: a score of 1450 on the SAT or 29 on the ACT?

**Answer**

1. Using Google Sheets, we can answer this question with the formula "=NORM.DIST(990, 1060, 195, TRUE)". A score of 990 would fall at the 36 th percentile.

2. Using the formula $z = \frac{x-\mu}{s}$, we get $z = \frac{27-21}{5} = 1.2$.

3. Let's compare the values using both percentiles and $z$-values:

Percentiles: Using "=NORM.DIST(1450, 1060, 195, TRUE)" we find that an SAT score of 1450 is at the 98th percentile. Meanwhile, by entering "=NORM.DIST(29, 21, 5 , TRUE)" we see that an ACT score of 29 is around the 95th percentile. Since it's at a higher percentile, we can conclude that an SAT score of 1450 is better than an ACT score of 29.

$z$-scores: Using the formula, we see that the $z$-score for an SAT score of 1450 is $z = \frac{1450-1060}{195} = 2$ , while the $z$-score for an ACT score of 29 is $z = \frac{29-21}{5} = 1.6$. Since it has a higher $z$-score, an SAT score of 1450 is better than an ACT score of 29.

### Your Turn 8.6.1

According to the Graduate Management Admission Council (GMAC), the mean score on the GMAT (an entrance exam for graduate schools in business management) is 565, with standard deviation 116. For the LSAT (an entrance exam for law schools), Kaplan informs us that the mean score is 150 with standard deviation 10.

1. What is the $z$-score for a GMAT score of 715?
2. At what percentile is an LSAT score of 166?
3. Which is better: a GMAT score of 650 or an LSAT score of 161?

## Coin flipping

In the opening of The Normal Distribution, we saw that the number of heads we get when we flip a coin 100 times is distributed normally. It can be shown that if $n$ is the number of flips, then the mean of that distribution is $\frac{n}{2}$ and the standard deviation is $\frac{\sqrt{n}}{2}$ (as long as $n \geq$). So, for 100 flips, the mean of the distribution is 50 and the standard deviation is 5. In that opening example, one of our early runs gave us 70 heads in 100 flips, which we noted seemed unusual. Using the normal distribution, we can identify exactly how unusual that really is. Using Google Sheets, the formula “=NORM.DIST(70, 50, 5, TRUE)” gives us 0.999968, which is the 99.997th percentile! How is that useful? Suppose you need to test whether a coin is fair, and so you flip it 100 times. While we might be suspicious if we get 70 heads out of the 100 flips, we now have a numerical measure for how unusual that is: If the coin were fair, we would expect to see 70 heads (or more) only $100 - 99.9968 = 0.0032\%$ of the time. That’s really unlikely! Analysis like this is related to *hypothesis testing*, an important application of statistics in the sciences and social sciences.

### Example 8.6.2: Flipping a Coin

Let’s say we flip a coin 64 times and count the number of heads.

1. What would be the mean of the corresponding distribution?
2. What would be the standard deviation of the corresponding distribution?
3. Suppose we got 25 heads, which seems a little low. At what percentile would 25 heads fall?

**Answer**

1. Since $n = 64$, the mean is $\frac{64}{2} = 32$.

2. Again using $n = 64$, we get a standard deviation of $\frac{\sqrt{64}}{2} = 4$.

3. Using "=NORM.DIST(25, 32, 4, TRUE)", we see that 25 heads is at the 4th percentile. Reading and Interpreting Scatter Plots

**Your Turn 8.6.2**

You flip a coin 144 times and count the number of heads.

1. What is the mean of the corresponding distribution?
2. What is the standard deviation of the corresponding distribution?
3. You flipped 81 heads. At what percentile does that fall?

## Analyzing Data That Are Normally Distributed

Whenever we're working with a dataset that has a distribution that looks symmetric and bell-shaped, we can use techniques associated with the normal distribution to analyze the data.

**Example 8.6.3: Using Normal Techniques to Analyze Data**

The data in "AvgSAT" contains the average SAT score for students attending every institution of higher learning in the United States for which data is available. We previously created a histogram for these data:

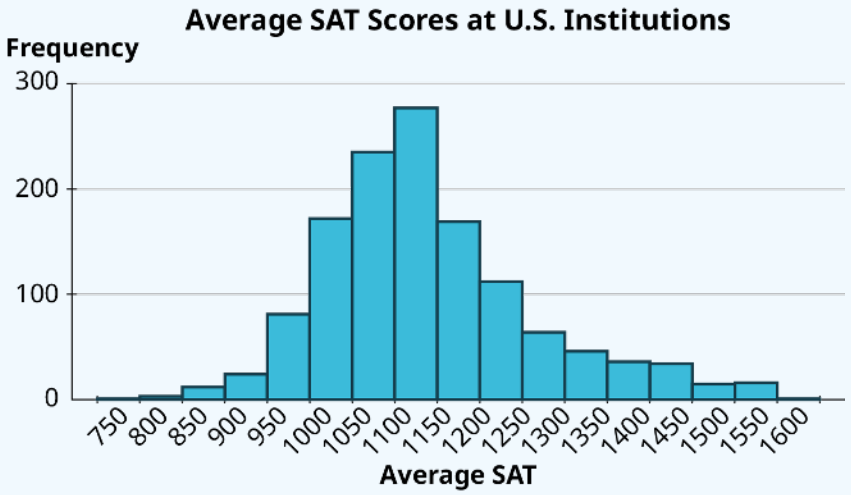



Figure 8.6.2 (data source: https://data.ed.gov)

This distribution is fairly symmetric (it's just a little right-skewed) and bell-shaped, so we can use normal distribution techniques to analyze the data.

1. What is the mean of these average SAT scores?
2. What is the standard deviation of these SAT scores?
3. Using the answers to the previous two questions, use NORM.DIST in Google Sheets to estimate at what percentile the University at Buffalo in New York (average SAT: 1250) falls.
4. Use PERCENTRANK to find the actual percentile of the University at Buffalo, and see how close the estimate in the previous question came.

**Answer**

1. Using the AVERAGE function in Google Sheets, we find that the mean is 1141.174.
2. Using the STDEV function, we get that the standard deviation is 125.517.
3. Entering "=NORM.DIST(1250, 1141, 125.517, TRUE)" into Google Sheets, we estimate that the University at Buffalo is at the 81st percentile.
4. Using PERCENTRANK, we find that the actual percentile is the 84th. These are close!

### Your Turn 8.6.3

Again using the data in "AvgSAT", find the average SAT score of a school at the 35th percentile in two ways: using NORM.INV and using PERCENTILE.

### Who Knew?: Political Meddling Exposed

The normal distribution pops up in some unusual places. Recently, a team at Duke University has been using statistics to help identify partisan gerrymandering, where electoral districts have been carefully drawn in a way that benefits one political party over another. In their analysis, they found that hypothetical election results in randomly drawn districts are normally distributed. By using techniques similar to the ones we used above, they can quantify precisely how biased a particular electoral map is by finding the percentile rank of the actual election result on the normal distribution of the hypothetical results. You can find out more about their work at the "Quantifying Gerrymandering" site here.

## Check Your Understanding

For the following problems, recall that the SAT exam has mean 1060 and standard deviation 195, and that composite scores on the ACT have mean 21 and standard deviation 5.

1. At what percentile would an SAT score of 940 fall? Round to the nearest whole number.
2. What score would be at the 67th percentile on the ACT?
3. Which is a better score: 1300 on the SAT or 27 on the ACT?

For the following problems, recall that if we flip a coin at least 20 times, the distribution of the number of heads is approximately normal with mean equal to half the number of flips and standard deviation equal to half of the square root of the number of flips.

4. Suppose we flip a coin 120 times. What are the mean and standard deviation of the corresponding distribution of heads?
5. Let's say you flip 70 heads in 120 flips. At what percentile would that fall?
6. How many heads would be at the 30th percentile? Round to the nearest whole number.

For the following problems, use the data in "World Tax," which gives the tax revenue of many countries of the world in 2017, expressed as a percentage of their gross domestic products. You can assume that the distribution is approximately normal (but you can make a histogram to check, if you want).

7. What tax revenue percentage falls at the third quartile? Answer this question using Google Sheets in two ways: using PERCENTILE and using NORM.INV.
8. What tax revenue percentage falls at the 20th percentile? Answer this question using Google Sheets in two ways: using PERCENTILE and using NORM.INV.
9. At what percentile does the United Kingdom (25.62%, found on row 46 in the spreadsheet) fall? Answer this question using Google Sheets in two ways: using PERCENTRANK and using NORM.DIST.
10. At what percentile does Kiribati (21.97%, found on row 62 in the spreadsheet) fall? Answer this question using Google Sheets in two ways: using PERCENTRANK and using NORM.DIST.

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# 8.7: Chapter Review

## SECTION 9.5 PROBLEM SET: CHAPTER REVIEW

1. A coin is tossed five times. Find the following
   a. P(2 heads and 3 tails)
   b. P(at least 4 tails)
2. A dandruff shampoo helps 80% of the people who use it. If 10 people apply this shampoo to their hair, what is the probability that 6 will be dandruff free?
3. A baseball player has a .250 batting average. What is the probability that he will have 2 hits in 4 times at bat?
4. Suppose that 60% of the voters in California intend to vote Democratic in the next election. If we choose five people at random, what is the probability that at least four will vote Democratic?
5. A basketball player has a .70 chance of sinking a basket on a free throw. What is the probability that he will sink at least 4 baskets in six shots?
6. During an archery competition, Stan has a 0.8 chance of hitting a target. If he shoots three times, what is the probability that he will hit the target all three times?
7. A company finds that one out of four new applicants overstate their work experience. If ten people apply for a job at this company, what is the probability that at most two will overstate their work experience?
8. A missile has a 70% chance of hitting a target. How many missiles should be fired to make sure that the target is destroyed with a probability of .99 or more?
9. Jar I contains 4 red and 5 white marbles, and Jar II contains 2 red and 4 white marbles. A jar is picked at random and a marble is drawn. Draw a tree diagram and find,
   a. P(Marble is red)
   b. P(It is white given that it came from Jar II)
   c. P(It came from Jar II knowing that the marble drawn is white)
10. Suppose a test is given to determine if a person is infected with HIV. If a person is infected with HIV, the test will detect it in 90% of the cases; and if the person is not infected with HIV, the test will show a positive result 3% of the time. If we assume that 2% of the population is actually infected with HIV, what is the probability that a person obtaining a positive result is actually infected with HIV?
11. A car dealer's inventory consists of 70% cars and 30% trucks. 20% of the cars and 10% of the trucks are used vehicles. If a vehicle chosen at random is used, find the probability that it is a car.
12. Two machines make all the products in a factory, with the first machine making 30% of the products and the second 70%. The first machine makes defective products 3% of the time and the second machine 5% of the time.
    a. Overall what percent of the products made are defective?
    b. If a defective product is found, what is the probability that it was made on the second machine?
    c. If it was made on the second machine, what is the probability that it is defective?
13. An instructor in a finite math course estimates that a student who does his homework has a 90% of chance of passing the course, while a student who does not do the homework has only a 20% chance of passing the course. It has been determined that 60% of the students in a large class do their homework.
    a. What percent of all the students will pass?
    b. If a student passes, what is the probability that he did the homework?
14. Cars are produced at three factories. Factory I produces 10% of the cars and it is known that 2% are defective. Factory II produces 20% of the cars and 3% are defective. Factory III produces 70% of the cars and 4% of those are defective. A car is chosen at random. Find the following probabilities:
    a. P(The car is defective)
    b. P(The car came from Factory III | the car is defective)
15. A stock has a 50% chance of a 10% gain, a 30% chance of no gain, and otherwise it will lose 8%. Find the expected return.
16. A game involves rolling a pair of dice. One receives the sum of the face value of both dice in dollars. How much should one be willing to pay to roll the dice to make the game fair?
17. A roulette wheel consists of numbers 1 through 36, 0, and 00. If the wheel shows an odd number you win a dollar, otherwise you lose a dollar. If you play the game ten times, what is your expectation?

18. A student takes a 100-question multiple-choice exam in which there are four choices to each question. If the student is just guessing the answers, what score can he expect?
19. Mr. Shaw invests 50% of his money in stocks, 30% in mutual funds, and the remaining 20% in bonds. If the annual yield from stocks is 10%, from mutual funds 12%, and from bonds 7%, what percent return can Mr. Shaw expect on his money?
20. An insurance company is planning to insure a group of surgeons against medical malpractice. Its research shows that two surgeons in every fifteen are involved in a medical malpractice suit each year where the average award to the victim is $450,000. How much minimum annual premium should the insurance company charge each doctor?
21. In an evening finite math class of 30 students, it was discovered that 5 students were of age 20, 8 students were about 25 years old, 10 students were close to 30, 4 students were 35, 2 students were 40 and one student 55. What is the average age of a student in this class?
22. Jar I contains 4 marbles of which one is red, and Jar II contains 6 marbles of which 3 are red. Katy selects a jar and then chooses a marble. If the marble is red, she gets paid 3 dollars, otherwise she loses a dollar. If she plays this game ten times, what is her expected payoff?
23. Jar I contains 1 red and 3 white, and Jar II contains 2 red and 3 white marbles. A marble is drawn from Jar I and put in Jar II. Now if one marble is drawn from Jar II, what is the probability that it is a red marble?
24. Let us suppose there are three traffic lights between your house and the school. The chance of finding the first light green is 60%, the second 50%, and the third 30%. What is the probability that on your way to school, you will find at least two lights green?
25. Sonya has just earned her law degree and is planning to take the bar exam. If her chance of passing the bar exam is 65% on each try, what is the probability that she will pass the exam in at least three tries?
26. Every time a particular baseball player is at bat, his probability of getting a hit is .3, his probability of walking is .1, and his probability of being struck out is .4. If he is at bat three times, what is the probability that he will get two hits and one walk?
27. Jar I contains 4 marbles of which none are red, and Jar II contains 6 marbles of which 4 are red. Juan first chooses a jar and then from it he chooses a marble. After the chosen marble is replaced, Mary repeats the same experiment. What is the probability that at least one of them chooses a red marble?
28. Andre and Pete are two tennis players with equal ability. Andre makes the following offer to Pete: We will not play more than four games, and anytime I win more games than you, I am declared a winner and we stop. Draw a tree diagram and determine Andre's probability of winning.

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